Physics Letters B 756 (2016) 303–308

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

The photon angular momentum controversy: Resolution of a conflict between laser optics and particle physics

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ARTICLE INFO

Article history:
Received 15 January 2016
Received in revised form 5 March 2016
Accepted 8 March 2016
Available online 10 March 2016
Editor: A. Ringwald

Keywords:
Photon
Angular momentum
Laser optics
Particle physics

ABSTRACT

The claim some years ago, contrary to all textbooks, that the angular momentum of a photon (and gluon) can be split in a gauge-invariant way into an orbital and spin term, sparked a major controversy in the Particle Physics community, exacerbated by the realization that many different forms of the angular momentum operators are, in principle, possible. A further cause of upset was the realization that the gluon polarization in a nucleon, a supposedly physically meaningful quantity, corresponds only to the gauge-variant gluon spin derived from Noether's theorem, evaluated in a particular gauge. On the contrary, Laser Physicists have, for decades, been happily measuring physical quantities which correspond to photon orbital and spin angular momentum evaluated in a particular gauge. This paper reconciles the two points of view, and shows that it is the gauge invariant version of the canonical angular momentum which agrees with the results of a host of laser optics experiments.

A major controversy has raged in Particle Physics recently as to whether the angular momentum (AM) of a photon, and *a fortiori* a gluon, can be split into physically meaningful, i.e. measurable, spin and orbital parts. The combatants in this controversy (for access to the controversy literature see the reviews by Leader and Lorcé [1] and Wakamatsu [2]) seem, largely, to be unaware of the fact that Laser Physicists have been measuring the spin and orbital angular momentum of laser beams for decades! (for access to the laser literature see the reviews of Bliokh and Nori [3], Franke-Arnold, Allen and Padgett [4] and Allen, Padgett and Babiker [5]). My aim is to reconcile these apparently conflicting points of view. Throughout this paper, unless explicitly stated, I will be discussing only free fields.

I shall first consider QED, where $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{A}$ are field operators, and as is customary, employ rationalized Gaussian units. It is usually stated that the momentum density in the electromagnetic field (known, in QED, as the *Belinfante* version) is proportional to the Poynting vector, i.e.

$$\boldsymbol{P}_{\rm bel} = \int d^3x\, \boldsymbol{p}_{\rm bel}(x) \qquad \boldsymbol{p}_{\rm bel}(x) = \boldsymbol{E}\times\boldsymbol{B} \tag{1}$$

and it is therefore eminently reasonable that the AM should be given by

$$\boldsymbol{J}_{\rm bel} = \int d^3x\, \boldsymbol{j}_{\rm bel}(x), \tag{2}$$

where the Belinfante AM density is

$$\boldsymbol{j}_{\rm bel} = \boldsymbol{r}\times(\boldsymbol{E}\times\boldsymbol{B}). \tag{3}$$

Although this expression has the structure of an orbital AM, *i.e.* $\boldsymbol{r}\times\boldsymbol{p}$, it is, in fact, the *total* photon angular momentum density. On the other hand, application of Noether's theorem to the rotationally invariant Lagrangian yields the *Canonical* version which has a spin plus orbital part

$$\boldsymbol{J}_{\rm can} = \int d^3x\, \boldsymbol{j}_{\rm can} = \int d^3x\,[\boldsymbol{l}_{\rm can} + \boldsymbol{s}_{\rm can}] \tag{4}$$

where the canonical densities are

$$\boldsymbol{s}_{\rm can} = \boldsymbol{E}\times\boldsymbol{A} \quad \text{and} \quad \boldsymbol{l}_{\rm can} = E^i(\boldsymbol{x}\times\nabla)A^i \tag{5}$$

but, clearly, each term is gauge non-invariant.

Textbooks have long stressed a "theorem" that such a split cannot be made gauge invariant. Hence the controversial reaction when Chen, Lu, Sun, Wang and Goldman [6] claimed that such a split *can* be made. They introduce fields $\boldsymbol{A}_{\rm pure}$ and $\boldsymbol{A}_{\rm phys}$, with

$$\boldsymbol{A} = \boldsymbol{A}_{\rm pure} + \boldsymbol{A}_{\rm phys} \tag{6}$$

