A consistent framework for valuation under collateralization, credit risk and funding costs

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by

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I dedicate this thesis to my family.

Abstract

We develop a consistent, arbitrage-free framework for valuing derivative trades with collateral, counterparty credit risk, and funding costs. This is achieved by modifying the payout cash-flows for the trade position. The framework is flexible enough to accommodate actual trading complexities such as asymmetric collateral and funding rates, replacement close-out, and rehypothecation of posted collateral. We show also how the traditional self-financing condition is adjusted to reflect the new market realities. The generalized valuation equation takes the form of a forward-backward SDE or semi-linear PDE. Nevertheless, it may be recast as a set of iterative equations which can be efficiently solved by our proposed least-squares Monte Carlo algorithm. We numerically implement the case of an equity option and show how its valuation changes when including the above effects. We also discuss the financial impact of the proposed valuation framework and of nonlinearity more generally. This is fourfold: Firstly, the valuation equation is only based on observable market rates, leaving the value of a derivatives transaction invariant to any theoretical risk-free rate. Secondly, the presence of funding costs and default close-out makes the valuation problem a recursive and nonlinear one. Thus, credit and funding risks are non-separable in general, and despite common practice in banks, the related CVA, DVA, and FVA cannot be treated as purely additive adjustments without running the risk of double counting. To quantify the valuation error that can be attributed to double counting, we introduce a nonlinearity valuation adjustment (NVA) and show that its magnitude can be significant under asymmetric funding rates and replacement close-out at default. Thirdly, as trading parties cannot observe each others liquidity policies nor their respective funding costs, the bilateral nature of a derivative price

breaks down. Finally, valuation becomes aggregation-dependent and portfolio values cannot simply be added up. This has operational consequences for banks, calling for a holistic, consistent approach across trading desks and asset classes.

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Notations

$\mathcal{G}_t$	Filtration that models the flow of information of the whole market.
$\mathcal{F}_t$	Default-free market filtration.
$\mathcal{H}_t$	Filtration generated by the default events.
τ_C, τ_I	Default times of the counterparty and the investor respectively.
τ	The first-to-default time, $\tau_I \wedge \tau_C$.
F_t, H_t, C_t	Funding cash account, risky-asset account and collateral account.
$\varepsilon_{C,\tau}$, $\varepsilon_{I,\tau}$	The close-out amounts on the investor's and the counterparty's default.
$\Pi(t,u)$	The sum of the discounted contractual cash-flows from t to u.
$\gamma(t,u;C)$	The sum of the discounted margining costs over the period $(t,u]$.
$\theta_\tau(C,\varepsilon)$	The on-default cash-flow.
$\varphi(t,u;F)$	The sum of the discounted funding costs over the period $(t,u]$.
λ_t	The first-to-default intensity.
c^+, c^-	The positive and negative collateral interest rates respectively.
$\tilde{c}_t(T)$	The effective funding rates from t to T, $c_t^-(T)\mathbf{1}_{\{C_t<0\}} + c_t^+(T)\mathbf{1}_{\{C_t>0\}}$.
f^+, f^-	The borrowing and lending rates respectively.
$\tilde{f}_t(T)$	The effective funding rates from t to T, $f_t^-(T)\mathbf{1}_{\{F_t<0\}} + f_t^+(T)\mathbf{1}_{\{F_t>0\}}$.
h^+, h^-	The risky asset borrowing and lending rates respectively.
$\tilde{h}_t(T)$	The effective hedging rates from t to T, $h_t^-(T)\mathbf{1}_{\{H_t<0\}} + h_t^+(T)\mathbf{1}_{\{H_t>0\}}$.
N^C, N^I	The initial margin accounts posted by the counterparty and the investor.
M_t	The variation margin account.
$f^{N,C}$, $f^{N,I}$	The funding rates of the initial margin accounts for the counterparty and the investor respectively.

Chapter 1

Introduction

Recent years have seen an unprecedented interest among banks in understanding the risks and associated costs of running a derivatives business. In the wake of the financial crisis in 2007-2008, dealers and financial institutions have been forced to rethink how they value and hedge contingent claims traded either in the over the counter (OTC) market or through central clearing house (CCPs). OTC derivatives are bilateral financial contracts negotiated between two default-risky entities. Yet, prior to the crisis, institutions tended to ignore the credit risk of high-quality rated counterparties, but as recent history has shown this was a particularly dangerous assumption. Moreover, as banks became reluctant to lend to each other with the crisis rumbling through the Western economies, the spread between the rate on overnight indexed swaps (OISs) and the LIBOR rate blew up.

To keep up with this sudden change of game, dealers today make a number of adjustments when they book OTC trades. The credit valuation adjustment (CVA) corrects the price for the expected costs to the dealer due to the possibility that the counterparty may default, while the so-called debt valuation adjustment (DVA) is a correction for the expected benefits to the dealer due to his own default risk. The latter adjustment has the controversial effect that the dealer can book a profit as his default risk increases and is very hard (if not impossible) to hedge. Finally, dealers often adjust the price for the costs of funding the trade. In the industry, this practice is known as a liquidity and funding valuation adjustment (LVA, FVA).

When a derivatives desk executes a deal with a client, it hedges the trade with other dealers in the market, posts or receives collateral, and additionally receives or pays interest on the posted collateral. This involves borrowing or lending money and other assets. Classical derivatives pricing theory rests on the assumption that one can borrow and lend at a unique risk-free rate of interest, a theoretical risk-free rate that is proxied by a number of market rates. The seminal work of Black-Scholes-Merton showed that in this case an option on equity can be replicated by a portfolio of equity and risk-free debt over any short period of time. Prior to the crisis, this assumption may have been reasonable with banks funding their hedging strategies at LIBOR. However, with drastically increasing spreads emerging as the crisis took hold, it became apparent that LIBOR is contaminated by credit risk (besides fraud risk) and as such is an imperfect proxy of the risk-free rate. While overnight rates have replaced LIBOR as proxies for the risk-free rate, it would be preferable for a pricing framework not to feature theoretical rates in the final valuation equations.

Recent headlines such as J.P. Morgan's results in January 2014 underscores the sheer importance of accounting for funding valuation adjustment. Michael Rapoport reports on January 14, 2014 in the Wall Street Journal:

"[...] So what is a funding valuation adjustment, and why did it cost J.P. Morgan Chase $1.5 billion? The giant bank recorded a $1.5 billion charge in its fourth-quarter earnings announced Tuesday because of the adjustment – the result of a complex change in J.P. Morgans approach to valuing some of the derivatives on its books. J.P. Morgan was persuaded to make the FVA [Funding Valuation Adjustment] change by an industry migration toward such a move, the bank said in an investor presentation. A handful of other large banks, mostly in the U.K. and Europe, have already made a similar change.

When dealing with funding costs, one may take a single deal (micro) or homogeneous (macro) cost view. In the micro view, funding costs are determined at deal level. This means that the trading desk may borrow funds at a different rate than at which it can invest funds, and the rates may vary across deals even in the same desk. In a slightly more aggregate cost view, average funding spreads are applied to all deals yet the spread on borrowing funds may still be different from that on lending. Finally, if we turn to the macro and symmetric view, funding costs of borrowing and lending are assumed the same and a common funding spread is applied across all deals. Clearly, the treasury department of a bank plays an active part in the micro approach and works as an operational center, while in the macro approach it takes more the role of a supporting function for the trading business. In this work we stay as general as possible and adopt a micro cost view. Naturally, the macro view is just a special case of the micro view. This will be implicit in making the otherwise exogenously assigned funding rates a function of the specific deal value. One should also notice that the specific treasury model one adopts also impacts the presence of credit risk, and in particular of DVA, on the funding policy. This effect is occasionally referred to as DVA2, but we will not adopt such terminology here.

Despite its general market acceptance, the practice of including an adjustment for funding costs has stirred quite some controversy among academics and practitioners (see the debate following Hull and White [53]). At the center of this controversy is the issue that funding-contingent pricing becomes subjective due to asymmetric information. The particular funding policy chosen by the client is not (fully) known to the dealer, and vice versa. As a result, the price of the deal may be different to either of the two parties. Theoretically, this should mean that the parties would never close the deal. However, in reality, the dealer may not be able to recoup his full funding costs from the client, yet traders say that funding risk was the key factor driving bid-ask spreads wider during the crisis.

The introduction of funding risk makes the pricing problem highly recursive and nonlinear. The price of the deal depends on the trader's funding strategies in future paths, while to determine the future funding strategies we need to know the deal price itself in future paths. This recursive structure was also discovered in the studies of Pallavicini et al. [60], Crépey [39] and Burgard and Kjaer [35], yet the feature is neglected in the common approach of adding a funding spread to the discount curve. The inherent nonlinearity manifests itself in the valuation equations by taking the form of a forward-backward stochastic differential equation (in short, FBSDE) or a semi-linear partial differential equation (in short, PDE).

In this thesis we develop an arbitrage-free framework for consistent valuation of collateralized as well as uncollateralized trades under counterparty credit risk, collateral margining and funding costs. The need to consistently account for the changed trading conditions in the valuation of derivatives is stressed by the sheer size of the OTC market. Indeed, despite the crisis and the previously neglected risks, the size of derivatives markets remains staggering, the market value of outstanding OTC derivative contracts equaled \$24.7 trillion by the end of 2012 with a whopping \$632.6 trillion in notional value (as stated in Bank for International Settlements, 2013 [4]). Adopting the risk-neutral valuation principle, we derive a general pricing equation for an OTC derivative deal where the new or previously neglected types of risks (CVA, DVA, collateral and funding costs) are included simply as modifications of the payout cash-flows. This approach can also be tailored to address trading through a central clearing house (CCP) with variation and initial margins as investigated in Brigo and Pallavicini [30]. In addition, we address the current market practices in accordance with the guidelines of the International Swaps and Derivatives Association without assuming restrictive constraints on the collateral margining procedures and close-out netting rules. In particular, we allow for asymmetric collateral and funding rates as well as exogenously given liquidity policies and hedging strategies. We also discuss rehypothecation of collateral guarantees and risk-free/replacement close-out conventions.

To explore valuation under funding costs concretely, we show how the general pricing equation can be cast as a set of iterative equations that can be conveniently solved by means of least-squares Monte Carlo (see for example, Carrier [36], Longstaff and Schwartz [58], Tilley [71] and Tsitsiklis and Van Roy [72]) and we propose an efficient simulation algorithm. Additionally, we derive a continuous-time approximation of the solution of the pricing equation as well as the associated FBSDE and semi-linear PDE. We study the existence and uniqueness problems for both the FBSDE and the semi-linear PDE cases. Moreover, we present an invariance theorem showing that the risk-free rate disappears from the funding inclusive PDE, which implies that the valuation of the trade depends no longer on some unobservable risk-free rates. In other words, valuation is purely based on observable market rates. The invariance theorem first appeared implicitly in Pallavicini et al. [60] and later in Brigo et al. [22, 23].

Valuation under funding risk poses a significantly more complex and computationally challenging problem than standard CVA and DVA computations (except for possibly CVA/DVA under replacement close-out), since it requires forward simulation and backward induction at the same time. In addition, FVA does not take the form of a simple additive term as appears to be commonly assumed by market participants. More fundamentally, this means that, by its very nature, identifying FVA with DVA is generally wrong, and only under restrictive assumptions would the two concepts collapse into one. Funding and Credit costs do not split up in a purely additive way. A consequence of this is that valuation becomes aggregation-dependent as portfolios prices do not simply add up. It is therefore difficult for banks to create CVA and FVA desks with separate and clearcut responsibilities. Nevertheless, banks often make such simplifying assumptions when accounting for the various price adjustments. This can be done, however, only at the expense of tolerating a degree of double counting in the different valuation adjustments.

In order to study such double counting, we introduce a nonlinearity valuation adjustment (in short NVA) to quantify the valuation error that one makes when treating CVA, DVA, and FVA as separate, additive terms. In particular, we examine the financial error of neglecting nonlinearities such as asymmetric borrowing and lending funding rates and substituting the replacement close-out at default by the more stylized risk-free close-out. We analyze the large scale implications of nonlinearity of the valuation equations: non-separability of risks, aggregation dependence in valuation, and local pricing measures as opposed to universal ones. Finally, our numerical results confirm that NVA and asymmetric funding rates can have a nontrivial impact on the valuation of financial derivatives. More generally, nonlinearity implies organizational challenges which we point out in the conclusion.

Literature Review In terms of available literature in this area, several studies have analysed the various valuation adjustments separately, but few have tried to build a valuation approach that consistently takes counterparty credit risk, collateralization and funding costs into account. Under unilateral default risk, i.e. when only one party is defaultable, Brigo and Masetti [24] consider valuation of derivatives with CVA, whereas particular applications of their approach are given in Brigo and Pallavicini [28], Brigo and Chourdakis [19], and Brigo et al [27]; see Brigo et al. [26] for a summary. Bilateral default risk appears in Bielecki and Rutkowski [7], Brigo and Capponi [16], Brigo et al. [31], and Gregory [51] who evaluate both the CVA and DVA of a derivative deal.

The fundamental impact of collateralization on default risk and on the credit valuation adjustment and debit valuation adjustment has been investigated in Cherubini [38] and more recently in Brigo et al. [17] and Brigo et al. [18]. The works of [17, 18] look at the CVA and DVA gap risk under several collateralization strategies, with or without rehypothecation, as a function of the margining frequency with wrong way risk and with possible instantaneous contagion. Minimum threshold amounts and minimum transfer amounts are also considered. We also cite Brigo et al. [26] for a list of frequently asked questions on the subject.

Assuming no default risk, Piterbarg [68] provides an initial analysis of derivative transactions under collateralization and funding risk in a stylized Black-Scholes economy. Yet, the introduction of collateral in a world without default risk is questionable since its main purpose is to mitigate such a risk. Moreover, the study does not consider the nonlinearities due to replacement close-out nor asymmetric funding rates. Fujii et al. [50] analyses the consequences of multi-currency features in collateral proceedings. The basic implications of funding in presence of default risk have been considered in Morini and Prampolini [59], see also Fries [49] and Castagna [37].

The above works constitute a beginning for the funding costs literature. However, these references focus only on simple financial products, such as zero-coupon bonds or loans, and do not offer the level of generality needed to include all the required features in a consistent framework that can be used to manage complex products. Thus, a general theory under the new risks is still missing. The most comprehensive attempts are those of Burgard and Kjaer [34, 35], Crépey [39–41], Pallavicini et al. [60, 61] and Brigo et al. [21–23]. Nonetheless, as [34, 35] resort to a PDE approach, their results are constrained to low dimensions. They neglect the hidden complexities of collateral modelling and mark-to-market discontinuities at default, and also do not state explicitly the funding assumptions for all the assets in their replicating portfolio. The approach in the series [39–41] is more general, using backwards stochastic differential equations, although it does not allow for credit instruments in the deal portfolio. We note that the papers [60, 61], [21–23] follow the same level of generality of Crépey [40] and address the inherent nonlinearity of the valuation problem, as well as considering different models of funding policy and still accounting for CVA, DVA collateralization and rehypothecation. Wu [74] studies the pricing problem of cash collateralized derivatives trades when credit and funding risks are present, and proposes a PDE representation for the derivatives price, which is solved as a Feynman-Kac formula. However, the author assumes that the margin account earns the CSA rate for positive balance or costs the CSA rate plus a spread for negative balance, discarding the funding benefit, which is obviously not realistic.

Brigo and Pallavicini [30] deal with the same general framework tailored to Central Counterparties Clearing (CCPs) and standard Credit Support Annex (CSA) trades with variation and initial margins. The paper Biffis et al. [9] studies longevity swaps under credit risk, collateralization and funding costs. Despite being applied to the specific and atypical asset class of longevity derivatives, the paper is one of the first to develop a comprehensive approach to an extended pricing framework addressing previously neglected risks.

Besides the above papers, as a testimony to the increasing effort in this research area, books have started to include funding costs analysis; see for example Kenyon and Stamm [56], Brigo et al. [26] and Crépey et al. [42].

In this thesis, we continue the work in [22, 23], which follows the work of [60, 61] and consider a general pricing framework for OTC deals that fully and consistently takes collateralization, counterparty credit risk, and funding risk into account. The valuation framework is conceptually simple and intuitive in contrast to previous attempts. It is based on the celebrated risk-neutral valuation principle and the new risks are included simply by adjusting the payout cash-flows of the deal. The valuation equation takes the form of an FBSDE or a semi-linear PDE. We show that the traditional self-financing condition can be adjusted to address the new market realities. We present a numerical case study that extends the benchmark theory of Black-Scholes for equity call options to credit gap risk (CVA/DVA after collateralization), collateralization and funding costs. We find that the precise patterns of funding-inclusive values depend on a number of factors, including the asymmetry between borrowing and lending rates. We stress such inputs in order to analyse their impact on the funding inclusive price. Our numerical results confirm that funding risk impacts non-trivially the deal price and that nonlinearity valuation adjustment can be relevant as well.

To summarize, the financial implications of our valuation framework are fourfold:

- Valuation is invariant to any theoretical risk-free rate and only based on observable market rates.
- Valuation is a nonlinear problem under asymmetric funding and replacement close-out at default, making funding and credit risks non-separable.
- Valuation is no longer bilateral because counterparties cannot observe each others liquidity policies nor their respective funding costs.
- Valuation is aggregation-dependent and portfolio values can no longer simply be added up.

1.1 Specific contribution

In the following, we briefly list the main contributions of the thesis.

1. Numerical case studies of the consistent valuation framework. This work can be found in Chapter 5 and was originally carried out in Brigo et al. [21, 23].
2. Introduction and numerical study of the nonlinearity valuation adjustment. This work can be found in Chapter 5 and was carried out also in Brigo et al. [22].

3. Analysis of the self-financing condition of the new funding inclusive valuation framework. The related work can be found in Chapter 4. A common mistake in the literature and the correct formulation can be found also in Brigo et al. [14].

4. Analysis of the formulation of the funding inclusive forward-backward SDE and semi-linear PDE and the existence and uniqueness problem of their solutions. The related work was conducted in Chapter 3. A preprint associated to this research is in preparation.

1.2 Credit and debit valuation adjustment

In this section, we introduce a fundamental valuation framework for over the counter (OTC) trades including the counterparty risk valuation adjustment, which forms the foundation of the study we are going to conduct when more market realities are introduced.

Due to the large amount of financial contracts that are traded over the counter, the importance of the credit quality of a counterparty is fundamental, and counterparty credit risk is introduced when evaluating a derivative contract. Basel II defines the *counterparty credit risk* as the risk that a counterparty in a financial contract will default prior to the final settlement of the transaction and fail to make the future obligatory payments. When investing in default risky assets, market participants will charge a risk premium to account for the counterparty credit risk. As a result, the value of a contingent claim with a defaultable counterparty will be smaller than the value of the same claim with a non-defaultable counterpary. Our goal in this section is to discuss how to quantify such a difference, and introduce a general arbitrage-free valuation framework taking into account the counterparty credit risk.

Probabilistic Framework We postulate the following probabilistic assumption throughout the section. Let $T \in \mathbb{R}^+$ be the expiry time of the derivative deal. Consider a filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{Q})$, where $\mathbb{Q}$ is the risk-neutral probability measure, Ω represents the set of all possible outcomes of the random experiment, and the σ-algebra $\mathcal{G}$ represents the set of events $A \in \Omega$. The filtration $(\mathcal{G}_t)_{t\in[0,T]}$ models the flow of information of the whole market up to time t, including default. The default time τ is defined on this probability space and is a $\mathcal{G}$-stopping time. This space is endowed with a right-continuous and complete sub-filtration $(\mathcal{F}_t)_{t\in[0,T]}$ called the default-free market filtration, which represents all the observable market quantities except for default events. Therefore, we have $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ with $\mathcal{H}_t = \sigma(\{\tau \le u\}, u \le t)$ being the right-continuous filtration generated by the default events. Moreover, we adopt the notational convention that $\mathbb{E}_t$ is the risk-neutral expectation conditional on the filtration $\mathcal{G}_t$.

1.2.1 Unilateral counterparty risk

Let's first consider the valuation problem where only one of the two parties is defaultable. Taking the viewpoint of the default-free party, the problem is to compute the adjustment to the default-free price of the deal when entering a financial trade with a counterparty that has a positive probability of defaulting before the maturity of the trade. Such adjustment is called *unilateral credit valuation adjustment* (in short UCVA). UCVA has been studied for example by Sorensen and Bollier in [70] and by Bielecki and Rutkowski in [7]. Brigo and Masetti in [24] considered pricing with UCVA under netting, whereas Cherubini [38] discussed UCVA with collateral in some stylized cases.

General pricing formula Consider the case when the investor is default-free. At the time of default $\tau < T$, the investor will calculate the *net present value* (in short NPV) being the residual value of the deal until maturity. If the NPV is negative to the investor, the investor will pay in full the NPV to the defaulted conterparty. If the NPV is positive to the investor, the investor will face a loss and receive only a recovery fraction of the NPV, denoted as R. In the case where no default happens, i.e. $\tau > T$, the derivative trade is the same as in the default-free case.

We denote by $\Pi(t, T)$ the sum of the discounted (at the risk-free rate) payoff at time $t < T$ happening over the time period $(t, T]$. We have immediately

$$NPV(\tau) = \mathbb{E}_\tau[\Pi(\tau, T)].$$

The defaultable derivative price denoted as $\bar{V}_t$ at time $t < \tau$ can be calculated as

$$\bar{V}_t = \mathbb{E}_t \left\{ \mathbf{1}_{\{\tau > T\}} \Pi(t, T) + \mathbf{1}_{\{\tau < T\}} \left[\Pi(t, \tau) + D(t, \tau) \Big(R(NPV(\tau))^+ - (-NPV(\tau))^+ \Big) \right] \right\},$$

where we use the short hand nations $\mathcal{X}^+ := \max\{\mathcal{X}, 0\}$ and $\mathcal{X}^- := \min\{\mathcal{X}, 0\}$. $D(t, u)$ is the risk-free (and funding-free) discount factor, given by the ratio

$$D(t, u) = \frac{B_t}{B_u}, \tag{1.1}$$

where

$$\mathrm{d}B_t = r_t B_t dt$$

is the bank account driven by the risk-free instantaneous interest rate r and associated to the risk-neutral measure $\mathbb{Q}$. The rate r is assumed to be $(\mathcal{F}_t)_{t \in [0,T]}$ adapted.

Applying the properties of conditional expectations, the following result can be proved.

Proposition 1.2.1 (General unilateral counterparty risk pricing formula [24]). *At time $t < \tau$, the price of the derivative trade under counterparty credit risk is*

$$\bar{V}_t = \mathbb{E}_t[\,\Pi(t)\,] - L_{\text{GD}}\,\mathbb{E}_t\left[\,\mathbf{1}_{\{t < \tau \le T\}} D(t, \tau)(NPV(\tau))^+\,\right],$$

where $L_{\text{GD}} = 1 - R$ is the loss given default, and the recovery rate R is assumed to be deterministic.

We notice that the counterparty risk adjusted deal price consists of the default-free deal price and an adjustment term which is called credit valuation adjustment denoted by CVA given as

$$\text{CVA}(t) = \text{LGD}\,\mathbb{E}_t\left[\,\mathbf{1}_{\{t < \tau \le T\}} D(t, \tau)(NPV(\tau))^+\,\right]. \tag{1.2}$$

The CVA term takes the form of an option price, more specifically, a call option on the NPV with zero strike, only when default happens ($\tau \leq T$). Including couterparty credit risk into valuation makes the pricing model dependent, even when the original valuation is model independent (for instance an interest rate swap).

However, it has been made clear that the unilateral assumption of the counterparty credit risk was not realistic during the financial crisis. If both parties in a transaction may default, the counterparty risk adjustment becomes a bilateral one. Indeed, the bilateral nature of counterparty risk has been recognised by market participants.

1.2.2 Arbitrage-free valuation of bilateral counterparty risk

In this section, we consider the valuation problem when the default possibility of the investor is included, in contrary to the previous section. The bilateral counterparty risk was first introduced by Duffie and Huang in [46], where a valuation model accounting for the default of both parties was presented. Bielecki and Rutkowski [7] gave a general formula for bilateral counterparty risk evaluation and Brigo and Capponi [16] developed a pricing formula for bilateral counterparty risk valuation adjustment (BCVA) considering the sequence of default events to avoid double counting.

The introduction of bilateral counterparty risk brings in the symmetry: the counterparty risk adjustment to the investor is the opposite of that to the counterparty. The adjustment driven by the default of the party who calculates the deal value is called *debit valuation adjustment* (in short DVA) which is a positive value added on to the deal price. However, DVA has some counter-intuitive features, as it increases the mark-to-market when the party's credit quality worsens.

We denote by τ_I and τ_C the default times of the investor and counterparty, respectively, both being $\mathcal{G}$-stopping times. Recall that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, and the right-continuous filtration generated by the default events either of the investor or of the counterparty $\mathcal{H}_t = \sigma\left(\{\tau_I \leq u\} \vee \{\tau_C \leq u\}, u \leq t\right)$. The stopping time $\tau = \min\{\tau_C, \tau_I\}$ is the first to default time.

Unlike the unilateral counterparty risk pricing, we need to consider at the first default time τ which party is the defaulting party, namely, $\tau = \tau_I$ or $\tau = \tau_C$. Let's define the following six events:

$$\begin{aligned} A &= \{\tau_I \le \tau_C \le T\} \quad B = \{\tau_I \le T \le \tau_C\} \quad C = \{\tau_C \le \tau_I \le T\} \\ D &= \{\tau_C \le T \le \tau_I\} \quad E = \{T \le \tau_I \le \tau_C\} \quad F = \{T \le \tau_C \le \tau_I\}. \end{aligned} \tag{1.3}$$

Remark 1.2.2. *Throughout this thesis, we assume that there is no simultaneous default. In practice, the case when two parties of a transaction default simultaneously is very rare, and the liquidation procedures in such a case are not clear. Therefore, we postulate that the probability of the event $\tau_I = \tau_C$ is zero - the default times of the two entities are never equal. Of course, the two default times can be very close. One party can default very closely to the other party with high probability.*

Taking the point of view of the investor, at $t < T$ the adjusted price of a deal including bilateral counterparty risk can be calculated as

$$\begin{aligned} \bar{V}_t =& \mathbb{E}_t \Big\{ \mathbf{1}_{\{E \cup F\}} \Pi(t,T) \\ &+ \mathbf{1}_{\{C \cup D\}} \Big[\Pi(t,\tau_C) + D(t,\tau_C) \Big(R_C (NPV(\tau_C))^+ - (-NPV(\tau_C))^+ \Big) \Big] \\ &+ \mathbf{1}_{\{A \cup B\}} \Big[\Pi(t,\tau_I) + D(t,\tau_I) \Big((NPV(\tau_I))^+ - R_I (-NPV(\tau_I))^+ \Big) \Big] \Big\}, \end{aligned}$$

where R_C and R_I denote respectively the recovery fraction of the counterparty and the investor. If there is no default event (only the events E and F happen), the above pricing formula reduces to the classical risk-neutral valuation formula. If the investor is taken as default-free (event A and B will not happen), the pricing formula will reduce to the unilateral counterparty risk pricing case.

An application of the properties of conditional expectation yields the following result for bilateral counterparty risk pricing.

Proposition 1.2.3 (General bilateral counterparty risk pricing formula [15]). *At time $t < \tau$, the price of the derivative trade under bilateral counterparty risk is*

$$\begin{aligned} \bar{V}_t = \mathbb{E}_t[\,\Pi(t,T)\,] &- L_{GD_C}\, \mathbb{E}_t\big[\, \mathbf{1}_{\{C \cup D\}} D(t,\tau_C)(NPV(\tau_C))^+ \,\big] \\ &+ L_{GD_I}\, \mathbb{E}_t\big[\, \mathbf{1}_{\{A \cup B\}} D(t,\tau_I)(-NPV(\tau_I))^+ \,\big]\,, \end{aligned} \tag{1.4}$$

where $\text{L}_{\text{GD}_C} = 1 - R_C$ *and* $\text{L}_{\text{GD}_I} = 1 - R_I$ *are the loss given default to the counterparty and the investor, and the recovery rates are assumed to be deterministic.*

Again, we see that the adjusted price is the sum of the risk-free deal price, the price of a short call option position if the counterparty defaults first, and the price of a long put option position if the investor defaults first, where both options have zero strikes. The adjustment added to the risk-free price to account for the counterparty credit risk is the so called bilateral counterparty risk valuation adjustment (BCVA), which can be either positive or negative. It is composed of a credit valuation adjusted term and a debit valuation adjustment term, denoted by C_{VA} and D_{VA} respectively. Taking the viewpoint of the investor, these terms can be expressed as

$$\begin{aligned} BCVA(t) &= \text{D}_{\text{VA}}(t) - \text{C}_{\text{VA}}(t), \\ \text{C}_{\text{VA}}(t) &= \text{L}_{\text{GD}_C}\, \mathbb{E}_t\big[\, \mathbf{1}_{\{C \cup D\}} D(t, \tau_C)(NPV(\tau_C))^+ \,\big]\,, \\ \text{D}_{\text{VA}}(t) &= \text{L}_{\text{GD}_I}\, \mathbb{E}_t\big[\, \mathbf{1}_{\{A \cup B\}} D(t, \tau_I)(-NPV(\tau_I))^+ \,\big]\,. \end{aligned} \tag{1.5}$$

Unlike the unilateral credit valuation adjustment, when considering bilateral counterparty risk, both parties can agree on the adjustment term being added to the default-free deal price. Indeed, the CVA of the investor is the DVA of the counterparty, whereas the CVA of the counterparty is the DVA of the investor.

The inclusion of DVA has been controversial. One objection is that one party books positive mark-to-market when its credit quality worsens. The defaulting party would gain if it defaults and to price this component might appear unusual. Moreover, the hedging of DVA is very difficult as the institution cannot sell CDS protection on their own name. In practice, the hedging is done by selling protection on some highly correlated names.

In the following sections, we analyse some practical consequences of the bilateral counterparty risk valuation adjustment.

1.2.3 Close-out convention

When closing a deal, we have a choice of which close-out convention to use: the risk-free close-out, or the replacement close-out. The close-out amount is the net present value (NPV) of the deal that is computed when the first default happens for default settlement. A risk-free close-out amount is the NPV calculated when assuming the surviving party to be default-free, whereas a replacement close-out amount is the NPV computed by taking into account the credit quality of the surviving party upon default of the first entity. Clearly, such a replacement close-out can be different from a risk-free one, and the counterparty risk valuation adjustment can change dramatically depending on the choice of close-out convention. From the perspective of valuation continuity, the replacement close-out is consistent with the counterparty risk valuation adjustment, whereas the risk-free close-out introduces a dependence of counterparty risk for a pre-default evaluation but discards any future obligation for the surviving party on the default event.

Risk-free Close-out Under a risk-free close-out, upon the first default event, the default probability for the surviving party will not be considered, so the NPV is computed as a risk-free price of the deal at the first default time, namely,

$$NPV(\tau_i) = \mathbb{E}_{\tau_i}[\,\Pi(\tau_i, T)\,], \quad \text{for } i \in \{I, C\}.$$

Therefore, the bilateral counterparty risk pricing formula at any time $t < \tau$ reads,

$$\begin{aligned} \bar{V}_t = \mathbb{E}_t[\,\Pi(t,T)\,] &- \mathrm{L{\scriptstyle GD}}_C\, \mathbb{E}_t\big[\,\mathbf{1}_{\{C\cup D\}} D(t,\tau_C)\,(\mathbb{E}_{\tau_C}[\,\Pi(\tau_C,T)\,])^+\,\big] \\ &+ \mathrm{L{\scriptstyle GD}}_I\; \mathbb{E}_t\big[\,\mathbf{1}_{\{A\cup B\}} D(t,\tau_I)\,(-\mathbb{E}_{\tau_I}[\,\Pi(\tau_I,T)\,])^+\,\big]\,, \end{aligned} \tag{1.6}$$

Replacement Close-out When replacement close-out is used in the settlement of a default, we take into account the default risk of the surviving party. In other words, the DVA of the surviving party needs to be included in the NPV calculation,

$$NPV(\tau_C) = \mathbb{E}_{\tau_C}[\,\Pi(\tau_C,T)\,] + \mathrm{D{\scriptstyle VA}}_I(\tau_C), \quad NPV(\tau_I) = \mathbb{E}_{\tau_I}[\,\Pi(\tau_I,T)\,] + \mathrm{D{\scriptstyle VA}}_C(\tau_I),$$

where DVA_i denotes the debit valuation adjustment viewed from the perspective of party i, $i \in \{I, C\}$, defined as (1.5). Therefore, the bilateral counterparty risk pricing formula at any time $t < \tau$ reads,

$$\begin{aligned}\bar{V}_t = &\mathbb{E}_t\left[\mathbf{1}_{\{E\cup F\}}\Pi(t,T)\right] \\ &+ \mathbb{E}_t\Big\{\mathbf{1}_{\{C\cup D\}}\Big[\Pi(t,\tau_C) + D(t,\tau_C)\Big(R_C\left(\mathbb{E}_{\tau_C}[\Pi(\tau_C,T)] + \text{DVA}_I(\tau_C)\right)^+ \\ &- \left(-\mathbb{E}_{\tau_C}[\Pi(\tau_C,T)] - \text{DVA}_I(\tau_C)\right)^+\Big)\Big]\Big\} \\ &+ \mathbb{E}_t\Big\{\mathbf{1}_{\{A\cup B\}}\Big[\Pi(t,\tau_I) + D(t,\tau_I)\Big(\left(-\mathbb{E}_{\tau_I}[\Pi(\tau_I,T)] - \text{DVA}_C(\tau_I)\right)^+ \\ &- R_I\left(\mathbb{E}_{\tau_I}[\Pi(\tau_I,T)] + \text{DVA}_C(\tau_I)\right)^+\Big)\Big]\Big\}.\end{aligned}$$

The above formula was also expressed in [25] for $t < \tau$, as

$$\begin{aligned}\bar{V}_t = &\mathbb{E}_t[\Pi(t,T)] \\ &+ \mathbb{E}_t\Big\{\mathbf{1}_{\{C\cup D\}}D(t,\tau_C)\Big[\text{DVA}_I(\tau_C) - \text{LGD}_C\left(\mathbb{E}_{\tau_C}[\Pi(\tau_C,T)] - \text{DVA}_I(\tau_C)\right)^+\Big]\Big\} \\ &+ \mathbb{E}_t\Big\{\mathbf{1}_{\{A\cup B\}}D(t,\tau_I)\Big[\text{LGD}_I\left(\text{DVA}_C(\tau_I) - \mathbb{E}_{\tau_I}[\Pi(\tau_I,T)]\right)^+ - \text{DVA}_C(\tau_I)\Big]\Big\}.\end{aligned} \tag{1.7}$$

Risk-free or Replacement? The risk-free close-out does not require an assessment of the default probability of the surviving party, and all contracts with the same payoff will have the same close-out value, which to a large extent, simplifies the valuation problem. Due to the computational simplicity, risk-free close-out seems more preferable. However, intuitively, replacement close-out is more fair. The replacement close-out is consistent with the counterparty risk valuation adjustment as the counterparty risk is considered throughout the valuation. Brigo and Morini [25] provided a quantitative analysis of the consequences of the different close-out conventions. They showed that for a simple derivative, assuming risk-free close-out, the bilateral counterparty risk adjustment formula is not consistent with the market practice on uncollateralized claims which, on the contrary, can be avoided with replacement close-out. The paper also studied the effects of the two close-out conventions in terms of default contagion. If a replacement close-out is used, there will be lower recovery for creditors. If, on the other hand, a risk-free close-out is used, there will be unexpected losses affecting also the debtors of the defaulted entity, which is at odds with standard counterparty risk for products like bonds and loans.

1.2.4 Double counting: first to default

A certain degree of double counting can be included when a bilateral counterparty risk inclusive valuation is carried out. One possible issue is when the first to default close-out proceeding is neglected. In the industry, a simplification of bilateral counterparty risk valuation adjustment is sometimes used, see for example Picoult [67]. This simplified expression allows one to consider the bilateral counterparty risk as a simple combination of the unilateral counterparty risk which may seem to be desirable. However, the approach does not conform with the fact that when the first default event happens, close-out proceedings start and the transaction is closed. Moreover, it ignores the default dependence between the two parties in the transaction.

A Simplified Formula We first recall that the full bilateral counterparty risk pricing formula is given in (1.4) and the counterparty risk valuation adjustments are defined in (1.5). If a risk-free close-out is in force, the bilateral counterparty risk pricing follows equation (1.6), whereas for the replacement close-out case, one should use the pricing formula (1.7).

In the unilateral case, only one of the two parties is considered to be defaultable. We write the unilateral credit valuation adjustment (UCVA) and unilateral debit valuation adjustment (UDVA) as

$$\begin{aligned} \mathrm{UC}\textsc{va}_I(t) &= \mathrm{L}\textsc{gd}_C\, \mathbb{E}_t\big[\, \mathbf{1}_{\{\tau_C \le T\}} D(t,\tau_C)(NPV(\tau_C))^+ \,\big]\,, \qquad (1.8)\\ \mathrm{UD}\textsc{va}_I(t) &= \mathrm{L}\textsc{gd}_I\, \mathbb{E}_t\big[\, \mathbf{1}_{\{\tau_I \le T\}} D(t,\tau_I)(-NPV(\tau_I))^+ \,\big]\,, \end{aligned}$$

where the $NPV(\tau)$ is calculated as the residual risk-free price of the deal as the survival party is default free. Notice that the unilateral credit valuation adjustment evaluated by the investor due to the default possibility of the counterparty is equivalent to the debit valuation adjustment of the counterparty due to its own default risk, precisely,

$$\mathrm{UC}_{\mathrm{VA}_I}(t) = \mathrm{UD}_{\mathrm{VA}_C}(t).$$

We now consider a simplified case of the bilateral counterparty risk adjustment where the first to default is neglected. More precisely, instead of considering all of the six events (1.3), we only study the following three scenarios:

$$\{\tau_C \le T\}, \quad \{\tau_I \le T\} \quad \text{and} \quad \{T < \min\{\tau_I, \tau_C\}\}.$$

In this case, each adjusted term is computed as if only one of the two parties is defaultable since the first to default is no longer checked. Therefore, the dependence between the defaults of the two parties is not required in determining the adjusted price.

Risk-free Close-out In the case of risk-free close-out, the simplified adjusted price, denoted as $\bar{V}_t^*$, can be computed as

$$\begin{aligned}\bar{V}_t^* &= \mathbb{E}_t[\,\Pi(t,T)\,] - \mathrm{L}_{\mathrm{GD}_C}\,\mathbb{E}_t\big[\,\mathbf{1}_{\{\tau_C \le T\}} D(t,\tau_C)\,(\mathbb{E}_{\tau_C}[\,\Pi(\tau_C,T)\,])^+\,\big] \\ &\qquad + \mathrm{L}_{\mathrm{GD}_I}\,\mathbb{E}_t\big[\,\mathbf{1}_{\{\tau_I \le T\}} D(t,\tau_I)\,(-\mathbb{E}_{\tau_I}[\,\Pi(\tau_I,T)\,])^+\,\big] \\ &= \mathbb{E}_t[\,\Pi(t,T)\,] - \mathrm{UC}_{\mathrm{VA}_I}(t) + \mathrm{UC}_{\mathrm{VA}_C}(t).\end{aligned}$$

The bilateral counterparty risk valuation adjustment in this case is simply the difference of two UCVA, which dramatically simplifies the calculation and gives great practical advantage.

