

INTEGRABILITY OF DOMINATED DECOMPOSITIONS ON THREE-DIMENSIONAL MANIFOLDS

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ABSTRACT. We investigate the integrability of 2-dimensional invariant distributions (tangent sub-bundles) which arise naturally in the context of dynamical systems on 3-manifolds. In particular we prove unique integrability of dynamically dominated and volume dominated Lipschitz continuous invariant decompositions as well as distributions with some other regularity conditions.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let M be a smooth manifold and $E \subset TM$ a distribution of tangent hyperplanes. A basic question concerns the (unique) integrability of the distribution E , i.e. the existence at every point of a (unique) local embedded submanifold everywhere tangent to E . For one-dimensional distributions it follows from classical results on the existence and uniqueness of solutions of ODE's that regularity conditions suffice: existence is always guaranteed for continuous distributions and uniqueness for Lipschitz continuous distributions. For higher dimensional distributions the situation is more complicated and regularity conditions alone cannot guarantee integrability, indeed there exists arbitrarily smooth distributions which are not integrable [18]. It turns out however, that if the distributions are realized as $D\varphi$ -invariant distributions for some diffeomorphisms φ then some conditions can be formulated which imply integrability. More precisely, let M be a Riemannian 3-manifold, $\varphi : M \rightarrow M$ a C^2 diffeomorphism and $E \oplus F$ a continuous $D\varphi$ -invariant tangent bundle decomposition. For definiteness we shall always assume, without loss of generality, that $\dim(E) = 2$ and $\dim(F) = 1$. We state our results in the following subsections.

1.1. Dynamical domination and robust transitivity. A diffeomorphism φ is *transitive* if there exists a dense orbit, and *robustly transitive* if any C^1 sufficiently close diffeomorphism is also transitive. A $D\varphi$ -invariant decomposition $E \oplus F$ is *dynamically dominated* if there exists a Riemannian metric such that

$$(1.1) \quad \frac{\|D\varphi_x|_{E_x}\|}{\|D\varphi_x|_{F_x}\|} < 1$$

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for all $x \in M$. Then we have the following result.

Theorem 1. *Let M be a Riemannian 3-manifold, $\varphi : M \rightarrow M$ a volume-preserving or robustly transitive C^2 diffeomorphism and $E \oplus F$ a $D\varphi$ -invariant, Lipschitz, dynamically dominated decomposition. Then E is uniquely integrable.*

Remark 1.1. Our dynamical domination condition is usually referred to in the literature simply as *domination*, we use this non-standard terminology to avoid confusion in view of the fact that we will introduce below another form of domination. We remark also that the dynamical domination condition is usually formulated with the co-norm $m(D\varphi_x|_{F_x}) := \min_{v \in F, v \neq 0} \|D\varphi_x(v)\|/\|v\|$ instead of $\|D\varphi_x|_{F_x}\|$ but of course the two definitions are equivalent when F is one-dimensional, as here. We will also occasionally call E , the dominated bundle.

Remark 1.2. We mention that robustly transitive diffeomorphisms always admit a dynamically dominated decomposition [11], so the main assumption in this case is that the decomposition is Lipschitz.

1.2. Volume domination. We will obtain Theorem 1 as a special case of the following more general result which replaces the volume preservation and the robust transitivity condition with a volume “domination” condition. A $D\varphi$ -invariant decomposition $E \oplus F$ is *volume dominated* if there exists a Riemannian metric such that

$$(1.2) \quad \frac{|\det(D\varphi_x|_{E_x})|}{|\det(D\varphi_x|_{F_x})|} < 1$$

for all $x \in M$. Then we have the following result.

Theorem 2. *Let M be a Riemannian 3-manifold, $\varphi : M \rightarrow M$ a C^2 diffeomorphism and $E \oplus F$ a $D\varphi$ -invariant, Lipschitz continuous, dynamically and volume dominated, decomposition. Then E is uniquely integrable.*

Theorems 1 and 2 extend analogous statements in [19] obtained using different arguments, in arbitrary dimension but under the assumption that the decomposition is C^1 . They also extend previous results of Burns and Wilkinson [10], Hammerlindl and Hertz-Hertz-Ures [13, 27] and Parwani [23] who prove analogous results¹ for respectively C^2 , C^1 and Lipschitz distributions under the assumption of *center-bunching* or *2-partial hyperbolicity*:

$$(1.3) \quad \frac{\|D\varphi_x|_{E_x}\|^2}{\|D\varphi_x|_{F_x}\|} < 1$$

for every $x \in M$. In the 3-dimensional setting condition (1.3) clearly implies volume domination and is therefore more restrictive. In Section 1.3

¹In some of the references mentioned, the relevant results are not always stated in the same form as given here but may be derived from related statements and the technical arguments. In some cases the setting considered is that of partially hyperbolic diffeomorphisms with a tangent bundle decomposition of the form $E^s \oplus E^c \oplus E^u$ where E^s is uniformly contracting and E^u uniformly expanding. In this setting, one considers the integrability of the sub-bundles $E^{sc} = E^s \oplus E^c$ and $E^{cu} = E^c \oplus E^u$ and it is not always completely clear to what extent the existence of a uniformly expanding sub-bundle is relevant to the arguments. We emphasize that the setting we consider here does not require the invariant distribution E to contain any further invariant sub-bundle.

we sketch an example of a diffeomorphism and an invariant distribution E which does not satisfy condition (1.3) but does satisfy the dynamical domination and volume domination assumptions we require in Theorem 2. This particular example is uniquely integrable by construction and so is not a “new” example, but helps to justify the observation that our conditions are indeed less restrictive than center-bunching (1.3).

