ON INVARIANCE OF PLURIGENERA FOR FOLIATIONS ON SURFACES

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Abstract. We show that if $(X_t, \mathcal{F}_t)_{t\in\Delta}$ is a family of foliations with reduced singularities on a smooth family of surfaces, then invariance of plurigenera $h^0(X_t, mK_{\mathcal{F}_t})$ holds for sufficiently large m. On the other hand, we provide examples on which the result fails, for small values of m.

Contents

1. Introduction

The aim of this paper is to study invariance of plurigenera for foliations on algebraic surfaces.

Both the study of the plurigenera of a manifold and the theory of foliations play a major role in birational geometry. Indeed, on the one hand Siu's Theorem on invariance of plurigenera [Siu98, Siu02] represents one of the most celebrated results in higher dimensional geometry. The result states that if $\mathcal{X} \to \Delta$ is a smooth family of projective manifolds over the disk Δ with fibre X_t at $t \in \Delta$, then the plurigenera $h^0(X_t, \mathcal{O}_{X_t}(mK_{X_t}))$ does not depend on $t \in \Delta$ (see also [Pău07]). This generalizes the well-known fact that the genus of a smooth curve is

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constant under smooth deformations. Apart of its own interest, Siu's proof introduces new methods, such as new extension theorems, which had a major impact on some of the recent developments in birational geometry.

On the other hand, thanks to the work of Miyaoka [Miy87], the theory of foliations plays an important role in the Minimal Model Program in dimension three, as it is needed to solve some of the crucial cases of the abundance conjecture for threefolds (see also [K+92, Chapt. 9]). It is therefore natural to ask whether the classical results in birational geometry, such as invariance of plurigenera, hold in the more general theory of foliations.

If X is a smooth surface, a (singular) foliation $\mathcal{F}$ on X corresponds to a saturated invertible subsheaf $T_{\mathcal{F}} \subseteq T_X$. Its canonical divisor $K_{\mathcal{F}}$ is the divisor associated to the dual of $T_{\mathcal{F}}$. If $\mathcal{F}$ has reduced singularities then the Kodaira dimension $\kappa(\mathcal{F})$ coincides with the Kodaira dimension of its canonical divisor $\kappa(K_{\mathcal{F}})$ (see section 2 for more details).

Remarkably, McQuillan [McQ08] was able to reproduce some of the main results of the minimal model program for surfaces to the case of foliations. Similarly, Brunella [Bru01] showed that if $(X_t, \mathcal{F}_t)_{t\in\Delta}$ is a family of foliations on surfaces with reduced singularities, then the Kodaira dimension $\kappa(\mathcal{F}_t)$ does not depend on t. The following step is therefore to understand to what extent the dimension $h^0(X_t, mK_{\mathcal{F}_t})$ depends on $t \in \Delta$.

The goal of this paper is to provide an answer to the question above. More specifically, we prove:

Theorem 1.1. *Let $(X_t, \mathcal{F}_t)_{t\in\Delta}$ be a family of foliations with reduced singularities.*

1. *If $\kappa(\mathcal{F}_t) = 0$ for any $t \in \Delta$, then for any $m \in \mathbb{N}$ the dimension $h^0(X_t, \mathcal{O}_{X_t}(mK_{\mathcal{F}_t}))$ does not depend on $t \in \Delta$;*
2. *If $\kappa(\mathcal{F}_t) = 1$ and $\mathcal{F}_t$ is induced by an elliptic fibration then for any sufficiently large positive integer m the dimension $h^0(X_t, \mathcal{O}_{X_t}(mK_{\mathcal{F}_t}))$ does not depend on $t \in \Delta$;*
3. *If $\kappa(\mathcal{F}_t) = 1$ and $\mathcal{F}_t$ is not induced by an elliptic fibration for any $t \in \Delta$ then for any positive integer m the dimension $h^0(X_t, \mathcal{O}_{X_t}(mK_{\mathcal{F}_t}))$ does not depend on $t \in \Delta$; and*
4. *If $\kappa(\mathcal{F}_t) = 2$ for any $t \in \Delta$, then for any sufficiently large positive integer m the dimension $h^0(X_t, \mathcal{O}_{X_t}(mK_{\mathcal{F}_t}))$ does not depend on $t \in \Delta$.*

Note that by Brunella's result above, for each family of foliations $(X_t, \mathcal{F}_t)_{t\in\Delta}$ with non-negative Kodaira dimension, either $\kappa(\mathcal{F}_t) = 0$ for any $t \in \Delta$, or $\kappa(\mathcal{F}_t) = 1$ for any $t \in \Delta$ or $\kappa(\mathcal{F}_t) = 2$ for any $t \in \Delta$. Furthermore we show that, in cases (2) and (4), the invariance of $h^0(X_t, mK_{\mathcal{F}_t})$ fails for small values of m. Indeed, we provide examples of families of foliations such that $h^0(X_t, K_{\mathcal{F}_t})$ is not constant as $t \in \Delta$.

Regarding foliations of Kodaira dimension 0, in [Per05] Pereira has shown that if $\mathcal{F}$ is a foliation on a smooth surface X such that $\kappa(\mathcal{F}) = 0$, then the smallest positive integer k such that $h^0(X, kK_{\mathcal{F}}) = 1$ belongs to the set $\{1, 2, 3, 4, 5, 6, 8, 10, 12\}$.

In the proof of the theorem above, we extensively use McQuillan's results on the minimal model program for foliation on surfaces and the classification of the singularities of the canonical model of a foliation with pseudo-effective canonical divisor.

As an immediate consequence of Theorem 1.1 and Brunella's result above, we obtain the following:

Corollary 1.2. *Let $(X_t, \mathcal{F}_t)_{t\in\Delta}$ be a family of foliations with reduced singularities. Then, for any sufficiently large positive integer m, the dimension $h^0(X_t, \mathcal{O}_{X_t}(mK_{\mathcal{F}_t}))$ is constant for all $t \in \Delta$.*

2. Preliminary results

We work over the field of complex numbers $\mathbb{C}$. We refer to [KM98] for some of the notations and basic results in birational geometry.

A *foliation* $\mathcal{F}$ on a n-dimensional smooth projective variety X is given by a coherent subsheaf $T_{\mathcal{F}}$ of the tangent bundle T_X of X which is closed under the Lie bracket and is such that the quotient $T_X/T_{\mathcal{F}}$ is torsion free. We denote by $\Omega_{\mathcal{F}} = T^*_{\mathcal{F}}$ the *cotangent sheaf* of $\mathcal{F}$ and by $K_{\mathcal{F}} = c_1(\Omega_{\mathcal{F}})$ the *canonical divisor* of $\mathcal{F}$. The *singular locus* of $\mathcal{F}$, denoted by Sing $\mathcal{F}$, is defined as the set of points of X on which $T_X/T_{\mathcal{F}}$ is not locally free. By Frobenius Theorem, around any point $p \in X$ outside the singular locus of $\mathcal{F}$, the germ of $T_{\mathcal{F}}$ coincides with the relative tangent bundle of a germ of a smooth fibration $X \supseteq U \to \mathbb{C}^q$, where $p \in U$. The *dimension* of $\mathcal{F}$ is defined as $n - q$.

We now recall some basic facts about foliations over a surface (see [Bru00] for more details). Let X be a smooth surface. A foliation $\mathcal{F}$ of dimension 1 on X is given by a short exact sequence

$$0 \to T_{\mathcal{F}} \to T_X \to \mathcal{I}_Z N_{\mathcal{F}} \to 0$$

where $T_{\mathcal{F}}$ and $N_{\mathcal{F}}$ are line bundles and $\mathcal{I}_Z$ is an ideal sheaf supported on a finite set. The line bundle $T_{\mathcal{F}}$ is the *tangent bundle* of $\mathcal{F}$. The line bundle $N_{\mathcal{F}}$ is the *normal bundle* of $\mathcal{F}$, while its dual $N^*_{\mathcal{F}}$ is the *conormal bundle*. The support of $\mathcal{O}_X/\mathcal{I}_Z$ coincides with Sing $\mathcal{F}$.

Equivalently, a foliation on X is the data of $\{(U_i, v_i)\}_{i\in I}$ where $\{U_i\}_{i\in I}$ is a covering of X, v_i is a vector field on U_i with only isolated zeroes and there exist $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ such that

$$v_i|_{U_i\cap U_j} = g_{ij} v_j|_{U_i\cap U_j} \qquad \text{for each } i, j \in I.$$

The cocycle g_{ij} defines $K_{\mathcal{F}}$ as a line bundle. A curve $C \subseteq X$ is said to be $\mathcal{F}$-*invariant* if the inclusion $T_{\mathcal{F}}|_C \to T_X|_C$ factors through T_C.

If p is a singular point of $\mathcal{F}$ and v is a vector field that defines $\mathcal{F}$ around p, then the eigenvalues of the linear part $(Dv)(p)$ are defined up to multiplication by a non-zero constant. The point p is a *reduced singularity* if at least one of the eigenvalues of $(Dv)(p)$ is non-zero and their quotient is not a positive rational number.

Alternatively, a foliation $\mathcal{F}$ on a smooth surface X can be locally defined by a holomorphic 1-form ω with isolated zeroes (see [Bru00, pag. 19]).

Let $\pi\colon \tilde{X} \to X$ be a proper birational morphism between smooth surfaces and let E be the exceptional curve. Then the foliation $\mathcal{F}$ induces a foliation on $\tilde{X}\setminus E$ which can be extended to a foliation on $\tilde{X}$ with isolated singularities. We denote this foliation by $\pi^*\mathcal{F}$.

Theorem 2.1 (Seidenberg)**.** [Bru00, Chapt. 1, Thm. 1] *Let $\mathcal{F}$ be a foliation on a smooth surface X. Then for any $p \in \operatorname{Sing}(\mathcal{F})$, there exists a sequence of blow-ups $\pi\colon \tilde{X} \to X$ over p such that the foliation $\pi^*\mathcal{F}$ has only reduced singularities in a neighborhood of $\pi^{-1}(p)$.*

Given a foliation $\mathcal{F}$ on a smooth surface X, we define the *Kodaira dimension of* $\mathcal{F}$ as the Kodaira dimension of $K_{\pi^*\mathcal{F}}$ where π is as in Theorem 2.1 and we denote it by $\kappa(\mathcal{F})$. It is easy to check that $\kappa(\mathcal{F})$ does not depend on the resolution π. In particular, we say that $\mathcal{F}$ is of *general type* if $\kappa(\mathcal{F}) = 2$. Foliations of general type appeared several times in the literature, e.g. if $X = \mathbb{P}^2$, then the foliations of general type on X were studied by Pereira in [Per02].

In this paper, we consider families of foliations defined over a smooth family of surfaces.

Definition 2.2. [Bru01, Def. 1] A *family of foliations with reduced singularities* $(X_t, \mathcal{F}_t)_{t\in\Delta}$ (or a family of foliations, for short) is the data of

- a smooth morphism $\pi\colon \mathcal{X} \to \Delta$, where $\mathcal{X}$ is a smooth complex variety and Δ is the complex disc, whose fibres X_t are projective surfaces, for all $t \in \Delta$;
- a foliation $\mathcal{F}$ of dimension 1 on $\mathcal{X}$ such that
  1. $\mathcal{F}$ is tangent to the fibres of π;
  2. the singular set $\operatorname{Sing}\mathcal{F}$ of $\mathcal{F}$ is of pure codimension 2 in $\mathcal{X}$ and cuts every fibre in a finite set; and
  3. for any $t \in \Delta$, the foliation $\mathcal{F}_t = \mathcal{F}|_{X_t}$ is a foliation whose singularities $\operatorname{Sing}\mathcal{F}|_{X_t} = \operatorname{Sing}\mathcal{F} \cap X_t$ are reduced.

Note that 2.2(1) is needed to ensure that $\mathcal{F}_t$ is a foliation of dimension 1 on X_t for all $t \in \Delta$ and 2.2(2) is needed to ensure the existence of a canonical divisor of the foliation $K_{\mathcal{F}}$ on $\mathcal{X}$ such that

$$K_{\mathcal{F}}|_{X_t} = K_{\mathcal{F}_t}$$

for any $t \in \Delta$. Note also that invariance of the Kodaira dimension does not hold without hypothesis 2.2(3). Indeed, if the singularities of $\mathcal{F}_t$ are not reduced for all $t \in \Delta$, then invariance of plurigenera fails, as shown by the example in [Bru01, p. 114]. Furthermore, Example 3.8 shows that the invariance of the Kodaira dimension does not hold without the equality $\operatorname{Sing}\mathcal{F}|_{X_t} = \operatorname{Sing}\mathcal{F} \cap X_t$: in the example, we describe a family of foliations $\mathcal{F}_t$, induced by a foliation $\mathcal{F}$ on a smooth family of surfaces $\mathcal{X} \to \Delta$ such that $\mathcal{F}_t$ has reduced singularities for any $t \in \Delta$ and such that there exists a curve $C \subseteq X_{t_0} \cap \operatorname{Sing}\mathcal{F}$, for some $t_0 \in \Delta$.

On the other hand, under the assumptions above, we have:

Theorem 2.3. [Bru01, Thm. 1] *Let $(X_t, \mathcal{F}_t)_{t\in\Delta}$ be a family of foliations on surfaces with reduced singularities. Then the Kodaira dimension $\kappa(\mathcal{F}_t)$ does not depend on $t \in \Delta$.*

Let X be a smooth surface. We will assume that a curve $C \subseteq X$ is reduced and compact. We say that C has *normal crossing singularities* if locally, with respect to the Euclidean topology, around each point $p \in C$, the curve C is a union of smooth curves meeting transversally. In particular a *nodal* curve is an irreducible curve C with normal crossing singularities.

For any curve $C \subseteq X$, the *arithmetic Euler characteristic* of C is given by

$$\chi(C) = -K_X \cdot C - C^2.$$

Note that if C is smooth, then it coincides with the usual Euler characteristic.

Let $\mathcal{F}$ be a foliation on X and let $p \in C$ be a point. If none of the components of C is $\mathcal{F}$-invariant, we define the *index of tangency* of $\mathcal{F}$ to C at p as follows. Let $\{f = 0\}$ be a local equation of C around p, let v be a local holomorphic vector field generating $\mathcal{F}$ around p. Then,

$$\operatorname{tang}(\mathcal{F}, C, p) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{X,p}}{< f, v(f) >}$$

where $v(f)$ is the Lie derivative of f along v. We have $\operatorname{tang}(\mathcal{F}, C, p) = 0$ except on the finite subset of points of C where $\mathcal{F}$ is not transverse to C. Thus, we define

$$\operatorname{tang}(\mathcal{F}, C) = \sum_{p \in C} \operatorname{tang}(\mathcal{F}, C, p).$$

Proposition 2.4. [Bru00, Chapt. 2, Prop. 2] *Let $\mathcal{F}$ be a foliation on a smooth surface X. Let C be a curve on X whose components are not $\mathcal{F}$-invariant. Then*

$$c_1(N_{\mathcal{F}}) \cdot C = \chi(C) + \operatorname{tang}(\mathcal{F}, C) \qquad \text{and} \qquad K_{\mathcal{F}} \cdot C = -C \cdot C + \operatorname{tang}(\mathcal{F}, C).$$

We now consider a curve C whose components are all $\mathcal{F}$-invariant. If $p \in C$ is a singular point of $\mathcal{F}$, $\{f = 0\}$ is a local equation for C at p and ω is a holomorphic 1-form that defines $\mathcal{F}$ around p, then we may write

$$g\omega = hdf + f\eta,$$

for some holomorphic 1-form η and holomorphic functions g, h such that h and f are coprime. We define

$$Z(\mathcal{F}, C, p) = \text{vanishing order of } \left.\frac{h}{g}\right|_C \quad \text{at } p$$

and

$$CS(\mathcal{F}, C, p) = \text{residue of } -\left.\frac{1}{h}\eta\right|_C \quad \text{at } p.$$

If p is a reduced singularity, then $Z(\mathcal{F}, C, p) \geq 0$ by [Bru99]. Let

$$Z(\mathcal{F}, C) = \sum_{p \in \operatorname{Sing}(\mathcal{F}) \cap C} Z(\mathcal{F}, C, p) \qquad \text{and} \qquad CS(\mathcal{F}, C) = \sum_{p \in \operatorname{Sing}(\mathcal{F}) \cap C} CS(\mathcal{F}, C, p).$$

Proposition 2.5. *Let $\mathcal{F}$ be a foliation on a smooth surface X and let C be a curve on X whose components are $\mathcal{F}$-invariant.*

Then

(1) $c_1(N_{\mathcal{F}}) \cdot C = C^2 + Z(\mathcal{F}, C)$;
(2) $K_{\mathcal{F}} \cdot C = -\chi(C) + Z(\mathcal{F}, C)$; *and*
(3) $C^2 = CS(\mathcal{F}, C)$.

