ASYMPTOTICS OF FORWARD IMPLIED VOLATILITY

by

Patrick Francois Springfield Roome

Department of Mathematics
Imperial College London
London SW7 2AZ
United Kingdom

Submitted to Imperial College London
for the degree of
Doctor of Philosophy

2015

Declaration

I the undersigned hereby declare that the work presented in this thesis is my own. When material from other authors has been used, these have been duly acknowledged. This thesis has not previously been presented for this or any other PhD examinations.

Patrick Francois Springfield Roome

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"Divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever."

Niels Hendrik Abel, 1828

Abstract

We study asymptotics of forward-start option prices and the forward implied volatility smile using the theory of sharp large deviations (and refinements). In Chapter 1 we give some intuition and insight into forward volatility and provide motivation for the study of forward smile asymptotics. We numerically analyse no-arbitrage bounds for the forward smile given calibration to the marginal distributions using (martingale) optimal transport theory. Furthermore, we derive several representations of forward-start option prices, analyse various measure-change symmetries and explore asymptotics of the forward smile for small and large forward-start dates.

In Chapter 2 we derive a general closed-form expansion formula (including large-maturity and 'diagonal' small-maturity asymptotics) for the forward smile in a large class of models including the Heston and Schöbel-Zhu stochastic volatility models and time-changed exponential Lévy models. In Chapter 3 we prove that the out-of-the-money small-maturity forward smile explodes in the Heston model and a separate model-independent analysis shows that the at-the-money small-maturity limit is well defined for any Itô diffusion. Chapter 4 provides a full characterisation of the large-maturity forward smile in the Heston model. Although the leading-order decay is provided by a fairly classical large deviations behaviour, the algebraic expansion providing the higher-order terms depends highly on the parameters, and different powers of the maturity come into play.

Classical (Itô diffusions) stochastic volatility models are not able to capture the steepness of small-maturity (spot) implied volatility smiles. Models with jumps, exhibiting small-maturity exploding smiles, have historically been proposed as an alternative. A recent breakthrough was made by Gatheral, Jaisson and Rosenbaum [74], who proposed to replace the Brownian driver of the instantaneous volatility by a short-memory fractional Brownian motion, which is able to capture the short-maturity steepness while preserving path continuity. In Chapter 5 we suggest a different route, randomising the Black-Scholes variance by a CEV-generated distribution, which allows us to modulate the rate of explosion (through the CEV exponent) of the implied volatility for small maturities. The range of rates includes behaviours similar to exponential Lévy models and fractional stochastic volatility models. As a by-product, we make a conjecture on the small-maturity forward smile asymptotics of stochastic volatility models, in exact agreement with the results in Chapter 3 for Heston.

I dedicate this thesis to the memory of my father.

Acknowledgements

First and foremost, I would like to thank my supervisor, Dr. Antoine Jacquier. It has been an honour to be his first PhD student. His guidance, support and encouragement were vital throughout my PhD and his passion and insight into the subject was inspirational. I remember leaving his office numerous times (usually after several hours of discussions) with a renewed sense of motivation and excitement.

The Imperial Mathematical Finance department always made PhD students welcome and provided a number of avenues for learning and growth. I was given the opportunity to present my work to members of the group on several occasions. The atmosphere was stimulating and discussions and questions about my work moved me in the right direction.

I would like to thank my co-authors, Dr. Stefano De Marco (Ecole Polytechnique) and Mr. Hamza Guennoun (Ecole Polytechnique). I would also like to thank Prof. Archil Gulisashvili (Ohio University) for a stimulating and insightful discussion on my research during one of his Imperial visits. Thanks to Prof. Damiano Brigo and Dr. Claude Martini for agreeing to be my PhD examiners: I very much appreciate the time and effort they will put into my thesis.

Before the PhD I was a quant at Bank of America Merrill Lynch. I was very fortunate to have the opportunity to work with Dr. Leif Andersen. He supported and encouraged my decision to do a full-time PhD and I am very grateful for his advice and guidance.

I would very much like to thank my parents and family for their support and always encouraging me to follow my dreams. Finally, I would like to thank my fiancée Daphne for her endless love, kindness and support.

Epigraph

"Only that at times like this,
when you're directionless in a dark wood,
trust to the abstract deductive...
Leap
like a knight of faith
into the arms of Peano, Leibniz, Hilbert, L'Hôpital.
You will be lifted up."

David Foster Wallace

Contents

Notations

$\mathbb{R}^*$	$\mathbb{R} \setminus \{0\}$.
$\mathbb{R}_+^*$	$(0, \infty)$.
$\mathbb{N}$	$\{1, 2, 3, ...\}$.
A^o	Interior of a set A in $\mathbb{R}$.
$\overline{A}$	Closure of a set A in $\mathbb{R}$.
$\mathrm{Law}(Z)$	Law of the random variable Z.
$\mathrm{Supp}(Z)$	Support of the random variable Z.
$\mathcal{N}$	Cumulative distribution function of the standard Gaussian distribution.
$\mathrm{BS}(k, \Sigma^2, \tau)$	Black-Scholes price of a call option with log-strike k, volatility Σ and maturity τ.
$C(k, \tau)$	Call option price under a given model with log-strike k.
$C(k, t, \tau)$	Type-I forward-start call option price under a given model with log-strike k.
$C^{\mathrm{II}}(k, t, \tau)$	Type-II forward-start call option price under a given model with log-strike k.
$\sigma_\tau(k)$	Spot implied volatility under a given model with log-strike k.
$\sigma_{t,\tau}(k)$	Type-I forward implied volatility under a given model with log-strike k.
$\widetilde{\sigma}_{t,\tau}(k)$	Type-II forward implied volatility under a given model with log-strike k.
$\mathbb{P}, \bar{\mathbb{P}}, \widetilde{\mathbb{P}}, \mathbb{P}^*$	Risk-neutral, share-price, stopped-share-price and forward measures respectively.
$\mathbb{E}, \bar{\mathbb{E}}, \widetilde{\mathbb{E}}, \mathbb{E}^*$	Expectations under the measures above respectively.
$\dfrac{f(\varepsilon)}{g(\varepsilon)} \sim 1$	$\lim\limits_{\varepsilon \to 0} \dfrac{f(\varepsilon)}{g(\varepsilon)} = 1$.
$g(\varepsilon) = \mathcal{O}(f(\varepsilon))$	There exists $\varepsilon_0, b > 0$ such that $\lvert g(\varepsilon)\rvert < bf(\varepsilon)$ for all $\varepsilon < \varepsilon_0$.
$g(\varepsilon) = o(f(\varepsilon))$	For all $b > 0$ there exists $\varepsilon_0(b)$ such that $\lvert g(\varepsilon)\rvert < bf(\varepsilon)$ for all $\varepsilon < \varepsilon_0(b)$.
x^+	$\max\{0, x\}$ for $x \in \mathbb{R}$.
$\mathrm{sgn}(p)$	1 if $p \geq 0$ and -1 otherwise.
$L^1(\mathbb{R})$	The set of integrable functions on $\mathbb{R}$.
$B_b(\mathbb{R})$	The set of bounded measurable functions on $\mathbb{R}$.
$\Re(z), \Im(z)$	Real and imaginary part of a complex number z.
$(\mathcal{F}f)(u)$	Fourier transform $\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ux} f(x)\mathrm{d}x$ of a function $f \in L^1$.
$(\mathcal{F}^{-1}h)(x)$	Inverse Fourier transform $\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}ux} h(u)\mathrm{d}u$ of a function $h \in L^1$.
$(f * g)(x)$	Convolution of two functions $\int_{\mathbb{R}} f(x-y)g(y)\mathrm{d}y$ where $f, g \in L^1$.

Finally, for a sequence of sets $(\mathcal{D}_\varepsilon)_{\varepsilon>0}$ in $\mathbb{R}$, we may, for convenience, use the notation $\lim_{\varepsilon \downarrow 0} \mathcal{D}_\varepsilon$, by which we mean the following (whenever both sides are equal): $\liminf_{\varepsilon \downarrow 0} \mathcal{D}_\varepsilon := \bigcup_{\varepsilon>0} \bigcap_{s \leq \varepsilon} \mathcal{D}_s = \bigcap_{\varepsilon>0} \bigcup_{s \leq \varepsilon} \mathcal{D}_s =: \limsup_{\varepsilon \downarrow 0} \mathcal{D}_\varepsilon$.

Chapter 1

Introduction

In this thesis we consider an asset price process $\left(S_t = \mathrm{e}^{X_t}\right)_{t\geq 0}$ with $X_0 = 0$, defined on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with a given risk-neutral measure $\mathbb{P}$, and assume that interest rates and dividends are zero. In the Black-Scholes-Merton (BSM) model, the dynamics of the logarithm of the asset price are given by

$$\mathrm{d}X_t = -\frac{1}{2}\Sigma^2 \mathrm{d}t + \Sigma \mathrm{d}W_t, \tag{1.0.1}$$

where $\Sigma > 0$ is the instantaneous volatility and W is a standard Brownian motion. In this model the price of a European call option with strike e^k and maturity $\tau > 0$ is given by the famous BSM formula [25, 124]:

$$\mathrm{BS}(k, \Sigma^2, \tau) := \mathcal{N}\left(-\frac{k}{\Sigma\sqrt{\tau}} + \frac{\Sigma\sqrt{\tau}}{2}\right) - \mathrm{e}^k \mathcal{N}\left(-\frac{k}{\Sigma\sqrt{\tau}} - \frac{\Sigma\sqrt{\tau}}{2}\right). \tag{1.0.2}$$

For a given market or model price $C(\tau, k)$ of a European call option with strike e^k and maturity τ we define the spot implied volatility $\sigma_\tau(k)$ as the unique solution to the equation $C(\tau, k) = \mathrm{BS}(k, \sigma_\tau^2(k), \tau)$. Implied volatility is the quoting mechanism used in option markets and serves as a useful metric to compare options with different strikes and maturities.

For any $t, \tau > 0$ and $k \in \mathbb{R}$, we define a Type-I and Type-II forward-start call option with forward-start date t, maturity τ and strike e^k as a European option with the following payoffs,

$$\text{Type-I:} \quad \left(\mathrm{e}^{X_\tau^{(t)}} - \mathrm{e}^k\right)^+, \quad X_\tau^{(t)} := X_{t+\tau} - X_t, \tag{1.0.3}$$

$$\text{Type-II:} \quad \left(\mathrm{e}^{X_{t+\tau}} - \mathrm{e}^{X_t + k}\right)^+, \tag{1.0.4}$$

where the forward-start process $X_\tau^{(t)}$ is defined pathwise. In the BSM model (1.0.1) a Type-I and Type-II forward-start option are both worth $\mathrm{BS}(k, \Sigma^2, \tau)$. For a given market or model price of a Type-I (resp. Type-II) forward-start call option $C(t, \tau, k)$ (resp. $C^{\mathrm{II}}(t, \tau, k)$) with strike e^k, forward-start date t and maturity τ we define the Type-I (resp. Type-II) forward implied volatility smile $\sigma_{t,\tau}(k)$ (resp. $\widetilde{\sigma}_{t,\tau}(k)$) as the unique solution to the equation

$$C(t,\tau,k) = \mathrm{BS}(k, \sigma^2_{t,\tau}(k), \tau), \tag{1.0.5}$$

$$\text{resp. } C^{\mathrm{II}}(t,\tau,k) = \mathrm{BS}(k, \widetilde{\sigma}^2_{t,\tau}(k), \tau). \tag{1.0.6}$$

Note that since C and C^{II} take values in the set $(0,1)$ and $\mathrm{BS}(k,\cdot,\tau)$ is strictly increasing from 0 to 1, there always exists a unique solution to equations (1.0.5) and (1.0.6). Both definitions of the forward smile are generalisations of the spot implied volatility smile since they reduce to the spot smile when $t = 0$.

The literature on spot implied volatility asymptotics is extensive and has drawn upon a wide range of mathematical techniques. In particular, small-maturity asymptotics have historically received wide attention due to earlier results from the 1980s on expansions of the heat kernel [16]. PDE methods for continuous-time diffusions [22, 83, 132], large deviations [51, 52, 62], saddlepoint methods [64], Malliavin calculus [17, 113] and differential geometry [72, 84] are among the main methods used to tackle the small-maturity case. Extreme strike asymptotics arose with the seminal paper by Roger Lee [116] and have been further extended by Benaim and Friz [14, 15] and in [51, 52, 68, 79, 81]. Comparatively, large-maturity asymptotics have only been studied in [63, 65, 95, 97, 146] using large deviations and saddlepoint methods. Fouque et al. [66] have also successfully introduced perturbation techniques in order to study slow and fast mean-reverting stochastic volatility models. Models with jumps (including Lévy processes), studied in the above references for large maturities and extreme strikes, 'explode' in small time, in a precise sense investigated in [3, 4, 61, 125, 127, 145].

A collection of implied volatility smiles over a time horizon $(0,T]$ is also known to be equivalent to the marginal distributions of the asset price process over $(0,T]$. Implied volatility asymptotics have therefore provided a set of tools to analytically understand the marginal distributions of a model and their relationships to market observable quantities such as volatility smiles. However many models can calibrate to implied volatility smiles (static information) with the same degree of precision and produce radically different prices and risk sensitivities for exotic securities. This can usually be traced back to a complex and often non-transparent dependence on transitional probabilities or equivalently on model-generated dynamics of the smile. The dynamics of the smile is therefore a key model risk associated with these products and any model used for pricing and risk management should produce realistic dynamics that are in line with trader expectations and historical dynamics. One metric that can be used to understand the dynamics of implied volatility smiles ([23] calls it a 'global measure' of the dynamics of implied volatilities) is to use the forward smile defined above. The forward smile is also a market-defined quantity and naturally extends the notion of the spot implied volatility smile. Forward-start options also serve as natural hedging instruments for several exotic securities (such as Cliquets, Ratchets and Napoleons; see [71, Chapter 10]) and are therefore worth investigating.

The literature on asymptotics of forward-start options and the forward smile is sparse. Glasserman and Wu [76] use different notions of forward volatilities to assess their predictive values in determining future option prices and future implied volatility. Keller-Ressel [109] studies the forward smile asymptotic when the forward-start date t becomes large (τ fixed) and Bompis [27] produces an expansion for the forward smile in local volatility models with bounded diffusion coefficient. Finally, empirical results on the forward smile have been carried out by practitioners in Balland [10], Bergomi [23], Bühler [36] and Gatheral [71].

This chapter lays the groundwork for the thesis: we introduce the main tools, give some intuition and insight into forward volatility and provide motivation for the study of forward smile asymptotics. In Section 1.1 we numerically analyse no-arbitrage bounds for the forward smile given calibration to the marginal distributions at maturities t and $t+\tau$ using (martingale) optimal transport theory. We try and answer questions such as is it reasonable to 'lock-in' (replicate) forward volatility using European options? Section 1.2 provides a brief overview of large deviations theory, Watson's lemma and the Laplace method and Section 1.3 details some of the main models and their properties that will be needed in the thesis. In Section 1.4 we look at pricing forward-start options: this entails an analysis of measure-change symmetries and several representations of forward-start options prices. Section 1.5 explores asymptotics of the forward smile and forward-start options for small and large-forward start dates and in Section 1.6 we give an outline of the structure of the thesis.

1.1 No-arbitrage bounds for the forward smile given marginals

Since the seminal paper of Hobson [89], an important literature developed on model-free super(sub)-hedging of multi-dimensional derivative products given a set of European option hedging instruments. The key observation is that the model-free super(sub)-hedging cost is closely related to the Skorokhod Embedding problem; see the survey papers of Obłój [129] and Hobson [90].

Recently, this problem has been addressed using the (martingale) version of optimal transport theory (see [13]). More specifically, under the assumption that European call option prices with all possible strikes are known for a given set of maturities (i.e. the marginal distributions of the asset price are known at these times), optimal transport yields a set of tools to study the no-arbitrage price range of a derivative product consistent with these marginal distributions. The primal problem endeavours to find the supremum (and infimum) of a derivative product price over the set of joint martingale measures (transport plans). The dual problem (equivalent to the primal problem under certain conditions) seeks to find the 'best' super(sub)-replicating portfolio for the derivative security. The dual formulation has a natural financial interpretation and can be cast as an (infinite) linear programme; numerical techniques for solving this LP have been explored in [85].

Forward-start options (Type-I and Type-II) are some of the the simplest products amenable to these techniques. The upper bound price for the at-the-money Type-II forward-start straddle has been found in [92]; in particular, the support of the optimal martingale measure is a binomial tree. But, unfortunately the optimal measure or associated super-hedging portfolio is not given analytically. The (martingale) optimal transference plan for the lower bound price of the at-the-money Type-II forward-start straddle has been characterised (semi-) analytically in [91]; the transference plan (the support of which is a trinomial tree) is found by solving a set of coupled ODE's. In [37] the authors study the change of numeraire in these two-dimensional optimal transport problems and show (as a corollary, under certain conditions) that the lower bound for the Type-I at-the-money forward-start straddle is also attained by the Hobson-Klimmek transference plan.

In this section we numerically study the no-arbitrage bounds of the Type-II forward-start straddle. Section 1.1.1 formulates the linear programme for the optimal transport problem. Section 1.1.2 details our no-arbitrage discretisation of the support of the marginal distributions which results in a consistent primal and dual problem for each discretisation and a robsut numerical result. Section 1.1.4 computes upper and lower bounds given (i) lognormal marginal distributions and (ii) marginal distributions generated from a Heston model (1.3.2). In the lower bound at-the-money case we numerically solve the (coupled) ODE's associated with the Hobson-Klimmek transference plan (numerical implementation given in Section 1.1.3) and show that it is in striking agreement with the LP solution of the dual problem. In Section 1.1.5 we numerically solve the primal problem and give the optimal transport plans for a range of strikes. Although, the transport plans are known for the at-the-money case [91, 92], they are not known for other strikes. We show here that the optimal transference plans are more subtle in these cases and appear to be a combination of the lower and upper bound at-the-money plans. The optimal transport plan gives insight into the key model risk for this product. Intuitively, the extremal measure exploits this risk to produce the maximum (or minimum) value of the product. The key model risk for forward-start options appears to be the exposure of the product to the kurtosis of the conditional distribution of the asset price process; see Sections 1.1.3 and 1.1.5 and [91, 92].

In the examples explored in Section 1.1.4 the range of forward smiles consistent with the marginal laws is large (even in the simple case that the marginal distributions are lognormal). Using European vanilla options to 'lock-in' (replicate) forward volatility or hedge forward volatility dependent claims seems illusory. Forward-start options should be seen as fundamental building blocks for exotic pricing and not decomposable (or approximately decomposable) into European options. Models used for forward volatility dependent exotics should have the capability of calibration to forward-start option prices and at a minimum should produce realistic forward smiles that are consistent with trader expectations and observable prices. The asymptotic results developed in this thesis allow one to study both of these points.

1.1.1 Problem formulation

Let μ and ν denote the distributions of S_t and $S_{t+\tau}$ for $t, \tau > 0$; we suppose they have common finite mean equal to 1, are supported on $[0, \infty)$, and are absolutely continuous with respect to the Lebesgue measure. We say that the bivariate law ζ is a martingale coupling and $\zeta \in \mathcal{M}(\mu, \nu)$ if ζ has marginals μ and ν and $\int_y (y-x)\zeta(\mathrm{d}x, \mathrm{d}y) = 0$ for each $x \in \mathbb{R}_+$. In order to ensure that $\mathcal{M}(\mu, \nu)$ is non-empty we assume that μ and ν are in convex order (and we denote $\mu \preceq \nu$), namely that they have equal means and satisfy $\int (y-x)^+ \mu(\mathrm{d}y) \le \int (y-x)^+ \nu(\mathrm{d}y)$ for all $x \in \mathbb{R}_+$ (See [142]). Our objective is to find the tightest possible bounds consistent with the marginal distributions for the Type-II forward-start straddle payoff $|S_{t+\tau} - KS_t|$ with $K > 0$. To this end we define our primal problem:

$$\underline{\mathcal{P}}(\mu, \nu) := \inf_{\zeta \in \mathcal{M}(\mu,\nu)} \int |y - Kx| \zeta(\mathrm{d}x, \mathrm{d}y), \quad \overline{\mathcal{P}}(\mu, \nu) := \sup_{\zeta \in \mathcal{M}(\mu,\nu)} \int |y - Kx| \zeta(\mathrm{d}x, \mathrm{d}y). \tag{1.1.1}$$

We now define our sets of sub and super-replicating portfolios:

$$\underline{Q} := \left\{ (\psi_0, \psi_1, \delta) \in L^1(\mu) \times L^1(\nu) \times B_b(\mathbb{R}) : h(x,y) \le |y - Kx|, \text{for all } x, y \in \mathbb{R}_+ \right\},$$

$$\overline{Q} := \left\{ (\psi_0, \psi_1, \delta) \in L^1(\mu) \times L^1(\nu) \times B_b(\mathbb{R}) : h(x,y) \ge |y - Kx|, \text{for all } x, y \in \mathbb{R}_+ \right\},$$

where $h(x, y) := \psi_1(y) + \psi_0(x) + \delta(x)(y - x)$. Clearly if $(\psi_0, \psi_1, \delta) \in \underline{Q}$ $(\in \overline{Q})$ then $\int |y - Kx| \zeta(\mathrm{d}x, \mathrm{d}y) \ge (\le) \int \psi_0(x)\mu(\mathrm{d}x) + \int \psi_1(y)\nu(\mathrm{d}y)$ by the martingale property. Our dual problem is then defined as the supremum (infimum) over all sub (super)-replicating portfolios:

$$\begin{aligned} \int |y - Kx| \zeta(\mathrm{d}x, \mathrm{d}y) &\ge \sup_{(\psi_0,\psi_1,\delta)\in \underline{Q}} \left\{ \int \psi_0(x)\mu(\mathrm{d}x) + \int \psi_1(y)\nu(\mathrm{d}y) \right\} =: \underline{\mathcal{D}}(\mu, \nu), \\ \int |y - Kx| \zeta(\mathrm{d}x, \mathrm{d}y) &\le \inf_{(\psi_0,\psi_1,\delta)\in \overline{Q}} \left\{ \int \psi_0(x)\mu(\mathrm{d}x) + \int \psi_1(y)\nu(\mathrm{d}y) \right\} =: \overline{\mathcal{D}}(\mu, \nu). \end{aligned} \tag{1.1.2}$$

In [13, Theorem 1 and Corollary 1.1], the authors proved (actually for a more general class of payoff functions) that there is no duality gap, namely that $\underline{\mathcal{P}}(\mu, \nu) = \underline{\mathcal{D}}(\mu, \nu)$ and $\overline{\mathcal{P}}(\mu, \nu) = \overline{\mathcal{D}}(\mu, \nu)$. However, the optimal values may not be attained in the dual problems, as proved in [13, Proposition 4.1]. In [91] and [92] the authors showed that in the at-the-money case ($K = 1$), with an additional dispersion assumption on the measures μ and ν (see Assumption 1.1.5 below), the optimal values of the dual problems (1.1.2) are actually attained. We record these results in the following theorem:

Theorem 1.1.1. *The set equalities $\underline{\mathcal{D}}(\mu, \nu) = \underline{\mathcal{P}}(\mu, \nu)$ and $\overline{\mathcal{D}}(\mu, \nu) = \overline{\mathcal{P}}(\mu, \nu)$ hold, and the primal optima in* (1.1.1) *are attained: there exist martingale measures $\mathbb{Q}_L$ and $\mathbb{Q}_U$ in $\mathcal{M}(\mu, \nu)$ such that $\underline{\mathcal{P}}(\mu, \nu) = \mathbb{E}^{\mathbb{Q}_L}|S_{t+\tau} - KS_t|$ and $\overline{\mathcal{P}}(\mu, \nu) = \mathbb{E}^{\mathbb{Q}_U}|S_{t+\tau} - KS_t|$. Furthermore, under Assumption 1.1.5, the infimum and supremum in the dual problems* (1.1.2) *are attained when $K = 1$.*

Our objective is to discretise the primal (1.1.1) and dual problems (1.1.2) and solve them as linear programming problems.

1.1.2 No-Arbitrage discretisation of the primal and dual problems

Let $t > 0$ be some given time horizon, S_t the random variable describing the stock price at time t, and μ the law of S_t. Fix $N \in \mathbb{N}$ with $N > 1$ and suppose that we are given a set $\mathrm{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$ of points $0 < x_1 < x_2 < ... < x_N$ in the support of μ, and a discrete distribution q with atom q_i at the point x_i. We wish to find a discrete distribution p, close to q, and that matches (at least) some moments of μ, in particular satisfying the martingale condition $\langle \mathrm{p}, \mathrm{x} \rangle = 1$. Suppose we want to match the first l moments of μ, for some $l \le N$. Let $T : \mathbb{R}_+ \to \mathbb{R}_+^l$ be given by $T(x) := (x, x^2, ..., x^l)$ and define the moment vector $\mathrm{T} := \int_{\mathbb{R}_+} T(x)\mu(\mathrm{d}x) \in \mathbb{R}^l$. Such a matching condition is not necessarily consistent, however, with a given set of (European) option prices. In order to ensure that the discrete density re-prices the given options, we add a second layer: Borwein, Choksi and Maréchal [28] suggested to recover discrete probability distributions from observed market prices of European call options by minimising the Kullback-Leibler divergence to the uniform distribution (they also comment that any prior distribution can be chosen). In particular given the law μ of S_t and a set of European call options $\Pi = (\Pi_1, \dots, \Pi_M)$ maturing at t with strikes $K_1, \dots, K_M$, we can solve the following minimisation problem:

$$\min_{\{\mathrm{p} \in [0,1]^N : \|\mathrm{p}\|_1 = 1\}} \sum_{i=1}^N p_i \log \frac{p_i}{q_i}, \qquad \text{subject to } \left(\sum_{i=1}^N G(x_i) p_i, \sum_{i=1}^N T(x_i) p_i \right) = (\Pi, \mathrm{T}). \qquad (1.1.3)$$

for some prior discrete distribution q, and where $G(x) := ((x - K_1)_+, \dots, (x - K_M)_+)$ denotes the payoff vector of the options. Note that the first component of $\sum_{i=1}^N T(x_i) p_i = \mathrm{T}$ is nothing else than the martingale condition. It must be noted that if the full marginal distribution μ of S_t is known, any finite subset of European options can be chosen above and the price vector can be defined as $\Pi := \int_{\mathbb{R}_+} G(x)\mu(\mathrm{d}x)$. In particular the solution to this problem can be obtained as a modification of the solution in [144], which itself is based on arguments by Borwein and Lewis [29, Corollary 2.6]:

$$p_i = \frac{q_i \mathrm{e}^{\langle \lambda^*, (G(x_i), T(x_i)) \rangle}}{\sum_{j=1}^N q_j \mathrm{e}^{\langle \lambda^*, (G(x_i), T(x_i)) \rangle}}, \qquad (1.1.4)$$

where

$$\lambda^* := \operatorname*{argmin}_{\lambda \in \mathbb{R}^{l+M}} \left[- \langle \lambda, (\Pi, \mathrm{T}) \rangle + \log \left(\sum_{j=1}^N q_j \mathrm{e}^{\langle \lambda, (G(x_i), T(x_i)) \rangle} \right) \right],$$

and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. We can now specify the following Algorithm where in step (iii) we solve for the discrete probability vector p using (1.1.4).

Algorithm 1.1.2.

(i) Several choices are possible for the N points $0 < x_1 < ... < x_N$; for instance:

(a) *Binomial*: Let Σ denote the at-the-money lognormal volatility (for European options maturing at t). Set $\delta := t/(N-1)$, $u := 1 + \left(\mathrm{e}^{\delta\Sigma^2} - 1\right)^{1/2}$, $d := 1 - \left(\mathrm{e}^{\delta\Sigma^2} - 1\right)^{1/2}$, and $x_i := u^{i-1} d^{N-i}$;

(b) *Gauss-Hermite*: $x_i := \mathrm{e}^{x_i^{\mathrm{H}}}$, where $x_1^{\mathrm{H}}, ..., x_N^{\mathrm{H}}$ are the nodes of an N-point Gauss-Hermite quadrature.

(ii) For the discrete distribution q, we can follow several routes:

(a) assume that μ admits a density f_μ. Then, for $i = 1, \dots, N$, set $q_i := f_\mu(x_i)/\sum_{j=1}^N f_\mu(x_j)$;

(b) alternatively, for $i = 1, ..., N$, let $q_i := \mu([x_{i-1}, x_i))$ (with $x_0 = 0$);

(iii) Compute the discretised measure p through (1.1.4).

Remark 1.1.3.

(i) As pointed out by Tanaka and Toda [144] the choice of discretisation points $x_1, \dots, x_N$ in Algorithm 1.1.2 is dictated by the quadrature rule used to approximate integrals $\int_{\mathbb{R}_+} T(x)\mathrm{d}f_\mu(x) \approx \sum_{i=1}^N w(x_i)T(x_i)f_\mu(x_i)$. Weights $w(x_i)$ are chosen in accordance with a given quadrature rule (e.g. Gauss-Hermite) and in the case of Algorithm 1.1.2 the weights are chosen to be constant $w(x_i) = 1/\sum_{i=1}^N f_\mu(x_i)$ for all $i = 1, \dots, N$.

(ii) The discrete primal and dual LP solution using this method produces accurate and robust results with just a few points. However, since we only match a finite number of call options for each maturity, our discrete measures will not necessarily be in convex order (only approximately). Other authors [9] do construct discrete measure approximations that are guaranteed to be in convex order.

1.1.2.1 Primal and dual formulation

We focus here on the primal and dual formulation for the upper bound; an analogous formulation holds for the lower bound. We use Algorithm 1.1.2 to approximate S_t and $S_{t+\tau}$ by discrete random variables $\widetilde{S}_t^m$ and $\widetilde{S}_{t+\tau}^n$ with finite supports $\{x_1, x_2, ..., x_m\}$ and $\{y_1, y_2, ..., y_n\}$ and $m, n > 1$. The atoms μ_i and ν_j at x_i and y_j are given following Algorithm 1.1.2, and the linear programme for the primal problem then reads

$$\overline{\mathcal{P}}(\mu, \nu) := \max_\zeta \sum_{i,j} \zeta_{i,j}|y_j - Kx_i|,$$

subject to the constraints $\sum_j \zeta_{i,j} = \mu_i$, $\sum_i \zeta_{i,j} = \nu_j$, $\sum_j \zeta_{i,j}(x_i - y_j) = 0$ and $\zeta_{i,j} \geq 0$. For the dual problem, denote the call option prices on $\widetilde{S}_t^m$ by $\widetilde{C}(t, K) := \mathbb{E}(\widetilde{S}_t^m - K)^+ = \sum_{i=i^*}^m (x_i - K)\mu_i$, where $i^* := \inf\{1 \leq i \leq m; x_i > K\}$ and $\widetilde{C}(t, K) = 0$ if $x_m \leq K$. Define the forward finite-difference operator for $i = 1, ..., m-1$ as follows

$$\mathrm{D}\psi_0(x_i) := \frac{\psi_0(x_{i+1}) - \psi_0(x_i)}{x_{i+1} - x_i},$$

and we then have the following lemma:

Lemma 1.1.4. *The following representation holds:*

$$\psi_0(\widetilde{S}_t^m) = \psi_0(x_1) + \mathrm{D}\psi_0(x_1)\left(\widetilde{S}_t^m - x_1\right) + \sum_{i=2}^{m-1}\left(\mathrm{D}\psi_0(x_i) - \mathrm{D}\psi_0(x_{i-1})\right)\left(\widetilde{S}_t^m - x_i\right)^+. \quad (1.1.5)$$

Proof. The proof is by induction. The representation clearly holds for $\widetilde{S}_t^m = x_1$. We now suppose that it holds for $\widetilde{S}_t^m = x_j$ and show that it is then true for $\widetilde{S}_t^m = x_{j+1}$, where $1 < j < m-1$. Consider

$$\psi_0(x_1) + \mathrm{D}\psi_0(x_1)\left(x_{j+1} - x_1\right) + \sum_{i=2}^{m-1}\left(\mathrm{D}\psi_0(x_i) - \mathrm{D}\psi_0(x_{i-1})\right)\left(x_{j+1} - x_i\right)^+.$$

Inserting the induction hypothesis into this expression yields

$$\psi_0(x_j) + \mathrm{D}\psi_0(x_1)\left(x_{j+1} - x_j\right) + \sum_{i=2}^{m-1}\left(\mathrm{D}\psi_0(x_i) - \mathrm{D}\psi_0(x_{i-1})\right)\left\{\left(x_{j+1} - x_i\right)^+ - \left(x_j - x_i\right)^+\right\}. \quad (1.1.6)$$

But we have that

$$\begin{aligned}
&\sum_{i=2}^{m-1}\left(\mathrm{D}\psi_0(x_i) - \mathrm{D}\psi_0(x_{i-1})\right)\left\{\left(x_{j+1} - x_i\right)^+ - \left(x_j - x_i\right)^+\right\} \\
&= \left(x_{j+1} - x_j\right)\sum_{i=2}^{j}\left(\mathrm{D}\psi_0(x_i) - \mathrm{D}\psi_0(x_{i-1})\right) = \left(x_{j+1} - x_j\right)\left(\mathrm{D}\psi_0(x_j) - \mathrm{D}\psi_0(x_1)\right),
\end{aligned}$$

where the last line follows since the sum is telescoping. Inserting this into (1.1.6) yields

$$\psi_0(x_{j+1}) = \psi_0(x_1) + \mathrm{D}\psi_0(x_1)\left(x_{j+1} - x_1\right) + \sum_{i=2}^{m-1}\left(\mathrm{D}\psi_0(x_i) - \mathrm{D}\psi_0(x_{i-1})\right)\left(x_{j+1} - x_i\right)^+.$$

□

With obvious notation (1.1.5) can be re-written as $\psi_0(\widetilde{S}_t^m) = w_1 + w_2[\widetilde{S}_t^m - x_1] + \sum_{i=2}^{m-1} w_{i+1}(\widetilde{S}_t^m - x_i)^+$, and using the martingale property (which is ensured by Algorithm 1.1.2) we have

$$\mathbb{E}(\psi_0(\widetilde{S}_t^m)) = w_1 + w_2[1 - x_1] + \sum_{i=2}^{m-1} w_{i+1}\widetilde{C}(t, x_i).$$

An analogous formulation holds for $\widetilde{S}_{t+\tau}^n$ and we define call option prices on $\widetilde{S}_{t+\tau}^n$ by $\widetilde{C}(t+\tau, K)$. Let now $z := (v_0, w_1^0, ..., w_{m-1}^0, w_1^1, ..., w_{n-1}^1, \delta(x_1), ..., \delta(x_m))^\top$ and denote the set

$$\chi := \{(x, y) : x \in \mathrm{Supp}(\widetilde{S}_t^m), y \in \mathrm{Supp}(\widetilde{S}_{t+\tau}^n)\}.$$

The dual problem then reads

$$\overline{\mathcal{D}}(\mu, \nu) := \min_z \left\{v_0 + w_1^0 + w_1^1 + \sum_{i=2}^{m-1} w_i^0 \widetilde{C}(t, x_i) + \sum_{i=2}^{n-1} w_i^1 \widetilde{C}(t+\tau, y_i)\right\},$$

subject to the constraints

$$v_0 + w_1^0 x + w_1^1 y + \sum_{i=2}^{m-1} w_i^0 (x - x_i)^+ + \sum_{i=2}^{n-1} w_i^1 (y - y_i)^+ + \delta(x)(y - x) \geq |y - Kx|, \quad (1.1.7)$$

for all $(x, y) \in \chi$. The dual problem has $2m + n - 1$ unknowns and both the primal and dual are exact and consistent results for the discretisations given in Algorithm 1.1.2 (which provides distributions converging to those of S_t and $S_{t+\tau}$). The importance of incorporating the martingale conditions $\mathbb{E}(S_t) = \mathbb{E}(S_{t+\tau}) = 1$ into the discretisation is critical. This is easily seen in the following example for the primal problem, which will also translate into an issue for the dual.

Suppose that S_t can take value 0.75 or 1.25 each with 50% probability and $S_{t+\tau}$ can take value 0.5 or 1.5 each with 50% probability. Note that $\mathbb{E}(S_t) = \mathbb{E}(S_{t+\tau}) = 1$. We consider the primal problem. The constraints $\sum_j \zeta_{i,j} = \mu_i$ and $\sum_j \zeta_{i,j}(x_i - y_j) = 0$ fully determine the probabilities $\zeta_{1,1} = \zeta_{2,2} = 3/8$ and $\zeta_{1,2} = \zeta_{2,1} = 1/8$. The final constraints $\sum_i \zeta_{i,j} = \nu_j$ are only true if $\nu_1 = \nu_2 = 0.5$ or $\mathbb{E}(S_{t+\tau}) = 1$. Otherwise, there will be no solution to this LP. This stresses the importance of a consistent no-arbitrage discretisation of the problem.

1.1.2.2 Approximation of the dual

We further choose our strikes to be in the region $[0.3, 2]$ and therefore ignore all x_i and y_i in the sums for which $x_i \notin [0.3, 2]$ and $y_i \notin [0.3, 2]$. Finally, similar to [85] we decompose the delta hedge over a finite dimensional basis $(e_i)_{i=1}^{m_b}$, $\delta(\widetilde{S}_t^m) \approx \sum_{i=1}^{m_b} w_i^b e_i(\widetilde{S}_t^m)$, where we let the e_i be a polynomial basis and m_b is much smaller than m. We note that one can also use the above algorithm with the cutting-plane method outlined in [85].

1.1.3 Primal solution for the at-the-money case

In [91] the authors derived the lower bound optimal martingale transport plan for the at-the-money ($K = 1$) forward-start straddle. The following dispersion assumption on the marginal measures μ and ν (readily satisfied in all examples presented here, see Figures 1.1 and 1.3) is fundamental for their analysis:

Assumption 1.1.5. The marginal distributions μ and ν are such that the support of $\eta := (\mu - \nu)^+$ is given by an interval $[a, b]$ with $0 < a < b$ and the support of $\gamma := (\nu - \mu)^+$ is given by $\mathbb{R}_+ \setminus [a, b]$. The corresponding densities will be denoted by f_μ, f_ν, f_η and f_γ.

Define $\Delta_f(z) := \int_0^z f_\nu(u)\mathrm{d}u - \int_0^z f_\mu(u)\mathrm{d}u$ for all $z \geq 0$. Then Assumption 1.1.5 is equivalent [37, Lemma 5.1] to Δ_f having a single maximiser. Assumption 1.1.5 imposes constraints on the tail behaviour of the difference between the two laws μ and ν, and is clearly satisfied in the Black-Scholes case.

1.1.3.1 Structure of the transport plan

The key risk for an at-the-money forward-start straddle is that a long position is equivalent to being short the kurtosis of the conditional distribution of the underlying asset (see for example [92]).

Therefore to produce the lowest possible price it seems reasonable to require a transport plan that maximises the kurtosis of the conditional distribution. This is indeed the structure of the solution in [91]. We leave as much common mass ($\mu \wedge \nu$) in place and then map the residual mass η on $[a, b]$ to the tails of the distribution γ via two decreasing functions $p : [a, b] \to [0, a]$ and $q : [a, b] \to [b, \infty)$. Using the martingale condition and the fact that the mass of η equals that of γ, [91] derives a system of coupled differential equations for (p, q):

$$p'(x) = \frac{q(x) - x}{q(x) - p(x)} \frac{f_\mu(x) - f_\nu(x)}{f_\mu(p(x)) - f_\nu(p(x))}, \qquad q'(x) = \frac{x - p(x)}{q(x) - p(x)} \frac{f_\mu(x) - f_\nu(x)}{f_\mu(q(x)) - f_\nu(q(x))}, \tag{1.1.8}$$

with boundary conditions $p(b) = 0$, $q(b) = b$, $p(a) = a$ and $q(a) = +\infty$.

1.1.3.2 Implementation

The RHS of the two equations in (1.1.8) are undefined at the boundary points. An application of L'Hôpital's rule shows that $\lim_{x\uparrow b} q'(x) = -1$ if $f'_\mu(b) \neq f'_\nu(b)$, which is a reasonable assumption in practice, as will be illustrated in Section 1.1.4. On the other hand $\lim_{x\uparrow b} p'(x)$ depends on the marginal measures μ and ν. For instance, in the lognormal example in Section 1.1.4, we find that $p'(x) = \mathcal{O}\left(\mathrm{e}^{\alpha(\log p(x))^2}\right)$ for some $\alpha > 0$ as $x \uparrow b$ and $\lim_{x\uparrow b} p'(x) = -\infty$ (see for example Figure 1.1(b)). On the other hand if for example $f_\mu(0) \neq f_\nu(0)$ or $f'_\mu(0) \neq f'_\nu(0)$ then $\lim_{x\uparrow b} p'(x) = 0$.

In order to circumvent these issues so that we can apply the Runge-Kutta method to solve these ODEs, we introduce the following pre-processing step: fix a small $\delta > 0$ (in our experiments we choose $\delta = 0.001$), integrate both sides of (1.1.8) over $[b - \delta, b]$ and then approximate the RHS by using the rectangle rule for the integral with the unknown values $p^* := p(b - \delta)$ and $q^* := q(b - \delta)$. This yields the following simultaneous equations for p^* and q^* which we numerically solve:

$$\begin{aligned} p^* &= -\delta \frac{q^* - b + \delta}{q^* - p^*} \frac{f_\mu(b - \delta) - f_\nu(b - \delta)}{f_\mu(p^*) - f_\nu(p^*)}, \\ q^* &= b - \delta \frac{b - \delta - p^*}{q^* - p^*} \frac{f_\mu(b - \delta) - f_\nu(b - \delta)}{f_\mu(q^*) - f_\nu(q^*)}. \end{aligned}$$

These equations can easily be reduced into one root search: the first equation gives

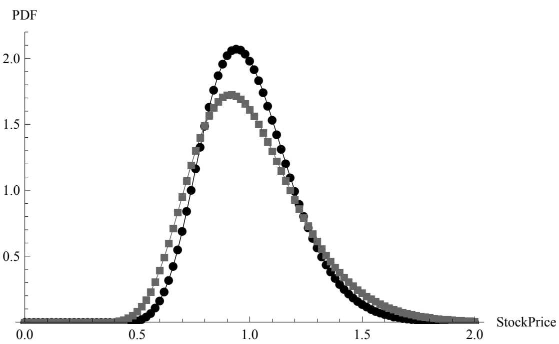
$$q^* = \frac{(p^*)^2 \left[f_\mu(p^*) - f_\nu(p^*)\right] + \delta(b - \delta) \left[f_\mu(b - \delta) - f_\nu(b - \delta)\right]}{p^* \left[f_\mu(p^*) - f_\nu(p^*)\right] + \delta \left[f_\mu(b - \delta) - f_\nu(b - \delta)\right]},$$

which in turn can be plugged into the second equation to solve for p^*. The pair (p, q) in (1.1.8) is then solved for $x \in [a, b - \delta]$ using standard Runge-Kutta methods with the new boundary conditions $p(b - \delta) = p^*, q(b - \delta) = q^*$. The at-the-money forward-start straddle price is then given by (see [91, page 8])

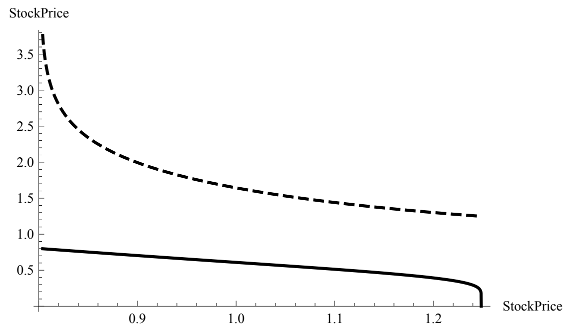
$$\int_a^b \frac{2(x - p(x))(q(x) - x)}{q(x) - p(x)} f_\eta(x) \mathrm{d}x.$$

1.1.4 Numerical analysis of the no-arbitrage bounds

We test here the numerical methods in Sections 1.1.2 and 1.1.3 on two examples. Let $\mathcal{N}(\mu, \Sigma^2)$ denote the Gaussian distribution with mean μ and variance Σ^2. First we assume that $\log(S_t) \sim \mathcal{N}(-\Sigma^2 t/2, \Sigma^2 t)$ and $\log(S_{t+\tau}) \sim \mathcal{N}(-\Sigma^2(t+\tau)/2, \Sigma^2(t+\tau))$ with $\Sigma = 0.2$, $t = 1$ and $\tau = 0.5$. Clearly a candidate martingale coupling is the Black-Scholes model (1.0.1) with volatility Σ and in this case the forward volatility is constant and equal to Σ. In Figure 1.1(a) we plot the distributions of S_t and $S_{t+\tau}$ and the corresponding lower bound at-the-money transport maps from Section 1.1.3. In Figure 1.2 we plot the lower and upper bounds for the Type-II forward smile (defined in (1.0.6)). The lower bound at-the-money case using the Hobson-Klimmek solution and the LP dual solution are virtually identical (6.95% vs 6.98%), giving credibility to both approaches. Note that even in this simple case the range of possible forward smiles consistent with the marginal laws is large though.



(a) Densities.



(b) Transport maps.

Figure 1.1: In (a) circles plot the 1 year lognormal density and squares plot the 1.5 year lognormal density. In (b) the solid line (dashed line) is the function p (q) in (1.1.8).

We next generate our marginal distributions for expiries $t = 1$ and $t+\tau = 1.5$ using the Heston stochastic volatility model (Section 1.3.1.1) and the model parameters: $v = \theta = 0.07$, $\kappa = 1$, $\xi = 0.4$ and $\rho = -0.8$. The (spot) implied volatility smiles and corresponding densities are displayed in Figure 1.3. In Figure 1.4 we plot the Heston forward smile consistent with the marginals (computed using the inverse Fourier transform representation in Lemma 1.4.7 and a simple root search to find the Type-II forward volatility) and the lower and upper bounds for the forward smile. As in the previous example the Hobson-Klimmek solution (7.77%) and the LP dual solution (7.80%) for the lower-bound at-the-money case are virtually identical. In Figure 1.5(a) we plot the payoff of our option prices in the super-hedge. We enter into positions that go long convexity for the 1.5 year maturity and go short convexity for the 1 year maturity, which intuitively makes sense.

In both examples the range of forward smiles consistent with the marginal laws is large. Using European options to 'lock-in' (replicate) forward volatility or hedge forward volatility dependent

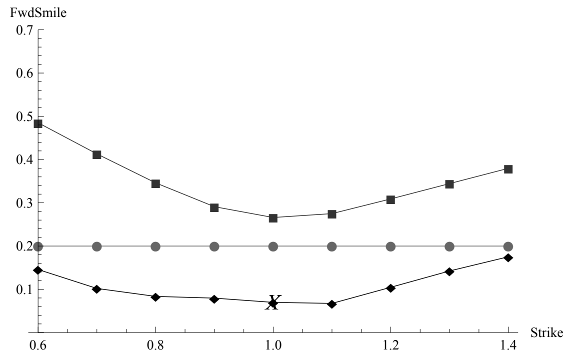
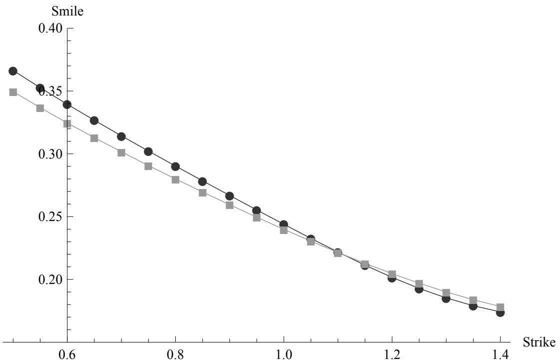
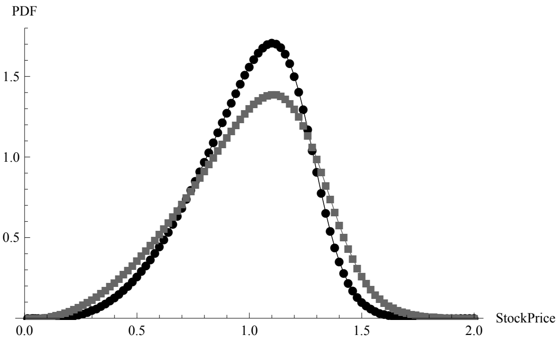


Figure 1.2: The circles represent the Black-Scholes forward volatility consistent with the marginals; squares and diamonds are the lower and upper bounds found by solving the LP dual problem (Section 1.1.2) and X is the primal solution for the lower bound at-the-money case using the Hobson-Klimmek solution (Section 1.1.3).



(a) Spot Implied Volatility.



(b) Densities.

Figure 1.3: (a): Circles (squares) represent the 1 year (1.5 year) spot implied volatility. (b): circles (squares) represents the corresponding marginal densities.

claims seems illusory. Forward-start options should be seen as fundamental building blocks for exotic pricing and not decomposable (or approximately decomposable) into European options. Models used for forward volatility dependent exotics should have the capability of calibration to forward-start option prices and at a minimum should produce realistic forward smiles that are consistent with trader expectations and observable prices. The asymptotic results developed in this thesis allow one to study both of these points.

1.1.5 Numerical analysis of the transport plans

As mentioned in Section 1.1.3, the key risk for the at-the-money forward-start straddle is that a long position is equivalent to being short the kurtosis of the conditional distribution. The solution in the lower bound case (under Assumption 1.1.5) was detailed in Section 1.1.3, where—intuitively—the

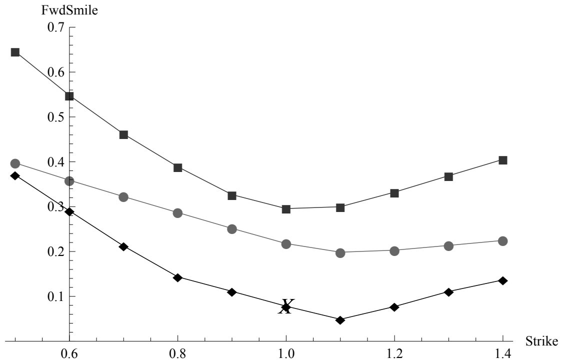
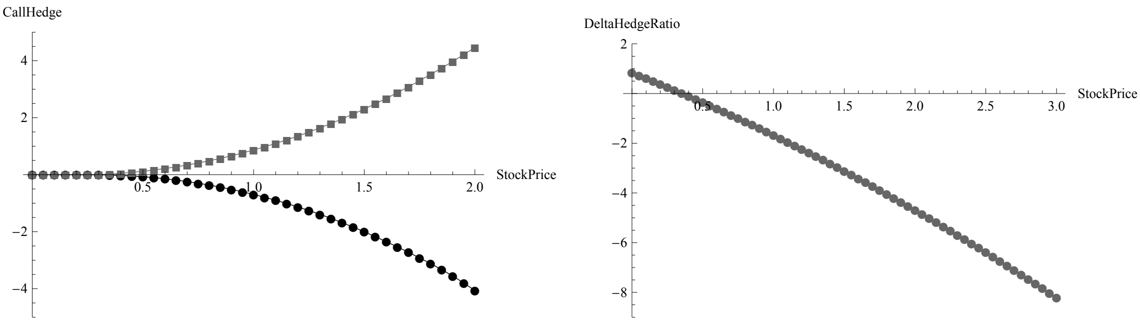


Figure 1.4: Circles represent the Heston forward volatility consistent with the marginals, squares and diamonds the lower and upper bounds found by solving the LP problem (Section 1.1.2), and X is the primal Hobson-Klimmek solution for the lower bound at-the-money case (Section 1.1.3).



(a) Option hedge.

(b) Delta-hedge ratio.

Figure 1.5: (a): Circles (squares) represent the payoff of our 1 year maturity (1.5 year maturity) option prices in the superhedge as a function of x (y). (b): approximation of the delta hedge ratio $x \mapsto \delta(x)$; the strike of the forward-start option is at-the-money.

transport plan maximises the kurtosis of the conditional distribution. In the upper bound case (see [92]) the support of the transport plan is concentrated on a binomial map with no mass being left in place, i.e. all the mass of μ gets mapped to ν via two increasing functions $R, S : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $R(x) \leq x \leq S(x)$. Intuitively in this case the solution minimises the kurtosis of the conditional distribution.

For out-of-the-money options the situation is more subtle. As the strike moves further away from the money, a long option position becomes longer the kurtosis of the conditional distribution. Intuitively one would expect the transport plan to be some combination of the lower and upper at-the-money transport plans discussed above. In this section, using the lognormal example of Section 1.1.4, we numerically solve for the transport plans using the LP primal formulation in Section 1.1.2 and make qualitative conjectures concerning the structure of the transport plans.

In Figure 1.6 we numerically compute the transport maps R, S for the at-the-money upper

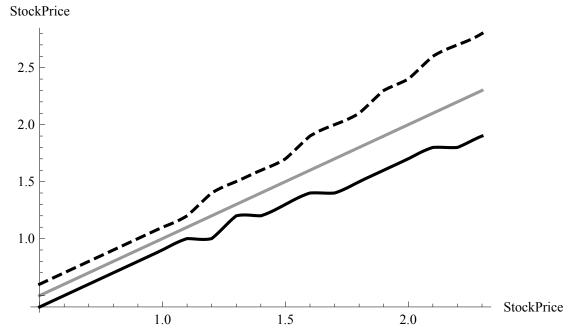
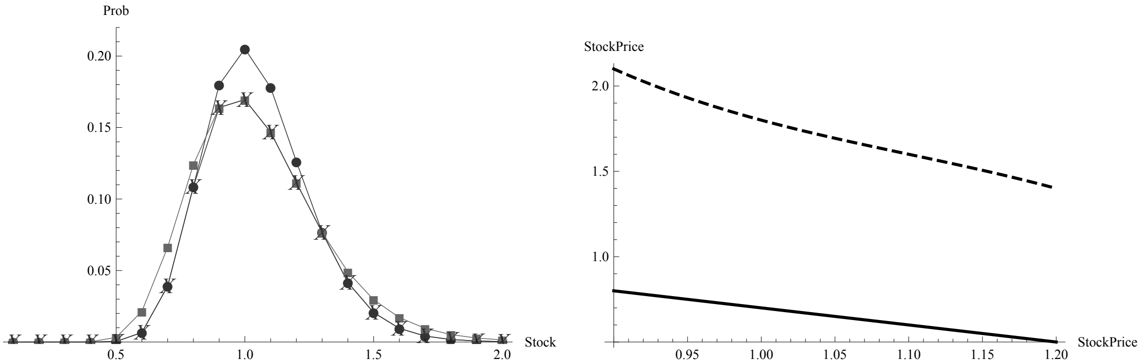


Figure 1.6: The dashed and dark lines are the transport maps for the upper bound at-the-money case and the grey line is the identity map. The horizontal axis is S_t and the vertical one is $S_{t+\tau}$.



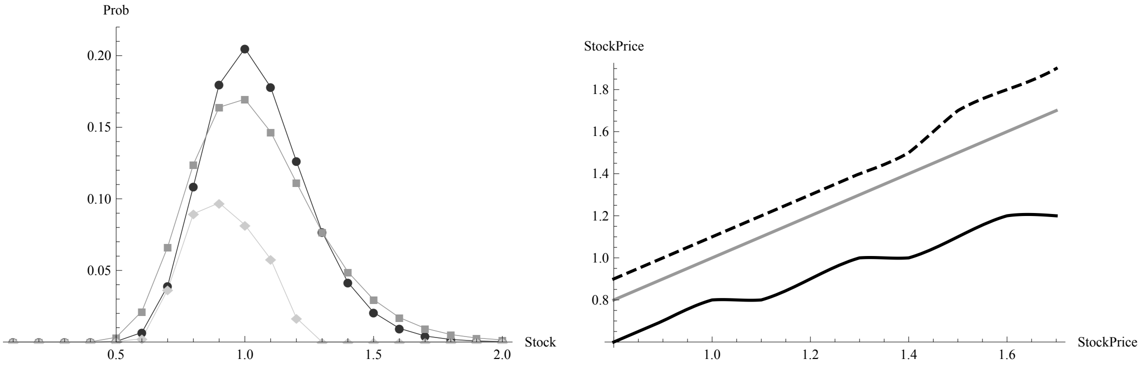
(a) Mass in place $K = 1$: Lower Bound.

(b) Transport Maps $K = 1$: Lower Bound.

Figure 1.7: (a): discretisation of the μ-measure (circles), the ν-measure (squares) and the amount of mass that must be left in place (X's) in the transport plan for the at-the-money ($K = 1$) lower bound case. (b): transport maps for the residual mass: the axes are labelled as in Figure 1.6.

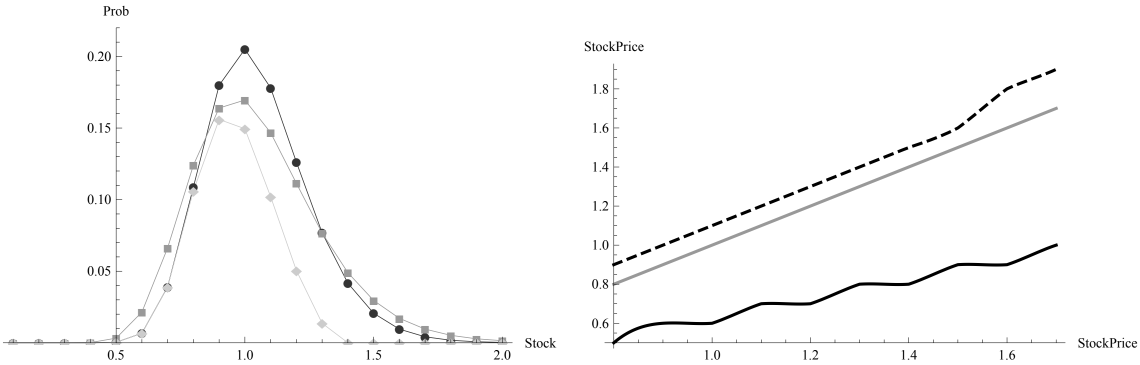
bound case. In this case no mass is left in place in the transport plan. In Figure 1.7 we numerically compute the transport plan for the at-the-money lower bound case. The figures are in striking agreement with Hobson-Klimmek: as much mass as possible is left in place and the residual mass is mapped to the tails of the distribution via two decreasing functions. Note the agreement with the transport maps in Figure 1.1(b). In this case the forward volatility is 6.92% matching the Hobson-Klimmek analytical solution and the numerical solution of the dual.

Figures 1.8 and 1.9 illustrate the transport plan for the upper bound case and strikes $K = 0.7$ and $K = 0.9$. As the strike decreases from at-the-money, more and more mass is left in place (starting from the left tail), and the residual mass of μ is mapped to ν via two increasing functions; one maps the residual mass to the left tail of ν while the other maps the residual mass to the right tail of ν. For strikes greater than at-the-money a mirror-image transport plan emerges where more and more mass is left in place (starting from the right tail) and again the residual mass of μ is



(a) Mass in place $K = 0.9$: Upper Bound. (b) Transport Maps $K = 0.9$: Upper Bound.

Figure 1.8: (a): discretisation of the measures μ (circles), ν (squares) and the amount of mass that must be left in place (diamonds) in the transport plan for the $K = 0.9$ upper bound case. (b): transport maps for the residual mass: the axes are labelled as in Figure 1.6.

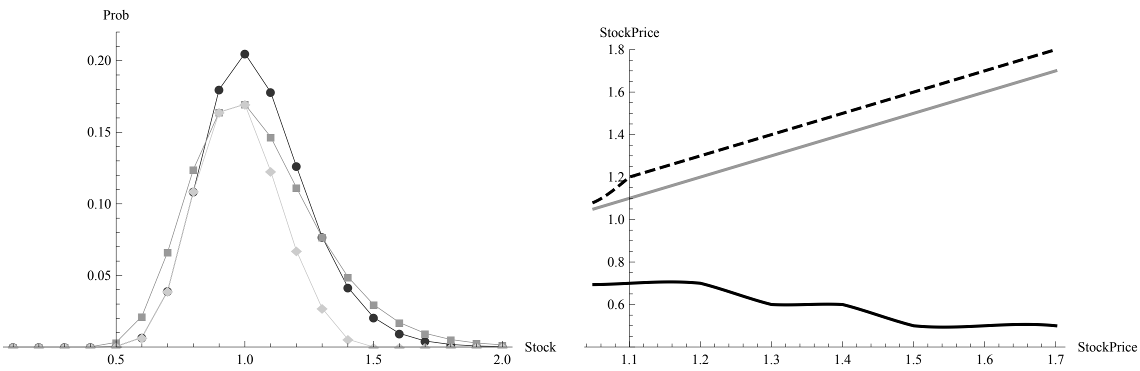


(a) Mass in place $K = 0.7$: Upper Bound. (b) Transport Maps $K = 0.7$: Upper Bound.

Figure 1.9: (a): discretisation of the measures μ (circles), ν (squares) and the amount of mass that must be left in place (diamonds) in the transport plan for the $K = 0.7$ upper bound case. (b): transport maps for the residual mass: the axes are labelled as in Figure 1.6.

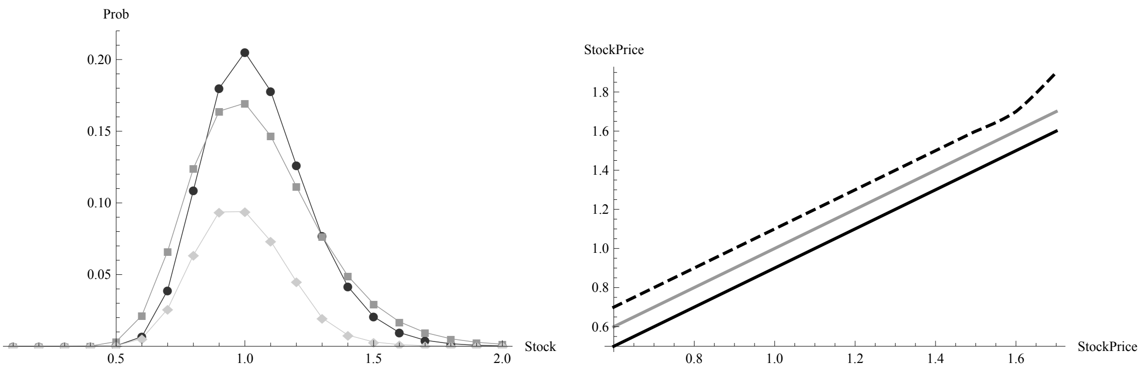
mapped to ν via two increasing functions (for brevity we omit the plots).

Figures 1.10 and 1.11 illustrate the transport plan for the lower bound case and strikes $K = 1.05$ and $K = 1.3$. As the strike increases from at-the-money, less and less mass is left in place (removing mass first from the right tail) and the residual mass of μ is mapped to ν via two functions: one maps the residual mass to the left tail of ν, the other maps the residual mass to the right tail of ν. These functions appear to be increasing for large strikes (Figure 1.11(b)), but since the transport maps are decreasing for the at-the-money strike (Figure 1.7(b)), for strikes close to the money these maps could be decreasing 1.10(b). For strikes lower than the money a mirror-image transport plan emerges where less and less mass stays in place (removing mass first from the left tail) and again



(a) Mass in place $K = 1.05$: Lower Bound. (b) Transport Maps $K = 1.05$: Lower Bound.

Figure 1.10: (a): discretisation of the measures μ (circles), ν (squares) and the amount of mass that must be left in place (diamonds) in the transport plan for the $K = 1.05$ lower bound case. (b): transport maps for the residual mass: the axes are labelled as in Figure 1.6.



(a) Mass in place $K = 1.3$: Lower Bound. (b) Transport Maps $K = 1.3$: Lower Bound.

Figure 1.11: (a): discretisation of the measures μ (circles), ν (squares) and the amount of mass left in place (diamonds) in the transport plan for the $K = 1.3$ lower bound case. (b): transport maps for the residual mass: the axes are labelled as in Figure 1.6.

the residual mass of μ is mapped to ν via two functions (for brevity we omit the plots).

1.2 Large deviations theory and the Laplace method

We provide here a brief review of large deviations and the Gärtner-Ellis theorem. The Gärtner-Ellis theorem is a key result in the theory of (finite-dimensional) large deviations. Extending the results of Cràmer [45] for sequences of random variables not necessarily independent and identically distributed (iid), it provides a large deviations framework based solely on the knowledge of the cumulant generating function (cgf) of the sequence. For a detailed account of these, the interested reader should consult [48]. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random variables in $\mathbb{R}$, with law μ_n and cumulant generating function $\Lambda_n(u) \equiv \log \mathbb{E}(\mathrm{e}^{uX_n})$.

Definition 1.2.1. The sequence X_n is said to satisfy a large deviations principle with speed n and rate function I if for each Borel mesurable set $E \subset \mathbb{R}$,

$$-\inf_{x\in E^o} I(x) \leq \liminf_{n\uparrow\infty} \frac{1}{n} \log \mathbb{P}\left(X_n \in E\right) \leq \limsup_{n\uparrow\infty} \frac{1}{n} \log \mathbb{P}\left(X_n \in E\right) \leq -\inf_{x\in \bar{E}} I(x).$$

The rate function $I : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, by definition, is a lower semi-continuous, non-negative and not identically infinite function such that the level sets $\{x \in \mathbb{R} : I(x) \leq \alpha\}$ are closed for all $\alpha \geq 0$. It is said to be a *good rate function* when these level sets are compact (in $\mathbb{R}$). Intuitively the large deviations principle characterises the tail probabilities in terms of exponential upper and lower bounds. If in addition I is continuous on $\bar{E}$ then the LDP simplifies to

$$\lim_{n\uparrow\infty} \frac{1}{n} \log \mathbb{P}\left(X_n \in E\right) = -\inf_{x\in E} I(x).$$

Before stating the main theorem, we need one more concept:

Definition 1.2.2. Let $\Lambda : \mathbb{R} \to (-\infty, +\infty]$ be a convex function, and $\mathcal{D}_\Lambda := \{u \in \mathbb{R} : \Lambda(u) < \infty\}$ its effective domain. It is said to be essentially smooth if

- The interior $\mathcal{D}_\Lambda^o$ is non-empty;
- Λ is differentiable throughout $\mathcal{D}_\Lambda^o$;
- Λ is steep: $\lim\limits_{n\uparrow\infty} |\Lambda'(u_n)| = \infty$ whenever (u_n) is a sequence in $\mathcal{D}_\Lambda^o$ converging to a boundary point of $\mathcal{D}_\Lambda^o$.

Assume now that the limiting cumulant generating function $\Lambda(u) := \lim_{n\uparrow\infty} n^{-1}\Lambda_n(nu)$, exists as an extended real number for all $u \in \mathbb{R}$, and let $\mathcal{D}_\Lambda$ denote its effective domain. Let $\Lambda^* : \mathbb{R} \to \mathbb{R}_+$ denote its (dual) Fenchel-Legendre transform, via the variational formula $\Lambda^*(x) \equiv \sup_{\lambda\in D_\Lambda}\{\lambda x - \Lambda(\lambda)\}$. Then the following holds:

Theorem 1.2.3 (Gärtner-Ellis theorem)**.** *If the origin lies in the interior of $\mathcal{D}_\Lambda$ and if Λ is lower semicontinuous and essentially smooth, then the sequence $(X_n)_n$ satisfies a large deviations principle with rate function Λ^*.*

The key assumptions are that the pointwise (rescaled) limit of the cgf satisfies some convexity property and becomes steep at the boundaries of its effective domain; this in turns implies that the rate function governing the large deviations, defined as the topological dual, is also convex.

When convexity breaks down, no general result is known, and large deviations may or may not hold; the classical example [48, Remark (d), page 46] is that of the sequence $(Z_n)_{n\in\mathbb{N}}$ distributed as exponential random variables with parameter n. It is immediate to see that $\Lambda(u) :=$

$\lim_{n\uparrow\infty} n^{-1}\log\mathbb{E}(\mathrm{e}^{nuZ_n}) = 0$ if $u < 1$ and is infinite otherwise. This clearly violates the assumptions of the Gärtner-Ellis theorem; however, a simple computation reveals that the conclusion of the latter still holds, namely that a large deviations principle exists, with speed n and rate function $\Lambda^*(x) := \sup_u(ux - \Lambda(u)) = x$ if $x \geq 0$, and infinity otherwise. Dembo and Zeitouni [49] and Bryc and Dembo [35]—in the context of quadratic functionals of Gaussian processes—have proposed a way to bypass this absence of convexity issue by making the change of measure (key tool in the proof of the Gärtner-Ellis theorem) dependent on n. Bercu and Rouault [21] and Bercu, Coutin and Savy [19] exploited this insight to obtain sharp large deviation estimates for the Ornstein-Uhlenbeck process and fractional Ornstein-Uhlenbeck process respectively. More recently, O'Brien [130] and Comman [41] have strengthened this theorem, by partially relaxing the steepness and convexity assumptions. In a general infinite-dimensional setting, Bryc's Theorem [34] (see also [48, Chapter 4.4]), or 'Inverse Varadhan's lemma', allows for large deviations with non convex rate functions. One of the hypotheses this theorem relies on is an exponential tightness requirement on the family of random variables under consideration, which is not always easy to verify. However, several examples have been dug out which do not fall into this framework, such as in the setting of random walks with interface [57], occupation measures of Markov chains [88], the on/off Weibull sojourn process [55], or m-variate von Mises statistics [58].

From a probabilistic point of view, this thesis deals with deriving large deviation estimates in cases where the assumptions of the Gärtner-Ellis theorem are violated: Chapters 3 and 5 provide examples where the limiting cgfs are in fact zero on their effective domains (completely degenerate) and Chapter 4 provides an example where the steepness assumption of the limiting cgf is violated. In all cases, however, a large deviations principle still holds.

To finish the section we now recall some classical results in asymptotics of integrals that will be required in the thesis. The following theorems, Watson's lemma and the Laplace method, are taken from [131, Theorem 3.2 and Theorem 8.1] and [82, Remark 2.1 and Equation 2.9].

Theorem 1.2.4 (Watson's lemma). *Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function such that*

$$f(y) \sim \sum_{s=0}^{\infty} a_s y^{(s+\lambda-\mu)/\mu}, \quad \text{as } y \downarrow 0, \tag{1.2.1}$$

where $\lambda, \mu > 0$. Then the following asymptotic holds as τ tends to infinity:

$$\int_0^{\infty} \mathrm{e}^{-\tau y} f(y)\mathrm{d}y \sim \sum_{s=0}^{\infty} \Gamma\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{\tau^{(s+\lambda)/\mu}},$$

provided that the integral converges throughout its range for sufficiently large τ.

Remark 1.2.5. The following is taken from [126, Exercise 2.7, Chapter 2.3]: If

$$f(y) = \sum_{s=0}^{N} a_s y^{(s+\lambda-\mu)/\mu} + \mathcal{O}\left(y^{(N+1+\lambda-\mu)/\mu}\right),$$

for some $N \in \mathbb{N} \cup \{0\}$ as y tends to zero then

$$\int_0^\infty \mathrm{e}^{-\tau y} f(y)\mathrm{d}y = \sum_{s=0}^{N} \Gamma\left(\frac{s+\lambda}{\mu}\right) \frac{a_s}{\tau^{(s+\lambda)/\mu}} + \mathcal{O}\left(\frac{1}{\tau^{(N+1+\lambda)/\mu}}\right), \quad \text{as } \tau \uparrow \infty.$$

Convergence of the integral at $y = 0$ for all τ is assured by (1.2.1). A sufficient condition for convergence of the integral is that $f(y) = \mathcal{O}(\mathrm{e}^{cy})$ for some $c > 0$ as y tends to infinity. Next we state the Laplace method. Note that the result holds true if either a or b are infinite below.

Theorem 1.2.6 (Laplace method)**.** *Suppose $\phi : [a, b] \to \mathbb{R}$ has a unique absolute minimum at some $y_0 \in [a, b]$ and is three times continuously differentiable in a neighbourhood of y_0 and $f : [a, b] \to \mathbb{R}$ is continuously differentiable in a neighbourhood of y_0. Then the following asymptotics hold as ε tends to zero:*

$$\int_a^b f(y)\mathrm{e}^{-\phi(y)/\varepsilon}\mathrm{d}y = \begin{cases} \mathrm{e}^{-\phi(y_0)/\varepsilon} f(y_0)\sqrt{\dfrac{2\pi\varepsilon}{\phi''(y_0)}}\left(1 + \mathcal{O}(\varepsilon)\right), & \text{if } a < y_0 < b \text{ and } \phi''(y_0) > 0, \\ \mathrm{e}^{-\phi(a)/\varepsilon} \dfrac{f(a)}{\varepsilon\phi'(a)}\left(1 + \mathcal{O}(\varepsilon)\right), & \text{if } y_0 = a, \\ -\mathrm{e}^{-\phi(b)/\varepsilon} \dfrac{f(b)}{\varepsilon\phi'(b)}\left(1 + \mathcal{O}(\varepsilon)\right), & \text{if } y_0 = b, \end{cases}$$

provided that the integral converges absolutely for sufficiently small ε.

1.3 Models and forward moment generating functions

The forward cumulant generating function (cgf), defined as the cgf of the forward price process $X_\tau^{(t)}$ (defined in (1.0.3)) will be key in the forthcoming analysis. In this section we introduce some of the main models analysed in the thesis, derive their forward cumulant generating functions and list a few important properties. In Section 1.3.1 we focus on stochastic volatility models—in particular the Heston and Schöbel-Zhu models—and in Section 1.3.2 we look at time-changed exponential Lévy models.

1.3.1 Stochastic volatility models

We will consider specific examples of the general stochastic volatility model where the log stock price process follows,

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}V_t\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t, & X_0 = 0, \\ \mathrm{d}V_t &= h_0(V_t)\mathrm{d}t + h_1(V_t)\mathrm{d}B_t, & V_0 = v > 0, \\ \mathrm{d}\langle W, B\rangle_t &= \rho\mathrm{d}t, \end{aligned} \tag{1.3.1}$$

with $|\rho| < 1$, $(W_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ are two standard Brownian motions and $h_0, h_1 : \mathbb{R}_+ \to \mathbb{R}$ are functions chosen such that SDE admits a unique strong solution and $V_t \geq 0$ for all $t \geq 0$, $\mathbb{P}$-almost surely. For example, h_0 and h_1 can be chosen to satisfy the Yamada-Watanabe conditions [106, Proposition 2.13, page 291]). In the next two sections we will consider the Heston and Schöbel-Zhu models.