The techniques we employ here are similar to those of Parwani but to relax the center-bunching condition one needs a more careful analysis of the behaviour of certain Lie brackets, this is carried out in Section 3.

A more sophisticated version of our arguments also yields an alternative sufficient condition for integrability which is related to bundles which are Lipschitz along a transversal direction. Since the conditions are somewhat technical and it is not completely clear if they are satisfied by any natural examples we have “relegated” the precise formulation and proof to the Appendix. We do think nevertheless that both the statement and the techniques used are of some independent interest and discuss this further below.

Remark 1.3. The assumption that the diffeomorphisms in the Theorems above are C^2 is necessary for the arguments we use in the proofs. In the proof of Theorem 2, we need to be able to compute the Lie brackets of iterates of certain sections from E by $D\varphi$. For this reason $D\varphi$ needs to be C^1 to keep the regularity of a section along the orbit of a initial point p .

1.3. Volume Domination versus 2 Partially Hyperbolic. In this section we are going to sketch the construction of some non-trivial examples which satisfy the volume domination condition (1.2) but not the center-bunching condition (1.3). This is a variation of the “derived from Anosov” construction due to Mañé [22] (see [30] for the volume preserving case, which is what we use here). We are very grateful to Raúl Ures for suggesting and explaining this construction to us. Consider the matrix

$$\begin{pmatrix} -3 & 0 & 2 \\ 1 & 2 & -3 \\ 0 & -1 & 1 \end{pmatrix}$$

This matrix has determinant 1 and has integer coefficients therefore induces a volume preserving toral automorphism on \mathbb{T}^3 . It is Anosov since its eigenvalues are $r_1 \sim -0.11, r_2 \sim 3.11, r_3 \sim -3.21$. Note that $r_1 r_2 / r_3 < 1$ but $r_2^2 / r_3 > 1$. Hence (1.2) is satisfied but (1.3) is not. Now take a fixed point p and a periodic point q and a neighbourhood U of p so that forward iterates of q never intersect U . One can apply Mané’s construction to perturb the map on U as to obtain a new partially hyperbolic automorphism of \mathbb{T}^3 which is still volume preserving. Such a perturbation is not a small one and therefore one can not claim integrability of the new system trivially by using standard theorems as in [16]. Since the perturbation is performed on U and orbit of q never intersects U , the perturbation does not change the splitting and the contraction and expansion rates around q and in particular (1.3) is still not satisfied on the orbit of q . Yet the new example is volume preserving therefore it is necessarily the case that (1.2) is satisfied.

2. STRATEGY AND OVERVIEW OF THE PROOF

We will first show that Theorem 1 is a special case of Theorem 2. We will consider the volume preserving setting and the robustly transitive setting separately. We then discuss the proof of Theorem 2.

2.1. Volume preserving implies volume domination. We show that when φ is volume preserving, dynamical domination implies volume domination. Indeed, notice that $|\det(D\varphi_x|_{F_x})| = \|D\varphi_x|_{F_x}\|$ since F is one-dimensional, so the difference between dynamical domination and volume domination consists of the difference between $\|D\varphi_x|_{E_x}\|$ and $|\det(D\varphi_x|_{E_x})|$. These two quantities are in general essentially independent of each other; indeed considering the singular value decomposition of $D\varphi_x|_{E_x}$ and letting $s_1 \leq s_2$ denote the two singular values (since we assume E is 2-dimensional), we have that $\|D\varphi_x|_{E_x}\| = s_2$ and $|\det(D\varphi_x|_{E_x})| = s_1 s_2$. If $\|D\varphi_x|_{E_x}\| = s_2 < 1$ then we have a straightforward inequality $|\det(D\varphi_x|_{E_x})| = s_1 s_2 < s_2 = \|D\varphi_x|_{E_x}\|$ but this is of course not necessarily the case in general. However there is a relation in the volume preserving setting as this implies $|\det D\varphi_x|_E| \cdot |\det D\varphi_x|_F| = 1$ and so (1.1) implies $|\det D\varphi_x|_F| > 1$ (arguing by contradiction, $|\det D\varphi_x|_F| = \|D\varphi_x|_F\| \leq 1$ would imply $|\det D\varphi_x|_E| \geq 1$ by the volume preservation, and this would imply $\|D\varphi_x|_E\|/\|D\varphi_x|_F\| \geq 1$ which would contradict (1.1)). Dividing the equation $|\det D\varphi_x|_E| \cdot |\det D\varphi_x|_F| = 1$ through by $(|\det D\varphi_x|_F|)^2$ we get (1.2).

2.2. Robust transitivity implies volume domination. It is shown in [11] that if φ is a C^1 robustly transitive diffeomorphism of a compact 3-manifold then it admits a dominated splitting such that at least one of the following two conditions hold: *i*) φ admits a splitting of the form $E^s \oplus E^c \oplus E^u$ and there exists a Riemannian metric such that $\|D\varphi|_{E_x^s}\| < \|D\varphi|_{E_x^c}\| < \|D\varphi|_{E_x^u}\|$ and $\|D\varphi|_{E_x^s}\| < 1 < \|D\varphi|_{E_x^u}\|$; *ii*) either φ or φ^{-1} admits a splitting of the form $E \oplus F$ for which $\dim(E) = 2$ and E is volume contracting, i.e. there exists a Riemannian metric such that $|\det(D\varphi|_{E_x})| < 1 < \|D\varphi|_{F_x}\|$ for all $x \in M$. Both of these conditions easily imply volume domination for E .

