

Global Well-posedness and scattering for the fourth order nonlinear Schrödinger equations with small data in modulation and Sobolev spaces

Michael Ruzhansky[†], Baoxiang Wang[‡] and Hua Zhang[‡]

[†]*Department of Mathematics, Imperial College London, Queen's Gate 180, London SW72AZ, U.K.*

E-mail: m.ruzhansky@imperial.ac.uk

[‡]*LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China,*

Emails: wbx@math.pku.edu.cn, zhanghuamaths@163.com

October 26, 2014

Abstract

The local well-posedness with small data in $H^s(\mathbb{R}^n)$ ($s \geq 3 + \max(n/2, 1_+)$) for the Cauchy problem of the fourth order nonlinear Schrödinger equations with the third order derivative nonlinear terms were obtained by Huo and Jia [17]. In this paper we show its global well-posedness with small data and in the modulation spaces $M_{2,1}^{7/2}$ and in Sobolev spaces $H^{n+/2+7/2}$. For a special nonlinear term containing only one third order derivative, we can show its is global well-posedness in $M_{2,1}^{1/2}$ and $H^{(n+1_+)/2}$.

Keywords: Global well-posedness, Fourth order nonlinear Schrödinger equations, Small initial data, Modulation spaces.

MSC 2010: 35 Q 55

1 Introduction

In this paper, we consider the Cauchy problem for the fourth order nonlinear Schrödinger equations with the third order derivative nonlinearities (4NLS)

$$iu_t = \Delta^2 u - \varepsilon \Delta u + F((\partial_x^\alpha u)_{|\alpha| \leq 3}, (\partial_x^\alpha \bar{u})_{|\alpha| \leq 3}), \quad u(0, x) = u_0(x), \quad (1.1)$$

where $\varepsilon \in \{-1, 0, 1\}$, u is a complex valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^n$; $\Delta = \partial_{x_1}^2 + \dots + \partial_{x_n}^2$, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$; $F: \mathbb{C}^{\frac{1}{3}n^3 + 2n^2 + \frac{11}{3}n + 2} \rightarrow \mathbb{C}$ is an analytical function at origin

$$F(z) = P(z_1, \dots, z_{\frac{1}{3}n^3 + 2n^2 + \frac{11}{3}n + 2}) = \sum_{m+1 \leq |\beta| < \infty} c_\beta z^\beta, \quad |c_\beta| \leq C^{|\beta|}, \quad (1.2)$$

$2 \leq m < \infty$, $m \in \mathbb{N}$. The 4NLS, including its special forms, arises in deep water wave dynamics, plasma physics, optical communications; cf. [10, 12, 22, 23] and vortex filaments [32]. Taking $\varepsilon = 0$ and $F((\partial_x^\alpha u)_{|\alpha| \leq 3}(\partial_x^\alpha \bar{u})_{|\alpha| \leq 3}) = |u|^2 u$, we get

$$i\partial_t u - \Delta^2 u - |u|^2 u = 0, \quad (1.3)$$

it was introduced by Karpman [22], Karpman and Shagalov [23] which describes the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Another example of 4NLS has the following nonlinearity

$$\begin{aligned} F(u, u_x, \bar{u}_x, u_{xx}, \bar{u}_{xx}) = & -\frac{1}{2}|u|^2 u + \lambda_1 |u|^4 u \\ & + \lambda_2 (\partial_x u)^2 \bar{u} + \lambda_3 |\partial_x u|^2 u + \lambda_4 u^2 \partial_x^2 \bar{u} + \lambda_5 |u|^2 \partial_x^2 u. \end{aligned} \quad (1.4)$$

This equation is related to vortex filament [32].

The 4NLS with different nonlinearities was studied by several authors, one can see [1, 6, 15, 16, 17, 22, 23, 33, 34, 30] and references therein. In [33] and [16], by using the method of Fourier restriction norm, Segata, Huo and Jia obtained that (1.4) is local well-posed in $H^s(\mathbf{R})$ ($s \geq 1/2$). In [15], the authors obtained that (1.1) is local well-posed when the nonlinear term contains the second order derivative nonlinearities. Recently, Huo and Jia [17] established the local well posedness of (1.1) with small initial data in $H^{3+\max(n/2, 1_+)}$ by using the dyadic $X^{s,b}$ type spaces developed by Tataru [36] and Ionescu and Kenig [18].

Roughly speaking, every nonlinear dispersive equation has a critical Sobolev space H^{s_c} so that it has ill-posed phenomena in H^s if $s < s_c$. One can further ask whether there exist a class of initial data out of H^{s_c} for which it is still well-posed. To answer this question, modulation spaces seem very useful tools. Recalling the sharp embedding (see [35, 38, 40]) $M_{2,1}^{s_c} \subset H^{s_c}$, $M_{2,1}^s \not\subset H^{s_c}$ if $s < s_c$, it will be interesting if we can establish the global well-posedness for the initial data in $M_{2,1}^s$ with $s < s_c$. In fact, there are some recent works which have been devoted to the study of the well-posedness for a class of nonlinear evolution equation in modulation spaces; cf. [2, 3, 5, 7, 8, 9, 20, 24, 25, 26, 31, 39, 40, 41, 42]. Our main goal of this paper is to study the global well-posedness of 4NLS in modulation spaces $M_{2,1}^{3+1/2}$. By establishing frequency-locally and time-globally smooth effects for the solutions of (1.1) in anisotropic Lebesgue spaces, together with the maximal functions estimates and Strichartz estimates, we obtain that (1.1) is global well-posed in modulation space $M_{2,1}^{3+1/2}$ (so in H^s , $s > 3 + n/2 + 1/2$) with small data. Moreover, we consider a special nonlinearity like $\partial_{x_i}^3(|u|^m u)$ and show (1.1) is global well-posed in modulation space $M_{2,1}^{1/m}$ (so in H^s , $s > n/2 + 1/m$) with small data.

1.1 Main results

We denote $(x_j)_{j \neq i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ and by $L_{x_i; (x_j)_{j \neq i}, t}^{p_1, p_2} := L_{x_i; (x_j)_{j \neq i}, t}^{p_1, p_2}(\mathbb{R}^{1+n})$ the anisotropic Lebesgue space for which the norm is defined as

$$\|f\|_{L_{x_i; (x_j)_{j \neq i}, t}^{p_1, p_2}} = \left\| \|f\|_{L_{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n, t}^{p_2}(\mathbb{R}^n)} \right\|_{L_{x_i}^{p_1}(\mathbb{R})}. \quad (1.5)$$

Let $Q_k = \{\xi : -1/2 \leq \xi_i - k_i < 1/2, i = 1, \dots, n\}$. We roughly write $\square_k = \mathcal{F}^{-1} \chi_{Q_k} \mathcal{F}$, where \mathcal{F} (\mathcal{F}^{-1}) denotes the (inverse) Fourier transform on \mathbb{R}^n , χ_E denotes the characteristic function on E . The exact definition of \square_k will be given in Section 1.2. Modulation spaces $M_{p,q}^s$ were introduced by Feichtinger [11] and one can refer to [13] for their basic properties. The modulation space $M_{2,1}^s$ can be equivalently defined in the following way (cf. [40, 41, 42]):

$$\|f\|_{M_{2,1}^s} = \sum_{k \in \mathbb{Z}^n} \langle k \rangle^s \|\mathcal{F}f\|_{L^2(Q_k)}, \quad (1.6)$$

where $\langle k \rangle = 1 + |k|$. For any Banach function spaces X defined in $\mathbb{R} \times \mathbb{R}^n$, we will use the function spaces $\ell_{\square}^{1,s}(X)$, $\ell_{\square^i}^{1,s}(X)$ which contains all of the functions $f(t, x)$ so that the following (semi-)norm is finite, respectively:

$$\|f\|_{\ell_{\square}^{1,s}(X)} := \sum_{k \in \mathbb{Z}^n} \langle k \rangle^s \|\square_k f\|_X. \quad (1.7)$$

$$\|f\|_{\ell_{\square^i}^{1,s}(X)} := \sum_{k \in \mathbb{Z}_i^n} \langle k_i \rangle^s \|\square_k f\|_X, \quad \mathbb{Z}_i^n = \{k \in \mathbb{Z}^n : |k_i| = \max_{1 \leq j \leq n} |k_j| > 100\}. \quad (1.8)$$

For simplicity, we denote $\ell_{\square}^1(X) = \ell_{\square}^{1,0}(X)$.

Theorem 1.1 *Let $n \geq 1$, $\varepsilon = 1$, $2 \leq m < \infty$, $m > 4/n$. Assume that $u_0 \in M_{2,1}^{3+1/m}$ and $\|u_0\|_{M_{2,1}^{3+1/m}} \leq \delta$ for some small $\delta > 0$. Then (1.1) has a unique global solution $u \in C(\mathbb{R}, M_{2,1}^{3+1/m}) \cap D$, where*

$$\|u\|_D = \sum_{\alpha=0,3} \sum_{i,l=1}^n \|\partial_{x_l}^{\alpha} u\|_{\ell_{\square^i}^{1,3/2+1/m}(L_{x_i; (x_j)_{j \neq i}, t}^{\infty, 2}) \cap \ell_{\square^i}^1(L_{x_i; (x_j)_{j \neq i}, t}^{m, \infty}) \cap \ell_{\square}^{1,1/m}(L_t^{\infty} L_x^2 \cap L_{t,x}^{2+m})}. \quad (1.9)$$

Moreover, the scattering operator of (1.1) carries a zero neighborhood in $C(\mathbb{R}, M_{2,1}^{3+1/m})$ into $C(\mathbb{R}, M_{2,1}^{3+1/m})$.

In Theorem 1.1, if $u_0 \in M_{2,1}^s$ with $s > 3 + 1/m$, then we have $u \in C(\mathbb{R}, M_{2,1}^s)$. If nonlinearity F takes the following form:

$$iu_t = \Delta^2 u - \varepsilon \Delta u + \sum_{i=1}^n \sum_{|\alpha| \leq 3} \lambda_{i,\alpha} \partial_{x_i}^{\alpha} (|u|^{\kappa_i} u), \quad u(0, x) = u_0(x), \quad (1.10)$$

Theorem 1.2 *Let $n \geq 1$, $\varepsilon = 1$, $\kappa_i \geq 2$, $\kappa_i > 4/n$, $\kappa_i \in 2\mathbb{N}$, $\lambda_i \in \mathbb{C}$, $\kappa = \min_{1 \leq i \leq n} \kappa_i$. Assume that $u_0 \in M_{2,1}^{1/\kappa}$ and $\|u_0\|_{M_{2,1}^{1/\kappa}} \leq \delta$ for some small $\delta > 0$. Then (1.10) has a unique global solution $u \in C(\mathbb{R}, M_{2,1}^{1/\kappa}) \cap D_2$, where*

$$\|u\|_{D_2} = \sum_{i=1}^n \|u\|_{\ell_{\square^i}^{1,3/2+1/\kappa}(L_{x_i;(x_j)_{j \neq i,t}}^{\infty,2}) \cap \ell_{\square^i}^1(L_{x_i;(x_j)_{j \neq i,t}}^{\kappa,\infty}) \cap \ell_{\square^i}^{1,1/\kappa}(L_t^\infty L_x^2 \cap L_{t,x}^{2+\kappa})}, \quad (1.11)$$

Moreover, the scattering operator of (1.10) carries a zero neighborhood in $C(\mathbb{R}, M_{2,1}^{1/\kappa})$ into $C(\mathbb{R}, M_{2,1}^{1/\kappa})$.

Remark 1.3 For the case $\varepsilon = 0$, Theorems 1.1 and 1.2 also hold if we add conditions $m > 8/n$ and $\kappa > 8/n$, respectively. Theorem 1.1 covers the initial data $u_0 \in H^{n+/2+3+1/m}$ and Theorem 1.2 contains the initial data $u_0 \in H^{n+/2+1/\kappa}$ as special cases.

1.2 Notations

Let $c \leq 1$, $C > 1$ denote positive universal constants, which can be different at different places, $a \lesssim b$ stands for $a \leq Cb$, $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$. We write $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$. For $k = (k_1, \dots, k_n)$, we write $k_{\max} = \max_{1 \leq l \leq n} |k_l|$. We write p' as the dual number of $p \in [1, \infty]$, i.e., $1/p + 1/p' = 1$. Let $\mathbb{C}, \mathbb{R}, \mathbb{N}$ and \mathbb{Z} stand for the sets of complex number, reals, positive integers and integers, respectively. We will use Lebesgue spaces $L^p := L^p(\mathbb{R}^n)$, $\|\cdot\|_p := \|\cdot\|_{L^p}$, Sobolev spaces $H^s = (I - \Delta)^{-s/2} L^2$. Some properties of these function spaces can be found in [4, 37]. For any $1 \leq k < n$, we denote by $\mathcal{F}_{x_1, \dots, x_k}$ the partial Fourier transform:

$$(\mathcal{F}_{x_1, \dots, x_k} f)(\xi_1, \dots, \xi_k, x_{k+1}, \dots, x_n) = \int_{\mathbb{R}^k} e^{-i(x_1 \xi_1 + \dots + x_k \xi_k)} f(x) dx_1 \dots dx_k \quad (1.12)$$

and by $\mathcal{F}_{\xi_1, \dots, \xi_k}^{-1}$ the partial inverse Fourier transform. Since we need to treat the functions $f(t, x)$ defined in $\mathbb{R} \times \mathbb{R}^n$, $\mathcal{F}_{t,x}$ and $\mathcal{F}_{\tau, \xi}^{-1}$ can be defined in the same way as above. We always write $\mathcal{F} := \mathcal{F}_{x_1, \dots, x_n}$, $\mathcal{F}^{-1} := \mathcal{F}_{\xi_1, \dots, \xi_n}^{-1}$. $D_{x_i}^s = (-\partial_{x_i}^2)^{s/2} = \mathcal{F}_{\xi_i}^{-1} |\xi_i|^s \mathcal{F}_{x_i}$ expresses the partial Riesz potential in the x_i direction; $\partial_{x_i}^{-1} = \mathcal{F}_{\xi_i}^{-1} (i\xi_i)^{-1} \mathcal{F}_{x_i}$. We will use Nikol'skij's multiplier estimate; cf. [4, 37]. For any $r \in [1, \infty]$,

$$\|\mathcal{F}^{-1} \varphi \mathcal{F} f\|_r \leq C \|\varphi\|_{H^s} \|f\|_r, \quad s > n/2. \quad (1.13)$$

We will use the frequency-uniform decomposition operators (cf. [40, 41, 42]). Let $\{\sigma_k\}_{k \in \mathbb{Z}^n}$ be a smoothing function sequence satisfying

$$\text{supp } \sigma \subset [-3/4, 3/4]^n, \quad \sigma_k(\cdot) = \sigma(\cdot - k), \quad \sum_{k \in \mathbb{Z}^n} \sigma_k(\xi) \equiv 1, \quad \forall \xi \in \mathbb{R}^n. \quad (1.14)$$

Denote

$$\Upsilon_n = \{ \{ \sigma_k \}_{k \in \mathbb{Z}^n} : \{ \sigma_k \}_{k \in \mathbb{Z}^n} \text{ satisfies (1.14)} \}. \quad (1.15)$$

We denote Υ_n by Υ if there is no confusion. Let $\{\sigma_k\}_{k \in \mathbb{Z}^n} \in \Upsilon_n$ be a function sequence and

$$\square_k := \mathcal{F}^{-1} \sigma_k \mathcal{F}, \quad (1.16)$$

\square_k ($k \in \mathbb{Z}^n$) are said to be the frequency-uniform decomposition operators. For convenience, we will always use the following function sequence $\{\sigma_k\}_{k \in \mathbb{Z}^n}$:

Lemma 1.4 *Let $\{\eta_k\}_{k \in \mathbb{Z}} \in \Upsilon_1$. Assume that $\text{supp} \eta_k \subset [k - 2/3, k + 2/3]$. Denote*

$$\sigma_k(\xi) := \eta_{k_1}(\xi_1) \dots \eta_{k_n}(\xi_n), \quad k = (k_1, \dots, k_n) \in \mathbb{Z}^n, \quad (1.17)$$

then we have $\{\sigma_k\}_{k \in \mathbb{Z}^n} \in \Upsilon_n$.

For convenience, we also use the following notations

$$\tilde{\sigma}_k = \sum_{|\ell|_\infty \leq 1} \sigma_{k+\ell}, \quad |\ell|_\infty = \max_{1 \leq i \leq n} |\ell_i|, \quad k \in \mathbb{Z}^n. \quad (1.18)$$

It is easy to see that

$$\tilde{\sigma}_k \sigma_k = \sigma_k, \quad k \in \mathbb{Z}^n. \quad (1.19)$$

The rest of this paper is organized as follows. In Section 2 we consider some necessary estimates for the solutions of the linear fourth-order Schrödinger equations. More precisely, in Section 2.1 we show the smooth effect estimates of the solutions of the fourth order linear Schrödinger equation in anisotropic Lebesgue spaces with \square_k -decomposition. In Sections 2.2 we state the standard Strichartz estimate with \square_k -decomposition. In Section 2.3, the time-global maximal function estimates in the frame of frequency-uniform localization are obtained. Then we consider the interactions between the smooth effect, Strichartz and maximal function estimates in Section 2.4. In Sections 3 and 4, we prove our Theorems 1.2 and 1.1, respectively. In Sections 5, we consider the cases $\varepsilon = 0$ and sketch the proof of Remark 1.3. In Section 6 we consider the decay in the case $\varepsilon = -1$ and an unsolved question is proposed.

2 Linear Estimates via \square_k -decomposition

2.1 Smooth effects with \square_k -decomposition

The smooth effect estimates for a class of dispersive equations go back to the works of Kato, Constantin-Saut [6], Kenig-Ponce-Vega [28], Linares-Ponce [29]. For the fourth order Schrödinger equation, Hao-Hsiao-Wang [15] can deal with the second order derivative. In this paper, we will handle the third order derivatives. The use of the anisotropic Lebesgue

spaces follows Linares-Ponce [29], where they first applied the $L_{x_1}^\infty L_{x_2,t}^2$ to handle the derivative term for the Davey-Stewartson equation in 2D. We will always denote

$$S(t) = e^{it(\Delta - \Delta^2)} = \mathcal{F}^{-1} e^{-it(|\xi|^4 + |\xi|^2)} \mathcal{F}, \quad \mathcal{A}f(t, x) = \int_0^t S(t - \tau) f(\tau, x) d\tau.$$

If there is no explanation, we will assume that $\{\sigma_k\}$ is as in Lemma 1.4.

