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Short Note

Extension of the Random-Effects Regression Algorithm to Account for the Effects of Nonlinear Site Response

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Abstract The random-effects regression algorithm, made popular within engineering seismology by Abrahamson and Youngs (1992), is arguably the most commonly used approach for developing empirical ground-motion models. The original presentation of this algorithm relates to the most simple application of a far more general mixed-effects model formulation. In recent years, it has become increasingly common to incorporate nonlinear site response effects within empirical, or semi-empirical, ground-motion models, but the original random-effects algorithm does not apply to cases in which the random effects enter the model in a nonlinear manner. This article presents a more general algorithm for fitting mixed-effects models that can accommodate nonlinear site effects (among other effects). The presented algorithm deliberately mirrors that of Abrahamson and Youngs (1992) but allows for the treatment of far more elaborate variance structures.

Introduction

The application of regression algorithms that partition the overall variability of ground motions among interevent and intraevent components are now common place in engineering seismology. Within the community, these algorithms are commonly referred to as random-effects approaches, but within the statistical literature they would be referred to more generally as mixed-effects models.

A brief overview of the evolution of mixed-effects approaches within engineering seismology has recently been presented by Stafford (2014), and it was noted therein that the most commonly used algorithm is that of Abrahamson and Youngs (1992), which builds upon the earlier work of Brillinger and Preisler (1984, 1985).

However, in recent years, the inclusion within ground-motion models of functional terms that reflect the effects of nonlinear site response has become more common. As shown by Chiou and Youngs (2008), these nonlinear effects complicate the variance structure within a regression analysis, and this has resulted in some confusion about how one should interpret the various components of the variance, as discussed by Al Atik and Abrahamson (2010). An important issue that has not yet received attention, however, is the fact that the standard algorithm of Abrahamson and Youngs (1992) cannot be directly applied in the case in which nonlinear site response is included within the ground-motion model (assuming a traditional treatment of these effects).

The purpose of this short note is to demonstrate how the commonly used algorithm of Abrahamson and Youngs (1992) can be modified in order to make it appropriate for models that have a more complex variance structure. In particular, the article focuses upon the treatment of nonlinear site effects.

The algorithm of Abrahamson and Youngs (1992) is used to define the parameters of the generic model:

$$\mathbf{y}_i = \boldsymbol{\mu}(\mathbf{X}_i; \boldsymbol{\beta}) + b_i + \boldsymbol{\varepsilon}_i, \tag{1}$$

in which the subscript i denotes that this expression holds for the records from an individual event. The $\mathbf{y}_i$ is an $n_i \times 1$ vector of the observed (usually logarithmic) motions for this event, and $\boldsymbol{\mu}$ represents the functional form of the model and takes the $n_i \times p$ dimension matrix $\mathbf{X}_i$ of predictor variables for the event. The number of observations from the ith event is denoted by n_i. The term b_i is the random effect for the event and is assumed to be distributed according to a normal distribution with zero mean and variance τ^2, and the $\boldsymbol{\varepsilon}_i$ is an $n_i \times 1$ vector of residual errors that are also assumed to be zero-mean normally distributed with a variance of σ^2. The overall set of parameters that must be estimated for this model are the elements of $\boldsymbol{\beta}$ (the fixed effects) as well as the variance components τ and σ.

The algorithm itself involves the following steps:

1. Obtain starting estimates of the model parameters $\boldsymbol{\beta}$ using a fixed-effects regression analysis.
2. For this $\boldsymbol{\beta}$, maximize the log-likelihood function in equation (2) in order to obtain τ and σ:

$$\ln \mathcal{L} = -\frac{N}{2}\ln(2\pi) - \frac{1}{2}\ln|\mathbf{C}| - \frac{1}{2}(\mathbf{y}-\boldsymbol{\mu})^T\mathbf{C}^{-1}(\mathbf{y}-\boldsymbol{\mu}). \tag{2}$$

   Note the covariance matrix $\mathbf{C}$ is a block-diagonal matrix defined only in terms of τ and σ.
3. Given $\boldsymbol{\beta}$, τ, and σ, compute the values of b_i for each event using equation (3), in which $\mathbf{y}_{ij}$ denotes the jth

element of the vector $\mathbf{y}_i$ and $\mathbf{X}_{ij}$ denotes the jth row of the matrix $\mathbf{X}_i$:

$$b_i = \frac{\tau^2 \sum_{j=1}^{n_i} \mathbf{y}_{ij} - \boldsymbol{\mu}(\mathbf{X}_{ij}; \boldsymbol{\beta})}{n_i \tau^2 + \sigma^2}. \tag{3}$$

4. Obtain an updated estimate of $\boldsymbol{\beta}$ using a fixed-effects regression analysis on the adjusted observations $\mathbf{y}_i \to \mathbf{y}_i - b_i$.
5. Repeat steps 2–4 until the log-likelihood function is maximized.

A detailed discussion of how the inclusion of nonlinear site response influences the variance components of a ground-motion model has been provided by Al Atik and Abrahamson (2010), and what is important to take from that discussion for now is the fact that one might formulate a regression model in a manner that leads to nonconstant values of τ and σ that cannot be used directly within equation (3) for estimating the random effects for each event. For that reason, Abrahamson and Silva (2007, pp. 167) state the following (in which their equation 4-24 corresponds to equation 3 above).