In addition, if $\mathcal{F}$ admits only reduced singularities, then C has only normal crossing singularities.

Formula (3) in Proposition 2.5 is usually referred as the *Camacho-Sad formula*.

Proof. (1) and (2) are implied by [Bru00, Chapt. 2, Prop. 3] and [CS82] implies (3) (see also [Bru00, Chapt. 3, Thm. 2]).

Finally, if $\mathcal{F}$ admits only reduced singularities then [Bru00, pag. 12] implies that C has only normal crossing singularities. □

2.1. **Minimal model program and Zariski decomposition for foliations.** In [McQ08], McQuillan has developed a minimal model program for foliations on surfaces. We now recall some of the main results.

Definition 2.6. [Bru00, Chapt. 5] Let $\mathcal{F}$ be a foliation on a smooth surface X and which admits only reduced singularities. We say that a curve C in X is $\mathcal{F}$-*exceptional* if

(1) C is a smooth rational curve of self-intersection -1;
(2) the contraction of C to a point p gives a new foliation $\overline{\mathcal{F}}$ such that p is either a regular point or a reduced singular point for $\overline{\mathcal{F}}$.

In particular, the foliation $\mathcal{F}$ is said to be *relatively minimal* if $\mathcal{F}$ admits only reduced singularities and there are no $\mathcal{F}$-exceptional curves on X.

If $\mathcal{F}$ is a foliation on X with reduced singularities, then there exists a birational morphism $X \to X'$ onto a smooth surface X' such that the induced foliation $\mathcal{F}'$ on X' is relatively minimal. Note that if $\pi\colon X \to \overline{X}$ is the contraction of an $\mathcal{F}$-exceptional curve C onto a point p and $\overline{\mathcal{F}}$ is the induced foliation on $\overline{X}$ then by [Bru00, pag. 72], it follows that either $\mathcal{F}$ is regular at p and $K_{\mathcal{F}} = \pi^* K_{\overline{\mathcal{F}}} + C$ or $\mathcal{F}$ is singular at p and $K_{\mathcal{F}} = \pi^* K_{\overline{\mathcal{F}}}$. Thus, we have that

$$h^0(X, \mathcal{O}_X(mK_{\mathcal{F}})) = h^0(X', \mathcal{O}_{X'}(mK_{\mathcal{F}'}))$$

for all positive integers m.

By the following result, $\mathcal{F}$-exceptional curves can be extended locally in a family:

Lemma 2.7. *Let $(X_t, \mathcal{F}_t)_{t\in\Delta}$ be a family of foliations and let $t_0 \in \Delta$ be such that $\mathcal{F}_{t_0}$ admits an $\mathcal{F}_{t_0}$-exceptional curve $E_{t_0} \subseteq X_{t_0}$. Then there exists a neighborhood $t_0 \in U \subseteq \Delta$ and a smooth hypersurface $E \subseteq \pi^{-1}(U)$ transverse to the fibres of π such that $E_t = E \cap X_t$ is an $\mathcal{F}_t$-exceptional curve for any $t \in U$.*

In particular, if $s \in \Delta$ then there exists a birational morphism $\nu\colon \mathcal{X}_U \to \mathcal{X}'_U$ over U which defines a factorization

$$\pi\colon \mathcal{X}_U \xrightarrow{\nu} \mathcal{X}'_U \xrightarrow{\pi'} U$$

and such that the foliation $\mathcal{F}'$ induced on $\mathcal{X}'_U$ is relatively minimal on X'_s (i.e. $\mathcal{F}'_s$ is relatively minimal) and the family $(X'_t, \mathcal{F}'_t)_{t\in U}$ is still a family of foliations with reduced singularities.

Proof. The existence of U and E is guaranteed by [Bru01, Lemme 2]. Thus, after possibly shrinking U, there exists a birational morphism $\varepsilon\colon \mathcal{X} \to \mathcal{Y}$ which contracts E and a smooth morphism $\pi'\colon \mathcal{Y} \to \Delta$ such that $\pi = \pi' \circ \varepsilon$ (see [FM94, Chapt. 1, Proof of Thm. 1.16] for a similar argument). For any $t \in U$, let $Y_t = \pi'^{-1}(t)$. Since $E|_{X_t}$ is $\mathcal{F}_t$-exceptional, the induced foliation on Y_t admits reduced singularities for all $t \in U$. Thus, the claim follows after repeating the argument finitely many times. □

We now consider a relatively minimal foliation $\mathcal{F}$ on a surface X such that $K_{\mathcal{F}}$ is pseudo-effective. Then, we denote the *Zariski decomposition* of $K_{\mathcal{F}}$ by

$$K_{\mathcal{F}} = P + N$$

where P is the positive part and N is the negative part of $K_{\mathcal{F}}$. McQuillan shows that there exists a contraction $X \to X'$ onto a surface X' with Kawamata log terminal singularities, which contracts all the curves contained in the support of N. More precisely, we say that a curve $C = \cup_{i=1}^r C_i$ in X is an *Hirzebruch-Jung string* if $C_1, \ldots, C_r$ are smooth rational curves such that, for all $i, j \in \{1, \ldots, r\}$, we have $C_i^2 \leq -2$, $C_i \cdot C_j = 1$ if $|i - j| = 1$ and $C_i \cdot C_j = 0$ if $|i - j| > 1$. Note that any Hirzebruch-Jung string on a smooth projective surface can be contracted onto a surface with cyclic quotient singularities (see [BHPdV04, Chapt. 3, Thm. 5.1 and Prop. 5.3]). We define:

Definition 2.8. [Bru00, Chapt. 8] Given a foliation $\mathcal{F}$ on a surface X, we say that a curve C is an *$\mathcal{F}$-chain* if

1. $C = \cup_{i=1}^r C_i$ is an Hirzebruch-Jung string;
2. each irreducible component C_i of C is $\mathcal{F}$-invariant;
3. the singularities of $\mathcal{F}$ along C are all reduced; and
4. $Z(\mathcal{F}, C_1) = 1$ and $Z(\mathcal{F}, C_i) = 2$ for any $i \geq 2$.

Theorem 2.9. [McQ08, Thm. 2, Prop. III.1.2], [Bru00, Chapt. 8, Thm. 1 and Addendum p. 109] *Let $\mathcal{F}$ be a relatively minimal foliation on a smooth surface X, such that $K_{\mathcal{F}}$ is pseudo-effective. Let $K_{\mathcal{F}} = P + N$ be the Zariski decomposition.*

Then the support of N is a disjoint union of maximal $\mathcal{F}$-chains and $\lfloor N \rfloor = 0$. In particular, there exists a contraction $X \to X'$ onto a surface X' with Kawamata log terminal singularities, which contracts all the curves in the support of N.

The following result is also due to McQuillan:

Lemma 2.10. [McQ08, Lemma IV.3.1], [Bru00, Chapt. 8, Thm. 2, and Chapt. 9, Thm. 2] *Let $\mathcal{F}$ be a foliation on a smooth surface X. Assume that $\mathcal{F}$ admits only reduced singularities and that $\kappa(\mathcal{F}) = 0$. Let $K_{\mathcal{F}} = P + N$ be the Zariski decomposition of $K_{\mathcal{F}}$.*

Then P is a torsion divisor.

By a result of Brunella, the Zariski decomposition of the canonical divisor of a foliation is well behaved in families:

Proposition 2.11. [Bru01, Prop. 1 and 2] *Let $(X_t, \mathcal{F}_t)_{t\in\Delta}$ be a family of foliations of non-negative Kodaira dimension. Then there exists an effective $\mathbb{Q}$-divisor N on $\mathcal{X}$ such that N does not contain any fibre of $\pi\colon \mathcal{X} \to \Delta$ and $N_t = N|_{X_t}$ is the negative part of the Zariski decomposition of $K_{\mathcal{F}_t}$ for any $t \in \Delta$.*

In addition, if there exists $s \in \Delta$ such that $(X_s, \mathcal{F}_s)$ is relatively minimal, then there exists an open set $s \in U \subseteq \Delta$ such that the irreducible components $E_1, \dots, E_k$ of N meet the surfaces X_t transversally in distinct rational curve $E_1|_{X_t}, \dots, E_k|_{X_t}$.

As a consequence, we have:

Lemma 2.12. *Let $(X_t, \mathcal{F}_t)_{t\in\Delta}$ be a family of foliations of non-negative Kodaira dimension. Let N_t be the negative part of the Zariski decomposition of $K_{\mathcal{F}_t}$.*

Then, the least common multiple of the denominators of N_t does not depend on $t \in \Delta$.

Proof. Let m_t be the least common multiple of the denominators of N_t. We will prove that m_t is locally constant. Fix $s \in \Delta$. Let $U \subseteq \Delta$ be the neighborhood of s and

$$\pi\colon \mathcal{X}_U \xrightarrow{\nu} \mathcal{X}'_U \xrightarrow{\pi'} U$$

be the factorization of π whose existence is guaranteed by Lemma 2.7. Let $(X'_t, \mathcal{F}'_t)_{t\in U}$ be the induced family of foliations.

By Proposition 2.11, modulo shrinking U, there exists an effective $\mathbb{Q}$-divisor N' on $\mathcal{X}'_U$ such that for any $t \in U$, $N'_t = N|_{X'_t}$ is the negative part of the Zariski decomposition of $K_{\mathcal{F}_t}$ and the components of N'_t meet the fibres of π' transversally. Thus, the least common multiple m'_t of the denominators of N'_t does not depend on $t \in U$.

On the other hand, for any $t \in U$ the foliation $\mathcal{F}'_t$ has only reduced singularities and in particular there exists an effective integral exceptional divisor E_t such that

$$K_{\mathcal{F}_t} = \nu_t^* K_{\mathcal{F}'_t} + E_t.$$

It follows that $N_t = \nu_t^* N'_t + E_t$ and the least common multiple of the denominators of N_t coincides with m'_t. Thus, the claim follows. □

Let $\mathcal{F}$ be a foliation on a smooth surface X. Assume that $\mathcal{F}$ is of general type and let

$$K_{\mathcal{F}} = P + N$$

be the Zariski decomposition of $K_{\mathcal{F}}$. We want to show that if C is a curve such that $P \cdot C = 0$, then C is $\mathcal{F}$-invariant. We assume first that C is such that $K_{\mathcal{F}} \cdot C = 0$ and assume by contradiction that C is not $\mathcal{F}$-invariant. Then, Proposition 2.4 implies

$$0 = K_{\mathcal{F}} \cdot C = -C^2 + \operatorname{tang}(\mathcal{F}, C) \geq -C^2$$

which is a contradiction because a big divisor cannot have intersection zero with a movable curve. We now consider the general case. To this end, we consider a variation of [McQ08, Lemma III.1.1]:

Proposition 2.13. *Let $\mathcal{F}$ be a relatively minimal foliation on a smooth surface X such that $K_{\mathcal{F}}$ is pseudo-effective. Let $\varepsilon\colon X \to Y$ be the contraction of all the components of the negative part of the Zariski decomposition of $K_{\mathcal{F}}$. Let C be a curve on X which is not $\mathcal{F}$-invariant, let $\overline{C}$ be its image in Y and let $\overline{K} = \varepsilon_* K_{\mathcal{F}}$.*

Then
$$(\overline{K} + \overline{C}) \cdot \overline{C} \geq 0.$$

We first prove the following:

Lemma 2.14. *Let $\varepsilon\colon X \to Y$ be a birational morphism between surfaces with only Kawamata log terminal singularities and assume that ε contracts a chain of rational curves $F_1, \dots, F_k$. Let L be a $\mathbb{Q}$-Cartier divisor on X and let C be a curve on X such that*

1. *$(L + C) \cdot C \geq 0$;*
2. *$C \cdot F_i$ is a non-negative integer for any $i = 1, \dots, k$;*
3. *$F_1^2 \leq -1$, $F_i^2 \leq -2$ and $F_{i-1} \cdot F_i = 1$ for any $i = 2, \dots, k$; and*
4. *$-1 \leq L \cdot F_1 < 0$ and $L \cdot F_i = 0$ for any $i = 2, \dots, k$.*

Let $\overline{L} = \varepsilon_* L$ and $\overline{C} = \varepsilon_* C$. Then $(\overline{L} + \overline{C}) \cdot \overline{C} \geq 0$.

Proof. We may write
$$L + C = \varepsilon^*(\overline{L} + \overline{C}) - G$$
for some ε-exceptional divisor G. If $G \geq 0$ then the claim follows immediately. Therefore we may assume that G is not effective and by the Negativity Lemma, there exists $i \in \{1, \dots, k\}$ such that $G \cdot F_i > 0$ and the coefficient of G along F_i is negative. Thus, $(L + C) \cdot F_i < 0$ and in particular we must have $i = 1$ and $C \cdot F_1 = 0$. We may write $G = -\alpha F_1 + G_1$ for some $\alpha > 0$ and some ε-exceptional divisor G_1 whose support does not contain F_1.

There exists a morphism $f\colon X \to X_1$ which contracts only the curve F_1 and such that ε factors through f. Let $L_1 = f_* L$ and $C_1 = f_* C$. Since $L \cdot F_1 < 0$, we have $L = f^* L_1 + \beta F_1$ for some $\beta > 0$ and since $C \cdot F_1 = 0$ we have $C = f^* C_1$. In particular,
$$(L_1 + C_1) \cdot C_1 = (L + C) \cdot C \geq 0.$$
Let $F_i' = f_* F_i$ for $i = 2, \dots, k$. Then $C_1 \cdot F_i' = C \cdot F_i$ for all $i = 2, \dots, k$ and $L_1 \cdot F_i' = 0$ for any $i = 3, \dots, k$. We may write $f^* F_2 = \gamma F_1 + F_2$ for some $\gamma > 0$. Then
$$0 = F_1 \cdot f^* F_2' = F_1 \cdot (\gamma F_1 + F_2) = -e_1 \cdot \gamma + 1,$$
where $e_1 = -F_1^2 \geq 1$. Thus, $\gamma = 1/e_1$. We have,
$$F_{i-1}' \cdot F_i' = F_{i-1} \cdot F_i = 1 \qquad \text{and} \qquad {F_i'}^2 = F_i^2$$
for any $i = 3, \dots, k$. Moreover
$${F_2'}^2 = F_2 \cdot (\gamma F_1 + F_2) = 1/e_1 + F_2^2 \leq 1 + F_2^2 \leq -1.$$
Finally,
$$L_1 \cdot F_2' = L \cdot f^* F_2' = L \cdot (\gamma F_1 + F_2) = 1/e_1 L \cdot F_1 \in [-1, 0).$$
Thus, the claim follows by induction on k. □

Proof of Proposition 2.13. We proceed by induction on the number of connected components of the exceptional locus of ε. By Theorem 2.9, any connected component of the exceptional locus of ε is a maximal $\mathcal{F}$-chain $F_1, \ldots, F_k$. Let $L = K_{\mathcal{F}}$. Then Proposition 2.4 implies that $(L + C) \cdot C \geq 0$. On the other hand, (2) of Proposition 2.5 implies that $L \cdot F_1 = -1$ and $L \cdot F_i = 0$ for any $i = 2, \ldots, k$. Thus, if $\varepsilon' \colon X \to Y'$ is the contraction of $F_1, \ldots, F_k$, then Lemma 2.14 implies that $(\varepsilon'_* L + \varepsilon'_* C) \cdot \varepsilon'_* C \geq 0$.

Thus, the claim follows by induction, by proceeding as above for each connected component of the exceptional locus of ε. □

Proposition 2.15. *Let $\mathcal{F}$ be a relatively minimal foliation of general type on a smooth surface X. Let $K_{\mathcal{F}} = P + N$ be the Zariski decomposition of $K_{\mathcal{F}}$ and let C be a curve such that $P \cdot C = 0$. Then C is $\mathcal{F}$-invariant.*

Proof. Assume that C is not $\mathcal{F}$-invariant. By Theorem 2.9, there exists a proper birational morphism $\varepsilon \colon X \to Y$ onto a normal surface Y and whose exceptional locus coincides with the support of N. Let $\overline{K} = \varepsilon_* K_{\mathcal{F}}$ and $\overline{C} = \varepsilon_* C$. The Negativity Lemma implies that $P = \varepsilon^* \overline{K}$. Thus,

$$\overline{K} \cdot \overline{C} = \varepsilon^* \overline{K} \cdot C = P \cdot C = 0.$$

Since $\overline{K}$ is big, we have $\overline{C}^2 < 0$. On the other hand, Proposition 2.13 implies

$$\overline{C}^2 = (\overline{K} + \overline{C}) \cdot \overline{C} \geq 0$$

that is a contradiction. Thus, C is $\mathcal{F}$-invariant. □

The following Theorem is proved in [McQ08].