Proposition 2.1 *For any $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, $|k_i| = k_{\max}$, $i = 1, \dots, n$. We have*

$$\left\| \square_k D_{x_i}^{3/2} S(t) u_0 \right\|_{L_{x_i; (x_j)_{j \neq i}, t}^{\infty, 2}} \lesssim \left\| \square_k u_0 \right\|_{L^2}. \quad (2.1)$$

Proof. It suffices to prove the case $i = 1$. For convenience, we write $\bar{z} = (z_2, \dots, z_n)$. By Plancherel's identity, we have

$$\begin{aligned} & \left\| \square_k D_{x_1}^{3/2} S(t) u_0 \right\|_{L_{x_1; \bar{x}, t}^{\infty, 2}} \\ &= \left\| \int \sigma_k(\xi) |\xi_1|^{\frac{3}{2}} e^{-it(|\xi|^4 + |\xi|^2)} \widehat{u_0}(\xi) e^{ix_1 \xi_1} d\xi_1 \right\|_{L_{x_1}^\infty L_{\bar{\xi}, t}^2} \\ &\lesssim \left\| \int \eta_{k_1}(\xi_1) |\xi_1|^{\frac{3}{2}} e^{-it(|\xi|^4 + |\xi|^2)} \widehat{u_0}(\xi) e^{ix_1 \xi_1} d\xi_1 \right\|_{L_{x_1}^\infty L_{\bar{\xi}, t}^2} := L. \end{aligned}$$

We estimate L according to the size of k_1 . In the case $k_1 \geq 1$, we have $\xi_1 > 0$ for $\xi_1 \in \text{supp} \eta_{k_1}(\cdot)$. Changing variable $\theta = |\xi|^4 + |\xi|^2$, $d\xi_1 = \frac{1}{2}(2|\xi|^2 + 1)^{-1} \xi_1^{-1} d\theta$ for $\xi_1 > 0$. Using Plancherel's identity, we have

$$\begin{aligned} L &= \left\| \int \eta_{k_1}(\xi_1) \xi_1^{\frac{3}{2}} e^{-it\theta} \widehat{u_0}(\xi) e^{ix_1 \xi_1} d\xi_1 \right\|_{L_{x_1}^\infty L_{\bar{\xi}, t}^2} \\ &\lesssim \left\| \int \eta_{k_1}(\xi_1(\theta)) \xi_1^{\frac{3}{2}}(\theta) e^{-it\theta} \widehat{u_0}(\xi(\theta)) e^{ix_1 \xi_1(\theta)} (2|\xi|^2 + 1)^{-1} \xi_1^{-1}(\theta) d\theta \right\|_{L_{x_1}^\infty L_{\bar{\xi}, t}^2} \\ &\lesssim \left\| \eta_{k_1}(\xi_1(\theta)) \xi_1^{\frac{1}{2}}(\theta) \widehat{u_0}(\xi(\theta)) (2|\xi|^2 + 1)^{-1} \right\|_{L_\theta^2 L_{\bar{\xi}}^2} \\ &\lesssim \left\| \eta_{k_1}(\xi_1(\theta)) \xi_1^{\frac{1}{2}} \widehat{u_0}(\xi) (2|\xi|^2 + 1)^{-1} (2|\xi|^2 + 1)^{1/2} \xi_1^{\frac{1}{2}} \right\|_{L_\xi^2} \\ &= \left\| \eta_{k_1}(\xi_1(\theta)) \xi_1 (2|\xi|^2 + 1)^{-1/2} \widehat{u_0}(\xi) \right\|_{L_\xi^2} \\ &\lesssim \|u_0\|_{L^2}. \end{aligned}$$

In the case $k_1 \leq -1$, the situation is identical to the case $k_1 \geq 1$. If $k_1 = 0$, we have $\xi_1 \in [-2/3, 2/3]$ if $\xi_1 \in \text{supp} \eta_0(\cdot)$.

$$L \leq \left\| \int_0^{2/3} \eta_0(\xi_1) |\xi_1|^{\frac{3}{2}} e^{-it(|\xi|^4 + |\xi|^2)} \widehat{u_0}(\xi) e^{ix_1 \xi_1} d\xi_1 \right\|_{L_{x_1}^\infty L_{\bar{\xi}, t}^2}$$

$$+ \left\| \int_{-2/3}^0 \eta_0(\xi_1) |\xi_1|^{\frac{3}{2}} e^{-it(|\xi|^4 + |\xi|^2)} \widehat{u}_0(\xi) e^{ix_1 \xi_1} d\xi_1 \right\|_{L_{x_1}^\infty L_{\xi, t}^2} := L_1 + L_2.$$

Similar to the case $k_1 \geq 1$, we can get the estimates of L_1 and L_2 . Collecting those estimates as in the above, we get the result, as desired. \square

The dual version of (2.1) is the following

Proposition 2.2 *For any $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, $|k_i| = k_{\max}$, $i = 1, \dots, n$, we have*

$$\left\| \square_k \partial_{x_i}^3 \mathcal{A} f \right\|_{L_t^\infty L_x^2} \lesssim \left\| \square_k D_{x_i}^{3/2} f \right\|_{L_{x_i; (x_j)_{j \neq i}, t}^{1,2}}. \quad (2.2)$$

Now we consider the inhomogeneous Cauchy problem

$$iu_t = \Delta^2 u - \Delta u + f(t, x), \quad u(0, x) = 0. \quad (2.3)$$

Proposition 2.3 *For any $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, $|k_i| = k_{\max}$, $i = 1, \dots, n$, the solution $u(t, x)$ of (2.3) satisfies*

$$\left\| \square_k \partial_{x_i}^3 u \right\|_{L_{x_i; (x_j)_{j \neq i}, t}^{\infty, 2}} \lesssim \left\| \square_k f \right\|_{L_{x_i; (x_j)_{j \neq i}, t}^{1,2}} \quad (2.4)$$

Proof. It suffices to consider the case $i = 1$. We write

$$u = \mathcal{F}_{\tau, \xi}^{-1} \frac{1}{\tau - |\xi|^4 - |\xi|^2} (\mathcal{F}_{t, x} f)(\tau, \xi).$$

Therefore, we have

$$\partial_{x_1}^3 u = \mathcal{F}_{\tau, \xi}^{-1} \frac{\xi_1^3}{|\xi|^4 + |\xi|^2 - \tau} \mathcal{F}_{t, x} f. \quad (2.5)$$

We show that

$$\left\| \mathcal{F}_{\xi_1}^{-1} \frac{\eta_{k_1}(\xi_1) \xi_1^3}{|\xi|^4 + |\xi|^2 - \tau} \mathcal{F}_{x_1} f \right\|_{L_{x_1}^\infty L_{\xi, \tau}^2} \lesssim \left\| \mathcal{F}_{\xi_1}^{-1} \eta_{k_1}(\xi_1) \mathcal{F}_{x_1} f \right\|_{L_{x_1}^1 L_{\xi, \tau}^2} \quad (2.6)$$

Using Young's inequality, it suffices to prove that

$$\sup_{x_1, \tau, \xi_j (j \neq 1)} \left| \mathcal{F}_{\xi_1}^{-1} \frac{\sigma_k(\xi) \xi_1^3}{|\xi|^4 + |\xi|^2 - \tau} \right| \lesssim 1. \quad (2.7)$$

We give the proof of (2.7) according to $\tau > 0$ or $\tau \leq 0$. When $|k_1| = k_{\max}$, we have $|\xi_1| \sim \max_{j=1, \dots, n} |\xi_j|$ for $\xi \in \text{supp } \sigma_k$. Now we consider the case $\tau \leq 0$. If $k_1 \neq 0$, we have

$$\sup_{x_1, \tau, \xi_j (j \neq 1)} \left| \mathcal{F}_{\xi_1}^{-1} \frac{\sigma_k(\xi) \xi_1^3}{|\xi|^4 + |\xi|^2 - \tau} \right| \lesssim \left| \int_{k_1 - \frac{3}{4}}^{k_1 + \frac{3}{4}} \frac{1}{\xi_1} d\xi_1 \right| \lesssim 1$$

If $k_1 = 0$, noticing that $\tau < 0$, we have from Young's and Nikol'skij's multiplier inequalities,

$$\begin{aligned}
& \sup_{x_1, \tau, \xi_j (j \neq 1)} \left| \mathcal{F}_{\xi_1}^{-1} \frac{\sigma_k(\xi) \xi_1^3}{|\xi|^4 + |\xi|^2 - \tau} \right| \\
& \lesssim \left| \int e^{ix_1 \xi_1} \eta_{k_1}(\xi_1) \frac{\xi_1^3}{|\xi|^4 + |\xi|^2 - \tau} d\xi_1 \right| \\
& \lesssim \left\| \mathcal{F}_{\xi_1}^{-1} \frac{1}{\xi_1} \right\|_{L_{x_1}^\infty} \left\| \mathcal{F}_{\xi_1}^{-1} \frac{\xi_1^4 \eta_{k_1}(\xi_1)}{|\xi|^4 + |\xi|^2 - \tau} \right\|_{L_{x_1}^1} \\
& \lesssim \left\| \frac{\xi_1^4 \eta_{k_1}(\xi_1)}{|\xi|^4 + |\xi|^2 - \tau} \right\|_{L_{\xi_1}^2}^{1/2} \left\| \partial_{\xi_1} \frac{\xi_1^4 \eta_{k_1}(\xi_1)}{|\xi|^4 + |\xi|^2 - \tau} \right\|_{L_{\xi_1}^2}^{1/2} \\
& \lesssim 1.
\end{aligned}$$

When $\tau > 0$ we can choose $\tau_2 = -\frac{1}{2} + \sqrt{\frac{1}{4} + \tau} > 0$ such that

$$|\xi|^4 + |\xi|^2 - \tau = (|\xi|^2 - \tau_2)(|\xi|^2 + \tau_2 + 1) \quad (2.8)$$

We have,

$$\begin{aligned}
\mathcal{F}_{\xi_1}^{-1} \frac{\sigma_k(\xi) \xi_1^3}{|\xi|^4 + |\xi|^2 - \tau} &= \mathcal{F}_{\xi_1}^{-1} \frac{\sigma_k(\xi) \xi_1^3}{(|\xi|^2 - \tau_2)(|\xi|^2 + \tau_2 + 1)} \\
&= \mathcal{F}_{\xi_1}^{-1} \frac{\sigma_k(\xi) \xi_1^3}{(\xi_1^2 + |\bar{\xi}|^2 - \tau_2)(\xi_1^2 + |\bar{\xi}|^2 + \tau_2 + 1)}. \quad (2.9)
\end{aligned}$$

When $|\bar{\xi}|^2 - \tau_2 \geq 0$, we can treat the case $k_1 \neq 0$ like above. If $k_1 = 0$, it is easy to see that the RHS of (2.9) can be controlled by

$$\int |\xi_1| \eta_0(\xi_1) d\xi_1 \leq 2.$$

Next, we consider the case $|\bar{\xi}|^2 - \tau_2 < 0$. Let $A^2 := A(\bar{\xi}, \tau)^2 = -(|\bar{\xi}|^2 - \tau_2)$ and $B^2 := B(\bar{\xi}, \tau)^2 = |\bar{\xi}|^2 + \tau_2 + 1$. We get

$$\begin{aligned}
& \mathcal{F}_{\xi_1}^{-1} \frac{\xi_1^3 \eta_{k_1}(\xi_1)}{(\xi_1^2 + |\bar{\xi}|^2 - \tau_2)(\xi_1^2 + |\bar{\xi}|^2 + \tau_2 + 1)} \quad (2.10) \\
&= \mathcal{F}_{\xi_1}^{-1} \frac{\xi_1}{\xi_1^2 - A^2} \frac{\xi_1^2}{\xi_1^2 + B^2} \eta_{k_1}(\xi_1) \\
&= \frac{1}{2} \mathcal{F}_{\xi_1}^{-1} \left(\frac{1}{\xi_1 + A} + \frac{1}{\xi_1 - A} \right) \frac{\xi_1^2}{\xi_1^2 + B^2} \eta_{k_1}(\xi_1) \\
&:= I + II
\end{aligned}$$

For the estimate of I , we have

$$I = \frac{1}{2} \mathcal{F}_{\xi_1}^{-1} \frac{\eta_{k_1}(\xi_1)}{\xi_1 + A} - \frac{1}{2} \mathcal{F}_{\xi_1}^{-1} \frac{B^2 \eta_{k_1}(\xi_1)}{(\xi_1 + A)(\xi_1^2 + B^2)} = I_1 + I_2.$$

Due to $\mathcal{F}^{-1}(1/\xi)$ is the sign function, we see that I_1 is bounded. For I_2 , by changing variables, it suffices to show

$$\sup_{x_1} \left| \mathcal{F}_{\xi_1}^{-1} \frac{1}{1+\xi_1} \frac{1}{1+D^2\xi_1^2} \right| \lesssim 1, \quad (2.11)$$

where $D = A/B$. Using the fact that $\mathcal{F}(e^{-|x|}) = C/(1+|\xi|^2)$, we have

$$\begin{aligned} \left\| \mathcal{F}_{\xi_1}^{-1} \frac{1}{1+\xi_1} \frac{1}{1+D^2\xi_1^2} \right\|_{L_{x_1}^\infty} &= \left\| \mathcal{F}_{\xi_1}^{-1} \left(\frac{1}{1+\xi_1} \right) * \mathcal{F}_{\xi_1}^{-1} \left(\frac{1}{1+D^2\xi_1^2} \right) \right\|_{L_{x_1}^\infty} \\ &\lesssim \left\| \mathcal{F}_{\xi_1}^{-1} \left(\frac{1}{1+\xi_1} \right) \right\|_{L_{x_1}^\infty} \left\| \mathcal{F}_{\xi_1}^{-1} \left(\frac{1}{1+D^2\xi_1^2} \right) \right\|_{L_{x_1}^1} \\ &\lesssim \left\| \mathcal{F}_{\xi_1}^{-1} \left(\frac{1}{1+\xi_1} \right) \right\|_{L_{x_1}^\infty} \left\| \frac{1}{D} e^{-\frac{x_1}{D}} \right\|_{L_{x_1}^1} \lesssim 1. \end{aligned}$$

The part II is similar to I , so we get the result.

In general, the solution $u(t, x)$ as above may not vanish at $t = 0$. However, using the Parseval identity, we can show that

$$u(0, x) = u(t, x)|_{t=0} = C \int_{-\infty}^{\infty} (S(s)\text{sgn}(s)\mathcal{F}f)(s, x) ds.$$

Combining it with (2.2), one has that $\square_k S(t)\partial_{x_1}^3 u(0, x) \in L^2$. Thus, by (2.1), the function $v(t)$,

$$v(t) := u(t) - S(t)u(0, \cdot) = -i \int_0^t S(t-\tau)f(\tau) d\tau$$

is the solution of (2.3) and satisfies the estimate (2.4). Therefore, we obtain the result, as desired. \square

Lemma 2.4 ([39]) *For any $\sigma \in \mathbb{R}$ and $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ with $|k_i| \geq 4$, we have*

$$\|\square_k D_{x_i}^\sigma u\|_{L_{x_1, (x_j)_{j \neq 1, t}}^{p_1, p_2}} \lesssim \langle k_i \rangle^\sigma \|\square_k u\|_{L_{x_1, (x_j)_{j \neq 1, t}}^{p_1, p_2}}. \quad (2.12)$$

Replacing $D_{x_i}^\sigma$ by $\partial_{x_i}^\sigma$ ($\sigma \in \mathbb{N}$), the above inequality holds for all $k \in \mathbb{Z}^n$.

Both sides of (2.12) are equivalent in the case $|k_i| \geq 4$. In view of Propositions 2.3, 2.2 and Lemma 2.4, we have

Proposition 2.5 *For any $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ and $|k_i| = k_{\max}, i = 1, \dots, n$, we have*

$$\|\square_k \partial_{x_i}^3 \mathcal{A}f\|_{L_{x_i, (x_j)_{j \neq i, t}}^{\infty, 2}} \lesssim \|\square_k f\|_{L_{x_i, (x_j)_{j \neq i, t}}^{1, 2}}, \quad (2.13)$$

$$\|\square_k \partial_{x_i}^3 \mathcal{A}f\|_{L_t^\infty L_x^2} \lesssim \langle k_{\max} \rangle^{3/2} \|\square_k f\|_{L_{x_i, (x_j)_{j \neq i, t}}^{1, 2}}. \quad (2.14)$$

2.2 Strichartz estimates with \square_k -decomposition

In this subsection we consider the Strichartz estimates for the solutions of the fourth order Schrödinger equations by using a standard method combined with the \square_k -decomposition operators. We need the following lemma (see [37, 42]).

Lemma 2.6 *Let $\Omega \subset \mathbb{R}^n$ be a compact set with $\text{diam } \Omega < 2R$, $0 < p \leq q \leq \infty$. Then there exists a constant $C > 0$, which depends only on p, q such that*

$$\|f\|_q \leq CR^{n(1/p-1/q)}\|f\|_p, \quad \forall f \in L_\Omega^p,$$

where $L_\Omega^p = \{f \in \mathcal{S}'(\mathbb{R}^n) : \text{supp } \hat{f} \subset \Omega, \|f\|_p < \infty\}$.

We also need the following Lemma.

