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CONSTRAINED VISCOSITY SOLUTION TO THE HJB EQUATION ARISING IN PERPETUAL AMERICAN EMPLOYEE STOCK OPTIONS PRICING

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Abstract. We consider the valuation of a block of perpetual ESOs and the optimal exercise decision for an employee endowed with them and with trading restrictions. A fluid model is proposed to characterize the exercise process. The objective is to maximize the overall discount returns for the employee through exercising the options over time. The optimal value function is defined as the grant-date fair value of the block of options, and is then shown by the dynamic programming principle to be a continuous constrained viscosity solution to the associated Hamilton-Jacobi-Bellman (HJB) equation, which is a fully nonlinear second order elliptic partial differential equation (PDE) in the plane. We prove the comparison principle and the uniqueness. The numerical simulation is discussed and the corresponding optimal decision turns out to be a threshold-style strategy. These results provide an appropriate method to estimate the cost of the ESOs for the company and also offer favorable suggestions on selecting right moments to exercise the options over time for the employee.

1. **Introduction.** In recent years, employee stock options (ESOs) have been extensively used by companies as a form of compensation or reward to the employees globally. An ESO is usually a call option issued by a company on its common stock, granting the holder the right to buy a certain number of shares of the underlying stock at a predetermined price, called the strike price, during a certain period of time. In most cases, this period lasts several years. When the stock price goes up, the holder can exercise the options to buy the stock at the strike price and then sell it at the market price, thereby keeping the difference as profit. Obviously ESOs serve as the incentive to the employees, encouraging them to strive for the benefits of the company, boosting the stock price so that they can get more profit from exercising these options.

With the cost of ESOs becoming increasingly significant to the companies in the past decades, since 2004 it has been required by the Financial Accounting

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Standards Board that all the companies should estimate and report the grant-date fair value of ESOs issued, which gives rise to the desire for a reasonable method to evaluate ESOs. Meanwhile the employees need directions in exercising so as to make the maximal profits. Consequently the discussion about the valuation and related optimal strategy has become a focus in mathematical research of finance, thereby covered by an extensive literature.

Furthermore, it's worth pointing out that compared to standardized exchange-traded options, ESOs have several unique features in different aspects (see [10]). In general, ESOs are American-style call options, i.e., they can be exercised at any time before expiration, with a long maturity ranging from 5 to 10 years, which much exceeds that of standardized options. In addition, for the most part ESOs involve a vesting period from the grant date, during which employees are prohibited from exercising any of the options, in order to maintain their incentive effect for the financial benefits of the company. On top of that, the transfer and hedging restrictions are also remarkable features which need handling with care. In most cases, employees are forbidden either to transfer ESOs or to short sell the company stock to hedge against their positions in those options. Hence they should exercise ESOs before expiration or just leave them worthless at expiration, leading to an appealing for instructions on how to work out the optimal strategy in order to maximize the returns through exercising over time. Besides, other prominent features include job termination risk, i.e. the risk of getting fired or leaving the company voluntarily in the duration of ESOs, and a list of flexible contract items. In conclusion, all these features result in the non-standardized ESOs' operating in an incomplete market, which causes the failure of the standard valuation methods for pricing options in a complete market.

A variety of approaches have been proposed in the literature to get useful insights and fruitful results into this problem. Earlier researches (see [2, 5, 6, 9]) are devoted to studying the optimal exercise strategy under the assumption that the employee would exercise the whole block of options at a single date. In this case, the optimal strategy is independent of the quantity of options she holds, which turns out to contradict the empirical evidence in which employees prefer distributed exercising over time, rather than at a single date. By virtue of utility function measuring personal risk preference, [7] establishes a multi-period model to examine the exercise policy for a risk-averse employee under the discrete time framework. [11] makes use of numerical examples based on utility models to illustrate the optimal exercise boundary which relies on a group of factors, particularly the number of options being held.

In this paper we consider the valuation of perpetual ESOs which can be exercised at any time from the grant date on. The employee is prohibited from trading on the underlying stock and there is a restriction on the instant exercise rate, which results in an incomplete market. The stochastic optimal control approach is applied to evaluate the block of ESOs and to find the optimal exercise policy for the employee.

Treating the number of options as continuous, we adopt a fluid model to characterize the exercise process and restrict the exercise rate not to exceeding an upper bound. It's justified by the common perspective of companies that if a large quantity of ESOs is exercised in a short period of time, the market stock price would probably be depressed and causing harm to the company.

Our objective is to maximize the expected overall discount returns of ESOs through exercising the options over time for the employee. To the best of our knowledge, all existing literature concerning ESOs including the aforementioned papers aim at maximizing the employee's expected accumulated utility attained by exercising the options, thereby leading to the associated optimal exercise policy based on the employee's risk preference. The unique feature of our model is that instead of pursuing utility maximization as in most literature, we target at maximizing the overall discount exercise returns, which naturally can be regarded as the grant-date fair value of the block of ESOs. As a result, with this stochastic optimal control problem solved, the value of ESOs and the corresponding optimal exercise strategy can be determined at the same time.

We derive the HJB equation that is a fully nonlinear PDE of second order with two variables and characterize the value function as its viscosity solution. Unlike the usual cases, the boundary conditions are not Dirichlet or Neumann type. In fact, the optimization process is terminated once the employee has exercised all ESOs held, which puts a constraint on the state process. So we study the value function under the constrained viscosity solution framework.

The rest of this paper is organized as follows. Section 2 formulates the pricing model to characterize the valuation process as a stochastic optimal control problem and gives the definition of the value function and the associated HJB equation. Section 3 shows that the value function is the constrained viscosity solution of the HJB equation. Section 4 discusses the comparison principle and the uniqueness of the constrained viscosity solution. Section 5 considers some limit cases. Section 6 exploits a numerical simulation method to obtain the approximation of the value function, determines the optimal policy which emerges in threshold style, and presents numerical examples to illustrate the impact of varying parameters on the optimal policy with some financial explanations. Section 7 concludes.

2. **Problem Formulation.**

$(\Omega, \mathcal{F}, P)$ is a complete probability space with a natural filtration $\{\mathcal{F}_t\}_{0\leq t<\infty}$ generated by a standard Brownian motion $\{W_t\}$. Let X_t denote the stock price of the company at time t, following a geometric Brownian motion

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x \tag{1}$$

where positive constants μ, σ represent the expected stock return rate and volatility respectively.

Consider an employee who is granted a total number N shares of perpetual American ESOs with the strike price K at time 0.

Let Y_t denote the aggregated number of options she has exercised up to time t, which is driven by the following differential equation

$$dY_t = u_t dt, \quad Y_0 = y, \tag{2}$$

where the exercise rate u_t is our control variable, restricted in the control set $\Gamma = [0, \lambda]$ with constant $\lambda > 0$. In section 5, we will discuss the limitation as $\lambda \to \infty$ to understand what would happen in the limit case. Obviously, $\{Y_t\}_{t\geq 0}$ is a non-negative non-decreasing right-continuous process. Let $S = (0, \infty) \times (0, N)$. Then at any time $t \in [0, \infty)$, the pair of state variables (X_t, Y_t) should belong to the state space $\bar{S}$.

Definition 2.1. A control $u.$ is admissible with respect to the initial value $(x, y) \in \bar{S}$ if and only if (i) $u.$ is $\{\mathcal{F}\}_t$-adapted; (ii) $u_t \in \Gamma$ for all $t \geq 0$; (iii) the corresponding state process $(X_t, Y_t) \in \bar{S}$ for all $t \geq 0$. Denote by $\mathcal{A} = \mathcal{A}(x, y)$ the set of all admissible controls.

The admissibility requires that for all $t \geq 0$, the control can only take values in the control set Γ, depending on the available information up to time t, rather than the indeterminate future, and meanwhile guarantees the state trajectory $(X_t, Y_t) \in \bar{S}$, especially $0 \leq Y_t \leq N$. In fact, once Y_τ attains N at time τ, it results in $u_t = 0$ for all $t > \tau$. This kind of optimal control is called state space constraints control.

The expected discounted total payoff associated with strategy $u. \in \mathcal{A}$ is defined by

$$J(x, y; u.) = E\left[\int_0^\infty e^{-\rho t} (X_t - K)^+ u_t dt \mid X_0 = x, Y_0 = y\right], \tag{3}$$

where ρ is the discount rate satisfying $\rho > \mu > 0$ and $G^+ = \max(G, 0)$. The parameter ρ serves as a time scale factor, affecting the time horizon of exercising the whole block of options. Moreover, larger ρ encourages quicker exercise actions with permissible exercise rate.