> *If the standard deviations are not constant (e.g., the data are heteroskedastic), then eq. 4-24 is modified to use the mean value of τ and σ for each event. For just magnitude dependence, this is just $\tau(M)$ and $\sigma(M)$. If there is also an amplitude dependence (e.g., due to non-linear site response), the mean value of τ and σ is computed based on the sampling of recordings for each event.*

Although this is an apparently straightforward solution to this problem, it is technically not a correct solution. To demonstrate this, a more generic framework for the regression analysis can be used (and will be presented in the following sections), but it is important to first understand more about the structure of the variance components in the case that the ground-motion model includes nonlinear site response effects.

Impact of Nonlinear Site Response

As shown by Al Atik and Abrahamson (2010), the ground motion recorded at the Earth's surface can be viewed as the result of some motion arriving at a particular horizon beneath the surface that is then modified as it propagates through the near-surface materials. This reference horizon is often assumed to correspond to the underlying bedrock at a site, but this is not necessarily the case. We do, however, need to assume that the seismic waves have only interacted with the propagating medium in a linear manner when they arrive at this reference horizon. Under this condition, we can then assume that the variations that we should expect from earthquake to earthquake at this horizon are representative of the interevent variability of the source motions.

In a manner consistent with the treatment of Chiou and Youngs (2008), Al Atik and Abrahamson (2010) show that the logarithmic surface motion $y^S(T)$ can be decomposed in terms of the predicted logarithmic motion at the reference horizon $\hat{y}^R(T)$, the predicted site amplification $\ln \widehat{AF}[T|\hat{y}^R_{ij}(T_0)]$, and some error components (or residual contributions). These error components relate to event-to-event effects η^R_i, intraevent effects on motions at the reference horizon level ξ^R_{ij}, and variability in the site amplification ζ_{ij} and the effects of their propagation when the overall nonlinear functional form is linearized about the median-predicted reference motion, $\hat{y}^R_{ij}(T_0)$:

$$\begin{aligned} y^S_{ij}(T) = {} & \hat{y}^R_{ij}(T) + \ln \widehat{AF}[T|\hat{y}^R_{ij}(T_0)] \\ & + \frac{\partial \ln AF[T|\hat{y}^R_{ij}(T_0)]}{\partial y^R_{ij}(T_0)} [\eta^R_i(T_0) + \xi^R_i(T_0)] + \eta^R_i(T) \\ & + \xi^R_{ij}(T) + \zeta_{ij}(T). \end{aligned} \tag{4}$$

In this framework, the site amplification function is defined for a given response period in terms of the amplitude of motions at the level of the reference horizon for potentially another response period T_0 (although T_0 can be made equal to T as well).

The terms $\eta^R_i(T_0)$ and $\xi^R_{ij}(T_0)$ can be represented as functions of variates at the actual response period of interest T using the relations in equations (5) and (6):

$$\eta^R_i(T_0) = \rho_{\eta/\tau}(T, T_0) \frac{\tau(T_0)}{\tau(T)} \eta^R_i(T) \tag{5}$$

and

$$\xi^R_{ij}(T_0) = \rho_{\xi/\sigma}(T, T_0) \frac{\sigma(T_0)}{\sigma(T)} \xi^R_{ij}(T). \tag{6}$$

The overall expression for the observed surface motions can then be recast into the following form that is a linear combination of the random variables $\eta^R_i(T)$, $\xi^R_i(T)$, and $\zeta_{ij}(T)$:

$$\begin{aligned} y^S_{ij}(T) = {} & \hat{y}^R_{ij}(T) + \ln \widehat{AF}[T|\hat{y}^R_{ij}(T_0)] \\ & + \left[1 + \frac{\partial \ln AF[T|\hat{y}^R_{ij}(T_0)]}{\partial y^R_{ij}(T_0)} \rho_{\eta/\tau}(T, T_0) \frac{\tau(T_0)}{\tau(T)}\right] \eta^R_i(T) \\ & + \left[1 + \frac{\partial \ln AF[T|\hat{y}^R_{ij}(T_0)]}{\partial y^R_{ij}(T_0)} \rho_{\xi/\sigma}(T, T_0) \frac{\sigma(T_0)}{\sigma(T)}\right] \xi^R_i(T) \\ & + \zeta_{ij}(T). \end{aligned} \tag{7}$$

In the case in which we predict the site amplification at response period T, in terms of reference horizon motions also at a period of T (so that $T = T_0$), this expression simplifies to become

$$\begin{aligned} y^S_{ij}(T) = {} & \hat{y}^R_{ij}(T) + \ln \widehat{AF}[T|\hat{y}^R_{ij}(T)] \\ & + \left[1 + \frac{\partial \ln AF[T|\hat{y}^R_{ij}(T)]}{\partial y^R_{ij}(T)}\right] \eta^R_i(T) \\ & + \left[1 + \frac{\partial \ln AF[T|\hat{y}^R_{ij}(T)]}{\partial y^R_{ij}(T)}\right] \xi^R_i(T) + \zeta_{ij}(T). \end{aligned} \tag{8}$$

Although the actual equation is still relatively complicated, the regression framework that is implied by this formulation is significantly improved over the case that T_0 is considered. In fact, the previous model that requires consideration of T_0 presents some very significant barriers. For example, some studies (e.g., Abrahamson and Silva, 2007) previously stated that the correlation coefficients $\rho_{\eta/\tau}$ and $\rho_{\xi/\sigma}$ can be computed from the interevent and intraevent residuals, respectively. However, when one considers the framework presented above, this is not strictly true. In fact, the correlation $\rho_{\xi/\sigma}(T, T_0)$ is supposed to reflect the correlation between $\xi_{ij}^R(T)$ and $\xi_{ij}^R(T)$, which are both unobserved except for the special case in which there is no site response.