Lemma 2.7 ([27]) *Let P be a real polynomial in \mathbb{R}^n , $HP(\xi) = \det(\frac{\partial^2 P}{\partial \xi_i \partial \xi_j})$, $\gamma > 0$. We define for $(x, t) \in \mathbb{R}^n \times \mathbb{R}$*

$$W_\gamma(t)u_0(x) = \int_{\mathbb{R}^n} e^{i(tP(\xi)+x\xi)} |HP(\xi)|^{\gamma/2} \hat{u}_0(\xi) d\xi,$$

we have

$$\|W_\gamma(t)u_0\|_{2/(1-\gamma)} \leq c|t|^{-\gamma n/2} \|u_0\|_{2/(1+\gamma)}.$$

Let $P(\xi) = -(|\xi|^4 + |\xi|^2)$, $HP(\xi) = \det(\frac{\partial^2 P}{\partial \xi_i \partial \xi_j})$. It is easy to see that

$$|HP(\xi)| \sim (1 + |\xi|^{2n}).$$

If we take $\gamma = 1 - 2/p$, $p \geq 2$ in Lemma 2.7, then we obtain

$$\|S(t)u_0\|_p \lesssim |t|^{n(1/p-1/2)} \|(I - \Delta)^{n(1/p-1/2)} u_0\|_{p'}. \quad (2.15)$$

Replacing u_0 by $\square_k u_0$ in (2.15) and using Nikol'skij's multiplier estimate, we have

$$\|\square_k S(t)u_0\|_p \lesssim (1 + |t|)^{n(1/p-1/2)} \|\square_k u_0\|_{p'}. \quad (2.16)$$

Following Section 5 in [39], we have the following

Lemma 2.8 *Let $2 \leq p < \infty$, $\gamma \geq 2 \vee \gamma(p)$,*

$$\frac{2}{\gamma(p)} = n \left(\frac{1}{2} - \frac{1}{p} \right).$$

Then we have

$$\begin{aligned} \|S(t)\varphi\|_{\ell_\square^1(L_t^\gamma L_x^p)} &\lesssim \|\varphi\|_{M_{2,1}}, \\ \|\mathcal{A}f\|_{\ell_\square^1(L_t^\gamma L_x^p) \cap \ell_\square^1(L_t^\infty L_x^2)} &\lesssim \|f\|_{\ell_\square^1(L_t^{\gamma'} L_x^{p'})}. \end{aligned}$$

In particular, if $2 + 4/n \leq p < \infty$, then we have

$$\begin{aligned} \|S(t)\varphi\|_{\ell_{\square}^1(L_{t,x}^p)} &\lesssim \|\varphi\|_{M_{2,1}(\mathbb{R}^n)}, \\ \|\mathcal{A}f\|_{\ell_{\square}^1(L_{t,x}^p) \cap \ell_{\square}^1(L_t^\infty L_x^2)} &\lesssim \|f\|_{\ell_{\square}^1(L_{t,x}^{p'})}. \end{aligned}$$

If the initial data and the function f are localized in one frequency, we immediately have

Proposition 2.9 *Let $2 \leq r < \infty$, $2/\gamma(r) = n(1/2 - 1/r)$ and $\gamma > \gamma(r) \vee 2$. We have*

$$\|\square_k S(t)u_0\|_{L_t^\gamma L_x^r} \lesssim \|\square_k u_0\|_{L^2(\mathbb{R}^n)}, \quad (2.17)$$

$$\|\square_k \mathcal{A}f\|_{L_t^\infty L_x^2 \cap L_t^\gamma L_x^r} \lesssim \|\square_k f\|_{L_t^{\gamma'} L_x^{r'}}. \quad (2.18)$$

In particular, if $2 + 4/n \leq p < \infty$, then we have

$$\|\square_k S(t)\varphi\|_{L_{t,x}^p} \lesssim \|\square_k \varphi\|_2, \quad (2.19)$$

$$\|\square_k \mathcal{A}f\|_{L_{t,x}^p \cap L_t^\infty L_x^2} \lesssim \|\square_k f\|_{L_{t,x}^{p'}}. \quad (2.20)$$

2.3 Maximal function estimates

In [18], Ionescu and Kenig considered the maximal function estimates for the Schrödinger semi-group $V(t) = e^{it\Delta}$ and the integral operator $\int_0^t V(t-\tau) \cdot d\tau$ in the space $L_{x_i; (x_j)_{j \neq i}, t}^{2, \infty}$. Combining their idea and the \square_k -decomposition operators, we obtain that

Proposition 2.10 *Let $4/n < q \leq \infty$, $q \geq 2$ and $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, we have*

$$\|\square_k S(t)u_0\|_{L_{x_i; (x_j)_{j \neq i}, t}^{q, \infty}} \lesssim \langle k \rangle^{1/q} \|\square_k u_0\|_{L^2}. \quad (2.21)$$

Proof. Recall that $\bar{x} = (x_2, \dots, x_n)$. By Lemma 1.4, we write $\square_k := \mathcal{F}^{-1} \eta_{k_1}(\xi_1) \sigma_{\bar{k}}(\bar{\xi}) \mathcal{F}$. By symmetry, it suffices to consider the case $i = 1$.

$$\|\square_k S(t)u_0\|_{L_{x_1; \bar{x}, t}^{q, \infty}} \lesssim \langle k \rangle^{1/q} \|\square_k u_0\|_{L^2}.$$

By a standard TT^* method, it suffices to show that

$$\left\| \int_{\mathbb{R}^n} e^{ix\xi} e^{-it(|\xi|^4 + |\xi|^2)} \sigma_k(\xi) d\xi \right\|_{L_{x_1; \bar{x}, t}^{q/2, \infty}} \lesssim \langle k \rangle^{2/q}. \quad (2.22)$$

We divide the proof into the cases $|k| \leq C$ and $|k| \geq C$. If $|k| \geq C$, using Lemma 2.7, we have

$$\|\mathcal{F}^{-1} e^{-it(|\xi|^4 + |\xi|^2)} \sigma_k(\xi)\|_{L_x^\infty} \lesssim \langle k \rangle^{-n} |t|^{-n/2}. \quad (2.23)$$

Using Lemma 2.6, we have

$$\|\square_k S(t) \mathcal{F}^{-1} \sigma_k\|_{L_{t,x}^\infty} \lesssim \|\square_k S(t) \mathcal{F}^{-1} \sigma_k\|_{L_t^\infty L_x^2} \lesssim 1. \quad (2.24)$$

Combining (2.23) with (2.24), we have

$$|\square_k S(t) \mathcal{F}^{-1} \sigma_k| \lesssim (1 + \langle k \rangle^2 |t|)^{-n/2}. \quad (2.25)$$

If $|x_1| \geq 16|t|\langle k \rangle^3 + 1$, integrating by part, we have

$$\left| \int_{\mathbb{R}} e^{i(x_1 \xi_1 - t(|\xi|^4 + |\xi|^2))} \eta_{k_1}(\xi_1) d\xi_1 \right| \lesssim |x_1|^{-2} \lesssim (|x_1| + 1)^{-2}.$$

Hence, we have

$$|\square_k S(t) \mathcal{F}^{-1} \sigma_k| \lesssim (|x_1| + 1)^{-2}. \quad (2.26)$$

If $|x_1| \leq 16|t|\langle k \rangle^3 + 1$, from (2.25) we have

$$|\square_k S(t) \mathcal{F}^{-1} \sigma_k| \lesssim (1 + \langle k \rangle^2 |t|)^{-n/2} \lesssim (1 + |x_1| \langle k \rangle^{-1})^{-n/2}. \quad (2.27)$$

Collecting (2.26) and (2.27), we obtain that

$$\sup_{\bar{x}, t} |\square_k S(t) \mathcal{F}^{-1} \sigma_k| \lesssim (|x_1| + 1)^{-2} + (1 + |x_1| \langle k \rangle^{-1})^{-n/2}. \quad (2.28)$$

Taking $L_{x_1}^{q/2}$ in both sides of (2.28), we obtain (2.22). If $|k| \leq C$, the proof is similar and we omit the details of the proof. \square

The dual version of Proposition 2.10 is

Proposition 2.11 *For $2 \leq q \leq \infty, q > 4/n$ and $k \in \mathbb{Z}^n, i = 1, \dots, n$, we have*

$$\left\| \square_k \int S(t - \tau) f(\tau) d\tau \right\|_{L_t^\infty L_x^2} \lesssim \langle k \rangle^{1/q} \|\square_k f\|_{L_{x_i; (x_j)_{j \neq i, t}}^{q', 1}}. \quad (2.29)$$

Proposition 2.12 *For $2 \leq q \leq \infty, q > 4/n$ and $k = (k_1, \dots, k_n) \in \mathbb{Z}^n, |k_i| \geq 100, i = 1, \dots, n$, we have*

$$\|\square_k \mathcal{A} f\|_{L_{x_i; (x_j)_{j \neq i, t}}^{q, \infty}} \lesssim \langle k_i \rangle^{-3/2+1/q} \|\square_k f\|_{L_{x_i; (x_j)_{j \neq i, t}}^{1, 2}}. \quad (2.30)$$

Proof. It suffices to consider the case $i = 1$. Denote

$$u = \mathcal{F}_{\tau, \xi}^{-1} \frac{1}{|\xi|^4 + |\xi|^2 - \tau} \mathcal{F}_{t, x} f. \quad (2.31)$$

One has that

$$\square_k u = \mathcal{F}_{\tau, \xi}^{-1} \frac{1}{|\xi|^4 + |\xi|^2 - \tau} (\mathcal{F}_{t, x} \square_k f)(\tau, \xi). \quad (2.32)$$

For the sake of convenience, we denote

$$\mathbb{E}_1 = \{\tau \leq -1/4\}, \quad \mathbb{E}_2 = \{-1/4 \leq \tau \leq |\bar{\xi}|^2(|\bar{\xi}|^2 + 1)\}, \quad \mathbb{E}_3 = \{\tau \geq |\bar{\xi}|^2(|\bar{\xi}|^2 + 1)\}$$

and $a = a(\tau, \bar{\xi}) = \sqrt{\sqrt{\tau + 1/4} - |\bar{\xi}|^2 - 1/2}$. Let us observe the decomposition of $|\xi|^4 + |\xi|^2 - \tau$,

$$|\xi|^4 + |\xi|^2 - \tau = \begin{cases} (|\xi|^2 + 1/2)^2 + (-\tau - 1/4), & (\bar{\xi}, \tau) \in \mathbb{E}_1, \\ (|\xi|^2 + 1/2 + \sqrt{\tau + 1/4})(|\xi|^2 + 1/2 - \sqrt{\tau + 1/4}), & (\bar{\xi}, \tau) \in \mathbb{E}_2, \\ (|\xi|^2 + 1/2 + \sqrt{\tau + 1/4})(\xi_1 - a)(\xi_1 + a), & (\bar{\xi}, \tau) \in \mathbb{E}_3. \end{cases} \quad (2.33)$$

We denote

$$\square_k u_i = \mathcal{F}_{\tau, \xi}^{-1} \frac{\chi_{\mathbb{E}_i}(\bar{\xi}, \tau)}{|\xi|^4 + |\xi|^2 - \tau} (\mathcal{F}_{t, x} \square_k f)(\tau, \xi). \quad (2.34)$$

First, we estimate $\square_k u_3$. One has the following decomposition:

$$\begin{aligned} \frac{\chi_{\mathbb{E}_3}(\bar{\xi}, \tau)}{|\xi|^4 + |\xi|^2 - \tau} &= -\frac{\chi_{\mathbb{E}_3}(\bar{\xi}, \tau)}{2\sqrt{\tau + 1/4}(|\xi|^2 + 1/2 + \sqrt{\tau + 1/4})} \\ &+ \frac{\chi_{\mathbb{E}_3}(\bar{\xi}, \tau)}{4a\sqrt{\tau + 1/4}} \left(\frac{1}{\xi_1 - a} - \frac{1}{\xi_1 + a} \right) := \sum_{j=1}^3 A_j(\tau, \xi). \end{aligned} \quad (2.35)$$

Let η_k be as in (1.17). We denote $\tilde{\eta}_{k_1}(\xi_1) = \sum_{|\ell| \leq 10} \eta_{k_1 + \ell}(\xi_1)$. According to the above decomposition, one can rewrite $\square_k u_3$ as

$$\begin{aligned} \square_k u_3 &= \mathcal{F}_{\tau, \xi}^{-1} \frac{\chi_{\mathbb{E}_3}(\bar{\xi}, \tau) \tilde{\eta}_{k_1}(a)}{|\xi|^4 + |\xi|^2 - \tau} (\mathcal{F}_{t, x} \square_k f)(\tau, \xi) \\ &+ \mathcal{F}_{\tau, \xi}^{-1} \frac{\chi_{\mathbb{E}_3}(\bar{\xi}, \tau) (1 - \tilde{\eta}_{k_1}(a))}{|\xi|^4 + |\xi|^2 - \tau} (\mathcal{F}_{t, x} \square_k f)(\tau, \xi) \\ &= \sum_{j=1}^3 \mathcal{F}_{\tau, \xi}^{-1} \chi_{\mathbb{E}_3}(\bar{\xi}, \tau) A_j(\tau, \xi) \tilde{\eta}_{k_1}(a) (\mathcal{F}_{t, x} \square_k f)(\tau, \xi) \\ &+ \mathcal{F}_{\tau, \xi}^{-1} \frac{\chi_{\mathbb{E}_3}(\bar{\xi}, \tau) (1 - \tilde{\eta}_{k_1}(a))}{|\xi|^4 + |\xi|^2 - \tau} (\mathcal{F}_{t, x} \square_k f)(\tau, \xi) \\ &:= I + II + III + IV. \end{aligned} \quad (2.36)$$

Case 1. We consider the case $k_1 \geq 100$. First, we estimate *II*. Let $\tilde{\sigma}_k$ be as in (1.19). Noticing that $\mathcal{F}_{\xi_1}^{-1} \xi_1^{-1} = \text{sgn}(x_1)$, we have

$$II = \int_{\mathbb{R}^{1+n}} \frac{e^{it\tau + i\bar{x}\bar{\xi}} \chi_{\mathbb{E}_3}(\bar{\xi}, \tau)}{4a\sqrt{1/4 + \tau}} \tilde{\sigma}_k(\bar{\xi}) \tilde{\eta}_{k_1}(a) \widehat{\square_k f}(y_1)(\tau, \bar{\xi}) e^{i(x_1 - y_1)a} \text{sgn}(x_1 - y_1) d\bar{\xi} d\tau dy_1. \quad (2.37)$$

By change of variable $\xi_1 = a(\tau, \bar{\xi})$ and putting $\tilde{\rho}_k(\xi) = \tilde{\sigma}_k(\bar{\xi}) \tilde{\eta}_{k_1}(\xi_1)$, one sees that

$$II = \int dy_1 \text{sgn}(x_1 - y_1) \int_{\mathbb{R}^n (\xi_1 > 0)} e^{it(|\xi|^4 + |\xi|^2)} e^{i(x_1 - y_1)\xi_1 + i\bar{x}\bar{\xi}} \tilde{\rho}_k(\xi) \widehat{\square_k f}(y_1)(|\xi|^4 + |\xi|^2, \bar{\xi}) d\xi. \quad (2.38)$$

Applying Proposition 2.10, we have

$$\begin{aligned}
\|II\|_{L_{x_1; \bar{x}, t}^{q, \infty}} &\leq \int dy_1 \left\| \int_{\mathbb{R}^n (\xi_1 > 0)} e^{it(|\xi|^4 + |\xi|^2)} e^{i(x_1 - y_1)\xi_1 + i\bar{x}\bar{\xi}} \tilde{\rho}_k(\xi) \widehat{\square_k f}(y_1)(|\xi|^4 + |\xi|^2, \bar{\xi}) d\xi \right\|_{L_{x_1; \bar{x}, t}^{q, \infty}} \\
&\lesssim \langle k_1 \rangle^{1/q} \int \left\| \tilde{\rho}_k(\xi) \widehat{\square_k f}(y_1)(|\xi|^4 + |\xi|^2, \bar{\xi}) \right\|_{L_{\xi}^2} dy_1 \\
&\lesssim \langle k_1 \rangle^{1/q-3/2} \|\square_k f\|_{L_{x_1; \bar{x}, t}^{1,2}}. \tag{2.39}
\end{aligned}$$

Since $k_1 > 0$, III is easier to handle than II and it has the same upper bound as in (2.39). Now we estimate IV . Let us write

$$K(x_1, a, \bar{\xi}) = \chi_{\mathbb{E}_3}(\bar{\xi}, \tau)(1 - \tilde{\eta}_{k_1}(a)) \int \frac{\sum_{|\ell| \leq 1} \eta_{k_1 + \ell}(\xi_1) e^{ix_1 \xi_1}}{|\xi|^4 + |\xi|^2 - \tau} d\xi_1. \tag{2.40}$$

It is easy to see that

$$IV = \int dy_1 \int_{\mathbb{R}^n} e^{it\tau + i\bar{x}\bar{\xi}} \tilde{\sigma}_{\bar{k}}(\bar{\xi}) \widehat{\square_k f}(y_1)(\tau, \bar{\xi}) K(x_1 - y_1, a, \bar{\xi}) d\bar{\xi} d\tau dy_1. \tag{2.41}$$

By Young's, Hölder's and Minkowski's inequalities and Plancherel's identity,

$$\begin{aligned}
\|IV\|_{L_{x_1; \bar{x}, t}^{q, \infty}} &\leq \left\| \int \|\tilde{\sigma}_{\bar{k}}(\bar{\xi}) \widehat{\square_k f}(y_1)(\tau, \bar{\xi}) K(x_1 - y_1, a, \bar{\xi})\|_{L_{\xi, \tau}^1} dy_1 \right\|_{L_{x_1}^q} \\
&\leq \|\square_k f\|_{L_{x_1; \bar{x}, t}^{1,2}} \|\tilde{\sigma}_{\bar{k}}(\bar{\xi}) K(x_1, a, \bar{\xi})\|_{L_{x_1; \bar{\xi}, \tau}^{q,2}} \\
&\lesssim \|\square_k f\|_{L_{x_1; \bar{x}, t}^{1,2}} \|\tilde{\sigma}_{\bar{k}}(\bar{\xi}) K(x_1, a, \bar{\xi})\|_{L_{\bar{\xi}}^\infty L_\tau^2 L_{x_1}^q}. \tag{2.42}
\end{aligned}$$

Integrating by part, we see that

$$|K(x_1, a, \bar{\xi})| \lesssim \frac{\chi_{\mathbb{E}_3}(\bar{\xi}, \tau)(1 - \tilde{\eta}_{k_1}(a))}{1 + |x_1|} \sum_{j=0,1} \int_{|\xi_1 - k_1| \leq 3} |\partial_{\xi_1}^j (|\xi|^4 + |\xi|^2 - \tau)^{-1}| d\xi_1. \tag{2.43}$$

It follows that

$$\begin{aligned}
&\|\tilde{\sigma}_{\bar{k}}(\bar{\xi}) K(x_1, a, \bar{\xi})\|_{L_{\bar{\xi}}^\infty L_\tau^2 L_{x_1}^q} \\
&\lesssim \sup_{|\xi - k|_\infty \leq 3} \sum_{j=0,1} \left\| \chi_{\mathbb{E}_3}(\bar{\xi}, \tau)(1 - \tilde{\eta}_{k_1}(a)) \partial_{\xi_1}^j (|\xi|^4 + |\xi|^2 - \tau)^{-1} \right\|_{L_\tau^2}. \tag{2.44}
\end{aligned}$$

Noticing that $|a - \xi_1| \geq 1$ in the support set of $(1 - \tilde{\eta}_{k_1}(a)) \chi_{\{|\xi_1 - k_1| \leq 3\}} \partial_{\xi_1}^j (|\xi|^4 + |\xi|^2 - \tau)^{-1}$, in view of the third decomposition in (2.33), we see that there is no singularity in the integration of (2.44) and a simple calculation yields

$$\|\tilde{\sigma}_{\bar{k}}(\bar{\xi}) K(x_1, a, \bar{\xi})\|_{L_{\bar{\xi}}^\infty L_\tau^2 L_{x_1}^q} \lesssim |k_1|^{-3/2}. \tag{2.45}$$

