Non-critical $d = 2$ Gravities and Integrable Models[*]

K.S. Stelle

The Blackett Laboratory
Imperial College
Prince Consort Road
London SW7 2BZ

ABSTRACT

We review the origin of anomaly-induced dynamics in theories of $d = 2$ gravity from a BRST viewpoint and show how quantum canonical transformations may be used to solve the resulting Liouville or Toda models for the anomalous modes.

Two-dimensional worldsheet gravity models coupled to non-critical matter systems provide a very useful workshop for investigating the way in which induced dynamics can arise from anomalies. In this article, we shall first review the way in which such anomalous dynamics can arise within the context of BRST quantization. Then we shall present a technique for solving the resulting anomalous quantum system by canonical transformations, implemented by intertwining operators. We shall use these techniques to find the wavefunctions for the minisuperspace limits of Liouville and Toda $d = 2$ gravities. The integrable-model developments discussed in this article are adapted from Ref. [1].

We start from the action for a set of $d = 2$ scalar fields x^a, $a = 1, \ldots, D$, coupled to worldsheet gravity,

$$I = -\tfrac{1}{2} \int d^2\sigma \sqrt{-\gamma} \gamma^{ij} \partial_i x^a \partial_j x^b \eta_{ab}. \tag{1}$$

We pick a worldsheet parametrization, using light-cone indices $i, j = +, -$,

$$\gamma_{ij} = e^\omega \begin{pmatrix} \tilde{h} & 1 \\ 1 & h \end{pmatrix}, \tag{2}$$

---

[*] Contribution to the Proceedings of the 6th Regional Conference on Mathematical Physics, Islamabad, Pakistan, February 1994.

so that for the contravariant metric density one has

$$\sqrt{-\gamma}\gamma^{ij} = (1 - h\tilde{h})^{-\frac{1}{2}} \begin{pmatrix} -h & 1 \\ 1 & -\tilde{h} \end{pmatrix},\tag{3}$$

and the Weyl invariance of classical two-dimensional gravity is expressed by the fact that the conformal factor e^{ω} drops out in (3). This invariance is of course subject to anomalies, which we next shall discuss using the BRST formalism.

Anomalous Dynamics

In discussing the anomalies, we shall treat $\sigma^- = \sigma - \tau$ as the "evolution" coordinate on the worldsheet. Admittedly, this overlooks the fact that the surfaces $\sigma^- = $ const. are not actually proper Cauchy surfaces, because there exist some physical trajectories that do not cross them, but we shall not be concerned with this subtlety here. We shall, on the other hand, be more careful with the gauge fixing. Treating σ^- as the evolution coordinate and σ^+ as a "spatial" coordinate means that the gauge symmetry for h, $\delta h = \partial_- k + \ldots$, is similar to the transformation of the time component A_0 of the Maxwell gauge field, and requires a derivative gauge-fixing term [2], imposed by a Lagrange multiplier, $\int \pi \partial_- h$. On the other hand, we shall treat the left-moving sector gauge field $\tilde{h}$ as analogous to A_3 in Maxwell theory, so we shall consider the gauge-fixing term $\tilde{\pi}\tilde{h}$ to be acceptable. Similarly, we shall need to impose gauge-fixing for the Weyl symmetry, using the gauge-fixing term $\pi_\omega \omega$. When these gauge conditions are all satisfied, the world-sheet metric takes the "chiral light-cone gauge" form

$$\gamma_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & h \end{pmatrix},\tag{4}$$

giving the simple form $-\int d^2\sigma[(\partial_+ x^a \partial_- x^b - h\partial_+ x^a \partial_+ x^b)\eta_{ab}$ for the scalar-field action. Corresponding to these gauge-fixing conditions, we shall need to introduce (antighost,ghost) pairs (b, c), $(\tilde{b}, \tilde{c})$, (b_ω, c_ω). The corresponding gauge-fixed action taken together with the ghost action, which we shall denote by S, then has a tree-level BRST residuum of the original gauge symmetry,

$$\delta x^a = c\partial_+ x^a + \tilde{c}\partial_- x^a,$$
$$\delta c = c\partial_+ c + \tilde{c}\partial_- c, \qquad \delta\tilde{c} = \tilde{c}\partial_-\tilde{c} + c\partial_+\tilde{c},$$
$$\delta c_\omega = c\partial_+ c_\omega + \tilde{c}\partial_- c_\omega,\tag{5}$$

etc. This tree-level BRST symmetry can be encoded in the standard way [3] by including sources for the BRST variations of all fields,

$$\Sigma = S + \int K_{q^A}\delta q^A$$
$$= S + \int K_{x^a}(c\partial_+ x^a + \tilde{c}\partial_- x^a) + K_C(c\partial_+ c + \tilde{c}\partial_- c) + \ldots,\tag{6}$$

2

where the generalized index A runs over all of the fields of the theory, including the ghosts and antighosts. Including sources for the variations in this way allows us to write the tree-level BRST invariance simply as

$$(\Sigma, \Sigma) = 0, \tag{7}$$

where the Batalin-Vilkovisky antibracket [4] is defined by

$$(A, B) = \int \frac{\delta A}{\delta q^A} \frac{\delta B}{\delta K_{q^A}} + \frac{\delta A}{\delta K_{q^A}} \frac{\delta B}{\delta q^A}. \tag{8}$$

Note that the ghost number of the (A, B) antibracket is one more than the sum of the ghost numbers of A and B, where ghost number is defined by $G(c, \tilde{c}, c_\omega) = 1$, $G(b, \tilde{b}, b_\omega) = -1$. Upon quantization, the extended classical action Σ becomes the tree-level limit of the quantum effective action $\Gamma = \hbar^0 \Sigma + \hbar^1 \Gamma^{(1)} + \hbar^2 \Gamma^{(2)} + \ldots$. If the BRST symmetry (5) were unbroken at the full quantum level, one would expect to have the quantum Ward identity $(\Gamma, \Gamma) = 0$. However, this identity is disturbed by the presence of anomalies, giving instead the anomalous Ward identity

$$(\Gamma, \Gamma) = \Delta. \tag{9}$$

The anomaly Δ on the right-hand side of Eq. (9) is a local expression at lowest order, and at higher orders one encounters "dressings" of the anomalies that have already occurred at lower orders, owing to their presence in subdiagrams. These dressings constitute the expected quantum corrections to Green functions with an operator insertion. To express this more precisely, one should really include a source K_Δ for every local expression Δ occurring in (9) and write the right-hand side of (9) as $\delta\Gamma/\delta K_\Delta \equiv \Delta \cdot \Gamma$. At order $\hbar^n$, if one subtracts out the anticipated nonlocal dressings of lower-order anomalies, the residual anomalous terms of order $\hbar^n$ will be local. For our present purposes, it is sufficient to note that the structure of Eq. (9), together with the locality condition, yields the Wess-Zumino consistency condition on the anomalies. At the one-loop order, use of Eqs (7) and (8) together with the Jacobi identity $(A, (B, C)) + (B, (C, A)) + (C, (A, B)) \equiv 0$ yields the consistency condition

$$(\Sigma, \Delta) = 0. \tag{10}$$

Note that this condition is the same as that which governs the structure of renormalization counterterms, but here it is considered at ghost number one instead of zero. In fact, for the anomalies we are strictly interested in those solutions of (10) that *cannot* be removed by renormalization; the true anomalies are solutions of (10) such that

$$\Delta \neq (\Sigma, Y) \tag{11}$$

for any local functional Y of ghost number zero. Eqs (10) and (11) constitute the familiar cohomology problem of identifying the potential anomalies for a theory. Anomaly

functionals are representatives of cohomology classes and may be changed in form by the addition of counterterms to the action, but given (11) they cannot be completely removed. Once an anomaly has occurred in a theory, however, counterterms noninvariant under its corresponding symmetry may then occur. Thus, anomalies and their associated induced renormalizations need to be considered together.

