Moments at "discontinuous signals" with applications: model reduction for hybrid systems and phasor transform for switching circuits

Giordano Scarciotti and Alessandro Astolfi

***Abstract*— We provide an overview of the theory and applications of the notion of moment at "discontinuous interpolation signals", *i.e.* the moments of a system for input signals that do not satisfy a differential equation. After introducing the theoretical framework, which makes use of an integral matrix equation in place of a Sylvester equation, we discuss some applications: the model reduction problem for linear systems at discontinuous signals, the model reduction problem for hybrid systems and the discontinuous phasor transform for the analysis of circuits powered by discontinuous sources.**

I. Introduction

The model reduction problem consists in finding a simplified description of a complex model in specific operating conditions maintaining at the same time certain properties [1]. Model reduction by moment matching is a method that has the objective of determining a reduced order model which has a steady-state output response equal to the one of the system to be reduced for the same class of input signals [2]. In fact, the moments of a linear system are in one-to-one relation with the steady-state response of a particular interconnected system. In this paper we provide an overview of the results generated from the notion of moment at "discontinuous signals" introduced in [3], [4]. The origin of this notion is motivated by a large number of applications in which standard operating conditions are associated to non-continuous input signals, such as pulse width modulated waves in power converters. The generality of the signals we are able to interpolate (Section II), which include signals generated by time-invariant systems, time-varying systems, nonlinear systems and hybrid systems, is pointed out. The theoretical framework is introduced (Section III) and the notion of moment at discontinuous signals is defined using an integral matrix equation. We then cover several applications in which this new notion has been, or can be, used. First, we give (Section IV) a family of reduced order models by moment matching at discontinuous signals. Second, we extend (Section V) the theory of model reduction by moment matching to hybrid systems. Third, we give (Section VI) the notion of discontinuous phasor transform and show the use of this new analytical tool for the analysis of switching circuits.

II. A general class of signals

We begin with introducing the definition of explicit and implicit models and we present the general class of signal generators in explicit form. We show that this general class is a powerful description which includes the signals generated by very different classes of systems.

Definition 1: Let x, with $x(t) \in \mathbb{R}^n$, be the state of a dynamical system Σ. Let u, with $u(t) \in \mathbb{R}^m$, be the input of Σ. Let t_0 and $x_0 = x(t_0)$ be the initial time and the initial state, respectively. If there exists a function $\phi : \mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n$ such that

$$x(t) = \phi(t, t_0, x_0, u), \tag{1}$$

for all $t \geq t_0$, we call equation (1) the *representation in explicit form* [5], or the *explicit model*, of Σ.
Assume $\phi(t, t_0, x_0, u)$ has a continuous derivative with respect to t for every t_0, x_0 and u, and there exists a function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ continuous for each t over $\mathbb{R}^n \times \mathbb{R}^m$ such that

$$\dot{x} = f(x, u). \tag{2}$$

We call the differential equation (2) the *representation in implicit form* [6], or the *implicit model*, of Σ.
Consider a signal generators in explicit form described by the equation

$$\omega(t) = \Lambda(t)\omega(0), \qquad u = L\omega, \tag{3}$$

with $\Lambda(t) \in \mathbb{R}^{\nu\times\nu}$ such that $\Lambda(0) = I$. Note that (3) provides a very general class of models which contains the implicit model

$$\dot{\omega} = S\omega, \qquad u = L\omega, \tag{4}$$

but that describes several other signal generators. For instance, it can represent signals generated by a time-varying system of the form

$$\dot{\omega} = S(t)\omega, \qquad u = L\omega, \tag{5}$$

in which case $\Lambda(t)$ is the transition matrix associated to (5) [7, Section 3].
Equation (3) can also represent a signal generator described by some class of hybrid systems of the form

$$\begin{aligned} \dot{\omega}(t,k) &= S\omega(t,k), & u_c &= L_c\omega, \\ \omega^+ &= \omega(t,k+1) = J\omega(t,k), & u_d &= L_d\omega, \end{aligned} \tag{6}$$

which jumps and flows on some hybrid time domain. Note also that any periodic signal, linear with respect the initial condition, can be described by (3) adding the property

$$\Lambda(t) = \Lambda(t-T), \quad t \geq T, \tag{7}$$

where T is the period of the signal u. Not only any periodic signal can be represented with (3), but (3) can be regarded as