We estimate I . Let us write

$$H(x_1, a, \bar{\xi}) = \chi_{\mathbb{E}_3}(\bar{\xi}, \tau) \tilde{\eta}_{k_1}(a) \int \frac{\sum_{|\ell| \leq 1} \eta_{k_1 + \ell}(\xi_1) e^{ix_1 \xi_1}}{2\sqrt{\tau + 1/4}(|\xi|^2 + 1/2 + \sqrt{\tau + 1/4})} d\xi_1. \tag{2.46}$$

It is easy to see that

$$I = \int dy_1 \int_{\mathbb{R}^n} e^{it\tau + i\bar{x}\bar{\xi}} \tilde{\sigma}_{\bar{k}}(\bar{\xi}) \widehat{\square_k f}(y_1)(\tau, \bar{\xi}) H(x_1 - y_1, a, \bar{\xi}) d\bar{\xi} d\tau dy_1. \quad (2.47)$$

Similar to (2.42), by Young's, Hölder's and Minkowski's inequalities and Plancherel's identity,

$$\|I\|_{L_{x_1, \bar{x}, t}^{q, \infty}} \lesssim \|\square_k f\|_{L_{x_1, \bar{x}, t}^{1, 2}} \|\tilde{\sigma}_{\bar{k}}(\bar{\xi}) H(x_1, a, \bar{\xi})\|_{L_{\bar{\xi}}^{\infty} L_{\tau}^2 L_{x_1}^q}. \quad (2.48)$$

Integrating by part, we see that

$$|H(x_1, a, \bar{\xi})| \lesssim \frac{\chi_{\mathbb{E}_3}(\bar{\xi}, \tau) \tilde{\eta}_{k_1}(a)}{(1 + |x_1|) \sqrt{\tau + 1/4}} \sum_{j=0,1} \int_{|\xi_1 - k_1| \leq 3} |\partial_{\xi_1}^j (|\xi|^2 + 1/2 + \sqrt{\tau + 1/4})^{-1}| d\xi_1. \quad (2.49)$$

It follows that

$$\begin{aligned} & \|\tilde{\sigma}_{\bar{k}}(\bar{\xi}) H(x_1, a, \bar{\xi})\|_{L_{\bar{\xi}}^{\infty} L_{\tau}^2 L_{x_1}^q} \\ & \lesssim \sup_{|\xi - k|_{\infty} \leq 3} \sum_{j=0,1} \left\| \frac{\chi_{\mathbb{E}_3}(\bar{\xi}, \tau) \tilde{\eta}_{k_1}(a)}{\sqrt{\tau + 1/4}} \sum_{j=0,1} |\partial_{\xi_1}^j (|\xi|^2 + 1/2 + \sqrt{\tau + 1/4})^{-1}| \right\|_{L_{\tau}^2}. \end{aligned} \quad (2.50)$$

Noticing that $|a - k_1| \leq 20$ in the support set of $\tilde{\eta}_{k_1}(a)$, we see that $\sqrt{\tau + 1/4} \gtrsim k_1^2$ and there is no singularity in the integration of (2.61) and a simple calculation yields

$$\|\tilde{\sigma}_{\bar{k}}(\bar{\xi}) H(x_1, a, \bar{\xi})\|_{L_{\bar{\xi}}^{\infty} L_{\tau}^2 L_{x_1}^q} \lesssim |k_1|^{-2}. \quad (2.51)$$

Case 2. We consider the case $k_1 \leq -100$. This case is similar to the case $k_1 \geq 100$ and we omit the details of the proof.

Finally, we need to estimate $\square_k u_1$ and $\square_k u_2$. By (2.33), we see that there is no singularity in $(|\xi|^4 + |\xi|^2 - \tau)^{-1}$ if $|\xi_1| \geq 1$ and $(\bar{\xi}, \tau) \in \mathbb{E}_1 \cup \mathbb{E}_2$. So, one can use the same way as in the estimates of I in (2.46)–(2.51) to have the desired result. This finishes the proof. \square

In Proposition 2.12 we show that $\square_k \mathcal{A} : L_{x_1; (x_j)_{j \neq 1}, t}^{1, 2} \rightarrow L_{x_1; (x_j)_{j \neq 1}, t}^{2, \infty}$. Moreover, we can show that $\mathcal{A} : L_{x_1; (x_j)_{j \neq 1}, t}^{1, 2} \rightarrow L_{x_2; (x_j)_{j \neq 2}, t}^{2, \infty}$ if $|k_2| = k_{\max} \gg 1$.

Proposition 2.13 *For $2 \leq q \leq \infty$, $q > 4/n$ and $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, $|k_i| = k_{\max} \geq 100$, $h, i = 1, \dots, n$, $h \neq i$, we have*

$$\|\square_k \mathcal{A} f\|_{L_{x_h; (x_j)_{j \neq h}, t}^{q, \infty}} \lesssim \langle k_i \rangle^{-3/2 + 1/q} \|\square_k f\|_{L_{x_i; (x_j)_{j \neq i}, t}^{1, 2}}. \quad (2.52)$$

Proof. First, we consider the case $q = 2$, $h = 1$ and $i = 2$. We will follow the same ideas as in the proof of Proposition 2.12 and omit the details of the proof if the estimates are completely the same ones as in Proposition 2.12. For the sake of convenience, we denote $\tilde{\xi} = (\xi_1, \xi_3, \dots, \xi_n)$,

$$\mathbb{F}_1 = \{\tau \leq -1/4\}, \quad \mathbb{F}_2 = \{-1/4 \leq \tau \leq |\tilde{\xi}|^2(|\tilde{\xi}|^2 + 1)\}, \quad \mathbb{F}_3 = \{\tau \geq |\tilde{\xi}|^2(|\tilde{\xi}|^2 + 1)\}$$

and $b = b(\tau, \tilde{\xi}) = \sqrt{\sqrt{\tau + 1/4} - |\tilde{\xi}|^2} - 1/2$. Let us observe the decomposition of $|\xi|^4 + |\xi|^2 - \tau$,

$$|\xi|^4 + |\xi|^2 - \tau = \begin{cases} (|\xi|^2 + 1/2)^2 + (-\tau - 1/4), & (\tilde{\xi}, \tau) \in \mathbb{F}_1, \\ (|\xi|^2 + 1/2 + \sqrt{\tau + 1/4})(|\xi|^2 + 1/2 - \sqrt{\tau + 1/4}), & (\tilde{\xi}, \tau) \in \mathbb{F}_2, \\ (|\xi|^2 + 1/2 + \sqrt{\tau + 1/4})(\xi_2 - b)(\xi_2 + b), & (\tilde{\xi}, \tau) \in \mathbb{F}_3. \end{cases} \quad (2.53)$$

We denote

$$\square_k v_i = \mathcal{F}_{\tau, \xi}^{-1} \frac{\chi_{\mathbb{F}_i}(\tilde{\xi}, \tau)}{|\xi|^4 + |\xi|^2 - \tau} (\mathcal{F}_{t, x} \square_k f)(\tau, \xi). \quad (2.54)$$

First, we estimate $\square_k v_3$. One has the following decomposition:

$$\begin{aligned} \frac{\chi_{\mathbb{F}_3}(\tilde{\xi}, \tau)}{|\xi|^4 + |\xi|^2 - \tau} &= -\frac{\chi_{\mathbb{F}_3}(\tilde{\xi}, \tau)}{2\sqrt{\tau + 1/4}(|\xi|^2 + 1/2 + \sqrt{\tau + 1/4})} \\ &+ \frac{\chi_{\mathbb{F}_3}(\tilde{\xi}, \tau)}{4b\sqrt{\tau + 1/4}} \left(\frac{1}{\xi_2 - b} - \frac{1}{\xi_2 + b} \right) := \sum_{j=1}^3 B_j(\tau, \xi). \end{aligned} \quad (2.55)$$

Let η_k be as in Lemma 1.4. We denote $\tilde{\eta}_{k_2}(\xi_2) = \sum_{|\ell| \leq 10} \eta_{k_2 + \ell}(\xi_2)$. According to the above decomposition, one can rewrite $\square_k v_3$ as

$$\begin{aligned} \square_k v_3 &= \sum_{j=1}^3 \mathcal{F}_{\tau, \xi}^{-1} \chi_{\mathbb{F}_3}(\tilde{\xi}, \tau) B_j(\tau, \xi) \tilde{\eta}_{k_2}(b) (\mathcal{F}_{t, x} \square_k f)(\tau, \xi) \\ &+ \mathcal{F}_{\tau, \xi}^{-1} \frac{\chi_{\mathbb{F}_3}(\tilde{\xi}, \tau) (1 - \tilde{\eta}_{k_2}(b))}{|\xi|^4 + |\xi|^2 - \tau} (\mathcal{F}_{t, x} \square_k f)(\tau, \xi) \\ &:= I + II + III + IV. \end{aligned} \quad (2.56)$$

We only consider the case $k_2 \geq 100$ and the case $k_2 \leq -100$ can be handled in an analogous way. The estimate of *II* and *III* follow the same way as in Proposition 2.12 by exchanging the roles of ξ_1 and ξ_2 . Now we estimate *I* and *IV*. In view of Young's, Hölder's and Minkowski's inequalities and Plancherel's identity,

$$\begin{aligned} \|\mathcal{F}_{\tau, \xi}^{-1} m(\xi, \tau) (\mathcal{F}_{t, x} \square_k f)\|_{L_{x_1; \bar{x}, t}^{2, \infty}} &\lesssim \|\mathcal{F}_{\xi_1}^{-1} m(\xi, \tau) \mathcal{F}_{t, x} \square_k f\|_{L_{x_1}^2 L_{\xi, \tau}^1} \\ &\lesssim \|m(\xi, \tau) \mathcal{F}_{t, x} \square_k f\|_{L_{\xi, \tau}^1 L_{\xi_1}^2} \end{aligned}$$

$$\begin{aligned}
&\lesssim \|\mathcal{F}_{t,x} \square_k f\|_{L_{\xi_2}^\infty L_{(\xi_j)_{j \neq 2}, \tau}^2} \|m\|_{L_{\xi_2}^1 L_{\xi_3, \dots, \xi_n, \tau}^2 L_{\xi_1}^\infty} \\
&\lesssim \|\square_k f\|_{L_{x_2}^1 L_{(x_j)_{j \neq 2}, t}^2} \|m\|_{L_{\xi_2}^1 L_{\xi_3, \dots, \xi_n, \tau}^2 L_{\xi_1}^\infty}.
\end{aligned} \tag{2.57}$$

Let $\tilde{\sigma}_k$ be as in (1.19). Putting

$$m(\xi, \tau) = \frac{\chi_{\mathbb{F}_3}(\tilde{\xi}, \tau) \tilde{\eta}_{k_2}(a) \tilde{\sigma}_k(\xi)}{2\sqrt{\tau + 1/4} (|\xi|^2 + 1/2 + \sqrt{\tau + 1/4})}, \tag{2.58}$$

one sees that $\sqrt{\tau + 1/4} \gtrsim k_2^2$ in the support set of m . It follows that

$$m(\xi, \tau) \lesssim \frac{\chi_{\{\tau \geq -1/4\}} \tilde{\sigma}_k(\xi)}{k_2^4 + \tau + 1/4}, \tag{2.59}$$

by (2.59), we immediately have

$$\|m\|_{L_{\xi_2}^1 L_{\xi_3, \dots, \xi_n, \tau}^2 L_{\xi_1}^\infty} \lesssim |k_2|^{-2}. \tag{2.60}$$

It follows from (2.57), (2.58) and (2.60) that

$$I \lesssim |k_2|^{-2} \|\square_k f\|_{L_{x_2}^1 L_{(x_j)_{j \neq 2}, t}^2}. \tag{2.61}$$

Taking

$$m(\xi, \tau) = \frac{\chi_{\mathbb{F}_3}(\tilde{\xi}, \tau) \tilde{\sigma}_k(\xi) (1 - \tilde{\eta}_{k_2}(b))}{|\xi|^4 + |\xi|^2 - \tau} \tag{2.62}$$

It follows that for $(\xi, \tau) \in \text{supp } m$,

$$||\xi|^4 + |\xi|^2 - \tau| = ||\xi|^2 + 1/2 + \sqrt{\tau + 1/4}(\xi_2 - b)(\xi_2 + b)| \gtrsim \langle k \rangle^2 |k_2|. \tag{2.63}$$

By the definition of b , if $(|\tilde{\xi}|^2 + 1/2) \leq \sqrt{\tau + 1/4}/2$, then we have

$$||\xi|^4 + |\xi|^2 - \tau| \geq \sqrt{\tau + 1/4}(\xi_2 + b) \gtrsim (\tau + 1/4)^{3/4}, \quad (\xi, \tau) \in \text{supp } m. \tag{2.64}$$

If $(|\tilde{\xi}|^2 + 1/2) \geq \sqrt{\tau + 1/4}/2$, noticing that $k_2 = k_{\max} \geq 100$, we have

$$\xi_2 \geq k_2 - 3 \gtrsim |k| + 1 \gtrsim (|\tilde{\xi}|^2 + 1)^{1/2} \gtrsim (\tau + 1/4)^{1/4} \quad (\xi, \tau) \in \text{supp } m, \tag{2.65}$$

which also implies that

$$||\xi|^4 + |\xi|^2 - \tau| \geq \sqrt{\tau + 1/4}(\xi_2 + b) \gtrsim (\tau + 1/4)^{3/4}, \quad (\xi, \tau) \in \text{supp } m. \tag{2.66}$$

So, in view of (2.63) and (2.66), we have

$$||\xi|^4 + |\xi|^2 - \tau| \gtrsim (k_2^4 + \tau + 1/4)^{3/4}, \quad (\xi, \tau) \in \text{supp } m. \tag{2.67}$$

It follows from (2.62) and (2.67) that

$$m(\xi, \tau) \lesssim \frac{\chi_{\{\tau+1/4 \geq 0\}} \tilde{\sigma}_k(\xi)}{(k_2^4 + \tau + 1/4)^{3/4}} \quad (2.68)$$

by (2.68), we immediately have

$$\|m\|_{L_{\xi_2}^1 L_{\xi_3, \dots, \xi_n, \tau}^2 L_{\xi_1}^\infty} \lesssim |k_2|^{-1}. \quad (2.69)$$

It follows from (2.57), (2.58) and (2.60) that

$$I \lesssim |k_2|^{-1} \|\square_k f\|_{L_{x_2}^1 L_{(x_j)_{j \neq 2, t}}^2}. \quad (2.70)$$

To conclude the proof of the case $q = 2$, we need to estimate $\square_k v_1$ and $\square_k v_2$. By (2.53), we see that there is no singularity in $(|\xi|^4 + |\xi|^2 - \tau)^{-1}$ if $|\xi_2| \geq 1$ and $(\bar{\xi}, \tau) \in \mathbb{F}_1 \cup \mathbb{F}_2$. So, one can use the same way as in the estimates of I to have the desired result.

If $q > 2$, noticing that $\|\square_k u\|_{L_{t,x}^\infty} \lesssim \|\square_k u\|_{L_t^\infty L_x^2}$ uniformly holds for all $k \in \mathbb{Z}^n$ and making an interpolation between (2.14) in Proposition 2.5 and the maximal function estimates at $q = 2$, we immediately have the result, as desired. \square

Collecting Propositions 2.12 and 2.13, we have

Proposition 2.14 *For $2 \leq q \leq \infty, q > 4/n$ and $k = (k_1, \dots, k_n) \in \mathbb{Z}^n, |k_i| = k_{\max}, i = 1, \dots, n$, we have*

$$\|\square_k \partial_{x_i}^3 \mathcal{A} f\|_{L_{x_h; (x_j)_{j \neq h, t}}^{q, \infty}} \lesssim \langle k_{\max} \rangle^{3/2+1/q} \|\square_k f\|_{L_{x_i; (x_j)_{j \neq i, t}}^{1, 2}}. \quad (2.71)$$

2.4 Main results on Linear estimates with \square_k -decomposition

In this subsection we establish some interaction estimates between the Strichartz, maximal function and smoothing effect estimates. In particular, it seems necessary to handle the case when the partial derivatives and the anisotropic Lebesgue spaces have different directions, say $\|\partial_{x_1}^3 \square_k \mathcal{A} f\|_{L_{x_2; (x_j)_{j \neq 2, t}}^{2, \infty}}$, $|k_3| = k_{\max}$, since the nonlinearity contains different partial derivatives. For convenience to later applications, we summarize the main conclusions of this section as the following:

Theorem 2.15 *For $4/n \leq p < \infty, 2 \leq q < \infty, q > 4/n$ and $k = (k_1, \dots, k_n) \in \mathbb{Z}^n, |k_i| = k_{\max}, h, i, \ell = 1, \dots, n$. We have*

$$\left\| D_{x_i}^{3/2} \square_k S(t) u_0 \right\|_{L_{x_i; (x_j)_{j \neq i, t}}^{\infty, 2}} \lesssim \|\square_k u_0\|_{L^2(\mathbb{R}^n)}, \quad (2.72)$$

$$\|\square_k S(t) u_0\|_{L_{x_i; (x_j)_{j \neq i, t}}^{q, \infty}} \lesssim \langle k_{\max} \rangle^{1/q} \|\square_k u_0\|_{L^2(\mathbb{R}^n)}, \quad (2.73)$$

$$\|\square_k S(t) u_0\|_{L_t^\infty L_x^2 \cap L_{t,x}^{2+p}} \lesssim \|\square_k u_0\|_{L^2}. \quad (2.74)$$

If $k_{\max} \geq 100$, then

$$\|\square_k \partial_{x_\ell}^3 \mathcal{A} f\|_{L_{x_i; (x_j)_{j \neq i, t}}^{\infty, 2}} \lesssim \|\square_k f\|_{L_{x_i; (x_j)_{j \neq i, t}}^{1, 2}}, \quad (2.75)$$

$$\|\square_k \partial_{x_\ell}^3 \mathcal{A} f\|_{L_{x_h; (x_j)_{j \neq h, t}}^{q, \infty}} \lesssim \langle k_{\max} \rangle^{3/2+1/q} \|\square_k f\|_{L_{x_i; (x_j)_{j \neq i, t}}^{1, 2}}, \quad (2.76)$$

$$\|\square_k \partial_{x_\ell}^3 \mathcal{A} f\|_{L_t^\infty L_x^2 \cap L_{t,x}^{2+p}} \lesssim \langle k_{\max} \rangle^{3/2} \|\square_k f\|_{L_{x_i; (x_j)_{j \neq i, t}}^{1, 2}}. \quad (2.77)$$

$$\|\square_k \partial_{x_\ell}^3 \mathcal{A} f\|_{L_{x_i; (x_j)_{j \neq i, t}}^{\infty, 2}} \lesssim \langle k_{\max} \rangle^{3/2} \|\square_k f\|_{L_{t,x}^{(2+p)/(1+p)}}, \quad (2.78)$$

$$\|\square_k \partial_{x_\ell}^3 \mathcal{A} f\|_{L_{x_i; (x_j)_{j \neq i, t}}^{q, \infty}} \lesssim \langle k_{\max} \rangle^{3+1/q} \|\square_k f\|_{L_{t,x}^{(2+p)/(1+p)}}, \quad (2.79)$$

$$\|\square_k \mathcal{A} f\|_{L_t^\infty L_x^2 \cap L_{t,x}^{2+p}} \lesssim \|\square_k f\|_{L_{t,x}^{(2+p)/(1+p)}}. \quad (2.80)$$

Moreover, (2.75), (2.77), (2.78) and (2.79) hold for $k_{\max} \leq 100$ if $i = \ell$.