1.3.1.1 Heston

In the Heston model the (log) stock price process is the unique strong solution to the following SDEs:

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}V_t\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t, \qquad X_0 = 0, \\ \mathrm{d}V_t &= \kappa(\theta - V_t)\,\mathrm{d}t + \xi\sqrt{V_t}\mathrm{d}B_t, \quad V_0 = v > 0, \\ \mathrm{d}\langle W, B\rangle_t &= \rho\mathrm{d}t, \end{aligned} \tag{1.3.2}$$

with $\kappa > 0$, $\xi > 0$, $\theta > 0$ and $|\rho| < 1$ and $(W_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ are two standard Brownian motions. The Feller SDE for the variance process has a unique strong solution by the Yamada-Watanabe conditions [106, Proposition 2.13, page 291]). The X process is a stochastic integral of the V process and is therefore well defined. The Feller condition, $2\kappa\theta \geq \xi^2$, ensures that the origin is unattainable. Otherwise the origin is regular (hence attainable) and strongly reflecting (see [107, Chapter 15]). We do not require the Feller condition in our analysis since we work with the forward cgf of X which is always well defined. The density of the instantaneous variance in the Heston model is known in closed-form. Set

$$\zeta_{\mathrm{H}}(y) := \exp\left(-\frac{1}{2\beta_t}\left(y + v\mathrm{e}^{-\kappa t}\right)\right)\frac{1}{2\beta_t}\left(\frac{y}{v\mathrm{e}^{-\kappa t}}\right)^{\mu/2-1/2}\mathrm{I}_{\mu-1}\left(\mathrm{e}^{-\kappa t/2}\frac{\sqrt{vy}}{\beta_t}\right)1\!\!1_{y\geq 0}, \tag{1.3.3}$$

where I_η is the modified Bessel function of the first kind of order η [1, Section 9.6] and

$$\mu := 2\kappa\theta/\xi^2, \quad \beta_t := \frac{\xi^2}{4\kappa}\left(1 - \mathrm{e}^{-\kappa t}\right). \tag{1.3.4}$$

In the Heston model the probability density function of the variance process observed at time t then reads [103, Proposition 6.3.2.1]

$$\mathbb{P}(V_t \in \mathrm{d}y) = \zeta_{\mathrm{H}}(y)\mathrm{d}y.$$

We recall that in the Heston model the joint cumulant generating function of the pair (X_τ, V_τ) is given by [46, Lemma 2.1]

$$\log\mathbb{E}(\mathrm{e}^{uX_\tau + wV_\tau}) = A(u, w, \tau) + B(u, w, \tau)v \tag{1.3.5}$$

defined for all (u, w) such that the rhs exists and where

$$\begin{aligned} A(u,w,\tau) &:= \frac{\kappa\theta}{\xi^2}\left((\kappa - \rho\xi u - d(u))\,\tau - 2\log\left(\frac{1-\gamma(u,w)\exp(-d(u)\tau)}{1-\gamma(u,w)}\right)\right), \\ B(u,w,\tau) &:= \frac{\kappa - \rho\xi u - d(u) - \big(\kappa - \rho\xi u + d(u)\big)\gamma(u,w)\exp(-d(u)\tau)}{\xi^2(1-\gamma(u,w)\exp(-d(u)\tau))}, \\ d(u) &:= \left((\kappa - \rho\xi u)^2 + u(1-u)\xi^2\right)^{1/2}, \quad \gamma(u,w) := \frac{\kappa - \rho\xi u - d(u) - \xi^2 w}{\kappa - \rho\xi u + d(u) - \xi^2 w}. \end{aligned} \tag{1.3.6}$$

We end the section by deriving the forward cgf:

Lemma 1.3.1. *The Heston* (1.3.2) *forward cgf reads* ($X_\tau^{(t)}$ *defined in* (1.0.3))

$$\log \mathbb{E}\left(\mathrm{e}^{uX_\tau^{(t)}}\right) = A(u,\tau) + \frac{B(u,\tau)}{1-2\beta_t B(u,\tau)} v\mathrm{e}^{-\kappa t} - \frac{2\kappa\theta}{\xi^2}\log\left(1-2\beta_t B(u,\tau)\right), \tag{1.3.7}$$

defined for all u *such that the rhs exists and where*

$$A(u,\tau) := A(u,0,\tau), \quad B(u,\tau) := B(u,0,\tau), \quad \gamma(u) := \gamma(u,0). \tag{1.3.8}$$

Proof. For any $t>0$, the random variable V_t in (1.3.2) is distributed as β_t times a non-central chi-square random variable with 2μ degrees of freedom and non-centrality parameter $v\mathrm{e}^{-\kappa t}/\beta_t$ [103, Remark 6.3.2.2]. It follows that the corresponding mgf is given by

$$\Lambda_t^V(u) := \mathbb{E}\left(\mathrm{e}^{uV_t}\right) = \exp\left(\frac{v\mathrm{e}^{-\kappa t}u}{1-2\beta_t u}\right)(1-2\beta_t u)^{-\mu}, \qquad \text{for all } u < \frac{1}{2\beta_t}. \tag{1.3.9}$$

Using (1.3.5) and the tower property for expectations then yields the forward cgf:

$$\log \mathbb{E}\left(\mathrm{e}^{uX_\tau^{(t)}}\right) = \log \mathbb{E}\left(\mathbb{E}\left(\mathrm{e}^{uX_\tau^{(t)}}|\mathcal{F}_t\right)\right) = A(u,\tau) + \log \Lambda_t^V(B(u,\tau)).$$

□

1.3.1.2 Schöbel-Zhu

The Schöbel-Zhu (SZ) stochastic volatility model [138] is an extension to non-zero correlation of the Stein & Stein [141] model in which the logarithmic spot price process $(X_t)_{t\geq 0}$ satisfies the following SDEs:

$$\begin{aligned}
\mathrm{d}X_t &= -\frac{1}{2}\sigma_t^2 \mathrm{d}t + \sigma_t \mathrm{d}W_t, & X_0 &= x_0 \in \mathbb{R},\\
\mathrm{d}\sigma_t &= \kappa(\theta - \sigma_t)\mathrm{d}t + \frac{1}{2}\xi \mathrm{d}B_t, & \sigma_0 &= \sqrt{v} > 0,\\
\mathrm{d}\langle W, B\rangle_t &= \rho \mathrm{d}t,
\end{aligned} \tag{1.3.10}$$

where κ, θ and ξ are strictly positive real numbers, $\rho \in (-1,1)$ and W and B are two standard Brownian motions. The volatility process $(\sigma_t)_{t\geq 0}$ is Gaussian and hence both SDEs are well defined. In order to specify the forward cgf we define the following functions:

$$\begin{aligned}
A(u,\tau) &:= A_1(u,\tau) + \frac{2\kappa^2\theta^2(\chi(u)-d(u))}{d(u)^3\xi^2}A_2(u,\tau),\\
A_1(u,\tau) &:= \frac{1}{2}(\chi(u)-d(u))\tau - \frac{1}{2}\log\left(\frac{1-\gamma(u)\exp(-2d(u)\tau)}{1-\gamma(u)}\right),\\
A_2(u,\tau) &:= \chi(u)(d(u)\tau - 2) + d(u)(d(u)\tau - 1) + 2\mathrm{e}^{-d(u)\tau}\frac{2\chi(u) + \frac{d(u)^2 - 2\chi(u)^2}{\chi(u)+d(u)}\mathrm{e}^{-d(u)\tau}}{1-\gamma(u)\mathrm{e}^{-2d(u)\tau}},\\
B_1(u,\tau) &:= \frac{4\kappa\theta}{\xi^2}\frac{\chi(u)-d(u)}{d(u)}\frac{(1-\exp(-d(u)\tau))^2}{1-\gamma(u)\exp(-2d(u)\tau)},\\
B_2(u,\tau) &:= \frac{2(\chi(u)-d(u))}{\xi^2}\frac{1-\exp(-2d(u)\tau)}{1-\gamma(u)\exp(-2d(u)\tau)},
\end{aligned}$$

and

$$M(r,p,q) := \frac{1}{2}\left(\frac{p^2r^2}{1-2rq} - \log(1-2rq)\right), \quad \beta_t := \frac{\xi^2}{8\kappa}\left(1-\mathrm{e}^{-2\kappa t}\right), \quad \chi(u) := \kappa - \frac{\rho\xi u}{2},$$
$$d(u) := \left(\chi(u)^2 + (1-u)\frac{u}{4}\xi^2\right)^{1/2}, \quad \gamma(u) := \frac{\chi(u)-d(u)}{\chi(u)+d(u)}, \quad \mu_t := \sqrt{v}\mathrm{e}^{-\kappa t} + \theta\left(1-\mathrm{e}^{-\kappa t}\right).$$

Although we may use the same names of variables and functions as for Heston (Section 1.3.1.1), they may have a different definition here. We shall require the following lemma, which follows from [8, Equation 29.6].

Lemma 1.3.2. *If $\mathcal{Z} \sim \mathcal{N}(0,1)$ and $(p,q) \in \mathbb{R}^2$, then $\log \mathbb{E}\left(\mathrm{e}^{u\left(p\mathcal{Z}+q\mathcal{Z}^2\right)}\right) = M(u,p,q)$, whenever $uq < 1/2$.*

Lemma 1.3.3. *In the Schöbel-Zhu model (1.3.10) the forward cgf reads*

$$\log \mathbb{E}\left(\mathrm{e}^{uX_\tau^{(t)}}\right) = A(u,\tau)+B_1(u,\tau)\mu_t+B_2(u,\tau)\mu_t^2+M\Big(1, \sqrt{\beta_t}\left(B_1(u,\tau)+2B_2(u,\tau)\mu_t\right), B_2(u,\tau)\beta_t\Big),$$

defined for all u such that the rhs exists.

Proof. Conditioning on the filtration $(\mathcal{F}_t)_{t\geq 0}$ and using the tower property we find

$$\Lambda(u) = \log \mathbb{E}\left[\mathbb{E}\left(\mathrm{e}^{uX_\tau^{(t)}}|\mathcal{F}_t\right)\right] = A(u,\tau) + \log \mathbb{E}\left[\exp\left(B_1(u,\tau)\sigma_t + B_2(u,\tau)\sigma_t^2\right)\right],$$

where we have used the Schöbel-Zhu cgf from [105]. Since $\sigma_t \sim \mathcal{N}(\mu_t, \beta_t)$, we obtain

$$\begin{aligned}\Lambda(u) &= A(u,\tau) + \log \mathbb{E}\left(\mathrm{e}^{B_1(u,\tau)\sigma_t + B_2(u,\tau)\sigma_t^2}\right) \\ &= A(u,\tau) + B_1(u,\tau)\mu_t + B_2(u,\tau)\mu_t^2 + \log \mathbb{E}\left(\mathrm{e}^{\left(B_1(u,\tau)\sqrt{\beta_t}+2B_2(u,\tau)\sqrt{\beta_t}\mu_t\right)\mathcal{Z}+(B_2(u,\tau)\beta_t)Z^2}\right),\end{aligned}$$

with $\mathcal{Z} \sim \mathcal{N}(0,1)$, and the lemma follows directly from Lemma 1.3.2. □

1.3.2 Time-changed exponential Lévy models

Let N be a Lévy process with cgf given by $\log \mathbb{E}\left(\mathrm{e}^{uN_t}\right) = t\phi(u)$ for $t \geq 0$ and $u \in \mathcal{K}_\phi := \{u \in \mathbb{R} : |\phi(u)| < \infty\}$. We consider models where $X := (N_{V_t})_{t\geq 0}$ pathwise and the time-change is given by $V_t := \int_0^t v_s \mathrm{d}s$ with v being a strictly positive process independent of N. We shall consider the two following examples:

$$\mathrm{d}v_t = \kappa\left(\theta - v_t\right)\mathrm{d}t + \xi\sqrt{v_t}\mathrm{d}B_t, \tag{1.3.11}$$

$$\mathrm{d}v_t = -\lambda v_t \mathrm{d}t + \mathrm{d}J_t, \tag{1.3.12}$$

with $v_0 = v > 0$ and $\kappa, \xi, \theta, \lambda > 0$. Here B is a standard Brownian motion and J is a compound Poisson subordinator with exponential jump size distribution and Lévy exponent $l(u) := \lambda\delta u/(\alpha - u)$ for all $u < \alpha$ with $\delta > 0$ and $\alpha > 0$. In (1.3.11), v is a Feller diffusion and in (1.3.12), it is a Γ-OU process. Although we may use the same names of variables and functions as for Sections 1.3.1.1 and 1.3.1.2, they may have a different definition here. We now derive the forward cgfs when v follows (1.3.11) and (1.3.12).

Lemma 1.3.4. *If v follows* (1.3.11) *then the forward cgf reads*

$$\log \mathbb{E}\left(\mathrm{e}^{uX_\tau^{(t)}}\right) = A(\phi(u),\tau) + \frac{B(\phi(u),\tau)}{1-2\beta_t B(\phi(u),\tau)} v\mathrm{e}^{-\kappa t} - \frac{2\kappa\theta}{\xi^2}\log\left(1-2\beta_t B(\phi(u),\tau)\right), \quad (1.3.13)$$

defined for all u such that the rhs exists and where

$$\begin{aligned} A(u,\tau) &:= \frac{\kappa\theta}{\xi^2}\left((\kappa-d(u))\tau - 2\log\left(\frac{1-\gamma(u)\mathrm{e}^{-d(u)\tau}}{1-\gamma(u)}\right)\right), \\ B(u,\tau) &:= \frac{\kappa-d(u)}{\xi^2}\frac{1-\mathrm{e}^{-d(u)\tau}}{1-\gamma(u)\mathrm{e}^{-d(u)\tau}}, \\ d(u) &:= \left(\kappa^2-2u\xi^2\right)^{1/2}, \qquad \gamma(u) := \frac{\kappa-d(u)}{\kappa+d(u)}, \qquad \beta_t := \frac{\xi^2}{4\kappa}\left(1-\mathrm{e}^{-\kappa t}\right). \end{aligned} \quad (1.3.14)$$

Proof. By conditioning on $(V_u)_{t\leq u\leq t+\tau}$ and using the independence of the time-change and the Lévy process we have $\mathbb{E}\left(\mathrm{e}^{u(X_{t+\tau}-X_t)}\right) = \mathbb{E}\left(\mathrm{e}^{\phi(u)\int_t^{t+\tau} v_s \mathrm{d}s}\right)$. Using [44, page 476] and the tower property we compute (A and B given in (1.3.14))

$$\mathbb{E}\left(\mathrm{e}^{u(X_{t+\tau}-X_t)}\right) = \mathbb{E}\left[\mathbb{E}\left(\mathrm{e}^{\phi(u)\int_t^{t+\tau} v_s \mathrm{d}s}|\mathcal{F}_t\right)\right] = \mathrm{e}^{A(\phi(u),\tau)}\mathbb{E}\left(\mathrm{e}^{B(\phi(u),\tau)v_t}\right), \quad (1.3.15)$$

and using the mgf for v in (1.3.9) yields the forward cgf. □

Lemma 1.3.5. *If v follows* (1.3.12) *then the forward cgf reads*

$$\log \mathbb{E}\left(\mathrm{e}^{uX_\tau^{(t)}}\right) = A(\phi(u),\tau) + B(\phi(u),\tau)v\mathrm{e}^{-\lambda t} + \delta\log\left(\frac{B(\phi(u),\tau)-\mathrm{e}^{t\lambda}\alpha}{\mathrm{e}^{t\lambda}(B(\phi(u),\tau)-\alpha)}\right), \quad (1.3.16)$$

defined for all u such that the rhs exists and where

$$A(u,\tau) := \frac{\lambda\delta}{\alpha\lambda-u}\left[u\tau + \alpha\log\left(1-\frac{u}{\alpha\lambda}\left(1-\mathrm{e}^{-\lambda\tau}\right)\right)\right], \qquad B(u,\tau) := \frac{u}{\lambda}\left(1-\mathrm{e}^{-\lambda\tau}\right). \quad (1.3.17)$$

Proof. Equality (1.3.15) also holds here with A and B defined in (1.3.17) (see [44, page 488]). The mgf for v in this case is given by [44, page 482]

$$\log \mathbb{E}\left(\mathrm{e}^{uv_t}\right) = uv\mathrm{e}^{-\lambda t} + \delta\log\left(\frac{u-\alpha\mathrm{e}^{\lambda t}}{(u-\alpha)\mathrm{e}^{\lambda t}}\right), \qquad \text{for all } u<\alpha,$$

and the result follows. □

1.4 Pricing forward-start options

In this Section we focus on pricing forward-start options. In Section 1.4.1 we introduce various changes of measures in order to understand the relationship between the Type-I and Type-II forward smile. As an application we show that the Type-II Heston forward smile can be read directly from the Type-I Heston forward smile. In the spot ($t=0$) case the left wing ($k<0$) of the Heston implied volatility can be read directly from the right wing ($k>0$). In the forward case ($t>0$) we show that this is no longer necessarily the case. In Section 1.4.2 we develop different representations for forward-start option prices. Each of these representations is useful in its own right and gives different intuition on the forward smile.

Let us suppose first that $(X_s)_{s\geq 0}$ has stationary increments. Then clearly $\mathbb{E}(\mathrm{e}^{X_{t+\tau}-X_t}-\mathrm{e}^k)^+ = \mathbb{E}(\mathrm{e}^{X_\tau}-\mathrm{e}^k)^+$ and there is no term structure for the forward implied volatility in these models. Exponential Lévy models fall into this class and this property is contrary to the forward implied volatility surface observed in the market. Non-stationary increments is therefore necessary in order to capture a more realistic forward volatility term structure. We record this result as a lemma:

Lemma 1.4.1. *Let $k \in \mathbb{R}$ and $t, \tau > 0$. If $(X_s)_{s\geq 0}$ has stationary increments then $\sigma_{t,\tau}(k) = \sigma_\tau(k)$.*

Suppose now that $(\mathrm{e}^{X_s})_{s\geq 0}$ is a $(\mathbb{P}, \mathcal{F}_s)-$martingale and $(X_s)_{s\geq 0}$ has independent increments. The next lemma shows us that the Type-I forward smile will then be the same as the Type-II forward smile.

Lemma 1.4.2. *Let $k \in \mathbb{R}$ and $t, \tau > 0$. If $(\mathrm{e}^{X_s})_{t\geq 0}$ is a $(\mathbb{P}, \mathcal{F}_s)-$martingale and $(X_s)_{s\geq 0}$ has independent increments then $\sigma_{t,\tau}(k) = \widetilde{\sigma}_{t,\tau}(k)$.*

Proof. Using the independent increment assumption and the martingale property we find that

$$\begin{aligned}\mathbb{E}\left(\mathrm{e}^{X_{t+\tau}} - \mathrm{e}^k\mathrm{e}^{X_t}\right)^+ &= \mathbb{E}\left(\mathrm{e}^{X_t}\left(\mathrm{e}^{X_{t+\tau}-X_t} - \mathrm{e}^k\right)^+\right) = \mathbb{E}\left(\mathrm{e}^{X_t}\right)\mathbb{E}\left(\mathrm{e}^{X_{t+\tau}-X_t} - \mathrm{e}^k\right)^+ \\ &= \mathbb{E}\left(\mathrm{e}^{X_{t+\tau}-X_t} - \mathrm{e}^k\right)^+.\end{aligned}$$

□

Exponential Lévy models satisfy this property, but the independent increment property is not a necessary condition for equality of the Type-I and II forward smile in a model. In stochastic volatility models the independent increment assumption is not true, but when the instantaneous correlation is zero, the Type-I and II forward smiles are equal (Proposition 1.4.4 below). Consider for example the Heston (1.3.2) model and let us see if the joint mgf factorises in a neighbourhood of the origin. Using the cgf in (1.3.7) we find that

$$\mathbb{E}\left(\mathrm{e}^{u(X_{t+\tau}-X_t)}\mathrm{e}^{wX_t}\right) = \mathbb{E}\left(\mathbb{E}\left(\mathrm{e}^{u(X_{t+\tau}-X_t)}|\mathcal{F}_t\right)\mathrm{e}^{wX_t}\right) = \mathrm{e}^{A(u,\tau)}\mathbb{E}\left(\mathrm{e}^{V_tB(u,\tau)+wX_t}\right),$$

for all $(u, w) \in \mathbb{R}^2$ such that the expectations exist and are finite. Using (1.3.5) we see that the joint mgf factorises if and only if the following two equations are satisfied:

$$A(w,t) - \frac{2\kappa\theta}{\xi^2}\log(1 - 2\beta_t B(u,\tau)) = A(w, B(u,\tau), t), \tag{1.4.1}$$

$$B(w,t) + \frac{B(u,\tau)\mathrm{e}^{-\kappa t}}{1 - 2\beta_t B(u,\tau)} = B(w, B(u,\tau), t), \tag{1.4.2}$$

with all functions defined in (1.3.4), (1.3.6) and (1.3.8). It can be easily checked (numerically or otherwise) that these equations do not hold in general (even if the correlation is null). However, Proposition 1.4.4 below shows us that the Type-I and II forward smiles are the same when the instantaneous correlation is null.

1.4.1 Measure-change symmetries

In this section we will assume that the asset price process $(\mathrm{e}^{X_s})_{s\geq 0}$ is a $(\mathbb{P},\mathcal{F}_s)$-martingale. This implies that for any $a\geq 0$ the process $(\mathrm{e}^{X_{a+s}-X_a})_{s\geq 0}$ is also a $(\mathbb{P},\mathcal{F}_{a+s})$-martingale. For any $t,\tau>0$ we then define the following measures:

$$\bar{\mathbb{P}}(A) := \mathbb{E}\left(\mathrm{e}^{X_t}1\!\!1_A\right), \qquad \text{for every } A\in\mathcal{F}_t, \tag{1.4.3}$$

$$\widetilde{\mathbb{P}}(A) := \mathbb{E}\left(\mathrm{e}^{X_t}1\!\!1_A\right), \qquad \text{for every } A\in\mathcal{F}_{t+\tau}, \tag{1.4.4}$$

$$\mathbb{P}^*(A) := \mathbb{E}\left(\mathrm{e}^{X_{t+\tau}-X_t}1\!\!1_A\right), \qquad \text{for every } A\in\mathcal{F}_{t+\tau}. \tag{1.4.5}$$

We will call $\bar{\mathbb{P}}$, $\widetilde{\mathbb{P}}$ and $\mathbb{P}^*$ the share-price measure, the stopped-share-price measure and the forward measure respectively. We will let M denote our model when the asset price is given by $(\mathrm{e}^{X_u})_{u\geq 0}$ under the risk-neutral measure $\mathbb{P}$, $\bar{M}$ denote our model when the asset price is given by $(\mathrm{e}^{-X_u})_{u\geq 0}$ under the share-price measure $\bar{\mathbb{P}}$, $\widetilde{M}$ denote our model when the asset price is given by $(\mathrm{e}^{X_u})_{u\geq 0}$ under the stopped-share-price measure $\widetilde{\mathbb{P}}$ and let M^* denote our model when the asset price is given by $(\mathrm{e}^{-X_u})_{u\geq 0}$ under the forward measure $\mathbb{P}^*$.

In the results in this section we will use a superscript to indicate the model under which the Type-I or II forward smile is computed. So for example, $\sigma_{t,\tau}^{\widetilde{M}}$ (resp. $\widetilde{\sigma}_{t,\tau}^{\widetilde{M}}$), denotes the unique solution to the equation

$$\begin{aligned}\widetilde{\mathbb{E}}\left(\mathrm{e}^{X_{t+\tau}-X_t}-\mathrm{e}^k\right)^+ &= \mathrm{BS}(k,\sigma_{t,\tau}^{\widetilde{M}}(k)^2,\tau),\\ \text{resp. } \widetilde{\mathbb{E}}\left(\mathrm{e}^{X_{t+\tau}}-\mathrm{e}^{X_t+k}\right)^+ &= \mathrm{BS}(k,\widetilde{\sigma}_{t,\tau}^{\widetilde{M}}(k)^2,\tau),\end{aligned}$$

with BS given in (1.0.2). Note that the lhs takes values within the set $(0,1)$ and so a unique solution always exists. Similar definitions hold for the other forward implied volatilities in models $M,\bar{M}$ and M^*. We now give the main result of the section.

Proposition 1.4.3. *Suppose that $(\mathrm{e}^{X_s})_{s\geq 0}$ is a $(\mathbb{P},\mathcal{F}_s)$-martingale. Then for all $k\in\mathbb{R}$,*

(i) $\sigma_{t,\tau}^{\widetilde{M}}(k)=\widetilde{\sigma}_{t,\tau}^{M}(k)$;

(ii) $\sigma_{t,\tau}^{M}(-k)=\widetilde{\sigma}_{t,\tau}^{\bar{M}}(k)=\sigma_{t,\tau}^{M^*}(k)$.

Proof. We first prove (i). We can write the value of our Type-II forward-start call option as

$$\begin{aligned}&\mathrm{BS}(k,\widetilde{\sigma}_{t,\tau}^{M}(k)^2,\tau)=\mathbb{E}\left(\mathrm{e}^{X_{t+\tau}}-\mathrm{e}^{k+X_t}\right)^+\\ &=\mathbb{E}\left(\mathrm{e}^{X_t}\left(\mathrm{e}^{X_{t+\tau}-X_t}-\mathrm{e}^k\right)^+\right)=\widetilde{\mathbb{E}}\left(\mathrm{e}^{X_{t+\tau}-X_t}-\mathrm{e}^k\right)^+=\mathrm{BS}(k,\sigma_{t,\tau}^{\widetilde{M}}(k)^2,\tau),\end{aligned} \tag{1.4.6}$$

and the result follows since $\mathrm{BS}(k,\cdot,\tau)$ (defined in (1.0.2)) is strictly increasing in the variance parameter for fixed τ and k. Now we prove (ii). Let $k\in\mathbb{R}$. Then

$$\mathbb{E}\left(\mathrm{e}^{-k}-\mathrm{e}^{X_{t+\tau}-X_t}\right)^+=\mathrm{e}^{-k}\bar{\mathbb{E}}\left(\mathrm{e}^{-X_{t+\tau}}-\mathrm{e}^{-X_t}\mathrm{e}^k\right)^+=\mathrm{e}^{-k}\mathbb{E}^*\left(\mathrm{e}^{-X_{t+\tau}+X_t}-\mathrm{e}^k\right)^+. \tag{1.4.7}$$

Using the definition of forward implied volatility and the BSM formula for a put and call option, the first equality implies

$$
\begin{aligned}
&\mathrm{e}^{-k}\mathcal{N}\left(\frac{-k}{\sigma^M_{t,\tau}(-k)\sqrt{\tau}}+\frac{\sigma^M_{t,\tau}(-k)\sqrt{\tau}}{2}\right)-\mathcal{N}\left(\frac{-k}{\sigma^M_{t,\tau}(-k)\sqrt{\tau}}-\frac{\sigma^M_{t,\tau}(-k)\sqrt{\tau}}{2}\right)\\
&=\mathrm{e}^{-k}\bar{\mathbb{E}}\left(\mathrm{e}^{-X_{t+\tau}}-\mathrm{e}^{-X_t}\mathrm{e}^k\right)^+\\
&=\mathrm{e}^{-k}\mathcal{N}\left(\frac{-k}{\widetilde{\sigma}^{\bar{M}}_{t,\tau}(k)\sqrt{\tau}}+\frac{\widetilde{\sigma}^{\bar{M}}_{t,\tau}(k)\sqrt{\tau}}{2}\right)-\mathcal{N}\left(\frac{-k}{\widetilde{\sigma}^{\bar{M}}_{t,\tau}(k)\sqrt{\tau}}-\frac{\widetilde{\sigma}^{\bar{M}}_{t,\tau}(k)\sqrt{\tau}}{2}\right).
\end{aligned}
$$

Since the lhs and rhs are strictly increasing in the volatility parameter, we have that $\sigma^M_{t,\tau}(-k)=\widetilde{\sigma}^{\bar{M}}_{t,\tau}(k)$. Using the second equality in (1.4.7), analogous arguments show that $\sigma^M_{t,\tau}(-k)=\widetilde{\sigma}^{\bar{M}}_{t,\tau}(k)=\sigma^{M^*}_{t,\tau}(k)$. □

In the case that $t=0$ (i.e. spot implied volatility asymptotics), then the Type-I and II smiles are the same and Proposition 1.4.3 reduces to $\sigma_\tau(-k)=\sigma^{\bar{M}}_\tau(k)$ for all $k\in\mathbb{R}$, which was shown for example in [116, Theorem 4.1]. The next result shows that the Type-I and II forward smiles are the same in uncorrelated stochastic volatility models:

Proposition 1.4.4. *If the instantaneous correlation is null in the general stochastic volatility model* (1.3.1) *and* $(\mathrm{e}^{X_s})_{s\geq 0}$ *is a* $(\mathbb{P},\mathcal{F}_s)$*-martingale then the Type-I forward smile is the same as the Type-II forward smile.*

Proof. In view of (1.4.6) it is enough to show that for all $k\in\mathbb{R}$,

$$
\widetilde{\mathbb{E}}\left(\mathrm{e}^{X_{t+\tau}-X_t}-\mathrm{e}^k\right)^+=\mathbb{E}\left(\mathrm{e}^{X_{t+\tau}-X_t}-\mathrm{e}^k\right)^+. \tag{1.4.8}
$$

When the correlation is null, then under the stopped-share-price measure $\widetilde{\mathbb{P}}$ the dynamics of (X,V) in (1.3.1) are given by

$$
\begin{aligned}
\mathrm{d}X_u &= \left(-\tfrac{1}{2}V_u+V_u\mathbb{1}_{u\leq t}\right)\mathrm{d}u+\sqrt{V_u}\mathrm{d}W_u, && X_0=0,\\
\mathrm{d}V_u &= h_0(V_u)\mathrm{d}u+h_1(V_u)\mathrm{d}B_u, && V_0=v>0,\\
\mathrm{d}\left\langle W,B\right\rangle_u &= 0.
\end{aligned}
$$

But, under both $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ we have that

$$
\exp\left(X_{t+\tau}-X_t\right)=\exp\left(-\frac{1}{2}\int_t^{t+\tau}V_s\mathrm{d}s+\int_t^{t+\tau}\sqrt{V_s}\mathrm{d}W_s\right),
$$

and when $\rho=0$ the dynamics of V are the same under both $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ and so (1.4.8) holds. □

In order to see some of these results in action, let us apply them to the Heston model (1.3.2). First we define the following constants:

$$
\widetilde{\kappa}:=\kappa-\rho\xi,\quad\text{and}\quad\widetilde{\theta}=\frac{\kappa\theta}{\kappa-\rho\xi}.
$$

We let $\mathcal{H}(v,\kappa,\theta,\xi,\rho)$ be the Heston model in (1.3.2), which corresponds to the model M above. Define a Heston model with modified parameters by $\mathcal{H}_1 = \mathcal{H}(v,\widetilde{\kappa},\widetilde{\theta},\xi,-\rho)$. Let $\mathcal{H}_2$ denote a Heston model that is given by $\mathcal{H}$ over the period $[0,t)$ and $\mathcal{H}_1$ over the period $[t,t+\tau]$. Finally, let $\mathcal{H}_3$ denote a Heston model that is given by $\mathcal{H}_1$ over the period $[0,t]$ and $\mathcal{H}$ over the period $(t,t+\tau]$. We now have the following corollary:

Corollary 1.4.5. *In Heston* (1.3.2), $\sigma^{\mathcal{H}_3}_{t,\tau}(k) = \widetilde{\sigma}^{\mathcal{H}}_{t,\tau}(k)$ *and* $\sigma^{\mathcal{H}}_{t,\tau}(-k) = \widetilde{\sigma}^{\mathcal{H}_1}_{t,\tau}(k) = \sigma^{\mathcal{H}_2}_{t,\tau}(k)$, *for all* $k \in \mathbb{R}$.

Remark 1.4.6. Note that this result applies even when $\kappa - \rho\xi < 0$, since even in this case $(\exp(-X_s))_{s\geq 0}$ is a $(\bar{\mathbb{P}},\mathcal{F}_s)$-martingale (see (1.4.9) below and [24, Proposition 5.1]).

Proof. In Heston $(\mathrm{e}^{X_s})_{s\geq 0}$ is a $(\mathbb{P},\mathcal{F}_s)$-martingale [5, Proposition 2.5]. Straightforward computations reveal that in Heston under $\bar{\mathbb{P}}$ we have that

$$\begin{aligned} \mathrm{d}(-X_u) &= -\frac{1}{2}V_u\mathrm{d}u + \sqrt{V_u}\mathrm{d}W_u, && X_0 = 0,\\ \mathrm{d}V_u &= \widetilde{\kappa}\left(\widetilde{\theta} - V_u\right)\mathrm{d}u + \xi\sqrt{V_u}\mathrm{d}B_u, && V_0 = v > 0,\\ \mathrm{d}\left\langle W,B\right\rangle_u &= -\rho\mathrm{d}u, \end{aligned} \tag{1.4.9}$$

which implies that $\mathcal{H}_1$ is the same as $\bar{M}$. The corollary then follows from Proposition 1.4.3 if we can show that it is sufficient to use $\mathcal{H}_2$ and $\mathcal{H}_3$ in place of M^* and $\widetilde{M}$ respectively. In Heston we have the following dynamics under $\widetilde{\mathbb{P}}$,

$$\begin{aligned} \mathrm{d}X_u &= \left(-\tfrac{1}{2}V_u + V_u\mathbb{1}_{u\leq t}\right)\mathrm{d}u + \sqrt{V_u}\mathrm{d}W_u, && X_0 = 0,\\ \mathrm{d}V_u &= \left(\kappa\theta - \kappa V_u + \rho\xi V_u\mathbb{1}_{u\leq t}\right)\mathrm{d}u + \xi\sqrt{V_u}\mathrm{d}B_u, && V_0 = v > 0,\\ \mathrm{d}\left\langle W,B\right\rangle_u &= \rho\mathrm{d}u, \end{aligned}$$

and the following dynamics under $\mathbb{P}^*$,

$$\begin{aligned} \mathrm{d}(-X_u) &= \left(-\tfrac{1}{2}V_u + V_u\mathbb{1}_{u\leq t}\right)\mathrm{d}u + \sqrt{V_u}\mathrm{d}W_u, && X_0 = 0,\\ \mathrm{d}V_u &= \left(\kappa\theta - \kappa V_u + \rho\xi V_u\mathbb{1}_{u\geq t}\right)\mathrm{d}u + \xi\sqrt{V_u}\mathrm{d}B_u, && V_0 = v > 0,\\ \mathrm{d}\left\langle W,B\right\rangle_u &= -\rho\mathrm{d}u. \end{aligned}$$

Due to (1.4.6) and (1.4.7) we see that the dynamics of X over $[0,t]$ are irrelevant for pricing and that a measure change only has an effect on pricing through a change of the variance dynamics. The proof is concluded by noting that $\mathcal{H}_2$ and $\mathcal{H}_3$ change the variance dynamics in the same way as M^* and $\widetilde{M}$ respectively. □

In Heston, one can directly read the Type-II forward smile from the Type-I forward smile (and visa-versa) after a transformation of parameters ($\mathcal{H}$ and $\mathcal{H}_1$) and provided that $\widetilde{\kappa} > 0$. Note that if $t \neq 0$ or $\rho \neq 0$ then $\mathcal{H}_2$ is not time-homogeneous. Therefore, if one has Type-I forward smile asymptotics for say the right wing ($k > 0$) in a time-homogeneous Heston model, then this is not sufficient to determine Type-I forward smile asymptotics for the left wing ($k < 0$). This is in contrast to the spot smile case ($t = 0$), where this feature is true for Heston spot smile asymptotics.

1.4.2 Representations of forward-start option prices

1.4.2.1 Inverse Fourier transform representation

A closed-form formula for forward-start options in the Black-Scholes-Merton model was originally derived in [136]. The pricing of forward-start options in the Heston model was first considered in [93], [112] and [118]. In [112] the authors derived a formula for forward-start options in Heston that involves two two-dimensional integrations and as such is not computationally efficient and will not be considered in this section. For comparison the approach suggested in [93] involves a single one-dimensional Fourier transform inversion.

The payoff of a Type-I forward-start option is a European option on the quantity $\mathrm{e}^{X_{t+\tau}-X_t}$. As first shown in [93] if one has access to the forward characteristic function of the price process then one can use the entire arsenal of efficient European option inverse Fourier transform methods [40, 115, 117] to price Type-I forward-start options. Define $A_{t,\tau,X} := \{u \in \mathbb{R} : \mathbb{E}\left[\mathrm{e}^{u(X_{t+\tau}-X_t)}\right] < \infty\}$ and set $\Lambda_{t,\tau,X} := \{z \in \mathbb{C} : -\Im(z) \in A_{t,\tau,X}\}$. We define the forward characteristic function $\phi_{t,\tau} : \mathbb{C} \to \mathbb{C}$ of $(X_t)_{t\geq 0}$ as

$$\phi_{t,\tau}(z) := \mathbb{E}\left[\mathrm{e}^{\mathtt{i}z(X_{t+\tau}-X_t)}\right] \quad \text{for all } z \in \Lambda_{t,\tau,X}. \tag{1.4.10}$$

In order to price our Type-I forward-start options we now use the European option inverse Fourier transform representation in [115, Theorem 5.1], but with the forward characteristic function defined in (1.4.10). Efficient pricing then boils down to finding the forward characteristic functions in various models. We record this result as a lemma, which will be used to numerically calculate the forward smile.

Lemma 1.4.7. *Assume that* $1 \in A^o_{t,\tau,X}$. *Then for any* $\alpha \in \mathbb{R}$ *such that* $\alpha + 1 \in A^o_{t,\tau,X}$ *we have the following inverse Fourier transform representation for a Type-I forward-start option:*

$$\begin{aligned}\mathbb{E}\left(\mathrm{e}^{X_{t+\tau}-X_t} - \mathrm{e}^k\right)^+ = \phi_{t,\tau}(-\mathtt{i})\mathbb{1}_{\{-1<\alpha<0\}} + \left(\phi_{t,\tau}(-\mathtt{i}) - \mathrm{e}^k\phi_{t,\tau}(0)\right)\mathbb{1}_{\{\alpha<-1\}} + \frac{\phi_{t,\tau}(-\mathtt{i})}{2}\mathbb{1}_{\{\alpha=0\}} \\ + \left(\phi_{t,\tau}(-\mathtt{i}) - \frac{\mathrm{e}^k}{2}\right)\mathbb{1}_{\{\alpha=-1\}} + \frac{1}{\pi}\int_{0-\mathtt{i}\alpha}^{+\infty-\mathtt{i}\alpha} \Re\left(\mathrm{e}^{\mathtt{i}zk}\frac{\phi_{t,\tau}(z-\mathtt{i})}{\mathtt{i}z - z^2}\right)\mathrm{d}z.\end{aligned}$$

Using (1.4.6) we know that a Type-II forward-start call option can be wriiten as a Type-I forward-start call option with the last expectation calculated under the stopped-share-price measure (1.4.4). The importance of this result is that we can now use Lemma 1.4.7 for pricing, but with the forward characteristic function calculated under the stopped-share-price measure. Using (1.4.7) our Type-II forward-start call option can also be written as a Type-I forward-start put option on the asset price process $(\mathrm{e}^{-X_t})_{t\geq 0}$ under the share-price measure (1.4.3). If one has access to the forward characteristic function in the share-price measure then one can use put-call parity and a slight modification of Lemma 1.4.7 (replace $X_{t+\tau} - X_t$ with $X_t - X_{t+\tau}$ and $\phi_{t,\tau}(u)$ with $\phi_{t,\tau}(u)$) for pricing. Similar comments apply to the forward measure in (1.4.3) using (1.4.7).

Efficient pricing of Type-I and II forward-start options is therefore reduced to finding the forward characteristic function in the risk-neutral, stopped-share-price, share-price or forward measure. In the Heston model $(\mathrm{e}^{X_s})_{s\geq 0}$ is a true martingale [5, Proposition 2.5] and all forward characteristic functions (or forward moment generating functions) are available in closed-form. The forward moment generating function under the risk-neutral measure was given in Lemma 1.3.1. We give the forward cumulant generating function under the stopped-share-price measure below, which we will need later:

Lemma 1.4.8. *Under the stopped-share-price measure* (1.4.4) *the forward Heston cgf reads*

$$\log \widetilde{\mathbb{E}}\left(\mathrm{e}^{uX_\tau^{(t)}}\right) = A(u,\tau) + \frac{B(u,\tau)}{1-2\widetilde{\beta}_t B(u,\tau)} v\mathrm{e}^{-\widetilde{\kappa}t} - \frac{2\kappa\theta}{\xi^2}\log\left(1-2\widetilde{\beta}_t B(u,\tau)\right),$$

for all u such that the rhs exists, where A and B are defined in (1.3.8)*, $\widetilde{\beta}_t := \frac{\xi^2}{4\widetilde{\kappa}}(1-\mathrm{e}^{-\widetilde{\kappa}t})$ and $\widetilde{\kappa} := \kappa - \xi\rho$.*

Proof. Under the stopped-share-price measure (1.4.4) the Heston dynamics are given by

$$\begin{aligned} \mathrm{d}X_u &= \left(-\tfrac{1}{2}V_u + V_u \mathbb{1}_{u\leq t}\right)\mathrm{d}u + \sqrt{V_u}\mathrm{d}W_u, && X_0 = 0,\\ \mathrm{d}V_u &= \left(\kappa\theta - \kappa V_u + \rho\xi V_u \mathbb{1}_{u\leq t}\right)\mathrm{d}u + \xi\sqrt{V_u}\mathrm{d}B_u, && V_0 = v > 0,\\ \mathrm{d}\left\langle W,B\right\rangle_u &= \rho \mathrm{d}u. \end{aligned}$$

Using the tower property for expectations, it is now straightforward to compute

$$\widetilde{\mathbb{E}}\left(\mathrm{e}^{u(X_{t+\tau}-X_t)}\right) = \widetilde{\mathbb{E}}\left(\widetilde{\mathbb{E}}\left(\mathrm{e}^{u(X_{t+\tau}-X_t)}|\mathcal{F}_t\right)\right) = \widetilde{\mathbb{E}}\left(\mathrm{e}^{A(u,\tau)+B(u,\tau)V_t}\right) = \mathrm{e}^{A(u,\tau)}\widetilde{\Lambda}_t^V(B(u,\tau)),$$

where $\widetilde{\Lambda}_t^V(u) = \exp\left(\frac{uv\exp(-\widetilde{\kappa}t)}{1-2\widetilde{\beta}_t u}\right)(1-2\widetilde{\beta}_t u)^{-2\kappa\theta/\xi^2}$, for all $u < 1/(2\widetilde{\beta}_t)$. □

1.4.2.2 Mixing formula in stochastic volatility models

By performing a Cholesky decomposition of the covariance matrix in our general stochastic volatility model (1.3.1), we can write the forward increment as

$$X_{t+\tau} - X_t = U_\tau^{(t)} - \frac{1-\rho^2}{2}\int_t^{t+\tau} V_s \mathrm{d}s + \sqrt{1-\rho^2}\int_t^{t+\tau}\sqrt{V_s}\mathrm{d}Z_s,$$

where

$$U_\tau^{(t)} := -\frac{\rho^2}{2}\int_t^{t+\tau} V_s\mathrm{d}s + \rho\int_t^{t+\tau}\sqrt{V_s}\mathrm{d}B_s,$$

and $(B_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ are two independent Brownian motions. By conditioning on the filtration generated by B up to time $t+\tau$ we find that (see [66, page 79] or [53, page 28])

$$\begin{aligned} \mathbb{E}(\mathrm{e}^{X_{t+\tau}-X_t} - \mathrm{e}^k)^+ &= \mathbb{E}\left(\mathbb{E}\left(\left(\mathrm{e}^{X_{t+\tau}-X_t}-\mathrm{e}^k\right)^+|\mathcal{F}_{t+\tau}^B\right)\right)\\ &= \mathbb{E}\left(\mathrm{e}^{U_\tau^{(t)}}\mathrm{BS}\left(\frac{k}{U_\tau^{(t)}}, \tau^{-1}(1-\rho^2)\int_t^{t+\tau}V_s\mathrm{d}s, \tau\right)\right), \end{aligned}$$

with BS defined in (1.0.2). The key point here is that now only one Brownian motion path has to be generated in order to price forward-start options. Forward-start options depend on the joint distribution of $\left(U_\tau^{(t)}, \int_t^{t+\tau} V_s \mathrm{d}s\right)$; for correlations close to zero, the forward-start option is a non-linear payoff of the forward variance $\int_t^{t+\tau} V_s \mathrm{d}s$.

1.4.2.3 Non-stationary representation in stochastic volatility models

In our general stochastic volatility model (1.3.1), the forward-start option price is given by

$$\mathbb{E}(\mathrm{e}^{X_{t+\tau}-X_t} - \mathrm{e}^k)^+ = \mathbb{E}(g_{k,\tau}(V_t)),$$

where $g_{k,\tau} \in \mathcal{C}^2(\mathbb{R}_+)$ is given by $g_{k,\tau}(x) := \mathbb{E}((\mathrm{e}^{X_{t+\tau}-X_t} - \mathrm{e}^k)^+/V_t = x)$. Using the replication formula in [39, Equation 2] we then have the representation (with $\bar{V}_t := \mathbb{E}(V_t) - v$)

$$\begin{aligned}
&\mathbb{E}\left(\mathrm{e}^{X_{t+\tau}-X_t} - \mathrm{e}^k\right)^+ \\
&= g_{k,\tau}(v) + g'_{k,\tau}(v)\bar{V}_t + \int_0^v g''_{k,\tau}(q)\mathbb{E}(q - V_t)^+\mathrm{d}q + \int_v^\infty g''_{k,\tau}(q)\mathbb{E}(V_t - q)^+\mathrm{d}q \\
&= \mathbb{E}\left(\mathrm{e}^{X_\tau} - \mathrm{e}^k\right)^+ + g'_{k,\tau}(v)\bar{V}_t + \int_0^v g''_{k,\tau}(q)\mathbb{E}(q - V_t)^+\mathrm{d}q + \int_v^\infty g''_{k,\tau}(q)\mathbb{E}(V_t - q)^+\mathrm{d}q.
\end{aligned}$$

Note that $g'_{k,\tau}(v)$ is simply the Vega for a standard European call option in the model (1.3.1) with maturity τ and log-strike k and one would expect this term to be positive. In the Heston model (1.3.2) for example, we have that $\bar{V}_t = (1 - \mathrm{e}^{-\kappa t})(\theta - v)$ and the sign of $\bar{V}$ depends on the relative values of the long-term mean reversion level θ and the initial variance v. The two integrals and $g'_{k,\tau}(v)\bar{V}_t$ account completely for the non-stationarity (t-dependence) of the forward smile over the spot smile. The integrals are weighted calls and puts on the instantaneous variance at time t. The weights represent the volatility convexity of a standard τ-maturity option in the general stochastic volatility model (1.3.1) with log-strike k and evaluated with an initial variance of q. Intuitively one would expect that (just as in the BSM model), the volatility convexity is positive except for a small region around at-the-money ($k = 0$) where it is negative. Therefore (at least intuitively for now) the out-of-the-money forward smile is larger than the corresponding out-of-the-money spot smile as long as $\bar{V}$ is not sufficiently negative. Similarly, the at-the-money forward volatility is lower than the corresponding at-the-money spot volatility as long as $\bar{V}$ is not sufficiently positive.

1.4.2.4 Random initial variance representation in stochastic volatility models

Consider the forward price process $X_\tau^{(t)} := X_{t+\tau} - X_\tau$ in the general stochastic volatility model (1.3.1) and fix $t > 0$. Then $X_0^{(t)} = 0$ and the only statistic relevant from the dynamics (X, V) in (1.3.1) over $[0, t]$ for determining $X_\cdot^{(t)}$ is the value of the variance process at the forward-start date t. We record this result in the following lemma:

Lemma 1.4.9. *In the model* (1.3.1) *the forward price process* $X^{(t)}$ *solves the following system of SDEs:*

$$\begin{aligned} \mathrm{d}X_\tau^{(t)} &= -\frac{1}{2}Y_\tau^{(t)}\mathrm{d}\tau + \sqrt{Y_\tau^{(t)}}\mathrm{d}W_\tau, & X_0^{(t)} &= 0, \\ \mathrm{d}Y_\tau^{(t)} &= h_0\left(Y_\tau^{(t)}\right)\mathrm{d}t + h_1\left(Y_\tau^{(t)}\right)\mathrm{d}B_\tau, & Y_0^{(t)} &\sim Law(V_t), \\ \mathrm{d}\left\langle W,B\right\rangle_\tau &= \rho\mathrm{d}\tau, \end{aligned} \tag{1.4.11}$$

where $Y_0^{(t)}$ *is independent to the Brownian motions* $(W_\tau)_{\tau\geq 0}$ *and* $(B_\tau)_{\tau\geq 0}$.

This lemma makes it clear that forward-start options in stochastic volatility models are European options on a stock price with similar dynamics to (1.3.1), but the initial variance is a random variable sampled from the variance distribution at the forward-start date. The SDE (1.4.11) is an example of a diffusion in a random environment. In Chapter 5 we will propose that models of the form (1.4.11) are used to directly model the stock price. In this framework the distribution of the initial variance is chosen to match observed steep small-maturity implied volatility smiles.

1.5 Small and large forward-start dates

In the next two subsections we will study forward smile asymptotics for fixed maturity $\tau > 0$ as the forward-start date t tends to zero or infinity. Results are not presented in great detail: the goal of the section is to develop intuition on the forward smile and many of the properties discovered will be rigorously proven in subsequent chapters.

We recall that $C(\tau, k)$ is a market or model price of a call option with maturity τ and log-strike k. In this section we will let $C(\tau, k)$ be the value in the Heston model (1.3.2) unless otherwise stated. We will use the notation $C(\tau, k; v)$ and $\sigma_\tau(k, v)$ to make it explicit that Heston call option prices and spot implied volatilities depend on the initial variance v. For ease of computations we define for fixed $k \in \mathbb{R}$ and $\tau > 0$, the function

$$\varphi_{\mathrm{BS}}(\Sigma) := \mathrm{BS}(k, \Sigma^2, \tau), \tag{1.5.1}$$

and by definition we then have that $C(t, \tau, k) = \varphi_{\mathrm{BS}}(\sigma_{t,\tau}(k))$ and $C(\tau, k; v) = \varphi_{\mathrm{BS}}(\sigma_\tau(k, v))$.

1.5.1 Small forward-start dates

In Lemma 1.5.1 below we first prove a tail estimate and then move onto the main result of the section, an asymptotic expansion for the Heston forward smile for small forward-start dates.

Lemma 1.5.1. *For fixed* $\tau, L > 0$ *there exists* $\delta > 0$ *such that the following tail estimate holds in Heston* (1.3.2) *as* $t \downarrow 0$*:*

$$\int_L^\infty C(\tau, k; y)\zeta_{\mathrm{H}}(y)\mathrm{d}y = \mathcal{O}\left(\exp\left(-\frac{\delta}{t}\right)\right),$$

with ζ_{H} *defined in* (1.3.3).

Proof. Combining Lemma 5.3.3 with ζ_{H} defined in (1.3.3) we see that

$$\zeta_{\mathrm{H}}(y) \leq \alpha_0 \beta_t^{-\mu} y^{\mu-1} \exp\left(-\frac{1}{2\beta_t}\left(\sqrt{y}-\sqrt{v}\mathrm{e}^{-\kappa t/2}\right)^2\right) \leq \alpha_0 \beta_t^{-\mu} y^{\mu-1} \exp\left(-\frac{y}{2\beta_t}\right),$$

where $\alpha_0 > 0$ is a constant independent of y and t. Also since $C(\tau, k; y) \leq 1$ we obtain

$$\int_L^\infty C(\tau, k; y)\zeta_{\mathrm{H}}(y)\mathrm{d}y \leq \int_L^\infty \zeta_{\mathrm{H}}(y)\mathrm{d}y \leq \alpha_0 \beta_t^{-\mu} \int_L^\infty y^{\mu-1} \exp\left(-\frac{y}{2\beta_t}\right)\mathrm{d}y.$$

The integral in the last inequality can be solved analytically to obtain

$$\int_L^\infty C(\tau, k; y)\zeta_{\mathrm{H}}(y)\mathrm{d}y \leq \alpha_0 2^\mu \Gamma\left(\mu, \frac{L}{2\beta_t}\right),$$

where $\Gamma(a; x) \equiv \int_x^\infty y^{a-1}\mathrm{e}^{-y}\mathrm{d}y$ is the incomplete Gamma function. Since β_t (in (1.3.4)) tends to zero as t tends to zero the asymptotic expansion [1, page 263],

$$\Gamma\left(\mu, \frac{L}{2\beta_t}\right) = \exp\left(-\frac{L}{2\beta_t}\right)\left(\frac{L}{\beta_t}\right)^\mu \left(\frac{2^{1-\mu}\beta_t}{L} + \mathcal{O}(\beta_t^2)\right),$$

holds as t tends to zero and the result follows after using the expansion $\beta_t = \xi^2 t/4 + \mathcal{O}(t^2)$ for small t. □

The following proposition gives the asymptotics of the forward smile in Heston as the forward-start date tends to zero. Since the asymptotic depends on the Heston spot implied volatility σ_τ, results on asymptotics of the spot implied volatility can be recycled here to obtain forward smile asymptotics.

Proposition 1.5.2. *For fixed $\tau > 0$, the following asymptotic holds in Heston* (1.3.2) *as $t \downarrow 0$ (φ_{BS} defined in* (1.5.1)*):*

$$\begin{aligned}\sigma_{t,\tau}(k) &= \sigma_\tau(k, v) + \left(\partial_v \sigma_\tau(k, v)\left((v-\theta)\kappa + \frac{1}{2}\frac{\partial_{\Sigma\Sigma}\varphi_{\mathrm{BS}}(\sigma_\tau(k, v))}{\partial_\Sigma \varphi_{\mathrm{BS}}(\sigma_\tau(k, v))}\partial_v \sigma_\tau(k, v) v \xi^2\right)\right. \\ &\left. + \frac{1}{2}\partial_{vv}\sigma_\tau(k, v) v \xi^2\right) t + \mathcal{O}(t^{3/2}).\end{aligned} \tag{1.5.2}$$

Proof. The Heston forward-start call option price is given by

$$C(t, \tau, k) = \mathbb{E}\left\{\mathbb{E}\left(\mathrm{e}^{X_{t+\tau}-X_t} - \mathrm{e}^k\right)^+ / V_t\right)\right\} = \int_0^\infty C(\tau, k; y)\zeta_{\mathrm{H}}(y)\mathrm{d}y, \tag{1.5.3}$$

where ζ_{H} (defined in (1.3.3)) is the density of the instantaneous variance at time t. Set

$$C_{\mathrm{Low}}(t, \tau, k) := \int_0^L C(\tau, k; y)\zeta_{\mathrm{H}}(y)\mathrm{d}y. \tag{1.5.4}$$

We break the integral in (1.5.3) into a compact part and a tail part as follows,

$$C(t, \tau, k) = C_{\mathrm{Low}}(t, \tau, k) + \int_L^\infty C(\tau, k; y)\zeta_{\mathrm{H}}(y)\mathrm{d}y = C_{\mathrm{Low}}(t, \tau, k) + \mathcal{O}\left(\exp\left(-\frac{\delta}{t}\right)\right), \tag{1.5.5}$$

for some $\delta > 0$ as t tends to zero. The final line follows from Lemma 1.5.1 and we set $L > v$. As z tends to infinity we have the following asymptotic expansion for the modified Bessel function of the first kind [1, Section 9.7.1]:

$$\mathrm{I}_\nu(z) = \frac{\mathrm{e}^z}{\sqrt{2\pi z}}\left(1 - \frac{1}{2z}(\nu^2 - \frac{1}{4}) + \mathcal{O}\left(\frac{1}{z^2}\right)\right). \tag{1.5.6}$$

Since β_t (defined in (1.3.4)) tends to zero as t tends to zero, straightforward computations yield the following asymptotic for the density as t tends to zero (μ defined in (1.3.4)):

$$\zeta_{\mathrm{H}}(y) = \exp\left(-\frac{2\left(\sqrt{y}-\sqrt{v}\right)^2}{\xi^2 t} + \frac{\kappa(v-y)}{\xi^2}\right)\frac{y^{\frac{2\mu-3}{4}}v^{\frac{1-2\mu}{4}}}{\xi\sqrt{2\pi t}}\left(1+\left(\frac{\kappa\mu}{2}+\frac{\xi^2(1-2\mu)(2\mu-3)}{32\sqrt{yv}}-\frac{\kappa^2\left(y+\sqrt{yv}+v\right)}{6\xi^2}\right)t+\mathcal{O}(t^2)\right).$$

Using (1.5.4) we have the following expansion

$$C_{\mathrm{Low}}(t,\tau,k) = v^{1/4-\mu/2}\xi^{-1}(2\pi t)^{-1/2}\Big(I_0(t)+I_1(t)t+\mathcal{O}\left(t^2\right)\Big),$$

as t tends to zero and where we set

$$I_0(t) := \int_0^L C(\tau,k;y)y^{\frac{2\mu-3}{4}}\exp\left(-\frac{2\left(\sqrt{y}-\sqrt{v}\right)^2}{\xi^2 t}+\frac{\kappa(v-y)}{\xi^2}\right)\mathrm{d}y$$

and

$$I_1(t) := \int_0^L C(\tau,k;y)y^{\frac{2\mu-3}{4}}\exp\left(-\frac{2\left(\sqrt{y}-\sqrt{v}\right)^2}{\xi^2 t}+\frac{\kappa(v-y)}{\xi^2}\right)\left(\frac{\kappa\mu}{2}+\frac{\xi^2(1-2\mu)(2\mu-3)}{32\sqrt{yv}}-\frac{\kappa^2\left(y+\sqrt{yv}+v\right)}{6\xi^2}\right)\mathrm{d}y.$$

A tedious but straightforward application of the Laplace method (Theorem 1.2.6) for I_0 and I_1 out to order $\mathcal{O}(t)$ then yields the following simple expansion as t tends to zero:

$$C_{\mathrm{Low}}(t,\tau,k) = C(\tau,k;v)+\left(\partial_v C(\tau,k;v)(v-\theta)\kappa+\frac{1}{2}\partial_{vv}C(\tau,k;v)v\xi^2\right)t+\mathcal{O}(t^{3/2}). \tag{1.5.7}$$

Combining this with (1.5.5) yields

$$C(t,\tau,k) = C(\tau,k;v)+\left(\partial_v C(\tau,k;v)(v-\theta)\kappa+\frac{1}{2}\partial_{vv}C(\tau,k;v)v\xi^2\right)t+\mathcal{O}(t^{3/2}). \tag{1.5.8}$$

Using the fact that the Black-Scholes vega for a call option is strictly positive implies that (φ_{BS} defined in (1.5.1))

$$\begin{aligned}\sigma_{t,\tau}(k) &= \varphi_{\mathrm{BS}}^{-1}\left(C(\tau,k;v)+\left(\partial_v C(\tau,k;v)(v-\theta)\kappa+\frac{1}{2}\partial_{vv}C(\tau,k;v)v\xi^2\right)t+\mathcal{O}(t^{3/2})\right)\\ &= \sigma_\tau(k,v)+\partial_\Sigma\varphi_{\mathrm{BS}}^{-1}\left(C(\tau,k;v)\right)\left(\partial_v C(\tau,k;v)(v-\theta)\kappa+\frac{1}{2}\partial_{vv}C(\tau,k;v)v\xi^2\right)t+\mathcal{O}(t^{3/2}),\end{aligned}$$

as t tends to zero, where in the second line we used the fact that by definition $\varphi_{\mathrm{BS}}^{-1}\left(C(\tau,k;v)\right) = \sigma_\tau(k,v)$. The asymptotic (1.5.2) then follows after using the following manipulations

$$\begin{aligned}\partial_\Sigma\varphi_{\mathrm{BS}}^{-1}\left(C(\tau,k;v)\right) &= \frac{1}{\partial_\Sigma\varphi_{\mathrm{BS}}\left(\varphi_{\mathrm{BS}}^{-1}\left\{C(\tau,k;v)\right\}\right)} = \frac{1}{\partial_\Sigma\varphi_{\mathrm{BS}}\left(\sigma_\tau(k,v)\right)},\\ \partial_v C(\tau,k;v) &= \partial_\Sigma\varphi_{\mathrm{BS}}\left(\sigma_\tau(k,v)\right)\partial_v\sigma_\tau(k,v),\\ \partial_{vv}C(\tau,k;v) &= \partial_{\Sigma\Sigma}\varphi_{\mathrm{BS}}\left(\sigma_\tau(k,v)\right)\left(\partial_v\sigma_\tau(k,v)\right)^2+\partial_\Sigma\varphi_{\mathrm{BS}}\left(\sigma_\tau(k,v)\right)\partial_{vv}\sigma_\tau(k,v),\end{aligned}$$

and the definition of φ_{BS}. □

The asymptotic (1.5.8) is remarkably simple (one would not guess this from using the Laplace method) and hints at a deeper structure. In the rest of this section we let $C(t,\tau,k)$ and $C(\tau,k;v)$ be the forward-start option and call option price in the general stochastic volatility model (1.3.1). We have defined $C(\tau,k;v)$ to be the price of a re-normalised (asset price is always one) call option with fixed maturity τ and initial instantaneous variance v. Therefore an application of Itô's lemma yields

$$C(\tau,k;V_t) = C(\tau,k;v) + \int_0^t \partial_v C(\tau,k;V_s)\mathrm{d}V_s + \frac{1}{2}\int_0^t \partial_{vv}C(\tau,k;V_s)\mathrm{d}\left\langle V,V\right\rangle_s,$$

and taking expectations we find that

$$\mathbb{E}[C(\tau,k;V_t)] = C(\tau,k;v) + \mathbb{E}\left[\int_0^t \partial_v C(\tau,k;V_s)h_0(V_s)\mathrm{d}s\right] + \frac{1}{2}\mathbb{E}\left[\int_0^t \partial_{vv}C(\tau,k;V_s)\mathrm{d}\left\langle V,V\right\rangle_s\right].$$

The lhs is just our forward-start option and we get the expression

$$C(t,\tau,k) = C(\tau,k;v) + \mathbb{E}\left[\int_0^t \partial_v C(\tau,k;V_s)h_0(V_s)\mathrm{d}s\right] + \frac{1}{2}\mathbb{E}\left[\int_0^t \partial_{vv}C(\tau,k;V_s)h_1^2(V_s)\mathrm{d}s\right].$$

Now doing an Itô-Taylor expansion we approximate the integrands by

$$\begin{aligned}\int_0^t \partial_v C(\tau,k;V_s)h_0(V_s)\mathrm{d}s &= \partial_v C(\tau,k;v)h_0(v)t + \mathcal{R}_1,\\ \int_0^t \partial_{vv} C(\tau,k;V_s)h_1(V_s)^2\mathrm{d}s &= \partial_{vv} C(\tau,k;v)h_1(v)^2 t + \mathcal{R}_2.\end{aligned}$$

We make the assumption here that h_0 and h_1 are functions such that the remainders have the property

$$\mathbb{E}(\mathcal{R}_1) = \mathbb{E}(\mathcal{R}_2) = \mathcal{O}(t^2). \tag{1.5.9}$$

This implies that

$$\begin{aligned}\mathbb{E}\left[\int_0^t \partial_v C(\tau,k;V_s)h_0(V_s)\mathrm{d}s\right] &= \partial_v C(\tau,k;v))h_0(v)t + \mathcal{O}(t^2),\\ \mathbb{E}\left[\int_0^t \partial_{vv} C(\tau,k;V_s)h_1(V_s)^2\mathrm{d}s\right] &= \partial_{vv} C(\tau,k;v)h_1(v)^2 t + \mathcal{O}(t^2),\end{aligned}$$

which agrees exactly with (1.5.8) in the Heston case. We leave the precise study of property (1.5.9) for future research and state the following proposition:

Proposition 1.5.3. *If the model* (1.3.1) *satisfies* (1.5.9) *then for fixed* $\tau > 0$, *the following asymptotic holds as* $t \downarrow 0$:

$$\begin{aligned}\sigma_{t,\tau}(k) &= \sigma_\tau(k,v) + \left(\partial_v\sigma_\tau(k,v)\left(h_0(v) + \frac{1}{2}\frac{\partial_{\Sigma\Sigma}\varphi_{\mathrm{BS}}(\sigma_\tau(k,v))}{\partial_\Sigma\varphi_{\mathrm{BS}}(\sigma_\tau(k,v))}\partial_v\sigma_\tau(k,v)h_1(v)^2\right)\right.\\ &\quad\left.+\frac{1}{2}\partial_{vv}\sigma_\tau(k,v)v\xi^2\right)t + \mathcal{O}(t^{3/2}),\end{aligned} \tag{1.5.10}$$

with φ_{BS} *defined in* (1.5.1) *and* $\sigma_\tau(k,v)$ *is the implied volatility in the model* (1.3.1).

We can use (1.5.2) and (1.5.10) to gain intuition on the forward smile. If the model (1.3.1) has a small-maturity limit for the spot implied volatility,

$$\lim_{\tau \to 0} \sigma_\tau(k, v) \equiv \sigma_0(k),$$

with $0 < \sigma_0(k) < \infty$ for all $k \in \mathbb{R}$, then straightforward computations yield

$$\frac{\partial_{\Sigma\Sigma} \varphi_{\mathrm{BS}}(\sigma_\tau(k, v))}{\partial_\Sigma \varphi_{\mathrm{BS}}(\sigma_\tau(k, v))} \sim \frac{k^2}{\sigma_0(k)^3 \tau},$$

as τ tends to zero. This implies (intuitively) that the forward smile explodes (becomes more convex for out-of-the-money options) as the maturity tends to zero. In Chapter 3 we will rigorously prove this result in Heston and state a conjecture for general stochastic volatility models in Chapter 5.

1.5.2 Large forward-start dates

In view of (1.5.3) we want to understand what happens to our Heston density ζ_{H} (1.3.3) as t tends to infinity. As z tends to zero we have the following asymptotic expansion for the modified Bessel function of the first kind [1, Section 9.6.10]:

$$\mathrm{I}_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu + 1)} \left(1 + \frac{z^2}{2(2\nu + 2)} + \mathcal{O}(z^4)\right). \tag{1.5.11}$$

This implies that the density has the following asymptotics as t tends to infinity (μ defined in (1.3.4)),

$$\zeta_{\mathrm{H}}(y) = \frac{\mathrm{e}^{-y/(2\beta_\infty)} y^{\mu-1}}{(2\beta_\infty)^\mu \Gamma(\mu)} (1 + \mathcal{O}(\mathrm{e}^{-\kappa t})),$$

where $\beta_\infty := \xi^2/(4\kappa)$. This is obviously the stationary distribution of the variance process, a Gamma distribution with shape parameter μ and scale parameter $2\beta_\infty$. Using (1.5.3) we see that forward-start options are given by

$$C(t, \tau, k) = \int_0^\infty C(\tau, k; y) \frac{\mathrm{e}^{-y/(2\beta_\infty)} y^{\mu-1}}{(2\beta_\infty)^\mu \Gamma(\mu)} \mathrm{d}y + \mathcal{O}(\mathrm{e}^{-\kappa t}),$$

as t tends to infinity. The interpretation of the first term is that it is the value of a Heston call option where the instantaneous variance is first sampled from the stationary distribution of the variance process. This result (for affine stochastic volatility models) was obtained by Keller-Ressel [109].

1.6 Structure of thesis

We conclude the introduction with a brief overview of the structure of the thesis. In Chapter 2 we derive a general closed-form expansion formula for forward-start options and the forward implied volatility smile in a large class of models including the Heston and Schöbel-Zhu stochastic volatility models and time-changed exponential Lévy models. This general result includes large-maturity asymptotics and so-called 'diagonal' small-maturity asymptotics, i.e. asymptotics for small forward start dates and small-maturities.

In Chapter 3 we investigate the asymptotics of forward-start options and the forward implied volatility smile in the Heston model as the maturity approaches zero. We prove that the forward smile for out-of-the-money options explodes and compute a closed-form high-order expansion detailing the rate of the explosion. In the at-the-money case a separate model-independent analysis shows that the small-maturity limit is well defined for any Itô diffusion. Chapter 4 provides a full characterisation of the large-maturity forward implied volatility smile in the Heston model. Although the leading order decay is provided by a fairly classical large deviations behaviour, the algebraic expansion providing the higher-order terms highly depends on the parameters, and different powers of the maturity come into play.

Classical (Itô diffusions) stochastic volatility models are not able to capture the steepness of small-maturity implied volatility smiles. Jumps, in particular exponential Lévy and affine models, which exhibit small-maturity exploding smiles, have historically been proposed to remedy this (see [145] for an overview). A recent breakthrough was made by Gatheral, Jaisson and Rosenbaum [74], who proposed to replace the Brownian driver of the instantaneous volatility by a short-memory fractional Brownian motion, which is able to capture the short-maturity steepness while preserving path continuity. In Chapter 5 we suggest a different route, randomising the Black-Scholes variance by a CEV-generated distribution, which allows us to modulate the rate of explosion (through the CEV exponent) of the implied volatility for small maturities. The range of rates includes behaviours similar to exponential Lévy models and fractional stochastic volatility models. As a by-product, we make a conjecture on the small-maturity forward smile asymptotics of stochastic volatility models, in exact agreement with the results in Chapter 3 for the Heston model.

Throughout the thesis we will identify a number of cases of degenerate large deviations behaviour. We will discover that these cases unlock fundamental dynamical properties of the model and we will relate them back to important empirical observations and conjectures made by practitioners.

Chapter 2

A general asymptotic formula for the forward smile

2.1 Introduction

In this chapter we consider a continuous-time stochastic process $(Y_\varepsilon)_{\varepsilon>0}$[1] and prove—under some assumptions on its characteristic function—an expansion for European option prices on $\exp(Y_\varepsilon)$ of the form

$$\mathbb{E}\left(\mathrm{e}^{Y_\varepsilon f(\varepsilon)} - \mathrm{e}^{kf(\varepsilon)}\right)^+ = \mathcal{I}(k,c,\varepsilon) + \alpha_0(k,c)\mathrm{e}^{-\Lambda^*(k)/\varepsilon + kf(\varepsilon)}\left(c\sqrt{\varepsilon}\mathbb{1}_{\{c>0\}} + \varepsilon^{3/2} f(\varepsilon)\mathbb{1}_{\{c=0\}}\right)\left[1 + \alpha_1(k,c)\varepsilon + \mathcal{O}\left(\varepsilon^2\right)\right],$$

as ε tends to zero, for some (explicit) functions α_0, α_1 and a residue term $\mathcal{I}$ (Theorem 2.2.4 and Corollary 2.2.5). Here f is a positive, continuous function satisfying $\varepsilon f(\varepsilon) = c + \mathcal{O}(\varepsilon)$ for some $c \geq 0$ as ε tends to zero, and Λ^* is a large deviations rate function. Setting $Y_\varepsilon \equiv X_{\varepsilon\tau}^{(\varepsilon t)}$ and $f(\varepsilon) \equiv 1$ or $Y_\varepsilon \equiv \varepsilon X_{\tau/\varepsilon}^{(t)}$ and $f(\varepsilon) \equiv \varepsilon^{-1}$ yields 'diagonal' small-maturity (Corollary 2.2.6) and large-maturity (Corollary 2.2.9) expansions of forward-start option prices ($X_\tau^{(t)}$ defined in (1.0.3)). These results also apply when the forward-start date is null ($t = 0$), and we then recover—and improve—the asymptotics in [60, 62, 63, 64, 65, 95]. The diagonal small-maturity re-scaling is necessary in order to obtain non-degenerate small-maturity asymptotics.

We also translate these results into closed-form asymptotic expansions for the forward implied volatility smile (Type-I and Type-II). In Section 2.3, we provide explicit examples for the Heston, Schöbel-Zhu and time-changed exponential Lévy processes. Section 2.4 provides numerical evidence supporting the practical relevance of these results and we leave the proofs of the main results to Section 2.5.

[1] We remark that we do not assume that $(Y_\varepsilon)_{\varepsilon>0}$ is continuous nor that $\exp(Y_\varepsilon)$ is necessarily a martingale.

2.2 General Results

This section gathers the main notations of the chapter as well as the general results. The main result is Theorem 2.2.4, which provides an asymptotic expansion—up to virtually any arbitrary order—of option prices on a given process (Y_ε), as ε tends to zero. This general formulation allows us, by suitable scaling, to obtain both small-time (Section 2.2.2.1) and large-time (Section 2.2.2.2) expansions.

2.2.1 Notations and main theorem

2.2.1.1 Notations and preliminary results

Let (Y_ε) be a stochastic process with re-normalised cumulant generating function (cgf)

$$\Lambda_\varepsilon(u) := \varepsilon \log \mathbb{E}\left[\exp\left(\frac{uY_\varepsilon}{\varepsilon}\right)\right], \qquad \text{for all } u \in \mathcal{D}_\varepsilon := \{u \in \mathbb{R} : |\Lambda_\varepsilon(u)| < \infty\}. \tag{2.2.1}$$

We further define $\mathcal{D}_0 := \lim_{\varepsilon\downarrow 0} \mathcal{D}_\varepsilon$ and now introduce the main assumptions of the chapter.

Assumption 2.2.1.

(i) **Expansion property:** For each $u \in \mathcal{D}_0^o$ the following Taylor expansion holds as $\varepsilon \downarrow 0$ [2]:

$$\Lambda_\varepsilon(u) = \sum_{i=0}^{2} \Lambda_i(u)\varepsilon^i + \mathcal{O}(\varepsilon^3); \tag{2.2.2}$$

(ii) **Differentiability:** There exists $\varepsilon_0 > 0$ such that the map $(\varepsilon, u) \mapsto \Lambda_\varepsilon(u)$ is of class $\mathcal{C}^\infty$ on $(0, \varepsilon_0) \times \mathcal{D}_0^o$;

(iii) **Non-degenerate interior:** $0 \in \mathcal{D}_0^o$;

(iv) **Essential smoothness:** Λ_0 is strictly convex and essentially smooth (Def. 1.2.2) on $\mathcal{D}_0^o$;

(v) **Tail error control:** For any fixed $p_r \in \mathcal{D}_0^o \backslash \{0\}$,

(a) $\Re\left(\Lambda_\varepsilon\left(\mathtt{i}p_i + p_r\right)\right) = \Re\left(\Lambda_0\left(\mathtt{i}p_i + p_r\right)\right) + \mathcal{O}(\varepsilon)$, for any $p_i \in \mathbb{R}$;

(b) the function $L : \mathbb{R} \ni p_i \mapsto \Re\left(\Lambda_0\left(\mathtt{i}p_i + p_r\right)\right)$ has a unique maximum at zero, is bounded away from $L(0)$ as $|p_i|$ tends to infinity and is of class $\mathcal{C}^3(\mathbb{R})$;

(c) there exist $\varepsilon_1, p_i^* > 0$ such that for all $|p_i| \geq p_i^*$ and $\varepsilon \leq \varepsilon_1$ there exists M (independent of p_i and ε) such that $\Re\left[\Lambda_\varepsilon(\mathtt{i}p_i + p_r) - \Lambda_0(\mathtt{i}p_i + p_r)\right] \leq M\varepsilon$.