Replacement Close-out The main assumption of the simplified formula is that each term is computed as in the unilateral case (only one of the parties can default), and the default party would be the one that defaults first in the full bilateral case. Therefore, when replacement close-out is used, the simplified adjusted price can be calculated as

$$
\begin{aligned}
\bar{V}_t^* = & \mathbb{E}_t[\,\Pi(t,T)\,] \\
& + \mathbb{E}_t\left\{\mathbf{1}_{\{\tau_C \le T\}} D(t,\tau_C)\Big[\mathrm{D}\textsc{va}_I(\tau_C) - \mathrm{L}\textsc{gd}_C(\mathbb{E}_{\tau_C}[\,\Pi(\tau_C,T)\,] - \mathrm{D}\textsc{va}_I(\tau_C))^+\Big]\right\} \\
& + \mathbb{E}_t\left\{\mathbf{1}_{\{\tau_I \le T\}} D(t,\tau_I)\Big[\mathrm{L}\textsc{gd}_I(\mathrm{D}\textsc{va}_C(\tau_I) - \mathbb{E}_{\tau_I}[\,\Pi(\tau_I,T)\,])^+ - \mathrm{D}\textsc{va}_C(\tau_I)\Big]\right\} \\
= & \mathbb{E}_t[\,\Pi(t,T)\,] \\
& - \mathbb{E}_t\left\{\mathbf{1}_{\{\tau_C \le T\}} D(t,\tau_C)\mathrm{L}\textsc{gd}_C\Big(\mathbb{E}_{\tau_C}[\,\Pi(\tau_C,T)\,]\Big)^+\right\} \\
& + \mathbb{E}_t\left\{\mathbf{1}_{\{\tau_I \le T\}} D(t,\tau_I)\mathrm{L}\textsc{gd}_I\Big(-\mathbb{E}_{\tau_I}[\,\Pi(\tau_I,T)\,]\Big)^+\right\} \\
= & \mathbb{E}_t[\,\Pi(t)\,] - \mathrm{UC}\textsc{va}_I(t,T) + \mathrm{UC}\textsc{va}_C(t,T),
\end{aligned}
$$

which is identical to the simplified formula with risk-free close-out. Here, the second equality holds because each term is calculated as if only one of the two names is defaultable, so the DVA associated to the default-free party will be zero.

The Impact of the Simplification In the full adjusted pricing formula, six events (1.3) are considered. However, in the simplified version, we only considered three cases: $\{\tau_C \le T\}$, $\{\tau_I \le T\}$ and $\{T < \min\{\tau_I,\tau_C\}\}$. Since $E \cup F = \{T < \min\{\tau_I,\tau_C\}\}$, events A, C, D are contained in $\{\tau_C \le T\}$, and events A, B, C are contained in $\{\tau_I \le T\}$, the double counting is then obvious. For $t < \min\{\tau_I,\tau_C\}$, we have

$$
\begin{aligned}
\mathbf{1}_{\{A\cup B\}} - \mathbf{1}_{\{\tau_I \le T\}} &= -\mathbf{1}_{\{\tau_C<\tau_I<T\}}, \\
\mathbf{1}_{\{C\cup D\}} - \mathbf{1}_{\{\tau_C \le T\}} &= -\mathbf{1}_{\{\tau_I<\tau_C<T\}}.
\end{aligned}
$$

Therefore, the difference of the full bilateral counterparty risk valuation and the simplified version is

$$
\begin{aligned}
\bar{V}_t - \bar{V}_t^* = & \mathbb{E}_t\left\{\mathbf{1}_{\{\tau_I<\tau_C<T\}} D(t,\tau_C)\mathrm{L}\textsc{gd}_C\Big(\mathbb{E}_{\tau_C}[\,\Pi(\tau_C,T)\,]\Big)^+\right\} \\
& - \mathbb{E}_t\left\{\mathbf{1}_{\{\tau_C<\tau_I<T\}} D(t,\tau_I)\mathrm{L}\textsc{gd}_I\Big(-\mathbb{E}_{\tau_I}[\,\Pi(\tau_I,T)\,]\Big)^+\right\}.
\end{aligned}
$$

As we see, the difference is due to the so called second to default term.

The impact of the first to default time was closely studied in Brigo et al. [12], where the authors considered the errors caused by the simplification in two simple products: a zero coupon bond and an equity forward contract, and presented a number of cases where the simplified formula differs considerably from the full formula.

1.2.5 Wrong way risk

The full valuation formula in (1.4) depends on the joint distribution of the default times and the underlying asset. *Wrong way risk* (in short WWR) is the risk the investor has when the underlying portfolio and the default of the counterparty are "correlated" in the worst possible way from the investor's perspective. Wrong way risk has been studied in the literature in different asset classes. For example, Brigo and Tarenghi [32, 33] analyse the counterparty risk valuation adjustment on Equity, Brigo and Masetti [24] examine CVA with netting, and Brigo et al. [27] consider the particular case of an Equity Return Swap with counterparty risk. Counterparty risk valuation adjustment on commodities with WWR is analysed in Brigo and Bakkar [11]. In interest rate products, Brigo and Pallavicini [28] incorporate the WWR in a stochastic intensity model which is correlated with the multi-factor short rate process driving the interest rate dynamics, whereas Brigo et al. [31] introduce methods for bilateral counterparty risk adjustment and allow also for correlation between the default times of the investor and counterparty. Brigo and Chourdakis [19] consider counterparty risk for Credit Default Swaps (CDS) in the presence of correlation between default of the counterparty and default of the CDS reference credit, but the paper only deals with unilateral and asymmetric counterparty credit risk. Crepey et al. [43] model wrong way risk for CDS with counterparty credit risk using a Markov chain copula model, whereas Lipton and Sepp [57] introduce a structural model with jumps. Brigo and Capponi [15] include the default risk of the investor by using a trivariate copula function on the default times to model default dependence.

Other subjects related to the counterparty risk adjusted pricing problem include collateralization and close-out netting rules, which will be closely studied in the next chapter.

1.3 Structure of the thesis

The thesis is organized as follows. Chapter 2 describes the general pricing framework with collateralization, credit, debit and funding valuation adjustments. We derive an iterative solution of the pricing equation as well as a continuous time approximation. In Chapter 3 we discuss the consistent valuation framework in a continuous time setting. We give both FBSDE and semi-linear PDE expressions for the deal price in the consistent framework and discuss the existence and uniqueness conditions in each case. The invariance theorem stating that the pricing framework does not depend on the theoretical risk-free rate is given at the end of the chapter. Chapter 4 addresses an important problem with the self-financing condition used in a derivative pricing framework with funding, collateral and discounting. We give the correct derivation by specifying the gain processes, the price processes and the dividend processes. We also show how the traditional self-financing condition is adjusted in our market settings. Chapter 5 describes a least-square Monte Carlo algorithm and provides numerical results on deal positions in European call options on equity under the benchmark model of Black and Scholes. A nonlinearity valuation adjustment (NVA) is introduced and computed. Chapter 6 provides details about how the consistent valuation framework can be tailored to address other market realities. We discuss a valuation framework which includes also the costs/benefits from assets borrowing/lending. We then study when the trade is cleared by a central clearing house (CCP) or governed by bilateral Credit Support Annex (CSA) with variation and initial margins. Moreover, we consider the case when the margin period of risk is included. We conclude the thesis in the last section and hint at the financial implications.

1.4 Preprints and published material

We list here all the published papers and the working papers related to the thesis:

1. [13] Illustrating a problem in the self-financing condition in two 2010-2011 papers on funding, collateral and discounting (2012). D. Brigo, C, Buescu, A. Pallavicini and Q.D. Liu. Preprint available at arXiv:1207.2316 or ssrn:2103121.

2. [14] A note on the self-financing condition for funding, collateral and discounting (2015). D. Brigo, C, Buescu, A. Pallavicini and Q.D. Liu. International Journal of Theoretical and Applied Finance, 18(2), 1550011.

3. [21] Nonlinear valuation under collateral, credit risk and funding costs: A numerical case study extending Black-Scholes (2014). D. Brigo, Q.D. Liu, A. Pallavicini and D. Sloth. Preprint available at arXiv:1404.7314 or ssrn:2430696.

4. [23] Nonlinear valuation under margining and funding costs with residual credit risk: A unified approach (2015). D. Brigo, Q.D. Liu, A. Pallavicini and D. Sloth. Handbook in Fixed-Income Securities, Wiley.

5. [22] Nonlinear valuation adjustment: nonlinear valuation under collateralization, credit risk and funding costs (2015). D. Brigo, Q.D. Liu, A. Pallavicini and D. Sloth. Forthcoming in Challenges in Derivative Markets, Springer Proceedings in Mathematics and Statistics, Springer.

6. No-arbitrage bounds for the forward smile given marginals. S. Badikov, A. Jacquier, Q.D. Liu and P. Roome. In progress. This work is not related to this thesis, but was carried out during the PhD programme.

Chapter 2

Consistent Valuation Framework

In this chapter we develop a general arbitrage-free valuation framework for over the counter (OTC) derivative deals. The chapter clarifies how the traditional pre-crisis derivative price is consistently adjusted to reflect the new market realities of counterparty credit risk, collateralization and funding risk.

We refer to the two parties of an OTC deal as the investor or dealer ("I") on one side, and the counterparty or client ("C") on the other side. Recall the probabilistic set-up in section 1.2. Fixing a finite time horizon of the deal, $T \in \mathbb{R}^+$, we define our risk-neutral pricing model on a filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,T]}, \mathbb{Q})$, where $\mathbb{Q}$ is the risk-neutral probability measure. The filtration $(\mathcal{G}_t)_{t\in[0,T]}$ models the flow of information of the whole market, including credit, such that the default times of the investor τ_I and the counterparty τ_C are $\mathcal{G}$-stopping times, and we have $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ with the default-free market filtration given by $(\mathcal{F}_t)_{t\in[0,T]}$ and the filtration generated by the default events denoted by $\mathcal{H}_t = \sigma\left(\{\tau_I \leq u\} \vee \{\tau_C \leq u\},\, u \leq t\right)$. Throughout this work, we adopt the notational convention that $\mathbb{E}_t$ is the risk-neutral expectation conditional on the filtration $\mathcal{G}_t$ (unless otherwise specified), while $\mathbb{E}_{\tau_i}$ denotes the conditional risk-neutral expectation given the stopped filtration $\mathcal{G}_{\tau_i}$ for $i \in \{I, C\}$. Moreover, we exclude the possibility of simultaneous defaults for simplicity, and define the time of the first default event among the two parties as the stopping time,

$$\tau := (\tau_I \wedge \tau_C).$$

In the sequel we adopt the view of the investor and consider the cash-flows and consequences of the deal from his perspective. In other words, when we price the deal we obtain the value of the position to the investor. As we will see, with funding risk this price will often not just be the value of the deal to the counterparty with an opposite sign.

The gist of the valuation framework is conceptually simple and rests neatly on the classical finance disciplines of risk-neutral pricing and discounting cash-flows. When a dealer enters into a derivative deal with a client, a number of cash-flows are exchanged, and just like valuation of any other financial claim, discounting these cash in- or outflows gives us a deal price. Post-crisis market practice distinguishes four different types of cash-flow streams occurring once a trading position has been entered:

(i) Cash-flows coming directly from the derivative contract such as payoffs, coupons, dividends, etc. We denote by $\Pi(t,T)$ the sum of all the discounted (at the risk-free rate) cash-flows of a given contract happening over the time period $(t,T]$, where Π is an arbitrary càdlàg process with finite variation. The process Π models all discounted cash-flows which are either paid out from or added to the wealth of a contract. This is where classical derivatives pricing would usually stop and the price of a derivative contract with maturity T would be given by

$$V_t = \mathbb{E}_t\left[\,\Pi(t,T)\right].$$

This price assumes no credit risk of the parties involved and no risk of funding the trade.

However, present-day market practice requires the price to be adjusted by taking further cash-flow transactions into account:

(ii) Cash-flows required by collateral margining. If the deal is collateralized, cash-flows happen in order to maintain a collateral account that in the case of default will be used to cover any losses. $\gamma(t,T;C)$ denotes the sum of the discounted (at the risk-free rate) margining costs over the period $(t,T]$ with C denoting the collateral account.

(iii) Cash-flows exchanged once a default event has occurred. We let $\theta_\tau(C, \varepsilon)$ denote the on-default cash-flow with ε being the residual value of the claim traded at default.

Lastly, (iv) cash-flows required for funding the deal. We denote the sum of the discounted (at the risk-free rate) funding costs over the period $(t, T]$ by $\varphi(t, T; F)$ with F being the cash account needed for funding the deal.

Collecting all the terms and taking expectation under the risk-neutral pricing measure, we obtain a consistent adjusted price $\bar{V}$ of a derivative deal taking into account counterparty credit risk, collateral margining costs and funding costs

$$\begin{aligned}\bar{V}_t(C, F) = \mathbb{E}_t \big[\Pi(t, T \wedge \tau) + \gamma(t, T \wedge \tau; C) + \varphi(t, T \wedge \tau; F) \\ + \mathbf{1}_{\{t<\tau<T\}} D(t, \tau) \theta_\tau(C, \varepsilon) \big] ,\end{aligned} \tag{2.1}$$

where $D(t, \tau)$ is the risk-free discount factor defined in (1.1).

By using a risk-neutral valuation approach, we see that only the payout needs to be adjusted under collateralization, counterparty credit and funding risks. In the following sections we will expand the terms in (2.1) and carefully discuss how to compute them.

2.1 Valuation under collateralization and close-out netting

The ISDA master agreement is the most commonly used framework for full and flexible documentation of OTC derivative transactions and is published by the International Swaps and Derivatives Association ([54]). Once agreed between two parties, the master agreement sets out standard terms that apply to all deals entered into between those parties. The ISDA master agreement lists two tools to mitigate counterparty credit risk: collateralization and close-out netting. *Collateralization of a dea*l means that the party which is out-of-the-money is required to post collateral – usually cash, government securities or highly rated bonds – corresponding to the amount payable by that party in the case of a default event. The credit support annex (CSA) to the ISDA master agreement defines the rules under which the collateral is posted or transferred between counterparties. *Close-out netting* means that in the case of default all transactions with the counterparty under the ISDA master agreement are consolidated into a single net obligation which then forms the basis for any recovery settlements.

Risk-neutral evaluation of counterparty risk taking into account the collateralization and close-out netting rules can be difficult due to the complexity of clauses. Early literature dealing with collateral inclusive pricing commonly assume that the collateral is a risk-free asset such as in Alavian et al. [1] and Assefa et al. [3]. Brigo et al. in [18] generalized an arbitrage-free framework for valuation including collateralization and possible rehypothecation.

The purpose of this section is to develop a model independent formula for OTC deals, inclusive of collateralization mitigation and close-out netting convention, before we introduce the funding risk. We will analyse the margining costs required by the collateralization, taking into account the counterparty credit effects and netting rules at close-out.

2.1.1 Collateral convention and margin account

Collateralization of a deal usually happens according to a margining procedure. Such a procedure involves that both parties post collateral amounts to or withdraw collateral amounts from the collateral account C according to their current exposure on a pre-fixed time-grid $\{t_1, \ldots, t_n = T\}$ during the life of the deal. We define the *collateral account* C_t at $t \in [0, T]$ to be a stochastic process adapted to the filtration $\mathcal{G}_t$. The terms of the margining procedure may, furthermore, include independent amounts, minimum transfer amounts, thresholds, etc., as described in [18]. However, here we adopt a general description of the margining procedure that does not rely on the particular terms chosen by the parties.

Collateral accounts in general can be any type of assets (both defaultable and risk-free), which can be liquidated at the default time. In this thesis, we assume that the collateral account is a risk-free cash account. Furthermore, we postulate that for each new deal, a new collateral account is opened, and when a default event happens or when the maturity of the trade is reached, the collateral account is closed. In particular, $C_t = 0$ for all $t \leq 0$ and $C_t = 0$ for all $t \geq T$. Upon the closure of the collateral account, any remaining collateral held by the collateral taker will be returned to the collateral provider.

Without loss of generality, we consider a collateral account C held by the investor. Moreover, we assume that the investor is the collateral taker when $C_t > 0$ and the collateral provider when $C_t < 0$. The CSA ensures that the collateral taker remunerates the account C at an accrual rate. If the investor is the collateral taker, he remunerates the collateral account by the accrual rate $c_t^+(T)$, while if he is the collateral provider, the counterparty remunerates the account at the rate $c_t^-(T)$. $c_t^+(T)$ and $c_t^-(T)$ are $\mathcal{F}_t$-adapted, thus are also $\mathcal{G}_t$-adapted processes. To fix ideas, let us resort to a toy example.

Example. (Collateral example) *Suppose we are at time t_1 during the trade of an equity return swap, and that the swap value is negative now with a mark to market equal to $\$2M$ (2 Million USD) in favour of our counterparty. In this case we have $C_{t_1} = -\$2M$ in the collateral account, posted by us in the past margining activity. Since $C_{t_1} = -\$2M < 0$, we receive collateral interest c^-, say 1% annually, on this. With simple compounding we will receive $1\% \times \$2M \times 1/250 = \$80$ on this day. Now time moves to the next day at $t_2 = t_1 + 1/250$, the market swings heavily, the mark to market turns around and goes to $+\$2M$ to us, or $-\$2M$ for the counterparty, so that we expect to receive $\$2M$ in collateral while we take back the amount we had posted previously from the collateral account. As we receive collateral, $C_{t_2} = \$2M$, we now have to pay interest c^+ to the counterparty, say 1.2% annually, so that we pay $1.2\% \times \$2M \times 1/250 = \$96$ as interest over the next day. Clearly we see that the asymmetry of rates makes the interest on collateral posted or received on an opposite mark to market not symmetric. We will return to this example later on.*

More generally, to understand the cash-flows originating from collateralization of the deal, let us consider the consequences of the margining procedure to the investor. At the first margin date, say t_1, the investor opens the account and posts collateral if he is out of the money, i.e. if $C_{t_1} < 0$, which means that the counterparty is the collateral taker. On each of the following margin dates t_j, $j \in \{2, \ldots, n-1\}$, the investor posts collateral according to his exposure as long as $C_{t_j} < 0$. As collateral taker, the counterparty pays interest on the collateral at the accrual rate $c^-_{t_j}(t_{j+1})$ between the following margin dates t_j and t_{j+1}. We assume that interest accrued on the collateral is saved into the account and thereby directly included in the margining procedure and the close-out. Finally, if $C_{t_n} < 0$ on the last margin date t_n, the investor closes the collateral account given no default event has occurred in between. Similarly, for positive values of the collateral account ($C_{t_j} > 0$), the investor is instead the collateral taker. The counterparty faces corresponding cash-flows at each margin date, and is entitled to interest payments from the investor at the rate $c^+_{t_j}(t_{j+1})$ for the associated margin period. If we do not take into account the default events and sum up all the discounted margining cash-flows of the investor and the counterparty occurring within the time interval $[t, (T \wedge \tau)]$, we obtain the following expression for the margining cash-flows denoted as $\Gamma(t, T \wedge \tau; C)$.

$$\begin{aligned}\Gamma(t, T \wedge \tau; C) = &\sum_{j=1}^{n-1} \mathbf{1}_{\{t_j < \tau\}} \left(D(t, t_j) C_{t_j} - D(t, t_{j+1}) \mu(t_j, t_{j+1}) \right) \\ &+ \sum_{j=1}^{n-1} \mathbf{1}_{\{t_j < \tau \ < t_{j+1}\}} D(t, t_{j+1}) \mu(t_j, t_{j+1}) \,,\end{aligned} \tag{2.2}$$

where

$$\mu(t_j, t_{j+1}) := \frac{C^-_{t_j}}{P^{c^-}_{t_j}(t_{j+1})} + \frac{C^+_{t_j}}{P^{c^+}_{t_j}(t_{j+1})},$$

denotes the value of the collateral account accrued from date t_j to date t_{j+1} as required by the CSA, and the (collateral) zero-coupon bond is defined as $P^{c^\pm}_t(T) := [1 + (T-t) c^{\pm}{}_t(T)]^{-1}$.

We use the short-hand notation $\mathcal{X}^+ := \max(\mathcal{X}, 0)$ and $\mathcal{X}^- := \min(\mathcal{X}, 0)$, so for a random variable X we have $\mathcal{X} = \mathcal{X}^+ + \mathcal{X}^-$.

Moreover, we assume that the probability of default at a particular time is zero. In other words, the distribution of the default times τ is assumed to be continuous so that $\mathbb{Q}(\tau = u) = 0$ for all $u \geq 0$. We define the effective accrual collateral rate

$\tilde{c}_t(T)$[1] as

$$\tilde{c}_t(T) := c_t^-(T)\mathbf{1}_{\{C_t<0\}} + c_t^+(T)\mathbf{1}_{\{C_t>0\}}. \tag{2.3}$$

From (2.2) the price of the collateral margining cash-flows is obtained by taking the risk-neutral expectation to the discounted cash-flows

$$\mathbb{E}_t[\,\Gamma(t,T\wedge\tau;C)\,] = \mathbb{E}_t\big[\,\gamma(t,T\wedge\tau;C) + \mathbf{1}_{\{\tau<T\}}D(t,\tau)C_{\tau^-}\,\big]\ ,$$

where we introduce the pre-default value C_{τ^-} of the collateral account as

$$C_{\tau^-} := \sum_{j=1}^{n-1} \mathbf{1}_{\{t_j<\tau<t_{j+1}\}} C_{t_j} \frac{P_\tau(t_{j+1})}{P^{\tilde{c}}_{t_j}(t_{j+1})}\ , \tag{2.4}$$

and the collateral margining costs $\gamma(t,T\wedge\tau;C)$ entering (2.1) are defined as

$$\gamma(t,T\wedge\tau;C) := \sum_{j=1}^{n-1} \mathbf{1}_{\{t\leqslant t_j<(T\wedge\tau)\}} D(t,t_j) C_{t_j} \left(1 - \frac{P_{t_j}(t_{j+1})}{P^{\tilde{c}}_{t_j}(t_{j+1})}\right)\ , \tag{2.5}$$

with the zero-coupon bond $P^{\tilde{c}}_t(T) := [1 + (T-t)\tilde{c}_t(T)]^{-1}$, and the risk-free zero coupon bond, related to the risk-free rate r, given by $P_t(T)$.

The pre-default value of the collateral account may be different from the actual value of the collateral account at default since part or all of the collateral may be rehypothecated. In accordance with the CSA, this pre-default value of the collateral account is used to compute the netted exposure at close-out. In particular, we will first net the exposure against the pre-default value of the collateral $C_{\tau-}$, and then treat any remaining collateral as an unsecured claim.

Let α_j be the year fraction between t_j and t_{j+1}. If we adopt a first order expansion (for small c and r), we can approximate

$$\gamma(t,T\wedge\tau;C) \approx \sum_{j=1}^{n-1} \mathbf{1}_{\{t\leqslant t_j<(T\wedge\tau)\}} D(t,t_j) C_{t_j} \alpha_j \left(r_{t_j}(t_{j+1}) - \tilde{c}_{t_j}(t_{j+1})\right)\ , \tag{2.6}$$

[1] We stress the slight abuse of notation here: A plus and minus sign does not indicate that the rates are positive or negative parts of some other rate, but instead it tells which rate is used to accrue interest on the collateral according to the sign of the collateral account.

where with a slight abuse of notation we call $\tilde{c}_t(T)$ and $r_t(T)$ the continuously (as opposed to simple) compounded interest rates associated with the bonds $P^{\tilde{c}}$ and P. This last expression clearly shows a cost of carry structure for collateral costs. If C is positive to the investor, then the investor is holding collateral and will have to pay (hence the minus sign) an interest c^+, while receiving the natural growth r for cash, since we are in a risk-neutral world. In the opposite case, if the investor posts collateral, the collateral account value C is negative to the investor and the investor receives interest c^- while paying the risk-free rate, as should happen when one shorts cash in a risk-neutral world.

A crucial role in collateral procedures is played by rehypothecation. We discuss rehypothecation and its inherent liquidity risk in the following.

Rehypothecation Liquidity Risk Often the CSA grants the collateral taker relatively unrestricted use of the collateral for his liquidity and trading needs until it is returned to the collateral provider. This unrestricted use includes selling collateral to a third party and lending or selling the collateral under a "repo" agreement. Rehypothecation is widespread as a practice, since it can lower the costs of remuneration of the provided collateral. However, while without rehypothecation the collateral provider can expect to get any excess collateral returned after honoring the amount payable on the deal, if rehypothecation is allowed, the collateral provider has to face the risk of losing a fraction or all of the excess collateral in case of default on the collateral taker's part. Indeed, when the collateral is rehypothecated, the collateral taker leaves the collateral provider as an unsecured creditor with respect to the collateral reimbursement.

We denote the recovery fraction on the rehypothecated collateral by R'_I when the investor is the collateral taker and by R'_C when the counterparty is the collateral taker. Recall that we defined in Chapter 1 that the general recovery fraction on the market value of the deal that the investor receives in the case of default of the counterparty is denoted by R_C, while R_I is the recovery fraction received by the counterparty if the investor defaults. All such quantities are defined on a unit notional. The collateral provider typically has precedence over other creditors of the defaulting party in getting back any excess capital, which means $R_I \leqslant R'_I \leqslant 1$ and $R_C \leqslant R'_C \leqslant 1$. If no rehypothecation is allowed and the collateral is kept safely in a segregated account, we have that $R'_I = R'_C = 1$. We do not rule out the case where the collateral losses are treated as standard unsecured debit losses upon default events, namely, when $R' = R$.

Example. (Collateral example continued) *As in our previous case, suppose we have posted* $\$2M$ *in the collateral account and we receive 1% annual interest from the counterparty. If the counterparty can rehypothecate collateral, they can use this collateral and post it in another trade, and gain interest on that. This will offset the* $\$80$ *daily cost they pay us and ease their funding costs. Clearly we may also like to rehypothecate collateral when we receive it, for the same reason. Suppose however that the mark to market moves in our favour at* t_2 *as in the previous example to* $+\$2M$*. We move to* t_2 *but before the counterparty may post* $\$2M$ *collateral it defaults. This is the worst case. Not only we face a loss on our mark to market, since there is no collateral in the account to cover our loss, but also when we go to the account to take back at least the collateral we posted the previous day, we find that we can only get a recovery of that since the counterparty had rehypothecated our collateral and has now defaulted. We may only receive a recovery on that collateral. If the recovery rate* R'_C *is* 20%*, we will receive back only* $20\% \times \$2M$*, namely* $\$400,000$*, losing* $\$2M$ *on the mark to market of today and* $\$1.6M$ *on the collateral posted yesterday.*

Detailed analysis for the impact of rehypothecation on the pricing of counterparty risk was carried out in Brigo et al. [18] for interest-rate derivatives and in Brigo et al. [17] for credit derivatives.

Perfect Collateralization Let's consider a special case for the counterparty risk valuation including collateralization–perfect collateralization. By *perfect collateralization*, we mean when the collateral margining is done in continuous time, with continuous mark-to-market of the deal upon default events. Under perfect collateralization, the collateral account is defined as the sum of the mark-to-market of the deal and the collateral margining costs at any time $t < T$, namely,

$$C_t := \mathbb{E}_t[\,\Pi(t,T) + \gamma(t,T;C)\,]\,,$$

and the close-out amount is equivalent to the collateral price,

$$\varepsilon_{C,\tau} = \varepsilon_{I,\tau} = C_\tau.$$

Here $\varepsilon_{I,\tau}$ is the close-out amount on the counterparty's default priced at time τ by the investor and $\varepsilon_{C,\tau}$ is the close-out amount computed by the counterparty if the investor defaults.

Recall the adjusted price of a deal before introducing the funding risk is calculated by taking the risk-neutral expectation of the contractual cash-flows and the margining cash-flows, written as

$$\bar{V}_t(C) = \mathbb{E}_t\big[\,\Pi(t,T\wedge\tau) + \gamma(t,T\wedge\tau;C) + \mathbf{1}_{\{t<\tau<T\}}D(t,\tau)C_\tau\,\big]\,. \tag{2.7}$$

Under the assumption of perfect collateralization, the above expression can be simplified as

$$\bar{V}_t(C) = \mathbb{E}_t[\,\Pi(t,T) + \gamma(t,T;C)\,] = C_t. \tag{2.8}$$

If we consider two adjacent margining dates in the discrete setting t_j and t_{j+1}, with $1 \le j \le n-1$, $t_n = T$, by substituting the expression for margining cash-flows into equation (2.8) up to maturity, we get

$$\bar{V}_{t_j}(C) = \frac{P^{\tilde{c}}_{t_j}(t_{j+1})}{P_{t_j}(t_{j+1})}\mathbb{E}_{t_j}\big[\,D(t_j,t_{j+1})\bar{V}_{t_{j+1}}(C) + \Pi(t_j,t_{j+1})\,\big]\,, \qquad \bar{V}_{t_n}(C) = 0.$$

Making use of the recursive nature, we can write

$$\bar{V}_t(C) = \mathbb{E}_t\left[\sum_{j=1}^{n-1}\Pi(t_j,t_{j+1})D(t,t_j)\prod_{i=1}^{j}\frac{P^{\tilde{c}}_{t_i}(t_{i+1})}{P_{t_i}(t_{i+1})}\right].$$

In the perfect collateralization case, collateral margining and mark-to-market are assumed to be carried out continuously in time. Taking the limit of the above expression, one has

$$\bar{V}_t(C) = \mathbb{E}_t\left[\int_t^T e^{-\int_t^s \tilde{c}_u du}\Pi(s, s+ds)\right]. \tag{2.9}$$

Therefore, the adjusted price of a deal under perfect collateralization is obtained by taking the expected value of the future cash flows discounted at the effective collateral rate $\tilde{c}$. One interesting result is that the risk-free rate r disappears from the valuation formula.

2.1.2 Close-out netting rules

In case of default all terminated transactions under the ISDA master agreement with a given counterparty are netted and consolidated into a single claim. This also includes any posted collateral to back the transactions. In this context the close-out amount plays a central role in calculating the on-default cash-flows. The *close-out amount* is the costs or losses that the surviving party incurs when replacing the terminated deal with an economic equivalent. Clearly, the close-out amount is not symmetric to the two parties, since the size of these costs will depend on which party survives. We define the close-out amount as

$$\varepsilon_\tau := \mathbf{1}_{\{\tau=\tau_C<\tau_I\}}\varepsilon_{I,\tau} + \mathbf{1}_{\{\tau=\tau_I<\tau_C\}}\varepsilon_{C,\tau}\,, \tag{2.10}$$

where $\varepsilon_{I,\tau}$ is the close-out amount on the counterparty's default priced at time τ by the investor and $\varepsilon_{C,\tau}$ is the close-out amount computed by the counterparty if the investor defaults. Bearing in mind that the investor and the counterparty may evaluate close-out amount differently, we always consider the deal from the investor's viewpoint in terms of the sign of the cash-flows involved. This means that if the close-out amount $\varepsilon_{I,\tau}$ as measured by the investor is positive, the investor is a creditor of the counterpaty, while if it is negative, the investor is a debtor of the counterparty. Analogously, if the close-out amount $\varepsilon_{C,\tau}$ to the counterparty but viewed from the investor is positive, the investor is a creditor of the counterparty, and if it is negative, the investor is a debtor to the counterparty.

We note that the ISDA documentation is, in fact, not very specific in terms of how to actually calculate the close-out amount. Since 2009 ISDA has allowed for the possibility to switch from a risk-free close-out rule to a replacement close-out rule that includes the DVA of the surviving party in the recoverable amount. Parker and McGarry [65] and Weeber and Robson [73] show how a wide range of values of the close-out amount can be produced within the terms of ISDA. We refer to Brigo et al. [18] and the references therein for further discussions on these issues. Here, we adopt the approach of [18] listing the cash-flows of all the various scenarios that can occur if default happens. Our aim is to determine the present value of all cash flows taking into account the collateral margining procedures and close-out netting rules upon default events.

We start by considering all possible scenarios that may arise upon the first default event. For example, if the counterparty defaults first ($\varepsilon_\tau = \varepsilon_{I,\tau}$), one of the following four different scenarios can happen:

1. If the investor has a positive exposure on the default of the counterparty and the counterparty has posted collateral to the investor, then the on-default cash-flow is given as the investor's exposure netted by any available collateral

$$\theta^1_\tau(\varepsilon_{I,\tau}) = \mathbf{1}_{\{\tau=\tau_C<T\}}\mathbf{1}_{\{\varepsilon_\tau>0\}}\mathbf{1}_{\{C_{\tau-}>0\}}\left(R_C(\varepsilon_\tau - C_{\tau-})^+ + (\varepsilon_\tau - C_{\tau-})^-\right) .$$

2. If the investor has a positive exposure on counterparty's default but the investor has posted collateral to the counterparty, then the investor suffers a loss on the whole exposure and on the collateral if it has been rehypothecated

$$\theta^2_\tau(\varepsilon_{I,\tau}) = \mathbf{1}_{\{\tau=\tau_C<T\}}\mathbf{1}_{\{\varepsilon_\tau>0\}}\mathbf{1}_{\{C_{\tau-}<0\}}\left(R_C\varepsilon_\tau - R'_C C_{\tau-}\right) .$$

3. If the investor has a negative exposure towards the counterparty and the counterparty has posted collateral to the investor, then the investor returns the

collateral and pays the full exposure to the creditors of the counterparty

$$\theta^3_\tau(\varepsilon_{I,\tau}) = \mathbf{1}_{\{\tau=\tau_C<T\}}\mathbf{1}_{\{\varepsilon_\tau<0\}}\mathbf{1}_{\{C_{\tau-}>0\}}\left(\varepsilon_\tau - C_{\tau-}\right) .$$

4. If the investor has a negative exposure but has posted collateral to the counterparty, then the exposure is netted with the posted collateral and the investor pays any remaining exposure or receives any excess collateral

$$\theta^4_\tau(\varepsilon_{I,\tau}) = \mathbf{1}_{\{\tau=\tau_C<T\}}\mathbf{1}_{\{\varepsilon_\tau<0\}}\mathbf{1}_{\{C_{\tau-}<0\}}\left((\varepsilon_\tau - C_{\tau-})^- + R'_C(\varepsilon_\tau - C_{\tau-})^+\right) .$$

Similarly, we can list the cash-flows exchanged under all the scenarios in the case when the investor defaults first, where $\varepsilon_\tau = \varepsilon_{C,\tau}$:

5. If the close-out amount to the counterparty is positive from the investor's point of view on the default of the investor, and the collateral is posted by the counterparty to the investor, then the investor receives the remaining exposure netted by the collateral, or pays any excess collateral

$$\theta^5_\tau(\varepsilon_{C,\tau}) = \mathbf{1}_{\{\tau=\tau_I<T\}}\mathbf{1}_{\{\varepsilon_\tau>0\}}\mathbf{1}_{\{C_{\tau-}>0\}}\left((\varepsilon_\tau - C_{\tau-})^+ + R'_I(\varepsilon_\tau - C_{\tau-})^-\right) .$$

6. If the counterparty has a negative exposure, i.e. $\varepsilon_\tau > 0$ from the investor's point of view, and the investor has posted collateral to the counterparty, then the counterparty pays the full exposure to the investor and returns the collateral

$$\theta^6_\tau(\varepsilon_{C,\tau}) = \mathbf{1}_{\{\tau=\tau_I<T\}}\mathbf{1}_{\{\varepsilon_\tau>0\}}\mathbf{1}_{\{C_{\tau-}<0\}}(\varepsilon_\tau - C_{\tau-}) .$$

7. If the close-out amount to the counterparty is negative seen by the investor, and the counterparty has posted collateral to the investor, then the counterparty would face a loss on the exposure and a possible loss on the collateral if it has been rehypothecated

$$\theta^7_\tau(\varepsilon_{C,\tau}) = \mathbf{1}_{\{\tau=\tau_I<T\}}\mathbf{1}_{\{\varepsilon_\tau<0\}}\mathbf{1}_{\{C_{\tau-}>0\}}(R_I\varepsilon_\tau - R'_I C_{\tau-}) .$$

8. If the counterparty have a positive exposure, i.e. $\varepsilon_\tau < 0$, viewed by the investor, and the investor has posted collateral to the counterparty, then the on-default cash-flow is given as the exposure netted by any available collateral

$$\theta^8_\tau(\varepsilon_{C,\tau}) = \mathbf{1}_{\{\tau=\tau_I<T\}}\mathbf{1}_{\{\varepsilon_\tau<0\}}\mathbf{1}_{\{C_{\tau-}<0\}}\left((\varepsilon_\tau - C_{\tau-})^+ + R_I(\varepsilon_\tau - C_{\tau-})^-\right) .$$

If we aggregate all these cash-flows and the pre-default value of collateral account, we have the discounted on-default cash-flow given as

$$\begin{aligned}\theta_\tau(C,\varepsilon) = C_{\tau^-} &+ \theta^1_\tau(\varepsilon_{I,\tau}) + \theta^2_\tau(\varepsilon_{I,\tau}) + \theta^3_\tau(\varepsilon_{I,\tau}) + \theta^4_\tau(\varepsilon_{I,\tau}) \\ &+ \theta^5_\tau(\varepsilon_{C,\tau}) + \theta^6_\tau(\varepsilon_{C,\tau}) + \theta^7_\tau(\varepsilon_{C,\tau}) + \theta^8_\tau(\varepsilon_{C,\tau}).\end{aligned}$$

Combining the above cash-flow with the contractual cash flows and the cash flows from the collateral margining, and taking expectation under the risk-neutral pricing measure, we reach the following proposition.

Proposition 2.1.1. *The collateral inclusive bilateral counterparty risk valuation adjusted pricing formula (without considering funding and investing costs) at time $t < \tau$ is given by*

$$\bar{V}_t(C) = \mathbb{E}_t\left[\,\Pi(t, T\wedge\tau) + \gamma(t, T\wedge\tau; C) + \mathbf{1}_{\{t<\tau<T\}}D(t,\tau)\theta_\tau(C,\varepsilon)\,\right], \tag{2.11}$$

where the on-default cash-flow is defined as

$$\begin{aligned}\theta_\tau(C,\varepsilon) :=\quad & \mathbf{1}_{\{\tau=\tau_C<\tau_I\}}\left(\varepsilon_{I,\tau} - \mathrm{L{\scriptstyle GD}}_C(\varepsilon^+_{I,\tau} - C^+_{\tau-})^+ - \mathrm{L{\scriptstyle GD}}'_C(\varepsilon^-_{I,\tau} - C^-_{\tau-})^+\right) \\ &+ \mathbf{1}_{\{\tau=\tau_I<\tau_C\}}\left(\varepsilon_{C,\tau} - \mathrm{L{\scriptstyle GD}}_I(\varepsilon^-_{C,\tau} - C^-_{\tau-})^- - \mathrm{L{\scriptstyle GD}}'_I(\varepsilon^+_{C,\tau} - C^+_{\tau-})^-\right) .\end{aligned} \tag{2.12}$$

We define the loss-given-default as $\mathrm{L{\scriptstyle GD}}_C := 1 - R_C$, and the collateral loss-given-default due to rehypothecation as $\mathrm{L{\scriptstyle GD}}'_C := 1 - R'_C$.