In the present case, the solutions to the consistency condition may be written, after setting the variation sources K_{q^A} to zero,

$$\Delta = \alpha \int d^2\sigma \partial_+^2 h c_\omega + \beta \int d^2\sigma c_\omega. \tag{12}$$

Only the α coefficient is strictly an anomaly in the sense of Eq. (11), *i.e.*, of not being removable by renormalization. However, once the α coefficient is nonzero, the Weyl symmetry preventing the occurrence of the β coefficient as a renormalization counterterm is absent. The β coefficient then arises as a finite residuum after renormalization. The coefficients α and β of the potential anomalies in (12) depend on the central charge of the scalar fields x^a, and take the values $\alpha = (c-26)/24$, $\beta = \mu_0^2(c-2)/24$, where μ_0 is an infrared regularization mass [5]. Further finite counterterms could be added to shift the anomaly away from the Weyl symmetry (with ghost c_w) into the general coordinate symmetries $(c, \tilde{c})$, but we shall find it more appropriate to leave them in the form (12).

Given the presence of BRST anomalies in a theory, one may adopt one of two approaches to studying the resulting dynamics. One may proceed by a direct consideration of the correlation functions implied by the anomalous Ward identity (9). Within the context of BRST quantization, such a direct approach to the anomalies has not been widely adopted. However, a related procedure of holding off from integrating over the gauge fields in the generating-functional path integral until after the anomalies have appeared from the integrals over scalar "matter" fields has been used to show the existence of a hidden $SL(2, \mathbb{R})$ symmetry in the case of anomalous Liouville gravity [6]. A more frequently-adopted way to extract the anomalous dynamics is to eliminate the anomalies by compensation. In this procedure, an extra field is introduced into the theory, frequently in the context of a symmetry-preserving regularization. Classically, this extra field drops out of the theory by virtue of the as yet unbroken gauge symmetries, but in the regularized theory this field has non-trivial couplings and this fact allows for the possibility of a residual non-trivial coupling for it in the renormalized quantum theory. It should be emphasized that in such a compensation procedure, no anomalies actually occur in the BRST symmetry because the extra field allows for a removal of the potential anomalies by finite renormalizations. But the finite renormalizations have the effect of stopping the compensator from decoupling as it did at the classical level. The anomalous dynamics is then the dynamics of this extra field.

In the case of $d = 2$ gravity theories with the conformal anomalies (12), for a compensation procedure one introduces a classically-decoupling scalar mode $\phi(\sigma^+, \sigma^-)$ as an extra

"conformal" mode for the two-dimensional metric, writing

$$\hat{\gamma}_{ij} = e^{2\phi}\gamma_{ij}, \tag{13}$$

and then rewriting the action with the replacement of γ_{ij} by $\hat{\gamma}_{ij}$. Of course, the ϕ field classically drops out owing to the original Weyl invariance of the action (1). Including ϕ into the formalism produces the BRST transformations

$$\delta\phi = \delta_{\text{g.c.}} + \delta_\omega\phi;$$
$$\delta_{\text{g.c.}}\phi = c\partial_+\phi + \tilde{c}\partial_-\phi \qquad \delta_\omega\phi = -\tfrac{1}{2}c_\omega. \tag{14}$$

Writing the anomaly (12) in a manifestly generally-covariant fashion, one has

$$\Delta = \tfrac{1}{24}\int d^2\sigma\sqrt{-\hat{\gamma}}[(c-26)R(\hat{\gamma}) + (c-2)\mu_0^2]c_\omega. \tag{15}$$

One may verify that this manifestly-covariant form reduces to the form (12) upon use of the gauge-fixed form of the metric (4). The presence of the extra field ϕ changes our earlier discussion, however, in that (15) is no longer cohomologically non-trivial and can be eliminated by a finite counterterm in the action. To see this, we note a lemma that holds in $d = 2$,

$$\sqrt{-\hat{\gamma}}R(\hat{\gamma}) = \sqrt{-\gamma}R(\gamma) - 2\sqrt{-\gamma}\nabla^2\phi, \tag{16}$$

so that

$$\Delta = -\tfrac{1}{12}\int d^2\sigma[(c-26)(R(\gamma) - 2\nabla^2\phi) + (c-2)\mu_0^2 e^{2\phi}]\delta_\omega\phi. \tag{17}$$

Thus, with ϕ included, Δ is now cohomologically trivial because it may be eliminated by the finite local counterterm

$$\delta\Sigma = \tfrac{1}{12}\int d^2\sigma\sqrt{-\gamma}[(c-26)(\gamma^{ij}\partial_i\phi\partial_j\phi + R(\gamma)\phi) + \tfrac{1}{2}(c-2)\mu_0^2 e^{2\phi}]. \tag{18}$$

Note that by a redefinition of the ϕ field, $\phi \rightarrow \phi + \text{constant}$, one may alter the scale of the coefficient μ_0. As a result, the specific value of this coefficient is not of physical importance; only the fact that its value is nonvanishing is important. Although the counterterm (18) removes the anomaly from the BRST symmetry, it violates the classical decoupling of the ϕ field, which is now the expression of the anomaly in the compensated formalism, and this gives rise to new dynamics not apparent in the original classical theory. Note that for a subcritical matter central charge, $c < 26$, the action (18) for the scalar field ϕ is that for a positive-norm mode.

To continue further using the compensated-anomaly approach, one would like to treat the anomaly-induced ϕ field from the start as an extra regular scalar field in the action. This necessitates some changes to the above picture. For one thing, the kinetic term for ϕ in (18)

does not have a standard normalization. More serious is the fact that the functional measure for the ϕ mode (*i.e.* the measure for the quantum-mechanical inner product) is originally that for the conformal part of a worldsheet metric (13), and is not the standard translation-invariant measure for an ordinary scalar field. It is much more convenient for calculation to treat this field as an ordinary scalar. Doing so causes it in turn to make a further contribution of 1 to the total central charge, arising from anomalous diagrams in which ϕ occurs in loops. For our present purposes, we shall simply follow the argument of Ref. [7] and shall posit that the action for the ϕ mode is of the form (18), but with renormalized coefficients. Going over to a Euclidean signature for the worldsheet now, we thus shall work with

$$I_\phi = \int d^2 z \sqrt{\gamma} (\tfrac{1}{2}\gamma^{ij}\partial_i\phi\partial_j\phi + QR(\gamma)\phi + \tfrac{1}{2}\mu^2 e^{\lambda\phi}), \tag{19}$$

where $z = e^{\tau + i\sigma}$.

The coefficients Q and λ need to be fixed by the requirement of anomaly cancellation, for we shall still require the ϕ mode to eliminate the anomalies by compensation. When it is treated as an ordinary field with a translation-invariant functional metric, the ϕ field contributes an amount $c_\phi = 1 + 12Q^2$ to the central charge. Thus, the central charge condition for anomaly cancellation after the change in the functional metric is [8]

$$c_x + c_\phi = c_x + 1 + 12Q^2 = 26, \tag{20}$$

fixing the value of the "background charge" coefficient Q for a given set of "matter" scalar fields x^a. Similarly, requiring the cancellation of anomalies arising from the presence of the potential term $e^{\lambda\phi}$, which threatens to produce additional ϕ-dependent anomalies beyond those controlled by the central charge, one derives [9] that λ take the value

$$\lambda = Q - \sqrt{Q^2 - 2} = \tfrac{1}{12}(\sqrt{25 - c} - \sqrt{1 - c_x}). \tag{21}$$

This expression shows that imaginary values for the Liouville potential coefficient occur for $c_x > 1$, revealing one well-known aspect of the "$c = 1$ barrier," that is also clearly seen in the matrix model-treatment of non-critical string theories. The proper handling of cases for $c_x > 1$ remains an important open problem.