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http://dx.doi.org/10.1016/j.physletb.2016.03.023

where

$$\nabla \times \boldsymbol{A}_{\text{pure}} = \boldsymbol{0}, \quad \text{and} \quad \nabla \cdot \boldsymbol{A}_{\text{phys}} = 0 \tag{7}$$

which are, of course, exactly the same fields as in the Helmholz decomposition into longitudinal and transverse components[1]

$$\boldsymbol{A}_{\text{pure}} \equiv \boldsymbol{A}_{\parallel} \qquad \boldsymbol{A}_{\text{phys}} \equiv \boldsymbol{A}_{\perp}. \tag{8}$$

Chen et al. then obtain

$$\boldsymbol{J}_{\text{chen}} = \underbrace{\int d^3x\, \boldsymbol{E} \times \boldsymbol{A}_{\perp}}_{\boldsymbol{S}_{\text{chen}}} + \underbrace{\int d^3x\, E^i (\boldsymbol{x} \times \nabla) A^i_{\perp}}_{\boldsymbol{L}_{\text{chen}}} \tag{9}$$

and since $\boldsymbol{A}_{\perp}$ and $\boldsymbol{E}$ are unaffected by gauge transformations, they appear to achieved the impossible. The explanation is that the "theorem" referred to above applies to *local* fields, whereas $\boldsymbol{A}_{\perp}$ is, in general, *non-local*. In fact

$$\boldsymbol{A}_{\perp}(x) = \boldsymbol{A}(x) - \frac{1}{4\pi}\nabla \int d^3x' \, \frac{\nabla' \cdot \boldsymbol{A}(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}. \tag{10}$$

In all three versions of AM just mentioned, the integrands differ by terms of the general form $\nabla \cdot \boldsymbol{f}$, where $\boldsymbol{f}$ is some function of the fields, so that the integrated versions differ by surface terms at infinity, and thus agree with each other if the fields vanish at infinity. For classical fields, to state that a field vanishes at infinity, is physically meaningful, but what does it mean to say an operator vanishes at infinity? This issue is rarely addressed in the literature on the angular momentum controversy and the most recent serious analysis of this question seems to be that of Lowdon [7], utilizing axiomatic field theory. I shall comment later on his conclusions.

Now the key question is: what is the physical relevance of the various $\boldsymbol{S}$ operators? Can they be considered as genuine spin operators for the electromagnetic field? A genuine spin operator should satisfy the following commutation relations (for an interacting theory these should only hold as ETCs i.e. as Equal Time Commutators)

$$[\, S^i \,,\, S^j \,] = i\hbar \epsilon^{ijk} S^k. \tag{11}$$

But to check these conditions, manifestly, one must know the fundamental commutation relations between the fields and their conjugate momenta i.e. the quantization conditions imposed when quantizing the original classical theory, yet to the best of my knowledge, with only one exception [8], none of the papers in the controversy actually state what fundamental commutation relations they are assuming. Thus the expressions alone for the operators $\boldsymbol{S}$ are insufficient.

Failure to emphasize the importance of the commutation relations in a gauge theory can lead to misleading conclusions. It must be remembered that the quantization of a gauge theory proceeds in three steps:

(1) One starts with a gauge-invariant *classical* Lagrangian.
(2) One chooses a gauge.
(3) One imposes quantization conditions which are compatible with the gauge choice.

I shall comment on just two cases. In covariant quantization (cq) [9–11], for example in the Fermi gauge, one takes

$$[\, \dot{A}^i(\boldsymbol{x}, t) \,,\, A^j(\boldsymbol{y}, t)\,] = -i\delta^{ij}\delta(\boldsymbol{x} - \boldsymbol{y}), \tag{12}$$

and then the Hilbert space of photon states has an indefinite metric.

Quantizing in the Coulomb gauge one uses transverse quantization (tq) (see e.g. [12])

$$[\, \dot{A}^i(\boldsymbol{x}, t) \,,\, A^j(\boldsymbol{y}, t)\,] = -i\delta^{\perp}_{ij}(\boldsymbol{x} - \boldsymbol{y}) \tag{13}$$

$$\equiv -i \int \frac{d^3k}{(2\pi)^3} \left(\delta_{ij} - \frac{k_i k_j}{\boldsymbol{k}^2} \right) e^{i\boldsymbol{k}\cdot(\boldsymbol{x} - \boldsymbol{y})} \tag{14}$$

and the Hilbert space is positive-semidefinite.