Theorem 2.16. [McQ08, Thm. 1 III.3.2, Remark III.2.2] *Let $\mathcal{F}$ be a relatively minimal foliation of general type on a smooth surface X. Let $K_{\mathcal{F}} = P + N$ be the Zariski decomposition of $K_{\mathcal{F}}$ and let Z be the union of all the curves C such that $P \cdot C = 0$. Then Z is the union of:*

1. *the support of N;*
2. *disjoint chains of rational curves none of which is contained in the support of N;*
3. *cycles Γ of rational curves such that* $\mathrm{Sing}(\mathcal{F}) \cap \Gamma$ *coincides with the singular locus of Γ; and*
4. *single rational nodal curves Γ such that* $\mathrm{Sing}(\mathcal{F}) \cap \Gamma$ *coincides with the singular locus of Γ.*

Moreover, a chain $\mathcal{C}$ of type (2) *is either disjoint from* $\mathrm{Supp} N$ *or there exist exactly two connected components of* $\mathrm{Supp} N$*, each of which consists of a smooth rational curve E_i of self-intersection -2, with $i = 1, 2$, and such that both E_1 and E_2 meet $\mathcal{C}$ transversally along the same tail C of $\mathcal{C}$ on the points p_1 and p_2, so that the intersection is transverse and $N|_C = \frac{1}{2} p_1 + \frac{1}{2} p_2$.*

Remark 2.17. Under the same assumptions as in Theorem 2.16, it follows by Proposition 2.15 that any curve C such that $P \cdot C = 0$ is $\mathcal{F}$-invariant. Since any point lying in the intersection of two $\mathcal{F}$-invariant curves is a singular point, the cycles of rational curves and the nodal curve appearing in Theorem 2.16 do not meet any other $\mathcal{F}$-invariant curve. In particular they do not meet any component of Supp$N$.

Definition 2.18. A connected component of type (3) and (4) of Theorem 2.16 is called an *elliptic Gorenstein leaf* (e.g.l. for short).

The following theorem due to McQuillan will play a key role in the proof of our main results:

Theorem 2.19. [McQ08, Thm. IV.2.2] *Let $\mathcal{F}$ be a relatively minimal foliation on a smooth surface X. Let Γ be an elliptic Gorenstein leaf. Then $K_{\mathcal{F}}|_{\Gamma}$ is not a torsion divisor.*

2.2. **Foliations and fibrations.** When a surface is endowed with a fibration, the study of foliations on the variety becomes simpler. In particular two types of foliations play a key role in the case of Kodaira dimension one: foliations induced by fibrations and foliations transverse to fibrations.

2.2.1. *Elliptic fibrations.* We first recall some of the basic notions for the canonical bundle formula for an elliptic fibration (e.g. see [Amb04, pag. 236] for more details). Let X be a smooth surface and let $f\colon X\to C$ be an elliptic fibration onto a curve C. Let $f'\colon X'\to C$ be the relatively minimal elliptic fibration associated to f, obtained by blowing-down any possible sequence of vertical (-1)-curves. Thus, we obtain a diagram:

$$\begin{array}{ccc} X & \xrightarrow{\varepsilon} & X' \\ & \searrow^{f} & \downarrow{\scriptstyle f'} \\ & & C. \end{array}$$

The *discriminant* of f is defined by

$$B_C=\sum_{p\in C}(1-\gamma_p)p \tag{2.1}$$

where, for any $p\in C$, γ_p denotes the *log canonical threshold* of X' with respect to f'^*p:

$$\gamma_p=\sup\{t\in\mathbb{R}_{>0}\mid (X',tf'^*p)\text{ is log canonical}\}.$$

Then $K_{X'/C}=f'^*(M_C+B_C)$ where M_C is a $\mathbb{Q}$-divisor on C which denotes the *moduli part* in the canonical bundle formula of f [Kod64]. In particular, $\deg M_C\geq 0$ and the equality holds if and only if f is isotrivial.

We may write,

$$B_C=B'_C+\sum\frac{m_p-1}{m_p}p \tag{2.2}$$

where the sum runs over the points p such that the fibre F of f' over p is a multiple fibre of multiplicity m_p, that is, $f'^*p=m_pF_{red}$, where F_{red} is the reduced divisor associated to f'^*p. Then

$$L=M_C+B'_C$$

is an integral divisor on C. Thus, if we denote by $\{M_C\}$ the fractional part of M_C, then

$$\mathrm{Supp}\{M_C\} = \mathrm{Supp} B'_C. \tag{2.3}$$

Furthermore, by [Kod64], it follows that

$$12M_C \text{ is Cartier and } |12M_C| \text{ is base point free.} \tag{2.4}$$

We will often use Kodaira's classification of the singular fibres of an elliptic fibration. In particular, if $p \in C$ is such that the fibre f^*p is singular and $b_p = 1 - \gamma_p$ is the coefficient of B_C along p, then the fibre $f^{-1}(p)$ is of one of the following types:

$$mI_b, I_b^*, II, II^*, III, III^*, IV, IV^*$$

and the corresponding values of b_p are:

$$1 - \frac{1}{m}, \frac{1}{2}, \frac{1}{6}, \frac{5}{6}, \frac{1}{4}, \frac{3}{4}, \frac{1}{3}, \frac{2}{3}.$$

Remark 2.20. Let X be a smooth surface and let $f\colon X \to C$ be an elliptic fibration onto a curve C and assume that all the fibres of f have support with only normal crossing singularities. Let $\varepsilon\colon X \to X'$ be the relative minimal elliptic surface associated to f and let $f'\colon X' \to C$ be the induced fibration. We may write $K_X = \varepsilon^* K_{X'} + E$ where $E = \sum a_D D \geq 0$ is an ε-exceptional divisor. Let $p \in C$. We may write

$$f^*p = \sum_{D \subseteq f^{-1}(p)} l_D D.$$

Then the log canonical threshold γ_p is computed on X and it is equal to

$$\gamma_p = \min \left\{ \frac{1 + a_D}{l_D} \,\middle|\, D \subseteq f^{-1}(p) \right\}.$$

We say that a prime divisor $D \subseteq f^{-1}(p)$ *computes the log canonical threshold* at p if $\frac{1+a_D}{l_D} = \gamma_p$.

2.2.2. *Foliations induced by fibrations.* Let X be a smooth surface and let $f\colon X \to C$ be a fibration onto a smooth curve. The fibration induces a foliation $\mathcal{F}$ whose leaves are contained in the fibres of f. The canonical divisor of $\mathcal{F}$ is (cf. [Bru00, Chapt. 2, Section 3])

$$K_{\mathcal{F}} \;=\; K_{X/C} + \sum (1 - l_D) D \tag{2.5}$$

where the sum runs over all the irreducible curves D contracted by f and l_D denotes the ramification order of f along D, i.e. for any $p \in C$ we have

$$f^*p = \sum_{D \subseteq f^{-1}(p)} l_D D.$$

Remark 2.21. Let $\mathcal{F}$ be a foliation on a smooth surface X induced by a fibration $f\colon X\to C$ onto a curve C. Then $\mathcal{F}$ admits only reduced singularities if and only if all the fibres of f have support with only normal crossing singularities. Indeed if $\mathcal{F}$ is reduced then by Proposition 2.5 all the $\mathcal{F}$-invariant curves have support with normal crossing singularities. The other direction follows from an explicit computation.

If a foliation $\mathcal{F}$ is induced by an elliptic fibration, then we can give a precise description of the Zariski decomposition of $K_{\mathcal{F}}$.

Lemma 2.22. *Let X be a smooth surface and let $f\colon X\to C$ be a fibration onto a smooth curve. Let $\mathcal{F}$ be the foliation induced by f.*

(1) *For any $p\in C$, there exists a neighborhood $\mathcal{U}$ of $f^{-1}(p)$ such that*

$$K_{\mathcal{F}}|_{\mathcal{U}}\sim_{\mathbb{Q}}(K_X+F_{red})|_{\mathcal{U}}$$

where F_{red} denotes the reduced divisor associated to f^*p.

(2) *Moreover, if f is an elliptic fibration, $\mathcal{F}$ admits only reduced singularities, $K_{\mathcal{F}}$ is pseudo-effective and $K_{\mathcal{F}}=P+N$ is the Zariski decomposition of $K_{\mathcal{F}}$, then*

$$P=f^*M_C$$

where M_C is the moduli part in the canonical bundle formula.

Proof. Let $p\in C$ and let $\mathcal{U}$ be a sufficiently small neighborhood of $f^{-1}(p)$. If l_D denotes the ramification order of f along D for all $D\subseteq f^{-1}(p)$, then (2.5) implies

$$K_{\mathcal{F}}|_{\mathcal{U}}\sim_{\mathbb{Q}}(K_X+\sum_{D\subseteq f^{-1}(p)}(1-l_D)D)|_{\mathcal{U}}=(K_X+F_{red})|_{\mathcal{U}}$$

and (1) follows.

Let us assume now that f is as in (2). Let $\varepsilon\colon X\to X'$ be the relative minimal elliptic surface associated to f and let $f'\colon X'\to C$ be the induced fibration.

Then $K_{X'/C}=f'^*(M_C+B_C)$ where M_C is the moduli part in the canonical bundle formula and B_C is the discriminant. Since $\mathcal{F}$ has only reduced singularities, Proposition 2.5 implies that the fibres of f have support with only normal crossing singularities. Thus, as in Remark 2.20, the log canonical threshold γ_p is computed on X so that

$$\gamma_p=\min\left\{\frac{1+a_D}{l_D}\,\middle|\, D\subseteq f^{-1}(p)\right\}$$

where $E\geq 0$ is an ε-exceptional divisor such that $K_X=\varepsilon^*K_{X'}+E$ and $E=\sum a_D D$.

Then (2.5) implies that

$$\begin{aligned}K_{\mathcal{F}}&=K_{X/C}+\sum(1-l_D)D\\&=f^*(M_C+B_C)+E+\sum(1-l_D)D.\end{aligned}$$

We claim that

$$\Psi=f^*(B_C)+E+\sum(1-l_D)D$$

is an effective divisor whose support does not contain any fibre. Indeed, for any $p \in C$ and for any prime divisor $D \subseteq f^{-1}(p)$, the coefficient of Ψ along D is

$$1 + a_D - l_D\gamma_p \geq 0$$

and the equality holds for any D which computes the log canonical threshold (cf. Remark 2.20). It follows that $\Psi \geq 0$ and, for any $p \in C$, there exists a prime divisor $D \subseteq f^{-1}(p)$ which is not contained in the support of Ψ. Thus, the claim follows.

Since SuppΨ does not contain any fibre, we have that $\Psi = N$ and $P = f^*M_C$. Thus, (2) follows. □

2.2.3. *Foliations transverse to a fibration.* Let X be a smooth surface and let $f\colon X \to C$ be a fibration onto a curve C. Let $\mathcal{F}$ be a foliation on X which is *transverse* to f, that is, such that the general fibre F of f is not $\mathcal{F}$-invariant and $K_{\mathcal{F}} \cdot F = 0$. Thus, there exists an effective divisor $D_{\tan}$ (cf. [Bru97, p. 573, Lemme 4]), whose support is contained in the set of $\mathcal{F}$-invariant curves contained in the fibres of f and such that

$$K_{\mathcal{F}} = f^*K_C + D_{\tan} + \sum (l_D - 1)D \tag{2.6}$$

where the sum runs over all the irreducible curves D contracted by f and l_D denotes the ramification order of f along D, i.e. for any $p \in C$ we have

$$f^*p = \sum_{D \subseteq f^{-1}(p)} l_D D.$$

Let $D_f = \sum (l_D - 1)D$. Since $D_{\tan} + D_f$ is contained in fibres of f, its Zariski decomposition is

$$D_{\tan} + D_f = \ f^*\theta + \bar{N} \tag{2.7}$$

where $\bar{N}$ is the negative part of the Zariski decomposition and θ is the largest effective $\mathbb{Q}$-divisor such that $D_{\tan} + D_f - f^*\theta$ is effective. In particular,

$$K_{\mathcal{F}} = f^*(K_C + \theta) + \overline{N}.$$

Since the support of $\overline{N}$ is contained in fibres of f but it does not contain any of its fibres, it follows that if $K_{\mathcal{F}}$ is pseudo-effective and $K_{\mathcal{F}} = P + N$ is its Zariski decomposition then $K_C + \theta$ has non-negative degree and $\bar{N} = N$.

Lemma 2.23. *With the notation introduced above, let $\theta = \sum_{q \in C} \theta_q q$ and let $p \in C$ be such that $\theta_p \in \mathbb{Q} \setminus \mathbb{Z}$. Then the support of $\bar{N}$ contains all the components D of f^*p that are reduced (i.e. such that $l_D = 1$).*

Proof. We may write

$$f^*p = \sum l_D D \qquad D_{\tan} = \sum a_D D \qquad \text{and} \qquad \bar{N} = \sum c_D D.$$

Let $\bar{D}$ be a reduced component of f^*p. Then (2.7) implies

$$a_{\bar{D}} = \theta_p + c_{\bar{D}}.$$

Since $a_{\bar{D}}$ is an integer and θ_p is not, it follows that $c_{\bar{D}} \neq 0$, as claimed. □

Definition 2.24. A foliation $\mathcal{F}$ on a smooth surface X and which is transverse to a fibration $f\colon X \to C$ whose general fibre is a rational curve is called a *Riccati foliation*.

Definition 2.25. A foliation $\mathcal{F}$ on a smooth surface X and which is transverse to an elliptic fibration $f\colon X \to C$ is called a *turbulent foliation*.

Remark 2.26. If $\mathcal{F}$ is a turbulent foliation associated to an elliptic fibration $f\colon X \to C$, then f is isotrivial [Bru00, pag. 64]. In particular, it follows that f does not admit fibres of type I_b and I_b^* for $b \geq 1$ [Bru00, pag. 68].

Similarly to (1) of Lemma 2.22, we have:

Proposition 2.27. [Bru00, pag. 69-70]. *Let X be a smooth surface and let $\mathcal{F}$ be a turbulent foliation on X which is transverse to the elliptic fibration $f\colon X \to C$. Assume that $\mathcal{F}$ admits only reduced singularities. Let $p \in C$ and let F_{red} be the reduced divisor associated to f^*p.*

Then F_{red} has normal crossing singularities and there exists a neighborhood $\mathcal{U}$ of $f^{-1}(p)$ such that

$$K_{\mathcal{F}}|_{\mathcal{U}} \sim_{\mathbb{Q}} (K_X + F_{red})|_{\mathcal{U}}.$$

Corollary 2.28. *Let X be a smooth surface which admits an elliptic fibration $f\colon X \to C$ and let $\varepsilon\colon X \to X'$ be the relative minimal fibration associated to f with induced fibration $f'\colon X' \to C$. Let $\mathcal{F}$ be either a turbulent foliation on X which is transverse to f or the foliation induced by f. Let B_C be the discriminant of f and let B'_C be the $\mathbb{Q}$-divisor on C defined in* (2.2)*. Assume that $\mathcal{F}$ admits only reduced singularities and that $K_{\mathcal{F}}$ is pseudo-effective. Let $K_{\mathcal{F}} = P + N$ be the Zariski decomposition of $K_{\mathcal{F}}$ and assume that $\lfloor N \rfloor = 0$.*

Then

1. *$f(\mathrm{Supp}N) = \mathrm{Supp}B'_C$ and in particular $N = 0$ if and only if $B'_C = 0$;*
2. *if $p \in \mathrm{Supp}B'_C$ then the coefficient of B_C at p coincides with the coefficient of N along any reduced component E of f^*p which is not contained in the exceptional locus of ε;*
3. *if $p \in \mathrm{Supp}B'_C$ and $D \subseteq f^{-1}(p)$ is a prime divisor, then D is contained in the support of N if and only if it does not compute the log canonical threshold at p (cf. Remark 2.20); and*
4. *if $\mathcal{F}$ is turbulent and f^*p is a multiple fibre for some $p \in C$, then the reduced divisor associated to F is the union of a smooth curve of genus one and trees of $\mathcal{F}$-exceptional curves.*

Proof. We have $K_X = \varepsilon^* K_{X'} + E$ for some ε-exceptional divisor $E = \sum a_D D \geq 0$. By (1) of Lemma 2.22 and Proposition 2.27, it follows that for all $p \in C$, there exists a neighborhood $\mathcal{U}$ of $f^{-1}(p)$ such that

$$K_{\mathcal{F}}|_{\mathcal{U}} \sim_{\mathbb{Q}} (K_X + F_{red})|_{\mathcal{U}}$$

where F_{red} denotes the reduced divisor associated to $f^*p = \sum_{D\subseteq f^{-1}(p)} l_D D$. Since $\varepsilon^* K_{X'}|_{\mathcal{U}} \sim_{\mathbb{Q}} 0$, it follows that if γ_p is the log canonical threshold of X' with respect to f'^*p, then

$$K_{\mathcal{F}}|_{\mathcal{U}} \sim_{\mathbb{Q}} (\sum_{D\subseteq f^{-1}(p)} (a_D+1)D)|_{\mathcal{U}} \sim_{\mathbb{Q}} (\sum_{D\subseteq f^{-1}(p)} (a_D+1-\gamma_p l_D)D)|_{\mathcal{U}}.$$

On the other hand, we have that since $\mathcal{F}$ admits only reduced singularities, F_{red} has normal crossing singularities and as in Remark 2.20, we have

$$\gamma_p = \min\left\{ \frac{1+a_D}{l_D} \,\middle|\, D\subseteq f^{-1}(p) \right\}.$$

By (2) of Lemma 2.22 and (2.6) it follows that the Iitaka fibration of $K_{\mathcal{F}}$ factors through f. Thus, $P|_{\mathcal{U}} \sim_{\mathbb{Q}} 0$ and

$$N|_{\mathcal{U}} = (\sum_{D\subseteq f^{-1}(p)} (a_D+1-\gamma_p l_D)D)|_{\mathcal{U}}.$$

If $p\notin \operatorname{Supp} B'_C$ then $\gamma_p = \frac{1}{m_p}$, where m_p denotes the multiplicity of f'^*p. Thus, $\gamma_p l_D$ is a positive integer for each $D\subseteq f^{-1}(p)$. In particular, since by assumption $\lfloor N\rfloor = 0$, we have that $N|_{\mathcal{U}} = 0$.