Proof. The first two terms in the pricing formula (2.11) are straightforward. We focus on the simplification of the on-default cash-flow term which has been broken down into the pre-default collateral value and eight default cases. We now combine the cases step by step:

$$
\begin{aligned}
&\theta^1_\tau(\varepsilon_{I,\tau}) + \theta^2_\tau(\varepsilon_{I,\tau})\\
&= \mathbf{1}_{\{\tau_C<\tau_I\}}\mathbf{1}_{\{\varepsilon_{I,\tau}<0\}}\Big[\mathbf{1}_{\{C_{\tau^-}>0\}}(\varepsilon_{I,\tau} - C_{\tau^-})\\
&\quad + \mathbf{1}_{\{C_{\tau^-}<0\}}\Big((\varepsilon_{I,\tau} - C_{\tau^-})^- + R'_C(\varepsilon_{I,\tau} - C_{\tau^-})^+\Big)\Big]\\
&= \mathbf{1}_{\{\tau_C<\tau_I\}}\mathbf{1}_{\{\varepsilon_{I,\tau}<0\}}\Big[(\varepsilon_{I,\tau} - C_{\tau^-}) - \mathbf{1}_{\{C_{\tau^-}<0\}}\Big((\varepsilon_{I,\tau} - C_{\tau^-})^+ - R'_C(\varepsilon_{I,\tau} - C_{\tau^-})^+\Big)\Big]\\
&= \mathbf{1}_{\{\tau_C<\tau_I\}}\Big[\mathbf{1}_{\{\varepsilon_{I,\tau}<0\}}(\varepsilon_{I,\tau} - C_{\tau^-}) - \mathrm{L{\scriptstyle GD}}'_C(\varepsilon^-_{I,\tau} - C^-_{\tau^-})^+ - \mathbf{1}_{\{\varepsilon_{I,\tau}>0\}}\mathrm{L{\scriptstyle GD}}'_C C^-_{\tau^-}\Big];
\end{aligned}
$$

For case 3 and 4, we have

$$
\begin{aligned}
&\theta^3_\tau(\varepsilon_{I,\tau}) + \theta^4_\tau(\varepsilon_{I,\tau})\\
&= \mathbf{1}_{\{\tau_C<\tau_I\}}\mathbf{1}_{\{\varepsilon_{I,\tau}>0\}}\Big[\mathbf{1}_{\{C_{\tau^-}>0\}}\Big((\varepsilon_{I,\tau} - C_{\tau^-})^- + R_C(\varepsilon_{I,\tau} - C_{\tau^-})^+\Big)\\
&\quad + \mathbf{1}_{\{C_{\tau^-}<0\}}(R_C\varepsilon_{I,\tau} - R'_C C_{\tau^-})\Big]\\
&= \mathbf{1}_{\{\tau_C<\tau_I\}}\mathbf{1}_{\{\varepsilon_{I,\tau}>0\}}\Big[(R_C\varepsilon_{I,\tau} - R'_C C_{\tau^-}) + \mathbf{1}_{\{C_{\tau^-}>0\}}\Big((\varepsilon_{I,\tau} - C_{\tau^-}) - \mathrm{L{\scriptstyle GD}}_C(\varepsilon_{I,\tau} - C_{\tau^-})^+\\
&\quad - (R_C\varepsilon_{I,\tau} - R'_C C_{\tau^-})\Big)\Big]\\
&= \mathbf{1}_{\{\tau_C<\tau_I\}}\Big[\mathbf{1}_{\{\varepsilon_{I,\tau}>0\}}(R_C\varepsilon_{I,\tau} - R'_C C_{\tau^-}) + \mathbf{1}_{\{\varepsilon_{I,\tau}>0\}}\mathbf{1}_{\{C_{\tau^-}>0\}}(\mathrm{L{\scriptstyle GD}}_C\varepsilon_{I,\tau} - \mathrm{L{\scriptstyle GD}}'_C C_{\tau^-})\\
&\quad - \mathrm{L{\scriptstyle GD}}_C(\varepsilon^+_{I,\tau} - C^+_{\tau^-})^+ + \mathbf{1}_{\{C_{\tau^-}<0\}}\mathrm{L{\scriptstyle GD}}_C\varepsilon^+_{I,\tau}\Big].
\end{aligned}
$$

Combining the above 4 cases yields

$$
\begin{aligned}
&\theta^1_\tau(\varepsilon_{I,\tau}) + \theta^2_\tau(\varepsilon_{I,\tau}) + \theta^3_\tau(\varepsilon_{I,\tau}) + \theta^4_\tau(\varepsilon_{I,\tau})\\
&= \mathbf{1}_{\{\tau_C<\tau_I\}}\Big[\varepsilon_{I,\tau} - C_{\tau^-} - \mathrm{L{\scriptstyle GD}}_C(\varepsilon^+_{I,\tau} - C^+_{\tau^-})^+ - \mathrm{L{\scriptstyle GD}}'_C(\varepsilon^-_{I,\tau} - C^-_{\tau^-})^+\Big].
\end{aligned}
$$

Using a similar technique, we can calculate the sum of the rest of cash flows:

$$
\begin{aligned}
&\theta^5_\tau(\varepsilon_{C,\tau}) + \theta^6_\tau(\varepsilon_{C,\tau})\\
&= \mathbf{1}_{\{\tau_I<\tau_C\}}\Big[\mathbf{1}_{\{\varepsilon_{C,\tau}>0\}}(\varepsilon_{C,\tau} - C_{\tau^-}) - \mathrm{L{\scriptstyle GD}}'_I(\varepsilon^+_{C,\tau} - C^+_{\tau^-})^- - \mathbf{1}_{\{\varepsilon_{C,\tau}<0\}}\mathbf{1}_{\{C_{\tau^-}>0\}}\mathrm{L{\scriptstyle GD}}'_I C_{\tau^-}\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
&\theta^6_\tau(\varepsilon_{C,\tau}) + \theta^7_\tau(\varepsilon_{C,\tau}) \\
&\quad = \mathbf{1}_{\{\tau_I<\tau_C\}}\Big[\mathbf{1}_{\{\varepsilon_{C,\tau}<0\}}(R_I\varepsilon_{C,\tau} - R'_I C_{\tau^-}) + \mathbf{1}_{\{\varepsilon_{C,\tau}<0\}}\mathbf{1}_{\{C_{\tau^-}<0\}}(\mathrm{L{\scriptstyle GD}}_I\varepsilon_{C,\tau} - \mathrm{L{\scriptstyle GD}}'_I C_{\tau^-}) \\
&\qquad - \mathrm{L{\scriptstyle GD}}_I(\varepsilon^-_{C,\tau} - C^-_{\tau^-})^- + \mathbf{1}_{\{\varepsilon_{C,\tau}<0\}}\mathbf{1}_{\{C_{\tau^-}>0\}}\mathrm{L{\scriptstyle GD}}_I\varepsilon_{C,\tau}\Big].
\end{aligned}
$$

Therefore, the sum of the cash flows of cases 5-8 is

$$
\begin{aligned}
&\theta^5_\tau(\varepsilon_{C,\tau}) + \theta^6_\tau(\varepsilon_{C,\tau}) + \theta^7_\tau(\varepsilon_{C,\tau}) + \theta^8_\tau(\varepsilon_{C,\tau}) \\
&\quad = \mathbf{1}_{\{\tau_I<\tau_C\}}\Big[\varepsilon_{C,\tau} - C_{\tau^-} - \mathrm{L{\scriptstyle GD}}_I(\varepsilon^-_{C,\tau} - C^-_{\tau^-})^- - \mathrm{L{\scriptstyle GD}}'_I(\varepsilon^+_{I,\tau} - C^+_{\tau^-})^-\Big].
\end{aligned}
$$

Putting all terms together with the pre-default value of the collateral account, we arrive at expression (2.12) for the on-default cash-flow. □

If both parties agree on the exposure, namely $\varepsilon_{I,\tau} = \varepsilon_{C,\tau} = \varepsilon_\tau$, when we take the risk-neutral expectation in (2.12), we see that the price of the discounted on-default cash-flow,

$$
\mathbb{E}_t[\mathbf{1}_{\{t<\tau<T\}}D(t,\tau)\theta_\tau(C,\varepsilon)] = \mathbb{E}_t[\mathbf{1}_{\{t<\tau<T\}}D(t,\tau)\,\varepsilon_\tau] - \mathrm{C{\scriptstyle VA}}(t,T;C) + \mathrm{D{\scriptstyle VA}}(t,T;C), \tag{2.13}
$$

is the present value of the close-out amount reduced by the collateralized CVA and DVA terms

$$
\begin{aligned}
\mathrm{C{\scriptstyle VA}}(t,T;C) &:= \mathbb{E}_t\left[\mathbf{1}_{\{\tau=\tau_C<T\}}D(t,\tau)\Pi_{\mathrm{CVAcoll}}(\tau)\right], \\
\mathrm{D{\scriptstyle VA}}(t,T;C) &:= \mathbb{E}_t\left[\mathbf{1}_{\{\tau=\tau_I<T\}}D(t,\tau)\Pi_{\mathrm{DVAcoll}}(\tau)\right], \qquad (2.14)
\end{aligned}
$$

and

$$
\begin{aligned}
\Pi_{\mathrm{CVAcoll}}(s) &= \left(\mathrm{L{\scriptstyle GD}}_C(\varepsilon^+_{I,s} - C^+_{s-})^+ + \mathrm{L{\scriptstyle GD}}'_C(\varepsilon^-_{I,s} - C^-_{s-})^+\right) \geq 0, \\
\Pi_{\mathrm{DVAcoll}}(s) &= -\left(\mathrm{L{\scriptstyle GD}}_I(\varepsilon^-_{C,s} - C^-_{s-})^- + \mathrm{L{\scriptstyle GD}}'_I(\varepsilon^+_{C,s} - C^+_{s-})^-\right) \geq 0.
\end{aligned}
$$

Also, observe that if rehypothecation of the collateral is not allowed, the terms multiplied by LGD'_C and LGD'_I drop out of the CVA and DVA calculations.

In order to aid the reader, for the rest of the thesis, we will simplify the model by assuming that both parties agree on the exposure, i.e. $\varepsilon_{I,\tau} = \varepsilon_{C,\tau} = \varepsilon_\tau$, unless otherwise stated.

2.2 Valuation under funding risk

The hedging strategy that perfectly replicates the no-arbitrage price of a derivative is formed by a position in cash and a position in a portfolio of hedging instruments. When we talk about funding of a derivative deal, we essentially mean the cash position that is required to establish the hedging strategy, and with funding costs we refer to the costs of maintaining this cash position. If we denote the cash account by F and the risky-asset account by H, we get

$$\bar{V}_t = F_t + H_t\,.$$

In the classical Black-Scholes-Merton theory, the risky part H of the hedge would be a delta position in the underlying stock, whereas the risk-less part F would be a position in the risk-free bank account.

If the deal is collateralized, the margining procedure is included in the deal definition insuring that funding of the collateral is automatically taken into account. Moreover, if rehypothecation is allowed for the collateralized deal, the collateral taker can use the posted collateral as a funding source, and thereby reduce or maybe even eliminate the costs of funding the deal. Thus, we have the following two definitions of the funding account:

(i) If rehypothecation of the posted collateral is forbidden, we have

$$F_t := \bar{V}_t - H_t, \tag{2.15}$$

(ii) If such rehypothecation is allowed, then

$$F_t := \bar{V}_t - H_t - C_t. \tag{2.16}$$

By implication of (2.16) and (2.15) it is obvious that, at time t, if the funding account $F_t > 0$, the dealer needs to borrow cash to establish the hedging strategy. Correspondingly, if the funding account $F_t < 0$, the hedging strategy requires the dealer to invest surplus cash. Specifically, we assume the dealer enters a funding position on a discrete time-grid $\{t_1, \ldots, t_m\}$ during the life of the deal. Given two adjacent funding times t_j and t_{j+1}, for $1 \leq j \leq m-1$, the dealer enters a position in cash equal to F_{t_j} at time t_j. At time t_{j+1} the dealer redeems the position again and either returns the cash to the funder if it was a long cash position and pays funding costs on the borrowed cash, or he gets the cash back if it was a short cash position and receives funding benefits as interest on the invested cash. We assume that these funding costs and benefits are determined at the start date of each funding period and charged at the end of the period.

The price of the contracts used by the investor to fund the deal, without loss of generality, are assumed to be adapted processes. Let $P_t^{\tilde{f}}(T)$ represent the price of a borrowing (or lending) contract measurable at t where the dealer pays (or receives) one unit of cash at maturity $T > t$. We introduce the effective funding rate $\tilde{f}_t$ as a function: $\tilde{f}_t = f(t, F, H, C)$, assuming that it depends on the cash account F_t, hedging account H_t and collateral account C_t. Moreover, the zero-coupon bond corresponding to the effective funding rate is defined as

$$P_t^{\tilde{f}}(T) := [1 + (T - t)\tilde{f}_t(T)]^{-1}.$$

If we assume that the dealer hedges the derivatives position by trading in the spot market of the underlying asset(s), and the hedging strategy is implemented on the same time-grid as the funding procedure of the deal, the sum of discounted cash-flows from funding the hedging strategy during the life of the deal is equal to

$$\begin{aligned}
&\varphi(t, T\wedge\tau; F, H) \\
&\quad = \sum_{j=1}^{m-1} \mathbf{1}_{\{t\leqslant t_j<(T\wedge\tau)\}} D(t,t_j) \left(F_{t_j} - (F_{t_j}+H_{t_j}) \frac{P_{t_j}(t_{j+1})}{P_{t_j}^{\tilde{f}}(t_{j+1})} + H_{t_j} \frac{P_{t_j}(t_{j+1})}{P_{t_j}^{\tilde{f}}(t_{j+1})} \right) \\
&\quad = \sum_{j=1}^{m-1} \mathbf{1}_{\{t\leqslant t_j<(T\wedge\tau)\}} D(t,t_j) F_{t_j} \left(1 - \frac{P_{t_j}(t_{j+1})}{P_{t_j}^{\tilde{f}}(t_{j+1})} \right) . \qquad (2.17)
\end{aligned}$$

This is, strictly speaking, a discounted payout and the funding cost or benefit at time t is obtained by taking the risk neutral expectation of the above cash-flows. For a trading example giving more details on how the above formula for φ originates see [20].

As we can see from equation (2.17), the dependence of the hedging account dropped off from the funding procedure. For modelling convenience, we can define the effective funding rate $\tilde{f}_t$ faced by the dealer as

$$\tilde{f}_t(T) := f_t^-(T)\mathbf{1}_{\{F_t<0\}} + f_t^+(T)\mathbf{1}_{\{F_t>0\}} . \qquad (2.18)$$

A related framework would be to consider the hedging account H as being perfectly collateralized and to use the collateral to fund the hedging account, so that there is no funding cost associated with the hedging account.

As before with collateral costs, we may rewrite the cash flows for funding as a first order approximation in continuously compounded rates $\tilde{f}$ and r associated to the relevant bonds. We obtain

$$\varphi(t, T\wedge\tau; F) \approx \sum_{j=1}^{m-1} \mathbf{1}_{\{t\leqslant t_j<(T\wedge\tau)\}} D(t,t_j) F_{t_j} \alpha_j \left(r_{t_j}(t_{j+1}) - \tilde{f}_{t_j}(t_{j+1}) \right) , \qquad (2.19)$$

where α_j is the time fraction between t_j and t_{j+1}. To help clarify the process, we look at the following toy example.

Example. (Equity call option example) *We consider a one period Binomial model for the stock price where the current stock price is* $S_0 = \$100$ *and moves to* S_1*, which can be either \$125 or \$75 in one years time. Let us consider a one year* *expiry call option on this stock with strike \$100. The option payoff* V_1 *will then be either \$25 or \$0 at expiry. We now investigate what happens at each time step if we sell this option.*

At time 0 *we receive cash* V_0 *from selling the option and we put cash* C_0 *into the collateral account. We borrow* ΔS_0 *(*$\$\Delta 100$*) amount of cash from the treasury and buy* Δ *units of the stock to set up the delta hedging portfolio. At this step, the risky-asset account is given by* $H_0 = \Delta S_0$ *and the funding cash account is given by* $F_0 = V_0 - C_0 - H_0$*.*

Now we consider the cash-flows in one year. We pay V_1 *to the counterparty to close the deal. We sell the stock and receive* ΔS_1 *(either* $\$\Delta 125$ *or* $\$\Delta 75$ *depending on the state of the world). Finally, we get back the collateral and receive interest at the collateral rate* c^+ *resulting in a net cash-flow of* $C_0(1+c^+)$*. The net amount that is borrowed from the treasury is given by* F_0*, which accrues at the funding borrowing rate* f^+ *and generates a cash-flow of* $(V_0 - C_0 - H_0)(1 + f^+)$*.*

The sum of all the net cash-flows yields

$$
\begin{aligned}
&(V_0 - C_0 - H_0)(1 + f^+) - V_1 + C_0(1 + c^+) + \Delta S_1 \\
&= (V_1 - V_0) - f^+ V_0 + (f^+ - c^+)C_0 - \Delta(S_1 - S_0) - f^+ H_0,
\end{aligned}
$$

which needs to be zero over this period in order to avoid arbitrage and replicate the option. We therefore obtain the following two equations in our binomial model:

$$(25 - V_0) - f^+ V_0 + (f^+ - c^+)C_0 - \Delta(125 - 100) - f^+ \Delta 100 = 0, \tag{2.20}$$

$$(0 - V_0) - f^+ V_0 + (f^+ - c^+)C_0 - \Delta(75 - 100) - f^+ \Delta 100 = 0. \tag{2.21}$$

If we assume perfect collateralization, i.e. $C_0 = V_0$*, and that the collateral rate is* $c^+ = 1\%$ *and the borrowing rate is* $f^+ = 1.2\%$*, we find that the derivative price* V_0 *is* $\$12.97$ *and the delta hedge is* $\Delta = 0.5$*.*

We should also mention that, occasionally, we may include the effects of repo markets or stock lending in our framework. In general, we may borrow/lend the cash needed to establish H from/to our treasury, and we may then use the risky asset in H for repo or stock lending/borrowing in the market. This means that we could include the funding costs and benefits coming from this use of the risky asset. In this chapter we assume that the bank Treasury automatically recognizes this benefit/cost to us at the same rate $\tilde{f}$ used for cash, but for a more general framework involving repo rate $\tilde{h}$ see Chapter 6 for a quick discussion and see for example [20, 61] for a more detailed analysis.

The particular positions entered by the dealer to either borrow or invest cash according to the sign and size of the funding account depend on the bank's liquidity policy. In the following we discuss two possible cases: One that the dealer can fund at rates set by the bank's treasury department, and another that the dealer goes to the market directly and funds his trades at the prevailing market rates. As a result, the funding rates and therefore the funding effect on the price of a derivative deal depends intimately on the chosen liquidity policy.

2.2.1 Treasury funding

If the dealer funds the hedge through the bank's treasury department, at time t, the treasury determines the funding rates $f_t^{\pm}$ faced by the dealer. We assume an average of funding costs and benefits is applied across all deals, regardless of the specific deal. This leads to two curves as functions of maturity: one for borrowing funds f^+ and one for lending funds f^-.

After entering a funding position F_{t_j} at time t_j, for $1 \leq j \leq m-1$, the dealer faces the following discounted cash-flow

$$\Phi_j(t_j, t_{j+1}; F) := -N_{t_j} D(t_j, t_{j+1}) \,, \tag{2.22}$$

with

$$N_{t_j} := \frac{F_{t_j}^-}{P_{t_j}^{f^-}(t_{j+1})} + \frac{F_{t_j}^+}{P_{t_j}^{f^+}(t_{j+1})} \,.$$

Under this liquidity policy, the treasury – and not the dealer himself – is in charge of debt valuation adjustments due to funding-related positions. Also, being entities of the same institution, both the dealer and the treasury disappear in case of the default of the institution without any further cash-flows being exchanged and we can neglect the effects of funding in this case. So, when default risk is considered, this leads to the following definition of the funding cash flows

$$\bar{\Phi}_j(t_j, t_{j+1}; F) := \mathbf{1}_{\{\tau > t_j\}} \Phi_j(t_j, t_{j+1}; F) \,.$$

Thus, the risk-neutral price of the cash-flows due to the funding positions entered at time t_j is

$$\mathbb{E}_{t_j}\left[\, \bar{\Phi}_j(t_j, t_{j+1}; F) \,\right] = -\mathbf{1}_{\{\tau > t_j\}} \left(F^-_{t_j} \frac{P_{t_j}(t_{j+1})}{P^{f^-}_{t_j}(t_{j+1})} + F^+_{t_j} \frac{P_{t_j}(t_{j+1})}{P^{f^+}_{t_j}(t_{j+1})} \right) .$$

If we consider a sequence of such funding operations at each time t_j during the life of the deal, we can define the sum of the cash-flows coming from all the borrowing and lending positions opened by the dealer to hedge the trade up to the first-default event as follows,

$$\begin{aligned} \varphi(t, T \wedge \tau; F) &:= \sum_{j=1}^{m-1} \mathbf{1}_{\{t \leqslant t_j < (T \wedge \tau)\}} D(t, t_j) \left(F_{t_j} + \mathbb{E}_{t_j}\left[\, \bar{\Phi}_j(t_j, t_{j+1}; F) \,\right] \right) \\ &= \sum_{j=1}^{m-1} \mathbf{1}_{\{t \leqslant t_j < (T \wedge \tau)\}} D(t, t_j) \left(F_{t_j} - F^-_{t_j} \frac{P_{t_j}(t_{j+1})}{P^{f^-}_{t_j}(t_{j+1})} - F^+_{t_j} \frac{P_{t_j}(t_{j+1})}{P^{f^+}_{t_j}(t_{j+1})} \right) . \end{aligned}$$

In terms of the effective funding rate, this expression collapses to

$$\varphi(t, T \wedge \tau; F) := \sum_{j=1}^{m-1} \mathbf{1}_{\{t \leqslant t_j < (T \wedge \tau)\}} D(t, t_j) F_{t_j} \left(1 - \frac{P_{t_j}(t_{j+1})}{P^{\tilde{f}}_{t_j}(t_{j+1})} \right) , \tag{2.23}$$

where the zero-coupon bond corresponding to the effective funding rate is defined as $P^{\tilde{f}}_t(T) := [1 + (T-t)\tilde{f}_t(T)]^{-1}$. This is, strictly speaking, a discounted payout and the funding cost or benefit at time t is obtained by taking the risk-neutral expectation of the above cash-flows.

As before with collateral costs, we may rewrite the cash flows for funding as a first order approximation in continuously compounded rates $\tilde{f}$ and r associated to the relevant bonds. We obtain

$$\varphi(t, T \wedge \tau; F) \approx \sum_{j=1}^{m-1} \mathbf{1}_{\{t \leqslant t_j < (T \wedge \tau)\}} D(t, t_j) F_{t_j} \alpha_j \left(r_{t_j}(t_{j+1}) - \tilde{f}_{t_j}(t_{j+1}) \right) . \tag{2.24}$$

2.2.2 Market funding

If the dealer funds the hedging strategy directly in the market – and not through the bank's treasury – the funding rates are determined by prevailing market conditions and are often deal specific. This means that the rate f^+ the dealer can borrow funds at may be different from the rate f^- at which funds can be invested. Moreover, these rates may differ across deals depending on the deals' notional, maturity structures, dealer-client relationship, and so forth. Similar to the liquidity policy of treasury funding, we assume a deal's funding operations are closed down in the case of default. Furthermore, as the dealer now operates directly on the market, he needs to include a debit valuation adjustment due to his funding positions when he marks-to-market his trading books. For simplicity, we assume that the funder in the market is default-free so no funding CVA needs to be accounted for. The discounted cash-flows from the borrowing or lending position, incorporated with the default probability of the dealer, between two adjacent funding times t_j and t_{j+1}, for $1 \leq j \leq m-1$, is given by

$$\begin{aligned} \bar{\Phi}_j(t_j, t_{j+1}; F) := {} & \mathbf{1}_{\{\tau > t_j\}} \mathbf{1}_{\{\tau_I > t_{j+1}\}} \Phi_j(t_j, t_{j+1}; F) \\ & - \mathbf{1}_{\{\tau > t_j\}} \mathbf{1}_{\{\tau_I < t_{j+1}\}} (\text{L}\textsc{gd}_I \varepsilon^-_{F,\tau_I} - \varepsilon_{F,\tau_I}) D(t_j, \tau_I) , \end{aligned}$$

or equivalently,

$$\begin{aligned} \bar{\Phi}_j(t_j, t_{j+1}; F) := {} & \mathbf{1}_{\{\tau > t_j\}} \mathbf{1}_{\{\tau_I > t_{j+1}\}} \Phi_j(t_j, t_{j+1}; F) \\ & - \mathbf{1}_{\{\tau > t_j\}} \mathbf{1}_{\{\tau_I < t_{j+1}\}} (R_I \varepsilon^-_{F,\tau_I} + \varepsilon^+_{F,\tau_I}) D(t_j, \tau_I) , \end{aligned}$$

where $\Phi_j(t_j, t_{j+1}; F)$ is as defined in (2.22), and $\varepsilon_{F,t}$ is the close-out amount calculated by the funder on the dealer's default as

$$\varepsilon_{F,\tau_I} := -N_{t_j} P_{\tau_I}(t_{j+1}) \,.$$

To price this funding cash-flow, we take the expectation under the risk-neutral probability measure

$$\begin{aligned}
&\mathbb{E}_{t_j}\left[\,\bar{\Phi}_j(t_j, t_{j+1}; F)\,\right] \\
&\quad = \mathbb{E}_{t_j}\left[-\mathbf{1}_{\{\tau>t_j\}}\mathbf{1}_{\{\tau_I>t_{j+1}\}}D(t_j,t_{j+1})\left(\frac{F^-_{t_j}}{P^{f^-}_{t_j}(t_{j+1})}+\frac{F^+_{t_j}}{P^{f^+}_{t_j}(t_{j+1})}\right)\right. \\
&\qquad\left. -\mathbf{1}_{\{\tau>t_j\}}\mathbf{1}_{\{\tau_I<t_{j+1}\}}D(t_j,\tau_I)\left(\frac{F^-_{t_j}}{P^{f^-}_{t_j}(t_{j+1})}+R_I\frac{F^+_{t_j}}{P^{f^+}_{t_j}(t_{j+1})}\right)P_{\tau_I}(t_{j+1})\right].
\end{aligned}$$

Since $P_{\tau_I}(t_{j+1}) = \mathbb{E}_{\tau_I}[\,D(\tau_I, t_{j+1})\,]$, we can write the above expectation as

$$\begin{aligned}
&\mathbb{E}_{t_j}\left[\,\bar{\Phi}_j(t_j, t_{j+1}; F)\,\right] \\
&\quad = \mathbb{E}_{t_j}\left\{-\mathbf{1}_{\{\tau>t_j\}}\mathbf{1}_{\{\tau_I>t_{j+1}\}}D(t_j,t_{j+1})\left(\frac{F^-_{t_j}}{P^{f^-}_{t_j}(t_{j+1})}+\frac{F^+_{t_j}}{P^{f^+}_{t_j}(t_{j+1})}\right)\right. \\
&\qquad\left. -\mathbb{E}_{\tau_I}\left[\mathbf{1}_{\{\tau>t_j\}}\mathbf{1}_{\{\tau_I<t_{j+1}\}}D(t_j,t_{j+1})\left(\frac{F^-_{t_j}}{P^{f^-}_{t_j}(t_{j+1})}+R_I\frac{F^+_{t_j}}{P^{f^+}_{t_j}(t_{j+1})}\right)\right]\right\} \\
&\quad = -\mathbb{E}_{t_j}\left\{\mathbf{1}_{\{\tau>t_j\}}D(t_j,t_{j+1})\left[\mathbf{1}_{\{\tau_I>t_{j+1}\}}\left(\frac{F^-_{t_j}}{P^{f^-}_{t_j}(t_{j+1})}+\frac{F^+_{t_j}}{P^{f^+}_{t_j}(t_{j+1})}\right)\right.\right. \\
&\qquad\left.\left. +\ \mathbf{1}_{\{\tau_I<t_{j+1}\}}\left(\frac{F^-_{t_j}}{P^{f^-}_{t_j}(t_{j+1})}+R_I\frac{F^+_{t_j}}{P^{f^+}_{t_j}(t_{j+1})}\right)\right]\right\} \\
&\quad = -\mathbb{E}_{t_j}\left\{\mathbf{1}_{\{\tau>t_j\}}D(t_j,t_{j+1})\left(\frac{F^-_{t_j}}{P^{f^-}_{t_j}(t_{j+1})}+\frac{F^+_{t_j}}{P^{f^+}_{t_j}(t_{j+1})}(R_I+\mathbf{1}_{\{\tau_I>t_{j+1}\}}\mathrm{L{\scriptsize GD}}_I)\right)\right\}.
\end{aligned}$$

If we define the zero-coupon funding bond for borrowing cash adjusted for the dealer's credit risk as

$$\bar{P}^{f^+}_t(T) := \frac{P^{f^+}_t(T)}{\mathbb{E}^T_t\left[\,\mathrm{L{\scriptsize GD}}_I\mathbf{1}_{\{\tau_I>T\}}+R_I\,\right]}\,,$$

with the expectation on the right hand side being taken under the T-forward measure, we can write the risk-neutral price of the cash-flows due to the funding positions at time t_j as

$$\mathbb{E}_{t_j}\left[\,\bar{\Phi}_j(t_j,t_{j+1};F)\,\right] = -\mathbf{1}_{\{\tau>t_j\}}\left(F^-_{t_j}\frac{P_{t_j}(t_{j+1})}{P^{f^-}_{t_j}(t_{j+1})} + F^+_{t_j}\frac{P_{t_j}(t_{j+1})}{\bar{P}^{f^+}_{t_j}(t_{j+1})}\right).$$

Naturally, since the seniority could be different, one might assume a different recovery rate on the funding position than on the derivatives deal itself (see [39]). Extensions to this case are straightforward.

Now, summing up the discounted cash-flows from the sequence of funding operations through the life of the deal, we get

$$\begin{aligned}\varphi(t,T\wedge\tau;F) &:= \sum_{j=1}^{m-1}\mathbf{1}_{\{t\leqslant t_j<T\wedge\tau\}}D(t,t_j)\left(F_{t_j}+\mathbb{E}_{t_j}\left[\,\bar{\Phi}_j(t_j,t_{j+1};F)\,\right]\right)\\ &= \sum_{j=1}^{m-1}\mathbf{1}_{\{t\leqslant t_j<T\wedge\tau\}}D(t,t_j)\left(F_{t_j} - F^-_{t_j}\frac{P_{t_j}(t_{j+1})}{P^{f^-}_{t_j}(t_{j+1})} - F^+_{t_j}\frac{P_{t_j}(t_{j+1})}{\bar{P}^{f^+}_{t_j}(t_{j+1})}\right).\end{aligned}$$

Notice that, if we set $R_I = 1$ (so that $\text{L}_{\text{GD}_I} = 0$), we get that $\bar{P}^{f^+}_t(T)$ is equal to $P^{f^+}_t(T)$, and we recover the previous example.

To avoid cumbersome notation, we will not explicitly write $\bar{P}^{f^+}$ in the sequel, but just keep in mind that when the dealer funds directly in the market then P^{f^+} needs to be adjusted for funding DVA. Thus, in terms of the effective funding rate, we obtain (2.17).

2.3 General pricing equations for OTC contracts

In the previous section we analysed the discounted cash-flows of a derivative deal and we developed a framework for consistent valuation of such deals under collateralization, counterparty credit risk and funding risk. The arbitrage-free pricing framework is captured in the following theorem.

Theorem 2.3.1 (General pricing equation). *The consistent arbitrage-free price $\bar{V}_t(C,F)$ of collateralized OTC derivative deals with counterparty credit risk and* *funding costs takes the form*

$$\bar{V}_t(C,F) = \mathbb{E}_t\left[\Pi(t,T\wedge\tau) + \gamma(t,T\wedge\tau;C) + \varphi(t,T\wedge\tau;F) + \mathbf{1}_{\{t<\tau<T\}} D(t,\tau)\theta_\tau(C,\varepsilon)\right], \tag{2.25}$$

where

1. $\Pi(t,T\wedge\tau)$ *is the discounted cash-flows from the contract's payoff structure up to the first-default event.*

2. $\gamma(t,T\wedge\tau;C)$ *is the discounted cash-flows from the collateral margining procedure up to the first-default event and is defined in* (2.5)*.*

3. $\varphi(t,T\wedge\tau;F)$ *is the discounted cash-flows from funding the hedging strategy up to the first-default event and is defined in* (2.17)*.*

4. $\theta_\tau(C,\varepsilon)$ *is the on-default cash-flow with close-out amount* ε *and is defined in* (2.12)*.*

If funding and collateral margining costs are discarded, while collateral is retained for loss reduction at default, this pricing equation collapses to the formula derived in [18] for the price of a derivative under bilateral counterparty credit risk. If further collateral guarantees are dropped, the formula reduces to the bilateral credit valuation formula in Proposition 1.2.3.

While the pricing equation is conceptually clear – we simply take the expectation of the sum of all discounted cash-flows of the deal under the risk-neutral measure – solving the equation poses a recursive, nonlinear problem. The future paths of the effective funding rate $\tilde{f}$ depend on the future signs of the funding account F, i.e. whether we need to borrow or lend cash on each future funding date. At the same time, through the relations (2.16) and (2.15), the future sign and size of the funding account F depend on the adjusted price $\bar{V}$ of the deal which is the quantity we are trying to compute in the first place. One crucial implication of this recursive structure of the pricing problem is the fact that FVA is generally not just an additive adjustment term, in contrast to CVA and DVA. More importantly, the conjecture identifying the DVA of a deal with its funding is wrong in general. Only in the unrealistic setting where the dealer can fund an uncollateralized trade at equal borrowing and lending rates, i.e. $f^+ = f^-$, do we achieve the additive structure often assumed by practitioners. If the trade is collateralized, we need to impose even further restrictions as to how the collateral is linked to price of the trade $\bar{V}$.

Remark 2.3.2. The law of one price. *On the theoretical side, the pricing equation shakes the foundation of the celebrated Law of One Price prevailing in classical derivatives pricing. Clearly, if we assume no funding costs, the dealer and counterparty agree on the price of the deal as both parties can – at least theoretically – observe the credit risk of each other through CDS contracts traded in the market and the relevant market risks, thus agreeing on CVA and DVA. In contrast, introducing funding costs, they will not agree on the FVA for the deal due to asymmetric information. The parties cannot observe each others' liquidity policies nor their respective funding costs associated with a particular deal. As a result, the value of a deal position will not generally be the same to the counterparty as to the dealer just with an opposite sign. In principle, this should mean that the dealer and the counterparty would never close the trade, but in practice trades are executed as a simple consequence of the fundamental forces of supply and demand. Nevertheless, among dealers it is a general belief that funding costs were one of the main factors driving the bid-ask spreads wider during the recent financial crisis.*

Finally, as we adopt a risk-neutral valuation framework, we implicitly assume the existence of a risk-free interest rate. Indeed, since the valuation adjustments are included as additional cash-flows and not as ad-hoc spreads, all the cash-flows are discounted by the risk-free discount factor $D(t,T)$ in (2.25). Nevertheless, the risk-free rate is merely an instrumental variable of the general pricing equation. We clearly distinguish market rates from the theoretical risk-free rate avoiding the false claim that the over-night rates (e.g., EONIA) are risk-free. In fact, as we will show later in continuous time, if the dealer funds the hedging strategy of the trade through cash accounts available to him – whether as rehypothecated collateral or funds from the treasury, repo market, etc. – the risk-free rate vanishes from the pricing equation.

2.3.1 Discrete-time formulation

Our purpose in this section is to turn the recursive pricing equation (2.25) into a set of iterative equations that can be solved numerically. The use of the least-squares Monte Carlo methods is already standard in CVA and DVA calculations (for example in Brigo and Pallavicini [28]). To this end, we introduce the auxiliary function

$$\bar{\Pi}(t_j, t_{j+1}; C) := \Pi(t_j, t_{j+1} \wedge \tau) + \gamma(t_j, t_{j+1} \wedge \tau; C) + \mathbf{1}_{\{t_j < \tau < t_{j+1}\}} D(t_j, \tau)\theta_\tau(C, \varepsilon) \quad (2.26)$$

which defines the cash-flows of the deal occurring between time t_j and t_{j+1} adjusted for collateral margining costs and default risks. We stress the fact that the close-out amount used for calculating the on-default cash-flow still refers to a deal with maturity T. If we then solve pricing equation (2.25) at each funding date t_j in the time-grid $\{t_1, \ldots, t_n = T\}$, we obtain the deal price $\bar{V}$ at time t_j as a function of the deal price on the next consecutive funding date t_{j+1},

$$\bar{V}_{t_j} = \mathbb{E}_{t_j}\left[\bar{V}_{t_{j+1}} D(t_j, t_{j+1}) + \bar{\Pi}(t_j, t_{j+1}; C) \right] + \mathbf{1}_{\{\tau > t_j\}} \left(F_{t_j} - F^-_{t_j} \frac{P_{t_j}(t_{j+1})}{P^{f^-}_{t_j}(t_{j+1})} - F^+_{t_j} \frac{P_{t_j}(t_{j+1})}{P^{f^+}_{t_j}(t_{j+1})} \right),$$

Furthermore, we have the terminal condition $\bar{V}_{t_n} := 0$ on the final date $t_n = T$. Recall the definitions of the funding account in (2.16) if no rehypothecation of collateral is allowed and in (2.15) if rehypothecation is permitted. We can substitute the two equations of F and then solve the above equation for the positive and negative parts of the funding account respectively. The outcome is a discrete-time iterative solution of the recursive pricing equation, provided in the following theorem.

Theorem 2.3.3 (Discrete-time solution of the general pricing equation). *We may solve the full recursive pricing equation in Theorem 2.3.1 as a set of* *backward-iterative equations on the time-grid* $\{t_1, \ldots, t_n = T\}$ *with* $\bar{V}_{t_n} := 0$*. For* $\tau < t_j$*, we have*

$$\bar{V}_{t_j} = 0,$$

while for $\tau > t_j$*, we have*

(i) if re-hypothecation is forbidden:

$$\left(\bar{V}_{t_j} - H_{t_j}\right)^{\pm} = P_{t_j}^{\tilde{f}}(t_{j+1}) \left(\mathbb{E}_{t_j}^{t_{j+1}}\left[\bar{V}_{t_{j+1}} + \frac{\bar{\Pi}(t_j, t_{j+1}; C) - H_{t_j}}{D(t_j, t_{j+1})}\right]\right)^{\pm},$$

(ii) if re-hypothecation is allowed:

$$\left(\bar{V}_{t_j} - C_{t_j} - H_{t_j}\right)^{\pm} = P_{t_j}^{\tilde{f}}(t_{j+1}) \left(\mathbb{E}_{t_j}^{t_{j+1}}\left[\bar{V}_{t_{j+1}} + \frac{\bar{\Pi}(t_j, t_{j+1}; C) - C_{t_j} - H_{t_j}}{D(t_j, t_{j+1})}\right]\right)^{\pm},$$

where the expectations are taken under the $\mathbb{Q}^{t_{j+1}}$*-forward measure.*

The $\pm$ sign in the theorem is supposed to stress the fact that the sign of the funding account, which determines the effective funding rate, depends on the sign of the conditional expectation. Further intuition may be gained by moving to the continuous time setting which is the case we will now turn to.