Generic Mixed-Effects Model Formulation

Rather than working with the regression model of equation (1), it is preferable to use the more general representation shown as

$$\mathbf{y}_i = \boldsymbol{\mu}(\mathbf{X}_i; \boldsymbol{\beta}) + \mathbf{Z}_i\mathbf{b}_i + \boldsymbol{\varepsilon}_i, \tag{9}$$

in which $\mathbf{X}_i$ is an $n_i \times p$ dimension matrix of predictors that combines with the vector (not necessarily p-dimensional) of fixed-effects regression coefficients $\boldsymbol{\beta}$ to estimate the mean logarithmic motions $\boldsymbol{\mu} \equiv \boldsymbol{\mu}(\mathbf{X}_i; \boldsymbol{\beta})$. In a similar manner, $\mathbf{Z}_i$ is an $n_i \times q$ dimension matrix of predictors for the q-dimensional vector of random effects $\mathbf{b}_i$. Finally, the n_i-dimensional vector $\boldsymbol{\varepsilon}_i$ contains the residual errors. The subscript i here denotes that this expression is relevant for the ith group. In the development of ground-motion models, this ith group is simply the ith earthquake.

The theoretical properties of models of this form have been studied in detail by Pinheiro and Bates (2000), and the interested reader is directed to that text for a far more rigorous treatment than what is provided here. The focus of the present article is to extract just the pertinent characteristics of these quite elaborate models to render them applicable for the most common cases of interest in engineering seismology.

As opposed to the framework considered by Abrahamson and Youngs (1992), the framework of equation (9) allows for multiple random effects within the vector $\mathbf{b}_i$. It should be noted that although the random effects are represented in a linear manner here, the random effects can be included within the nonlinear model formulation in $\boldsymbol{\mu}$. However, as shown in the previous section and was done by Al Atik and Abrahamson (2010), it then becomes necessary to linearize the model using a first-order expansion so that the linear form of equation (9) is satisfied.

Now, the random effects $\mathbf{b}_i$ and the residual errors $\boldsymbol{\varepsilon}_i$ are assumed to be zero-mean random vectors that are distributed according to the multivariate normal distributions

$$\mathbf{b}_i \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Psi}) \quad \text{and} \quad \boldsymbol{\varepsilon}_i \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I}_{n_i}). \tag{10}$$

Pinheiro and Bates (2000) provide a very detailed theoretical description of how mixed-effects models are defined and fitted from a computational perspective. In this description, they state that the model parameters and random effects can be estimated using a pseudodata approach in which the original data and response matrices are augmented by pseudodata.

When using this augmented pseudodata, the governing likelihood function is given by

$$\mathcal{L}(\boldsymbol{\beta}, \boldsymbol{\theta}, \sigma^2|\mathbf{y}) = \prod_{i=1}^{M} \frac{\text{abs}|\boldsymbol{\Delta}|}{(2\pi\sigma^2)^{n_i/2}} \int \frac{1}{(2\pi\sigma^2)^{q/2}} \exp(-\|\tilde{\mathbf{y}}_i - \tilde{\boldsymbol{\mu}}_i - \tilde{\mathbf{Z}}_i\mathbf{b}_i\|^2/2\sigma^2)d\mathbf{b}_i, \tag{11}$$

in which the $\tilde{\mathbf{y}}_i$, $\tilde{\boldsymbol{\mu}}_i$, and $\tilde{\mathbf{Z}}_i$ are the augmented vectors and matrices defined by

$$\tilde{\mathbf{y}}_i = \begin{Bmatrix} \mathbf{y}_i \\ \mathbf{0} \end{Bmatrix}, \qquad \tilde{\boldsymbol{\mu}}_i = \begin{Bmatrix} \boldsymbol{\mu}_i \\ \mathbf{0} \end{Bmatrix} \quad \text{and} \quad \tilde{\mathbf{Z}}_i = \begin{bmatrix} \mathbf{Z}_i \\ \Delta \end{bmatrix}, \tag{12}$$

and the term $\boldsymbol{\Delta}$ is referred to as the relative precision factor and is any matrix that satisfies the condition

$$\frac{\boldsymbol{\Psi}^{-1}}{1/\sigma^2} = \boldsymbol{\Delta}^T\boldsymbol{\Delta}. \tag{13}$$

Although this likelihood function appears far more complex than that shown previously within the definition of the Abrahamson and Youngs (1992) algorithm, it is actually an equivalent formulation. It should also be noted that the use of the augmented pseudodata is effectively a clever way to remove a nested product operation from the likelihood function.

The conditional modes of the random effects are then defined from the expression

$$\hat{\mathbf{b}}_i = (\tilde{\mathbf{Z}}_i^T\tilde{\mathbf{Z}}_i)^{-1}\tilde{\mathbf{Z}}_i^T(\tilde{\mathbf{y}}_i - \tilde{\boldsymbol{\mu}}_i). \tag{14}$$

As mentioned previously, within the field of empirical ground-motion modeling, it is common to consider just a single random effect that accounts for the variations in ground motions leaving the source of an earthquake of some magnitude. This source variation is often attributed to variations in dynamic stress drop from event to event but also to the specific features of the dynamic evolution during a rupture, among other things. In this particular case, the term $\mathbf{Z}_i\mathbf{b}_i$ consists of a column vector of length n_i (that is $\mathbf{Z}_i = \mathbf{1}_{n_i}$) being multiplied by a scalar constant (which is $\mathbf{b}_i = b_i$). However, Stafford (2014) demonstrated that the structure of $\mathbf{Z}_i\mathbf{b}_i$ can be far more complex than this and can be used to reflect our understanding of how ground motions might vary from place to place.