From the standpoint of conformal field theory, the anomaly cancellation condition (21) may be understood as the requirement that the Liouville potential be an operator with (left,right) chiral weights equal to (1,1). The weight of an operator $e^{\alpha\phi}$ is obtained by taking an operator product of this together with the holomorphic stress tensor

$$T = T_{zz} = -\tfrac{1}{2}(\partial\phi)^2 - Q\partial^2\phi, \tag{22}$$

where the Q term comes from the background charge term in the action (19). The resulting operator-product expansion contains the singular terms

$$T(z)e^{\alpha\phi(w,\bar{w})} \sim \frac{\Delta_\alpha e^{\alpha\phi(w,\bar{w})}}{(z - w)^2} + \frac{\partial(e^{\alpha\phi(w,\bar{w})})}{(z - w)}, \tag{23}$$

6

where $\Delta_\alpha = -\frac{1}{2}\alpha(\alpha + 2Q)$, so that the value for λ selected in (21) gives $\Delta_\alpha = 1$ as required.

Higher-spin Worldsheet Symmetries

Two-dimensional theories admit a much greater variety of consistent gauge symmetries than is possible in higher dimensions. The gauge fields in two-dimensional "gravity" theories all have enough local symmetries so that a naïve count of their continuous degrees of freedom gives the result zero. Correspondingly, there is no natural local candidate for a kinetic action for the spin-two gauge field, because the natural Einstein-Hilbert Lagrangian in $d = 2$ becomes just the Euler-number density. However, there is a natural anomaly-induced dynamics for the $d = 2$ metric, as we have seen. There is nothing in the derivation of this anomalous dynamics that restricts one to consideration of the spin-two metric, however. Indeed, it is now well-known that there exist consistent closed quantum algebras with infinitely many different combinations of spin-two and higher-spin generators. When one works in the chiral light-cone gauge (4) for the worldsheet metric, the residual chiral symmetry for the spin-two gauge field h is the Virasoro algebra. Extensions of the Virasoro algebra that include higher-spin generators are known generically as W algebras. Here, we shall be concerned principally with the most basic such extension, the W_3 algebra [10],

$$T(z)T(w) \sim \frac{\partial T}{(z - w)} + \frac{2T}{(z - w)^2} + \frac{\frac{1}{2}c}{(z - w)^4} \tag{24a}$$

$$T(z)W(w) \sim \frac{\partial W}{z - w} + \frac{3W}{(z - w)^2} \tag{24b}$$

$$\begin{aligned} W(z)W(w) \sim{} & \frac{1}{(z - w)}\Big(\frac{1}{15}\partial^3 T + \frac{16}{22 + 5c}\partial\Lambda\Big) \\ & + \frac{1}{(z - w)^2}\Big(\frac{3}{10}\partial^2 T + 2\frac{16}{22 + 5c}\Lambda\Big) \\ & + \frac{\partial T}{(z - w)^3} + \frac{2T}{(z - w)^4} + \frac{\frac{1}{3}c}{(z - w)^6}, \end{aligned} \tag{24c}$$

where Λ is a composite operator,

$$\Lambda(z) =: TT : (z) - \frac{3}{10}\partial^2 T(z), \tag{25}$$

in which the colons denote normal ordering.

The W_3 algebra has a central-charge structure that is determined by the central charge in its Virasoro subalgebra. In order to realize local W_3 symmetry, gauge fields of spin two (the h component of the usual metric in the gauge (4)) and spin three (generally denoted B) are needed. Upon gauge fixing, both of these symmetries require ghosts and both the spin-two and spin-three ghosts contribute to the central charge that must be canceled by the compensating scalars $\phi_{1,2}$ and the "matter" scalar fields x^a. In the W_3 case, the central charge

that needs to be canceled is [11] $c_{\text{gh}} = -26 - 74 = -100$, where the -74 contribution comes from the spin-three ghosts. It turns out that, given the requirement to cancel independent anomalies in both the spin-two and spin-three currents, there is no "critical" set of free fields x^a for which all the anomalies cancel, not even for 100 scalars. Thus, a compensating-field mechanism similar to that discussed above in the Liouville gravity case is essential. Realizations with arbitrary numbers of x^a fields exist [12]; the simplest realization is the original one [13], with no x^a fields, but with two compensating fields ϕ_1 and ϕ_2. An anomaly-free realization requires both of these fields to have background charges; their chiral stress-tensor is just the sum

$$T = T_1 + T_2 = [-\tfrac{1}{2}(\partial\phi_1)^2 - Q_1\partial^2\phi_1] + [-\tfrac{1}{2}(\partial\phi_2)^2 - Q_2\partial^2\phi_2]. \tag{26}$$

For this stress tensor, one has a central-charge contribution $c = c_1 + c_2 = 100$ for the values

$$Q_1 = \sqrt{\frac{49}{8}} \qquad Q_2 = \sqrt{\frac{49}{24}}; \tag{27}$$

this set leads to a cancellation of anomalies both in the spin-two stress-tensor algebra and also in the algebra of the spin-three generator, whose tree-level limit is given by

$$W = \tfrac{1}{3}(\partial\phi_1)^3 + Q_1\partial\phi_1\partial^2\phi_1 + \tfrac{1}{3}\partial^3\phi_1 + 2\partial\phi_1 T_2 + Q_1\partial T_2. \tag{28}$$

Note that the second compensating field ϕ_2 occurs in the spin-three current only through its stress-tensor T_2. This feature persists when quantum corrections to the realization (26) are taken into account, so that one may identify ϕ_2 as the compensating field for the Virasoro subalgebra, while ϕ_1 is the compensator for the spin-three sector.

As in the Liouville case, once the local W_3 symmetry has been broken by anomalies, new counterterms become possible in the theory and after the corresponding renormalizations, one should have finite residual interaction terms generalizing the Liouville potential $e^{\lambda\phi}$. Following the same logic of demanding the cancellation of potential compensating-field-dependent anomalies [9], or by demanding that the the corresponding operators be of weight (1,1) with respect to the full W_3 algebra [14], one obtains the allowed generalizations of the Liouville potential,

$$V_1 = e^{-\frac{3}{7}Q_1\phi_1 + \frac{3}{7}Q_2\phi_2} \tag{29a}$$

$$V_2 = e^{-\frac{6}{7}Q_2\phi_2}. \tag{29b}$$

As in the Liouville case, the magnitudes of the coefficients of these potentials may be altered by constant shifts of $\phi_{1,2}$, so the only physically-meaningful aspect of the coefficients of these potentials is their non-vanishing. The potentials (29) taken together with the kinetic terms for $\phi_{1,2}$ describe an A_2 Toda field theory.

Minisuperspace Approximation

In the presence of interaction potentials such as the Liouville potential $e^{\lambda\phi}$ or the Toda potentials (29), the dynamics of the compensating modes is discretely changed with respect to the dynamics of free fields, even thought the Liouville and Toda theories are integrable field theories. We shall review how some of these differences come about, concentrating on the center-of-mass modes of the compensating fields, which are the ones most affected by the potentials. Inclusion of the oscillating-string modes may then be carried out consistently within the context of perturbation theory, after the non-perturbative dynamics of the center-of-mass modes has been understood. The separation of the center-of-mass modes is known as the "minisuperspace approximation," where reference is made to a subspace of the "superspace" configuration space of metric states, and not to supersymmetry. In the Liouville case, one splits up the field ϕ as follows:

$$\phi(\tau, \sigma) = \frac{2}{\lambda} q(\tau) + \phi^{\text{osc}}(\tau, \sigma), \tag{30}$$

where $\phi^{\text{osc}}(\tau, \sigma)$ is defined to satisfy $\oint d\sigma \phi^{\text{osc}}(\tau, \sigma) = 0$, and the coefficient in front of $q(\tau)$ is for convenience of normalization in the minisuperspace action. Recall that λ is given in terms of Q by (21). (In most of the following, we shall concentrate on the case of "pure" Liouville gravity, for which $\frac{2}{\lambda} = -\frac{6Q}{5}$). In the Toda case, the splitup is (specializing to the two-field pure Toda gravity case, for which the background charges are as given in (27))