Assumption 2.2.1(i) implies that the functions $\lim_{\varepsilon\downarrow 0} \partial_\varepsilon^i \Lambda_\varepsilon(u)$ exist on $\mathcal{D}_0^o$ for $i = 0, 1, 2$. Assumption 2.2.1(ii) could be relaxed to $\mathcal{C}^6((0, \varepsilon_0) \times \mathcal{D}_0^o)$, but this hardly makes any difference in practice and does, however, render some formulations awkward. If the expansion (2.2.2) holds

[2] The abuse of notation between Λ_ε and Λ_i should not yield any confusion.

up to some higher order $n \geq 3$, one can in principle show that both forward-start option prices and the forward implied volatility expansions below hold to order n as well. However expressions for the coefficients of higher order are extremely cumbersome and scarcely useful in practice. Assumption 2.2.1(v) is a technical condition (readily satisfied by practical models) required to show that the dependence of option prices on the tails of the characteristic function of the asset price is exponentially small (see Lemma 2.5.3 and Appendix A for further details). We do not require this condition to be satisfied at $p_r = 0$ since this corresponds to an option strike at which our main result does not hold anyway ($k = \Lambda_{0,1}(0)$ in Theorem 2.2.4 below). We note that this assumption is not required if one is only interested in the leading-order behaviour of option prices and forward implied volatility. Strictly speaking, we have only defined the function Λ_ε on (part of) the real line. It is however possible to extend it to a strip in the complex plane, and we refer the reader to the proof of Lemma 2.5.1 for more details. Assumption 2.2.1(iv) is the key property that needs to be checked in practical computations and can be violated by well known models under certain parameter configurations (see Section 2.3.1.2 for an example).

Define now the function $\Lambda^* : \mathbb{R} \to \mathbb{R}_+$ as the Fenchel-Legendre transform of Λ_0:

$$\Lambda^*(k) := \sup_{u \in \mathcal{D}_0} \{uk - \Lambda_0(u)\}, \qquad \text{for all } k \in \mathbb{R}. \tag{2.2.3}$$

For ease of exposition in the paper we will use the notation

$$\Lambda_{i,l}(u) := \partial_u^l \Lambda_i(u) \quad \text{for } l \geq 1, i = 0, 1, 2. \tag{2.2.4}$$

The following lemma gathers some immediate properties of the functions Λ^* and $\Lambda_{i,l}$ which will be needed later.

Lemma 2.2.2. *Under Assumption 2.2.1, the following properties hold:*

(i) For any $k \in \mathbb{R}$, there exists a unique $u^*(k) \in \mathcal{D}_0^o$ such that

$$\Lambda_{0,1}(u^*(k)) = k, \tag{2.2.5}$$

$$\Lambda^*(k) = u^*(k)k - \Lambda_0(u^*(k)); \tag{2.2.6}$$

(ii) Λ^* is strictly convex and differentiable on $\mathbb{R}$;

(iii) if $a \in \mathcal{D}_0^o$ is such that $\Lambda_0(a) = 0$, then $\Lambda^*(k) > ak$ for all $k \in \mathbb{R} \setminus \{\Lambda_{0,1}(a)\}$ and $\Lambda^*(\Lambda_{0,1}(a)) = a\Lambda_{0,1}(a)$.

Proof.

(i) By Assumption 2.2.1(iv), $\Lambda_{0,1}$ is a strictly increasing differentiable function from $-\infty$ to ∞ on $\mathcal{D}_0$.

(ii) By (i), $\partial_k \Lambda^*$ is the inverse of the function $\Lambda_{0,1}$ on $\mathbb{R}$. In particular $\partial_k \Lambda^*$ is strictly increasing on $\mathbb{R}$.

(iii) Since $\Lambda_{0,1}$ is strictly increasing, $\Lambda_{0,1}(a) = k$ if and only if $u^*(k) = a$ and then $\Lambda^*(\Lambda_{0,1}(a)) = a\Lambda_{0,1}(a)$ using (2.2.6). Using the definition (2.2.3) with $a \in \mathcal{D}_0^o$ and $\Lambda_0(a) = 0$ gives $\Lambda^*(k) \geq ak$. Since Λ^* is strictly convex from (ii) it follows that $\Lambda^*(k) > ak$ for all $k \in \mathbb{R} \setminus \{\Lambda_{0,1}(a)\}$.

□

Remark 2.2.3. The saddlepoint u^* is not always available in closed-form, but can be computed via a simple root-finding algorithm. Furthermore, a Taylor expansion around any point can be computed iteratively in terms of the derivatives of Λ_0. For instance, around $k = 0$, we can write

$$u^*(k) = u^*(0) + \frac{k}{\Lambda_{0,2}(u^*(0))} - \frac{1}{2}\frac{\Lambda_{0,3}(u^*(0))}{\Lambda_{0,2}(u^*(0))^3}k^2 + \mathcal{O}(k^3).$$

A precise example can be found in the proof of Corollary 2.3.2.

The last tool we need is a (continuous) function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that there exists $c \geq 0$ for which

$$f(\varepsilon)\varepsilon = c + \mathcal{O}(\varepsilon), \quad \text{as } \varepsilon \text{ tends to zero.} \tag{2.2.7}$$

This function will play the role of rescaling the strike of European options and will give us the flexibility to deal with both small- and large-time behaviours. Finally, for any $b \geq 0$ we now define the functions $A_b, \bar{A}_b : \mathbb{R} \setminus \{\Lambda_{0,1}(0), \Lambda_{0,1}(b)\} \times \mathbb{R}_+^* \to \mathbb{R}$ by

$$\begin{aligned}
\bar{A}_b(k,\varepsilon) &:= \frac{b\sqrt{\varepsilon}\mathbb{1}_{\{b>0\}} + \varepsilon^{3/2}f(\varepsilon)\mathbb{1}_{\{b=0\}}}{u^*(k)\,(u^*(k)-b)\,\sqrt{2\pi\Lambda_{0,2}(u^*(k))}},\\
A_b(k,\varepsilon) &:= 1 + \Upsilon(b,k)\varepsilon + \frac{u^*(k)(\varepsilon f(\varepsilon)-b)}{(u^*(k)-b)\,b}\mathbb{1}_{\{b>0\}} + \frac{\varepsilon f(\varepsilon)}{u^*(k)}\mathbb{1}_{\{b=0\}},
\end{aligned}$$

where $\Upsilon : \mathbb{R}_+ \times \mathbb{R}\backslash\{\Lambda_{0,1}(0), \Lambda_{0,1}(b)\} \to \mathbb{R}$ is given by

$$\begin{aligned}
\Upsilon(b,k) := \Lambda_2 &- \frac{5\Lambda_{0,3}^2}{24\Lambda_{0,2}^3} + \frac{4\Lambda_{1,1}\Lambda_{0,3} + \Lambda_{0,4}}{8\Lambda_{0,2}^2} - \frac{\Lambda_{1,1}^2 + \Lambda_{1,2}}{2\Lambda_{0,2}} - \frac{\Lambda_{0,3}}{2\,(u^*(k)-b)\,\Lambda_{0,2}^2}\\
&- \frac{\Lambda_{0,3}}{2u^*(k)\Lambda_{0,2}^2} - \frac{\Lambda_{1,1}\,(b-2u^*(k))+3}{u^*(k)\,(u^*(k)-b)\,\Lambda_{0,2}} - \frac{b^2}{u^*(k)^2\,(u^*(k)-b)^2\,\Lambda_{0,2}}.
\end{aligned} \tag{2.2.8}$$

For ease of notation we write Λ_i and $\Lambda_{i,l}$ in place of $\Lambda_i\,(u^*(k))$ and $\Lambda_{i,l}\,(u^*(k))$. The domains of definition of A_b and $\bar{A}_b$ exclude the set $\{\Lambda_{0,1}(0), \Lambda_{0,1}(b)\} = \{k \in \mathbb{R} : u^*(k) \in \{0, b\}\}$. For all k in this domain, $\Lambda_{0,2}(u^*(k)) > 0$ by Assumption 2.2.1(iv), so that A_b and $\bar{A}_b$ are both well defined real-valued functions.

2.2.1.2 Main theorem and corollaries

The following theorem on asymptotics of option prices is the main result of the chapter. A quick glimpse at the proof of Theorem 2.2.4 in Section 2.5.1 shows that this result can be extended to any arbitrary order.

Theorem 2.2.4. *Let $(Y_\varepsilon)_{\varepsilon>0}$ satisfy Assumption 2.2.1, and $f:\mathbb{R}_+\to\mathbb{R}_+$ be a function satisfying* (2.2.7) *with constant $c\in\mathcal{D}_0^o\cap\mathbb{R}_+$. Then the following expansion holds for all $k\in\mathbb{R}\backslash\{\Lambda_{0,1}(0),\Lambda_{0,1}(c)\}$ as $\varepsilon\downarrow 0$:*

$$\mathrm{e}^{-\Lambda^*(k)/\varepsilon+kf(\varepsilon)+\Lambda_1}\bar{A}_c(k,\varepsilon)\left[A_c(k,\varepsilon)+\mathcal{O}\left(\varepsilon^2\right)\right]=\begin{cases}\mathbb{E}\left(\mathrm{e}^{Y_\varepsilon f(\varepsilon)}-\mathrm{e}^{kf(\varepsilon)}\right)^+, & \text{if } k>\Lambda_{0,1}(c),\\ \mathbb{E}\left(\mathrm{e}^{kf(\varepsilon)}-\mathrm{e}^{Y_\varepsilon f(\varepsilon)}\right)^+, & \text{if } k<\Lambda_{0,1}(0),\\ -\mathbb{E}\left(\mathrm{e}^{Y_\varepsilon f(\varepsilon)}\wedge\mathrm{e}^{kf(\varepsilon)}\right), & \text{if } k\in(\Lambda_{0,1}(0),\Lambda_{0,1}(c)).\end{cases}$$

The expansion does not hold at the strikes $\{\Lambda_{0,1}(0),\Lambda_{0,1}(c)\}$ (these points are not in the strip of regularity of the Fourier transforms) and a different method needs to be used to obtain asymptotics. This is done in the large-maturity case for the Heston model in Chapter 4. Using Put-Call parity, the theorem can also be read as an expansion for European Call options (or for Put options) for all strikes, except at the two points $\Lambda_{0,1}(0)$ and $\Lambda_{0,1}(c)$:

Corollary 2.2.5. *Under the assumptions of Theorem 2.2.4, we have, for $k\in\mathbb{R}\backslash\{\Lambda_{0,1}(0),\Lambda_{0,1}(c)\}$, as $\varepsilon\downarrow 0$:*

$$\begin{aligned}\mathbb{E}\left(\mathrm{e}^{Y_\varepsilon f(\varepsilon)}-\mathrm{e}^{kf(\varepsilon)}\right)^+ &= \mathrm{e}^{\Lambda_\varepsilon(f(\varepsilon)\varepsilon)/\varepsilon}\mathbb{1}_{\{k<\Lambda_{0,1}(c)\}}-\mathrm{e}^{kf(\varepsilon)}\mathbb{1}_{\{k<\Lambda_{0,1}(0)\}}\\ &\quad+\mathrm{e}^{-\Lambda^*(k)/\varepsilon+kf(\varepsilon)+\Lambda_1}\bar{A}_c(k,\varepsilon)\left[A_c(k,\varepsilon)+\mathcal{O}\left(\varepsilon^2\right)\right].\end{aligned}$$

2.2.2 Forward-start option asymptotics

We now specialise Theorem 2.2.4 to forward-start option asymptotics. For a stochastic process $(X_t)_{t\geq 0}$, we recall (Definition (1.0.3)), that for any $t\geq 0$, we define (pathwise) the forward price process $(X_\tau^{(t)})_{\tau\geq 0}$ by $X_\tau^{(t)}:=X_{t+\tau}-X_t$.

2.2.2.1 Diagonal small-maturity asymptotics

We first consider asymptotics when both t and τ are small, which we term *diagonal small-maturity asymptotics*. Set $(Y_\varepsilon):=(X_{\varepsilon\tau}^{(\varepsilon t)})$ and $f\equiv 1$. Then $c=0$ and the following corollary follows from Theorem 2.2.4:

Corollary 2.2.6. *If $(X_{\varepsilon\tau}^{(\varepsilon t)})_{\varepsilon>0}$ satisfies Assumption 2.2.1, then the following holds for all $k\in\mathbb{R}\backslash\{\Lambda_{0,1}(0)\}$, as $\varepsilon\downarrow 0$:*

$$\frac{\mathrm{e}^{-\Lambda^*(k)/\varepsilon+k+\Lambda_1}\varepsilon^{3/2}}{u^*(k)^2\sqrt{2\pi\Lambda_{0,2}}}\left(1+\left(\Upsilon(0,k)+\frac{1}{u^*(k)}\right)\varepsilon+\mathcal{O}\left(\varepsilon^2\right)\right)=\begin{cases}\mathbb{E}\left(\mathrm{e}^{X_{\varepsilon\tau}^{(\varepsilon t)}}-\mathrm{e}^k\right)^+, & \text{if } k>\Lambda_{0,1}(0),\\ \mathbb{E}\left(\mathrm{e}^k-\mathrm{e}^{X_{\varepsilon\tau}^{(\varepsilon t)}}\right)^+, & \text{if } k<\Lambda_{0,1}(0).\end{cases}$$

In the Black-Scholes model, all the quantities above can be computed explicitly and we obtain the following lemma. Note the exact agreement here and in Lemma D.0.10 (when $\widetilde{\tau}\equiv 1$) which was derived using independent methods.

Corollary 2.2.7. *In the BSM model* (1.0.1) *the following expansion holds for all $k \in \mathbb{R}^*$, as $\varepsilon \downarrow 0$:*

$$\frac{\mathrm{e}^{k/2-k^2/(2\Sigma^2\tau\varepsilon)}\left(\Sigma^2\tau\varepsilon\right)^{3/2}}{k^2\sqrt{2\pi}}\left[1-\left(\frac{3}{k^2}+\frac{1}{8}\right)\Sigma^2\tau\varepsilon+\mathcal{O}(\varepsilon^2)\right]=\begin{cases}\mathbb{E}\left(\mathrm{e}^{X_{\varepsilon\tau}^{(\varepsilon t)}}-\mathrm{e}^k\right)^+, & \text{if } k>0,\\ \mathbb{E}\left(\mathrm{e}^k-\mathrm{e}^{X_{\varepsilon\tau}^{(\varepsilon t)}}\right)^+, & \text{if } k<0.\end{cases}$$

Proof. For the rescaled (forward) process $(X_{\varepsilon\tau}^{(\varepsilon t)})_{\varepsilon>0}$ in the BSM model (1.0.1) we have $\Lambda_\varepsilon(u) = \Lambda_0(u) + \varepsilon\Lambda_1(u)$ for $u \in \mathbb{R}$, where $\Lambda_0(u) = u^2\sigma^2\tau/2$ and $\Lambda_1(u) = -u\sigma^2\tau/2$. It follows that $\Lambda_{0,1}(u) = u\sigma^2\tau$, $\Lambda_{0,2}(u) = \sigma^2\tau$ and $\Lambda_{1,1}(u) = -\sigma^2\tau/2$. For any $k \in \mathbb{R}$, $u^*(k) := k/(\sigma^2\tau)$ is the unique solution to the equation $\Lambda_{0,1}(u^*(k)) = k$ and $\Lambda^*(k) = k^2/(2\sigma^2\tau)$. Λ_0 is essentially smooth and strictly convex on $\mathbb{R}$ and the BSM model satisfies the other conditions in Assumption 2.2.1. Since $\Lambda_{0,1}(0) = 0$, the result follows from Corollary 2.2.6. □

It is natural to wonder why we considered diagonal small-maturity asymptotics and not the small-maturity asymptotic of $\sigma_{t,\tau}$ for fixed $t > 0$. In this case it turns out that in many cases of interest (stochastic volatility models, time-changed exponential Lévy models), the forward smile blows up to infinity (except at-the-money) as τ tends to zero. However under the assumptions given above, this degenerate behaviour does not occur in the diagonal small-maturity regime (Corollary 2.2.6). In the Heston case, this explosive behaviour will be studied in Chapter 3. More generally this will be explored in Chapter 5, but we can provide a preliminary conjecture explaining the origin of this behaviour. Consider a two-state Markov-chain $\mathrm{d}X_t = -\frac{1}{2}V\mathrm{d}t + \sqrt{V}\mathrm{d}W_t$, starting at $X_0 = 0$, where W is a standard Brownian motion and where V is independent of W and takes value V_1 with probability $p \in (0, 1)$ and value $V_2 \in (0, V_1)$ with probability $1 - p$. Conditioning on V and by the independence assumption, we have

$$\mathbb{E}\left(\mathrm{e}^{u(X_{t+\tau}-X_t)}\right) = p\mathrm{e}^{V_1u\tau(u-1)/2} + (1-p)\mathrm{e}^{V_2u\tau(u-1)/2}, \qquad \text{for all } u \in \mathbb{R}.$$

Consider now the small-maturity regime where $\varepsilon = \tau$, $f(\varepsilon) \equiv 1$ and $Y_\varepsilon := X_\varepsilon^{(t)}$ for a fixed $t > 0$. In this case an expansion for the re-scaled cgf in (2.2.2) as τ tends to zero is given by

$$\Lambda_\varepsilon(u) = \tau\log\mathbb{E}\left(\mathrm{e}^{u(X_{t+\tau}-X_t)/\tau}\right) = \frac{V_1}{2}u^2 + \tau\log\left(p\mathrm{e}^{-V_1u/2}\right) + \tau\mathcal{O}\left(\mathrm{e}^{-u^2(V_1-V_2)/(2\tau)}\right),$$

for all $u \in \mathbb{R}$. Since $V_1 > V_2$ the remainder tends to zero exponentially fast as $\tau \downarrow 0$. The assumptions of Theorem 2.2.4 are clearly satisfied and a simple calculation shows that $\lim_{\tau\downarrow 0}\sigma_{t,\tau}(k) = \sqrt{V_1}$. This example naturally extends to n-state Markov chains, and a natural conjecture is that the small-maturity forward smile does not blow up if and only if the quadratic variation of the process is bounded. In practice, most models have unbounded quadratic variation (see examples in Section 2.3), and hence the diagonal small-maturity asymptotic is a natural scaling.

2.2.2.2 Large-maturity asymptotics

We now consider large-maturity asymptotics, when τ is large and t is fixed. Consider $(Y_\varepsilon) := (\varepsilon X_{1/\varepsilon}^{(t)})$, $\varepsilon := 1/\tau$ and $f(\varepsilon) \equiv 1/\varepsilon$ (so that $c = 1$). Theorem 2.2.4 then applies and we obtain the following expansion for forward-start options:

Corollary 2.2.8. *If* $(\tau^{-1}X_\tau^{(t)})_{\tau>0}$ *satisfies Assumption 2.2.1 with* $\varepsilon=\tau^{-1}$ *and* $1\in\mathcal{D}_0^o$*, then the following expansion holds for all* $k\in\mathbb{R}\backslash\{\Lambda_{0,1}(0),\Lambda_{0,1}(1)\}$ *as* $\tau\uparrow\infty$*:*

$$\frac{\mathrm{e}^{-\tau(\Lambda^*(k)-k)+\Lambda_1}\tau^{-1/2}}{u^*(k)\left(u^*(k)-1\right)\sqrt{2\pi\Lambda_{0,2}}}\left(1+\frac{\Upsilon(1,k)}{\tau}+\mathcal{O}\left(\frac{1}{\tau^2}\right)\right)$$
$$=\begin{cases}\mathbb{E}\left(\mathrm{e}^{X_\tau^{(t)}}-\mathrm{e}^{k\tau}\right)^+, & \text{if } k>\Lambda_{0,1}(1),\\ \mathbb{E}\left(\mathrm{e}^{k\tau}-\mathrm{e}^{X_\tau^{(t)}}\right)^+, & \text{if } k<\Lambda_{0,1}(0),\\ -\mathbb{E}\left(\mathrm{e}^{X_\tau^{(t)}}\wedge\mathrm{e}^{k\tau}\right), & \text{if } k\in(\Lambda_{0,1}(0),\Lambda_{0,1}(1)).\end{cases}$$

In the Black-Scholes model, all the quantities above can be computed in closed form, and we obtain:

Corollary 2.2.9. *In the BSM model* (1.0.1) *the following expansion holds for all* $k\in\mathbb{R}\backslash\left\{-\Sigma^2/2,\Sigma^2/2\right\}$ *as* $\tau\uparrow\infty$*:*

$$\frac{\mathrm{e}^{-\tau\left(\left(k+\Sigma^2/2\right)^2/(2\Sigma^2)-k\right)}4\Sigma^3}{\left(4k^2-\Sigma^4\right)\sqrt{2\pi\tau}}\left(1-\frac{4\Sigma^2\left(\Sigma^4+12k^2\right)}{\left(4k^2-\Sigma^4\right)^2\tau}+\mathcal{O}\left(\frac{1}{\tau^2}\right)\right)$$
$$=\begin{cases}\mathbb{E}\left(\mathrm{e}^{X_\tau^{(t)}}-\mathrm{e}^{k\tau}\right)^+, & \text{if } k>\frac{1}{2}\Sigma^2,\\ \mathbb{E}\left(\mathrm{e}^{k\tau}-\mathrm{e}^{X_\tau^{(t)}}\right)^+, & \text{if } k<-\frac{1}{2}\Sigma^2,\\ -\mathbb{E}\left(\mathrm{e}^{X_\tau^{(t)}}\wedge\mathrm{e}^{k\tau}\right), & \text{if } k\in(-\frac{1}{2}\Sigma^2,\frac{1}{2}\Sigma^2).\end{cases}$$

Proof. Consider the process $(X_\tau^{(t)}/\tau)_{\tau>0}$ and set $\varepsilon=\tau^{-1}$. In the BSM model (1.0.1), $\Lambda_\varepsilon(u):=\tau^{-1}\log\mathbb{E}(\exp(uX_\tau^{(t)}))=\Lambda_0(u)=\frac{1}{2}\Sigma^2u(u-1)$. Thus $\Lambda_{0,1}(u)=\Sigma^2\left(u-1/2\right)$ and $\Lambda_{0,2}(u)=\Sigma^2$. For any $k\in\mathbb{R}$, $\Lambda_{0,1}(u^*(k))=k$ has a unique solution $u^*(k)=1/2+k/\Sigma^2$ and hence $\Lambda^*(k)=\left(k+\Sigma^2/2\right)^2/(2\Sigma^2)$. Λ_0 is essentially smooth and strictly convex on $\mathbb{R}$ and Assumption 2.2.1 is satisfied. Since $\{0,1\}\subset\mathcal{D}_0^o$ the result follows from Corollary 2.2.8. □

2.2.3 Forward smile asymptotics

We now translate the forward-start option expansions above into asymptotics of the forward implied volatility smile $k\mapsto\sigma_{t,\tau}(k)$, which was defined in (1.0.5).

2.2.3.1 Diagonal small-maturity forward smile

We first focus on the diagonal small-maturity case. For $i = 0, 1, 2$ we define the functions $v_i : \mathbb{R}^* \times \mathbb{R}_+ \times \mathbb{R}_+^* \to \mathbb{R}$ by

$$\begin{aligned}
v_0(k,t,\tau) &:= \frac{k^2}{2\tau\Lambda^*(k)}, \\
v_1(k,t,\tau) &:= \frac{v_0(k,t,\tau)^2\tau}{k}\left[1+\frac{2}{k}\log\left(\frac{k^2\mathrm{e}^{\Lambda_1(u^*(k))}}{u^*(k)^2\sqrt{\Lambda_{0,2}(u^*(k))}\left(\tau v_0(k,t,\tau)\right)^{3/2}}\right)\right], \\
v_2(k,t,\tau) &:= \frac{2\tau^2 v_0^3(k,t,\tau)}{k^2}\left(\frac{3}{k^2}+\frac{1}{8}\right)+\frac{2\tau v_0^2(k,t,\tau)}{k^2}\left(\Upsilon(0,k)+\frac{1}{u^*(k)}\right) \\
&\quad +\frac{v_1^2(k,t,\tau)}{v_0(k,t,\tau)}-\frac{3\tau}{k^2}v_0(k,t,\tau)v_1(k,t,\tau),
\end{aligned} \tag{2.2.9}$$

where Λ^*, u^*, $\Lambda_{i,l}$, Υ are defined in (2.2.3), (2.2.5), (2.2.4), (2.2.8). The diagonal small-maturity forward smile asymptotic is now given in the following proposition, proved in Section 2.5.1.

Proposition 2.2.10. *Suppose that $(X_{\varepsilon\tau}^{(\varepsilon t)})_{\varepsilon>0}$ satisfies Assumption 2.2.1 and that $\Lambda_{0,1}(0) = 0$ (defined in (2.2.4)). The following expansion then holds for all $k \in \mathbb{R}^*$ as ε tends to zero:*

$$\sigma_{\varepsilon t,\varepsilon\tau}^2(k) = v_0(k,t,\tau) + v_1(k,t,\tau)\varepsilon + v_2(k,t,\tau)\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right). \tag{2.2.10}$$

When $\Lambda_{0,1}(0) = 0$ then $\Lambda^*(k) > 0$ for $k \in \mathbb{R}^*$ and $\Lambda^*(0) = 0$ from Assumption 2.2.1(iii) and Lemma 2.2.2(iii) (with $a = 0 \in \mathcal{D}_0^o$) so that v_0 is always strictly positive, and all the v_i $(i = 0, 1, 2)$ are well defined on $\mathbb{R}^*$.

2.2.3.2 Large-maturity forward smile

In the large-maturity case, define for $i = 0, 1, 2$, the functions $v_i^\infty : \mathbb{R}\backslash\{\Lambda_{0,1}(0), \Lambda_{0,1}(1)\} \times \mathbb{R}_+ \to \mathbb{R}$ by

$$v_0^\infty(k,t) := \begin{cases} 2\left(2\Lambda^*(k) - k - 2\sqrt{\Lambda^*(k)(\Lambda^*(k)-k)}\right), & \text{if } k \in \mathbb{R}\backslash\left[\Lambda_{0,1}(0), \Lambda_{0,1}(1)\right], \\ 2\left(2\Lambda^*(k) - k + 2\sqrt{\Lambda^*(k)(\Lambda^*(k)-k)}\right), & \text{if } k \in \left(\Lambda_{0,1}(0), \Lambda_{0,1}(1)\right), \end{cases} \tag{2.2.11}$$

and

$$\begin{aligned}
v_1^\infty(k,t) &:= \frac{8v_0^\infty(k,t)^2}{4k^2 - v_0^\infty(k,t)^2}\Big(\Lambda_1(u^*(k)) \\
&\quad +\log\left(\frac{4k^2 - v_0^\infty(k,t)^2}{4(u^*(k)-1)u^*(k)v_0^\infty(k,t)^{3/2}\sqrt{\Lambda_{0,2}(u^*(k))}}\right)\Big), \\
v_2^\infty(k,t) &:= \frac{4}{v_0^\infty(k,t)\left(v_0^\infty(k,t)^2 - 4k^2\right)^3}\Big[8k^4 v_1^\infty(k,t)v_0^\infty(k,t)^2\left(v_1^\infty(k,t)+6\right) \\
&\quad - 16k^6 v_1^\infty(k,t)^2 - 2\Upsilon(1,k)v_0^\infty(k,t)^3\left(v_0^\infty(k,t)^2 - 4k^2\right)^2 \\
&\quad - k^2 v_0^\infty(k,t)^4\left(96 + v_1^\infty(k,t)^2 + 8v_1^\infty(k,t)\right) - v_0^\infty(k,t)^6\left(v_1^\infty(k,t)+8\right)\Big].
\end{aligned} \tag{2.2.12}$$

The quantities Λ^*, u^*, $\Lambda_{i,l}$, Υ are defined in (2.2.3), (2.2.5), (2.2.4), (2.2.8). The large-maturity forward smile asymptotic is given in the following proposition, proved in Section 2.5.1. When $t = 0$ in (2.2.10) and (2.2.13) below, we recover—and improve—the asymptotics in [60, 62, 63, 64, 65, 95].

Proposition 2.2.11. *Suppose that* $(\tau^{-1}X_\tau^{(t)})_{\tau>0}$ *satisfies Assumption 2.2.1, with* $\varepsilon = \tau^{-1}$ *and that* $1 \in \mathcal{D}_0^o$ *and* $\Lambda_0(1) = 0$ *(all defined in Assumption 2.2.1). The following then holds for all* $k \in \mathbb{R}\backslash\{\Lambda_{0,1}(0), \Lambda_{0,1}(1)\}$ *as* τ *tends to infinity:*

$$\sigma_{t,\tau}^2(k\tau) = v_0^\infty(k,t) + v_1^\infty(k,t)\tau^{-1} + v_2^\infty(k,t)\tau^{-2} + \mathcal{O}(\tau^{-3}). \tag{2.2.13}$$

Remark 2.2.12. It is interesting to note that the (strict) martingale property ($\Lambda_0(1) = 0$) is only required in Proposition 2.2.11 and not in Proposition 2.2.10 and Theorem 2.2.4.

Since $\{0,1\} \subset \mathcal{D}_0^o$ and $\Lambda_0(1) = \Lambda_0(0) = 0$, we always have $\Lambda^*(k) \geq \max(0,k)$ from Lemma 2.2.2(iii). One can also check that $0 < v_0^\infty(k,t) < 2|k|$ for $k \in \mathbb{R}\backslash[\Lambda_{0,1}(0), \Lambda_{0,1}(1)]$ and $v_0^\infty(k,t) > 2|k|$ for $k \in (\Lambda_{0,1}(0), \Lambda_{0,1}(1))$. This implies that the functions v_i^∞ $(i = 0,1,2)$ are always well defined. By Assumption 2.2.1 and Lemma 2.2.2(iii) we have $\Lambda^*(\Lambda_{0,1}(0)) = 0$. Again from Lemma 2.2.2(iii) this implies that $\Lambda^*(\Lambda_{0,1}(1)) = \Lambda_{0,1}(1)$. Hence $v_0^\infty(\cdot,t)$ is continuous on $\mathbb{R}$ with $v_0^\infty(\Lambda_{0,1}(1),t) = 2\Lambda_{0,1}(1)$ and $v_0^\infty(\Lambda_{0,1}(0),t) = -2\Lambda_{0,1}(0)$. The functions $v_1^\infty(\cdot,t)$ and $v_2^\infty(\cdot,t)$ are undefined on $\{\Lambda_{0,1}(0), \Lambda_{0,1}(1)\}$. However, it can be shown that since Λ_0 is strictly convex (Assumption 2.2.1) and $\Lambda_0(1) = 0$ all limits are well defined and hence both functions can be extended by continuity to $\mathbb{R}$. For example, using Taylor expansions in neighbourhoods of these points yields:

$$\lim_{k\to p} v_1^\infty(k,t) = 2 - 2\sqrt{\frac{v_0^\infty(p,t)}{\Lambda_{0,2}(u^*(p))}}\left(1 + \mathrm{sgn}(p)\left(\frac{\Lambda_{0,3}(u^*(p))}{6\Lambda_{0,2}(u^*(p))} - \Lambda_{1,1}(u^*(p))\right)\right),$$

for $p \in \{\Lambda_{0,1}(0), \Lambda_{0,1}(1)\}$, which, for $t = 0$, agrees with Theorem 4.4.1 and [65, Equation (3.2)] for the specific case of the Heston model (1.3.2).

2.2.3.3 Type-II forward smile

As mentioned on Page 13, another type of forward-start option has been considered in the literature. We show here that the forward implied volatility expansions proved above carry over in this case with some minor modifications. The following corollary shows how the Type-II forward smile $\widetilde{\sigma}_{t,\tau}$ (defined in (1.0.6)) can be incorporated into our framework. The proof follows directly from Proposition 1.4.3(i) and is therefore omitted.

Corollary 2.2.13. *If* $\left(\mathrm{e}^{X_t}\right)_{t\geq 0}$ *is an* $(\mathcal{F}_t)$*-martingale under* $\mathbb{P}$*, then Propositions 2.2.10 and 2.2.11 hold for the Type-II forward smile* $\widetilde{\sigma}_{t,\tau}$ *with the cgf* (2.2.1) *calculated under* $\widetilde{\mathbb{P}}$ *(defined in* (1.4.4)*).*

2.3 Applications

2.3.1 Heston

In this section, we apply our general results to the Heston model (1.3.2). In Section 2.3.1.1 we focus on diagonal small-maturity asymptotics and in Section 2.3.1.2 we focus on large-maturity asymptotics.

2.3.1.1 Diagonal Small-Maturity Heston Forward Smile

The objective of this section is to apply Proposition 2.2.10 to the Heston forward smile, namely

Proposition 2.3.1. *In Heston, Corollary 2.2.6 and Proposition 2.2.10 hold with* $\mathcal{D}_0 = \mathcal{K}_{t,\tau}$, $\Lambda_0 = \Xi$, $\Lambda_1 = L$.

This proposition is proved in Section 2.5.2.1, and all the functions therein are defined as follows. We set $\Xi : \mathcal{K}_{t,\tau} \times \mathbb{R}_+ \times \mathbb{R}_+^* \to \mathbb{R}$ as

$$\Xi(u,t,\tau) := \frac{uv}{\xi\left(\bar\rho\cot\left(\frac{1}{2}\xi\bar\rho\tau u\right) - \rho\right) - \frac{1}{2}\xi^2 tu}, \qquad \mathcal{K}_{t,\tau} := \left\{u \in \mathbb{R} : \Xi(u,0,\tau) < \frac{2v}{\xi^2 t}\right\}, \tag{2.3.1}$$

with $\bar\rho := \sqrt{1-\rho^2}$ and the functions $L, L_0, L_1 : \mathcal{K}_{t,\tau} \times \mathbb{R}_+ \times \mathbb{R}_+^* \to \mathbb{R}$ are defined as

$$\begin{aligned}
L(u,t,\tau) &:= L_0(u,\tau) + \Xi(u,t,\tau)^2\left(\frac{vL_1(u,\tau)}{\Xi(u,0,\tau)^2} - \frac{\kappa\xi^2 t^2}{4v}\right) - \Xi(u,t,\tau)\kappa t \\
&\quad - \frac{2\kappa\theta}{\xi^2}\log\left(1 - \frac{\Xi(u,0,\tau)\xi^2 t}{2v}\right), \\
L_0(u,\tau) &:= \frac{\kappa\theta}{\xi^2}\left((\mathtt{i}\xi\rho - d_0)\mathtt{i}\tau u - 2\log\left(\frac{1-g_0\mathrm{e}^{-\mathtt{i}d_0\tau u}}{1-g_0}\right)\right), \\
L_1(u,\tau) &:= \frac{\exp(-\mathtt{i}d_0\tau u)}{\xi^2\left(1-g_0\mathrm{e}^{-\mathtt{i}d_0\tau u}\right)}\Big[(\mathtt{i}\xi\rho - d_0)\mathtt{i}d_1\tau u + (d_1-\kappa)\left(1-\mathrm{e}^{\mathtt{i}d_0\tau u}\right) \\
&\quad + \frac{(\mathtt{i}\xi\rho - d_0)\left(1-\mathrm{e}^{-\mathtt{i}d_0\tau u}\right)(g_1 - \mathtt{i}d_1 g_0\tau u)}{1-g_0\mathrm{e}^{-\mathtt{i}d_0\tau u}}\Big],
\end{aligned} \tag{2.3.2}$$

with

$$d_0 := \xi\bar\rho, \quad d_1 := \frac{\mathtt{i}\left(2\kappa\rho - \xi\right)}{2\bar\rho}, \qquad g_0 := \frac{\mathtt{i}\rho - \bar\rho}{\mathtt{i}\rho + \bar\rho} \qquad \text{and} \qquad g_1 := \frac{2\kappa - \xi\rho}{\xi\bar\rho\left(\bar\rho + \mathtt{i}\rho\right)^2}.$$

For any $t \geq 0, \tau > 0$ the functions L_0 and L_1 are well defined real-valued functions for all $u \in \mathcal{K}_{t,\tau}$ (see Remark 2.5.10 for technical details). Also since $\Xi(0,t,\tau)/\Xi(0,0,\tau) = 1$, L is well defined at $u = 0$. In order to gain some intuition on the role of the Heston parameters on the forward smile we expand (2.2.10) around the at-the-money point in terms of the log-strike k:

Corollary 2.3.2. *The following expansion holds for the Heston forward smile as* ε *and* k *tend to zero:*

$$\sigma^2_{\varepsilon t,\varepsilon\tau}(k) = v + \varepsilon\nu_0(t,\tau) + \left(\frac{\rho\xi}{2} + \varepsilon\nu_1(t,\tau)\right)k + \left(\frac{(4-7\rho^2)\xi^2}{48v} + \frac{\xi^2 t}{4\tau v} + \varepsilon\nu_2(t,\tau)\right)k^2 + \mathcal{O}(k^3) + \mathcal{O}(\varepsilon^2).$$

The corollary is proved in Section 2.5.2.1, and the functions appearing in it are defined as follows:

$$\begin{aligned}
\nu_0(t,\tau) &:= \frac{\tau}{48}\left(24\kappa\theta + \xi^2\left(\rho^2-4\right) + 12v(\xi\rho - 2\kappa)\right) - \frac{t}{4}\left(\xi^2 + 4\kappa\left(v-\theta\right)\right), \\
\nu_1(t,\tau) &:= \frac{\rho\xi\tau}{24v}\left[\xi^2\left(1-\rho^2\right) - 2\kappa\left(v+\theta\right) + \xi\rho v\right] + \frac{\rho\xi^3 t}{8v}, \\
\nu_2(t,\tau) &:= \Big[80\kappa\theta\left(13\rho^2-6\right) + \xi^2\left(521\rho^4 - 712\rho^2 + 176\right) + 40\rho^2 v\left(\xi\rho - 2\kappa\right)\Big]\frac{\xi^2\tau}{7680v^2} \\
&\quad - \frac{\xi^2 t}{192v^2}\Big[4\kappa\theta\left(16-7\rho^2\right) + \left(7\rho^2-4\right)\left(9\xi^2+4\kappa v\right)\Big] \\
&\quad + \frac{\xi^2 t^2}{32\tau v^2}\Big(4\kappa\left(v-3\theta\right) + 9\xi^2\Big).
\end{aligned} \tag{2.3.3}$$

Remark 2.3.3. The following remarks should convey some practical intuition about the results above:

(i) For $t=0$ this expansion perfectly lines up with [64, Corollary 4.3].

(ii) Corollary 2.3.2 implies $\sigma_{\varepsilon t,\varepsilon\tau}(0) = \sigma_{0,\varepsilon\tau}(0) - \frac{\varepsilon t}{8\sqrt{v}}\left(\xi^2 + 4\kappa(v-\theta)\right) + \mathcal{O}(\varepsilon^2)$, as $\varepsilon \downarrow 0$. For small enough ε, the spot at-the-money volatility is higher than the forward if and only if $\xi^2 + 4\kappa(v-\theta) > 0$. In particular, when $v \geq \theta$, the difference between the forward at-the-money volatility and the spot one is increasing in the forward-start date and volatility of variance ξ. In Figure 2.2 we plot this effect using $\theta = v$ and $\theta > v + \xi^2/(4\kappa)$. The relative values of v and θ impact the level of the forward smile vs spot smile.

(iii) For practical purposes, we can deduce some information on the forward skew by loosely differentiating Corollary 2.3.2 with respect to k:

$$\partial_k \sigma_{\varepsilon t,\varepsilon\tau}(0) = \frac{\xi\rho}{4\sqrt{v}} + \frac{(4\nu_1(t,\tau)v - \xi\rho\nu_0(t,\tau))}{8v^{3/2}}\varepsilon + \mathcal{O}(\varepsilon^2).$$

(iv) Likewise an expansion for the Heston forward convexity as ε tends to zero is given by

$$\begin{aligned}
\partial_k^2 \sigma_{\varepsilon t,\varepsilon\tau}(0) = {} & \frac{\xi^2((2-5\rho^2)\tau + 6t)}{24\tau v^{3/2}} \\
& - \frac{\nu_0(t,\tau)\xi^2(3t + (1-4\rho^2)\tau) + 6\tau v(\rho\xi\nu_1(t,\tau) - 4\nu_2(t,\tau)v)}{24\tau v^{5/2}}\varepsilon + \mathcal{O}(\varepsilon^2),
\end{aligned}$$

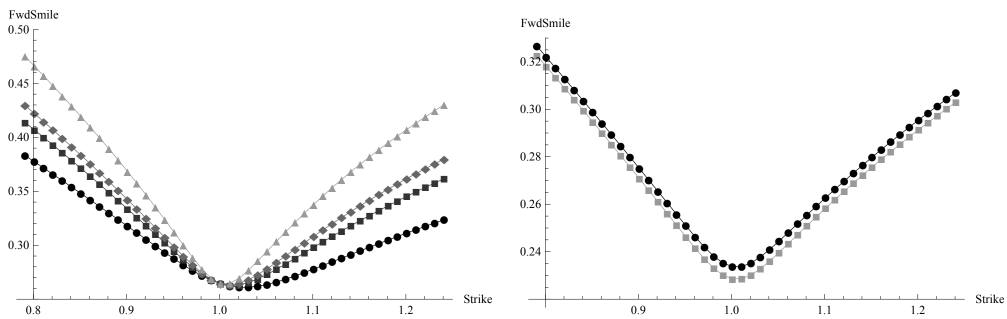
and in particular $\partial_k^2 \sigma_{\varepsilon t,\varepsilon\tau}(0) = \partial_k^2 \sigma_{0,\varepsilon\tau}(0) + \xi^2 t/(4\tau v^{3/2}) + \mathcal{O}(\varepsilon)$. For fixed maturity the forward convexity is always greater than the spot implied volatility convexity (see Figure 2.2) and this difference is increasing in the forward-start dates and volatility of variance. At zeroth order in ε the wings of the forward smile increase to arbitrarily high levels with decreasing maturity (see Figure 2.1(a)). This effect has been mentioned qualitatively by practitioners [36]. As it turns out for fixed $t > 0$ the Heston forward smile blows up to infinity (except at-the-money) as the maturity tends to zero, see Chapter 3 for details.

In the Heston model, $(\mathrm{e}^{X_t})_{t\geq 0}$ is a true martingale [5, Proposition 2.5]. Applying Corollary 2.2.13 with Lemma 1.4.8, giving the Heston forward cgf under the stopped-share-price measure (1.4.4), we derive the following asymptotic for the Type-II Heston forward smile $\widetilde{\sigma}_{t,\tau}$:

Corollary 2.3.4. *The diagonal small-maturity expansion of the Heston Type-II forward smile as ε and k tend to zero is the same as the one in Corollary 2.3.2 with ν_0, ν_1 and ν_2 replaced by $\widetilde{\nu}_0$, $\widetilde{\nu}_1$ and $\widetilde{\nu}_2$, where*

$$\widetilde{\nu}_0(t,\tau) := \nu_0(t,\tau) + \xi\rho v t, \qquad \widetilde{\nu}_1(t,\tau) := \nu_1(t,\tau),$$
$$\widetilde{\nu}_2(t,\tau) := \nu_2(t,\tau) + \frac{\rho\xi^3 t}{48v}\left(7\rho^2 - 4\right) - \frac{\rho\xi^3 t^2}{8v\tau}.$$

Its proof is analogous to the proofs of Proposition 2.3.1 and Corollary 2.3.2, and is therefore omitted. Note that when $\rho = 0$ or $t = 0$, $\nu_i = \widetilde{\nu}_i$ $(i = 1, 2, 3)$, and the Heston forward smiles Type-I and Type-II are the same, in exact agreement with Proposition 1.4.4.



(a) Small-maturity forward smile explosion.

(b) Type I vs Type II forward smile.

Figure 2.1: (a): Forward smiles with forward-start date $t = 1/2$ and maturities $\tau = 1/6, 1/12, 1/16, 1/32$ given by circles, squares, diamonds and triangles respectively using the Heston parameters $(v, \theta, \kappa, \rho, \xi) = (0.07, 0.07, 1, -0.6, 0.5)$ and the asymptotic in Proposition 2.3.1. (b): Type I (circles) vs Type 2 (squares) forward smile with $t = 1/2$, $\tau = 1/12$ and the Heston parameters $(v, \theta, \kappa, \rho, \xi) = (0.07, 0.07, 1, -0.2, 0.34)$ using Corollaries 2.3.2 and 2.3.4.

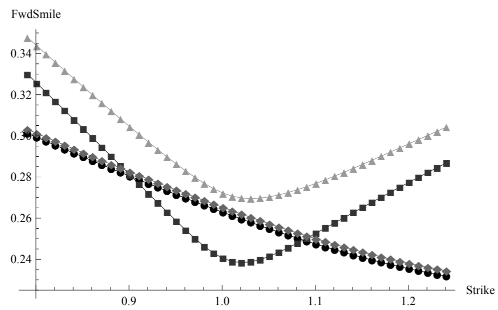


Figure 2.2: Forward smile vs spot smile with $v = \theta$ and $\theta > v + \xi^2/(4\kappa)$. Circles $(t = 0, \tau = 1/12)$ and squares $(t = 1/2, \tau = 1/12)$ use the Heston parameters $v = \theta = 0.07$,$\kappa = 1$, $\rho = -0.6$, $\xi = 0.3$. Diamonds $(t = 0, \tau = 1/12)$ and triangles $(t = 1/2, \tau = 1/12)$ use the same parameters but with $\theta = 0.1$. Plots use the asymptotic in Proposition 2.3.1.

2.3.1.2 Large-maturity Heston forward smile

Our main result here is Proposition 2.3.5, which is an application of Proposition 2.2.11 to the Heston forward smile. In order to apply this result we first define a number of regimes depending on the Heston correlation. Define the real numbers ρ_- and ρ_+ by

$$\rho_\pm := \frac{\mathrm{e}^{-2\kappa t}\left(\xi(\mathrm{e}^{2\kappa t}-1) \pm (\mathrm{e}^{\kappa t}+1)\sqrt{16\kappa^2\mathrm{e}^{2\kappa t}+\xi^2(1-\mathrm{e}^{\kappa t})^2}\right)}{8\kappa}, \tag{2.3.4}$$

and note that $-1 \le \rho_- < 0 < \rho_+$ with $\rho_\pm = \pm 1$ if and only if $t = 0$. We now define the large-maturity regimes:

$$\begin{array}{lll} \mathfrak{R}_1: & \textit{Good correlation regime:} & \rho_- \le \rho \le \min(\rho_+, \kappa/\xi); \\ \mathfrak{R}_2: & \textit{Asymmetric negative correlation regime:} & -1 < \rho < \rho_- \text{ and } t > 0; \\ \mathfrak{R}_3: & \textit{Asymmetric positive correlation regime:} & \rho_+ < \rho < 1 \text{ and } t > 0; \\ & \mathfrak{R}_{3a}: & \rho \le \kappa/\xi; \\ & \mathfrak{R}_{3b}: & \rho > \kappa/\xi; \\ \mathfrak{R}_4: & \textit{Large correlation regime:} & \kappa/\xi < \rho \le \min(\rho_+, 1). \end{array} \tag{2.3.5}$$

In the standard case $t = 0$, $\mathfrak{R}_1$ corresponds to $\kappa \ge \rho\xi$ and $\mathfrak{R}_4$ is its complement. We now define the following quantities:

$$u_\pm := \frac{\xi - 2\kappa\rho \pm \eta}{2\xi(1-\rho^2)} \quad \text{and} \quad u^*_\pm := \frac{\psi \pm \nu}{2\xi(\mathrm{e}^{\kappa t}-1)}, \tag{2.3.6}$$

with

$$\eta := \sqrt{\xi^2(1-\rho^2)+(2\kappa-\rho\xi)^2}, \qquad \nu := \sqrt{\psi^2 - 16\kappa^2\mathrm{e}^{\kappa t}}, \qquad \psi := \xi(\mathrm{e}^{\kappa t}-1) - 4\kappa\rho\mathrm{e}^{\kappa t}, \tag{2.3.7}$$

as well as the interval $\mathcal{K}_{\mathrm{H}} \subset \mathbb{R}$ defined in Table 2.1. Note that ν defined in (2.3.7) is a well defined

	$\mathfrak{R}_1$	$\mathfrak{R}_2$	$\mathfrak{R}_{3a}$	$\mathfrak{R}_{3b}$	$\mathfrak{R}_4$
$\mathcal{K}_{\mathrm{H}}$	$[u_-, u_+]$	$[u_-, u^*_+)$	$(u^*_-, u_+]$	$(u^*_-, 1]$	$(u_-, 1]$

Table 2.1: Limiting domains in each large-maturity regime.

real number for all $\rho \in [-1, \rho_-] \cup [\rho_+, 1]$. We define the real-valued functions V and H from $\mathcal{K}_{\mathrm{H}}$ to $\mathbb{R}$ by

$$V(u) := \frac{\kappa\theta}{\xi^2}(\kappa - \rho\xi u - d(u)) \quad \text{and} \quad H(u) := \frac{V(u)v\mathrm{e}^{-\kappa t}}{\kappa\theta - 2\beta_t V(u)} - \frac{2\kappa\theta}{\xi^2}\log\left(\frac{\kappa\theta - 2\beta_t V(u)}{\kappa\theta\,(1-\gamma(u))}\right), \tag{2.3.8}$$

with d, β_t and γ defined in (1.3.6), (1.3.4) and (1.3.8). For any $k \in \mathbb{R}$ the (saddlepoint) equation $V'(q^*(k)) = k$ has a unique solution $q^*(k) \in (u_-, u_+)$:

$$q^*(k) := \frac{\xi - 2\kappa\rho + (\kappa\theta\rho + k\xi)\,\eta\,\left(k^2\xi^2 + 2k\kappa\theta\rho\xi + \kappa^2\theta^2\right)^{-1/2}}{2\xi\,(1-\rho^2)}. \tag{2.3.9}$$

Further let $V^* : \mathbb{R} \to \mathbb{R}_+$ denote the Fenchel-Legendre transform of V:

$$V^*(k) := \sup_{u \in \mathcal{K}_{\mathrm{H}}} \{uk - V(u)\}, \qquad \text{for all } k \in \mathbb{R}. \tag{2.3.10}$$

In Regime $\mathfrak{R}_1$, V^* is given in closed-form as $V^*(k) = q^*(k)k - V(q^*(k))$. (See Lemma 4.2.1 for further details). The following proposition gives the large-maturity Heston forward smile in Regime $\mathfrak{R}_1$ (the good correlation regime), and its proof is postponed to Section 2.5.2.2.

Proposition 2.3.5. *If $\rho_- \leq \rho \leq \min(\rho_+, \kappa/\xi)$, then Corollary 2.2.8 and Proposition 2.2.11 hold with $\Lambda_0 = V$, $\Lambda^* = V^*$, $u^* = q^*$, $\Lambda_1 = H$, $\Lambda_2 = 0$ and $\mathcal{D}_0 = [u_-, u_+]$.*

Remark 2.3.6.

(i) In the Heston model there is no t-dependence for v_0^∞ in (2.2.13) since V^* does not depend on t. Therefore under the conditions of the proposition, the limiting (zeroth order) smile is exactly of SVI form (see [73]).

(ii) For all other regimes in (2.3.5) the essential smoothness property in Assumption 2.2.1(iv) is not satisfied (and $1 \notin \mathcal{K}_{\mathrm{H}}^o$ in Regimes $\mathfrak{R}_{3b}$ and $\mathfrak{R}_4$) and a different strategy needs to be employed to derive a sharp large deviations result for large-maturity forward-start options. We leave this analysis for Chapter 4.

(iii) $t = 0$ implies that $\rho_\pm = \pm 1$ and Proposition 2.3.5 extends the large-maturity asymptotics in [63, 65].

(iv) For practical purposes, note that $\rho \in [0, \min(1/2, \kappa/\xi)]$ is always satisfied under the assumptions of the proposition.

(v) Even though the function V^* does not depend on t, $\rho_\pm$ and the function H do (see the at-the-money example below). That said, to zeroth order and correlation close to zero, the large-maturity forward smile is the same as the large-maturity spot smile. This is a very different result compared to the Heston small-maturity forward smile (see Remark 2.3.3(iv)), where large differences emerge between the forward smile and the spot smile at zeroth order.

We now give an example illustrating some of the differences between the Heston large-maturity forward smile and the large-maturity spot smile due to first-order differences in the asymptotic (2.2.13). This ties in with Remark 2.3.6(v). Specifically we look at the forward at-the-money volatility which, when using Proposition 2.3.5 with $\rho_- \leq \rho \leq \min(\rho_+, \kappa/\xi)$, satisfies

$\sigma^2_{t,\tau}(0) = v_0^\infty(0) + v_1^\infty(0,t)/\tau + \mathcal{O}\left(1/\tau^2\right)$, as τ tends to infinity, with

$$\begin{aligned} v_0^\infty(0) &= \frac{4\theta\kappa(\eta - 2\kappa + \xi\rho)}{\xi^2\left(1-\rho^2\right)}, \\ v_1^\infty(0,t) &= \frac{16\kappa v\left(\rho\xi - 2\kappa + \eta\right)}{\Delta\xi^2} + \frac{16\kappa\theta}{\xi^2}\log\left(\frac{\Delta \mathrm{e}^{-\kappa t}\left(2\kappa - \xi\rho + \left(1-2\rho^2\right)\eta\right)}{8\kappa\left(1-\rho^2\right)^2\eta}\right) \\ &\quad - 8\log\left(\frac{\xi\left(1-\rho^2\right)^{3/2}\sqrt{\eta\left(2\xi\rho - 4\kappa + 2\eta\right)}}{\left(\xi\left(1-2\rho^2\right) - \rho(\eta - 2\kappa)\right)\left(\rho(\eta-2\kappa)+\xi\right)}\right); \end{aligned}$$

η is defined in (2.3.7) and $\Delta := 2\kappa\left(1+\mathrm{e}^{\kappa t}\left(1-2\rho^2\right)\right) - \left(1-\mathrm{e}^{\kappa t}\right)\left(\rho\xi+\eta\right)$. To get an idea of the t-dependence of the at-the-money forward volatility we set $\rho = 0$ (since Proposition 2.3.5 is valid for correlations near zero) and perform a Taylor expansion of $v_1^\infty(0,t)$ around $t = 0$: $v_1^\infty(0,t) = v_1^\infty(0,0) + \left(\frac{2\theta}{1+\sqrt{1+\xi^2/4\kappa^2}} - v\right)t + \mathcal{O}\left(t^2\right)$. When $v \geq \theta$ then at this order the large τ-maturity forward at-the-money volatility is lower than the corresponding large τ-maturity at-the-money implied volatility and this difference is increasing in t and in the ratio ξ/κ. This is similar in spirit to Remark 2.3.3(ii) for the small-maturity Heston forward smile.

2.3.2 Schöbel-Zhu

In this Section we apply our general results to the Schöbel-Zhu (SZ) stochastic volatility model (1.3.10). The forward cgf for the SZ model was derived in Lemma 1.3.3. The analysis in this section is similar to the diagonal small-maturity Heston one and we therefore omit the proofs, only highlighting the similarities and differences between the two models.

Proposition 2.3.7. *In the Schöbel-Zhu model Corollary 2.2.6 and Proposition 2.2.10 hold with* $\mathcal{D}_0 = \mathcal{K}_{t,\tau}$ *and* $\Lambda_0 = \Xi$*, where* $\mathcal{K}_{t,\tau}$ *and* Ξ *are defined in 2.3.1.*

At zeroth order in ε the SZ diagonal small-maturity forward smile is the same as in Heston modulo a re-scaling of the volatility of volatility. The first-order asymptotic is used in Corollary 2.3.8 below to highlight differences with the Heston model. In order to gain some intuition on the role of the Schöbel-Zhu parameters on the forward smile we expand our solution (to first order in ε) around the at-the-money point in terms of the log-strike k.

Corollary 2.3.8. *The following expansion holds for the Schöbel-Zhu forward smile as* ε *and* k *tend to zero:*

$$\sigma^2_{\varepsilon t,\varepsilon\tau}(k) = v + \varepsilon\nu_0(t,\tau) + \left(\frac{\xi\rho}{2} + \varepsilon\nu_1(t,\tau)\right)k + \left(\frac{(4-7\rho^2)\xi^2}{48v} + \frac{\xi^2 t}{4\tau v} + \varepsilon\nu_2(t,\tau)\right)k^2 + \mathcal{O}(k^3) + \mathcal{O}(\varepsilon^2),$$

where

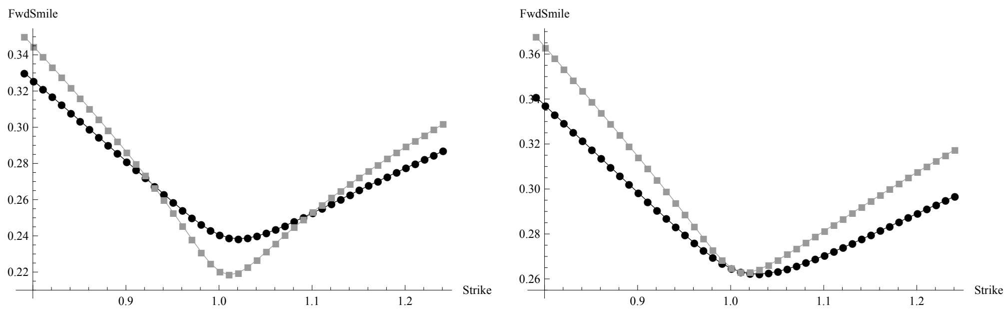
$$\nu_0(t,\tau) := \tau\left(\frac{1}{48}\xi^2\left(\rho^2+2\right)+\kappa\theta\sqrt{v}+\frac{1}{4}v(\xi\rho-4\kappa)\right)+2\kappa t\sqrt{v}\left(\theta-\sqrt{v}\right),$$
$$\nu_1(t,\tau) := \frac{\rho\xi\tau\left(\xi^2\left(1-2\rho^2\right)-8\kappa v+2\xi\rho v\right)}{48v}+\frac{\xi^3\rho t}{8v},$$
$$\nu_2(t,\tau) := \Big(\left(521\rho^4-452\rho^2+56\right)\xi^2+480\kappa\theta\sqrt{v}\left(2\rho^2-1\right)+40\rho^2 v(\rho\xi-4\kappa)\Big)\frac{\xi^2\tau}{7680v^2}$$
$$-\frac{\xi^2 t}{48v^2}\Big(\left(14\rho^2-5\right)\xi^2+2\kappa\theta\sqrt{v}\left(10-7\rho^2\right)+2\kappa v\left(7\rho^2-4\right)\Big)+\frac{\xi^2t^2}{16\tau v^2}\Big(3\xi^2$$
$$+4\kappa\sqrt{v}\left(\sqrt{v}-2\theta\right)\Big).$$

Remark 2.3.9. At this order we can make the following remarks concerning the SZ forward smile:

(i) Remark 2.3.3(iv) for the Heston forward smile also applies to the SZ forward smile.

(ii) The forward ATM volatility has a different dependence on the volatility of volatility ξ in Heston and SZ. In Heston (Remark 2.3.3(ii)), $\sigma_{\varepsilon t,\varepsilon\tau}(0)-\sigma_{0,\varepsilon\tau}(0)$ is decreasing in ξ. In SZ, Corollary 2.3.8 implies $\sigma_{\varepsilon t,\varepsilon\tau}(0)=\sigma_{0,\varepsilon\tau}(0)+(\theta-\sqrt{v})\kappa t\varepsilon+\mathcal{O}(\varepsilon^2)$, as $\varepsilon\downarrow 0$, which does not depend on ξ (up to an error of order $\mathcal{O}(\varepsilon^2)$). For realistic parameter choices ($\rho\le 0$) the Heston ATM forward volatility is decreasing in ξ while (for example when $\xi>2v$) it is increasing in ξ in SZ and the impact is small. This effect is illustrated in Figure 2.3.

An analysis analogous to that of the Heston model can be conducted for the large-maturity SZ forward smile, but we omit it here for brevity.



(a) Heston forward smile ξ dependence

(b) SZ forward smile ξ dependence

Figure 2.3: Here $t=1/2$ and $\tau=1/12$ and we apply Corollaries 2.3.8 and 2.3.2. Circles use the Heston parameters $v=0.07,\theta=0.07,\kappa=1,\rho=-0.6,\xi=0.3$ and SZ parameters $v=0.07,\theta=\sqrt{0.07},\kappa=1,\rho=-0.6,\xi=0.3$. Squares use the same parameters but with $\xi=0.4$.

2.3.3 Time-changed exponential Lévy

Due to Lemma 1.4.1, the forward smile in exponential Lévy models is time-homogeneous in the sense that $\sigma_{t,\tau}$ does not depend on t. This is not necessarily true in time-changed exponential

Lévy models as we shall now see. We shall consider here the two examples in Section 1.3.2 and we briefly recall the set-up. We let N be a Lévy process with cgf given by $\log \mathbb{E}\left(\mathrm{e}^{uN_t}\right) = t\phi(u)$ for $t \geq 0$ and $u \in \mathcal{K}_\phi := \{u \in \mathbb{R} : |\phi(u)| < \infty\}$. We consider models where $X := (N_{V_t})_{t\geq 0}$ pathwise and the time-change is given by $V_t := \int_0^t v_s \mathrm{d}s$ with v being a strictly positive process independent of N. We shall consider the two examples where v is a Feller diffusion (1.3.11) and where it is a Γ-OU process (1.3.12). We now define the functions $\widehat{V}$ and $\widehat{H}$ from $\widehat{\mathcal{K}}_\infty$ to $\mathbb{R}$, and the functions $\widetilde{V}$ and $\widetilde{H}$ from $\widetilde{\mathcal{K}}_\infty$ to $\mathbb{R}$ by

$$
\begin{aligned}
\widehat{V}(u) &:= \frac{\kappa\theta}{\xi^2}\left(\kappa - \sqrt{\kappa^2 - 2\phi(u)\xi^2}\right), \\
\widehat{H}(u) &:= \frac{\widehat{V}(u)v\mathrm{e}^{-\kappa t}}{\kappa\theta - 2\beta_t\widehat{V}(u)} - \frac{2\kappa\theta}{\xi^2}\log\left(\frac{\kappa\theta - 2\beta_t\widehat{V}(u)}{\kappa\theta\left(1-\gamma(\phi(u))\right)}\right), \\
\widetilde{V}(u) &:= \frac{\phi(u)\lambda\delta}{\alpha\lambda - \phi(u)}, \\
\widetilde{H}(u) &:= \frac{\lambda\alpha\delta}{\alpha\lambda - \phi(u)}\log\left(1 - \frac{\phi(u)}{\alpha\lambda}\right) + \frac{\phi(u)v\mathrm{e}^{-\lambda t}}{\lambda} + d\log\left(\frac{\phi(u) - \alpha\lambda\mathrm{e}^{\lambda t}}{\mathrm{e}^{t\lambda}(\phi(u) - \alpha\lambda)}\right),
\end{aligned}
\tag{2.3.11}
$$

where we set

$$
\widehat{\mathcal{K}}_\infty := \left\{u : \phi(u) \leq \kappa^2/(2\xi^2)\right\}, \qquad \text{and} \qquad \widetilde{\mathcal{K}}_\infty := \{u : \phi(u) < \alpha\lambda\}; \tag{2.3.12}
$$

ϕ is the Lévy exponent of N, β_t and γ are defined in (1.3.14) and the other model parameters are given in (1.3.11) and (1.3.12). The following proposition—proved in Section 2.5.3—is the main result of the section.

Proposition 2.3.10. *Suppose that ϕ is essentially smooth (Assumption 2.2.1(iv)), strictly convex and of class $\mathcal{C}^\infty$ on $\mathcal{K}_\phi^o$ with $\{0,1\} \subset \mathcal{K}_\phi^o$ and $\phi(1) = 0$. Then Corollary 2.2.8 and Proposition 2.2.11 hold:*

(i) when v follows (1.3.11)*, with $\Lambda_0 = \widehat{V}$, $\Lambda_1 = \widehat{H}$, $\Lambda_2 = 0$ and $\mathcal{D}_0 = \widehat{\mathcal{K}}_\infty$;*

(ii) when v follows (1.3.12)*, with $\Lambda_0 = \widetilde{V}$, $\Lambda_1 = \widetilde{H}$, $\Lambda_2 = 0$ and $\mathcal{D}_0 = \widetilde{\mathcal{K}}_\infty$;*

(iii) when $v_t \equiv 1$, with $\Lambda_0 = \phi$, $\Lambda_1 = 0$, $\Lambda_2 = 0$ and $\mathcal{D}_0 = \mathcal{K}_\phi$.

Remark 2.3.11.

(i) If $(B_t)_{t\geq 0}$ is a standard Brownian motion then the uncorrelated Heston model (1.3.2) can be represented as $N_t := -t/2 + B_t$ time-changed by an integrated Feller diffusion (1.3.11). With $\phi(u) \equiv u(u-1)/2$ and $\mathcal{K}_\phi = \mathbb{R}$, Proposition 2.3.10(i) agrees with Proposition 2.3.5.

(ii) The zeroth order large-maturity forward smile is the same as its corresponding zeroth order large-maturity spot smile and differences only emerge at first order. It seems plausible that this will always hold if there exists a stationary distribution for v and if v is independent of the Lévy process N;

(iii) Case (iii) in the proposition corresponds to the standard exponential Lévy case (without time-change).

We now use Proposition 2.3.10 to highlight the first-order differences in the large-maturity forward smile (2.2.13) and the corresponding spot smile. If v follows (1.3.11) then a Taylor expansion of v_1^∞ in (2.2.12) around $t = 0$ gives

$$v_1^\infty(t,k) = v_1^\infty(0,k) + \frac{8v_0^\infty(k)^2}{4k^2 - v_0^\infty(k)^2}\widehat{V}(u^*(k))\left(\frac{\xi^2 v\widehat{V}(u^*(k))}{2\theta^2\kappa^2} + 1 - \frac{v}{\theta}\right)t + \mathcal{O}(t^2),$$

for all $k \in \mathbb{R} \setminus \{\widehat{V}'(0), \widehat{V}'(1)\}$. Using simple properties of v_0^∞ and $\widehat{V}$ we see that the large-maturity forward smile is lower than the corresponding spot smile for $k \in (\widehat{V}'(0), \widehat{V}'(1))$ (which always includes the at-the-money) if $v \geq \theta$. The forward smile is higher than the corresponding spot smile for $k \in \mathbb{R}\backslash(\widehat{V}'(0), \widehat{V}'(1))$ (OTM options) if $v \leq \theta$, and these differences are increasing in ξ/κ and t. This effect is illustrated in Figure 2.4 and $k \in (\widehat{V}'(0), \widehat{V}'(1))$ corresponds to strikes in the region $(0.98, 1.02)$ in the figure.

If v follows (1.3.12) then a simple Taylor expansion of $v_1^\infty(\cdot, k)$ in (2.2.12) around $t = 0$ gives

$$v_1^\infty(t,k) = v_1^\infty(0,k) + \frac{8v_0^\infty(k)^2}{4k^2 - v_0^\infty(k)^2}\frac{\phi(u^*(k))\left[\lambda(\delta - \alpha v) + v\phi(u^*(k))\right]}{\alpha\lambda - \phi(u^*(k))}t + \mathcal{O}(t^2),$$

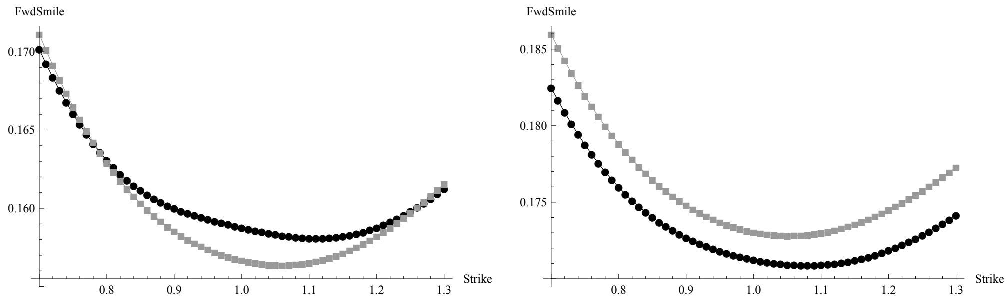
for all $k \in \mathbb{R} \setminus \{\widetilde{V}'(0), \widetilde{V}'(1)\}$. Similarly we deduce that the large-maturity forward smile is lower than the corresponding spot smile for $k \in (\widetilde{V}'(0), \widetilde{V}'(1))$ if $v \geq \delta/\alpha$. The forward smile is higher than the corresponding spot smile for $k \in \mathbb{R}\backslash(\widetilde{V}'(0), \widetilde{V}'(1))$ (OTM options) if $v \leq \delta/\alpha$, and these differences are increasing in t.

If v follows (1.3.11) (respectively (1.3.12)) then the stationary distribution is a gamma distribution with mean θ (resp. δ/α), see [44, page 475 and page 487]. The above results seem to indicate that the differences in level between the large-maturity forward smile and the corresponding spot smile depend on the relative values of v_0 and the mean of the stationary distribution of the process v. This is also similar to Remark 2.3.3(ii) and the analysis below Remark 2.3.6 for the Heston forward smile. These observations are also independent of the choice of ϕ indicating that the fundamental quantity driving the non-stationarity of the large-maturity forward smile over the corresponding spot implied volatility smile is the choice of time-change.

In the Variance-Gamma model [122], $\phi(u) \equiv \mu u + C\log\left(\frac{GM}{(M-u)(G+u)}\right)$, for $u \in (-G, M)$, with $C > 0$, $G > 0$, $M > 1$ and $\mu := -C\log\left(\frac{GM}{(M-1)(G+1)}\right)$ ensures that $(\mathrm{e}^{X_t})_{t\geq 0}$ is a true martingale ($\phi(1) = 0$). Clearly ϕ is essentially smooth, strictly convex and infinitely differentiable on $(-G, M)$ with $\{0,1\} \subset (-G, M)$; therefore Proposition 2.3.10 applies. For Proposition 2.3.10(iii), the solution to $\phi'(u^*(k)) = k$ is $u^*(\mu) = (M-G)/2$ and

$$u_\pm^*(k) = \frac{-2C - (G-M)(k-\mu) \pm \sqrt{4C^2 + (G+M)^2(k-\mu)^2}}{2(k-\mu)} \quad \text{for all } k \neq \mu.$$

The sign condition $(M-u)(G+u) > 0$ imposes $-2C \pm \sqrt{4C^2 + (G+M)^2(k-\mu)^2} > 0$ for all $k \neq \mu$. Hence u_+^* (continuous on the whole real line) is the only valid solution and the rate function is then given in closed-form as $\Lambda^*(k) = ku_+^*(k) - \phi(u_+^*(k))$ for all real k.



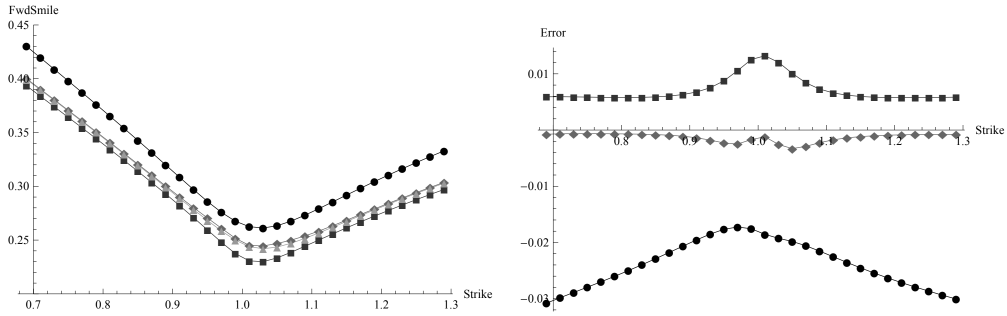
(a) Feller time-change: forward smile vs spot smile $v > \theta$.

(b) Feller time-change: forward smile vs spot smile $v < \theta$.

Figure 2.4: Circles represent $t = 0$ and $\tau = 2$ and squares represent $t = 1/2$ and $\tau = 2$ using a Variance-Gamma model time-changed by a Feller diffusion and the asymptotic in Proposition 2.3.10. In (a) the parameters are $C = 58.12$, $G = 50.5$, $M = 69.37$, $\kappa = 1.23$, $\theta = 0.9$, $\xi = 1.6$, $v = 1$ and (b) uses the same parameters but with $\theta = 1.1$.

2.4 Numerics

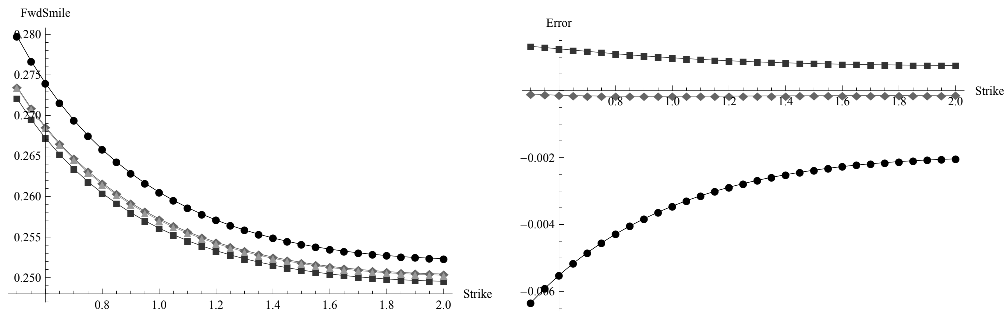
We compare here the true forward smile in various models and the asymptotics developed in Propositions 2.2.10 and 2.2.11. We calculate forward-start option prices using the inverse Fourier transform representation in Lemma 1.4.7 and a global adaptive Gauss-Kronrod quadrature scheme. We then compute the forward smile $\sigma_{t,\tau}$ using a simple root search and compare it to the zeroth, first and second order asymptotics given in Propositions 2.2.10 and 2.2.11 for various models. In Figure 2.5 we compare the Heston diagonal small-maturity asymptotic in Proposition 2.3.1 with the true forward smile. Figure 2.6 tests the accuracy of the Heston large-maturity asymptotic from Proposition 2.3.5. In order to use this proposition we require $\rho_- \leq \rho \leq \min(\rho_+, \kappa/\xi)$ with $\rho_\pm$ defined in (2.3.6). For the parameter choice in the figure we have $\rho_- = -0.65$ and the condition is satisfied. Finally in Figure 2.7 we consider the Variance Gamma model time-changed by a Γ-OU process using Proposition 2.3.10. Results are in line with expectations and the higher the order of the asymptotic the closer we match the true forward smile.



(a) Heston diagonal small-maturity vs Fourier inversion.

(b) Errors

Figure 2.5: In (a) circles, squares and diamonds represent the zeroth, first and second order asymptotics respectively in Proposition 2.3.1 and triangles represent the true forward smile using Fourier inversion. In (b) we plot the differences between the true forward smile and the asymptotic. Here, $t = 1/2$, $\tau = 1/12$, $v = 0.07$, $\theta = 0.07$, $\kappa = 1$, $\xi = 0.34$, $\rho = -0.8$.



(a) Heston Large-Maturity vs Fourier Inversion.

(b) Errors

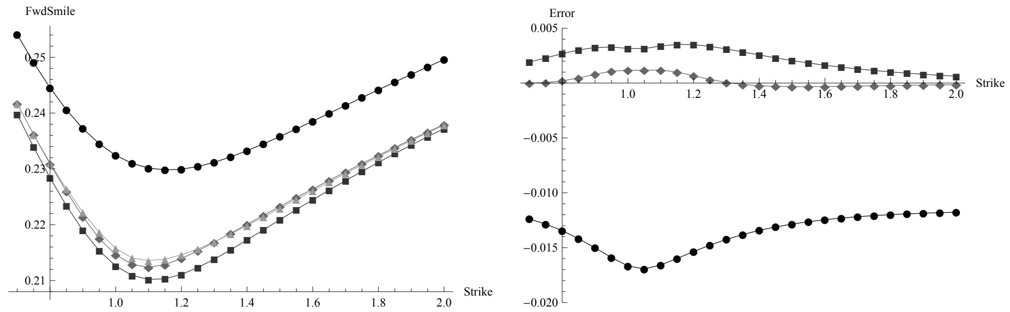
Figure 2.6: In (a) circles, squares and diamonds represent the zeroth, first and second order asymptotics respectively in Proposition 2.3.5 and triangles represent the true forward smile using Fourier inversion. In (b) we plot the differences between the true forward smile and the asymptotic. Here, $t = 1$, $\tau = 5$, $v = 0.07$, $\theta = 0.07$, $\kappa = 1.5$, $\xi = 0.34$, $\rho = -0.25$.