The algorithm of Abrahamson and Youngs (1992) is subsumed within the more generic framework just presented. For the traditional case of a single additive random effect, the covariance matrix of the random effects is simply $\boldsymbol{\Psi} = \tau^2$, which then implies that

$$\frac{\boldsymbol{\Psi}^{-1}}{1/\sigma^2} = \boldsymbol{\Delta}^T\boldsymbol{\Delta} = \frac{1/\tau^2}{1/\sigma^2} = \left(\frac{\sigma}{\tau}\right)^2 \Rightarrow \boldsymbol{\Delta} = \frac{\sigma}{\tau}. \tag{15}$$

Because we also have $\mathbf{Z}_i = \mathbf{1}_{n_i}$, it is easy to show that the term $\tilde{\mathbf{Z}}_i^T\tilde{\mathbf{Z}}_i$ is given by $n_i + (\sigma/\tau)^2$ and that

$$(\tilde{\mathbf{Z}}_i^T\tilde{\mathbf{Z}}_i)^{-1} = \frac{1}{n_i + (\sigma/\tau)^2}. \quad (16)$$

In addition, the term $\tilde{\mathbf{Z}}_i^T(\tilde{\mathbf{y}}_i - \tilde{\boldsymbol{\mu}}_i) = \sum_j^{n_i} \mathbf{y}_{ij} - \boldsymbol{\mu}_{ij}$. Therefore, we have our estimates of $\mathbf{b}_i \equiv b_i$, given by the expression

$$\hat{b}_i = \frac{\sum_j^{n_i} \mathbf{y}_{ij} - \boldsymbol{\mu}_{ij}}{n_i + (\sigma/\tau)^2} = \frac{\tau^2 \sum_j^{n_i} y_{ij} - \mu_{ij}}{n_i\tau^2 + \sigma^2}, \quad (17)$$

which is exactly equivalent to the expression used by Abrahamson and Youngs (1992), and it can be appreciated that the generic framework presented above is therefore able to reduce to the special case considered by Abrahamson and Youngs (1992).

Extension for Nonlinear Site Response

To relate our extended expression for the surface motions, including the influence of the site response to this generic regression framework, it is necessary to recognize the following equivalencies, starting with $\mathbf{y}_i \equiv y_{ij}^S(T)$:

$$\boldsymbol{\mu}_i = \boldsymbol{\mu}_i(\mathbf{X}_i;\boldsymbol{\beta}) = \hat{\mathbf{y}}_i^R(T) + \ln\widehat{AF}[T|\hat{\mathbf{y}}_i^R(T_0)], \quad (18)$$

$$\mathbf{Z}_i\mathbf{b}_i = \left[\mathbf{1}_{n_i} + \frac{\partial \ln AF[T|\hat{\mathbf{y}}_i^R(T_0)]}{\partial \mathbf{y}_i^R(T_0)}\rho_{\eta/\tau}(T,T_0)\frac{\tau(T_0)}{\tau(T)}\right]\eta_i^R(T), \quad (19)$$

and

$$\boldsymbol{\varepsilon}_i = \left[\mathbf{1}_{n_i} + \frac{\partial \ln AF[T|\hat{\mathbf{y}}_i^R(T_0)]}{\partial \mathbf{y}_i^R(T_0)}\rho_{\xi/\sigma}(T,T_0)\frac{\sigma(T_0)}{\sigma(T)}\right]\xi_i^R(T) + \boldsymbol{\zeta} i(T). \quad (20)$$

For this elaborate case, the covariance matrix for the random effects is still just given by $\boldsymbol{\Psi} = \tau^2$, which has important implications for the apparent confusion discussed by Al Atik and Abrahamson (2010). However, the $\mathbf{Z}_i$ vector is no longer $\mathbf{1}_{n_i}$, but is instead defined as

$$\mathbf{Z}_i = \mathbf{1}_{n_i} + \frac{\partial \ln AF[T|\hat{\mathbf{y}}_i^R(T_0)]}{\partial \mathbf{y}_i^R(T_0)}\rho_{\eta/\tau}(T,T_0)\frac{\tau(T_0)}{\tau(T)}, \quad (21)$$

and the variance of $\boldsymbol{\varepsilon}_i$ has now changed completely from its original $\sigma^2 I_{n_i}$ to

$$\mathrm{var}(\boldsymbol{\varepsilon}_i) = \sigma^2\boldsymbol{\Lambda}_i = \sigma^2\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{n_i} \end{bmatrix}, \quad (22)$$

in which the various λ_j terms are defined from

$$\lambda_j = \left(1 + \frac{\partial \ln AF[T|\hat{\mathbf{y}}_{ij}^R(T_0)]}{\partial \mathbf{y}_{ij}^R(T_0)}\rho_{\xi/\sigma}(T,T_0)\frac{\sigma(T_0)}{\sigma(T)}\right)^2 + \frac{\sigma^2_{\ln AF}(T)}{\sigma^2(T)}. \quad (23)$$

This complex variance for the residual errors means that the regression model must be adjusted to account for the fact that the residual errors are no longer independently and identically distributed (i.i.d.). In order to recast this problem so that the residual errors are i.i.d., the generic regression model must be whitened by premultiplying the terms of the regression equation by the transpose of $\boldsymbol{\Lambda}_i^{-1/2}$.