$$\phi_1(\tau, \sigma) = -\frac{4Q_1}{7} q_1(\tau) + \phi_1^{\text{osc}}(\tau, \sigma) \tag{31a}$$

$$\phi_2(\tau, \sigma) = -\frac{4Q_2}{7} (2q_2(\tau) - q_1(\tau)) + \phi_2^{\text{osc}}(\tau, \sigma), \tag{31b}$$

where again $\oint d\sigma \phi_{1,2}^{\text{osc}}(\tau, \sigma) = 0$. In extracting the σ-independent modes in (30,31), we are using the original (τ, σ) coordinates of a cylindrical string worldsheet instead of the complex $z = e^{\tau + i\sigma}$ coordinates generally used in conformal field theory. The change from the coordinate z to the coordinate $w = \ln z$ is effected by a conformal transformation on the worldsheet, generated by the stress tensor $T(z)$. Owing to the background-charge terms Q in (22,24), the transformations of the compensating fields are not quite those of ordinary scalars, but instead give, in the Liouville case,

$$\partial_z \phi \to (\partial_w \phi - Q) z^{-1}, \tag{32}$$

so that in the transition from z to w the momentum carried by a ϕ state is modified according to $p_\phi \to p_\phi(w) = p_\phi(z) - iQ$. This shift must be taken into account in comparing free-field states constructed using conformal-field theory with the minisuperspace wavefunctions that we shall discuss. Of course, the overall wavefunction in a string theory including the compensating modes ϕ or $\phi_{1,2}$ will be subject to the constraints following from varying h and

9

B in the action. These constraints impose "mass-shell" conditions that include contributions from the background charges. For our present purposes, however, we shall be content to treat such questions at the string-theory level in a perturbative fashion once we have understood the dynamics of the interacting compensating-field sectors in isolation. In particular, we shall be interested in finding the wavefunctions for the minisuperspace modes q and $q_{1,2}$.

Canonical Transformations for Integrable Models

The Liouville and Toda systems that emerge as the Lagrangians of the anomalous modes in ordinary and W-string theories are famous examples of integrable systems. They are integrable at the classical level because they possess sufficiently large symmetry algebras to give conserved quantities corresponding to all the degrees of freedom. This does not guarantee that these systems remain integrable at the quantum level, although this does in fact prove to be the case. Many different approaches have been followed in studying these problems. Owing to the importance of vertex operators in string theory, much effort has been expended on the promotion of exponentials of field operators to their analogues at the quantum level, taking into account the requirements of locality and covariance. One should mention along these lines especially the work of Gervais, Neveu and collaborators [15], together with the recent work of [16]. Here, we shall follow a different line of attack in concentrating on the actual wavefunctions of the Liouville and Toda theories. At the present stage of development of this approach, we shall confine our attention to the minisuperspace level discussed above. We shall aim to derive the wavefunctions of these theories by applying canonical transformations to map these interacting models onto corresponding free-field theories. In the process, a characteristic feature shall emerge: these canonical transformations are multi-valued (a feature also important in the approaches of [15,16]), so the relation to free-field theories is modified by the need to take a quotient of these free theories by the Weyl groups of the corresponding interacting Toda systems. The Weyl group symmetry plays a crucial rôle in the structure of the resulting integral representations for the wavefunctions.

We begin with the Liouville case. The classical Liouville Hamiltonian is

$$H_{\mathrm{L}} = \tfrac{1}{2}(p^2 + e^{2q}), \tag{33}$$

and in the following we shall let the evolution parameter be denoted by by $\tau = t$. The equations of motion following from (33) are integrable, since for the one variable q, we have a corresponding conserved quantity, namely H_{L} itself. This equality of the numbers of conserved quantities and independent variables persists also at the full field-theory level, owing to the infinite-dimensional Virasoro symmetry of the model. The general solution to the classical equations of motion following from (33) may be written

$$q = -\ln\left(\frac{1}{\tilde{p}}\cosh(\tilde{q}(t))\right), \tag{34}$$

10

where $\tilde{q} = \tilde{p}(t - t_0)$, and $\tilde{p}$ and t_0 are the two expected integration constants of the motion. Writing the general classical solution in this way suggests the following canonical transformation, in which $(\tilde{q}, \tilde{p})$ are now interpreted as a new pair of phase-space variables:

$$e^{-q} = \frac{1}{\tilde{p}} \cosh \tilde{q} \qquad (35a)$$

$$p = -\tilde{p} \tanh \tilde{q}. \qquad (35b)$$

The usefulness of this canonical transformation is that in the new $(\tilde{q}, \tilde{p})$ variables, the Hamiltonian becomes

$$\tilde{H}_{\mathrm{L}} = \tfrac{1}{2}\tilde{p}^2, \qquad (36)$$

which makes it plain that in the $(\tilde{q}, \tilde{p})$ variables we have a free system.

An important feature of the transformation from (33) to (36) is that the canonical transformation between them has a branch structure. The free Hamiltonian (36) has reflection symmetry in momentum space, $\tilde{p} \to -\tilde{p}$; this has the consequence that the inverse map from $(\tilde{q}, \tilde{p})$ to (q, p) is *two-to-one*. Both free-variable motions $(\tilde{q}, \tilde{p})$ and $(-\tilde{q}, -\tilde{p})$ correspond to the same solution of the interacting system (q, p). Clearly, for real p, Eq. (35a) cannot be solved for real $\tilde{p} < 0$, but for that case there is another canonical transformation that maps to a free system, obtained by flipping the signs of $\tilde{q}$ and $\tilde{p}$ in (35). Consequently, the general transformation to the free system can be written $e^{-q} = \frac{1}{|\tilde{p}|} \cosh \tilde{q}$; $p = -\tilde{p} \tanh \tilde{q}$, which makes the branch structure transparent. The Z_2 transformation on the free variables can be identified with the Weyl group of the underlying $A_1 = SL(2, \mathbb{R})$ group of the Liouville theory.

At the quantum level, one has to contend with the non-commuting nature of field operators. Nonetheless, one can still find a canonical transformation at the quantum level for the Liouville case if one first takes care to split up the overall transformation between the interacting and the free theories into small substeps, each of which remains canonical even when taking account of operator ordering and also which has a clear effect on quantum wavefunctions. Letting the overall generator of the transformation be denoted C, the canonical transformation may be written

$$C H_{\mathrm{L}} C^{-1} = \tilde{H}_{\mathrm{L}}, \qquad (37)$$

from which it is clear that what we are looking for is an operator that *intertwines* between the free and interacting Hamiltonians. The technique of constructing canonical transformations as intertwining operators has been developed by Anderson in Refs [17].