There is an important physical consequence of this difference in quantization procedures. Gauge transformations on field *operators* almost universally utilize *classical* functions i.e.

$$\boldsymbol{A}(x) \to \boldsymbol{A}(x) + \nabla\alpha(x) \tag{15}$$

where $\alpha(x)$ is a "c-number" function. Clearly this transformation cannot alter the commutators. Or, put another way, gauge transformations are canonical transformations and therefore are generated by unitary operators, which do not alter commutation relations. This means that one cannot go from say Canonically quantized QED to Coulomb gauge quantized QED via a gauge transformation. This point was emphasized by Lautrup [9], who explains that although the theories are physically identical at the classical level, it is necessary to demonstrate that the physical predictions, meaning scattering amplitudes and cross-sections, are the same in the different quantum versions. This is also stressed by Cohen-Tannoudji, Dupont-Roc and Grynberg [13] on the basis that also the Hilbert spaces of the different quantum versions are incompatible.

It is not difficult to show that the canonical $\boldsymbol{S}_{\text{can}}$ with covariant quantization i.e. $\boldsymbol{S}^{\text{cq}}_{\text{can}}$ satisfies Eq. (11) and so is a genuine spin operator. However it is not gauge invariant. I shall comment on this presently.

For the Chen et al. case, since we are dealing with free fields, the parallel component of the electric field is zero i.e. $\boldsymbol{E}_{\parallel} = 0$ so that $\boldsymbol{J}_{\text{chen}}$ becomes

$$\boldsymbol{J}_{\text{chen}} = \int d^3x\, \boldsymbol{E}_{\perp} \times \boldsymbol{A}_{\perp} + \int d^3x\, E^i_{\perp} (\boldsymbol{x} \times \nabla) A^i_{\perp}. \tag{16}$$

But this is exactly the expression for $\boldsymbol{J}$, first discussed in [13], and later studied in great detail, with transverse quantization, by van Enk and Nienhuis (vE–N) in their classic paper [14], which, together with [15], is often the basis for statements about spin and orbital angular momentum in Laser Optics i.e. one has

$$\boldsymbol{S}^{\text{tq}}_{\text{chen}} \equiv \boldsymbol{S}^{\text{tq}}_{\text{vE–N}} = \int d^3x\, \boldsymbol{E}_{\perp} \times \boldsymbol{A}_{\perp} \tag{17}$$

and

$$\boldsymbol{L}^{\text{tq}}_{\text{chen}} \equiv \boldsymbol{L}^{\text{tq}}_{\text{vE–N}} = \int d^3x\, E^i_{\perp} (\boldsymbol{x} \times \nabla) A^i_{\perp}. \tag{18}$$

Now it is clear that $\boldsymbol{S}^{\text{tq}}_{\text{vE–N}}$, and $\boldsymbol{L}^{\text{tq}}_{\text{vE–N}}$, which are gauge invariant, are exactly the same as the gauge-variant canonical versions evaluated *in the Coulomb gauge*. For this reason, following [1], we shall henceforth refer to the Chen et al. = van Enk–Nienhaus operators as the Gauge Invariant Canonical (gic) operators. Thus

$$\boldsymbol{J}_{\text{gic}} = \boldsymbol{L}_{\text{gic}} + \boldsymbol{S}_{\text{gic}} = \int d^3x\, [\boldsymbol{l}_{\text{gic}} + \boldsymbol{s}_{\text{gic}}] \tag{19}$$

where the densities are

$$\boldsymbol{s}_{\text{gic}} = \boldsymbol{E}_{\perp} \times \boldsymbol{A}_{\perp} \quad \text{and} \quad \boldsymbol{l}_{\text{gic}} = E^i_{\perp} (\boldsymbol{x} \times \nabla) A^i_{\perp}. \tag{20}$$

[1] Indeed the only reason for the new nomenclature was Chen et al.'s intention to extend these ideas to QCD.

Now van Enk and Nienhuis show that the commutation relations for $\boldsymbol{S}^{tq}_{gic}$ are very peculiar and not at all like those in Eq. (11).[2] They demonstrate that

$$[\, S^{tq,\,i}_{gic}\,,\, S^{tq,\,j}_{gic}\,] = 0\,! \tag{21}$$

and stress that the components of $\boldsymbol{S}^{tq}_{gic}$ *cannot* therefore be considered as the components of a genuine spin vector in general. Moreover, they are careful to refer to this operator as the 'spin' *in inverted commas* (and similarly $\boldsymbol{L}^{tq}_{gic}$, is referred to as the 'orbital angular momentum'), but it seems that later papers on Laser Optics have not bothered to respect this convention.