2.3. Volume domination implies integrability. From now on we concentrate on Theorem 2 and reduce it to a key technical Proposition. We fix an arbitrary point $x_0 \in M$ and a local chart $(\mathcal{U}, x^1, x^2, x^3)$ centered at x_0 . We can assume (up to change of coordinates) that $\partial/\partial x^i, i = 1, 2, 3$ are transverse to E and thus we can define linearly independent vector fields X and Y , which span E and are of the form

$$X = \frac{\partial}{\partial x^1} + a \frac{\partial}{\partial x^3} \quad Y = \frac{\partial}{\partial x^2} + b \frac{\partial}{\partial x^3}.$$

where a and b are Lipschitz functions. Notice that it follows from the form of the vector fields X, Y that at every point of differentiability the Lie bracket is well defined and lies in the x^3 direction, i.e.

$$[X, Y] = c \frac{\partial}{\partial x^3}$$

for some L^∞ function c . In Section 3 we will prove the following

Proposition 2.1. *There exists $C > 0$ such that for every $k > 1$ and $x \in \mathcal{U}$, if the distribution E is differentiable at x then we have*

$$\|[X, Y]_x\| \leq C \frac{|\det(D\varphi_x^k|_{E_x})|}{|\det(D\varphi_x^k|_{F_x})|}.$$

Substituting the volume domination condition (1.2) into the estimate in Proposition 2.1, we get that the right hand side converges to 0 as $k \rightarrow \infty$, and therefore $\|[X, Y]_x\| = 0$ and so the distribution E is involutive at every point x at which it is differentiable. Theorem 2 is then an immediate consequence of the following general result of Simić [28] which holds in arbitrary dimension and a generalization of a well-known classical result of Frobenius proving unique integrability for involutive C^1 distributions.

Theorem ([28]). *Let E be an m dimensional Lipschitz distribution on a smooth manifold M . If for every point $x_0 \in M$, there exists a local neighbourhood \mathcal{U} and a local Lipschitz frame $\{X_i\}_{i=1}^m$ of E in \mathcal{U} such that for almost every point $x \in \mathcal{U}$, $[X_i, X_j]_x \in E_x$, then E is uniquely integrable.*

Remark 2.2. We mention that there are some versions of Proposition 2.1 in the literature for C^1 distributions and giving an estimate of the the $\|[X, Y]_x\| \leq |D\varphi_x^k|_{E_x}|^2/m|D\varphi_x^k|_{F_x}|$, see e.g. [27, 23]. For this quantity to go to zero, one needs the center bunching assumption (1.3). In our proposition, through more careful analysis, we relax the condition of center bunching (1.3) to volume domination (1.2).

3. LIE BRACKET BOUNDS

This section is devoted to the proof of Proposition 2.1, which is now the only missing component in the proof of Theorems 1 and 2. As a first step in the proof, we reduce the problem to that of estimating the norm of a certain projection of the bracket of an orthonormal frame. More specifically, let π denote the orthogonal projection (with respect to the Lyapunov metric which orthogonalizes the bundles E and F) onto F .

Lemma 3.1. *There exists a constant $C_1 > 0$ such that if $\{Z, W\}$ is an orthonormal Lipschitz frame for E and differentiable at $x \in \mathcal{U}$ then we have*

$$\|[X, Y]_x\| \leq C_1 \|\pi[Z, W]_x\|.$$

Proof. Notice that since F and $\frac{\partial}{\partial x_3}$ are transverse to E , then one has that $K_1 \leq \|\pi \frac{\partial}{\partial x_3}\| \leq K_2$ for some constants $K_1, K_2 > 0$. Moreover since $\|\pi[X, Y]\| = |c| \cdot \|\pi \partial / \partial x^3\|$ and $\|[X, Y]\| = |c|$ then it is sufficient to get an upper bound for $\|\pi[X, Y]\|$. Writing X, Y in the local orthonormal frame $\{Z, W\}$ we have

$$X = \alpha_1 Z + \alpha_2 W \quad \text{and} \quad Y = \beta_1 Z + \beta_2 W.$$

By bilinearity of the Lie bracket and the fact that $\pi(Z) = \pi(W) = 0$ since π is a projection along E , straightforward calculation gives

$$\|\pi[X, Y]\| = |\alpha_1 \beta_2 - \alpha_2 \beta_1| \cdot \|\pi[Z, W]\|$$

By orthonormality of $\{Z, W\}$, we have $|\alpha_i| \leq \|X\|, |\beta_i| \leq \|Y\|$ and since these are uniformly bounded, the same is true for $|\alpha_1 \beta_2 - \alpha_2 \beta_1|$ and so we get the result. \square

By Lemma 3.1 it is sufficient to obtain an upper bound for the quantity $\|\pi[Z, W]\|$ for some Lipschitz orthonormal frame. In particular we can (and do) choose Lipschitz orthonormal frames $\{Z, W\}$ of E such that for every $x \in \mathcal{U}$ and every $k \geq 1$ we have

$$\|D\varphi_x^k Z\| \|D\varphi_x^k W\| = |\det(D\varphi^k|_E)|.$$

For these frames will we prove the following.

Lemma 3.2. *There exists $C_2 > 0$ such that for every $k \geq 1$ and $x \in \mathcal{U}$, if the distribution E is differentiable at x we have*

$$\|\pi[Z, W]_x\| \leq C_2 \frac{|\det(D\varphi_x^k|_{E_x})|}{\|D\varphi_x^k|_{F_x}\|}.$$

Combining Lemma 3.2 and Lemma 3.1 and letting $C = C_1 C_2$ we get:

$$\|[X, Y]_x\| \leq C_1 \|\pi[Z, W]_x\| \leq C_1 C_2 \frac{|\det(D\varphi_x^k|_{E_x})|}{\|D\varphi_x^k|_{F_x}\|} = C \frac{|\det(D\varphi_x^k|_{E_x})|}{|\det(D\varphi_x^k|_{F_x})|}$$

which is the desired bound in Proposition 2.1 and therefore completes its proof.

