A note on the electrical equivalent of the moment theory

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***Abstract*— In this short note the relation between the moments of a linear system and the phasors of an electric circuit is discussed. We show that the phasors are a special case of moments and we prove that the components of the solution of a Sylvester equation are the phasors of the currents of the system. We point out several directions in which the phasor theory can be extended using recent generalizations of the moment theory that can benefit the analysis of circuits and power electronics.**

I. INTRODUCTION

The phasor transform represents a powerful and flexible mathematical tool which has been used for the study of the steady-state behavior of circuits powered by sinusoidal sources [1], [2]. The phasor transform greatly simplifies the dynamic analysis because it changes integro-differential equations into algebraic equations, which are computationally and analytically more easily solvable.

This note originates from the observation that a *phasor* is what in the model reduction theory is known as the *moment* of a system. The model reduction problem consists in finding a simplified description of a dynamical system maintaining at the same time specific properties. It has been extensively studied exploiting a plethora of methods, among which there are, precisely, the moment matching methods (for an extensive literature review on model reduction see [3]). In this paper it is shown that the phasors of an electric circuit are the components of the unique solution of a Sylvester equation (which are the moments of the linear system describing the circuit, see *e.g.* [4]). The Sylvester equation itself is proved to be the phasor transformed system describing the electric circuit. Exploiting this equivalence, the instantaneous power and the average power are defined utilizing the moments.

The reason that justifies the interest in this equivalence is that the description of moment has been recently generalized beyond linear systems, see *e.g.* [4], [5], [6], [7]. In particular in [8] and [9], the moments have been generalized to systems driven by discontinuous signal generators. Thus, exploiting the equivalence discussed, and exploiting the results given in [8], a new, more general, phasor transform can be given to deal with discontinuous signals (this is the subject of a forthcoming paper [8]). The new phasor transform is expected to be useful in power electronics. This is described as *a branch of electrical and electronic engineering concerned with the analysis, simulation, design, manufacture, and application of switching-mode power converters* [10], [11], [12]. Since power converters are operated with discontinuous signals, this note may have great impact on applications.

The rest of the paper is organized as follows. This section continues with a precise formulation of our problem and aim. Then the definition of moments as given in [4] is recalled. In Section II the equivalence between moments and phasors is proved and the definition of power is given in terms of moments. In Section III the use of the results are illustrated by means of an example. Section IV contains some concluding remarks and future directions of research.

Notation. We use standard notation. $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers; $\mathbb{C}_{<0}$ denotes the set of complex numbers with negative real part. The symbol I denotes the identity matrix and $\sigma(A)$ denotes the spectrum of the matrix $A \in \mathbb{R}^{n\times n}$. The symbol $\Re[z]$ indicates the real part of the complex number z, $\Im[z]$ denotes its imaginary part and j denotes the imaginary unit. The symbol ϵ_k indicates the vector with the k-th element equal to 1 and with all the other elements equal to 0. Given a function $f(t)$, $\overline{F}(\omega)$ represents its phasor at ω, whereas $<f(t)>$ indicates its time average.

A. Problem and aim

All currents and voltages in linear circuits are described by linear differential equations of the form

$$a_n \frac{d^n}{dt^n} f + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} f + \cdots + a_0 f = u, \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ represents a current or voltage, $u(t) \in \mathbb{R}$ is a current or voltage source and $a_i \in \mathbb{R}$, with $i = 0, \ldots, n$. Without loss of generality we assume that $a_n \neq 0$. In the analysis of circuits is of interest to study the steady-state response (provided this exstis) of the system, which intuitively can be described like the response of the system when the transient response has become negligible. If the input u is a sinusoidal signal of amplitude a_u, angular frequency ω and phase ϕ, then a classical tool for the steady-state analysis of (1) is the phasor of $f(t)$. This is usually introduced by means of the inverse phasor transform which is defined in its simplest form as

$$f(t) = \Re\left[\overline{F} e^{j\omega t}\right], \tag{2}$$

where[1] $\overline{F} : \mathbb{C} \to \mathbb{C}$ is called the phasor of $f(t)$.

In this note we revisit the notion of phasor. In particular we show that the phasors of an electric circuit are the moments at $j\omega$ of the linear system describing the circuit. This result

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[1] Phasors are frequency dependent. We omit the argument ω from the phasor $\overline{F}(\omega)$ for ease of notation.

has the potential to be extended beyond the use of sources described by complex exponential. In fact, exploiting the results given in [8], [9], it should be possible to extend the notion of phasor to a general class of input signals including discontinuous signals, such as square waves and triangular waves.
For the sake of clarity, we now define what we mean with phasor transform of a linear system.