In the Liouville case, one decomposes the transformation into the following sequence of subtransformations [1]:

$$
\begin{array}{llll}
[\text{L1}] & \mathcal{P}_{\ln q} : & q \mapsto \ln q & p \mapsto qp \\
[\text{L2}] & \mathcal{I} : & q \mapsto p & p \mapsto -q \\
[\text{L3}] & p^{-1} : & q \mapsto p^{-1}qp = q + \mathrm{i}p^{-1} & p \mapsto p \\
[\text{L4}] & \mathcal{P}_{\sinh q} : & q \mapsto \sinh q & p \mapsto \frac{1}{\cosh q}p.
\end{array}
\qquad (38)
$$

It may be verified that each of the subtransformations in (38) is canonical in the quantum sense of preserving the canonical commutation relation $[p, q] = -\mathrm{i}$. Transformations [L1] and [L4] are point transformations, and have a straightforward action on Schrödinger representation wavefunctions, for $q \mapsto q'$, $\psi(q) \mapsto \psi(q')$. Transformation [L2] is implemented on wavefunctions by a Fourier transformation. Transformation [L3] is implemented on Schrödinger representation wavefunctions by indefinite integration in the argument q and multiplication by $-\mathrm{i}$. The overall transformation after combining [L1–L4] may be written

$$e^{-q} = \frac{1}{\tilde{p}} \cosh \tilde{q} \tag{39a}$$

$$p = -\tanh(\tilde{q})\tilde{p}, \tag{39b}$$

showing that, remarkably, the quantum canonical transformation is actually one of the simple ordering choices for the operators in (35). Corresponding to [L1–L4], we have the sequence of transformed Hamiltonians:

$$
\begin{aligned}
2H_{\mathrm{L}} &= p^2 + e^{2q} \\
[\mathrm{L}1] \quad &\mapsto (qp)^2 + q^2 = q^2 p^2 - \mathrm{i}qp + q^2 \\
[\mathrm{L}2] \quad &\mapsto p^2 q^2 + \mathrm{i}pq + p^2 \\
[\mathrm{L}3] \quad &\mapsto pq^2 p + \mathrm{i}qp + p^2 = (1+q^2)p^2 - \mathrm{i}qp = \left[(1+q^2)^{\frac{1}{2}}p\right]^2 \\
[\mathrm{L}4] \quad &= \tilde{p}^2 = 2\tilde{H}_{\mathrm{L}}.
\end{aligned} \tag{40}
$$

The generator C of the overall transformation (39) intertwines between H_{L} and $\tilde{H}_{\mathrm{L}}$, as we have seen. Using C^{-1}, we may obtain an eigenfunction of the interacting Hamiltonian by operating on a free-Hamiltonian eigenfunction $\tilde{\psi}_k(\tilde{q}) = \exp(\mathrm{i}k\tilde{q})$. Since C intertwines between H_{L} and $\tilde{H}_{\mathrm{L}}$, the resulting interacting-theory wavefunction must have the same eigenvalue, $\frac{1}{2}k^2$, as for the free wavefunction. The inverse intertwining operator is, from (38),

$$C^{-1} = \mathcal{P}_{e^q}\, \mathcal{I}^{-1}\, p\, \mathcal{P}_{\mathrm{arcsinh}\, q}. \tag{41}$$

In this way, one obtains

$$
\begin{aligned}
\psi_k(q) &\sim \frac{k}{\sqrt{2\pi}} \int_0^\infty dy\, e^{-\frac{\mathrm{i}}{2}e^q(y - y^{-1})} y^{\mathrm{i}k-1} \\
&= \frac{2k}{\sqrt{2\pi}} e^{\frac{\pi k}{2}} K_{\mathrm{i}k}(e^q),
\end{aligned} \tag{42}
$$

where $K_{\mathrm{i}k}$ is a modified Bessel function. Now we have to face the issue of normalization. The transformation (37) is canonical but is not unitary. As a consequence, normalization is not preserved; another way of expressing this is that the transformation has a non-trivial action also on the quantum-mechanical inner product. In order to have a properly-normalized

12

Liouville wavefunction with respect to the standard quantum-mechanical inner product, a normalization factor must be supplied. The final result, normalized to a delta function $\delta(k - k')$, is

$$\psi_k(q) = \frac{1}{\pi}\sqrt{2k\sinh(\pi k)}\, K_{\mathrm{i}k}(e^q). \tag{43}$$

In this result, this we note two related features. First, as a result of the symmetry of the modified Bessel function in its $\mathrm{i}k$ index, the Z_2 Weyl-group symmetry is now manifest in the interacting Liouville wavefunction, $i.e.$ $\psi_k(q) = \psi_{-k}(q)$. Second, the zero-eigenvalue wavefunction for $k = 0$, which was an acceptable delta-function normalizable wavefunction for the free Hamiltonian $\tilde{H}_{\mathrm{L}}$, drops out of the normalizable spectrum for the interacting Hamiltonian H_{L}.

Now consider the case of W_3 gravity. In the minisuperspace approximation, with the parametrization (31) for the center-of-mass modes, the Hamiltonian becomes

$$H_{\mathrm{T}} = \tfrac{1}{3}(p_1^2 + p_2^2 + p_1 p_2) + e^{2q_1 - q_2} + e^{2q_2 - q_1}. \tag{44}$$

In addition to the Hamiltonian, we have also the spin-three generator (28), whose minisuperspace limit is

$$W_{\mathrm{T}} = \tfrac{1}{18}(2p_1 + p_2)(2p_2 + p_1)(p_1 - p_2) + \tfrac{1}{2}(2p_2 + p_1)e^{2q_1 - q_2} - \tfrac{1}{2}(2p_1 + p_2)e^{2q_2 - q_1}. \tag{45}$$

The existence of these two first integrals and the consequent equality of the numbers of conservation laws and degrees of freedom makes Toda mechanics classically integrable. Following the pattern of the Liouville discussion, the classical solution leads to a canonical transformation over to free-field phase-space variables $(\tilde{q}_i, \tilde{p}_i)$, $i = 1, 2$:

$$e^{-q_1} = \frac{1}{\tilde{p}_1(\tilde{p}_1 - \tilde{p}_2)}e^{\tilde{q}_1} + \frac{1}{\tilde{p}_2(\tilde{p}_1 - \tilde{p}_2)}e^{\tilde{q}_2} + \frac{1}{\tilde{p}_1\tilde{p}_2}e^{-\tilde{q}_1 - \tilde{q}_2} \tag{46a}$$

$$e^{-q_2} = \frac{1}{\tilde{p}_1(\tilde{p}_1 - \tilde{p}_2)}e^{-\tilde{q}_1} + \frac{1}{\tilde{p}_2(\tilde{p}_1 - \tilde{p}_2)}e^{-\tilde{q}_2} + \frac{1}{\tilde{p}_1\tilde{p}_2}e^{\tilde{q}_1 + \tilde{q}_2} \tag{46b}$$

$$(2p_1 + p_2)e^{-q_1} = -\frac{(2\tilde{p}_1 - \tilde{p}_2)}{\tilde{p}_1(\tilde{p}_1 - \tilde{p}_2)}e^{\tilde{q}_1} - \frac{(2\tilde{p}_2 - \tilde{p}_1)}{\tilde{p}_2(\tilde{p}_1 - \tilde{p}_2)}e^{\tilde{q}_2} + \frac{(\tilde{p}_1 + \tilde{p}_2)}{\tilde{p}_1\tilde{p}_2}e^{-\tilde{q}_1 - \tilde{q}_2} \tag{46c}$$

$$(2p_2 + p_1)e^{-q_2} = \frac{(2\tilde{p}_1 - \tilde{p}_2)}{\tilde{p}_1(\tilde{p}_1 - \tilde{p}_2)}e^{-\tilde{q}_1} + \frac{(2\tilde{p}_2 - \tilde{p}_1)}{\tilde{p}_2(\tilde{p}_1 - \tilde{p}_2)}e^{-\tilde{q}_2} - \frac{(\tilde{p}_1 + \tilde{p}_2)}{\tilde{p}_1\tilde{p}_2}e^{\tilde{q}_1 + \tilde{q}_2}. \tag{46d}$$

The transformations (46) yield a free Hamiltonian and also a purely cubic version of the spin-three conserved quantity (45):

$$\tilde{H}_{\mathrm{T}} = \tfrac{1}{3}(\tilde{p}_1^2 + \tilde{p}_2^2 - \tilde{p}_1\tilde{p}_2) \tag{47}$$

$$\tilde{W}_{\mathrm{T}} = \tfrac{1}{18}(2\tilde{p}_1 - \tilde{p}_2)(2\tilde{p}_2 - \tilde{p}_1)(\tilde{p}_1 + \tilde{p}_2). \tag{48}$$

13

As in the Liouville case, the map between the interacting and free theories has a branch structure, now described by the Weyl group for the A_2 Toda theory, which is a symmetry of the free-theory invariants (47,48). In this case, the Weyl group is the discrete group S_3, whose six elements are generated by a threefold rotation

$$M: \qquad (\tilde{q}_1, \tilde{q}_2; \tilde{p}_1, \tilde{p}_2) \to (-\tilde{q}_1 - \tilde{q}_2, \tilde{q}_1; -\tilde{p}_2, \tilde{p}_1 - \tilde{p}_2), \qquad (49)$$

and a twofold reflection

$$R: \qquad (\tilde{q}_1, \tilde{q}_2; \tilde{p}_1, \tilde{p}_2) \to (\tilde{q}_2, \tilde{q}_1; \tilde{p}_2, \tilde{p}_1). \qquad (50)$$

As a result, the map from the free variables $(\tilde{q}_i, \tilde{p}_i)$ to the interacting variables (q_i, p_i) is six-to-one. Just as in the Liouville case, where all the *distinct* motions in the interacting theory are obtained from momenta $\tilde{p} > 0$, so in the Toda case all the distinct motions of the interacting theory are obtained by mapping from free-theory momenta that lie in the *principle Weyl chamber*: $\tilde{p}_1 > \tilde{p}_2 > 0$.