To prove Lemma 3.2, observe first that for every $y \in M$ there exist 2 orthonormal Lipschitz vector fields A_y, B_y that span E in a neighborhood of y and by compactness we can suppose that we have finitely many pairs, say $(A_1, B_1), \dots, (A_\ell, B_\ell)$ of such vector fields which together cover the whole manifold. We denote by \mathcal{U}_i the domain where the vector fields A_i, B_i are defined and let

$$C_2 := \sup\{|\pi[A_i, B_i](x)| : 1 \leq i \leq \ell \text{ and almost every } x \in \mathcal{U}_i\}.$$

Note this constant C_2 is finite. In fact, by the standard fact that Lipschitz functions have weak differential which is essentially bounded (or L^∞), then for every $i \in \{1, \dots, \ell\}$ the function $|\pi[A_i, B_i]|$ is bounded. To complete the proof we will use the following observation.

Lemma 3.3. *For any Lipschitz orthonormal local frame $\{Z, W\}$ for E which is differentiable at $x \in M$, we have*

$$|\pi[Z, W]| \leq C_2$$

Proof. Write $Z = \alpha_1 A_i + \alpha_2 B_i$ and $W = \beta_1 A_i + \beta_2 B_i$ for some $1 \leq i \leq \ell$. Using the bilinearity of the Lie bracket and the fact that $\pi(A_i) = \pi(B_i) = 0$ we get $|\pi[Z, W]| = |\alpha_1 \beta_2 - \alpha_2 \beta_1| |\pi[A_i, B_i]|$. Since $\{A_i, B_i\}$ and $\{Z, W\}$ are both orthonormal frames, we have $|\alpha_1 \beta_2 - \alpha_2 \beta_1| = 1$, and so we get result. \square

Proof of lemma 3.2. For $k > k_0$ and $x \in \mathcal{U}$ such that E is differentiable at x , Let

$$\tilde{Z}(\varphi^k x) = \frac{D\varphi_x^k Z}{\|D\varphi_x^k Z\|} \quad \text{and} \quad \tilde{W}(\varphi^k x) = \frac{D\varphi_x^k W}{\|D\varphi_x^k W\|}$$

Recall that $D\varphi_x^k(E_x) = E_{\varphi^k(x)}$. Therefore, since Z, W span E in a neighborhood of x , then \tilde{Z}, \tilde{W} span E in a neighbourhood of $\varphi^k(x)$ and in particular $\pi(\tilde{Z}) = \pi(\tilde{W}) = 0$. Therefore we get

$$(3.1) \quad \|\pi[D\varphi^k Z, D\varphi^k W]\| = |\det(D\varphi^k|_{E^{(k)}})| \|\pi[\tilde{Z}, \tilde{W}]\|$$

Note that $\|\pi[D\varphi^k Z, D\varphi^k W]\| = \|\pi D\varphi^k[Z, W]\|$. Then by the invariance of the bundles we have

$$(3.2) \quad \|\pi D\varphi^k[Z, W]\| = \|D\varphi^k \pi[Z, W]\|.$$

Since F is one dimensional,

$$(3.3) \quad \|D\varphi^k \pi[Z, W]\| = \|D\varphi^k|_F\| \|\pi[Z, W]\|$$

Combining (3.2) and (3.3) we get

$$\|D\varphi^k|_F\| \|\pi[Z, W]\| = \|\pi D\varphi^k[Z, W]\|.$$

Putting this into equation (3.1) and using the fact that $\|\pi[\tilde{Z}, \tilde{W}]\|$ is uniformly bounded by lemma 3.3 one gets

$$\|\pi[Z, W]\| \leq C_2 \frac{|\det(D\varphi^k|_E)|}{\|D\varphi^k|_F\|}$$

This concludes the proof of Lemma 3.2. \square

APPENDIX A. SEQUENTIAL TRANSVERSAL REGULARITY

A counter example in [26] shows that the Lipschitz regularity condition in our Theorems cannot be fully relaxed, without additional assumptions, in order to guarantee unique integrability. Nevertheless the kind of techniques we use lead naturally to the formulation of a somewhat unorthodox regularity condition, which we call “sequentially transversal Lipschitz regularity”. The main reason that we choose to present this result is that the techniques used in the proof, especially those in Section A.4, generalize naturally to yield continuous Frobenius-type theorems, such as those given in forthcoming papers [21, 29]; we also believe that there are some interesting questions to be pursued regarding the relation between Lipschitz regularity and transversal Lipschitz regularity, we discuss these in Section A.2 below. We will give a “detailed sketch” of the arguments concentrating mostly on techniques which are novel, the full arguments can be found in a previous version of this paper [20].

A.1. Definition and statement of result. As above, let M be a 3-manifold, $\varphi : M \rightarrow M$ be a C^2 diffeomorphism, and $E \oplus F$ a continuous $D\varphi$ -invariant tangent bundle decomposition with $\dim(E) = 2$. We say that E is *sequentially transversally Lipschitz* if there exists a C^1 line bundle Z , everywhere transverse to E , and a C^1 distribution $E^{(0)}$ such that the sequence of C^1 distributions $\{E^{(k)}\}_{k>1}$ given by

$$(A.1) \quad E_x^{(k)} = D\varphi_{\varphi^k x}^{-k} E_{\varphi^k x}^{(0)}, \forall x \in M, k > 1$$

are equi-Lipschitz along Z , i.e. there exists $K > 0$ such that for every $x, y \in M$ close enough and belonging to the same integral curve of Z , and every $k \geq 0$, we have $\angle(E_x^{(k)}, E_y^{(k)}) \leq Kd(x, y)$.