The objective of the employee is to maximize the expected discounted function from stock excierse. The value function is defined by

$$v(x, y) = \sup_{u. \in \mathcal{A}(x,y)} J(x, y; u.). \tag{4}$$

Define operators $\mathcal{L}$ and $\mathcal{B}$ of the value function

$$\mathcal{L}v = \mu x v_x + \frac{\sigma^2}{2} x^2 v_{xx} - \rho v, \tag{5}$$

$$\mathcal{B}v = v_y + (x - K)^+. \tag{6}$$

The HJB equation, for the optimal control problem, is

$$\mathcal{L}v + \max_{u \in \Gamma}(u\mathcal{B}v) = 0, \quad (x, y) \in (0, \infty) \times (0, N) \tag{7}$$

which is equivalent to

$$\mathcal{L}v + \lambda(\mathcal{B}v)^+ = 0, \quad (x, y) \in (0, \infty) \times (0, N) \tag{8}$$

Remark 1. If the value function $v \in C^{2,1}$, one can show that it is a classic solution to the HJB equation (8). We, however, in general do not know the smoothness of the value function. We therefore use the concept of viscosity solution and prove that the value function is the unique continuous constrained viscosity solution to the HJB equation (8) in sections 3 and 4.

The Dirichlet boundary condition naturally follows from the definition of value function,

$$v(0, y) = 0, \quad 0 \leq y \leq N. \tag{9}$$

At $y = 0$, N, we prescribe the following constrained boundary conditions

$$\mathcal{L}v + \max_{u \in \Gamma}(u\mathcal{B}v) \geq 0$$

in the viscosity sense, which means that the pair of state variables (X_t, Y_t) should belong to the state space $\bar{S}$. In section 3, we will provide a rigorous definition of constrained viscosity solution to the HJB equation associated with the above boundary conditions.

3. **Value Function and Constrained Viscosity Solution.** In this section we focus on the value function and show it is a constrained viscosity solution of related HJB equation (7). We first illustrate some properties of the value function to prepare for the further study.

Lemma 3.1. *The following assertions hold.*
(i) *For each $x \in [0,\infty)$, $v(x,y)$ is non-increasing in y;*
(ii) *For each $y \in [0,N]$, $v(x,y)$ is non-decreasing in x;*
(iii) *$v(x,y)$ is Lipschitz continuous in (x,y) and*

$$|v(x_1,y_1)-v(x_2,y_2)| \leq \min\left\{\frac{\lambda}{\rho-\mu}, 2(N-y_1)\right\}|x_1-x_2|+(2x_2+1)|y_1-y_2|, \quad (10)$$

for any $(x_i,y_i) \in [0,\infty)\times[0,N]$ $(i=1,2)$.

Proof. For a certain $x \in [0,\infty)$, suppose $0 \leq y_1 \leq y_2 \leq N$ and we have $\mathcal{A}(x,y_2) \subset \mathcal{A}(x,y_1)$. Let $u. \in \mathcal{A}(x_2,y)$, then $u. \in \mathcal{A}(x_2,y)$ implying

$$v(x,y_1) \geq J(x,y_1;u.) = J(x,y_2;u.).$$

Since $u.$ is arbitrary, it follows $v(x,y_1) \geq v(x,y_2)$ which means (i) holds.

Similarly, for a certain $y \in [0,N]$, suppose $0 \leq x_1 \leq x_2 < \infty$ and we have $\mathcal{A}(x_1,y) = \mathcal{A}(x_2,y)$. Denote by X_t^i the solution of (1) with initial values $X_0 = x_i$ for $i=1,2$. Then

$$X_t^i = x_i e^{\left(\mu-\frac{\sigma^2}{2}\right)t+W_t}, \quad (11)$$

with expectations $E(X^i) = x_i e^{\mu t}$, $i=1,2$.

For any $u. \in \mathcal{A}(x_1,y) = \mathcal{A}(x_2,y)$, we have

$$\begin{aligned}
J(x_1,y;u.) =& E\left[\int_0^\infty e^{-\rho t}\left(X_t^1-K\right)^+ u_t dt\right]\\
=& E\left[\int_0^\infty e^{-\rho t}\left(x_1 e^{\left(\mu-\frac{\sigma^2}{2}\right)t+W_t}-K\right)^+ u_t dt\right]\\
\leq& E\left[\int_0^\infty e^{-\rho t}\left(x_2 e^{\left(\mu-\frac{\sigma^2}{2}\right)t+W_t}-K\right)^+ u_t dt\right]\\
=& E\left[\int_0^\infty e^{-\rho t}\left(X_t^2-K\right)^+ u_t dt\right]\\
=& J(x_2,y;u.)
\end{aligned} \quad (12)$$

Due to the arbitrariness of $u.$, taking the supreme on both sides yields $v(x_1,y) \leq v(x_2,y)$ which justifies (ii).

We proceed to prove (iii) by showing that $v(x,y)$ is Lipschitz continuous in both x and y.

On one hand, given $y \in [0,N]$, let $x_1,x_2 \in [0,\infty)$. Without loss of generality, we suppose $x_1 \leq x_2$. Then from (12), for any $u. \in \mathcal{A}(x_1,y) = \mathcal{A}(x_2,y)$, it follows $J(x_1,y;u.) \leq J(x_2,y;u.)$. Define a stopping time $\tau = \inf\{t \geq 0 : \int_0^t u_s ds = N-y\}$. In fact, $u_s = 0$ for all $s \geq \tau$. Thus

$$\begin{aligned}
J(x_2,y;u.)-J(x_1,y;u.) =& E\int_0^\tau e^{-\rho t}\left[\left(X_t^2-K\right)^+-\left(X_t^1-K\right)^+\right]u_t dt\\
\leq& E\int_0^\tau e^{-\rho t}\left(X_t^2-X_t^1\right)u_t dt
\end{aligned} \quad (13)$$

Noting that $0 \le u_t \le \lambda$ for all $t \ge 0$, we have

$$\begin{aligned} E\int_0^\tau e^{-\rho t}\left(X_t^2 - X_t^1\right) u_t dt \le & \lambda(x_2-x_1)\int_0^\tau e^{-\rho t} E e^{\left(\mu-\frac{\sigma^2}{2}\right)t+W_t} dt \\ \le & \lambda(x_2-x_1)\int_0^\infty e^{-\rho t} e^{\mu t} dt \\ = & \frac{\lambda}{\rho-\mu}(x_2-x_1). \end{aligned} \tag{14}$$

In addition, using integration by parts, we get

$$\begin{aligned} E\int_0^\tau e^{-\rho t}\left(X_t^2 - X_t^1\right) u_t dt = & E\int_0^\tau e^{-\rho t}\left(X_t^2 - X_t^1\right) d\left(\int u_s ds\right) \\ = & E\left[\int_0^\tau u_s ds \cdot e^{-\rho\tau}\left(X_\tau^2 - X_\tau^1\right)\right] \\ & + (\rho-\mu) E\left[\int_0^\tau\left(\int_0^t u_s ds\right) e^{-\rho t}\left(X_t^2 - X_t^1\right) dt\right] \end{aligned} \tag{15}$$

Since $\int_0^\tau u_s ds = N - y$ and $\rho > \mu$, it follows

$$E\left[\int_0^\tau u_s ds \cdot e^{-\rho\tau}\left(X_\tau^2 - X_\tau^1\right)\right] \le (N-y) E\left[e^{-\rho\tau}\left(X_\tau^2 - X_\tau^1\right)\right] \le (N-y)(x_2-x_1) \tag{16}$$

and

$$\begin{aligned} E\left[\int_0^\tau\left(\int_0^t u_s ds\right) e^{-\rho t}\left(X_t^2 - X_t^1\right) dt\right] \le & (N-y) E\left[\int_0^\tau e^{-\rho t}\left(X_t^2 - X_t^1\right) dt\right] \\ \le & (N-y) E\left[\int_0^\infty e^{-\rho t}\left(X_t^2 - X_t^1\right) dt\right] \\ = & (N-y)\frac{(x_2-x_1)}{\rho-\mu}. \end{aligned} \tag{17}$$

Substituting (16), (17) into (15) yields

$$E\int_0^\tau e^{-\rho t}\left(X_t^2 - X_t^1\right) u_t dt \le 2(N-y)(x_2-x_1). \tag{18}$$

Hence it follows from (13), (14) and (18),

$$J(x_2, y; u.) - J(x_1, y; u.) \le \min\left\{\frac{\lambda}{\rho-\mu}, 2(N-y)\right\}(x_2-x_1).$$

For the arbitrariness of $u.$, we obtain

$$v(x_2, y) - v(x_1, y) \le \min\left\{\frac{\lambda}{\rho-\mu}, 2(N-y)\right\}(x_2-x_1). \tag{19}$$

which indicates $v(x,y)$ is Lipschitz continuous in x.

On the other hand, given $x \in [0,\infty)$, let $y_1, y_2 \in [0,N]$. Without loss of generality, we assume $0 \le y_1 \le y_2 \le N$.