2.5 Proofs

2.5.1 Proofs of Section 2.2

2.5.1.1 Proof of Theorem 2.2.4

Our proof relies on several steps and is based on so-called sharp large deviations tools. We first —as in classical large deviations theory—define an asymptotic measure-change allowing for weak convergence of a rescaled version of $(Y_\varepsilon)_{\varepsilon>0}$. In Lemma 2.5.1 we derive the asymptotics of the characteristic function of this rescaled process under this new measure. The limit is a Gaussian



(a) Γ-OU time-change large-maturity / Fourier inversion.

(b) Errors

Figure 2.7: In (a) circles, squares and diamonds represent the zeroth, first and second order asymptotics respectively in Proposition 2.3.10 and triangles represent the true forward smile using Fourier inversion for a variance gamma model time-changed by a Γ-OU process. In (b) we plot the differences between the true forward smile and the asymptotic. We use $t = 1$ and $\tau = 3$ with the parameters $C = 6.5$, $G = 11.1$, $M = 33.4$, $v = 1$, $\alpha = 0.6$, $d = 0.6$, $\lambda = 1.8$.

characteristic function making all forthcoming computations analytically tractable. We then write the option price as an expectation of the rescaled process under the new measure (see (2.5.11)), and prove an inverse Fourier transform representation (Lemma 2.5.4) for sufficiently small ε. Splitting the integration domain (Equation (2.5.18)) of this inverse Fourier transform in two (compact interval and tails), (a) we integrate term by term the compact part, and (b) we show that Assumption 2.2.1(v) implies that the tail part is exponentially small (Lemma 2.5.3). We now start the analysis and define such a change of measure by

$$\frac{\mathrm{d}\mathbb{Q}_{k,\varepsilon}}{\mathrm{d}\mathbb{P}} = \exp\left(\frac{u^*(k)Y_\varepsilon}{\varepsilon} - \frac{\Lambda_\varepsilon\left(u^*(k)\right)}{\varepsilon}\right), \tag{2.5.1}$$

with $u^*(k)$ defined in (2.2.5). By Lemma 2.2.2(i), $u^*(k) \in \mathcal{D}_0^o$ for all $k \in \mathbb{R}$ and so $|\Lambda_\varepsilon\left(u^*(k)\right)|$ is finite for ε small enough since $\mathcal{D}_0 = \lim_{\varepsilon\downarrow 0}\{u \in \mathbb{R} : |\Lambda_\varepsilon(u)| < \infty\}$. Also $\mathrm{d}\mathbb{Q}_{k,\varepsilon}/\mathrm{d}\mathbb{P}$ is almost surely strictly positive and $\mathbb{E}\left(\mathrm{d}\mathbb{Q}_{k,\varepsilon}/\mathrm{d}\mathbb{P}\right) = 1$. Therefore (2.5.1) is a valid measure change for all $k \in \mathbb{R}$. We define the random variable

$$Z_{k,\varepsilon} := (Y_\varepsilon - k)/\sqrt{\varepsilon} \tag{2.5.2}$$

and set the characteristic function $\Phi_{Z_{k,\varepsilon}} : \mathbb{R} \to \mathbb{C}$ of $Z_{k,\varepsilon}$ in the $\mathbb{Q}_{k,\varepsilon}$-measure as follows

$$\Phi_{Z_{k,\varepsilon}}(u) = \mathbb{E}^{\mathbb{Q}_{k,\varepsilon}}\left(\mathrm{e}^{\mathrm{i}uZ_{k,\varepsilon}}\right). \tag{2.5.3}$$

Recall from Section 2.2 that $\Lambda_i := \Lambda_i(u^*(k))$ and $\Lambda_{i,l} := \partial_u^l \Lambda_i(u)\big|_{u=u^*(k)}$; we first start with the following important technical lemma.

Lemma 2.5.1. *The following expansion holds as $\varepsilon \downarrow 0$:*

$$\Phi_{Z_{k,\varepsilon}}(u) = \mathrm{e}^{-\frac{\Lambda_{0,2}u^2}{2}}\Big(1 + \eta_1(u)\sqrt{\varepsilon} + \left(\frac{\eta_1^2(u)}{2} + \eta_2(u)\right)\varepsilon + \left(\frac{\eta_1^3(u)}{6} + \eta_1(u)\eta_2(u) + \eta_3(u)\right)\varepsilon^{3/2} + \mathcal{R}(u,\varepsilon)\Big),$$

with the functions η_i, $i = 1, 2, 3$ defined in (2.5.6) *and $\mathcal{R}(u,\varepsilon) = \mathcal{O}(\varepsilon^2)$. Furthermore, for $|u| \leq \varepsilon^{-1/6}$, the remainder can be written $\mathcal{R}(u,\varepsilon) = \max(1, |u|^{12})\mathcal{O}(\varepsilon^2)$ where $\mathcal{O}(\varepsilon^2)$ is uniform in u.*

Remark 2.5.2. By Lévy's Convergence Theorem [147, Page 185, Theorem 18.1], $Z_{k,\varepsilon}$ defined in (2.5.2) converges weakly to a normal random variable with mean 0 and variance $\Lambda_{0,2}$ in the $\mathbb{Q}_{k,\varepsilon}$-measure as ε tends to zero.

Proof. Using the measure change in (2.5.1) we write

$$\begin{aligned}
\log \Phi_{Z_{k,\varepsilon}}(u) &= \log \mathbb{E}^{\mathbb{P}}\left(\frac{\mathrm{d}\mathbb{Q}_{k,\varepsilon}}{\mathrm{d}\mathbb{P}}\mathrm{e}^{\mathtt{i}uZ_{k,\varepsilon}}\right) \\
&= \log \mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{u^*(k)Y_\varepsilon}{\varepsilon} - \frac{\Lambda_\varepsilon(u^*(k))}{\varepsilon}\right)\exp\left(\mathtt{i}u\sqrt{\varepsilon}\left(\frac{Y_\varepsilon}{\varepsilon}\right) - \frac{\mathtt{i}ku}{\sqrt{\varepsilon}}\right)\right] \\
&= -\frac{1}{\varepsilon}\Lambda_\varepsilon\left(u^*(k)\right) - \frac{\mathtt{i}uk}{\sqrt{\varepsilon}} + \log \mathbb{E}^{\mathbb{P}}\left[\exp\left(\left(\frac{Y_\varepsilon}{\varepsilon}\right)\left(\mathtt{i}u\sqrt{\varepsilon} + u^*(k)\right)\right)\right] \\
&= -\frac{\mathtt{i}uk}{\sqrt{\varepsilon}} + \frac{1}{\varepsilon}\left(\Lambda_\varepsilon\left(\mathtt{i}u\sqrt{\varepsilon} + u^*(k)\right) - \Lambda_\varepsilon\left(u^*(k)\right)\right). \qquad (2.5.4)
\end{aligned}$$

Since Λ_ε is analytic [119, Theorem 7.1.1] on the set $\{z \in \mathbb{C} : \Re(z) \in \mathcal{D}_0^o\}$ for ε small enough, we have the Taylor expansion

$$\log \Phi_{Z_{k,\varepsilon}}(u) = -\frac{\mathtt{i}uk}{\sqrt{\varepsilon}} + \frac{1}{\varepsilon}\sum_{n=1}^{5}\Lambda_\varepsilon^{(n)}(u^*(k))\frac{(\mathtt{i}u\sqrt{\varepsilon})^n}{n!} + \frac{(\mathtt{i}u\sqrt{\varepsilon})^6}{6!}\frac{1}{\varepsilon}\Lambda_\varepsilon^{(6)}(u^*(k) + \mathtt{i}A),$$

with $A \in (-|u|\sqrt{\varepsilon}, |u|\sqrt{\varepsilon})$ and where we have used the Lagrange form of the remainder in Taylor's theorem. By [128, Theorem 1.8.5] the asymptotic for Λ_ε in 2.2.2 can be differentiated with respect to u due to Assumption 2.2.1(ii) and therefore we write

$$\begin{aligned}
\log \Phi_{Z_{k,\varepsilon}}(u) = -\frac{\mathtt{i}uk}{\sqrt{\varepsilon}} &+ \frac{1}{\varepsilon}\sum_{n=1}^{5}(\Lambda_{0,n} + \Lambda_{1,n}\varepsilon + \Lambda_{2,n}\varepsilon^2)\frac{(\mathtt{i}u\sqrt{\varepsilon})^n}{n!} \\
&+ \frac{1}{\varepsilon}\sum_{n=1}^{5}\mathcal{O}(\varepsilon^3)\frac{(\mathtt{i}u\sqrt{\varepsilon})^n}{n!} + \frac{(\mathtt{i}u\sqrt{\varepsilon})^6}{6!}\frac{1}{\varepsilon}\Lambda_\varepsilon^{(6)}(u^*(k) + \mathtt{i}A).
\end{aligned}$$

We now set $|u|\sqrt{\varepsilon} \leq 1$ and note that $A \in [-1, 1]$. Since Λ_ε is analytic, the function $U : \mathbb{R} \ni x \mapsto |\Lambda_\varepsilon^{(6)}(u^*(k) + \mathtt{i}x)|$ is continuous on the compact set $[-1, 1]$ and attains its maximum at some point on this set. Again by [128, Theorem 1.8.5] and Assumption 2.2.1(ii) we have that $\Lambda_\varepsilon^{(6)}(u^*(k) + \mathtt{i}A) = \Lambda_0^{(6)}(u^*(k) + \mathtt{i}A) + \mathcal{O}(\varepsilon)$, as ε tends to zero. The function $V : \mathbb{R} \ni x \mapsto |\Lambda_0^{(6)}(u^*(k) + \mathtt{i}x)|$ is continuous on the compact set $[-1, 1]$ and attains its maximum at some point on this set. Hence $\frac{(\mathtt{i}u\sqrt{\varepsilon})^6}{6!}\frac{1}{\varepsilon}\Lambda_\varepsilon^{(6)}(u^*(k) + \mathtt{i}A) = |u|^6\mathcal{O}(\varepsilon^2)$ where the remainder $\mathcal{O}(\varepsilon^2)$ is uniform in

u. Further $\frac{1}{\varepsilon}\sum_{n=1}^{5}\mathcal{O}(\varepsilon^3)\frac{(\mathtt{i}u\sqrt{\varepsilon})^n}{n!} = \mathcal{O}(\varepsilon^2)$. We therefore write for $|u|\sqrt{\varepsilon} \le 1$ (and using (2.2.5)):

$$\begin{aligned}\log \Phi_{Z_{k,\varepsilon}}(u) &= -\Lambda_{0,2}\frac{u^2}{2} + \frac{1}{\varepsilon}\sum_{n=3}^{5}\Lambda_{0,n}\frac{(\mathtt{i}u\sqrt{\varepsilon})^n}{n!} + \sum_{n=1}^{3}\Lambda_{1,n}\frac{(\mathtt{i}u\sqrt{\varepsilon})^n}{n!} + \mathtt{i}\Lambda_{2,1}u\varepsilon^{3/2} + \max(1,|u|^6)\mathcal{O}(\varepsilon^2)\\ &= -\frac{1}{2}\Lambda_{0,2}u^2 + \eta_1(u)\sqrt{\varepsilon} + \eta_2(u)\varepsilon + \eta_3(u)\varepsilon^{3/2} + \max(1,|u|^6)\mathcal{O}(\varepsilon^2), \end{aligned} \tag{2.5.5}$$

where the remainder $\mathcal{O}(\varepsilon^2)$ is uniform in u and where we define the functions

$$\begin{aligned}\eta_1(u) &:= \mathtt{i}u\Lambda_{1,1} - \frac{\mathtt{i}u^3}{6}\Lambda_{0,3}, \quad \eta_2(u) := -\frac{u^2}{2}\Lambda_{1,2} + \frac{u^4}{24}\Lambda_{0,4},\\ \eta_3(u) &:= \mathtt{i}u\Lambda_{2,1} - \frac{\mathtt{i}u^3}{6}\Lambda_{1,3} + \frac{\mathtt{i}u^5}{120}\Lambda_{0,5}.\end{aligned} \tag{2.5.6}$$

Note that the $\mathcal{O}(\varepsilon^2)$ terms in the sum can be absorbed into the remainder since the powers of u are smaller than the u in the remainder term. The Lagrange form of the remainder in Taylor's theorem yields $e^x = 1 + x + e^{\zeta}\frac{x^2}{2}$ for any x and some $\zeta \in [-|x|,|x|]$; since that all terms in (2.5.5) but the first one are bounded for $|u| \le \varepsilon^{-1/6}$,

$$\begin{aligned}\Phi_{Z_{k,\varepsilon}}(u) =& e^{-\frac{\Lambda_{0,2}u^2}{2}}\Bigg(1 + \eta_1(u)\sqrt{\varepsilon} + \left(\frac{\eta_1^2(u)}{2} + \eta_2(u)\right)\varepsilon + \left(\frac{\eta_1^3(u)}{6} + \eta_1(u)\eta_2(u) + \eta_3(u)\right)\varepsilon^{3/2}\\ &+ \max(1,|u|^{12})\mathcal{O}(\varepsilon^2)\Bigg).\end{aligned}$$

□

We prove now that under Assumption 2.2.1(v) the tail integral $\left|\int_{|u|>\varepsilon^{-1/6}}\Phi_{Z_{k,\varepsilon}}(u)\overline{C_{\varepsilon,k}(u)}\mathrm{d}u\right|$ is exponentially small, where $\Phi_{Z_{k,\varepsilon}}$ is defined in (2.5.3) and $C_{\varepsilon,k} : \mathbb{R} \to \mathbb{C}$ is defined by

$$C_{\varepsilon,k}(u) := \frac{\varepsilon^{3/2}f(\varepsilon)}{(u^*(k) - \mathtt{i}u\sqrt{\varepsilon})\,(u^*(k) - \varepsilon f(\varepsilon) - \mathtt{i}u\sqrt{\varepsilon})}. \tag{2.5.7}$$

Note that its complex conjugate is then given by

$$\overline{C_{\varepsilon,k}(u)} = \frac{\varepsilon^{3/2}f(\varepsilon)}{(u^*(k) + \mathtt{i}u\sqrt{\varepsilon})\,(u^*(k) - \varepsilon f(\varepsilon) + \mathtt{i}u\sqrt{\varepsilon})}, \tag{2.5.8}$$

and the simple bounds follow:

$$\left|\overline{C_{\varepsilon,k}(u)}\right| \le \min\left(\frac{\sqrt{\varepsilon}f(\varepsilon)}{u^2}, \frac{\varepsilon^{3/2}f(\varepsilon)}{|u^*(k)(u^*(k) - \varepsilon f(\varepsilon))|}\right). \tag{2.5.9}$$

Therefore the tail estimates (using the change of variable $z = u\sqrt{\varepsilon}$)

$$\begin{aligned}\left|\int_{|u|>1/\sqrt{\varepsilon}}\Phi_{Z_{k,\varepsilon}}(u)\overline{C_{\varepsilon,k}(u)}\mathrm{d}u\right| &\le \frac{1}{\sqrt{\varepsilon}}\int_{|z|>1}\left|\Phi_{Z_{k,\varepsilon}}(z/\sqrt{\varepsilon})\right|\left|\overline{C_{\varepsilon,k}(z/\sqrt{\varepsilon})}\right|\mathrm{d}z \\ &\le \varepsilon f(\varepsilon)\int_{|z|>1}\frac{\mathrm{d}z}{z^2} < \infty,\end{aligned} \tag{2.5.10}$$

$$\begin{aligned}\left|\int_{\varepsilon^{-\frac{1}{6}}<|u|<\varepsilon^{-\frac{1}{2}}}\Phi_{Z_{k,\varepsilon}}(u)\overline{C_{\varepsilon,k}(u)}\mathrm{d}u\right| &\le \frac{1}{\sqrt{\varepsilon}}\int_{\varepsilon^{\frac{1}{3}}<|z|<1}\left|\Phi_{Z_{k,\varepsilon}}(z/\sqrt{\varepsilon})\right|\left|\overline{C_{\varepsilon,k}(z/\sqrt{\varepsilon})}\right|\mathrm{d}z\\ &\le \frac{2\varepsilon f(\varepsilon)(1-\varepsilon^{1/3})}{|u^*(k)(u^*(k) - \varepsilon f(\varepsilon))|} < \infty,\end{aligned}$$

are finite for sufficiently small ε since $f(\varepsilon)\varepsilon = c+\mathcal{O}(\varepsilon)$ and $u^*(k) \notin \{0, c\}$. We now proceed to show that Assumption 2.2.1(v) allows us to further conclude that these terms are in fact exponentially small:

Lemma 2.5.3. *There exists $\beta > 0$ such that the tail estimate $\left|\int_{|u|>\varepsilon^{-1/6}} \Phi_{Z_{k,\varepsilon}}(u)\overline{C_{\varepsilon,k}(u)}\mathrm{d}u\right| = \mathcal{O}(\mathrm{e}^{-\beta/\varepsilon^{1/3}})$ holds for all $k \notin \{\Lambda_{0,1}(0), \Lambda_{0,1}(c)\}$ as ε tends to zero.*

Proof. We break the proof into two parts. We first show that $\left|\int_{|u|>\varepsilon^{-1/2}} \Phi_{Z_{k,\varepsilon}}(u)\overline{C_{\varepsilon,k}(u)}\mathrm{d}u\right| = \mathcal{O}(\mathrm{e}^{-\alpha/\varepsilon})$ for some $\alpha > 0$ and then that $\left|\int_{\varepsilon^{-1/6}<|u|<\varepsilon^{-1/2}} \Phi_{Z_{k,\varepsilon}}(u)\overline{C_{\varepsilon,k}(u)}\mathrm{d}u\right| = \mathcal{O}(\mathrm{e}^{-\beta/\varepsilon^{1/3}})$ for some $\beta > 0$.

Using (2.5.4), $\Phi_{Z_{k,\varepsilon}}(u) = \exp\left[-\frac{\mathtt{i}uk}{\sqrt{\varepsilon}} + \frac{1}{\varepsilon}\left(\Lambda_\varepsilon\left(\mathtt{i}u\sqrt{\varepsilon} + u^*(k)\right) - \Lambda_\varepsilon\left(u^*(k)\right)\right)\right]$. Let $\mathcal{R}(\varepsilon, z) \equiv \mathcal{R}_0(\varepsilon, z) + \mathcal{R}_1(\varepsilon)$, with $\mathcal{R}_0(\varepsilon, z) := \frac{1}{\varepsilon}\left[\Re\left(\Lambda_\varepsilon\left(\mathtt{i}z + u^*(k)\right)\right) - \Re\left(\Lambda_0\left(\mathtt{i}z + u^*(k)\right)\right)\right]$ and $\mathcal{R}_1(\varepsilon) := \frac{1}{\varepsilon}\left[\Lambda_0\left(u^*(k)\right) - \Lambda_\varepsilon\left(u^*(k)\right)\right]$, so that

$$|\Phi_{Z_{k,\varepsilon}}(z/\sqrt{\varepsilon})| = \exp\left[\frac{1}{\varepsilon}\left(\Re\left(\Lambda_0\left(\mathtt{i}z + u^*(k)\right)\right) - \Lambda_0\left(u^*(k)\right)\right) + \mathcal{R}(\varepsilon, z)\right].$$

Set $F(z) := \Re\left(\Lambda_0\left(\mathtt{i}z + u^*(k)\right)\right) - \Lambda_0\left(u^*(k)\right)$. Using (2.5.9) the tail estimate is then given by

$$\begin{aligned}\left|\int_{|u|>1/\sqrt{\varepsilon}} \Phi_{Z_{k,\varepsilon}}(u)\overline{C_{\varepsilon,k}(u)}\mathrm{d}u\right| &\leq \frac{1}{\sqrt{\varepsilon}}\int_{|z|>1} \left|\Phi_{Z_{k,\varepsilon}}(z/\sqrt{\varepsilon})\right| \left|\overline{C_{\varepsilon,k}(z/\sqrt{\varepsilon})}\right| \mathrm{d}z \\ &\leq \varepsilon f(\varepsilon)\int_{|z|>1} \mathrm{e}^{F(z)/\varepsilon+\mathcal{R}(\varepsilon,z)}\frac{\mathrm{d}z}{z^2}.\end{aligned}$$

Consider first the case $z > 1$:

$$\begin{aligned}\int_{z>1} \mathrm{e}^{F(z)/\varepsilon+\mathcal{R}(\varepsilon,z)}\frac{\mathrm{d}z}{z^2} &= \mathbb{1}_{\{p_i^*>1\}}\int_1^{p_i^*} \mathrm{e}^{F(z)/\varepsilon+\mathcal{R}(\varepsilon,z)}\frac{\mathrm{d}z}{z^2} + \int_{z>\max(p_i^*,1)} \mathrm{e}^{F(z)/\varepsilon+\mathcal{R}(\varepsilon,z)}\frac{\mathrm{d}z}{z^2} \\ &\leq \frac{(p_i^*-1)^+\mathrm{e}^{F(\widetilde{p}_i)/\varepsilon+\mathcal{R}(\varepsilon,\widetilde{p}_i)}}{\widetilde{p}_i^2} + \int_{z>p_i^*} \mathrm{e}^{F(z)/\varepsilon+\mathcal{R}(\varepsilon,z)}\frac{\mathrm{d}z}{z^2},\end{aligned}$$

where the first integral on the rhs follows from the extreme value theorem which implies that the integrand attains its maximum on $[1, p_i^*]$ at some point $\widetilde{p}_i$ and the inequality for the second integral on the rhs follows since the integrand is positive. Using Assumption 2.2.1(v)(c), for $z > p_i^*$ there exists $\varepsilon_1 > 0$ and M (independent of z) such that $\mathcal{R}_0(\varepsilon, z) < M$ for $\varepsilon < \varepsilon_1$. In particular for $\varepsilon < \varepsilon_1$ we have

$$\int_{z>1} \mathrm{e}^{F(z)/\varepsilon+\mathcal{R}(\varepsilon,z)}\frac{\mathrm{d}z}{z^2} \leq \frac{(p_i^*-1)^+\mathrm{e}^{F(\widetilde{p}_i)/\varepsilon+\mathcal{R}(\varepsilon,\widetilde{p}_i)}}{\widetilde{p}_i^2} + \mathrm{e}^{M+\mathcal{R}_1(\varepsilon)}\int_{z>p_i^*} \mathrm{e}^{F(z)/\varepsilon}\frac{\mathrm{d}z}{z^2}.$$

From Assumption 2.2.1(i), both $\mathcal{R}_1(\varepsilon)$ and $\mathcal{R}(\varepsilon, \widetilde{p}_i)$ are of order $\mathcal{O}(1)$. By a similar argument to (2.5.10) the integral on the rhs is finite and we now use the Laplace method. Since F is continuous, has a unique maximum at $z = 0$ and is bounded away from zero as $|z|$ tends to infinity (Assumption 2.2.1(v)(b)) there exists $z_+^* > 0$ such that $F(z_+^*) > F(z)$ for $z > z_+^*$; hence

$$\int_{z>p_i^*} \mathrm{e}^{F(z)/\varepsilon}\frac{\mathrm{d}z}{z^2} \leq \int_{z>\min(p_i^*,z_+^*)} \mathrm{e}^{F(z)/\varepsilon}\frac{\mathrm{d}z}{z^2} \leq \frac{(z_+^*-p_i^*)^+\mathrm{e}^{F(z_+)/\varepsilon}}{z_+^2} + \int_{z>z_+^*} \mathrm{e}^{F(z)/\varepsilon}\frac{\mathrm{d}z}{z^2},$$

where again the final step follows from the extreme value theorem: if $z_+^* > p_i^*$ the integrand attains its maximum on $[p_i^*, z_+^*]$ at z_+. Since the contribution of the last integral is centred around $z = z_+^*$ as $\varepsilon \downarrow 0$, the Laplace method with concentration at the boundary yields (see Theorem 1.2.6, and using the fact that $F \in \mathcal{C}^3(\mathbb{R})$ by Assumption 2.2.1(v)(b))

$$\int_{z>z_+^*} \mathrm{e}^{F(z)/\varepsilon} \frac{\mathrm{d}z}{z^2} \sim -\frac{\varepsilon \mathrm{e}^{2F(z_+^*)/\varepsilon}}{2F'(z_+^*)(z_+^*)^2}.$$

A similar argument holds for $z < -1$ and therefore $\left|\int_{|u|>\varepsilon^{-1/2}} \Phi_{Z_{k,\varepsilon}}(u)\overline{C_{\varepsilon,k}(u)}\mathrm{d}u\right| = \mathcal{O}(\mathrm{e}^{-\alpha/\varepsilon})$, for some $\alpha > 0$. We now consider the case $\varepsilon^{-1/6} < |u| < \varepsilon^{-1/2}$. Using (2.5.9) this tail estimate is given by

$$\begin{aligned}\left|\int_{\varepsilon^{-\frac{1}{6}}<|u|<\varepsilon^{-\frac{1}{2}}} \Phi_{Z_{k,\varepsilon}}(u)\overline{C_{\varepsilon,k}(u)}\mathrm{d}u\right| &\leq \frac{1}{\sqrt{\varepsilon}} \int_{\varepsilon^{\frac{1}{3}}<|z|<1} \left|\Phi_{Z_{k,\varepsilon}}(z/\sqrt{\varepsilon})\right| \left|\overline{C_{\varepsilon,k}(z/\sqrt{\varepsilon})}\right| \mathrm{d}z \\ &\leq \frac{\varepsilon f(\varepsilon)}{|u^*(k)(u^*(k)-\varepsilon f(\varepsilon))|} \int_{\varepsilon^{1/3}<|z|<1} \mathrm{e}^{F(z)/\varepsilon+\mathcal{R}(\varepsilon,z)}\mathrm{d}z.\end{aligned}$$

Let us now estimate the last integral, and, for simplicity consider only the positive side $(\varepsilon^{1/3}, 1)$. Since $F \in \mathcal{C}^3(\mathbb{R})$ has a unique maximum at the origin (Assumption 2.2.1(v)(b)) and $F''(0) = -\Lambda_0''(u^*(k)) < 0$ (Assumption 2.2.1(iv)), then it is strictly decreasing in an open neighbourhood $(0,\eta) \subset (0,1)$ of it. Take now $\varepsilon > 0$ small enough so that $\varepsilon^{1/3} \in (0,\eta)$. The extreme value theorem and the fact that $\mathcal{R}(\varepsilon, z) = \mathcal{O}(1)$ implies that

$$\begin{aligned}\int_{(\varepsilon^{1/3},\eta)} \mathrm{e}^{F(z)/\varepsilon+\mathcal{R}(\varepsilon,z)}\mathrm{d}z &\leq \mathrm{e}^{F(\varepsilon^{1/3})/\varepsilon} \max_{z\in(\varepsilon^{1/3},\eta)} \mathrm{e}^{\mathcal{R}(\varepsilon,z)} \left(\eta - \varepsilon^{1/3}\right) \\ &\leq \mathrm{e}^{F(\varepsilon^{1/3})/\varepsilon} \max_{z\in[0,1]} \mathrm{e}^{\mathcal{R}(\varepsilon,z)} \leq M\mathrm{e}^{F(\varepsilon^{1/3})/\varepsilon} = M\mathrm{e}^{-\Lambda_{0,2}/(2\varepsilon^{1/3})+\mathcal{O}(1)},\end{aligned}$$

for some $M > 0$. The final equality follows from the expansion $F(\varepsilon^{1/3}) = \Re\left(\Lambda_0\left(\mathtt{i}\varepsilon^{1/3} + u^*(k)\right)\right) - \Lambda_0\left(u^*(k)\right) = -\Lambda_{0,2}\varepsilon^{2/3}/2 + \mathcal{O}(\varepsilon)$. Now, on $(\eta, 1)$, the function F might not be decreasing but has a maximum, say at $z_\eta \in [\eta, 1]$, and hence, similarly, there exists a constant $m > 0$ such that

$$\int_{(\eta,1)} \mathrm{e}^{F(z)/\varepsilon+\mathcal{R}(\varepsilon,z)}\mathrm{d}z \leq m\mathrm{e}^{-|F(z_\eta)|/\varepsilon}.$$

Since $F(z_\eta) < 0$ does not depend on ε, the result follows. □

With these preliminary results, we can now move on to the actual proof of Theorem 2.2.4. For $j = 1, 2, 3$, let us define the functions $g_j : \mathbb{R}_+^2 \to \mathbb{R}_+$ by

$$g_j(x,y) := \begin{cases} (x-y)^+, & \text{if } j = 1, \\ (y-x)^+, & \text{if } j = 2, \\ \min(x,y), & \text{if } j = 3. \end{cases}$$

Using the definition of the $\mathbb{Q}_{k,\varepsilon}$-measure in (2.5.1) the option prices in Theorem 2.2.4 can be written as

$$\mathbb{E}\left[g_j\left(\mathrm{e}^{Y_\varepsilon f(\varepsilon)},\mathrm{e}^{kf(\varepsilon)}\right)\right]=\mathrm{e}^{\frac{1}{\varepsilon}\Lambda_\varepsilon(u^*(k))}\mathbb{E}^{\mathbb{Q}_{k,\varepsilon}}\left[\mathrm{e}^{-\frac{u^*(k)}{\varepsilon}Y_\varepsilon}g_j\left(\mathrm{e}^{Y_\varepsilon f(\varepsilon)},\mathrm{e}^{kf(\varepsilon)}\right)\right]$$
$$=\mathrm{e}^{-\frac{1}{\varepsilon}[ku^*(k)-\Lambda_\varepsilon(u^*(k))]}\mathbb{E}^{\mathbb{Q}_{k,\varepsilon}}\left[\mathrm{e}^{-\frac{u^*(k)}{\varepsilon}(Y_\varepsilon-k)}g_j\left(\mathrm{e}^{Y_\varepsilon f(\varepsilon)},\mathrm{e}^{kf(\varepsilon)}\right)\right]. \quad (2.5.11)$$

By the expansion in Assumption 2.2.1(i) and Equality (2.2.6) we immediately have

$$\exp\left(-\frac{1}{\varepsilon}\left(ku^*(k)-\Lambda_\varepsilon\left(u^*(k)\right)\right)\right)=\exp\left(-\frac{1}{\varepsilon}\Lambda^*(k)+\Lambda_1+\Lambda_2\,\varepsilon+\mathcal{O}\left(\varepsilon^2\right)\right). \quad (2.5.12)$$

From the definition of the random variable $Z_{k,\varepsilon}$ in (2.5.2) we obtain

$$\mathbb{E}^{\mathbb{Q}_{k,\varepsilon}}\left[\mathrm{e}^{-\frac{u^*(k)}{\varepsilon}(Y_\varepsilon-k)}g_j\left(\mathrm{e}^{Y_\varepsilon f(\varepsilon)},\mathrm{e}^{kf(\varepsilon)}\right)\right]=\mathrm{e}^{kf(\varepsilon)}\mathbb{E}^{\mathbb{Q}_{k,\varepsilon}}\left[\tilde{g}_j(Z_{k,\varepsilon})\right],$$

where for $j=1,2,3$, we define the modified payoff functions $\tilde{g}_j:\mathbb{R}\to\mathbb{R}_+$ by

$$\tilde{g}_j(z):=\mathrm{e}^{-u^*(k)z/\sqrt{\varepsilon}}g_j(\mathrm{e}^{z\sqrt{\varepsilon}f(\varepsilon)},1). \quad (2.5.13)$$

Assuming (for now) that $\tilde{g}_j\in L^1(\mathbb{R})$, we have for any $u\in\mathbb{R}$,

$$\left(\mathcal{F}\tilde{g}_j\right)(u):=\int_{-\infty}^{\infty}\tilde{g}_j(z)\mathrm{e}^{\mathtt{i}uz}\mathrm{d}z=\int_{-\infty}^{\infty}\exp\left(-\frac{u^*(k)z}{\sqrt{\varepsilon}}\right)g_j\left(\mathrm{e}^{z\sqrt{\varepsilon}f(\varepsilon)},1\right)\mathrm{e}^{\mathtt{i}uz}\mathrm{d}z,$$

for $j=1,2,3$. For $j=1$ we can write ($C_{\varepsilon,k}$ defined in (2.5.7))

$$\int_{-\infty}^{\infty}\tilde{g}_1(z)\mathrm{e}^{\mathtt{i}uz}\mathrm{d}z=\left[\frac{\mathrm{e}^{z\left(\sqrt{\varepsilon}f(\varepsilon)-u^*(k)/\sqrt{\varepsilon}+\mathtt{i}u\right)}}{\sqrt{\varepsilon}f(\varepsilon)-u^*(k)/\sqrt{\varepsilon}+\mathtt{i}u}\right]_0^\infty-\left[\frac{\mathrm{e}^{z\left(-u^*(k)/\sqrt{\varepsilon}+\mathtt{i}u\right)}}{-u^*(k)/\sqrt{\varepsilon}+\mathtt{i}u}\right]_0^\infty=C_{\varepsilon,k}(u),$$

which is valid for $u^*(k)>\varepsilon f(\varepsilon)$. For ε sufficiently small and by the definition of f in (2.2.7) this holds for $u^*(k)>c$. For $j=2$ we can write

$$\int_{-\infty}^{\infty}\tilde{g}_2(z)\mathrm{e}^{\mathtt{i}uz}\mathrm{d}z=\left[\frac{\mathrm{e}^{z\left(-u^*(k)/\sqrt{\varepsilon}+\mathtt{i}u\right)}}{-u^*(k)/\sqrt{\varepsilon}+\mathtt{i}u}\right]_{-\infty}^0-\left[\frac{\mathrm{e}^{z\left(\sqrt{\varepsilon}f(\varepsilon)-u^*(k)/\sqrt{\varepsilon}+\mathtt{i}u\right)}}{\sqrt{\varepsilon}f(\varepsilon)-u^*(k)/\sqrt{\varepsilon}+\mathtt{i}u}\right]_{-\infty}^0=C_{\varepsilon,k}(u),$$

which is valid for $u^*(k)<0$ as ε tends to zero. Finally, for $j=3$ we have

$$\int_{-\infty}^{\infty}\tilde{g}_3(z)\mathrm{e}^{\mathtt{i}uz}\mathrm{d}z=\int_{-\infty}^{0}\mathrm{e}^{-\frac{u^*(k)}{\sqrt{\varepsilon}}z}g_3\left(\mathrm{e}^{z\sqrt{\varepsilon}f(\varepsilon)},1\right)\mathrm{e}^{\mathtt{i}uz}\mathrm{d}z+\int_{0}^{\infty}\mathrm{e}^{-\frac{u^*(k)}{\sqrt{\varepsilon}}z}g_3\left(\mathrm{e}^{z\sqrt{\varepsilon}f(\varepsilon)},1\right)\mathrm{e}^{\mathtt{i}uz}\mathrm{d}z$$
$$=\left[\frac{\exp\left(z\left(\sqrt{\varepsilon}f(\varepsilon)-u^*(k)/\sqrt{\varepsilon}+\mathtt{i}u\right)\right)}{\sqrt{\varepsilon}f(\varepsilon)-u^*(k)/\sqrt{\varepsilon}+\mathtt{i}u}\right]_{-\infty}^0+\left[\frac{\exp\left(z\left(-u^*(k)/\sqrt{\varepsilon}+\mathtt{i}u\right)\right)}{-u^*(k)/\sqrt{\varepsilon}+\mathtt{i}u}\right]_0^\infty$$
$$=-C_{\varepsilon,k}(u),$$

which is valid for $0<u^*(k)<\varepsilon f(\varepsilon)$. For ε sufficiently small and by the assumption on f in (2.2.7) this is true for $0<u^*(k)<c$. In this context $u^*(k)$ comes out naturally in the analysis as a classical dampening factor. Note that in order for these strips of regularity to exist we require that $\{0,c\}\subset\mathcal{D}_0^o$, as assumed in the theorem. By the strict convexity and essential smoothness property in Assumption 2.2.1(iv) we have

$$\begin{aligned}0<u^*(k)<c &\quad \text{if and only if} \quad \Lambda_{0,1}(0)<k<\Lambda_{0,1}(c),\\ u^*(k)<0 &\quad \text{if and only if} \quad k<\Lambda_{0,1}(0),\\ u^*(k)>c &\quad \text{if and only if} \quad k>\Lambda_{0,1}(c).\end{aligned} \quad (2.5.14)$$

The following technical lemma allows us to write the transformed option price as an inverse Fourier transform. Recall that $C_{\varepsilon,k}$ is given in (2.5.7), its complex conjugate in (2.5.8) and $\tilde{g}_j$ in (2.5.13).

Lemma 2.5.4. *There exists $\varepsilon_1^* > 0$ such that for all $\varepsilon < \varepsilon_1^*$ and all $k \in \mathbb{R}\backslash\{\Lambda_{0,1}(0), \Lambda_{0,1}(c)\}$, we have ($\bar{a}$ denoting the complex conjugate of $a \in \mathbb{C}$)*

$$\mathbb{E}^{\mathbb{Q}_{k,\varepsilon}}\left[\tilde{g}_j(Z_{k,\varepsilon})\right] = \begin{cases} \frac{1}{2\pi}\int_{\mathbb{R}} \Phi_{Z_{k,\varepsilon}}(u)\overline{C_{\varepsilon,k}(u)}\mathrm{d}u, & \text{if } j=1, u^*(k) > c, \\ \frac{1}{2\pi}\int_{\mathbb{R}} \Phi_{Z_{k,\varepsilon}}(u)\overline{C_{\varepsilon,k}(u)}\mathrm{d}u, & \text{if } j=2, u^*(k) < 0, \\ -\frac{1}{2\pi}\int_{\mathbb{R}} \Phi_{Z_{k,\varepsilon}}(u)\overline{C_{\varepsilon,k}(u)}\mathrm{d}u, & \text{if } j=3, 0 < u^*(k) < c. \end{cases} \tag{2.5.15}$$

The proof of Lemma 2.5.4 proceeds in two steps: we first prove that the integrand in the right-hand side of Equality (2.5.15) belongs to $L^1(\mathbb{R})$ (and hence the integral is well defined), and we then prove that this very equality holds. The first step is contained in the following lemma.

Lemma 2.5.5. *There exists $\varepsilon_0^* > 0$ such that $\int_{\mathbb{R}} |\Phi_{Z_{k,\varepsilon}}(u)\overline{C_{\varepsilon,k}(u)}|\mathrm{d}u < \infty$ for all $\varepsilon < \varepsilon_0^*$ and $k \in \mathbb{R}\backslash\{\Lambda_{0,1}(0), \Lambda_{0,1}(c)\}$.*

Proof. Using the simple bounds in (2.5.9) we compute

$$\begin{aligned}\int_{\mathbb{R}} \left|\Phi_{Z_{k,\varepsilon}}(u)\overline{C_{\varepsilon,k}(u)}\right| \mathrm{d}u &= \int_{|u|\le 1/\sqrt{\varepsilon}} \left|\Phi_{Z_{k,\varepsilon}}(u)\overline{C_{\varepsilon,k}(u)}\right| \mathrm{d}u + \int_{|u|>1/\sqrt{\varepsilon}} \left|\Phi_{Z_{k,\varepsilon}}(u)\overline{C_{\varepsilon,k}(u)}\right| \mathrm{d}u \\ &\le \frac{\varepsilon^{3/2} f(\varepsilon)}{|u^*(k)(u^*(k)-\varepsilon f(\varepsilon))|}\int_{|u|\le 1/\sqrt{\varepsilon}} \left|\Phi_{Z_{k,\varepsilon}}(u)\right| \mathrm{d}u + \varepsilon f(\varepsilon)\int_{|z|>1}\frac{\mathrm{d}z}{z^2} \\ &\le \frac{2\varepsilon f(\varepsilon)}{|u^*(k)(u^*(k)-\varepsilon f(\varepsilon))|} + \varepsilon f(\varepsilon)\int_{|z|>1}\frac{\mathrm{d}z}{z^2}.\end{aligned}$$

The quantity on the rhs is finite for ε small enough since $\varepsilon f(\varepsilon) = c + \mathcal{O}(\varepsilon)$ and $u^*(k) \notin \{0, c\}$. □

We now move on to the proof of Lemma 2.5.4. We only look at the case $j = 1$, the other cases being completely analogous. We denote the convolution of two functions $f, h \in L^1(\mathbb{R})$ by $(f * g)(x) := \int_{\mathbb{R}} f(x-y)g(y)\mathrm{d}y$, and recall that $(f * g) \in L^1(\mathbb{R})$. For such functions, we denote the Fourier transform by $(\mathcal{F}f)(u) := \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ux} f(x)\mathrm{d}x$ and the inverse Fourier transform by $(\mathcal{F}^{-1}h)(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}ux} h(u)\mathrm{d}u$.

With $\tilde{g}_j$ defined in (2.5.13), the $\mathbb{Q}_{k,\varepsilon}$-measure in (2.5.1) and the random variable $Z_{k,\varepsilon}$ in (2.5.2), we have

$$\mathbb{E}^{\mathbb{Q}_{k,\varepsilon}}\left[\tilde{g}_j(Z_{k,\varepsilon})\right] = \int_{\mathbb{R}} q_j(k/\sqrt{\varepsilon} - y)p(y)\mathrm{d}y = (q_j * p)(k/\sqrt{\varepsilon}),$$

with $q_j(z) \equiv \tilde{g}_j(-z)$ and p denoting the density of $Y_\varepsilon/\sqrt{\varepsilon}$. On the strips of regularity given in (2.5.14) we know there exists $\varepsilon_0 > 0$ such that $q_j \in L^1(\mathbb{R})$ for $\varepsilon < \varepsilon_0$. Since p is a density, $p \in L^1(\mathbb{R})$, and therefore

$$\mathcal{F}(q_j * p)(u) = \mathcal{F}q_j(u)\mathcal{F}p(u). \tag{2.5.16}$$

We note that $\mathcal{F}q_j(u) \equiv \mathcal{F}\tilde{g}_j(-u) \equiv \overline{\mathcal{F}\tilde{g}_j(u)}$ and hence

$$\mathcal{F}q_j(u)\mathcal{F}p(u) \equiv \mathrm{e}^{\mathtt{i}uk/\sqrt{\varepsilon}}\Phi_{Z_{k,\varepsilon}}(u)\overline{C_{\varepsilon,k}}(u). \tag{2.5.17}$$

Thus by Lemma 2.5.5 there exists an $\varepsilon_1 > 0$ such that $\mathcal{F}q_j\mathcal{F}p \in L^1(\mathbb{R})$ for $\varepsilon < \varepsilon_1$. By the inversion theorem [137, Theorem 9.11] this then implies from (2.5.16) and (2.5.17) that for $\varepsilon < \min(\varepsilon_0, \varepsilon_1)$:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}_{k,\varepsilon}}\left[\tilde{g}_j(Z_{k,\varepsilon})\right] &= (q_j * p)(k/\sqrt{\varepsilon}) = \mathcal{F}^{-1}\left(\mathcal{F}q_j(u)\mathcal{F}p(u)\right)(k/\sqrt{\varepsilon}) \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-\mathtt{i}uk/\sqrt{\varepsilon}}\mathcal{F}q_j(u)\mathcal{F}p(u)\mathrm{d}u = \frac{1}{2\pi}\int_{\mathbb{R}}\Phi_{Z_{k,\varepsilon}}(u)\overline{C_{\varepsilon,k}(u)}\mathrm{d}u.
\end{aligned}$$

Remark 2.5.6. There exists $\varepsilon_0 > 0$ such that for the strips of regularity given in (2.5.14), the modified payoffs $\tilde{g}_j$ are in $L^2(\mathbb{R})$ for $\varepsilon < \varepsilon_0$. If there further exists $\varepsilon_1 > 0$ such that $\Phi_{Z_{k,\varepsilon}} \in L^2(\mathbb{R})$ for $\varepsilon < \varepsilon_1$ then we can directly apply Parseval's Theorem [77, Theorem 13E] for $\varepsilon < \min(\varepsilon_0, \varepsilon_1)$ and we obtain the same result as in Lemma 2.5.4. This requires though a stronger tail assumption compared to 2.2.1(v)(c).

We now consider the integral appearing in Lemma 2.5.4. For $\varepsilon > 0$ small enough, we can split the integral as

$$\begin{aligned}
\int_{\mathbb{R}}\Phi_{Z_{k,\varepsilon}}(u)\overline{C_{\varepsilon,k}(u)}\mathrm{d}u &= \int_{|u|<\varepsilon^{-1/6}}\Phi_{Z_{k,\varepsilon}}(u)\overline{C_{\varepsilon,k}(u)}\mathrm{d}u + \int_{|u|\geq\varepsilon^{-1/6}}\Phi_{Z_{k,\varepsilon}}(u)\overline{C_{\varepsilon,k}(u)}\mathrm{d}u \\
&= \int_{|u|<\varepsilon^{-1/6}}\exp\left(-\frac{\Lambda_{0,2}u^2}{2}\right)H(\varepsilon,u)\mathrm{d}u + \mathcal{O}\left(\mathrm{e}^{-\beta/\varepsilon^{1/3}}\right),
\end{aligned} \tag{2.5.18}$$

for some $\beta > 0$ by Lemma 2.5.3, and using also Lemma 2.5.1 for the first integral. The function $H : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{C}$ is defined as $H(\varepsilon, u) := \exp(\Lambda_{0,2}u^2/2)\Phi_{Z_{k,\varepsilon}}(u)\overline{C_{\varepsilon,k}(u)}$. As ε tends to zero, the function $\overline{C_{\varepsilon,k}}$ (given in (2.5.8)) satisfies

$$\overline{C_{\varepsilon,k}}(u) = \frac{f(\varepsilon)\epsilon^{3/2}}{u^*(k)^2}\left(1 + h_1(u,0)\sqrt{\varepsilon} + h_2(u,0)\varepsilon + h_3(u,0)\varepsilon^{3/2} + \frac{\varepsilon f(\varepsilon)}{u^*(k)} - \frac{3\mathtt{i}u}{u^*(k)^2}\varepsilon^{3/2}f(\varepsilon) + \mathcal{O}(\varepsilon^2)\right),$$

with h_i defined in (2.5.19), so that Lemma 2.5.1 and a Taylor expansion of H around $\varepsilon = 0$ for $c = 0$ and $|u| \leq \varepsilon^{-1/6}$ yield

$$\begin{aligned}
H(\varepsilon,u) = \frac{f(\varepsilon)\varepsilon^{3/2}}{u^*(k)^2}\Big[&1 + \widetilde{h}_1(u,0)\sqrt{\varepsilon} + \widetilde{h}_2(u,0)\varepsilon + \widetilde{h}_3(u,0)\varepsilon^{3/2} + \frac{\varepsilon f(\varepsilon)}{u^*(k)} \\
&+ \left(\frac{\eta_1(u)}{u^*(k)} - \frac{3\mathtt{i}u}{u^*(k)^2}\right)\varepsilon^{3/2}f(\varepsilon) + \max(1,|u|^{12})\mathcal{O}(\varepsilon^2)\Big],
\end{aligned}$$

where $\mathcal{O}(\varepsilon^2)$ is uniform in u and where we define the following functions:

$$\begin{aligned}
h_1(u,c) &:= \frac{\mathtt{i}u}{u^*(k)-c}\left(\frac{c}{u^*(k)} - 2\right), \qquad h_2(u,c) := -\frac{u^2\left(c^2 - 3cu^*(k) + 3u^*(k)^2\right)}{u^*(k)^2\left(u^*(k)-c\right)^2}, \\
h_3(u,c) &:= \frac{\mathtt{i}u^3\left(4u^*(k)^3 - c^3 + 4c^2u^*(k) - 6cu^*(k)^2\right)}{u^*(k)^3\left(u^*(k)-c\right)^3}, \\
\widetilde{h}_1(u,c) &:= \eta_1(u) + h_1(u,c), \qquad \widetilde{h}_2(u,c) := \frac{\eta_1^2(u)}{2} + \eta_2(u) + h_2(u,c) + \eta_1(u)h_1(u,c), \\
\widetilde{h}_3(u,c) &:= h_2(u,c)\eta_1(u) + h_1(u,c)\left(\frac{\eta_1^2(u)}{2} + \eta_2(u)\right) + \frac{\eta_1^3(u)}{6} + \eta_2(u)\eta_1(u) \\
&\quad + \eta_3(u) + h_3(u,c),
\end{aligned} \tag{2.5.19}$$

with the η_i for $i = 1, 2, 3$, defined in (2.5.6). Analogously a Taylor expansion around $\varepsilon = 0$ for $c > 0$ gives

$$\overline{C_{\varepsilon,k}(u)} = \frac{c\sqrt{\varepsilon}}{u^*(k)\,(u^*(k)-c)}\left\{1 + h_1(u,c)\sqrt{\varepsilon} + h_2(u,c)\varepsilon + h_3(u,c)\varepsilon^{3/2} + \frac{u^*(k)(\varepsilon f(\varepsilon)-c)}{c\,(u^*(k)-c)}\right.$$
$$\left. - \frac{2\mathtt{i}uu^*(k)\sqrt{\varepsilon}(\varepsilon f(\varepsilon)-c)}{c\,(u^*(k)-c)^2} + \mathcal{O}(\varepsilon^2)\right\},$$

from which we deduce an expansion for H, whenever $|u| \le \varepsilon^{-1/6}$:

$$H(\varepsilon,u) = \frac{c\sqrt{\varepsilon}}{u^*(k)\,(u^*(k)-c)}\left\{1 + \widetilde{h}_1(u,c)\sqrt{\varepsilon} + \widetilde{h}_2(u,c)\varepsilon + \widetilde{h}_3(u,c)\varepsilon^{3/2} + \frac{u^*(k)(\varepsilon f(\varepsilon)-c)}{c\,(u^*(k)-c)}\right.$$
$$\left. + \frac{u^*(k)\sqrt{\varepsilon}(\varepsilon f(\varepsilon)-c)}{c\,(u^*(k)-c)}\left(\eta_1(u) - \frac{2\mathtt{i}u}{u^*(k)-c}\right) + \max(1,|u|^{12})\mathcal{O}(\varepsilon^2)\right\},$$

where $\mathcal{O}(\varepsilon^2)$ is uniform in u. We will shortly be integrating H against a zero-mean Gaussian characteristic function over $\mathbb{R}$ and as such all odd powers of u will have a null contribution. In particular note that the polynomials

$$\eta_1, \quad \widetilde{h}_1, \quad \widetilde{h}_3, \quad \left(\frac{\eta_1(u)}{u^*(k)} - \frac{3\mathtt{i}u}{(u^*(k))^2}\right)\varepsilon^{3/2}f(\varepsilon), \quad \frac{u^*(k)\sqrt{\varepsilon}(\varepsilon f(\varepsilon)-c)}{c\,(u^*(k)-c)}\left(\eta_1(u) - \frac{2\mathtt{i}u}{u^*(k)-c}\right)$$

are odd functions of u and hence have zero contribution. The major quantity is $\widetilde{h}_2$, which we can rewrite as $\widetilde{h}_2(u,c) = \widetilde{h}_{2,1}(c)u^2 + \widetilde{h}_{2,2}(c)u^4 - \frac{1}{72}\Lambda_{0,3}^2 u^6$, where

$$\widetilde{h}_{2,1}(c) := -\frac{h_1(u,c)\Lambda_{1,1}}{\mathtt{i}} - \frac{\Lambda_{1,1}^2 + \Lambda_{1,2}}{2} + h_2(1,c), \quad \widetilde{h}_{2,2}(c) := \frac{h_1(u,c)\Lambda_{0,3}}{6\mathtt{i}} + \frac{\Lambda_{1,1}\Lambda_{0,3}}{6} + \frac{\Lambda_{0,4}}{24}.$$

Let

$$\phi_\varepsilon(c) \equiv \frac{c\sqrt{\varepsilon}\mathbb{1}_{\{c>0\}} + \varepsilon^{3/2}f(\varepsilon)\mathbb{1}_{\{c=0\}}}{u^*(k)\,(u^*(k)-c)}.$$

Using simple properties of moments of a Gaussian random variable we finally compute the following

$$\int_{|u|<\varepsilon^{-1/6}} \exp\left(-\frac{\Lambda_{0,2}u^2}{2}\right) H(\varepsilon,u)\mathrm{d}u$$
$$= \phi_\varepsilon(c)\left[\int_{|u|<\varepsilon^{-1/6}} \mathrm{e}^{-\frac{1}{2}\Lambda_{0,2}u^2}\left(1 + \widetilde{h}_2(u,c) + \frac{u^*(k)(\varepsilon f(\varepsilon)-c)}{c\,(u^*(k)-c)}\mathbb{1}_{\{c>0\}} + \frac{\varepsilon f(\varepsilon)}{u^*(k)}\mathbb{1}_{\{c=0\}}\right)\mathrm{d}u + \mathcal{O}(\varepsilon^2)\right]$$
$$= \phi_\varepsilon(c)\left[\int_{\mathbb{R}} \mathrm{e}^{-\frac{1}{2}\Lambda_{0,2}u^2}\left(1 + \widetilde{h}_2(u,c) + \frac{u^*(k)(\varepsilon f(\varepsilon)-c)}{c\,(u^*(k)-c)}\mathbb{1}_{\{c>0\}} + \frac{\varepsilon f(\varepsilon)}{u^*(k)}\mathbb{1}_{\{c=0\}}\right)\mathrm{d}u + \mathcal{O}(\varepsilon^2)\right]$$
$$= \phi_\varepsilon(c)\sqrt{\frac{2\pi}{\Lambda_{0,2}}}\left(1 + \frac{\widetilde{h}_{2,1}(c)}{\Lambda_{0,2}} + \frac{3\widetilde{h}_{2,2}(c)}{\Lambda_{0,2}^2} - \frac{5\Lambda_{0,3}^2}{24\Lambda_{0,2}^3} + \frac{u^*(k)(\varepsilon f(\varepsilon)-c)}{c\,(u^*(k)-c)}\mathbb{1}_{\{c>0\}} + \frac{\varepsilon f(\varepsilon)}{u^*(k)}\mathbb{1}_{\{c=0\}} + \mathcal{O}(\varepsilon^2)\right).$$

The third line follows from the Laplace method (Theorem 1.2.6), applied to the two integrals $\int_{\varepsilon^{-1/6}}^{+\infty}(\cdots)\mathrm{d}u$ and $\int_{-\infty}^{-\varepsilon^{-1/6}}(\cdots)\mathrm{d}u$, where the concentration is at the boundary points of the domains, so that the tail estimate $|u| > \varepsilon^{-1/6}$ is exponentially small, and hence is absorbed in the $\mathcal{O}(\varepsilon^2)$ term. Combining this with (2.5.18), Lemma 2.5.4, (2.5.11), (2.5.12) and property (2.5.14), the theorem follows.

2.5.1.2 Proof of Propositions 2.2.10 and 2.2.11

Gao and Lee [69] have obtained representations for asymptotic implied volatility for small and large-maturity regimes in terms of the assumed asymptotic behaviour of certain option prices, outlining the general procedure for transforming option price asymptotics into implied volatility asymptotics. The same methodology can be followed to transform our forward-start option asymptotics (Corollary 2.2.6 and Corollary 2.2.8) into forward smile asymptotics. In the proofs of Proposition 2.2.10 and Proposition 2.2.11 we hence assume for brevity the existence of an ansatz for the forward smile asymptotic and solve for the coefficients. We refer the reader to [69] for the complete methodology.

Proof of Proposition 2.2.10. Substituting the ansatz

$$\sigma^2_{\varepsilon t,\varepsilon\tau}(k) = v_0(k,t,\tau) + v_1(k,t,\tau)\varepsilon + v_2(k,t,\tau)\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right),$$

into Corollary 2.2.7, we get that forward-start option prices have the asymptotics

$$\begin{aligned}\mathbb{E}\left(\mathrm{e}^{X^{(\varepsilon t)}_{\varepsilon\tau}} - \mathrm{e}^k\right)^+ \mathbb{1}_{\{k>0\}} + \mathbb{E}\left(\mathrm{e}^k - \mathrm{e}^{X^{(\varepsilon t)}_{\varepsilon\tau}}\right)^+ \mathbb{1}_{\{k<0\}} \\ = \exp\left(-\frac{k^2}{2\tau v_0\varepsilon} + \frac{k^2 v_1}{2\tau v_0^2} + \frac{k}{2}\right)\frac{(v_0\varepsilon\tau)^{3/2}}{k^2\sqrt{2\pi}}\left(1 + \gamma\varepsilon + \mathcal{O}\left(\varepsilon^2\right)\right),\end{aligned}$$

where we set

$$\gamma(k,t,\tau) := -\tau\left(\frac{3}{k^2} + \frac{1}{8}\right)v_0 + \frac{k^2 v_2}{2\tau v_0^2} - \frac{k^2 v_1^2}{2\tau v_0^3} + \frac{3v_1}{2v_0}.$$

The result follows after using $\Lambda_{0,1}(0) = 0$ and equating orders with the general formula in Corollary 2.2.6. □

Proof of Proposition 2.2.11. Substituting the ansatz

$$\sigma^2_{t,\tau}(k) = v_0^\infty(k,t) + v_1^\infty(k,t)\tau^{-1} + v_2^\infty(k,t)\tau^{-2} + \mathcal{O}\left(\tau^{-3}\right),$$

into Corollary 2.2.9 we obtain the following asymptotic expansions for forward-start options:

$$\begin{aligned}\mathbb{E}\left(\mathrm{e}^{X^{(t)}_\tau} - \mathrm{e}^{k\tau}\right)^+ \mathbb{1}_{\mathrm{A}} - \mathbb{E}\left(\mathrm{e}^{X^{(t)}_\tau} \wedge \mathrm{e}^{k\tau}\right)\mathbb{1}_{\mathrm{B}} + \mathbb{E}\left(\mathrm{e}^{k\tau} - \mathrm{e}^{X^{(t)}_\tau}\right)^+ \mathbb{1}_{\mathrm{C}} \\ = \exp\left(-\tau\left(\frac{k^2}{2v_0} - \frac{k}{2} + \frac{v_0}{8}\right) + \frac{v_1 k^2}{2v_0^2} - \frac{v_1}{8}\right)\frac{4\tau^{-1/2}v_0^{3/2}}{\left(4k^2 - v_0^2\right)\sqrt{2\pi}}\left(1 + \frac{\gamma^\infty}{\tau} + \mathcal{O}\left(\frac{1}{\tau^2}\right)\right),\end{aligned}$$

where

$$\mathrm{A} := \left\{k > \frac{\sigma^2_{t,\tau}(k)}{2}\right\}, \quad \mathrm{B} := \left\{-\frac{\sigma^2_{t,\tau}(k)}{2} < k < \frac{\sigma^2_{t,\tau}(k)}{2}\right\}, \quad \mathrm{C} := \left\{k < -\frac{\sigma^2_{t,\tau}(k)}{2}\right\}, \tag{2.5.20}$$

and

$$\gamma^\infty(k,t) := \frac{\left(12k^2 + v_0^2\right)\left(4k^2 v_1 - v_0^2\left(v_1 + 8\right)\right)}{2v_0\left(v_0^2 - 4k^2\right)^2} - \frac{v_1^2 k^2}{2v_0^3} + \frac{v_2 k^2}{2v_0^2} - \frac{v_2}{8}.$$

We obtain the expressions for v_1^∞ and v_2^∞ by equating orders with the formula in Corollary 2.2.8. However it is not clear which is the correct root for the zeroth order term v_0^∞. In order to do so, we have to match the domains in (2.5.20) and in Corollary 2.2.8. Indeed, suppose that we choose the roots according to v_0^∞ in (2.2.11). For τ sufficiently large the condition $k > \sigma^2_{t,\tau}(k)/2$ is equivalent to $k > v_0^\infty(k,t)/2$. Now for $k > \Lambda_{0,1}(1)$ or $k < \Lambda_{0,1}(0)$, the definition of v_0^∞ in (2.2.11) implies

$$k > \sigma^2_{t,\tau}(k)/2 \qquad \text{if and only if} \qquad \sqrt{(\Lambda^*(k)-k)^2 + k\,(\Lambda^*(k)-k)} > \Lambda^*(k) - k, \tag{2.5.21}$$

which is always true since $\Lambda^*(k) > k$ by Lemma 2.2.2(iii). Now, for $k \in (\Lambda_{0,1}(0), \Lambda_{0,1}(1))$, the definition of v_0^∞ in (2.2.11) implies

$$k > \sigma^2_{t,\tau}(k)/2 \qquad \text{if and only if} \qquad -\sqrt{(\Lambda^*(k)-k)^2 + k\,(\Lambda^*(k)-k)} > \Lambda^*(k) - k, \tag{2.5.22}$$

which never holds. By the assumption in the Proposition 2.2.11 and Assumption 2.2.1 we have $\{0,1\} \subset \mathcal{D}_0^o$ and $\Lambda_0(0) = \Lambda_0(1) = 0$. The differentiability and strict convexity of Λ_0 (Assumption 2.2.1(iv)) then imply $\Lambda_{0,1}(0) < 0$ and $\Lambda_{0,1}(1) > 0$. Since $v_0^\infty > 0$ we can ignore the case $k < \Lambda_{0,1}(0) < 0$ and hence $k > \sigma^2_{t,\tau}(k)/2$ if and only if $k > \Lambda_{0,1}(1)$. Similarly the definition of v_0^∞ in (2.2.11) implies that for τ large enough,

$$-\sigma^2_{t,\tau}(k)/2 < k < \sigma^2_{t,\tau}(k)/2 \quad \text{if and only if} \quad \Lambda_{0,1}(0) < k < \Lambda_{0,1}(1),$$

and

$$k < -\sigma^2_{t,\tau}(k)/2 \quad \text{if and only if} \quad k < \Lambda_{0,1}(0).$$

This lines up the domains in (2.5.20) with the domains in Corollary 2.2.8. Had we specified the roots in any other way, it is easy to check that a contradiction would have occurred. □

2.5.2 Proofs of Section 2.3.1

We now let $(X_t)_{t\geq 0}$ be the Heston process satisfying the SDE (1.3.2). The Heston forward cgf was derived in Lemma 1.3.1. In the next two subsections we develop the tools needed to apply Propositions 2.2.10 and 2.2.11 to the Heston model.

2.5.2.1 Proofs of Section 2.3.1.1

We consider here the Heston diagonal small-maturity process $(X^{(\varepsilon t)}_{\varepsilon\tau})_{\varepsilon>0}$ with X defined in (1.3.2) and $(X^{(t)}_{\tau})_{\tau>0}$ in (1.0.3). The forward rescaled cgf Λ_ε in (2.2.1) is easily determined from (1.3.7).

In this subsection, we prove Proposition 2.3.1. For clarity, the proof is divided into the following steps:

(i) In Lemma 2.5.7 we show that $\mathcal{D}_0 = \mathcal{K}_{t,\tau}$ and $0 \in \mathcal{D}_0^o$;

(ii) In Lemma 2.5.9 we show that the Heston diagonal small-maturity process has an expansion of the form given in Assumption 2.2.1 with $\Lambda_0 = \Xi$ and $\Lambda_1 = L$, where Ξ and L are defined in (2.3.1) and (2.3.2);

(iii) In Lemma 2.5.11 we show that Ξ is strictly convex and essentially smooth on $\mathcal{D}_0^o$, i.e. Assumption 2.2.1(iv);

(iv) The map $(\varepsilon, u) \mapsto \Lambda_\varepsilon(u)$ is of class $\mathcal{C}^\infty$ on $\mathbb{R}_+^* \times \mathcal{D}_0^o$, $\Lambda_{0,1}(0) = 0$ and Assumption 2.2.1(v) is also satisfied.

Lemma 2.5.7. *For the Heston diagonal small-maturity process we have* $\mathcal{D}_0 = \mathcal{K}_{t,\tau}$ *and* $0 \in \mathcal{D}_0^o$ *with* $\mathcal{K}_{t,\tau}$ *defined in* (2.3.1) *and* $\mathcal{D}_0$ *defined in Assumption 2.2.1.*

Proof. For any $t > 0$, the random variable V_t in (1.3.2) is distributed as β_t times a non-central chi-square random variable with $4\kappa\theta/\xi^2 > 0$ degrees of freedom and non-centrality parameter $\lambda = v\mathrm{e}^{-\kappa t}/\beta_t > 0$. It follows that the corresponding mgf is given by

$$\Lambda_t^V(u) := \mathbb{E}\left(\mathrm{e}^{uV_t}\right) = \exp\left(\frac{\lambda\beta_t u}{1-2\beta_t u}\right)(1-2\beta_t u)^{-2\kappa\theta/\xi^2}, \qquad \text{for all } u < \frac{1}{2\beta_t}.$$

The re-normalised Heston forward cgf Λ_ε is then computed as

$$\begin{aligned}\mathrm{e}^{\Lambda_\varepsilon(u)/\varepsilon} &= \mathbb{E}\left[\mathrm{e}^{\frac{u}{\varepsilon}(X_{\varepsilon t+\varepsilon\tau}-X_{\varepsilon t})}\right] = \mathbb{E}\left[\mathbb{E}\left(\mathrm{e}^{\frac{u}{\varepsilon}(X_{\varepsilon t+\varepsilon\tau}-X_{\varepsilon t})}|\mathcal{F}_{\varepsilon t}\right)\right] = \mathbb{E}\left(\mathrm{e}^{A\left(\frac{u}{\varepsilon},\varepsilon\tau\right)+B\left(\frac{u}{\varepsilon},\varepsilon\tau\right)V_{\varepsilon t}}\right)\\ &= \mathrm{e}^{A\left(\frac{u}{\varepsilon},\varepsilon\tau\right)}\Lambda_{\varepsilon t}^V\left(B\left(u/\varepsilon,\varepsilon\tau\right)\right),\end{aligned}$$

which agrees with (1.3.7). This only makes sense in some effective domain $\mathcal{K}_{\varepsilon t,\varepsilon\tau} \subset \mathbb{R}$. The cgf for $V_{\varepsilon t}$ is well defined in $\mathcal{K}_{\varepsilon t}^V := \{u \in \mathbb{R} : B(u/\varepsilon, \varepsilon\tau) < \frac{1}{2\beta_{\varepsilon t}}\}$, and hence $\mathcal{K}_{\varepsilon t,\varepsilon\tau} = \mathcal{K}_{\varepsilon t}^V \cap \mathcal{K}_{\varepsilon\tau}^H$, where $\mathcal{K}_{\varepsilon\tau}^H$ is the effective domain of the (spot) Heston cgf. Consider first $\mathcal{K}_{\varepsilon\tau}^H$ for small ε. From [5, Proposition 3.1] if $\xi^2(u/\varepsilon-1)u/\varepsilon > (\kappa-\xi\rho u/\varepsilon)^2$ then the explosion time $\tau_H^*(u) := \sup\{t \geq 0 : \mathbb{E}(\mathrm{e}^{uX_t}) < \infty\}$ of the Heston cgf is

$$\begin{aligned}\tau_H^*\left(\frac{u}{\varepsilon}\right) = \frac{2}{\sqrt{\xi^2(u/\varepsilon-1)u/\varepsilon-(\kappa-\rho\xi u/\varepsilon)^2}}&\Bigg(\pi\mathbb{1}_{\{\rho\xi u/\varepsilon-\kappa<0\}}\\ &+\arctan\left(\frac{\sqrt{\xi^2(u/\varepsilon-1)u/\varepsilon-(\kappa-\rho\xi u/\varepsilon)^2}}{\rho\xi u/\varepsilon-\kappa}\right)\Bigg).\end{aligned}$$

Recall the following Taylor series expansions, for x close to zero:

$$\begin{aligned}\arctan\left(\frac{1}{\rho\xi u/x-\kappa}\sqrt{\xi^2\left(\frac{u}{x}-1\right)\frac{u}{x}-\left(\kappa-\xi\rho\frac{u}{x}\right)^2}\right) &= \operatorname{sgn}(u)\arctan\left(\frac{\bar\rho}{\rho}\right)+\mathcal{O}(x), \quad \text{if } \rho\neq 0,\\ \arctan\left(-\frac{1}{\kappa}\sqrt{\xi^2\left(\frac{u}{x}-1\right)\frac{u}{x}-\kappa^2}\right) &= -\frac{\pi}{2}+\mathcal{O}(x), \qquad \text{if } \rho = 0.\end{aligned}$$

As ε tends to zero $\xi^2(u/\varepsilon-1)u/\varepsilon > (\kappa-\rho\xi u/\varepsilon)^2$ is satisfied since $\xi^2 > \xi^2\rho^2$ and hence

$$\tau_H^*\left(\frac{u}{\varepsilon}\right) = \begin{cases}\dfrac{\varepsilon}{\xi|u|}\left(\pi\mathbb{1}_{\{\rho=0\}}+\dfrac{2}{\bar\rho}\left(\pi\mathbb{1}_{\{\rho u\leq 0\}}+\operatorname{sgn}(u)\arctan\left(\dfrac{\bar\rho}{\rho}\right)\right)\mathbb{1}_{\{\rho\neq 0\}}+\mathcal{O}(\varepsilon)\right), & \text{if } u\neq 0,\\ \infty, & \text{if } u = 0.\end{cases}$$

Therefore, for ε small enough, we have $\tau_H^*\left(\frac{u}{\varepsilon}\right) > \varepsilon\tau$ for all $u \in (u_-, u_+)$, where

$$\begin{aligned}u_- &:= \frac{2}{\bar\rho\xi\tau}\arctan\left(\frac{\bar\rho}{\rho}\right)\mathbb{1}_{\{\rho<0\}} - \frac{\pi}{\xi\tau}\mathbb{1}_{\{\rho=0\}} + \frac{2}{\bar\rho\xi\tau}\left(\arctan\left(\frac{\bar\rho}{\rho}\right)-\pi\right)\mathbb{1}_{\{\rho>0\}},\\ u_+ &:= \frac{2}{\bar\rho\xi\tau}\left(\arctan\left(\frac{\bar\rho}{\rho}\right)+\pi\right)\mathbb{1}_{\{\rho<0\}} + \frac{\pi}{\xi\tau}\mathbb{1}_{\{\rho=0\}} + \frac{2}{\bar\rho\xi\tau}\arctan\left(\frac{\bar\rho}{\rho}\right)\mathbb{1}_{\{\rho>0\}}.\end{aligned}$$

So as ε tends to zero, $\mathcal{K}^H_{\varepsilon\tau}$ shrinks to (u_-, u_+). Regarding $\mathcal{K}^V_{\varepsilon t}$, we have (see (2.5.25) for details on the expansion computation) $\beta_{\varepsilon t} B(u/\varepsilon, \varepsilon\tau) = \frac{\xi^2 t}{4v}\Xi(u,0,\tau) + \mathcal{O}(\varepsilon)$ for any $u \in (u_-, u_+)$, with Ξ defined in (2.3.1). Therefore $\lim_{\varepsilon\downarrow 0}\mathcal{K}^V_{\varepsilon t} = \{u \in \mathbb{R} : \Xi(u,0,\tau) < \frac{2v}{\xi^2 t}\}$ and hence $\lim_{\varepsilon\downarrow 0}\mathcal{K}_{\varepsilon t,\varepsilon\tau} = \{u \in \mathbb{R} : \Xi(u,0,\tau) < \frac{2v}{\xi^2 t}\} \cap (u_-, u_+)$. It is easily checked that $\Xi(u,0,\tau)$ is strictly positive except at $u = 0$ where it is zero, $\Xi'(u,0,\tau) > 0$ for $u > 0$, $\Xi'(u,0,\tau) < 0$ for $u < 0$ and that $\Xi(u,0,\tau)$ tends to infinity as u approaches $u_\pm$. Since v and ξ are strictly positive and $t \geq 0$ it follows that $\{u \in \mathbb{R} : \Xi(u,0,\tau) < 2v/(\xi^2 t)\} \subseteq (u_-, u_+)$ with equality only if $t = 0$. So $\mathcal{D}_0$ is an open interval around zero and the lemma follows with $\mathcal{D}_0 = \mathcal{K}_{t,\tau}$. □

Remark 2.5.8. For $u \in \mathbb{R}^*$ the inequality $0 < \Xi(u,0,\tau) < 2v/(\xi^2 t)$ is equivalent to $\Xi(u,t,\tau) \in (0,\infty)$. In Lemma 2.5.9 below we show that Ξ is the limiting cgf of the rescaled Heston forward cgf and so the condition for the limiting forward domain is equivalent to ensuring that the limiting forward cgf does not blow up and is strictly positive except at $u = 0$ where it is zero.

Lemma 2.5.9. *For any $t \geq 0$, $\tau > 0$, $u \in \mathcal{K}_{t,\tau}$, the expansion $\Lambda_\varepsilon(u) = \Xi(u,t,\tau) + L(u,t,\tau)\varepsilon + \mathcal{O}(\varepsilon^2)$ holds as ε tends to zero, where $\mathcal{K}_{t,\tau}$, Ξ and L are defined in* (2.3.1), (2.3.1) *and* (2.3.2) *and Λ_ε is the rescaled cgf in Assumption 2.2.1 for the Heston diagonal small-maturity process $(X^{(\varepsilon t)}_{\varepsilon\tau})_{\varepsilon>0}$.*

Remark 2.5.10. For any $u \in \mathcal{K}_{t,\tau}$, Lemma 2.5.7 implies that $\Lambda_\varepsilon(u)$ is a finite number for any $\varepsilon > 0$. Therefore L defined in (2.3.2) and used in Lemma 2.5.9 is a real-valued function on $\mathcal{K}_{t,\tau}$.