On the other hand, if $p\in \operatorname{Supp} B'_C$ then by the classification of the singular fibres of an elliptic fibration, there exists a component D_0 of F_{red} such that $l_{D_0} = 1$ and $a_{D_0} = 0$. Thus (1) and (3) follow. If $E\subseteq f^{-1}(p)$ is a reduced component of f^*p which is not contained in the exceptional locus of ε, we have that $a_E = 0$ and $l_E = 1$. In particular, the coefficient of N along E is $1-\gamma_p$. Thus (2) follows.

Assume now that $\mathcal{F}$ is turbulent and f^*p is a multiple fibre for some $p\in C$ and let F_{red} be the reduced divisor associated to F. Suppose that F_{red} is not irreducible. By Remark 2.26, $f^{-1}(p)$ is not of type I_b or I_b^* for $b\geq 1$. Then by the classification of the singular fibres of an elliptic fibration, f^*p contains a smooth curve of genus 1 in its support and there exists a (-1)-curve E_0 contained in $f^{-1}(p)$ which meets $F_{red}-E_0$ transversally in either one or two points. Proposition 2.27 implies

$$K_{\mathcal{F}}\cdot E_0 = (K_X+F_{red})\cdot E_0 = -2+(F_{red}-E_0)\cdot E_0 \leq 0.$$

In particular, Proposition 2.4 implies that E_0 is $\mathcal{F}$-invariant, as otherwise

$$K_{\mathcal{F}}\cdot E_0 = -E_0^2 + \operatorname{tang}(\mathcal{F},E_0) \geq 1.$$

Thus, Proposition 2.5 implies that $Z(\mathcal{F},E_0) = (F_{red}-E_0)\cdot E_0 \leq 2$ and E_0 is $\mathcal{F}$-exceptional [Bru00, pag. 72]. Let $g\colon X\to Y$ be the contraction of E_0 and let $\mathcal{F}'$ be the induced foliation on Y. Then, after replacing X by Y and $\mathcal{F}$ by $\mathcal{F}'$ and repeating the same argument as above finitely many times, we may assume that there exists a (-1)-curve E_0 contained in $f^{-1}(p)$ such that $(F_{red}-E_0)\cdot E_0 = 1$ and in particular $K_{\mathcal{F}} = g^*K_{\mathcal{F}'}+E_0$. Thus, E_0 is contained in the support of $\lfloor N\rfloor$, a contradiction. Thus, F_{red} is a smooth curve of genus one and (4) follows. $\square$

2.2.4. *Isotrivial fibrations.* Given a smooth surface X, we consider an isotrivial fibration

$$\psi\colon X \to D$$

over a curve D and whose general fibre F has genus greater than one. Then there exists a curve G and a finite group Γ acting on G and F such that, if we consider the diagonal action of Γ on $G \times F$, then $D = G/\Gamma$, X is birational to $(G \times F)/\Gamma$ and the induced diagram

$$\begin{array}{ccc} X & \dashrightarrow & (G \times F)/\Gamma \\ & \searrow^{\psi} & \downarrow h \\ & & D = G/\Gamma. \end{array}$$

is commutative. [Ser96, Prop 2.2] implies that

$$h^1(X, \mathcal{O}_X) = g(G/\Gamma) + g(F/\Gamma). \tag{2.8}$$

Let $\varepsilon\colon Y \to (G \times F)/\Gamma$ be the minimal resolution and let $f\colon Y \to D$ be the induced morphism. Then the exceptional locus of ε is a disjoint union of Hirzebruch-Jung strings, each of which meets the strict transform of a fibre of ψ transversally in one point [Ser96, Theorem 2.1].

Remark 2.29. With the notation introduced above, we claim that $f\colon Y \to D$ is the minimal fibration so that each fibre has support with only normal crossing singularities, which means that if $f'\colon Y' \to D$ is any fibration from a smooth surface Y' which is birational to $f\colon Y \to D$, i.e. there exists a birational map $\eta\colon Y' \dashrightarrow Y$ such that $f \circ \eta = f'$, and such that each fibre of f' has support with only normal crossing singularities, then there exists a proper birational morphism $Y' \to Y$ which defines a factorization of f'.

Indeed, let $q\colon W \to Y$ and $q'\colon W \to Y'$ be proper birational morphisms from a smooth surface W which resolve the indeterminacy of η, i.e. $\eta \circ q' = q$. It is enough to show that the exceptional locus of q' is contained in the exceptional locus of $\varepsilon \circ q$, which implies that there exists a proper birational morphism $Y' \to (G \times F)/\Gamma$ and the claim follows from the fact that Y is the minimal resolution of $(G \times F)/\Gamma$.

Let $h\colon (G \times F)/\Gamma \to D$ be the induced moprhism and let us assume by contradiction that the exceptional locus of q' is not contained in the exceptional locus of $\varepsilon \circ q$. It follows that there exists $p \in D$ such that if E is the strict transform in Y' of the support E' of the fibre $h^{-1}(p)$ then E is q'-exceptional. In particular E is a rational curve. Since F has genus greater than one, there are at least three singular points of $(G \times F)/\Gamma$ along E' which coincide with the branch points of the induced finite morphism $F \to E'$. Thus, there exist at least three Hirzebruch-Jung strings inside $f^{-1}(p)$ intersecting E and in particular the fibre $f'^{-1}(p)$ does not have support with only normal crossing, a contradiction. Thus, the claim follows.

Let us assume now that any fibre of $\psi\colon X \to D$ has support with only normal crossing singularities. Remark 2.29 implies that ψ factors through a proper birational morphism $X \to (G \times F)/\Gamma$. Let $C = F/\Gamma$. Then the induced morphism $\varphi\colon X \to C$ is a fibration which is called *transverse* to ψ. Note that φ is also an isotrivial fibration with general fibre isomorphic to G. From now on, we assume that the genus of G is greater than one.

As above, we denote by $\varepsilon\colon Y \to (G\times F)/\Gamma$ the minimal resolution and by $\psi'\colon Y\to D$ and $\varphi'\colon Y\to C$ the induced morphisms. By Remark 2.29, there exists a proper birational morphism $\nu\colon X\to Y$.

For any $p\in D$ and for any prime divisor $E\subseteq \psi^{-1}(p)$, we denote by l_E the ramification index of ψ along E, so that $\psi^*p=\sum l_E E$. Thus, we define $D_\psi=\sum(l_E-1)E$. Similarly, we define D_φ, $D_{\psi'}$ and $D_{\varphi'}$. Note that $D_{\varphi'}=\nu_*D_\varphi$ and $D_{\psi'}=\nu_*D_\psi$.

Theorem 2.30. [Ser96, Thm. 2.1(i) and Thm. 4.1] *With the notation introduced above, we have that the support of any fibre of φ and ψ has support with only normal crossing singularities.*

Further, if Z is the reduced divisor on Y whose support coincides with the exceptional locus of ε, we have

$$K_Y=\varphi'^*K_C+D_{\varphi'}+\psi'^*K_D+D_{\psi'}+Z.$$

Lemma 2.31. *With the notation introduced above, we have that the morphism φ' is the Iitaka fibration of $K_{Y/D}-D_{\psi'}$ and the morphism ψ' is the Iitaka fibration of $K_{Y/C}-D_{\varphi'}$.*

Proof. [Ser96, Prop. 3.1 and Prop. 5.1] implies that

$$\kappa(K_{Y/D}-D_{\psi'})=1.$$

Let G' be the general fibre of φ'. Then, if Z is the reduced divisor on Y whose support coincides with the exceptional locus of ε, Theorem 2.30 implies

$$G'\cdot(K_{Y/D}-D_{\psi'})=G'\cdot(\varphi'^*K_C+D_{\varphi'}+Z)=0.$$

Thus, it follows that φ' is the Iitaka fibration of $K_{Y/D}-D_{\psi'}$. Similarly, ψ' is the Iitaka fibration of $K_{Y/C}-D_{\varphi'}$. ☐

Thus, given an isotrivial fibration $\psi\colon X\to D$ as above and the associated transverse fibration $\varphi\colon X\to C$, there are two foliations on X which are naturally associated to ψ. Indeed, we denote by $\mathcal{F}$ the foliation induced by ψ and by $\mathcal{G}$ the foliation induced by φ. By (2.5), we have

$$K_{\mathcal{F}}=K_{X/D}-D_\psi \qquad\text{and}\qquad K_{\mathcal{G}}=K_{X/C}-D_\varphi. \tag{2.9}$$

In particular, Lemma 2.31 and Proposition 2.4 imply that $\operatorname{tang}(\mathcal{F},G)=0$ and therefore $\mathcal{F}$ is transverse to φ. Similarly, $\mathcal{G}$ is transverse to ψ. By Remark 2.21 and Theorem 2.30, $\mathcal{F}$ and $\mathcal{G}$ have reduced singularities.

Lemma 2.32. *With the notation introduced above, we have that*

1. *$\mathcal{F}$ and $\mathcal{G}$ are relatively minimal if and only if X is the minimal resolution of $(G\times F)/\Gamma$;*
2. *the support of the negative part in the Zariski decomposition of $K_{\mathcal{F}}$ coincides with the exceptional locus of the morphism $X\to(G\times F)/\Gamma$; and*

(3) *if* $p \in D$ *then there exists a unique component* E *of the fibre* ψ^*p *which is not* $\mathcal{F}$*-invariant and all the other components are contained in the exceptional locus of the morphism* $X \to (G \times F)/\Gamma$.

Proof. Remark 2.29 implies that the minimal resolution of $(G \times F)/\Gamma$ is the minimal fibration so that each fibre has support with only normal crossing singularities. Thus, by Remark 2.21, (1) follows.

We now show (2). We first assume that $\mathcal{F}$ is relatively minimal. By (1), we have that $\varepsilon\colon X \to (G \times F)/\Gamma$ is the minimal resolution. Let $C_1, \ldots, C_m$ be a maximal Hirzebruch-Jung string contained in the exceptional locus of ε. Then, only one tail, say C_m intersects the rest of the fibre and, in particular, we have that $C_1, \ldots, C_m$ are $\mathcal{F}$-invariant and such that $Z(\mathcal{F}, C_1) = 1$ and $Z(\mathcal{F}, C_i) = 2$ if $i = 2, \ldots, m$. Proposition 2.5 implies that $K_{\mathcal{F}} \cdot C_1 = -1$ and $K_{\mathcal{F}} \cdot C_i = 0$ for $i = 2, \ldots, m$. Let $\varepsilon'\colon X \to X'$ be the birational morphism which contracts $C_2, \ldots, C_m$. Then there exists a Cartier divisor K on X' such that $K_{\mathcal{F}} = \varepsilon'^* K$. We have $K \cdot \varepsilon'_* C_1 = K_{\mathcal{F}} \cdot C_1 = -1$ and in particular $\varepsilon'_* C_1$ is contained in the support of the negative part of the Zariski decomposition of K. Thus, $C_1, \ldots, C_m$ are also contained in the support of the negative part of the Zariski decomposition of $K_{\mathcal{F}}$ and the claim follows.

We now assume that $\mathcal{F}$ is not relatively minimal. Let $\varepsilon\colon Y \to (G \times F)/\Gamma$ be the minimal resolution. By Remark 2.29, there exists a birational morphism $\nu\colon X \to Y$. By (1), it follows that ν is not an isomorphism. We prove the claim by induction on the number of blow-ups in ν. We can factor $\nu = \mu \circ \nu_1$ where $\nu_1\colon X \to X_1$ is the contraction of a single $\mathcal{F}$-exceptional curve E_1. Let $\mathcal{F}_1$ be the induced foliation on X_1 and let $K_{\mathcal{F}_1} = P_1 + N_1$ be the Zariski decomposition of $K_{\mathcal{F}_1}$. By induction, the support of N_1 is the exceptional locus of $\varepsilon \circ \mu$. Since $\mathcal{F}_1$ has reduced singularities, there exists $\alpha \geq 0$ such that $K_{\mathcal{F}} = \nu_1^* K_{\mathcal{F}_1} + \alpha E_1$. Thus, it is enough to show that E_1 is contained in the support of the negative part of the Zariski decomposition of $K_{\mathcal{F}}$. If $\alpha > 0$, then the claim follows immediately. If $\alpha = 0$, then $K_{\mathcal{F}} = \nu_1^* K_{\mathcal{F}_1}$ and the support of the negative part of the Zariski decomposition of $K_{\mathcal{F}}$ is the preimage of the support of N_1. Since $K_{\mathcal{F}} \cdot E_1 = 0$, the centre of E_1 in X_1 is a singular point of $\mathcal{F}_1$ and in particular, it is contained in the intersection of two components of a fibre of the induced morphism $X_1 \to D$. Since the fibres of $(G \times F)/\Gamma \to D = G/\Gamma$ are irreducible, at least one of these components is $(\varepsilon \circ \mu)$-exceptional. Thus, the claim follows.

Finally, (3) follows from the fact that any fibre of the morphism $(G \times F)/\Gamma \to D = F/\Gamma$ is irreducible and its strict transform on X is the only component in the fibre $\psi^* p$ which is not $\mathcal{F}$-invariant. □

2.2.5. *Foliations of Kodaira dimension one.* We conclude this section with the following characterisation of foliations with reduced singularities of Kodaira dimension one, due to Mendes and McQuillan [Men00, McQ08]:

Theorem 2.33. *Let* $\mathcal{F}$ *be a foliation with reduced singularities on a smooth surface and such that* $\kappa(\mathcal{F}) = 1$. *Then* $\mathcal{F}$ *is one of the following:*

(1) *a Riccati foliation;*

(2) *a turbulent foliation;*
(3) *a foliation induced by a non-isotrivial elliptic fibration; or*
(4) *a foliation induced by an isotrivial fibration of genus ≥ 2.*

3. Proof of the main results

Let $(X_t, \mathcal{F}_t)_{t\in\Delta}$ be a family of foliations on surfaces with reduced singularities. Our goal is to prove that for any sufficiently large positive integer m, the plurigenera $h^0(X_t, mK_{\mathcal{F}_t})$ does not depend on $t \in \Delta$. By Theorem 2.3, we can analyse separately the three cases

(a) $\kappa(\mathcal{F}_t) = 0$ for all $t \in \Delta$;
(b) $\kappa(\mathcal{F}_t) = 1$ for all $t \in \Delta$; and
(c) $\kappa(\mathcal{F}_t) = 2$ for all $t \in \Delta$.

3.1. **Kodaira dimension 0.** We begin with case (a) above:

Proposition 3.1. *Let $(X_t, \mathcal{F}_t)_{t\in\Delta}$ be a family of foliations such that $\kappa(\mathcal{F}_t) = 0$ for any $t \in \Delta$. Then for any positive integer m, the dimension $h^0(X_t, mK_{\mathcal{F}_t})$ does not depend on $t \in \Delta$.*

Proof. Let $K_{\mathcal{F}_t} = P_t + N_t$ be the Zariski decomposition of $K_{\mathcal{F}_t}$. By Lemma 2.12, the Cartier index m_0 of N_t does not depend on $t \in \Delta$. By Lemma 2.10, the Cartier divisor $m_0 P_t$ is a torsion divisor. In particular, the torsion index l does not depend on $t \in \Delta$. Hence, for all $t \in \Delta$ we have

$$h^0(X_t, mK_{\mathcal{F}_t}) = \begin{cases} 0 & \text{if } m_0 l \text{ does not divide } m; \\ 1 & \text{otherwise.} \end{cases}$$

Thus, the claim follows. □

3.2. **Kodaira dimension 1.** We now consider case (b) above. Given a family of foliations $(X_t, \mathcal{F}_t)_{t\in\Delta}$ of Kodaira dimension 1, we first prove that the foliations $\mathcal{F}_t$ are all of the same type: either $\mathcal{F}_t$ is Riccati for all t, or turbulent for all t, or induced by a non-isotrivial elliptic fibration for all t, or by an isotrivial fibration of curves of genus greater than 1 for all t (see Proposition 3.3). Then, we show that, for any $t \in \Delta$, there exists a fibration $\varphi_t \colon X_t \to C_t$ onto a curve C_t such that $\varphi_t^* \delta_t$ is the positive part of the Zariski decomposition of $K_{\mathcal{F}_t}$ for some ample $\mathbb{Q}$-divisor δ_t on C_t such that the degree of δ_t and the genus of C_t do not depend on $t \in \Delta$. Finally, for every such case, we show the invariance of the plurigenera $h^0(X_t, mK_{\mathcal{F}_t})$ for a sufficiently large positive integer m.