Definition 1: Consider a linear, single-input, single-output, continuous-time, system described by the equations

$$\dot{x} = Ax + Bu, \qquad y = Cx, \tag{3}$$

with[2] $x(t) \in \mathbb{C}^n$, $u(t) \in \mathbb{C}$, $y(t) \in \mathbb{C}$, $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times 1}$ and $C \in \mathbb{C}^{1\times n}$. The *phasor transform of the linear system (3)* for the sourse $u(t) = a_u e^{j(\omega t+\phi)}$, with $a_u \in \mathbb{R}$, $\omega \in \mathbb{R}$ and $\phi \in \mathbb{R}$, is

$$\begin{aligned} \overline{X} j\omega e^{j\omega t} &= A\overline{X}e^{j\omega t} + Ba_u e^{j\phi} e^{j\omega t}, \\ \overline{Y} e^{j\omega t} &= H\overline{X}e^{j\omega t}. \end{aligned} \tag{4}$$

Remark 1: In the following we apply Kirchhoff's Voltage Law. We assume that the sources are voltage sources and the state variables and the output are currents. As a consequence we show that the moments are the phasors of the currents. An equivalent analysis based on the Kirchhoff's Current Law can be derived.

B. Moments of linear systems

In this section we recall the notion of moment for linear systems as presented in [4]. Consider system (3), let

$$W(s) = C(sI - A)^{-1}B$$

be the associated transfer function and assume that (3) is minimal, *i.e.* controllable and observable.

Definition 2: Let $s_i \in \mathbb{C}$, with $s_i \notin \sigma(A)$. The 0*-moment of system (3) at* s_i is the complex number $\eta_0(s_i) = C(s_iI - A)^{-1}B$. The *k-moment of system (3) at* s_i is the complex number

$$\eta_k(s_i) = \frac{(-1)^k}{k!}\left[\frac{d^k}{ds^k}(C(sI - A)^{-1}B)\right]_{s=s_i},$$

with $k \geq 1$ integer.

In [4] (see also [13] and [14]), a characterization of the moments of system (3) has been given in terms of the solution of a Sylvester equation as follows.

Lemma 1: [4] Consider system (3), $s_i \in \mathbb{C}$ and suppose $s_i \notin \sigma(A)$, for all $i = 1, \ldots, \eta$. There exists a one-to-one relation between the moments $\eta_0(s_1), \ldots, \eta_{k_1-1}(s_1), \ldots, \eta_0(s_\eta), \ldots, \eta_{k_\eta-1}(s_\eta)$ and the matrix $C\Pi$, where Π is the unique solution of the Sylvester equation

$$A\Pi + B\Gamma = \Pi\Sigma, \tag{5}$$

[2]Usually the state $x(t)$ of a dynamical system represents real quantities. However, herein we use the complex domain because the quantities involved in the phasor analysis are complex valued.

with $\Sigma \in \mathbb{C}^{\nu\times\nu}$ any non-derogatory matrix with characteristic polynomial

$$p(s) = \prod_{i=1}^{\eta}(s - s_i)^{k_i}, \tag{6}$$

where $\nu = \sum_{i=1}^{\eta} k_i$, and Γ is such that the pair (Γ, Σ) is observable.

In [4] it has also been noted that the moments of system (3) are in one-to-one relation with the well-defined steady-state response of the output of the interconnection between a signal generator with dynamic matrix Σ and output matrix Γ (with the properties described in Lemma 1) and system (3). This interpretation of the notion of moment relies upon the center manifold theory [15], it has the advantage that it can be extended to nonlinear systems and it is of particular interest for the aims of this paper.

Theorem 1: [4] Consider system (3), $s_i \in \mathbb{C}$ and suppose $s_i \notin \sigma(A)$, for all $i = 1, \ldots, \eta$, and $\sigma(A) \in \mathbb{C}_{<0}$. Let $\Sigma \in \mathbb{C}^{\nu\times\nu}$ be any non-derogatory matrix with characteristic polynomial (6). Consider the interconnection of system (3) with the system

$$\dot{\zeta} = \Sigma\zeta, \qquad u = \Gamma\zeta, \tag{7}$$

with Γ and $\zeta(0)$ such that the triple $(\Gamma, \Sigma, \zeta(0))$ is minimal. Then there exists a one-to-one relation between the moments $\eta_0(s_1), \ldots, \eta_{k_1-1}(s_1), \ldots, \eta_0(s_\eta), \ldots, \eta_{k_\eta-1}(s_\eta)$ and the steady-state response of the output y of such interconnected system.