Proof. All expansions below for d, γ and β_t defined in (1.3.4),(1.3.6) and (1.3.8) hold for any $u \in \mathcal{K}_{t,\tau}$:

$$\begin{aligned} d\left(\frac{u}{\varepsilon}\right) &= \frac{1}{\varepsilon}\left(\kappa^2\varepsilon^2 + u\varepsilon(\xi - 2\kappa\rho) - u^2\xi^2(1-\rho^2)\right)^{1/2} = \frac{\mathrm{i}u}{\varepsilon}d_0 + d_1 + \mathcal{O}(\varepsilon), \\ \gamma\left(\frac{u}{\varepsilon}\right) &= \frac{\kappa\varepsilon - \rho\xi u - \mathrm{i}ud_0 - d_1\varepsilon + \mathcal{O}(\varepsilon^2)}{\kappa\varepsilon - \rho\xi u + \mathrm{i}ud_0 + d_1\varepsilon + \mathcal{O}(\varepsilon^2)} = g_0 - \frac{\mathrm{i}\varepsilon}{u}g_1 + \mathcal{O}(\varepsilon^2), \\ \beta_{\varepsilon t} &= \frac{1}{4}\xi^2 t\varepsilon - \frac{1}{8}\kappa\xi^2 t^2\varepsilon^2 + \mathcal{O}(\varepsilon^3), \end{aligned} \tag{2.5.23}$$

where

$$d_0 := \bar\rho\xi\,\mathrm{sgn}(u), \quad d_1 := \frac{\mathrm{i}\,(2\kappa\rho - \xi)\,\mathrm{sgn}(u)}{2\bar\rho}, \quad g_0 := \frac{\mathrm{i}\rho - \bar\rho\,\mathrm{sgn}(u)}{\mathrm{i}\rho + \bar\rho\,\mathrm{sgn}(u)}, \quad g_1 := \frac{(2\kappa - \xi\rho)\,\mathrm{sgn}(u)}{\xi\bar\rho\,(\bar\rho + \mathrm{i}\rho\,\mathrm{sgn}(u))^2},$$

with $\bar\rho := \sqrt{1-\rho^2}$ and $\mathrm{sgn}(u) = 1$ if $u \geq 0$, -1 otherwise. From the definition of A in (1.3.8) we obtain

$$\begin{aligned} A\left(\frac{u}{\varepsilon}, \varepsilon\tau\right) &= \frac{\kappa\theta}{\xi^2}\left(\left(\kappa - \rho\xi u/\varepsilon - d(u/\varepsilon)\right)\varepsilon\tau - 2\log\left(\frac{1 - \gamma(u/\varepsilon)\exp\left(-d(u/\varepsilon)\varepsilon\tau\right)}{1 - \gamma(u/\varepsilon)}\right)\right) \\ &= L_0(u,\tau) + \mathcal{O}(\varepsilon), \end{aligned} \tag{2.5.24}$$

where L_0 is defined in (2.3.2). Substituting the asymptotics for d and γ above we further obtain

$$\frac{1 - \exp\left(-d(u/\varepsilon)\varepsilon\tau\right)}{1 - \gamma(u/\varepsilon)\exp\left(-d(u/\varepsilon)\varepsilon\tau\right)} = \frac{1 - \exp\left(-\mathrm{i}ud_0\tau - \varepsilon d_1\tau + \mathcal{O}(\varepsilon^2)\right)}{1 - \left(g_0 - \mathrm{i}\varepsilon g_1/u + \mathcal{O}(\varepsilon^2)\right)\exp\left(-\mathrm{i}ud_0\tau - \varepsilon d_1\tau + \mathcal{O}(\varepsilon^2)\right)},$$

and therefore using the definition of B in (1.3.8) we obtain

$$\begin{aligned} B\left(\frac{u}{\varepsilon},\varepsilon\tau\right) &= \frac{\kappa-\rho\xi u/\varepsilon-d(u/\varepsilon)}{\xi^2}\frac{1-\exp\left(-d\left(u/\varepsilon\right)\varepsilon\tau\right)}{1-\gamma\left(u/\varepsilon\right)\exp\left(-d\left(u/\varepsilon\right)\varepsilon\tau\right)} \\ &= \frac{\Xi(u,0,\tau)}{v\varepsilon}+L_1(u,\tau)+\mathcal{O}(\varepsilon), \end{aligned} \tag{2.5.25}$$

with L_1 defined in (2.3.2) and Ξ in (2.3.1). Combining (2.5.23) and (2.5.25) we deduce

$$\beta_{\varepsilon t}B\left(u/\varepsilon,\varepsilon\tau\right) = \frac{\xi^2 t\Xi(u,0,\tau)}{4v}+\left(\frac{L_1(u,\tau)\xi^2 t}{4}-\frac{\Xi(u,0,\tau)\kappa\xi^2t^2}{8v}\right)\varepsilon+\mathcal{O}(\varepsilon^2), \tag{2.5.26}$$

and therefore as ε tends to zero,

$$\begin{aligned} \frac{\varepsilon B(u/\varepsilon,\varepsilon\tau)v\mathrm{e}^{-\kappa\varepsilon t}}{1-2\beta_{\varepsilon t}B(u/\varepsilon,\varepsilon\tau)} &= \frac{\left[\Xi(u,0,\tau)+vL_1(u,\tau)\varepsilon+\mathcal{O}\left(\varepsilon^2\right)\right]\left(1-t\kappa\xi+\mathcal{O}(\varepsilon^2)\right)}{1-\xi^2t\Xi(u,0,\tau)/2v+\left(\Xi(u,0,\tau)\kappa\xi^2t^2/4v-L_1(u,\tau)\xi^2t/2\right)\varepsilon+\mathcal{O}\left(\varepsilon^2\right)} \\ &= \Xi(u,t,\tau)+\left(\Xi(u,t,\tau)^2\left(\frac{vL_1(u,\tau)}{\Xi(u,0,\tau)^2}-\frac{\kappa\xi^2t^2}{4v}\right)-\kappa t\Xi(u,t,\tau)\right)\varepsilon+\mathcal{O}(\varepsilon^2). \end{aligned} \tag{2.5.27}$$

Again using (2.5.26) we have

$$-\frac{2\kappa\theta\varepsilon}{\xi^2}\log\left(1-2\beta_{\varepsilon t}B\left(u/\varepsilon,\varepsilon\tau\right)\right) = -\frac{2\kappa\theta}{\xi^2}\log\left(1-\frac{\Xi(u,0,\tau)\xi^2t}{2v}\right)\varepsilon+\mathcal{O}(\varepsilon^2). \tag{2.5.28}$$

Recalling from Lemma 1.3.1 that

$$\Lambda_\varepsilon(u)=\varepsilon A\left(u/\varepsilon,\varepsilon\tau\right)+\frac{\varepsilon B\left(u/\varepsilon,\varepsilon\tau\right)}{1-2\beta_{\varepsilon t}B\left(u/\varepsilon,\varepsilon\tau\right)}v\mathrm{e}^{-\kappa\varepsilon t}-\frac{2\kappa\theta\varepsilon}{\xi^2}\log\left(1-2\beta_{\varepsilon t}B\left(u/\varepsilon,\varepsilon\tau\right)\right),$$

the lemma follows by combining (2.5.24), (2.5.27) and (2.5.28). □

Lemma 2.5.11. *For all $t\geq 0$, $\tau>0$, Ξ (given in* (2.3.1)*) is convex and essentially smooth on $\mathcal{K}_{t,\tau}$, defined in* (2.3.1).

Proof. The first derivative of Ξ is given, after simplification, by

$$\frac{\partial\Xi(u,t,\tau)}{\partial u}=\frac{\Xi(u,t,\tau)}{u}\left[1+\frac{\Xi(u,t,\tau)}{v}\left(\frac{\xi^2t}{2}+\frac{1}{2}\xi^2\bar{\rho}^2\tau\csc^2\left(\frac{1}{2}\bar{\rho}\xi\tau u\right)\right)\right].$$

Any sequence tending to the boundary satisfies $\Xi(u,0,\tau)\to 2v/\xi^2t$ which implies $\Xi(u,t,\tau)\uparrow\infty$ from Remark 2.5.8 and hence $|\partial\Xi(u,t,\tau)/\partial u|\uparrow\infty$. Therefore $\Xi(\cdot,t,\tau)$ is essentially smooth. Now,

$$\frac{\partial^2\Xi(u,t,\tau)}{\partial u^2}=\frac{\xi^2}{2}\Xi(u,t,\tau)\frac{\left(t+\bar{\rho}^2\tau\csc^2(\psi_u)\right)^2}{\left(\rho+\frac{1}{2}\xi tu-\bar{\rho}\cot(\psi_u)\right)^2}+\frac{v+\bar{\rho}^2\tau v\left(1-\psi_u\cot(\psi_u)\right)\csc^2(\psi_u)}{\left(\rho+\frac{1}{2}\xi tu-\bar{\rho}\cot(\psi_u)\right)^2},$$

where $\psi_u:=\bar{\rho}\xi\tau u/2$. For $u\in\mathcal{K}_{t,\tau}\backslash\{0\}$, we have $\Xi(u,t,\tau)>0$ and $\Xi(0,t,\tau)=0$ from Remark 2.5.8. Also we have the inequality that $1-\theta/2\cot\left(\theta/2\right)\geq 0$ for $\theta\in(-2\pi,2\pi)$, so that Ξ is strictly convex on $\mathcal{K}_{t,\tau}$. □

As detailed in the beginning of this subsection, this concludes the proof of Proposition 2.3.1. We now prove the forward implied volatility expansions, namely Corollary 2.3.2.

Proof of Corollary 2.3.2. We fix $t \geq 0, \tau > 0$ and for ease of computations set $\Xi(u) := \Xi(u, t, \tau)$. We first look for a Taylor expansion of $u^*(k)$ around $k = 0$ using the saddlepoint equation $\Xi'(u^*(k)) = k$. Differentiating this equation iteratively and setting $k = 0$ (and using $u^*(0) = 0$) gives an expansion for u^* in terms of the derivatives of Ξ. In particular, $\Xi''(0,)u^{*'}(0) = 1$ and $\Xi'''(0)(u^{*'}(0))^2 + \Xi''(0)u^{*''}(0) = 0$, which implies that $u^{*'}(0) = 1/\Xi''(0)$ and $u^{*''}(0) = -\Xi'''(0)/\Xi''(0)^3$. From the explicit expression of Ξ in (2.3.1), we then obtain

$$\begin{aligned} u^*(k) = \frac{k}{\tau v} - \frac{3\xi\rho}{4\tau v^2}k^2 + \frac{\xi^2\left(\left(19\rho^2-4\right)\tau - 12t\right)}{24\tau^2 v^3}k^3 + \frac{5\xi^3\rho\left(48t + \left(16-37\rho^2\right)\tau\right)}{192\tau^2 v^4}k^4 \\ + \frac{\xi^4\left(1080t^2 + \left(2437\rho^4 - 1604\rho^2 + 112\right)\tau^2 - 180\left(27\rho^2-4\right)\tau t\right)}{1920\tau^3 v^5}k^5 + \mathcal{O}(k^6). \end{aligned}$$

Using this series expansion and the fact that $\Lambda^*(k) = u^*(k)k - \Xi(u^*(k))$, the corollary follows from tedious but straightforward Taylor expansions of v_0 and v_1 defined in (2.2.9). □

2.5.2.2 Proofs of Section 2.3.1.2

In this section, we prove the large-maturity asymptotics for the Heston model. Let $\varepsilon = \tau^{-1}$ and consider the Heston process $(\tau^{-1}X_\tau^{(t)})_{\tau>0}$ with $(X_t)_{t>0}$ defined in (1.3.2) and $(X_\tau^{(t)})_{\tau>0}$ defined in (1.0.3). Specifically Λ_ε defined in (2.2.1) is then given by $\Lambda_\varepsilon(u) = \tau^{-1}\mathbb{E}(\mathrm{e}^{uX_\tau^{(t)}})$, and for ease of notation we set

$$\Lambda_\tau^{(t)}(u) = \Lambda_\varepsilon(u) \quad \text{for all } u \in \mathcal{D}_\varepsilon. \tag{2.5.29}$$

We prove here Proposition 2.3.5 in several steps:

(i) In Proposition 2.5.14 we show that $\mathcal{D}_0 = \mathcal{K}_\mathrm{H}$ and if $\rho < \kappa/\xi$ then $\{0,1\} \subset \mathcal{K}_\mathrm{H}^o$;

(ii) Lemma 2.5.15 proves the expansion of Assumption 2.2.1 with $\Lambda_0 = V$, $\Lambda_1 = H$, $\Lambda_2 = 0$;

(iii) By Lemma 2.5.12 and Proposition 2.5.14, V is strictly convex and essentially smooth on $\mathcal{K}_\mathrm{H}^o$ if $\rho_- \leq \rho \leq \min(\rho_+, \kappa/\xi)$; see also Remark 2.3.6(ii);

(iv) The map $(\varepsilon, u) \mapsto \Lambda_\varepsilon(u)$ is of class $\mathcal{C}^\infty$ on $\mathbb{R}_+^* \times \mathcal{K}_\mathrm{H}^o$, Assumption 2.2.1(v) is also satisfied and $V(1) = 0$ from Lemma 2.5.12;

(v) u^* can be computed in closed-form and is given by q^* in (2.3.9).

(vi) A direct application of Proposition 2.2.11 completes the proof.

The following lemma recalls some elementary facts (see also [63, 97]) about the function V in (2.3.8), which will be used throughout the section. We then proceed with a technical result needed in the proof of Proposition 2.5.14.

Lemma 2.5.12. *The function* V *in* (2.3.8) *is* $\mathcal{C}^\infty$*, strictly convex and essentially smooth on the open interval* (u_-, u_+) *(defined in* (2.3.6)*) and* $V(0) = 0$*,* $u_- < 0$ *and* $\lim_{u\downarrow u_-} V(u)$ *and* $\lim_{u\uparrow u_+} V(u)$ *are both finite. Furthermore* $V(1) = 0$ *if* $\rho \leq \kappa/\xi$ *and* $V(1) < 0$ *if* $\rho > \kappa/\xi$*. Finally, if* $\rho \leq \kappa/\xi$*, then* $u_+ \geq 1$ *with* $u_+ = 1$ *if and only if* $\rho = \kappa/\xi$ *.*

Lemma 2.5.13. *Let* $\rho_\pm$ *be defined as in* (2.3.6) *and* β_t *in* (1.3.4)*. Assume further that* $t > 0$ *and define the functions* g_+ *and* g_- *by*

$$g_\pm(\rho) := (2\kappa - \rho\xi) \pm \rho\sqrt{\xi^2\left(1-\rho^2\right) + (2\kappa - \rho\xi)^2 - \frac{\xi^2(1-\rho^2)}{\beta_t}}.$$

(i) The inequalities $\rho_- \in (-1, 0)$ *and* $\rho_+ > 1/2$ *always hold; if* $\kappa/\xi > \rho_+$ *and* $t \neq 0$*, then* $\rho_+ < 1$*; finally* $\rho_+ = 1$ *(and* $\rho_- = -1$*) if and only if* $t = 0$*;*

(ii) the inequality $g_+(\rho) > 0$ *holds if and only if* $\rho_+ < 1$ *and* $\rho \in (\rho_+, 1)$*;*

(iii) the inequality $g_-(\rho) > 0$ *holds if and only if* $\rho \in (-1, \rho_-)$*;*

(iv) let $u^*_\pm$ *be as in* (2.3.6) *and* $t > 0$*. Then* $u^*_+ > 1$ *if* $\rho \leq \rho_-$*, and* $u^*_- < 0$ *if* $\rho \geq \rho_+$*.*

Proof. We first prove Lemma 2.5.13(i). The double inequality $-1 < \rho_- < 0$ is equivalent to

$$\frac{\xi - (8\kappa + \xi)\mathrm{e}^{2\kappa t}}{\mathrm{e}^{\kappa t} + 1} < -\sqrt{16\kappa^2\mathrm{e}^{2\kappa t} + \xi^2\left(1 - \mathrm{e}^{\kappa t}\right)^2} < \xi\left(1 - \mathrm{e}^{\kappa t}\right).$$

The upper bound clearly holds, and the lower bound follows from the identity

$$\sqrt{16\kappa^2\mathrm{e}^{2\kappa t} + \xi^2\left(1 - \mathrm{e}^{\kappa t}\right)^2} = \sqrt{\frac{\left(\xi - (8\kappa + \xi)\mathrm{e}^{2\kappa t}\right)^2}{\left(\mathrm{e}^{\kappa t} + 1\right)^2} - \frac{16\kappa\mathrm{e}^{2\kappa t}\left(\mathrm{e}^{\kappa t} - 1\right)\left(\kappa + \xi + \xi\mathrm{e}^{\kappa t} + 3\kappa\mathrm{e}^{\kappa t}\right)}{\left(\mathrm{e}^{\kappa t} + 1\right)^2}}.$$

We now prove that $\rho_+ > 1/2$. From (2.3.6) this is equivalent to

$$\sqrt{16\kappa^2\mathrm{e}^{2\kappa t} + \xi^2\left(1 - \mathrm{e}^{\kappa t}\right)^2} > \frac{4\xi + (\kappa - 4\xi)\mathrm{e}^{2\kappa t}}{4\left(\mathrm{e}^{\kappa t} + 1\right)}.$$

The result follows by rearranging the left-hand side as

$$\sqrt{16\kappa^2\mathrm{e}^{2\kappa t} + \xi^2\left(1 - \mathrm{e}^{\kappa t}\right)^2}$$
$$= \sqrt{\frac{\left(4\xi + (\kappa - 4\xi)\mathrm{e}^{2\kappa t}\right)^2}{16\left(\mathrm{e}^{\kappa t} + 1\right)^2} + \frac{\kappa\mathrm{e}^{2\kappa t}\left(8\xi\left(\mathrm{e}^{2\kappa t} - 1\right) + \kappa\left(512\mathrm{e}^{\kappa t} + 255\mathrm{e}^{2\kappa t} + 256\right)\right)}{16\left(\mathrm{e}^{\kappa t} + 1\right)^2}}.$$

Assume now $\kappa/\xi > \rho_+$. The inequality $\rho_+ < 1$ is equivalent to

$$\sqrt{16\kappa^2\mathrm{e}^{2\kappa t} + \xi^2\left(1 - \mathrm{e}^{\kappa t}\right)^2} < \frac{\xi + (8\kappa - \xi)\mathrm{e}^{2\kappa t}}{\mathrm{e}^{\kappa t} + 1},$$

or

$$\sqrt{\frac{\left(\xi + (8\kappa - \xi)\mathrm{e}^{2\kappa t}\right)^2}{\left(\mathrm{e}^{\kappa t} + 1\right)^2} - \frac{16\kappa\mathrm{e}^{2\kappa t}\left(\mathrm{e}^{\kappa t} - 1\right)\left(\kappa - \xi\left(\mathrm{e}^{\kappa t} + 1\right) + 3\kappa\mathrm{e}^{\kappa t}\right)}{\left(\mathrm{e}^{\kappa t} + 1\right)^2}} < \frac{\xi + (8\kappa - \xi)\mathrm{e}^{2\kappa t}}{\mathrm{e}^{\kappa t} + 1}. \quad (2.5.30)$$

This statement is true if $\kappa - \xi\left(\mathrm{e}^{\kappa t}+1\right) + 3\kappa \mathrm{e}^{\kappa t} > 0$ and if the rhs is positive, which follow from the obvious inequalities

$$\frac{\mathrm{e}^{\kappa t}+1}{3\mathrm{e}^{\kappa t}+1} < \frac{1}{2} < \frac{\kappa}{\xi}.$$

We now prove Lemma 2.5.13(ii). The equation $g_+(\rho) = 0$ implies (by squaring and rearranging the terms):

$$4\kappa(\rho^2-1)\left(4\kappa \mathrm{e}^{2\kappa t}\rho^2 + \xi(1-\mathrm{e}^{2\kappa t})\rho - \kappa(1+2\mathrm{e}^{\kappa t}+\mathrm{e}^{2\kappa t})\right) = 0.$$

The roots of this equation are ± 1 and $\rho_\pm$ defined in (2.3.6). The two possible positive roots are $\{\rho_+, 1\}$ and the two possible negative ones are $\{\rho_-, -1\}$. Clearly $g_+(-1) = 0$. Straightforward computations show that $g'_+(-1) < 0$ and $g'_+(0) > 0$. Since g_+ is continuous on $(-1, 0)$ with $g_+(0) < 0$, it cannot have a single root in this interval, and $\rho_- \in (-1, 0)$ (by Lemma 2.5.13(i)) is hence not a valid root. Consider now $\rho \in (0, 1]$. From Lemma 2.5.13(i) the only possible roots are 1 and ρ_+. Now $g_+(1) = 2\kappa - \xi + |2\kappa - \xi|$. If $\kappa/\xi > 1/2$ then $g_+(1) > 0$ and hence ρ_+ is the unique root of g_+ in $(0, 1)$. Assume now that $\kappa/\xi \le 1/2$, which implies $g_+(1) = 0$. Either $g'_+(1) \ge 0$ or $g'_+(1) < 0$. Since $g_+(0) < 0$, the first case implies that g_+ has zero or more than two roots in $(0, 1)$. If it has zero roots, then clearly $g_+(\rho) < 0$ for $\rho \in (0, 1)$. More than two roots yields a contradiction with the fact that ρ_+ is the only possible root on $(0, 1)$. Now, Inequality (2.5.30) implies that $\rho_+ < 1$ if and only if $\kappa/\xi > (\mathrm{e}^{\kappa t}+1)/(3\mathrm{e}^{\kappa t}+1)$, which is equivalent to $g'_+(1) < 0$. Therefore in the case $\kappa/\xi \le 1/2$, the only possible scenario is $g'_+(1) < 0$, where g_+ has a unique root $\rho_+ \in (0, 1)$. In summary, on the interval $[-1, 1]$, $g_+(\rho) > 0$ if and only if $\rho \in (\rho_+, 1)$ and $\rho_+ < 1$. The proof of (iii) is analogous to the proof of (ii) and we omit it for brevity.

We now prove Lemma 2.5.13(iv). From (2.3.6) write $\nu = z(\rho)^{1/2}$, where $z(\rho) := \xi^2 - 2\mathrm{e}^{\kappa t}\left(8\kappa^2 - 4\kappa\xi\rho + \xi^2\right) + \mathrm{e}^{2\kappa t}(\xi - 4\kappa\rho)^2$. The two numbers u^*_- and u^*_+ in (2.3.6) are well defined in $\mathbb{R}$ if and only if $z(\rho) \ge 0$ and $t > 0$. The two roots of this polynomial are given by $\chi_\pm := \frac{1}{4\kappa}\left[\mathrm{e}^{-\kappa t}\left(\xi(\mathrm{e}^{\kappa t}-1) \pm 4\kappa \mathrm{e}^{\kappa t/2}\right)\right]$. We now claim that $\rho_- \le \chi_-$ and $\rho_+ \ge \chi_+$. From the expression of ρ_- given in (2.3.6), the inequality $\rho_- \le \chi_-$ can be rearranged as

$$-\sqrt{\xi^2 + 16\kappa^2\mathrm{e}^{2\kappa t} - 2\xi^2\mathrm{e}^{\kappa t} + \xi^2\mathrm{e}^{2\kappa t}} \le \frac{\xi - 2\xi\mathrm{e}^{\kappa t} + \xi\mathrm{e}^{2\kappa t} - 8\kappa\mathrm{e}^{3\kappa t/2}}{\mathrm{e}^{\kappa t}+1}.$$

The claim then follows from the identity

$$\sqrt{\xi^2 + 16\kappa^2\mathrm{e}^{2\kappa t} - 2\xi^2\mathrm{e}^{\kappa t} + \xi^2\mathrm{e}^{2\kappa t}} = \sqrt{\frac{4\mathrm{e}^{\kappa t}\left(\mathrm{e}^{\kappa t}-1\right)^2\left(\xi + 2\kappa\mathrm{e}^{\kappa t/2}\right)^2}{\left(\mathrm{e}^{\kappa t}+1\right)^2} + \frac{\left(\xi - 2\xi\mathrm{e}^{\kappa t} + \xi\mathrm{e}^{2\kappa t} - 8\kappa\mathrm{e}^{3\kappa t/2}\right)^2}{\left(\mathrm{e}^{\kappa t}+1\right)^2}}.$$

Analogous manipulations imply $\rho_+ \ge \chi_+$, and hence $z(\rho)$ is a well-defined real number for $\rho \in [-1, \rho_-] \cup [\rho_+, 1]$.

The claim $u^*_- < 0$ is equivalent to

$$-\sqrt{\xi^2 - 2\mathrm{e}^{\kappa t}\left(8\kappa^2 - 4\kappa\xi\rho + \xi^2\right) + \mathrm{e}^{2\kappa t}(\xi - 4\kappa\rho)^2} < \xi\left(1 - \mathrm{e}^{\kappa t}\right) + 4\kappa\rho\mathrm{e}^{\kappa t},$$

which holds as soon as $\xi\left(1-\mathrm{e}^{\kappa t}\right)+4\kappa\rho\mathrm{e}^{\kappa t}>0$, or $\rho>\frac{\xi}{4\kappa}\left(1-\mathrm{e}^{-\kappa t}\right)$. Therefore for any $\rho\geq\rho_+$, $u_-^*<0$ if and only if $\rho_+>\frac{\xi}{4\kappa}\left(1-\mathrm{e}^{-\kappa t}\right)$. This simplifies to

$$\sqrt{\xi^2+16\kappa^2\mathrm{e}^{2\kappa t}-2\xi^2\mathrm{e}^{\kappa t}+\xi^2\mathrm{e}^{2\kappa t}}>\frac{\xi\left(\mathrm{e}^{\kappa t}-1\right)^2}{\mathrm{e}^{\kappa t}+1},$$

which also reads

$$\sqrt{\frac{4\mathrm{e}^{\kappa t}\left(4\kappa^2\mathrm{e}^{\kappa t}\left(\mathrm{e}^{\kappa t}+1\right)^2+\xi^2\left(\mathrm{e}^{\kappa t}-1\right)^2\right)}{\left(\mathrm{e}^{\kappa t}+1\right)^2}+\frac{\xi^2\left(\mathrm{e}^{\kappa t}-1\right)^4}{\left(\mathrm{e}^{\kappa t}+1\right)^2}}>\frac{\xi\left(\mathrm{e}^{\kappa t}-1\right)^2}{\mathrm{e}^{\kappa t}+1},$$

and this is clearly true. Now straightforward manipulations show that the inequality $u_+^*>1$ is equivalent to

$$\sqrt{\left(\xi\left(\mathrm{e}^{\kappa t}-1\right)+4\kappa\rho\mathrm{e}^{\kappa t}\right)^2-16\kappa\mathrm{e}^{\kappa t}\left(\kappa+\xi\rho\left(\mathrm{e}^{\kappa t}-1\right)\right)}>\xi\left(\mathrm{e}^{\kappa t}-1\right)+4\kappa\rho\mathrm{e}^{\kappa t},$$

which is true if $\rho<-\dfrac{\kappa}{\xi\left(\mathrm{e}^{\kappa t}-1\right)}$ or $\rho<-\dfrac{\xi\left(1-\mathrm{e}^{-\kappa t}\right)}{4\kappa}$. And of course the claim $\left(u_+^*>1 \text{ if } \rho\leq\rho_-\right)$ holds if

$$\rho_-<-\frac{\kappa}{\xi\left(\mathrm{e}^{\kappa t}-1\right)} \qquad \text{or} \qquad \rho_-<-\frac{\xi\left(1-\mathrm{e}^{-\kappa t}\right)}{4\kappa}. \tag{2.5.31}$$

The first inequality, which can be re-written as

$$-\sqrt{\frac{16\kappa^2\mathrm{e}^{3\kappa t}\left(\xi^2\left(\mathrm{e}^{\kappa t}-1\right)^2\left(\mathrm{e}^{\kappa t}+1\right)-4\kappa^2\mathrm{e}^{\kappa t}\right)}{\xi^2\left(\mathrm{e}^{2\kappa t}-1\right)^2}+\left(\frac{\xi^2(1-\mathrm{e}^{\kappa t})(1-\mathrm{e}^{2\kappa t})+8\kappa^2\mathrm{e}^{2\kappa t}}{\xi(\mathrm{e}^{\kappa t}+1)(1-\mathrm{e}^{\kappa t})}\right)^2} \\ <\frac{\xi^2(1-\mathrm{e}^{\kappa t})(1-\mathrm{e}^{2\kappa t})+8\kappa^2\mathrm{e}^{2\kappa t}}{\xi(\mathrm{e}^{\kappa t}+1)(1-\mathrm{e}^{\kappa t})},$$

holds if $\xi^2\left(\mathrm{e}^{\kappa t}-1\right)^2\left(\mathrm{e}^{\kappa t}+1\right)-4\kappa^2\mathrm{e}^{\kappa t}>0$, or

$$\frac{\left(\mathrm{e}^{\kappa t}-1\right)^2\left(1+\mathrm{e}^{-\kappa t}\right)}{4}>\frac{\kappa^2}{\xi^2}.$$

Quick manipulations turn the second inequality in (2.5.31) into

$$-\sqrt{\frac{4\mathrm{e}^{\kappa t}\left(4\kappa^2\mathrm{e}^{\kappa t}\left(\mathrm{e}^{\kappa t}+1\right)^2-\xi^2\left(\mathrm{e}^{\kappa t}-1\right)^2\left(2\mathrm{e}^{\kappa t}+1\right)\right)}{\left(\mathrm{e}^{\kappa t}+1\right)^2}+\frac{\xi^2\left(2\mathrm{e}^{\kappa t}-3\mathrm{e}^{2\kappa t}+1\right)^2}{\left(\mathrm{e}^{\kappa t}+1\right)^2}} \\ <\frac{\xi\left(2\mathrm{e}^{\kappa t}-3\mathrm{e}^{2\kappa t}+1\right)}{\mathrm{e}^{\kappa t}+1}.$$

Again this trivially holds if $4\kappa^2\mathrm{e}^{\kappa t}\left(\mathrm{e}^{\kappa t}+1\right)^2-\xi^2\left(\mathrm{e}^{\kappa t}-1\right)^2\left(2\mathrm{e}^{\kappa t}+1\right)>0$, which is in turn equivalent to

$$\frac{\kappa^2}{\xi^2}>\frac{\left(\mathrm{e}^{\kappa t}-1\right)^2\left(2+\mathrm{e}^{-\kappa t}\right)}{4\left(\mathrm{e}^{\kappa t}+1\right)^2}.$$

Since

$$\frac{\left(\mathrm{e}^{\kappa t}-1\right)^2\left(2+\mathrm{e}^{-\kappa t}\right)}{4\left(\mathrm{e}^{\kappa t}+1\right)^2}<\frac{\left(\mathrm{e}^{\kappa t}-1\right)^2\left(1+\mathrm{e}^{-\kappa t}\right)}{4},$$

is clearly true, it follows that for any valid choice of parameters either inequality (or both) in (2.5.31) holds, and the claim follows. □

We now use Lemma 2.5.13 to compute the large-maturity cgf effective limiting domain for the forward price process $(\tau^{-1}X_\tau^{(t)})_{\tau>0}$. This is of fundamental importance since in the large-maturity case (unlike the diagonal small-maturity case) we need to find conditions on the parameters of the model such that the limiting cgf is essentially smooth (Assumption 2.2.1(iv)) on the interior of its effective domain.

Proposition 2.5.14. *Let $\varepsilon = \tau^{-1}$ and consider the large-maturity Heston forward process $(\tau^{-1}X_\tau^{(t)})_{\tau>0}$. Then $\mathcal{D}_0 = \mathcal{K}_{\mathrm{H}}$ and if $\rho < \kappa/\xi$ then $\{0,1\} \subset \mathcal{D}_0^o$ with $\mathcal{K}_{\mathrm{H}}$ and $\mathcal{D}_0$ defined in Table 2.1 and in Assumption 2.2.1.*

Proof. The tower property yields

$$\mathbb{E}\left(\mathrm{e}^{u(X_{t+\tau}-X_t)}\right) = \mathbb{E}\left[\mathbb{E}\left(\mathrm{e}^{u(X_{t+\tau}-X_t)}|\mathcal{F}_t\right)\right] = \mathbb{E}\left(\mathrm{e}^{A(u,\tau)+B(u,\tau)V_t}\right) = \mathrm{e}^{A(u,\tau)}\mathbb{E}\left(\mathrm{e}^{B(u,\tau)V_t}\right),$$

with A and B defined in (1.3.8). For any fixed $t \geq 0$ we require that

$$\mathbb{E}\left(\mathrm{e}^{u(X_{t+\tau}-X_t)}|\mathcal{F}_t\right) < \infty \quad \text{for all} \quad \tau > 0. \tag{2.5.32}$$

If $\kappa \geq \rho\xi$ then due to [97, Proposition 2.3] we know that (2.5.32) is satisfied when $u \in [u_-, u_+]$, with $u_- < 0$ and $u_+ \geq 1$ ($u_\pm$ defined in (2.3.6) with $u_+ = 1$ if and only if $\kappa = \rho\xi$). If $\kappa < \rho\xi$ then due to [97, Proposition 2.3] we know that (2.5.32) is satisfied when $u \in [u_-, 1]$ with $u_- < 0$. Further we require that

$$\mathbb{E}\left(\mathrm{e}^{B(u,\tau)V_t}\right) < \infty, \quad \text{for all } \tau > 0. \tag{2.5.33}$$

Now denote $\mathcal{K}_V := \{u \in \mathbb{R} : \mathbb{E}(\mathrm{e}^{B(u,\tau)V_t}) < \infty, \text{ for all } \tau > 0\}$. Then if $\kappa \geq \rho\xi$, the domain of the limiting forward cgf is given by $\mathcal{K}_{\mathrm{H}} = [u_-, u_+] \cap \mathcal{K}_V$ and if $\kappa < \rho\xi$ then $\mathcal{K}_{\mathrm{H}} = [u_-, 1] \cap \mathcal{K}_V$. Condition (2.5.33) is equivalent to $B(u,\tau) < 1/(2\beta_t)$ for all $\tau > 0$. Note that $[0,1] \subset \mathcal{K}_{\mathrm{H}}$ by the martingale condition. For fixed $u \in \mathbb{R}$,

$$\frac{\partial B(u,\tau)}{\partial \tau} = \frac{2u(u-1)d(u)^2\mathrm{e}^{d(u)\tau}}{\left(\kappa - \kappa\mathrm{e}^{d(u)\tau} + \xi\rho u\left(\mathrm{e}^{d(u)\tau}-1\right) - d(u)\left(\mathrm{e}^{d(u)\tau}+1\right)\right)^2},$$

so that for any $u \notin [0,1]$, $B(u,\cdot)$ is strictly increasing. Therefore

$$\mathcal{K}_V = \left\{u \in \mathbb{R} : \lim_{\tau\uparrow\infty} B(u,\tau) < \frac{1}{2\beta_t}\right\}. \tag{2.5.34}$$

We have $\lim_{\tau\uparrow\infty} B(u,\tau) = \xi^{-2}(\kappa - \rho\xi u - d(u))$. So the condition is equivalent to $\kappa - \rho\xi u - d(u) < 2\kappa/(1-\mathrm{e}^{-\kappa t})$. If $\rho \leq 0$ ($\rho \geq 0$) and $u \leq 0$ ($u \geq 0$) then $\kappa - \rho\xi u - d(u) \leq \kappa - \rho\xi u \leq \kappa < \frac{2\kappa}{1-\mathrm{e}^{-\kappa t}}$, and the condition in (2.5.34) is always satisfied. So if $\rho = 0$, $\mathcal{K}_{\mathrm{H}} = [u_-, u_+]$. If $\rho < 0$ ($\rho > 0$), then $\mathbb{R}_- \subset \mathcal{K}_V$ ($\mathbb{R}_+ \subset \mathcal{K}_V$), and hence $\mathcal{K}_{\mathrm{H}}$ contains $[u_-, 0]$ ($[0, u_+]$ if $0 < \rho \leq \kappa/\xi$ or $[0,1]$ if $\rho > \kappa/\xi$). Now suppose that $\rho < 0$ and $u > 0$. The condition in (2.5.34) (V given in (2.3.8)) is equivalent to $V(u) < \kappa\theta/(2\beta_t)$. From Lemma 2.5.12, on $(0, u_+]$, the function V attains its maximum at u_+. Using the properties in Lemma 2.5.12, there exists $u_+^* \in (1, u_+)$ solving the equation

$$\frac{V(u_+^*)}{\kappa\theta} = \frac{1}{2\beta_t}, \tag{2.5.35}$$

if and only if $g_-(\rho) > 0$ (g_- defined in Lemma 2.5.13), which is equivalent (see Lemma 2.5.13) to $-1 < \rho < \rho_-$ and $t > 0$. The solution to (2.5.35) has two roots u_-^* and u_+^* defined in (2.3.6), and the correct solution here is u_+^* by Lemma 2.5.13(iv). So if $\rho_- \le \rho < 0$ then $\mathcal{K}_{\mathrm{H}} = [u_-, u_+]$. If $-1 < \rho < \rho_-$ and $t > 0$ then $\mathcal{K}_{\mathrm{H}} = [u_-, u_+^*)$. Analogous arguments show that for $0 < \rho \le \min(\kappa/\xi, \rho_+)$, we have $\mathcal{K}_{\mathrm{H}} = [u_-, u_+]$. If $\rho_+ < \rho < \min(\kappa/\xi, 1)$ and $t > 0$ then $\mathcal{K}_{\mathrm{H}} = (u_-^*, u_+]$, with $u_- < u_-^* < 0$. Finally if $\rho > \kappa/\xi$ and $\rho > \rho_+$ then $\mathcal{K}_{\mathrm{H}} = (u_-^*, 1]$ and if $\rho > \kappa/\xi$ and $\rho \le \rho_+$ then $\mathcal{K}_{\mathrm{H}} = (u_-, 1]$. □

The following lemma provides the asymptotic behaviour of the forward cgf $\Lambda_\tau^{(t)}$ defined in (2.5.29) as τ tends to infinity.

Lemma 2.5.15. *The following expansion holds (V,H and d given in* (2.3.8) *and* (1.3.6)*):*

$$\Lambda_\tau^{(t)}(u) = \begin{cases} V(u) + \tau^{-1}H(u)\left(1 + \mathcal{O}\left(\mathrm{e}^{-d(u)\tau}\right)\right), & \text{for all } u \in \mathcal{K}_{\mathrm{H}} \setminus \{1\}, \text{ as } \tau \uparrow \infty, \\ 0, & \text{for } u = 1 \text{ and all } \tau > 0. \end{cases}$$

Remark 2.5.16.

(i) When $\rho > \kappa/\xi$ ($\mathfrak{R}_{3b}$ and $\mathfrak{R}_4$ in (2.3.5)), we have $\lim_{u\uparrow 1} \Lambda_\tau^{(t)}(u) = V(1) \neq 0$, so that the limit is not continuous at the right boundary $u = 1$. For $\rho \le \kappa/\xi$ we always have $V(1) = H(1) = 0$ and $1 \in \mathcal{K}_{\mathrm{H}}^o$ for $\rho < \kappa/\xi$.

(ii) For all $u \in \mathcal{K}_{\mathrm{H}}^o$, $d(u) > 0$, so that the remainder goes to zero exponentially fast as τ tends to infinity.

Proof of Lemma 2.5.15. First note that $\Lambda_\tau^{(t)}(1) = 0$ for all $\tau > 0$ since the asset price process $(\mathrm{e}^{X_t})_{t>0}$ is a true martingale [5, Proposition 2.5]. From the definition of $\Lambda_\tau^{(t)}$ in (2.5.29) and the Heston forward cgf given in (1.3.7) we immediately obtain the following asymptotics as τ tends to infinity:

$$A(u,\tau) = \tau V(u) - \frac{2\kappa\theta}{\xi^2}\log\left(\frac{1}{1-\gamma(u)}\right) + \mathcal{O}\left(\mathrm{e}^{-d(u)\tau}\right), \qquad B(u,\tau) = \frac{V(u)}{\kappa\theta} + \mathcal{O}\left(\mathrm{e}^{-d(u)\tau}\right),$$

where A and B are defined in (1.3.8), V in (2.3.8) and d and γ in (1.3.6) and (1.3.8). In particular this implies that $\frac{B(u,\tau)}{1-2\beta_t B(u,\tau)} = \frac{V(u)}{\theta\kappa - 2\beta_t V(u)} + \mathcal{O}\left(\mathrm{e}^{-d(u)\tau}\right)$ and $\log(1 - 2\beta_t B(u,\tau)) = \log\left(1 - \frac{2\beta_t V(u)}{\theta\kappa}\right) + \mathcal{O}\left(\mathrm{e}^{-d(u)\tau}\right)$, which are well defined for all $u \in \mathcal{K}_{\mathrm{H}}^o$. We therefore obtain

$$H(u) = \frac{V(u)}{\kappa\theta - 2\beta_t V(u)} v\mathrm{e}^{-\kappa t} - \frac{2\kappa\theta}{\xi^2}\log\left(1 - \frac{2\beta_t V(u)}{\kappa\theta}\right) - \frac{2\kappa\theta}{\xi^2}\log\left(\frac{1}{1-\gamma(u)}\right),$$

and the lemma follows from straightforward simplifications. □

2.5.3 Proofs of Section 2.3.3

We consider here the two examples of time-changed exponential Lévy models given in Section 1.3.2. The forward cgf's were derived in Lemma 1.3.4 and Lemma 1.3.5.

Proof of Proposition 2.3.10. We show that Proposition 2.2.11 is applicable given the assumptions of Proposition 2.3.10. Consider case (i). The expansion for $\Lambda_\tau^{(t)}$ defined in (2.5.29) is straightforward and analogous to Lemma 2.5.15. In particular we establish that

$$\Lambda_\tau^{(t)}(u) = \widehat{V}(u) + \frac{\widehat{H}(u)}{\tau}\left(1 + \mathcal{O}\left(\mathrm{e}^{-d(\phi(u))\tau}\right)\right), \quad \text{for all } u \in \widehat{\mathcal{K}}_\infty^o, \text{ as } \tau \text{ tends to infinity},$$

where the functions $\widehat{V}$, $\widehat{H}$, d and the domain $\widehat{\mathcal{K}}_\infty$ are defined in (2.3.11), (1.3.14) and (2.3.12). Since ϕ is essentially smooth and strictly convex on $\mathcal{K}_\phi$ and $\widehat{\mathcal{K}}_\infty \subseteq \mathcal{K}_\phi$, then the limiting cgf $\Lambda_0 = \widehat{V}$ is essentially smooth and strictly convex on $\widehat{\mathcal{K}}_\infty$. The map $(\varepsilon, u) \mapsto \Lambda_\varepsilon(u)$ (defined in (2.5.29)) is of class $\mathcal{C}^\infty$ on $\mathbb{R}_+^* \times \widehat{\mathcal{K}}_\infty^o$ since ϕ is of class $\mathcal{C}^\infty$ on $\widehat{\mathcal{K}}_\infty^o$ and Assumption 2.2.1(v) is also satisfied. Since $\phi(1) = 0$ we have that $\widehat{V}(1) = 0$ and $\{0, 1\} \subset \widehat{\mathcal{K}}_\infty^o$. It remains to be checked that the limiting domain is in fact given by $\widehat{\mathcal{K}}_\infty$. We first note that by conditioning on $(V_u)_{t \le u \le t+\tau}$ and using the independence of the time-change and the Lévy process we have $\mathbb{E}\left(\mathrm{e}^{u(X_{t+\tau}-X_t)}\right) = \mathbb{E}\left(\mathrm{e}^{\phi(u)\int_t^{t+\tau} v_s \mathrm{d}s}\right)$ and so any u in the limiting domain must satisfy $\phi(u) < \infty$. Using [44, page 476] and the tower property we compute

$$\begin{aligned}\mathbb{E}\left(\mathrm{e}^{u(X_{t+\tau}-X_t)}\right) &= \mathbb{E}\left[\mathbb{E}\left(\mathrm{e}^{\phi(u)\int_t^{t+\tau} v_s \mathrm{d}s} | \mathcal{F}_t\right)\right] = \mathbb{E}\left(\mathrm{e}^{A(\phi(u),\tau) + B(\phi(u),\tau)v_t}\right) \\ &= \mathrm{e}^{A(\phi(u),\tau)}\mathbb{E}\left(\mathrm{e}^{B(\phi(u),\tau)v_t}\right), \end{aligned} \tag{2.5.36}$$

with A and B given in (1.3.14). Further from (1.3.9) we have

$$\log \mathbb{E}\left(\mathrm{e}^{uv_t}\right) = \frac{uv\mathrm{e}^{-\kappa t}}{1 - 2\beta_t u} - \frac{2\kappa\theta}{\xi^2}\log\left(1 - 2\beta_t u\right), \qquad \text{for all } u < \frac{1}{2\beta_t}.$$

Following a similar argument to the proof of Proposition 2.5.14 we can show that for any $t \ge 0$, $B(\phi(u), \tau) < 1/(2\beta_t)$ is always satisfied for each $\tau > 0$. This follows from the independence of the Lévy process N and the time-change. We also require that for any $t \ge 0$, $\mathbb{E}\left(\mathrm{e}^{\phi(u)\int_t^{t+\tau} v_s \mathrm{d}s} | \mathcal{F}_t\right) < \infty$, for every $\tau > 0$. Here we use [5, Corollary 3.3] with zero correlation to find that we require $\phi(u) \le \kappa^2/(2\xi^2)$. It follows that $\widehat{\mathcal{K}}_\infty = \left\{u : \phi(u) \le \kappa^2/(2\xi^2)\right\}$.

Regarding case (ii), arguments analogous to case (i) hold and we focus on showing that the limiting domain is $\widetilde{\mathcal{K}}_\infty$. Using [44, page 488] Equality (2.5.36) also holds with A and B defined in (1.3.17). Since we require that for any $t \ge 0$, $\mathbb{E}\left(\mathrm{e}^{\int_t^{t+\tau} v_s \mathrm{d}s \phi(u)} | \mathcal{F}_t\right) < \infty$, for every $\tau > 0$ we have $\phi(u) < \alpha\lambda$. Using [44, page 482] we also have

$$\log \mathbb{E}\left(\mathrm{e}^{uv_t}\right) = uv\mathrm{e}^{-\lambda t} + \delta \log\left(\frac{u - \alpha \mathrm{e}^{\lambda t}}{(u - \alpha)\mathrm{e}^{\lambda t}}\right), \qquad \text{for all } u < \alpha.$$

But it is straightforward to show that $\phi(u) < \alpha\lambda$ implies $B(\phi(u), \tau) < \alpha$ for any $\tau > 0$ and it follows that $\widetilde{\mathcal{K}}_\infty = \{u : \phi(u) < \alpha\lambda\}$. Case (iii) is straightforward and omitted. □

Chapter 3

The small-maturity Heston forward smile

3.1 Introduction

In Chapter 2 we derived small and large-maturity forward smile asymptotics for a general class of models including the Heston model (1.3.2). However, these results only apply to the so-called diagonal small-maturity regime, i.e. the behaviour (as ε tends to zero) of the process $(X^{(\varepsilon t)}_{\varepsilon \tau})_{\varepsilon \geq 0}$ (defined in (1.0.3)). The conjecture, stated in Chapter 2 (see for example Remark 2.3.3(iv)), is that for fixed $t > 0$ the Heston forward smile explodes to infinity (except at-the-money) as τ tends to zero.

In this chapter we confirm this conjecture and give a high-order expansion for the forward smile. The main result (Theorem 3.4.1) is that the small-maturity Heston forward smile explodes according to the following asymptotic: $\sigma^2_{t,\tau}(k) = \mathfrak{N}_0(k,t)\tau^{-1/2} + \mathfrak{N}_1(k,t)\tau^{-1/4} + o\left(\tau^{-1/4}\right)$ for $k \in \mathbb{R}^*$ and $t > 0$ as τ tends to zero. Here the forward smile, $\sigma_{t,\tau}$, is defined in (1.0.3) and $\mathfrak{N}_0(\cdot, t)$ and $\mathfrak{N}_1(\cdot, t)$ are even continuous functions (over $\mathbb{R}$) with $\mathfrak{N}_0(0,t) = \mathfrak{N}_1(0,t) = 0$ and independent of the Heston correlation. In the at-the-money case ($k = 0$) a separate model-independent analysis (Lemma 3.4.3 and Theorem 3.4.4) shows that the small-maturity limit is well defined and $\lim_{\tau \searrow 0} \sigma_{t,\tau}(0) = \mathbb{E}(\sqrt{V_t})$ holds for any well-behaved diffusion where V_t is the instantaneous variance at time t. This exploding nature is consistent with empirical observations in [36] and the diagonal small-maturity asymptotic from Chapter 2.

The chapter is structured as follows. In Section 3.2 we introduce the notion of a forward time-scale and characterise it in the Heston model. In Section 3.3 we state the main result on small-maturity asymptotics of forward-start options in the Heston model. Section 3.4 tackles the forward implied volatility asymptotics: Section 3.4.1 translates the results of Section 3.3 into out-of-the-money forward smile asymptotics, and Section 3.4.2 presents a model-independent result for the at-the-money forward implied volatility. Section 3.5 provides numerical evidence supporting the asymptotics derived in the chapter and the main proofs are gathered in Section 3.6. In this chapter we will always assume that forward-start date is greater than zero ($t > 0$) unless otherwise stated.

3.2 Forward time-scales

In this section we introduce the notion of a forward time-scale and characterise it in the Heston model (1.3.2). In the BSM model (1.0.1) the time-scale is given by $h(t) \equiv t$, which is related to the quadratic variation of the driving Brownian motion. For diffusions (such as Heston) the (spot) time-scale is the same, which implies that the spot smile has a finite (non-zero) small-maturity limit. In the forward case, this however no longer remains true. Stochastic volatility models (eg. Heston) exhibit different time-scales to the BSM model leading to different asymptotic regimes for the forward smile relative to the spot smile. As we will show below (Lemma 3.2.3), all re-scalings of the Heston model lead to limiting cumulant generating functions (cgf's) that are all zero on their domains of definition. But the forward time-scale is the only choice that leads to the limiting cgf being zero on a bounded domain. This is one of the key properties that allows us to derive sharp large deviation results even though at first sight this zero limit appears trivial and non-consequential. We define the re-normalised forward cgf by ($X_\tau^{(t)}$ defined in (1.0.3))

$$\Lambda_\tau^{(t)}(u,a) := a \log \mathbb{E}\left(\mathrm{e}^{uX_\tau^{(t)}/a}\right), \quad \text{for all } u \in \mathcal{D}_{t,\tau}, \tag{3.2.1}$$

where $\mathcal{D}_{t,\tau} := \{u \in \mathbb{R} : |\Lambda_\tau^{(t)}(u,a)| < \infty\}$. With this definition the domain $\mathcal{D}_{t,\tau}$ will depend on a, but it will be clear from the context which choice of a we are using. Recall that the Heston forward cgf (with $a = 1$) was derived in Lemma 1.3.1.

Definition 3.2.1. We define a (small-maturity) *forward time-scale* as a continuous function $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{\tau \searrow 0} h(\tau) = 0$ and $\Lambda(u) := \lim_{\tau \searrow 0} \Lambda_\tau^{(t)}(u, h(\tau))$ produces a non-trivial pointwise limit. We shall say that a (pointwise) limit is trivial if it is null on $\mathbb{R}$ or null at the origin and infinite on $\mathbb{R}^*$.

Remark 3.2.2.

(i) The forward time-scale is unique up to scaling. If h is a forward time-scale then the family of functions αh for any $\alpha > 0$ are also forward times-scales.

(ii) In the BSM model the forward time-scale is $h(\tau) \equiv \tau$.

(iii) A forward time-scale may not exist for a model. For example, consider exponential Lévy models with bounded domain for the Lévy exponent. The only non-trivial limit occurs when $h \equiv 1$, which does not satisfy Definition 3.2.1 and so a forward-time scale does not exist.

(iv) If $(X_\tau^{(t)})_{\tau\geq 0}$ satisfies a large deviations principle [48, Section 1.2] with speed h and assuming further some tail condition (see [48, Theorem 4.3.1]), then h is the forward time-scale for the model by Varadhan's lemma.

(v) Diffusion models have the same spot time-scale ($t = 0$) as the BSM model, namely $h(\tau) \equiv \tau$ (see for example [22]). This is not necessarily true in the forward case as we will shortly see.

In order to characterise the Heston forward time-scale we require the following lemma, proved in Section 3.6.1.

Lemma 3.2.3. *Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function such that $\lim_{\tau\searrow 0} h(\tau) = 0$ and $a \in \mathbb{R}_+^*$. The following limits hold for the Heston forward cgf as τ tends to zero with β_t defined in (1.3.4):*

(i) If $h(\tau) \equiv a\sqrt{\tau}$ then $\lim_{\tau\searrow 0} \Lambda_\tau^{(t)}(u, h(\tau)) = 0$, for all $|u| < a/\sqrt{\beta_t}$ and is infinite otherwise;

(ii) if $\sqrt{\tau}/h(\tau) \nearrow \infty$ then $\lim_{\tau\searrow 0} \Lambda_\tau^{(t)}(u, h(\tau)) = 0$, for $u = 0$ and is infinite otherwise;

(iii) if $\sqrt{\tau}/h(\tau) \searrow 0$ then $\lim_{\tau\searrow 0} \Lambda_\tau^{(t)}(u, h(\tau)) = 0$, for all $u \in \mathbb{R}$.

As it turns out all limits are zero on their domains of definition, but using $h(\tau) \equiv \sqrt{\tau}$ produces the only (up to a constant multiplicative factor) non-trivial zero limit. It follows that $\tau \mapsto \sqrt{\tau}$ is the Heston forward time-scale. Let now $\Lambda : \mathcal{D}_\Lambda = (-1/\sqrt{\beta_t}, 1/\sqrt{\beta_t}) \to \mathbb{R}$ be the pointwise limit (with $\beta_t := \xi^2 \left(1 - \mathrm{e}^{-\kappa t}\right)/(4\kappa)$) from Lemma 3.2.3, i.e. satisfying $\Lambda(u) = 0$ for $u \in \mathcal{D}_\Lambda$ and infinity otherwise. Further we define the function $\Lambda^* : \mathbb{R} \to \mathbb{R}_+$ as the Fenchel-Legendre transform of Λ:

$$\Lambda^*(k) := \sup_{u\in\mathcal{D}_\Lambda} \{uk - \Lambda(u)\}, \qquad \text{for all } k \in \mathbb{R}. \tag{3.2.2}$$

Lemma 3.2.4. *The function Λ^* defined in (3.2.2) is characterised explicitly as $\Lambda^*(k) = |k|/\sqrt{\beta_t}$ for all $k \in \mathbb{R}$.*

Proof. Clearly $\Lambda^*(0) = 0$. Now suppose that $k > 0$. Then $\Lambda^*(k) = \sup_{u\in\mathcal{D}_\Lambda} \{uk\} = k/\sqrt{\beta_t}$. A similar result holds for $k < 0$ and the result follows. □

3.3 Small-maturity forward-start option asymptotics

In this section we state the main result on small-maturity forward-start option asymptotics. First we need to define a number of functions. All functions below are real-valued and defined on $\mathbb{R}^*$. We recall that $\mathrm{sgn}(u) = 1$ if $u \geq 0$ and -1 otherwise.

$$\begin{cases} a_0(k) := \dfrac{\mathrm{sgn}(k)}{\sqrt{\beta_t}}, \quad a_1(k) := -\dfrac{a_0(k)\sqrt{v}\mathrm{e}^{-\kappa t/2}}{2\sqrt{|k|}\beta_t^{1/4}}, \quad a_2(k) := -\dfrac{\kappa\theta}{k\xi^2} - \dfrac{\widehat{B}_1(a_0(k))}{a_0(k)}, \\ a_3(k) := \dfrac{2\beta_t a_1^3(k)}{\xi^4 v^2}\Big[\xi^2 v\beta_t \mathrm{e}^{\kappa t}\left(|k|\xi^2\beta_t^{\frac{1}{2}}\widehat{B}_1(a_0(k)) - k\xi^2\widehat{B}_1'(a_0(k)) - \kappa\theta\right) \\ +(2\kappa\theta\beta_t\mathrm{e}^{\kappa t})^2 - \dfrac{\xi^4 v^2}{16}\Big], \end{cases} \tag{3.3.1}$$

where

$$\widehat{B}_1(u) := \frac{u}{4}\Big(u^2\rho\xi - 2\Big); \tag{3.3.2}$$

$$\zeta(k) := \frac{2\sqrt{v}\mathrm{e}^{-\kappa t/2}}{e_0(k)^{3/2}}, \qquad r(k) := \frac{a_1^2(k)}{2} - \frac{\kappa\theta}{|k|\xi^2\sqrt{\beta_t}}, \tag{3.3.3}$$

$$\begin{cases} e_0(k) & := -2a_1(k)/a_0(k), \\ e_1(k) & := -2\beta_t r(k), \\ e_2(k) & := -2\beta_t\left(a_1(k)a_2(k) + a_0(k)a_3(k) + a_1(k)\widehat{B}_1'(a_0(k))\right), \end{cases} \tag{3.3.4}$$

$$\begin{cases} \psi_0(k) & := \frac{a_0(k)v\mathrm{e}^{-\kappa t}}{e_0^3(k)}\Big(e_0^2(k) + a_0(k)\beta_t\left[3a_1(k)e_0(k) - 2a_0(k)e_1(k)\right]\Big), \\ \psi_1(k) & := -4a_0(k)v\beta_t\mathrm{e}^{-\kappa t}/e_0^4(k) \\ \psi_2(k) & := \frac{v\mathrm{e}^{-\kappa t}}{2e_0^4(k)}\Big(4a_0(k)\beta_t[3a_0(k)e_1(k) - 4a_1(k)e_0(k)] - 5e_0^2(k)\Big), \\ \psi_3(k) & := 8v\beta_t\mathrm{e}^{-\kappa t}/e_0^5(k), \\ \psi_4(k) & := \frac{v\mathrm{e}^{-\kappa t}}{2e_0^3(k)}\left(\frac{e_1^2(k) - e_0(k)e_2(k)}{\beta_t} - 2a_0(k)a_1(k)e_0(k)e_1(k) + 2e_0^2(k)r(k)\right), \end{cases} \tag{3.3.5}$$

$$\begin{cases} \phi_2^a(k) & := \psi_2(k) - \frac{1}{2}\psi_0^2(k) - \frac{4\kappa\theta\beta_t}{\xi^2}\frac{2\kappa\theta+\xi^2}{e_0^2(k)\xi^2} - \frac{4\kappa\theta\beta_t}{\xi^2}\frac{a_0(k)\psi_0(k)}{e_0(k)}, \\ \phi_2^b(k) & := \psi_3(k) - \psi_0(k)\psi_1(k) - \frac{4\kappa\theta\beta_t}{\xi^2}\frac{a_0(k)\psi_1(k)}{e_0(k)}, \\ \phi_2^c(k) & := -\psi_1^2(k)/2, \end{cases} \tag{3.3.6}$$

$$z_1(k) := \psi_4(k) - a_3(k)k - \frac{2\kappa\theta}{\xi^2}\frac{e_1(k)}{e_0(k)}, \quad p_1(k) := e_0(k) + \frac{\phi_2^a(k)}{\zeta^2(k)} + \frac{3\phi_2^b(k)}{\zeta^4(k)} + \frac{15\phi_2^c(k)}{\zeta^6(k)}, \tag{3.3.7}$$

$$\begin{cases} c_0(k) & := 2|a_1(k)k|, \qquad c_1(k) := \frac{v\mathrm{e}^{-\kappa t}}{e_0(k)}\left(a_0(k)a_1(k) - \frac{e_1(k)}{2\beta_t e_0(k)}\right) - a_2(k)k, \\ c_2(k) & := e_0(k)^{-2\kappa\theta/\xi^2}, \quad c_3(k) := z_1(k) + p_1(k). \end{cases} \tag{3.3.8}$$

We now state the main result of the section, i.e. an asymptotic expansion formula for forward-start option prices as the remaining maturity tends to zero. The proof is given in Section 3.6.4.

Theorem 3.3.1. *The following expansion holds for forward-start option prices for all $k \in \mathbb{R}^*$ as τ tends to zero:*

$$\begin{aligned}\mathbb{E}\left(\mathrm{e}^{X_\tau^{(t)}} - \mathrm{e}^k\right)^+ &= \left(1 - \mathrm{e}^k\right)\mathbb{1}_{\{k<0\}} \\ &+ \exp\left(-\frac{\Lambda^*(k)}{\sqrt{\tau}} + \frac{c_0(k)}{\tau^{1/4}} + c_1(k) + k\right)\frac{\beta_t\tau^{\left(7/8-\theta\kappa/(2\xi^2)\right)}c_2(k)}{\zeta(k)\sqrt{2\pi}}\left(1 + c_3(k)\tau^{1/4} + o\left(\tau^{1/4}\right)\right),\end{aligned}$$

where Λ^* is characterised in Lemma 3.2.4, $c_0, \ldots, c_3$ in (3.3.8)*, ζ in* (3.3.3) *and β_t in* (1.3.4).

Remark 3.3.2.

(i) We have $\Lambda^*(k) > 0$ and $c_0(k) > 0$ for all $k \in \mathbb{R}^*$. Also note that Λ^* is piecewise linear as opposed to being strictly convex in the BSM model, see Lemma 3.3.4 below.

(ii) The forward time-scale $\sqrt{\tau}$ results in out-of-the-money forward-start options decaying as τ tends to zero at leading order with a rate of $\exp(-1/\sqrt{\tau})$ as opposed to a rate of $\exp(-1/\tau)$ in the BSM model.

(iii) The fact that the limiting forward cgf is non-steep (trivially zero on a bounded interval) results in a different asymptotic regime for higher order terms compared to the BSM model. In particular we have a $\tau^{1/4}$ dependence as opposed to a τ dependence in the BSM model and the introduction of the parameter dependent term $\tau^{(7/8-\theta\kappa/(2\xi^2))}$. The implications of this parameter dependent term for forward-smile asymptotics will be discussed further in Remark 3.4.2(vii).

(iv) The asymptotic expansion is given in closed-form and can in principle be extended to arbitrary order using the methods given in the proof.

As an immediate consequence of Theorem 3.3.1 we have the following corollary, which provides an example of a family of random variables for which the limiting re-scaled cumulant generating function is zero (on its effective domain) but a large deviation principle still holds. This is to be compared to the Gärtner-Ellis theorem (Theorem 1.2.3) which requires the limiting cgf to be at least steep at the boundaries of its effective domain for an LDP to hold.

Corollary 3.3.3. $\left(X_\tau^{(t)}\right)_{\tau\geq 0}$ *satisfies an LDP with speed* $\sqrt{\tau}$ *and good rate function* Λ^* *as* τ *tends to zero.*

Proof. The proof of Theorem 3.3.1 holds with only minor modifications for digital options, which are equivalent to probabilities of the form $\mathbb{P}\left(X_\tau^{(t)} \leq k\right)$ or $\mathbb{P}\left(X_\tau^{(t)} \geq k\right)$. One can then show that $\lim_{\tau\searrow 0}\sqrt{\tau}\log\mathbb{P}\left(X_\tau^{(t)} \leq k\right) = -\inf\{\Lambda^*(x), x \leq k\}$. Note that of course this infimum is null whenever $k > 0$. Consider now an open interval of the real line of the form (a, b). Since $(a, b) = (-\infty, b) \setminus (-\infty, a]$, then by continuity of the function Λ^* and its properties given in Lemma 3.2.4, we immediately obtain that

$$\lim_{\tau\searrow 0}\sqrt{\tau}\log\mathbb{P}\left(X_\tau^{(t)} \in (a,b)\right) = -\inf_{x\in(a,b)}\Lambda^*(x).$$

Since any Borel set of the real line can be written as a (countable) union / intersection of open intervals, the corollary follows from the definition of the large deviations principle, see Definition 1.2.1. □

In order to translate the forward-start option results into forward smile asymptotics we require a similar expansion for the BSM model. The following lemma is a direct consequence of Corollary 2.2.7 and the proof is therefore omitted.

Lemma 3.3.4. *In the BSM model* (1.0.1) *the following expansion holds for all* $k \in \mathbb{R}^*$ *as* $\tau \downarrow 0$*:*

$$\mathbb{E}\left(\mathrm{e}^{X_\tau^{(t)}} - \mathrm{e}^k\right)^+ = \left(1-\mathrm{e}^k\right)\mathbb{1}_{\{k<0\}} + \frac{\mathrm{e}^{k/2-k^2/(2\Sigma^2\tau)}\left(\Sigma^2\tau\right)^{3/2}}{k^2\sqrt{2\pi}}\left[1-\left(\frac{3}{k^2}+\frac{1}{8}\right)\Sigma^2\tau + o(\tau)\right].$$

3.4 Small-maturity forward smile asymptotics

3.4.1 Out-of-the-money forward implied volatility

We now translate the small-maturity forward-start option asymptotics into forward smile asymptotics. Define the functions $\mathfrak{N}_i : \mathbb{R}^* \times \mathbb{R}_+^* \to \mathbb{R}$ $(i = 0, 1, 2, 3)$ by

$$\begin{aligned}
\mathfrak{N}_0(k,t) &:= \frac{k^2}{2\Lambda^*(k)} = \frac{\sqrt{\beta_t}|k|}{2}, \quad \mathfrak{N}_1(k,t) := \frac{2c_0(k)\mathfrak{N}_0^2(k,t)}{k^2} = \frac{\mathrm{e}^{-\kappa t/2}\beta_t^{1/4}\sqrt{v|k|}}{2},\\
\mathfrak{N}_2(k,t) &:= \frac{2\mathfrak{N}_0^2(k,t)}{k^2}\log\left(\frac{\mathrm{e}^{c_1(k)}c_2(k)\beta_t k^2}{\zeta(k)\mathfrak{N}_0^{3/2}(k,t)}\right) + \frac{\mathfrak{N}_0^2(k,t)}{k} + \frac{\mathfrak{N}_1^2(k,t)}{\mathfrak{N}_0(k,t)},\\
\mathfrak{N}_3(k,t) &:= \frac{\mathfrak{N}_0(k,t)}{k^2}\Big(2c_3(k)\mathfrak{N}_0(k,t) - 3\mathfrak{N}_1(k,t)\Big) + \frac{\mathfrak{N}_1(k,t)}{\mathfrak{N}_0(k,t)}\left(2\mathfrak{N}_2(k,t) - \frac{\mathfrak{N}_1^2(k,t)}{\mathfrak{N}_0(k,t)}\right),
\end{aligned}$$

with Λ^* characterised in Lemma 3.2.4, $c_0, \ldots, c_3$ in (3.3.8), ζ in (3.3.3) and β_t in (1.3.4). On $\mathbb{R}^*$, $\Lambda^*(k) > 0$ and so $\mathfrak{N}_0(k,t) > 0$. Further $c_0(k) > 0$ and so $\mathfrak{N}_1(k,t) > 0$. Also $c_2(k) > 0$ and $\zeta(k) > 0$ so that $\mathfrak{N}_2$ is a well defined real-valued function. The following theorem—proved in Section 3.6.4—is the main result of the section.

Theorem 3.4.1. *The following expansion holds for the forward smile for all* $k \in \mathbb{R}^*$ *as* τ *tends to zero:*

$$\sigma_{t,\tau}^2(k) = \begin{cases} \dfrac{\mathfrak{N}_0(k,t)}{\tau^{1/2}} + \dfrac{\mathfrak{N}_1(k,t)}{\tau^{1/4}} + o\left(\dfrac{1}{\tau^{1/4}}\right), & \text{if } 4\kappa\theta \neq \xi^2,\\ \dfrac{\mathfrak{N}_0(k,t)}{\tau^{1/2}} + \dfrac{\mathfrak{N}_1(k,t)}{\tau^{1/4}} + \mathfrak{N}_2(k,t) + \mathfrak{N}_3(k,t)\tau^{1/4} + o\left(\tau^{1/4}\right), & \text{if } 4\kappa\theta = \xi^2. \end{cases}$$

Remark 3.4.2.

(i) Note that $\mathfrak{N}_0(k,t)$ and $\mathfrak{N}_1(k,t)$ are strictly positive for all $k \in \mathbb{R}^*$, so that the Heston forward smile blows up to infinity (except ATM) as τ tends to zero.

(ii) Both $\mathfrak{N}_0(\cdot,t)$ and $\mathfrak{N}_1(\cdot,t)$ are even functions and correlation-independent quantities so that for small maturities the Heston forward smile becomes symmetric (in log-strikes) around the at-the-money point. Consequently, if one believes that the small-maturity forward smile should be downward sloping (similar to the spot smile) then the Heston model should not be chosen. This small-maturity 'U-shaped' effect for the Heston forward smile has been mentioned qualitatively by practitioners; see [36].

(iii) We use the notation $f \sim g$ to mean $f/g = 1$ as $\tau \to 0$. Then in Heston we have $\sigma_{t,\tau}^2 \sim \sqrt{\beta_t}|k|/(2\sqrt{\tau})$ and in exponential Lévy models with Lévy measure ν satisfying $\operatorname{supp}\nu = \mathbb{R}$ we have $\sigma_{t,\tau}^2 \sim -k^2/(2\tau\log\tau)$ [145, Page 21]. We therefore see that the small-maturity exponential Lévy smile blows up at a much quicker rate than the Heston forward smile.

(iv) We have $\lim_{k\to 0}\mathfrak{N}_0(k,t) = \mathfrak{N}_0(0,t) = 0$ and $\lim_{k\to 0}\mathfrak{N}_1(k,t) = \mathfrak{N}_1(0,t) = 0$. Higher-order terms are not necessarily continuous at $k = 0$. For example (when $4\kappa\theta = \xi^2$) we have $\lim_{k\to 0}\mathfrak{N}_2(k,t) = +\infty$.

(v) The at-the-money forward implied volatility ($k = 0$) asymptotic is not covered by Theorem 3.4.1 and a separate analysis is needed for this case (see Section 3.4.2). In particular the proof fails since in this case the key function $u_\tau^*(0)$ (defined through equation (3.6.5)) does not converge to a boundary point, but rather to zero as τ tends to zero (see the proof of Lemma 3.6.3).

(vi) It does not make sense to consider the limit of our asymptotic result for fixed $k \in \mathbb{R}^*$ as t tends to zero since for $t = 0$ using the forward time-scale $h(\tau) \equiv \sqrt{\tau}$ will produce a trivial limiting cgf and hence none of the results will carry over. The time scale in the spot case is $h(\tau) \equiv \tau$; see [62]. Our result is only valid in the forward (not spot) smile case.

(vii) As seen in the proof, due to the term $\tau^{7/8-\theta\kappa/(2\xi^2)}$ in the forward-start option asymptotics in Theorem 3.3.1, one can only specify the small-maturity forward smile to arbitrary order if $4\kappa\theta = \xi^2$. If this is not the case then such an expansion for the forward smile only holds up to order $\mathcal{O}(1/\tau^{1/4})$. Let $\sigma_t := \sqrt{V_t}$ be given by the dynamics $\mathrm{d}\sigma_t = -\frac{\kappa\sigma_t}{2}\mathrm{d}t + \frac{\xi}{2}\mathrm{d}W_t$, with $\sigma_0 = \sqrt{v}$. This corresponds to a specific case of the Schöbel-Zhu stochastic volatility model (Section 1.3.1.2). In this case V then corresponds to the Heston model with the parameters related to each other by the equality $4\kappa\theta = \xi^2$. So as the Heston volatility dynamics deviate from Gaussian volatility dynamics a certain degeneracy occurs such that one cannot specify high order forward smile asymptotics in the small-maturity case. Interestingly, a similar degeneracy occurs when studying the tail probability of the stock price. As proved in [51], the square-root behaviour of the variance process induces some singularity and hence a fundamentally different behaviour when $4\kappa\theta \neq \xi^2$.

3.4.2 At-the-money forward implied volatility

The analysis above excluded the at-the-money case $k = 0$. We show below that this case has a very different behaviour and can be studied with a much simpler machinery. In this section, we shall denote the future implied volatility $\sigma_t(k,\tau)$ as the implied volatility corresponding to a European call/put option with strike e^k, maturity τ, observed at time t. We first start with the following model-independent lemma, bridging the gap between the at-the-money future implied volatility $\sigma_t(0,\tau)$ and the forward implied volatility $\sigma_{t,\tau}(0)$. Note that a similar result—albeit less general—was derived in [114]. We shall denote by $\mathbb{E}_0$ the expectation (under the given risk-neutral probability measure) with respect to $\mathcal{F}_0$, the filtration at time zero.

Lemma 3.4.3. *Let $t > 0$. Assume that there exists $n \in \mathbb{N}^*$ such that the expansion $\sigma_t(0,\tau) = \sum_{j=0}^n \sigma_j(t)\tau^j + o(\tau^n)$ holds and that $\mathbb{E}_0(\sigma_j(t)) < \infty$ for $j = 0, ..., n$. If the at-the-money forward implied volatility satisfies $\sigma_{t,\tau}(0) = \sum_{j=0}^n \bar{\sigma}_j(t)\tau^j + o(\tau^n)$, then $\bar{\sigma}_j(t) = \mathbb{E}_0(\sigma_j(t))$ for all $j = 0, \ldots, n$.*

Proof. In the Black-Scholes model (1.0.1), we know that for any $t \geq 0$, $\tau > 0$, the price at time t of a (re-normalised) European call option with maturity $t+\tau$ is given by $\mathrm{BS}(k, \Sigma^2, \tau) = \mathbb{E}\left[\left(S_{t+\tau}/S_t - \mathrm{e}^k\right)^+ |\mathcal{F}_t\right]$, and its at-the-money expansion as the maturity τ tends to zero reads (see [64, Corollary 3.5])

$$\mathrm{BS}(k, \Sigma^2, \tau) = \frac{1}{\sqrt{2\pi}}\left(\Sigma\sqrt{\tau} - \frac{\Sigma^3\tau^{3/2}}{24} + \mathcal{O}\left(\Sigma^5\tau^{5/2}\right)\right).$$

We keep the Σ dependence in the $\mathcal{O}(\ldots)$ to highlight the fact that, when Σ depends on τ (such as $\Sigma = \sigma_t(0,\tau)$), one has to be careful not to omit some terms. Now, for a given martingale model for the stock price S, we shall denote by $C_t(k,\tau)$ the price at time t of a European call option with payoff $\left(S_{t+\tau}/S_t - \mathrm{e}^k\right)^+$ at time $t+\tau$. The future implied volatility $\sigma_t(k,\tau)$ is then the unique solution to $\mathrm{BS}(k, \sigma_t^2(k,\tau), \tau) = C_t(k,\tau)$. For at-the-money $k = 0$, we obtain the following expansion for short maturity τ:

$$C_t(0,\tau) = \frac{1}{\sqrt{2\pi}}\left(\sigma_0(t)\sqrt{\tau} + \left(\sigma_1(t) - \frac{\sigma_0^3(t)}{24}\right)\tau^{3/2} + \mathcal{O}\left(\tau^{5/2}\right)\right), \tag{3.4.1}$$

where we have used here the expansion assumed for $\sigma_t(0,\tau)$. Note also that the coefficients $\sigma_j(t)$ are random variables. We follow the probabilistic version of the $\mathcal{O}$ notation detailed in [102, Section 5], namely the random remainder R_τ is $\mathcal{O}_P(\tau^{5/2})$ as τ tends to zero if and only if for any $\varepsilon > 0$ there exist a constant $c_\varepsilon > 0$ and a threshold $\tau_\varepsilon > 0$ for which $\mathbb{P}\left(|R_\tau| \leq c_\varepsilon\tau^{5/2}\right) > 1 - \varepsilon$ for all $\tau < \tau_\varepsilon$. For brevity we abuse the notations slightly here and write $\mathcal{O}$ instead of $\mathcal{O}_P$. Now, the forward-start European call option (at inception) in the Black-Scholes model reads

$$\mathbb{E}_0\left[\left(\frac{S_{t+\tau}}{S_t} - \mathrm{e}^k\right)^+\right] = \mathcal{N}(d_+(\Sigma,\tau)) - \mathrm{e}^k\mathcal{N}(d_-(\Sigma,\tau)) = \mathrm{BS}(k, \Sigma^2, \tau),$$

where $d_\pm(\Sigma,\tau) := (-k \pm \Sigma^2\tau/2)/(\Sigma\sqrt{\tau})$. For a given model, we recall that $C(k,t,\tau)$ is the price of a Type-I forward-start European call option. By definition of the forward implied volatility $\sigma_{t,\tau}(k)$, we have $C(k,t,\tau) = \mathrm{BS}(k, \sigma_{t,\tau}^2(k), \tau)$. For at-the-money $k = 0$, it follows that

$$C(0,t,\tau) = \mathrm{BS}(0, \sigma_{t,\tau}^2(0), \tau) = \mathcal{N}(d_+(\sigma_{t,\tau}(0),\tau)) - \mathcal{N}(d_-(\sigma_{t,\tau}(0),\tau)).$$

Using the assumed expansion for $\sigma_{t,\tau}(0)$ we similarly obtain (as in (3.4.1))

$$C(0,t,\tau) = \frac{1}{\sqrt{2\pi}}\left(\bar{\sigma}_0(t)\sqrt{\tau} + \left(\bar{\sigma}_1(t) - \frac{\bar{\sigma}_0^3(t)}{24}\right)\tau^{3/2} + \mathcal{O}\left(\tau^{5/2}\right)\right). \tag{3.4.2}$$

Note that now the coefficients $\bar{\sigma}_j(t)$ are not random variables, but simple constants. Recall now that

$$C(k,t,\tau) := \mathbb{E}_0\left[\left(\frac{S_{t+\tau}}{S_t} - \mathrm{e}^k\right)^+\right] = \mathbb{E}_0\left\{\mathbb{E}\left[\left(\frac{S_{t+\tau}}{S_t} - \mathrm{e}^k\right)^+ |\mathcal{F}_t\right]\right\} = \mathbb{E}_0\left(C_t(k,\tau)\right). \quad (3.4.3)$$

Combining this with (3.4.1) and (3.4.2), we find that $\bar{\sigma}_j(t) = \mathbb{E}_0(\sigma_j(t))$ for $j = 0, 1$. The higher-order terms for the expansion can be proved analogously and the lemma follows. □

We now apply this to the Heston model. Recall the definition of the Kummer (confluent hypergeometric) function $\mathrm{M} : \mathbb{C}^3 \to \mathbb{R}$:

$$\mathrm{M}(\alpha, \mu, z) := \sum_{n\geq 0} \frac{(\alpha)_n}{(\mu)_n}\frac{z^n}{n!}, \quad \mu \neq 0, -1, ...,$$

where the Pochhammer symbol is defined by $(\alpha)_n := \alpha(\alpha+1)\cdots(\alpha+n-1)$ for $n \geq 1$ and $(\alpha)_0 = 1$. For any $p > -2\kappa\theta/\xi^2$ and $t > 0$ we define

$$\Delta(t,p) := 2^p \beta_t^p \exp\left(-\frac{v\mathrm{e}^{-\kappa t}}{2\beta_t}\right)\frac{\Gamma\left(2\kappa\theta/\xi^2 + p\right)}{\Gamma\left(2\kappa\theta/\xi^2\right)}\mathrm{M}\left(\frac{2\kappa\theta}{\xi^2} + p, \frac{2\kappa\theta}{\xi^2}, \frac{v\mathrm{e}^{-\kappa t}}{2\beta_t}\right), \quad (3.4.4)$$

with β_t defined in (1.3.4). This function is related to the moments of the Feller diffusion (see [56, Theorem 2.4]): for any $t > 0$, $\mathbb{E}[V_t^p] = \Delta(t,p)$ if $p > -2\kappa\theta/\xi^2$ and is infinite otherwise. Note in particular that $\lim_{t\searrow 0}\Delta(t,p) = v^p$ (see [1, 13.1.4 page 504]). The Heston forward at-the-money volatility asymptotic is given in the following theorem.