The generic regression model is now represented by

$$\mathbf{y}_i^* = \boldsymbol{\mu}_i^* + \mathbf{Z}_i^*\mathbf{b}_i + \boldsymbol{\varepsilon}_i^*, \quad (24)$$

in which the new terms $\mathbf{y}_i^*$, $\boldsymbol{\mu}_i^*$, $\mathbf{Z}_i^*$, and $\boldsymbol{\varepsilon}_i^*$ are related to their earlier counterparts through

$$\mathbf{y}_i^* = (\boldsymbol{\Lambda}_i^{-1/2})^T\mathbf{y}_i, \qquad \boldsymbol{\mu}_i^* = (\boldsymbol{\Lambda}_i^{-1/2})^T\boldsymbol{\mu}_i,$$
$$\mathbf{Z}_i^* = (\boldsymbol{\Lambda}_i^{-1/2})^T\mathbf{Z}_i \quad \text{and} \quad \boldsymbol{\varepsilon}_i^* = (\boldsymbol{\Lambda}_i^{-1/2})^T\boldsymbol{\varepsilon}_i. \quad (25)$$

The random effects and residual error terms are now distributed according to

$$\mathbf{b}_i \sim \mathcal{N}(\mathbf{0},\boldsymbol{\Psi}) \quad \text{and} \quad \boldsymbol{\varepsilon}_i^* \sim \mathcal{N}(\mathbf{0},\sigma^2\mathbf{I}) \quad (26)$$

due to the fact that

$$\mathrm{var}(\boldsymbol{\varepsilon}_i^*) = E[\boldsymbol{\varepsilon}_i^{*T}\boldsymbol{\varepsilon}_i^*] = [\boldsymbol{\varepsilon}_i^T\boldsymbol{\Lambda}_i^{-1/2}(\boldsymbol{\Lambda}_i^{-1/2})^T\boldsymbol{\varepsilon}_i] = E[\boldsymbol{\varepsilon}_i^T\boldsymbol{\Lambda}_i^{-1}\boldsymbol{\varepsilon}_i] = E[\sigma^2\mathbf{I}_{n_i}]. \quad (27)$$

With this whitened form of the regression model, the estimates of the random effects can be found again through the use of augmented vectors and matrices using pseudodata:

$$\tilde{\mathbf{y}}_i^* = \begin{Bmatrix} \mathbf{y}_i^* \\ \mathbf{0} \end{Bmatrix}, \quad \tilde{\boldsymbol{\mu}}_i^* = \begin{Bmatrix} \boldsymbol{\mu}_i^* \\ \mathbf{0} \end{Bmatrix} \quad \text{and} \quad \tilde{\mathbf{Z}}_i^* = \begin{bmatrix} \mathbf{Z}_i^* \\ \boldsymbol{\Delta} \end{bmatrix}. \quad (28)$$

The conditional modes of the random effects are then defined as

$$\hat{\mathbf{b}}_i = (\tilde{\mathbf{Z}}_i^{*T}\tilde{\mathbf{Z}}_i^*)^{-1}\tilde{\mathbf{Z}}_i^{*T}(\tilde{\mathbf{y}}_i^* - \tilde{\boldsymbol{\mu}}_i^*). \quad (29)$$

In this general framework, it helps to understand a little more clearly what the nature of $\boldsymbol{\Lambda}_i$, its inverse, and square roots are. Although we could see earlier that the individual elements of $\boldsymbol{\Lambda}_i$ are rather elaborate terms (λ_j), the matrix itself is diagonal. Therefore, its inverse is simply given by

$$\boldsymbol{\Lambda}^{-1} = \begin{bmatrix} 1/\lambda_1 & 0 & \cdots & 0 \\ 0 & 1/\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\lambda_{n_i} \end{bmatrix}, \quad (30)$$

and the square root of this matrix is

$$\boldsymbol{\Lambda}^{-1/2} = \begin{bmatrix} 1/\sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & 1/\sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\sqrt{\lambda_{n_i}} \end{bmatrix}. \quad (31)$$

Thus, the premultiplication by the formidable looking term $(\boldsymbol{\Lambda}_i^{-1/2})^T$ amounts to a relatively innocuous operation involving a scalar multiplication on each element of our vectors.

As was done previously when demonstrating the equivalence of this general approach with that of Abrahamson and Youngs (1992), an expression for the random effects can be obtained from consideration of the form of the two main terms in equation (29).

The first term requires that we have defined $\tilde{\mathbf{Z}}_i^*$. In the present case, this is a column vector that has individual elements defined by

$$\tilde{\mathbf{Z}}_i^* = \begin{bmatrix} \left(1 + \frac{\partial \ln AF[T|\hat{\mathbf{y}}_{i1}^R(T_0)]}{\partial \mathbf{y}_{i1}^R(T_0)} \rho_{\eta/\tau}(T,T_0) \frac{\tau(T_0)}{\tau(T)}\right) / \sqrt{\lambda_1} \\ \left(1 + \frac{\partial \ln AF[T|\hat{\mathbf{y}}_{i2}^R(T_0)]}{\partial \mathbf{y}_{i2}^R(T_0)} \rho_{\eta/\tau}(T,T_0) \frac{\tau(T_0)}{\tau(T)}\right) / \sqrt{\lambda_2} \\ \vdots \\ \left(1 + \frac{\partial \ln AF[T|\hat{\mathbf{y}}_{in_i}^R(T_0)]}{\partial \mathbf{y}_{in_i}^R(T_0)} \rho_{\eta/\tau}(T,T_0) \frac{\tau(T_0)}{\tau(T)}\right) / \sqrt{\lambda_{n_i}} \\ \sigma(T)/\tau(T) \end{bmatrix}, \quad (32)$$