2.3.2 Continuous-time formulation

Let us consider a continuous-time approximation of the general pricing equation. In the following, we assume that rehypothecation is allowed, but similar results hold if this is not the case. This implies that collateral margining, funding procedures and hedging strategies are executed in continuous time. By taking the time limit, we have the following expressions for the discounted cash-flow streams of the deal

$$\begin{aligned}
\Pi(t, T \wedge \tau) &= \int_t^{T\wedge\tau} D(t,s)\Pi(s, s + \mathrm{d}s), \\
\gamma(t, T \wedge \tau; C) &= \int_t^{T\wedge\tau} (r_s - \tilde{c}_s) C_s D(t,s)\mathrm{d}s, \\
\varphi(t, T \wedge \tau; F) &= \int_t^{T\wedge\tau} (r_s - \tilde{f}_s) F_s D(t,s)\mathrm{d}s,
\end{aligned}$$

where as before, $\Pi(t, t + \mathrm{d}t)$ is the payoff coupon process of the derivative contract and r_t is the risk-free rate. These last two equations can be immediately derived by looking at the approximations given in equations (2.6) and (2.24).

Then, putting all the above terms together with the on-default cash-flow as in Theorem 2.3.1, and substituting the funding account as defined in (2.16), the recursive pricing equation yields

$$\begin{aligned}\bar{V}_t = &\int_t^T \mathbb{E}_t\big[\, D(t,s)\,\big(\mathbf{1}_{\{s<\tau\}}\Pi(s,s+\mathrm{d}s) + \mathbf{1}_{\{\tau\in\mathrm{d}s\}}\theta_s(C,\varepsilon)\big)\,\big] \\ &+ \int_t^T \mathbb{E}_t\big[\,\mathbf{1}_{\{s<\tau\}}(r_s - \tilde{c}_s)C_s D(t,s)\,\big]\,\mathrm{d}s + \int_t^T \mathbb{E}_t\Big[\,\mathbf{1}_{\{s<\tau\}}(r_s - \tilde{f}_s)F_s D(t,s)\,\Big]\,\mathrm{d}s.\end{aligned} \tag{2.27}$$

By recalling equation (2.13), we can write the following

Proposition 2.3.4. *The value $\bar{V}_t$ of the claim under credit gap risk, collateral and funding costs can be written as*

$$\bar{V}_t = V_t - \mathrm{CVA}_t + \mathrm{DVA}_t + \mathrm{LVA}_t + \mathrm{FVA}_t \tag{2.28}$$

where V_t is the price of the deal when there is no credit risk, no collateral, and there is no funding costs. LVA is a liquidity valuation adjustment accounting for the costs/benefits of collateral margining, FVA is the funding costs/benefits for the deal hedging strategy, and CVA/DVA are the familiar credit and debit valuation adjustments after collateralization. These different adjustments can be written by rewriting formula (2.27). One obtains

$$V_t = \int_t^T \mathbb{E}_t\Big\{D(t,s)1_{\{\tau>s\}}\Big[\Pi(s,s+ds) + 1_{\{\tau\in ds\}}\varepsilon_s\Big]\Big\} \tag{2.29}$$

and the valuation adjustments

$$\begin{aligned}
\mathrm{CVA}_t &= \int_t^T \mathbb{E}_t\Big\{ D(t,s)\mathbf{1}_{\{\tau>s\}}\mathbf{1}_{\{s=\tau_C<\tau_I\}}\Pi_{CVAcoll}(s)\Big\}\mathrm{d}s \\
\mathrm{DVA}_t &= \int_t^T \mathbb{E}_t\Big\{ D(t,s)\mathbf{1}_{\{\tau>s\}}\mathbf{1}_{\{s=\tau_I<\tau_C\}}\Pi_{DVAcoll}(s)\Big\}\mathrm{d}s \\
\mathrm{LVA}_t &= \int_t^T \mathbb{E}_t\Big\{ D(t,s)1_{\{\tau>s\}}(r_s-\tilde{c}_s)C_s\Big\}\mathrm{d}s \\
\mathrm{FVA}_t &= \int_t^T \mathbb{E}_t\Big\{ D(t,s)1_{\{\tau>s\}}(r_s-\tilde{f}_s)F_s\Big\}\mathrm{d}s
\end{aligned}$$

As usual, CVA and DVA are both positive, while LVA and FVA can be either positive or negative. Notice that if $\tilde{c}$ equals the risk-free rate, LVA vanishes. FVA vanishes if the funding rate $\tilde{f}$ is equal to the risk-free rate.

The proof is immediate. Notice that equation (2.29) simplifies further under a risk-free close-out

$$\varepsilon_\tau = V_\tau.$$

Indeed, in such a case the presence of τ does not alter the present value. In fact one can show that, since the unwinding at τ happens at the fair price V_τ and with recovery one, this is equivalent, in terms of valuation at time t, to valuing the whole deal:

$$V_t = \int_t^T \mathbb{E}_t\Big\{ D(t,s)\Pi(s,s+\mathrm{d}s)\Big\}.$$

Remark 2.3.5. Separability? *As pointed out earlier and in Pallavicini et al. [60], valuation formula (2.28) is not really splitting risk components in different terms. For example, to determine $\tilde{f}$, one needs to know future signs of F and hence future $\bar{V}$'s. Such future $\bar{V}$'s depend on all risks together, and so does $\tilde{f}$. Hence interpreting the $\tilde{f}$-dependent term FVA as a pure funding adjustment is misleading. Similarly, if we adopt a replacement close-out at default where $\varepsilon_\tau = \bar{V}_{\tau-}$, then the CVA and DVA terms will depend on future $\bar{V}$'s, and hence on all other risks as well. Therefore, CVA is no longer a pure credit valuation adjustment. If enforcing the above separation a posteriori after solving the total equation for $\bar{V}$, then one has to be careful in interpreting this as a real split of risks.*

A further point concerns the presence of the short rate r_t in the terms above. Since r_t is a theoretical rate with no direct market counterpart, this decomposition is not ideal. To implement the pseudo-decomposition, one would have to proxy r_t with a real market rate, such as an overnight rate.

2.3.3 Formulation under the market filtration $\mathcal{F}$

Recalling equation (2.16), we may rewrite equation (2.27) as

$$\begin{aligned}\bar{V}_t = &\int_t^T \mathbb{E}_t\big[\,\big(\mathbf{1}_{\{s<\tau\}}\Pi(s,s+\mathrm{d}s) + \mathbf{1}_{\{\tau\in\mathrm{d}s\}}\theta_s(C,\varepsilon)\big)\,D(t,s)\,\big]\\ &+\int_t^T \mathbb{E}_t\Big[\,\mathbf{1}_{\{s<\tau\}}(\tilde{f}_s - \tilde{c}_s)C_s D(t,s)\,\Big]\,\mathrm{d}s\\ &+\int_t^T \mathbb{E}_t\Big[\,\mathbf{1}_{\{s<\tau\}}(r_s - \tilde{f}_s)\,\big(\bar{V}_s - H_s\big)\,D(t,s)\Big]\,\mathrm{d}s.\end{aligned}$$

We now adopt an immersion hypothesis and switch to the default-free market filtration $(\mathcal{F}_t)_{t\geq 0}$. This step implicitly assumes a separable structure of our complete filtration $(\mathcal{G}_t)_{t\geq 0}$. In other words, $\mathcal{G}_t$ is generated by the pure default-free market filtration $\mathcal{F}_t$ and by the filtration generated by all the relevant default times monitored up to t.

We can easily describe an event which belongs to the σ-algebra $\mathcal{G}_t$ on the set $\{t<\tau\}$. For example, if the event $A\in\mathcal{G}_t$, then there exists some event $B\in\mathcal{F}_t$ such that $A\cap\{t<\tau\} = B\cap\{t<\tau\}$. Therefore, for any $\mathcal{G}_t$-measurable random variable X, there exists an $\mathcal{F}_t$-measurable random variable x, such that $\mathbf{1}_{\{t<\tau\}}X = \mathbf{1}_{\{t<\tau\}}x$.

The following lemma in Bielecki and Rutkowski [7] (Section 5.1) explains how the information expressed by filtration $\mathcal{G}$ relates to the one expressed by filtration $\mathcal{F}$.

Lemma 2.3.6. *For any integrable $\mathcal{F}$-measurable random variable X and $t\in[0,T]$, we have*

$$\mathbb{E}_t\big[\,\mathbf{1}_{\{t<\tau\}}X\,\big] = \mathbf{1}_{\{t<\tau\}}\frac{\mathbb{E}\left[\mathbf{1}_{\{t<\tau\}}X|\mathcal{F}_t\right]}{\mathbb{E}\left[\mathbf{1}_{\{t<\tau\}}|\mathcal{F}_t\right]}.$$

An extension of the above lemma when dealing with a predictable process instead of a simple random variable can be found in Bielecki et al. [6], as follows:

Lemma 2.3.7. *Suppose that φ_s is a $\mathcal{G}$-predictable process. Consider a default time τ with $\mathcal{F}$-intensity λ. The following equality then holds:*

$$\mathbb{E}_t\left[\int_t^{\tau\wedge T}\varphi_s\mathrm{d}s\right] = \mathbf{1}_{\{t<\tau\}}\,\mathbb{E}\left[\int_t^T e^{-\int_t^s\lambda_u du}\,\widetilde{\varphi}_s\mathrm{d}s \mid \mathcal{F}_t\right], \tag{2.30}$$

where $\widetilde{\varphi}_s$ is an $\mathcal{F}_s$-measurable variable such that $\mathbf{1}_{\{s<t\}}\varphi_s = \mathbf{1}_{\{s<t\}}\widetilde{\varphi}_s$.

We now assume that the basic portfolio cash flows $\Pi(0,t)$ are $\mathcal{F}_t$-measurable and that default times of all parties are conditionally independent given filtration $\mathcal{F}$ (see also Brigo and Pallavicini [30] for the full details in the present set-up). The above lemmas allow us to rewrite the previous price equation under the default-free market filtration $\mathcal{F}$ as

$$\begin{aligned}\bar{V}_t = \mathbf{1}_{\{\tau>t\}}&\int_t^T \mathbb{E}_t[\,(\Pi(s,s+\mathrm{d}s)+\lambda_s\theta_s(C,\varepsilon)\mathrm{d}s)\,D(t,s;r+\lambda)|\mathcal{F}\,]\\ &+\mathbf{1}_{\{\tau>t\}}\int_t^T \mathbb{E}_t\Big[\,(\tilde{f}_s-\tilde{c}_s)C_sD(t,s;r+\lambda)|\mathcal{F}\Big]\,\mathrm{d}s\\ &+\mathbf{1}_{\{\tau>t\}}\int_t^T \mathbb{E}_t\Big[\,(r_s-\tilde{f}_s)\left(\bar{V}_s-H_s\right)D(t,s;r+\lambda)|\mathcal{F}\Big]\,\mathrm{d}s,\end{aligned} \tag{2.31}$$

where λ_t is the first-to-default intensity and the discount factor is defined as $D(t,s;\xi) := e^{-\int_t^s\xi_u\mathrm{d}u}$ for some $\mathcal{F}$-adapted process ξ_u. Notice that we use the same notations for the processes under both filtrations for notational simplicity.

We need to stress that the above continuous formulation is not a pricing formula for the deal. The recursive nature of our consistent valuation framework is hidden in the fact that the paths of the effective funding rates $\tilde{f}$ depend on the future signs of the funding account F, which is defined as $\bar{V}-C-H$. Although in the equations, it looks like the different adjustments are achieved as additive decompositions, in fact the dependence of the future adjusted price $\bar{V}$ is still present and there is no real decomposition. Nevertheless, in the case of symmetric funding rates, we have $f^+ = f^-$, and the dependence no longer exists. However, such an assumption that one can borrow and fund at the same rate is unrealistic.

We will continue the discussion of the consistent valuation framework in continuous time in later chapters.

Chapter 3

Funding Inclusive Valuation in a Continuous Time Setting

In this chapter, we continue the study of a consistent valuation framework including credit risk, collateralization and funding risk in the continuous time setting using both a forward-backward stochastic differential equation approach and a semi-linear partial differential equation approach.

3.1 FBSDE approach

Forward-backward stochastic differential equations (FBSDEs) have been widely used for pricing and optimization problems in mathematical finance. The earliest version of such an FBSDE was introduced by Bismut [10] in 1973, with a decoupled form, namely, a system of a usual (forward) stochastic differential equation and a (linear) backward stochastic differential equation (in short, BSDE). In 1983, Bensoussan [5] proved the well-posedness of general linear BSDEs by using the martingale representation theorem. The first well-posedness result for nonlinear BSDEs was proved in 1990 by Pardoux and Peng [62], while studying the general Pontryagin-type maximum principle for stochastic optimal controls. El Karoui et al. [48] in 1997 provided an overview for the use of BSDEs in the field. In this section, we continue the study of a consistent valuation framework in continuous time, adopting a BSDE approach.

Throughout this section, we consider a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$, where

$\{(\mathcal{F}_t), t \in [0.T]\}$ is the filtration generated by the Brownian motion W, and $\mathbb{Q}$ is the risk-neutral probability measure.

3.1.1 Introduction to FBSDEs

In this section we will introduce some usual assumptions and important theorems on forward-backward stochastic differential equations (in short FBSDEs).

Consider the triplet $(X, Y, Z) = \{(X_t, Y_t, Z_t), t \in [0, T]\}$ of square-integrable $\mathcal{F}_T$-adapted processes with values in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$, satisfying the following FBSDE:

$$\mathrm{d}X_t = \mu(t, X_t, Y_t, Z_t)\mathrm{d}t + \sigma(t, X_t, Y_t, Z_t)\mathrm{d}W_t, \qquad X_0 = x_0, \tag{3.1}$$

$$\mathrm{d}Y_t = -B(t, X_t, Y_t, Z_t)\mathrm{d}t + Z_t\mathrm{d}W_t, \qquad Y_T = \Psi(X_T), \tag{3.2}$$

or, equivalently,

$$X_t = x_0 + \int_0^t \mu(s, X_s, Y_s, Z_s)\mathrm{d}s + \int_0^t \sigma(s, X_s, Y_s, Z_s)\mathrm{d}W_s,$$

$$Y_t = \Psi(X_T) + \int_t^T B(s, X_s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s\mathrm{d}W_s,$$

where the coefficient mappings are given by $\mu : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \mapsto \mathbb{R}^n$, $\sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \mapsto \mathbb{R}^{m\times d}$, $B : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \mapsto \mathbb{R}^m$ and the terminal condition is given by the map $\Psi : \mathbb{R}^n \mapsto \mathbb{R}^m$. The process X is the *forward component* of the FBSDE and $B(t, X_t, Y_t, Z_t)$ is the so called *driver of the FBSDE*.

The above FBSDE is called a *coupled* FBSDE. When the solution (Y, Z) of the BSDE (3.2) does not interfere with the forward component (3.1), or more precisely, $\mu(t, X_t, Y_t, Z_t) = \mu(t, X_t)$ and $\sigma(t, X_t, Y_t, Z_t) = \sigma(t, X_t)$, the FBSDE is said to be decoupled.

The existence and uniqueness of the solution to the system (3.1) and (3.2) was first addressed by Pardoux and Peng in [62], after which an extensive literature has been published on this topic. Here we adopt the results in [62] for the case of a decoupled FBSDE.

Assumption 1. *The coefficient functions are continuous with respect to* $(x, y, z) \in$

$\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$, *and satisfy: There exist constants* $K > 0$ *and* $p \geq 1/2$, *such that for all* $t, x, x_1, x_2, y_1, y_2, z_1, z_2$,

(i) $|\mu(t, x_1) - \mu(t, x_2)| + |\sigma(t, x_1) - \sigma(t, x_2)| \leq K\,|x_1 - x_2|\,;$

(ii) $|\mu(t, x)| + |\sigma(t, x)| \leq K\,(1 + |x|)\,;$

(iii) $|B(t, x, y_1, z_1) - B(t, x, y_2, z_2)| \leq K\,(|y_1 - y_2| + |z_1 - z_2|)\,;$

(iv) $|B(t, x, 0, 0)| + |\Psi(x)| \leq K(1 + |x|^p)$.

Notice that under the above assumption, the forward SDE (3.1) has a unique strong solution X_t (see Karatzas and Shreve [55]). We follow the existence and uniqueness study in [62], and have the following theorem.

Theorem 3.1.1 (Exisitence and uniqueness of decoupled FBSDEs). *If Assumption 1 is in force, then there exists a unique adapted triplet* (X, Y, Z) *which solves the FBSDE system* (3.1) *and* (3.2).

In the case of a coupled FBSDE, we follow the study in Delarue [44] and have the following results.

Assumption 2. *The coefficient functions are continuous with respect to* $(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$, *and satisfy: There exist constants* $K > 0$ *and* $\lambda > 0$ *such that for all* $t, x, x_1, x_2, y, y_1, y_2, z, z_1, z_2$,

(i) $|\mu(t, x_1, y_1, z_1) - \mu(t, x_2, y_2, z_2)| + |\sigma(t, x_1, y_1, z_1) - \sigma(t, x_2, y_2, z_2)|$
$\leq K\,(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|)\,;$

(ii) $|\mu(t, x, y, z)| + |\sigma(t, x, y, z)| + |B(t, x, y, z)| + |\Psi(x)| \leq K\,(1 + |y| + |z|)\,;$

(iii) $|B(t, x, y_1, z_1) - B(t, x, y_2, z_2)| + |\Psi(x_1) - \Psi(x_2)|$
$\leq K\,(|x_1 - x_2| + |y_1 - y_2| + |z_1 - z_2|)\,;$

(iv) $\forall \zeta \in \mathbb{R}^n$, $\langle \zeta, a(t, x, y)\zeta\rangle \geq \lambda|\zeta|^2$, *where* $\langle\,,\,\rangle$ *is the euclidean scalar product on* $\mathbb{R}^n$ *and the function* a *is defined as follows on* $[0, T] \times \mathbb{R}^n \times \mathbb{R}^m$,

$$\forall (t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m, \quad a(t, x, y) = \sigma(t, x, y)\sigma^*(t, x, y).$$

Theorem 3.1.2 (Exisitence and uniqueness of coupled FBSDEs). *If Assumption 2 is satisfied, then there exits a unique solution* (X, Y, Z) *to the FBSDE system* (3.1) *and* (3.2).

3.1.2 Consistent valuation framework in terms of FBSDE

In this section we derive the FBSDE expression for the collateralized, credit and funding risks inclusive consistent valuation equation built up in Chapter 2. We then discuss the existence and uniqueness of the solution to such an FBSDE.

Derivation of the FBSDE for the Funding Equation

Let's start by looking at the continuous time pricing equation derived in Chapter 2, at time $t < \tau$,

$$
\begin{aligned}
\bar{V}_t = &\int_t^T \mathbb{E}_t[\,(\Pi(s, s+\mathrm{d}s) + \lambda_s\theta_s(C,\varepsilon)\mathrm{d}s)\, D(t,s;r+\lambda)|\mathcal{F}\,] \\
&+ \int_t^T \mathbb{E}_t\Big[\,(\tilde{f}_s - \tilde{c}_s)C_s D(t,s;r+\lambda)|\mathcal{F}\Big]\,\mathrm{d}s \\
&+ \int_t^T \mathbb{E}_t\Big[\,(r_s - \tilde{f}_s)\left(\bar{V}_s - H_s\right) D(t,s;r+\lambda)|\mathcal{F}\Big]\,\mathrm{d}s,
\end{aligned}
$$

where the discount factor is defined as $D(t,s;\xi) := e^{-\int_t^s \xi_u \mathrm{d}u}$. Multiplying both sides of the above equation by $D(0,t;r+\lambda)$ gives us

$$
\begin{aligned}
\bar{V}_t D(0,t;r+\lambda) = &\int_t^T \mathbb{E}_t\Big[\,(r_s - \tilde{f}_s)\left(\bar{V}_s - H_s\right) D(0,s;r+\lambda)|\mathcal{F}\Big]\,\mathrm{d}s \\
+ \int_t^T \mathbb{E}_t\Big[\,&\Big(\Pi(s,s+\mathrm{d}s) + \lambda_s\theta_s(C,\varepsilon) + (\tilde{f}_s - \tilde{c}_s)C_s\Big)\, D(0,s;r+\lambda)|\mathcal{F}\Big]\,\mathrm{d}s. \quad (3.3)
\end{aligned}
$$

Our aim is to obtain a BSDE expression for $\bar{V}_t$, so we introduce the following process

$$
\begin{aligned}
X_t := \int_0^t \Big(&\Pi(s,s+\mathrm{d}s) + \lambda_s\theta_s(C,\varepsilon) + (\tilde{f}_s - \tilde{c}_s)C_s \\
&+ (r_s - \tilde{f}_s)\left(\bar{V}_s - H_s\right)\Big) D(0,s;r+\lambda)\mathrm{d}s.
\end{aligned}
$$

We can now construct an $\mathcal{F}$-martingale by adding X_t to both sides of (3.3), and we have

$$\bar{V}_t D(0,t;r+\lambda) + X_t = \mathbb{E}_t[\,X_T|\mathcal{F}\,]\,. \tag{3.4}$$

We define $\mathcal{M}_t = \mathbb{E}_t[\,X_T|\mathcal{F}\,]$. Differentiating both sides of (3.4) with respect to t yields

$$-(r_t+\lambda_t)D(0,t;r+\lambda)\bar{V}_t\mathrm{d}t + D(0,t;r+\lambda)\mathrm{d}\bar{V}_t + \mathrm{d}X_t = \mathrm{d}\mathcal{M}_t.$$

Now substitute X_t into the above equation and we have

$$-(r_t+\lambda_t)\bar{V}_t\mathrm{d}t+\mathrm{d}\bar{V}_t+\Pi(t,t+\mathrm{d}t)+\left[\lambda_t\theta_t(C,\varepsilon)+(\tilde{f}_t-\tilde{c}_t)C_t+(r_t-\tilde{f}_t)\left(\bar{V}_t-H_t\right)\right]\mathrm{d}t$$
$$= \mathrm{d}\mathcal{M}_t/D(0,t;r+\lambda). \tag{3.5}$$

Let $\mathrm{d}\mathcal{M}'_t = \mathrm{d}\mathcal{M}_t/D(0,t;r+\lambda)$. Since the process $(\mathcal{M}_t)_{t\geq 0}$ is a closed $\mathcal{F}$-martingale under the risk-neutral probability measure, $\int_0^t \mathrm{d}\mathcal{M}'_u$ is a local $\mathcal{F}_t$-martingale. Assuming that $\int_0^t \mathrm{d}\mathcal{M}'_u$ is adapted to the Brownian filtration $\sigma(W)$, we can then apply the martingale representation theorem, and write $\int_0^t \mathrm{d}\mathcal{M}'_u = \int_0^t Z_u \mathrm{d}W_u$ for Z_u being a $\sigma(W)$-predictable process.

Recall that the on-default cash-flow $\theta_t(C,\varepsilon)$ depends on the close-out amount ε and the collateral amount C. If we adopt a replacement close-out rule, the close-out amount is equivalent to the adjusted full deal price, i.e. $\varepsilon_t = \bar{V}_t$. Additionally, if we assume that the collateral amount is a function of the full deal price, the on-default cash-flow is then a function of the full deal price. More precisely, we can denote $\theta_t(C,\varepsilon) = \theta(t,\bar{V}_t)$ for some adapted function θ.

Moreover, we define a process π by

$$\pi_t \mathrm{d}t = \Pi(t,t+\mathrm{d}t), \tag{3.6}$$

for a small time period $\mathrm{d}t$, where π_t is assumed to be $\mathcal{F}_t$-adapted. We can then rewrite equation (3.5) as follows,

$$\mathrm{d}\bar{V}_t = -\left[\pi_t - \lambda_t\bar{V}_t + \tilde{f}_t(C_t-\bar{V}_t+H_t) - \tilde{c}_tC_t - r_tH_t + \lambda_t\theta(t,\bar{V}_t)\right]\mathrm{d}t + Z_t\mathrm{d}W_t. \tag{3.7}$$

Furthermore, assume that the price process $\bar{V}_t$ satisfies the following smoothness assumptions:

$$\bar{V}_t = \bar{V}(t, S_t) \qquad \text{and} \qquad \bar{V}_t \in \mathcal{C}^{1,2}\left([0,T] \times \mathbb{R}^+\right), \tag{3.8}$$

and also that the underlying risk factor follows the SDE

$$\mathrm{d}S_t = \mu(t, S_t, \bar{V}_t)\mathrm{d}t + \sigma(t, S_t, \bar{V}_t)\mathrm{d}W_t.$$

We can then apply Itô's formula to $\bar{V}_t$ yielding

$$\mathrm{d}\bar{V}_t = \frac{\partial \bar{V}}{\partial t}\mathrm{d}t + \mu(t, S_t, \bar{V}_t)\frac{\partial \bar{V}}{\partial S}\mathrm{d}t + \frac{1}{2}\sigma(t, S_t, \bar{V}_t)^2\frac{\partial^2 \bar{V}}{\partial S^2}\mathrm{d}t + \sigma(t, S_t, \bar{V}_t)\frac{\partial \bar{V}}{\partial S}\mathrm{d}W_t. \tag{3.9}$$

Now, we match the $\mathrm{d}W_t$ terms in (3.9) and in (3.7) and we have that

$$Z_t = \sigma(t, S_t, \bar{V}_t)\frac{\partial \bar{V}}{\partial S}.$$

This leads to the following theorem:

Theorem 3.1.3. *Define a stochastic process $Y_t = \bar{V}_t$. Suppose there exists a four dimensional deterministic linear function H such that the hedging process H_t satisfies $H_t = H(t, S_t, Y_t, Z_t)$. Assume that the close-out amount is taken to be the adjusted full price of the deal, i.e. $\varepsilon_t = \bar{V}_t$, and also that the collateral account is proportional to the adjusted deal price such that $C_t = \alpha_t \bar{V}_t$, for some $\mathcal{F}_t$-predictable process $\alpha_t \leq 1$ a.s.. The consistent funding inclusive valuation framework introduced in Chapter 2 can then be expressed in terms of the following (coupled) forward-backward stochastic differential equation (FBSDE) :*

$$\begin{aligned}
&\mathrm{d}S_t = \mu(t, S_t, Y_t)\mathrm{d}t + \sigma(t, S_t, Y_t)\mathrm{d}W_t, \qquad S_0 = s_0,\\
&\mathrm{d}Y_t = -B(t, S_t, Y_t, Z_t)\mathrm{d}t + Z_t\mathrm{d}W_t, \qquad Y_T = 0, \qquad\qquad (3.10)\\
&B(t, S_t, Y_t, Z_t) = \pi_t + \lambda_t\theta(t, Y_t) - \lambda_t Y_t + \tilde{f}_t(\alpha_t - 1)Y_t - \tilde{c}_t\alpha_t Y_t - (r_t - \tilde{f}_t)H(t, S_t, Y_t, Z_t),
\end{aligned}$$

where s_0 is the initial underlying stock price, and $B(t, S_t, Y_t, Z_t)$ is the driver of the

FBSDE.

Furthermore, if the stock price follows the Black-Scholes dynamics, and delta hedging is used, i.e. $H_t = \frac{\partial \bar{V}}{\partial S} S_t$*, then the funding inclusive valuation (decoupled) FBSDE can be written as:*

$$\begin{aligned} \mathrm{d}S_t &= r_t S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t, \qquad S_0 = s_o, \\ \mathrm{d}Y_t &= -B(t, S_t, Y_t, Z_t)\mathrm{d}t + Z_t \mathrm{d}W_t, \qquad Y_T = 0, \\ B(t, S_t, Y_t, Z_t) &= \pi_t - \lambda_t Y_t + \tilde{f}_t \left((\alpha_t - 1)Y_t + \frac{Z_t}{\sigma_t} \right) - \tilde{c}_t \alpha_t Y_t - \frac{r_t}{\sigma_t} Z_t + \lambda_t \theta(t, Y_t). \end{aligned} \tag{3.11}$$

Proof. The proof of the coupled FBSDE expression is immediate from the derivation above. In the decoupled case, because $Z_t = \sigma(t, S_t, \bar{V}_t)\frac{\partial \bar{V}}{\partial S} = \sigma_t S_t \frac{\partial \bar{V}}{\partial S}$, we have that the delta hedging account is $H_t = S_t \frac{\partial \bar{V}}{\partial S} = Z_t/\sigma_t$. The result is then straightforward after substituting the expression for H_t (in terms of Z_t) into the driver of the BSDE of the coupled case. □

Remark 3.1.4. *In most realistic cases the dynamic of the underlying asset will have a risk-neutral type drift and a volatility part that will lead to a decoupled structure.*

3.1.3 Existence and uniqueness of solution to the funding inclusive FBSDE

For both the decoupled and the coupled cases, we postulate the following assumption.

Assumption 3. *The coefficients in the forward components are uniformly Lipschitz-continuous in time.*

Under Assumption 3, (i) and (ii) in Assumption 1 and (i) in Assumption 2 hold. In fact this is a very standard assumption since it guarantees the existence and uniqueness of the underlying price process S_t.

Recall that the terminal conditions of the FBSDEs of (3.10) and (3.11) are both $\Psi(S_T) = Y_T = 0$. The following result shows that under some assumptions on the market rates, the drivers of the FBSDEs are uniformly Lipschitz-continuous, and so there exists a unique solution. Note that in the case of the coupled FBSDE (3.10), we have to deal with the stochastic dependence of the rates involved in the driver. We postulate that all the rates depend only on t, S_t, Y_t, namely,

$$\tilde{f}_t = f(t, S_t, Y_t), \qquad \tilde{c}_t = c(t, S_t, Y_t), \qquad \lambda_t = \lambda(t, S_t, Y_t), \qquad r_t = r(t, S_t, Y_t).$$

Theorem 3.1.5. *Suppose Assumption 3 is satisfied. If all the rates $\tilde{f}$, $\tilde{c}$ and r in the decoupled case are bounded, and functions f, c, λ, r in the coupled case are deterministic and bounded, then there exist unique solutions for both the coupled and the decoupled FBSDEs defined in Theorem 3.1.3.*

Proof. Let's first consider the decoupled FBSDE (3.11). Since under Assumption 3 (i, ii) in Assumption 1 hold, we show that the driver (3.11) is Lipschitz-continuous and has linear growth, so that Assumption 1 is satisfied.

Let's first look at the driver of the FBSDE (3.11). It can be rewritten as

$$\begin{aligned}
B(t, S_t, Y_t, Z_t) =& \pi_t - \lambda_t Y_t - \frac{r_t}{\sigma_t} Z_t + f_t^+ \left((\alpha_t - 1) Y_t + \frac{Z_t}{\sigma_t} \right)^+ \\
&+ f_t^- \left((\alpha_t - 1) Y_t + \frac{Z_t}{\sigma_t} \right)^- - c_t^+ \alpha_t (Y_t)^+ - c_t^- \alpha_t (Y_t)^- + \lambda_t \theta(t, Y_t) \\
=& \pi_t - \lambda_t Y_t - \frac{r_t}{\sigma_t} Z_t + f_t^+ \left((\alpha_t - 1) Y_t + \frac{Z_t}{\sigma_t} \right) - c^+ \alpha_t Y_t \\
&+ \left(f_t^- - f_t^+ \right) \left((\alpha_t - 1) Y_t + \frac{Z_t}{\sigma_t} \right)^- - (c_t^- - c_t^+) \alpha_t (Y_t)^- + \lambda_t \theta(t, Y_t) \\
=& \pi_t + \left(f_t^+ (\alpha_t - 1) - c_t^- \alpha_t - \lambda_t \right) Y_t + \frac{(f_t^+ - r_t)}{\sigma_t} Z_t \\
&+ \left(f_t^- - f_t^+ \right) \left((\alpha_t - 1) Y_t + \frac{Z_t}{\sigma_t} \right)^- - (c_t^- - c_t^+) \alpha_t (Y_t)^- + \lambda_t \theta(t, Y_t).
\end{aligned}$$

We know that the sum of Lipschitz-continuous functions is still Lipschitz-continuous. Split $B(t, S_t, Y_t, Z_t)$ into the sum of four terms, i.e. $B(t, S_t, Y_t, Z_t) = B_1(t, S_t, Y_t, Z_t) +$

$B_2(t, S_t, Y_t, Z_t) + B_3(t, S_t, Y_t, Z_t) + B_4(t, S_t, Y_t, Z_t)$, with

$$
\begin{aligned}
B_1(t, S_t, Y_t, Z_t) &= \pi_t + \left(f_t^+(\alpha_t - 1) - c_t^- \alpha_t - \lambda_t\right) Y_t + \frac{(f_t^+ - r_t)}{\sigma_t} Z_t, \\
B_2(t, S_t, Y_t, Z_t) &= \left(f_t^- - f_t^+\right) \left((\alpha_t - 1)Y_t + \frac{Z_t}{\sigma_t}\right)^-, \\
B_3(t, S_t, Y_t, Z_t) &= (c_t^- - c_t^+)\alpha_t (Y_t)^-, \\
B_4(t, S_t, Y_t, Z_t) &= \lambda_t \theta(t, Y_t).
\end{aligned}
$$

Let's first look at a general function of the form

$$
\widetilde{B}(t, s, y, z) := \widetilde{a}_t y - \widetilde{b}_t z, \tag{3.12}
$$

where the coefficients $\widetilde{a}_t$ and $\widetilde{b}_t$ are bounded. For any t, s, y_1, y_2, z_1, z_2,

$$
\begin{aligned}
&|\widetilde{B}(t, s, y_1, z_1) - \widetilde{B}(t, s, y_2, z_2)| \\
&= |(\widetilde{a}_t y_1 - \widetilde{b}_t z_1) - (\widetilde{a}_t y_2 - \widetilde{b}_t z_2)| \\
&\leq |\widetilde{a}_t||y_1 - y_2| + |\widetilde{b}_t||z_1 - z_2| \\
&= \max\{|\widetilde{a}_t|, |\widetilde{b}_t|\} \left(\left| \frac{\widetilde{a}_t}{\max\{|\widetilde{a}_t|, |\widetilde{b}_t|\}} \right| |y_1 - y_2| + \left| \frac{\widetilde{b}_t}{\max\{|\widetilde{a}_t|, |\widetilde{b}_t|\}} \right| |z_1 - z_2| \right) \\
&\leq \max\{|\widetilde{a}_t|, |\widetilde{b}_t|\}(|y_1 - y_2| + |z_1 - z_2|).
\end{aligned}
$$

Define a constant $K > 0$ such that $K = \max\{|\widetilde{a}_t|, |\widetilde{b}_t|\}$. We have

$$
\left| \widetilde{B}(t, s, y_1, z_1) - \widetilde{B}(t, s, y_2, z_2) \right| \leq K(|y_1 - y_2| + |z_1 - z_2|),
$$

and so functions of the form (3.12) are Lipschitz-continuous. Since B_1 is of the form (3.12), and $\tilde{f}_t$, $\tilde{c}_t$, r_t and λ_t are bounded, $B_1(t, s, y, z)$ is then K-Lipschitz-continuous with

$$
K = \max\left(\left| \frac{f_t^+ - r_t}{\sigma_t} \right|, \left| f_t^+(\alpha_t - 1) - c_t^- \alpha_t - \lambda_t \right| \right).
$$

Next, let's look at functions of the form

$$\widehat{B}(t,s,y,z) := \widehat{A}_t(\widehat{a}_t y - \widehat{b}_t z)^-, \tag{3.13}$$

where $\widehat{A}_t$, $\widehat{a}_t$ and $\widehat{b}_t$ are all bounded. For any t, s, y_1, y_2, z_1, z_2, let's consider the following cases:

(1) If $\widehat{a}_t y_1 - \widehat{b}_t z_1 < 0$ and $\widehat{a}_t y_2 - \widehat{b}_t z_2 > 0$, then

$$\begin{aligned}
&\left|\widehat{B}(t,s,y_1,z_1) - \widehat{B}(t,s,y_2,z_2)\right| = \left|\widehat{A}_t(\widehat{a}_t y_1 - \widehat{b}_t z_1)\right| \\
&\leq \left|\widehat{A}_t(\widehat{a}_t y_1 - \widehat{b}_t z_1) - \widehat{A}_t(\widehat{a}_t y_2 - \widehat{b}_t z_2)\right| \leq |\widehat{A}_t\widehat{a}_t||y_1 - y_2| + |\widehat{A}_t\widehat{b}_t||z_1 - z_2| \\
&= \max\{|\widehat{A}_t\widehat{a}_t|, |\widehat{A}_t\widehat{b}_t|\}\left(\left|\frac{\widehat{A}_t\widehat{a}_t}{\max\{|\widehat{A}_t\widehat{a}_t|, |\widehat{A}_t\widehat{b}_t|\}}\right||y_1 - y_2| \right. \\
&\qquad\qquad\qquad\qquad\qquad \left. + \left|\frac{\widehat{A}_t\widehat{b}_t}{\max\{|\widehat{A}_t\widehat{a}_t|, |\widehat{A}_t\widehat{b}_t|\}}\right||z_1 - z_2|\right) \\
&\leq \max\{|\widehat{A}_t\widehat{a}_t|, |\widehat{A}_t\widehat{b}_t|\}(|y_1 - y_2| + |z_1 - z_2|).
\end{aligned}$$

Therefore, $\widehat{B}(t,s,y,z)$ is K-Lipschitz-continuous with $K = \max\{|\widehat{A}_t\widehat{a}_t|, |\widehat{A}_t\widehat{b}_t|\}$.

(2) If $\widehat{a}_t y_1 - \widehat{b}_t z_1 < 0$ and $\widehat{a}_t y_2 - \widehat{b}_t z_2 < 0$, $\widehat{B}(t,s,y,z)$ is of the form (3.12) and is then Lipschitz-continuous.

(3) The case when $\widehat{a}_t y_1 - \widehat{b}_t z_1 > 0$ and $\widehat{a}_t y_2 - \widehat{b}_t z_2 < 0$ is similar to (1) and $\widehat{B}(t,s,y,z)$ is Lipschitz-continuous.

(4) If $\widehat{a}_t y_1 - \widehat{b}_t z_1 > 0$ and $\widehat{a}_t y_2 - \widehat{b}_t z_2 > 0$,

$$|\widehat{B}(t,s,y_1,z_1) - \widehat{B}(t,s,y_2,z_2)| = 0.$$

$\widehat{B}(t,s,y,z)$ is obviously Lipschitz-continuous.