Despite all these peculiarities it is claimed, correctly, that the spin and angular momentum of certain types of laser beam can and are regularly measured.[3] So the key question is how is this to be reconciled with the above, where, on the one hand, we have $\boldsymbol{S}^{cq}_{can}$ which looks like a genuine spin operator, but which is not gauge invariant and, on the other hand, $\boldsymbol{S}^{tq}_{vE\text{-}N}$, which in no way resembles a spin operator, but which is at least gauge invariant.

It was shown in [8] that $(\boldsymbol{S}^{cq}_{can} \cdot \boldsymbol{P}/|\boldsymbol{P}|)$, where $\boldsymbol{P}$ is the momentum operator, measures helicity and that its matrix elements between arbitrary physical photon states are gauge invariant. A key step in this proof was to consider the action of $(\boldsymbol{S}^{cq}_{can} \cdot \boldsymbol{P}/|\boldsymbol{P}|)$ on the physical photon state $|\,\boldsymbol{k},\, j\,\rangle$ with transverse polarization. Provided the operators are normal ordered one has

$$(\boldsymbol{S}^{cq}_{can} \cdot \boldsymbol{P}/|\boldsymbol{P}|)\,|\,\boldsymbol{k},\, j\,\rangle = \hat{k}^i\,[\, S^{cq,\,i}_{can}\,,\, a^\dagger(\boldsymbol{k},\, j)\,]\,|\text{vac}\rangle \tag{22}$$

and the commutator is then evaluated using the covariant quantization conditions. Acting on a state of helicity λ one eventually finds that $(\boldsymbol{S}^{cq}_{can} \cdot \boldsymbol{P}/|\boldsymbol{P}|)$ measures helicity:

$$(\boldsymbol{S}^{cq}_{can} \cdot \boldsymbol{P}/|\boldsymbol{P}|)\,|\boldsymbol{k}, \lambda\rangle = \lambda\hbar\,|\boldsymbol{k}, \lambda\rangle. \tag{23}$$

For the case of the helicity based on $\boldsymbol{S}^{tq}_{gic}$ the analogous commutator has to be evaluated using the transverse commutation conditions Eq. (13), but it turns out that the terms $k_i k_j$ don't contribute, so that also $\boldsymbol{S}^{tq}_{gic} \cdot \boldsymbol{P}/|\boldsymbol{P}|$ measures helicity i.e.

$$(\boldsymbol{S}^{tq}_{gic} \cdot \boldsymbol{P}/|\boldsymbol{P}|)\,|\,\boldsymbol{k},\, \lambda\,\rangle = \lambda\hbar\,|\,\boldsymbol{k},\, \lambda\,\rangle. \tag{24}$$

In summary, only the *helicity*, based either on $\boldsymbol{S}_{can}$ or on $\boldsymbol{S}_{gic}$, is physically meaningful as a measure of angular momentum. But, interestingly, as van-Enk and Nienhaus [14] show, the other components of $\boldsymbol{s}_{gic}$, though not angular momenta, are nevertheless measurable quantities. We shall see this concretely in the classical discussion which follows, where, it should be borne in mind that, unlike the QED situation, it is straightforward to compare expressions in different gauges. Somewhat surprisingly, it will be seen later that despite the failure to satisfy genuine angular momentum commutation rules *all* the components of $\boldsymbol{S}_{gic}$, as suggested by Eq. (41), are involved in the transfer of angular momentum to probes inserted in laser beams [3,16,17].

I turn now to the key question which has remained unresolved in the particle physics discussions, namely, which of the AM densities $\boldsymbol{j}_{bel}$, $\boldsymbol{j}_{can}$ or $\boldsymbol{j}_{gic}$ is relevant physically. Contrary to the opinion expressed in [1], where it is argued that it is simply a matter of taste, and to [7], which favours the Belinfante version, I shall argue that the laser experiments clearly indicate that it is $\boldsymbol{j}_{gic}$ which plays a direct role in the interaction of classical EM waves with matter and that the Belinfante expression is definitely unacceptable. The criticism that a density should not depend on a non-local field $\boldsymbol{A}_\perp$ does not apply to the situation of most interest, namely when dealing with monochromatic free fields with time dependence $e^{-i\omega t}$, since then $\boldsymbol{E} = \boldsymbol{E}_\perp = -\dot{\boldsymbol{A}}_\perp$ so that

$$\boldsymbol{A}_\perp = -\frac{i}{\omega}\,\boldsymbol{E} \tag{25}$$

is a local field.