G. Scarciotti and A. Astolfi are with the Dept. of Electrical and Electronic Engineering, Imperial College London, London, SW7 2AZ, UK, E-mail: gs3610@ic.ac.uk.
A. Astolfi is also with DICII, University of Rome "Tor Vergata", Via del Politecnico 1, 00133 Rome, Italy, E-mail: a.astolfi@ic.ac.uk.

"the signal itself" generated by all the other representations. For instance, if we consider a square wave, then $\Lambda(t) = \sqcap(t)$, irrespective of the class of systems that generated the signal. For instance, $\sqcap(t)$ can be generated by a nonlinear system, *i.e.* $\sqcap(t) = \mathrm{sign}(\sin(t))$. Or it can be generated by the hybrid system (6). In fact, considering the case of periodic jumps, with period T, yields

$$\Lambda(t) = J^{\lfloor \frac{t}{T} \rfloor} e^{St}. \tag{8}$$

It is evident that the characterization of the moments for the explicit signal generator (3) would solve the problem of model reduction by moment matching for many different classes of input signals.

III. Integral definition of moment

Consider a linear, single-input, single-output, continuous-time, minimal, system described by the equations

$$\dot{x} = Ax + Bu, \qquad y = Cx, \tag{9}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$. To guarantee that the steady-state response of system (9) driven by (3), upon which the description of moment is based, exists, we need to introduce further hypotheses on the class of input signals (3).

Assumption 1: The vector $\omega(t)$ defined in equation (3) has a strictly proper Laplace transform with non-negative poles.

Assumption 2: The matrix valued function $\Lambda(t)$ is non-singular for all $t \geq 0$.

Assume now that there exists a set $\mathcal{T} \subset \mathbb{R}_{\geq 0}$ in which $\Lambda(t)$ is differentiable with respect to t and consider the time-varying system described by the equation

$$\dot{z}(t) = G(t)^\top z(t), \tag{10}$$

with $G(t) = -\dot{\Lambda}(t)\Lambda(t)^{-1}$. Let $\Phi(t)$ be the transition matrix of system (10).

Assumption 3: The function $G(t)$ is piecewise continuous with respect to t. Moreover, there exist $T \geq 0$ and a polynomial $p(t)$ such that $||\Phi(t)^\top|| \leq p(t)$ for all $t \geq T$.

Assumption 4: The triple $(L, \Lambda, \omega(0))$ is minimal, see [4].

Remark 1: Assumptions 1, 2, 3 and 4 are "mild" assumptions and the role of each assumption is explained in [4]. Note that they are satisfied by a general class of discontinuous periodic signals, see [8].

We report now the definition of moment as derived in [4].

Theorem 1: [4] Consider system (9) and the signal generator (3). Assume Assumptions 2 and 3 hold, $\sigma(A) \subset \mathbb{C}_{<0}$ and $\Lambda(t)$ is almost everywhere differentiable. Let

$$\Pi(t) = \left(e^{At}\Pi(0) + \int_0^t e^{A(t-\tau)} BL\Lambda(\tau) d\tau \right) \Lambda(t)^{-1}, \tag{11}$$

be a family of matrix valued functions parametrized in $\Pi(0) \in \mathbb{R}^{n\times\nu}$. Then there exists a unique $\Pi_\infty(0)$ such that, for any $\Pi(0)$, $\lim_{t\to+\infty} \Pi(t) - \Pi_\infty(t) = 0$, where $\Pi_\infty(t)$ is the solution of (11) with $\Pi(0) = \Pi_\infty(0)$. Moreover, if $x(0) = \Pi_\infty(0)\omega(0)$ then $x(t) - \Pi_\infty(t)\omega(t) = 0$ for all $t \geq 0$, and the set $\mathcal{M}_\infty = \{(x, \omega) \in \mathbb{R}^{n+\nu} \mid x(t) = \Pi_\infty(t)\omega(t)\}$ is attractive.