Combining the four observations, we can conclude that functions of the form (3.13) are Lipschitz-continuous. Notice that both the terms $B_2(t,s,y,z)$ and $B_3(t,s,y,z)$ in the driver of the funding FBSDE are of the form (3.13) so are both Lipschitz-continuous functions. Moreover, since $\varepsilon_I = \varepsilon_C = Y_t$, the on-default cash-flow can be expressed as

$$\theta(t, Y_t) = Y_t - \mathbb{Q}(\tau^1 = \tau_C < \tau_I)LGD_C(Y_t)^+ + \mathbb{Q}(\tau^1 = \tau_I < \tau_C)LGD_I(-Y_t)^+.$$

Thus,

$$\begin{aligned} B_4(t, s, y, z) &= \lambda_t \left[Y_t - \mathbb{Q}(\tau^1 = \tau_C < \tau_I)LGD_C(Y_t)^+ + \mathbb{Q}(\tau^1 = \tau_I < \tau_C)LGD_I(-Y_t)^+\right] \\ &= \lambda_t Y_t + \lambda_t \mathbb{Q}(\tau^1 = \tau_C < \tau_I)LGD_C(-Y_t)^- - \lambda_t \mathbb{Q}(\tau^1 = \tau_I < \tau_C)LGD_I(Y_t)^-. \end{aligned}$$

Because λ_t is bounded, the first term $\lambda_t Y_t$ is Lipschitz, and the second and third terms are of the form (3.13) which was shown to be Lipschitz. So $B_4(t, s, y, z)$ is also Lipschitz-continuous.

Collecting everything, we can now conclude that the driver $B(t, s, y, z)$ of the funding FBSDE (3.11) is Lipschitz-continuous in Y_t and Z_t. Moreover, since all the rates are bounded, $B(t, s, y, z)$ has linear growth. According to Theorem 3.1.1, there exists a unique solution (S_t, Y_t, Z_t) to FBSDE (3.11).

In the case of the coupled FBSDE (3.10), since Assumption 3 is in force and all the rates are bounded, (i, iv) in Assumption 2 are satisfied. Moreover, since all functions of the rates are deterministic and bounded, we can then adopt the previous result for the decoupled case and conclude that (ii, iii) in Assumption 2 are satisfied. According to Theorem 3.1.2, there then exists a unique solution to the coupled FBSDE (3.10). □

3.2 Semi-linear PDE approach

There exists a very strong link between FBSDEs and quasi-linear parabolic systems of partial differential equations (in short PDEs). Following the study in [62], Peng in [66] gave a probabilistic formula for the given solution of a system of parabolic partial differential equations. Pardoux and Peng then in [63] showed that a given function expressed in terms of the solution to the BSDE solves a certain system of parabolic PDEs. Their results generalised the well known Feynman-Kac formula. Later, in Pardoux and Tang [64], the authors deduced that, under certain assumptions, the solution of the FBSDE provides a viscosity solution to the quasi-linear parabolic PDE.

A PDE of order m is called *quasi-linear* if it is linear in the derivatives of order m with coefficients that depend on the independent variables and derivatives of the unknown function or order strictly less than m. Quasi-linear PDEs are categorised into two: Semi-linear and Non-semilinear. A quasi-linear PDE where the coefficients of derivatives of order m are functions of the independent variables only is called a *semi-linear PDE*.

In this section, we continue the study of the consistent valuation framework in the continuous time setting. We find the consistent valuation equation takes the form of a semi-linear PDE. Our aim is to obtain such a semi-linear PDE form of the funding inclusive valuation equation, find the condition of the existence and uniqueness of the solution to the PDE and study the properties of the solution.

3.2.1 From a FBSDE to a semi-linear PDE

In this section, we focus on the decoupled case. We postulate that $\tilde{f}_t = f(t, S_t, Y_t)$, $\tilde{c}_t = c(t, S_t, Y_t)$, $\lambda_t = \lambda(t, S_t, Y_t)$ and $r_t = r(t, S_t, Y_t)$, where functions f, c, λ, r are deterministic and bounded. We also assume that the price process $\bar{V}$ has sufficient smoothness, as required in (3.8). Let's now look at equations (3.7) and (3.9). Keep in mind that in the decoupled case, we have $\mu(t, S_t, Y_t, Z_t) = \mu(t, S_t)$ and $\sigma(t, S_t, Y_t, Z_t) = \sigma(t, S_t)$. Equating the drift and the diffusion terms of $\mathrm{d}\bar{V}$ in the equations, we obtain the following for $\tau > t$ (for the ease of notation, from now on we denote $H_t = H(t, S_t, Y_t, Z_t)$),

$$\begin{aligned}&\pi_t - \lambda_t \bar{V}_t + \tilde{f}_t(C_t - \bar{V}_t + H_t) - \tilde{c}_t C_t - r_t H_t + \lambda_t \theta(t, \bar{V}_t) + \frac{\partial \bar{V}}{\partial t} + \mu(t, S_t)\frac{\partial \bar{V}}{\partial S} \\ &+ \frac{1}{2}\sigma(t, S_t)^2 \frac{\partial^2 \bar{V}}{\partial S^2} = 0, \quad Z_t = \sigma(t, S_t)\frac{\partial \bar{V}}{\partial S}.\end{aligned} \tag{3.14}$$

Therefore, the adjusted deal price $\bar{V}$ satisfies the following semi-linear PDE for all $(t, s) \in [0, T] \times \mathbb{R}^+$:

$$\begin{aligned}
&\partial_t \nu(t,s) + \mu(t,s)\partial_s \nu(t,s) + \frac{1}{2}\sigma(t,s)^2 \partial_s^2 \nu(t,s) + B\left(t, s, \nu(t,s), (\partial_s \nu \sigma)(t,s)\right) = 0, \\
&\nu(T,s) = 0,
\end{aligned} \tag{3.15}$$

where $B\left(t, s, \nu(t,s), (\partial_s \nu \sigma)(t,s)\right)$ is the driver of the FBSDE defined in Theorem 3.1.3.

3.2.2 Existence and uniqueness of the solution to the funding inclusive PDEs

In this section we seek a link between the funding inclusive FBSDE and PDE (3.15) without postulating the smoothness assumption (3.8) for the price process $\bar{V}$. Let's now return to the FBSDE (3.10) in a decoupled case where $\mu(t, S_t, Y_t, Z_t) = \mu(t, S_t)$ and $\sigma(t, S_t, Y_t, Z_t) = \sigma(t, S_t)$. For $(t, s) \in [0, T] \times \mathbb{R}^+$, let $\{S_u^{t,s}; t \le u \le T\}$ denote the diffusion process S on the time interval $[t, T]$, starting at time t from the point s. More precisely, for $t \le u \le T$,

$$\begin{aligned}
\mathrm{d}S_u^{t,s} &= \mu(u, S_u^{t,s})\mathrm{d}u + \sigma(u, S_u^{t,s})\mathrm{d}W_u, \\
S_t^{t,s} &= s, \\
\mathrm{d}Y_u^{t,s} &= -B(u, S_u^{t,s}, Y_u^{t,s}, Z_u^{t,s})\mathrm{d}u + Z_u^{t,s}\mathrm{d}W_u, \\
Y_T^{t,s} &= 0.
\end{aligned} \tag{3.16}$$

In this section we shall discuss the existence and uniqueness of viscosity, weak and classical solutions to the semi-linear parabolic PDE associated to the above FBSDE.

We recall that in Section 3.1 we showed that under the assumptions in Theorem 3.1.5, the funding inclusive valuation FBSDE (3.16) has a unique solution. El Karoui et al. in [47] showed that the solution $(S_u^{t,s}, Y_u^{t,s}, Z_u^{t,s})$ to FBSDE (3.16) is Markovian in the sense that these processes can be expressed through deterministic functions of u and $S_u^{t,s}$. More precisely,

Theorem 3.2.1. *Under Assumption 1, there exists two measurable deterministic functions $\nu(t,s)$ and $d(t,s)$ such that the solution $(S_u^{t,s}, Y_u^{t,s}, Z_u^{t,s})$ of FBSDE* (3.16) *is given by*

$$\forall u \leq T, \quad Y_u^{t,s} = \nu(u, S_u^{t,s}) \quad \text{and} \quad Z_u^{t,s} = d(u, S_u^{t,s})\sigma(u, S_u^{t,s}).$$

We now show that the FBSDE's solution is a viscosity solution of some non-linear PDE.

Viscosity solution

In order to avoid restrictive assumptions on the coefficients in (3.16), we will consider the PDE (3.15) in the viscosity sense. We start with defining the notion of viscosity solution of (3.15).

Definition 3.2.1. *Let $\nu \in \mathcal{C}([0,T] \times \mathbb{R}^+)$ satisfy $\nu(T,s) = 0$, $s \in \mathbb{R}^+$. ν is said to be a **viscosity subsolution** (resp. **supersolution**) of* (3.15) *if for any $\varphi \in \mathcal{C}^{1,2}((0,T) \times \mathbb{R}^+)$ and $(t,s) \in (0,T) \times \mathbb{R}^+$ such that $\varphi(t,s) = \nu(t,s)$ and (t,s) is a minimum of $\varphi - \nu$,*

$$\partial_t \varphi(t,s) + \mu(t,s)\partial_s \varphi(t,s) + \frac{1}{2}\sigma(t,s)^2 \partial_s^2 \varphi(t,s) + B\left(t, s, \varphi(t,s), (\partial_s \varphi \sigma)(t,s)\right) \geq 0$$

$$(\text{resp. } \partial_t \varphi(t,s) + \mu(t,s)\partial_s \varphi(t,s) + \frac{1}{2}\sigma(t,s)^2 \partial_s^2 \varphi(t,s) + B\left(t, s, \varphi(t,s), (\partial_s \varphi \sigma)(t,s)\right) \leq 0).$$

*$\nu \in \mathcal{C}([0,T] \times \mathbb{R}^+)$ is called a **viscosity solution** of* (3.15) *if it is both a viscosity subsolution and supersolution of* (3.15).

We now have the following results due to Pardoux and Peng [63].

Theorem 3.2.2. *If the mapping $s \mapsto B(t,s,0,0)$ is continuous and Assumption 1 is satisfied, then $\nu(t,s) := Y_t^{t,s}$ is a viscosity solution of PDE* (3.15).

We have proved in Theorem 3.1.5 that when the coefficients in the forward components are uniformly Lipschitz-continuous in time (Assumption 3), and the rates $\tilde{f}$, $\tilde{c}$ and r are all bounded, Assumption 1 is in place. Hence, according to

Theorem 3.2.2 it is easy to see that the funding inclusive valuation PDE (3.15) associated to FBSDE (3.16) has a viscosity solution.

For a more complete theory, one should also study the uniqueness of the solution to the semi-linear PDE (3.15). However, the uniqueness of the viscosity solution is by no means trivial. Moreover, to consider a hedging strategy, one needs to compute the derivative of the deal price process with respect to the underlying price process, which does not exist when the deal price process is a viscosity solution. Therefore, from a practical point of view, we will not include the uniqueness discussion here.

Regular solution

In this section we are interested in the relationship between FBSDE (3.16) and regular solutions of the semi-linear PDE (3.15).

Classical solution We will first study the classical solution. $\mathcal{L}^p$-estimates of the diffusion solution as well as its first and second derivatives are required in order to show the solution to the semi-linear PDE is classical. The following theorem in El Karoui et al. [47] gives a probabilistic interpretation for solutions of the semi-linear PDE (3.15) using the solution of the Markovian FBSDE (3.16).

Theorem 3.2.3. *(El Karoui et al. [47]) Suppose that Assumption 1 is in force and that the functions μ, σ and B are $\mathcal{C}^3$ with bounded derivatives. Then:*

(i) $(\nu(u, S_u^{t,s}), \partial_s \nu(u, S_u^{t,s})\sigma(u, S_u^{t,s}))$ *is the solution of the FBSDE* (3.16) *in the time interval* $[u, T]$ *if* $\nu \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R}^+)$ *is a classical solution of PDE* (3.15)*. In addition, for any* $t < T$, $\nu(u, s) = Y_t^{t,s}$.

(ii) If $(S_u^{t,s}, Y_u^{t,s}, Z_u^{t,s})$ *is the unique solution of the FBSDE* (3.16)*, then* $\nu(t, s) := Y_t^{t,s}$, $0 \le t \le T$, $s \in \mathbb{R}^+$ *belongs to* $\mathcal{C}([0, T] \times \mathbb{R}^+)$ *and is a classical solution of the PDE* (3.15)*.*

Weak solution As we can see, only under some very restrictive smoothness assumptions on the coefficients does the semi-linear PDE (3.15) have a classical solution. If we assume that the driver B of the FBSDE (3.16) is merely a Lipschitz function, we need to consider the solution in a weak sense. We first introduce the following Hilbert space:

$$\mathcal{H} := \left\{ \nu \in \mathcal{L}^2\left([0,T]\times\mathbb{R}^+\right) \mid \nabla\nu\sigma \in \mathcal{L}^2\left([0,T]\times\mathbb{R}^+\right) \right\}.$$

Here, we adopt the convention of a weak solution in [47]. We say that a solution $\nu \in \mathcal{H}$ of PDE (3.15) is a weak solution if the following relation holds for all $\varphi \in \mathcal{C}_c^{1,\infty}([0,T]\times\mathbb{R}^+)$:

$$\int_u^T (\nu(t,\cdot), \partial_t\varphi(t,\cdot))\,\mathrm{d}t + (\nu(u,\cdot), \varphi(u,\cdot)) + \int_u^T \mathcal{E}(\nu(t,\cdot), \varphi(t,\cdot))\mathrm{d}t$$
$$= \int_u^T (B(t,\cdot,\nu(t,\cdot),(\nabla\nu\sigma)(t,\cdot)), \varphi(t,\cdot))\,\mathrm{d}t,$$

where $(\nu,\varphi) = \int_{\mathbb{R}^+}\nu(x)\varphi(x)\mathrm{d}x$ is the scalar product in $\mathcal{L}^2$ and

$$\mathcal{E}(\nu,\varphi) = \int_{\mathbb{R}^+}\left[(\nabla\nu\sigma)(\nabla\varphi\sigma) + \varphi\nabla\left(\left(\frac{1}{2}\sigma^*\nabla\sigma + \mu\right)\varphi\right)\right]\mathrm{d}x$$

is the *energy* of the system associated with the PDE.

The following theorem in [47] gives the weak Feynman-Kac's formula for the solution of PDE (3.15).

Theorem 3.2.4. *Assume that functions μ and σ are $\mathcal{C}^2$ and $\mathcal{C}^3$ respectively, and with bounded derivatives. Further, suppose that the function B is uniformly Lipschitz in (y,z) with Lipschitz constant K, i.e.*

$$|B(u,s,y_1,z_1) - B(u,s,y_2,z_2)| \leq K(|y_1-y_2|+|z_1-z_2|).$$

Then there exists a unique weak solution $\nu \in \mathcal{H}$ of the PDE (3.15)*. Moreover, $\nu(t,s) = Y_t^{t,s}$ and $\nabla\nu\sigma = Z_t^{t,s}$, where $\{(S_u^{t,s}, Y_u^{t,s}, Z_u^{t,s}), t \leq u \leq T\}$ is the solution of FBSDE* (3.16) *and*

$$Y_u^{t,s} = \nu(u, S_u^{t,s}), \qquad Z_u^{t,s} = (\nabla\nu\sigma)(u, S_u^{t,s}).$$

3.2.3 Invariance of valuation with respect to the short rate

In this section, we show that the value of a deal does not depend explicitly on the theoretical risk-free rate. We start by assuming that the underlying S_t is a tradable asset. In the convention of the decoupled case (3.11), the underlying follows the Black-Scholes type dynamic. According to the standard no-arbitrage theory, the drift of the underlying asset is the risk-free rate under the risk-neutral probability measure, more precisely, for $t \leq u \leq T$,

$$\begin{aligned} \mathrm{d}S_u^{t,s} &= r(u)S_u^{t,s}\mathrm{d}u + \sigma(u)S_u^{t,s}\mathrm{d}W_u, \\ S_t^{t,s} &= s. \end{aligned} \tag{3.17}$$

If we assume that the smoothness assumption (3.8) of the price process is satisfied, namely, $\bar{V}(t, S_t) = Y_t^{t,s} \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^+)$, we can rewrite equations (3.14), such that $\bar{V}$ satisfies the following equations for $t < \tau$,

$$\begin{aligned} &\partial_t \nu(t,s) + r_t s \partial_s \nu(t,s) + \frac{1}{2}\sigma(t)^2 s^2 \partial_s^2 \nu(t,s) + \tilde{f}_t \left((\alpha - 1) - \tilde{c}_t \alpha - \lambda_t\right) \nu(t,s) \\ &- \left(r_t - \tilde{f}_t\right) H(t, s, \nu(t,s), Z_t) + \pi_t + \lambda_t \theta(t, \nu(t,s)) = 0, \quad Z_t = \sigma(t) s \partial_s \nu(t,s). \end{aligned}$$

Observe that $\partial_s \nu(t,s)s = \frac{Z_t^{t,s}}{\sigma(t)}$ is the delta-hedging process. Therefore, we choose $H(t,s,\nu(t,s),Z_t) = \partial_s \nu(t,s)s = \frac{Z_t}{\sigma(t)}$. For the ease of notation, from now on we denote $H_t = H(t,s,\nu(t,s),Z_t)$ and $\theta_t = \theta(t,\nu(t,s))$. The equation (3.15) collapses to

$$\begin{aligned} &(\partial_t - \tilde{f}_t - \lambda_t + \mathcal{L}_t^{\tilde{f}})\nu(t,s) + (\tilde{f}_t - \tilde{c}_t)C_t + \lambda_t \theta_t + \pi_t = 0, \\ &\nu(T,s) = 0, \end{aligned} \tag{3.18}$$

for $(t,s) \in [0,T] \times \mathbb{R}^+$, where the infinitesimal generator $\mathcal{L}_t^{\tilde{f}}$ is defined as follows,

$$\mathcal{L}_t^{\tilde{f}} \nu(t,s) := \tilde{f}_t H_t + \mathcal{L}_t^2 \nu(t,s) := \tilde{f}_t H_t + \frac{1}{2}\sigma(t,s)^2 \frac{\partial^2 \nu}{\partial s^2}.$$

We can see that the pre-default PDE no longer depends on the risk-free rate r_t. Equation (3.18) may be solved numerically as in Crépey [40]. On the other hand, we can also express the pre-default PDE as an expectation, as we show in the following theorem:

Theorem 3.2.5 (**Continuous-time solution of the general pricing equation**). *If we assume collateral rehypothecation and delta-hedging, then the funding inclusive adjusted deal price can be expressed as*

$$\bar{V}_t = \int_t^T \mathbb{E}_t^{\tilde{f}} \left[\left(\pi_s + \lambda_s \theta_s + (\tilde{f}_s - \tilde{c}_s) C_s \right) D(t,s;\tilde{f}+\lambda) \mid \mathcal{F} \right] \mathrm{d}s. \tag{3.19}$$

where the expectation is taken under the pricing measure $\mathbb{Q}^{\tilde{f}}$ for which the underlying risk factors grow at the effective funding rate $\tilde{f}$ if no dividend is paid.

Proof. Assume $\nu(t, X_t)$ is a solution to the PDE (3.18), with boundary condition

$$\nu(T, X_T) = 0,$$

where X_t satisfies the SDE

$$\mathrm{d}X_t = \tilde{f}_t X_t \mathrm{d}t + \sigma_t X_t \mathrm{d}W_t^{\tilde{f}},$$

with $W_t^{\tilde{f}}$ being the Brownian motion under the pricing measure $\mathbb{Q}^{\tilde{f}}$ where the underlying risk factor X grows at the rate $\tilde{f}$.

For $s > t$, $s \leq T$, define a process

$$Y_s = \int_t^s \left[(\tilde{f}_u - \tilde{c}_u) C_u + \lambda_u \theta_u + \pi_u \right] D(t,u;\tilde{f}+\lambda)\mathrm{d}u + D(t,s;\tilde{f}+\lambda)\nu(s, X_s).$$

If we define $F(u, X_u, C_u, \bar{V}_u) = (\tilde{f}_u - \tilde{c}_u)C_u + \lambda_u \theta_u + \pi_u$, then we can write

$$Y_s = \int_t^s F(u, X_u, C_u, \bar{V}_u) D(t,u;\tilde{f}+\lambda)\mathrm{d}u + D(t,s;\tilde{f}+\lambda)\nu(s, X_s).$$

Differentiating Y_s, we get

$$\begin{aligned}
\mathrm{d}Y_s &= \mathrm{d}D(t,s;\tilde{f}+\lambda)\nu(s,X_s) + D(t,s;\tilde{f}+\lambda)\mathrm{d}\nu(s,X_s) + \mathrm{d}D(t,s;\tilde{f}+\lambda)\mathrm{d}\nu(s,X_s) \\
&\quad + \mathrm{d}\left(\int_t^s F(u,X_u,C_u,\bar{V}_u)D(t,u;\tilde{f}+\lambda)\mathrm{d}u\right) \\
&= D(t,s;\tilde{f}+\lambda)\left(\partial_s\nu_s + \tilde{f}_s X_s \frac{\partial \nu_s}{\partial X_s} + \frac{1}{2}\sigma_s^2 X_s^2 \frac{\partial^2 \nu_s}{\partial X_s^2} - (\tilde{f}_s+\lambda_s)\nu_s + F(s,X_s,C_s,\bar{V}_s)\right)\mathrm{d}s \\
&\quad + D(t,s;\tilde{f}+\lambda)\sigma_s X_s \frac{\partial \nu_s}{\partial X_s}\mathrm{d}W_s^{\tilde{f}}.
\end{aligned}$$

Since ν_s is a solution to the PDE (3.18), the ds term can be cancelled and we obtain

$$Y_T = Y_t + \int_t^T D(t,s;\tilde{f}+\lambda)\sigma_s X_s \frac{\partial \nu_s}{\partial X_s}\mathrm{d}W_s^{\tilde{f}}.$$

Therefore, the process Y is a continuous local martingale. If we take the conditional expectation with respect to the filtration $\mathcal{F}_t$, we have

$$\mathbb{E}_t^{\tilde{f}}[Y_T \mid \mathcal{F}] = \mathbb{E}_t^{\tilde{f}}[Y_t \mid \mathcal{F}] = \nu(t,X_t).$$

The solution to the PDE (3.18) is therefore

$$\begin{aligned}
\nu(t,X_t) &= \mathbb{E}_t^{\tilde{f}}\left[\int_t^T F(s,X_s,C_s,\bar{V}_s)D(t,s;\tilde{f}+\lambda)\mathrm{d}s + D(t,T;\tilde{f}+\lambda)\nu(T,X_T) \mid \mathcal{F}\right] \\
&= \mathbb{E}_t^{\tilde{f}}\left[\int_t^T \left(\pi_s + \lambda_s\theta_s + (\tilde{f}_s - \tilde{c}_s)C_s\right)D(t,s;\tilde{f}+\lambda)\mathrm{d}s \mid \mathcal{F}\right].
\end{aligned}$$

□

Theorem 3.2.5 decomposes the deal price $\bar{V}$ into three intuitive terms. The first term is the value of the deal cash flows, discounted at funding plus credit. The second term is the price of the on-default cash-flow in excess of the collateral, which includes the CVA and DVA of the deal after collateralization. The last term collects the cost of collateralization. In addition, we see that any dependence on the hedging strategy H can be dropped by taking all expectations under the pricing measure $\mathbb{Q}^{\tilde{f}}$. At this point it is very important to appreciate once again that $\tilde{f}$ depends on F, and hence on $\bar{V}$.

Remark 3.2.6. (Deal dependent pricing measure, local risk neutral measures). *Since the pricing measure depends on $\tilde{f}$ which in turn depends on the very value $\bar{V}$ we are trying to compute, we have that the pricing measure becomes deal dependent. Every deal or portfolio has a different pricing measure.*

Finally, we stress once again a very important invariance result that first appeared in [61] and [26]. The proof of the following theorem is immediate by inspection and follows directly from our analysis.

Theorem 3.2.7. (Invariance of the valuation equation with respect to the short rate r_t). *Equations (3.18) or (3.19) for valuation under credit, collateral and funding costs are completely governed by market rates; there is no dependence on a risk-free rate r_t and the final price is invariant to it. This confirms our earlier conjecture that the risk-free rate is merely an instrumental variable of our valuation framework and we do not, in fact, need to know the value of such a rate.*

Chapter 4

The Self-financing Condition

In this chapter, we present an outline of the derivation of the self-financing condition used in a derivative pricing framework with the presence of funding risk and collateral margining. The derivation is done in a way that clarifies the structure of the relevant funding accounts. This clarification is achieved by properly distinguishing between price processes, dividend processes and gains processes. Without this explicit distinction the resulting self-financing condition can be erroneous, as we illustrate in the case of two papers: Piterbarg [68] and Burgard and Kjaer [35] in the first section. We then in the second section follow a study carried out by Bielecki and Rutkowski [8] and show how the adjusted self-financing condition fits in our funding inclusive valuation framework.

4.1 A common mistake

This section is an update of the papers [13] and [14]. In this section we address an important problem with the self-financing condition used in the derivative pricing framework in [68] and [35]. In the first paper, the self-financing condition is equivalent to assuming that the equity position is self-financing on its own without including the cash position. In the second paper, the self-financing condition is equivalent to assuming that a sub-portfolio is self-financing on its own, contrary to the assumption that the whole portfolio is self-financing. The error stems from a failure in applying the stochastic Leibnitz rule and is present even in mainstream textbooks such as Hull [52] (see also a discussion with explicit calculations in Shreve [69] Exercise 4.10). It is important to highlight the issue not only because [68] is highly quoted at industrial conferences worldwide in the practitioners' space, while [35] has received extensive exposure, but also to provide a useful tool in the flourishing research on the valuation of the cost of funding.

We then provide an alternative derivation for the funding formula using the self-financing condition. We show that the final result in [68] is correct, even if the related self-financing condition is not. In the process, we raise a further question on the appropriateness of the replication approach described in that paper.

4.1.1 The self-financing condition and the problem in [68]

In the traditional derivative pricing in the Black-Scholes setting, replication is achieved by borrowing/lending at the risk-free rate. In modern practice funding costs are an important consideration in replication, especially when considering repo, unsecured funding and collateral accounts with different rates, as was done in the paper by Piterbarg [68].

Integral to the replication argument used for derivative pricing in [68] is the use of a self-financing trading strategy, and with respect to this we highlight the following problem. (We note that the same problem affects the proofs in [35]).

Formula (2) in [68] reads, for the portfolio Π that replicates the derivative V (here we use identical notation to the paper [68]):

$$V(t) = \Pi(t) = \Delta(t)S(t) + \gamma(t) \tag{4.1}$$

where S is the "price process" of the underlying asset, and γ is the "cash amount split among a number of accounts [...]".

Then [68] continues in reference to "equation (2)", which is (4.1) above:

> "On the other hand, from (2), by the self-financing condition
>
> $$\mathrm{d}\gamma(t) = \mathrm{d}V(t) - \Delta(t)\ \mathrm{d}S(t) \tag{4.2}$$
>
> [...]"

We argue that the above formulation of the self-financing condition is wrong. It is enough to directly differentiate both sides of equation (4.1) to obtain:

$$\mathrm{d}V(t) = \mathrm{d}(\Delta(t)S(t)) + \mathrm{d}\ \gamma(t) \tag{4.3}$$

and combine this last equation (4.3) with (4.2) to obtain:

$$\mathrm{d}(\Delta(t)S(t)) = \Delta(t)\ \mathrm{d}S(t) \quad \text{(wrong)}. \tag{4.4}$$

In light of the assumptions in [68], equation (4.4) is wrong because it would imply that the position in the risky asset S is self-financing on its own, in that the change in the total value of the position, namely $\mathrm{d}(\Delta(t)S(t))$, is funded by the asset market movements alone: $\Delta(t)\ \mathrm{d}S(t)$.

A further consequence of the above error follows immediately from the stochastic Leibnitz rule, leading to:

$$\mathrm{d}\Delta_t = 0 \quad \text{(wrong)}, \tag{4.5}$$

and indeed if equity needs to be self-financing on its own, the only possibility is that the amount of equity is constant (there is no re-balancing of the single position).

We briefly point out also that, in the reference book Hull [52], equation (14.12) and (14.13) yield exactly the same problem we are discussing here.

4.1.2 The self-financing condition and the problem in [35]

Burgard and Kjaer in [35] consider credit risk in addition to funding costs by allowing corporate bonds of the two parties of the derivative transaction in the replicating portfolio. However, the same problem affects their self-financing condition.

Specifically, in that work it is stated explicitly that the portfolio consisting of the stock S, the bond P_B of party B, the bond P_C of party C and $\beta(t)$ cash is self-financing. This portfolio value can be written as in the first equation following equation (3.2) of [35], namely (we use identical notation):

$$-\hat{V}(t) = \Pi(t) = \delta(t)S(t) + \alpha_B(t)P_B(t) + \alpha_C(t)P_C(t) + \beta(t). \tag{4.6}$$

The self-financing condition stated in equation (3.3) of [35] reads:

$$-\mathrm{d}\hat{V}(t) = \delta(t)\mathrm{d}S(t) + \alpha_B(t)\mathrm{d}P_B(t) + \alpha_C(t)\mathrm{d}P_C(t) + \mathrm{d}\beta(t). \tag{4.7}$$

The reader can clearly see where the problem is: if we now differentiate equation (4.6) and equate the resulting equation with (4.7) we immediately obtain:

$$\mathrm{d}\left(\delta(t)S(t) + \alpha_B(t)P_B(t) + \alpha_C(t)P_C(t)\right) = \delta(t)\mathrm{d}S(t) + \alpha_B(t)\mathrm{d}P_B(t) + \alpha_C(t)\mathrm{d}P_C(t) \quad \text{(wrong)}. \tag{4.8}$$

This is wrong because it implies that the portfolio of the three assets:

$$S,\ P_B,\ P_C$$

is self-financing, which is clearly at odds with [35] stating instead that the entire portfolio:

$$S,\ P_B,\ P_C,\ \text{Cash account}$$

is self-financing.

This is the same problem that afflicts the self-financing condition in [68], except that here it is distributed across more than one asset.

4.1.3 Presentation of the correct formulation in the framework of [68]

Since the derivation of the result is important, as it provides a description of the funding account and of the funding strategy, we believe it is appropriate at this point to illustrate the proper formulation of the self-financing condition in the case of [68].

We point out that we do not discuss the assumptions in [68], not least because of some inconsistencies that we could not reconcile. Specifically, it is mentioned that the spread between the funding rate and the short (CSA) rate can be thought of as stochastic, and its dynamics follow a one-factor Gaussian model in the related example. Since the short (CSA) rate and the repo rate are assumed to be deterministic, it follows that at least the funding rate is stochastic. Indeed, the replicating portfolio contains a certain amount of cash $V(t) - C(t)$ borrowed from (lent to) the treasury desk at the unsecured funding rate r_F. However, this funding rate is driven by the Brownian motion W_F driving the funding spread, which is distinct from the Brownian motion W driving the stock price (the two are assumed to be correlated in the example of this article). But this distinction is neither made clear, nor taken into account in the replication argument.

In light of this we rather concentrate on the correct formulation of the self-financing condition in the framework of [68], but further assuming deterministic rates to avoid the potential inconsistencies described above. For a more comprehensive framework going beyond [68] assumptions and including explicit default modeling, collateral modeling, rehypothecation and debit valuation adjustments we refer the reader elsewhere, for example Pallavicini et al. [60] or Crépey [39], and especially to Bielecki and Rutkowski [8] who analyze the matter from a rigorous point of view in the specific context of replication (see also Antonov and Bianchetti [2]).

One of the problems in the above derivation is that it does not distinguish between gains processes, price processes and dividend processes, and not doing so brings about the error highlighted in equation (4.4). We present below the correct formulation (for the full theory see Duffie [45]).

We start with a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{Q})$, with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by a Brownian motion W. We consider a stock with price dynamics given by a geometric Brownian motion with deterministic coefficients:

$$\mathrm{d}S_t = \mu(t)S_t\mathrm{d}t + \sigma(t)S_t\mathrm{d}W_t. \tag{4.9}$$

The financial market consists of a vector $A = \{A_i : 1 \leq i \leq n\}$ of traded assets available for hedging claims. At time $t \geq 0$ each asset A_i has a price $P_t^{A_i}$ and has had since inception a cumulative dividend $D_t^{A_i}$. The *gains process* of asset A_i is defined in terms of the price and cumulative dividend processes by:

$$G_t^{A_i} := P_t^{A_i} + D_t^{A_i}, \quad t \geq 0. \tag{4.10}$$

A *trading strategy* θ is an $\mathcal{F}_t$-predictable square-integrable stochastic process having as components the numbers of units of each asset held at time t:

$$\theta_t = (\theta_t^1, \theta_t^2, \ldots, \theta_t^n).$$

The time t value of the portfolio that results from employing the strategy θ is given as:

$$V_t^\theta := \theta_t^1 P_t^{A_1} + \ldots + \theta_t^n P_t^{A_n}, \tag{4.11}$$

while the gains (profits/losses) generated by holding a position θ_t^i in asset A_i is $\theta_t^i \mathrm{d}G_t^{A_i}$. By relation (4.10) this is coming from changes in the price of the asset and from changes in the cumulative dividend process (new dividends).

A trading strategy θ is *self-financing* if:

$$V_t^\theta = V_0^\theta + G_t^\theta, \quad t \geq 0, \tag{4.12}$$

where the gains process associated with θ is:

$$\mathrm{d}G_t^\theta := \theta_t^1 \mathrm{d}G_t^{A_1} + \ldots + \theta_t^n \mathrm{d}G_t^{A_n}, \quad G_0^\theta = 0. \tag{4.13}$$

This implies in particular that the only change in the value of the portfolio comes from the change in the gains process associated with the strategy θ, namely,

$$\mathrm{d}V_t^\theta = \mathrm{d}G_t^\theta, \tag{4.14}$$

or, using (4.11) and (4.13), from the changes in the gains processes of the assets:

$$\mathrm{d}(\theta_t^1 P_t^{A_1} + \ldots + \theta_t^n P_t^{A_n}) = \theta_t^1 \mathrm{d}G_t^{A_1} + \ldots + \theta_t^n \mathrm{d}G_t^{A_n}.$$

However, in general it is true that:

$$\mathrm{d}(\theta_t^1 P_t^{A_1} + \ldots + \theta_t^n P_t^{A_n}) \neq \theta_t^1 \mathrm{d}P_t^{A_1} + \ldots + \theta_t^n \mathrm{d}P_t^{A_n}. \tag{4.15}$$

Indeed, the trading strategy θ being self-financing does not imply that:

$$\mathrm{d}G_t^{A_i} = \mathrm{d}P_t^{A_i}$$

for the single asset i. Clearly, if all single assets have null dividend process, it follows that the self-financing condition for the strategy implies an equality in (4.15), but more generally this does not hold.

Finally, a claim Y with payoff V_t^Y at time $0 \leq t \leq T$ is *replicated* by the strategy θ if the value V^θ of the portfolio corresponding to this strategy satisfies:

$$V_t^Y = V_t^\theta, \tag{4.16}$$

for all $t \geq 0$ and up to the claim maturity T.

We now apply the above framework to the setup in [68]. Here the vector A of assets characterizing the financial market has the following components: A_1 is a repo contract for the risky stock given in (4.9), A_2 is a collateral account (like a cash account), and A_3 is the funding account (opened for example with the internal treasury). The collateral account is used to post an amount related to the claim value, and the funding account is used to borrow/invest as necessary to replicate the claim, so the only source of randomness in the assets of this market is the one-dimensional Brownian motion W driving the stock price (4.9).

The price processes for these assets are denoted by:

$$P_t^{A_1} = 0, \quad P_t^{A_2} = C_t, \quad P_t^{A_3} = \alpha_t, \quad t \geq 0, \tag{4.17}$$

where C_t is a market observable collateral requirement at time t and α_t is to be determined below.

From the point of view of an investor holding these assets, the incoming stock dividends $r_D(t)S_t\mathrm{d}t$ and the outgoing repo interest payments $r_R(t)S_t\mathrm{d}t$ are accounted for in the gains process $G_t^{A_1}$ associated with the repo on stock, so as to maintain the zero price $P_t^{A_1}$ of the repo contract. It leads to the following gains processes:

$$\begin{aligned}
\mathrm{d}G_t^{A_1} &= \mathrm{d}S_t + (r_D(t) - r_R(t))S_t\mathrm{d}t, \\
\mathrm{d}G_t^{A_2} &= r_C(t)C_t\mathrm{d}t, \\
\mathrm{d}G_t^{A_3} &= r_F(t)\alpha_t\mathrm{d}t,
\end{aligned} \tag{4.18}$$

where $r_D(t)$ is the rate at which stock dividends are paid at time t, $r_R(t)$ is the short rate at time t on funding secured via repo, $r_C(t)$ is the short rate at time t of cash/collateral, and $r_F(t)$ is the time t short rate for unsecured funding.

The dividend processes for the three assets are written using (4.10) as follows:

$$\begin{aligned}
\mathrm{d}D_t^{A_1} &= \mathrm{d}G_t^{A_1} - \mathrm{d}P_t^{A_1} = \mathrm{d}S_t + (r_D(t) - r_R(t))S_t\mathrm{d}t, \quad && D_0^{A_1} = 0, \\
\mathrm{d}D_t^{A_2} &= \mathrm{d}G_t^{A_2} - \mathrm{d}P_t^{A_2} = r_C(t)C_t\mathrm{d}t - \mathrm{d}C_t, && D_0^{A_2} = 0, \\
\mathrm{d}D_t^{A_3} &= \mathrm{d}G_t^{A_3} - \mathrm{d}P_t^{A_3} = r_F(t)\alpha_t\mathrm{d}t - \mathrm{d}\alpha_t, && D_0^{A_3} = 0.
\end{aligned} \tag{4.19}$$

We seek a trading strategy θ to replicate a derivative Y with time t value V_t^Y. The strategy $\theta = (\theta^1, \theta^2, \theta^3)$ is chosen to be, for some yet unknown process Δ:

$$\theta_t^1 = \Delta_t, \quad \theta_t^2 = 1, \quad \theta_t^3 = 1. \tag{4.20}$$

(We remark that here the single asset dividend processes are not null, so (4.15) reminds us that attention is needed in devising the correct setup.)

The time t value V_t^θ of the replicating portfolio obtained from strategy θ given by (4.20) is obtained from (4.11) with the prices (4.17):

$$V_t^\theta = \Delta_t\, 0 + 1\, C_t + 1\, \alpha_t. \tag{4.21}$$

This gives a funding account price of $\alpha_t = V_t^\theta - C_t = V_t^Y - C_t$ ($V^Y = V^\theta$ by (4.16)).

Contrast the replicating condition (4.21) with (4.1): the replicating portfolio has interests in the stock price via the dividend process (4.19) of the repo, but doesn't hold the stock.