Theorem 3. *Let M be a Riemannian 3-manifold, $\varphi : M \rightarrow M$ a C^2 diffeomorphism and $E \oplus F$ a $D\varphi$ -invariant, sequentially transversally Lipschitz, dynamically and volume dominated, decomposition. Then E is uniquely integrable.*

A.2. Relation Between Lipschitzness and Transversal Lipschitzness. Before starting the sketch of the proof of Theorem 3 we discuss some general questions concerning the relationships between various forms of Lipschitz regularity. We say that a sub-bundle E is *transversally Lipschitz* if there exists a C^1 line bundle Z , everywhere transverse to E , along which E is Lipschitz. The relations between Lipschitz, transversally Lipschitz, and sequentially transversally Lipschitz are not clear in general. For example it is easy to see that sequentially transversally Lipschitz implies transversally Lipschitz but we have not been able to show that transversally Lipschitz, or even Lipschitz, implies sequentially transversally Lipschitz. Nevertheless certain equivalence may exist under certain forms of dominations for bundles which occur as invariant bundles for diffeomorphisms. We formulate the following question:

Question 1. *Suppose $E \oplus F$ is a $D\varphi$ -invariant decomposition satisfying (1.3). Then is E transversally Lipschitz if and only if it is Lipschitz ?*

One reason why we believe this question is interesting is that that transversal Lipschitz regularity is a-priori strictly weaker than full Lipschitz regularity. Thus a positive answer to this question would imply that transversal Lipschitzness of center-bunched dominated systems becomes in particular, by Theorem 2, a criterion for their unique integrability. More generally, a positive answer to this question would somehow be saying that one only needs some domination condition and transversal regularity to prevent E from demonstrating pathological behaviours such as non-integrability or non-Lipschitzness.

The notion of sequential transversal regularity and the result of Theorem 3 may play a role in a potential solution to the question above. Indeed, if E is sequentially transversally Lipschitz and volume dominated, then by Theorem 3 it is uniquely integrable. Then, under the additional assumption of centre-bunching, by arguments derived from theory of normal hyperbolicity (see [16]) it is possible to deduce that E is Lipschitz along its foliation \mathcal{F} . We also know that there is a complementary transversal foliation given by integral curves of Z along which E is sequentially Lipschitz and therefore Lipschitz. This implies that E is Lipschitz.

Thus center-bunching and sequential transverse regularity implies Lipschitz. The missing link would just be to show that if E is transversally Lipschitz along a direction then it is also sequentially transversally Lipschitz along that direction. This would yield a positive answer to the question.

A.3. General philosophy and strategy of proof. Since our distribution is no longer Lipschitz we are not able to apply any existing general involutivity/integrability result, such as that of Simić quoted above². Instead we will have to essentially construct the required integral manifolds more or less explicitly “by hand”.

²Some notion of Lie bracket can be formulated in lower regularity, see for example [9, Proposition 3.1], but it is not clear to us how to obtain a full unique integrability result using these ideas.

The standard approach for this kind of construction is the so-called *graph transform* method, see [16], which takes full advantage of certain hyperbolicity conditions and consists of "pulling back" a sequence of manifolds and showing that the sequence of pull-backs converges to a geometric object which can be shown to be a unique integral manifold of the distribution. This method goes back to Hadamard and has been used in many different settings but, generally, cannot be applied in the partially hyperbolic or dominated decomposition setting where the dynamics is allowed to have a wide range of dynamical behaviour and it is therefore impossible to apply any graph transform arguments to E^{sc} under our assumptions. This is perhaps one of the main reasons why this setting has proved so difficult to deal with.

The strategy we use here can be seen as a combination of the Frobenius/Simić involutivity approach and the Hadamard graph transform method. Rather than approximating the desired integral manifold by a sequence of manifolds we approximate the continuous distribution E by a sequence $\{E^{(k)}\}$ of C^1 distributions obtained dynamically by "pulling back" a suitably chosen initial distribution. Since these approximate distributions are C^1 , the Lie brackets of C^1 vector fields in $E^{(k)}$ can be defined. *If the $E^{(k)}$ were involutive*, then each one would admit an integral manifold $\mathcal{E}^{(k)}$ and it is fairly easy to see that these converge to an integral manifold of the original distribution E . However this is generally not the case and we need a more sophisticated argument to show that the distributions $E^{(k)}$ are "asymptotically involutive" in a particular sense which will be defined formally below. For each k we will construct an "approximate" local center-stable manifold $\mathcal{W}^{(k)}$ which is not an integral manifold of $E^{(k)}$ (because the $E^{(k)}$ are not necessarily involutive) but is "close" to being integral manifolds. Further estimates, using also the asymptotic involutivity of the distributions $E^{(k)}$, then allow us to show that these manifolds converge to an integral manifold of the distribution E . We will then use a separate argument to obtain uniqueness, taking advantage of a result of Hartman.