Note that the function $\Pi_\infty(t)$ solves the differential equation

$$\dot{\Pi}(t) = A\Pi(t) + BL - \Pi(t)\dot{\Lambda}(t)\Lambda(t)^{-1}, \tag{12}$$

with the initial condition $\Pi(0) = \Pi_\infty(0)$, for all $t \in \mathcal{T}$.

Definition 2: Consider system (9) and the signal generator (3). Suppose Assumptions 1, 2 and 3 hold and $\sigma(A) \subset \mathbb{C}_{<0}$. The function $C\Pi_\infty(t)$, where $\Pi_\infty(t)$ is the solution of equation (11) with $\Pi(0) = \Pi_\infty(0)$, is defined as the *moment of system (9) at* Λ.

Corollary 1: [4] Consider the interconnection of system (9) with the signal generator (3). Suppose Assumptions 1, 2, 3 and 4 hold and $\sigma(A) \subset \mathbb{C}_{<0}$. Then the moment of (9) at Λ coincides with the steady-state response of the output of the interconnected system (9)-(3).

The choice of defining the moment of (9) as in Definition 2 is justified by the equivalence, when an implicit model of (3) is available, between the new and the classical definition of moment (see [4, Theorem 3], [9]).

The determination of the initial condition $\Pi_\infty(0)$ is simplified if particular classes of systems are considered. For instance we may consider periodic signals, since they are of special interest for applications [3], [4], [8].

Corollary 2: [4] Consider system (9) and the signal generator (3). Assume Assumptions 2 and 3 hold and $\sigma(A) \subset \mathbb{C}_{<0}$. If for (3) the property (7) holds, then $\Pi_\infty(t)$ is periodic and equation (11) becomes

$$\Pi_\infty(t) = \left(I - e^{AT}\right)^{-1} \left[\int_{t-T}^t e^{A(t-\tau)} BL\Lambda(\tau) d\tau \right] \Lambda(t)^{-1}. \tag{13}$$

Remark 2: Exploiting the periodicity of the steady-state, $\Pi_\infty(t)$ defined in (13) has to be computed only over a period. This can be done off-line and the obtained values can then be used on-line for any time interval, considerably reducing the complexity of determining $\Pi_\infty(t)$.

IV. Model reduction at discontinuous signals

With this characterization of moment we can define a family of reduced order models for the linear system (9) at the input signals generated by (3) with the property (7). Note that the property (7) can be removed, as shown in [4].

Proposition 1: [4] Consider system (9) and the signal generator (3) with the property (7). Suppose Assumptions 1, 2, 3 and 4 hold and $\sigma(A) \subset \mathbb{C}_{<0}$. Then the system

$$\dot{\xi} = F\xi + Gu, \qquad \psi = C\Pi_\infty(t)P_\infty(t)^{-1}\xi(t), \tag{14}$$

with $\xi(t) \in \mathbb{R}^\nu$, $F \in \mathbb{R}^{\nu\times\nu}$, $G \in \mathbb{R}^{\nu\times\nu}$ and $\Pi_\infty(t)$ defined in (13), is a *model of system (9) at* (Λ, L), if $\sigma(F) \subset \mathbb{C}_{<0}$ and

$$P_\infty(t) = \left(I - e^{FT}\right)^{-1} \left[\int_{t-T}^t e^{F(t-\tau)} GL\Lambda(\tau) d\tau \right] \Lambda(t)^{-1}, \tag{15}$$

is non-singular for all $t \in \mathbb{R}_{\geq 0}$.

V. Model reduction for hybrid systems

First of all note that the problem of model reduction of linear systems at input signals generated by hybrid systems is already solved by Proposition 1, considering the matrix Λ defined in (8). An alternative formulation based on hybrid output regulation, but still inspired by the results of [4], has been given in [10]. Instead, herein we consider the more general problem of model reduction of hybrid systems at input signals generated by hybrid systems, extending in this way the results in the literature.