It's clear that $\mathcal{A}(x,y_2) \subset \mathcal{A}(x,y_1)$. From the definition of the value function (4), for $|y_1-y_2| \ge 0$, there exists $u. \in \mathcal{A}(x,y_1)$ such that

$$v(x,y_1) \le J(x,y_1;u.) + |y_1-y_2|, \tag{20}$$

and

$$y_1 + \int_0^\infty u_t dt = N. \tag{21}$$

Set stopping time $\tau = \inf\{t > 0 : y_1 + \int_0^t u_s ds = y_2\}$ and define a control

$$\bar{u}_t = \begin{cases} 0, & 0 \le t \le \tau, \\ u_t, & t > \tau. \end{cases}$$

It's easy to see that $\bar{u}_\cdot \in \mathcal{A}(x, y_2)$. Thus

$$\begin{aligned} |J(x, y_1; u_\cdot) - J(x, y_2; \bar{u}_\cdot)| =& |E \int_0^\tau e^{-\rho t}(X_t - K)^+ u_t dt| \\ \le& E \int_0^\tau e^{-\rho t} X_t u_t dt \end{aligned}$$

Using integration by parts, we have

$$\begin{aligned} E \int_0^\tau e^{-\rho t} X_t u_t dt =& E \int_0^\tau e^{-\rho t} X_t d \left(\int u_s ds \right) \\ =& E \left(\int_0^\tau u_s ds \cdot e^{-\rho\tau} X_\tau \right) + (\rho - \mu) E \left[\int_0^\tau \left(\int_0^t u_s ds \right) e^{-\rho t} X_t dt \right] \end{aligned}$$

Note that $\int_0^\tau u_t dt = y_2 - y_1$ and $\rho > \mu$. It follows,

$$E \left(\int_0^\tau u_s ds \cdot e^{-\rho\tau} X_\tau \right) \le (y_2 - y_1) E(e^{-\rho\tau} X_\tau) \le (y_2 - y_1)x$$

and

$$\begin{aligned} E \left[\int_0^\tau \left(\int_0^t u_s ds \right) e^{-\rho t} X_t dt \right] \le& (y_2 - y_1) E \left(\int_0^\tau e^{-\rho t} X_t dt \right) \\ \le& (y_2 - y_1) E \left(\int_0^\infty e^{-\rho t} X_t dt \right) \\ =& (y_2 - y_1) \frac{x}{\rho - \mu}. \end{aligned}$$

Hence,

$$|J(x, y_1; u_\cdot) - J(x, y_2; \bar{u}_\cdot)| \le 2x|y_1 - y_2|. \tag{22}$$

Combining (20) and (22), we get

$$\begin{aligned} v(x, y_1) \le& J(x, y_1; u_\cdot) + |y_1 - y_2| \\ \le& J(x, y_2; \bar{u}_\cdot) + 2x|y_1 - y_2| + |y_1 - y_2| \\ \le& v(x, y_2) + (2x + 1)|y_1 - y_2|. \end{aligned}$$

From the property in (ii),

$$|v(x, y_1) - v(x, y_2)| \le (2x + 1)|y_1 - y_2|. \tag{23}$$

By virtue of (19) and (23), for any $(x_1, y_1), (x_2, y_2) \in [0, \infty) \times [0, N]$, we have

$$\begin{aligned} |v(x_1, y_1) - v(x_2, y_2)| \le& |v(x_1, y_1) - v(x_2, y_1)| + |v(x_2, y_1) - v(x_2, y_2)| \\ \le& \min \left\{ \frac{\lambda}{\rho - \mu}, 2(N - y_1) \right\} |x_1 - x_2| + (2x_2 + 1)|y_1 - y_2|. \end{aligned}$$

This completes the proof. $\square$

Now we introduce the sets of semicontinuous functions in $\bar{S}$ and give the definitions for viscosity solutions afterwards. Define

$$USC(\bar{S}) = \{v : \bar{S} \to \mathbb{R} \cup \{-\infty\} \mid v \text{ is upper semicontinuous}\},$$
$$LSC(\bar{S}) = \{v : \bar{S} \to \mathbb{R} \cup \{+\infty\} \mid v \text{ is lower semicontinuous}\}.$$

Definition 3.2. (Viscosity Supersolution and Subsolution)
(i) $w(x,y) \in LSC(S)$ is a supersolution of (7) in S if and only if

$$\mathcal{L}\varphi + \max_{u\in\Gamma}(u\mathcal{B}\varphi)\big|_{(x_0,y_0)} \le 0, \tag{24}$$

whenever $\varphi(x,y) \in C^{2,1}$ and $w(x,y) - \varphi(x,y)$ has a local minimum at $(x_0,y_0) \in S$ with $w(x_0,y_0) = \varphi(x_0,y_0)$;
(ii) $w(x,y) \in USC(\bar{S})$ is a subsolution of (7) in $(0,\infty) \times [0,N]$ if and only if

$$\mathcal{L}\varphi + \max_{u\in\Gamma}(u\mathcal{B}\varphi)\big|_{(x_0,y_0)} \ge 0, \tag{25}$$

whenever $\varphi(x,y) \in C^{2,1}$ and $w(x,y) - \varphi(x,y)$ has a local maximum at $(x_0,y_0) \in (0,\infty) \times [0,N]$ with $w(x_0,y_0) = \varphi(x_0,y_0)$.

The definition of the constrained viscosity solution follows.

Definition 3.3. A continuous function w is a constrained viscosity solution of (7) if it is both a viscosity supersolution of (7) in S and a viscosity subsolution of (7) in $(0,\infty) \times [0,N]$.

Remark 2. In the definition of the viscosity subsolution, the minima (x_0,y_0) may lie on the $y = 0, N$. This means that w is a viscosity solution in S and a viscosity subsolution on the $y = 0, N$.

We have the following result for the value function.

Theorem 3.4. *The value function $v(x,y)$ is a continuous constrained viscosity solution of (7), which satisfies (9).*

Proof. We first show that $v(x,y)$ is a viscosity supersolution.

Let the test function $\varphi(x,y) \in C^{2,1}(S)$ such that $v - \varphi$ attains its local minimum at $(x_0,y_0) \in S$ and, without loss of generality, $v(x_0,y_0) = \varphi(x_0,y_0)$. Then there exists a neighborhood $N(x_0,y_0) \in S$ of the point (x_0,y_0) satisfying

$$v(x,y) \ge \varphi(x,y),\ \forall (x,y) \in N(x_0,y_0). \tag{26}$$

Let (X_t, Y_t) be the solution of (1), (2) with $(X_0,Y_0) = (x_0,y_0)$ and the control $u. \in \mathcal{A}(x_0,y_0)$. Define a stopping time τ by

$$\tau = \inf\{t : (X_t, Y_t) \notin N(x_0,y_0\}. \tag{27}$$

For $h > 0$, by dynamic programming principle,

$$\begin{aligned} v(x_0,y_0) = &\sup_{u.\in\mathcal{A}(x_0,y_0)} E\left[\int_0^{h\wedge\tau} e^{-\rho t}(X_t - K)^+ u_t dt + e^{-\rho(h\wedge\tau)} v(X_{h\wedge\tau}, Y_{h\wedge\tau})\right] \\ \ge &E\int_0^{h\wedge\tau} e^{-\rho t}(X_t - K)^+ u_t dt + Ee^{-\rho(h\wedge\tau)} v(X_{h\wedge\tau}, Y_{h\wedge\tau}). \end{aligned}$$

Using (26), it leads to

$$\varphi(x_0,y_0) \ge E\int_0^{h\wedge\tau} e^{-\rho t}(X_t - K)^+ u_t dt + Ee^{-\rho(h\wedge\tau)} \varphi(X_{h\wedge\tau}, Y_{h\wedge\tau}).$$

Subtracting $\varphi(x_0, y_0)$ from both sides and applying Ito's formula, we get

$$0 \geq E \int_0^{h\wedge\tau} e^{-\rho t} \left[\mathcal{L}\varphi(X_t, Y_t) + u_t \mathcal{B}\varphi(X_t, Y_t)\right] dt. \tag{28}$$

For a fixed $w \in \Gamma$, we can choose a control $u^0_\cdot \in \mathcal{A}(x_0, y_0)$ such that

$$w = \lim_{t\to 0} u^0_t.$$

Then use this control $u^0_\cdot$ in (28) and divide both sides by h. Since w is arbitrary, taking $h \to 0$ yields

$$\max_{w\in\Gamma} \left[\mathcal{L}\varphi(x, y) + w\mathcal{B}\varphi(x, y)\right] \leq 0$$

which is equivalent to (25) implying $v(x, y)$ is a supersolution.

It remains to show that $v(x, y)$ is a subsolution.