We begin with the following basic and more general result:

Lemma 3.2. *Let $\pi \colon \mathcal{X} \to \Delta$ be a smooth family of surfaces. For any $t \in \Delta$, let $X_t = \pi^{-1}(t)$. Let P be a nef $\mathbb{Q}$-divisor such that $P|_{X_t}$ is of Kodaira dimension 1 for any $t \in \Delta$ and let $P_t = P|_{X_t}$.*

Then, for any $t \in \Delta$, there exists a fibration $\varphi_t \colon X_t \to C_t$ onto a smooth curve C_t and an ample $\mathbb{Q}$-divisor δ_t on C_t such that $P_t = \varphi_t^* \delta_t$.

Further, if $P_t \cdot K_{X_t} \leq 0$ for some $t \in \Delta$ then the degree of δ_t, the genus of the general fibre of φ_t and the genus of C_t do not depend on $t \in \Delta$.

Proof. By assumption, we have that P_t is semi-ample for all $t \in \Delta$ and there exists a fibration $\varphi_t\colon X_t \to C_t$ onto a smooth curve C_t and an ample $\mathbb{Q}$-divisor δ_t on C_t such that $P_t = \varphi_t^*\delta_t$.

Pick $s \in \Delta$ and let F be a general smooth fibre of φ_s. We first assume that $K_{X_s} \cdot P_s < 0$. Then, for all $t \in \Delta$, the general fibre of φ_t is a smooth rational curve and, in particular, $K_{X_s} \cdot F = -2$. Thus, for all $t \in \Delta$, we have that $P \cdot K_{\mathcal{X}} \cdot X_t = -2 \deg \delta_t$, which implies that the degree of δ_t does not depend on $t \in \Delta$. Further, for all $t \in \Delta$, we have $R^1\varphi_{t*}\mathcal{O}_{X_t} = 0$ and by the Leray spectral sequence we have that

$$g(C_t) = h^1(C_t, \varphi_{t*}\mathcal{O}_{X_t}) = h^1(X_t, \mathcal{O}_{X_t}).$$

Thus, the claim follows.

We now assume that $K_{X_s} \cdot P_s = 0$. Then, the general fibre of φ_t is an elliptic curve for all $t \in \Delta$. Let us assume that there exists a (-1)-curve E_s on X_s which is contained in a fibre of φ_s. If N_{E_s/X_s} denotes the normal bundle of E_s in X_s then $h^1(E_s, N_{E_s/X_s}) = 0$ and by [Kod63, Theorem 1], after possibly shrinking Δ, there exists a smooth surface E in $\mathcal{X}$ which intersects the fibres of π transversally and such that $E|_{X_s} = E_s$. Thus, there exists a birational morphism $\varepsilon\colon \mathcal{X} \to \mathcal{X}'$ which contracts E and a smooth morphism $\pi'\colon \mathcal{X}' \to \Delta$ such that $\pi = \pi' \circ \varepsilon$ (see also [FM94, Chapt. 1, Prop. 1.20] for a similar argument). Let $Q = \varepsilon_*P$. Since $P_s \cdot E_s = 0$, it follows that $P = \varepsilon^*Q$ and therefore we may replace $\mathcal{X}$ by $\mathcal{X}'$ and P by Q. Thus, after finitely many steps, we may assume that there are no (-1)-curves on X_s which are contained in a fibre of φ_s. Since X_s is relatively minimal, it follows that $K^2_{X_s} = 0$. Thus, $K^2_{X_t} = 0$ for all $t \in \Delta$ and X_t is also relatively minimal.

Let B_{C_t} be the discriminant of φ_t (cf. (2.1)), let M_{C_t} be the moduli part of φ_t and let B'_{C_t} as in (2.2), for all $t \in \Delta$. As in [FM94, Chapt. I, Prop. 7.1], if $\varphi_s\colon X_s \to C_s$ has k multiple fibres, of multiplicities $m_1, \dots, m_k$ for some $s \in \Delta$, then every surface X_t has exactly k multiple fibres of multiplicities $m_1, \dots, m_k$. By [BHPdV04, Chapt. V, Prop.12.2], we have that $\deg(M_{C_t} + B'_{C_t}) = \chi(X_t, \mathcal{O}_{X_t})$ does not depend on $t \in \Delta$. Thus, it follows that the degree of $M_{C_t} + B_{C_t}$ does not depend on $t \in \Delta$. Since $K_{X_t} = \varphi_t^*(K_{C_t} + M_{C_t} + B_{C_t})$, invariance of plurigenera implies that the genus of C_t is constant.

We now show that, after possibly shrinking Δ, the divisor P is semi-ample over Δ, i.e. there exists a family of curves $\rho\colon \mathcal{C} \to \Delta$ and a morphism $q\colon \mathcal{X} \to \mathcal{C}$ such that π factors through q and $P \equiv q^*A$ for some ample $\mathbb{Q}$-divisor A on $\mathcal{C}$. To this end, we distinguish three cases. If $\kappa(X_t) > 0$ then [FM94, Chapt. I, Thm. 7.11 (iii)] implies the existence of the family $\rho\colon \mathcal{C} \to \Delta$ and the morphism $q\colon \mathcal{X} \to \mathcal{C}$ as above. Let A be any relatively ample $\mathbb{Q}$-divisor on $\mathcal{C}$ and let $P' = q^*A$. Then, after possibly rescaling A, we may assume that $P|_s \equiv P'|_s$. Thus, $(P - P') \cdot H \cdot X_t = 0$ for any ample divisor H on $\mathcal{X}_t$ and for all $t \in \Delta$. It follows that, after possibly shrinking Δ, we have that $P \equiv P'$ and the claim follows.

Thus, we may assume that $\kappa(X_t) \leq 0$ and, in particular, $g(C_t) \leq 1$ for all $t \in \Delta$. We first assume that $g(C_t) = 0$ for all $t \in \Delta$. After possibly shrinking Δ, we may assume that the stable base locus of P is contained in the fibre $X_0 = \pi^{-1}(0)$ and in particular, if $\mathcal{X}^* = \mathcal{X} \setminus X_0$, then $P|_{\mathcal{X}^*}$ is semi-ample. Thus, there exists a factorization

$$\mathcal{X}^* \xrightarrow{\varphi} \mathcal{C}^* \xrightarrow{\nu} \Delta \setminus \{0\}$$

where, for each $t \in \Delta\setminus\{0\}$, we have $C_t = \nu^{-1}(t)$. We may assume that $P \geq 0$. Let $x \in X_0$ be a general point. After possibly shrinking Δ, we may assume that there exists a one-dimensinoal subvariety $\Gamma \subseteq \mathcal{X}$ such that the induced morphism $\pi|_\Gamma \colon \Gamma \to \Delta$ is an isomorphism and Γ meets X_t transversally for each $t \in \Delta$. Let $\Gamma^* = \Gamma \setminus X_0$ and let $R^* = \varphi^{-1}(\varphi(\Gamma^*))$. Then, for each $t \in \Delta \setminus \{0\}$, we have that $R^* \cap X_t$ is a fibre of φ_t. Let R be the closure of R^* in $\mathcal{X}$. After possibly shrinking Δ, we may assume that R does not intersect the support of P. There exists a positive rational number α such that if $Q = \alpha R$, we have that $P|_{X_0} \equiv Q|_{X_0}$ for some $\mathbb{Q}$-divisor $Q \geq 0$ whose support does not intersect the support of P. As above, it follows that after possibly shrinking Δ further, we have that $P \equiv Q$. Moreover, since $g(C_t) = 0$, there exists a positive integer m such that $mP|_{X_t} \sim mQ|_{X_t}$ for all $t \in \Delta$. In particular, $mP \sim mQ$. Since the support of P and the support of Q do not intersect, it follows that the relative Iitaka fibration associated to P is a proper morphism over Δ, i.e. P is semi-ample over Δ. Thus, the claim follows.

We now assume that $g(C_t) = 1$, and in particular $\kappa(X_t) = 0$ for all $t \in \Delta$. By the Enrique-Kodaira classification of algebraic surfaces, it follows that X_t is either hyperelliptic for all $t \in \Delta$ or an abelian surface for all $t \in \Delta$. In the first case, [FM94, pag. 130, Remark 2)] implies the existence of the family $\rho\colon \mathcal{C} \to \Delta$ and we can conclude as above. Let us assume now that X_t is an abelian surface for all $t \in \Delta$. By [MFK94, Thm 6.14] (see also [FC90, pag. 89]), we may assume that $\mathcal{X}$ is an abelian scheme over Δ. In particular, there exists a morphism $\beta\colon \mathcal{X} \to \mathcal{X}$ over Δ, such that its restriction to X_t coincides with the inverse morphism on X_t for all $t \in \Delta$ and, for each section $\gamma\colon \Delta \to \mathcal{X}$ of π, we may define a morphism $t_\gamma\colon \mathcal{X} \to \mathcal{X}$ over Δ, whose restriction on X_t coincides with the translation by $\gamma(t)$ on X_t for all $t \in \Delta$. Thus, the theorem of squares (e.g. [BL04, Theorem 2.3.3]) implies that if Q is any Cartier divisor on $\mathcal{X}$ and if $\gamma' = \beta \circ \gamma$, then $2Q \sim t_\gamma^* Q \otimes t_{\gamma'}^* Q$. In particular, if Q is effective, then after possibly shrinking Δ, we have that Q is semi-ample over Δ. Thus, also in this case, the claim follows.

We now show that the degree of δ_t does not depend on $t \in \Delta$. We have that $P \equiv q^*A$ for some relatively ample $\mathbb{Q}$-divisor A on $\mathcal{C}$. Note that since the fibres of π are reduced, also the fibres of ρ are reduced. Thus, we have that

$$\deg \delta_t = \deg A|_{\rho^{-1}(t)}$$

does not depend on $t \in \Delta$, as claimed. □

We now show that the type of the foliation is preserved. A similar argument also appeared in [Bru01, pag. 130].

Proposition 3.3. *Let $(X_t, \mathcal{F}_t)_{t\in\Delta}$ be a family of foliations such that $\kappa(\mathcal{F}_t) = 1$ for any $t \in \Delta$. Then all the $\mathcal{F}_t$ are of the same type, i.e. there are 4 possibilities:*

(1) *$\mathcal{F}_t$ is a Riccati foliation transverse to a fibration $\varphi_t\colon X_t \to C_t$ for all $t \in \Delta$ (cf. Definition 2.24);*
(2) *$\mathcal{F}_t$ is a turbulent foliation transverse to a fibration $\varphi_t\colon X_t \to C_t$ for all $t \in \Delta$ (cf. Definition 2.25);*
(3) *$\mathcal{F}_t$ is induced by a non-isotrivial elliptic fibration $\varphi_t\colon X_t \to C_t$ for all $t \in \Delta$;*

(4) *$\mathcal{F}_t$ is not Riccati or turbulent and it is induced by an isotrivial fibration of genus ≥ 2 for all $t \in \Delta$.*

Further, in the cases (1), (2) *and* (3) *the genus $g(C_t)$ of the curve C_t does not depend on $t \in \Delta$.*

Proof. We first prove that the class of the foliation $\mathcal{F}_t$ does not depend on $t \in \Delta$. Fix $s \in \Delta$. By Lemma 2.7, we may assume that $\mathcal{F}_s$ is relatively minimal. Let $N \geq 0$ be the $\mathbb{Q}$-divisor on $\mathcal{X}$ whose existence is guaranteed by Proposition 2.11, such that, if we denote $P = K_{\mathcal{F}} - N$, $P_t = P|_{X_t}$ and $N_t = N|_{X_t}$ then $K_{\mathcal{F}_t} = P_t + N_t$ is the Zariski decomposition of $K_{\mathcal{F}_t}$ for all $t \in \Delta$. By Theorem 2.9, we have that $\lfloor N_s \rfloor = 0$. By Proposition 2.11, after possibly shrinking Δ, we may assume that each irreducible component of N meet the surfaces X_t transversally in a rational curve and that $\lfloor N_t \rfloor = 0$ for all $t \in \Delta$.

Let $t \in \Delta$. By Theorem 2.33, $\mathcal{F}_t$ is one of the following:

(1) a Riccati foliation;
(2) a turbulent foliation;
(3) a foliation induced by a non-isotrivial elliptic fibration; or
(4) a foliation which is not Riccati or turbulent and it is induced by an isotrivial fibration of genus ≥ 2.

For all $t \in \Delta$, let $\varphi_t \colon X_t \to C_t$ be the Iitaka fibration of P_t. Note that, by (2.6), if $\mathcal{F}_t$ is of type (1), (2) then it is transverse to φ_t and by (2) of Proposition 2.22, if $\mathcal{F}_t$ is of type (3) then it is induced by φ_t. If $\mathcal{F}_t$ is of type (4) and F is the general fibre of φ_t, since φ_t is the Iitaka fibration of $K_{\mathcal{F}_t}$, it follows that $K_{\mathcal{F}_t} \cdot F = 0$. Since any $\mathcal{F}_t$-invariant curve passing through the general point of X_t is of genus greater than one, Proposition 2.5 implies that F is not $\mathcal{F}_t$-invariant and therefore Proposition 2.4 implies that $\mathcal{F}_t$ is transverse to φ_t. By assumption $\mathcal{F}_t$ is not Riccati or turbulent, and in particular the genus of F is greater than one.

The intersection $K_{X_t} \cdot P|_{X_t} = K_{\mathcal{X}} \cdot P \cdot X_t$ does not depend on $t \in \Delta$. If $K_{X_t} \cdot P|_{X_t} < 0$ for all $t \in \Delta$ then $\mathcal{F}_t$ is of type (1) and if $K_{X_t} \cdot P|_{X_t} > 0$ for all $t \in \Delta$ then $\mathcal{F}_t$ is of type (4). Thus, we may assume that φ_t is an elliptic fibration and $\mathcal{F}_t$ is either turbulent or induced by a non-isotrivial elliptic fibration.

Let B_{C_t} be the discriminant of φ_t (cf. (2.1)), let M_{C_t} be the moduli part of φ_t and let B'_{C_t} as in (2.2), for all $t \in \Delta$. By [BHPdV04, Chapt. V, Prop.12.2], we have that $\deg(M_{C_t} + B'_{C_t}) = \chi(X_t, \mathcal{O}_{X_t})$ does not depend on $t \in \Delta$.

For any $t \in \Delta$ and for any $p \in \mathrm{Supp} B'_{C_t}$, (1) and (3) of Corollary 2.28 imply that there exists a prime divisor $D \subseteq \varphi_t^{-1}(p) \cap \mathrm{Supp} N$ which does not compute the log canonical threshold at p (cf. Remark 2.20). We claim that the number of points in $\mathrm{Supp} B'_{C_t}$ does not depend on $t \in \Delta$. Indeed, if $\mathcal{C}, \mathcal{C}'$ are connected components of $\mathrm{Supp} N$ such that $\mathcal{C}|_{X_t}, \mathcal{C}'|_{X_t}$ are contained in two different fibres of φ_t for some $t \in \Delta$, then it is sufficient to show that $\mathcal{C}|_{X_t}, \mathcal{C}'|_{X_t}$ are chains of rational curves which are contained in two different fibres of φ_t for all $t \in \Delta$. Indeed, assume by contradiction that there exists $t_0 \in \Delta$ and $\mathcal{C}_1, \ldots, \mathcal{C}_h, \mathcal{C}_{h+1}, \ldots, \mathcal{C}_k$ connected components of $\mathrm{Supp} N$ such that

- $\mathcal{C}_i|_{X_t}$ is contained in F_1^t for $i = 1, \ldots, h$ and

- $\mathcal{C}_i|_{X_t}$ is contained in F_2^t for $i = h+1, \dots, k$

where F_1^t, F_2^t are fibres of φ_t and $F_1^t \neq F_2^t$ for $t \neq t_0$ and $F_1^{t_0} = F_2^{t_0}$. Since $\lfloor N_t \rfloor = 0$ for all $t \in \Delta$, Lemma 2.23 implies that the chains $\mathcal{C}_i|_{X_t}$ contain all the reduced components of F_1^t for $i = 1, \dots, h$ for $t \neq t_0$. It follows that the same condition is preserved for $t = t_0$. By (1) of Corollary 2.28, the fibre F_2^t is not a multiple fibre for all $t \neq t_0$ and since φ_t is a fibration by elliptic curves, there exists a reduced component of F_2^t, for all $t \neq t_0$. Thus, there exists a component of a chain $\mathcal{C}_i$ with $i = h+1, \dots, k$ that degenerates to a rational curve with multiple coefficient, which is a contradiction because, by Proposition 2.11, the chains $\mathcal{C}_i$ meet the fibres of π transversally. Thus, it follows that the number of the points in the support of B'_{C_t} does not depend on $t \in \Delta$.