Note that by replacing $\alpha_t = V_t^Y - C_t$ into equations (4.17), (4.18) and (4.19) the dynamics of the price, gains and dividend processes can be expressed in terms of the market observable quantities:

$$S_t,\ C_t,\ V_t^Y.$$

Then the gains process (4.13) associated with θ is:

$$\mathrm{d}G_t^\theta = \Delta_t \left[\mathrm{d}S_t + (r_D(t) - r_R(t))S_t \mathrm{d}t\right] + 1\left[r_C(t)C_t \mathrm{d}t\right] + 1\left[r_F(t)(V_t^\theta - C_t)\mathrm{d}t\right]. \quad (4.22)$$

The self-financing condition for the strategy θ can be obtained using (4.14), with $\mathrm{d}G_t^\theta$ given by the equation (4.22) above.

We have thus proved that for the market in [68] with assets: repo on stock, collateral account and unsecured funding account, the self-financing condition for the trading strategy θ that replicates a claim Y requires:

$$\mathrm{d}V_t^\theta = \Delta_t \left[\mathrm{d}S_t + (r_D(t) - r_R(t))S_t \mathrm{d}t\right] + 1\left[r_C(t)C_t \mathrm{d}t\right] + 1\left[r_F(t)(V_t^\theta - C_t)\mathrm{d}t\right]. \quad (4.23)$$

On the other hand, assuming that the payoff can be written as $V_t^Y = v^Y(t, S)$ for some $\mathcal{C}^{1,2}$ function v^Y, Itô's formula gives:

$$\mathrm{d}v^Y(t, S_t) = v_t^Y(t, S_t)\mathrm{d}t + \frac{\partial v^Y}{\partial S}\mathrm{d}S_t + \frac{1}{2}\frac{\partial^2 v^Y}{\partial S^2}\mathrm{d}(S_t)^2. \quad (4.24)$$

Equating the right hand sides of the equations (4.23) and (4.24) gives:

$$\begin{aligned} v_t^Y(t, S_t) + \frac{1}{2}\frac{\partial^2 v^Y}{\partial S^2}\sigma(t)^2 S_t^2 &= \frac{\partial v^Y}{\partial S}(t, S_t)(r_D(t) - r_R(t))S_t \\ &\quad + r_C(t)C_t + r_F(t)(v^Y(t, S_t) - C_t), \\ \Delta_t &= \frac{\partial v^Y}{\partial S}(t, S_t). \end{aligned} \quad (4.25)$$

With (4.25) we have now completely identified the strategy (4.20) that replicates the claim, so the pricing derivation may continue as in [68] with their "equation (3)".

Summarizing, the wrong self-financing condition implies that the portfolio without cash is self-financing on its own (see (4.4) and (4.8)). Fortunately, the error can be corrected and the correct self-financing condition can be derived rigorously. We showed how to do this in the setup of [68], casting light on the necessary distinction between dividend processes, price processes and gains processes, and obtaining the correct self-financing condition (4.23).

While the original published versions of [68] and [35] have the problems we have signaled in this article, the authors of [68] and [35] have updated the online versions of their articles with our proposed corrections following personal communication with us at the time the research report [13] (upon which the present section is based) appeared.

Nonetheless, we think it is necessary and useful to specify the setting explicitly in this section, since a sound funding theory depends crucially on financing costs, and the violation of the self-financing condition is an important problem.

4.2 Self-financing condition in our framework

In the following, we will show how the traditional self-financing condition is adjusted to reflect the new market realities of funding risk and collateralization in our framework. By adjusting the self-financing condition, we can derive our consistent valuation framework using only market observable quantities. The analysis carried out in this section is inspired by the work conducted by Bielecki and Rutkowski in [8], where the authors provided a theoretical underpinning for a unified framework for the nonlinear approach to hedging and pricing of OTC contracts. However, the default events and close-out conventions were not discussed.

Our goal is to derive the consistent valuation framework set up in the previous chapters using the adjusted self-financing condition. We start by giving the classical definition of a self-financing strategy.

We denote Π' all contractual cash flows directly generated by a derivative from the point of view of the investor (without discounting). Note that $\Pi'(t,T) = \frac{\Pi(t,T)}{D(t,T)}$, and Π' is an arbitrary càdlàg process of finite variation. A trading strategy (φ, Π') will be used to replicate all cash flows Π', where $\varphi = (\xi, \psi)$ is a dynamic replicating portfolio that consists of a risky asset S and a cash account B. For model simplicity, we assume that there is only one risky asset in this set-up that pays no dividend.

Definition 4.2.1. (Self-financing strategy) *A trading strategy* (φ, Π') *is said to be self-financing if the wealth process given by*

$$V_t(\varphi, \Pi') := \xi_t S_t + \psi_t B_t, \tag{4.26}$$

satisfies

$$\mathrm{d}V_t(\varphi, \Pi') = \xi_t \mathrm{d}S_t + \psi_t \mathrm{d}B_t - \mathrm{d}\Pi'_t. \tag{4.27}$$

We assume that the account will be empty at the end of the derivative trade, in particular, we have $V_T(\varphi) = 0$.

4.2.1 Different lending and borrowing rates

As a first adjustment to the classical pricing framework, we introduce different unsecured borrowing and lending rates. Denoting the cash account in the replicating strategy (φ, Π') associated with a contract Π' by an $\mathcal{F}$-adapted stochastic process F, we have

$$F_t = V_t(\varphi) - \xi_t S_t. \tag{4.28}$$

Let's now introduce different lending and borrowing cash accounts:

$$F_t = F_t \mathbf{1}_{\{F_t \geq 0\}} + F_t \mathbf{1}_{\{F_t < 0\}} = F_t^+ + F_t^-, \tag{4.29}$$

where F_t^+ stands for the cash value the investor needs to borrow in order to establish the replicating strategy, and F_t^- represents the surplus cash value from the replicating strategy that can be used for assets lending. We use the short-hand notation $\mathcal{X}^+ := \max(\mathcal{X}, 0)$ and $\mathcal{X}^- := \min(\mathcal{X}, 0)$.

We denote by B_t^l and B_t^b the strictly positive cash account processes corresponding to the lending and borrowing accounts respectively. Formally, we postulate that

$$\psi_t^l B_t^l = (V_t(\varphi) - \xi_t S_t)^-, \qquad \psi_t^b B_t^b = (V_t(\varphi) - \xi_t S_t)^+.$$

where for all $t \in [0,T]$,

$$\psi_t^l = (B_t^l)^{-1}(V_t(\varphi) - \xi_t S_t)^- \leq 0, \qquad \psi_t^b = (B_t^b)^{-1}(V_t(\varphi) - \xi_t S_t)^+ \geq 0. \tag{4.30}$$

We assume that at any time simultaneous lending and borrowing is prohibited, namely, $\psi_t^l \psi_t^b = 0$.

We say that the wealth process of a portfolio $\varphi = (\xi, \psi^l, \psi^b)$ given by

$$V_t(\varphi) = \xi_t S_t + \psi_t^l B_t^l + \psi_t^b B_t^b \tag{4.31}$$

is self-financing if the following condition is satisfied:

$$\mathrm{d}V_t(\varphi) = \xi_t \mathrm{d}S_t + \psi_t^l \mathrm{d}B_t^l + \psi_t^b \mathrm{d}B_t^b - \mathrm{d}\Pi_t'. \tag{4.32}$$

If we further assume that the account processes B^b and B^l are absolutely continuous, and the corresponding borrowing and lending rates are denoted by $\mathcal{F}$-adapted processes f^+ and f^- respectively (one may assume $f_t^+ \geq f_t^-$ for all $t \in [0,T]$ to avoid arbitrage opportunity), the following relations hold:

$$\frac{\mathrm{d}B_t^l}{B_t^l} = f_t^- \mathrm{d}t, \qquad \frac{\mathrm{d}B_t^b}{B_t^b} = f_t^+ \mathrm{d}t,$$

and the dynamic of the wealth process $V_t(\varphi)$ (4.32), can be re-written as

$$\mathrm{d}V_t(\varphi) = \xi_t \mathrm{d}S_t + f_t^-(V_t(\varphi) - \xi_t S_t)^- \mathrm{d}t + f_t^+(V_t(\varphi) - \xi_t S_t)^+ \mathrm{d}t - \mathrm{d}\Pi_t'. \tag{4.33}$$

4.2.2 Trading strategies under collateralization

When a deal is collateralized, we need to consider our problem in both scenarios when rehypothecation is forbidden and allowed. As introduced in Chapter 2, when rehypothecation is forbidden, the collateral will be kept in segregated accounts, whereas if rehypothecation is allowed, the collateral can be used as a source of funding by the collateral taker. Again, we assume that the collateral to be a risk-free cash account. We denote the cash collateral account by a stochastic process C, with

$$C_t = C_t \mathbf{1}_{\{C_t \geq 0\}} + C_t \mathbf{1}_{\{C_t < 0\}} = C_t^+ + C_t^-,$$

where C_t^+ stands for the cash value of collateral received, whereas C_t^- represents the cash value of collateral posted, both from the investor's point of view for any $0 \leq t < T$. As before we set $C_T = 0$.

Collateral trading with segregated accounts

If rehypothecation is not allowed, cash collateral will be placed in segregated accounts. The collateral amount that is received can not be used for trading. The wealth process $V(\varphi)$ of the collateralized trading strategy (φ, Π', C) should not explicitly depend on the collateral process. Therefore, the value of the replicating portfolio is

$$V_t(\varphi) = \xi_t S_t + \psi_t^l B_t^l + \psi_t^b B_t^b,$$

which is the same as (4.31). However, the self-financing condition reads

$$\mathrm{d}V_t(\varphi) = \xi_t \mathrm{d}S_t + \psi_t^l \mathrm{d}B_t^l + \psi_t^b \mathrm{d}B_t^b - \mathrm{d}\gamma_t - \mathrm{d}\Pi_t', \tag{4.34}$$

where

$$\psi_t^l B_t^l = (V_t(\varphi) - \xi_t S_t)^-, \qquad \psi_t^b B_t^b = (V_t(\varphi) - \xi_t S_t)^+,$$

and we denote the costs/benefits from the collateral margining account as γ_t.

In order to determine $\mathrm{d}\gamma_t$, we examine carefully what happens during collateral margining. We assume that the collateral processes are absolutely continuous with accrual rates c_t^+ when $C_t \geq 0$ and c_t^- when $C_t < 0$. Both the rates c^+ and c^- are $\mathcal{F}$-adapted processes. At time $t \in [0, T)$, if the investor is the collateral provider, i.e. $C_t < 0$, he needs to borrow cash for the collateral amount C_t^- from the cash borrowing account B_t^b at the borrowing rate f_t^+, and receives interest from the counterparty on the posted collateral at the accrual rate c_t^-. On the other hand, if the investor is the collateral taker, i.e. $C_t > 0$, he needs to pay the counterparty interest at the collateral accrual rate c_t^+, and (perhaps unlikely in practice) receives from the collateral custodian interest for the positive cash amount C_t^+ in the segregated account at some rate denoted by an $\mathcal{F}$-adapted process $r_t^{C,+}$. Formally, we have the following expression for $\mathrm{d}\gamma_t$,

$$\mathrm{d}\gamma_t = (f_t^+ - c_t^-)C_t^- \mathrm{d}t + (r_t^{C,+} - c_t^+)C_t^+ \mathrm{d}t. \tag{4.35}$$

Collateral trading with rehypothecation

For the case when rehypothecation is allowed, the collateral taker has the right to use the posted collateral for his liquidity and trading needs. In other words, at time $t \in [0, T)$, when the investor is the collateral taker, he is granted an unrestricted use of the full collateral amount C_t^+ received from the counterparty and pays interest on C_t^+ at collateral accrual rate c_t^+. However, if the investor is the collateral provider, namely $C_t < 0$, he is entitled to interest payments on the posted collateral amount C_t^- at collateral rate c_t^-.

In the case of collateral rehypothecation, the posted collateral can be used to reduce the funding costs of the collateral taker. The cash account for the trading strategy is then defined as

$$F_t = V_t(\varphi) - \xi_t S_t - C_t. \tag{4.36}$$

Moreover, the wealth process $V(\varphi)$ reads

$$V_t(\varphi) = \xi_t S_t + \psi_t^l B_t^l + \psi_t^b B_t^b + C_t, \tag{4.37}$$

and the dynamics of $V_t(\varphi)$ follows

$$\mathrm{d}V_t(\varphi) = \xi_t \mathrm{d}S_t + \psi_t^l \mathrm{d}B_t^l + \psi_t^b \mathrm{d}B_t^b - \mathrm{d}\gamma_t - \mathrm{d}\Pi_t', \tag{4.38}$$

where

$$\psi_t^l B_t^l = (V_t(\varphi) - \xi_t S_t - C_t)^-, \qquad \psi_t^b B_t^b = (V_t(\varphi) - \xi_t S_t - C_t)^+.$$

We assume that the cash collateral posted by the investor is also funded by the same cash accounts which fund the replicating strategy (at rates f^- and f^+ as defined before).

If the collateral account processes are absolutely continuous, the costs/benefits of the margining account is then given by

$$\mathrm{d}\gamma_t = -c_t^- C_t^- \mathrm{d}t - c_t^+ C_t^+ \mathrm{d}t. \tag{4.39}$$

Assume that the borrowing and lending account processes are absolutely continuous. Adopting the notations $\tilde{f}$ and $\tilde{c}$ from Chapter 2 for the effective funding rate (2.18) and the effective collateral accrual rate (2.3) respectively, the above dynamics of the wealth process $V_t(\varphi)$ for the rehypothecation case can be expressed as

$$\mathrm{d}V_t(\varphi) = \xi_t \mathrm{d}S_t + \tilde{f}_t(V_t(\varphi) - \xi_t S_t - C_t)\mathrm{d}t + \tilde{c}_t C_t \mathrm{d}t - \mathrm{d}\Pi'_t, \tag{4.40}$$

or equivalently,

$$\mathrm{d}V_t(\varphi) = \xi_t \mathrm{d}S_t + \tilde{f}_t(V_t - \xi_t S_t)\mathrm{d}t - (\tilde{f}_t - \tilde{c}_t)C_t \mathrm{d}t - \mathrm{d}\Pi'_t. \tag{4.41}$$

Let's now discuss how the adjusted self-financing condition (4.41) can be used to derive our consistent pricing equation.

4.2.3 Funding risk inclusive pricing formula

From now on, we assume that rehypothecation is allowed, but analogous results hold for the case where collateral is placed in segregated accounts.

We consider a replicating portfolio (φ, Π', C) associated with a contract Π'. The wealth process $V(\varphi)$ is given in (4.37) and we have that the dynamic of $V(\varphi)$ is given in (4.41).

In order to derive the consistent valuation framework set up in the previous chapters, we now define a new process $\tilde{V}_t(\varphi)$, associated with an arbitrary self-financing trading strategy φ, as follows,

$$\tilde{V}_t(\varphi) := V_t(\varphi) + B_t^\lambda \int_0^t \left[(\lambda_s - \tilde{f}_s)(V_s(\varphi) - H_s) + (\tilde{f}_s - \tilde{c}_s)C_s + \pi_s \right] (B_s^\lambda)^{-1} \mathrm{d}s, \tag{4.42}$$

where $H_t = \xi_t S_t$ is the hedging account. We use the notation $\pi_t \mathrm{d}t = \Pi(t, t + \mathrm{d}t)$ with $\Pi'(t,T)D(t,T) = \int_t^T D(t,s)\Pi(s, s+\mathrm{d}s)$, and B_t^λ is an arbitrary process of finite variation, such that for $t \geq 0$,

$$\mathrm{d}B_t^\lambda = \lambda_t B_t^\lambda \mathrm{d}t, \qquad B_0^\lambda > 0.$$

Assumption 4. *There exists a probability measure $\mathbb{Q}^\lambda$ such that the process S/B^λ is a $\mathbb{Q}^\lambda$-local martingale.*

Notice that if the rate λ is the risk-free rate, the measure $\mathbb{Q}^\lambda$ is then the risk-neutral probability measure and Assumption 4 is satisfied in a typical market model (such as Black-Scholes model).

Proposition 4.2.1. *The process $\tilde{V}(\varphi)/B^\lambda$ with $\tilde{V}(\varphi)$ defined in* (4.42) *is a $\mathbb{Q}^\lambda$-local martingale.*

Proof. To prove $\tilde{V}(\varphi)/B^\lambda$ is a $\mathbb{Q}^\lambda$-local martingale, it is sufficient to show that

$$\mathrm{d}\left(\frac{\tilde{V}_t(\varphi)}{B_t^\lambda}\right) = \xi_t \mathrm{d}\left(\frac{S_t}{B_t^\lambda}\right).$$

By applying Itô's formula to both sides of the above equation, we obtain (we drop φ from the notation in the following proof for notational simplicity)

$$\frac{1}{B_t^\lambda}(\mathrm{d}\tilde{V}_t - \tilde{V}_t \frac{\mathrm{d}B_t^\lambda}{B_t^\lambda}) = \frac{\xi_t}{B_t^\lambda}(\mathrm{d}S_t - S_t \frac{\mathrm{d}B_t^\lambda}{B_t^\lambda}),$$

which is equivalent to showing that

$$\mathrm{d}\tilde{V}_t - \lambda_t \tilde{V}_t \mathrm{d}t = \xi_t(\mathrm{d}S_t - \lambda_t S_t \mathrm{d}t).$$

Applying Itô's formula to $\tilde{V}_t$ yields

$$\mathrm{d}\tilde{V}_t = \mathrm{d}V_t + (\lambda_t - \tilde{f}_t)(V_t - H_t)\mathrm{d}t + (\tilde{f}_t - \tilde{c}_t)C_t\mathrm{d}t + \mathrm{d}\Pi'_t + \lambda_t(\tilde{V}_t - V_t)\mathrm{d}t.$$

Now substituting the self-financing condition property of (φ, Π', C), i.e. equation (4.41), we obtain

$$\begin{aligned}\mathrm{d}\tilde{V}_t - \lambda_t\tilde{V}_t\mathrm{d}t =& \xi_t\mathrm{d}S_t + \tilde{f}_t(V_t - \xi_t S_t)\mathrm{d}t - (\tilde{f}_t - \tilde{c}_t)C_t\mathrm{d}t - \mathrm{d}\Pi'_t + (\lambda_t - \tilde{f}_t)(V_t - H_t)\mathrm{d}t \\ &+ (\tilde{f}_t - \tilde{c}_t)C_t\mathrm{d}t + \mathrm{d}\Pi'_t + \lambda_t\tilde{V}_t\mathrm{d}t - \lambda_t V_t\mathrm{d}t - \lambda_t\tilde{V}_t\mathrm{d}t \\ =& \xi_t\mathrm{d}S_t + \lambda_t(V_t - H_t)\mathrm{d}t - \lambda_t V_t \\ =& \xi_t\mathrm{d}S_t - \lambda_t\xi_t S_t\mathrm{d}t\end{aligned}$$

as we were required to show. □

Using the result in Proposition 4.2.1 and the terminal condition $V_T(\varphi) = 0$, we establish the following theorem.

Theorem 4.2.2. *Suppose that Assumption 4 is in force. In the case of collateral rehypothecation and delta-hedging, the price of a derivative contract with contractual cash-flow* Π' *that is replicated by a self-financing trading strategy* (φ, Π', C) *is given by*

$$V_t(\varphi) = \mathbb{E}_t^\lambda\left[B_t^\lambda \int_t^T \Big((\lambda_s - \tilde{f}_s)(V_s(\varphi) - H_s) + (\tilde{f}_s - \tilde{c}_s)C_s + \pi_s\Big)(B_s^\lambda)^{-1}\mathrm{d}s \mid \mathcal{F}\right]. \tag{4.43}$$

where the expectation is taken under some pricing measure $\mathbb{Q}^\lambda$ *for which the underlying risk factors grow at the rate* λ *if no dividend is paid.*

Proof. According to Proposition 4.2.1 the process $\tilde{V}_t(\varphi)/B_t^\lambda$ defined as (4.42) is a $\mathbb{Q}^\lambda$-local martingale. Therefore, for $t \in [0, T]$,

$$\tilde{V}_t(\varphi)/B_t^\lambda = \mathbb{E}_t^\lambda\left[\tilde{V}_T(\varphi)/B_T^\lambda \mid \mathcal{F}\right],$$

which implies that

$$\frac{\tilde{V}_t(\varphi)}{B_t^\lambda} = \frac{V_t(\varphi)}{B_t^\lambda} + \int_0^t \left[(\lambda_s - \tilde{f}_s)(V_s(\varphi) - H_s) + (\tilde{f}_s - \tilde{c}_s)C_s + \pi_s\right](B_s^\lambda)^{-1}\mathrm{d}s$$
$$= \mathbb{E}_t^\lambda \left[\frac{V_T(\varphi)}{B_T^\lambda} + \int_0^T \left[(\lambda_s - \tilde{f}_s)(V_s(\varphi) - H_s) + (\tilde{f}_s - \tilde{c}_s)C_s + \pi_s\right](B_s^\lambda)^{-1}\mathrm{d}s \mid \mathcal{F}\right].$$

Since $V_T(\varphi) = 0$, we can obtain

$$V_t^H(\varphi) = \mathbb{E}_t^\lambda \left[B_t^\lambda \int_t^T \left((\lambda_s - \tilde{f}_s)(V_s(\varphi) - H_s) + (\tilde{f}_s - \tilde{c}_s)C_s + \pi_s\right)(B_s^\lambda)^{-1}\mathrm{d}s \mid \mathcal{F}\right].$$

□

We notice that if we set the rate λ in Theorem 4.2.2 to be the risk-free rate r_t, the pricing equation (4.43) becomes

$$V_t(\varphi) = \mathbb{E}_t \left[\int_t^T \left((r_s - \tilde{f}_s)(V_s(\varphi) - H_s) + (\tilde{f}_s - \tilde{c}_s)C_s + \pi_s\right) D(t,s)\mathrm{d}s \mid \mathcal{F}\right],$$

which is the same as the pricing equation (2.31) in continuous time without credit default events.

If, however, we set λ equal to the funding rate $\tilde{f}$, the valuation equation (4.43) becomes

$$V_t(\varphi, \Pi', C) = \mathbb{E}_t^{\tilde{f}} \left[\int_t^T \left((\tilde{f}_s - \tilde{c}_s)C_s + \pi_s\right) D(t,s;\tilde{f})\,\mathrm{d}s \mid \mathcal{F}\right],$$

which coincides with the result in Theorem 3.2.5 without the introduction of counterparty default risk (and the close-out conventions).

4.2.4 Funding risk inclusive PDE

Again, we consider the wealth process $V_t(\varphi)$ corresponding to a self-financing replicating strategy (φ, Π', C) associated with a contract Π'. The dynamic of the wealth process $V_t(\varphi)$ follows

$$\mathrm{d}V_t(\varphi) = \xi_t \mathrm{d}S_t + \tilde{f}_t(V_t - \xi_t S_t)\mathrm{d}t - (\tilde{f}_t - \tilde{c}_t)C_t\mathrm{d}t - \mathrm{d}\Pi'_t, \tag{4.44}$$

with terminal condition $V_T(\varphi) = 0$.

Now assume that the price process $V_t(\varphi)$ satisfies the smoothness Assumption 3.8, and also that the underlying risk factor S follows the following SDE (we drop φ from the notation here for simplicity)

$$\mathrm{d}S_t = \mu(t, S_t)\mathrm{d}t + \sigma(t, S_t)\mathrm{d}W_t.$$

We can then apply Itô's formula to $V_t(\varphi)$ giving us

$$\mathrm{d}V_t = \frac{\partial V}{\partial t}\mathrm{d}t + \mu(t, S_t)\frac{\partial V}{\partial S}\mathrm{d}t + \frac{1}{2}\sigma(t, S_t)^2\frac{\partial^2 V}{\partial S^2}\mathrm{d}t + \sigma(t, S_t)\frac{\partial V}{\partial S}\mathrm{d}W_t. \tag{4.45}$$

Equating the right hand sides of equations (4.44) and (4.45), we can write

$$\left[\frac{\partial V}{\partial t} - \xi_t\mu(t, S_t) + \mu(t, S_t)\frac{\partial V}{\partial S} + \frac{1}{2}\sigma(t, S_t)^2\frac{\partial^2 V}{\partial S^2} - \tilde{f}_t V_t + \tilde{f}_t\xi_t S_t \right.$$
$$\left. +(\tilde{f}_t - \tilde{c}_t)C_t + \pi_t\right] dt + \left[-\sigma(t, S_t)\xi_t + \sigma(t, S_t)\frac{\partial V}{\partial S}\right]\mathrm{d}W_t = 0.$$

Since the coefficient of the diffusion term needs to be zero, we have

$$\xi_t = \frac{\partial V}{\partial S},$$

substituting which into the the drift term yielding,

$$\frac{\partial V}{\partial t} - \tilde{f}_t V_t + \tilde{f}_t\frac{\partial V}{\partial S}S_t + \frac{1}{2}\sigma(t, S_t)^2\frac{\partial^2 V}{\partial S^2} + (\tilde{f}_t - \tilde{c}_t)C_t + \pi_t = 0.$$

Observe that $\frac{\partial V}{\partial S}S_t$ is the delta-hedging process. If we define the hedging account $H_t = \frac{\partial V}{\partial S}S_t$, the price process of the hedging portfolio V satisfies the following semi-linear PDE,

$$\begin{aligned} &\frac{\partial \nu(t,s)}{\partial t} - \tilde{f}_t\nu(t,s) + \tilde{f}_t H_t + \frac{1}{2}\sigma(t,s)^2\frac{\partial^2\nu(t,s)}{\partial s^2} + (\tilde{f}_t - \tilde{c}_t)C_t + \pi_t = 0, \\ &\nu(T,s) = 0. \end{aligned} \tag{4.46}$$

Comparing the above PDE to the pre-default PDE (3.18) we obtained previously, we see that the two semi-linear PDEs are identical if there is no counterparty credit risk (hence nor the on-default cash-flow).

Chapter 5

Numerical Results

This chapter provides a numerical study of the valuation framework outlined in the previous chapters. We investigate the impact of funding risk on the price of a derivatives deal under default risk and collateralization. Also, we analyse the valuation error of ignoring nonlinearties of the general valuation problem. Specifically, to quantify this error, we introduce the concept of a nonlinearity valuation adjustment (in short, NVA). A generalized least-squares Monte Carlo algorithm is proposed inspired by the simulation methods of Carriere [36], Longstaff and Schwartz [58], Tilley [71], and Tsitsiklis and Van Roy [72] for pricing American-style options. As the purpose is to understand the fundamental implications of funding risk, we focus on relatively simple deal positions in European call options. However, the Monte Carlo method we propose below can be applied to more complex derivative contracts, including derivatives with bilateral payments.

5.1 Monte Carlo algorithm

Recall the recursive structure of the general pricing equation: The deal price depends on the funding decisions, while the funding strategy depends on the future price itself. The intimate relationship among the key quantities makes the pricing problem computationally challenging.

We consider K default scenarios during the life of the deal – either obtained by simulation, bootstrapped from empirical data, or assumed in advance. For each first-to-default time τ corresponding to a default scenario, we compute the price of the deal $\bar{V}$ under collateralization, close-out netting and funding costs. The first step of our simulation method entails simulating a large number of sample paths N of the underlying risk factors X. We simulate these paths on the time-grid $\{t_1, \ldots, t_m = T^*\}$ with step size $\Delta t = t_{j+1} - t_j$ from the assumed dynamics of the risk factors. T^* is equal to the final maturity T of the deal or the consecutive time-grid point following the first-default time τ, whichever occurs first. For simplicity, we assume the time periods for funding and hedging decisions and collateral margin payments coincide with the simulation time grid.

Given the set of simulated paths, we solve the funding strategy recursively in a dynamic programming fashion. Starting one period before T^*, we compute for each simulated path the funding decision F and the deal price $\bar{V}$ according to the set of backward-inductive equations of Theorem 2.3.3. The algorithm then proceeds recursively until time zero. Ultimately, the total price of the deal is computed as the probability weighted average of the individual prices obtained in each of the K default scenarios.

The conditional expectations in the backward-inductive funding equations are approximated by across-path regressions based on least squares estimation similar to Longstaff and Schwartz [58]. We regress the present value of the deal price at time t_{j+1}, the adjusted payout cash flow between t_j and t_{j+1}, the collateral account and funding account at time t_j on basis functions ψ of realizations of the underlying risk factors at time t_j across the simulated paths. To keep notation simple, let us assume that we are exposed to only one underlying risk factor, e.g. a stock price. Extensions to higher dimensions are straightforward. Specifically, the conditional expectations in the iterative equations of Theorem 2.3.3, taken under the risk-neutral measure, are equal to

$$\mathbb{E}_{t_j}\left[\Xi_{t_j}(\bar{V}_{t_{j+1}})\right] = \alpha^*_{t_j}\,\psi(X_{t_j}), \tag{5.1}$$

where we have defined $\Xi_{t_j}(\bar{V}_{t_{j+1}}) := D(t_j, t_{j+1})\bar{V}_{t_{j+1}} + \bar{\Pi}(t_j, t_{j+1}; C) - C_{t_j} - H_{t_j}$. Note the C_{t_j} term drops out if rehypothecation is not allowed. The usual least-squares estimator of α is then given by

$$\hat{\alpha}_{t_j} := \left[\psi(X_{t_j})\psi(X_{t_j})^*\right]^{-1} \psi(X_{t_j})\, \Xi_{t_j}(\bar{V}_{t_{j+1}}). \tag{5.2}$$

Orthogonal polynomials such as Chebyshev, Hermite, Laguerre, and Legendre may all be used as basis functions for evaluating the conditional expectations. We find, however, that simple power series are quite effective and that the order of the polynomials can be kept relatively small. In fact, linear or quadratic polynomials, i.e. $\psi(X_{t_j}) = (\mathbf{1}, X_{t_j}, X_{t_j}^2)^*$, are often enough.

Further complexities are added, as the dealer may – realistically – decide to hedge the full deal price $\bar{V}$. Now, the hedge H itself depends on the funding strategy through $\bar{V}$, while the funding decision depends on the hedging strategy. This added recursion requires that we solve the funding and hedging strategies simultaneously. For example, if the dealer applies a delta-hedging strategy, we can write, heuristically,

$$H_{t_j} = \frac{\partial \bar{V}}{\partial X}\Big|_{t_j} X_{t_j} \approx \frac{\bar{V}_{t_{j+1}} - (1 + \Delta t_j \tilde{f}_{t_j})\bar{V}_{t_j}}{X_{t_{j+1}} - (1 + \Delta t_j \tilde{f}_{t_j})X_{t_j}}\, X_{t_j}. \tag{5.3}$$

We obtain, in the case of rehypothecation, the following system of nonlinear equations

$$\begin{cases} F_{t_j} - \frac{P_{t_j}^{\tilde{f}}(t_{j+1})}{P_{t_j}(t_{j+1})}\, \mathbb{E}_{t_j}\left[\Xi_{t_j}(\bar{V}_{t_{j+1}})\right] = 0, \\ H_{t_j} - \frac{\bar{V}_{t_{j+1}} - (1+\Delta t_j \tilde{f}_{t_j})\bar{V}_{t_j}}{X_{t_{j+1}} - (1+\Delta t_j \tilde{f}_{t_j})X_{t_j}}\, X_{t_j} = 0, \\ \bar{V}_{t_j} = F_{t_j} + C_{t_j} + H_{t_j}, \end{cases} \tag{5.4}$$

where all matrix operations are on an element-by-element basis. An analogous result holds when rehypothecation of the posted collateral is forbidden.

For each period and for each simulated path, we find the funding and hedging decisions by solving this system of equations, given the funding and hedging strategies for all future periods until the end of the deal. We apply a simple Newton-Raphson method to solve the system of nonlinear equations numerically, but instead of using the exact Jacobian, we approximate it by finite differences. As an initial guess, we use the Black-Scholes delta position

$$H^0_{t_j} = \Delta^{BS}_{t_j} X_{t_j}.$$

The convergence is quite fast and only a small number of iterations are needed in practice. Finally, if the dealer decides to hedge only the risk-free price of the deal, i.e. the classical derivative price V, the pricing problem collapses to a much simpler one. The hedge H no longer depends on the funding decision and can be computed separately and the numerical solution of the nonlinear equation system can be avoided altogether.

In the following we apply our valuation framework to the case of a stock or equity index option. Nevertheless, the methodology extends fully to any other derivatives transaction. For instance, applications to interest rate swaps can be found in Brigo and Pallavicini [29] and [30]. We now fully specify our modeling setup.

5.2 Case outline

Let S_t denote the price of some stock or equity index and assume it evolves according to a geometric Brownian motion $dS_t = rS_t dt + \sigma S_t dW_t$ where W is a standard Brownian motion under the risk neutral measure. The risk-free interest rate r is 100 bps, the volatility σ is 25%, and the current price of the underlying is $S_0 = 100$. The European call option is in-the-money and has strike $K = 80$. The maturity T of the deal is 3 years and, in the full case, we assume that the investor delta-hedges the deal according to (5.3). The usual default-free funding-free and collateral-free Black-Scholes price V_0 of the call option deal is given by

$$V_t = S_t\Phi(d_1(t)) - Ke^{-r(T-t)}\Phi(d_2(t)), \qquad d_{1,2} = \frac{\ln(S_t/K) + (r \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},$$

for $t = 0$ and is $V_0 = 28.9$ with our choice of inputs. As usual, Φ is the cumulative distribution function of the standard normal random variable. In the usual setting the hedge would not be (5.3) but a classical delta-hedging strategy based on $\Phi(d_1(t))$.

We consider two simple discrete probability distributions of default. Both parties of the deal are considered default risky but can only default at year 1 or at year 2. The localized joint default probabilities are provided in the matrices below. The rows denote the default time of the investor, while the columns denote the default times of the counterparty. For example, in matrix D_{low} the event $(\tau_I = 2yr, \tau_C = 1yr)$ has a 3% probability and the first-to-default time is 1 year. Simultaneous defaults are introduced and we determine the close-out amount by a random draw from a uniform distribution. If the random number is above 0.5, we compute the close-out as if the counterparty defaulted first, and vice versa.

For the first to default distribution, we have a low dependence between the default risk of the counterparty and the default risk of the investor

$$D_{\text{low}} = \begin{array}{c} \\ 1yr \\ 2yr \\ n.d. \end{array} \begin{array}{c} \begin{array}{ccc} 1yr & 2yr & n.d. \end{array} \\ \begin{pmatrix} 0.01 & 0.01 & 0.03 \\ 0.03 & 0.01 & 0.05 \\ 0.07 & 0.09 & 0.70 \end{pmatrix} \end{array}, \qquad \tau_K(D_{\text{low}}) = 0.21, \tag{5.5}$$

where $n.d.$ means no default and τ_K denotes the rank correlation as measured by Kendall's tau. In the second case, we have a high dependence between the two parties' default risks,

$$D_{\text{high}} = \begin{array}{c} \\ 1yr \\ 2yr \\ n.d. \end{array} \begin{array}{c} \begin{array}{ccc} 1yr & 2yr & n.d. \end{array} \\ \begin{pmatrix} 0.09 & 0.01 & 0.01 \\ 0.03 & 0.11 & 0.01 \\ 0.01 & 0.03 & 0.70 \end{pmatrix} \end{array}, \qquad \tau_K(D_{\text{high}}) = 0.83. \tag{5.6}$$

Note also that the distributions are skewed in the sense that the counterparty has a higher default probability than the investor. The loss given default is 50% for both the investor and the counterparty and the loss on any posted collateral is considered the same. The collateral rates are chosen to be equal to the risk-free rate. We assume that the collateral account is equal to the risk-free price of the deal at each margin date, i.e. $C_t = V_t$. This is reasonable as the dealer and client will be able to agree on this price, in contrast to $\bar{V}_t$ due to asymmetric information. Also, choosing the collateral this way has the added advantage that the collateral account C works as a control variate, reducing the variance of the least-squares Monte Carlo estimator of the deal price.

5.3 Preliminary analysis without credit risk and with symmetric funding rates

To provide some ball-park figures on the effect of funding risk, we first look at the case without default risk and without collateralization of the deal. We compare our Monte Carlo approach to four alternative (simplified) approaches:

(i) Simple discounting of the risk-free Black-Scholes price with a symmetric funding rate $\hat{f} = f^+ = f^-$. We obtain

$$V_t^{(i)} = e^{-\hat{f}T}\left(S_t\Phi(d_1(t)) - Ke^{-r(T-t)}\Phi(d_2(t))\right),$$

assuming a continuously compounded funding rate.

(ii) The Black-Scholes price where both discounting and the growth of the underlying happens at the symmetric funding rate

$$V_t^{(ii)} = \left(S_t\Phi(g_1(t)) - Ke^{-\hat{f}(T-t)}\Phi(g_2(t))\right), \quad g_{1,2} = \frac{\ln(S_t/K) + (\hat{f} \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

(iii) Simple discounting of the forward price with the symmetric funding rate. This approach can be justified by the fact that the price of a deep out-of-the-money call option will approximately be equal to that of a forward contract.

$$V_t^{(iii)} = S_t - Ke^{-\hat{f}(T-t)}.$$

(iv) We use the above FVA formula in Proposition 2.3.4 with some approximations. Since in a standard Black-Scholes setting $F_t = -Ke^{-r(T-t)}\Phi(d_2(t))$, we

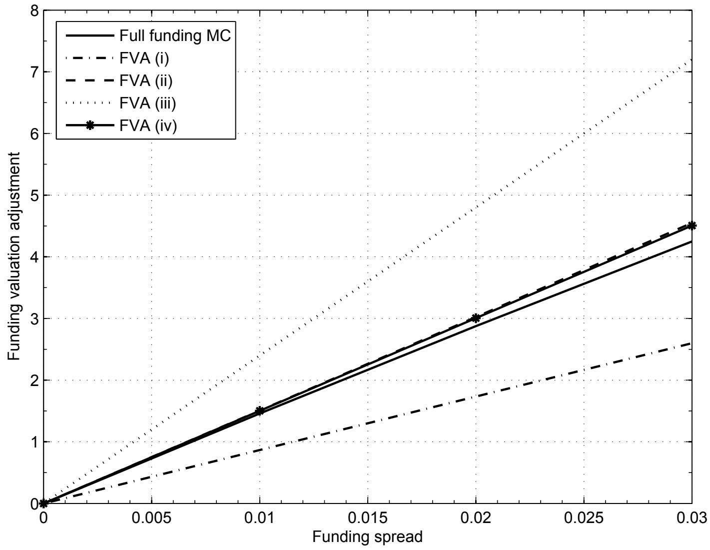

Figure 5.1: Funding valuation adjustment of a long call position as a function of symmetric funding spreads $s_f := \hat{f} - r$ with $\hat{f} := f^+ = f^-$. The adjustments are computed under the assumption of no default risk nor collateralization.

compute

$$\mathrm{FVA}^{(iv)} = (r - \hat{f}) \int_0^T \mathbb{E}_0 \left\{ e^{-rs}[F_s] \right\} ds = (\hat{f} - r) K e^{-rT} \int_0^T \mathbb{E}_0 \left\{ \Phi(d_2(s)) \right\} ds.$$

We illustrate the four approaches in the case of an equity call option (long position). Moreover, let the funding valuation adjustment in each case be defined by $\mathrm{FVA}^{(i,ii,iii,iv)} = V^{(i,ii,iii,iv)} - V$. Figure 5.1 plots the resulting funding valuation adjustment with credit and collateral switched off under the four different approaches and under the full valuation approach. Recall that if the funding rate is equal to the risk-free rate, the value of the call option collapses to the Black-Scholes price and the funding valuation adjustment is zero.