Let $\varphi(x, y) \in C^{2,1}((0,\infty) \times [0, N])$ be a test function such that $v - \varphi$ attains its local maximum at $(x_0, y_0) \in (0,\infty) \times [0, N]$ and $v(x_0, y_0) = \varphi(x_0, y_0)$. Similarly we can find a neighborhood $N(x_0, y_0) \subset (0,\infty) \times [0, N]$ of (x_0, y_0) such that

$$v(x, y) \leq \varphi(x, y),\ \forall (x, y) \in N(x_0, y_0). \tag{29}$$

For any $h > 0$, there exists a control process $u^h_\cdot \in \mathcal{A}(x_0, y_0)$ such that

$$v(x_0, y_0) \leq h^2 + E \int_0^{h\wedge\tau} e^{-\rho t}(X_t - K)^+ u^h_t dt + Ee^{-\rho(h\wedge\tau)} v(X_{h\wedge\tau}, Y_{h\wedge\tau}),$$

where τ is a stopping time given by

$$\tau = \inf\{t : (X_t, Y_t) \notin N(x_0, y_0)\} \tag{30}$$

and (X_t, Y_t) is driven by (1), (2) with $(X_0, Y_0) = (x_0, y_0)$. Using (29), we get

$$\varphi(x_0, y_0) \leq h^2 + E \int_0^{h\wedge\tau} e^{-\rho t}(X_t - K)^+ u^h_t dt + Ee^{-\rho(h\wedge\tau)} \varphi(X_{h\wedge\tau}, Y_{h\wedge\tau}).$$

By Ito's formula,

$$0 \leq h^2 + E \int_0^{h\wedge\tau} e^{-\rho t} \left[\mathcal{L}\varphi(X_t, Y_t) + u^h_t \mathcal{B}\varphi(X_t, Y_t)\right] dt.$$

Finally sending $h \to 0$ yields

$$0 \leq \mathcal{L}\varphi(x, y) + \max_{w\in\Gamma} \{w\mathcal{B}\varphi(x, y)\},$$

which means (24) holds, namely $v(x, y)$ is the subsolution.

So far we've completed the proof by concluding that $v(x, y)$ is the constrained viscosity solution of (7). □

4. **Comparison Principle and Uniqueness.**

In this section, we prove the comparison principle which enables us to verify the uniqueness of the viscosity solution.

Denote by $\mathbf{M}$ the set of symmetric 2×2 matrices and define $F : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \times \mathbf{M} \to \mathbb{R}$,

$$F(X, r, p, M) = \mu x p_1 + \frac{\sigma^2}{2} x^2 m_{11} - \rho r + \max_{u\in\Gamma} u \cdot \left\{p_2 + (x - K)^+\right\} \tag{31}$$

where

$$X = (x, y),\ p = (p_1, p_2),\ M = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix}.$$

To prove the comparison principle, we need an alternative definition of constrained viscosity solution in terms of the notions of semijets as below.

Definition 4.1. $v : \bar{S} \to \mathbb{R}$ is a function.
The second order superjet(subjet) $J^{2,+}v(\hat{X})(J^{2,-}v(\hat{X}))$ at $\hat{X} \in \bar{S}$ is the set of $(p, M) \in \mathbb{R}^2 \times \mathbf{M}$ satisfying,

$$v(X) \leq (\geq) v(\hat{X}) + \langle p, X - \hat{X}\rangle + \frac{1}{2}\langle M(X - \hat{X}), X - \hat{X}\rangle + o(|X - \hat{X}|^2), \quad \text{as } \bar{S} \ni X \to \hat{X}.$$

Further, the closure $\bar{J}^{2,+}v(X)(\bar{J}^{2,-}v(X))$ is defined as the set of $(p, M) \in \mathbb{R}^2 \times \mathbf{M}$ satisfying,

$$\exists (X_n, v(X_n), p_n, M_n) \in \bar{S} \times \mathbb{R} \times \mathbb{R}^2 \times \mathbf{M} \to (X, v(X), p, M), \quad \text{as } n \to \infty,$$

where $(p_n, M_n) \in J^{2,+}v(X_n)(J^{2,-}v(X_n))$ for all n.

For the convenience in proof, we give another version of definitions for the viscosity subsolution and supersolution which are equivalent to Definition 3.1.

Definition 4.2. *(i)* $w(X) \in USC(\bar{S})$ is a subsolution of (7) in $\bar{S}$ if and only if

$$F(X, v(X), p, M) \geq 0$$

for $X \in \bar{S}$ and $(p, M) \in J^{2,+}v(X)$.
(ii) $w(X) \in LSC(S)$ is a supersolution of (7) in S if and only if

$$F(X, v(X), p, M) \leq 0$$

for $X \in S$ and $(p, M) \in J^{2,-}v(X)$.

In addition, we restate the following proposition from [4] to apply to our case in the proof of the comparison principle thereafter.

Proposition 1. *Let a locally compact set $\Omega \in \mathbb{R}^2$ and $v_i \in USC(\Omega)$ for $i = 1, \cdots, k$. Suppose $\phi(X_1, \cdots, X_k)$ is twice continuously differentiable(locally) in $\Omega \times \cdots \times \Omega$ and the function $v_1(X_1) + \cdots + v_k(X_k) - \phi(X_1, \cdots, X_k)$ attains a local maximum at $(\hat{X}_1, \cdots, \hat{X}_k)$ in $\Omega \times \cdots \times \Omega$. Then for each $\varepsilon > 0$, we can find $M_i \in \mathbf{M}$ satisfying,*

$$(D_{X_i}\phi(\hat{X}_1, \cdots, \hat{X}_k), M_i) \in \bar{J}^{2,+}v_i(\hat{X}_i) \text{ for } i = 1, \cdots, k,$$

$$-\left(\frac{1}{\varepsilon} + ||A||\right) I \leq \begin{pmatrix} M_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & M_k \end{pmatrix} \leq A + \varepsilon A^2$$

where the symmetric matrix $A = D^2\phi(\hat{X}_1, \cdots, \hat{X}_k)$ and its norm is given by $||A|| = \sup\{|\langle A\xi, \xi\rangle| : |\xi| \leq 1\}$.

At this stage, we are ready to prove the comparison principle for the viscosity subsolution and supersolution.

Theorem 4.3. *Let $\underline{v} \in USC(\bar{S})$ is a subsolution of (7) in $\bar{S}$ and $\bar{v} \in LSC(S)$ is a supersolution of (7) in S. Furthermore, suppose $\underline{v}, -\bar{v}$ satisfying the following conditions:*

1. *$\underline{v}, -\bar{v}$ grow at most linearly in X, i.e. there exists a constant $C > 0$ such that,*

$$\underline{v}(X), \ -\bar{v}(X) \leq C(1 + |X|) \text{ for } X \in \bar{S}, \tag{32}$$

2. *$\underline{v}(0, y) \leq \bar{v}(0, y)$,*
3. *$\rho > 2\mu + \sigma^2$.*

Then we have $\underline{v} \leq \bar{v}$ in $\bar{S}$.

Proof. Following the idea in [4], assume, for contradiction, that there exists $X^* \in \bar{S}$ such that,

$$\underline{v}(X^*) - \bar{v}(X^*) \geq 2\delta > 0, \quad \text{for some } \delta > 0. \tag{33}$$

Set $\partial S = l_1 \cup l_2 \cup l_3$ where

$$l_1 = \{(x,0)|\ 0 \leq x < \infty\},\ l_2 = \{(0,y)|\ 0 \leq y \leq N\},\ l_3 = \{(x,N)|\ 0 < x < \infty\}.$$

Let $\vec{n} = (0,1)$ denote the outer normal vector of the unique constrained boundary l_3 and define $\Phi(X_1, X_2) = \underline{v}(X_1) - \bar{v}(X_2) - \phi(X_1, X_2)$ where

$$\phi(X_1, X_2) = |\alpha(X_1 - X_2) - \varepsilon\vec{n}|^2 + \varepsilon(|X_1|^2 + |X_2|^2)$$

for $X_1, X_2 \in \bar{S}$ and $\alpha > 1, 0 < \varepsilon < 1$.

Due to the upper semicontinuity of $\Phi(X_1, X_2)$ and (32), we can find (X_1^α, X_2^α) satisfying

$$\sup_{\bar{S}\times\bar{S}} \Phi(X_1, X_2) = \Phi(X_1^\alpha, X_2^\alpha) := G^\alpha < \infty.$$

In the above notions we keep ε fixed and emphasize the dependence on α.