Moreover, by the classification of singular fibres of an elliptic fibration, there exists a reduced component E of $\varphi_t^* p$ which is not contained in the exceptional locus of the relative minimal fibration $X_t \to X'_t$ associated to $\varphi_t \colon X_t \to C_t$. Thus, (2) of Corollary 2.28 implies that $\deg B'_{C_t}$ does not depend on $t \in \Delta$. In particular, $\deg M_{C_t}$ does not depend on $t \in \Delta$. Remark 2.26 implies that $\deg M_{C_t} = 0$ if and only if $\mathcal{F}_t$ is turbulent. Thus, either $\mathcal{F}_t$ is turbulent for all $t \in \Delta$ or $\mathcal{F}_t$ is induced by a non-isotrivial elliptic fibration for all $t \in \Delta$.

Finally, Lemma 3.2 implies that in the cases (1), (2) and (3), the genus of the curve C_t does not depend on $t \in \Delta$. □

Proposition 3.4. *Let $(X_t, \mathcal{F}_t)_{t\in\Delta}$ be a family of foliations induced by non-isotrivial elliptic fibrations $\varphi_t \colon X_t \to C_t$ over a curve C_t of genus g. Then for any sufficiently large positive intger m, the dimension $h^0(X_t, mK_{\mathcal{F}_t})$ does not depend on $t \in \Delta$.*

In Example 3.9, we show that in general the claim does not hold if $m = 1$.

Proof. We want to show that for any sufficiently large positive integer m, the dimension $h^0(X_t, mK_{\mathcal{F}_t})$ is locally constant. Fix $s \in \Delta$. By Lemma 2.7, we may assume that $\mathcal{F}_s$ is relatively minimal. Let $N \geq 0$ be the $\mathbb{Q}$-divisor on $\mathcal{X}$ whose existence is guaranteed by Proposition 2.11, such that, if we denote $P = K_{\mathcal{F}} - N$, $P_t = P|_{X_t}$ and $N_t = N|_{X_t}$ then $K_{\mathcal{F}_t} = P_t + N_t$ is the Zariski decomposition of $K_{\mathcal{F}_t}$ for all $t \in \Delta$. By Theorem 2.9, we have that $\lfloor N_s \rfloor = 0$. By Proposition 2.11, after possibly shrinking Δ, we may assume that each irreducible component of N meet the surfaces X_t transversally in a rational curve and that $\lfloor N_t \rfloor = 0$ for all $t \in \Delta$.

For any $t \in \Delta$, let $\varepsilon_t \colon X_t \to X'_t$ be the minimal elliptic fibration associated to φ_t and let $\varphi'_t \colon X'_t \to C_t$ be the induced morphism. Let B_{C_t} and M_{C_t} be the discriminant and the moduli part in the canonical bundle formula of φ_t (cf. (2.1)). Then, Lemma 2.22 implies that

$$K_{X_t} = \varphi_t^*(K_{C_t} + M_{C_t} + B_{C_t}) + E_t \qquad \text{and} \qquad P_t = \varphi_t^*(M_{C_t}),$$

where $E_t \geq 0$ is ε_t-exceptional for each $t \in \Delta$.

Then Lemma 3.2 implies that

$$\deg(K_{C_t} + M_{C_t} + B_{C_t}) \qquad \text{and} \qquad \deg(M_{C_t})$$

do not depend on $t \in \Delta$. By Proposition 3.3 the genus of C_t does not depend on $t \in \Delta$. Thus, also $\deg B_{C_t}$ does not depend on $t \in \Delta$.

For any $t \in \Delta$, let B'_{C_t} be the $\mathbb{Q}$-divisor on C_t defined as in (2.2). By (2.3), we have

$$\mathrm{Supp}\{M_{C_t}\} = \mathrm{Supp} B'_{C_t}.$$

As in the proof of Proposition 3.3, there exists $c_1, \ldots, c_k \in (0,1)$ and, for any $t \in \Delta$, there exist distinct $p_1^t, \ldots, p_k^t \in C_t$ such that

$$B'_{C_t} = \sum_{i=1}^{k} (1 - c_i) p_i^t.$$

Since $M_{C_t} + B'_{C_t}$ is integral,

$$\{M_{C_t}\} = \sum_{i=1}^{k} c_i p_i^t.$$

Thus, for any positive integer m the degree of $\lfloor mM_{C_t} \rfloor$, does not depend on $t \in \Delta$ and we may choose m sufficiently large so that $\deg\lfloor mM_{C_t} \rfloor \geq 2g - 1$. For all $t \in \Delta$, we have

$$h^0(X_t, \mathcal{O}_{X_t}(\lfloor mP_t \rfloor)) = h^0(C_t, \mathcal{O}_{C_t}(\lfloor mM_{C_t} \rfloor)) = \deg\lfloor mM_{C_t} \rfloor + 1 - g(C_t)$$

and the claim follows. □

Proposition 3.5. *Let $(X_t, \mathcal{F}_t)_{t \in \Delta}$ be a family of Riccati or turbulent foliations transverse to the fibration $\varphi_t \colon X_t \to C_t$ for all $t \in \Delta$. Then for any positive integer m, the dimension $h^0(X_t, mK_{\mathcal{F}_t})$ does not depend on $t \in \Delta$.*

Proof. We want to show that $h^0(X_t, mK_{\mathcal{F}_t})$ is locally constant. Fix $s \in \Delta$. By Lemma 2.7, we may assume that $\mathcal{F}_s$ is relatively minimal. Let $N \geq 0$ be the $\mathbb{Q}$-divisor on $\mathcal{X}$ whose existence is guaranteed by Proposition 2.11, such that, if we denote $P = K_{\mathcal{F}} - N$, $P_t = P|_{X_t}$ and $N_t = N|_{X_t}$ then $K_{\mathcal{F}_t} = P_t + N_t$ is the Zariski decomposition of $K_{\mathcal{F}_t}$ for all $t \in \Delta$. By Theorem 2.9, we have that $\lfloor N_s \rfloor = 0$. By Proposition 2.11, after possibly shrinking Δ, we may assume that N meets the fibres of π transversally in chains of rational curves and that $\lfloor N_t \rfloor = 0$ for all $t \in \Delta$. By (2.6), there exist effective divisors $D^t_{\tan}$ and D_{φ_t} contained in the fibres of φ_t which are $\mathcal{F}_t$-invariant and such that

$$K_{\mathcal{F}_t} = \varphi_t^* K_{C_t} + D^t_{\tan} + D_{\varphi_t}.$$

As in (2.7), there exists a $\mathbb{Q}$-divisor θ_t on C_t such that $D^t_{\tan} + D_{\varphi_t} = \varphi_t^* \theta_t + N_t$ and $P_t = \varphi_t^*(K_{C_t} + \theta_t)$, for any $t \in \Delta$. By Lemma 3.2, the degree of $K_{C_t} + \theta_t$ is constant and, by Proposition 3.3, the genus of the curve C_t does not depend on $t \in \Delta$. It follows that also the degree of θ_t is constant.

For all $t \in \Delta$, we may write

$$\theta_t = \sum_{p \in C_t} \theta_{p,t} p.$$

We may assume $s = 0$. We want to show that the components of $\varphi_t^* p$ such that the coefficient $\theta_{p,t}$ is not integral do not meet as t approaches 0. Let $p \in C_t$ be such that $\theta_{p,t}$ is rational but not integral, for some $t \in \Delta$. We claim that either $\varphi_t^* p$ is a multiple fibre whose support is a smooth curve $\tilde{F}$ of genus one or it contains a reduced component. Indeed if $\mathcal{F}_t$ is a Riccati foliation, then the claim follows from the fact that every fibre of φ_t admits a reduced component. On the other hand, if $\mathcal{F}_t$ is a turbulent foliation, then the claim follows from (4) of Corollary 2.28 and the fact that, by classification of singular fibres of elliptic fibrations, any singular fibre, which is not a multiple fibre, contains a reduced component.

As in [FM94, Chapt. I, Prop. 7.1], if $\varphi_0\colon X_0 \to C_0$ has k multiple fibres, of multiplicities $m_1, \dots, m_k$, then every surface X_t has exactly k multiple fibres of multiplicities $m_1, \dots, m_k$. It follows that multiple fibres cannot meet. We now assume that $\varphi_t^* p$ is not a multiple fibre and it contains a reduced component. By Lemma 2.23, $\mathrm{Supp} N_t$ contains all the reduced components of $\varphi_t^* p$. Thus, since N meets X_t transversally for all $t \in \Delta$, it follows that the components of $\varphi_t^* p$ such that the coefficient $\theta_{p,t}$ is not integral do not meet as t approaches 0. Indeed the same argument as in Proposition 3.3 goes through because since φ_t is a fibration by rational or elliptic curves, every fibre has a reduced component.

By Proposition 3.3, the genus $g(C_t)$ does not depend on $t \in \Delta$. Thus,

$$h^0(X_t, mK_{\mathcal{F}_t}) = h^0(X_t, \lfloor mP_t \rfloor) = h^0(C_t, mK_{C_t} + \lfloor m\theta_t \rfloor) = \deg(mK_{C_t} + \lfloor m\theta_t \rfloor) + 1 - g(C_t)$$

does not depend on $t \in \Delta$ and the claim follows. □

Proposition 3.6. *Let $(X_t, \mathcal{F}_t)_{t\in\Delta}$ be a family of foliations which are not Riccati or turbulent and they are induced by isotrivial fibrations of genus $g \geq 2$. Then for any positive integer m, the dimension $h^0(X_t, mK_{\mathcal{F}_t})$ does not depend on $t \in \Delta$.*

Proof. We want to show that $h^0(X_t, mK_{\mathcal{F}_t})$ is locally constant. Fix $s \in \Delta$. By Lemma 2.7, we may assume that $\mathcal{F}_0$ is relatively minimal. Let $N \geq 0$ be the $\mathbb{Q}$-divisor on $\mathcal{X}$ whose existence is guaranteed by Proposition 2.11, such that, if we denote $P = K_{\mathcal{F}} - N$, $P_t = P|_{X_t}$ and $N_t = N|_{X_t}$ then $K_{\mathcal{F}_t} = P_t + N_t$ is the Zariski decomposition of $K_{\mathcal{F}_t}$ for all $t \in \Delta$. By Theorem 2.9, we have that $\lfloor N_0 \rfloor = 0$. By Proposition 2.11, after possibly shrinking Δ, we may assume that each irreducible component of N meet the surfaces X_t transversally in a rational curve and that $\lfloor N_t \rfloor = 0$ for all $t \in \Delta$.

By Remark 2.29, for all $t \in \Delta$, there exists a proper birational morphism

$$\alpha_t\colon X_t \to W_t := (G_t \times F_t)/\Gamma_t$$

where F_t and G_t are smooth curves such that the genus of F_t is greater than one and Γ_t is a finite group acting on F_t and G_t. By Theorem 2.30, for any $t \in \Delta$, there exist two fibrations $\varphi_t\colon X_t \to C_t$ and $\psi_t\colon X_t \to D_t$ over the curves $C_t = F_t/\Gamma_t$ and $D_t = G_t/\Gamma_t$ with fibres having support with only normal crossing singularities and such that $\mathcal{F}_t$ is induced by ψ_t and it is transverse to φ_t. Since $\mathcal{F}_t$ is not Riccati nor turbulent, also the curve G_t is of genus greater than one for all $t \in \Delta$.

We now show that $\mathcal{F}_t$ is relatively minimal for all $t \in \Delta$. Indeed, by (2) of Lemma 2.32, the support of N_t coincides with the exceptional locus of α_t. By (1) of Lemma 2.32, the exceptional locus of α_t contains a (-1)-curve if and only if the foliation is not relatively minimal. Since $\mathcal{F}_s$ is relatively minimal, it follows that the support of N_s is a union of Hirzebruch-Jung strings and, in particular, the support of N_s does not contain any (-1)-curve. Therefore the support of N_t does not contain any (-1)-curve for all $t \in \Delta$. Thus, $\mathcal{F}_t$ is relatively minimal and X_t is the minimal resolution of W_t for all $t \in \Delta$.

Let $E_1, \dots, E_k$ be the irreducible components of N and, for any $t \in \Delta$, let

$$E_i^t = E_i|_{X_t}.$$

By Theorem 2.9, E_i^t is a smooth rational curve. In particular, the self-intersection $(E_i^t)^2$ is a negative number which does not depend on $t \in \Delta$. Since, by (2) of Lemma 2.32, the exceptional locus of α_t coincides with the support of N_t, it follows that there exists an effective $\mathbb{Q}$-divisor Θ on $\mathcal{X}$, whose coefficients are contained in $(0,1)\cap\mathbb{Q}$ and whose support is contained in the support of N, such that

$$K_{X_t} = \alpha_t^* K_{W_t} - \Theta|_{X_t}$$

for any $t \in \Delta$. In particular, $(\mathcal{X}, \Theta)$ is Kawamata log terminal, and since K_{W_t} is ample for all $t \in \Delta$, it follows by the base point free theorem [KM98, Thm. 3.24] (see also [KM98, Example 2.17]) that $K_{\mathcal{X}} + \Theta$ is semi-ample over Δ, i.e. if we define

$$R(\mathcal{X}/\Delta, K_{\mathcal{X}} + \Theta) = \bigoplus_{m \geq 0} \pi_* \mathcal{O}_{\mathcal{X}}(\lfloor m(K_{\mathcal{X}} + \Theta) \rfloor)$$

and

$$\mathcal{W} = \text{Proj } R(\mathcal{X}/\Delta, K_{\mathcal{X}} + \Theta) \tag{3.1}$$

denotes the log canonical model of $K_{\mathcal{X}} + \Theta$ over Δ, then, there exists a proper birational morphism $\alpha \colon \mathcal{X} \to \mathcal{W}$ which defines a factorization of π and such that, if we denote $\Delta^* = \Delta \backslash \{0\}$, then, after possibly shrinking Δ, the induced morphism $X_t \to \alpha(X_t)$ coincides with α_t, for all $t \in \Delta^*$. Note that, by the Negativity Lemma, we have that $P = \alpha^* Q$, where $Q = \alpha_* P$ is a nef $\mathbb{Q}$-divisor on $\mathcal{W}$.

We claim that, after possibly shrinking Δ, the $\mathbb{Q}$-divisor P is numerically equivalent to a semi-ample over Δ, i.e. there exists a family of curves $\mathcal{C} \to \Delta$ and a morphism $\varphi \colon \mathcal{X} \to \mathcal{C}$ which defines a factorization of π, and such that $P \equiv \varphi^* H$ for some relatively ample $\mathbb{Q}$-divisor H on $\mathcal{C}$. To this end, we may and will perform a base change $\tau \colon \Delta \to \Delta$ which is totally ramified over the origin. Indeed, we may replace $\pi \colon \mathcal{X} \to \Delta$ by the family of surfaces $\pi' \colon \mathcal{X}' \to \Delta$ obtained by base change and P by the pull-back of P on $\mathcal{X}'$. We will do this, without changing notation.

Lemma 2.31 and (2.9) imply that φ_t is the Iitaka fibration of $K_{\mathcal{F}_t}$ for all $t \in \Delta$. Let $Z = \cup E_i$ be the support of N. By (2) of Lemma 2.32, $Z|_{X_t}$ coincides with the exceptional divisor of α_t. Thus, (2.9) and Theorem 2.30 imply that

$$K_{X_t} - K_{\mathcal{F}_t} + Z|_{X_t} = K_{X_t} - K_{X_t/D_t} + D_{\psi_t} + Z|_{X_t} = K_{X_t/C_t} - D_{\varphi_t}$$

and, therefore, Lemma 2.31 implies that ψ_t is the Iitaka fibration of $(K_{\mathcal{X}} - K_{\mathcal{F}} + Z)|_{X_t}$ for all $t \in \Delta$. It follows that, after possibly shrinking Δ, we may assume that, if $\mathcal{X}^* = \mathcal{X} \setminus X_0$, we have two factorisations

$$\mathcal{X}^* \xrightarrow{\varphi'} \mathcal{C}^* \to \Delta^* \qquad \text{and} \qquad \mathcal{X}^* \xrightarrow{\psi'} \mathcal{D}^* \to \Delta^*$$

such that for all $t \in \Delta^*$ the restrictions of φ' and ψ' to X_t coincide with φ_t and ψ_t. After possibly shrinking Δ again, we may assume that there exist two families of curves $p' \colon F^* \to \Delta^*$ and $q' \colon G^* \to \Delta^*$ such that for all $t \in \Delta^*$, we have that F_t is isomorphic to $p'^{-1}(t)$ and G_t is isomorphic to $q'^{-1}(t)$. Finally, we may assume that the group Γ_t does not depend on $t \in \Delta^*$. We will denote it by Γ.