II. Electrical equivalent of the moment theory

In this section we show that the theory of moments developed to solve the model reduction problem has an electrical equivalent. In particular we prove that the phasors of an electric circuit, as defined in (2), are the moments of the system describing the circuit when a single complex interpolation point is selected. Then, exploiting the equivalence between moments and phasors, we define the power using the moments. Revisiting these already known results is instrumental to lie the foundation for a future extension of the notion of phasor to non-conventional sources: knowing the relation between moments and power is essential to give a physical meaning to newly defined quantities. To streamline the presentation, we introduce the following definition.

Definition 3: The system (3) and the generator (7) are said to be in the *mixed convention* if the matrices A, B and C have real entries and the matrices Γ and Σ have complex entries.

Note that in the mixed convention, for all integers k with $2 \leq 2k \leq n$, the component x_{2k} of x is a current i_k, whereas the component x_{2k-1} of x is the integral $\int_{t_0}^{t} i_k(\tau)d\tau$.

A. Equivalence between moments and phasors

Herein we show that writing the phasor transform of a linear electric circuit is equivalent to writing the associated Sylvester equation. Moreover, the components of the solution of this Sylvester equation are the phasors of all the currents (and of the integrals of the currents) in the circuit.

Proposition 1: Consider the source $u(t) = a_u e^{j(\omega t+\phi)}$, with $a_u \in \mathbb{R}$, $\omega \in \mathbb{R}$ and $\phi \in \mathbb{R}$, and assume $j\omega \notin \sigma(A)$. The phasor transform of system (3) written in the mixed convention coincides with the Sylvester equation (5) with $\Sigma = j\omega$ and $\Gamma = a_u e^{j\phi}$. The components of Π, which is the unique solution of equation (5), are the phasors of the currents and of the integrals of the currents in the circuit.

Proof: We first compute the phasor transform of system (3) for the source $u = a_u e^{j(\omega t+\phi)}$, namely

$$\overline{X} j\omega e^{j\omega t} = A\overline{X} e^{j\omega t} + B a_u e^{j\phi} e^{j\omega t},$$
$$\overline{Y} e^{j\omega t} = C\overline{X} e^{j\omega t},$$

by Definition 1. Since $e^{j\omega t} \neq 0$ for all $t \in \mathbb{R}$, it can be canceled out yielding

$$\overline{X} j\omega = A\overline{X} + B a_u e^{j\phi},$$
$$\overline{Y} = C\overline{X}.$$

Thus the phasors of all the currents (and their integrals) in the circuit and the phasor of the output current are given by

$$\overline{X} = (j\omega I - A)^{-1} B a_u e^{j\phi},$$
$$\overline{Y} = C(j\omega I - A)^{-1} B a_u e^{j\phi},$$

respectively. Consider now the signal generator (7) with

$$\Sigma = j\omega, \quad \Gamma = a_u e^{j\phi}.$$

The associated Sylvester equation (5) is

$$A\Pi + B a_u e^{j\phi} = \Pi j\omega,$$

which, if $j\omega \notin \sigma(A)$, has the unique solution

$$\Pi = (j\omega I - A)^{-1} B a_u e^{j\phi},$$

which concludes the proof. ■

Corollary 1: The phasor of the output response y of system (3) is the moment of the system at $j\omega$, namely $\overline{Y} = C\Pi$. The *inverse phasor transform* of the output current y of system (3) is

$$y(t) = \Re\left[C\Pi e^{\Sigma t}\right].$$

Proof: The first claim follows noting that the phasor of the output response of system (3) is given by

$$\overline{Y} = C\Pi = C(j\omega I - A)^{-1} B a_u e^{\phi}.$$

To prove the second claim we note that

$$\Re\left[C\Pi e^{\Sigma t}\right] = \Re\left[\overline{Y} e^{j\omega t}\right],$$

which is the inverse phasor transform of the output response of system (3). ■

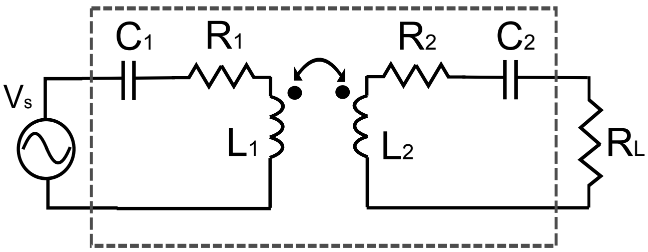

Fig. 1. Equivalent circuit of a wireless power transfer system with two coils.