Lemma 2.16 Let $k = (k_1, \dots, k_n)$ with $|k_m| = k_{\max} > 100$, $1 \leq p, q \leq \infty$. Then we have

$$\|\square_k \partial_{x_\ell}^3 f\|_{L_{x_i; (x_j)_{j \neq i, t}}^{q,p}} \lesssim \|\square_k \partial_{x_m}^3 f\|_{L_{x_i; (x_j)_{j \neq i, t}}^{q,p}}. \quad (2.81)$$

Proof. We have for any p, q ,

$$\begin{aligned} \|\square_k \partial_{x_\ell}^3 f\|_{L_{x_i; (x_j)_{j \neq i, t}}^{q,p}} &\lesssim \sum_{|l_\ell|, |l_m| \leq 1} \left\| \mathcal{F}_{\xi_\ell, \xi_m}^{-1} \left(\left(\frac{\xi_\ell}{\xi_m} \right)^3 \eta_{k_\ell + l_\ell}(\xi_\ell) \eta_{k_m + l_m}(\xi_m) \right) \right\|_{L^1(\mathbb{R}^2)} \\ &\quad \times \|\square_k \partial_{x_m}^3 f\|_{L_{x_i; (x_j)_{j \neq i, t}}^{q,p}} \\ &\lesssim \|\square_k \partial_{x_m}^3 f\|_{L_{x_i; (x_j)_{j \neq i, t}}^{q,p}}. \end{aligned} \quad (2.82)$$

□

Remark 2.17 Since $|k_m|$ is maximal in all of $|k_i|$, Lemma 2.16 can be regarded as ∂_{x_m} is the maximal directional derivative in all of ∂_{x_i} by considering the frequency localized functions $\square_k f$.

In order to show Theorem 2.15, we need the following

Proposition 2.18 Let $2 \leq r < \infty$, $2/\gamma(r) = n(1/2 - 1/r)$ and $\gamma > \gamma(r) \vee 2$. We have

$$\|\square_k \partial_{x_i}^3 \mathcal{A} f\|_{L_t^\gamma L_x^r} \lesssim \langle k_{\max} \rangle^{3/2} \|\square_k f\|_{L_{x_i; (x_j)_{j \neq i, t}}^{1, 2}}, \quad (2.83)$$

$$\|\square_k \partial_{x_i}^3 \mathcal{A} f\|_{L_{x_i; (x_j)_{j \neq i, t}}^{\infty, 2}} \lesssim \langle k_{\max} \rangle^{3/2} \|\square_k f\|_{L_t^{\gamma'} L_x^{r'}}, \quad (2.84)$$

and for $2 \leq q < \infty$, $q > 4/n$, $\alpha = 0, 3$,

$$\|\square_k \partial_{x_i}^\alpha \mathcal{A} f\|_{L_{x_i; (x_j)_{j \neq i, t}}^{q, \infty}} \lesssim \langle k_{\max} \rangle^{\alpha+1/q} \|\square_k f\|_{L_t^{\gamma'} L_x^{r'}}. \quad (2.85)$$

Proof. Denote

$$\mathcal{L}_k(f, \psi) = \int \left(\square_k \int S(t - \tau) f(\tau) d\tau, \psi(t) \right) dt. \quad (2.86)$$

By duality and maximal function estimate (2.21),

$$\begin{aligned} |\mathcal{L}_k(f, \psi)| &\leq \|\square_k f\|_{L_{x_1}^{q'} L_{\bar{x}, t}^1} \sum_{|\ell|_\infty \leq 1} \left\| \square_{k+\ell} \int S(\tau - t) \psi(t) dt \right\|_{L_{x_1}^q L_{\bar{x}, t}^\infty} \\ &\leq \|\square_k f\|_{L_{x_1}^{q'} L_{\bar{x}, t}^1} \sum_{|\ell|_\infty \leq 1} \int \|\square_{k+\ell} S(\tau - t) \psi(t)\|_{L_{x_1}^q L_{\bar{x}, \tau}^\infty} dt \\ &\lesssim \langle k \rangle^{1/q} \|\square_k f\|_{L_{x_1}^{q'} L_{\bar{x}, t}^1} \|\psi\|_{L_t^1 L_x^2}. \end{aligned} \quad (2.87)$$

By duality, we immediately have

$$\left\| \square_k \int S(t - \tau) f(\tau) d\tau \right\|_{L_t^\infty L_x^2} \lesssim \langle k \rangle^{1/q} \|\square_k f\|_{L_{x_1}^{q'} L_{\bar{x}, t}^1}. \quad (2.88)$$

Again, by duality, Strichartz estimate (2.18) and (2.88),

$$\begin{aligned} |\mathcal{L}_k(f, \psi)| &\leq \left\| \square_k \int S(-\tau) f(\tau) d\tau \right\|_2 \left\| \square_k \int S(-t) \psi(t) dt \right\|_2 \\ &\leq \langle k \rangle^{1/q} \|f\|_{L_{x_1}^{q'} L_{\bar{x}, t}^1} \|\square_k \psi\|_{L_t^{\gamma'} L_x^{r'}}, \end{aligned} \quad (2.89)$$

which implies (2.85) in the case $q > 2$ or $r > 2$. In the case $q = r = 2$, (2.85) also holds in view of (2.21). By Lemmas 2.8, 2.4 and Proposition 2.5,

$$\mathcal{L}_k(\partial_{x_1}^3 f, \psi) \lesssim \langle k_{\max} \rangle^{3/2} \|\square_k f\|_{L_{x_1; (x_j)_{j \neq 1}, t}^{1,2}} \|\psi\|_{L_t^{\gamma'} L_x^{r'}}. \quad (2.90)$$

Hence, we obtain (2.83). For (2.84), when $r > 2$, we get it by exchanging the roles of f and ψ in (2.90). If $r = 2$, we obtain (2.84) by using the 3/2-order smooth effect (2.1) of $S(t)$. \square

Proof of Theorem 2.15. (2.72), (2.73) and (2.74) have been shown in (2.1), (2.21) and (2.17), respectively. In view of Lemma 2.16, if $|k_i| = k_{\max} > 100$, then

$$\|\square_k \partial_{x_\ell}^3 \mathcal{A} f\|_{L_{x_i; (x_j)_{j \neq i}, t}^{\infty, 2}} \lesssim \|\square_k \partial_{x_i}^3 \mathcal{A} f\|_{L_{x_i; (x_j)_{j \neq i}, t}^{\infty, 2}}, \quad (2.91)$$

$$\|\square_k \partial_{x_\ell}^3 \mathcal{A} f\|_{L_{x_h; (x_j)_{j \neq h}, t}^{q, \infty}} \lesssim \|\square_k \partial_{x_i}^3 \mathcal{A} f\|_{L_{x_h; (x_j)_{j \neq h}, t}^{q, \infty}}, \quad (2.92)$$

$$\|\square_k \partial_{x_\ell}^3 \mathcal{A} f\|_{L_t^\infty L_x^2 \cap L_{t, x}^{2+p}} \lesssim \|\square_k \partial_{x_i}^3 \mathcal{A} f\|_{L_t^\infty L_x^2 \cap L_{t, x}^{2+p}}. \quad (2.93)$$

Combining (2.91) with (2.13), (2.92) with (5.10), (2.83) with (2.93), we have (2.75), (2.76), (2.77), respectively. Similarly, by (2.84), (2.85) and Lemma 2.16, we can obtain (2.78) and (2.79). (2.79) follows from (2.20).

Finally, if $\ell = i$, (2.75), (2.77), (2.78) and (2.79) hold for $|k_i| \leq 100$ has been shown in previous subsections. \square

3 Proof of Theorem 1.2

For later purpose, we establish some lemmas related to nonlinear mapping estimates. The ideas follow from [39]. For $i = 1, \dots, n$ and $N \in \mathbb{N}$, we denote

$$\begin{aligned}\mathbb{B}_{i,1}^{(N)} &:= \{k^{(1)}, \dots, k^{(N)} \in \mathbb{Z}^n : |k_i^{(1)}| \vee \dots \vee |k_i^{(N)}| > 100\}, \\ \mathbb{B}_{i,2}^{(N)} &:= \{k^{(1)}, \dots, k^{(N)} \in \mathbb{Z}^n : |k_i^{(1)}| \vee \dots \vee |k_i^{(N)}| \leq 100\}.\end{aligned}$$

Lemma 3.1 *If $s \in \mathbb{R}_+$, $N \geq 3$, then we have*

$$\begin{aligned}& \left\| \sum_{\mathbb{B}_{1,1}^{(N)}} \square_{k^{(1)}} u_1 \dots \square_{k^{(N)}} u_N \right\|_{\ell_{\square^1}^{1,s}(L_{x_1:(x_j)_{j \neq 1},t}^{1,2})} \\ & \lesssim \sum_{\eta=1}^N \|u_\eta\|_{\cap_{h=1}^n \ell_{\square^h}^{1,s}(L_{x_h:(x_j)_{j \neq h},t}^{\infty,2})} \prod_{i=1, i \neq \eta}^N \|u_i\|_{\cap_{h=1}^n \ell_{\square^h}^1(L_{x_h:(x_j)_{j \neq h},t}^{N-1,\infty})}.\end{aligned}\quad (3.1)$$

Proof. In view of the support property of $\widehat{\square_k u}$, we see that

$$\square_k (\square_{k^{(1)}} u_1 \dots \square_{k^{(N)}} u_N) = 0, \quad \text{if } |k - k^{(1)} - \dots - k^{(N)}| \geq N + 2. \quad (3.2)$$

Hence,

$$\begin{aligned}& \left\| \sum_{\mathbb{B}_{1,1}^{(N)}} \square_{k^{(1)}} u_1 \dots \square_{k^{(N)}} u_N \right\|_{\ell_{\square^1}^{1,s}(L_{x_1:(x_j)_{j \neq 1},t}^{1,2})} \\ & \lesssim \sum_{k \in \mathbb{Z}_1^n} \langle k_1 \rangle^s \sum_{\mathbb{B}_{1,1}^{(N)}} \|\square_{k^{(1)}} u_1 \dots \square_{k^{(N)}} u_N\|_{L_{x_1:(x_j)_{j \neq 1},t}^{1,2}} \chi_{\{|k - k^{(1)} - \dots - k^{(N)}| \leq N+1\}}.\end{aligned}\quad (3.3)$$

Case I. $|k_1^{(1)}| = \max_{1 \leq m \leq N} |k_1^{(m)}|$ and $|k_1^{(1)}| = \max_{1 \leq i \leq n} |k_i^{(1)}|$. First, we assume that N is an odd integer. By Hölder's inequality, we have

$$\|\square_{k^{(1)}} u_1 \dots \square_{k^{(N)}} u_N\|_{L_{x_1:(x_j)_{j \neq 1},t}^{1,2}} \leq \|\square_{k^{(1)}} u_1\|_{L_{x_1:(x_j)_{j \neq 1},t}^{\infty,2}} \prod_{m=2}^N \|\square_{k^{(m)}} u_m\|_{L_{x_1:(x_j)_{j \neq 1},t}^{N-1,\infty}}.$$

$|k - k^{(1)} - \dots - k^{(N)}| \leq N + 1$ implies that $|k_1 - k_1^{(1)} - \dots - k_1^{(N)}| \leq N + 1$. We have

$$\left\| \sum_{\mathbb{B}_{1,1}^{(N)}, |k_1^{(1)}| = \max_{1 \leq m \leq N} |k_1^{(m)}| \vee \max_{1 \leq i \leq n} |k_i^{(1)}|} \square_{k^{(1)}} u_1 \dots \square_{k^{(N)}} u_N \right\|_{\ell_{\square^1}^{1,s}(L_{x_1:(x_j)_{j \neq 1},t}^{1,2})}$$

$$\lesssim \sum_{k^{(1)} \in \mathbb{Z}^n, |k_1^{(1)}| > 100} \langle k_1^{(1)} \rangle^s \|\square_{k^{(1)}} u_1\|_{L_{x_1:(x_j)_{j \neq 1}, t}^{\infty, 2}} \prod_{m=2}^N \sum_{k^{(m)} \in \mathbb{Z}^n} \|\square_{k^{(m)}} u_m\|_{L_{x_1:(x_j)_{j \neq 1}, t}^{N-1, \infty}}.$$

The case $|k_1^{(2)}| = \max(|k_1^{(1)}|, |k_1^{(2)}|)$ can be handled in an analogous way as above. Hence, we obtain the result, as desired.

Case II. $|k_1^{(1)}| = \max_{1 \leq m \leq N} |k_1^{(m)}|$ and $|k_2^{(1)}| = \max_{1 \leq i \leq n} |k_i^{(1)}|$. By Hölder's inequality, we have

$$\begin{aligned} \left\| \prod_{m=1}^N \square_{k^{(m)}} u_m \right\|_{L_{x_1:(x_j)_{j \neq 1}, t}^{1, 2}} &\leq \left\| \prod_{m=1}^{(N+1)/2} \square_{k^{(m)}} u_m \right\|_{L_{x, t}^2} \left\| \prod_{m=(N+3)/2}^N \square_{k^{(m)}} u_m \right\|_{L_{x_1:(x_j)_{j \neq 1}, t}^{2, \infty}} \\ &\leq \|\square_{k^{(1)}} u_1\|_{L_{x_2:(x_j)_{j \neq 2}, t}^{\infty, 2}} \prod_{m=2}^{(N+1)/2} \|\square_{k^{(m)}} u_m\|_{L_{x_2:(x_j)_{j \neq 2}, t}^{N-1, \infty}} \\ &\quad \times \prod_{m=(N+3)/2}^N \|\square_{k^{(m)}} u_m\|_{L_{x_1:(x_j)_{j \neq 1}, t}^{N-1, \infty}} \\ &\leq \|\square_{k^{(1)}} u_1\|_{L_{x_2:(x_j)_{j \neq 2}, t}^{\infty, 2}} \prod_{m=2}^N \|\square_{k^{(m)}} u_m\|_{\cap_{h=1, 2} L_{x_h:(x_j)_{j \neq h}, t}^{N-1, \infty}}. \end{aligned}$$

This implies that

$$\begin{aligned} &\left\| \sum_{\mathbb{B}_{1,1}^{(N)}, |k_1^{(1)}| = \max_{1 \leq m \leq N} |k_1^{(m)}|, |k_2^{(1)}| = \max_{1 \leq i \leq n} |k_i^{(1)}|} \square_{k^{(1)}} u_1 \dots \square_{k^{(N)}} u_N \right\|_{\ell_{\square^1}^{1, s}(L_{x_1:(x_j)_{j \neq 1}, t}^{1, 2})} \\ &\lesssim \sum_{k^{(1)} \in \mathbb{Z}^n, |k_2^{(1)}| > 100} \langle k_2^{(1)} \rangle^s \|\square_{k^{(1)}} u_1\|_{L_{x_2:(x_j)_{j \neq 2}, t}^{\infty, 2}} \prod_{m=2}^N \sum_{k^{(m)} \in \mathbb{Z}^n} \|\square_{k^{(m)}} u_m\|_{\cap_{h=1, 2} L_{x_h:(x_j)_{j \neq h}, t}^{N-1, \infty}}. \end{aligned}$$

Case III. $|k_1^{(1)}| = \max_{1 \leq m \leq N} |k_1^{(m)}|$ and $|k_r^{(1)}| = \max_{1 \leq i \leq n} |k_i^{(1)}|$ for some $r = 3, \dots, n$.

Using the same way as in Case II, we can deduce the result and the details are omitted.