Theorem 3.4.4. *The following expansion holds for the forward at-the-money volatility as τ tends to zero:*

$$\sigma_{t,\tau}(0) = \begin{cases} \Delta\left(t, \frac{1}{2}\right) + o(1), & \text{if } 4\kappa\theta \leq \xi^2, \\ \Delta\left(t, \frac{1}{2}\right) + \frac{\Delta\left(t, -\frac{1}{2}\right)}{4}\left(\kappa\theta + \frac{\xi^2(\rho^2-4)}{24}\right)\tau + \frac{\Delta\left(t, \frac{1}{2}\right)}{8}(\rho\xi - 2\kappa)\tau + o(\tau), & \text{if } 4\kappa\theta > \xi^2. \end{cases}$$

Remark 3.4.5.

(i) As opposed to the out-of-the-money case, the small-maturity limit here is well defined.

(ii) Combining Lemma 3.4.3 and [22], $\lim_{\tau\searrow 0}\sigma_{t,\tau}(0) = \mathbb{E}(\sqrt{V_t})$ holds for any well-behaved stochastic volatility model (S, V).

(iii) The proof does not allow one to conclude any information about higher order terms in Heston for the case $4\kappa\theta \leq \xi^2$. A different method would need to be used to compute higher order asymptotics in this case.

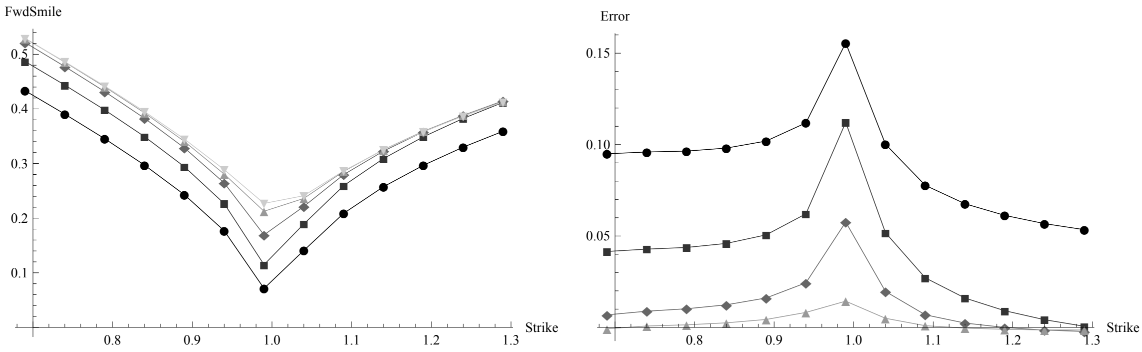
Proof of Theorem 3.4.4. In Heston we recall from Corollary 2.3.2 the asymptotic $\sigma_t^2(0,\tau) = V_t + \left(\frac{\kappa\theta}{2} + \frac{\xi^2}{48}\left(\rho^2 - 4\right) + \frac{V_t}{4}(\rho\xi - 2\kappa)\right)\tau + o(\tau)$, and so for small τ we have $\sigma_t(0,\tau) = \sigma_0(t) + \sigma_1(t)\tau + o(\tau)$, with $\sigma_0(t) := \sqrt{V_t}$ and $\sigma_1(t) := \frac{1}{4\sqrt{V_t}}\left(\kappa\theta + \frac{\xi^2}{24}(\rho^2 - 4)\right) + \frac{\sqrt{V_t}}{8}(\rho\xi - 2\kappa)$. Lemma 3.4.3 and (3.4.4) conclude the proof. □

3.5 Numerics

We first compare the true Heston forward smile and the asymptotics developed in the paper. We calculate forward-start option prices using the inverse Fourier transform representation in Lemma 1.4.7 and a global adaptive Gauss-Kronrod quadrature scheme. We then compute the forward smile $\sigma_{t,\tau}$ with a simple root-finding algorithm. The Heston model parameters are given by $\rho = -0.8$, $\xi = 0.52$, $\kappa = 1$ and $v = \theta = 0.07$ unless otherwise stated in the figures. In Figures 3.1 and 3.2 we compare the true forward smile using Fourier inversion and the asymptotic in Theorem 3.4.1. It is clear that the small-maturity asymptotic has very different features relative to "smoother" asymptotics derived in Chapter 2. This is due to the introduction of the forward time-scale and to the fact that the limiting cgf is not steep. Note also from Remark 3.4.2(iv) that the asymptotics in Theorem 3.4.1 can approach zero or infinity as the strike approaches at-the-money. This appears to be a fundamental feature of non-steep asymptotics; numerically this implies that the asymptotic may break down for strikes in a region around the at-the-money point. In Figure 3.3 we compare the true at-the-money forward volatility using Fourier inversion and the asymptotic in Lemma 3.4.3. Results are in line with expectations and the at-the-money asymptotic is more accurate than the out-of-the-money asymptotic. This is because the at-the-money forward volatility (unlike the out-of-the-money case) has a well defined limit as τ tends to zero. In Figure 3.4 we use these results to gain intuition on how the Heston forward smile explodes for small maturities. In Section 2.3.1.1 we derived a diagonal small-maturity asymptotic expansion for the Heston forward smile valid for small forward start-dates and small maturities. In order for the small-maturity asymptotic in this chapter to be useful, there needs to be a sufficient amount of variance of variance at the forward-start date. Practically this means that the asymptotic performs better as one increases the forward-start date. On the other hand the diagonal-small maturity asymptotic expansion is valid for small forward-start dates. In this sense these asymptotics complement each other. Figure 3.5 shows the consistency of these two results for small forward-start date and maturity.

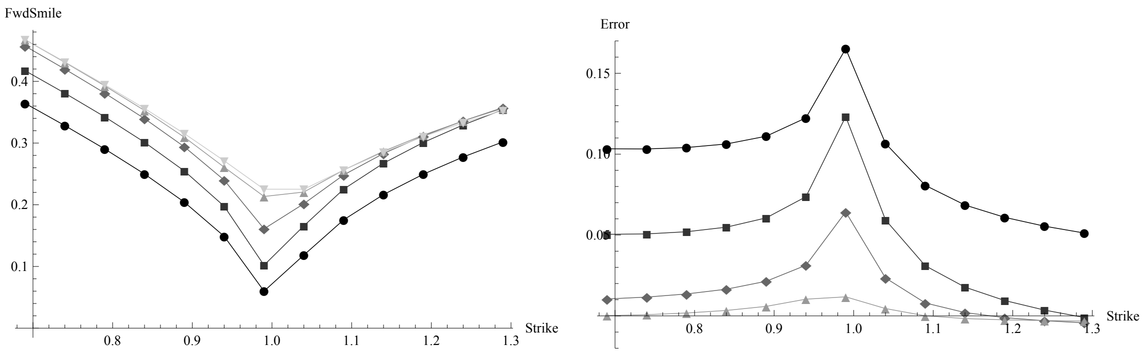
3.6 Proof of Theorems 3.3.1 and 3.4.1

We split the proof of Theorems 3.3.1 and 3.4.1 into several parts, from Section 3.6.1 to Section 3.6.4 below. In Section 3.6.1 we develop the necessary tools to characterise the small-maturity Heston forward cgf domain and derive the Heston forward time-scale (Lemma 3.2.3). In Section 3.6.2 we use the forward time-scale to define a time-dependent asymptotic measure-change and derive expansions for fundamental auxiliary functions needed in the analysis. In Section 3.6.3 we derive the asymptotics of the characteristic function of a re-scaled version of the forward price process $(X_\tau^{(t)})$ under the asymptotic measure-change defined in Section 3.6.2. This section also uses Fourier trans-



(a) Asymptotic vs Fourier inversion. (b) Errors.

Figure 3.1: Here $t = 1$ and $\tau = 1/24$. In (a) circles, squares, diamonds and triangles represent the zeroth, first, second and third-order asymptotics respectively and backwards triangles represent the true forward smile using Fourier inversion. In (b) we plot the errors.



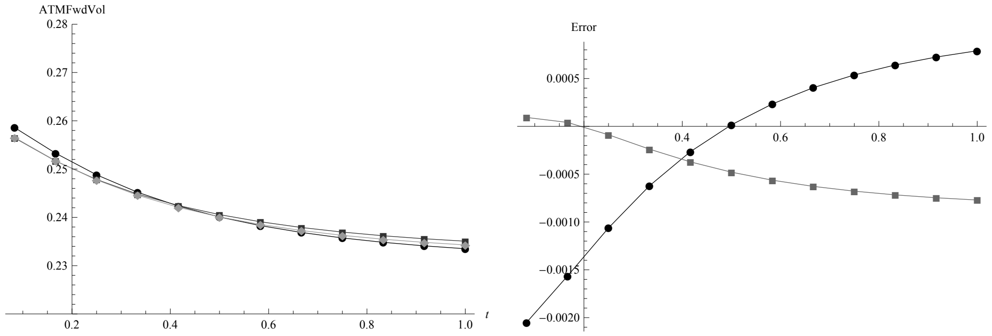
(a) Asymptotic vs Fourier inversion. (b) Errors.

Figure 3.2: Here $t = 1$ and $\tau = 1/12$. In (a) circles, squares, diamonds and triangles represent the zeroth, first, second and third-order asymptotics respectively and backwards triangles represent the true forward smile using Fourier inversion. In (b) we plot the errors.

form methods to derive asymptotics of important expectations using this characteristic function expansion. Section 3.6.4 finally puts all the pieces together and proves Theorems 3.3.1 and 3.4.1.

3.6.1 Heston forward time-scale

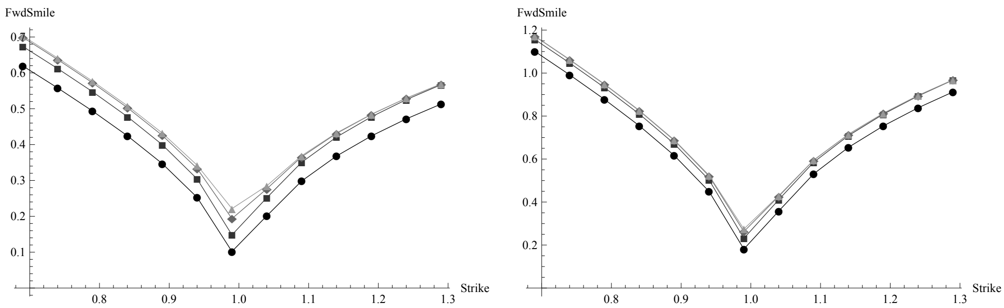
We recall that the Heston forward cgf was derived in Lemma 1.3.1. The first step in our analysis is to characterise the forward time-scale in the Heston model. In order to achieve this we first need to understand the limiting behaviour of a re-scaled version of the B function in (1.3.8) that plays a fundamental role in the analysis below. The following lemma shows that using $h(\tau) \equiv \sqrt{\tau}$ as a time-scale produces the only non-trivial limit for the re-scaled B function. We then immediately



(a) At-the-money asymptotic vs Fourier inversion.

(b) Errors.

Figure 3.3: Plot of the forward at-the-money volatility ($\tau = 1/12$) as a function of the forward-start date t. The Heston parameters are $\rho = -0.6$, $\kappa = 1$, $\xi = 0.4$ and $v = \theta = 0.07$. In (a) circles, squares and diamonds are the zeroth-order, the first-order and the true forward at-the-money volatility.



(a) $t = 1$, $\tau = 1/100$.

(b) $t = 1$, $\tau = 1/1000$.

Figure 3.4: Circles, squares, diamonds and triangles represent the zeroth, first, second and third-order asymptotics respectively.

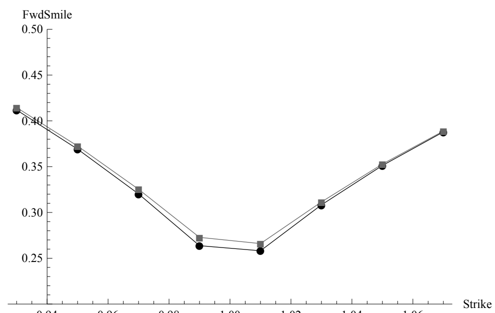


Figure 3.5: Here we compare the small-maturity third-order asymptotic (circles) to the diagonal small-maturity second-order asymptotic of Chapter 2 (squares) for $t = 1/12$ and $\tau = 1/1000$.

prove Lemma 3.2.3 which characterises the forward time-scale in the Heston model.

Lemma 3.6.1. *Let* $h : \mathbb{R}_+ \to \mathbb{R}_+$ *be a continuous function such that* $\lim_{\tau \searrow 0} h(\tau) = 0$ *and* $a \in \mathbb{R}_+^*$. *The following limit then holds for* B *in* (1.3.8) *for all* $u \in \mathbb{R}^*$*:*

$$\lim_{\tau\to 0} B(u/h(\tau),\tau) = \begin{cases} \textit{undefined}, & \textit{if} \quad \tau/h(\tau) \nearrow \infty, \\ +\infty, & \textit{if} \quad h(\tau) \equiv a\tau, \\ +\infty, & \textit{if} \quad \sqrt{\tau}/h(\tau) \nearrow \infty \quad \textit{and} \quad \tau/h(\tau) \searrow 0, \\ 0, & \textit{if} \quad \sqrt{\tau}/h(\tau) \searrow 0, \\ u^2/(2a^2), & \textit{if} \quad h(\tau) \equiv a\tau^{1/2}. \end{cases}$$

Proof. As τ tends to zero we have the following asymptotic behaviours for the functions d and γ defined in (1.3.6) and (1.3.8):

$$\begin{aligned} d\,(u/h(\tau)) &= \frac{1}{h(\tau)}\left(\kappa^2 h(\tau)^2 + uh(\tau)\,(\xi - 2\kappa\rho) - \bar{\rho}^2\xi^2u^2\right)^{1/2} \\ &= \frac{\mathtt{i}u}{h(\tau)}d_0 + d_1 + \mathcal{O}(h(\tau)), \\ \gamma\,(u/h(\tau)) &= \frac{\kappa h(\tau) - \rho\xi u - \mathtt{i}ud_0 - d_1 h(\tau) + \mathcal{O}\left(h(\tau)^2\right)}{\kappa h(\tau) - \rho\xi u + \mathtt{i}ud_0 + d_1 h(\tau) + \mathcal{O}\left(h(\tau)^2\right)} \\ &= g_0 - \frac{\mathtt{i}h(\tau)}{u}g_1 + \mathcal{O}\left(h(\tau)^2\right), \end{aligned} \tag{3.6.1}$$

where we have set ($\bar{\rho} := \sqrt{1-\rho^2}$)

$$d_0 := \bar{\rho}\xi\,\mathrm{sgn}(u), \quad d_1 := \frac{\mathtt{i}\,(2\kappa\rho - \xi)\,\mathrm{sgn}(u)}{2\bar{\rho}}, \quad g_0 := \frac{\mathtt{i}\rho - \bar{\rho}\,\mathrm{sgn}(u)}{\mathtt{i}\rho + \bar{\rho}\,\mathrm{sgn}(u)} \quad g_1 := \frac{(2\kappa - \rho\xi)\,\mathrm{sgn}(u)}{\xi\bar{\rho}\,(\bar{\rho} + \mathtt{i}\rho\,\mathrm{sgn}(u))^2}.$$

First let $\tau/h(\tau) \to \infty$. Then $\exp\left(-d\,(u/h(\tau))\,\tau\right) = \exp\left(-\mathtt{i}\tau\bar{\rho}\xi|u|/h(\tau) + \mathcal{O}\,(\tau)\right)$, and so the limit is undefined (complex infinity). Next let $\tau/h(\tau) \equiv 1/a$. Using (3.6.1) we see that

$$B(u/h(\tau),\tau) = -\left(\frac{u\rho + \mathtt{i}\bar{\rho}|u|}{\xi h\,(\tau)}\right)\frac{1 - \mathrm{e}^{-\mathtt{i}\xi\bar{\rho}|u|/a}}{1 - g_0\mathrm{e}^{-\mathtt{i}\xi\bar{\rho}|u|/a}} + \mathcal{O}\,(1) = a\zeta(u/a)/h(\tau) + \mathcal{O}\,(1)\,,$$

where $\zeta(u) := u\left(\bar{\rho}\xi\cot\left(u\xi\bar{\rho}/2\right) - \rho\xi\right)^{-1}$, which is strictly positive for $u \in \mathbb{R}^*$ and $\zeta(0) = 0$. It follows that the limit in this case is infinite. Next let $\tau/h(\tau) \to 0$. Here we can write

$$\begin{aligned} B(u/h(\tau),\tau) &= \left(-\frac{\rho u + \mathtt{i}\bar{\rho}|u|}{\xi h(\tau)} + \mathcal{O}\,(1)\right)\left(\left(\frac{1}{g_0 - 1} + \mathcal{O}\,(h(\tau))\right)\left(\frac{-\mathtt{i}\tau\bar{\rho}\xi|u|}{h(\tau)} + \mathcal{O}\,(\tau)\right)\right. \\ &\qquad \left. + \mathcal{O}\left(\left(\frac{\tau}{h(\tau)}\right)^2\right)\right) \\ &= \left(\frac{\rho u + \mathtt{i}\bar{\rho}|u|}{\xi}\right)\left(\frac{1}{g_0 - 1}\right)\frac{\mathtt{i}\tau\bar{\rho}\xi|u|}{h(\tau)^2} + \mathcal{O}\left(\tau/h(\tau)\right) \\ &= \frac{u^2}{2}\left(\frac{\sqrt{\tau}}{h\,(\tau)}\right)^2 + \mathcal{O}\left(\tau/h(\tau)\right). \end{aligned} \tag{3.6.2}$$

If $\sqrt{\tau}/h(\tau)$ tends to infinity, so does $B(u/h(\tau),\tau)$. When $\sqrt{\tau}/h(\tau)$ tends to zero then $B(u/h(\tau),\tau)$ does as well. If $\sqrt{\tau}/h(\tau)$ converges to a constant $1/a$, then $B(u/h(\tau),\tau)$ converges to $u^2/(2a^2)$, and the lemma follows. □

Proof of Lemma 3.2.3. For any $t > 0$, the random variable V_t in (1.3.2) is distributed as β_t (defined in (1.3.4)) times a non-central chi-square random variable with $4\kappa\theta/\xi^2 > 0$ degrees of freedom and non-centrality parameter $\lambda = v\mathrm{e}^{-\kappa t}/\beta_t > 0$. It follows that the corresponding moment generating function is given by

$$\Lambda_t^V(u) := \mathbb{E}\left(\mathrm{e}^{uV_t}\right) = \exp\left(\frac{\lambda\beta_t u}{1-2\beta_t u}\right)(1-2\beta_t u)^{-2\kappa\theta/\xi^2}, \qquad \text{for all } u < \frac{1}{2\beta_t}.$$

The re-normalised Heston forward cumulant generating function is then computed as (A and B defined in (1.3.8))

$$\begin{aligned}\Lambda_\tau^{(t)}(u, h(\tau))/h(\tau) &= \log\mathbb{E}\left[\mathrm{e}^{u(X_{t+\tau}-X_t)/h(\tau)}\right] = \log\mathbb{E}\left[\mathbb{E}\left(\mathrm{e}^{u(X_{t+\tau}-X_t)/h(\tau)}|\mathcal{F}_t\right)\right] \\ &= \log\mathbb{E}\left(\mathrm{e}^{A(u/h(\tau),\tau)+B(u/h(\tau),\tau)V_t}\right) = A\left(u/h(\tau),\tau\right) + \log\Lambda_t^V\left(B\left(u/h(\tau),\tau\right)\right),\end{aligned}$$

which agrees with (1.3.7) when $h(\tau) \equiv 1$. This is only valid in some effective domain $\mathcal{D}_{t,\tau} \subset \mathbb{R}$. The mgf for V_t is well defined in $\mathcal{D}_{t,\tau}^V := \{u \in \mathbb{R} : B\left(u/h(\tau),\tau\right) < 1/(2\beta_t)\}$, and hence $\mathcal{D}_{t,\tau} = \mathcal{D}_{t,\tau}^V \cap \mathcal{D}_\tau$, where $\mathcal{D}_\tau$ is the effective domain of the (spot) re-normalised Heston cgf. Consider first $\mathcal{D}_\tau$ for small τ. From [5, Proposition 3.1] if $\xi^2(u/h(\tau)-1)u/h(\tau) > (\kappa - \rho\xi u/h(\tau))^2$ then the explosion time $\tau^*(u) := \sup\{t \geq 0 : \mathbb{E}(\mathrm{e}^{uX_t}) < \infty\}$ of the Heston mgf is

$$\begin{aligned}\tau_H^*\left(u/h(\tau)\right) = &\frac{2}{\sqrt{\xi^2(u/h(\tau)-1)u/h(\tau) - (\kappa-\rho\xi u/h(\tau))^2}}\left\{\pi\mathbb{1}_{\{\rho\xi u/h(\tau)-\kappa<0\}}\right. \\ &\left.+ \arctan\left(\frac{\sqrt{\xi^2(u/h(\tau)-1)u/h(\tau) - (\kappa-\rho\xi u/h(\tau))^2}}{\rho\xi u/h(\tau)-\kappa}\right)\right\}.\end{aligned}$$

Recall the following Taylor series expansions, for x close to zero:

$$\begin{aligned}\arctan\left(\frac{1}{\rho\xi u/x-\kappa}\sqrt{\xi^2\left(\frac{u}{x}-1\right)\frac{u}{x} - \left(\kappa - \xi\rho\frac{u}{x}\right)^2}\right) &= \mathrm{sgn}(u)\arctan\left(\frac{\bar{\rho}}{\rho}\right) + \mathcal{O}\left(x\right), \quad \text{if } \rho \neq 0, \\ \arctan\left(-\frac{1}{\kappa}\sqrt{\xi^2\left(\frac{u}{x}-1\right)\frac{u}{x} - \kappa^2}\right) &= -\frac{\pi}{2} + \mathcal{O}(x), \qquad \text{if } \rho = 0.\end{aligned}$$

As τ tends to zero $\xi^2(u/h(\tau)-1)u/h(\tau) > (\kappa-\rho\xi u/h(\tau))^2$ is satisfied since $\rho^2 < 1$ and hence

$$\tau_H^*\left(\frac{u}{h(\tau)}\right) = \begin{cases} \frac{h(\tau)}{\xi|u|}\left(\pi\mathbb{1}_{\{\rho=0\}} + \frac{2}{\bar{\rho}}\left(\pi\mathbb{1}_{\{\rho u\leq 0\}} + \mathrm{sgn}(u)\arctan\left(\frac{\bar{\rho}}{\rho}\right)\right)\mathbb{1}_{\{\rho\neq 0\}} + \mathcal{O}\left(h(\tau)\right)\right), & \text{if } u \neq 0, \\ \infty, & \text{if } u = 0.\end{cases}$$

Therefore, for τ small enough, we have $\tau_H^*\left(u/h(\tau)\right) > \tau$ for all $u \in \mathbb{R}$ if $\tau/h\left(\tau\right)$ tends to zero and $\tau_H^*\left(u/h(\tau)\right) > \tau$ for all $u \in (u_-, u_+)$ if $h\left(\tau\right) \equiv a\tau$, where

$$\begin{aligned}u_- &:= \frac{2a}{\bar{\rho}\xi\tau}\arctan\left(\frac{\bar{\rho}}{\rho}\right)\mathbb{1}_{\{\rho<0\}} - \frac{\pi a}{\xi\tau}\mathbb{1}_{\{\rho=0\}} + \frac{2a}{\bar{\rho}\xi\tau}\left(\arctan\left(\frac{\bar{\rho}}{\rho}\right) - \pi\right)\mathbb{1}_{\{\rho>0\}}, \\ u_+ &:= \frac{2a}{\bar{\rho}\xi\tau}\left(\arctan\left(\frac{\bar{\rho}}{\rho}\right) + \pi\right)\mathbb{1}_{\{\rho<0\}} + \frac{\pi a}{\xi\tau}\mathbb{1}_{\{\rho=0\}} + \frac{2a}{\bar{\rho}\xi\tau}\arctan\left(\frac{\bar{\rho}}{\rho}\right)\mathbb{1}_{\{\rho>0\}}.\end{aligned}$$

If $\tau/h(\tau)$ tends to infinity, then $\tau_H^*\left(u/h(\tau)\right) \leq \tau$ for all $u \in \mathbb{R}^*$. We are also required to find $\mathcal{D}_{t,\tau}^V$ for small τ. Using Lemma 3.6.1 we see that if $h(\tau) \equiv a\tau^{1/2}$ then $\lim_{\tau\searrow 0}\mathcal{D}_{t,\tau}^V = \{u \in \mathbb{R} : |u| < a/\sqrt{\beta_t}\}$.

By the limit of a set we precisely mean the following:

$$\liminf_{\tau \searrow 0} \mathcal{D}^V_{t,\tau} := \bigcup_{\tau>0} \bigcap_{s\leq\tau} \mathcal{D}^V_{t,s} = \bigcap_{\tau>0} \bigcup_{s\leq\tau} \mathcal{D}^V_{t,s} =: \limsup_{\tau \searrow 0} \mathcal{D}^V_{t,\tau}.$$

If $\tau^{1/2}/h(\tau)$ tends to infinity then $\lim_{\tau\searrow 0} \mathcal{D}^V_{t,\tau} = \{0\}$ and if it tends to zero, then $\lim_{\tau\searrow 0} \mathcal{D}^V_{t,\tau} = \mathbb{R}$. The limiting domains in the lemma follow after taking the appropriate intersections. Next we move on to the limits. We only consider the cases where $h(\tau) \equiv a\tau^{1/2}$ and where $\tau^{1/2}/h(\tau)$ tends to zero since these are the only cases for which the forward cumulant generating function is defined. Using (3.6.2) we see as τ tends to zero

$$\log\left(1 - 2\beta_t B\left(u/h(\tau), \tau\right)\right) = \frac{B(u/h(\tau), \tau) v e^{-\kappa t}}{1 - 2\beta_t B(u/h(\tau), \tau)} = \begin{cases} \mathcal{O}(1), & \text{if} \quad h(\tau) \equiv a\tau^{1/2}, \\ \mathcal{O}(\tau/h(\tau)), & \text{if} \quad \sqrt{\tau}/h(\tau) \searrow 0. \end{cases}$$

The lemma follows from this and the fact that the function A in (1.3.8) satisfies $A(u/h(\tau), \tau) = \mathcal{O}\left((\tau/h(\tau))^2\right)$. □

3.6.2 Asymptotic time-dependent measure-change

In this section we define the fundamental asymptotic time-dependent measure-change in (3.6.6) and derive expansions for critical functions related to this measure-change. In order to proceed with this program we first need to prove some technical lemmas. We use our forward time-scale and define the following rescaled quantities:

$$\Lambda^{(t)}_\tau(u) := \Lambda^{(t)}_\tau(u, \sqrt{\tau}), \quad \widehat{A}(u) := A(u/\sqrt{\tau}, \tau), \quad \widehat{B}(u) := B(u/\sqrt{\tau}, \tau), \tag{3.6.3}$$

with $\Lambda^{(t)}_\tau$, A and B defined in (3.2.1) and (1.3.8) respectively. The following lemma gives the asymptotics of the re-scaled quantities $\widehat{A}$, $\widehat{B}$ as τ tends to zero:

Lemma 3.6.2. *The following expansions hold for all $u \in \mathcal{D}_\Lambda$ as τ tends to zero ($\widehat{B}_1$ was defined in (3.3.2)):*

$$\widehat{B}(u) = \frac{u^2}{2} + \widehat{B}_1(u)\sqrt{\tau} + \mathcal{O}(\tau), \qquad \widehat{A}(u) = \frac{u^2\kappa\theta\tau}{4} + \mathcal{O}(\tau^{3/2}). \tag{3.6.4}$$

Proof. From the definition of A in (1.3.8) and the asymptotics in (3.6.1) with $h(\tau) \equiv \sqrt{\tau}$ we obtain

$$\begin{aligned} \widehat{A}(u) := A\left(u/\sqrt{\tau}, \tau\right) &= \frac{\kappa\theta}{\xi^2}\left(\left(\kappa - \frac{\rho\xi u}{\sqrt{\tau}} - d\left(\frac{u}{\sqrt{\tau}}\right)\right)\tau \right. \\ &\qquad \left. -2\log\left(\frac{1-\gamma(u/\sqrt{\tau})\exp\left(-d(u/\sqrt{\tau})\tau\right)}{1-\gamma(u/\sqrt{\tau})}\right)\right) \\ &= \frac{\kappa\theta}{\xi^2}\left(\left(\kappa - \frac{\rho\xi u}{\sqrt{\tau}} - \frac{\mathtt{i}ud_0}{\sqrt{\tau}} - d_1 + \mathcal{O}(\sqrt{\tau})\right)\tau \right. \\ &\qquad \left. -2\log\left(\frac{1-(g_0 - \mathtt{i}\sqrt{\tau}g_1/u + \mathcal{O}(\tau))\exp\left(-\mathtt{i}ud_0\sqrt{\tau} - d_1\tau + \mathcal{O}(\tau^{3/2})\right)}{1-(g_0 - \mathtt{i}\sqrt{\tau}g_1/u + \mathcal{O}(\tau))}\right)\right) \\ &= u^2\theta\kappa\tau/4 + \mathcal{O}(\tau^{3/2}). \end{aligned}$$

Substituting the asymptotics for d and γ in (3.6.1) we further obtain

$$\frac{1-\exp\left(-d(u/\sqrt{\tau})\tau\right)}{1-\gamma(u/\sqrt{\tau})\exp\left(-d(u/\sqrt{\tau})\tau\right)} = \frac{1-\exp\left(-\mathtt{i}ud_0\sqrt{\tau}-d_1\tau+\mathcal{O}(\tau^{3/2})\right)}{1-(g_0-\mathtt{i}\sqrt{\tau}g_1/u+\mathcal{O}(\tau))\exp\left(-\mathtt{i}ud_0\sqrt{\tau}-d_1\tau+\mathcal{O}(\tau^{3/2})\right)},$$

and therefore using the definition of B in (1.3.8) we obtain

$$\begin{aligned}\widehat{B}(u) := B\left(\frac{u}{\sqrt{\tau}},\tau\right) &= \frac{\kappa-\rho\xi u/\sqrt{\tau}-d(u/\sqrt{\tau})}{\xi^2}\frac{1-\exp\left(-d\left(u/\sqrt{\tau}\right)\tau\right)}{1-\gamma\left(u/\sqrt{\tau}\right)\exp\left(-d\left(u/\sqrt{\tau}\right)\tau\right)}\\ &= -\frac{\rho\xi u+\mathtt{i}ud_0}{\xi^2}\frac{\mathtt{i}ud_0}{1-g_0}+\widehat{B}_1(u)\sqrt{\tau}+\mathcal{O}(\tau) = \frac{u^2}{2}+\widehat{B}_1(u)\sqrt{\tau}+\mathcal{O}(\tau).\end{aligned}$$

□

It is still not clear what benefit the forward time-scale has given us since the limiting cgf is still degenerate. Firstly, even though the limiting cgf is zero on a bounded interval, the re-scaled forward cgf for fixed $\tau>0$ is still steep on the domain of definition which implies the existence of a unique solution $u_\tau^*(k)$ to the equation

$$\partial_u\Lambda_\tau^{(t)}(u_\tau^*(k)) = k. \tag{3.6.5}$$

Further as τ tends to zero, $u_\tau^*(k)$ converges to $1/\sqrt{\beta_t}$ when $k>0$ and to $-1/\sqrt{\beta_t}$ when $k<0$ (see Lemma 3.6.3 below). The key observation is that the forward time-scale ensures finite boundary points for the effective domain, which in turn implies finite limits for $u_\tau^*(k)$. This is critical to the asymptotic analysis that follows and it will become clear that if any other time-scale were to be used the analysis would break down. The following lemma shows that our definition (3.6.5) of $u_\tau^*(k)$ is exactly what we need to conduct an asymptotic analysis in this degenerate case.

Lemma 3.6.3. *For any $k\in\mathbb{R}$, $\tau>0$, the equation* (3.6.5) *admits a unique solution $u_\tau^*(k)$; as τ tends to zero, it converges to $1/\sqrt{\beta_t}$ ($-1/\sqrt{\beta_t}$) when $k>0$ ($k<0$), to zero when $k=0$, and $u_\tau^*(k)\in\mathcal{D}_\Lambda^o$ for τ small enough.*

Proof. We first start by the following claims, which can be proved using the convexity of the forward moment generating function and tedious computations; we shall not however detail these lengthy computations here for brevity, but Figure 3.6 below provides a visual help (see also Appendix B).

(i) For any $\tau>0$, the map $\partial_u\Lambda_\tau^{(t)}:\mathcal{D}_{t,\tau}\to\mathbb{R}$ is strictly increasing and the image of $\mathcal{D}_{t,\tau}$ by $\partial_u\Lambda_\tau^{(t)}$ is $\mathbb{R}$;

(ii) For any $\tau>0$, $u_\tau^*(0)>0$ and $\lim_{\tau\searrow 0}u_\tau^*(0)=0$, i.e. the unique minimum of $\Lambda_\tau^{(t)}$ converges to zero;

(iii) For each $u\in\mathcal{D}_\Lambda^o$, $\partial_u\Lambda_\tau^{(t)}(u)$ converges to zero as τ tends to zero.

Now, choose $k>0$ (analogous arguments hold for $k<0$). It is clear from (i) that (3.6.5) admits a unique solution. Since $\lim_{\tau\searrow 0}\mathcal{D}_{t,\tau}=\mathcal{D}_\Lambda$, then there exists $\tau_1>0$ such that for any $\tau<\tau_1$,

$u_\tau^*(k) \in \mathcal{D}_\Lambda^o$. Note further that (i) and (ii) imply $u_\tau^*(k) > 0$. From (iii) there exists $\tau_2 > 0$ such that the sequence $(u_\tau^*(k))_{\tau>0}$ is strictly increasing as τ goes to zero for $\tau < \tau_2$. Now let $\tau^* = \min(\tau_1, \tau_2)$ and consider $\tau < \tau^*$. Then $u_\tau^*(k)$ is bounded above by $1/\sqrt{\beta_t}$ (because $u_\tau^*(k) \in \mathcal{D}_\Lambda^o$) and therefore converges to a limit $L \in [0, 1/\sqrt{\beta_t}]$. Suppose that $L \neq 1/\sqrt{\beta_t}$. Since $s \mapsto u_s^*(k)$ is strictly increasing as s tends to zero (and $s < \tau^*$), and $\partial_u \Lambda_\tau^{(t)}$ is strictly increasing we have $\partial_u \Lambda_\tau^{(t)}(u_\tau^*(k)) \leq \partial_u \Lambda_\tau^{(t)}(L)$; Combining this and (iii) yields

$$\lim_{\tau \searrow 0} \partial_u \Lambda_\tau^{(t)}(u_\tau^*(k)) \leq \lim_{\tau \searrow 0} \partial_u \Lambda_\tau^{(t)}(L) = 0 \neq k,$$

which contradicts the assumption $k > 0$. Therefore $L = 1/\sqrt{\beta_t}$ and the lemma follows. □

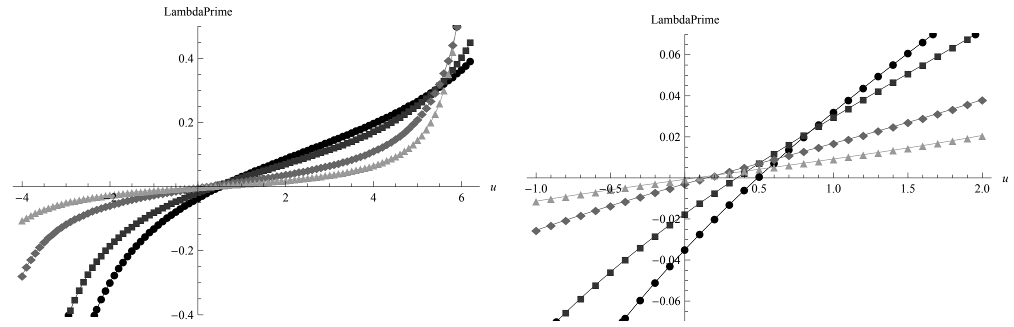


Figure 3.6: Plot of $u \mapsto \partial_u \Lambda_\tau^{(t)}(u)$ for different values of τ. Circles, squares, diamonds and triangles represent $\tau = 1, 1/2, 1/12, 1/50$. The forward-start date is $t = 1$ and the Heston model parameters are $v = \theta = 0.07$, $\xi = 0.4$, $\rho = -0.6$, $\kappa = 1$. The limiting domain is $(-1/\sqrt{\beta_t}, 1/\sqrt{\beta_t}) \approx (-6.29, 6.29)$. The right plot is a zoomed version of the left graph.

For sufficiently small τ we introduce a time-dependent change of measure by

$$\frac{\mathrm{d}\mathbb{Q}_{k,\tau}}{\mathrm{d}\mathbb{P}} := \exp\left(u_\tau^*(k) X_\tau^{(t)}/\sqrt{\tau} - \Lambda_\tau^{(t)}(u_\tau^*(k))/\sqrt{\tau}\right). \tag{3.6.6}$$

By Lemma 3.6.3, $u_\tau^*(k) \in \mathcal{D}_\Lambda^0$ for τ small enough and so $|\Lambda_\tau^{(t)}(u_\tau^*)|$ is finite since $\mathcal{D}_\Lambda = \lim_{\tau \searrow 0}\{u \in \mathbb{R} : |\Lambda_\tau^{(t)}(u)| < \infty\}$. Also $\mathrm{d}\mathbb{Q}_{k,\tau}/\mathrm{d}\mathbb{P}$ is almost surely strictly positive and by definition $\mathbb{E}[\mathrm{d}\mathbb{Q}_{k,\tau}/\mathrm{d}\mathbb{P}] = 1$. Therefore (3.6.6) is a valid measure change for all $k \in \mathbb{R}^*$ and sufficiently small τ. Equation (3.6.5) can be written explicitly as

$$\frac{\sqrt{\tau}\mathrm{e}^{-\kappa t}}{k\xi^2}\left[\xi^2 \mathrm{e}^{\kappa t} \widehat{A}'(u_\tau^*)\left(1 - 2\widehat{B}(u_\tau^*)\beta_t\right)^2 + \widehat{B}'(u_\tau^*)\left(4\kappa\theta\beta_t \mathrm{e}^{\kappa t}(1 - 2\widehat{B}(u_\tau^*)\beta_t) + \xi^2 v\right)\right] = \left(1 - 2\widehat{B}(u_\tau^*)\beta_t\right)^2, \tag{3.6.7}$$

with $\widehat{A}$ and $\widehat{B}$ defined in (3.6.3). We now use this to derive an asymptotic expansion for u_τ^* as τ tends to zero.

Lemma 3.6.4. *The expansion $u_\tau^*(k) = a_0(k) + a_1(k)\tau^{1/4} + a_2(k)\tau^{1/2} + a_3(k)\tau^{3/4} + \mathcal{O}(\tau)$ holds for all $k \in \mathbb{R}^*$ as τ tends to zero, with a_0, a_1, a_2 and a_3 defined in* (3.3.1).

Proof. Existence and uniqueness was proved in Lemma 3.6.3 and so we assume the result as an ansatz. Consider $k > 0$. From Lemma 3.6.3 it is clear that $a_0(k) = 1/\sqrt{\beta_t}$. The ansatz and Lemma 3.6.2 then imply the following asymptotics as τ tends to zero (we drop here the k-dependence):

$$\begin{aligned}
\widehat{B}(u_\tau^*) &= \frac{1}{2\beta_t} + a_0 a_1 \tau^{1/4} + r\tau^{1/2} + \left(a_1 a_2 + a_0 a_3 + a_1 \widehat{B}_1'(a_0)\right)\tau^{3/4} + \mathcal{O}(\tau),\\
\widehat{B}'(u_\tau^*) &= a_0 + a_1\tau^{1/4} + (a_2 + \widehat{B}_1'(a_0))\tau^{1/2} + \mathcal{O}(\tau^{3/4}),\\
\widehat{A}'(u_\tau^*) &= \frac{1}{2}\kappa\theta a_0 \tau + \mathcal{O}(\tau^{5/4}),
\end{aligned} \tag{3.6.8}$$

where $r \equiv r(k) := a_0 a_2 + \widehat{B}_1(a_0) + a_1^2/2 = a_1^2/2 - \kappa\theta/(|k|\xi^2\sqrt{\beta_t})$ is defined in (3.3.3). We substitute these asymptotics into (3.6.7) and solve at each order. At the $\tau^{1/4}$ order we have two solutions, $a_1(k) = \pm\sqrt{v}\mathrm{e}^{-\kappa t/2}/(2\sqrt{k}\beta_t^{3/4})$ and we choose the negative root so that $u_\tau^* \in \mathcal{D}_\Lambda^o$ for τ small enough. In a straightforward, yet tedious, manner we continue the procedure and iteratively solve at each order (the next two equations are linear in a_2 and a_3) to derive the asymptotic expansions in the lemma. An analogous treatment holds in the case $k < 0$.

To complete the proof (and make the ansatz approach above rigorous) we need to show the existence of this expansion for $u_\tau^*(k)$. Fix $k \in \mathbb{R}^*$ and set $f_k(u,\tau) := \partial_u \Lambda_\tau^{(t)}(u) - k$. Now let $\bar{\tau} > 0$. From Lemma 3.6.3 we know that there exists a solution $u_{\bar{\tau}}^*(k)$ to the equation $f_k(u_{\bar{\tau}}^*(k), \bar{\tau}) = 0$ and the strict convexity of the forward cgf implies $\partial_u f_k(u_{\bar{\tau}}^*(k), \bar{\tau}) > 0$. Further, the two-dimensional map $f_k : \mathcal{D}_{t,\tau}^o \times \mathbb{R}_+^* \to \mathbb{R}$ is analytic (see [119, Theorem 7.1.1]). It follows by the Implicit Function Theorem [108, Theorem 8.6, Chapter 0] that $\tau \mapsto u_\tau^*(k)$ is analytic in some neighbourhood around $\bar{\tau}$. Since this argument holds for all $\bar{\tau} > 0$, this function is also analytic on $\mathbb{R}_+^*$. Also from Lemma 3.6.3 we know that $\lim_{\tau\searrow 0} u_\tau^*(k) = \mathrm{sgn}(k)/\sqrt{\beta_t}$. Since we computed the Taylor series expansion consistent with this limit and the expansion is unique, it follows that $u_\tau^*(k)$ admits this representation. □

In the forthcoming analysis we will be interested in the asymptotics of

$$e_\tau(k) := \left(1 - 2\widehat{B}(u_\tau^*(k))\beta_t\right)\tau^{-1/4}, \tag{3.6.9}$$

as τ tends to zero. Since $(1 - 2\widehat{B}(u_\tau^*(k))\beta_t)$ converges to zero, it is not immediately clear that e_τ has a well defined limit. But we can use the asymptotics in (3.6.8) to deduce the following lemma:

Lemma 3.6.5. *The expansion $e_\tau(k) = e_0(k) + e_1(k)\tau^{1/4} + e_2(k)\tau^{1/2} + \mathcal{O}(\tau^{3/4})$ holds for all $k \in \mathbb{R}^*$ as τ tends to zero, where e_0, e_1 and e_2 are defined in* (3.3.4).

Proof. We substitute the asymptotics for $\widehat{B}(u_\tau^*)$ in (3.6.8) into the definition of e_τ in (3.6.9) and the lemma follows after simplification. □

3.6.3 Characteristic function asymptotics

We now define the random variable $Z_{\tau,k} := \left(X_\tau^{(t)} - k\right)/\tau^{1/8}$ and the characteristic function $\Phi_{\tau,k} : \mathbb{R} \to \mathbb{C}$ of $Z_{\tau,k}$ in the $\mathbb{Q}_{k,\tau}$-measure in (3.6.6) as

$$\Phi_{\tau,k}(u) := \mathbb{E}^{\mathbb{Q}_{k,\tau}}\left(\mathrm{e}^{\mathtt{i}uZ_{\tau,k}}\right). \tag{3.6.10}$$

Define now the functions $\phi_1, \phi_2 : \mathbb{R}^* \times \mathbb{R} \to \mathbb{C}$ by

$$\phi_1(k,u) := \mathtt{i}u\left(\psi_0(k) + \frac{4a_0(k)\theta\kappa\beta_t}{e_0(k)\xi^2}\right) + \mathtt{i}u^3\psi_1(k), \qquad \phi_2(k,u) := u^2\phi_2^a(k) + u^4\phi_2^b(k) + u^6\phi_2^c(k),$$

with ψ_0, ψ_1 defined in (3.3.5), a_0, e_0 in (3.3.1), (3.3.4), and ϕ_2^a, ϕ_2^b and ϕ_2^c in (3.3.6). The following lemma provides the asymptotics of $\Phi_{\tau,k}$:

Lemma 3.6.6. *The following expansion holds for all $k \in \mathbb{R}^*$ as τ tends to zero (with ζ given in* (3.3.3)*):*

$$\Phi_{\tau,k}(u) = \mathrm{e}^{-\frac{1}{2}\zeta^2(k)u^2}\left(1 + \phi_1(k,u)\tau^{1/8} + \phi_2(k,u)\tau^{1/4} + \mathcal{O}\left(\tau^{3/8}\right)\right).$$

Remark 3.6.7. For any $k \in \mathbb{R}^*$, Lévy's Convergence Theorem [147, Page 185, Theorem 18.1] implies that $Z_{\tau,k}$ converges weakly to a normal random variable with zero mean and variance $\zeta^2(k)$ as τ tends to zero.

Proof. From the change of measure (3.6.6) and the forward cgf given in (1.3.7) we compute (we drop the k-dependence throughout) for small τ:

$$\begin{aligned}
\log\Phi_{k,\tau}(u) &= \log\mathbb{E}^{\mathbb{P}}\left(\frac{\mathrm{d}\mathbb{Q}_{k,\tau}}{\mathrm{d}\mathbb{P}}\mathrm{e}^{\mathtt{i}uZ_{k,\tau}}\right) = \log\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{u^*(k)X_\tau^{(t)}}{\sqrt{\tau}} - \frac{\Lambda_\tau^{(t)}(u_\tau^*)}{\sqrt{\tau}}\right)\exp\left(\frac{\mathtt{i}uX_\tau^{(t)}}{\tau^{1/8}} - \frac{\mathtt{i}uk}{\tau^{1/8}}\right)\right] \\
&= -\frac{1}{\sqrt{\tau}}\Lambda_\tau^{(t)}(u_\tau^*) - \frac{\mathtt{i}uk}{\tau^{1/8}} + \log\mathbb{E}^{\mathbb{P}}\left[\exp\left(\left(\frac{X_\tau^{(t)}}{\sqrt{\tau}}\right)\left(\mathtt{i}u\tau^{3/8} + u_\tau^*\right)\right)\right] \\
&= -\frac{\mathtt{i}uk}{\tau^{1/8}} + \frac{1}{\sqrt{\tau}}\left(\Lambda_\tau^{(t)}\left(\mathtt{i}u\tau^{3/8} + u_\tau^*\right) - \Lambda_\tau^{(t)}(u_\tau^*)\right).
\end{aligned} \tag{3.6.11}$$

Using the asymptotics in (3.6.8) we have that as τ tends to zero (we drop the k-dependence)

$$\begin{aligned}
\widehat{B}\left(u_\tau^* + \mathtt{i}u\tau^{3/8}\right) &= \frac{a_0^2}{2} + a_0a_1\tau^{1/4} + \mathtt{i}a_0\tau^{3/8}u + r\tau^{1/2} + \mathcal{O}(\tau^{5/8}), \\
\widehat{B}(u_\tau^*) &= \frac{a_0^2}{2} + a_0a_1\tau^{1/4} + r\tau^{1/2} + \mathcal{O}(\tau^{3/4}), \\
\widehat{B}\left(u_\tau^* + \mathtt{i}u\tau^{3/8}\right) - \widehat{B}(u_\tau^*) &= \mathtt{i}a_0u\tau^{3/8} + \mathtt{i}a_1u\tau^{5/8} - \frac{1}{2}u^2\tau^{3/4} + \mathcal{O}(\tau^{7/8}),
\end{aligned} \tag{3.6.12}$$

where $r \equiv r(k) := a_0a_2 + \widehat{B}_1(a_0) + a_1^2/2 = a_1^2/2 - \kappa\theta/(|k|\xi^2\sqrt{\beta_t})$ is defined in (3.3.3). Similarly for small τ,

$$\begin{aligned}
\widehat{A}\left(u_\tau^* + \mathtt{i}u\tau^{3/8}\right) &= \frac{\kappa\theta}{4}a_0^2\tau + \frac{\kappa\theta}{2}a_0a_1\tau^{5/4} + \frac{\mathtt{i}\kappa\theta}{2}a_0u\tau^{11/8} + \mathcal{O}(\tau^{3/2}), \\
\widehat{A}(u_\tau^*) &= \frac{\kappa\theta}{4}a_0^2\tau + \frac{\kappa\theta}{2}a_0a_1\tau^{5/4} + \mathcal{O}(\tau^{3/2}), \\
\widehat{A}\left(u_\tau^* + \mathtt{i}u\tau^{3/8}\right) - \widehat{A}(u_\tau^*) &= \frac{\mathtt{i}\kappa\theta}{2}a_0u\tau^{11/8} + \mathcal{O}(\tau^{3/2}).
\end{aligned} \tag{3.6.13}$$

We now use e_τ defined in (3.6.9) to re-write the term

$$\frac{\widehat{B}\left(u_\tau^* + \mathtt{i}u\tau^{3/8}\right) v\mathrm{e}^{-\kappa t}}{1 - 2\beta_t \widehat{B}\left(u_\tau^* + \mathtt{i}u\tau^{3/8}\right)} = \frac{v\mathrm{e}^{-\kappa t}\tau^{-1/4}\widehat{B}\left(u_\tau^* + \mathtt{i}u\tau^{3/8}\right)}{e_\tau - 2\beta_t\tau^{-1/4}\left(\widehat{B}\left(u_\tau^* + \mathtt{i}u\tau^{3/8}\right) - \widehat{B}\left(u_\tau^*\right)\right)},$$

and then use the asymptotics in (3.6.12) and Lemma 3.6.5 to find that for small τ

$$\begin{aligned}
&\frac{v\mathrm{e}^{-\kappa t}\tau^{-1/4}\widehat{B}\left(u_\tau^* + \mathtt{i}u\tau^{3/8}\right)}{e_\tau - 2\beta_t\tau^{-1/4}\left(\widehat{B}\left(u_\tau^* + \mathtt{i}u\tau^{3/8}\right) - \widehat{B}\left(u_\tau^*\right)\right)} \\
&= \frac{v\mathrm{e}^{-\kappa t}\tau^{-1/4}\left(a_0^2/2 + a_0a_1\tau^{1/4} + \mathtt{i}a_0\tau^{3/8}u + r\tau^{1/2} + \mathcal{O}(\tau^{5/8})\right)}{e_0 + e_1\tau^{1/4} + e_2\tau^{1/2} + \mathcal{O}(\tau^{3/4}) - 2\beta_t\tau^{-1/4}\left(\mathtt{i}a_0u\tau^{3/8} + \mathtt{i}a_1u\tau^{5/8} - \frac{1}{2}u^2\tau^{3/4} + \mathcal{O}(\tau^{7/8})\right)} \\
&= \frac{v\mathrm{e}^{-\kappa t}a_0^2}{2e_0}\tau^{-1/4} + \frac{v\mathrm{e}^{-\kappa t}\mathtt{i}a_0^3u\beta_t}{e_0^2}\tau^{-1/8} + v\mathrm{e}^{-\kappa t}\left(\frac{a_0a_1}{e_0} - \frac{a_0^2e_1}{2e_0^2}\right) - \frac{\zeta^2u^2}{2} \\
&\quad + (\mathtt{i}u\psi_0 + \mathtt{i}u^3\psi_1)\tau^{1/8} + (\psi_4 + \psi_2u^2 + \psi_3u^4)\tau^{1/4} + \mathcal{O}(\tau^{3/8}),
\end{aligned} \tag{3.6.14}$$

with ζ and $\psi_0, \ldots, \psi_4$ defined in (3.3.3) and (3.3.5). From the definition of a_0, e_0 and β_t we note the simplification

$$\frac{\mathtt{i}v\mathrm{e}^{-\kappa t}a_0^3(k)u\beta_t}{e_0^2(k)\tau^{1/8}} = \frac{\mathtt{i}uk}{\tau^{1/8}}. \tag{3.6.15}$$

Similarly we find that as τ tends to zero

$$\begin{aligned}
\frac{\widehat{B}\left(u_\tau^*\right)v\mathrm{e}^{-\kappa t}}{1 - 2\beta_t\widehat{B}\left(u_\tau^*\right)} &= \frac{v\mathrm{e}^{-\kappa t}\tau^{-1/4}\widehat{B}\left(u_\tau^*\right)}{e_\tau} = \frac{v\mathrm{e}^{-\kappa t}\tau^{-1/4}\left(a_0^2/2 + a_0a_1\tau^{1/4} + r\tau^{1/2} + \mathcal{O}(\tau^{3/4})\right)}{e_0 + e_1\tau^{1/4} + e_2\tau^{1/2} + \mathcal{O}(\tau^{3/4})} \\
&= \frac{a_0^2v\mathrm{e}^{-\kappa t}}{2e_0}\tau^{-1/4} + v\mathrm{e}^{-\kappa t}\left(\frac{a_0a_1}{e_0} - \frac{a_0^2e_1}{2e_0^2}\right) + \psi_4\tau^{1/4} + \mathcal{O}(\tau^{1/2}).
\end{aligned} \tag{3.6.16}$$

Again we use e_τ defined in (3.6.9) to re-write the term

$$\exp\left[\frac{2\kappa\theta}{\xi^2}\log\left(\frac{1 - 2\beta_t\widehat{B}\left(u_\tau^*\right)}{1 - 2\beta_t\widehat{B}\left(u_\tau^* + \mathtt{i}u\tau^{3/8}\right)}\right)\right] = \left(1 - \frac{2\beta_t\left(\widehat{B}\left(u_\tau^* + \mathtt{i}u\tau^{3/8}\right) - \widehat{B}\left(u_\tau^*\right)\right)}{e_\tau\tau^{1/4}}\right)^{-2\kappa\theta/\xi^2},$$

and then use the asymptotics in (3.6.12) and Lemma 3.6.5 to find that for small τ

$$\begin{aligned}
&\left(1 - \frac{2\beta_t\left(\widehat{B}\left(u_\tau^* + \mathtt{i}u\tau^{3/8}\right) - \widehat{B}\left(u_\tau^*\right)\right)}{e_\tau\tau^{1/4}}\right)^{-\frac{2\kappa\theta}{\xi^2}} = \left(1 + \frac{2\mathtt{i}a_0\beta_tu}{e_0}\tau^{1/8} - \frac{4a_0^2u^2\beta_t^2}{e_0^2}\tau^{1/4} + \mathcal{O}(\tau^{3/8})\right)^{\frac{2\kappa\theta}{\xi^2}} \\
&= 1 + \frac{4\mathtt{i}\kappa\theta a_0\beta_tu}{e_0\xi^2}\tau^{1/8} - \frac{4\kappa\theta a_0^2\beta_t^2u^2\left(2\kappa\theta + \xi^2\right)}{\xi^4e_0^2}\tau^{1/4} + \mathcal{O}(\tau^{3/8}).
\end{aligned} \tag{3.6.17}$$

Using (3.6.11) with definition (3.6.3) and (1.3.7), property (3.6.15) and the asymptotics in (3.6.13), (3.6.14), (3.6.16) and (3.6.17) we finally calculate the characteristic function for small τ as

$$\begin{aligned}
\Phi_{k,\tau}(u) = \exp&\left(-\frac{\zeta^2u^2}{2} + (\mathtt{i}u\psi_0 + \mathtt{i}u^3\psi_1)\tau^{1/8} + (\psi_2u^2 + \psi_3u^4)\tau^{1/4} + \mathcal{O}(\tau^{3/8})\right)\Big(1 \\
&+ \frac{4\mathtt{i}\kappa\theta a_0\beta_tu}{e_0\xi^2}\tau^{1/8} - \frac{4\kappa\theta a_0^2\beta_t^2u^2\left(2\kappa\theta + \xi^2\right)}{\xi^4e_0^2}\tau^{1/4} + \mathcal{O}(\tau^{3/8})\Big),
\end{aligned}$$

with $\psi_0,\dots,\psi_3$ defined in (3.3.3), (3.3.5), and so the lemma follows from the following decomposition

$$\Phi_{k,\tau}(u) = \exp\left(-\frac{\zeta^2u^2}{2}\right)\Bigg\{1+\mathtt{i}\left(u\left(\psi_0+\frac{4\kappa\theta a_0\beta_t}{e_0\xi^2}\right)+u^3\psi_1\right)\tau^{1/8}+$$
$$\left[\left(\psi_2-\frac{\psi_0^2}{2}-\frac{4\kappa\theta a_0^2\beta_t^2\left(2\kappa\theta+\xi^2\right)}{\xi^4e_0^2}-\frac{4\kappa\theta a_0\beta_t}{e_0\xi^2}\psi_0\right)u^2+\left(\psi_3-\psi_0\psi_1-\psi_1\frac{4\kappa\theta a_0\beta_t}{e_0\xi^2}\right)u^4-\frac{u^6\psi_1^2}{2}\right]\tau^{\frac{1}{4}}$$
$$+\mathcal{O}(\tau^{\frac{3}{8}})\Bigg\}.$$

□

The following technical lemma will be needed in Section 3.6.4 where it will be used to give the leading order exponential decay of out-of-the-money forward-start options as τ tends to zero.

Lemma 3.6.8. *The following expansion holds for all $k\in\mathbb{R}^*$ as τ tends to zero:*

$$\mathrm{e}^{-ku_\tau^*/\sqrt{\tau}+\Lambda_\tau^{(t)}(u_\tau^*)/\sqrt{\tau}} = \mathrm{e}^{-\Lambda^*(k)/\sqrt{\tau}+c_0(k)/\tau^{1/4}+c_1(k)}\tau^{-\kappa\theta/(2\xi^2)}c_2(k)\left(1+z_1(k)\tau^{1/4}+\mathcal{O}(\tau^{1/2})\right),$$

where c_0, c_1 and c_2 are defined in (3.3.8), *Λ^* is characterised explicitly in Lemma 3.2.4 and z_1 is given in* (3.3.7).

Proof. We use the asymptotics in Lemma 3.6.4 and the characterisation of Λ^* in Lemma 3.2.4 to write for small τ (we drop the k-dependence)

$$\begin{aligned}\exp\left(-ku_\tau^*/\sqrt{\tau}\right) &= \exp\left(-a_0k/\sqrt{\tau}-a_1k/\tau^{1/4}-a_2k\right)\left(1-a_3k\tau^{1/4}+\mathcal{O}(\tau^{1/2})\right) \\ &= \exp\left(-\Lambda^*(k)/\sqrt{\tau}-a_1k/\tau^{1/4}-a_2k\right)\left(1-a_3k\tau^{1/4}+\mathcal{O}(\tau^{1/2})\right).\end{aligned} \tag{3.6.18}$$

Using the Heston forward cgf definition in (3.2.1), (3.6.3) and (1.3.7) we can write

$$\exp\left(\Lambda_\tau^{(t)}(u_\tau^*)/\sqrt{\tau}\right) = \exp\left(\widehat{A}(u_\tau^*)+\frac{\widehat{B}(u_\tau^*)v\mathrm{e}^{-\kappa t}}{1-2\beta_t\widehat{B}(u_\tau^*)}-\frac{2\kappa\theta}{\xi^2}\log(1-2\beta_t\widehat{B}(u_\tau^*))\right). \tag{3.6.19}$$

Using the definition of e_τ in (3.6.9) and the asymptotics in Lemma 3.6.5 we find that for small τ

$$\begin{aligned}\exp\left(-\frac{2\kappa\theta}{\xi^2}\log(1-2\beta_t\widehat{B}(u_\tau^*))\right) &= \tau^{-\kappa\theta/(2\xi^2)}e_\tau^{-\frac{2\kappa\theta}{\xi^2}} = \tau^{-\kappa\theta/(2\xi^2)}\left(e_0+e_1\tau^{1/4}+\mathcal{O}(\tau^{1/2})\right)^{-\frac{2\kappa\theta}{\xi^2}} \\ &= \tau^{-\kappa\theta/(2\xi^2)}e_0^{-2\kappa\theta/\xi^2}\left(1-\frac{2\kappa\theta e_1}{\xi^2e_0}\tau^{1/4}+\mathcal{O}(\tau^{1/2})\right).\end{aligned} \tag{3.6.20}$$

Then the lemma follows after using (3.6.18) and (3.6.19), the asymptotics in (3.6.20), (3.6.16) and (3.6.13) and the simplification $c_0(k)=v\mathrm{e}^{-\kappa t}/(2e_0(k)\beta_t)-a_1(k)k=2|a_1(k)k|$. □

We now demonstrate that $|C_{\tau,k}(u)\Phi_{\tau,k}(u)|$ is bounded for small τ by an integrable function where $\Phi_{\tau,k}$ is defined in (3.6.10) and

$$C_{\tau,k}(u) := \frac{\tau^{7/8}}{\left(u_\tau^*-\mathtt{i}\tau^{3/8}u\right)\left(u_\tau^*-\sqrt{\tau}-\mathtt{i}\tau^{3/8}u\right)}.$$

In particular we have the following lemma:

Lemma 3.6.9. *Fix $k \in \mathbb{R}^*$. There exists $\tau^* > 0$ and $d > 0$ such that for all $\tau < \min(\tau^*, 1)$ and $u \in \mathbb{R}$:*

$$|C_{\tau,k}(u)\Phi_{\tau,k}(u)| \leq d\mathbb{1}_{\{u\in[-1,1]\}} + u^{-2}\mathbb{1}_{\{u\in\mathbb{R}\setminus[-1,1]\}}.$$

Proof. First we note that

$$|C_{\tau,k}(u)|^2 = \frac{\tau^{7/4}}{\left((u_\tau^*)^2 + \tau^{6/8}u^2\right)\left(\left(u_\tau^* - \sqrt{\tau}\right)^2 + \tau^{6/8}u^2\right)} \leq \min\left(\frac{\tau^{7/4}}{(u_\tau^*)^2\left(u_\tau^* - \sqrt{\tau}\right)^2}, \frac{\tau^{1/4}}{u^4}\right),$$

and so we have that

$$|C_{\tau,k}(u)| \leq \min\left(\frac{\tau^{7/8}}{|u_\tau^*|\,|u_\tau^* - \sqrt{\tau}|}, \frac{\tau^{1/8}}{u^2}\right).$$

Using the expansions for u_τ^* in Lemma 3.6.4 we find that

$$\frac{\tau^{7/8}}{|u_\tau^*|\,|u_\tau^* - \sqrt{\tau}|} = \frac{\tau^{7/8}}{a_0^2} + \mathcal{O}(\tau^{9/8}).$$

Hence there exists a $\tau^* > 0, B > 0$ such that for all $\tau < \min(\tau^*, 1)$,

$$\frac{\tau^{7/8}}{|u_\tau^*|\,|u_\tau^* - \sqrt{\tau}|} \leq \frac{1}{a_0^2} + B,$$

and for $\tau < \min(\tau^*, 1)$ we then have that

$$|C_{\tau,k}(u)| \leq \min\left(\frac{1}{a_0^2} + B, \frac{1}{u^2}\right),$$

and the lemma follows after using $|\Phi_{\tau,k}(u)| \leq 1$. □

We now use the characteristic function expansion in Lemma 3.6.6 and Fourier transform methods to derive the asymptotics for the expectation (under the measure (3.6.6)) of the modified payoff on the re-scaled forward price process. This lemma will be critical for the analysis in Section 3.6.4.

Lemma 3.6.10. *The following expansion holds for all $k \in \mathbb{R}^*$ as τ tends to zero:*

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}_{\tau,k}}&\left[\mathrm{e}^{-u_\tau^* Z_{\tau,k}/\tau^{3/8}}\left(\mathrm{e}^{Z_{\tau,k}\tau^{1/8}} - 1\right)^+\right]\mathbb{1}_{\{k>0\}} + \mathbb{E}^{\mathbb{Q}_{\tau,k}}\left[\mathrm{e}^{-u_\tau^* Z_{\tau,k}/\tau^{3/8}}\left(1 - \mathrm{e}^{Z_{\tau,k}\tau^{1/8}}\right)^+\right]\mathbb{1}_{\{k<0\}}\\ &= \frac{\tau^{7/8}\beta_t}{\zeta(k)\sqrt{2\pi}}\left(1 + p_1(k)\tau^{1/4} + o\left(\tau^{1/4}\right)\right),\end{aligned}$$

where ζ is defined in (3.3.3)*, p_1 in* (3.3.7) *and β_t in* (1.3.4)*.*

Proof. We first consider $k > 0$ and drop the k-dependence for the functions below. We denote the Fourier transform $\mathcal{F}$ by $(\mathcal{F}f)(u) := \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ux} f(x)\mathrm{d}x$, for all $f \in L^2$, $u \in \mathbb{R}$. The Fourier transform of the payoff $\mathrm{e}^{-u_\tau^* Z_{\tau,k}/\tau^{3/8}}\left(\mathrm{e}^{Z_{\tau,k}\tau^{1/8}} - 1\right)^+$ is given by

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-u_\tau^* z/\tau^{3/8}}\left(\mathrm{e}^{z\tau^{1/8}} - 1\right)\mathrm{e}^{\mathrm{i}uz}\mathrm{d}z &= \left[\frac{\mathrm{e}^{z\left(\mathrm{i}u - u_\tau^*/\tau^{3/8} + \tau^{1/8}\right)}}{\left(\mathrm{i}u - u_\tau^*/\tau^{3/8} + \tau^{1/8}\right)}\right]_0^\infty - \left[\frac{\mathrm{e}^{z\left(\mathrm{i}u - u_\tau^*/\tau^{3/8}\right)}}{\left(\mathrm{i}u - u_\tau^*/\tau^{3/8}\right)}\right]_0^\infty\\ &= \frac{1}{\left(\mathrm{i}u - u_\tau^*/\tau^{3/8}\right)} - \frac{1}{\left(\mathrm{i}u - u_\tau^*/\tau^{3/8} + \tau^{1/8}\right)}\\ &= \frac{\tau^{7/8}}{\left(u_\tau^* - \mathrm{i}\tau^{3/8}u\right)\left(u_\tau^* - \sqrt{\tau} - \mathrm{i}\tau^{3/8}u\right)},\end{aligned}$$

if $u_\tau^* > \max(\tau^{1/2}, 0) = \tau^{1/2}$, which holds for τ small enough since u_τ^* converges to $a_0 > 0$ by Lemma 3.6.4. Due to Remark 3.6.7, Z_τ converges weakly to a Gaussian random variable and since the Gaussian density and the modified payoff are in L^2 we can use Parseval's Theorem [77, Page 48,Theorem 13E] for small enough τ to write

$$\mathbb{E}^{\mathbb{Q}_{\tau,k}}\left(\mathrm{e}^{-\frac{u_\tau^* Z_{\tau,k}}{\tau^{3/8}}}\left(\mathrm{e}^{Z_{\tau,k}\tau^{1/8}}-1\right)^+\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\tau^{7/8}\Phi_{\tau,k}(u)}{\left(u_\tau^*+\mathtt{i}\tau^{3/8}u\right)\left(u_\tau^*-\sqrt{\tau}+\mathtt{i}\tau^{3/8}u\right)}\mathrm{d}u, \quad (3.6.21)$$

where we have used that

$$\overline{\frac{\tau^{7/8}}{\left(u_\tau^*-\mathtt{i}\tau^{3/8}u\right)\left(u_\tau^*-\sqrt{\tau}-\mathtt{i}\tau^{3/8}u\right)}} = \frac{\tau^{7/8}}{\left(u_\tau^*+\mathtt{i}\tau^{3/8}u\right)\left(u_\tau^*-\sqrt{\tau}+\mathtt{i}\tau^{3/8}u\right)},$$

with $\overline{a}$ denoting the complex conjugate for $a \in \mathbb{C}$. Using the asymptotics of u_τ^* given in Lemma 3.6.4 we can Taylor expand for small τ to find that

$$\begin{aligned}\frac{\tau^{7/8}}{\left(u_\tau^*+\mathtt{i}\tau^{3/8}u\right)\left(u_\tau^*-\sqrt{\tau}+\mathtt{i}\tau^{3/8}u\right)} &= \frac{\tau^{7/8}}{a_0^2+2a_0a_1\tau^{1/4}+\mathcal{O}(\tau^{3/8})}\\ &= \frac{\tau^{7/8}}{a_0^2}\left(1-\frac{2a_1}{a_0}\tau^{1/4}+\mathcal{O}\left(\tau^{3/8}\right)\right). \end{aligned} \quad (3.6.22)$$

Finally combining (3.6.22) and the asymptotics of the characteristic function derived in Lemma 3.6.6 with (3.6.21) we find that for small τ

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\tau^{7/8}\Phi_{\tau,k}(u)}{\left(u_\tau^*+\mathtt{i}\tau^{3/8}u\right)\left(u_\tau^*-\sqrt{\tau}+\mathtt{i}\tau^{3/8}u\right)}\mathrm{d}u\\ &= \frac{\tau^{7/8}}{a_0^2 2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\varsigma^2u^2}{2}}\left(1+\phi_1(u,k)\tau^{1/8}+\left(\phi_2(u,k)-\frac{2a_1}{a_0}\right)\tau^{1/4}+\mathcal{O}\left(\tau^{3/8}\right)\right)\mathrm{d}u\\ &= \frac{\tau^{7/8}}{a_0^2 2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\varsigma^2u^2}{2}}\left(1+\left(u^2\phi_2^a+u^4\phi_2^b+u^6\phi_2^c-\frac{2a_1}{a_0}\right)\tau^{1/4}+\mathcal{O}\left(\tau^{3/8}\right)\right)\mathrm{d}u,\end{aligned}$$

where in the last line we have used that $\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\varsigma^2u^2}{2}}\phi_1(u,k)\mathrm{d}u = 0$, since ϕ_1 is an odd power of u. The result then follows by using the Lebesgue dominated convergence theorem (using Lemma 3.6.9) and simple moment formulae of the normal distribution. Fix now $k < 0$. The Fourier transform of the payoff $\mathrm{e}^{-u_\tau^* Z_{\tau,k}/\tau^{3/8}}\left(1-\mathrm{e}^{Z_{\tau,k}\tau^{1/8}}\right)^+$ is given by

$$\begin{aligned}\int_{-\infty}^{0}\mathrm{e}^{-u_\tau^* z/\tau^{3/8}}\left(1-\mathrm{e}^{z\tau^{1/8}}\right)\mathrm{e}^{\mathtt{i}uz}\mathrm{d}z &= \left[\frac{\mathrm{e}^{z\left(\mathtt{i}u-u_\tau^*/\tau^{3/8}\right)}}{\left(\mathtt{i}u-u_\tau^*/\tau^{3/8}\right)}\right]_{-\infty}^{0} - \left[\frac{\mathrm{e}^{z\left(\mathtt{i}u-u_\tau^*/\tau^{3/8}+\tau^{1/8}\right)}}{\left(\mathtt{i}u-u_\tau^*/\tau^{3/8}+\tau^{1/8}\right)}\right]_{-\infty}^{0}\\ &= \frac{1}{\left(\mathtt{i}u-u_\tau^*/\tau^{3/8}\right)} - \frac{1}{\left(\mathtt{i}u-u_\tau^*/\tau^{3/8}+\tau^{1/8}\right)}\\ &= \frac{\tau^{7/8}}{\left(u_\tau^*-\mathtt{i}\tau^{3/8}u\right)\left(u_\tau^*-\sqrt{\tau}-\mathtt{i}\tau^{3/8}u\right)},\end{aligned}$$

if $u_\tau^* < \min(\tau^{1/2}, 0) = 0$, which holds for τ small enough since u_τ^* converges to $a_0 < 0$ by Lemma 3.6.4. The rest of the proof is analogous to $k < 0$ above and we omit it for brevity. □

Remark 3.6.11. We have chosen to specify the remainder in the form $o(1/\tau^{1/4})$ instead of $\mathcal{O}(1/\tau^{3/8})$ since it can actually be shown that the term $\mathcal{O}(1/\tau^{3/8})$ is zero by extending the results in Lemma 3.6.6 and the next non-trivial term is $\mathcal{O}(1/\tau^{1/2})$. For brevity we omit this analysis.