Remark 2: Phasors are a very special case of moments of linear systems. In fact, they are the moments at the single interpolation point $j\omega$.

Remark 3: The higher order derivatives $\frac{d^n x}{dt^n}$ and the integral $\int_{t_0}^{t} x(\tau)d\tau$ are transformed in the phasor domain into $(j\omega)^n \overline{X} e^{j\omega}$ and $\frac{1}{j\omega}\overline{X} e^{j\omega}$, respectively. Similarly they are transformed in the "moment domain" into $\Sigma^n \Pi e^{j\omega}$ and $\Sigma^{-1}\Pi e^{j\omega}$, respectively.

B. Definition of power from the moments

The phasor analysis is useful to determine the instantaneous and average power absorbed by a load Z at steady-state. Exploiting the relation between phasors and moments we can define these two quantities with respect to the moments. The instantaneous power is defined as

$$p(t) = v(t)i(t) = \Re\left[\overline{V} e^{j\omega t}\right] \Re\left[\overline{I} e^{j\omega t}\right]. \quad (8)$$

Exploiting the properties of the real part operator we can write the instantaneous power as

$$p(t) = \frac{1}{2}\Re\left[\overline{V}^* \overline{I} e^{j\omega t} e^{-j\omega t}\right] + \frac{1}{2}\Re\left[\overline{V}\overline{I} e^{j2\omega t}\right]. \quad (9)$$

Using Euler's formula it can be proved that the average power

$$P = <p(t)> = \frac{1}{2}\Re\left[\overline{V}^* \overline{I}\right]$$

is equal to the first term of (9), and that the second term of (9) has zero average.
Since we proved that $\overline{I} = C\Pi$, the instantaneous power is described by

$$p(t) = \frac{1}{2}\Re\left[\frac{(C\Pi)^*}{Z} C\Pi\right] + \frac{1}{2}\Re\left[\frac{C\Pi}{Z} C\Pi e^{j2\omega t}\right]. \quad (10)$$

with Z the complex impedance in the relation $\overline{V} = Z\overline{I}$.

Remark 4: The equivalence between the average power P and $\frac{1}{2}\Re\left[\overline{V}^* \overline{I}\right]$ is not a definition. The relation is a consequence of the properties of the complex exponential, as highlighted in (9), and we may expect that this relation does not hold if the source is not a complex exponential.

III. An illustrative example

We present now a worked out example which shows how to apply this result from a computational point of view. Fig. 1 illustrates a wireless power transfer system [16] consisting of two coils. In this example we assume that a sinusoidal voltage source with an amplitude of V_s and an angular frequency of ω is applied to the transmitter coil on the input side. A load resistor R_L is connected to the receiving coil on the output side. By applying the Kirchhoff's Voltage Law to the two coils, we obtain the system of equations

$$\begin{aligned} R_1 i_1 + L_1 \frac{di_1}{dt} + \frac{1}{C_1}\int i_1 dt + M_{12}\frac{di_2}{dt} &= u(t), \\ M_{21}\frac{di_1}{dt} + R_{2L} i_2 + L_2\frac{di_2}{dt} + \frac{1}{C_2}\int i_2 dt &= 0, \end{aligned} \tag{11}$$

where i_1 and i_2 are the currents flowing in the coils 1 and 2, R_1 and R_2 are the resistances, $R_{2L} = R_2 + R_L$, L_1 and L_2 are the self-inductances, C_1 and C_2 are the capacitances and $M_{12} = M_{21}$ are the mutual inductances between the two coils. We are interested in determining the amplitude and phase of the steady-state current in the receiving coil, *i.e.* the phasor $\overline{I}_2$.