If $|k_1^{(2)}| = \max_{1 \leq m \leq N} |k_1^{(m)}|$, one can exchange the roles of $\square_{k^{(1)}} u_1$ and $\square_{k^{(2)}} u_2$, then repeat the procedures as in the above to prove the result, as desired. The other cases $|k_1^{(r)}| = \max_{1 \leq m \leq N} |k_1^{(m)}|$ for some $r = 3, \dots, N$ are also analogous to the above cases. \square

Remark 3.2 From the proof of Lemma 3.1, it is easily verified that for $\beta = 1, \dots, n$, we have

$$\begin{aligned} &\left\| \sum_{\mathbb{B}_{\beta, 1}^{(N)}} \square_{k^{(1)}} u_1 \dots \square_{k^{(N)}} u_N \right\|_{\ell_{\square^\beta}^{1, s}(L_{x_\beta:(x_j)_{j \neq \beta}, t}^{1, 2})} \\ &\lesssim \sum_{\eta=1}^N \|u_\eta\|_{\cap_{h=1}^n \ell_{\square^h}^{1, s}(L_{x_h:(x_j)_{j \neq h}, t}^{\infty, 2})} \prod_{i=1, i \neq \eta}^N \|\square_{k^{(i)}} u_i\|_{\cap_{h=1}^n \ell_{\square^h}^1(L_{x_h:(x_j)_{j \neq h}, t}^{N-1, \infty})}. \end{aligned} \quad (3.4)$$

Lemma 3.3 *If $1 \leq p, q, p_i, q_i \leq \infty$, satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_N}, \quad \frac{1}{q} = \frac{1}{q_1} + \dots + \frac{1}{q_N}$$

and for $j = 1, \dots, n$, we have

$$\left\| \sum_{\mathbb{B}_{j,2}^{(N)}} \square_{k^{(1)}} u_1 \dots \square_{k^{(N)}} u_N \right\|_{\ell_{\square_j}^1(L_t^q L_x^p)} \lesssim \sum_{\mathbb{B}_{j,2}^{(N)}} \prod_{i=1}^N \|\square_{k^{(i)}} u_i\|_{L_t^{q_i} L_x^{p_i}}. \quad (3.5)$$

Proof. By symmetry, it suffices to consider the case $j = 1$. In view of (3.2), we easily see that $|k_1| \leq C$ in the left side of (3.5). Using Hölder's inequality, we have

$$\begin{aligned} & \left\| \sum_{\mathbb{B}_{1,2}^{(N)}} \square_{k^{(1)}} u \dots \square_{k^{(N)}} u \right\|_{\ell_{\square_1}^1(L_t^q L_x^p)} \\ & \lesssim \sum_{k \in \mathbb{Z}_1^n} \sum_{\mathbb{B}_{1,2}^{(N)}} \|\square_{k^{(1)}} u \dots \square_{k^{(N)}} u\|_{L_t^q L_x^p \chi_{|k-k^{(1)}-\dots-k^{(N)}| \leq C}} \\ & \lesssim \sum_{\mathbb{B}_{1,2}^{(N)}} \|\square_{k^{(1)}} u \dots \square_{k^{(N)}} u\|_{L_t^q L_x^p} \\ & \lesssim \sum_{\mathbb{B}_{1,2}^{(N)}} \prod_{i=1}^N \|\square_{k^{(i)}} u\|_{L_t^{q_i} L_x^{p_i}}. \end{aligned} \quad (3.6)$$

The result follows. \square

Lemma 3.4 ([41]) *Let $s \geq 0$, $1 \leq p, p_i, \gamma, \gamma_i \leq \infty$ satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_N}, \quad \frac{1}{\gamma} = \frac{1}{\gamma_1} + \dots + \frac{1}{\gamma_N}. \quad (3.7)$$

Then

$$\|u_1 \dots u_N\|_{\ell_{\square}^{1,s}(L_t^\gamma L_x^p)} \lesssim \prod_{i=1}^N \|u_i\|_{\ell_{\square}^{1,s}(L_t^{\gamma_i} L_x^{p_i})}. \quad (3.8)$$

Proof of Theorem 1.2. Since the lower order derivative nonlinear terms are easier to handle than the third order ones, one can assume that the nonlinear term takes the form

$$F((\partial_x^\alpha u)_{|\alpha| \leq 3}, (\partial_x^\alpha \bar{u})_{|\alpha| \leq 3}) = \sum_{i=1}^n \lambda_i \partial_{x_i}^3 (|u|^{\kappa_i} u).$$

Denote $\kappa = \min(\kappa_1, \dots, \kappa_n)$ and

$$\|u\|_{F_i} = \|u\|_{\ell_{\square^i}^{1,3/2+1/\kappa}(L_{x_i;(x_j)_{j \neq i,t}}^{\infty,2})}, \quad \|u\|_F = \sum_{i=1}^n \|u\|_{F_i},$$

$$\begin{aligned}\|u\|_{G_i} &= \|u\|_{\ell_{\square}^1(L_{x_i;(x_j)_{j \neq i}, t}^{\kappa, \infty})}, & \|u\|_G &= \sum_{i=1}^n \|u\|_{G_i}, \\ \|u\|_S &= \|u\|_{\ell_{\square}^{1, 1/\kappa}(L_t^\infty L_x^2 \cap L_{t,x}^{2+\kappa})}.\end{aligned}$$

Put

$$D_2 := \{u \in \mathcal{S}' : \|u\|_{D_2} := \|u\|_F + \|u\|_G + \|u\|_S \leq \delta_0\}.$$

Since $\|u\|_{D_2} = \|\bar{u}\|_{D_2}$, we can replace $|u|^{\kappa_i} u$ by u^{κ_i+1} in our later proof. Considering the following mapping:

$$\mathcal{T} : u(t) \rightarrow S(t)u_0 - i\mathcal{A} \left(\sum_{i=1}^n \lambda_i \partial_{x_i}^3 u^{\kappa_i+1} \right),$$

we will show that $\mathcal{T} : D_2 \rightarrow D_2$ is a contraction mapping. Firstly, we consider $\|S(t)u_0\|_{D_2}$. For $\|u\|_F$, it suffices to control $\|\cdot\|_{F_1}$. By (2.72)–(2.74) in Theorem 2.15 we have

$$\begin{aligned}\|S(t)u_0\|_{F_1} &\lesssim \sum_{k \in \mathbb{Z}_1^n} \langle k_1 \rangle^{3/2+1/\kappa} \|\square_k D_{x_1}^{-3/2} u_0\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \sum_{k \in \mathbb{Z}_1^n} \langle k_1 \rangle^{1/\kappa} \|\square_k u_0\|_{L^2(\mathbb{R}^n)}.\end{aligned}\tag{3.9}$$

$$\|S(t)u_0\|_G + \|S(t)u_0\|_S \lesssim \sum_{k \in \mathbb{Z}^n} \langle k \rangle^{1/\kappa} \|\square_k u_0\|_{L^2(\mathbb{R}^n)}.$$

Secondly, we estimate $\|\mathcal{A}(\sum_{i=1}^n \lambda_i \partial_{x_i}^3 u^{\kappa_i+1})\|_{D_2}$. Using the frequency-uniform decomposition, we have

$$u^{\kappa_\ell+1} = \sum_{\mathbb{B}_{i,1}^{(\kappa_\ell+1)}} \square_{k^{(1)}} u \dots \square_{k^{(\kappa_\ell+1)}} u + \sum_{\mathbb{B}_{i,2}^{(\kappa_\ell+1)}} \square_{k^{(1)}} u \dots \square_{k^{(\kappa_\ell+1)}} u.\tag{3.10}$$

Using (2.75) and (2.78) in Theorem 2.15, we obtain that

$$\begin{aligned}\|\mathcal{A} \partial_{x_\ell}^3 u^{\kappa_\ell+1}\|_{F_1} &\lesssim \left\| \sum_{\mathbb{B}_{1,1}^{(\kappa_\ell+1)}} \square_{k^{(1)}} u \dots \square_{k^{(\kappa_\ell+1)}} u \right\|_{\ell_{\square}^{1, 3/2+1/\kappa} \left(L_{x_1;(x_j)_{j \neq 1}, t}^{1,2} \right)} \\ &\quad + \left\| \sum_{\mathbb{B}_{1,2}^{(\kappa_\ell+1)}} \square_{k^{(1)}} u \dots \square_{k^{(\kappa_\ell+1)}} u \right\|_{\ell_{\square}^1 \left(L_{x,t}^{(2+\kappa)/(1+\kappa)} \right)} := I + II,\end{aligned}$$

where in the estimates of II , we used the fact that $k_{\max} \lesssim 100N$. Using Lemma 3.1 and 3.3 to I and II , respectively, we obtain that

$$\|\mathcal{A} \partial_{x_\ell}^3 u^{\kappa_\ell+1}\|_{F_1} \lesssim \|u\|_F (\|u\|_G + \|u\|_S)^{\kappa_\ell} + \|u\|_S^{1+\kappa_\ell}.\tag{3.11}$$

The treatment of other terms in $\|\cdot\|_F$ is similar. Hence, we obtain that

$$\left\| \left(\mathcal{A} \left(\sum_{i=1}^n \lambda_i \partial_{x_i}^3 u^{\kappa_i+1} \right) \right) \right\|_F \lesssim \sum_{i=1}^n \left(\|u\|_F (\|u\|_G + \|u\|_S)^{\kappa_i} + \|u\|_S^{1+\kappa_i} \right). \quad (3.12)$$

For $G(\cdot)$, we have

$$\left\| \mathcal{A} \left(\sum_{j=1}^n \lambda_j \partial_{x_j}^3 u^{\kappa_j+1} \right) \right\|_G \lesssim \sum_{i=1}^n \left\| \mathcal{A} (\partial_{x_i}^3 u^{\kappa_i+1}) \right\|_{G_i}. \quad (3.13)$$

By symmetry of G_1, \dots, G_n , it suffices to consider the estimate of $\|\cdot\|_{G_1}$.

$$\|v\|_{G_1} \leq \left(\sum_{k \in \mathbb{Z}^n, k_{\max} > 100} + \sum_{k \in \mathbb{Z}^n, k_{\max} \leq 100} \right) \|\square_k v\|_{L_{x_1; (x_j)_{j \neq 1}, t}^{\kappa, \infty}} := \Gamma_1(v) + \Gamma_2(v). \quad (3.14)$$

Using (2.79) in Theorem 2.15 and then applying Lemma 3.4 we obtain that

$$\begin{aligned} \Gamma_2 \left(\mathcal{A} \left(\sum_{i=1}^n \lambda_i \partial_{x_i}^3 u^{\kappa_i+1} \right) \right) &\lesssim \sum_{i=1}^n \sum_{k^{(1)}, \dots, k^{(\kappa_i+1)} \in \mathbb{Z}^n} \left\| \square_{k^{(1)}} u \dots \square_{k^{(\kappa_i+1)}} u \right\|_{L_{t,x}^{(2+\kappa_i)/(1+\kappa_i)}} \\ &\lesssim \sum_{i=1}^n \|u\|_S^{\kappa_i+1}. \end{aligned} \quad (3.15)$$

For Γ_1 , we have

$$\begin{aligned} \Gamma_1(v) &\leq \left(\sum_{k \in \mathbb{Z}_1^n} + \dots + \sum_{k \in \mathbb{Z}_n^n} \right) \|\square_k v\|_{L_{x_1; (x_j)_{j \neq 1}, t}^{\kappa, \infty}} \\ &:= \Gamma_1^1(v) + \dots + \Gamma_1^n(v). \end{aligned} \quad (3.16)$$

Collecting the decomposition (3.10), (2.76) in Theorem 2.15, Lemma 3.1 and 3.3, we have

$$\begin{aligned} &\Gamma_1^1 \left(\mathcal{A} \left(\sum_{i=1}^n \lambda_i \partial_{x_i}^3 u^{\kappa_i+1} \right) \right) \\ &\lesssim \sum_{i=1}^n \left\| \sum_{\mathbb{B}_{1,1}^{(\kappa_i+1)}} \square_{k^{(1)}} u \dots \square_{k^{(\kappa_i+1)}} u \right\|_{\ell_{\square^1}^{1,3/2+1/\kappa} (L_{x_1; (x_j)_{j \neq 1}, t}^{1,2})} \\ &\quad + \sum_{i=1}^n \left\| \sum_{\mathbb{B}_{1,2}^{(\kappa_i+1)}} \square_{k^{(1)}} u \dots \square_{k^{(\kappa_i+1)}} u \right\|_{\ell_{\square^1}^1 (L_{t,x}^{(2+\kappa_i)/(1+\kappa_i)})} \\ &\lesssim \sum_{i=1}^n \left(\|u\|_F (\|u\|_G + \|u\|_S)^{\kappa_i} + \|u\|_S^{1+\kappa_i} \right) .. \end{aligned} \quad (3.17)$$

Using (2.76), we can change the estimate of $\Gamma_1^i(\cdot)$ for any $2 \leq i \leq n$ to a similar estimate as in $\Gamma_1^1(\cdot)$ and there hold the same upper bound as in (3.17). Now we estimate $\|\mathcal{A}\partial_{x_i}^3 u^{\kappa_i+1}\|_S$. Combining (2.18) with Lemma 2.4, we have

$$\begin{aligned} \|\square_k \partial_{x_i}^3 \mathcal{A} \partial_{x_i}^3 f\|_{L_t^\infty L_x^2 \cap L_{t,x}^{2+\kappa}} &\lesssim \|\square_k \partial_{x_i}^3 f\|_{L_{t,x}^{(2+\kappa)/(1+\kappa)}} \\ &\lesssim \langle k_i \rangle^3 \|\square_k f\|_{L_{t,x}^{(2+\kappa)/(1+\kappa)}}. \end{aligned} \quad (3.18)$$

Noticing that

$$\|v\|_S \leq \left(\sum_{k \in \mathbb{Z}^n, k_{\max} > 100} + \sum_{k \in \mathbb{Z}^n, k_{\max} \leq 100} \right) \langle k \rangle^{1/\kappa} \|\square_k v\|_{L_t^\infty L_x^2 \cap L_{t,x}^{2+\kappa}} := \|v\|_{S_1} + \|v\|_{S_2}.$$

From (3.18) and Hölder's inequality, we have

$$\begin{aligned} \left\| \left(\mathcal{A} \left(\sum_{i=1}^n \lambda_i \partial_{x_i}^3 u^{\kappa_i+1} \right) \right) \right\|_{S_2} &\leq \sum_{i=1}^n \sum_{k^{(1)}, \dots, k^{(\kappa_i+1)} \in \mathbb{Z}^n} \|\square_{k^{(1)}} u \dots \square_{k^{(\kappa_i+1)}} u\|_{L_{t,x}^{(2+\kappa)/(1+\kappa)}} \\ &\lesssim \sum_{i=1}^n \|u\|_S^{\kappa_i+1}. \end{aligned} \quad (3.19)$$

For $\|\cdot\|_{S_1}$, we have

$$\|v\|_{S_1} = \left(\sum_{k \in \mathbb{Z}_1^n} + \dots + \sum_{k \in \mathbb{Z}_n^n} \right) \langle k \rangle^{1/\kappa} \|\square_k v\|_{L_t^\infty L_x^2 \cap L_{t,x}^{2+\kappa}} := L_1(v) + \dots + L_n(v).$$

It suffices to consider $L_1(\cdot)$. Collecting (3.10), (3.18) and (2.77), we have

$$\begin{aligned} L_1(\mathcal{A}\partial_{x_1}^3 u^{\kappa_1+1}) &\lesssim \left\| \sum_{\mathbb{B}_{1,1}^{(\kappa_1+1)}} \square_{k^{(1)}} u \dots \square_{k^{(\kappa_1+1)}} u \right\|_{\ell_{\square 1}^{1,3/2+1/\kappa} (L_{x_1;(x_j)_{j \neq 1},t}^{1,2})} \\ &\quad + \left\| \sum_{\mathbb{B}_{1,2}^{(\kappa_1+1)}} \square_{k^{(1)}} u \dots \square_{k^{(\kappa_1+1)}} u \right\|_{\ell_{\square 1}^1 (L_{t,x}^{(2+\kappa)/(1+\kappa)})}. \end{aligned}$$

Using Lemma 3.1 and 3.3, we always have

$$L_1(\mathcal{A}\partial_{x_1}^3 u^{\kappa_1+1}) \lesssim \|u\|_F (\|u\|_G + \|u\|_S)^{\kappa_1} + \|u\|_S^{1+\kappa_1}. \quad (3.20)$$

Similarly, we have

$$L_1(\mathcal{A}\partial_{x_2}^3 u^{\kappa_2+1}) \lesssim \left\| \sum_{\mathbb{B}_{1,1}^{(\kappa_2+1)}} \square_{k^{(1)}} u \dots \square_{k^{(\kappa_2+1)}} u \right\|_{\ell_{\square 1}^{1,3/2+1/\kappa} (L_{x_1;(x_j)_{j \neq 1},t}^{1,2})}$$

$$+ \left\| \sum_{\mathbb{B}_{1,2}^{(\kappa_2+1)}} \square_{k(1)} u \dots \square_{k(\kappa_2+1)} u \right\|_{\ell_{\square_1}^1(L_{t,x}^{(2+\kappa)/(1+\kappa)})},$$

and

$$L_1(\mathcal{A}\partial_{x_2}^3 u^{\kappa_2+1}) \lesssim \|u\|_F (\|u\|_G + \|u\|_S)^{\kappa_2} + \|u\|_S^{1+\kappa_2}.$$

The treatment of other terms in $\|\cdot\|_S$ is similar. Therefore, we have shown that

$$\|\mathcal{T}u\|_{D_2} \lesssim \|u_0\|_{M_{2,1}^{1/\kappa}} + \sum_{i=1}^n \|u\|_{D_2}^{1+\kappa_i}. \quad (3.21)$$

Hence, Theorem 1.2 holds by a standard contraction mapping argument. \square

4 Proof of Theorem 1.1

We follow some ideas in the proof of Theorem 1.2 to show Theorem 1.1. Since the nonlinearity contains the general terms $(\partial_x^\alpha u)^\beta$ with $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| \leq 3$, $m+1 \leq |\beta| < \infty$, we choose the following space D as a resolution space:

$$D := \left\{ u \in \mathcal{S}' : \|u\|_D := \sum_{\alpha=0,3} \sum_{l=1}^3 \sum_{i,j=1}^n \varrho_l^{(i)}(\partial_{x_j}^\alpha u) \leq \delta \right\}.$$

where

$$\varrho_1^{(i)}(u) = \|u\|_{\ell_{\square^i}^{1,3/2+1/m}(L_{x_i;(x_j)_{j \neq i},t}^{\infty,2})},$$

$$\varrho_2^{(i)}(u) = \|u\|_{\ell_{\square^i}^1(L_{x_i;(x_j)_{j \neq i},t}^{m,\infty})},$$

$$\varrho_3^{(i)}(u) = \|u\|_{\ell_{\square^i}^{1,1/m}(L_t^\infty L_x^2 \cap L_{t,x}^{2+m})}.$$

However, Comparing with the estimates we have established, we hope the space as following:

$$D := \left\{ u \in \mathcal{S}' : \|u\|_D := \sum_{\alpha=0,3} \sum_{l=1}^3 \sum_{i,j=1}^n \varrho_l^{(i)}(\partial_{x_j}^\alpha u) \leq \delta \right\}.$$

However, using Lemma 2.4, Remark ?? and Sobolev imbedding theorem, we have

Since $\|u\|_D = \|\bar{u}\|_D$, we can assume that

$$\begin{aligned} F((\partial_x^\alpha u)_{|\alpha| \leq 3}, (\partial_x^\alpha \bar{u})_{|\alpha| \leq 3}) &= F((\partial_x^\alpha u)_{|\alpha| \leq 3}) \\ &= \sum_{m+1 \leq |\beta| < \infty} c_{\beta_0, \beta_1, \beta_2, \beta_3} u^{\beta_0} (\partial_x^{\alpha_1} u)^{\beta_1} (\partial_x^{\alpha_2} u)^{\beta_2} (\partial_x^{\alpha_3} u)^{\beta_3}, \end{aligned}$$

where $|\beta| = \beta_0 + |\beta_1| + |\beta_2| + |\beta_3|$, β_i , α_i are multi-indices and $|\alpha_i| = i$ ($i = 1, 2, 3$). For simplicity, we denote

$$u^{\beta_0} (\partial_x^{\alpha_1} u)^{\beta_1} (\partial_x^{\alpha_2} u)^{\beta_2} (\partial_x^{\alpha_3} u)^{\beta_3} := v_1 \dots v_R$$

Now we give the proof of Theorem 1.1. Considering the following mapping:

$$\mathcal{T} : u(t) \rightarrow S(t)u_0 - i\mathcal{A}F((\partial_x^\alpha u)_{|\alpha| \leq 3}, (\partial_x^\alpha \bar{u})_{|\alpha| \leq 3}),$$

we will show that $\mathcal{T} : D \rightarrow D$ is a contraction mapping.