Define the hybrid time domain $\mathcal{H} := \{(t,k) : t \in [kT,(k+1)T], k \in \mathbb{Z}\}$ and the set $\mathbb{D}$ of complex numbers with modulus strictly smaller than one, *i.e.* $\mathbb{D} := \{s \in \mathbb{C} : |s| < 1\}$. Consider a linear, single-input, single-output, minimal, hybrid system flowing and jumping according to $\mathcal{H}$ described by the equations

$$\dot{x} = A_c x + B_c u_c, \quad x^+ = A_d x + B_d u_d, \quad y = Cx, \quad (16)$$

with $x(t,k) \in \mathbb{R}^n$, $u_{\{c,d\}}(t,k) \in \mathbb{R}$, $y(t,k) \in \mathbb{R}$, $A_{\{c,d\}} \in \mathbb{R}^{n\times n}$, $B_{\{c,d\}} \in \mathbb{R}^{n\times 1}$ and $C \in \mathbb{R}^{1\times n}$.

Theorem 2: Consider the interconnection of system (16) with the hybrid generator (6). Assume $\sigma\left(A_d e^{A_c T}\right) \subset \mathbb{D}_{<1}$. The steady-state output response of the interconnected system is $C\Pi_\infty(t)\omega$, where Π_∞ is the unique solution of

$$\begin{array}{rcl} \dot{\Pi}_\infty &=& A_c\Pi_\infty - \Pi_\infty S + B_c L_c, \\ \Pi_\infty(T)J &=& A_d\Pi_\infty(0) + B_d L_d, \end{array} \quad (17)$$

if and only if $\sigma\left(Je^{ST}\right) \cap \sigma\left(A_d e^{A_c T}\right) = \emptyset$.

Proof: It is easy to show that the output response of the interconnected system is

$$y(t,k) = C\Pi_\infty(t)\omega(t,k) + CA_d^k e^{A_c t}(x(0,0) - \Pi_\infty(0)\omega(0,0)),$$

with $\Pi_\infty(t) = \left(e^{A_c(t)}\Pi_\infty(0) + \int_0^t e^{A_c(t-\tau)}B_c L_c e^{S\tau}d\tau\right)e^{-St}$, where $\Pi_\infty(0)$ is the solution of the Sylvester equation

$$\Pi(0)_\infty e^{-ST}J - e^{-A_c T}A_d\Pi_\infty(0) = = -\int_0^T e^{-A_c\tau}B_c L_c e^{S(\tau-T)}d\tau + e^{-A_c T}B_d L_d.$$

The solution of this equation is unique if and only if $\sigma\left(Je^{ST}\right) \cap \sigma\left(A_d e^{A_c T}\right) = \emptyset$. Moreover the transient term $CA_d^k e^{A_c t}(x(0,0) - \Pi_\infty(0)\omega(0,0))$ decays to zero since $\sigma\left(A_d e^{A_c T}\right) \subset \mathbb{D}_{<1}$, proving the claim. ■

Note that $\Pi_\infty(t)$ defined in (17) is exactly the same matrix defined in Theorem 1, for the special signal generator (6).

Definition 3: Consider system (16) and the signal generator (6). Assume $\sigma\left(Je^{ST}\right) \cap \sigma\left(A_d e^{A_c T}\right) = \emptyset$ and $\sigma\left(A_d e^{A_c T}\right) \subset \mathbb{D}_{<1}$. The function $C\Pi_\infty(t)$, where $\Pi_\infty(t)$ is the solution of equations (17) is defined as the *moment of system (16) at* (S, L_c, J, L_d).