Remark 5.3.1. (Current market practice for FVA). *It is important to realize in looking at Figure 5.1, that at the time of writing this thesis, for a simple call option most market players would adopt a methodology like (iv) or (ii). Even if borrowing or lending rates were different, most market players would average them and apply a common rate to borrowing and lending, in order to avoid nonlinearities. We will discuss the approximation error entailed in this symmetrization later when* *introducing the nonlinearity valuation adjustment. For the time being, we notice that method (iv) produces the same results as the quicker method (ii), that simply replaces the risk-free rate by the funding rate. In the simple case of a long position in a call option without credit and collateral, and with symmetric borrowing and lending rates, we can show that this method is sound since it stems directly from our rigorous Formula (3.19). We also see that both methods (ii) and (iv) are quite close to the full numerical method we adopt. Occasionally, the industry may adopt methods such as (i), but this is not recommended, as we can see from the results. Method (iii) is not accurate either, since it ignores optionality, and would not be used by the industry in a case like this. Overall industry-like methods such as (ii) or (iv) work well here, and there would be no need to implement the full machinery. However, once collateral and credit risk are in the picture, and once nonlinearities due to replacement close-out at default and asymmetry in borrowing and lending are present, there is no way we can keep using something like (ii) or (iv) and we need to implement the full methodology.*

5.4 Complete valuation under credit risk, collateral, and asymmetric funding

Let us now switch on credit risk and consider collateralized deals. The recursive structure of our simulation method makes the pricing problem particularly demanding in terms of computational time, so we are forced to choose a relatively small number of sample paths. We use 1,000 sample paths but, fortunately, the presence of collateral as a control variate mitigates large errors. In Tables 5.1-5.2 we conduct a ceteris paribus analysis of funding risk under counterparty credit risk and collateralization. Specifically, we investigate how the value of a deal changes for different values of the borrowing (lending) rate f^+ (f^-) while keeping the lending (borrowing) rate fixed to 100 bps. When both funding rates are equal to 100 bps the deal is funded at the risk-free rate and we are in the classical derivatives pricing setting.

Remark 5.4.1. (Potential arbitrage). *Note that if $f^+ < f^-$ arbitrage opportunities might be present, unless certain constraints are imposed on the funding policy of the treasury. Such constraints may look unrealistic and may be debated from viewpoint of arbitrageability, but since our point here is strictly to explore the impact of asymmetries in the funding equations we will still apply our framework to a few examples where $f^+ < f^-$.*

Table 5.1 reports the impact of changing funding rates for a call position when the posted collateral may not be used for funding the deal, i.e. rehypothecation is not allowed. First, for the long position, increasing the lending rate f^- while keeping the borrowing rate f^+ fixed causes an increase in the deal value. On the other hand, an increase in the borrowing rate while fixing the lending rate, decreases the value of the short position, i.e. the negative exposure of the investor increases. As a call option is just a one-sided contract, increasing the borrowing rate for a long position only has a minor impact. Recall that F is defined as the cash account needed as part of the derivative replication strategy or, analogously, the cash account required to fund the hedged derivative position. To hedge a long call, the investor goes short in delta position of the underlying asset and invests excess cash in the treasury at f^-. Correspondingly, to hedge the short position, the investor enters a long delta position in the stock and finances it by borrowing cash from the treasury at f^+, so changing the lending rate only has a small effect on the deal value. Finally, due to the presence of collateral, we observe an almost similar price impact of funding under the two different default distributions D_{low} and D_{high}.

Assuming cash collateral, we consider the case of rehypothecation and allow the investor and counterparty to use any posted collateral as a funding source. If the collateral is posted to the investor, this means it effectively reduces his costs of funding the delta-hedging strategy. As the payoff of the call is one-sided, the investor only receives collateral when he holds a long position in the call option. However, as he hedges this position by short-selling the underlying stock and lending the excess cash proceeds, the collateral adds to his cash lending position and increases the funding benefit of the deal. Analogously, if the investor has a short position, he

Table 5.1: Price impact of funding with default risk and collateralization

	Default risk, low[b]		Default risk, high[c]	
Funding[a]	Long	Short	Long	Short
Borrowing rate f^+				
100 bps	28.70 (0.15)	-28.72 (0.15)	29.06 (0.21)	-29.07 (0.21)
125 bps	28.53 (0.17)	-29.37 (0.18)	28.91 (0.21)	-29.70 (0.20)
150 bps	28.37 (0.18)	-30.02 (0.22)	28.75 (0.22)	-30.34 (0.20)
175 bps	28.21 (0.20)	-30.69 (0.27)	28.60 (0.22)	-30.99 (0.21)
200 bps	28.05 (0.21)	-31.37 (0.31)	28.45 (0.22)	-31.66 (0.25)
Lending rate f^-				
100 bps	28.70 (0.15)	-28.72 (0.15)	29.06 (0.21)	-29.07 (0.21)
125 bps	29.35 (0.18)	-28.56 (0.17)	29.69 (0.20)	-28.92 (0.21)
150 bps	30.01 (0.22)	-28.40 (0.18)	30.34 (0.20)	-28.76 (0.22)
175 bps	30.68 (0.27)	-28.23 (0.20)	31.00 (0.21)	-28.61 (0.22)
200 bps	31.37 (0.32)	-28.07 (0.39)	31.67 (0.25)	-28.46 (0.22)

Standard errors of the price estimates are given in parentheses.
[a] Ceteris paribus changes in one funding rate while keeping the other fixed to 100 bps.
[b] Based on the joint default distribution D_{low} with low dependence.
[c] Based on the joint default distribution D_{high} with high dependence.

posts collateral to the counterparty and a higher borrowing rate would increase his costs of funding the collateral he has to post, as well as his delta-hedge position. Table 5.2 reports the results for the short and long positions in the call option when rehypothecation is allowed. Figures 5.2-5.3 plot the values of collateralized long and short positions in the call option as a function of asymmetric funding spreads. In addition, Figure 5.4 reports the corresponding FVA defined as the difference between the full funding-inclusive deal price and the full deal price but symmetric funding rates equal to the risk-free rate. Recall that the collateral rates are equal to the risk-free rate, so the LVA collapses to zero in these examples.

This shows that funding asymmetry matters even under full collateralization when there is no repo market for the underlying stock. In practice, however, the dealer cannot hedge a long call by shorting a stock he does not own. Instead, he would first borrow the stock in a repo transaction and then sell it in the spot market. Similarly, to enter the long delta position needed to hedge a short call, the dealer could finance the purchase by lending the stock in a reverse repo transaction.

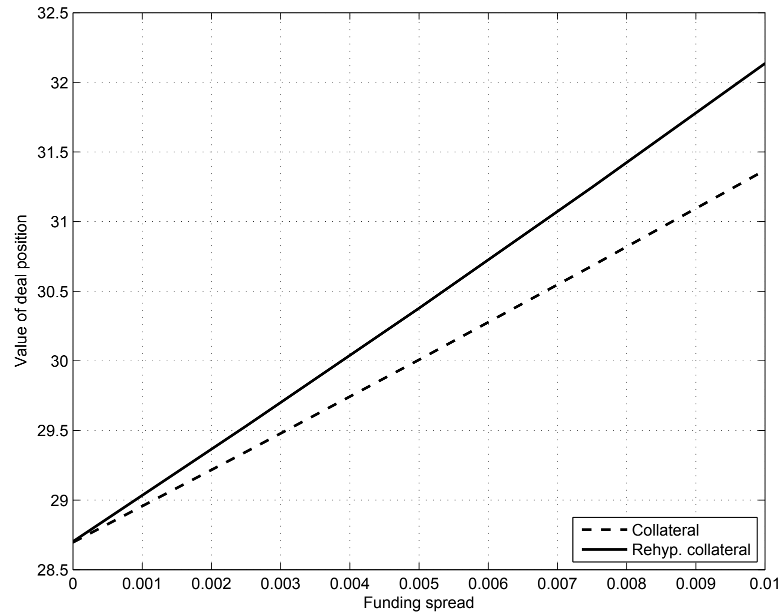

Figure 5.2: The value of a long call position for asymmetric funding spreads $s_f^- = f^- - r$, i.e. fixing $f^+ = r = 0.01$ and varying $f^- \in (0.01, 0.0125, 0.015, 0.0175, 0.02)$.

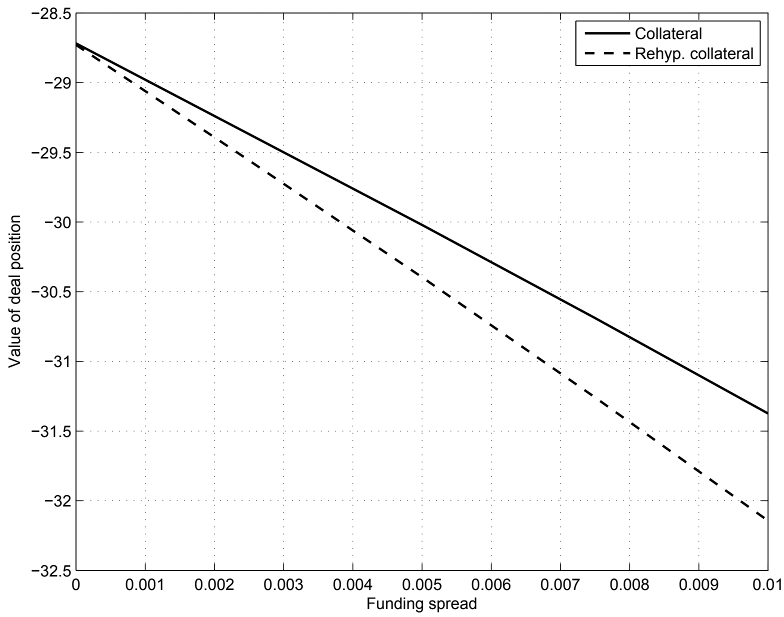

Figure 5.3: The value of a short call position for asymmetric funding spreads $s_f^+ = f^+ - r$, i.e. fixing $f^- = r = 0.01$ and varying $f^+ \in (0.01, 0.0125, 0.015, 0.0175, 0.02)$.

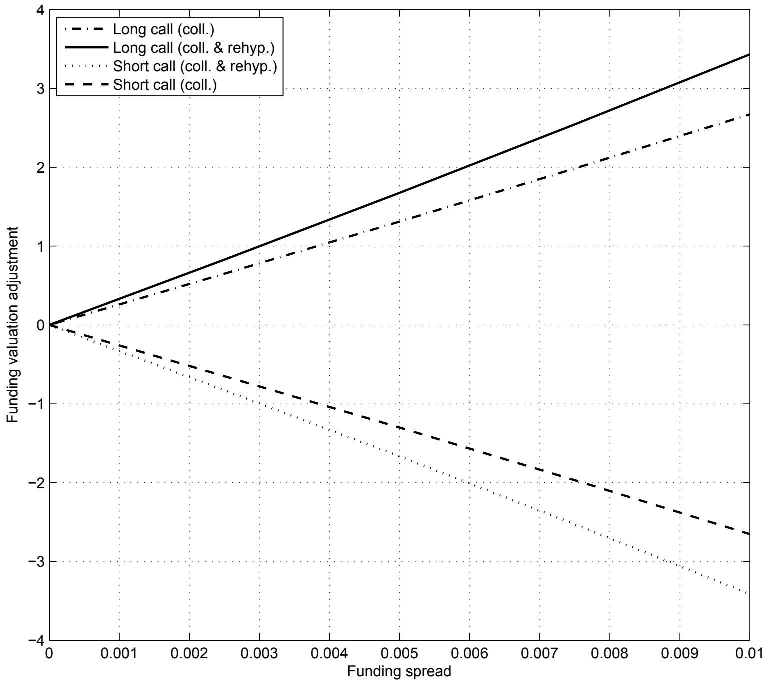

Figure 5.4: Funding valuation adjustment as a function of asymmetric funding spreads. The adjustments are computed under the presence of default risk and collateralization.

Table 5.2: Price impact of funding with default risk, collateralization, and rehypothecation

	Default risk, low[b]		Default risk, high[c]	
Funding[a]	Long	Short	Long	Short
Borrowing rate f^+				
100 bps	28.70 (0.15)	-28.73 (0.15)	29.07 (0.22)	-29.08 (0.22)
125 bps	28.55 (0.17)	-29.56 (0.19)	28.92 (0.22)	-29.89 (0.20)
150 bps	28.39 (0.18)	-30.40 (0.24)	28.77 (0.22)	-30.72 (0.20)
175 bps	28.23 (0.20)	-31.26 (0.30)	28.63 (0.22)	-31.56 (0.23)
200 bps	28.07 (0.22)	-32.14 (0.36)	28.48 (0.22)	-32.43 (0.29)
Lending rate f^-				
100 bps	28.70 (0.15)	-28.73 (0.15)	29.07 (0.22)	-29.08 (0.22)
125 bps	29.53 (0.19)	-28.57 (0.17)	29.07 (0.22)	-28.93 (0.22)
150 bps	30.38 (0.24)	-28.42 (0.18)	32.44 (0.29)	-28.78 (0.22)
175 bps	31.25 (0.30)	-28.26 (0.20)	36.19 (0.61)	-28.64 (0.22)
200 bps	32.14 (0.37)	-28.10 (0.22)	32.44 (0.29)	-28.49 (0.22)

Standard errors of the price estimates are given in parentheses.
[a] Ceteris paribus changes in one funding rate while keeping the other fixed to 100 bps.
[b] Based on the joint default distribution D_{low} with low dependence.
[c] Based on the joint default distribution D_{high} with high dependence.

Effectively, the delta hedging position in the underlying stock would be funded at the prevailing repo rate. Thus, once the delta hedge has to be executed through the repo market, there is no funding valuation adjustment (meaning any dependence on the funding rate $\tilde{f}$ drops out) given the deal is fully collateralized, but the underlying asset still grows at the repo rate. More detailed discussions are carried out later in Chapter 6. If there is no credit risk, this would leave us with the result of Piterbarg [68]. However, if the deal is not fully collateralized or the collateral cannot be rehypothecated, funding costs enter the picture even when there is a repo market for the underlying stock.

5.5 Nonlinearity valuation adjustment

In this last section we introduce a nonlinearity valuation adjustment, and to stay within the usual jargon of the business, we abbreviate it NVA. The NVA first introduced by Brigo et al. in [23] is defined as the difference between the true price

$\bar{V}$ and a version of $\bar{V}$ where nonlinearities have been approximated away through blunt symmetrization of rates and possibly a change in the close-out convention from a replacement close-out to a risk-free close-out. This entails a degree of double counting (both positive and negative interest). In some situations the positive and negative double counting will offset each other, but in other cases this may not happen. Moreover, as pointed out briefly in Chapter 1 section 1.2.4 (for a more detailed analysis, we refer the reader to Brigo et al. [12]), a further source of double counting might be neglecting the first-to-default time in the bilateral CVA/DVA valuation, which is done in a number of industry approximations.

Let $\hat{V}$ be the resulting price of our full pricing algorithm when we replace both f^+ and f^- by $\hat{f} := (f^+ + f^-)/2$. We adopt both a risk-free close-out and a replacement close-out at default, respectively, for this approximated price $\hat{V}$ in our valuation framework. A further simplification in $\hat{V}$ could be to neglect the first-to-default check in the close-out. We have the following definition

Definition 5.5.1. (Nonlinearity valuation adjustment, NVA) *The nonlinearity valuation adjustment (NVA) is defined as*

$$NVA_t := \bar{V}_t - \hat{V}_t,$$

where $\bar{V}$ denotes the full nonlinear deal value while $\hat{V}$ denotes an approximate linearized price of the deal.

As an illustration, we revisit the above example of an equity call option and analyze the NVA in a number of cases. The results are reported in Figures 5.5 and 5.6.

In both figures, we compare NVA under a risk-free close-out and under a replacement close-out. We can see that, depending on the direction of the symmetrization, NVA may be either positive or negative. As the funding spread increases, NVA grows in absolute value. In addition, adopting the replacement close-out amplifies the presence of double counting. Moreover, the NVA accounts for up to 8% of the full deal price $\bar{V}$ depending on the funding spread - a relevant figure in a valuation context.

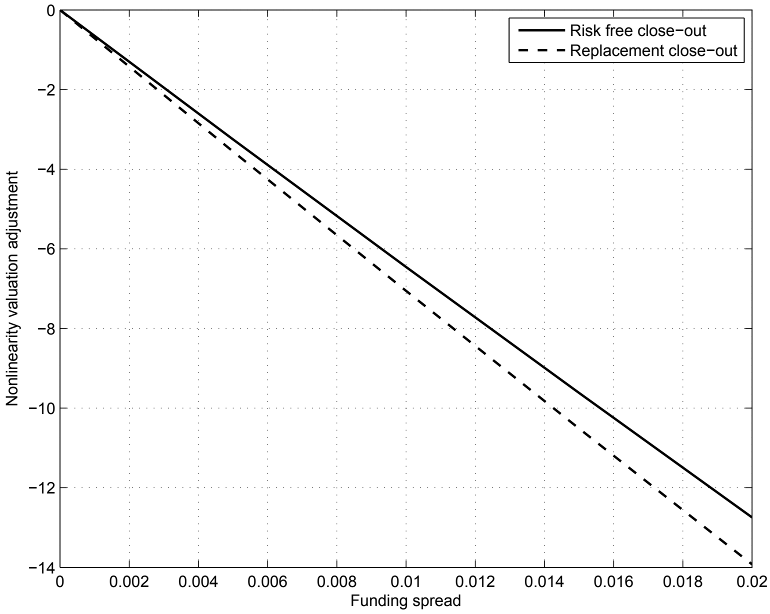

Figure 5.5: Nonlinearity valuation adjustment (in percentage of $\hat{V}$) for different funding spreads $s_f^+ = f^+ - f^- \in (0, 0.005, 0.01, 0.015, 0.02)$ and fixed $\hat{f} = (f^+ + f^-)/2 = 0.01$.

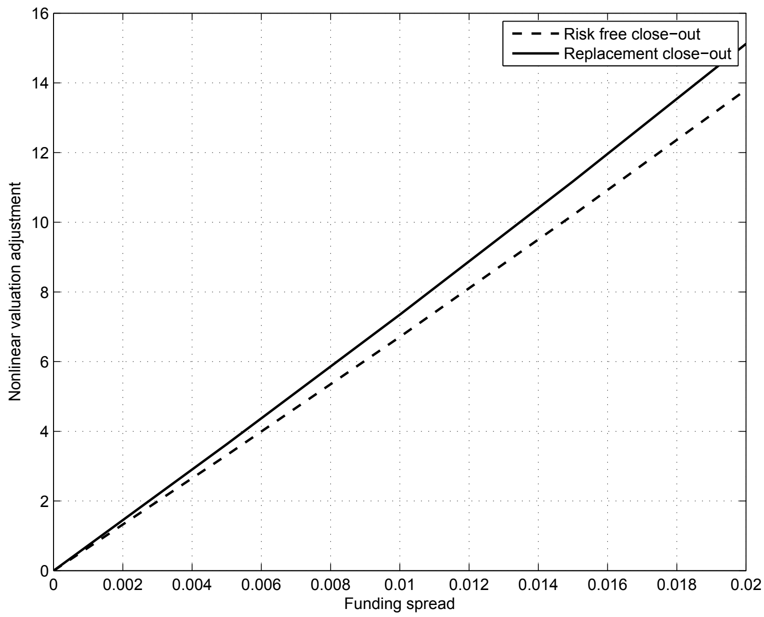

Figure 5.6: Nonlinearity valuation adjustment (in percentage of $\hat{V}$) for different funding spreads $s_f^- = f^- - f^+ \in (0, 0.005, 0.01, 0.015, 0.02)$ and fixed $\hat{f} = (f^+ + f^-)/2 = 0.01$.

Table 5.3: %N_{VA} with default risk, collateralization and rehypothecation

Funding Rates		Risk free			Replacement		
		$\%\widehat{\mathrm{N}_{\mathrm{VA}}}$	$\%\overline{\mathrm{N}_{\mathrm{VA}}}$	$\%\mathrm{N}_{\mathrm{VA}}$	$\%\widehat{\mathrm{N}_{\mathrm{VA}}}$	$\%\overline{\mathrm{N}_{\mathrm{VA}}}$	$\%\mathrm{N}_{\mathrm{VA}}$
s_f	$\hat{f}$						
0 bps	100 bps	0%	0%	0%	0%	0%	0%
25 bps	112.5 bps	1.65%	1.62 %	1.67%	1.79%	1.75%	1.81%
50 bps	125 bps	3.31%	3.20%	3.39%	3.58%	3.46%	3.68%
75 bps	137.5 bps	5.02%	4.78%	5.19%	5.39%	5.12%	5.61%
100 bps	150 bps	6.70%	6.28%	7.01%	7.24%	6.75%	7.62%

i. Funding spread $s_f = f \quad - f^+$.
ii. The prices of the call option are based on the joint default distribution D_{high} with high dependence.

Table 5.3 reports (a) $\%\widehat{\mathrm{N}_{\mathrm{VA}}}$ denoting the fraction of the approximated deal price $\hat{V}$ explained by NVA, (b) $\%\overline{\mathrm{N}_{\mathrm{VA}}}$ denoting the fraction of the full deal price $\bar{V}$, and (c) $\%\mathrm{N}_{\mathrm{VA}}$ denoting the fraction of the deal price with symmetric funding rates equal to the risk-free rate r explained by NVA. Notice that for those cases where we adopt a risk-free close-out at default, the results primarily highlight the double-counting error due to symmetrization of borrowing and lending rates.

We should finally point out that close-out nonlinearities play a limited role here, due to the absence of wrong way risk. An analysis of close-out nonlinearity under wrong way risk is under development.

Chapter 6

Extension and Conclusion

In this chapter, we describe how our model can be extended to address other market realities. We show with a few adjustments that the consistent valuation framework can be used to model trades where the trader implements a hedging strategy via the repo-market, and when the trade is cleared via a central clearing house or governed by a bilateral Credit Support Annex with variation and initial margins. Moreover, we explain how we can include the margin period risk into the model by listing all possible cash flows upon an early default event. The conclusion of the thesis is given at the end of the chapter.

6.1 Repo-Market

So far, we have more or less silently made the assumption that the dealer hedges the derivatives position by trading in the spot market of the underlying asset(s). Nonetheless, to be in business, the dealer might decide or even be forced to implement a hedging strategy that involves trading the underlying assets through stock-lending or repo markets or by entering other derivatives positions, e.g., (synthetic) forward contracts on the underlying risk factors. As a result, the dealer may incur additional costs or revenues which we obviously need to include when pricing the deal.

6.1.1 Incorporating the hedging costs

To address this issue, we introduce two general adapted processes: $h_t^-(T)$ is the rate of hedging revenue for lending risky assets from time t to T, while $h_t^+(T)$ is the hedging cost rate for asset borrowing. The corresponding effective hedging rate $\tilde{h}_t(T)$ is defined as

$$\tilde{h}_t(T) := h_t^-(T)\mathbf{1}_{\{H_t<0\}} + h_t^+(T)\mathbf{1}_{\{H_t>0\}}\,.$$

For example, if the dealer hedges in the stock-lending or repo market, we can apply the quoted repo rate as the hedging rate.

Another example is hedging by trading in collateralized markets, i.e. markets where only collateralized financial contracts are quoted. The money market falls in this category and contracts traded in this market are collateralized on a daily basis at the over-night rate. So, if the hedging strategy implies trading directly in the money market, the effective hedging cost is simply given by the collateral rate itself.

If we assume that the hedging strategy is implemented on the same time-grid as the funding procedure of the deal, we can sum both the funding and hedging costs in a single term. This leads us to redefine φ in (2.17) so it explicitly takes the dependence on the hedging strategy into account:

$$\begin{aligned}\varphi(t, T\wedge\tau; F, H) := &\sum_{j=1}^{m-1} \mathbf{1}_{\{t\leqslant t_j<(T\wedge\tau)\}} D(t,t_j) F_{t_j}\left(1-\frac{P_{t_j}(t_{j+1})}{P_{t_j}^{\tilde{f}}(t_{j+1})}\right) \\ &-\sum_{j=1}^{m-1} \mathbf{1}_{\{t\leqslant t_j<(T\wedge\tau)\}} D(t,t_j) H_{t_j}\left(\frac{P_{t_j}(t_{j+1})}{P_{t_j}^{\tilde{f}}(t_{j+1})}-\frac{P_{t_j}(t_{j+1})}{P_{t_j}^{\tilde{h}}(t_{j+1})}\right),\end{aligned} \tag{6.1}$$

where the zero-coupon (hedging) bond is defined as $P_t^{\tilde{h}}(T) := [1+(T-t)\tilde{h}_t(T)]^{-1}$. From now on, we assume that rehypothecation is allowed (i.e. $F_t = \bar{V}_t - H_t - C_t$). For the case when rehypothecation is forbidden, analogous results can be obtained.

If we take the continuous-time limit, we obtain

$$\begin{aligned}\varphi(t, T\wedge\tau; F, H) = &\int_t^{T\wedge\tau} (r_s - \tilde{f}_s)\left[\bar{V}_s(C,F) - C_s\right] D(t,s)\mathrm{d}s \\ &- \int_t^{T\wedge\tau}(r_s - \tilde{h}_s)H_s D(t,s)\mathrm{d}s\,,\end{aligned} \tag{6.2}$$

which assumes that funding and hedging of the deal takes place in continuous time.

Summing up all the cash-flow streams of the deal: the discounted contractual cash-flows Π, the on-default cash-flow θ, the collateral margining cash-flows γ and the new funding cash-flows including the costs and revenues from the repo market φ, we have

$$\begin{aligned}\bar{V}_t = &\int_t^T \mathbb{E}_t\big[\, \big(\mathbf{1}_{\{s<\tau\}}\Pi(s, s+\mathrm{d}s) + \mathbf{1}_{\{\tau\in\mathrm{d}s\}}\theta_s(C,\varepsilon)\big)\, D(t,s)\,\big] \\ &+ \int_t^T \mathbb{E}_t\Big[\,\mathbf{1}_{\{s<\tau\}}(\tilde{f}_s - \tilde{c}_s)C_s D(t,s)\,\Big]\,\mathrm{d}s \\ &+ \int_t^T \mathbb{E}_t\Big[\,\mathbf{1}_{\{s<\tau\}}\Big((r_s - \tilde{f}_s)\bar{V}_s - (r_s - \tilde{h}_s)H_s\Big)\, D(t,s)\Big]\,\mathrm{d}s.\end{aligned} \tag{6.3}$$

A simple trading example

We use a trading example in the report [20] as a justification of the above cash-flows. Suppose that a trader buys a call option on an equity asset S_t with strike K at time $t < T$:

1. The trader borrows $\bar{V}_t$ amount of cash from the treasury and buys the option.

2. He receives cash C_t from the counterparty as collateral, which is then given to the treasury.

In order to hedge the deal, he needs to hold a short position in the underlying asset, which requires him to repo-borrow the stock on the repo-market:

3. He borrows $H_t = \Delta_t S_t$ cash from the treasury as the guarantee for the repo-borrowing on the repo-market.

4. He then borrows Δ_t units of stock and posts cash H_t as guarantee.

5. The trader sells the stock for H_t and gives the amount back to the treasury.

At time t, the trader owes to the treasury the cash amount $\bar{V}_t - C_t$. Denote the underlying stock price at $t + \mathrm{d}t$ by $S_{t+\mathrm{d}t}$. Then at time $t + \mathrm{d}t$ the trader needs to return the borrowed stock on the repo-market and engages in the following actions:

6. The trader borrows cash $\Delta_t S_{t+\mathrm{d}t}$ from the treasury.

7. He buys Δ_t units of stock and returns to the repo-market to close the position.

8. He receives $H_t(1 + \tilde{h}_t \mathrm{d}t)$ amount of cash from the deposit.

9. He then pays $\Delta_t S_{t+\mathrm{d}t}$ cash back to the treasury.

The net value from the repo position is then given by

$$H_t(1 + \tilde{h}_t \mathrm{d}t) - \Delta_t S_{t+\mathrm{d}t} = -\Delta_t \mathrm{d}S_t + \tilde{h}_t H_t \mathrm{d}t. \tag{6.4}$$

10. The trader closes the derivative position, and receives cash $\bar{V}_{t+\mathrm{d}t}$, which is given back to the treasury.

The trader needs to pay the collateral with interest back to the counterparty. To do so,

11. He borrows $C_t(1 + \tilde{c}_t \mathrm{d}t)$ amount of cash from treasury.

12. He pays the collateral amount with interest back to the counterparty.

The trader's debt to the treasury at time $t + \mathrm{d}t$ is then given by

$$-\bar{V}_{t+\mathrm{d}t} + (\bar{V}_t - C_t)(1 + \tilde{f}_t \mathrm{d}t) + C_t(1 + \tilde{c}_t \mathrm{d}t) = -\mathrm{d}\bar{V}_t + \tilde{f}_t \bar{V}_t \mathrm{d}t + (\tilde{c}_t - \tilde{f}_t) C_t \mathrm{d}t. \tag{6.5}$$

The total amount of cash flows is (combining (6.4) and (6.5))

$$\mathrm{d}\bar{V}_t - \tilde{f}_t \bar{V}_t \mathrm{d}t - (\tilde{c}_t - \tilde{f}_t) C_t \mathrm{d}t - \Delta_t \mathrm{d}S_t + \tilde{h}_t H_t \mathrm{d}t.$$

Assuming that the contract pays a dividend during the time interval $[t, t{+}\mathrm{d}t]$ denoted as $\mathrm{d}D$, and we have $\mathbb{E}_t\big[\,\mathrm{d}\bar{V}_t\big] = r_t \bar{V}_t \mathrm{d}t - \mathrm{d}D$. No-arbitrage risk-neutral pricing requires the price of the above cash-flows to be equal to zero, so we write

$$\begin{aligned} 0 = \mathbb{E}_t\Big[\, \mathrm{d}\bar{V}_t - \tilde{f}_t\bar{V}_t\mathrm{d}t - (\tilde{c}_t - \tilde{f}_t)C_t\mathrm{d}t - \Delta_t\mathrm{d}S_t + \tilde{h}_t H_t\mathrm{d}t \Big] \\ = (r_t - \tilde{f}_t)\bar{V}_t\mathrm{d}t - (\tilde{c}_t - \tilde{f}_t)C_t\mathrm{d}t - \Delta_t(r_t - \tilde{h}_t)H_t\mathrm{d}t - \mathrm{d}D \\ = (r_t - \tilde{f}_t)F_t\mathrm{d}t + (r_t - \tilde{c}_t)C_t\mathrm{d}t + (\tilde{h}_t - \tilde{f}_t)H_t\mathrm{d}t - \mathrm{d}D. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathrm{d}D &= (r_t - \tilde{f}_t)F_t\mathrm{d}t + (r_t - \tilde{c}_t)C_t\mathrm{d}t + (\tilde{h}_t - \tilde{f}_t)H_t\mathrm{d}t \\ &= (r_t - \tilde{f}_t)\bar{V}_t + (\tilde{f}_t - \tilde{c}_t)C_t\mathrm{d}t - (r_t - \tilde{h}_t)H_t\mathrm{d}t, \end{aligned}$$

which coincides with the cash flows in (6.3) that come from the collateral margining, funding and hedging procedures.

6.1.2 Continuous time formulation

If we assume all needed technical conditions to be satisfied as in Section 2.3.2, we can switch from the filtration $\mathcal{G}$ to the default-free market filtration $\mathcal{F}$, and rewrite equation (6.3) as follows:

$$\begin{aligned} \bar{V}_t = \mathbf{1}_{\{\tau>t\}} \int_t^T \; \mathbb{E}_t[\, (\Pi(s, s+\mathrm{d}s) + \lambda_s\theta_s(C,\varepsilon)\mathrm{d}s)\, D(t,s;r+\lambda)|\mathcal{F}\,] \\ + \mathbf{1}_{\{\tau>t\}} \int_t^T \; \mathbb{E}_t\Big[\, (\tilde{f}_s - \tilde{c}_s)C_s D(t,s;r+\lambda)|\mathcal{F}\Big]\, \mathrm{d}s \\ + \mathbf{1}_{\{\tau>t\}} \int_t^T \; \mathbb{E}_t\Big[\, \Big((r_s - \tilde{f}_s)\bar{V}_s - (r_s - \tilde{h}_s)H_s\Big)\, D(t,s;r+\lambda)|\mathcal{F}\Big]\, \mathrm{d}s. \end{aligned} \tag{6.6}$$

We can now repeat the calculations in Chapter 3 to obtain the FBSDE and the semi-linear PDE for the continuous-time consistent pricing problem including hedging costs.

We start by introducing the following stochastic process,

$$X_t := \int_0^t \Big(\pi_s + \lambda_s\theta_s(C,\varepsilon) + (\tilde{f}_s - \tilde{c}_s)C_s + (r_s - \tilde{f}_s)\bar{V}_s - (r_s - \tilde{h}_s)H_s\Big) D(0,s;r+\lambda)\mathrm{d}s.$$

Now we multiply by $D(0,t;r+\lambda)$ and then add X_t to both sides of (6.6):

$$\bar{V}_t D(0,t;r+\lambda) + X_t = \mathbb{E}_t[\, X_T|\mathcal{F}\,]\,. \tag{6.7}$$

Define $\mathcal{M}_t = \mathbb{E}_t[\, X_T|\mathcal{F}\,]$, and differentiate both sides of (6.7) with respect to t yielding

$$\begin{aligned}-(r_t+\lambda_t)\bar{V}_t\mathrm{d}t + \mathrm{d}\bar{V}_t + \Big[\pi_t + \lambda_t\theta_t(C,\varepsilon) + (\tilde{f}_t - \tilde{c}_t)C_t + (r_t - \tilde{f}_t)\bar{V}_t - (r_t - \tilde{h}_t)H_t\Big]\mathrm{d}t \\ = \mathrm{d}\mathcal{M}_t/D(0,t;r+\lambda).\end{aligned}$$

Observe that $\int_0^t \mathrm{d}\mathcal{M}_t/D(0,t;r+\lambda)$ is a local $\mathcal{F}_t$-martingale. Applying the martingale representation theorem, we can write $\int_0^t \mathrm{d}\mathcal{M}_t/D(0,t;r+\lambda) = \int_0^t Z_u \mathrm{d}W_u$ for Z_u being a $\sigma(W)$-predictable process.

Define a stochastic process $Y_t = \bar{V}_t$. Suppose that there exists a deterministic linear function H such that $H_t = H(t,S_t,Y_t,Z_t)$. Moreover, we postulate that the close-out amount is taken to be the price of the deal, i.e. $\varepsilon_t = \bar{V}_t$, and also that the collateral account is a function of the adjusted deal price, so that the on-default cash-flow $\theta_t(C,\varepsilon)$ is a function of the derivative price $\bar{V}_t$, denoted as $\theta(Y_t)$ for some measurable function θ. The (coupled) FBSDE for the consistent pricing equation including funding and hedging costs is then given by

$$\begin{aligned}&\mathrm{d}S_t = \mu(t,S_t,Y_t)\mathrm{d}t + \sigma(t,S_t,Y_t)\mathrm{d}W_t, \qquad S_0 = s_0,\\ &\mathrm{d}Y_t = -B^{\tilde{h}}(t,S_t,Y_t,Z_t)\mathrm{d}t + Z_t\mathrm{d}W_t, \qquad Y_T = 0, \\ &B^{\tilde{h}}(t,S_t,Y_t,Z_t) = \pi_t + \lambda_t\theta(Y_t) - (\tilde{f}_t + \lambda_t)Y_t + (\tilde{f}_t - \tilde{c}_t)C_t - (r_t - \tilde{h}_t)H(t,S_t,Y_t,Z_t),\end{aligned} \tag{6.8}$$

with s_0 being the initial underlying stock price and $B^{\tilde{h}}(t,S_t,Y_t,Z_t)$ being the driver of the FBSDE.

Assume that Assumption 3 is in force and all the rates $\tilde{f}$, $\tilde{c}$, $\tilde{h}$ and r are bounded. Following analogous arguments to the proof of Theorem 3.1.5, there exists a unique solution to the above FBSDE (6.8).

We now focus on the decoupled case. We postulate that, as in section 3.2.1, $\tilde{f}_t =$

$f(t,S_t,Y_t)$, $\tilde{c}_t = c(t,S_t,Y_t)$, $\tilde{h}_t = h(t,S_t,Y_t)$, $\lambda_t = \lambda(t,S_t,Y_t)$ and $r_t = r(t,S_t,Y_t)$, where functions f, c, h, λ, r are deterministic and bounded. In the decoupled case, we have $\mu(t,S_t,Y_t,Z_t) = \mu(t,S_t)$ and $\sigma(t,S_t,Y_t,Z_t) = \sigma(t,S_t)$ for the forward component. If we further assume that the price process $\bar{V}$ has sufficient smoothness, as required in (3.8), we can apply Itô's formula to $\bar{V}_t$, and compare the drift and diffusion terms with the BSDE in (6.8). We obtain the following for $\tau > t$ (For the ease of notation, we denote $H_t = H(t,s,\nu(t,s),Z_t)$ in the following),

$$\begin{aligned}
&-\tilde{f}_t\bar{V}_t - \lambda_t\bar{V}_t + (\tilde{f}_t - \tilde{c}_t)C_t - (r_t - \tilde{h}_t)H_t + \pi_t + \lambda_t\theta_t(\bar{V}_t) + \frac{\partial \bar{V}}{\partial t} + \mu(t,S_t)\frac{\partial \bar{V}}{\partial S}\\
&+\frac{1}{2}\sigma(t,S_t)^2\frac{\partial^2\bar{V}}{\partial S^2} = 0, \qquad Z_t = \sigma(t,S_t)\frac{\partial \bar{V}}{\partial S}.
\end{aligned} \tag{6.9}$$

Hence, the adjusted deal price with hedging costs $\bar{V}$ satisfies the following semi-linear PDE for all $(t,s) \in [0,T] \times \mathbb{R}^+$:

$$\begin{aligned}
&\partial_t\nu(t,s) + \mu(t,s)\partial_s\nu(t,s) + \frac{1}{2}\sigma(t,s)^2\partial_s^2\nu(t,s) + B^{\tilde{h}}\left(t,s,\nu(t,s),(\partial_s\nu\sigma)(t,s)\right) = 0,\\
&\nu(T,s) = 0,
\end{aligned} \tag{6.10}$$

with $B^{\tilde{h}}\left(t,s,\nu(t,s),(\partial_s\nu\sigma)(t,s)\right)$ being the driver of the FBSDE defined in (6.8).