We start solving the problem with the phasor transform approach. Transforming the differential equations (11) we obtain the complex algebraic system

$$\begin{aligned} Z_1\overline{I}_1 + j\omega M_{12}\overline{I}_2 &= V_s, \\ j\omega M_{21}\overline{I}_1 + Z_2\overline{I}_2 &= 0, \end{aligned}$$

where $Z_1 = R_1 + j\omega L_1 - j\frac{1}{\omega C_1}$ and $Z_2 = R_{2L} + j\omega L_2 - j\frac{1}{\omega C_2}$. Solving with respect to $\overline{I}_2$ yields

$$\overline{I}_2 = \frac{-j\omega M_{21}}{Z_1 Z_2 + \omega^2 M_{21} M_{12}} V_s.$$

Now we compute the moment of system (11) at

$$\dot{\zeta} = \Sigma\zeta, \quad u = \Gamma\zeta. \tag{12}$$

Consider $\Gamma = V_s$ and $\Sigma = j\omega$ and the state

$$\begin{aligned} x_1(t) &= \int_{t_0}^{t} i_1(\tau)\, d\tau, \quad x_2(t) = i_1(t), \\ x_3(t) &= \int_{t_0}^{t} i_2(\tau)\, d\tau, \quad x_4(t) = i_2(t). \end{aligned}$$

System (11) can be represented by the first order system of differential equations (3) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{L_2}{C_1\hat{L}} & -\frac{R_1 L_2}{\hat{L}} & \frac{M_{12}}{C_2\hat{L}} & \frac{M_{12}R_{2L}}{\hat{L}} \\ 0 & 0 & 0 & 1 \\ \frac{M_{21}}{C_1\hat{L}} & \frac{R_1 M_{21}}{\hat{L}} & -\frac{L_1}{C_2\hat{L}} & -\frac{L_1 R_{2L}}{\hat{L}} \end{bmatrix}, \tag{13}$$

$$B = \begin{bmatrix} 0 & \frac{L_2}{\hat{L}} & 0 & -\frac{M_{21}}{\hat{L}} \end{bmatrix}^\top,$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\hat{L} = L_1 L_2 - M_{12}M_{21} \neq 0$. The solution of the Sylvester equation (5) is given by

$$\Pi = (j\omega - A)^{-1} B V_s,$$

and the moment of the system at Σ is given by

$$C\Pi = \epsilon_4 \Pi = \frac{j\omega^3 M_{21}}{D\hat{L}} V_s,$$

with

$$D = -\frac{\omega^2}{\hat{L}}\left(Z_1 Z_2 + \omega^2 M_{12} M_{21}\right)$$

the determinant of the matrix $(j\omega I - A)$. Hence $\overline{I}_2 = C\Pi$. In a similar way we can prove that

$$\begin{bmatrix} \Pi_1 & \Pi_2 & \Pi_3 & \Pi_4 \end{bmatrix}^\top = \begin{bmatrix} \frac{1}{j\omega}\overline{I}_1 & \overline{I}_1 & \frac{1}{j\omega}\overline{I}_2 & \overline{I}_2 \end{bmatrix}^\top.$$

IV. Conclusion and further research directions

In this short note we have proved and discussed the equivalence between the moment theory and the phasor theory. The importance of this result lies on the multiple directions in which the moment theory has been extended in recent years. The first relevant extension is the one presented in [8], [9]. Since power electronics is a field which deals with inherently discontinuous signals, it is of interest to extend the phasor theory to any periodic source which has the explicit[3] representation

$$\zeta(t) = \Lambda(t)\zeta(0), \quad u = \Gamma\zeta, \tag{14}$$

with $\Lambda(t)$ such that $\Lambda(t) = \Lambda(t-T)$ for $t \geq T$. This is a general representation of any periodic signal which is linear with respect to the initial condition. This class includes possibly discontinuous signals, such as square waves and triangular waves, which are of great interest in circuit analysis. Following the results in [8], [9] we expect that the components of the function

$$\Pi_\infty(t) = (I - e^{AT})^{-1}\left[\int_{t-T}^{t} e^{A(t-\tau)} B\Gamma\Lambda(\tau) d\tau\right]\Lambda(t)^{-1}, \tag{15}$$

are linked to the generalized phasors of the currents in system (3), and that a generalized inverse phasor transform of the output current $i(t)$ of system (3) would have the form

$$i(t) = \Re\left[\overline{I}(t)\Lambda(t)\right], \tag{16}$$

with $\overline{I}(t) = C\Pi_\infty(t)$. This topic is the subject of a forthcoming paper [19].

Moreover, the equivalence that we have established between moments and phasors lends itself to the extension of the phasor transform beyond the discontinuous linear framework. In fact, the moment theory has been extended to nonlinear systems [4], time-delay systems [7] and differential-algebraic systems [20]. New generalizations of the phasor analysis could be achieved exploiting the equivalence pointed out in this note.

[3] See [8], [9], [17], [18] for the definition of explicit and implicit forms. Briefly, a system in implicit form is described by an ordinary differential equation; a system in explicit form may not have a differential representation.

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