Proposition 2: Consider system (16) and the signal generator (6). Suppose $\sigma\left(Je^{ST}\right) \cap \sigma\left(A_d e^{A_c T}\right) = \emptyset$, $\sigma\left(A_d e^{A_c T}\right) \subset \mathbb{D}_{<1}$ and Assumption 4 hold. Then the system

$$\begin{array}{rcl} \dot{\xi} &=& (S - G_c L_c)\xi + G_c u_c, \\ \xi^+ &=& (J - G_d L_d)\xi + G_d u_d, \\ \psi &=& C\Pi_\infty(t)\xi(t), \end{array} \quad (18)$$

with $\xi(t,k) \in \mathbb{R}^\nu$, $\psi(t,k) \in \mathbb{R}$, $G_{\{c,d\}} \in \mathbb{R}^{\nu\times\not{k}}$ and $\Pi_\infty(t)$ the unique solution of (17), is a *model of system (16) at* (S, L_c, J, L_d), for any $G_{\{c,d\}}$ such that $\sigma\left(Je^{ST}\right) \cap \sigma\left((J - G_d L_d)e^{(S-G_c L_c)T}\right) = \emptyset$.

Proof: Under the hypotheses of the Proposition, the steady-state of the interconnection of system (16) and the signal generator (6) has a steady-state described by $C\Pi_\infty P_\infty^{-1}\omega$, with $P_\infty(t) = I$ the unique solution of

$$\begin{array}{rcl} \dot{P}_\infty &=& (S - G_c L_c)P_\infty - P_\infty S + G_c L_c \\ P_\infty(T)J &=& (J - G_d L_d)P_\infty(0) + G_d L_d. \end{array} \quad (19)$$

■

VI. Discontinuous phasor transform

Assume now that the state variables of the linear system (9) represent the currents and the integrals of the currents of an electrical circuit, *i.e.* they are obtained applying the Kirchhoff's Voltage Law. We assume also that the sources are voltage sources. In [8], see also [11], it has been shown that the phasors of a linear system are the moments of the system describing the circuit when a single complex interpolation point is selected. Hence, the integral definition of moment allows to define an extension of the phasor transform for discontinuous sources, *e.g.*, pulse width modulated waves.

Definition 4: The system (9) and the generator (3) are said to be in the *mixed convention* if the matrices A, B and C have real entries and the matrices L and Λ have complex entries.

We define now the phasor for sources described by (3).

Definition 5: [8] Consider system (9) and the signal generator (3) with the property (7). Assume Assumptions 1, 2, 3 and 4 hold, $\sigma(A) \subset \mathbb{C}_{<0}$ and $\Lambda(t)$ is almost everywhere differentiable. The components of the function $\Pi_\infty(t)$, defined in (13), are the *discontinuous phasors* of all the currents and of all the integrals of the currents in system (9) for the source $\Lambda(t)$. The *discontinuous inverse phasor transform* of the steady-state output current $i(t)$ of system (9) is

$$i(t) = \Re\left[\overline{I}(t)\Lambda(t)\right], \quad (20)$$

with $\overline{I}(t) = C\Pi_\infty(t)$.

Remark 3: Similarly to the sinusoidal case, the instantaneous currents are recovered multiplying the phasor with the source and taking the real part.

Remark 4: Differently from the sinusoidal case, the phasor $\overline{I}(t)$ is a time-dependent periodic function. Note that if $\Lambda(t)$ is sinusoidal, equation (13) defines the usual constant phasor and Π_∞ solves a Sylvester equation (see [11]).

Remark 5: The inverse phasor transform introduced in [12] is a particular case of the more general phasor transform we have introduced. In fact, that phasor transform is recovered when $\Lambda(t) = e^{j\omega(t)}$. Note moreover that in [12] the phasor itself, *i.e.* the direct phasor transform, is not defined.

Now that we have defined the discontinuous phasor and the discontinuous inverse phasor transform we extend the properties of the phasor circuit analysis describing the v-i characteristics of some common subcircuits which constitute power electronic devices. In fact, to be useful for applications we need to be able to compute the voltage across an inductor, capacitor and resistor given the phasor of the current which flows through these components. This is of paramount importance to be able to define the power and, more in general, to make this mathematical extension an accurate description of the physical quantities in the circuit. The expressions that relate voltage and current in an inductor, capacitor and resistor are, respectively,

$$v = \mathscr{L}\frac{di}{dt}, \qquad v = \frac{1}{\mathscr{C}}\int_0^t i\, d\tau, \qquad v = \mathscr{R}i. \tag{21}$$