Once again, it turns out to be possible to promote classical integrability into quantum integrability by factorizing the canonical transformation (46) into a sequence of subtransformations, each of which has a clear effect on wavefunctions [1]:

[T1] $\quad \dfrac{e^{\frac{\pi}{2}(p_1+p_2)}}{\Gamma(1-\mathrm{i}(p_1+p_2))}:\quad$ $\begin{cases} e^{q_1} \mapsto -e^{q_1}(p_1+p_2), & p_1 \mapsto p_1 \\ e^{q_2} \mapsto -e^{q_2}(p_1+p_2), & p_2 \mapsto p_2 \end{cases}$

[T2] $\quad \mathcal{P}_{(\ln q_1, \ln q_2)}:\quad$ $\begin{cases} q_1 \mapsto \ln q_1, & p_1 \mapsto q_1 p_1 \\ q_2 \mapsto \ln q_2, & p_2 \mapsto q_2 p_2 \end{cases}$

[T3] $\quad q_1^{-1} q_2^{-2}:\quad$ $\begin{cases} q_1 \mapsto q_1, & p_1 \mapsto p_1 - \frac{\mathrm{i}}{q_1} \\ q_2 \mapsto q_2, & p_2 \mapsto p_2 - \frac{\mathrm{i}}{q_2} \end{cases}$

[T4] $\quad \exp\left(-\mathrm{i}(\frac{q_1^2}{q_2}+\frac{q_2^2}{q_1})\right):\quad$ $\begin{cases} q_1 \mapsto q_1, & p_1 \mapsto p_1 - \frac{q_2^2}{q_1^2} + \frac{2q_1}{q_2} \\ q_2 \mapsto q_2, & p_2 \mapsto p_2 - \frac{q_1^2}{q_2^2} + \frac{2q_2}{q_1} \end{cases}$

[T5] $\quad \mathcal{I}_1 \mathcal{I}_2:\quad$ $\begin{cases} q_1 \mapsto p_1, & p_1 \mapsto -q_1 \\ q_2 \mapsto p_2, & p_2 \mapsto -q_2 \end{cases}$

[T6] $\quad \mathcal{P}_{(q_1-q_2+\frac{1}{q_1 q_2}, \frac{1}{q_1}-\frac{1}{q_2}+q_1 q_2)}:$ $\begin{cases} q_1 \mapsto q_1' = q_1 - q_2 + \frac{1}{q_1 q_2}, & p_1 \mapsto p_1' = \frac{1}{\det \frac{\partial q_i'}{\partial q_j}} \left(\frac{\partial q_2'}{\partial q_2} p_1 - \frac{\partial q_2'}{\partial q_1} p_2 \right) \\[2ex] q_2 \mapsto q_2' = \frac{1}{q_1} - \frac{1}{q_2} + q_1 q_2, & p_2 \mapsto p_2' = \frac{1}{\det \frac{\partial q_i'}{\partial q_j}} \left(\frac{\partial q_1'}{\partial q_1} p_2 - \frac{\partial q_1'}{\partial q_2} p_1 \right) \end{cases}$

[T7] $\quad \mathcal{P}_{(e^{q_1}, e^{q_2})}:\quad$ $\begin{cases} q_1 \mapsto e^{q_1}, & p_1 \mapsto e^{-q_1} p_1 \\ q_2 \mapsto e^{q_2}, & p_2 \mapsto e^{-q_2} p_2. \end{cases}$

$$(51)$$

Among the transformations composing this free-field map, we have a type not yet encountered, the "similarity" transformations embodied in [T1,T3,T4] (although, strictly speaking, the inverse-momentum transformations [L3] are also of this type). The coordinate similarity transformations [T3,T4], of the form $(p_i \mapsto p_i - f_{,i}(q_j),\ q_i \mapsto q_i)$ are generated by $e^{\mathrm{i}f(q_j)}$, transforming wavefunctions $\Psi(q_j)$ into $e^{\mathrm{i}f(q_j)}\Psi(q_j)$ [17]. Momentum versions such as [T1], of the form $(q_i \mapsto q_i + f_{,i}(p_j),\ p_i \mapsto p_i)$, are generated by $e^{\mathrm{i}f(p_j)} = \mathcal{I}e^{\mathrm{i}f(q_j)}\mathcal{I}^{-1}$. The sequence of steps evolving the interacting into the free Hamiltonian is

$$3H_{\mathrm{T}} = p_1^2 + p_2^2 + p_1 p_2 + 3e^{2q_1-q_2} + 3e^{2q_2-q_1}$$

$$[\text{T1}] \qquad \mapsto p_1^2 + p_2^2 + p_1 p_2 - 3(e^{2q_1-q_2} + e^{2q_2-q_1})(p_1 + p_2)$$

$$[\text{T2}] \qquad \mapsto (q_1 p_1)^2 + (q_2 p_2)^2 + q_1 p_1 q_2 p_2 - 3\left(\frac{q_1^2}{q_2} + \frac{q_2^2}{q_1}\right)(q_1 p_1 + q_2 p_2)$$

$$[\text{T3}] \qquad \mapsto (p_1 q_1)^2 + (p_2 q_2)^2 + p_1 q_1 p_2 q_2 - 3\left(\frac{q_1^2}{q_2} + \frac{q_2^2}{q_1}\right)(p_1 q_1 + p_2 q_2) \qquad (52)$$

$$[\text{T4}] \qquad \mapsto (p_1 q_1)^2 + (p_2 q_2)^2 + p_1 q_1 p_2 q_2 - 9 q_1 q_2 - 3 p_2 q_1^2 - 3 p_1 q_2^2$$

$$[\text{T5}] \qquad \mapsto (q_1 p_1)^2 + (q_2 p_2)^2 + q_1 p_1 q_2 p_2 - 9 p_1 p_2 + 3 q_2 p_1^2 + 3 q_1 p_2^2$$

$$[\text{T6}] \qquad \mapsto (q_1 p_1)^2 + (q_2 p_2)^2 - q_1 p_1 q_2 p_2$$

$$[\text{T7}] \qquad \mapsto p_1^2 + p_2^2 - p_1 p_2 = 3\tilde{H}_{\mathrm{T}}.$$