First, by Theorem 2.15 we have

$$\|S(t)u_0\|_D \lesssim \|u_0\|_{M_{2,1}^{3+1/m}}.$$

Secondly, for the estimates of $\varrho_1^{(i)}(\mathcal{A}\partial_{x_j}^\alpha(\cdot))$, $i, j = 1, \dots, n$, it suffices to estimate $\varrho_1^{(1)}(\mathcal{A}\partial_{x_1}^\alpha(\cdot))$. Indeed, by Lemma (2.16),

$$\varrho_1^{(1)}(\mathcal{A}\partial_{x_2}^\alpha(\cdot)) \lesssim \varrho_1^{(1)}(\mathcal{A}\partial_{x_1}^\alpha(\cdot)).$$

Using frequency-uniform decomposition, we have

$$\square_k(v_1 \dots v_R) = \sum_{\mathbb{B}_{i,1}^{(R)}} \square_k(\square_{k^{(1)}}v_1 \dots \square_{k^{(R)}}v_R) + \sum_{\mathbb{B}_{i,2}^{(R)}} \square_k(\square_{k^{(1)}}v_1 \dots \square_{k^{(R)}}v_R). \quad (4.1)$$

In view of (2.13) and (2.84), Lemma 3.1 and 3.3, we obtain that

$$\begin{aligned} \varrho_1^{(1)}(\mathcal{A}\partial_{x_1}^\alpha(v_1 \dots v_R)) &\lesssim \left\| \sum_{\mathbb{B}_{1,1}^{(R)}} \square_{k^{(1)}}v_1 \dots \square_{k^{(R)}}v_R \right\|_{\ell_{\square_1}^{1,3/2+1/m}(L_{x_1;(x_j)_{j \neq 1},t}^{1,2})} \\ &\quad + \left\| \sum_{\mathbb{B}_{1,2}^{(R)}} \square_{k^{(1)}}v_1 \dots \square_{k^{(R)}}v_R \right\|_{\ell_{\square_1}^1(L_{t,x}^{(R+1)/R})} \\ &\lesssim \|u\|_D^R. \end{aligned} \quad (4.2)$$

Thirdly, We estimate $\varrho_2^{(1)}(\mathcal{A}(v_1 \dots v_R))$ and $\varrho_3^{(1)}(\mathcal{A}(v_1 \dots v_R))$. In view of (2.80) and (2.85),

$$\sum_{j=2,3} \varrho_j^{(1)}(\mathcal{A}(v_1 \dots v_R)) \lesssim \|v_1 \dots v_R\|_{\ell_{\square}^{1,1/m}(L_{t,x}^{(2+m)/(1+m)})}. \quad (4.3)$$

Using Lemma 3.4, we see that the right hand side of (4.3) can be bounded by

$$\|v_1 \dots v_R\|_{\ell_{\square}^{1,1/m}(L_{t,x}^{(2+m)/(1+m)})} \lesssim \prod_{i=1}^R \varrho_3^{(1)}(v_i) \leq \|u\|_D^R. \quad (4.4)$$

Fourthly, we consider $\varrho_2^{(1)}(\mathcal{A}\partial_{x_1}^3(v_1\dots v_R))$.

$$\begin{aligned} & \varrho_2^{(1)}(\mathcal{A}\partial_{x_1}^3(v_1\dots v_R)) \\ & \lesssim \left(\sum_{k \in \mathbb{Z}^n, k_{\max} > 100} + \sum_{k \in \mathbb{Z}^n, k_{\max} \leq 100} \right) \|\square_k \mathcal{A}\partial_{x_1}^3(v_1\dots v_R)\|_{L_{x_1:(x_j)_{j \neq 1}, t}^{m, \infty}} \\ & := III + IV. \end{aligned} \quad (4.5)$$

By (2.85) and Lemma 3.4, we have

$$IV \lesssim \|v_1\dots v_R\|_{\ell_{\square}^1(L_{t,x}^{(2+m)/(1+m)})} \lesssim \|u\|_D^R. \quad (4.6)$$

It is easy to see that

$$\begin{aligned} III & \lesssim \left(\sum_{k \in \mathbb{Z}_1^n} + \dots + \sum_{k \in \mathbb{Z}_n^n} \right) \|\square_k \mathcal{A}\partial_{x_1}^3(v_1\dots v_R)\|_{L_{x_1:(x_j)_{j \neq 1}, t}^{m, \infty}} \\ & := \Upsilon_1(u) + \dots + \Upsilon_n(u). \end{aligned} \quad (4.7)$$

Using the same way as in the proof of Theorem 1.2, we have

$$\varrho_2^{(1)}(\mathcal{A}\partial_{x_1}^3(v_1\dots v_R)) \lesssim \|u\|_D^R. \quad (4.8)$$

Using (2.76), we see the estimate of $\rho_2^{(1)}(\mathcal{A}\partial_{x_i}^3(v_1\dots v_R))$ ($i = 2, \dots, n$) proceeds in the same way as the above. Finally, we consider $\varrho_3^{(1)}(\mathcal{A}\partial_{x_i}^3(v_1\dots v_R))$, as before, it suffices to consider $i = 1$. By Lemma 2.4, (2.18), we see that

$$\|\square_k \mathcal{A}\partial_{x_1}^3 f\|_{L_t^\infty L_x^2 \cap L_{t,x}^{2+m}} \lesssim \langle k_1 \rangle^3 \|\square_k f\|_{L_{t,x}^{(2+m)/(1+m)}}. \quad (4.9)$$

Hence, in view of (2.83) and (2.77), $\varrho_3^{(1)}(\mathcal{A}\partial_{x_1}^3(v_1\dots v_R))$ can be estimated by an analogous way as $\|u\|_S$ in Theorem 1.2. Collecting the estimates as in the above, we have shown that

$$\|\mathcal{T}u\|_D \lesssim \|u_0\|_{M_{2,1}^{3+1/m}} + \sum_{m+1 \leq R \leq M+1} \|u\|_D^R. \quad (4.10)$$

Therefore, the desired result holds by a standard contracting mapping argument.

5 The Case $\varepsilon = 0$

Our method can also treat the (1.1) and (1.10) when $\varepsilon = 0$. The case $\varepsilon = 0$ and $\varepsilon > 0$ has an essential difference in the lower frequency. Roughly speaking, the $L^1 \rightarrow L^\infty$ decay for the semi-group $S(t)$ is $t^{-n/2}$ in the lower frequency, however, the $L^1 \rightarrow L^\infty$ decay for the the semi-group $e^{it\Delta^2}$ is much slower than that of $S(t)$, which is only $t^{-n/4}$ in the lower frequency. So, we need to modify the Strichartz estimates, maximal function estimates in the case $\varepsilon = 0$. Denote

$$S_0(t) = e^{it\Delta^2} = \mathcal{F}^{-1} e^{-it|\xi|^4} \mathcal{F}, \quad \mathcal{A}_0 f(t, x) = \int_0^t S_0(t - \tau) f(\tau, x) d\tau.$$

5.1 Smoothing effect estimates

The treatments of the smoothing effect estimates for $S_0(t)$ and \mathcal{A}_0 are easier than those of $S(t)$ and \mathcal{A} . Noticing that only the smoothing effect estimates in the higher frequency are useful in this paper, there are no essential difference between $S_0(t)$ and $S(t)$ (also between \mathcal{A}_0 and \mathcal{A}) if we consider the smoothing effect estimates in higher frequency.

Proposition 5.1 *For any $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, $|k_i| = k_{\max} \geq 100$, $i = 1, \dots, n$. We have*

$$\left\| \square_k D_{x_i}^{3/2} S_0(t) u_0 \right\|_{L_{x_i; (x_j)_{j \neq i}, t}^{\infty, 2}} \lesssim \|\square_k u_0\|_{L^2}, \quad (5.1)$$

$$\left\| \square_k \partial_{x_i}^3 \mathcal{A}_0 f \right\|_{L_{x_i; (x_j)_{j \neq i}, t}^{\infty, 2}} \lesssim \|\square_k f\|_{L_{x_i; (x_j)_{j \neq i}, t}^{1, 2}}, \quad (5.2)$$

$$\left\| \square_k \partial_{x_i}^3 \mathcal{A}_0 f \right\|_{L_t^\infty L_x^2} \lesssim \langle k_{\max} \rangle^{3/2} \|\square_k f\|_{L_{x_i; (x_j)_{j \neq i}, t}^{1, 2}}. \quad (5.3)$$

5.2 Strichartz estimates

It is known that $S_0(t)$ satisfies the following $L^p - L^{p'}$ estimate:

$$\|S_0(t)f\|_p \lesssim |t|^{-n(1-2/p)/4} \|f\|_{p'}, \quad |t| \geq 1; \quad 2 \leq p \leq \infty, \quad (5.4)$$

Using an analogous procedure as in [42], we have

$$\|\square_k S_0(t)f\|_p \lesssim \sum_{|\ell|_\infty \leq 1} \|\square_{k+\ell} f\|_{p'}, \quad 2 \leq p \leq \infty. \quad (5.5)$$

Combining (5.4) with (5.5), we have

$$\|\square_k S_0(t)f\|_p \lesssim (1 + |t|)^{-n(1-2/p)/4} \sum_{|\ell|_\infty \leq 1} \|\square_{k+\ell} f\|_{p'}, \quad 2 \leq p \leq \infty. \quad (5.6)$$

Using (5.6) and following the procedure in [41], we get the following

Lemma 5.2 *Let $2 \leq p < \infty$, $\gamma \geq 2 \vee \gamma(p)$,*

$$\frac{4}{\gamma(p)} = n \left(\frac{1}{2} - \frac{1}{p} \right). \quad (5.7)$$

Then we have

$$\begin{aligned} \|S_0(t)\varphi\|_{\ell_{\square}^1(L_t^\gamma L_x^p)} &\lesssim \|\varphi\|_{M_{2,1}}, \\ \|\mathcal{A}_0 f\|_{\ell_{\square}^1(L_t^\gamma L_x^p) \cap \ell_{\square}^1(L_t^\infty L_x^2)} &\lesssim \|f\|_{\ell_{\square}^1(L_t^{\gamma'} L_x^{p'})}. \end{aligned}$$

In particular, if $2 + 8/n \leq p < \infty$, then we have

$$\begin{aligned} \|S_0(t)\varphi\|_{\ell_{\square}^1(L_{t,x}^p)} &\lesssim \|\varphi\|_{M_{2,1}}, \\ \|\mathcal{A}_0 f\|_{\ell_{\square}^1(L_{t,x}^p) \cap \ell_{\square}^1(L_t^\infty L_x^2)} &\lesssim \|f\|_{\ell_{\square}^1(L_{t,x}^{p'})}. \end{aligned}$$

5.3 Maximal function estimates

Recall that the maximal function estimates in Proposition 2.10 rely upon the decay speed of the semi-group $S(t)$. For the maximal function estimate of the semi-group $S_0(t)$, one can use the same way as in Proposition 2.10 in the higher frequency, since the time decays of $S(t)$ and $S_0(t)$ in the higher frequency are the same ones like $t^{-n/2}$ and both of them gain n -order derivatives. However, the decays of $S(t)$ and $S_0(t)$ in the lower frequency are $t^{-n/2}$ and $t^{-n/4}$, respectively. Similar to (2.27), if $|k| \leq C$ and $|x_1| \leq C|t| + 1$, using the time decay of $S_0(t)$, we can only obtain that

$$|\square_k S_0(t) \mathcal{F}^{-1} \sigma_k| \lesssim (1 + |t|)^{-n/4} \lesssim (1 + |x_1|)^{-n/4}. \quad (5.8)$$

So, in order to guarantee that the right hand side of (5.8) belongs to $L^{q/2}$, one needs the condition $q > 8/n$.

Proposition 5.3 *Let $8/n < q \leq \infty$, $q \geq 2$ and $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, we have*

$$\|\square_k S_0(t) u_0\|_{L^{q, \infty}_{x_i; (x_j)_{j \neq i}, t}} \lesssim \langle k \rangle^{1/q} \|\square_k u_0\|_{L^2}. \quad (5.9)$$

Proposition 5.4 *For $2 \leq q \leq \infty$, $q > 8/n$ and $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, $|k_i| = k_{\max} \geq 100$, $i = 1, \dots, n$, we have*

$$\|\square_k \partial_{x_i}^3 \mathcal{A} f\|_{L^{q, \infty}_{x_h; (x_j)_{j \neq h}, t}} \lesssim \langle k_{\max} \rangle^{3/2+1/q} \|\square_k f\|_{L^{1,2}_{x_i; (x_j)_{j \neq i}, t}}. \quad (5.10)$$

5.4 Main linear estimates in the case $\varepsilon = 0$

Theorem 5.5 *For $8/n \leq p < \infty$, $2 \leq q < \infty$, $q > 8/n$ and $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$, $|k_i| = k_{\max}$, $h, i, \ell = 1, \dots, n$. If $k_{\max} \geq 100$, then*

$$\|\square_k \partial_{x_\ell}^3 \mathcal{A} f\|_{L^{\infty, 2}_{x_i; (x_j)_{j \neq i}, t}} \lesssim \|\square_k f\|_{L^{1,2}_{x_i; (x_j)_{j \neq i}, t}}, \quad (5.11)$$

$$\|\square_k \partial_{x_\ell}^3 \mathcal{A} f\|_{L^{q, \infty}_{x_h; (x_j)_{j \neq h}, t}} \lesssim \langle k_{\max} \rangle^{3/2+1/q} \|\square_k f\|_{L^{1,2}_{x_i; (x_j)_{j \neq i}, t}}, \quad (5.12)$$

$$\|\square_k \partial_{x_\ell}^3 \mathcal{A} f\|_{L_t^\infty L_x^2 \cap L_{t,x}^{2+p}} \lesssim \langle k_{\max} \rangle^{3/2} \|\square_k f\|_{L^{1,2}_{x_i; (x_j)_{j \neq i}, t}}. \quad (5.13)$$

$$\|\square_k \partial_{x_\ell}^3 \mathcal{A} f\|_{L^{\infty, 2}_{x_i; (x_j)_{j \neq i}, t}} \lesssim \langle k_{\max} \rangle^{3/2} \|\square_k f\|_{L_{t,x}^{(2+p)/(1+p)}}, \quad (5.14)$$

$$\|\square_k \partial_{x_\ell}^3 \mathcal{A} f\|_{L^{q, \infty}_{x_i; (x_j)_{j \neq i}, t}} \lesssim \langle k_{\max} \rangle^{3+1/q} \|\square_k f\|_{L_{t,x}^{(2+p)/(1+p)}}, \quad (5.15)$$

$$\|\square_k \mathcal{A} f\|_{L_t^\infty L_x^2 \cap L_{t,x}^{2+p}} \lesssim \|\square_k f\|_{L_{t,x}^{(2+p)/(1+p)}}. \quad (5.16)$$

5.5 Sketch Proof of Remark 1.3

Using Theorem 5.5, we can repeat the procedures as in the proof of Theorem 1.2 to show that Remark 1.3 holds for the initial data in $M_{2,1}^s$. For the initial data in H^s , one can construct the following

$$\|f\|_{\ell_{\square}^{q,s}(X)} := \left(\sum_{k \in \mathbb{Z}^n} \langle k \rangle^{sq} \|\square_k f\|_X^q \right)^{1/q}, \quad (5.17)$$

$$\|f\|_{\ell_{\square^i}^{q,s}(X)} := \left(\sum_{k \in \mathbb{Z}_i^n} \langle k_i \rangle^{sq} \|\square_k f\|_X^q \right)^{1/q}, \quad (5.18)$$

$$\|u\|_{D_q} = \sum_{\alpha=0,3} \sum_{i,l=1}^n \|\partial_{x_l}^\alpha u\|_{\ell_{\square^i}^{q,s+3/2+1/m}(L_{x_i;(x_j)_{j \neq i},t}^{\infty,2}) \cap \ell_{\square^i}^1(L_{x_i;(x_j)_{j \neq i},t}^{m,\infty}) \cap \ell_{\square^i}^{q,s+1/m}(L_t^\infty L_x^2 \cap L_{t,x}^{2+m})}. \quad (5.19)$$

Then, using the same way as in the proof of Theorem 1.2, we can show that, for any $u_0 \in M_{2,q}^{s+1/m} \cap M_{2,1}^{3+1/m}$, (1.1) has a unique solution $u \in D_q$. Noticing that $H^{3+n/2+1/m+} = M_{2,2}^{3+n/2+1/m+} \subset M_{2,1}^{3+1/m}$, we immediately obtain that (1.1) is globally well-posed in $H^{3+n/2+1/m+}$.