Discussing the classical electrodynamics of laser fields, I shall follow custom and switch to SI units. *The only effect on all the previous formulae for momentum and AM densities is to multiply them by a factor* ϵ_0.

The real, monochromatic physical EM fields $(\mathcal{E}, \mathcal{B})$ are, as usual, expressed in terms of complex fields $(\boldsymbol{E}, \boldsymbol{B})$

$$\mathcal{E} = Re(\boldsymbol{E}) \qquad \boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}_0(\boldsymbol{r})\, e^{-i\omega t} \tag{26}$$

$$\mathcal{B} = Re(\boldsymbol{B}) \qquad \boldsymbol{B}(\boldsymbol{r}, t) = \boldsymbol{B}_0(\boldsymbol{r})\, e^{-i\omega t}. \tag{27}$$

We shall discuss the simplest physical case, where the force on, and the torque (about the centre of mass of a small neutral object), are given adequately in electric-dipole approximation (for a more general treatment see the papers of Chaumet and Nieto-Vesperinas [18,19], and also [16,17,20]),

$$\boldsymbol{F} = (\mathcal{P}.\nabla)\,\mathcal{E} + \dot{\mathcal{P}} \times \mathcal{B} \qquad \boldsymbol{\tau} = \mathcal{P} \times \mathcal{E} \tag{28}$$

where the induced electric dipole moment is given by

$$\mathcal{P} = Re[\alpha\,\boldsymbol{E}(\boldsymbol{r}, t)] \tag{29}$$

and the complex polarizability is

$$\alpha = \alpha_R + i\,\alpha_I. \tag{30}$$

First consider the force acting on the neutral dipole. It is known (for access to the literature see [21]) that the total force splits into two terms

$$\boldsymbol{F} = \boldsymbol{F}_{reactive} + \boldsymbol{F}_{dissipative} \tag{31}$$

where, for the cycle average, which I indicate by $\langle\,\rangle$,

$$\langle \boldsymbol{F}_{dissipative}\rangle = \frac{\alpha_I}{2} Im[E^{*i}\nabla E^i] \tag{32}$$

and for a classical electric dipole with momentum $\boldsymbol{P}_{dipole}$ it is $\boldsymbol{F}_{dissipative}$ that controls its rate of change of momentum (see Chapter V of [22])

$$\left\langle \frac{d\boldsymbol{P}_{dipole}}{dt} \right\rangle = \langle \boldsymbol{F}_{dissipative}\rangle. \tag{33}$$

Naturally, for the linear momentum, as for the AM, besides the Belinfante version Eq. (1), there exist also the gauge-variant canonical and gauge-invariant gic versions

$$\boldsymbol{P}_{can} = \epsilon_0 \int d^3x\, \mathcal{E}^i \nabla \mathcal{A}^i \tag{34}$$

and

$$\boldsymbol{P}_{gic} = \int d^3x\, \boldsymbol{p}_{gic} \quad \text{with} \quad \boldsymbol{p}_{gic} = \epsilon_0\, \mathcal{E}^i \nabla \mathcal{A}^i_\perp \tag{35}$$

and as in the AM case the three space-integrated versions are equal if the fields vanish at infinity.

Evaluating the cycle average, using Eq. (25), it turns out that

$$\langle \boldsymbol{F}_{dissipative}\rangle = \frac{\alpha_I \omega}{\epsilon_0}\,\langle \boldsymbol{p}_{gic}\rangle \tag{36}$$

so that it is the gauge-invariant canonical version that is physically relevant, and it is, of course, equal to the canonical version evaluated in the Coulomb gauge.

[2] Also $\boldsymbol{L}^{tq}_{gic}$ has peculiar commutation relations, but as expected, $\boldsymbol{J}^{tq}_{gic}$ behaves as a perfectly normal total angular momentum.

[3] Similar comments apply also to gluons.