From $\Phi(X_1^\alpha, X_2^\alpha) \geq \Phi(X^*, X^*)$, we get

$$G^\alpha \geq \underline{v}(X^*) - \bar{v}(X^*) - \varepsilon^2 - 2\varepsilon|X^*|^2 > \delta > 0$$

if ε is sufficiently small. Therefore,

$$\underline{v}(X_1^\alpha) - \bar{v}(X_2^\alpha) > \delta. \tag{34}$$

The inequality $\Phi(X_1^\alpha, X_2^\alpha) \geq \Phi(X^*, X^*)$ also reads

$$\begin{aligned} &|\alpha(X_1^\alpha - X_2^\alpha) - \varepsilon\vec{n}|^2 + \varepsilon(|X_1^\alpha|^2 + |X_2^\alpha|^2) \\ \leq &\underline{v}(X_1^\alpha) - \bar{v}(X_2^\alpha) - \underline{v}(X^*) + \bar{v}(X^*) + \varepsilon^2 + 2\varepsilon|X^*|^2 \\ \leq &C_\varepsilon(1 + |X_1^\alpha| + |X_2^\alpha|) \end{aligned} \tag{35}$$

in which (35) follows from the assumption (32). Hence we have $|X_1^\alpha|,\ |X_2^\alpha| \leq C_\varepsilon$ with the constant $C_\varepsilon > 0$ implying $X_1^\alpha \to Z_1^\varepsilon, X_2^\alpha \to Z_2^\varepsilon$(along a subsequence) as $\alpha \to \infty$. Observe that $|\alpha(X_1^\alpha - X_2^\alpha) - \varepsilon\vec{n}| \leq C_\varepsilon$ $(\alpha \to \infty)$, so we conclude $Z_1^\varepsilon = Z_2^\varepsilon$, rewritten as Z^ε and $\alpha(X_1^\alpha - X_2^\alpha) \to W^\varepsilon$ (along a subsequence) as $\alpha \to \infty$. Here Z^ε, W^ε depend on ε.

Firstly, we show $Z^\varepsilon \notin l_2$. We argue by contradiction and suppose $Z^\varepsilon \in l_2$. From the upper semicontinuity of function Φ, we have

$$\Phi(Z^\varepsilon, Z^\varepsilon) \geq \limsup_{\alpha\to\infty} \Phi(X_1^\alpha, X_2^\alpha) \geq \delta,$$

but

$$\Phi(Z^\varepsilon, Z^\varepsilon) = \underline{v}(Z^\varepsilon) - \bar{v}(Z^\varepsilon) - \varepsilon^2 - 2\varepsilon|Z^\varepsilon|^2 < 0.$$

This is a contradiction.

If $Z^\varepsilon \in l_3$, $Z^\varepsilon - \frac{\varepsilon}{\alpha}\vec{n} \in S$. Further using $\Phi(X_1^\alpha, X_2^\alpha) \geq \Phi(Z^\varepsilon, Z^\varepsilon - \frac{\varepsilon}{\alpha}\vec{n})$, we get

$$\begin{aligned} |\alpha(X_1^\alpha - X_2^\alpha) - \varepsilon\vec{n}|^2 \leq &\underline{v}(X_1^\alpha) - \bar{v}(X_2^\alpha) - \underline{v}(Z^\varepsilon) + \bar{v}(Z^\varepsilon - \frac{\varepsilon}{\alpha}\vec{n}) \\ &- \varepsilon\left(|X_1^\alpha|^2 + |X_2^\alpha|^2 - |Z^\varepsilon|^2 - |Z^\varepsilon - \frac{\varepsilon}{\alpha}\vec{n}|^2\right). \end{aligned}$$

Sending $\alpha \to \infty$ yields the right side smaller than 0 which indicates $\alpha(X_1^\alpha - X_2^\alpha) - \varepsilon\vec{n} = o(1)$. Thus $X_2^\alpha = X_1^\alpha - \frac{1}{\alpha}(\varepsilon\vec{n} + o(1)) \in S$ and $\lim_{\varepsilon\to 0} W^\varepsilon = 0$.

If $Z^\varepsilon \in S$, then for large α, $X_1^\alpha, X_2^\alpha \in S$. In order to apply Proposition 4.1 to derive the contradiction. We rewrite ϕ as

$$\phi(X_1, X_2) = [\alpha(x_1 - x_2)]^2 + [\alpha(y_1 - y_2) - \varepsilon]^2 + \varepsilon(x_1^2 + y_1^2 + x_2^2 + y_2^2).$$

Thus,

$$D_{X_1}\phi = \begin{pmatrix} 2\alpha^2(x_1-x_2)+2\varepsilon x_1 \\ 2\alpha[\alpha(y_1-y_2)-\varepsilon]+2\varepsilon y_1 \end{pmatrix}, \quad D_{X_2}\phi = \begin{pmatrix} -2\alpha^2(x_1-x_2)+2\varepsilon x_2 \\ -2\alpha[\alpha(y_1-y_2)-\varepsilon]+2\varepsilon y_2 \end{pmatrix}.$$

There exist $M_i^\alpha, M_2^\alpha \in \mathbf{M}$ such that

$$(D_{X_1}\phi(X_1^\alpha),\ M_1^\alpha) \in J^{2,+}\underline{v}(X_1^\alpha),$$
$$(-D_{X_2}\phi(X_2^\alpha),\ -M_2^\alpha) \in J^{2,-}\bar{v}(X_2^\alpha),$$

and

$$\begin{pmatrix} M_1^\alpha & 0 \\ 0 & M_2^\alpha \end{pmatrix} \le 2(3\alpha^2+2\varepsilon)\begin{pmatrix} I_2 & -I_2 \\ -I_2 & I_2 \end{pmatrix} + 2\left(\varepsilon+\frac{\varepsilon^2}{\alpha^2}\right)\begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} \tag{36}$$

where $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Set $M_i^\alpha = \begin{pmatrix} m_i^\alpha & * \\ * & * \end{pmatrix}$ and $X_i^\alpha = (x_i^\alpha, y_i^\alpha)$ $(i=1,2)$.

Since $\underline{v}$ is a subsolution of (7) in $\bar{S}$ and $\bar{v}$ is a supersolution of (7) in S. Note the form of F in (31) and Definition 4.2, by Proposition 4.1 we can find $\lambda_0 \in \Gamma$ such that

$$\begin{aligned} 0 \le& \mu x_1^\alpha \left[2\alpha^2(x_1^\alpha - x_2^\alpha)+2\varepsilon x_1^\alpha\right] + \frac{\sigma^2}{2}(x_1^\alpha)^2 m_1^\alpha - \rho\underline{v}(X_1^\alpha) \\ &+ \lambda_0\left\{2\alpha^2(y_1^\alpha - y_2^\alpha) - 2\alpha\varepsilon + 2\varepsilon y_1^\alpha + (x_1^\alpha - K)^+\right\}, \end{aligned} \tag{37}$$

$$\begin{aligned} 0 \ge& \mu x_2^\alpha \left[2\alpha^2(x_1^\alpha - x_2^\alpha)-2\varepsilon x_2^\alpha\right] - \frac{\sigma^2}{2}(x_2^\alpha)^2 m_2^\alpha - \rho\bar{v}(X_2^\alpha) \\ &+ \lambda_0\left\{2\alpha^2(y_1^\alpha - y_2^\alpha) - 2\alpha\varepsilon - 2\varepsilon y_2^\alpha + (x_2^\alpha - K)^+\right\}. \end{aligned} \tag{38}$$

Subtracting (38) from (37), we have

$$\begin{aligned} 0 \le& 2\mu\alpha^2\left(x_1^\alpha - x_2^\alpha\right)^2 + 2\mu\varepsilon\left[(x_1^\alpha)^2+(x_2^\alpha)^2\right] + \frac{\sigma^2}{2}\left[(x_1^\alpha)^2 m_1^\alpha + (x_2^\alpha)^2 m_2^\alpha\right] \\ &- \rho\left[\underline{v}(X_1^\alpha) - \bar{v}(X_2^\alpha)\right] + \lambda_0\left\{2\varepsilon(y_1^\alpha + y_2^\alpha) + (x_1^\alpha - K)^+ - (x_2^\alpha - K)^+\right\}. \end{aligned} \tag{39}$$

The inequality (36) yields

$$\begin{aligned} (x_1^\alpha)^2 m_1^\alpha + (x_2^\alpha)^2 m_2^\alpha =& \begin{pmatrix} x_1^\alpha & 0 & x_2^\alpha & 0 \end{pmatrix}\begin{pmatrix} M_1^\alpha & 0 \\ 0 & M_2^\alpha \end{pmatrix}\begin{pmatrix} x_1^\alpha & 0 & x_2^\alpha & 0 \end{pmatrix}^T \\ \le& 2(3\alpha^2+2\varepsilon)(x_1^\alpha - x_2^\alpha)^2 + 2\left(\varepsilon + \frac{\varepsilon^2}{\alpha^2}\right)\left[(x_1^\alpha)^2 + (x_2^\alpha)^2\right]. \end{aligned} \tag{40}$$

Then by substituting (40) into (39),

$$\begin{aligned} 0 \le& -\rho[\underline{v}(X_1^\alpha) - \bar{v}(X_2^\alpha)] + (2\mu+3\sigma^2)\alpha^2\left(x_1^\alpha - x_2^\alpha\right)^2 + 2\sigma^2\varepsilon\left(x_1^\alpha - x_2^\alpha\right)^2 \\ &+ (2\mu+\sigma^2)\varepsilon\left[(x_1^\alpha)^2+(x_2^\alpha)^2\right] + \sigma^2\frac{\varepsilon^2}{\alpha^2}\left[(x_1^\alpha)^2+(x_2^\alpha)^2\right] \\ &+ \lambda_0\left\{2\varepsilon(y_1^\alpha + y_2^\alpha) + (x_1^\alpha - K)^+ - (x_2^\alpha - K)^+\right\}. \end{aligned} \tag{41}$$