The generator C of the transformation (51), which intertwines between the interacting and free theories to yield $CH_{\mathrm{T}}C^{-1} = \tilde{H}_{\mathrm{T}}$ and $CW_{\mathrm{T}}C^{-1} = \tilde{W}_{\mathrm{T}}$, also gives the Toda wavefunction by acting on a free wavefunction, $\Psi_{k_1,k_2}(q_1, q_2) \sim C^{-1}e^{\mathrm{i}(k_1 q_1 + k_2 q_2)}$, with the result

$$\Psi_{k_1,k_2}(q_1, q_2) = \frac{N_{k_1 k_2}}{2\pi} e^{\pi k_1} \int_0^\infty du\, e^{q_1+q_2} u^{-2} e^{-u - (e^{2q_1-q_2} + e^{2q_2-q_1})u^{-1}} \times$$

$$\int_0^\infty dy_1 \int_0^\infty dy_2\, [\text{jac}]\, y_1^{\mathrm{i}k_1} y_2^{\mathrm{i}k_2} e^{-e^{q_1}(y_1+y_2+\frac{1}{y_1 y_2})u^{-1}} e^{-e^{q_2}(\frac{1}{y_1}+\frac{1}{y_2}+y_1 y_2)u^{-1}},$$

$$(53)$$

where

$$[\text{jac}] = \frac{1}{y_1 y_2}(y_1 - y_2)(y_2 - \frac{1}{y_1^2})(y_1 - \frac{1}{y_2^2}) \qquad (54)$$

and $N_{k_1 k_2}$ is a normalization factor. This result is manifestly convergent and falls away quickly under the Toda potential, so that the normalization factor $N_{k_1 k_2}$ is calculable as a convergent integral obtained using (53). As in the Liouville case, the result after normalization should be fully Weyl-group symmetric, but the zero momentum state $(0,0)$ is again not normalizable and so drops out of the spectrum. Thus Toda theory is also a theory without a vacuum state. The result (53) for the Toda wavefunction is of a different form from previous results obtained principally by reduction of wavefunctions on group manifolds [18], but these

earlier forms may also be obtained by modifications of the intertwining-operator procedure [1].

Vertex Operators versus States

Now let us return to one of the underlying issues of conformal field theory and of string theory, namely the relation between vertex operators and states, using the insights gained from the above canonical transformations. This relation is clear enough in the case of free-field theory, but it is worth re-examining carefully in the more complicated cases with interacting Liouville or Toda modes. The link between an operator $\mathcal{O}$ and its associated state $\psi_{\mathcal{O}}(\phi(\sigma))$ at some time τ is frequently written in string theory as a path integral,

$$\psi_{\mathcal{O}}\big(\phi(\sigma)\big) = \int_{\mathcal{D};\,\xi|_{\partial\mathcal{D}}=\phi(\sigma)} [d\xi(\sigma_i)] \; e^{-\mathrm{i}I_{\mathrm{L}}}\, \mathcal{O}(\xi), \tag{55}$$

where the point on the worldsheet at which $\mathcal{O}$ acts locally is taken to correspond to negative temporal infinity $\tau \to -\infty$, and the domain of integration $\mathcal{D}$ for the $[d\xi]$ integral is over all worldsheets bounded by an end loop $\partial\mathcal{D}$ corresponding to the evaluation time τ, on which Dirichlet boundary conditions $\xi|_{\partial\mathcal{D}} = \phi(\sigma)$ are imposed. In free-field theory, which has a Fock-space interpretation and a normalizable vacuum state $|0\rangle$, this reproduces the usual conformal-field-theory expression $|\mathcal{O}\rangle = \lim_{z\to 0} \mathcal{O}(\phi(z))|0\rangle$ for the state associated to $\mathcal{O}$.

In our interacting theories, we may use our canonical transformations to evaluate path integrals such as (55). We shall consider the Liouville state associated to a vertex operator $\mathcal{O} = e^{\alpha\phi(z)}$, but shall restrict our discussion to the minisuperspace limit $\phi(z) \to q(t)$ and to the tree level. The expression for $\psi_{\mathcal{O}}(q)$ becomes just the path-integral form of the evolution operator from t_0 to t applied to an initial wavefunction $\mathcal{O}(q(t_0))$ where $t_0 \to -\infty$,

$$\psi_{\mathcal{O}}\big(\phi(t)\big) = \lim_{t_0 \to -\infty} e^{-\mathrm{i}H_{\mathrm{L}}(t-t_0)}\mathcal{O}\big(\phi(t_0)\big). \tag{56}$$

Letting $\alpha = \mathrm{i}p$, so $\mathcal{O}_p = e^{\mathrm{i}pq}$, means starting off at $t_0 \to -\infty$ with a simple plane wave even though this is definitely not an eigenstate (43) of the Liouville theory. Most non-stationary state wavefunctions dissipate in quantum mechanics, so it requires special circumstances for such a construction to give any final standing wave. The evaluation of $\psi_{\mathcal{O}}$ may be done [1] by using the intertwining operator (41) to calculate the Liouville Green function,

$$G(z, w; \Delta t) = [C^{-1} e^{-\frac{\mathrm{i}}{2}\tilde{p}^2 \Delta t} C \delta(q - w)](z), \tag{57}$$

where $\Delta t = t - t_0$. In this way, one obtains the time-evolved wavefunction

$$\psi|_{\mathcal{O}_p}(q, t) = 2^{\mathrm{i}p} \int_{-\infty}^{\infty} \frac{dk}{(2\pi)^2}\, k e^{\pi k} K_{\mathrm{i}k}(e^q) \Gamma\left(\frac{\mathrm{i}(p+k)}{2}\right) \Gamma\left(\frac{\mathrm{i}(p-k)}{2}\right) e^{-\frac{\mathrm{i}}{2}k^2 \Delta t}. \tag{58}$$

16

The behavior of $\psi|_{\mathcal{O}_p}(q,t)$ as $\Delta t \to \infty$ may be evaluated by contour-integral methods [1]. Here, we shall just summarize the results. The situation depends importantly on whether p is real or imaginary. The occurrence of imaginary momenta in noncritical string theory is occasioned by the presence of background charges Q as in (19). In subcritical cases ($d < 26$ for the ordinary string), the background charges need to push the central charge of the compensating Liouville mode up above its canonical value of 1, and in consequence, as one can see from (20), the background charge is then real. In integrations in correlation functions over the constant mode ϕ_0 of the Liouville field, one then has at the tree level (where the Euler number of the worldsheet is 2) an extra factor $e^{2Q\phi_0}$, as one can see from (19). This produces an extra "background" term of $-2iQ$ in momentum-conservation delta functions, and makes the consideration of imaginary momenta unavoidable. The normalizability implications of such imaginary momenta in the "gravitational dressing" of string states are not in our view yet fully established.

For real p, there are different cases depending on whether $p \gtrless 0$:

Real $p < 0$: $\psi|_{\mathcal{O}_p} \xrightarrow[\Delta t \to \infty]{} 0$ like $(\Delta t)^{-3/2}$;

Real $p > 0$: $\psi|_{\mathcal{O}_p} \xrightarrow[\Delta t \to \infty]{} K_{ip}(e^q)$.

Thus, for real $p < 0$, the initial plane-wave wavefunction just dissipates in expected for the generic case. For real $p > 0$, however, the path-integral implementation of the operator-state map (55) does work as desired and one is left with an (improperly-normalized) standing wave proportional to a single Liouville eigenstate (43). The difference between the $p \gtrless 0$ cases may be understood heuristically in terms of the need to set up a superposition of incoming and outgoing plane waves in order to create a Liouville eigenstate. For $p > 0$, this is possible owing to the entirely reflective nature of the potential $e^{2\phi}$. In this case, one has at $t_0 \to -\infty$ an incoming wave that subsequently reflects and produces an outgoing wave, with the superposition eventually settling down as $\Delta t \to \infty$ to a Liouville eigenstate of the form (43). For $p < 0$, on the other hand, the initial wave is purely outgoing and so there is no way to generate the incoming wave that would be needed to create a stationary state, so the wavefunction just dissipates as generically expected, like $(\Delta t)^{-3/2}$.

For imaginary values $p = i\beta$, there are again two cases depending on whether $\beta \gtrless 0$:

Imaginary $p = i\beta$, $\beta < 0$: $\psi|_{\mathcal{O}_p} \xrightarrow[\Delta t \to \infty]{} 0$ like $(\Delta t)^{-3/2}$;

Imaginary $p = i\beta$, $\beta > 0$: $\psi|_{\mathcal{O}_p} \xrightarrow[\Delta t \to \infty]{} \sum_{n=0}^{[\beta/2]} c_n K_{\beta-2n}(e^q)$,

where $[\beta/2]$ is the integer part of $\beta/2$.