Next consider the torque about the centre of mass of the dipole. One finds that

$$\mathcal{P} = \alpha_R\, \boldsymbol{\mathcal{E}} - \frac{\alpha_I}{\omega}\, \dot{\boldsymbol{\mathcal{E}}} \tag{37}$$

so that

$$\boldsymbol{\tau} = \frac{\alpha_I}{\omega}\, \boldsymbol{\mathcal{E}} \times \dot{\boldsymbol{\mathcal{E}}}. \tag{38}$$

For the cycle average, one finds

$$\langle \boldsymbol{\tau} \rangle = \alpha_I [Re\boldsymbol{E}_0 \times Im\boldsymbol{E}_0]. \tag{39}$$

Now consider the cycle average of $\boldsymbol{s}_{\rm gic}$ given in Eq. (20)

$$\langle \boldsymbol{s}_{\rm gic} \rangle = \frac{1}{2\omega}\, \epsilon_0\, Im[\boldsymbol{E}^* \times \boldsymbol{E}] = \frac{1}{\omega}\, \epsilon_0\, [Re\boldsymbol{E}_0 \times Im\boldsymbol{E}_0] \tag{40}$$

so that from Eq. (39) follows the fundamental result

$$\langle \boldsymbol{\tau} \rangle = \frac{\alpha_I \omega}{\epsilon_0} \langle \boldsymbol{s}_{\rm gic} \rangle. \tag{41}$$

The physical torque is thus given by a gauge-invariant expression, as it ought to be, which coincides with the canonical version evaluated in the Coulomb gauge, in accordance with the entire discussion in [3] (see also [17] and [20]). At first sight it may seem odd that only the spin vector enters Eq. (41), but it should be remembered that $\boldsymbol{\tau}$ is the torque about the centre of mass of the dipole, whereas $\boldsymbol{L}$ is the orbital AM about the origin of the axis system.

Consider now the application of these results to lasers. In the foundation paper on laser angular momentum by Allen, Beijersbergen, Spreeuw and Woerdman [23] the AM is associated with the gauge invariant Belinfante version in Eq. (3). It is therefore important to review some of the properties of the AM density $\boldsymbol{j}_{\rm bel}$ and of the Belinfante linear momentum, whose density is proportional to the Poynting vector. Firstly, for a plane wave propagating in the Z-direction the helicity is the same as the z-component of the angular momentum,[4] and, as shown in Section 2.6.4 of [1], for a left-circularly polarized i.e. positive helicity beam, $\boldsymbol{j}_{{\rm bel},\, z} = 0$, whereas, *per photon*

$$\boldsymbol{j}_{{\rm can},\, z} = \boldsymbol{s}_{{\rm can},\, z} = \boldsymbol{j}_{{\rm gic},\, z} = \boldsymbol{s}_{{\rm gic},\, z} = \hbar \tag{42}$$

as intuitively expected. Moreover, this result is much more general: $\boldsymbol{j}_{\rm bel}$ obviously has zero component in the direction of the Belinfante field momentum density:

$$\boldsymbol{j}_{\rm bel} \cdot \boldsymbol{p}_{\bf bel} = \epsilon_0^2\, [\boldsymbol{r} \times (\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{B}})] \cdot (\boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{B}}) = 0. \tag{43}$$

Thus the Belinfante AM fails, whereas the gauge invariant canonical version succeeds, in correctly generating the helicity. Secondly, and this seems most surprising in light of the initial comments on the controversy given above, it will be seen presently that for a superposition of polarized plane waves, $\boldsymbol{j}_{\rm bel}$ splits into two terms apparently corresponding to orbital and spin angular momentum [23].

In their analysis Allen et al. utilize the paraxial approximation, which corresponds to keeping the first two terms in an expansion in terms of a parameter equal to the beam waist divided by the diffraction length [25], and apply it to a Laguerre–Gaussian laser mode, but their treatment is actually more general and applies to any monochromatic, axially-symmetric vortex beam of finite cross-section. In such a beam propagating in the Z-direction the complex electric field, in paraxial approximation and in the notation often used in laser papers, has the form[5]

$$\boldsymbol{E} = i\omega \Big(u(\boldsymbol{r}),\, v(\boldsymbol{r}),\, \frac{-i}{k}\big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\big) \Big) e^{i(kz-\omega t)} \tag{44}$$

where

$$\Big|\frac{\partial u}{\partial z}\Big| \ll k|u| \qquad \Big|\frac{\partial v}{\partial z}\Big| \ll k|v| \tag{45}$$

and all second derivatives and products of first derivatives are ignored. As in [3] I shall indicate relations that are valid in paraxial approximation by "$\simeq$", so for example $\omega \simeq kc$.