6 Time-decay in the case $\varepsilon = -1$

In Huo and Jia [17], they also considered the local well-posedness for the small data in $H^{3+\max(n/2,1+)}$ for the case $\varepsilon = -1$. It seems difficult to develop their local well-posedness results to global ones. The time decay in the case $\varepsilon = -1$ is a little bit complicated and very weak if we consider the $L^1 \rightarrow L^\infty$ estimates. The result of this section seems essentially known (cf. [1, 27, 14]), however, we are interested in its frequency-localized version and we consider its lower, medium and higher frequency separately. Let us consider the decay for the following fourth order Schrödinger semigroup:

$$S_{-1}(t) = e^{it(\Delta+\Delta^2)} = \mathcal{F}^{-1} e^{-it(|\xi|^4 - |\xi|^2)} \mathcal{F}.$$

Denote $P(|\xi|) = |\xi|^4 - |\xi|^2$. It is known that the decay behavior of $S_{-1}(t)$ heavily rely upon its singular points, i.e., $\nabla P(|\xi|) = 0$. Noticing that $P'(r) = 4r(r+1/\sqrt{2})(r-1/\sqrt{2})$, the singular points of $S_{-1}(t)$ are $\xi = 0$ and all of the points in the sphere $|\xi| = 1/\sqrt{2}$. It is known that, the Fourier transform for a radial f is also radial,

$$\widehat{f}(\xi) = 2\pi \int_0^\infty f(r) r^{n-1} (r|\xi|)^{-(n-2)/2} J_{\frac{n-2}{2}}(r|\xi|) dr, \quad (6.1)$$

where $J_m(r)$ is the Bessel function defined by

$$J_m(r) = \frac{(r/2)^m}{\Gamma(m+1/2)\pi^{1/2}} \int_{-1}^1 e^{irt} (1-t^2)^{m-1/2} dt, \quad m > -1/2.$$

If $m = -\frac{n-2}{2}$, $J_m(r)$ has the following property (see [21, Chapter 1, (1.5)])

$$r^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(r) = c_n \mathcal{R}(e^{ir} h(r)), \quad (6.2)$$

where h satisfies

$$|h^{(k)}(r)| \leq c_k (1+r)^{-\frac{n-1}{2}-k}. \quad (6.3)$$

It follows from (6.1) and (6.2) that

$$\widehat{f}(s) = c_n \pi \int_0^\infty f(r) r^{n-1} e^{-irs} \bar{h}(rs) dr + c_n \pi \int_0^\infty f(r) r^{n-1} e^{irs} h(rs) dr. \quad (6.4)$$

We denote by ψ a smooth cut-off function which equals 1 in the unit ball and equals 0 outside the ball $B(0, 2)$. $\varphi = \psi(\cdot) - \psi(2\cdot)$, $\Delta_j = \mathcal{F}^{-1} \varphi(2^{-j}\cdot) \mathcal{F}$. Considering the decay of $S_{-1}(t)u_0$, we divide it into the following three parts:

$$\begin{aligned} S_{-1}(t)u_0 &= \sum_{j=-10}^{10} \Delta_j S_{-1}(t)u_0 + \sum_{j < -10} \Delta_j S_{-1}(t)u_0 + \sum_{j > 10} \Delta_j S_{-1}(t)u_0 \\ &:= P_{\sim 1} S_{-1}(t)u_0 + P_{\ll 1} S_{-1}(t)u_0 + P_{\gg 1} S_{-1}(t)u_0. \end{aligned} \quad (6.5)$$

Roughly speaking, if $j < -10$, $P(r) \sim r^2$ plays a crucial role in $\Delta_j S_{-1}(t)u_0$; for $j > 10$, $P(r) \sim r^4$ contributes the main part in $\Delta_j S_{-1}(t)u_0$. According to [14], we have

$$\|P_{\ll 1} S_{-1}(t)u_0\|_\infty \leq (1+|t|)^{-n/2} \|u_0\|_1, \quad (6.6)$$

$$\|P_{\gg 1} \Delta_j S_{-1}(t)u_0\|_\infty \leq |t|^{-n/4} \|u_0\|_1, \quad j > 10. \quad (6.7)$$

$S_{-1}(t)$ is problematic in the medium frequency. $P(r)$ has a singular point $r = r_1 := 1/\sqrt{2}$ which corresponds the sphere $|\xi| = r_1$ in the support set of $\mathcal{F}(\sum_{j=-10}^{10} \Delta_j S_{-1}(t)u_0)$. Noticing that $P''(r) = 12(r+1/\sqrt{6})(r-1/\sqrt{6})$, we see that $r_2 := 1/\sqrt{6}$ is a zero point of $P''(r) = 0$, which corresponds the sphere $|\xi| = r_2$ in the support set of $\mathcal{F}(\sum_{j=-10}^{10} \Delta_j S_{-1}(t)u_0)$. In order to handle the singularity in the sphere $|\xi| = r_1$, we perform the same techniques as the point $\xi = 0$ and make a dyadic decomposition around the sphere $|\xi| = r_1$. Let us denote $\tilde{\psi}(\xi) = \psi(2^{-10}\xi) - \psi(2^{11}\xi)$, then $P_{\sim 1} = \mathcal{F}^{-1} \tilde{\psi} \mathcal{F}$. Denote $P_k = \mathcal{F}^{-1} \varphi_k(|\xi| - r_1) \mathcal{F}$, we have

$$\sum_{j=-10}^{10} \Delta_j S_{-1}(t)u_0 = \sum_{k \in \mathbb{Z}} P_{\sim 1} P_k S_{-1}(t)u_0. \quad (6.8)$$

By Young's inequality,

$$\|P_{\sim 1} P_k S_{-1}(t)u_0\|_\infty \lesssim \left\| \mathcal{F}^{-1} \left(\tilde{\psi} \varphi_k(|\xi| - r_1) e^{it(|\xi|^4 - |\xi|^2)} \right) \right\|_\infty \|u_0\|_1. \quad (6.9)$$

Using (6.4), we have

$$\mathcal{F}^{-1} \left(\tilde{\psi} \varphi_k(|\xi| - r_1) e^{it(|\xi|^4 - |\xi|^2)} \right)$$

$$\begin{aligned}
&= c_n \pi \int_0^\infty r^{n-1} \tilde{\psi}(r) \varphi_k(r-r_1) e^{itP(r)-ir|x|} \bar{h}(r|x) dr \\
&\quad + c_n \pi \int_0^\infty r^{n-1} \tilde{\psi}(r) \varphi_k(r-r_1) e^{itP(r)+ir|x|} h(r|x) dr := A_k(|x|) + B_k(|x|), \quad (6.10)
\end{aligned}$$

Let us rewrite $A_k(s)$ as

$$\begin{aligned}
A_k(s) &= c_n \pi \left(\int_{r_1}^\infty + \int_0^{r_1} \right) r^{n-1} \tilde{\psi}(r) \varphi_k(r-r_1) e^{itP(r)-irs} \bar{h}(rs) dr \\
&:= A_k^{(1)}(s) + A_k^{(2)}(s). \quad (6.11)
\end{aligned}$$

Now we consider the estimates of $A_k^{(1)}(s)$. Noticing that $A_k^{(1)}(s) = 0$ if $k > 12$, we can assume that $k \leq 12$. By changing of variables, we have

$$A_k^{(1)}(s) = 2^k c_n \pi e^{-ir_1 s} \int_{1/2}^2 F(\rho) e^{it2^{2k} P_1(\rho)} d\rho, \quad (6.12)$$

where

$$\begin{aligned}
F(\rho) &:= (r_1 + 2^k \rho)^{n-1} \tilde{\psi}(r_1 + 2^k \rho) \varphi(\rho) \bar{h}((r_1 + 2^k \rho)s), \\
P_1(\rho) &:= (t2^{2k})^{-1} (tP(r_1 + 2^k \rho) - 2^k \rho s).
\end{aligned}$$

One easily sees that,

$$|P_1'(\rho)| = |4(r_1 + 2^k \rho)(2r_1 + 2^k \rho)\rho - s/t2^k|,$$

Let $s \gg 1$. If $s \gg t2^k$ or $s \ll t2^k$, then we have

$$|F^{(k)}(\rho)| \lesssim 1, \quad |P_1'(\rho)| \sim 1, \quad |P_1^{(k)}(\rho)| \lesssim 1, \quad \forall \rho \in [1/2, 2].$$

Integrating by part we have

$$A_k^{(1)}(s) = 2^k (t2^{2k})^{-N} c_n \pi e^{ir_1 s} \int_{1/2}^2 e^{it2^{2k} P_1(\rho)} \frac{d}{d\rho} \left(\frac{1}{P_1'(\rho)} \cdots \frac{d}{d\rho} \left(\frac{1}{P_1'(\rho)} \frac{d}{d\rho} \left(\frac{F(\rho)}{P_1'(\rho)} \right) \right) \right) d\rho. \quad (6.13)$$

It follows that

$$|A_k^{(1)}(s)| \lesssim 2^k (t2^{2k})^{-N}. \quad (6.14)$$

If $s \gg 1$, $s \sim t2^k$, by Von de Corput Lemma and (6.3) we have

$$|A_k^{(1)}(s)| \lesssim 2^k (t2^{2k})^{-1/2} \int_{1/2}^2 |\partial_\rho F(\rho)| d\rho \lesssim 2^k (t2^{2k})^{-1/2} s^{-(n-1)/2} \lesssim 2^k (t2^{2k})^{-n/2}. \quad (6.15)$$

Moreover, we easily see that $|A_k^{(1)}(s)| \lesssim 2^k$. If $s \gg 1$, then it follows from that

$$|A_k^{(1)}(s)| \lesssim 2^k \min\{1, (t2^{2k})^{-n/2}\}. \quad (6.16)$$

If $s \lesssim 1$, one can rewrite $A_k^{(1)}(s)$ as

$$A_k^{(1)}(s) = 2^k c_n \pi e^{-ir_1 s} \int_{1/2}^2 F_1(\rho) e^{itP(r_1+2^k\rho)} d\rho, \quad (6.17)$$

where

$$F_1(\rho) := (r_1 + 2^k \rho)^{n-1} \tilde{\psi}(r_1 + 2^k \rho) \varphi(\rho) \bar{h}((r_1 + 2^k \rho)s) e^{-i2^k \rho s}.$$

Integrating by part, we have

$$|A_k^{(1)}(s)| \lesssim 2^k \min\{1, (t2^{2k})^{-n/2}\}. \quad (6.18)$$

Now we estimate $A_k^{(2)}(s)$. Noticing that $r_2 \in \text{supp}\varphi_k(r_1 - \cdot) \cap \text{supp}\tilde{\psi}$ if and only if $k = -1, -2$. When $k \neq -1, -2$, one can repeat the same procedures as in the above arguments to obtain that

$$|A_k^{(2)}(s)| \lesssim 2^k \min\{1, (t2^{2k})^{-n/2}\}. \quad (6.19)$$

When $k = -1, -2$, we have $P'(r) \sim t$ for all $\text{supp}\varphi_k(r_1 - \cdot) \cap \text{supp}\tilde{\psi}$. Hence, in the cases $s \gg t$ or $s \ll t$, integrating by part, we have

$$|A_k^{(2)}(s)| \lesssim \min\{1, t^{-N}\}, \quad \forall N \in \mathbb{N}.$$

If $s \sim t$, by von de Corput Lemma,

$$|A_k^{(2)}(s)| \lesssim t^{-1/3} s^{-(n-1)/2} \lesssim t^{-n/2+1/6}.$$

So, we have for $k = -1, -2$,

$$|A_k^{(2)}(s)| \lesssim \min\{1, t^{-n/2+1/6}\}. \quad (6.20)$$

In view of (6.19) and (6.20),

$$|A_k^{(2)}(s)| \lesssim 2^k \min\{1, (t2^{2k})^{-n/2+1/6}\}. \quad (6.21)$$

Summing A_k over all $k \leq 12$, for $n \geq 2$, we have shown the following estimates:

$$\|P_{\sim 1} S_{-1}(t) u_0\|_\infty \lesssim (1 + |t|)^{-1/2} \|u_0\|_1. \quad (6.22)$$

Roughly speaking, the time decay in the higher, lower and medium frequency are $t^{-n/4}$, $\langle t \rangle^{-n/2}$ and $\langle t \rangle^{-1/2}$, respectively. Using the method in this paper, it seems very difficult to develop Huo and Jia's local solutions in the case $\varepsilon = -1$ to the global ones.

Acknowledgments

This work is supported in part by the National Natural Science Foundation of China grant and by British-CSC Scholarship. The third named author is grateful to Zihua Guo for his enlightening discussions.

References

- [1] M. Ben-Artzi, H. Koch and J. C. Saut, Dispersion estimates for fourth order Schrödinger equations, *C. R. Acad. Sci. Paris Sér. I Math.*, **330** (2000), 87–92.
- [2] Á. Bényi; K. A. Okoudjou, Local well-posedness of nonlinear dispersive equations on modulation spaces. *Bull. Lond. Math. Soc.* **41** (2009), 549–558.
- [3] Á. Bényi; , K. Gröchenig; K. A. Okoudjou; L. G. Rogers, Unimodular Fourier multipliers for modulation spaces. *J. Funct. Anal.* **246** (2007), 366–384.
- [4] J. Bergh and J. Löfström, *Interpolation Spaces, An Introduction*, Springer-Verlag, 1976.
- [5] J. Chen, D. Fan and L. Sun, Asymptotic estimates for unimodular Fourier multipliers on modulation spaces. *Discrete Contin. Dyn. Syst.* **32** (2012), 467–485.
- [6] P. Constantin and J. C. Saut, Local smoothing properties of dispersive equations, *J. Amer. Math. Soc.*, **1** (1988), 413–446.
- [7] E. Cordero and F. Nicola, Metaplectic representation on Wiener amalgam spaces and applications to the Schrödinger equation. *J. Funct. Anal.* **254** (2008), 506–534.
- [8] E. Cordero and F. Nicola, Some new Strichartz estimates for the Schrödinger equation. *J. Differential Equations*, **245** (2008), 1945–1974.
- [9] E. Cordero and F. Nicola, Remarks on Fourier multipliers and applications to the wave equation. *J. Math. Anal. Appl.*, **353** (2009), 583–591.
- [10] K. B. Dysthe, Note on a modification to the nonlinear Schrödinger equation for application to deep water waves, *Proc. R. Soc. Lond.*, **A369** (1979) 105–114.
- [11] H. G. Feichtinger, Modulation spaces on locally compact Abelian group, Technical Report, University of Vienna, 1983.
- [12] G. Fibich, B. Ilan and G. Papanicolaou, Self-focusing with fourth order dispersion. *SIAM J. Appl. Math.* **62** (2002), 1437–1462.
- [13] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Birkhäuser, 2001.
- [14] Z. Guo, L. Peng and B. Wang, Decay estimates for a class of wave equations, *J. Funct. Anal.* **254** (2008), 1642–1660.
- [15] C. C. Hao, L. Hsiao and B. X. Wang, Wellposedness of the fourth order nonlinear Schrödinger equations in multi-dimension spaces, *J. Math. Anal. Appl.*, **328**(1) (2007) 58–83.
- [16] Z. H. Huo and Y. L. Jia, The Cauchy problem for the fourth-order nonlinear Schrödinger equation related to the vortex filament, *J. Diff. Eqns.*, **214** (2005) 1–35.
- [17] Z. H. Huo and Y. L. Jia, Well-posedness for the fourth-order nonlinear derivative Schrödinger equation in higher dimension, *J. Math. Pures Appl.*, **96** (2011), 190–206.
- [18] A. D. Ionescu and C. E. Kenig, Global well-posedness of the Benjamin-Ono equation in low-regularity spaces. *J. Amer. Math. Soc.* **20** (2007), no. 3, 753–798
- [19] A. Ionescu and C. E. Kenig, Low-regularity Schrödinger maps, II:Global well posedness in dimension $d \geq 3$, *Commun. Math. Physics.*, **271** (2007), 523–559.
- [20] T. Iwabuchi, Navier-Stokes equations and nonlinear heat equations in modulation spaces with negative derivative indices. *J. Differential Equations* **248** (2010), 1972C2002.
- [21] F. John, *Plane Waves and Spherical Means. Applied to partial differential equations*, Springer, 1981.
- [22] V. I. Karpman, Stabilization of soliton instabilities by higher-order dispersion: fourth order nonlinear Schrödinger-type equations, *Phys. Rev. E*, **53** (2) (1996) R1336–R1339.

- [23] V. I. Karpman and A. G. Shagalov, Stability of soliton described by nonlinear Schrödinger-type equations with higher-order dispersion, *Phys. D.* **144** (2000) 194–210.
- [24] K. Kato, M. Kobayashi and S. Ito, Representation on Schrödinger operator of a free particle via short time Fourier transform and its applications. *Tohoku Math. J.*, **64** (2012), 223–231.
- [25] K. Kato, M. Kobayashi and S. Ito, Estimates on modulation spaces for Schrödinger evolution operators with quadratic and sub-quadratic potentials. *J. Funct. Anal.* **266** (2014), 733–753.
- [26] T. Kato, The global Cauchy problems for the nonlinear dispersive equations on modulation spaces. *J. Math. Anal. Appl.* **413** (2014), 821–840.
- [27] C. E. Kenig, G. Ponce and L. Vega, Oscillatory integrals and regularity of dispersive equations, *Indiana Univ. Math. J.*, **40** (1991) 33–69.
- [28] C. E. Kenig, G. Ponce and L. Vega, Small solutions to nonlinear Schrödinger equations, *Ann. Inst. Henri Poincaré, Nonlinear Anal.*, **10** (1993) 255–288.
- [29] F. Linares and G. Ponce, On the Davey-Stewartson systems, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **10** (1993), 523–548.
- [30] B. Pausader, Global well-posedness for energy critical fourth-order Schrödinger equations in the radial case, *Dynamics of PDE*, **4** (3), (2007), 197–225.
- [31] F. Nicola, Phase space analysis of semilinear parabolic equations. *J. Funct. Anal.* **267** (2014), no. 3, 727–743.
- [32] L.S. Da Rios, On the motion of an unbounded fluid with a vortex filament of any shape, *Rend. Circ. Mat. Palermo.* 22 (1906) 117C135 (in Italian).
- [33] J. Segata, Well-posedness for the fourth order nonlinear Schrödinger type equation related to the vortex filament, *Diff. Integral Eqs.*, **16** (7) (2003) 841–864.
- [34] J. Segata, Remark on well-posedness for the fourth order nonlinear Schrödinger type equation, *Proc. Amer. Math. Soc.*, **132** (2004) 3559–3568.
- [35] M. Sugimoto and N. Tomita, The dilation property of modulation spaces and their inclusion relation with Besov spaces, *J. Funct. Anal.*, **248** (2007), 79–106.
- [36] D. Tataru, Local and global results for wave maps. I, *Comm. Partial Differential Equations*, **23** (1998), 1781–1793.
- [37] H. Triebel, *Theory of Function Spaces*, Birkhäuser-Verlag, 1983.
- [38] J. Toft, Continuity properties for modulation spaces, with applications to pseudo-differential calculus, *J. Funct. Anal.*, **207** (2004), 399–429.
- [39] B. X. Wang, L. J. Han and C. Y. Huang, Global smooth effects and well-posedness for the derivative nonlinear Schrödinger equation with small rough data, *Ann. Inst. H. Poincaré, AN*, **26** (2009), 2253–2281.
- [40] B. X. Wang and C. Y. Huang, Frequency-uniform decomposition method for the generalized BO, KdV and NLS equations, *J. Diff. Eqns.*, **239** (2007), 213–250.
- [41] B. X. Wang and H. Hudzik, The global Cauchy problem for the NLS and NLKG with small rough data, *J. Diff. Eqns.*, **232** (2007) 36–73.
- [42] B. X. Wang, L. F. Zhao and B. L. Guo, Isometric decomposition operators, function spaces $E_{p,q}^\lambda$ and applications to nonlinear evolution operators, *J. Funct. Anal.*, **233** (2006) 1–39.