Recall that we have

$$X_1^\alpha, X_2^\alpha \to Z^\varepsilon = (\bar{z}^\varepsilon, \tilde{z}^\varepsilon), \quad \alpha(X_1^\alpha - X_2^\alpha) \to W^\varepsilon = (\bar{w}^\varepsilon, \tilde{w}^\varepsilon) \text{ as } \alpha \to \infty.$$

and

$$|Z^\varepsilon| \le C_\varepsilon, \quad \lim_{\varepsilon\to 0} W^\varepsilon = 0. \tag{42}$$

Noting $\underline{v}(X^*) - \bar{v}(X^*) > 0$, thus by sending $\alpha \to \infty$ and $\varepsilon \to 0$, the inequality (35) leads us to,

$$\lim_{\varepsilon \to 0} 2\varepsilon |Z^\varepsilon|^2 \le \underline{v}(Z^\varepsilon) - \bar{v}(Z^\varepsilon). \tag{43}$$

Using (34), due to the upper semicontiuity of $\underline{v} - \bar{v}$, taking $\alpha \to \infty$ we get

$$\underline{v}(Z^\varepsilon) - \bar{v}(Z^\varepsilon) > \delta > 0. \tag{44}$$

Letting $\alpha \to \infty$, (41) yields,

$$0 \le -\rho[\underline{v}(Z^\varepsilon) - \bar{v}(Z^\varepsilon)] + (2\mu + 3\sigma^2)|W^\varepsilon|^2 + (2\mu + \sigma^2)2\varepsilon|Z^\varepsilon|^2 + 4\lambda_0 \varepsilon \tilde{z}^\varepsilon.$$

Now making use of (42)—(44) and the fact that $\rho - 2\mu - \sigma^2 > 0, \tilde{z}^\varepsilon \in [0, N]$, further taking $\varepsilon \to 0$, we obtain

$$0 \le -(\rho - 2\mu - \sigma^2)\delta < 0$$

which is a contradiction. □

Remark 3. In the inequality (41), all the terms are less than or equal to 0 as $\alpha \to \infty$ and $\varepsilon \to 0$ except for $(2\mu + \sigma^2)\varepsilon\left[(x_1^\alpha)^2 + (x_2^\alpha)^2\right]$. Thus we need a technical condition such as $\rho > 2\mu + \sigma^2$ to get the contradiction. We think this is not a necessary condition, so further research could be done to remove this condition.

Now we state the main theorem regarding the value function in the sense of the constrained viscosity solution.

Theorem 4.4. *The value function $v(x, y)$ is the unique continuous constrained viscosity solution of (7) in $\bar{S}$ that grows at most linearly in (x, y) and verify (9).*

Proof. Let v^1 and v^2 be two constrained viscosity solutions of (7). Since v^1 and v^2 are subsolution and supersolution respectively, by Lemma 4.1, we get $v^1 \le v^2$. On the contrary, we have $v^2 \le v^1$ since v^2 and v^1 are subsolution and supersolution respectively. So we conclude $v^1 = v^2$. □

5. **The Limit Case as $\lambda \to \infty$.** In this section, we consider the limitation as $\lambda \to \infty$ to understand what would happen in the limit case. For each $\lambda > 0$, $v^\lambda(x, y)$ is the value function of the previous optimization problem particularly with the control set $\Gamma^\lambda = [0, \lambda]$ and $\mathcal{A}^\lambda(x, y)$ corresponding to the admissible set given in Definition 2.1. From (7), the HJB equation governing $v^\lambda(x, y)$ is given by

$$\mathcal{L}v^\lambda + \max_{u \in \Gamma^\lambda} u \cdot \mathcal{B}v^\lambda = 0, \quad (x, y) \in [0, \infty) \times [0, N]. \tag{45}$$

Then we have the following helpful results.

Lemma 5.1. *(i) $v^\lambda(x, y)$ increases with respect to λ; (ii) $v^\lambda(x, y)$ is bounded for all $\lambda > 0$;*
(iii) $v^\lambda(x, y)$ converges to $v^\infty(x, y)$ pointwise as $\lambda \to \infty$.

Proof. *(i)* In fact, for any $0 < \lambda_1 < \lambda_2 < \infty$, we have $\mathcal{A}^{\lambda_1}(x, y) \subset \mathcal{A}^{\lambda_2}(x, y)$.
Recall the value function

$$v^{\lambda_i}(x, y) = \sup_{u. \in \mathcal{A}^{\lambda_i}(x,y)} E\left[\int_0^\infty e^{-\rho t}(X_t - K)^+ u_t dt\right], \quad i = 1, 2,$$

which implies $v^{\lambda_1} \le v^{\lambda_2}$.
(ii) For any $\varepsilon > 0$, we can find $\bar{u}. \in \mathcal{A}^\lambda(x, y)$ such that

$$v^\lambda(x, y) \le E\int_0^\tau e^{-\rho t}(X_t - K)^+ \bar{u}_t dt + \varepsilon$$

where the stopping time $\tau = \inf\{t \geq 0 : Y_t = N\}$ and X_t, Y_t are governed by (1), (2) with control $\bar{u}$..

Then using integration by parts, it follows

$$\begin{aligned} v^\lambda(x,y) \leq& E\int_0^\tau e^{-\rho t} X_t \bar{u}_t dt + \varepsilon \\ =& E\int_0^\tau e^{-\rho t} X_t d\left(\int \bar{u}_s ds\right) + \varepsilon \\ =& E\left(\int_0^\tau \bar{u}_s ds \cdot e^{-\rho\tau} X_\tau\right) + (\rho-\mu)E\left[\int_0^\tau \left(\int_0^t \bar{u}_s ds\right) e^{-\rho t} X_t dt\right] + \varepsilon \\ \leq& (N-y)E\left(e^{-\rho\tau}X_\tau\right) + (\rho-\mu)(N-y)E\left(\int_0^\tau e^{-\rho t} X_t dt\right) + \varepsilon \end{aligned} \tag{46}$$

Note that

$$E\left(e^{-\rho\tau}X_\tau\right) \leq x,$$

and

$$E\left(\int_0^\tau e^{-\rho t} X_t dt\right) \leq E\left(\int_0^\infty e^{-\rho t} X_t dt\right) \leq \frac{x}{\rho-\mu}.$$

Substituting them into (46) and taking $\varepsilon \to 0$, we obtain

$$v^\lambda(x,y) \leq 2x(N-y).$$

(iii) For any $(x,y) \in [0,\infty)\times[0,N]$, sending $\lambda \to \infty$, the convergence of $v^\lambda(x,y)$ as $\lambda \to \infty$ directly follows from (i), (ii). We denote it by $v^\infty(x,y)$. □

We proceed to give more insight into the limit case by the theorem below.

Theorem 5.2. *Let v^λ be the unique constrained viscosity solution of (45). Then $v^\lambda \to v^\infty$ as $\lambda \to \infty$ and v^∞ is the unique viscosity solution satisfying*

$$\max\{\mathcal{L}v^\infty, \mathcal{B}v^\infty\} = 0, \quad (x,y) \in [0,\infty)\times[0,N] \tag{47}$$

with boundary condition

$$v^\infty(0,y) = 0, \quad 0 \leq y \leq N \tag{48}$$

$$v^\infty(x,N) = 0, \quad 0 \leq x < \infty. \tag{49}$$

Remark 4. In fact, by sending $\lambda \to \infty$, we just remove the restriction on the exercise rate with all the other conditions maintained. It's conceivable that without this restriction the employee would simply select an optimal moment to exercise the whole block of options at a time. In this case, every single option is treated equally and thus can be viewed as a standard perpetual American option. Moreover, the value function should take the form

$$v^\infty(x,y) = (N-y)\bar{v}(x), \quad (x,y) \in [0,\infty)\times[0,N], \tag{50}$$

where $\bar{v}(x)$ represents the initial value of the corresponding standard perpetual American call option whose value has an analytical solution (see [8]).

To prove the previous theorem, we need this proposition (see [4]) which works well in the convergence analysis of viscosity solutions.

Proposition 2. *Let $\Omega \subset \mathbb{R}^N$ be locally compact, $v \in USC(\Omega)$, $z \in \Omega$, and $(p,M) \in J^{2,+}v(z)$. Suppose also that v_n is a sequence of upper semicontinuous functions on Ω satisfying*

1. *there exists $x_n \in \Omega$ such that $(x_n, v_n(x_n)) \to (z, v(z))$,*

2. if $x_n \in \Omega$ and $x_n \to z \in \Omega$, then $\overline{\lim_{n\to\infty}} v_n(x_n) \le v(z)$.