For the case of circularly polarization

$$v = i\sigma_z u \tag{46}$$

where $\sigma_z = \pm 1$ for left/right circular polarization, and in cylindrical coordinates $(\rho,\, \phi,\, z)$

$$u(\rho, \phi, z) = f(\rho, z) e^{il\phi}. \tag{47}$$

For the cylindrical components of the cycle averaged Belinfante momentum density one finds

$$\langle p_{\rm bel} \rangle_\rho \simeq -\epsilon_0 \omega\, Im\big(u\frac{\partial u^*}{\partial \rho}\big) \qquad \langle p_{\rm bel} \rangle_z \simeq \epsilon_0 k\omega |u|^2 \tag{48}$$

$$\langle p_{\rm bel} \rangle_\phi \simeq \epsilon_0 \omega \left[\frac{l}{\rho}|u|^2 - \frac{\sigma_z}{2}\frac{\partial |u|^2}{\partial \rho} \right]. \tag{49}$$

Most interesting is the z-component of the Belinfante AM density[6]

$$\langle j_{\rm bel} \rangle_z = [\boldsymbol{r} \times \langle p_{\rm bel} \rangle]_z = \rho \langle p_{\rm bel} \rangle_\phi \simeq \epsilon\omega \left[l|u|^2 - \frac{\sigma_z}{2}\rho \frac{\partial |u|^2}{\partial \rho} \right], \tag{50}$$

implying the unintuitive result that *per photon*

$$\langle j_{\rm bel} \rangle_z^{\rm photon} \simeq l\hbar - \frac{\sigma_z \hbar}{2|u|^2}\rho \frac{\partial |u|^2}{\partial \rho}. \tag{51}$$

On the contrary for the gauge invariant canonical version one finds

$$\langle l_{\rm gic} \rangle_z \simeq \epsilon_0 \omega l |u|^2 \qquad \langle s_{\rm gic} \rangle_z \simeq \epsilon_0 \omega \sigma_z |u|^2 \tag{52}$$

implying the beautiful result *per photon*

$$\langle l_{\rm gic} \rangle_z^{\rm photon} \simeq l\hbar \qquad \langle s_{\rm gic} \rangle_z^{\rm photon} \simeq \sigma_z \hbar. \tag{53}$$

Not surprisingly, if one integrates Eq. (50) over the beam cross-section, one obtains, *per photon*,

$$\langle J_{\rm bel} \rangle_z^{\rm photon} \Big|_{beam} \simeq l\hbar + \sigma_z \hbar. \tag{54}$$

However, crucially, for small enough dipoles the angular momentum absorbed depends on the *local* AM density, which, comparing Eqs. (51), (53) is quite different for the Belinfante and gic cases, even differing in sign between the beam axis and the beam periphery. The first semi-quantitative test of the above was made by Garcés-Chávez, McGloin, Padgett, Dulz, Schmitzer and Dholakia [26] who succeeded in studying the motion of a tiny particle

[4] Note that for a field which is a more general superposition of plane waves the helicity is no longer simply a component of the spin vector. See e.g. [24].

[5] Often the z-component is put equal to zero, but that gives zero for the Belinfante angular momentum, whereas the laser papers have the non-zero value obtained below.

[6] Note that this does not contradict Eq. (43) since $\boldsymbol{j}_{\rm bel}$ does not point along the Z-direction.

trapped at various radial distances ρ from the axis of a so-called Bessel beam. The transfer of orbital AM causes the particle to circle about the beam axis with a rotation rate Ω_{orbit} whereas the transfer of spin AM causes the particle to spin about its centre of mass with rotation rate Ω_{spin}. Given that, for a Bessel beam, $|u|^2 \propto 1/\rho$ one finds for the Belinfante case that

$$\Omega_{\text{orbit}} \propto 1/\rho^3 \qquad \text{and} \qquad \Omega_{\text{spin}} \propto 1/\rho, \tag{55}$$

which is precisely the behaviour found experimentally, apparently showing the Belinfante expressions are the correct physical ones. However, exactly the same functional dependence on ρ follows from the gic expressions. In fact this equivalence is not restricted to Bessel beams. It holds as long as $|u|^2$ follows a simple power law behaviour $|u|^2 \propto \rho^{-\beta}$. Since the absolute rotation rates depend upon detailed parameters which, according to the authors, were beyond experimental control, it would be incorrect to interpret these results as evidence in favour of the Belinfante expressions. Moreover, in an unpublished paper [27], Chen and Chen have argued that the dependence on l and σ_z, of the shift of the diffraction fringes, found by Ghai, Senthilkumaran and Sirohi [28] in the single slit diffraction of optical beams with a phase singularity, implies that the correct expression for the optical angular momentum density is the gic one. And, further, as summarized in the recent review [3] it is the canonical AM in the Coulomb gauge i.e. the gic AM that agrees with a wide range of experiments.