A.4. Almost involutive approximations. In this section we state and prove a generalization of Proposition 2.1 which formalizes the meaning of "almost" involutive. We consider the sequence of C^1 distributions $\{E^{(k)}\}_{k>1}$ as in the definition of sequential transverse regularity in (A.1). We fix a coordinate system $(x^1, x^2, x^3, \mathcal{U})$ so that $\partial/\partial x^i$ are all transverse to E and therefore to $E^{(k)}$ for k large enough since $E^{(k)} \rightarrow E$ uniformly in angle. Then thanks to this transversality assumption we can find vector fields defined on \mathcal{U} of the form

$$(A.2) \quad X^{(k)} = \frac{\partial}{\partial x^1} + a^{(k)} \frac{\partial}{\partial x^3} \quad \text{and} \quad Y^{(k)} = \frac{\partial}{\partial x^2} + b^{(k)} \frac{\partial}{\partial x^3}.$$

that span $E^{(k)}$ and converge to vector fields of the form

$$(A.3) \quad X = \frac{\partial}{\partial x^1} + a \frac{\partial}{\partial x^3} \quad \text{and} \quad Y = \frac{\partial}{\partial x^2} + b \frac{\partial}{\partial x^3}.$$

that span E for $a^{(k)}$, a , $b^{(k)}$, b everywhere non-vanishing functions. Moreover we can choose $\partial/\partial x^3$ to be the direction where sequential transversal Lipschitzness holds true (since it is already transversal to E) so we have the

property that there exists $C > 0$

$$(A.4) \quad \left| \frac{\partial a^{(k)}}{\partial x^3} \right| < C \quad \text{and} \quad \left| \frac{\partial b^{(k)}}{\partial x^3} \right| < C$$

for all k . It is easy to check that $[X^{(k)}, Y^{(k)}]$ lies in the $\partial/\partial x^3$ direction. As before we will have some estimates about how fast the Lie brackets of these vector fields decay to 0. For the following let $F^{(k)}$ be the continuous bundle which is orthogonal to $E^{(k)}$ with respect to Lyapunov metric on E so that $F^{(k)}$ goes to F in angle (since F is orthogonal to E with respect to the Lyapunov metric). We have the following analogue of Proposition 2.1

Proposition A.1. *There exists $C > 0$ such that for every $k > 1$ and $x \in \mathcal{U}$, we have*

$$\|[X^{(k)}, Y^{(k)}](x)\| \leq C \frac{|\det(D\varphi_x^k|_{E_x^{(k)}})|}{\|D\varphi_x^k|_{F_x^{(k)}}\|}.$$

Sketch of proof. The proof of Proposition A.1 is very similar to that of Proposition 2.1 and it is not hard to get the result with the difference that E and F in the right hand side of the estimate in Proposition 2.1 are replaced by $E^{(k)}$ and $F^{(k)}$. In this case we choose our collection of C^1 orthonormal collection of frames $\{Z^{(k)}, W^{(k)}\}$ of $E^{(k)}$ so that $\|D\varphi^k Z^{(k)}\| \|D\varphi^k W^{(k)}\| = \det(D\varphi^k|_{E^{(k)}})$. Then exactly as in lemma 3.3, to get an upper bound on $\|[X^{(k)}, Y^{(k)}]\|$, it is enough to bound $[Z^{(k)}, W^{(k)}]$. The proof of the inequality $\|[Z^{(k)}, W^{(k)}]\| \leq \det(D\varphi^k|_{E^{(k)}})/\|D\varphi^k|_{F^{(k)}}\|$ follows quite closely the proof of lemma 3.2 where the projection π is replaced by $\pi^{(k)}$ which the projection to $F^{(k)}$ along $E^{(k)}$ at relevant places. \square

The next, fairly intuitive but in fact quite technical, step is to replace the estimates on the approximations with estimates on the limit bundle.

Proposition A.2. *There exists $C > 0$ such that for every $k > 1$ and $x \in \mathcal{U}$, we have*

$$(A.5) \quad |\det(D\varphi_x^k|_{E_x^{(k)}})| \leq C |\det(D\varphi_x^k|_{E_x})|$$

and

$$(A.6) \quad \|D\varphi_x^k|_{F_x^{(k)}}\| \geq C \|D\varphi_x^k|_{F_x}\|$$

Sketch of proof. (A.6) is fairly easy since for any vector $v \notin E$, $|D\varphi^k v| \geq C D\varphi_x^k|_{F_x} |v|$ (since F has dimension 1). The real technical estimate is (A.5). One first needs to make the observation that there exists a cone $C(\alpha)$ of angle α around E such that $D\varphi^k E^{(k)} \subset C(\alpha)$. This is the main observation that allows to relate two determinants independent of k . Indeed given a basis v_1^k, u_1^k of $E^{(k)}$, then $K |\det(D\varphi_x^k|_{E_x^{(k)}})| = |D\varphi_x^k v_1^k \wedge D\varphi_x^k u_1^k| = |\Lambda(D\varphi_x^k) v_1^k \wedge u_1^k|$ where $\Lambda(D\varphi_x^k)$ is the induced action of $D\varphi_x^k$ on $TM \wedge TM$. But then $\Lambda(D\varphi_x^k)$ allows a dominated splitting of $TM \wedge TM$ whose invariant spaces are $E_1 = E \wedge E$, $E_2 = E \wedge F$ where E_1 is dominated by E_2 . We have that for all k , $E_1^{(k)} = \text{span}(v_1^k \wedge u_1^k)$ is a space which is inside a cone $C(\alpha)$ around E_1 . Therefore usual dominated splitting estimates give that

$|\Lambda(D\varphi_x^k)|_{E_1^{(k)}(x)}| \leq K|\Lambda(D\varphi_x^k)|_{E_1(x)}$ which proves the claim about determinants since $|\Lambda(D\varphi_x^k)|_{E_1(x)} = |\det(D\varphi_x^k|_{E_x})|$. \square