Since the moduli functor of stable curves with an action of a finite group is proper (e.g. see [Tuf93]), after a base change $\tau \colon \Delta \to \Delta$ totally ramified over the origin, we may find two families of curves $p \colon F \to \Delta$ and $q \colon G \to \Delta$ which extend the families F^* and G^* on the whole Δ. Moreover, the group Γ acts on F and G so that, for all $t \in \Delta^*$, the action on $p^{-1}(t)$ and $q^{-1}(t)$ coincides with the action of Γ_t on F_t and G_t respectively (see [vO06, Theorem 3.1] for a similar argument).

Let $\rho \colon \mathcal{W}' = (G \times_\Delta F)/\Gamma \to \Delta$. By abuse of notation, we continue to denote by $\mathcal{W}$, the pull-back of the log canonical model over Δ defined in (3.1) after the base change performed above. Note that $\mathcal{W}$ and $\mathcal{W}'$ are isomorphic over Δ^*, and by the separateness property of the moduli functor of stable pairs (e.g. see [HX13, Lemma 7.2]), it follows that $\mathcal{W}$ is isomorphic to $\mathcal{W}'$. In particular, there exists a proper birational morphism $\mathcal{X} \to \mathcal{W}'$, induced by $\alpha \colon \mathcal{X} \to \mathcal{W}$. Let $\mathcal{C} := F/\Gamma$ and let $\nu \colon \mathcal{C} \to \Delta$ be the induced morphism. Then, there exists a proper morphism $\varphi \colon \mathcal{X} \to \mathcal{C}$ which factors through $\mathcal{X} \to \mathcal{W}'$ and, after possibly shrinking Δ again, we may assume that there exists a relatively ample $\mathbb{Q}$-divisor H over Δ, and after rescaling H we have $P \equiv \varphi^* H$. Thus, the claim follows.

Since C_0 is the normalization of the curve $\nu^{-1}(0) \subseteq \mathcal{C}$, we have that $g(C_0) \le g(C_t)$ for all $t \in \Delta$ and, similarly, we have that $g(D_0) \le g(D_t)$. Since (2.8) implies that $h^1(X_t, \mathcal{O}_{X_t}) = g(C_t) + g(D_t)$, it follows that the genus of C_t and the genus of D_t do not depend on $t \in \Delta$.

Let $t \in \Delta$. By (2.6) there exists effective divisors $D^t_{\tan}$ and D_{φ_t} contained in fibres of φ_t such that

$$K_{\mathcal{F}_t} = \varphi_t^* K_{C_t} + D^t_{\tan} + D_{\varphi_t}.$$

Thus, (2.9) and Theorem 2.30 imply that

$$Z|_{X_t} = D^t_{\tan} \tag{3.2}$$

where, as above, Z denotes the support of N.

By (2.7), we have that $P_t = \varphi_t^*(K_{C_t} + \theta_t)$ for some $\mathbb{Q}$-divisor $\theta_t \ge 0$ on D_t such that $\deg(K_{C_t} + \theta_t)$ does not depend on $t \in \Delta$ and θ_t is the largest $\mathbb{Q}$-effective divisor such that $D^t_{\tan} + D_{\varphi_t} - \varphi_t^* \theta_t$ is effective. Since $g(C_t)$ is constant, we have that $\deg \theta_t = \deg \theta_0$ for all $t \in \Delta$.

We want to show that the points in the support of θ_t do not collide as t approaches 0. For any $t \in \Delta$, the support of θ_t is contained in $\varphi_t(\mathrm{Supp}(D^t_{\tan} + D_{\varphi_t}))$. Multiple fibres with smooth support do not collide nor they meet components corresponding to singular fibres because the self-intersection of the reduced part of a multiple fibre is nilpotent but non-trivial and the structure is preserved for all $t \in \Delta$. Since the components of N do not meet, (2) and (3) of Lemma 2.32 imply that the points in $\varphi_t(\mathrm{Supp} N_t)$ do not collide as t approaches 0. Thus, (3.2) implies the claim.

In particular, there exist positive rational numbers $c_1, \ldots, c_\ell$ such that for all $t \in \Delta$ there exist distinct points $p_{1,t}, \ldots, p_{\ell,t}$ which form irreducible curves on $\mathcal{C}$ and such that

$$\theta_t = \sum c_i p_{i,t} \qquad \text{for all } t \in \Delta. \tag{3.3}$$

In particular, the degree $\deg\lfloor m\theta_t \rfloor$ does not depend on $t \in \Delta$. Thus,

$$h^0(X_t, mK_{\mathcal{F}_t}) = h^0(X_t, \lfloor mP_t \rfloor) = h^0(C_t, mK_{C_t} + \lfloor m\theta_t \rfloor) = \deg(mK_{C_t} + \lfloor m\theta_t \rfloor) + 1 - g(C_t)$$

does not depend on $t \in \Delta$ and the claim follows. □

3.3. **Foliations of general type.** It remains to study the invariance of plurigenera for foliations of general type. The main ingredients used in the proof are Theorem 2.19, Theorem 2.16, Kawamata-Viehweg vanishing theorem and the study of the Zariski decomposition of $\lceil mP_t \rceil$.

Proposition 3.7. *Let $(X_t, \mathcal{F}_t)_{t\in\Delta}$ be a family of foliations such that $\kappa(\mathcal{F}_t) = 2$ for any $t \in \Delta$. Then for any sufficiently large positive integer m, the dimension $h^0(X_t, mK_{\mathcal{F}_t})$ does not depend on $t \in \Delta$.*

Proof. In the course of the proof, we denote by $\{D\}$ the fractional part of a $\mathbb{Q}$-divisor D. We divide the proof in 6 Steps:

Step 1. By Lemma 2.7, after possibly shrinking Δ, we may assume that $\mathcal{F}_0$ is relatively minimal, where $0 \in \Delta$ corresponds to the central fibre X_0. By Proposition 2.11, we may write

$$K_{\mathcal{F}} = P + N$$

such that for any $t \in \Delta$, if we denote $P_t = P|_{X_t}$ and $N_t = N|_{X_t}$ then

$$K_{\mathcal{F}_t} = P_t + N_t$$

is a Zariski decomposition of $K_{\mathcal{F}_t}$. After possibly shrinking Δ further, by Proposition 2.11 we may assume that each irreducible component of N meet the surfaces X_t transversally in a rational curve. In particular, we have

$$H^0(X_t, mK_{\mathcal{F}_t}) \cong H^0(X_t, \lceil mP_t \rceil),$$

for any $t \in \Delta$. It is enough to show that $h^0(X_t, \lceil mP_t \rceil) \geq h^0(X_0, \lceil mP_0 \rceil)$ for any $t \in \Delta$. We denote by i the Cartier index of P_t. By Lemma 2.12, i does not depend on $t \in \Delta$.

Step 2. Let $E_1, \ldots, E_k$ be the irreducible components of N and let

$$E_i^t = E_i|_{X_t}.$$

By Theorem 2.9, E_i^t is a smooth rational curve. In particular, the self-intersection $(E_i^t)^2$ is a negative number which does not depend on $t \in \Delta$.

Thus, if

$$\nu_t \colon X_t \to Y_t$$

is the contraction of SuppN_t, then there exists an effective $\mathbb{Q}$-divisor Θ on $\mathcal{X}$, whose coefficients are contained in $(0,1)\cap\mathbb{Q}$ and whose support is contained in the support of N, such that

$$K_{X_t}=\nu_t^*K_{Y_t}-\Theta|_{X_t},$$

for any $t\in\Delta$. We denote

$$\Theta_t=\Theta|_{X_t}$$

for any $t\in\Delta$.

Step 3. For any $t\in\Delta$ and for any positive integer m, let

$$\lceil mP_t\rceil=P_m^{(t)}+N_m^{(t)}$$

be the Zariski decomposition of $\lceil mP_t\rceil$. Note that

$$\lceil mP_t\rceil=mP_t+\{-mP_t\}$$

and therefore $N_m^{(t)}\le\{-mP_t\}$. In particular, the support of $N_m^{(t)}$ is contained in the support of N_t and

$$\lfloor N_m^{(t)}\rfloor=0$$

for any $t\in\Delta$. Note that the divisor $N_m^{(t)}$ is uniquely determined by the intersection of $\lceil mP_t\rceil$ with the components of SuppN_t. Thus,

$$N_m^{(t)}=N_{m'}^{(t)}$$

for any positive integers m,m' which are equal modulo i. In addition, there exist $\mathbb{Q}$-divisors N_m, P_m on $\mathcal{X}$ such that

$$\lceil mP\rceil=P_m+N_m$$

and

$$N_m|_{X_t}=N_m^{(t)},$$

for any $t\in\Delta$.

Step 4. Let $\Gamma_1,\dots,\Gamma_p$ be all the elliptic Gorenstein leaves contained in the central fibre X_0 (cf. Definition 2.18). By Theorem 2.16, there exist $\mathcal{C}_1,\dots,\mathcal{C}_q$ disconnected chains of rational curves in X_0 such that

$$\{C\subseteq X_0\mid P\cdot C=0\}=\bigcup_{i=1}^p\Gamma_i\cup\bigcup_{i=1}^q\mathcal{C}_i\cup\mathrm{Supp}N_0.$$

Let $Z=\sum_{i=1}^p\Gamma_i+\sum_{i=1}^q\mathcal{C}_i$.

By Remark 2.17, the curves $\Gamma_1,\dots,\Gamma_p$ do not intersect the support of N_0 and in particular

$$\mathcal{O}_{\cup\Gamma_i}(\lceil mP_0\rceil)=\mathcal{O}_{\cup\Gamma_i}(mK_{\mathcal{F}_0})$$

Thus, Theorem 2.19 implies that for any positive integer m the sheaf $\mathcal{O}_{\cup\Gamma_i}(\lceil mP_0\rceil)$ has degree zero and is not torsion. For each $i=1,\dots,p$, the curve Γ_i is Cohen-Macaulay with trivial dualizing sheaf and Serre duality implies that

$$h^1(\Gamma_i,\lceil mP_0\rceil)=h^0(\Gamma_i,-\lceil mP_0\rceil)=0.$$

Thus

$$h^0(\cup\Gamma_i, \lceil mP_0\rceil) = h^1(\cup\Gamma_i, \lceil mP_0\rceil) = 0. \tag{3.4}$$

Let $C \subseteq \cup\mathcal{C}_i$ be an irreducible component. By Theorem 2.16, there are two possibilities:

type A: $C \cap \mathrm{Supp}N_0 = \emptyset$; or

type B: $N_0|_C = \frac{1}{2}p_1 + \frac{1}{2}p_2$, with $p_1, p_2 \in C$ distinct points.

Moreover, if C is of type B, then p_i belong to a connected component E_i^C of $\mathrm{Supp}N_0$ which is a smooth rational curve of self-intersection -2, for $i = 1, 2$. In particular, the coefficient of Θ_0 along E_i^C is zero for $i = 1, 2$.

For any positive integer m, we have

$$\deg\lceil mP_0\rceil|_C = mK_{\mathcal{F}_0} \cdot C - \deg\lfloor mN_0|_C\rfloor = \begin{cases} 0 & \text{if } C \text{ is of type A} \\ 0 & \text{if } C \text{ is of type B and } m \text{ is even} \\ 1 & \text{if } C \text{ is of type B and } m \text{ is odd.} \end{cases}$$

Let h_i be positive integers so that

$$\lfloor N_m^{(0)} + \Theta_0 + \frac{1}{2}\sum(E_1^C + E_2^C)\rfloor = \sum h_i E_i^0$$

where the sum runs over all the curves $C \subseteq \cup\mathcal{C}_i$ of type B. We first prove the following:

Claim: We have $h^1(\cup\mathcal{C}_i, \lceil mP_0\rceil - \sum h_i E_i^0) = 0$.

Let C_0 be an irreducible component of $\cup\mathcal{C}_i$. If C_0 is of type A, then

$$\deg(\lceil mP_0\rceil - \sum h_i E_i^0)|_{C_0} = 0,$$

and in particular $h^1(C_0, \lceil mP_0\rceil - \sum h_i E_i^0) = 0$.

If C_0 is of type B and m is even, then the coefficient of mP_0 along $E_i^{C_0}$ is integral and in particular the coefficient of $N_m^{(0)}$ along $E_i^{C_0}$ is zero. Since also the coefficient of Θ_0 along $E_i^{C_0}$ is zero, we have that

$$\mathrm{coeff}_{E_i^{C_0}}(\lfloor N_m^{(0)} + \Theta_0 + \frac{1}{2}\sum(E_1^C + E_2^C)\rfloor) = 0 \qquad \text{for } i = 1, 2$$

and, in particular,

$$\deg(\lceil mP_0\rceil - \sum h_i E_i^0)|_{C_0} = 0.$$

Thus, also in this case, we have $h^1(C_0, \lceil mP_0\rceil - \sum h_i E_i^0) = 0$.

Finally, if C_0 is of type B and m is odd, similarly as above we have

$$\deg(\sum h_i E_i^0)|_{C_0} \in \{0, 1, 2\},$$

and in particular

$$\deg(\lceil mP_0\rceil - \sum h_i E_i^0)|_{C_0} \in \{1, 0, -1\}$$

and also in this case, we have $h^1(C_0, \lceil mP_0\rceil - \sum h_i E_i^0) = 0$.

Thus, if $\mathcal{C}_i = C_1 \cup \cdots \cup C_q$ is a connected component of $\cup \mathcal{C}_i$, then it is a chain of rational curves which admits at most one tail of type B, and all the other curves are of type A. Thus we have a short exact sequence

$$0 \to \mathcal{O}_{\mathcal{C}_i}(\lceil mP_0 \rceil - \sum h_i E_i^0) \to \bigoplus_{j=1}^{q} \mathcal{O}_{C_j}(\lceil mP_0 \rceil - \sum h_i E_i^0) \to \bigoplus_{j=1}^{q-1} \mathcal{O}_{C_j \cap C_{j+1}}(\lceil mP_0 \rceil - \sum h_i E_i^0) \to 0$$

such that the induced map

$$\bigoplus_{j=1}^{q} H^0(C_j, \lceil mP_0 \rceil - \sum h_i E_i^0) \to \bigoplus_{j=1}^{q-1} H^0(C_j \cap C_{j+1}, \lceil mP_0 \rceil - \sum h_i E_i^0)$$

is surjective. Since $h^1(C_j, \lceil mP_0 \rceil - \sum h_i E_i^0) = 0$ for any irreducible component C_j of $\mathcal{C}$, the claim follows.

Consider now the short exact sequence

$$0 \to \mathcal{O}_{X_0}\left(\lceil mP_0 \rceil - Z - \sum h_i E_i^0\right) \to \mathcal{O}_{X_0}\left(\lceil mP_0 \rceil - \sum h_i E_i^0\right) \to \mathcal{O}_{\cup \Gamma_i \cup \mathcal{C}_i}\left(\lceil mP_0 \rceil - \sum h_i E_i^0\right) \to 0.$$

Thus, (3.4) and the claim above yield the exact sequence

$$\begin{aligned} 0 \to \quad & H^0\left(X_0, \lceil mP_0 \rceil - Z - \textstyle\sum h_i E_i^0\right) \to \quad H^0\left(X_0, \lceil mP_0 \rceil - \textstyle\sum h_i E_i^0\right) \to H^0\left(\cup \mathcal{C}_i, \lceil mP_0 \rceil - \textstyle\sum h_i E_i^0\right) \to \\ \to \quad & H^1\left(X_0, \lceil mP_0 \rceil - Z - \textstyle\sum h_i E_i^0\right) \to \quad H^1\left(X_0, \lceil mP_0 \rceil - \textstyle\sum h_i E_i^0\right) \to 0 \end{aligned}$$

and

$$H^2\left(X_0, \lceil mP_0 \rceil - Z - \sum h_i E_i^0\right) \cong H^2\left(X_0, \lceil mP_0 \rceil - \sum h_i E_i^0\right).$$

Step 5. Set $\bar{P}_0 = \nu_{0*} P_0$. By the Negativity Lemma, it follows that $P_0 = \nu_0^* \bar{P}_0$.