For the linear momentum, on the other hand, it seems more difficult to prove experimentally that it is precisely the gic version that is the correct one, but there is solid experimental evidence that the Belinfante version does *not* describe the data. These experiments measure the optical forces acting in evanescent waves, and the first results were already reported in 1977 [29]! Recent measurements, using ultra-sensitive nano-cantilevers, confirm that the Belinfante version fails to describe the data [30], and suggest that the gic version works adequately.

In any event, I shall give a strong argument in favour of the gic version for photons. For the cycle averages one finds

$$\langle \boldsymbol{p}_{\text{bel}} \rangle = \langle \boldsymbol{p}_{\text{gic}} \rangle + \frac{\epsilon_0 \omega}{2} Im[(\boldsymbol{E} \cdot \nabla)\boldsymbol{E}^*] \tag{56}$$

and in the paraxial case under discussion this becomes

$$\langle \boldsymbol{p}_{\text{bel}} \rangle_{\text{paraxial}} = \langle \boldsymbol{p}_{\text{gic}} \rangle_{\text{paraxial}} - \frac{\epsilon_0 \omega \sigma_z}{2} \frac{\partial |u|^2}{\partial \rho} \hat{\phi}. \tag{57}$$

Following [22], assuming that the change of momentum of a small enough dipole is due to the momentum of the photons absorbed locally from the beam, I shall take the number of photons totally absorbed by the dipole per second to be given by $1/\hbar\omega$ times the rate of increase of the dipole's internal energy. For a small enough dipole in a paraxial beam I then find that Eqs. (36) and (33) are satisfied only if the average photon momentum is taken as

$$\langle \boldsymbol{p} \rangle^{\text{photon}} \Big|_{ave} \simeq \frac{1}{N} \langle \boldsymbol{p}_{\text{gic}} \rangle \tag{58}$$

where N is the number of photons per unit volume. A similar argument supports the gic version for the AM. Namely, assuming that the change in internal angular momentum of the dipole arises from photon absorption I find that Eq. (41) is satisfied only if

$$\langle \boldsymbol{s} \rangle^{\text{photon}} \Big|_{ave} \simeq \frac{1}{N} \langle \boldsymbol{s}_{\text{gic}} \rangle. \tag{59}$$

In summary, the angular momentum controversy, which has bedevilled particle physicists for some time, is resolved by a host of laser optics experiments which indicate that the Gauge Invariant Canonical linear momentum and angular momentum densities are the physically relevant ones, and that this is not simply a question of taste.[7] Moreover, although there does not exist a genuine spin vector for photons, the van Enk–Nienhuisen = Chen et al. = gic 'spin vector' plays a central role in Laser Optics and all of its components can, in principle, be measured, despite the fact that only one component, strictly speaking the helicity, is a genuine AM. For a paraxial beam propagating in the Z-direction one can show that the Z-component of the gic spin vector coincides with the gic helicity i.e. $\langle S_{\text{gic}} \rangle_z \simeq \langle \text{gic helicity} \rangle$, so this component is effectively a genuine AM. And finally, recognizing that the fundamental expressions are the gic ones, allows one to avoid the somewhat disturbing claim that what is physically measured corresponds to a gauge-variant quantity evaluated in a particular gauge, i.e. the Coulomb one.

In this paper we have not touched on the question of gluons and QCD, but the experience with photons suggests that also for gluons the analogue of the Gauge Invariant Canonical operators are the physically relevant ones. It remains a challenge, however, to find some consequences of this that could be tested experimentally.

I am grateful to K. Bliokh for making me aware of the vast literature on laser angular momentum, to S.M. Barnett, X-S. Chen, C. Lorcé, G. Nienhuis, M. Nieto-Vesperinas, A. Stewart and M. Wakamtsu for helpful comments, and to J. Qiu and R. Venugopalan for hospitality at Brookhaven National Laboratory. I thank the Leverhulme Trust for an Emeritus Fellowship.

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[7] This is in contradiction with Lowdon [7] who favours the Belinfante expressions. Starting with Belinfante he finds no reason to expect that the operator surface terms vanish, as would be necessary in order to make the Canonical AM equal to the Belinfante AM. However, if he had *started* with the Canonical AM he would have reached the opposite conclusion.

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