3.6.4 Option price and forward smile asymptotics

In this section we finally put all the pieces together from Sections 3.6.1 - 3.6.3 and prove Theorems 3.3.1 and 3.4.1.

Proof of Theorem 3.3.1. We use the time-dependent change of measure defined in (3.6.6) to write forward-start call option prices for all $k > 0$ as

$$\begin{aligned}\mathbb{E}\left(\mathrm{e}^{X_\tau^{(t)}} - \mathrm{e}^k\right)^+ &= \mathrm{e}^{\Lambda_\tau^{(t)}(u_\tau^*)/\sqrt{\tau}}\mathbb{E}^{\mathbb{Q}_{k,\tau}}\left[\mathrm{e}^{-u_\tau^* X_\tau^{(t)}/\sqrt{\tau}}\left(\mathrm{e}^{X_\tau^{(t)}} - \mathrm{e}^k\right)^+\right] \\ &= \mathrm{e}^{-\frac{ku_\tau^* - \Lambda_\tau^{(t)}(u_\tau^*)}{\sqrt{\tau}}}\mathbb{E}^{\mathbb{Q}_{k,\tau}}\left[\mathrm{e}^{-\frac{u_\tau^*}{\sqrt{\tau}}\left(X_\tau^{(t)} - k\right)}\left(\mathrm{e}^{X_\tau^{(t)}} - \mathrm{e}^k\right)^+\right] \\ &= \mathrm{e}^{-\frac{ku_\tau^* - \Lambda_\tau^{(t)}(u_\tau^*)}{\sqrt{\tau}}}\mathrm{e}^k\mathbb{E}^{\mathbb{Q}_{k,\tau}}\left[\mathrm{e}^{-\frac{u_\tau^* Z_{\tau,k}}{\tau^{3/8}}}\left(\mathrm{e}^{Z_{\tau,k}\tau^{1/8}} - 1\right)^+\right],\end{aligned}$$

with $Z_{\tau,k}$ defined on page 108. A similar result holds for forward-start put option prices for all $k < 0$. The theorem then follows by applying Lemma 3.6.8 and Lemma 3.6.10 and using put-call parity and that in the Heston model $(\mathrm{e}^{X_t})_{t\geq 0}$ is a true martingale [5, Proposition 2.5]. □

Proof of Theorem 3.4.1. The general machinery to translate option price asymptotics into implied volatility asymptotics has been fully developed by Gao and Lee [69]. We simply outline the main steps here. Assume the following ansatz for the forward implied volatility as τ tends to zero:

$$\sigma_{t,\tau}^2(k) = \frac{\mathfrak{N}_0(k,t)}{\sqrt{\tau}} + \frac{\mathfrak{N}_1(k,t)}{\tau^{1/4}} + \mathfrak{N}_2(k,t) + \mathfrak{N}_3(k,t)\tau^{1/4} + o(\tau^{1/4}).$$

Substituting this ansatz into the BSM asymptotics in Lemma 3.3.4 we then obtain

$$\begin{aligned}&\exp\left(-\frac{k^2}{2\sqrt{\tau}\mathfrak{N}_0} + \frac{k^2\mathfrak{N}_1}{2\tau^{1/4}\mathfrak{N}_0^2} - \frac{k^2\left(\mathfrak{N}_1^2 - \mathfrak{N}_0\mathfrak{N}_2\right)}{2\mathfrak{N}_0^3} + \frac{k}{2}\right)\frac{\tau^{3/4}\mathfrak{N}_0^{3/2}}{\sqrt{2\pi}k^2}\Big[1 \\ &\qquad + \left(\frac{k^2\left(\mathfrak{N}_1^3 - 2\mathfrak{N}_0\mathfrak{N}_1\mathfrak{N}_2 + \mathfrak{N}_0^2\mathfrak{N}_3\right)}{2\mathfrak{N}_0^4} + \frac{3\mathfrak{N}_1}{2\mathfrak{N}_0}\right)\tau^{1/4} + o(\tau^{1/4})\Big].\end{aligned}$$

Equating orders with Theorem 3.3.1 we solve for $\mathfrak{N}_0$ and $\mathfrak{N}_1$, but we can only solve for higher order terms if $\tau^{3/4} = \tau^{(7/8 - \theta\kappa/(2\xi^2))}$ or $4\kappa\theta = \xi^2$. □

Chapter 4

Large-maturity regimes of the Heston forward smile

4.1 Introduction

Under some conditions on the parameters, it was shown in Section 2.3.1.2 that the smooth behaviour of the pointwise limit $\lim_{\tau\uparrow\infty}\tau^{-1}\log\mathbb{E}(\mathrm{e}^{uX_\tau^{(t)}})$ in the Heston model (1.3.2) yielded an asymptotic behaviour for the forward smile (1.0.5) as $\sigma_{t,\tau}^2(k\tau) = v_0^\infty(k) + v_1^\infty(k,t)\tau^{-1} + \mathcal{O}(\tau^{-2})$, where $v_0^\infty(\cdot)$ and $v_1^\infty(\cdot,t)$ are continuous functions on $\mathbb{R}$. In particular for $t=0$ (spot smiles), we recovered the result in [63, 65] (also under some restrictions on the parameters). Interestingly, the limiting large-maturity forward smile v_0^∞ does not depend on the forward-start date t. A number of practitioners (eg. Balland [10]) have made the natural conjecture that the large-maturity forward smile should be the same as the large-maturity spot smile. The result above rigorously shows us that this indeed holds if and only if the Heston correlation is close enough to zero.

It is natural to ask what happens when these parameter restrictions are violated. We identify a number of regimes depending on the correlation and derive asymptotics in each regime. The main results (Theorems 4.3.1 and 4.4.1) state the following, as τ tends to infinity:

$$\mathbb{E}\left(\mathrm{e}^{X_\tau^{(t)}} - \mathrm{e}^{k\tau}\right)^+ = \mathcal{I}\left(k,\tau,V'(0),V'(1),\mathbb{1}_{\{\kappa<\rho\xi\}}\right) + \frac{\phi(k,t)}{\tau^\alpha}\mathrm{e}^{-\tau(V^*(k)-k)+\psi(k,t)\tau^\gamma}\left(1+\mathcal{O}\left(\tau^{-\beta}\right)\right),$$

$$\sigma_{t,\tau}^2(k\tau) = \mathfrak{N}_0^\infty(k,t) + \mathfrak{N}_1^\infty(k,t)\tau^{-\lambda} + \mathcal{R}(\tau,\lambda),$$

for any $k\in\mathbb{R}$, where $\mathcal{I}$ is some indicator function related to the intrinsic value of the option price, and α, γ, β, λ are strictly positive constants, depending on the level of the correlation. The remainder $\mathcal{R}$ decays to zero as τ tends to infinity. If $t=0$ (spot smiles) we recover and extend the results in [63, 65].

The chapter is structured as follows. In Section 4.2 we recall the different large-maturity regimes for the Heston model introduced in Section 2.3.1.2, which will drive the asymptotic behaviour of forward-start option prices and forward implied volatilities. In Section 4.3 we derive large-maturity forward-start option asymptotics in each regime and in Section 4.4 we translate these results into forward smile asymptotics, including extended SVI-type formulae (Section 4.4.1). Section 4.5 provides numerics supporting the asymptotics developed in the chapter and Section 4.6 gathers the proofs of the main results.

4.2 Large-maturity regimes

In this section we recall the large-maturity regimes introduced in (2.3.5) and some properties that will be needed throughout the chapter. Each regime is determined by the Heston correlation and yields fundamentally different asymptotic behaviours for large-maturity forward-start options and the corresponding forward smile. This is due to the distinct behaviour of the moment explosions of the forward price process $(X_\tau^{(t)})_{\tau>0}$ in each regime. The large-maturity regimes are given as follows (2.3.5):

$$
\begin{array}{lll}
\mathfrak{R}_1: & \textit{Good correlation regime:} & \rho_- \le \rho \le \min(\rho_+, \kappa/\xi); \\
\mathfrak{R}_2: & \textit{Asymmetric negative correlation regime:} & -1 < \rho < \rho_- \text{ and } t > 0; \\
\mathfrak{R}_3: & \textit{Asymmetric positive correlation regime:} & \rho_+ < \rho < 1 \text{ and } t > 0; \\
 & \mathfrak{R}_{3a}: & \rho \le \kappa/\xi; \\
 & \mathfrak{R}_{3b}: & \rho > \kappa/\xi; \\
\mathfrak{R}_4: & \textit{Large correlation regime:} & \kappa/\xi < \rho \le \min(\rho_+, 1).
\end{array}
$$

where the real numbers ρ_- and ρ_+ are defined in (2.3.4) and note that $-1 \le \rho_- < 0 < \rho_+$ with $\rho_\pm = \pm 1$ if and only if $t = 0$. In the standard case $t = 0$, $\mathfrak{R}_1$ corresponds to $\kappa \ge \rho\xi$ and $\mathfrak{R}_4$ is its complement. We recall the following quantities defined in (2.3.6):

$$
u_\pm := \frac{\xi - 2\kappa\rho \pm \eta}{2\xi(1-\rho^2)} \qquad \text{and} \qquad u_\pm^* := \frac{\psi \pm \nu}{2\xi(\mathrm{e}^{\kappa t} - 1)},
$$

with η, ν and ψ defined in (2.3.7). From the properties outlined in Lemmas 2.5.12, 2.5.13 and Proposition 2.5.14 we note that for $t > 0$, $u_+ > u_+^* > 1$ if $\rho \le \rho_-$ and $u_- < u_-^* < 0$ if $\rho \ge \rho_+$. Furthermore we always have $u_- < 0$ and if $\rho < \kappa/\xi$ then $u_+ \ge 1$ with $u_+ = 1$ if and only if $\rho = \kappa/\xi$. Recall the function V and H from $\mathcal{K}_{\mathrm{H}}$ to $\mathbb{R}$ given in (2.3.8):

$$
V(u) := \frac{\mu}{2}\left(\kappa - \rho\xi u - d(u)\right) \qquad \text{and} \qquad H(u) := \frac{V(u) v \mathrm{e}^{-\kappa t}}{\kappa\theta - 2\beta_t V(u)} - \mu \log\left(\frac{\kappa\theta - 2\beta_t V(u)}{\kappa\theta\left(1 - \gamma(u)\right)}\right),
$$

with d, β_t, μ and γ defined in (1.3.6), (1.3.4) and (1.3.8) and the limiting domain $\mathcal{K}_{\mathrm{H}}$ defined in Table 2.1 and given below for clarity. We notify the reader that the constant μ (defined in (1.3.4)) will be used extensively throughout the chapter. It is clear (see Lemma 2.5.12) that the function V is infinitely differentiable, strictly convex and essentially smooth on the open interval (u_-, u_+) and that $V(0) = 0$. Furthermore if $\rho \le \kappa/\xi$ then $V(1) = 0$ and if $\rho > \kappa/\xi$ then $V(1) < 0$.

	$\mathfrak{R}_1$	$\mathfrak{R}_2$	$\mathfrak{R}_{3a}$	$\mathfrak{R}_{3b}$	$\mathfrak{R}_4$
$\mathcal{K}_{\mathrm{H}}$	$[u_-,u_+]$	$[u_-,u_+^*)$	$(u_-^*,u_+]$	$(u_-^*,1]$	$(u_-,1]$

Limiting domains in each large-maturity regime.

The following lemma characterises V^* (defined in (2.3.10)) and can be proved using straightforward calculus. The proof is therefore omitted. As we will see in Section 4.3.1, the function V^* can be interpreted as a large deviations rate function for our problem.

Lemma 4.2.1. *Define the function* $W(k,u) \equiv uk - V(u)$ *for any* $(k,u) \in \mathbb{R} \times [u_-,u_+]$. *Then (*$q^*$ *defined in* (2.3.9)*)*

- $\mathfrak{R}_1$*:* $V^*(k) \equiv W(k,q^*(k))$ *on* $\mathbb{R}$*;*
- $\mathfrak{R}_2$*:* $V^*(k) \equiv W(k,q^*(k))$ *on* $(-\infty, V'(u_+^*)]$ *and* $V^*(k) \equiv W(k,u_+^*)$ *on* $(V'(u_+^*),+\infty)$*;*
- $\mathfrak{R}_{3a}$*:* $V^*(k) \equiv W(k,u_-^*)$ *on* $(-\infty, V'(u_-^*))$ *and* $V^*(k) \equiv W(k,q^*(k))$ *on* $[V'(u_-^*),+\infty)$*;*
- $\mathfrak{R}_{3b}$*:*
$$V^*(k) \equiv \begin{cases} W(k,u_-^*), & \text{on } (-\infty, V'(u_-^*)), \\ W(k,q^*(k)), & \text{on } [V'(u_-^*), V'(1)], \\ W(k,1), & \text{on } (V'(1),+\infty); \end{cases}$$
- $\mathfrak{R}_4$*:* $V^*(k) \equiv W(k,q^*(k))$ *on* $(-\infty, V'(1)]$ *and* $V^*(k) \equiv W(k,1)$ *on* $(V'(1),+\infty)$.

4.3 Forward-start option asymptotics

In order to specify the forward-start option asymptotics we need to introduce some functions and constants. As outlined in Theorem 4.3.1, each of them is defined in a specific regime and strike region where it is well defined and real-valued. In the formulae below, γ, β_t, μ are defined in (1.3.8) and (1.3.4), $u_\pm^*$ in (2.3.6), V, H in (2.3.8) and q^* in (2.3.9).

$$\begin{cases} a_1^\pm(k) &:= \mp\dfrac{2|k-V'(u_\pm^*)|}{\zeta_\pm^2(k)}, \qquad \widetilde{a}_1^\pm := \mp\left|\dfrac{\mathrm{e}^{-\kappa t}\kappa\theta v}{4V'(u_\pm^*)V''(u_\pm^*)\beta_t^2}\right|^{1/3}, \\ a_2^\pm(k) &:= \dfrac{\mu\mathrm{e}^{-\kappa t}}{16\beta_t^2}\dfrac{\xi^2 vV''(u_\pm^*) - 8\beta_t^2\mathrm{e}^{\kappa t}V'(u_\pm^*)\left(k-V'(u_\pm^*)\right)}{V'(u_\pm^*)\left(k-V'(u_\pm^*)\right)^2}, \\ \widetilde{a}_2^\pm &:= -\dfrac{(\kappa\theta\mathrm{e}^{-\kappa t})^{2/3}}{12\xi^2 v^{1/3}\beta_t^{4/3}}\dfrac{16V'(u_\pm^*)V''(u_\pm^*)\beta_t^2\mathrm{e}^{\kappa t} + \xi^2 vV'''(u_\pm^*)}{2^{1/3}|V'(u_\pm^*)|^{2/3}V''(u_\pm^*)^{5/3}}, \end{cases} \tag{4.3.1}$$

where

$$\zeta_\pm^2(k) := 4\beta_t\left(\frac{V'(u_\pm^*)(k-V'(u_\pm^*))^3}{\kappa\theta v\mathrm{e}^{-\kappa t}}\right)^{1/2}; \tag{4.3.2}$$

$$\begin{cases} e_0^\pm(k) := -2\beta_t a_1^\pm(k)V'(u_\pm^*), & e_1^\pm(k) := -\beta_t\left[V''(u_\pm^*)a_1^\pm(k)^2 + 2V'(u_\pm^*)a_2^\pm(k)\right], \\ \widetilde{e}_0^\pm := -2\beta_t\widetilde{a}_1^\pm V'(u_\pm^*), & \widetilde{e}_1^\pm := -\beta_t\left[V''(u_\pm^*)(\widetilde{a}_1^\pm)^2 + 2V'(u_\pm^*)\widetilde{a}_2^\pm\right], \end{cases} \tag{4.3.3}$$

$$\begin{cases} c_0^{\pm}(k) := -2a_1^{\pm}(k)\left(k - V'(u_{\pm}^*)\right), \quad c_2^{\pm}(k) := \left(\frac{\kappa\theta\left(1-\gamma(u_{\pm}^*)\right)}{e_0^{\pm}(k)}\right)^{\mu}, \\ c_1^{\pm}(k) := v\mathrm{e}^{-\kappa t}\left(\frac{a_1^{\pm}(k)V'(u_{\pm}^*)}{e_0^{\pm}(k)} - \frac{\kappa\theta e_1^{\pm}(k)}{2e_0^{\pm}(k)^2\beta_t}\right) - a_2^{\pm}(k)\left(k - V'(u_{\pm}^*)\right) \\ \qquad +\frac{1}{2}a_1^{\pm}(k)^2V''(u_{\pm}^*), \end{cases} \tag{4.3.4}$$

$$\begin{cases} \widetilde{c}_0^{\pm} := \frac{3}{2}(\widetilde{a}_1^{\pm})^2V''(u_{\pm}^*), \quad \widetilde{c}_2^{\pm} := \left(\frac{\kappa\theta\left(1-\gamma(u_{\pm}^*)\right)}{\widetilde{e}_0^{\pm}}\right)^{\mu}, \quad g_0 := \frac{v\mathrm{e}^{-\kappa t}V(1)}{\kappa\theta - 2\beta_t V(1)}, \\ \widetilde{c}_1^{\pm} := v\mathrm{e}^{-\kappa t}\left(\frac{\widetilde{a}_1^{\pm}V'(u_{\pm}^*)}{\widetilde{e}_0^{\pm}} - \frac{\kappa\theta\widetilde{e}_1^{\pm}}{2(\widetilde{e}_0^{\pm})^2\beta_t}\right) + \widetilde{a}_1^{\pm}\widetilde{a}_2^{\pm}V''(u_{\pm}^*) + \frac{(\widetilde{a}_1^{\pm})^3V'''(u_{\pm}^*)}{6}, \end{cases} \tag{4.3.5}$$

$$\phi_0(k) := \frac{1}{\sqrt{2\pi V''(q^*(k))}} \begin{cases} \frac{\exp\left(H(q^*(k))\right)}{q^*(k)(q^*(k)-1)}, & \text{if } k \in \mathcal{Q}, \\ \left(-1 - \operatorname{sgn}(k)\left(\frac{V'''(q^*(k))}{6V''(q^*(k))} - H'(q^*(k))\right)\right), & \text{if } k \in \mathcal{Q}^{\mathrm{c}}, \end{cases} \tag{4.3.6}$$

where

$$\mathcal{Q} = \mathbb{R} \setminus \{V'(0), V'(1)\}, \qquad \mathcal{Q}^{\mathrm{c}} = \{V'(0), V'(1)\}. \tag{4.3.7}$$

$$\begin{cases} \phi_{\pm}(k) & := \frac{c_2^{\pm}(k)\mathrm{e}^{c_1^{\pm}(k)}}{\zeta_{\pm}(k)u_{\pm}^*(u_{\pm}^*-1)\sqrt{2\pi}}, \qquad \widetilde{\phi}_{\pm} := \frac{\widetilde{c}_2^{\pm}\mathrm{e}^{\widetilde{c}_1^{\pm}}}{u_{\pm}^*(u_{\pm}^*-1)\sqrt{6\pi V''(u_{\pm}^*)}}, \\ \phi_2(k) & := \frac{-\mathrm{e}^{g_0}}{\Gamma(1+\mu)}\left(\frac{2\mu(\kappa-\rho\xi)^2(k-V'(1))}{\kappa\theta - 2\beta_t V(1)}\right)^{\mu}, \\ \phi_1 & := \frac{-\mathrm{e}^{g_0}}{2\Gamma(1+\mu/2)}\left(\frac{\mu(\kappa-\rho\xi)^2\sqrt{2V''(1)}}{\kappa\theta - 2\beta_t V(1)}\right)^{\mu}, \end{cases} \tag{4.3.8}$$

Since $u_-^* < 0$ and $u_+^* > 1$, we always have $V'(u_+^*) > 0$ and $V'(u_-^*) < 0$. Furthermore, $V''(u_{\pm}^*) > 0$ and one can show that $\gamma(u_{\pm}^*) \neq 1$; therefore all the functions and constants in (4.3.1),(4.3.2),(4.3.3),(4.3.4) and (4.3.5) are well defined and real-valued. ϕ_0 is well defined since $V''(q^*(k)) > 0$ and ϕ_2 and the constant ϕ_1 are well defined since $\kappa\theta - 2\beta_t V(1) > 0$. Finally define the following combinations and the function $\mathcal{I} : \mathbb{R} \times \mathbb{R}_+^* \times \mathbb{R}^3 \to \mathbb{R}$:

$$\begin{array}{llllll} \mathcal{H}_0 : & \alpha = \frac{1}{2}, & \beta = 1, & \gamma = 0, & \phi \equiv \phi_0, & \psi \equiv 0, \\ \widetilde{\mathcal{H}}_{\pm} : & \alpha = \frac{\mu}{3} - \frac{1}{2}, & \beta = \frac{1}{3}, & \gamma = \frac{1}{3}, & \phi \equiv \widetilde{\phi}_{\pm}, & \psi \equiv \widetilde{c}_0^{\pm}, \\ \mathcal{H}_{\pm} : & \alpha = \frac{\mu}{2} - \frac{3}{4}, & \beta = \frac{1}{2}, & \gamma = \frac{1}{2}, & \phi \equiv \phi_{\pm}, & \psi \equiv c_0^{\pm}, \\ \mathcal{H}_1 : & \alpha = -\frac{\mu}{2}, & \beta = \frac{1}{2}, & \gamma = 0, & \phi \equiv \phi_1, & \psi \equiv 0, \\ \mathcal{H}_2 : & \alpha = -\mu, & \beta = 1, & \gamma = 0, & \phi \equiv \phi_2, & \psi \equiv 0, \end{array} \tag{4.3.9}$$

$$\mathcal{I}(k,\tau,a,b,c) := \left(1 - \mathrm{e}^{k\tau}\right)\mathbb{1}_{\{k<a\}} + \mathbb{1}_{\{a<k<b\}} + c\mathbb{1}_{\{b\le k\}} + \frac{1-c}{2}\mathbb{1}_{\{k=b\}} + \left(1 - \frac{1}{2}\mathrm{e}^{k\tau}\right)\mathbb{1}_{\{k=a\}}. \tag{4.3.10}$$

We are now in a position to state the main result of the chapter, namely an asymptotic expansion for forward-start option prices in all regimes for all (log) strikes on the real line. The proof is obtained using Lemma 4.6.3 in conjunction with the asymptotics in Lemmas 4.6.12, 4.6.14, 4.6.17 and 4.6.18.

Theorem 4.3.1. *The following expansion holds for forward-start call options for all $k \in \mathbb{R}$ as τ tends to infinity:*

$$\mathbb{E}\left(\mathrm{e}^{X_\tau^{(t)}} - \mathrm{e}^{k\tau}\right)^+ = \mathcal{I}\left(k, \tau, V'(0), V'(1), \mathbb{1}_{\{\kappa<\rho\xi\}}\right) + \frac{\phi(k,t)}{\tau^\alpha}\mathrm{e}^{-\tau(V^*(k)-k)+\psi(k,t)\tau^\gamma}\left(1+\mathcal{O}\left(\tau^{-\beta}\right)\right),$$

where the functions ϕ, ψ and the constants α, β and γ are given by the following combinations[1]*:*

- $\mathfrak{R}_1$*:* $\mathcal{H}_0$ *for* $k \in \mathbb{R}$*;*
- $\mathfrak{R}_2$*:* $\mathcal{H}_0$ *for* $k \in (-\infty, V'(u_+^*))$*;* $\widetilde{\mathcal{H}}_+$ *for* $k = V'(u_+^*)$*;* $\mathcal{H}_+$ *for* $k \in (V'(u_+^*), +\infty)$*;*
- $\mathfrak{R}_{3a}$*:* $\mathcal{H}_-$ *for* $k \in (-\infty, V'(u_-^*))$*;* $\widetilde{\mathcal{H}}_-$ *for* $k = V'(u_-^*)$*;* $\mathcal{H}_0$ *for* $k \in (V'(u_-^*), +\infty)$*;*
- $\mathfrak{R}_{3b}$*:* $\mathcal{H}_-$ *for* $k \in (-\infty, V'(u_-^*))$*;* $\widetilde{\mathcal{H}}_-$ *for* $k = V'(u_-^*)$*;* $\mathcal{H}_0$ *for* $k \in (V'(u_-^*), V'(1))$*;* $\mathcal{H}_1$ *at* $k = V'(1)$*;* $\mathcal{H}_2$ *for* $k \in (V'(1), +\infty)$*;*
- $\mathfrak{R}_4$*:* $\mathcal{H}_0$ *for* $k \in (-\infty, V'(1))$*;* $\mathcal{H}_1$ *for* $k = V'(1)$*;* $\mathcal{H}_2$ *for* $k \in (V'(1), +\infty)$*;*

In order to highlight the symmetries appearing in the asymptotics, we shall at times identify an interval with the corresponding regime and combination in force. This slight abuse of notations should not however be harmful to the comprehension.

Remark 4.3.2.

(i) Under $\mathfrak{R}_1$, asymptotics for the large-maturity forward smile (for $k \in \mathbb{R} \setminus \{V'(0), V'(1)\}$) have been derived in Proposition 2.3.5.

(ii) For $t = 0$, large-maturity asymptotics have been derived in [63, 65] under $\mathfrak{R}_1$ and partially in [97] under $\mathfrak{R}_4$.

(iii) All asymptotic expansions are given in closed-form and can in principle be extended to arbitrary order.

(iv) When $\mathcal{H}_\pm$ and $\mathcal{H}_2$ are in force then $V^*(k) - k$ is linear in k as opposed to being strictly convex as in $\mathcal{H}_0$.

(v) If $\rho \leq \kappa/\xi$ then $V^*(k) - k \geq 0$ with equality if and only if $k = V'(1)$. If $\rho > \kappa/\xi$ then $V^*(k) - k \geq -V(1) > 0$. Since $\gamma \in [0, 1)$, the leading order decay term is given by $\mathrm{e}^{-\tau(V^*(k)-k)}$.

(vi) Under $\mathcal{H}_2$ (which only occurs when $\rho > \kappa/\xi$ for log-strikes strictly greater than $V'(1)$), forward-start call option prices decay to one as τ tends to infinity. This is fundamentally different than the large-strike behaviour in other regimes and in the BSM model (1.0.1), where call option prices decay to zero. This seemingly contradictory behaviour is explained as follows: as the maturity increases there is a positive effect on the price by an increase in

[1] whenever $\mathcal{H}_0$ is in force, the case $k = V'(a)$ is excluded if $v = \theta\Upsilon(a)$, with Υ defined in (4.6.35), for $a \in \{0, 1\}$.

the time value of the option and a negative effect on the price by increasing the strike of the forward-start call option. In standard regimes and for sufficiently large strikes the strike effect is more prominent than the time value effect in the large-maturity limit. Here, because of the large correlation, this effect is opposite: as the asset price increases, the volatility tends to increase driving the asset price to potentially higher levels. This gamma or time value effect outweighs the increase in the strike of the option.

(vii) In $\mathfrak{R}_4$, the decay rate $V^*(k) - k$ has a very different behaviour: the minimum achieved at $V'(1)$ is not zero and $V^*(k) - k$ is constant for $k \geq V'(1)$. There is limited information in the leading-order behaviour and important distinctions must therefore occur in higher-order terms. This is illustrated in Figures 4.5 and 4.6 where the first-order asymptotic is vastly superior to the leading order.

(viii) It is important to note that $u^*_\pm$ and V^* depend on the forward-start date t through (2.3.6) and the regime choice. However, in the uncorrelated case $\rho = 0$, $\mathfrak{R}_1$ always applies and V^* does not depend on t. The non-stationarity of the forward smile over the spot smile (at leading order) depends critically on how far the correlation is away from zero.

In order to translate these results into forward smile asymptotics (in the next section), we require a similar expansion for the Black-Scholes model (1.0.1). Define the functions $V^*_{\mathrm{BS}} : \mathbb{R}\times\mathbb{R}^*_+ \to \mathbb{R}$ and $\phi_{\mathrm{BS}} : \mathbb{R} \times \mathbb{R}^*_+ \times \mathbb{R} \to \mathbb{R}$ by $V^*_{\mathrm{BS}}(k, a) := (k + a/2)^2 /(2a)$ and

$$\phi_{\mathrm{BS}}(k, a, b) \equiv \frac{4a^{3/2}}{(4k^2 - a^2)\sqrt{2\pi}} \exp\left(b\left(\frac{k^2}{2a^2} - \frac{1}{8}\right)\right)\mathbb{1}_{\{k\neq\pm a/2\}} + \frac{b-2}{2\sqrt{2a\pi}}\mathbb{1}_{\{k=\pm a/2\}},$$

so that the following holds (see Corollary 2.2.9 and [65, Proposition 2.7]):

Corollary 4.3.3. *Let $a > 0$, $b \in \mathbb{R}$ and set $\Sigma^2 := a + b/\tau$ for τ large enough so that $a + b/\tau > 0$. In the BSM model (1.0.1) the following expansion then holds for any $k \in \mathbb{R}$ as τ tends to infinity (the function $\mathcal{I}$ is defined in (4.3.10)):*

$$\mathbb{E}\left(\mathrm{e}^{X^{(t)}_\tau} - \mathrm{e}^{k\tau}\right)^+ = \mathcal{I}\left(k, \tau, -\frac{a}{2}, \frac{a}{2}, 0\right) + \frac{\phi_{BS}(k, a, b)}{\tau^{1/2}}\mathrm{e}^{-\tau(V^*_{BS}(k,a)-k)}\left(1 + \mathcal{O}(\tau^{-1})\right).$$

4.3.1 Connection with large deviations

Although obvious from Theorem 4.3.1, we have so far not mentioned the notion of large deviations at all. The leading-order decay of the option price as the maturity tends to infinity gives rise to estimates for large-time probabilities; more precisely, by formally differentiating both sides with respect to the log-strike, one can prove, following a completely analogous proof to Corollary 3.3.3, that

$$-\lim_{\tau\uparrow\infty} \tau^{-1} \log \mathbb{P}\left(X^{(t)}_\tau \in B\right) = \inf_{z\in B} V^*(z),$$

for any Borel subset B of the real line, namely that $(X_\tau^{(t)}/\tau)_{\tau>0}$ satisfies a large deviations principle under $\mathbb{P}$ with speed τ and good rate function V^* as τ tends to infinity. We refer the reader to Section 1.2 and the excellent monograph [48] for more details on large deviations. The theorem actually states a much stronger result here since it provides higher-order estimates, coined 'sharp large deviations' in [21]. Now, classical methods to prove large deviations, when the moment generating function is known rely on the Gärtner-Ellis theorem (Theorem 1.2.3). In mathematical finance, one can consult for instance [62, 63, 95] for the small-and large-time behaviour of stochastic volatility models, and [134] for an overview. The Gärtner-Ellis theorem requires, in particular, the limiting cumulant generating function V to be steep at the boundaries of its effective domain. This is indeed the case in Regime $\mathcal{R}_1$, but fails to hold in other regimes. The standard proof of this theorem (as detailed in [48, Chapter 2, Theorem 2.3.6]) clearly holds in the open intervals of the real line where the function V is strictly convex, encompassing basically all occurrences of $\mathcal{H}_0$. The other cases, when V becomes linear, and the turning points $V'(0)$ and $V'(1)$, however have to be handled with care and solved case by case.

4.4 Forward smile asymptotics

We now translate the forward-start option asymptotics obtained above into asymptotics of the forward implied volatility smile. Let us first define the function $\mathfrak{N}_0^\infty : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ by

$$\mathfrak{N}_0^\infty(k,t) := 2\left(2V^*(k) - k + 2\mathcal{Z}(k)\sqrt{V^*(k)(V^*(k)-k)}\right), \quad \text{for all } k \in \mathbb{R}, t \in \mathbb{R}_+ \tag{4.4.1}$$

with $\mathcal{Z} : \mathbb{R} \to \{-1, 1\}$ defined by $\mathcal{Z}(k) \equiv \mathbb{1}_{\{k\in[V'(0),V'(1)]\}} + \mathrm{sgn}(\rho\xi - \kappa)\mathbb{1}_{\{k>V'(1)\}} - \mathbb{1}_{\{k<V'(0)\}}$ and V^* given in Lemma 4.2.1. Define the following combinations:

$$\begin{array}{lllll}
\mathcal{P}_0: & \chi \equiv \chi_0, & \eta \equiv 1, & \lambda = 1, & \mathcal{R}(\tau,\lambda) = \mathcal{O}(\tau^{-2\lambda}), \\
\widetilde{\mathcal{P}}_\pm: & \chi \equiv \widetilde{c}_0^\pm, & \eta \equiv 1, & \lambda = \frac{2}{3}, & \mathcal{R}(\tau,\lambda) = o(\tau^{-\lambda}), \\
\mathcal{P}_\pm: & \chi \equiv c_0^\pm, & \eta \equiv 1, & \lambda = \frac{1}{2}, & \mathcal{R}(\tau,\lambda) = \begin{cases} o\left(\tau^{-\lambda}\right), & \text{if } \mu \neq 1/2, \\ \mathcal{O}\left(\tau^{-2\lambda}\right), & \text{if } \mu = 1/2, \end{cases} \\
\mathcal{P}_1: & \chi \equiv 0, & \eta \equiv 0, & \lambda = 0, & \mathcal{R}(\tau,\lambda) = o(1).
\end{array}$$

Here $c_0^\pm$ and $\widetilde{c}_0^\pm$ are given in (4.3.4) and (4.3.5) and $\chi_0 : \mathbb{R} \setminus \{V'(0), V'(1)\} \to \mathbb{R}$ is defined by

$$\chi_0(k,t) \equiv H(q^*(k)) + \log\left(\frac{4k^2 - \mathfrak{N}_0^\infty(k,t)^2}{4(q^*(k)-1)q^*(k)\mathfrak{N}_0^\infty(k,t)^{3/2}\sqrt{V''(q^*(k))}}\right), \tag{4.4.2}$$

with V and H given in (2.3.8) and q^* in (2.3.9). We now state the main result of the section, namely an expansion for the forward smile in all regimes and (log) strikes on the real line. The proof is given in Section 4.6.6.

Theorem 4.4.1. *The following expansion holds for the forward smile as τ tends to infinity:*

$$\sigma^2_{t,\tau}(k\tau) = \mathfrak{N}_0^\infty(k,t) + \mathfrak{N}_1^\infty(k,t)\tau^{-\lambda} + \mathcal{R}(\tau,\lambda), \qquad \text{for any } k \in \mathbb{R},$$

where $\mathfrak{N}_1^\infty : \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ is defined by ($\mathcal{Q}$ given in (4.3.7))

$$\mathfrak{N}_1^\infty(k,t) := \begin{cases} \dfrac{8\mathfrak{N}_0^\infty(k,t)^2}{4k^2 - \mathfrak{N}_0^\infty(k,t)^2}\chi(k,t), & \text{if } k \in \mathcal{Q}, \\ 2\eta(k)\left[1 - \sqrt{\dfrac{\mathfrak{N}_0^\infty(k,t)}{V''(q^*(k))}}\left(1 + \operatorname{sgn}(k)\left(\dfrac{V'''(q^*(k))}{6V''(q^*(k))} - H'(q^*(k))\right)\right)\right], & \text{if } k \in \mathcal{Q}^c, \end{cases}$$

with the functions χ, η, the remainder $\mathcal{R}$ and the constant λ given by the following combinations[2]*:*

- $\mathfrak{R}_1$*:* $\mathcal{P}_0$ *for* $k \in \mathbb{R}$*;*
- $\mathfrak{R}_2$*:* $\mathcal{P}_0$ *for* $k \in (-\infty, V'(u_+^*))$*;* $\widetilde{\mathcal{P}}_+$ *for* $k = V'(u_+^*)$*;* $\mathcal{P}_+$ *for* $k \in (V'(u_+^*), +\infty)$*;*
- $\mathfrak{R}_{3a}$*:* $\mathcal{P}_-$ *for* $k \in (-\infty, V'(u_-^*))$*;* $\widetilde{\mathcal{P}}_-$ *for* $k = V'(u_-^*)$*;* $\mathcal{P}_0$ *for* $k \in (V'(u_-^*), +\infty)$*;*
- $\mathfrak{R}_{3b}$*:* $\mathcal{P}_-$ *for* $k \in (-\infty, V'(u_-^*))$*;* $\widetilde{\mathcal{P}}_-$ *for* $k = V'(u_-^*)$*;* $\mathcal{P}_0$ *for* $k \in (V'(u_-^*), V'(1))$*;* $\mathcal{P}_1$ *for* $k \in [V'(1), +\infty)$*;*
- $\mathfrak{R}_4$*:* $\mathcal{P}_0$ *for* $k \in (-\infty, V'(1))$*;* $\mathcal{P}_1$ *for* $k \in [V'(1), +\infty)$*.*

Remark 4.4.2.

(i) In the standard spot case $t = 0$, the large-maturity asymptotics of the implied volatility smile was derived in [63, 65] for $\mathfrak{R}_1$ only (i.e. assuming $\kappa > \rho\xi$).

(ii) The zeroth-order term $\mathfrak{N}_0^\infty$ is continuous on $\mathbb{R}$ (see also section 4.4.1), which is not necessarily true for higher-order terms. In $\mathfrak{R}_2$, $\mathfrak{R}_{3a}$ and $\mathfrak{R}_{3b}$, $\mathfrak{N}_1^\infty$ tends to either infinity or zero at the critical strikes $V'(u_+^*)$ and $V'(u_-^*)$ (this is discussed further in Section 4.5). In $\mathfrak{R}_1$, $\mathfrak{N}_1^\infty$ is continuous on the whole real line.

(iii) Straightforward computations show that $0 < \mathfrak{N}_0^\infty(k) < 2|k|$ for $k \in \mathbb{R}\setminus[V'(0), V'(1)]$, and $\mathfrak{N}_0^\infty(k) > 2|k|$ for $k \in (V'(0), V'(1))$, so that $\mathfrak{N}_1^\infty$ is well defined on $\mathbb{R}\setminus\{V'(0), V'(1)\}$. On $(-\infty, V'(u_-^*)) \cup (V'(u_+^*), \infty)$, $c_0^\pm > 0$, so that in Regimes $\mathfrak{R}_2$ on $(V'(u_+^*), \infty)$ and in $\mathfrak{R}_{3b}, \mathfrak{R}_{3b}$ on $(-\infty, V'(u_-^*))$, $\mathfrak{N}_1^\infty$ is always a positive adjustment to the zero-order term $\mathfrak{N}_0^\infty$; see Figure 4.1 for an example of this ‘convexity effect’.

(iv) In the practically relevant (on Equity markets) case of large negative correlation ($\mathcal{R}_2$), the additional convexity of the right wing of the forward smile is due to extreme positive moment explosions of the forward price process. This asymmetric feature of the Heston forward smile is a fundamental property of the model—not only for large-maturities. Quoting Bergomi [23] from an empirical analysis: “...the increased convexity (of the forward smile) with respect to today’s smile is larger for $k > 0$ than for $k < 0$...this is specific to the Heston model.”

[2] whenever $\mathcal{P}_0$ is in force, the case $k = V'(a)$ is excluded if $v = \theta\Upsilon(a)$, with Υ defined in (4.6.35), for $a \in \{0, 1\}$.

Theorem 4.4.1 displays varying levels of degeneration for high-order forward smile asymptotics. In $\mathfrak{R}_1$ one can in principle obtain arbitrarily high-order asymptotics. In $\mathfrak{R}_2$, $\mathfrak{R}_{3a}$ and $\mathfrak{R}_{3b}$ (with $\mathcal{P}_+$ or $\mathcal{P}_-$ in force) one can only specify the forward smile to arbitrary order if $\mu = 1/2$ (μ defined in (1.3.4)). If this is not the case then we can only specify the forward smile to first order. Now the dynamics of the Heston volatility $\sigma_t := \sqrt{V_t}$ is given by $\mathrm{d}\sigma_t = \left(\frac{2\mu-1}{8\sigma_t}\xi^2 - \frac{\kappa\sigma_t}{2}\right)\mathrm{d}t + \frac{\xi}{2}\mathrm{d}W_t$, with $\sigma_0 = \sqrt{v}$. If $\mu = 1/2$ then the volatility becomes Gaussian, which this corresponds to a specific case of the Schöbel-Zhu stochastic volatility model (see Section 1.3.1.2). So as the Heston volatility dynamics deviate from Gaussian volatility dynamics a certain degeneracy occurs such that one cannot specify high order forward smile asymptotics. Interestingly, a similar degeneracy occurs in Chapter 3 for exploding small-maturity Heston forward smile asymptotics and in [51] when studying the tail probability of the stock price. As proved in [51], the square-root behaviour of the variance process induces some singularity and hence a fundamentally different behaviour when $\mu \neq 1/2$. In $\mathfrak{R}_2$, $\mathfrak{R}_{3a}$ and $\mathfrak{R}_{3b}$ at the boundary points $V'(u^*_\pm)$ one cannot specify the forward smile beyond first order for any parameter configurations. This could be because these asymptotic regimes are extreme in the sense that they are transition points between standard and degenerate behaviours and therefore difficult to match with BSM forward volatility. Finally in $\mathfrak{R}_{3b}$ and $\mathfrak{R}_4$ for $k > V'(1)$ we obtain the most extreme behaviour, in the sense that one cannot specify the forward smile beyond zeroth order. This is however not that surprising since the large correlation regime has fundamentally different behaviour to the BSM model (see also Remark 4.3.2(vi)(vii)).

4.4.1 SVI-type limits

The so-called 'Stochastic Volatility Inspired' (SVI) parametrisation of the spot implied volatility smile was proposed in [70]. As proved in [73], under the assumption $\kappa > \rho\xi$, the SVI parametrisation turn out to be the true large-maturity limit for the Heston (spot) smile. We now extend these results to the large-maturity forward implied volatility smile. Define the following extended SVI parametrisation

$$\sigma^2_{\mathrm{SVI}}(k, a, b, r, m, s, i_0, i_1, i_2) := a + b\left(r(k-m) + i_0\sqrt{i_1(k-m)^2 + i_2(k-m) + i_0 s^2}\right),$$

for all $k \in \mathbb{R}$ and the constants

$$\begin{cases} \omega_1 &:= \dfrac{2\mu}{1-\rho^2}\left(\sqrt{\left(2\kappa+\xi^2-\rho\xi\right)^2 + \xi^2\left(1-\rho^2\right)} - \left(2\kappa+\xi^2-\rho\xi\right)\right), \qquad \omega_2 := \dfrac{\xi}{\kappa\theta}, \\ a_\pm &:= \dfrac{\kappa\theta}{2\left(u^*_\pm-1\right)u^*_\pm\beta_t}, \quad b_\pm := 4\sqrt{\left(u^*_\pm-1\right)u^*_\pm}, \quad r_\pm := \dfrac{2(2u^*_\pm-1)}{b_\pm}, \quad m_\pm := \left(u^*_\pm - \dfrac{1}{2}\right)a_\pm, \\ \widetilde{a} &:= -2\widetilde{m}, \quad \widetilde{b} := 4\sqrt{-\widetilde{m}}, \quad \widetilde{r} := \frac{1}{2\sqrt{-\widetilde{m}}}, \quad \widetilde{m} := \mu(\kappa-\rho\xi), \end{cases}$$

where $u_\pm^*$ is defined in (2.3.6) and β_t, μ in (1.3.4). Define the following combinations:

$$
\begin{array}{lllllllll}
\mathcal{S}_0: & a = \frac{\omega_1(1-\rho)^2}{2}, & b = \frac{\omega_1\omega_2}{2}, & r = \rho, & m = -\frac{\rho}{\omega_2}, & s = \frac{\sqrt{1-\rho^2}}{\omega_2}, & i_0 = 1, & i_1 = 1, & i_2 = 0, \\
\mathcal{S}_\pm: & a = a_\pm, & b = b_\pm, & r = r_\pm, & m = m_\pm, & s = \frac{1}{8}a_\pm, & i_0 = -1, & i_1 = 1, & i_2 = 0, \\
\mathcal{S}_1: & a = \widetilde{a}, & b = \widetilde{b}, & r = \widetilde{r}, & m = \widetilde{m}, & s = 0, & i_0 = 1, & i_1 = 0, & i_2 = 1.
\end{array}
$$

The proof of the following result follows from simple manipulations of the zeroth-order forward smile in Theorem 4.4.1 using the characterisation of V^* in Lemma 4.2.1.

Corollary 4.4.3. *The pointwise continuous limit* $\lim_{\tau\uparrow\infty} \sigma_{t,\tau}^2(k\tau) = \sigma_{\mathrm{SVI}}^2(k, a, b, r, m, s, i_0, i_1, i_2)$ *exists for* $k \in \mathbb{R}$ *with constants* a, b, r, m, s, i_0, i_1 *and* i_2 *given by*[3]*:*

- $\mathfrak{R}_1$*:* $\mathcal{S}_0$ *for* $k \in \mathbb{R}$*;*
- $\mathfrak{R}_2$*:* $\mathcal{S}_0$ *for* $k \in (-\infty, V'(u_+^*))$*;* $\mathcal{S}_+$ *for* $k \in [V'(u_+^*), +\infty)$*;*
- $\mathfrak{R}_{3a}$*:* $\mathcal{S}_-$ *for* $k \in (-\infty, V'(u_-^*)]$*;* $\mathcal{S}_0$ *for* $k \in (V'(u_-^*), +\infty)$*;*
- $\mathfrak{R}_{3b}$*:* $\mathcal{S}_-$ *for* $k \in (-\infty, V'(u_-^*)]$*;* $\mathcal{S}_0$ *for* $k \in (V'(u_-^*), V'(1))$*;* $\mathcal{S}_1$ *for* $k \in [V'(1), +\infty)$*;*
- $\mathfrak{R}_4$*:* $\mathcal{S}_0$ *for* $k \in (-\infty, V'(1))$*;* $\mathcal{S}_1$ *for* $k \in [V'(1), +\infty)$*.*

It is natural to conjecture [10] that the limiting forward smile $\lim_{\tau\uparrow\infty} \sigma_{t,\tau}$ is similar to the limiting spot smile $\lim_{\tau\uparrow\infty} \sigma_{0,\tau}$. Corollary 4.4.3 shows that this only holds under $\mathfrak{R}_1$. For the practically relevant case of the asymmetric regime $\mathfrak{R}_2$ when $\rho < \rho_-$, in Figure 4.1 we compare the two limits using the zeroth-order asymptotics in Corollary 4.4.3. At the critical log-strike $V'(u_+^*)$, the forward smile becomes more convex than the corresponding spot smile. Interestingly this asymmetric feature has been empirically observed by practitioners [23] and seems to be a fundamental feature of the Heston forward smile (not just for large maturities).

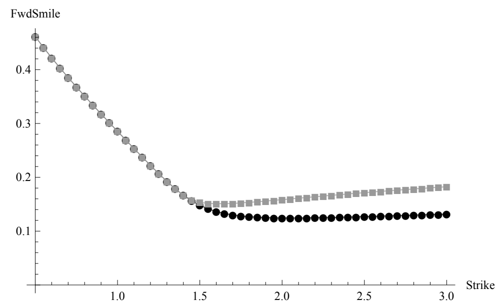


Figure 4.1: Here $t = 0.5, \tau = 2, v = \theta = 0.1, \kappa = 2, \xi = 1, \rho = -0.9$, so that $\mathfrak{R}_2$ applies. Circles correspond to the spot smile $K \mapsto \sigma_\tau(\log K)$ and squares to the forward smile $K \mapsto \sigma_{t,\tau}(\log K)$ using the zeroth-order asymptotics in Corollary 4.4.3. Here $\rho_- \approx -0.63$ and $\mathrm{e}^{2V'(u_+^*)} \approx 1.41$.

[3] whenever $\mathcal{S}_0$ is in force, the case $k = V'(a)$ is excluded if $v = \theta\Upsilon(a)$, with Υ defined in (4.6.35), for $a \in \{0, 1\}$.

4.5 Numerics

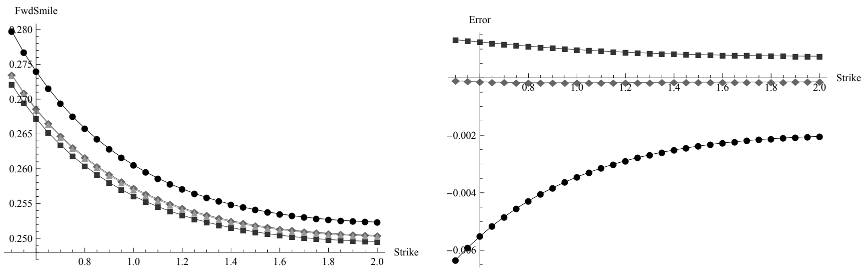
We first compare the true Heston forward smile and the asymptotics developed in the paper. We calculate forward-start option prices using the inverse Fourier transform representation in Lemma 1.4.7 and a global adaptive Gauss-Kronrod quadrature scheme. We then compute the forward smile $\sigma_{t,\tau}$ with a simple root-finding algorithm. In Figure 4.2 we compare the true forward smile using Fourier inversion and the asymptotic in Theorem 4.4.1(i) for the good correlation regime, which was derived in Proposition 2.3.5. In Figure 4.3 we compare the true forward smile using Fourier inversion and the asymptotic in Theorem 4.4.1(ii) for the asymmetric negative correlation regime. Higher-order terms are computed using the theoretical results above; these can in principle be extended to higher order, but the formulae become rather cumbersome; numerically, these higher-order computations seem to add little value to the accuracy anyway. In Figure 4.4 we compare the asymptotic in Theorem 4.4.1(ii) for the transition strike $k = V'(u_+^*)$. Results are all in line with expectations.

In the large correlation regime $\mathfrak{R}_4$, we find it more accurate to use Theorem 4.3.1 and then numerically invert the price to get the corresponding forward smile (Figures 4.5 and 4.6), rather than use the forward smile asymptotic in Theorem 4.4.1. As explained in Remark 4.3.2(vii) the leading-order accuracy of option prices in this regime is poor and higher-order terms embed important distinctions that need to be included. This also explains the poor accuracy of the forward smile asymptotic in Theorem 4.4.1 for the large correlation regime. As seen in the proof (Section 4.6.6), the leading-order behaviour of option prices is used to line up strike domains in the BSM and Heston model and then forward smile asymptotics are matched between the models. If the leading-order behaviour is poor, then regardless of the order of the forward smile asymptotic, there will always be a mismatch between the asymptotic forms and the forward smile asymptotic will be poor. Using the approach above bypasses this effect and is extremely accurate already at first order (Figures 4.5 and 4.6).

In all but $\mathfrak{R}_1$, higher-order terms can approach zero or infinity as the strike approaches the critical values ($V'(u_+^*)$ or $V'(1)$), separating the asymptotic regimes, and forward smile (and forward-start option price) asymptotics are not continuous there (apart from the zeroth-order term), see also Remark 4.4.2(ii). Numerically this implies that the asymptotic formula may break down for strikes in a region around the the critical strike. Similar features were observed in Section 3.5 where degenerate asymptotics were derived for the exploding small-maturity Heston forward smile.

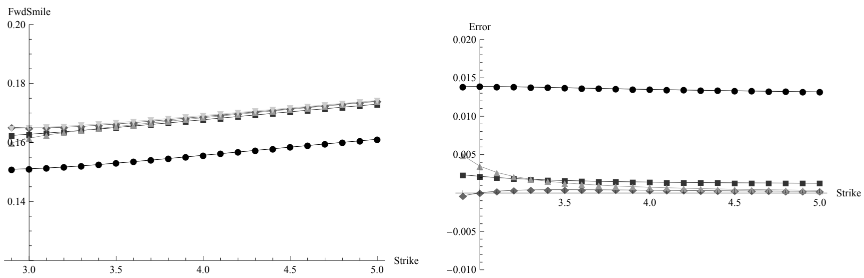
4.6 Proof of Theorems 4.3.1 and 4.4.1

This section is devoted to the proofs of the option price and implied volatility expansions in Theorems 4.3.1 and 4.4.1. We first start (Section 4.6.1) with some preliminary results on the



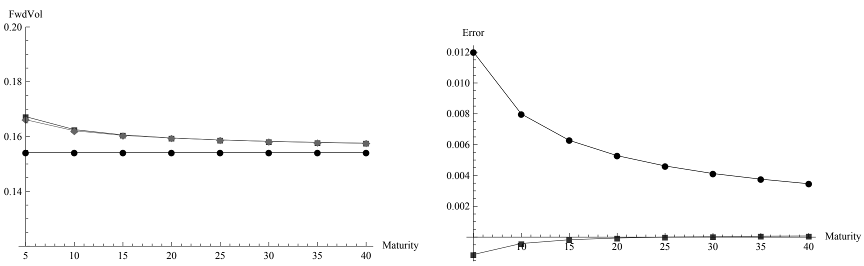
(a) Asymptotic vs Fourier Inversion. (b) Errors

Figure 4.2: **Good correlation regime $\mathfrak{R}_1$.** In (a) circles, squares and diamonds represent the zeroth-, first-and second-order asymptotics respectively and triangles represent the true forward smile. In (b) we plot the differences between the true forward smile and the asymptotic. Here $t = 1$, $\tau = 5$ and $v = 0.07$, $\theta = 0.07$, $\kappa = 1.5$, $\xi = 0.34$, $\rho = -0.25$.



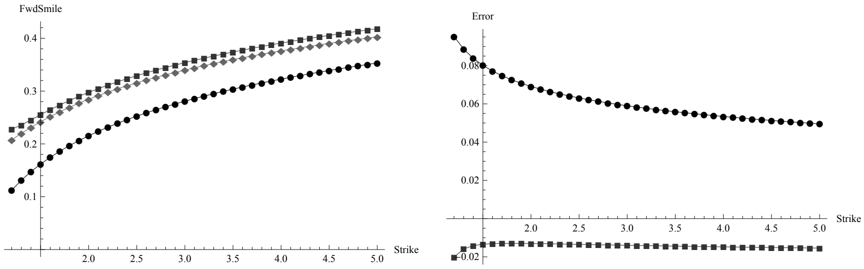
(a) Asymptotic vs Fourier inversion. (b) Errors.

Figure 4.3: **Asymmetric correlation regime $\mathfrak{R}_2$.** Here $t = 1$, $\tau = 5$ and $v = \theta = 0.07$, $\rho = -0.8$, $\xi = 0.65$ and $\kappa = 1.5$, which implies $\mathrm{e}^{V'(u_+^*)\tau} \approx 2.39$. In (a) circles, squares, diamonds and triangles represent the zeroth-, first-, second- and third-order asymptotics respectively and backwards triangles represent the true forward smile. In (b) we plot the errors.



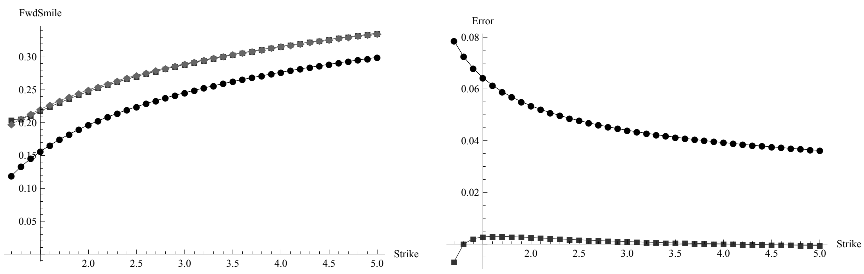
(a) Asymptotic vs Fourier inversion. (b) Errors.

Figure 4.4: **Asymmetric correlation regime $\mathfrak{R}_2$.** Here $t = 1$ and the Heston parameters are the same as in Figure 4.3. Circles and squares represent the zeroth- and first-order asymptotic and triangles represent the true forward smile. The horizontal axis is the maturity and the strike is equal $\mathrm{e}^{V'(u_+^*)\tau}$. In (b) we plot the errors.



(a) Asymptotic vs Fourier inversion. (b) Errors.

Figure 4.5: **Large correlation regime $\mathfrak{R}_4$.** Here $t = 0$, $\tau = 10$, $v = \theta = 0.07$, $\rho = 0.5$, $\xi = 0.6$, and $\kappa = 0.1$. Circles and squares represent the zeroth- and first-order asymptotic and triangles represent the true forward smile. Further $\mathrm{e}^{V'(1)\tau} \approx 1.06$.



(a) Asymptotic vs Fourier inversion. (b) Errors.

Figure 4.6: **Large correlation regime $\mathfrak{R}_4$.** Here $t = 0$, $\tau = 20$ and the Heston parameters are the same as in Figure 4.5. Circles and squares represent the zeroth- and first-order asymptotic and triangles represent the true forward smile.

behaviour of the cumulant generating function of the forward process $(X_\tau^{(t)})_{\tau>0}$, on which the proofs will rely. The remainder of the section is devoted to the different cases, as follows:

- Section 4.6.2 is the easy case, namely whenever the function V in (2.3.8) is strictly convex, corresponding to the behaviour $\mathcal{H}_0$, except at the points $V'(0)$ and $V'(1)$.
- In Section 4.6.3, we outline the general methodology we shall use in all other cases:
  - Section 4.6.4 tackles the cases $\mathcal{H}_\pm$, $\widetilde{\mathcal{H}}_\pm$ and $\mathcal{H}_2$, corresponding to the function V^* being linear;
  - Section 4.6.5 is devoted to the analysis at the points $V'(0)$ and $V'(1)$
- Section 4.6.6 translates the expansions for the option price into expansions for the forward implied volatility.

4.6.1 Forward cumulant generating function (cgf) expansion and limiting domain

For any $t \geq 0$, $\tau > 0$, define the re-normalised cgf of $X_\tau^{(t)}$ and its effective domain $\mathcal{D}_{t,\tau}$ by

$$\Lambda_\tau^{(t)}(u) := \tau^{-1} \log \mathbb{E}\left(\mathrm{e}^{uX_\tau^{(t)}}\right), \quad \text{for all } u \in \mathcal{D}_{t,\tau} := \{u \in \mathbb{R} : |\Lambda_\tau^{(t)}(u)| < \infty\}. \tag{4.6.1}$$

The Heston forward cgf was derived in Lemma 1.3.1, from which it is straightforward to determine $\Lambda_\tau^{(t)}$. We recall from Proposition 2.5.14 that for fixed $t \geq 0$, $\mathcal{D}_{t,\tau}$ converges (in the set sense) to $\mathcal{K}_{\mathrm{H}}$ defined in Table 2.1, as τ tends to infinity. Also the large-time expansion of $\Lambda_\tau^{(t)}$ was derived in Lemma 2.5.15.

4.6.2 The strictly convex case

Let $\overline{k} := \sup_{a\in\mathcal{K}_{\mathrm{H}}} V'(a)$ and $\underline{k} := \inf_{a\in\mathcal{K}_{\mathrm{H}}} V'(a)$. When $k \in (\underline{k}, \overline{k}) \setminus \{V'(0), V'(1)\}$, an analogous analysis to Theorem 2.2.4 and Propositions 2.2.11 and 2.3.5, essentially based on the strict convexity of V on $(\underline{k}, \overline{k})$, can be carried out and we immediately obtain the following results for forward-start option prices and forward implied volatilities (hence proving Theorems 4.3.1 and 4.4.1 when $\mathcal{H}_0$ holds):

Lemma 4.6.1. *The following expansions hold for all $k \in (\underline{k}, \overline{k}) \setminus \{V'(0), V'(1)\}$ as τ tends to infinity:*

$$\mathbb{E}\left(\mathrm{e}^{X_\tau^{(t)}} - \mathrm{e}^{k\tau}\right)^+ = \mathcal{I}\left(k, \tau, V'(0), V'(1), 0\right) + \frac{\phi_0(k,t)}{\tau^{1/2}} \mathrm{e}^{-\tau(V^*(k)-k)} \left(1 + \mathcal{O}\left(\tau^{-1}\right)\right),$$
$$\sigma_{t,\tau}^2(k\tau) = \mathfrak{N}_0^\infty(k,t) + \frac{8\mathfrak{N}_0^\infty(k,t)^2}{4k^2 - \mathfrak{N}_0^\infty(k,t)^2}\chi_0(k,t)\tau^{-1} + \mathcal{O}(\tau^{-2}),$$

with V^* given in Lemma 4.2.1, $\mathcal{I}$ and ϕ_0 in (4.3.10) *and* (4.3.6), $\mathfrak{N}_0^\infty$ *in* (4.4.1), χ_0 *in* (4.4.2) *and*

$$(\underline{k},\overline{k}) = \begin{cases} \mathbb{R}, & \textit{in } \mathfrak{R}_1, \\ (-\infty, V'(u_+^*)), & \textit{in } \mathfrak{R}_2, \\ (V'(u_-^*), +\infty), & \textit{in } \mathfrak{R}_{3a}, \\ (V'(u_-^*), V'(1)), & \textit{in } \mathfrak{R}_{3b}, \\ (-\infty, V'(1)), & \textit{in } \mathfrak{R}_4. \end{cases} \tag{4.6.2}$$

Proof. We sketch here a quick outline of the proof. For any $k \in (\underline{k},\overline{k})$, the equation $V'(q^*(k)) = k$ has a unique solution $q^*(k)$ by strict convexity arguments. Define the random variable $Z_{k,\tau} := (X_\tau^{(t)} - k\tau)/\sqrt{\tau}$; using Fourier transform methods analogous to Theorem 2.2.4 and Proposition 2.2.11 the option price reads, for large enough τ,

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{X_\tau^{(t)}} - \mathrm{e}^{k\tau}\right]^+ &= \mathcal{I}\left(k,\tau,V'(0),V'(1),0\right) \\ &\quad + \frac{\mathrm{e}^{-\tau(k(q^*(k)-1)-V(q^*(k)))}\mathrm{e}^{H(q^*(k))}}{2\pi}\int_{\mathbb{R}} \frac{\Phi_{\tau,k}(u)\sqrt{\tau}\mathrm{d}u}{[u-\mathtt{i}\sqrt{\tau}(q^*(k)-1)][u-\mathtt{i}\sqrt{\tau}q^*(k)]},\end{aligned}$$

where $\Phi_{\tau,k}(u) \equiv \mathbb{E}^{\widetilde{\mathbb{Q}}_{k,\tau}}(\mathrm{e}^{\mathtt{i}uZ_{k,\tau}})$ is the characteristic function of $Z_{k,\tau}$ under the new measure $\widetilde{\mathbb{Q}}_{k,\tau}$ defined by $\frac{\mathrm{d}\widetilde{\mathbb{Q}}_{k,\tau}}{\mathrm{d}\mathbb{P}} := \exp\left(q^*(k)X_\tau^{(t)} - \tau\Lambda_\tau^{(t)}(q^*(k)\right)$. Using Lemma 2.5.15, the proofs of the option price and the forward smile expansions are similar to those of Theorem 2.2.4 and Proposition 2.2.11 and Proposition 2.3.5. The exact representation of the set $(\underline{k},\overline{k})$ follows from the definition of $\mathcal{K}_{\mathrm{H}}$ in Table 2.1 and the properties of V. □

4.6.3 Other cases: general methodology

Suppose that $\overline{k}$ (defined in Section 4.6.2) is finite with $V'(\overline{u}) = \overline{k}$. We cannot define a change of measure (as in the proof of Lemma 4.6.1) by simply replacing $q^*(k) \equiv \overline{u}$ for $k \geq \overline{k}$ since the forward cgf $\Lambda_\tau^{(t)}$ explodes at these points as τ tends to infinity (see Figure 4.7). One of the objectives of the

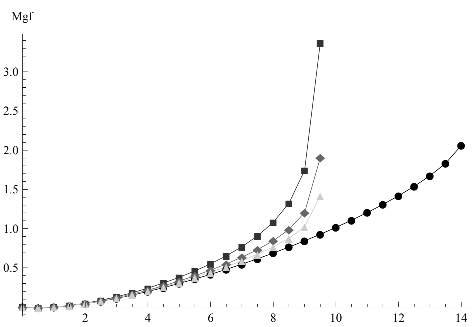


Figure 4.7: Regime $\mathfrak{R}_2$: Circles plot $u \mapsto V(u)$. Squares, diamonds and triangles plot $u \mapsto V(u) + H(u)/\tau$ with $t = 1$ and $\tau = 2, 5, 10$. Heston model parameters are $v = 0.07$, $\theta = 0.07$, $\rho = -0.8$, $\xi = 0.65$ and $\kappa = 1.5$. Also $\rho_- \approx -0.56$, $u_+^* \approx 9.72$ and $u_+ \approx 14.12$.

analysis is to understand the explosion rate of the forward cgf at these boundary points. The key observation is that just before infinity, the forward cgf $\Lambda_\tau^{(t)}$ is still steep on $\mathcal{D}_{t,\tau}^o$, and an analogous measure change to the one above can be constructed. We therefore introduce the time-dependent change of measure

$$\frac{\mathrm{d}\mathbb{Q}_{k,\tau}}{\mathrm{d}\mathbb{P}} := \exp\left(q_\tau^*(k)X_\tau^{(t)} - \tau\Lambda_\tau^{(t)}(q_\tau^*(k))\right), \tag{4.6.3}$$

where $q_\tau^*(k)$ is the unique solution to the equation $\partial_u\Lambda_\tau^{(t)}(q_\tau^*(k)) = k$ for $k \geq \overline{k}$. We shall also require that there exists $\tau_1 > 0$ such that $q_\tau^*(k) \in \mathcal{K}_{\mathrm{H}}^o$ for all $\tau > \tau_1$ and $q_\tau^* \uparrow \overline{u}$; therefore Lemma 2.5.15 holds, and we can ignore the exponential remainder ($d(u) > 0$ for all $u \in \mathcal{K}_{\mathrm{H}}^o$) so that the equation $\partial_u\Lambda_\tau^{(t)}(q_\tau^*(k)) = k$ reduces to [4]

$$V'\left(q_\tau^*(k)\right) + \tau^{-1}H'\left(q_\tau^*(k)\right) = k, \tag{4.6.4}$$

where we recall the function V and H given in (2.3.8):

$$V(u) := \frac{\mu}{2}\left(\kappa - \rho\xi u - d(u)\right) \qquad \text{and} \qquad H(u) := \frac{V(u)v\mathrm{e}^{-\kappa t}}{\kappa\theta - 2\beta_t V(u)} - \mu\log\left(\frac{\kappa\theta - 2\beta_t V(u)}{\kappa\theta\left(1-\gamma\left(u\right)\right)}\right),$$

with d, β_t, μ and γ defined in (1.3.6), (1.3.4) and (1.3.8). In the analysis below, we will also require $q_\tau^*(k)$ to solve (4.6.4) and to converge to other points in the domain (not only boundary points). This will be required to derive asymptotics under $\mathcal{H}_0$ for the strikes $V'(0)$ and $V'(1)$, where there are no moment explosion issues but rather issues with the non-existence of the limiting Fourier transform (see Section 4.6.5 for details). We therefore make the following assumption:

Assumption 4.6.2 (Large-maturity time-dependent saddlepoint)**.** There exists $\tau_1 > 0$ and a set $\mathcal{A} \subseteq \mathbb{R}$ such that for all $\tau > \tau_1$ and $k \in \mathcal{A}$, Equation (4.6.4) admits a unique solution $q_\tau^*(k)$ on $\mathcal{K}_{\mathrm{H}}^o$ satisfying $\lim_{\tau\uparrow\infty} q_\tau^*(k) =: q_\infty^* \in \overline{\mathcal{K}_{\mathrm{H}}} \cap (u_-, u_+)$.

Under this assumption $|\Lambda_\tau^{(t)}(q_\tau^*(k))|$ is finite for $\tau > \tau_1$ and $\mathcal{K}_{\mathrm{H}} = \lim_{\tau\uparrow\infty}\{u \in \mathbb{R} : |\Lambda_\tau^{(t)}(u)| < \infty\}$. Also $\mathrm{d}\mathbb{Q}_{k,\tau}/\mathrm{d}\mathbb{P}$ is almost surely strictly positive and by definition $\mathbb{E}[\mathrm{d}\mathbb{Q}_{k,\tau}/\mathrm{d}\mathbb{P}] = 1$. Therefore (4.6.3) is a valid measure change for sufficiently large τ and all $k \in \mathcal{A}$.