6.1.3 Invariance theorem

We now assume that the underlying S_t is a tradable asset and follows the Black-Scholes type dynamic. According to standard no-arbitrage theory, the drift of the underlying asset is the risk-free rate under the risk-neutral probability measure. Substituting the dynamics of the underlying asset (3.17) to the PDE (6.10), we have

$$\begin{aligned}
&\partial_t\nu(t,s) + r_t s\partial_s\nu(t,s) + \frac{1}{2}\sigma(t)^2s^2\partial_s^2\nu(t,s) + \left(\tilde{f}_t - \lambda_t\right)\nu(t,s) + \left(\tilde{f}_t - \tilde{c}_t\right)C_t\\
&-\left(r_t - \tilde{h}_t\right)H(t,s,\nu(t,s),Z_t) + \pi_t + \lambda_t\theta(\nu(t,s)) = 0, \quad Z_t = \sigma(t)s\partial_s\nu(t,s).
\end{aligned}$$

If we choose $H(t,s,\nu(t,s),Z_t) = \partial_s \nu(t,s)s = \frac{Z_t}{\sigma(t)}$ and assume that the investor adopts delta-hedging, equation (6.10) collapses to

$$\begin{aligned}&\left(\partial_t - \tilde{f}_t - \lambda_t + \mathcal{L}_t^{\tilde{h}}\right)\nu(t,s) + \left(\tilde{f}_t - \tilde{c}_t\right)C_t + \lambda_t\theta_t + \pi_t = 0,\\ &\nu(T,s) = 0,\end{aligned} \tag{6.11}$$

where the generator is defined as

$$\mathcal{L}_t^{\tilde{h}}\nu(t,s) := \tilde{h}_t H_t + \mathcal{L}_t^2\nu(t,s) := \tilde{h}_t H_t + \frac{1}{2}\sigma(t,s)^2\frac{\partial^2\nu}{\partial s^2}.$$

The above semi-linear PDE is the pre-default ($\tau > t$) PDE for the consistent pricing problem including hedging costs.

Again, we could solve this equation numerically, but we choose to apply the reasoning of Section 3.2.3 to reach a similar result as Theorem 3.2.5:

Corollary 6.1.1 (Continuous-time Solution with Hedging Costs). *Suppose that collateral rehypothecation is allowed and that delta-hedging is implemented by trading on a derivative market where the effective hedging rate is $\tilde{h}$. The consistent valuation equation in continuous time is then given by*

$$\bar{V}_t(C;F) = \int_t^T \mathbb{E}_t^{\tilde{h}}\left[\left(\pi_s + \lambda_s\theta_s + (\tilde{f}_s - \tilde{c}_s)C_s\right)D(t,s;\tilde{f}+\lambda) \mid \mathcal{F}\right]\mathrm{d}s, \tag{6.12}$$

where the expectation is taken under a pricing measure $\mathbb{Q}^{\tilde{h}}$ for which the underlying risk factors grow at the rate $\tilde{h}$ if no dividend is paid.

Proof. Assume that $\nu(t,S_t)$ is a solution to the PDE (6.11), with boundary condition $\nu(T,S_T) = 0$, where the process S_t satisfies the following SDE

$$\mathrm{d}S_t = \tilde{h}_t S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t^{\tilde{h}},$$

with $W_t^{\tilde{h}}$ being the Brownian motion under the pricing measure $\mathbb{Q}^{\tilde{h}}$.
For $t < s \le T$, we define a process

$$Y_s = \int_t^s F(u,S_u,C_u,\bar{V}_u)D(t,u;\tilde{f}+\lambda)\mathrm{d}u + D(t,s;\tilde{f}+\lambda)\nu(s,S_s),$$

where

$$F(u, S_u, C_u, \bar{V}_u) = (\tilde{f}_u - \tilde{c}_u)C_u + \lambda_u\theta_u + \pi_u.$$

Differentiating Y_s, we get

$$\begin{aligned}
\mathrm{d}Y_s &= \mathrm{d}D(t,s;\tilde{f}+\lambda)\nu(s,S_s) + D(t,s;\tilde{f}+\lambda)\mathrm{d}\nu(s,S_s) + \mathrm{d}D(t,s;\tilde{f}+\lambda)\mathrm{d}\nu(s,S_s) \\
&\quad + \mathrm{d}\left(\int_t^s F(u,S_u,C_u,\bar{V}_u)D(t,u;\tilde{f}+\lambda)\mathrm{d}u\right) \\
&= D(t,s;\tilde{f}+\lambda)\left(\partial_s\nu_s + \tilde{h}_s S_s\frac{\partial\nu_s}{\partial S_s} + \frac{1}{2}\sigma_s^2 S_s^2\frac{\partial^2\nu_s}{\partial S_s^2} - (\tilde{f}_s+\lambda_s)\nu_s + F(s,S_s,C_s,\bar{V}_s)\right)\mathrm{d}s \\
&\quad + D(t,s;\tilde{f}+\lambda)\sigma_s S_s\frac{\partial\nu_s}{\partial S_s}\mathrm{d}W_s^{\tilde{h}}.
\end{aligned}$$

Since ν_s is a solution to the PDE, the ds term in the above equation cancels out, leaving

$$Y_T = Y_t + \int_t^T D(t,s;\tilde{f}+\lambda)\sigma_s S_s\frac{\partial\nu_s}{\partial S_s}\mathrm{d}W_s^{\tilde{h}}.$$

Therefore, the process Y is a $(Q^{\tilde{h}}, \mathcal{F})$-local martingale. Taking the conditional expectation with respect to filtration $\mathcal{F}_t$, we have

$$\mathbb{E}_t^{\tilde{h}}[Y_T \mid \mathcal{F}] = \mathbb{E}_t^{\tilde{h}}[Y_t \mid \mathcal{F}] = \nu(t, S_t).$$

So the solution to the PDE is

$$\begin{aligned}
\bar{V}_t &= \nu(t,S_t) \\
&= \mathbb{E}_t^{\tilde{h}}\left[\int_t^T F(s,S_s,C_s,\bar{V}_s)D(t,s;\tilde{f}+\lambda)\mathrm{d}s + D(t,T;\tilde{f}+\lambda)\nu(T,S_T) \mid \mathcal{F}\right] \\
&= \mathbb{E}_t^{\tilde{h}}\left[\int_t^T \left(\pi_s + \lambda_s\theta_s + (\tilde{f}_s - \tilde{c}_s)C_s\right) D(t,s;\tilde{f}+\lambda)\mathrm{d}s \mid \mathcal{F}\right].
\end{aligned}$$

□

Analogous to the case of hedging in the spot market, we incorporate the additional hedging costs by altering the drift of the price processes of the underlying risk factors. Additionally, by handling hedging costs via a change of measure, we observe that the explicit dependence on H_t disappears from the pricing equation.

Moreover, the dependence on the risk-free rate r_t dropped out from the valuation equation as we found for the case without assets lending/borrowing.

Remark 6.1.1. (Invariance of the valuation equation with respect to the short rate r_t). *There is no dependence on a risk-free rate r_t in equations (6.11) or (6.12) for valuation under credit, collateral, funding and hedging costs. The valuation is completely governed by market rates and is invariant to r_t.*

6.2 CCP cleared or bilateral CSA trades with variation and initial margins

The growing attention on counterparty credit risk resulted in an increased number of operations moved from a bilateral OTC agreement under a Credit Support Annex (CSA) to a cleared trade through central clearing houses (CCPs), while most of the remaining contracts are traded under collateralization regulated by a CSA with variation and intial margins. The Tabb group estimated a 2 USD trillion liquidity impact lead by the full onset of CCPs. A CCP acting as a market participant interposes itself between two parties, takes the risk of the counterparty default and ensures the exchange of payments even in case of default. Brigo and Pallavicini in [30], for the first time, developed a comprehensive approach for pricing under CCP clearing, including variation and initial margins, gap credit risk and collateralization. In this section we explain, based on the study in [30], how the consistent pricing framework we set up in the previous chapters can be tailored to address trading through a CCP or via a bilateral CSA with initial and variation margins.

6.2.1 Variation and initial margins

When a client ("C") enters into a CCP cleared trade, he will trade with his clearing member denoted in the following as "I". There will be no direct obligation between each client. If the mark-to-market moves against one of the parties, this party will post collateral margins called variation margin (VM) which protects the clearing house against credit and market risk. The VM will be passed to the other party by the CCP and can be rehypothecated. The VM provider will receive interest on the posted cash collateral as is done in uncleared bilateral trades under CSA. Additionally, an initial margin (IM) is posted as needed to protect the CCP over additional risks, for example gap risk, wrong way risk, deteriorating quality of collateral and so forth, and it will be held in a segregated account by the CCP during the life of the trade. So the IM is a source of funding costs but does not generate a funding benefit.

In the case of bilateral CSA trades, when initial margins are posted to cover for additional risks which are not protected by variation margins, such as gap risk, the approach is similar to that of CCPs.

We now modify the pricing framework set up previously to address variation and initial margining. The total amount of collateral assets C_t exchanged by the margining procedure of a derivative trade with variation and initial margins at time $0 \leq t < T$ (taking the point of view of the investor/clearing member) is defined as

$$C_t := M_t + N_t^C + N_t^I, \quad N_t^C \geq 0, \quad N_t^I \leq 0, \tag{6.13}$$

where we denote the variation margin account as an adapted process M_t, the initial margin account posted by the counterparty as an adapted process N_t^C and the initial margin account posted by the investor as an adapted process N_t^I. Note that the initial margins are posted without netting to cover for the gap risk. However, when a derivative trade is cleared by a CCP, only the counterparty (or client) posts initial margin, the clearing member "I" does not, and we can set $N_t^I = 0$ in this case.

As in the classical theory, a derivative price can be perfectly replicated by a cash position F and a risky component of the hedging portfolio H, namely,

$$\bar{V}_t = F_t + H_t. \tag{6.14}$$

When rehypothecation is allowed, the variation margin can be used by the collateral taker as a source of funding to reduce the costs of funding the deal. In such case, we replicate the derivative price by means of the following

$$\bar{V}_t = F_t + H_t + M_t. \tag{6.15}$$

We assume that the variation margin can always be rehypothecated. The results for the case when rehypothecation is forbidden can be analogously obtained.

6.2.2 Funding costs under CCP clearing and bilateral CSA

In the case of CCP clearing or bilateral CSA with variation and initial margins, additional cash accounts are needed to implement the collateral margining procedures. The cash flows for the funding of the segregated initial margins, $N_s^C \geq 0$ posted by the counterparty and $N_s^I \leq 0$ posted by the investor, should be taken into account when we calculate the funding cash flows. If a party is posting the initial margin, he is facing extra costs to fund this collateral. On the contrary, the party that receives the initial margin may book a funding benefit, if he is allowed to invest this collateral.

Recall that when the investor implements its hedging strategy via repo market, the funding cash-flow φ is defined as (6.2). If we add the additional funding costs for the initial margins to (6.2), the funding cash-flow can be redefined as

$$\begin{aligned}\varphi(t, T\wedge\tau) := &\int_t^{T\wedge\tau} (r_s - \tilde{f}_s) F_s D(t,s)\mathrm{d}s - \int_t^{T\wedge\tau} (\tilde{f}_s - \tilde{h}_s) H_s D(t,s)\mathrm{d}s \\ &+ \int_t^{T\wedge\tau} (f_s^{N,C} - r_s) N_s^C D(t,s)\mathrm{d}s + \int_t^{T\wedge\tau} (f_s^{N,I} - r_s) N_s^I D(t,s)\mathrm{d}s\,,\end{aligned} \tag{6.16}$$

where we assume that funding and hedging procedures take place in continuous time. We denote processes $f^{N,C}$ and $f^{N,I}$ for the funding rates associated with the initial margin accounts for the counterparty and the investor respectively. Notice that the funding rates for the initial margin accounts $f^{N,C}$ and $f^{N,I}$, in principle, can be different from the funding rate $\tilde{f}$, because the initial margins are not in the funding netting set of the derivative. Moreover, if the initial margin funding rate $f^{N,I}$ is greater than the risk-free rate r_t, the funding adjustment term will act as a penalty for the investor. In the case when the investment of the received IM is not allowed, the initial margin funding rate $f^{N,C}$ is equal to the risk-free rate r_t, and there will be no adjustment to the derivative price.

We now replace the funding cash-flow in the pricing equation (6.3) with (6.16) to find that

$$\begin{aligned}
\bar{V}_t = & \int_t^T \mathbb{E}_t\big[\, \big(\mathbf{1}_{\{s<\tau\}}\Pi(s,s+\mathrm{d}s) + \mathbf{1}_{\{\tau\in \mathrm{d}s\}}\theta_s(C,\varepsilon)\big)\, D(t,s)\,\big] \\
& + \int_t^T \mathbb{E}_t\Big[\, \mathbf{1}_{\{s<\tau\}}(\tilde{f}_s - \tilde{c}_s)M_s D(t,s)\Big]\,\mathrm{d}s \\
& + \int_t^T \mathbb{E}_t\Big[\, \mathbf{1}_{\{s<\tau\}}\Big((r_s - \tilde{f}_s)\bar{V}_s - (r_s - \tilde{h}_s)H_s\Big)\, D(t,s)\Big]\,\mathrm{d}s \\
& + \int_t^T \mathbb{E}_t\big[\, \mathbf{1}_{\{s<\tau\}}\big((f_s^{N,C} - \tilde{c}_s)N_s^C + (f_s^{N,I} - \tilde{c}_s)N_s^I\big)\, D(t,s)\,\big]\,\mathrm{d}s. \qquad (6.17)
\end{aligned}$$

Again, assuming that all needed technical conditions are satisfied as in Section 2.3.2, we switch from the filtration $\mathcal{G}$ to the default-free market filtration $\mathcal{F}$, and rewrite the above equation (6.17) as

$$\begin{aligned}
\bar{V}_t = & \mathbf{1}_{\{\tau>t\}}\int_t^T \mathbb{E}_t\big[\, (\Pi(s,s+\mathrm{d}s) + \lambda_s\theta_s(C,\varepsilon)\mathrm{d}s)\, D(t,s;r+\lambda)|\mathcal{F}\,\big] \\
& + \mathbf{1}_{\{\tau>t\}}\int_t^T \mathbb{E}_t\Big[\, (\tilde{f}_s - \tilde{c}_s)M_s D(t,s;r+\lambda)|\mathcal{F}\Big]\,\mathrm{d}s \qquad (6.18) \\
& + \mathbf{1}_{\{\tau>t\}}\int_t^T \mathbb{E}_t\Big[\, \Big((r_s - \tilde{f}_s)\bar{V}_s - (r_s - \tilde{h}_s)H_s\Big)\, D(t,s;r+\lambda)|\mathcal{F}\Big]\,\mathrm{d}s \\
& + \mathbf{1}_{\{\tau>t\}}\int_t^T \mathbb{E}_t\big[\, \big((f_s^{N,C} - \tilde{c}_s)N_s^C + (f_s^{N,I} - \tilde{c}_s)N_s^I\big)\, D(t,s;r+\lambda)|\mathcal{F}\,\big]\,\mathrm{d}s.
\end{aligned}$$

The above pricing equation takes the form of an FBSDE.

6.2.3 FBSDE formulation

In the following, we repeat the calculations in Chapter 3 to obtain the FBSDE of the pricing problem including VM and IM. Firstly, we introduce the following process

$$X_t := \int_0^t \Big(\pi_s + \lambda_s\theta_s(C,\varepsilon) + (\tilde{f}_s - \tilde{c}_s)M_s + (r_s - \tilde{f}_s)\bar{V}_s - (r_s - \tilde{h}_s)H_s \\ + (f_s^{N,C} - \tilde{c}_s)N_s^C + (f_s^{N,I} - \tilde{c}_s)N_s^I\Big)D(0,s;r+\lambda)\mathrm{d}s.$$

We now construct an $\mathcal{F}$-martingale by multiplying $D(0,t;r+\lambda)$ and then adding X_t to both sides of (6.18):

$$\bar{V}_t D(0,t;r+\lambda) + X_t = \mathbb{E}_t[\,X_T|\mathcal{F}\,]\,. \tag{6.19}$$

We define $\mathcal{M}_t = \mathbb{E}_t[\,X_T|\mathcal{F}\,]$, and then differentiate both sides of (6.19) with respect to t,

$$-(r_t+\lambda_t)\bar{V}_t\mathrm{d}t + \mathrm{d}\bar{V}_t + \Big[\pi_t + \lambda_t\theta_t(C,\varepsilon) + (\tilde{f}_t - \tilde{c}_t)M_t + (r_t - \tilde{f}_t)\bar{V}_t - (r_t - \tilde{h}_t)H_t \\ + (f_t^{N,C} - \tilde{c}_t)N_t^C + (f_t^{N,I} - \tilde{c}_t)N_t^I\Big]\mathrm{d}t = \mathrm{d}\mathcal{M}_t/D(0,t;r+\lambda).$$

We see that the right hand-side $\int_0^t \mathrm{d}\mathcal{M}_t/D(0,t;r+\lambda)$ is a local $\mathcal{F}_t$-martingale. Assuming it is adapted to the Brownian filtration $\sigma(W)$, we can apply the martingale representation theorem, and write $\int_0^t \mathrm{d}\mathcal{M}_t/D(0,t;r+\lambda) = \int_0^t Z_u \mathrm{d}W_u$ for Z_u being a $\sigma(W)$-predictable process.

Define a stochastic process $Y_t = \bar{V}_t$. Suppose that there exists a deterministic linear function H such that $H_t = H(t,S_t,Y_t,Z_t)$. Moreover, we postulate that the on-default cash-flow $\theta(C,\varepsilon)$ is a function of the derivative price $\bar{V}_t$, i.e. $\theta(C,\varepsilon) = \theta(t,Y_t)$ for some measurable function θ. (In the following, we write θ_t instead of $\theta(t,Y_t)$ for the sake of notation simplification.) The funding risk inclusive valuation equation including variation and initial margins can be expressed in terms of the following (coupled) FBSDE:

$$\begin{aligned} \mathrm{d}S_t =& \mu(t,S_t,Y_t)\mathrm{d}t + \sigma(t,S_t,Y_t)\mathrm{d}W_t, \qquad S_0 = s_0, \\ \mathrm{d}Y_t =& -\Big[\pi_t + \lambda_t\theta_t - \left(\tilde{f}_t + \lambda_t\right)Y_t + \left(\tilde{f}_t - \tilde{c}_t\right)M_t - \left(r_t - \tilde{h}_t\right)H(t,S_t,Y_t,Z_t) \\ & + \left(f_t^{N,C} - \tilde{c}_t\right)N_t^C + \left(f_t^{N,I} - \tilde{c}_t\right)N_t^I\Big]\mathrm{d}t + Z_t\mathrm{d}W_t, \qquad Y_T = 0, \end{aligned} \tag{6.20}$$

where s_0 is the initial underlying stock price.

Suppose that Assumption 3 is satisfied and all the rates $\tilde{f}$, $\tilde{c}$, $\tilde{h}$, r, $f^{N,C}$ and $f^{N,I}$ are bounded. In the case of funding inclusive valuation with variation and initial margins we can prove the existence of a unique solution to the FBSDE (6.20) analogously to the proof of Theorem 3.1.5.

6.2.4 Semi-linear PDE

In the following, we consider the decoupled case, where the forward component is given as $(t < u)$

$$\mathrm{d}S_u = \mu(u,S_u)\mathrm{d}u + \sigma(u,S_u)\mathrm{d}W_u, \qquad S_t = s.$$

Assume that $\tilde{f}_t = f(t,S_t,Y_t)$, $\tilde{c}_t = c(t,S_t,Y_t)$, $\tilde{h}_t = h(t,S_t,Y_t)$, $\lambda_t = \lambda(t,S_t,Y_t)$, $r_t = r(t,S_t,Y_t)$, $f^{N,C} = f^C(t,S_t,Y_t)$ and $f^{N,I} = f^I(t,S_t,Y_t)$, where the functions f,c,h,λ,r,f^C and f^I are all deterministic and bounded. Moreover, we postulate that the price process $\bar{V}$ satisfy the smoothness assumption (3.8). Applying Itô's formula to $\bar{V}_t$ and comparing the drift and diffusion terms with the FBSDE (6.20), we obtain the following relations for $\tau > t$ (For ease of notation, we denote $H_t = H(t,s,\nu(t,s),Z_t)$),

$$\begin{aligned} & -\left(\tilde{f}_t + \lambda_t\right)\bar{V}_t + \left(\tilde{f}_t - \tilde{c}_t\right)M_t - \left(r_t - \tilde{h}_t\right)H_t + \left(f_t^{N,C} - \tilde{c}_t\right)N_t^C + \left(f_t^{N,I} - \tilde{c}_t\right)N_t^I \\ & + \pi_t + \lambda_t\theta_t + \frac{\partial \bar{V}}{\partial t} + \mu(t,S_t)\frac{\partial \bar{V}}{\partial S} + \frac{1}{2}\sigma(t,S_t)^2\frac{\partial^2 \bar{V}}{\partial S^2} = 0, \quad Z_t = \sigma(t,S_t)\frac{\partial \bar{V}}{\partial S}. \end{aligned} \tag{6.21}$$

In other words, the adjusted deal price with variation and initial margins $\bar{V}$ satisfies the following semi-linear PDE for all $(t,s) \in [0,T] \times \mathbb{R}^+$:

$$\begin{aligned}
&\partial_t \nu(t,s) + \mu(t,s)\partial_s \nu(t,s) + \frac{1}{2}\sigma(t,s)^2 \partial_s^2 \nu(t,s) + \pi_t + \lambda_t \theta_t - \left(\tilde{f}_t + \lambda_t\right)\nu(t,s) \\
&\quad + \left(\tilde{f}_t - \tilde{c}_t\right) M_t - \left(r_t - \tilde{h}_t\right) H_t + \left(f_t^{N,C} - \tilde{c}_t\right) N_t^C + \left(f_t^{N,I} - \tilde{c}_t\right) N_t^I = 0, \\
&\nu(T,s) = 0.
\end{aligned} \tag{6.22}$$

Now assume that the underlying S_u is a tradable asset and follows the Black-Scholes dynamic for $(t < u)$:

$$\mathrm{d}S_u = r_u S_u \mathrm{d}u + \sigma_u S_u \mathrm{d}W_u, \qquad S_t = s,$$

and also that the investor adopts delta-hedging, i.e. $H_t = \partial_s \nu(t,s)s = \frac{Z_t}{\sigma_t}$. We can then rewrite (6.22) as follows,

$$\begin{aligned}
&\partial_t \nu(t,s) + \tilde{h}_t H_t + \frac{1}{2}\sigma(t)^2 s^2 \partial_s^2 \nu(t,s) - \left(\tilde{f}_t + \lambda_t\right)\nu(t,s) + \left(\tilde{f}_t - \tilde{c}_t\right) M_t \\
&\quad + \left(f_t^{N,C} - \tilde{c}_t\right) N_t^C + \left(f_t^{N,I} - \tilde{c}_t\right) N_t^I = 0, \\
&\nu(T,s) = 0.
\end{aligned} \tag{6.23}$$

The above semi-linear PDE is the pre-default $(\tau > t)$ PDE for the pricing problem in the case of trading via CCP clearing or bilateral CSA with variation and initial margins.

Again, we notice that the risk-free rate r_t disappears in (6.23). The PDE is completely governed by market observable quantities as in the previous set-ups. Applying similar reasoning as before, we obtain the following result.

Corollary 6.2.1 (**Continuous-time Solution for CCP cleared or bilateral CSA trades with variation and initial margins**). *Assuming that rehypothecation is allowed and delta-hedging is used, we can solve the pricing problem in continuous time when trading via CCP clearing or bilateral CSA with variation and* *initial margins. We have*

$$\bar{V}_t(C;F) = \int_t^T \mathbb{E}_t^{\tilde{h}} \Big[\Big(\pi_s + \lambda_s \theta_s + \left(\tilde{f}_s - \tilde{c}_s \right) M_s + \left(f_t^{N,C} - \tilde{c}_t \right) N_t^C + \left(f_t^{N,I} - \tilde{c}_t \right) N_t^I \Big) \\ D(t,s;\tilde{f}+\lambda) \mid \mathcal{F} \Big] \,\mathrm{d}s, \tag{6.24}$$

where the expectation is taken under a pricing measure $\mathbb{Q}^{\tilde{h}}$ *where the underlying risk factors grow at the rate* $\tilde{h}$ *if no dividend is paid.*

We will not go into detail of the proof here as the reasoning is analogous to that of Corollary 6.1.1.

We see that there is no dependence on a risk-free rate r_t in equations (6.24) either. In other words, the final adjusted price is invariant to the theoretical risk-free rate r_t.

6.3 Margin period of risk

In the case of early default, the default procedure may take several days to be completed. The time elapsed between the default event and the completion of the close-out procedure is called the *margin period of risk*. During this time period, the mark-to-market of the derivative may change considerably, resulting in a large mismatch between the posted collateral and the exposure. Moreover, the surviving party may default during this period, which should also be considered when we compute the on-default cash-flow.

In this section, we continue our discussion for the CCP cleared or bilateral CSA trades following the analysis carried out in Brigo and Pallavicini [30] and discuss the cash flows occurring upon an early default event taking into consideration the margin period of risk.

6.3.1 On-default cash-flow

The case of trades where there was a bilateral CSA was discussed in Section 2.1.2. Here, we extend the study and consider also the initial margins that are posted to cover the additional risks. In the case of CCP cleared trades, if the client "C" defaults first, the clearing member "I" will take responsibility for the position as in a bilateral trade and evaluate the close-out amount.

Assume that the default procedure takes time δ to be completed. The default procedure can be considered as though the surviving party at the default time τ enters a deal with a cash-flow θ and maturity $\tau + \delta$. This cash-flow will depend on the close-out amount $\varepsilon_{\tau+\delta}$ (with consideration of the margin period risk) and the value of (the pre-default) variation and initial margin accounts denoted respectively as M_{τ^-}, $N^C_{\tau^-}$ and $N^I_{\tau^-}$.

Applying the results in Corollary 6.2.1, the adjusted price of such a deal denoted as ϑ at time τ can be expressed as (we take the conditional expectations under the filtration $\mathcal{G}_\tau$, using the same technique as in Section 2.3.2),

$$\vartheta_\tau := \mathbb{E}^{\tilde{h}}_\tau \left[\theta_{\tau+\delta} \left(\varepsilon_{\tau+\delta}, M_{\tau^-}, N^C_{\tau^-}, N^I_{\tau^-} \right) D(\tau, \tau+\delta; \tilde{f}^S) \right], \tag{6.25}$$

where the expectation is taken under the probability measure $\mathbb{Q}^{\tilde{h}}$, and $\tilde{f}^S$ is the effective funding rate of the surviving party that is funding such a deal.

The above pricing equation depends on the funding rates of both parties. However, practically, one cannot know the other party's liquidity policy. [30] approximates the discount factors by assuming that the payment takes place at the default time τ without modelling the funding rates of both parties and points out that the effects of this approximation are second order compared to the uncertainties of the recovery rates and close-out values. Therefore, instead of (6.25) we write

$$\vartheta_\tau := \mathbb{E}^{\tilde{h}}_\tau \left[\theta_{\tau+\delta} \left(\varepsilon_{\tau+\delta}, M_{\tau^-}, N^C_{\tau^-}, N^I_{\tau^-} \right) \right]. \tag{6.26}$$

Bare in mind that in the case of a CCP cleared trade we have $N^I_{\tau^-} = 0$.

6.3.2 Close-out netting rule

In order to determine the cash-flow $\theta_{\tau+\delta}$, we need to investigate the close-out netting rules. We repeat the analysis in [30] here and consider all possible scenarios that may happen upon the first default event. Starting with the case where the counterparty/client default first $\tau = \tau_C < \tau_I$, we analyse the following scenarios.

When $\varepsilon_{\tau+\delta} \geq 0$ **and** $M_{\tau^-} \geq 0$**,** the investor has a positive exposure and the counterparty has posted variation margin. We then have the following cases:

1. The exposure is netted with the variation and initial margins posted by the counterparty, but the collateral is not enough to cover the exposure, and the investor can get back his initial margin:

$$\mathbf{1}_{\{\varepsilon_{\tau+\delta} \geq M_{\tau^-} + N^C_{\tau^-}\}}(R_C(\varepsilon_{\tau+\delta} - M_{\tau^-} - N^C_{\tau^-}) - N^I_{\tau^-}).$$

2. The exposure is covered by the variation and initial margins. The investor does not face a loss and gets back his initial margin:

$$\mathbf{1}_{\{\varepsilon_{\tau+\delta} < M_{\tau^-} + N^C_{\tau^-}\}}(\varepsilon_{\tau+\delta} - M_{\tau^-} - N^C_{\tau^-} - N^I_{\tau^-}).$$

3. In the case where the exposure is completely covered by the variation margin, we need to consider two more scenarios:

   - The investor does not default or defaults after the margin period. He faces no loss and gets back his initial margin:

$$\mathbf{1}_{\{\varepsilon_{\tau+\delta} < M_{\tau^-}\}}\mathbf{1}_{\{\tau_I > \tau_C + \delta\}}(\varepsilon_{\tau+\delta} - M_{\tau^-} - N^C_{\tau^-} - N^I_{\tau^-}).$$

   - The investor defaults during the margin period. In this case, the investor's initial margin can be used to reduce losses:

$$\mathbf{1}_{\{\varepsilon_{\tau+\delta} < M_{\tau^-}\}}\mathbf{1}_{\{\tau_I \leq \tau_C + \delta\}}\big((\varepsilon_{\tau+\delta} - M_{\tau^-} - N^I_{\tau^-})^+ \\ + R'_I(\varepsilon_{\tau+\delta} - M_{\tau^-} - N^I_{\tau^-})^- - N^C_{\tau^-}\big)\,.$$

When $\varepsilon_{\tau+\delta} \geq 0$ **and** $M_{\tau^-} < 0$**,** the investor has a positive exposure and the investor has posted variation margin. We then have the following cases:

4. If the initial margin posted by the counterparty is not enough to cover the investor's exposure, the investor faces a loss and gets back the initial margin and the variation margin if it is not rehypothecated:

$$\mathbf{1}_{\{\varepsilon_{\tau+\delta} \geq N^C_{\tau^-}\}}(R_C(\varepsilon_{\tau+\delta} - N^C_{\tau^-}) - R'_C M_{\tau^-} - N^I_{\tau^-}).$$

5. If the initial margin is enough to cover the investor's exposure, the investor gets back his initial margin and does not suffer a loss unless the variation margin is rehypothecated:

$$\mathbf{1}_{\{\varepsilon_{\tau+\delta} < N^C_{\tau^-}\}}((\varepsilon_{\tau+\delta} - M_{\tau^-} - N^C_{\tau^-})^- - R'_C(\varepsilon_{\tau+\delta} - M_{\tau^-} - N^C_{\tau^-})^+ - N^I_{\tau^-}).$$

When $\varepsilon_{\tau+\delta} < 0$ **and** $M_{\tau^-} \geq 0$**,** the investor has a negative exposure and the counterparty has posted variation margin. We then have the following cases:

6. The counterparty expects to get back the variation and initial margins.

   - If the investor does not default or defaults after the margin period, the counterparty gets back the collateral in full:

   $$\mathbf{1}_{\{\tau_I > \tau_C + \delta\}}(\varepsilon_{\tau+\delta} - M_{\tau^-} - N^C_{\tau^-} - N^I_{\tau^-}).$$

   - If the investor defaults before the margin period, the counterparty may face a loss depending on where the collateral is rehypothecated:

   $$\mathbf{1}_{\{\tau_I < \tau_C + \delta\}} \Big(R_I(\varepsilon_{\tau+\delta} - N^I_{\tau^-})^- + R'_I \left((\varepsilon_{\tau+\delta} - N^I_{\tau^-})^+ - M_{\tau^-} \right)^- + \left((\varepsilon_{\tau+\delta} - N^I_{\tau^-})^+ - M_{\tau^-} \right)^+ - N^C_{\tau^-} \Big).$$

When $\varepsilon_{\tau+\delta} < 0$ **and** $M_{\tau^-} < 0$**,** the investor has a negative exposure and the investor has posted variation margin. The exposure is netted with the posted collateral unless the variation margin is rehypothecated, in which case the initial margin is used to reduce the losses. We then have the following cases:

7. If the initial margin is not enough to cover the losses due to the rehypothecation, the investor suffers a loss and gets back his initial margin:

$$\mathbf{1}_{\{\varepsilon_{\tau+\delta}-M_{\tau^-}\geq N^C_{\tau^-}\}}(R'_C(\varepsilon_{\tau+\delta}-M_{\tau^-}-N^C_{\tau^-})-N^I_{\tau^-}).$$

8. If the initial margin is enough to cover the losses due to the rehypothecation, the investor gets back his collateral in full:

$$\mathbf{1}_{\{\varepsilon_{\tau+\delta}-M_{\tau^-}<N^C_{\tau^-}\}}(\varepsilon_{\tau+\delta}-M_{\tau^-}-N^C_{\tau^-}-N^I_{\tau^-}).$$

9. If the investor has to pay a greater exposure, we consider the following two cases:

   - If the investor does not default or defaults after the margin period, the investor gets back his initial margin:

$$\mathbf{1}_{\{\varepsilon_{\tau+\delta}<M_{\tau^-}\mathbf{1}_{\{\tau_I>\tau_C+\delta\}}\}}(\varepsilon_{\tau+\delta}-M_{\tau^-}-N^C_{\tau^-}-N^I_{\tau^-}).$$

   - If the investor defaults before the margin period, the investor's initial margin can be used to reduce the losses:

$$\mathbf{1}_{\{\varepsilon_{\tau+\delta}<M_{\tau^-}\}}\mathbf{1}_{\{\tau_I<\tau_C+\delta\}}\Big((\varepsilon_{\tau+\delta}-M_{\tau^-}-N^I_{\tau^-})^+ \\ +R_I(\varepsilon_{\tau+\delta}-M_{\tau^-}-N^I_{\tau^-})^- - N^C_{\tau^-}\Big)\,.$$

Similarly, we can list the cash flows when the investor defaults first. If we sum up all the cash flows for all the possible scenarios, we can reach an expression for the on-default cash-flow $\theta_{\tau+\delta}$ (see [30] for more details). Substituting $\theta_{\tau+\delta}$ in (6.26) we see that the price of the on-default cash-flow can be expressed as follows,

$$\vartheta_\tau=\mathbb{E}^{\tilde{h}}_\tau\left[\varepsilon_{\tau+\delta}\right]-\mathrm{C}\textsc{va}(\tau,T;M,N^C,N^I)+\mathrm{D}\textsc{va}(\tau,T;M,N^C,N^I),\qquad(6.27)$$

where the first term is the replacement price of the deal, and it is reduced by collateralized CVA and DVA terms with

$$\begin{aligned}\mathrm{C}_{\mathrm{VA}}(\tau, T; M, N^C, N^I) &:= \mathbb{E}^{\tilde{h}}_{\tau}\left[\mathbf{1}_{\{\tau_C<\tau_I+\delta\}}\Pi_{\mathrm{CVAcoll}}(\tau)\right], \\ \mathrm{D}_{\mathrm{VA}}(\tau, T; M, N^C, N^I) &:= \mathbb{E}^{\tilde{h}}_{\tau}\left[\mathbf{1}_{\{\tau_I<\tau_C+\delta\}}\Pi_{\mathrm{CVAcoll}}(\tau)\right],\end{aligned} \tag{6.28}$$

and

$$\begin{aligned}\Pi_{\mathrm{CVAcoll}}(s) &= \Big(\mathrm{L}_{\mathrm{GD}_C}\left((\varepsilon_{\tau+\delta} - N^C_{\tau^-})^+ - M^+_{\tau^-}\right)^+ + \mathrm{L}_{\mathrm{GD}'_C}\left((\varepsilon_{\tau+\delta} - N^C_{\tau^-})^- - M^-_{\tau^-}\right)^+\Big), \\ \Pi_{\mathrm{DVAcoll}}(s) &= -\Big(\mathrm{L}_{\mathrm{GD}_I}\left((\varepsilon_{\tau+\delta} - N^I_{\tau^-})^- - M^-_{\tau^-}\right)^- + \mathrm{L}_{\mathrm{GD}'_I}\left((\varepsilon_{\tau+\delta} - N^I_{\tau^-})^+ - M^+_{\tau^-}\right)^-\Big).\end{aligned} \tag{6.29}$$

Observe that if rehypothecation of the collateral is not allowed, the terms multiplied by $\mathrm{L}_{\mathrm{GD}'_C}$ and $\mathrm{L}_{\mathrm{GD}'_I}$ drop out of the CVA and DVA calculations.

In the case where the trade is cleared by a CCP, only the client posts initial margin, so we can set $N^I = 0$ in equation (6.27). Moreover, if upon the default of a clearing member the transaction will be transferred to a backup clearing member, we can then assume that the loss given default for the clearing member is close to zero.

6.4 Conclusions and Financial Implications

We have developed a consistent framework for valuation of derivative trades under collateralization, counterparty credit risk, and funding costs. Based on no arbitrage, we derived a generalized pricing equation where CVA, DVA, LVA, and FVA are introduced by simply modifying the payout cash-flows of the trade. The framework is flexible enough to accommodate actual trading complexities such as asymmetric collateral and funding rates, replacement close-out, and rehypothecation of posted collateral. We also provided a detailed analysis of the adjusted self-financing condition that incorporates in the new market realities. Moreover, we presented an invariance theorem showing that the valuation framework does not depend on any theoretical risk-free rate, but is purely based on observable market rates.

The generalized valuation equation under credit, collateral and funding takes the form of a forward-backward SDE or a semi-linear PDE. We discussed the conditions under which such a forward-backward SDE or a semi-linear PDE has a unique solution.

The consistent valuation equation can also be recast as a set of iterative equations which can be efficiently solved by a proposed least-squares Monte Carlo algorithm. Our numerical results confirm that funding risk as well as asymmetries in borrowing and lending rates have a critical impact on the ultimate value of a derivatives transaction.

Introducing funding costs into the pricing equation makes the valuation problem recursive and nonlinear. The price of the deal depends on the trader's funding strategy, while to determine the funding strategy we need to know the deal price itself. Credit and funding risks are in general non-separable; this means that FVA is not an additive adjustment, let alone a discounting spread. Thus, despite being common practice among market participants, treating it as such comes at the cost of double counting. We introduce the nonlinearity valuation adjustment (NVA) to quantify the effect of double counting and we show that its magnitude can be significant under asymmetric funding rates and replacement close-out at default.

Furthermore, valuation under funding costs is no longer bilateral as the particular funding policy chosen by the dealer is not known to the client, and vice versa. As a result, the value of the trade will generally be different to the two counterparties. Conceptually, this should mean that the parties would never close the deal, but in reality dealers confirm that this was a key factor driving bid-ask spreads wider during the crisis.

Finally, valuation depends on the level of aggregation; asset portfolios cannot simply be priced separately and added up. Theoretically, valuation is conducted under deal or portfolio-dependent risk-neutral measures. This has clear operational consequences for financial institutions; it's difficult for banks to establish CVA and FVA desks with separate, clear-cut responsibilities. Instead, they should adopt a holistic, consistent valuation approach across all trading desks and asset classes. A trade should be priced on an appropriate aggregation-level to quantify the value it actually adds to the business. This, of course, leads us to the old distinction between price and value: Should funding costs be charged to the client or just included internally to determine the profitability of a particular trade? The relevance of this question is reinforced by the fact that the client has no direct control on the funding policy of the bank and therefore cannot influence any potential inefficiencies for which he or she would have to pay.

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