Let a be a positive integer. By Step 3, we have that

$$\operatorname{Supp} N_a^{(0)} \subseteq \operatorname{Supp} N_0.$$

Set $\bar{Z} = \nu_{0*} Z$. Thus

$$Z = \nu_0^* \bar{Z} - \frac{1}{2} \sum (E_1^C + E_2^C),$$

where, as above, the sum is over all the curve C of type B. Let $C_0 \subseteq X_0$ be an irreducible curve such that $P_0 \cdot C_0 = 0$. If C_0 is contained in the support of N_0, then

$$(K_{X_0} + \Theta_0) \cdot C_0 = 0 \qquad \text{and} \qquad (Z + \frac{1}{2} \sum (E_1^C + E_2^C)) \cdot C_0 = 0.$$

On the other hand, if C_0 is a smooth rational curve which is not contained in the support of N_0, then Theorem 2.16 and Remark 2.17 imply that

$$(\Theta_0 + Z + \frac{1}{2} \sum (E_1^C + E_2^C)) \cdot C_0 \leq C_0^2 + 2.$$

Finally, if C_0 is a single rational nodal curve, then

$$(K_{X_0} + Z) \cdot C_0 = 0 \qquad \text{and} \qquad \Theta_0 + \frac{1}{2} \sum (E_1^C + E_2^C)) \cdot C_0 = 0.$$

Thus, Theorem 2.16 implies that for each $C_0 \subseteq X_0$ irreducible curve such that $P_0 \cdot C_0 = 0$, we have

$$(K_{Y_0} + \bar{Z}) \cdot \nu_{0*}C_0 = (K_{X_0} + \Theta_0 + Z + \frac{1}{2}\sum(E_1^C + E_2^C)) \cdot C_0 \leq 0.$$

It follows that there exists a sufficiently large positive integer b such that

$$bi\bar{P}_0 - (K_{Y_0} + \bar{Z})$$

is big and nef.

We denote $m = a + bi$. In particular, by Step 3 we have that $N_a^{(0)} = N_m^{(0)}$. In addition,

$$\lceil mP_0 \rceil = biP_0 + \lceil aP_0 \rceil.$$

We have

$$\begin{aligned} & \lceil mP_0 \rceil - \sum \Gamma_i - \sum \mathcal{C}_i - \sum h_i E_i^0 \\ = \ & \lceil mP_0 \rceil - Z - \sum h_i E_i^0 \\ = \ & K_{X_0} + \nu_0^*(bi\bar{P}_0 - (K_{Y_0} + \bar{Z})) + \lceil aP_0 \rceil - \sum h_i E_i^0 + \Theta_0 + \tfrac{1}{2}\sum(E_1^C + E_2^C) \\ = \ & K_{X_0} + \nu_0^*(bi\bar{P}_0 - (K_{Y_0} + \bar{Z})) + P_a^{(0)} + N_a^{(0)} - \sum h_i E_i^0 + \Theta_0 + \tfrac{1}{2}\sum(E_1^C + E_2^C). \end{aligned}$$

Since $\sum h_i E_i^0 = \lfloor N_m^{(0)} + \Theta_0 + \frac{1}{2}\sum(E_1^C + E_2^C) \rfloor$, the divisor

$$N_a^{(0)} - \sum h_i E_i^0 + \Theta_0 + \frac{1}{2}\sum(E_1^C + E_2^C)$$

is an effective divisor with coefficients in the interval $(0, 1)$ and whose support is contained in the support of N_0. On the other hand, $\nu_0^*(bi\bar{P}_0 - (K_{Y_0} + \bar{Z})) + P_a^{(0)}$ is big and nef.

Thus, Kawamata-Viehweg vanishing theorem implies

$$H^j(X_0, \lceil mP_0 \rceil - Z - \sum h_i E_i^0) = 0 \qquad \text{for all } j > 0.$$

Then, by Step 4, we have

$$H^j(X_0, \lceil mP_0 \rceil - \sum h_i E_i^0) = 0 \qquad \text{for all } j > 0.$$

In particular,

$$h^0(X_0, \lceil mP_0 \rceil - \sum h_i E_i^0) = \chi(X_0, \lceil mP_0 \rceil - \sum h_i E_i^0).$$

Note that $\chi(X_t, \lceil mP_t \rceil - \sum h_i E_i^t)$ does not depend on $t \in \Delta$.

Step 6. In Step 4 we defined $\sum h_i E_i^0 = \lceil N_m^{(0)} + \Theta_0 + \frac{1}{2}\sum(E_1^C + E_2^C) \rceil$. Note that, since the coefficient of Θ_0 are contained in the interval $(0, 1)$, and for each curve C of type B, the curves E_1^C and E_2^C are not contained in the support of Θ_0, it follows that

$$\lfloor N_m^{(0)} + \Theta_0 + \frac{1}{2}\sum(E_1^C + E_2^C) \rfloor \leq \lceil N_m^{(0)} \rceil.$$

Thus, we have that

$$\sum h_i E_i^t \leq \lceil N_m^{(t)} \rceil$$

for any $t \in \Delta$. By the properties of the Zariski decomposition,

$$H^0(X_t, \lceil mP_t \rceil) \cong H^0(X_t, \lfloor \lceil mP_t \rceil - N_m^{(t)} \rfloor) = H^0(X_t, \lceil mP_t \rceil - \lceil N_m^{(t)} \rceil).$$

More precisely, for any effective divisor $E \leq \lceil N_m^{(t)} \rceil$,

$$H^0(X_t, \lceil mP_t \rceil) \cong H^0(X_t, \lceil mP_t \rceil - E).$$

In particular

$$h^0(X_t, \lceil mP_t \rceil) = h^0(X_t, \lceil mP_t \rceil - \sum h_i E_i^t)$$

for all $t \in \Delta$. Note that since P_t is big and nef for all $t \in \Delta$, if m is sufficiently large then Serre duality implies that

$$h^2(X_t, \lceil mP_t \rceil - \sum h_i E_i^t) = 0$$

for all $t \in \Delta$. Since

$$\lceil mP_t \rceil - \sum h_i E_i^t = (\lceil mP \rceil - \sum h_i E_i)|_{X_t}$$

for all $t \in \Delta$, Step 5 implies that

$$\begin{aligned} h^0(X_t, \lceil mP_t \rceil) &= h^0(X_t, \lceil mP_t \rceil - \sum h_i E_i^t) \\ &\geq \chi(X_t, \lceil mP_t \rceil - \sum h_i E_i^t) \\ &= \chi(X_0, \lceil mP_0 \rceil - \sum h_i E_i^0) \\ &= h^0(X_0, \lceil mP_0 \rceil - \sum h_i E_i^0) = h^0(X_0, \lceil mP_0 \rceil) \end{aligned}$$

for all $t \in \Delta$, concluding the proof of the Proposition. □

We are now ready to prove our main Theorem:

Proof of Theorem 1.1. Proposition 3.1 implies (1). Proposition 3.3 together with Propositions 3.4, 3.5 and 3.6 imply (2) and (3). Finally, Proposition 3.7 implies (4). □

3.4. **Some Examples.** We now show three examples on which invariance of plurigenera does not hold.

We begin by providing an example of a family of foliations which does not satisfy hypothesis (2) of Definition 2.2 and for which the Kodaira dimension is not constant:

Example 3.8. For $j = 1, 2$, let $f_{j,t_0} \in \mathbb{C}[x_0, x_1, x_2]$ be a homogeneous polynomial of degree four which is the product of a linear factor l and a factor c_j of degree 3 such that $\{c_j = 0\}$ is a smooth cubic

$$f_{j,t_0} = l \cdot c_j.$$

For $j = 1, 2$, let $f_{j,t_1} \in \mathbb{C}[x_0, x_1, x_2]$ be a homogeneous polynomial of degree four such that $\{f_{j,t_1} = 0\}$ is a smooth quartic. We may assume that f_{j,t_0} and f_{j,t_1} are such that

- for any $t \in \Delta \setminus \{0\}$ and $j = 1, 2$, if we define $f_{j,t} = t f_{j,t_0} + (1 - t) f_{j,t_1}$ then the curve $\{f_{j,t} = 0\}$ is a smooth quadric;
- for any $t \neq 0$ the curves $\{f_{1,t} = 0\}$ and $\{f_{2,t} = 0\}$ meet transversally;
- the curves $\{c_1 = 0\}$ and $\{c_2 = 0\}$ meet transversally in 9 points $p_1, \ldots, p_9$; and

- for any $t \in \Delta \setminus \{0\}$, the curves of the pencil

$$\mathcal{P}_t = \{uf_{1,t} + vf_{2,t} = 0 \,\big|\, [u:v] \in \mathbb{P}^1\}$$

  are all irreducible and reduced.

The base points of the pencils $\mathcal{P}_t$ form 16 curves $\mathcal{B}_1, \ldots, \mathcal{B}_{16}$ in $\mathbb{P}^2 \times \Delta$ meeting the fibres transversally. After possibly reordering the points $p_1, \ldots, p_9$, we may assume that there exists k such that the curves $\mathcal{B}_1, \ldots, \mathcal{B}_{16}$ pass through $p_{k+1}, \ldots, p_9$ and we may pick $\mathcal{B}_{17}, \ldots, \mathcal{B}_{16+k}$ smooth curves in $\mathbb{P}^2 \times \Delta$ meeting $\mathbb{P}^2 \times \{0\}$ transversally in $p_1, \ldots, p_k$.

Let

$$\mathcal{X} \xrightarrow{\varepsilon} \mathbb{P}^2 \times \Delta \to \Delta$$

be the blow-up of $\mathbb{P}^2 \times \Delta$ along the curves $\mathcal{B}_1, \ldots, \mathcal{B}_{16+k}$. Let $\pi\colon \mathcal{X} \to \Delta$ be the induced morphism with $X_t = \pi^{-1}(t)$ and let $\varepsilon_t\colon X_t \to \mathbb{P}^2$ be the induced morphism with exceptional divisors $E_1^t, \ldots, E_{16+k}^t$.

For any $t \in \Delta\setminus\{0\}$ we have a fibration $f_t\colon X_t \to \mathbb{P}^1$ whose general fibres are the strict transforms of the elements of $\mathcal{P}_t$. On X_0 we have a fibration $f_0\colon X_0 \to \mathbb{P}^1$ whose fibres are the curves in the pencil generated by c_1 and c_2. Let H be a hyperplane section in $\mathbb{P}^2$ and let $p \in \mathbb{P}^1$ be a general point. For any $t \neq 0$, the fibre f_t^*p is the strict transform of a curve in the pencil, thus

$$f_t^*p \sim \varepsilon_t^*(4H) - \sum_{i=1}^{16} E_i^t.$$

Let $\mathcal{F}_t$ be the foliation on X_t induced by the fibration f_t. Then for any $t \neq 0$

$$K_{\mathcal{F}_t} = K_{X_t/\mathbb{P}^1} = \varepsilon_t^* K_{\mathbb{P}^2} + \sum_{i=1}^{16+k} E_i^t + f_t^*(2p) \sim \sum_{i=17}^{16+k} E_i^t + \varepsilon_t^* H + f_t^* p.$$

In particular, $K_{\mathcal{F}_t}$ is big. On the other hand, if $t = 0$ then

$$f_0^*p \sim \varepsilon_0^*(3H) - \sum_{i=1}^{9} E_i^0.$$

and

$$K_{\mathcal{F}_0} = K_{X_0/\mathbb{P}^1} = \varepsilon_0^* K_{\mathbb{P}^2} + \sum_{i=1}^{16+k} E_i^0 + f_0^*(2p) \sim \sum_{i=10}^{16+k} E_i^0 + f_0^* p.$$

Then for $t = 0$ the canonical divisor $K_{\mathcal{F}_0}$ is not big.

Note that the foliation we obtain is induced by a fibration on each fibre, but there does not exist a fibration on $\mathcal{X}$ which induces the foliation.

We now show an example of a family of foliations $(X_t, \mathcal{F}_t)_{t\in\Delta}$ of Kodaira dimension one, which are induced by nonisotrivial elliptic fibrations and such that $h^0(X_t, \mathcal{O}_{X_t}(K_{\mathcal{F}_t}))$ is not constant, for $t \in \Delta$.

Example 3.9. Let C be a curve of genus at least 2. Let $p_{0,t}$, $p_{1,t}$ be two families of points on C, with $t \in \Delta$, such that $p_{0,0} = p_{1,0}$ and $p_{0,t} \neq p_{1,t}$ for any $t \neq 0$. The line bundle

$$\mathcal{L}_t = \mathcal{O}_C(K_C + p_{0,t} - p_{1,t})$$

defines an elliptic fibration

$$\varphi_t \colon X_t \to C$$

such that its moduli part is $M_{C,t} = K_C + p_{0,t} - p_{1,t}$ and with discriminant $B_{C,t} = 0$ for any $t \in \Delta$ (cf. (2.1)). Let $\mathcal{F}_t$ be the foliation associated to φ_t. By Remark 2.21, the singularities of $\mathcal{F}_t$ are reduced and therefore they define a family of foliations $(X_t, \mathcal{F}_t)_{t\in\Delta}$. We have

$$H^0(X_t, \mathcal{O}_{X_t}(K_{\mathcal{F}_t})) = H^0(C, \mathcal{O}_{C_t}(M_{C,t}))$$

and the latter vector space has dimension g if $t = 0$ and strictly less than g if $t \neq 0$.

Note that elliptic Gorenstein leaves (cf. Definition 2.18) never appear on foliations of general type induced by fibrations over a curve. Indeed, let $\mathcal{F}$ be a foliation of general type on a smooth surface X induced by a fibration $f \colon X \to C$ over a curve C. Assume by contradiction that there exists an e.g.l. Γ on X. Since Γ is $\mathcal{F}$-invariant, Γ is contained in a fibre F of f. Since the fibres of f are connected, Remark 2.17 implies that $\mathrm{Supp}(\Gamma) = \mathrm{Supp}(F)$. Since $K_{\mathcal{F}} \cdot \Gamma_i = 0$ for any curve contained in the support of Γ, it follows that $K_{\mathcal{F}} \cdot F = 0$, which is a contradiction because $K_{\mathcal{F}}$ is big and the fibres form a covering family.

Nevertheless, even for foliations of general type induced by fibrations, invariance of plurigenera does not hold for small values of m, as our example below shows.

Example 3.10. Let C be a smooth curve of genus $g \geq 2$. Let $\mathcal{L}$ be a line bundle on $\mathcal{C} = C \times \Delta$ such that, if we denote $L_t = c_1(\mathcal{L}|_{C\times\{t\}})$, then

$$L_0 = K_C \qquad \text{and} \qquad h^0(C, L_t) < h^0(C, K_C) = g \quad \text{ if } t \neq 0.$$

Let

$$\mathcal{E} = \mathcal{O}_{\mathcal{C}} \oplus \mathcal{O}_{\mathcal{C}} \oplus \mathcal{L}$$

and let $p \colon \mathcal{Z} = \mathbb{P}(\mathcal{E}) \to \mathcal{C}$. Let $\eta \colon \mathcal{Z} \to \Delta$ be the induced morphism. For any $t \in \Delta$, we denote by $p_t \colon Z_t := \eta^{-1}(t) \to C$ the restriction morphism. We have

$$K_{Z_t/C} = -3\xi_t + p_t^* L_t$$

where $\xi_t = c_1(\mathcal{O}_{\mathcal{E}_t}(1))$. Let $\xi = c_1(\mathcal{O}_{\mathcal{E}}(1))$. The linear system $|4\xi|$ is base point free and in particular the general element $\mathcal{X} \in |4\xi|$ is smooth. Let $\pi \colon \mathcal{X} \to \Delta$ be the induced morphism and let $X_t = \pi^{-1}(t)$. Then, the morphism $\mathcal{X} \to \mathcal{C}$ defines a regular foliation $\mathcal{F}$ on $\mathcal{X}$ such that if $\mathcal{F}_t$ is the restriction of $\mathcal{F}$ to X_t then (2.5) implies that

$$K_{\mathcal{F}_t} = K_{X_t/C} = (\xi_t + p_t^* L_t)|_{X_t}$$

for all $t \in \Delta$. Note that $\mathcal{F}_t$ is a foliation of general type. We want to show that $h^0(X_t, K_{\mathcal{F}_t})$ is not constant.

The dimension

$$h^0(Z_t, \xi_t + p_t^* L_t) = 2h^0(C, L_t) + h^0(C, 2L_t)$$

is not constant by our choice of $\mathcal{L}$. Therefore. it is enough to show that

$$h^0(X_t, K_{X_t/C}) = h^0(Z_t, \xi_t + p_t^* L_t).$$

Pick $t \in \Delta$. By the exact sequence obtained by restriction, we have

$$0 \to H^0(Z_t, \xi_t + p_t^* L_t) \to H^0(X_t, K_{X_t/C}) \to H^1(Z_t, K_{Z_t/C}).$$

On the other hand, by Serre duality

$$h^1(Z_t, K_{Z_t/C}) = h^2(Z_t, p_t^* K_C)$$

and the latter dimension is zero by the Leray spectral sequence. Thus, the claim follows.

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