which dictates that the term $\tilde{\mathbf{Z}}_i^{*T}\tilde{\mathbf{Z}}_i^*$ is defined by

$$\tilde{\mathbf{Z}}_i^{*T}\tilde{\mathbf{Z}}_i^* = \sum_j^{n_i} \frac{1}{\lambda_j}\left(1 + \frac{\partial \ln AF[T|\hat{\mathbf{y}}_{ij}^R(T_0)]}{\partial \mathbf{y}_{ij}^R(T_0)} \rho_{\eta/\tau}(T,T_0) \frac{\tau(T_0)}{\tau(T)}\right)^2 + \left(\frac{\sigma(T)}{\tau(T)}\right)^2. \quad (33)$$

The term $\tilde{\mathbf{Z}}_i^{*T}(\tilde{\mathbf{y}}_i^* - \tilde{\boldsymbol{\mu}}_i^*)$ is also more simple than it first appears due to the presence of the 0 values in the augmented entities. That is, we can drop the tilde from the various terms and also factor out the whitening term to obtain

$$\tilde{\mathbf{Z}}_i^{*T}(\tilde{\mathbf{y}}_i^* - \tilde{\boldsymbol{\mu}}_i^*) = \mathbf{Z}_i^{*T}(\mathbf{y}_i^* - \boldsymbol{\mu}_i^*) = \mathbf{Z}_i^T \boldsymbol{\Lambda}_i^{-1}(\mathbf{y}_i - \boldsymbol{\mu}_i). \quad (34)$$

This second term is then equivalent to a sum of weighted residuals:

$$\mathbf{Z}_i^T \boldsymbol{\Lambda}_i^{-1}(\mathbf{y}_i - \boldsymbol{\mu}_i) = \sum_j^{n_i} \frac{1}{\lambda_j}\left(1 + \frac{\partial \ln AF[T|\hat{\mathbf{y}}_{ij}^R(T_0)]}{\partial \mathbf{y}_{ij}^R(T_0)} \rho_{\eta/\tau}(T,T_0) \times \frac{\tau(T_0)}{\tau(T)}\right)(\mathbf{y}_{ij} - \boldsymbol{\mu}_{ij}). \quad (35)$$

Combining equations (33)–(35) through equation (29), the final expression for the estimates of the random effects is then given by

$$\hat{b}_i = \frac{\sum_j^{n_i} \frac{1}{\lambda_j}\left(1 + \frac{\partial \ln AF[T|\hat{\mathbf{y}}_{ij}^R(T_0)]}{\partial \mathbf{y}_{ij}^R(T_0)} \rho_{\eta/\tau}(T,T_0) \frac{\tau(T_0)}{\tau(T)}\right)(\mathbf{y}_{ij} - \boldsymbol{\mu}_{ij})}{\sum_j^{n_i} \frac{1}{\lambda_j}\left(1 + \frac{\partial \ln AF[T|\hat{\mathbf{y}}_{ij}^R(T_0)]}{\partial \mathbf{y}_{ij}^R(T_0)} \rho_{\eta/\tau}(T,T_0) \frac{\tau(T_0)}{\tau(T)}\right)^2 + \left(\frac{\sigma(T)}{\tau(T)}\right)^2}. \quad (36)$$

In the case in which $T = T_0$ the expression simplifies to become

$$\hat{b}_i = \frac{\sum_j^{n_i} \frac{1}{\lambda_j}\left(1 + \frac{\partial \ln AF[T|\hat{\mathbf{y}}_{ij}^R(T)]}{\partial \mathbf{y}_{ij}^R(T)}\right)(\mathbf{y}_{ij} - \boldsymbol{\mu}_{ij})}{\sum_j^{n_i} \frac{1}{\lambda_j}\left(1 + \frac{\partial \ln AF[T|\hat{\mathbf{y}}_{ij}^R(T)]}{\partial \mathbf{y}_{ij}^R(T)}\right)^2 + \left(\frac{\sigma(T)}{\tau(T)}\right)^2}. \quad (37)$$

What should be clear from inspection of either equation (36) or (37) is that neither of the expressions can be rearranged into a form in which the heteroskedasticity is accommodated through the use of the mean values of the inter- and intraevent standard deviations.