A.5. Almost Integral Manifolds. We will use the local frames $\{X^{(k)}, Y^{(k)}\}$ to define a family of local manifolds which we will then show converge to the required integral manifold of E . We emphasize that these are *not* in general integral manifolds of the approximating distribution $E^{(k)}$. We will use the relatively standard notation $e^{tX^{(k)}}$ to denote the flow at time $t \in \mathbb{R}$ of the vector field $X^{(k)}$. Then we let

$$\mathcal{W}_{x_0}^{(k)}(t, s) := e^{tX^{(k)}} \circ e^{sY^{(k)}}(x_0).$$

This map is well defined for all sufficiently small s, t so that the composition of the corresponding flows remains in the local chart \mathcal{U} in which the vector fields $X^{(k)}, Y^{(k)}$ are defined. Since the vector fields $X^{(k)}, Y^{(k)}$ are uniformly bounded in norm, choosing ϵ sufficiently small the functions $\mathcal{W}_{x_0}^{(k)}$ can be defined in a fixed domain $U_\epsilon = (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$ independent of k such that $\mathcal{W}_{x_0}^{(k)}(U_\epsilon) \subset \mathcal{U}$. By a direct application of the chain rule and the definition of $\mathcal{W}_{x_0}^{(k)}$, for every $(t, s) \in U_\epsilon$ we have

$$\tilde{X}^{(k)}(t, s) = \frac{\partial \mathcal{W}_{x_0}^{(k)}}{\partial t}(t, s) = X^{(k)}(\mathcal{W}_{x_0}^{(k)}(t, s))$$

and

$$\tilde{Y}^{(k)}(t, s) = \frac{\partial \mathcal{W}_{x_0}^{(k)}}{\partial s}(t, s) = (e^{tX^{(k)}})_* Y^{(k)}(\mathcal{W}_{x_0}^{(k)}(t, s)).$$

where for two vector fields V, Z and $t \in \mathbb{R}$, $(e^{tV})_* Z$ denotes that pushforward of Z by the flow of V defined by

$$[(e^{tV})_* Z]_p = D e_{e^{-tV}(p)}^{tV} Z_{e^{-tV}(p)}.$$

The following lemma gives a condition for this family of maps to have a convergent subsequence whose limits becomes a surface tangent to E :

Lemma A.3. *If $\tilde{X}^{(k)} \rightarrow X$ and $\tilde{Y}^{(k)} \rightarrow Y$ then the images of $\mathcal{W}_{x_0}^{(k)}$ are embedded submanifolds and this sequence of submanifolds has a convergent subsequence whose limit is an integral manifold of E .*

Proof. Since X and Y are linearly independent by assumption of convergence, the differential of the map $\mathcal{W}_{x_0}^{(k)}$ is invertible at every point $(t, s) \in U_\epsilon$, i.e. the partial derivatives $\partial \mathcal{W}_{x_0}^{(k)} / \partial s$ and $\partial \mathcal{W}_{x_0}^{(k)} / \partial t$ are linearly independent for every $(t, s) \in U_\epsilon$. Thus the maps $\mathcal{W}_{x_0}^{(k)}$ are embeddings and define submanifolds through x_0 (which are not in general integral manifolds of $E^{(k)}$). Moreover, since $X^{(k)}, Y^{(k)}$ have uniformly bounded norms, it follows by Proposition A.4 that $D\mathcal{W}_{x_0}^{(k)}$ has bounded norm uniformly in k and therefore the family $\{\mathcal{W}_{x_0}^{(k)}\}$ is a compact family in the C^1 topology. By the Arzela-Ascoli Theorem this family has a subsequence converging to some limit

$$(A.7) \quad \mathcal{W}_{x_0} : U_\epsilon \rightarrow \mathcal{U}.$$

We claim that $\mathcal{W}_{x_0}(U_\epsilon)$ is an integral manifold of E . Indeed, as $k \rightarrow \infty$, $X^{(k)} \rightarrow X, Y^{(k)} \rightarrow Y$ and $\{X, Y\}$ is a local frame of continuous vector fields

for E , in particular X, Y are linearly independent and span the distribution E . Moreover, by Proposition A.4, the partial derivatives $\partial\mathcal{W}_{x_0}^{(k)}/\partial t$ and $\partial\mathcal{W}_{x_0}^{(k)}/\partial s$ are converging uniformly to X and Y and therefore

$$\frac{\partial\mathcal{W}_{x_0}}{\partial t} = X \quad \text{and} \quad \frac{\partial\mathcal{W}_{x_0}}{\partial s} = Y.$$

This shows that $\mathcal{W}_{x_0}(U_\epsilon)$ is a C^1 submanifold and its tangent space coincides with E and thus $\mathcal{W}_{x_0}(U_\epsilon)$ is an integral manifold of E , thus proving integrability of E under these assumptions. \square

It therefore just remains to verify the assumptions of Lemma A.3, i.e. to show that the vectors $X^{(k)}$ and $(e^{tX^{(k)}})_*Y^{(k)}$ converge to X and Y . The first convergence is obviously true. Thus it remains to show the latter which we show in the next result, thus completing the proof of the existence of integral manifolds.