Utilizing the classical phasor transform

$$f(t) = \Re\left[\overline{F}e^{j\omega t}\right], \tag{22}$$

it can be proved that the relations

$$\overline{V} = j\omega\mathscr{L}\overline{I}, \qquad \overline{V} = \frac{1}{j\omega\mathscr{C}}\overline{I}, \qquad \overline{V} = \mathscr{R}\overline{I}, \tag{23}$$

hold. When the source is described by the generator (3), these relations may not hold anymore. Consider, for instance, a square wave, which is described by the sum of infinitely many frequencies ω_k. It is exactly for this inability to deal with this type of signals without approximations that Definition 5 has been introduced. In fact, exploiting the discontinuous phasor transform we obtain the following exact relations.

Theorem 3: [8] Consider the equations in (21). The relations

$$\begin{aligned} \overline{V}(t) &= \mathscr{L}\dot{\overline{I}}(t) + \mathscr{L}\frac{\dot{\Lambda}(t)}{\Lambda(t)}\overline{I}(t), \\ \dot{\overline{V}}(t) + \frac{\dot{\Lambda}(t)}{\Lambda(t)}\overline{V}(t) &= \frac{1}{\mathscr{C}}\overline{I}(t), \\ \overline{V} &= \mathscr{R}\overline{I}(t), \end{aligned} \tag{24}$$

hold.

Remark 6: If $\Lambda(t) = e^{j\omega t}$, then $\dot{\overline{I}}(t) = 0$, $\dot{\overline{V}}(t) = 0$, $\dot{\Lambda}(t)\Lambda(t)^{-1} = j\omega$ and (24) become the relations in (23).

Remark 7: In the mixed convention, the components with odd indeces of Π_∞, computed from (13), are those functions that multiplied by Λ give the steady-state of the integrals of the currents.

Using the phasor transform (20) the instantaneous power is defined as

$$p(t) = v(t)i(t) = \Re\left[\overline{V}(t)\Lambda(t)\right]\Re\left[\overline{I}(t)\Lambda(t)\right], \tag{25}$$

which, exploiting the properties of the real part operator, yields

$$p(t) = \frac{1}{2}\Re\left[\overline{V}(t)^*\overline{I}(t)\Lambda(t)\Lambda(t)^*\right] + \frac{1}{2}\Re\left[\overline{V}(t)\overline{I}(t)\Lambda(t)^2\right]. \tag{26}$$

As in the sinusoidal case the instantaneous power is separated in two terms: the average of the first term is equal to the average power, whereas the average of the second term is zero. However, differently from the sinusoidal case, the first term is not constant, in general, and thus it is not equal to the average power. Hence, the average power and the reactive power are defined as follows.

Definition 6: In the phasor domain identified by the phasor transform (20), the average power P_a and the reactive power Q are defined as

$$\begin{aligned} P_a &= \frac{1}{2}\left\langle\Re\left[\overline{V}(t)^*\overline{I}(t)\Lambda(t)\Lambda(t)^*\right]\right\rangle, \\ Q &= \frac{1}{2}\left\langle\Im\left[\overline{V}(t)^*\overline{I}(t)\Lambda(t)\Lambda(t)^*\right]\right\rangle, \end{aligned} \tag{27}$$

where $< \cdot >$ represents the time average operator.

Equations (27) are consistent with the usual definition of average power and reactive power in the complex exponential case. For the non-exponential case, equations (27) generalize the respective relations achievable with the phasor transform (22). Note that one can say more when specific signals are considered. For instance, if the input signal is a square wave, $P_a = \frac{1}{4}\Re\left[\overline{V}(t)^*\overline{I}(t)\right]$.

VII. Conclusion

We have provided an overview of the results originated from the notion of moment at "discontinuous interpolation signals". We have introduced the target class of signal generators and we have shown its generality. We have defined the notion of moment with an integral matrix equation and we have discussed some applications presented in the literature, *i.e.* the model reduction problem for linear systems at discontinuous signals and the discontinuous phasor transform for the analysis of circuits powered by discontinuous sources, and a new application, *i.e.* the model reduction problem for hybrid systems.

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