Then there exists $\hat{x}_n \in \Omega$, $(p_n, M_n) \in J^{2,+}v_n(\hat{x}_n)$ such that $(\hat{x}_n, v_n(\hat{x}_n), p_n, M_n) \to (z, v(z), p, M)$.

Next we show v^∞ is the unique viscosity solution of (47)—(49).

Proof of Theorem 5.2. Note that $S = (0,\infty) \times (0, N)$ and define

$$L(x,y,r,p,M) = \mu x p_1 + \frac{\sigma^2}{2} m_{11} - \rho r,$$
$$B(x,y,p) = p_2 + (x-K)^+,$$

where $M = (m_{ij})_{2\times 2} \in \mathbf{M}$ and $p = (p_1,p_2) \in \mathbb{R}^2$.

First, we claim v^∞ is a viscosity subsolution of (47). Given $(x,y) \in S$, $(p,X) \in J^{2,+}v^\infty(x,y)$, we have $v^\lambda(x,y) \to v^\infty(x,y)$ as $\lambda \to \infty$. Then for all $(x_\lambda, y_\lambda) \to (x,y)$,

$$\begin{aligned}
&\left|v^\lambda(x_\lambda,y_\lambda) - v^\infty(x,y)\right| \\
\le &\left|v^\lambda(x_\lambda,y_\lambda) - v^\lambda(x,y)\right| + \left|v^\lambda(x,y) - v^\infty(x,y)\right| \\
\le &K\left(|x_\lambda - x| + |y_\lambda - y|\right) + \left|v^\lambda(x,y) - v^\infty(x,y)\right| \to 0,\ (\lambda \to \infty).
\end{aligned}$$

From Proposition 2, we can find $(x_\lambda,y_\lambda) \in S, (p_\lambda, M_\lambda) \in J^{2,+}v^\lambda(x_\lambda,y_\lambda)$, such that

$$((x_\lambda,y_\lambda), v^\lambda(x_\lambda,y_\lambda), p_\lambda, M_\lambda) \to ((x,y), v^\infty(x,y), p, M),$$
$$L(x_\lambda,y_\lambda, v^\lambda(x_\lambda,y_\lambda), p_\lambda, M_\lambda) \to L(x,y,v^\infty(x,y),p,M),$$
$$B(x_\lambda,y_\lambda,p_\lambda) \to B(x,y,p).$$

Since v^λ is a subsolution of (45), it leads to

$$L(x_\lambda,y_\lambda,v^\lambda(x_\lambda,y_\lambda),p_\lambda,X_\lambda) + \lambda[B(x_\lambda,y_\lambda,p_\lambda)]^+ \ge 0. \tag{51}$$

To show v^∞ is a subsolution of (47), we need to verify

$$\max\{L(x,y,v^\infty(x,y),p,X), B(x,y,p)\} \ge 0. \tag{52}$$

If $B(x,y,p) \ge 0$, then (52) naturally holds. If $B(x,y,p) < 0$, we can find a sufficiently large Λ such that $B(x_\lambda,y_\lambda,p_\lambda) < 0$ for all $\lambda > \Lambda$. From (51), it follows $L(x_\lambda,y_\lambda,v^\lambda(x_\lambda,y_\lambda),p_\lambda,X_\lambda) \ge 0$. Sending $\lambda \to \infty$, we get $L(x,y,v^\infty(x,y),p,X) \ge 0$ implying (52) is true.

Analogously, we apply the lower semicontinuous version of Proposition 2 to show $v^\infty(x,y)$ is a supersolution of (47).

Given $(x,y) \in S$ and $(p,M) \in J^{2,-}v^\infty(x,y)$,there exists $(x_\lambda,y_\lambda) \in S$, $(p_\lambda,M_\lambda) \in J^{2,-}v^\lambda(x_\lambda,y_\lambda)$ such that

$$((x_\lambda,y_\lambda), v^\lambda(x_\lambda,y_\lambda), p_\lambda, M_\lambda) \to ((x,y), v(x,y), p, M)$$
$$L(x_\lambda,y_\lambda, v^\lambda(x_\lambda,y_\lambda), p_\lambda, M_\lambda) \to L(x,y,v(x,y),p,M)$$
$$B(x_\lambda,y_\lambda,p_\lambda) \to B(x,y,p).$$

Again, since v^λ is a supersolution of (45), we have

$$L(x_\lambda,y_\lambda,v^\lambda(x_\lambda,y_\lambda),p_\lambda,X_\lambda) + \lambda[B(x_\lambda,y_\lambda,p_\lambda)]^+ \le 0. \tag{53}$$

Thus,

$$L(x_\lambda,y_\lambda,v^\lambda(x_\lambda,y_\lambda),p_\lambda,X_\lambda) = -\lambda[B(x_\lambda,y_\lambda,p_\lambda)]^+ \le 0. \tag{54}$$

To showv^∞ is a supersolution of (47), we have to verify

$$\max\{L(x,y,v^\infty(x,y),p,M),B(x,y,p)\}\le 0. \tag{55}$$

In fact letting $\lambda\to\infty$ in (54) yields $L(x,y,v^\infty(x,y),p,M)\le 0$. On the other hand, if $B(x,y,p)>0$, we have $\lambda(B(x_\lambda,y_\lambda,p_\lambda))^+\to\infty$ as $\lambda\to\infty$. By virtue of (54), we obtain $L(x,y,v(x,y),p,X)=-\infty$ which is impossible. Therefore, $B(x,y,p)\le 0$ and (52) holds.

Finally (48) and (49) naturally follow from the boundary condition for v^λ and the uniqueness can be justified by similar arguments as we use in the previous analysis for v^λ. □

6. **Optimal Exercise Decision and Numerical Simulations.** In this section, we apply the finite difference method to numerically approximate the value function and the associated optimal control.

6.1. **The Optimal Exercise Decision.** Under the viscosity solution framework, the standard verification theorem in [3] enables us to define the optimal exercise rate $u^*(x,y)$ in terms of the value function $v(x,y)$.

Noting $\Gamma=[0,\lambda]$, from (7) it's straightforward to see that in order to attain the maximum, u should take value 0 whenever $\mathcal{B}v\le 0$, otherwise equal λ.

We define no-exercise region and exercise region as

$$NR:=\{(x,y):\ \mathcal{B}v(x,y)\le 0\}, \tag{56}$$

$$ER:=\{(x,y):\ \mathcal{B}v(x,y)>0\}, \tag{57}$$

It naturally leads us to define the optimal exercise rate $u^*(x,y)$ by

$$u^*(x,y)=\begin{cases}0, & \text{if } (x,y)\in NR,\\ \lambda, & \text{if } (x,y)\in ER.\end{cases} \tag{58}$$

In fact, the above control can be verified to be optimal, namely maximizing our objective function to achieve the value function. Also it's exactly the feedback control well known in dynamic programming theory.

The previous analysis sheds light on the dependence of the optimal exercise rate on the availability of the value function. However due to the difficulty in getting the analytical solution, we apply the numerical simulation approach to approximate the value function, the two separate regions and the optimal exercise decision.

6.2. **Numerical Scheme.** It is reasonable to deduce that the employee would exercise all the remaining options at the largest permissible exercise rate λ when the stock price has gone high enough, namely, beyond some large $M>0$. Under this assumption, it would take a period of $\frac{N-y}{\lambda}$ to fully exercise the options held.

Let $v^M(x,y)$ which is defined on the bounded domain $[0,M]\times[0,N]$ be the approximate solution of $v(x,y)$. It's easy to see that $v^M(x,y)\to v(x,y)$ as $M\to\infty$. We give another boundary condition on $x=M$ for $v^M(x,y)$,

$$\begin{aligned}v^M(M,y)=&E\left[\int_0^{\frac{N-y}{\lambda}} e^{-\rho t}\lambda(X_t-K)^+dt\big|X_0=M,Y_0=y\right]\\ =&\int_0^{\frac{N-y}{\lambda}}\lambda e^{-\rho t}E\left[(X_t-K)^+\big|\ X_0=M\right]dt\\ =&\lambda M\int_0^{\frac{N-y}{\lambda}} e^{-(\rho-\mu)t}Q(d_1)dt-\lambda K\int_0^{\frac{N-y}{\lambda}} e^{-\rho t}Q(d_2)dt\end{aligned} \tag{59}$$

in which

$$\begin{aligned} d_1 &= \frac{\ln \frac{M}{K} + (\mu + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}, \\ d_2 &= d_1 - \sigma\sqrt{t}, \\ Q(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\theta^2}{2}} d\theta. \end{aligned}$$

It would allow us to solve the numerical approximation for $v^M(x,y)$ by the following finite difference scheme.