Proposition A.4. *For all $t \in (-\epsilon, \epsilon)$ we have*

$$\lim_{k \rightarrow \infty} \|(e^{tX^{(k)}})_*Y^{(k)} - Y^{(k)}\| = 0.$$

Proof. To obtain this proposition one uses the following standard property of the pushforward (see proof of Proposition 2.6 in [1] for instance):

$$(A.8) \quad \frac{d}{dt}[(e^{tX^{(k)}})_*Y^{(k)} - Y^{(k)}] = (e^{tX^{(k)}})_*[X^{(k)}, Y^{(k)}]$$

together with the following proposition:

Proposition A.5. *There exists $C > 0$ such that for every $k \geq 1, x \in \mathcal{U}$ and $|t| \leq \epsilon$, we have*

$$\|(e^{tX^{(k)}})_*\frac{\partial}{\partial x^3}|_x\| = \exp \int_0^t \frac{\partial a^{(k)}}{\partial x^3} \circ e^{-\tau X^{(k)}}(x) d\tau$$

This latter proposition follows by integrating the equality

$$\frac{d}{dt}((e^{tX^{(k)}})_*\frac{\partial}{\partial x^3}|_x) = (e^{tX^{(k)}})_*[X^{(k)}, \frac{\partial}{\partial x^3}]|_x$$

Once this is established since we know that $\partial a^{(k)}/\partial x^3$ is uniformly bounded (A.4), we obtain that the effect of $(e^{tX^{(k)}})_*$ on $\partial/\partial x^3$ is bounded. Since $[X^{(k)}, Y^{(k)}]$ is a vector in this direction whose norm goes to 0 we directly obtain by equation (A.8) that $\frac{d}{dt}[(e^{tX^{(k)}})_*Y^{(k)} - Y^{(k)}]$ goes to 0 uniformly and hence by mean value theorem that $|(e^{tX^{(k)}})_*Y^{(k)} - Y^{(k)}|$ goes to 0 which is the proposition. \square

Remark A.6. From the proof, it is seen that the most crucial ingredient is for the approximations to satisfy the pushforward bound in Proposition A.4. One can generalize this observation to get geometric theorems about integrability of continuous sub-bundles, not just those arising in dynamical systems, with some additional assumptions such as the Lie brackets going to 0. This idea, which originated in this paper, is employed in forthcoming works [21, 29].

A.6. Uniqueness. To get uniqueness of the integral manifolds we will take advantage of a general result of Hartman which we state in a simplified version which is sufficient for our purposes.

Theorem 4 ([15], Chapter 5, Theorem 8.1). *Let $X = \sum_{i=1}^n X^i(t, p) \frac{\partial}{\partial x^i}$ be a continuous vector field defined on $I \times U$ where $U \subset \mathbb{R}^n$ and $I \subset \mathbb{R}$. Let $\eta_i = X^i(t, p)dt - dx^i$. If there exists a sequence of C^1 differential forms η_i^k such that $|\eta_i^k - \eta_i|_\infty \rightarrow 0$ and $d\eta_i^k$ are uniformly bounded then X is uniquely integrable on $I \times U$. Moreover on compact subset of $U \times I$ the integral curves are uniformly Lipschitz continuous with respect to the initial conditions.*

We recall that a two form being uniformly bounded is equivalent to each of its component is being uniformly bounded.

Corollary A.7. *Vector fields X and Y defined in (A.3) are uniquely integrable.*

Proof. We will give the proof for X , that for Y is exactly the same. Since X has the form

$$X = \frac{\partial}{\partial x^1} + a \frac{\partial}{\partial x^3},$$

its solutions always lie in the $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^3}$ plane. Therefore given a point (x_0^1, x_0^2, x_0^3) , it is sufficient to consider the restriction to such a plane. Then the C^1 differential 1-forms defined in Theorem 4 are

$$\eta_1 = dt - dx^1 \quad \eta_2 = a(x)dt - dx^3$$

where $x = (x^1, x_0^2, x^3)$, and for the approximations we can write

$$\eta_1^k = dt - dx^1 \quad \eta_2^k = a^{(k)}(x)dt - dx^3$$

where $a^{(k)}(x)$ are functions given in equation (A.2), again for some fixed x_0^2 . But then by sequential transversal Lipschitz assumption and choice of coordinates we have that $|\frac{\partial a^{(k)}}{\partial x^3}| < C$ for all k and

$$d\eta_1^k = 0 \quad d\eta_2^k = \frac{\partial a^{(k)}}{\partial x^3} dx^3 \wedge dt$$

(since we restrict to $x^2 = \text{const}$ planes) and the requirements of Theorem 4 are satisfied which proves that X is uniquely integrable. \square

Now we have that X and Y are uniquely integrable at every point. Assume there exist a point $p \in \mathcal{U}$ such that through p there exist two integral surfaces $\mathcal{W}_1, \mathcal{W}_2$. This means both surfaces are integral manifolds of E and in particular the restriction of X and Y to their tangent space are uniquely integrable vector fields. Therefore there exists ϵ_1 such that the integral curve $e^{tX}(p)$ for $|t| \leq \epsilon_1$ belongs to both surfaces. Now consider an integral curve of Y starting at the points of $e^{tX}(p)$, that is $e^{sY} \circ e^{tX}(p)$. For ϵ_1 small enough, there exists ϵ_2 small enough such that for every $|t| < \epsilon_1$ and $|s| < \epsilon_2$ this set is inside both surfaces since Y is also uniquely integrable (ϵ_i only depend on norms $|X|, |Y|$ and size of \mathcal{W}_i and therefore can be chosen uniformly independent of point). This set is a C^1 disk and therefore \mathcal{W}_1 and \mathcal{W}_2 coincide on an open domain. Applying this to every point $p \in U$ we obtain that through every point in U there passes a single local integral manifold. This concludes the proof of the uniqueness.

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