Thus, for $\beta < 0$ one finds again the generic case of a dissipating wavefunction. For $\beta > 0$, however, one is left in general with not one but a whole superposition of imaginary-momentum Liouville eigenfunctions. The implications of this have not been fully worked out, but the issue is important for the proper interpretation of Liouville correlation functions, which have generally been considered using the vertex-operator construction. The

phenomenon of having only one sign of momentum give rise to a Liouville eigenstate is known as the Seiberg bound [19].

Conclusions and Open Problems

The technique of solving Toda theory models via canonical transformations implemented by intertwining operators highlights the similarities and differences between these integrable models and the free-field theories that are the basis for conformal field theory. Although the intertwining-operator technique still remains to be applied at the full field-theory level, indications on how that may be done can be obtained by comparison to Bäcklund transformation methods [20] that have been successfully applied to Liouville field theory. The method of Refs [20] relies on an ansätz based upon the classical generating functional $F(q, \tilde{q})$ for the canonical transformation. In essence, that approach expresses the interacting-theory wavefunction as an integral transform involving this generating functional,

$$\psi(q) = \int d\tilde{q} \, e^{iF(q,\tilde{q})} \tilde{\psi}(\tilde{q}), \tag{59}$$

In promoting this transformation to the quantum case, one has to require that $e^{iF(q,\tilde{q})}$ satisfy an analogue of our intertwining condition (37),

$$H_{\mathrm{L}}(q, p) \, e^{\mathrm{i}F(q,\tilde{q})} = \tilde{H}_{\mathrm{L}}(\tilde{q}, \tilde{p}) \, e^{\mathrm{i}F(q,\tilde{q})}, \tag{60}$$

where the momenta are realized as derivatives in the Schrödinger representation. In the case of Liouville theory, the classical generator actually satisfies the condition (60) without further quantum corrections. This could be related to the fact that our quantum transformation (39) turns out to be one of the simple operator-ordering versions of the classical transformation (35). Whether this luck will persist in the more general Toda cases remains to be determined.

From the BRST point of view, an open problem remains the role of the ghost fields in the field-theoretic extension of the canonical transformations and in the Weyl-group structure of these transformations. In the cases of free-field Virasoro or W_3 gravities with minimal field content (*i.e.* just the fields ϕ or $\phi_{1,2}$), it is remarkable that when one includes the oscillator states a Weyl-multiplet structures persists in the spectra, corresponding to Z_2 or S_3 transformations of the center-of-mass mode momenta [21, 1]. But these Weyl-group multiplets involve states of non-trivial ghost structure, unlike the situation at the minisuperspace level that we have considered here. This suggests that in worldsheet gravity theories the Liouville- or Toda-theory aspects cannot be completely disentangled from the gauge-theory aspects of the problem. Another puzzle in the BRST context is the origin of hidden symmetries such as the $SL(2, \mathbb{R})$ Kač-Moody symmetry of the correlation functions [6], and how such symmetries might be related to the Weyl-group symmetries in the canonical-transformation approach. Overall, it seems that unraveling the mysteries of non-critical worldsheet gravity theories will require a more profound synthesis of these different approaches.

REFERENCES

[1] A. Anderson, B.E.W. Nilsson, C.N. Pope and K.S. Stelle, "The Multivalued Free-Field Maps of Liouville and Toda Gravities," `hep-th/9401007`, *Nucl. Phys.* **B**, in press.

[2] C. Teitelboim, *Phys. Rev.* **D25** (1982) 3159.

[3] J. Zinn-Justin, "Renormalization of Gauge Theories," in *Proc. Int. Summer Inst. for Theor. Physics (Bonn, 1974)*.

[4] I.A. Batalin and G.A. Vilkovisky, *Phys. Lett.* **102B** (1981) 27;
Phys. Rev. **D28** (1983) 2567.

[5] C.M. Hull and P.K. Townsend, *Nucl. Phys.* **B301**1988 197.

[6] A.M. Polyakov, *Phys. Lett.* **101B** (1981) 207; *Mod. Phys. Lett.* **A2** (1987) 893.

[7] F. David and E. Guitter, *Euro Phys. Lett.* **3** (1987) 1169;
F. David, *Mod. Phys. Lett.* **A3** (1988) 1651;
J. Distler and H. Kawai, *Nucl. Phys.* **B321** (1989) 509.

[8] E. D'Hoker, *Mod. Phys. Lett.* **A**, (1991) 745.

[9] J. Benn, *Phys. Lett.* **B332** (1994) 329.

[10] A.B. Zamolodchikov, *Teo. Mat. Fiz.* **65** (1985) 644.

[11] J. Thierry-Mieg, *Phys. Lett.* **197B** (1987) 368.

[12] L.J. Romans, *Nucl. Phys.* **B352** (1991) 829.

[13] V.A. Fateev and A.B. Zamolodchikov, *Nucl. Phys.* **B280** (1987) 644.

[14] S.R. Das, A. Dhar and S.K. Rama, *Mod. Phys. Lett.* **A6** (1991) 3055; *Int. J. Mod. Phys.* **A7** (1992) 2295.

[15] J.-L. Gervais and A. Neveu, *Nucl. Phys.* **B199** (1982) 59; **B209** (1982) 125; **B224** (1983) 329; **B238** (1984) 125; **B238** (1984) 396; **B257 [FS14]** (1985) 59; **B264** (1986) 557; *Commun. Math. Phys.* **100** (1985) 15; *Phys. Lett.* **B151** (1985) 271;
J.-L. Gervais, *Commun. Math. Phys.* **130** (1990) 257; **138** (1991) 301; *Phys. Lett.* **B243** (1990) 85; *Nucl. Phys.* **B391** (1993) 287;
J.-L. Gervais and J. Schnittger, *Nucl. Phys.* **B413** (1994) 433.

[16] H.J. Otto and G. Weigt, *Phys. Lett.* **B159** (1985) 341; *Z. Phys.* **C31** (1986) 219;
G. Weigt, *Phys. Lett.* **B277** (1992) 79; "Canonical quantization of the Liouville theory, quantum group structures and correlation functions," in *Pathways to fundamental theories*, Proc. Johns Hopkins Workshop on Current Problems in Particle Theory 16 (World Scientific, 1993);
Y. Kazama and H. Nicolai, *Int. J. Mod. Phys.* **A9** (1994) 667.

[17] A. Anderson and R. Camporesi, *Commun. Math. Phys.* **130** (1990) 61;
A. Anderson, *Phys. Lett.* **B319** (1993) 157; *Phys. Rev.* **D47** (1993) 4458; *Ann. Phys.* **232** (1994) 292.

[18] M.A. Olshanetsky and A.M. Perelomov, *Phys. Rep.* **94** (1983) 313.

[19] N. Seiberg, "Notes on quantum Liouville theory and quantum gravity," in *Common trends in mathematics and quantum field theory*, Proc. of the 1990 Yukawa International Seminar, *Prog. Theor. Phys.* Suppl. 102 (1991).

[20] T. Curtright, "Quantum Bäcklund transformations and conformal algebras," in *Differential Geometric Methods in Theoretical Physics*, eds L.L. Chau and W. Nahm (Plenum Press, NY, 1990);
T. Curtright and G.I. Ghandour, "Using Functional Methods to Compute Quantum Effects in the Liouville Model," in *Quantum Field Theory, Statistical Mechanics, Quantum Groups and Topology,* eds T. Curtright, L. Mezincescu and R. Nepomechie (World Scientific, Singapore, 1992).

[21] H. Lu, C.N. Pope, X.J. Wang and K.W. Xu, *Class. Quantum Grav.* **11** (1994) 967.