Let $H = (h,k)$ be a partition of the bounded domain $[0,M] \times [0,N]$ with h,k representing the step sizes for x,y respectively. Define $(x_i,y_j) = (ih,jk)$ for $i = 0,1,\cdots,m, j = 0,1,\cdots,n$, where $m = \frac{M}{h}$ and $n = \frac{N}{k}$. Let $v_{i,j}^M$ be the approximation for $v^M(x_i,y_j)$ with discretization operators $\mathcal{L}^H, B^H$ defined by

$$\begin{aligned} L^H v_{i,j}^M &= \mu i h \frac{v_{i+1,j}^M - v_{i,j}^M}{h} + \frac{\sigma^2}{2} i^2 h^2 \frac{v_{i+1,j}^M - 2v_{i,j}^M + v_{i-1,j}^M}{h^2} - \rho v_{i,j}^M, \\ B^H v_{i,j}^M &= \frac{v_{i,j+1}^M - v_{i,j}^M}{k} + (ih-K)^+. \end{aligned}$$

Substituting them into (8) yields,

$$\mathcal{L}^H v_{i,j}^M + \lambda(\mathcal{B}^H v_{i,j}^M)^+ = 0 \quad \text{for } i = 1,\cdots,m-1,\ j = 0,1,\cdots,n-1.$$

On the boundary, we have

$$\begin{aligned} v_{0,j}^M &= 0, \quad j = 0,1,\cdots,n, \\ v_{i,n}^M &= 0, \quad i = 0,1,\cdots,m. \end{aligned}$$

In addition, $v_{m,j}^M$ is given by (59) with $y = jk$ for $j = 0,1,\cdots,n$.

The non-linear iteration method is applied here and we resort to the iteration formula in [1] to deal with the non-linear term F^+. With the ith iterative result F^i known, it gives

$$\left(F^{i+1}\right)^+ = F^{i+1} I_{\{F^i>0\}}, \quad i = 0,1,\cdots$$

where the indication function $I_{\{F^i>0\}} = 1$ if $F^i > 0$, otherwise equals 0.

Consequently with the lth iterative solution $v^{M,l}$ obtained, we solve $v^{M,l+1}$ using the following equations,

$$\mathcal{L}^H v_{i,j}^{M,l+1} + \lambda(\mathcal{B}^H v_{i,j}^{M,l+1}) I_{\{\mathcal{B}^H v_{i,j}^{M,l}\}} = 0, \quad \text{for } i = 1,\cdots,m-1 \text{ and } j = 0,\cdots,n-1.$$

6.3. **Numerical Examples.** In the sequel, we demonstrate some numerical examples obtained using the above simulation scheme, trying to examine the value function and the corresponding optimal control through their approximations.

Data used for numerical tests are:

$$\mu = 0.1, \quad \sigma = 0.3, \quad \rho = 0.15, \quad \lambda = 1, \quad K = 2, \quad M = 5, \quad N = 30. \tag{60}$$

The graph of the value function $v(x,y)$ shown in Fig.1 confirms the results in Lemma 3.1. Clearly the more options the employee holds, corresponding to a smaller y, the more returns she could possibly gain from these options and results in a higher cost of these options for the company. It's the same with the case when the stock price x gets relatively high.

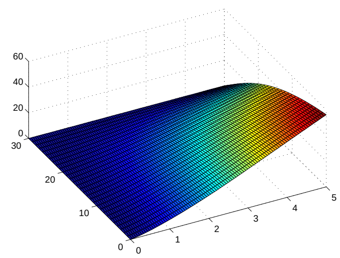

FIGURE 1. The value function $v(x, y)$

Now recall our discussion about the optimal exercise decision. It's clear that the optimal control u^* totally depends on $\mathcal{B}v$. Fig.2 shows that the threshold boundary specified by $\mathcal{B}v = 0$ separates the entire region into two parts.

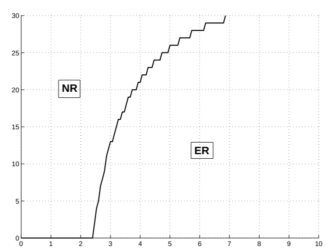

FIGURE 2. The regions NR, ER and the threshold boundary

The region NR to the left of the boundary corresponds to $u^* = 0$ where the employee is not supposed to exercise any option, but hold and wait. While in ER to the right, u^* should attain λ suggesting the employee exercise the options at the largest permissible rate at once.

Indeed it sheds light on the optimal exercise decision for practitioners in the financial market. For an employee holding $(N - y)$ shares of options, once the stock price goes beyond a specific level given by the boundary, usually termed a threshold, she should take action to exercise, otherwise take no action. Such threshold-style strategy is rather appealing for practitioners due to its simplicity to grasp and implement.

In addition, we are interested in the impact of varying model parameters on the optimal exercise decision, with more numerical examples to follow.

6.3.1. *Impact of varing ρ on u^*.* Let the discount factor ρ take values $0.15, 0.18, 0.21$ and fix the values of other parameters as in (60). In Fig.3, the region ER tends to become larger with the threshold boundary moving upward while ρ increases. In fact, a larger ρ implies deeper discount in the future, thus leading to earlier exercise for the employee.

6.3.2. *Impact of varing μ on u^*.* Fig.4 illustrates the threshold value for different y with the stock return rate $\mu = 0.1, 0.12, 0.14$ and others maintained. Obviously, a

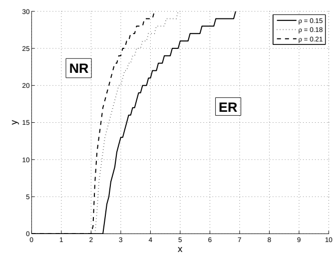

FIGURE 3. Impact of varing ρ on u^*

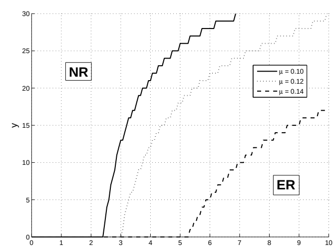

FIGURE 4. Impact of varing μ on u^*

larger μ encourages the employee to hold the options and wait more patiently since it suggests a stronger increasing capacity for the stock price.

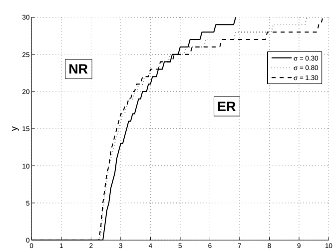

FIGURE 5. Impact of varing σ on u^*

6.3.3. *Impact of varing σ on u^*.* The shift of the threshold boundary is shown to be non-monotonic with respect to σ, as in Fig.5 where we take $\sigma = 0.3, 0.8, 1.1$. In essence, a relatively higher stock volatility σ implies both more opportunity and higher risk for the future stock mounting. When the employee still holds plenty of options, i.e. a small y, she would pay more attention to the potential high risk, resulting in more exercise pressure. Otherwise, with a few options yet to exercise,i.e. a large y, she would wait longer and expect more exercise returns due to a larger σ.

6.3.4. *Impact of varing λ on u^*.* Fig.6 demonstrates the relationship between the optimal control and the upper bound of the exercise rate. Intuitively, a larger λ

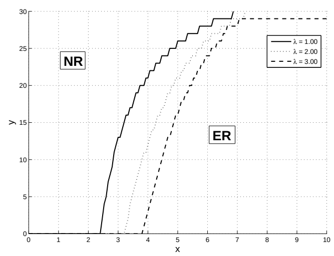

FIGURE 6. Impact of varing λ on u^*

gives the employee more freedom to choose exercise rate, which makes her more patient and expect more returns from exercising, thus leading to larger NR and smaller ER which is confirmed by Fig.6 with λ taking values $1, 2, 3$.

7. **Conclusions.** In this paper we consider the valuation and optimal exercise s-trategy of perpetual American employee stock options. Adopting a fluid model with restricted exercise rate to govern the exercise behavior, the value function is defined as the maximum of the expected overall discount returns realized through exercising the ESOs over time. This optimum value can be viewed as the initial cost of these ESOs for the company and thereby determines the optimal strategy for the employee. We derive the HJB equation governing the value function by the dynamic programming approach and stochastic analysis theory. Some properties of the value function are investigated in the sense of constrained viscosity solution. Due to the unavailability of the analytical solution to the HJB equation, we approximate the value function and the corresponding optimal control by numerical simulation. Furthermore, we analyze the impact of the varying parameters on the exercise decision accompanied by some financial explanations. The obtained results provide the reasonably estimated costs of ESOs for the company and the helpful suggestions for the employees on how to select right exercise moments to achieve most returns.

The nonstandard ESOs have involved many other outstanding features (see[10] for details), especially the risk that the employee would possibly get fired or leave the company voluntarily before the maturity of ESOs. It is interesting to investigate how such job termination risk would affect the employee's exercise behavior, which we hope to incorporate it into our future model. On the other hand, considering that most utility-based literature target at utility maximization to derive the optimal strategy for agents in an incomplete market, it is reasonable to shift our goal to study utility maximization strategy with trading constrains and seek other appropriate ways to give the fair price of ESOs. Much research and efforts are expected in this direction as well.

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