With these expressions for the conditional modes of the random effects now defined, the principle component of the updated regression algorithm that is yet to be defined is the covariance matrix that will enter into the computation of the log likelihood previously defined in equation (2). This covariance matrix can be developed by recognizing that in the most general case that has been considered herein, the random effects and residual errors have been distributed according to

$$\mathbf{b}_i \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Psi}) \quad \text{and} \quad \boldsymbol{\varepsilon}_i \sim \mathcal{N}(\mathbf{0}, \sigma^2\boldsymbol{\Lambda}_i). \quad (38)$$

With the variance components represented in this manner, the conditional distribution of logarithmic ground motions, given known fixed and random effects, can be defined as

$$(\mathbf{y}_i|\boldsymbol{\mu}_i, \mathbf{Z}_i, \boldsymbol{\beta}, \mathbf{b}_i) \sim \mathcal{N}(\mathbf{0}, \sigma^2\boldsymbol{\Lambda}_i). \quad (39)$$

In a similar manner, the marginal distribution of the motions given the unknown random effects is defined as

$$(\mathbf{y}_i|\boldsymbol{\mu}_i, \mathbf{Z}_i, \boldsymbol{\beta}) \sim \mathcal{N}(\mathbf{0}, \mathbf{Z}_i^T\boldsymbol{\Psi}\mathbf{Z}_i + \sigma^2\boldsymbol{\Lambda}_i). \quad (40)$$

Therefore, the covariance of the logarithmic motions for a given event is defined using

$$\mathrm{var}(\mathbf{y}_i|\mathbf{X}_i, \mathbf{Z}_i, \boldsymbol{\beta}) = \mathbf{Z}_i^T\boldsymbol{\Psi}\mathbf{Z}_i + \sigma^2\boldsymbol{\Lambda}_i = \mathbf{C}_i. \quad (41)$$

The likelihood for an entire dataset is the product of the likelihoods for the individual events. Therefore, the log likelihood is the sum of the logarithmic contributions from each event. The overall covariance matrix in this case is related to the event-specific covariance matrices by $\mathbf{C} = \mathbf{C}_1 \oplus \mathbf{C}_2 \oplus \ldots$ (in which $\oplus$ is the direct sum operator). The determinant of $\mathbf{C}$ is also defined by the product of the determinants of the $\mathbf{C}_i$. That is, $|\mathbf{C}| = \prod_{i=1}^M |\mathbf{C}_i|$, and the inverse of $\mathbf{C}$ is composed of the inverses of the individual block matrices that represent its block diagonal form.

The log-likelihood function above can therefore be written as

$$\ln \mathcal{L} = -\frac{N}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{M}\ln|\mathbf{C}_i| - \frac{1}{2}\sum_{i=1}^{M}(\mathbf{y}_i - \boldsymbol{\mu}_i)^T \mathbf{C}_i^{-1}(\mathbf{y}_i - \boldsymbol{\mu}_i). \quad (42)$$

The equations linking the overall covariance matrix to the covariance matrices for each event would require modification in the case in which random effects for site, or regional, terms were considered (Stafford, 2014).

New Regression Algorithm

Building upon the results of the previous section, it is now possible to propose the updated form of the random-effects regression algorithm of Abrahamson and Youngs (1992). The new algorithm is shown below.

1. Obtain starting estimates of the model parameters $\boldsymbol{\beta}$ using a fixed-effects regression analysis.
2. For this $\boldsymbol{\beta}$, maximize the log-likelihood function defined in equation (42) in order to obtain the elements of $\boldsymbol{\Psi}$ (corresponding to just τ when only nonlinear site response is considered) and σ.
3. Given $\boldsymbol{\beta}$, $\boldsymbol{\Psi}$ (or just τ), and σ, compute the values of $\mathbf{b}_i$ (or just b_i) for each event using equation (36) or equation (37), or use a more elaborate expression derived in a similar manner if multiple random effects are considered.
4. Obtain an updated estimate of $\boldsymbol{\beta}$ using a fixed-effects regression analysis on the adjusted observations $\mathbf{y}_i \rightarrow \mathbf{y}_i - \mathbf{Z}_i\mathbf{b}_i$ (or just $\mathbf{y}_i \rightarrow \mathbf{y}_i - \mathbf{Z}_i b_i$ when only nonlinear site response is considered).
5. Repeat steps 2–4 until the log-likelihood function is maximized.

Example Application

To briefly demonstrate the difference between the rigorous algorithm just presented and the approximate solution of Abrahamson and Silva (2007) discussed earlier, a simple example application is performed. The functional form used for this exercise is similar to that used by Stafford (2014) but includes nonlinear site response terms based upon those used by Chiou and Youngs (2008).

The functional expression for the reference motion is given as

$$\ln y_{\text{ref}} = \beta_1 + \beta_M M_{\text{w}} + [\beta_4 + \beta_5(M_{\text{w}} - 6.75)]\ln\sqrt{R_{\text{rup}}^2 + \beta_6^2} + \beta_7 F_{\text{nm}} + \beta_8 F_{\text{rv}} + b. \quad (43)$$

The nonlinear site response is a function of this reference motion and is defined as

$$\ln y = \ln y_{\text{ref}} + \beta_9 \ln\left(\frac{V_{\text{lim}}}{1100}\right) + \beta_{10}\{\exp[\beta_{11}(V_{\text{lim}} - 360)] - \exp[\beta_{11}(1100 - 360)]\}\ln\left(\frac{y_{\text{ref}} + \beta_{12}}{\beta_{12}}\right), \quad (44)$$

Table 1
Coefficients from This Study and the Approach of Abrahamson and Silva (2007)

Parameter	This Study*	Abrahamson and Silva (2007)*
β_1	2.7179 ± 0.5267	2.4267 ± 0.5438
β_2	−0.2145 ± 0.0871	−0.1576 ± 0.0915
β_3	−0.2176 ± 0.0866	−0.1603 ± 0.0913
β_4	−1.1666 ± 0.0398	−1.1841 ± 0.0419
β_5	0.2557 ± 0.0227	0.2434 ± 0.0236
β_6	5.3972 ± 0.8171	5.4670 ± 0.8372
β_7	0.1831 ± 0.0558	0.1649 ± 0.0561
β_8	0.1658 ± 0.0292	0.1587 ± 0.0294
β_9	−0.5410 ± 0.0357	−0.5434 ± 0.0357
β_{10}	−0.3812 ± 0.0932	−0.4258 ± 0.1091
β_{11}	−0.0039 ± 0.0011	−0.0032 ± 0.0011
β_{12}†	0.1	0.1
τ	0.4282 ± 0.0423‡	0.4304 ± 0.0427‡
σ	0.4736 ± 0.0106‡	0.4774 ± 0.0106‡

*The ± values represent standard errors in the estimated coefficients.
†The value of β_{12} is fixed at 0.1 in both cases.
‡The standard errors presented for τ and σ are mapped values given that the regression solves for $\ln(\tau^2)$ and $\ln(\sigma^2)$.

in which the term $V_{\text{lim}} = \min(V_{S30}, 1100)$. In this equation, the coefficient β_M is equal to β_2 in the case in which $M_{\text{w}} \leq 6.75$ and is equal to β_3 otherwise. The predictor variables are the moment magnitude M_{w}, the rupture distance R_{rup}, the average shear-wave velocity V_{S30}, and binary variables for normal F_{nm} and reverse F_{rv} events. The predicted quantity $\ln y$ is the logarithmic peak ground acceleration. The random effect b features in the expression for the reference motion, but this effect also appears nonlinearly within the site response terms. The database used for the regressions is a subset of the Next Generation Attenuation (NGA)-West database (see Data and Resources; Chiou *et al.*, 2008) described in detail elsewhere (e.g., Stafford and Bommer, 2009; Stafford *et al.*, 2009).

The model parameters obtained through application of the rigorous approach presented in this article as well as through the use of the approximate method of Abrahamson and Silva (2007) are presented in Table 1. These results have been obtained, in both cases, by fixing the value of $\sigma_{\ln AF}$ to 0.3. The specific values of the coefficients are not of particular interest and are only presented here to highlight the fact that the different algorithms influence all aspects of the model, the fixed effects, variance components, and random effects. Although, with that being said, the standard errors shown in Table 1 demonstrate that the obtained fixed-effects coefficients are not statistically different from each other for this particular example.

The random effects, $\mathbf{b}_i$, obtained from both algorithms are compared in Figure 1 by plotting these against the magnitude of the event from which they came. It is clear from this figure that the application of these two different approaches also leads to different estimates of the random effects. The differences that will be encountered in any particular case will depend upon the fraction of records that are influenced by nonlinear site effects, as predicted by the model.

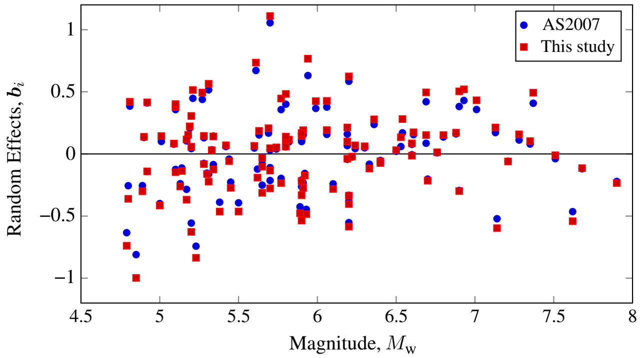

Figure 1. Comparison of random effects computed using the method of the present study and that of Abrahamson and Silva (2007; AS2007). The color version of this figure is available only in the electronic edition.

A point that is interesting to note is that Table 1 indicates that the variance components obtained from the new algorithm are smaller than for the Abrahamson and Silva (2007) approach. However, visual inspection of the random effects in Figure 1 appears to suggest the contrary. These differences can arise for a number of reasons. Small differences in the total residuals exist due to the slightly different fixed-effects coefficients, and the random effects are partitioned from these total residuals. The records for any given event are associated with differing degrees of modeled nonlinearity in the soil response. Events that generate motions strong enough to develop nonlinear site response are often widely recorded, and the more distant recordings tend to have linear response. Taking the mean of the variance components for all records from a given event can lessen the influence of the records with the strongest nonlinearity. Both of these causes depend upon the particular dataset and may result in positive or negative differences. However, the main reason for the systematic pattern of larger random effects for the method of this study can be most easily appreciated from consideration of equations (3) and (37) for the case in which an event is singly recorded. In this case, it can be shown that the ratio of random effects found using the mean of the interevent and intraevent standard deviations to those from this study is $1+\partial \ln AF[T|\hat{\mathbf{y}}_{ij}^{R}(T)]/\partial \mathbf{y}_{ij}^{R}(T)$. Given that the partial derivative is zero or negative (for the vast majority of cases in reality), the random effects computed using Abrahamson and Silva (2007) are smaller. This example serves as an important reminder that the interevent variance of a ground-motion model is not simply the variance of the random effects.

Conclusions

The inclusion of nonlinear site response effects within ground-motion models has resulted in the complexity of the variance structures of these models increasing considerably. A side effect of this increased complexity is that the most commonly used regression algorithm for fitting these models is no longer strictly valid. The more generic framework presented in this article allows for this common algorithm to be extended to handle the more elaborate variance structures that are now encountered during the development of sophisticated ground-motion models that incorporate nonlinear site response and other nonlinear effects.

Data and Resources

The ground-motion data was taken from the Pacific Earthquake Engineering Research (PEER) Next Generation Attenuation (NGA) database (http://peer.berkeley.edu/nga, last accessed June 2013).

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