Our next objective is to prove weak convergence of a rescaled version of the forward price process $(X_\tau^{(t)})_{\tau>0}$ under this new measure. To this end define the random variable $Z_{\tau,k,\alpha} := (X_\tau^{(t)} - k\tau)/\tau^\alpha$ for $k \in \mathcal{A}$ and some $\alpha > 0$, with characteristic function $\Phi_{\tau,k,\alpha} : \mathbb{R} \to \mathbb{C}$ under $\mathbb{Q}_{k,\tau}$:

$$\Phi_{\tau,k,\alpha}(u) := \mathbb{E}^{\mathbb{Q}_{k,\tau}}\left(\mathrm{e}^{\mathrm{i}uZ_{\tau,k,\alpha}}\right). \tag{4.6.5}$$

Define now the functions $D : \mathbb{R}_+^* \times \mathcal{A} \to \mathbb{R}$ and $F : \mathbb{R}_+^* \times \mathcal{A} \times \mathbb{R}_+^* \to \mathbb{R}$ by

$$\begin{aligned} D(\tau,k) &:= \exp\left[-\tau\Big(k(q_\tau^*(k)-1) - V(q_\tau^*(k))\Big) + H(q_\tau^*(k))\right], \\ F(\tau,k,\alpha) &:= \frac{1}{2\pi}\int_{\mathbb{R}} \Phi_{\tau,k,\alpha}(u)\overline{C_{\tau,k,\alpha}(u)}\mathrm{d}u, \end{aligned} \tag{4.6.6}$$

[4] A similar analysis can be conducted even if $q_\tau^*(k)$ is not eventually in the interior of the limiting domain, but then one will need to use the full cgf (not just the expansion) in (4.6.4).

where $C_{\tau,k,\alpha}:\mathbb{R}\to\mathbb{C}$ is given by

$$C_{\tau,k,\alpha}(u) := \frac{\tau^\alpha}{(u+\mathtt{i}\tau^\alpha(q_\tau^*-1))(u+\mathtt{i}\tau^\alpha q_\tau^*)}, \tag{4.6.7}$$

and $\overline{C_{\tau,k,\alpha}(u)}$ denotes the complex conjugate of $C_{\tau,k,\alpha}$, namely:

$$\overline{C_{\tau,k,\alpha}(u)} = \frac{\tau^\alpha}{(u-\mathtt{i}\tau^\alpha(q_\tau^*-1))(u-\mathtt{i}\tau^\alpha q_\tau^*)}. \tag{4.6.8}$$

The main result here is an asymptotic representation for forward-start option prices:

Lemma 4.6.3. *Under Assumption 4.6.2, there exists $\beta>0$ such that for all $k\in\mathcal{A}$, as $\tau\uparrow\infty$:*

$$\mathbb{E}\left(\mathrm{e}^{X_\tau^{(t)}}-\mathrm{e}^{k\tau}\right)^+ = \begin{cases} D(\tau,k)F(\tau,k,\alpha)\left(1+\mathcal{O}(\mathrm{e}^{-\beta\tau})\right), & \text{if } q_\tau^*(k)>1,\\ (1-\mathrm{e}^{k\tau})+D(\tau,k)F(\tau,k,\alpha)\left(1+\mathcal{O}(\mathrm{e}^{-\beta\tau})\right), & \text{if } q_\tau^*(k)<0,\\ 1+D(\tau,k)F(\tau,k,\alpha)\left(1+\mathcal{O}(\mathrm{e}^{-\beta\tau})\right), & \text{if } 0<q_\tau^*(k)<1. \end{cases} \tag{4.6.9}$$

The proof of Lemma 4.6.3 relies on the inverse Fourier representation given in Lemma 4.6.5 below. In order to prove this representation we first need to show that the integrand in the right-hand side of Equality (4.6.11) belongs to $L^1(\mathbb{R})$ (and hence the integral is well defined), which is the purpose of the following lemma:

Lemma 4.6.4. *There exists $\tau_0^*>0$ such that $\int_{\mathbb{R}}|\Phi_{\tau,k,\alpha}(u)\overline{C_{\tau,k,\alpha}(u)}|\mathrm{d}u<\infty$ for all $\tau>\tau_0^*$, $k\in\mathcal{A}$, $q_\tau^*(k)\notin\{0,1\}$.*

Proof. We compute:

$$\begin{aligned}\int_{\mathbb{R}}\left|\Phi_{\tau,k}(u)\overline{C_{\tau,k,\alpha}(u)}\right|\mathrm{d}u &= \int_{|u|\le\tau^\alpha}\left|\Phi_{\tau,k,\alpha}(u)\overline{C_{\tau,k,\alpha}(u)}\right|\mathrm{d}u+\int_{|u|>\tau^\alpha}\left|\Phi_{\tau,k,\alpha}(u)\overline{C_{\tau,k,\alpha}(u)}\right|\mathrm{d}u\\ &\le \frac{\tau^{-\alpha}}{|q_\tau^*(k)(q_\tau^*(k)-1)|}\int_{|u|\le\tau^\alpha}|\Phi_{\tau,k,\alpha}(u)|\,\mathrm{d}u+\int_{|u|>1}\frac{\mathrm{d}u}{u^2},\end{aligned} \tag{4.6.10}$$

where the inequality follows from the simple bounds

$$\left|\overline{C_{\tau,k,\alpha}(u)}\right|\le\frac{\tau^{-\alpha}}{|q_\tau^*(k)(q_\tau^*(k)-1)|},\ \text{for all } |u|\le\tau^\alpha \quad\text{and}\quad \left|\overline{C_{\tau,k,\alpha}(u)}\right|\le\frac{\tau^\alpha}{u^2}.$$

Finally (4.6.10) is finite since $q_\tau^*(k)\ne1$, $q_\tau^*(k)\ne0$ and $|\Phi_{\tau,k,\alpha}|\le1$. □

We denote the convolution of two functions $f,h\in L^1(\mathbb{R})$ by $(f*g)(x):=\int_{\mathbb{R}}f(x-y)g(y)\mathrm{d}y$, and recall that $(f*g)\in L^1(\mathbb{R})$. For such functions, we denote the Fourier transform by $(\mathcal{F}f)(u):=\int_{\mathbb{R}}\mathrm{e}^{\mathtt{i}ux}f(x)\mathrm{d}x$ and the inverse Fourier transform by $(\mathcal{F}^{-1}h)(x):=\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{e}^{-\mathtt{i}ux}h(u)\mathrm{d}u$. For $j=1,2,3$, let us define the functions $g_j:\mathbb{R}_+^2\to\mathbb{R}_+$ by

$$g_j(x,y):=\begin{cases}(x-y)^+, & \text{if } j=1,\\ (y-x)^+, & \text{if } j=2,\\ \min(x,y), & \text{if } j=3.\end{cases}$$

and define $\widetilde{g}_j:\mathbb{R}\to\mathbb{R}_+$ by $\widetilde{g}_j(z):=\exp\left(-q_\tau^*(k)z\tau^\alpha\right)g_j(\mathrm{e}^{z\tau^\alpha},1)$. Recall the $\mathbb{Q}_{k,\tau}$-measure defined in (4.6.3) and the random variable $Z_{k,\tau,\alpha}$ defined on page 129. We now have the following result:

Lemma 4.6.5. *There exists $\tau_1^* > 0$ such that for all $k \in \mathcal{A}$ and $\tau > \tau_1^*$:*

$$\mathbb{E}^{\mathbb{Q}_{k,\tau}}\left[\widetilde{g}_j(Z_{k,\tau,\alpha})\right] = \begin{cases} \frac{1}{2\pi}\int_{\mathbb{R}} \Phi_{\tau,k,\alpha}(u)\overline{C_{\tau,k,\alpha}(u)}\mathrm{d}u, & \text{if } j=1, q_\tau^*(k) > 1, \\ \frac{1}{2\pi}\int_{\mathbb{R}} \Phi_{\tau,k,\alpha}(u)\overline{C_{\tau,k,\alpha}(u)}\mathrm{d}u, & \text{if } j=2, q_\tau^*(k) < 0, \\ -\frac{1}{2\pi}\int_{\mathbb{R}} \Phi_{\tau,k,\alpha}(u)\overline{C_{\tau,k,\alpha}(u)}\mathrm{d}u, & \text{if } j=3, 0 < q_\tau^*(k) < 1. \end{cases} \tag{4.6.11}$$

Proof. Assuming (for now) that $\widetilde{g}_j \in L^1(\mathbb{R})$, we have for any $u \in \mathbb{R}$,

$$\left(\mathcal{F}\widetilde{g}_j\right)(u) := \int_{\mathbb{R}} \widetilde{g}_j(z)\mathrm{e}^{\mathrm{i}uz}\mathrm{d}z,$$

for $j = 1, 2, 3$. For $j = 1$ we can write

$$\int_0^\infty \mathrm{e}^{-q_\tau^* z\tau^\alpha}\left(\mathrm{e}^{z\tau^\alpha} - 1\right)\mathrm{e}^{\mathrm{i}uz}\mathrm{d}z = \left[\frac{\mathrm{e}^{z(\mathrm{i}u - q_\tau^*\tau^\alpha + \tau^\alpha)}}{(\mathrm{i}u - q_\tau^*\tau^\alpha + \tau^\alpha)}\right]_0^\infty - \left[\frac{\mathrm{e}^{z(\mathrm{i}u - q_\tau^*\tau^\alpha)}}{(\mathrm{i}u - q_\tau^*\tau^\alpha)}\right]_0^\infty = C_{\tau,k,\alpha}(u),$$

which is valid for $q_\tau^*(k) > 1$ with $C_{\tau,k,\alpha}$ in (4.6.7). For $j = 2$ we can write

$$\int_{-\infty}^0 \mathrm{e}^{-q_\tau^* z\tau^\alpha}\left(1 - \mathrm{e}^{z\tau^\alpha}\right)\mathrm{e}^{\mathrm{i}uz}\mathrm{d}z = \left[\frac{\mathrm{e}^{z(\mathrm{i}u - q_\tau^*\tau^\alpha)}}{(\mathrm{i}u - q_\tau^*\tau^\alpha)}\right]_{-\infty}^0 - \left[\frac{\mathrm{e}^{z(\mathrm{i}u - q_\tau^*\tau^\alpha + \tau^\alpha)}}{(\mathrm{i}u - q_\tau^*\tau^\alpha + \tau^\alpha)}\right]_{-\infty}^0 = C_{\tau,k,\alpha}(u),$$

which is valid for $q_\tau^*(k) < 0$. Finally, for $j = 3$ we have

$$\begin{aligned} \int_{\mathbb{R}} \mathrm{e}^{-q_\tau^* z\tau^\alpha}\left(\mathrm{e}^{z\tau^\alpha} \wedge 1\right)\mathrm{e}^{\mathrm{i}uz}\mathrm{d}z &= \int_{-\infty}^0 \mathrm{e}^{-q_\tau^* z\tau^\alpha}\mathrm{e}^{z\tau^\alpha}\mathrm{e}^{\mathrm{i}uz}\mathrm{d}z + \int_0^\infty \mathrm{e}^{-q_\tau^* z\tau^\alpha}\mathrm{e}^{\mathrm{i}uz}\mathrm{d}z \\ &= \left[\frac{\mathrm{e}^{z(\mathrm{i}u - q_\tau^*\tau^\alpha + \tau^\alpha)}}{(\mathrm{i}u - q_\tau^*\tau^\alpha + \tau^\alpha)}\right]_{-\infty}^0 + \left[\frac{\mathrm{e}^{z(\mathrm{i}u - q_\tau^*\tau^\alpha)}}{(\mathrm{i}u - q_\tau^*\tau^\alpha)}\right]_0^\infty = -C_{\tau,k,\alpha}(u), \end{aligned}$$

which is valid for $0 < q_\tau^*(k) < 1$. From the definition of the $\mathbb{Q}_{k,\tau}$-measure in (4.6.3) and the random variable $Z_{k,\tau,\alpha}$ on page 129 we have

$$\mathbb{E}^{\mathbb{Q}_{k,\tau}}\left[\widetilde{g}_j(Z_{\tau,k,\alpha})\right] = \int_{\mathbb{R}} r_j(k\tau^{1-\alpha} - y)p(y)\mathrm{d}y = (r_j * p)(k\tau^{1-\alpha}),$$

with $r_j(z) \equiv \widetilde{g}_j(-z)$ and p denoting the density of $X_\tau^{(t)}\tau^{-\alpha}$. On the strips of regularity derived above we know there exists $\tau_0 > 0$ such that $r_j \in L^1(\mathbb{R})$ for $\tau > \tau_0$. Since p is a density, $p \in L^1(\mathbb{R})$, and therefore

$$\mathcal{F}(r_j * p)(u) = \mathcal{F}r_j(u)\mathcal{F}p(u). \tag{4.6.12}$$

We note that $\mathcal{F}r_j(u) \equiv \mathcal{F}\widetilde{g}_j(-u) \equiv \overline{\mathcal{F}\widetilde{g}_j(u)}$ and hence

$$\mathcal{F}r_j(u)\mathcal{F}p(u) \equiv \mathrm{e}^{\mathrm{i}uk\tau^{1-\alpha}}\Phi_{\tau,k,\alpha}(u)\overline{C_{\tau,k,\alpha}(u)}. \tag{4.6.13}$$

Thus by Lemma 4.6.4 there exists $\tau_1 > 0$ such that $\mathcal{F}r_j\mathcal{F}p \in L^1(\mathbb{R})$ for $\tau > \tau_1$. By the inversion theorem [137, Theorem 9.11] this then implies from (4.6.12) and (4.6.13) that for $\tau > \max(\tau_0, \tau_1)$:

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}_{k,\varepsilon}}\left[\widetilde{g}_j(Z_{\tau,k,\alpha})\right] &= (r_j * p)(k\tau^{1-\alpha}) = \mathcal{F}^{-1}\left(\mathcal{F}r_j(u)\mathcal{F}p(u)\right)(k\tau^{1-\alpha}) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}uk\tau^{1-\alpha}}\mathcal{F}r_j(u)\mathcal{F}p(u)\mathrm{d}u = \frac{1}{2\pi}\int_{\mathbb{R}} \Phi_{\tau,k,\alpha}(u)\overline{C_{\tau,k,\alpha}(u)}\mathrm{d}u. \end{aligned}$$

□

We now move onto the proof of Lemma 4.6.3. We use our time-dependent change of measure defined in (4.6.3) to write our forward-start option price for $j = 1, 2, 3$ as

$$\mathbb{E}\left(g_j(\mathrm{e}^{X_\tau^{(t)}}, \mathrm{e}^{k\tau})\right) = \mathrm{e}^{-\tau\left[kq_\tau^*(k) - \Lambda_\tau^{(t)}(q_\tau^*(k))\right]}\mathrm{e}^{k\tau}\mathbb{E}^{\mathbb{Q}_{k,\tau}}\left[\widetilde{g}_j(Z_{\tau,k,\alpha})\right],$$

with $Z_{\tau,k,\alpha}$ defined on page 129. We now apply Lemma 4.6.5 and then convert to forward-start call option prices using Put-Call parity and that in the Heston model $(\mathrm{e}^{X_t})_{t\geq 0}$ is a true martingale [5, Proposition 2.5]. Finally the expansion for $\exp\left(-\tau\left(k(q_\tau^*(k)-1) - \Lambda_\tau^{(t)}(q_\tau^*(k))\right)\right)$ follows from Lemma 2.5.15.

Finally, to end the section, we shall also need the following result on the behaviour of the characteristic function of $Z_{\tau,k,\alpha}$:

Lemma 4.6.6. *Under Assumption 4.6.2 there exists $\beta > 0$ such that for any $k \in \mathcal{A}$ as $\tau \uparrow \infty$:*

$$\Phi_{\tau,k,\alpha}(u) = \mathrm{e}^{-\mathtt{i}uk\tau^{1-\alpha} + \tau\left(V\left(\mathtt{i}u\tau^{-\alpha}+q_\tau^*\right) - V(q_\tau^*)\right) + H\left(\mathtt{i}u\tau^{-\alpha}+q_\tau^*\right) - H(q_\tau^*)}\left(1 + \mathcal{O}(\mathrm{e}^{-\beta\tau})\right),$$

where the remainder is uniform in u.

Proof. Fix $k \in \mathcal{A}$. Analogous arguments to Appendix A yield that $\Re d\left(\mathtt{i}u\tau^{-\alpha} + a\right) > d(a)$ for any $a \in \mathcal{K}_{\mathrm{H}}^o$. Assumption 4.6.2 implies that for all $\tau > \tau_1$, $\Re d\left(\mathtt{i}u\tau^{-\alpha} + q_\tau^*(k)\right) > d(q_\tau^*(k))$. It also implies that $q_\infty^* < u_+$, and hence there exists $\delta > 0$ and $\tau_2 > 0$ such that $q_\tau^*(k) < u_+ - \delta$ for all $\tau > \tau_2$. Now, since d is strictly positive and concave on (u_-, u_+) and $d(u_-) = d(u_+) = 0$, we obtain $d(q_\tau^*(k)) > d(u_+ - \delta) > 0$. This implies that the quantities $\mathcal{O}\left(\exp\left[-d\left(\frac{\mathtt{i}u}{\tau^\alpha} + q_\tau^*(k)\right)\right)\tau\right]\right)$ and $\mathcal{O}\left(\mathrm{e}^{-d(q_\tau^*(k))\tau}\right)$ are all equal to $\mathcal{O}\left(\mathrm{e}^{-d(u_+-\delta)\tau}\right)$ for all $k \in \mathcal{A}$. Using the definition of $Z_{\tau,k,\alpha}$, the change of measure (4.6.3) and Lemma 2.5.15, we can write

$$\begin{aligned}
\log\Phi_{\tau,k,\alpha}(u) &= \log\mathbb{E}^{\mathbb{Q}_{k,\tau}}\left[\mathrm{e}^{\mathtt{i}uZ_{\tau,k,\alpha}}\right] = \log\mathbb{E}\left[\exp\left(q_\tau^*X_\tau - \tau\Lambda_\tau^{(t)}\left(q_\tau^*\right) + \frac{\mathtt{i}u}{\tau^\alpha}\left(X_\tau - k\tau\right)\right)\right] \\
&= -\mathtt{i}uk\tau^{1-\alpha} + \tau\left(\Lambda_\tau^{(t)}\left(\mathtt{i}u/\tau^\alpha + q_\tau^*\right) - \Lambda_\tau^{(t)}\left(q_\tau^*\right)\right) \\
&= -\frac{\mathtt{i}uk}{\tau^{\alpha-1}} + \tau\left[V\left(\frac{\mathtt{i}u}{\tau^\alpha} + q_\tau^*\right) - V(q_\tau^*)\right] + H\left(\frac{\mathtt{i}u}{\tau^\alpha} + q_\tau^*\right) - H\left(q_\tau^*\right) \\
&\quad + \mathcal{O}\left[\mathrm{e}^{-d\left(\mathtt{i}u\tau^{-\alpha}+q_\tau^*\right)\tau}\right] - \mathcal{O}\left(\mathrm{e}^{-d(q_\tau^*)\tau}\right) \\
&= -\mathtt{i}uk\tau^{1-\alpha} + \tau\left(V\left(\mathtt{i}u/\tau^\alpha + q_\tau^*\right) - V\left(q_\tau^*\right)\right) + H\left(\mathtt{i}u/\tau^\alpha + q_\tau^*\right) - H\left(q_\tau^*\right) \\
&\quad + \mathcal{O}\left(\mathrm{e}^{-d(u_+-\delta)\tau}\right).
\end{aligned}$$

Since $d(u_+ - \delta) > 0$ the remainder tends to zero exponentially fast as τ tends to infinity. The uniformity of the remainder follows from tedious, yet non-technical, computations showing that the absolute value of the difference between $\Phi_{\tau,k,\alpha}(u)$ and its approximation is bounded by a constant independent of u as τ tends to infinity (see Appendix C). □

4.6.4 Asymptotics in the case of extreme limiting moment explosions

We consider now the cases $\mathcal{H}_\pm$, $\widetilde{\mathcal{H}}_\pm$ and $\mathcal{H}_2$, corresponding to the rate function V^* being linear.

Lemma 4.6.7. *Assumption 4.6.2 is verified in the following cases:*

(i) $\mathfrak{R}_2$ *with* $\mathcal{A} = [V'(u_+^*), \infty)$ *and* $q_\infty^* = u_+^*$;

(ii) $\mathfrak{R}_{3a}$ *and* $\mathfrak{R}_{3b}$ *with* $\mathcal{A} = (-\infty, V'(u_-^*)]$ *and* $q_\infty^* = u_-^*$.

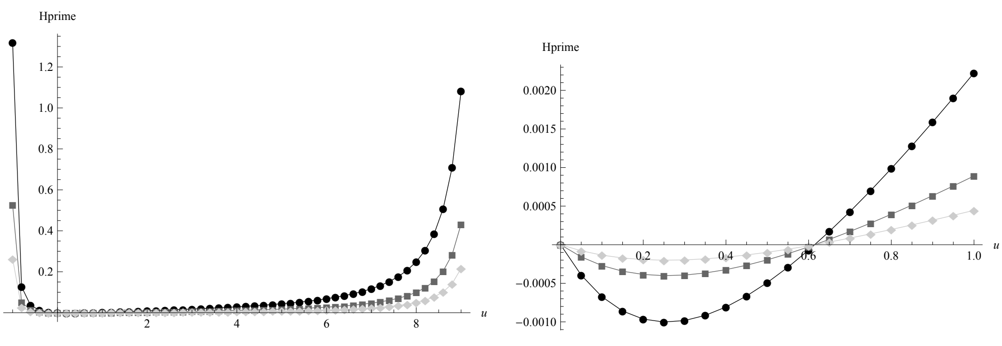
(iii) $\mathfrak{R}_{3b}$ *and* $\mathfrak{R}_4$ *with* $\mathcal{A} = (V'(1), \infty]$ *and* $q_\infty^* = 1$.

Proof. Consider Case (i) and re-write (4.6.4) as $H'(q_\tau^*(k))/\tau = k - V'(q_\tau^*(k))$. Let $k \geq V'(u_+^*)$; since V is strictly convex on (u_-, u_+), we have $H'(q_\tau^*(k))/\tau = k - V'(q_\tau^*(k)) \geq V'(u_+^*) - V'(q_\tau^*(k)) > 0$. We now show that H' has the necessary properties to prove the lemma. The following statements can be proven in a tedious yet straightforward manner (Figure 4.8 provides a visual help):

(i) There exists a $\bar{u} \in (0, u_+^*)$ such that $H'(\bar{u}) = 0$;

(ii) $H' : (\bar{u}, u_+^*) \to \mathbb{R}$ is strictly increasing and tends to infinity at u_+^*.

Therefore (i) and (ii) imply that a unique solution to (4.6.4) exists satisfying the conditions of the lemma with $q_\tau^*(k) \in (\bar{u}, u_+^*)$. Let $\tau_2 > \tau_1$ and suppose that $q_{\tau_2}^* < q_{\tau_1}^*$. Since the function H' is strictly increasing and positive on $(\bar{u}, u_+^*)$ this implies that $H'(q_{\tau_2}^*)/\tau_2 < H'(q_{\tau_1}^*)/\tau_2$ and using (4.6.4) we see that $k - V'(q_{\tau_2}^*) < \tau_1(k - V'(q_{\tau_1}^*))/\tau_2 < k - V'(q_{\tau_1}^*)$ and so $V'(q_{\tau_1}^*) < V'(q_{\tau_2}^*)$. The strict convexity of V then implies that $q_{\tau_1}^* < q_{\tau_2}^*$ which contradicts our assumption that $q_{\tau_2}^* < q_{\tau_1}^*$. Hence $q_\tau^*(k)$ is strictly increasing and bounded above by u_+^*, and therefore converges to a limit $L \in [\bar{u}, u_+^*]$. If $L \in [\bar{u}, u_+^*)$, then the continuity of V' and H' and the strict convexity of V implies that $\lim_{\tau\uparrow\infty} V'(q_\tau^*(k)) + H'(q_\tau^*(k))/\tau = V'(L) < V'(u_+^*) \leq k$, which is a contradiction. Therefore $L = u_+^*$, which proves Case (i). Cases (ii) and (iii) are analogous, and the lemma follows. ☐



(a) (b)

Figure 4.8: Plot of $u \mapsto H'(u)/\tau$ for different values of τ. Circles, Squares and diamonds represent $\tau = 2, 5, 10$. In (a) $u \in (-1.05, 9.72)$ and in (b) $u \in (0, 1)$. The Heston parameters are $v = 0.07$, $\theta = 0.07$, $\rho = -0.8$, $\xi = 0.65$ and $\kappa = 1.5$. Also $t = 1$, $\rho_- = -0.56$, $u_+^* = 9.72$ and $u_- = -1.05$.

In the following lemma we derive an asymptotic expansion for $q_\tau^*(k)$. This key result will allow us to derive asymptotics for the characteristic function $\Phi_{\tau,k,\alpha}$ as well as other auxiliary quantities needed in the analysis.

Lemma 4.6.8. *The following expansions hold for $q_\tau^*(k)$ as τ tends to infinity (μ defined in (1.3.4)):*

(i) In Regimes $\mathfrak{R}_2$, $\mathfrak{R}_{3a}$ and $\mathfrak{R}_{3b}$,

(a) under $\mathcal{H}_\pm$: $q_\tau^*(k) = u_\pm^* + a_1^\pm(k)\tau^{-1/2} + a_2^\pm(k)\tau^{-1} + \mathcal{O}\left(\tau^{-3/2}\right)$;

(b) under $\widetilde{\mathcal{H}}_\pm$: $q_\tau^*(k) = u_\pm^* + \widetilde{a}_1^\pm\tau^{-1/3} + \widetilde{a}_2^\pm\tau^{-2/3} + \mathcal{O}\left(\tau^{-1}\right)$;

(ii) In Regimes $\mathfrak{R}_{3b}$ and $\mathfrak{R}_4$,

(a) For $k > V'(1)$: $q_\tau^*(k) = 1 - \frac{\mu}{(k-V'(1))}\tau^{-1} + \mathcal{O}(\tau^{-2})$;

(b) For $k = V'(1)$: $q_\tau^*(k) = 1 - \tau^{-1/2}\sqrt{\frac{\mu}{V''(1)}} + \mathcal{O}\left(\tau^{-1}\right)$,

with $a_1^\pm$, $a_2^\pm$ and $a_3^\pm$ defined in (4.3.1) and $u_\pm^*$ in (2.3.6).

Proof. Consider Regime $\mathfrak{R}_2$ when $\mathcal{H}_+$ is in force, i.e. $k > V'(u_+^*)$, and fix such a k. Existence and uniqueness was proved in Lemma 4.6.7 and so we assume the result as an ansatz. This implies the following asymptotics as τ tends to infinity:

$$\begin{cases} V(q_\tau^*(k)) &= V(u_+^*) + \dfrac{a_1 V'(u_+^*)}{\sqrt{\tau}} + \left(\dfrac{a_1^2 V''(u_+^*)}{2} + a_2 V'(u_+^*)\right)\dfrac{1}{\tau} + \mathcal{O}\left(\dfrac{1}{\tau^{3/2}}\right), \\ V'(q_\tau^*(k)) &= V'(u_+^*) + \dfrac{a_1 V''(u_+^*)}{\sqrt{\tau}} + \left(\dfrac{a_1^2 V'''(u_+^*)}{2} + a_2 V''(u_+^*)\right)\dfrac{1}{\tau} + \mathcal{O}\left(\dfrac{1}{\tau^{3/2}}\right), \\ \gamma(q_\tau^*(k)) &= \gamma(u_+^*) + \dfrac{a_1 \gamma'(u_+^*)}{\sqrt{\tau}} + \left(\dfrac{a_1^2 \gamma''(u_+^*)}{2} + a_2 \gamma'(u_+^*)\right)\dfrac{1}{\tau} + \mathcal{O}\left(\dfrac{1}{\tau^{3/2}}\right), \\ \gamma'(q_\tau^*(k)) &= \gamma'(u_+^*) + \dfrac{a_1 \gamma''(u_+^*)}{\sqrt{\tau}} + \left(\dfrac{a_1^2 \gamma'''(u_+^*)}{2} + a_2 \gamma''(u_+^*)\right)\dfrac{1}{\tau} + \mathcal{O}\left(\dfrac{1}{\tau^{3/2}}\right). \end{cases} \tag{4.6.14}$$

We substitute this into (4.6.4) and solve at each order. At the $\tau^{-1/2}$ order we obtain

$$a_1^+(k) = \pm\frac{\mathrm{e}^{-\kappa t/2}}{2\beta_t}\sqrt{\frac{\kappa\theta v}{V'(u_+^*)\left(k - V'(u_+^*)\right)}},$$

which is well defined since $k - V'(u_+^*) > 0$ and $V'(u_+^*) > 0$. We choose the negative root since we require $q_\tau^* \in (0, u_+^*) \subset \mathcal{K}_{\mathrm{H}}^o$ for τ large enough. In a tedious yet straightforward manner we continue the procedure and iteratively solve at each order (the next equation is linear in a_2) to derive the asymptotic expansion in the lemma. The other cases follow from analogous arguments.

To complete the proof (and make the ansatz approach above rigorous) we need to show the existence of this expansion for $q_\tau^*(k)$. Fix $k > V'(u_+^*)$ and set $f_k(u,\tau) := V'(u) + H'(u)/\tau - k$. Now let $\bar{\tau} > 0$. From Lemma 4.6.7 we know that there exists a solution $q_{\bar{\tau}}^*(k)$ to the equation $f_k(q_{\bar{\tau}}^*(k), \bar{\tau}) = 0$ and the strict convexity of $V + H/\tau$ implies $\partial_u f_k(q_{\bar{\tau}}^*(k), \bar{\tau}) > 0$. Further, the two-dimensional map $f_k : \mathcal{K}_{\mathrm{H}}^o \times \mathbb{R}_+^* \to \mathbb{R}$ is analytic. It follows by the Implicit Function Theorem [108, Theorem 8.6, Chapter 0] that $\tau \mapsto q_\tau^*(k)$ is analytic in some neighbourhood around $\bar{\tau}$. Since this argument holds for all $\bar{\tau} > 0$, this function is also analytic on $\mathbb{R}_+^*$. Also from Lemma 4.6.7 we know that $\lim_{\tau \nearrow \infty} q_\tau^*(k) = u_+^*$. Since we computed the Taylor series expansion consistent with this limit and the expansion is unique, it follows that $q_\tau^*(k)$ admits this representation. □

We now derive asymptotic expansions for $\Phi_{\tau,k,\alpha}$. These expansions will be used in the next section to derive asymptotics for the function F in (4.6.6).

Lemma 4.6.9. *The following expansions hold as τ tends to infinity:*

(i) In Regimes $\mathfrak{R}_2$, $\mathfrak{R}_{3a}$ and $\mathfrak{R}_{3b}$,

(a) under $\mathcal{H}_\pm$: $\Phi_{\tau,k,3/4}(u) = \mathrm{e}^{-\zeta_\pm^2(k)u^2/2}\left(1 + \mathcal{O}\left(\tau^{-1/4}\right)\right)$;

(b) under $\widetilde{\mathcal{H}}_\pm$: $\Phi_{\tau,k,1/2}(u) = \mathrm{e}^{-3V''(u_\pm^*)u^2/2}\left(1 + \mathcal{O}\left(\tau^{-1/6}\right)\right)$;

(ii) In Regimes $\mathfrak{R}_{3b}$ and $\mathfrak{R}_4$,

(a) For $k > V'(1)$: $\Phi_{\tau,k,1}(u) = \mathrm{e}^{-\mathtt{i}u(k-V'(1)) - \frac{u^2 V''(1)}{2\tau}}\left(1 - \mathtt{i}u\frac{(k-V'(1))}{\mu}\right)^{-\mu}(1 + \mathcal{O}(\tau^{-1}))$;

(b) For $k = V'(1)$: $\Phi_{\tau,k,1/2}(u) = \mathrm{e}^{-\mathtt{i}u\sqrt{\mu V''(1)} - \frac{u^2 V''(1)}{2}}\left(1 - \mathtt{i}u\sqrt{\frac{V''(1)}{\mu}}\right)^{-\mu}(1 + \mathcal{O}(\tau^{-1/2}))$,

with $\Phi_{\tau,k,\alpha}$ defined in (4.6.5), *$\zeta_\pm^2$ in* (4.3.2) *and μ in* (1.3.4).

Remark 4.6.10.

(i) In Case (i)(a), $Z_{\tau,k,3/4}$ converges weakly to a centred Gaussian with variance $\zeta_\pm^2(k)$ when $\mathcal{H}_\pm$ holds.

(ii) In Case (i)(b), $Z_{\tau,k,1/2}$ converges weakly a centred Gaussian with variance $3V''(u_+)$ when $\widetilde{\mathcal{H}}_\pm$ holds.

(iii) In Case(ii)(a), $Z_{\tau,k,1}$ converges weakly to the zero-mean random variable $\Xi - \gamma$, where $\gamma := k - V'(1)$ and Ξ is a Gamma random variable with shape parameter μ and scale parameter $\beta := (k - V'(1))/\mu$. Note here that we specify the asymptotics with the Gaussian part $\exp(-u^2 V''(1)/(2\tau)$ so that we can apply Lemma 4.6.13 to compute large-maturity integral asymptotics later in the section.

(iv) In Case(ii)(b), $Z_{\tau,k,1/2}$ converges weakly to the zero-mean random variable $\Psi + \Xi$, where Ψ is Gaussian with mean $-\sqrt{\mu V''(1)}$ and variance $V''(1)$ and Ξ is Gamma-distributed with shape μ and scale $\sqrt{V''(1)/\mu}$.

We now prove Case (i)(a) in Regime $\mathfrak{R}_2$, as the proofs in all other cases are similar. In the forthcoming analysis we will be interested in the asymptotics of the function e_τ defined by

$$e_\tau(k) \equiv \sqrt{\tau}\left(\kappa\theta - 2\beta_t V(q_\tau^*(k))\right). \tag{4.6.15}$$

Under $\mathfrak{R}_2$, in Case (i)(a), $(\kappa\theta - 2\beta_t V(q_\tau^*))$ tends to zero as τ tends to infinity, so that it is not immediately clear what happens to e_τ for large τ. But the asymptotic behaviour of $V(q_\tau^*)$ in (4.6.14) and the definition (4.6.15) yield the following result:

Lemma 4.6.11. *Assume $\mathfrak{R}_2$ and $\mathcal{H}_+$. Then the expansion $e_\tau(k) = e_0^+(k) + e_1^+(k)\tau^{-1/2} + \mathcal{O}\left(\tau^{-1}\right)$ holds as τ tends to infinity, with e_0 and e_1 defined in* (4.3.3).

Proof of Lemma 4.6.9. Consider Regime $\mathfrak{R}_2$ when $\mathcal{H}_+$ is in force, i.e. $k > V'(u_+^*)$, and fix such a k, and for ease of notation drop the superscripts and k-dependence. Lemma 4.6.6 yields

$$\log \Phi_{\tau,k,3/4}(u) = -\mathtt{i}uk\tau^{1/4} + \tau\left(V\left(\frac{\mathtt{i}u}{\tau^{3/4}} + q_\tau^*\right) - V\left(q_\tau^*\right)\right) + H\left(\frac{\mathtt{i}u}{\tau^{3/4}} + q_\tau^*\right) - H\left(q_\tau^*\right) + \mathcal{O}(\tau^{-1/4}). \tag{4.6.16}$$

Using Lemma 4.6.8, we have the Taylor expansion (similar to (4.6.14))

$$V\left(q_\tau^* + \frac{\mathtt{i}u}{\tau^{3/4}}\right) = \frac{\kappa\theta}{2\beta_t} + \frac{a_1V'}{\sqrt{\tau}} + \frac{\mathtt{i}uV'}{\tau^{3/4}} + \left(\frac{V''a_1^2}{2} + V'a_2\right)\frac{1}{\tau} + \frac{\mathtt{i}ua_1V''}{\tau^{5/4}} + \mathcal{O}\left(\frac{1}{\tau^{3/2}}\right), \tag{4.6.17}$$

as τ tends to infinity, where V, V' and V'' are evaluated at u_+^*. Using (4.6.14) we further have

$$V\left(q_\tau^* + \mathtt{i}u/\tau^{3/4}\right) - V\left(q_\tau^*\right) = \frac{\mathtt{i}uV'(u_+^*)}{\tau^{3/4}} + \frac{\mathtt{i}ua_1V''(u_+^*)}{\tau^{5/4}} + \mathcal{O}\left(\frac{1}{\tau^{3/2}}\right), \tag{4.6.18}$$

$$\gamma\left(q_\tau^* + \mathtt{i}u/\tau^{3/4}\right) = \gamma(u_+^*) + \frac{a_1\gamma'(u_+^*)}{\sqrt{\tau}} + \frac{\mathtt{i}u\gamma'(u_+^*)}{\tau^{3/4}} + \mathcal{O}\left(\frac{1}{\tau}\right), \tag{4.6.19}$$

with γ defined in (1.3.8). We now study the behaviour of $H\left(\mathtt{i}u/\tau^{3/4} + q_\tau^*\right)$, where H is defined in (2.3.8). Using Lemma 4.6.11 and the expansion (4.6.18) for large τ, we first note that

$$e_\tau - 2\beta_t\sqrt{\tau}\left[V(q_\tau^* + \frac{\mathtt{i}u}{\tau^{3/4}}) - V(q_\tau^*)\right] = e_0 - \frac{2\beta_t\mathtt{i}uV'}{\tau^{1/4}} + \frac{e_1}{\sqrt{\tau}} - \frac{2\beta_t\mathtt{i}ua_1V''}{\tau^{3/4}} + \mathcal{O}\left(\frac{1}{\tau}\right), \tag{4.6.20}$$

with e_τ defined in (4.6.15). Together with (4.6.17), this implies

$$\begin{aligned}\frac{\mathrm{v}\mathrm{e}^{-\kappa t}V(q_\tau^* + \mathtt{i}u/\tau^{3/4})}{\kappa\theta - 2\beta_tV(q_\tau^* + \mathtt{i}u/\tau^{3/4})} &= \frac{\sqrt{\tau}\mathrm{v}\mathrm{e}^{-\kappa t}V(q_\tau^* + \mathtt{i}u/\tau^{3/4})}{e_\tau - 2\beta_t\sqrt{\tau}\left(V(q_\tau^* + \mathtt{i}u/\tau^{3/4}) - V(q_\tau^*)\right)} \\ &= \frac{\kappa\theta\mathrm{v}\mathrm{e}^{-\kappa t}\sqrt{\tau}}{2e_0\beta_t} + \frac{\mathtt{i}\kappa\theta u v e^{-\kappa t}V'\tau^{1/4}}{e_0^2} + v\mathrm{e}^{-\kappa t}\left(\frac{a_1V'}{e_0} - \frac{e_1\kappa\theta}{2e_0^2\beta_t}\right) \\ &\quad - \frac{\zeta_+^2u^2}{2} + \mathcal{O}\left(\frac{1}{\tau^{1/4}}\right),\end{aligned} \tag{4.6.21}$$

with ζ_+ defined in (4.3.2). Substituting e_0 in (4.3.3) into the second term in (4.6.21) we find

$$\frac{\mathtt{i}\kappa\theta u v e^{-\kappa t}V'}{e_0^2} = \mathtt{i}u\left(k - V'\right). \tag{4.6.22}$$

Following a similar procedure using e_τ we establish for large τ that

$$\frac{\mathrm{v}\mathrm{e}^{-\kappa t}V(q_\tau^*)}{\kappa\theta - 2\beta_tV(q_\tau^*)} = \frac{\kappa\theta\mathrm{v}\mathrm{e}^{-\kappa t}\sqrt{\tau}}{2e_0\beta_t} + \mathrm{v}\mathrm{e}^{-\kappa t}\left(\frac{a_1V'}{e_0} - \frac{e_1\kappa\theta}{2e_0^2\beta_t}\right) + \mathcal{O}\left(\frac{1}{\sqrt{\tau}}\right), \tag{4.6.23}$$

and combining (4.6.21), (4.6.22) and (4.6.23) we find that

$$\frac{V(q_\tau^* + \mathtt{i}u/\tau^{3/4})\mathrm{v}\mathrm{e}^{-\kappa t}}{\kappa\theta - 2\beta_t V(q_\tau^* + \mathtt{i}u/\tau^{3/4})} - \frac{V(q_\tau^*)\mathrm{v}\mathrm{e}^{-\kappa t}}{\kappa\theta - 2\beta_t V(q_\tau^*)} = \mathtt{i}u\,(k - V')\,\tau^{1/4} - \frac{\zeta_+^2 u^2}{2} + \mathcal{O}\left(\frac{1}{\tau^{1/4}}\right). \quad (4.6.24)$$

We now analyse the second term of $\exp(H(\mathtt{i}u/\tau^{3/4} + q_\tau^*) - H(q_\tau^*))$. We first re-write this term as (μ defined in (1.3.4) and γ in (1.3.8))

$$\begin{aligned}
&\exp\left(-\mu\log\left(\frac{\kappa\theta - 2\beta_t V(q_\tau^* + \mathtt{i}u/\tau^{3/4})}{\kappa\theta\left(1 - \gamma\left(q_\tau^* + \mathtt{i}u/\tau^{3/4}\right)\right)}\right) + \mu\log\left(\frac{\kappa\theta - 2\beta_t V(q_\tau^*)}{\kappa\theta\left(1 - \gamma\left(q_\tau^*\right)\right)}\right)\right)\\
&= \left(\left(\frac{\kappa\theta - 2\beta_t V(q_\tau^* + \mathtt{i}u/\tau^{3/4})}{\kappa\theta - 2\beta_t V(q_\tau^*)}\right)\left(\frac{1 - \gamma\left(q_\tau^* + \mathtt{i}u/\tau^{3/4}\right)}{1 - \gamma\left(q_\tau^*\right)}\right)^{-1}\right)^{-\mu},
\end{aligned}$$

and deal with each of the multiplicative terms separately. For the first term we re-write it as

$$\frac{\kappa\theta - 2\beta_t V(q_\tau^* + \mathtt{i}u/\tau^{3/4})}{\kappa\theta - 2\beta_t V(q_\tau^*)} = \frac{e_\tau - 2\beta_t\sqrt{\tau}\left(V(q_\tau^* + \mathtt{i}u/\tau^{3/4}) - V(q_\tau^*)\right)}{e_\tau}, \quad (4.6.25)$$

and then we use the asymptotics of e_τ in Lemma 4.6.11 and equation (4.6.20) to find that as τ tends to infinity,

$$\frac{\kappa\theta - 2\beta_t V(q_\tau^* + \mathtt{i}u/\tau^{3/4})}{\kappa\theta - 2\beta_t V(q_\tau^*)} = 1 + \mathcal{O}\left(\frac{1}{\tau^{1/4}}\right). \quad (4.6.26)$$

For the second term we use the asymptotics in (4.6.14) and (4.6.19) to find that for large τ

$$\left(\frac{1 - \gamma\left(q_\tau^* + \mathtt{i}u/\tau^{3/4}\right)}{1 - \gamma\left(q_\tau^*\right)}\right)^{-1} = \left(\frac{1 - \left(\gamma + a_1\gamma'/\sqrt{\tau} + \mathtt{i}u\gamma'/\tau^{3/4} + \mathcal{O}(1/\tau)\right)}{1 - (\gamma + a_1\gamma'/\sqrt{\tau} + \mathcal{O}(1/\tau))}\right)^{-1} = 1 + \mathcal{O}(1/\tau^{3/4}).$$

It then follows that for the second term of $\exp(H(\mathtt{i}u/\tau^{3/4} + q_\tau^*) - H(q_\tau^*))$ that for large τ we have

$$\exp\left(-\mu\log\left(\frac{\kappa\theta - 2\beta_t V(q_\tau^* + \mathtt{i}u/\psi_\tau)}{\kappa\theta\left(1 - \gamma\left(q_\tau^* + \mathtt{i}u/\psi_\tau\right)\right)}\right) + \mu\log\left(\frac{\kappa\theta - 2\beta_t V(q_\tau^*)}{\kappa\theta\left(1 - \gamma\left(q_\tau^*\right)\right)}\right)\right) = 1 + \mathcal{O}\left(\frac{1}{\tau^{1/4}}\right). \quad (4.6.27)$$

Further as τ tends to infinity, the equality (4.6.18) implies

$$\tau\left(V(q_\tau^* + \mathtt{i}u/\tau^{3/4}) - V(q_\tau^*)\right) = \mathtt{i}uV'(q^*)\tau^{1/4} + \mathcal{O}(\tau^{-1/4}). \quad (4.6.28)$$

Combining (4.6.24), (4.6.27) and (4.6.28) into (4.6.16) completes the proof. □

In order to derive complete asymptotic expansions we still need to derive expansions for D and F in (4.6.6). This is the purpose of this section. We first derive an expansion for D which gives the leading-order decay of large-maturity out-of-the-money options:

Lemma 4.6.12. *The following expansions hold as τ tends to infinity (μ defined in* (1.3.4)*):*

(i) In Regimes $\mathfrak{R}_2$, $\mathfrak{R}_{3a}$ and $\mathfrak{R}_{3b}$,

(a) under $\mathcal{H}_\pm$: $D(\tau,k) = \exp\left(-\tau(V^*(k) - k) + \sqrt{\tau}c_0^\pm(k) + c_1^\pm(k)\right)\tau^{\mu/2}c_2^\pm(k)(1 + \mathcal{O}(\tau^{-1/2}))$;

(b) under $\widetilde{\mathcal{H}}_\pm$: $D(\tau,k) = \exp\left(-\tau(V^*(k) - k) + \tau^{1/3}c_0^\pm + c_1^\pm\right)\tau^{\mu/3}c_2^\pm(1 + \mathcal{O}(\tau^{-1/3}))$;

(ii) In Regimes $\mathfrak{R}_{3b}$ *and* $\mathfrak{R}_4$*,*

(a) For $k > V'(1)$*:* $D(\tau, k) = \mathrm{e}^{-\tau(V^*(k)-k)+\mu+g_0} \left(\frac{2(k-V'(1))(\kappa-\rho\xi)^2}{(\kappa\theta-2V(1)\beta_t)}\right)^{\mu} \tau^{\mu}(1+\mathcal{O}(\tau^{-1}))$;

(b) For $k = V'(1)$*:* $D(\tau, k) = \mathrm{e}^{-\tau(V^*(k)-k)+\mu/2+g_0} \left(\frac{2(\kappa-\rho\xi)^2\sqrt{V''(1)\mu}}{(\kappa\theta-2V(1)\beta_t)}\right)^{\mu} \tau^{\mu/2}(1+\mathcal{O}(\tau^{-1/2}))$.

where c_0*,* c_1 *and* c_2 *in* (4.3.4)*,* g_0 *in* (4.3.5) *and* V^* *is characterised explicitly in Lemma 4.2.1.*

Proof. Consider Regime $\mathfrak{R}_2$ in Case(i)(a) (namely when $\mathcal{H}_+$ holds), and again for ease of notation drop the superscripts and k-dependence. We now use Lemma 4.6.8 and (4.6.14) to write for large τ:

$$\begin{aligned} \mathrm{e}^{-\tau(kq_\tau^*-V(q_\tau^*))} &= \exp\left[-\tau(ku_+^*-V(u_+^*)) - \sqrt{\tau}a_1(k-V') + r_0 - a_2k + \mathcal{O}(\tau^{-1/2})\right] \\ &= \mathrm{e}^{-\tau V^*(k)-\sqrt{\tau}a_1(k-V')+r_0-a_2k}\left[1+\mathcal{O}(\tau^{-1/2})\right], \end{aligned} \tag{4.6.29}$$

with $r_0 := \frac{1}{2}V''a_1^2 + V'a_2$ and where we have used the characterisation of V^* given in Lemma 4.2.1. We now study the asymptotics of $H(q_\tau^*)$. Using the definition of e_τ in (4.6.15) we write

$$\begin{aligned} \mathrm{e}^{H(q_\tau^*)} &= \exp\left(\frac{V(q_\tau^*)v\mathrm{e}^{-\kappa t}}{\kappa\theta - 2\beta_t V(q_\tau^*)}\right)\left[\frac{\kappa\theta-2\beta_tV(q_\tau^*)}{\kappa\theta\left(1-\gamma\left(q_\tau^*\right)\right)}\right]^{-\mu} \\ &= \tau^{\frac{\mu}{2}}\exp\left(\frac{V(q_\tau^*)v\mathrm{e}^{-\kappa t}}{\kappa\theta - 2\beta_t V(q_\tau^*)}\right)\left[\frac{e_\tau}{\kappa\theta\left(1-\gamma\left(q_\tau^*\right)\right)}\right]^{-\mu}, \end{aligned} \tag{4.6.30}$$

and deal with each of these terms in turn. Now by (4.6.23) we have, as τ tends to infinity,

$$\frac{v\mathrm{e}^{-\kappa t}V(q_\tau^*)}{\kappa\theta-2\beta_tV(q_\tau^*)} = \frac{\kappa\theta v\mathrm{e}^{-\kappa t}\sqrt{\tau}}{2e_0\beta_t} + v\mathrm{e}^{-\kappa t}\left(\frac{a_1V'}{e_0} - \frac{e_1\kappa\theta}{2e_0^2\beta_t}\right) + \mathcal{O}\left(\frac{1}{\sqrt{\tau}}\right). \tag{4.6.31}$$

Using the asymptotics of e_τ given in Lemma 4.6.11 and those of γ in (4.6.14) we find

$$\begin{aligned} \left(\frac{e_\tau}{\kappa\theta\left(1-\gamma\left(q_\tau^*\right)\right)}\right)^{-\mu} &= \left(\frac{e_0+e_1/\sqrt{\tau}+\mathcal{O}\left(1/\tau\right)}{\kappa\theta\left(1-\gamma\right)+\kappa\theta a_1\gamma'/\sqrt{\tau}+\mathcal{O}\left(1/\tau\right)}\right)^{-\mu} \\ &= \left(\frac{\kappa\theta\left(1-\gamma\right)}{e_0}\right)^{\mu}\left(1+\mathcal{O}\left(\frac{1}{\sqrt{\tau}}\right)\right). \end{aligned} \tag{4.6.32}$$

Using the definition of e_0 in (4.3.3), note the simplification $-a_1(k-V') + \frac{\kappa\theta v\mathrm{e}^{-\kappa t}}{2e_0\beta_t} = -2a_1(k-V')$. Combining this, (4.6.29), (4.6.30), (4.6.31) and (4.6.32) we find that

$$D(\tau,k) := \mathrm{e}^{-\tau\left(k(q_\tau^*-1)-\Lambda_\tau^{(t)}(q_\tau^*)\right)} = \exp\left(-\tau(V^*(k)-k)+\sqrt{\tau}c_0^+ + c_1^+\right)\tau^{\mu/2}c_2^+(1+\mathcal{O}(\tau^{-1/2})),$$

with c_0^+, c_1^+ and c_2^+ in (4.3.4). All other cases follows in an analogous fashion and this completes the proof. □

In Lemma 4.6.14 below we provide asymptotic expansions for the function F in (4.6.6). However, we first need the following technical result, the proof of which can be found in [21, Lemma 7.3]. Let p denote the density of a Gamma random variable with shape λ and scale ν, and $\widehat{p}$ the corresponding characteristic function:

$$p(x) \equiv \frac{1}{\Gamma(\lambda)\nu^\lambda}x^{\lambda-1}\mathrm{e}^{-x/\nu}\mathbb{1}_{\{x>0\}}, \qquad \widehat{p}(u) \equiv (1-\mathtt{i}\nu u)^{-\lambda}. \tag{4.6.33}$$

Lemma 4.6.13. *The following expansion holds as τ tends to infinity:*

$$\int_{\mathbb{R}} \exp\left(-\mathtt{i}\gamma u - \frac{\sigma^2 u^2}{2\tau}\right) u^\beta \widehat{p}(\gamma u)\mathrm{d}u = \sum_{r=0}^{w} \frac{2\pi\sigma^{2r}}{\mathtt{i}^\beta \gamma^{2r+\beta+1} 2^r r! \tau^r} p^{(2r+\beta)}(1) + \mathcal{O}\left(\frac{1}{\tau^{w+1}}\right),$$

with $\gamma, \nu, \lambda \in \mathbb{R}_+^*$, $\beta \in \mathbb{N} \cup \{0\}$, $w \in \mathbb{N}$ and $p^{(n)}$ denoting the n-th derivative of the Gamma density p.

Lemma 4.6.14. *The following expansions hold as τ tends to infinity (with $\zeta_\pm$ in* (4.3.2)*, $u_\pm^*$ in* (2.3.6) *and μ in* (1.3.4)*):*

(i) In Regimes $\mathfrak{R}_2$, $\mathfrak{R}_{3a}$ and $\mathfrak{R}_{3b}$,

(a) under $\mathcal{H}_\pm$: $F(\tau, k, 3/4) = \frac{\tau^{-3/4}}{\zeta_\pm(k) u_+^* (u_\pm^* - 1)\sqrt{2\pi}}(1 + \mathcal{O}(\tau^{-1/2}))$;

(b) under $\widetilde{\mathcal{H}}_\pm$: $F(\tau, k, 1/2) = \frac{\tau^{-1/2}}{u_\pm^*(u_\pm^* - 1)\sqrt{6\pi V''(u_\pm^*)}}(1 + \mathcal{O}(\tau^{-1/3}))$;

(ii) In Regimes $\mathfrak{R}_{3b}$ and $\mathfrak{R}_4$,

(a) For $k > V'(1)$: $F(\tau, k, 1) = -\frac{\mathrm{e}^{-\mu}\mu^\mu}{\Gamma(1+\mu)}(1 + \mathcal{O}(\tau^{-1}))$;

(b) For $k = V'(1)$: $F(\tau, k, 1/2) = -\frac{\mathrm{e}^{-\mu/2}(\mu/2)^{\mu/2}}{2\Gamma(1+\mu/2)}(1 + \mathcal{O}(\tau^{-1/2}))$.

Proof. First we consider Regime $\mathfrak{R}_2$ under $\mathcal{H}_+$ in Case (i)(a). Using the asymptotics of q_τ^* given in Lemma 4.6.8, we can Taylor expand for large τ to obtain $\overline{C(\tau, k, 3/4)} = \frac{\tau^{-3/4}}{(u_+^* - 1)u_+^*}(1 + \mathcal{O}(\tau^{-1/2}))$, Combining this with the characteristic function asymptotics in Lemma 4.6.9 we find that for large τ,

$$\begin{aligned} F(\tau, k, 3/4) &= \frac{1}{2\pi\tau^{3/4}\left(u_+^* - 1\right) u_+^*} \int_{\mathbb{R}} \exp\left(-\frac{\zeta_+^2(k) u^2}{2}\right)(1 + \mathcal{O}(\tau^{-1/4}))\mathrm{d}u \\ &= \frac{1}{\sqrt{2\pi}|\zeta_\pm(k)|\tau^{3/4}\left(u_+^* - 1\right) u_+^*}\left(1 + \mathcal{O}(\tau^{-1/4})\right), \end{aligned}$$

where the second line follows from simple properties of the normal distribution. By extending the analysis to higher order, the $\mathcal{O}(\tau^{-1/4})$ term is actually zero and the next non-trivial term is $\mathcal{O}(\tau^{-1/2})$. For brevity we omit the analysis and we give the remainder as $\mathcal{O}(\tau^{-1/2})$ in the lemma. All other cases in (i) follow from analogous arguments to above and we now move onto Case (ii)(a). Using the asymptotics of q_τ^* in Lemma 4.6.8 we have $\overline{C(\tau, k, 1)} = -\left(\frac{\mu}{\nu(k)} - \mathtt{i}u\right)^{-1} + \mathcal{O}(\tau^{-1}) = \frac{-\nu(k)}{\mu}\left(1 - \frac{\mathtt{i}u\nu(k)}{\mu}\right)^{-1} + \mathcal{O}(\tau^{-1})$, where we set $\nu(k) := k - V'(1)$. Using this and the characteristic function asymptotics in Lemma 4.6.9 we see that as τ tends to infinity:

$$\begin{aligned} F(\tau, k, 1) &= \frac{-\nu}{2\pi\mu}\int_{-\infty}^{\infty} \exp\left(-\mathtt{i}u\nu - \frac{u^2 V''(1)}{2\tau}\right)\left(1 - \frac{\mathtt{i}u\nu}{\mu}\right)^{-1-\mu} \mathrm{d}u\left[1 + \mathcal{O}\left(\tau^{-1}\right)\right], \\ &= \left(-\frac{\mathrm{e}^{-\mu}\mu^\mu}{\Gamma(1+\mu)} + \mathcal{O}\left(\tau^{-1}\right)\right)\left[1 + \mathcal{O}\left(\tau^{-1}\right)\right], \end{aligned}$$

where the second line follows from Lemma 4.6.13. We now prove (ii)(b). Using the asymptotics of q_τ^* for large τ in Lemma 4.6.8, we obtain $\overline{C(\tau, k, 1/2)} = \frac{1}{a_1(1 + \mathtt{i}u/a_1)} + \mathcal{O}(\tau^{-1/2})$, with $a_1 = -\sqrt{\frac{\mu}{V''(1)}}$.

Using this and the characteristic function asymptotics in Lemma 4.6.9 we have the following expansion for large τ:

$$F(\tau, k, 1/2) = \frac{1}{2\pi a_1}\int_{\mathbb{R}} \frac{\exp\left(\mathtt{i}ua_1 V''(1) - \frac{1}{2}u^2 V''(1)\right)}{(1+\mathtt{i}u/a_1)^{1+\mu}}\mathrm{d}u\left(1 + \mathcal{O}\left(\tau^{-1/2}\right)\right).$$

Let n and $\widehat{n}$ denote the Gaussian density and characteristic function with zero mean and variance $V''(1)$. Using (4.6.33), we have

$$\begin{aligned}\int_{\mathbb{R}} \mathrm{e}^{-\mathtt{i}\omega u}\widehat{n}(u)\widehat{p}(u)\mathrm{d}u = 2\pi\mathcal{F}^{-1}(\widehat{n}(u)\widehat{p}(u))(\omega) &= 2\pi\mathcal{F}^{-1}(\mathcal{F}(n * p)) \\ &= 2\pi\int_0^\infty n(\omega - y)p(y)\mathrm{d}y,\end{aligned} \tag{4.6.34}$$

so that

$$\frac{1}{2\pi a_1}\int_{\mathbb{R}} \frac{\exp\left(\mathtt{i}ua_1 V''(1) - \frac{1}{2}u^2 V''(1)\right)}{(1+\mathtt{i}u/a_1)^{1+\mu}}\mathrm{d}u = \frac{1}{a_1}\int_0^\infty n(-a_1 V''(1) - y)p(y)\mathrm{d}y.$$

This integral can now be computed in closed-form and the result follows after simplification using the definition of a_1 and the duplication formula for the Gamma function. □

4.6.5 Asymptotics in the case of non-existence of the limiting Fourier transform

In this section, we are interested in the cases where $k \in \{V'(0), V'(1)\}$ whenever $\mathcal{H}_0$ is in force, which corresponds to all the regimes except $\mathcal{R}_{3b}$ and $\mathcal{R}_4$ at $V'(1)$. In these cases, the limiting Fourier transform is undefined at these points. We show here however that the methodology of Section 4.6.3 can still be applied, and we start by verifying Assumption 4.6.2. The following quantity will be of primary importance:

$$\Upsilon(a) := 1 + \frac{a\rho\xi}{\kappa - \rho\xi}\mathrm{e}^{\kappa t}, \tag{4.6.35}$$

for $a \in \{0, 1\}$, and it is straightforward to check that Υ is well defined whenever $\mathcal{H}_0$ is in force.

Lemma 4.6.15. *Let $a \in \{0,1\}$ and assume that $v \neq \theta\Upsilon(a)$. Then, whenever $\mathcal{H}_0$ holds, Assumption 4.6.2 is satisfied with $\mathcal{A} = \{V'(a)\}$ and $q^*_\infty = a$. Additionally, if $v < \theta\Upsilon(a)$, then there exists $\tau_1^* > 0$ such that $q^*_\tau(k) < 0$ if $a = 0$ and $q^*_\tau(k) > 1$ if $a = 1$ for all $\tau > \tau_1^*$, and if $v > \theta\Upsilon(a)$, then there exists $\tau_1^* > 0$ such that $q^*_\tau(k) \in (0,1)$ for all $\tau > \tau_1^*$.*

Remark 4.6.16. When $v = \theta\Upsilon(a)$ for $a \in \{0,1\}$ then $q^*_\tau(V'(a)) = a$ for all $\tau > 0$. In particular, the Fourier transform is always undefined for all $\tau > 0$ and the methodology in this section cannot be applied.

Proof. Recall that the function H is defined in (2.3.8). We first prove the lemma in the case $a = 0$, in which case $\Upsilon(0) = 1$. Note that $H'(0) > 0 (< 0)$ if and only if $v/\theta < 1 (> 1)$ and $H'(0) = 0$ if and only if $v = \theta$. Now let $k = V'(0)$ and $v < \theta$ and consider the equation $H'(u)/\tau = V'(0) - V'(u)$. Since H' is continuous H' is strictly positive in some neighbourhood of zero. In order for the right-hand side to be positive we require our solution to be in $(-\delta_0, 0)$ for some $\delta_0 > 0$ since V is strictly convex. So let $\delta_1 \in (-\delta_0, 0)$. With the right-hand side locked at $V'(0) - V'(\delta_1) > 0$ we then adjust τ accordingly so that $H'(\delta_1)/\tau_1 = V'(0) - V'(\delta_1)$. We then set $u_{\tau_1} = \delta_1$. It is clear that for $\tau > \tau_1$ there always exists a unique solution to this equation and furthermore q_τ^* is strictly increasing and bounded above by zero. The limit has to be zero otherwise the continuity of V' and H' implies $\lim_{\tau\uparrow\infty} V'(q_\tau^*) + H'(q_\tau^*)/\tau = V'(\lim_{\tau\uparrow\infty} q_\tau^*) < V'(0)$, a contradiction. A similar analysis holds for $v > \theta$ and in this case q_τ^* converges to zero from above. When $v = \theta$ then $q_\tau^* = 0$ for all $\tau > 0$ (i.e. it is a fixed point). Analogous arguments hold for $k = V'(1)$: $H'(1) > 0 (< 0)$ if and only if $v/\theta > \Upsilon(1)$ $(< \Upsilon(1))$ and $H'(1) = 0$ if and only if $v/\theta = \Upsilon(1)$. If $v/\theta > \Upsilon(1)$ $(< \Upsilon(1))$ then q_τ^* converges to 1 from below (above) and when $v/\theta = \Upsilon(1)$, $q_\tau^* = 1$ for all $\tau > 0$. □

We now provide expansions for q_τ^* and the characteristic function $\Phi_{\tau,k,1/2}$. Define the following quantities:

$$\alpha_0 := \frac{2\mathrm{e}^{-\kappa t}(v-\theta)\kappa}{\theta((2\kappa-\xi)^2 + 4\kappa\xi(1-\rho^2))}, \quad \alpha_1 := \frac{2\mathrm{e}^{-\kappa t}(\kappa-\rho\xi)^2}{\kappa\theta((2\kappa-\xi)^2 + 4\kappa\xi(1-\rho^2))}(\theta\Upsilon(1) - v). \tag{4.6.36}$$

The proofs are analogous to Lemma 4.6.8 and 4.6.9 and omitted. Note that the asymptotics are in agreement with the properties of $q_\tau^*(k)$ in Lemma 4.6.15.

Lemma 4.6.17. *Let $a \in \{0,1\}$ and assume that $v \neq \theta\Upsilon(a)$. When $k = V'(a)$, the following expansions hold as τ tends to infinity:*

$$q_\tau^*(k) = a + \alpha_a \tau^{-1} + \mathcal{O}\left(\tau^{-2}\right), \qquad D(\tau, k) = \mathrm{e}^{\tau V'(a)(1-a)}\left(1 + \mathcal{O}\left(\tau^{-1}\right)\right),$$
$$\Phi_{\tau,k,1/2}(u) = \mathrm{e}^{-\frac{1}{2}u^2 V''(a)}\left(1 + \left(\mathtt{i}\alpha_a u V''(a) - \frac{\mathtt{i}u^3 V'''(a)}{6} + \mathtt{i}uH'(a)\right)\tau^{-1/2} + \mathcal{O}(\tau^{-1})\right).$$

In the lemma below we now provide expansions for F in (4.6.6):

Lemma 4.6.18. *Let $a \in \{0,1\}$ and assume that $v \neq \theta\Upsilon(a)$. Then the following expansions hold as τ tends to infinity (with a_0 given in* (4.6.36)*):*

$$F\left(\tau, V'(a), \frac{1}{2}\right) = \frac{\mathbb{1}_{\{a=1\}} - \mathbb{1}_{\{a=0\}}\mathrm{sgn}(\alpha_0)}{2} + \frac{\left[-1 + \mathrm{sgn}(-a)\left(\frac{V'''(a)}{6V''(a)} - H'(a)\right)\right]}{\sqrt{2\pi\tau V''(a)}}\left(1 + \mathcal{O}(\tau^{-1})\right).$$

Proof. Define the following functions from $\mathbb{R}^* \times \{0,1\}$ to $\mathbb{R}$:

$$\begin{cases} \varpi_1(w,a) &:= \mathrm{e}^{w^2 V''(a)/2}\pi\left[2\mathcal{N}(w\sqrt{V''(a)}) - 1 - \mathrm{sgn}(w)\right], \\ \varpi_2(w,a) &:= -\sqrt{\dfrac{2\pi}{V''(a)}} + \mathrm{e}^{w^2 V''(a)/2}\pi w\left[1 + \mathrm{sgn}(w) - 2\mathcal{N}\left(w\sqrt{V''(a)}\right)\right], \\ \varpi_3(w,a) &:= \dfrac{\sqrt{2\pi}(w^2 V''(a) - 1)}{(V''(a))^{3/2}} - 2\pi w^2|w|\exp\left(\dfrac{w^2 V''(a)}{2}\right)\mathcal{N}\left(-|w|\sqrt{V''(a)}\right), \\ \varpi(w,a) &:= \dfrac{\varpi_1(w,a)}{2\pi} + \dfrac{1}{2\pi\sqrt{\tau}}\left((\alpha_a V''(a) + H'(a))\varpi_2(w,a) + \dfrac{V'''(a)\varpi_3(w,a)}{6}\right). \end{cases} \tag{4.6.37}$$

Consider the case $a = 0$. Set $P(u) := \mathtt{i}\alpha_0 u V''(0) - \mathtt{i}u^3 V'''(0)/6 + \mathtt{i}uH'(0)$ and note that $\overline{C(u,\tau,1/2)} := \frac{1}{\left(-\mathtt{i}u - q^*_\tau\sqrt{\tau}\right)} - \frac{1}{\left(-\mathtt{i}u - q^*_\tau\sqrt{\tau}+\sqrt{\tau}\right)}$. Using Lemma 4.6.17 and the definition of F in (4.6.6):

$$F(\tau, V'(0), 1/2) = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-V''(0)u^2/2}\overline{C(u,\tau)}(1 + P(u)\tau^{-1/2} + \mathcal{O}(\tau^{-1}))\mathrm{d}u. \tag{4.6.38}$$

We cannot now simply Taylor expand $\overline{C(u,\tau,1/2)}$ for small τ and integrate term by term since in the limit $\overline{C(u,\tau,1/2)}$ is not L^1. This was the reason for introducing the time dependent term $q^*_\tau(V'(0))$ so that the Fourier transform exists for any $\tau > 0$. Indeed, we easily see that $\overline{C(u,\tau,1/2)} = -\mathtt{i}/u + \mathcal{O}(\tau^{-1/2})$. We therefore integrate these terms directly and then compute the asymptotics as τ tends to infinity. Note first that since $|\overline{C(u,\tau,1/2)}| = \mathcal{O}(1)$, then $\overline{C(u,\tau,1/2)}(1+P(u)\tau^{-1/2}+\mathcal{O}(\tau^{-1})) = \overline{C(u,\tau,1/2)}(1+P(u)\tau^{-1/2})+\mathcal{O}(\tau^{-1})$. Further for any $w \neq 0$, $\int_{\mathbb{R}} \mathrm{e}^{-V''(0)u^2/2}\frac{1}{-\mathtt{i}u-w}\mathrm{d}u = \varpi_1(w,0)$, $\int_{\mathbb{R}} \mathrm{e}^{-V''(0)u^2/2}\frac{\mathtt{i}u}{-\mathtt{i}u-w}\mathrm{d}u = \varpi_2(w,0)$ and $\int_{\mathbb{R}} \mathrm{e}^{-V''(0)u^2/2}\frac{\mathtt{i}u^3}{-\mathtt{i}u-w}\mathrm{d}u = \varpi_3(w,0)$. These integrals can be solved by applying the same method in (4.6.34) and identifying $\mathrm{e}^{-V''(0)u^2/2}$ as a Gaussian characteristic function and $\frac{u^n}{\mathtt{i}u/w+1}$ as the Fourier transform of $\partial^n_x \exp(x/w)/w$ for $n \in \mathbb{N}\cup\{0\}$. Now using the definition of ϖ in (4.6.37) we then obtain

$$F(\tau, V'(0), 1/2) = \varpi\left(q^*_\tau\sqrt{\tau}, 0\right) - \varpi\left((q^*_\tau - 1)\sqrt{\tau}, 0\right) + \mathcal{O}\left(\tau^{-1}\right).$$

Using Lemma 4.6.17 and asymptotics of the cumulative normal distribution function we compute:

$$\varpi\left(q^*_\tau\sqrt{\tau},0\right) = \varpi\left(\alpha_0\tau^{-1/2} + \mathcal{O}\left(\tau^{-3/2}\right),0\right) = -\frac{\mathrm{sgn}(\alpha_0)}{2} - \frac{6H'(0)V''(0) - V'''(0)}{6\sqrt{2\pi}(V''(0))^{3/2}\sqrt{\tau}} + \mathcal{O}\left(\tau^{-1}\right),$$
$$\varpi((q^*_\tau - 1)\sqrt{\tau},0) = \varpi\left(-\sqrt{\tau} + \alpha_0\tau^{-1/2} + \mathcal{O}\left(\tau^{-3/2}\right),0\right) = \frac{1}{\sqrt{2\pi V''(0)\tau}} + \mathcal{O}\left(\tau^{-1}\right).$$

The case $a = 1$ is analogous using $\varpi(\cdot, 1)$ and the lemma follows after using the Lebesgue dominated convergence theorem (analogously to Lemma 3.6.9 and Lemma 3.6.10) . □

Remark 4.6.19. Consider $\mathfrak{R}_{3b}$ and $\mathfrak{R}_4$ with $k = V'(1)$ in Section 4.6.4. Here also $q^*_\tau(k)$ tends to 1 and it is natural to wonder why we did not encounter the same issues with the limiting Fourier transform as we did in the present section. The reason this was not a concern was that the speed of convergence ($\tau^{-1/2}$) of q^*_τ to 1 was the same as that of the random variable $Z_{\tau,k,1/2}$ to its limiting value. Intuitively the lack of steepness of the limiting cgf was more important than any issues with the limiting Fourier transform. In the present section steepness is not a concern, but again in the limit the Fourier transform is not defined. This becomes the dominant effect since $q^*_\tau(k)$ converges to 1 at a rate of τ^{-1} while the re-scaled random variable $Z_{\tau,k,1/2}$ converges to its limit at the rate $\tau^{-1/2}$.

4.6.6 Forward smile asymptotics: Proof of Theorem 4.4.1

The general machinery to translate option price asymptotics into implied volatility asymptotics has been fully developed by Gao and Lee [69]. We simply outline the main steps here. There are two main steps to determine forward smile asymptotics: (i) choose the correct root for the zeroth-order term in order to line up the domains (and hence functional forms) in Theorem 4.3.1 and Corollary 4.3.3; (ii) match the asymptotics.

We illustrate this with a few cases from Theorem 4.4.1. Consider $\mathfrak{R}_{3b}$ and $\mathfrak{R}_4$ with $k > V'(1)$. We have asymptotics for forward-start call option prices for $k > V'(1)$ in Theorem 4.3.1 that decay to one as τ tends to infinity. The only BSM regime in Corollary 4.3.3 where this holds (asymptotics decay to one) is where $k \in (-\Sigma^2/2, \Sigma^2/2)$. We now substitute our asymptotics for Σ and at leading order we have the requirement: $k > V'(1)$ implies that $k \in (-\mathfrak{N}_0(k)/2, \mathfrak{N}_0(k)/2)$. We then need to check that this holds only for the correct root $\mathfrak{N}_0$ used in the theorem. Note that we only use the leading order condition here since if $k \in (-\mathfrak{N}_0(k)/2, \mathfrak{N}_0(k)/2)$ then there will always exist a $\tau_1 > 0$ such that $k \in (-\mathfrak{N}_0(k)/2 + o(1), \mathfrak{N}_0(k)/2 + o(1))$, for $\tau > \tau_1$. Suppose now that we choose the root not as given in Theorem 4.4.1. Then for the upper bound we get the condition $kV(1) > 0$. Since $V(1) < 0$ we require $V'(1) < 0$ and then this only holds for $V'(1) < k < 0$. This already contradicts $k > V'(1)$ but let us continue since it may be true for a more limited range of k. The lower bound gives the condition $(k - V(1))k > 0$. But the upper bound implied that we needed $V'(1) < k < 0$ and so further $k < V'(1)$. Therefore $V'(1) < k < V(1)$ but this can never hold since simple computations show that $V'(1) > V(1)$. Now let's choose the root according to the theorem. For the upper bound we get the condition $-\sqrt{(V^* - k)^2 + k(V^*(k) - k)} < V^*(k) - k = -V(1) > 0$ and this is always true. For the lower bound we get the condition $-\sqrt{(V^* - k)^2 + k(V^*(k) - k)} < V^*(k) = k - V(1)$ and this is always true for $k > V'(1)$ since $V'(1) > V(1)$. This shows that we have chosen the correct root for the zeroth-order term and we then simply match asymptotics for higher order terms.

As a second example consider $\mathfrak{R}_2$ and $k > V'(u_+^*)$ in Theorem 4.4.1. Substituting the ansatz $\sigma_{t,\tau}^2(k\tau) = \mathfrak{N}_0^\infty(k) + \mathfrak{N}_{1,+}^\infty(k,t)\tau^{-1/2} + \mathfrak{N}_{2,+}^\infty(k,t)\tau^{-1} + \mathcal{O}(\tau^{-3/2})$ into the BSM asymptotics for forward-start call options in Corollary 4.3.3, we find

$$\mathbb{E}\left(\mathrm{e}^{X_\tau^{(t)}} - \mathrm{e}^{k\tau}\right)^+ = \exp\left(-\alpha_0^\infty\tau + \alpha_1^\infty\sqrt{\tau} + \alpha_2^\infty\right)\frac{4\mathfrak{N}_0^{3/2}}{\sqrt{2\pi\tau}\,(4k^2 - \mathfrak{N}_0^2)}\left(1 + \mathcal{O}\left(\tau^{-1/2}\right)\right),$$

where $\alpha_0^\infty := \frac{k^2}{2\mathfrak{N}_0} - \frac{k}{2} + \frac{\mathfrak{N}_0}{8}$, and $\alpha_1^\infty := \mathfrak{N}_1\frac{4k^2 - \mathfrak{N}_0^2}{8\mathfrak{N}_0^2}$ and α_2^∞ is a constant, the exact value does not matter here. We now equate orders with Theorem 4.3.1. At the zeroth order we get two solutions and since $V'(u_+^*) > V(1)$, we choose the negative root such that the domains match in Corollary 4.3.3 and Theorem 4.3.1 for large τ (using similar arguments as above). At the first order we solve for $\mathfrak{N}_1^\infty$. But now at the second order, we can only solve for higher order terms if $\mu = 1/2$ due to the term $\tau^{\mu/2-3/4} = \tau^{-1/2}$ in the forward-start option asymptotics in Theorem 4.3.1. All other cases follow analogously.

Chapter 5

Black-Scholes in a CEV random environment: a new approach to smile modelling

5.1 Introduction

We propose a simple model with continuous paths for stock prices that allows for small-maturity explosion of the spot implied volatility smile. It is indeed a well-documented fact on Equity markets (see for instance [71, Chapter 5]) that standard (Itô) stochastic volatility models with continuous paths are not able to capture the observed steepness of the left wing of the smile when the maturity becomes small. To remedy this, several authors have suggested the addition of jumps, either in the form of an independent Lévy process or within the more general framework of affine diffusions. Jumps (in the stock price dynamics) imply an explosive behaviour for the small-maturity smile and are better able to capture the observed steepness of the small-maturity spot implied volatility smile. In particular, Tankov [145] showed that, for exponential Lévy models with Lévy measure supported on the whole real line, the squared implied volatility smile explodes as $\sigma_\tau^2(k) \sim -k^2/(2\tau \log \tau)$, as the maturity τ tends to zero, where k represents the log-moneyness.

Gatheral, Jaisson and Rosenbaum [74] have recently been revisiting stochastic volatility models, where the instantaneous variance process is driven by a fractional Brownian motion. They suggest that the Hurst exponent should not be used as an indicator of the historical memory of the volatility, but rather as an additional parameter to be calibrated to the volatility surface. Their study reveals that $H \in (0, 1/2)$ (in fact $H \approx 0.1$ in their calibration results), indicating short memory of the volatility, thereby contradicting decades of time series analyses. By considering a specific fractional uncorrelated volatility model, directly inspired by the fractional version of the

Heston model [42, 87], Guennoun, Jacquier and Roome [78] provide a theoretical justification of this result. They show in particular that, when $H \in (0, 1/2)$, the implied volatility explodes as $\sigma_\tau^2(k) \sim y_0 \tau^{H-1/2}/\Gamma(H+3/2)$ as τ tends to zero (where y_0 is the initial instantaneous variance).

In this chapter we propose an alternative framework: we suppose that the stock price follows a standard Black-Scholes model; however the instantaneous variance, instead of being constant, is sampled from a continuous distribution. We first derive some general properties, interesting from a financial modelling point of view, and devote a particular attention to a particular case of it, where the variance is generated from independent CEV dynamics: Assume that interest rates and dividends are null, and let S denote the stock price process starting at $S_0 = 1$, the solution to the stochastic differential equation $\mathrm{d}S_\tau = S_\tau \sqrt{\mathcal{V}} \mathrm{d}W_\tau$, for $\tau \geq 0$, where W is a standard Brownian motion. Here, $\mathcal{V}$ is a random variable, which we assume to be distributed as $\mathcal{V} \sim Y_t$, for some $t > 0$, where Y is the unique strong solution of the CEV dynamics $\mathrm{d}Y_u = \xi Y_u^p \mathrm{d}B_u$, $Y_0 > 0$ where $p \in \mathbb{R}$, $\xi > 0$ and B is an independent Brownian motion (see Section 5.2.1 for precise statements). The main result of this chapter (Theorem 5.2.3) is that the implied volatility generated from this model exhibits the following behaviour as the maturity τ tends to zero:

$$\sigma_\tau^2(k) \sim \begin{cases} \dfrac{2(1-p)}{3-2p} \left(\dfrac{k^2 \xi^2 (1-p) t}{2\tau} \right)^{1/(3-2p)}, & \text{if } p < 1, \\ \dfrac{k^2 \xi^2 t}{\tau (\log \tau)^2}, & \text{if } p = 1, \\ \dfrac{k^2}{2(2p-1)\tau |\log \tau|}, & \text{if } p > 1, \end{cases} \tag{5.1.1}$$

for all $k \neq 0$. Sampling the initial variance from the CEV process at time t induces different time scales for small-maturity spot smiles, thereby providing flexibility to match steep small-maturity smiles. For $p > 1$, the explosion rate is the same as exponential Lévy models, and the case $p \leq 1/2$ mimics the explosion rate of fractional stochastic volatility models. The CEV exponent p therefore allows the user to modulate the short-maturity steepness of the smile.

We are not claiming here that this model should come as a replacement of fractional stochastic volatility models or exponential Lévy models, notably because its dynamic structure looks too simple at first sight. However, we believe it can act as an efficient building block for more involved models, in particular for stochastic volatility models with initial random distribution for the instantaneous variance. While we leave these extensions for future research, we shall highlight how our model comes naturally into play when pricing forward-start options in stochastic volatility models. In Chapter 3 we proved that the small-maturity forward implied volatility smile explodes in the Heston model when the remaining maturity (after the forward-start date) becomes small. This explosion rate corresponds precisely to the case $p = 1/2$ in (5.1.1). This in particular shows that the key quantity determining the explosion rate is the (right tail of the) variance distribution at the forward-start date (here corresponding to t).

The chapter is structured as follows: in Sections 5.2.1 and 5.2.2 we introduce our model and relate it to other existing approaches. In Section 5.2.3 we use the cumulant generating function to derive extreme strike asymptotics (for some special cases) and show why this approach is not readily applicable for small and large-maturity asymptotics. Sections 5.2.4 and 5.2.5 detail the main results, namely the small and large-maturity asymptotics of option prices and the corresponding implied volatility. Section 5.2.6 provides numerics and Section 5.2.7 describes the relationship between our model and the pricing of forward-start options in stochastic volatility models. Section 5.2.7 also includes a conjecture on the small-maturity forward smile in stochastic volatility models. Finally, the proofs of the main results are gathered in Section 5.3.

5.2 Model and main results

5.2.1 Model description

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s\geq 0}, \mathbb{P})$ supporting a standard Brownian motion, and let $(Z_s)_{s\geq 0}$ denote the solution to the following stochastic differential equation:

$$\mathrm{d}Z_s = -\frac{1}{2}\mathcal{V}\mathrm{d}s + \sqrt{\mathcal{V}}\mathrm{d}W_s, \qquad Z_0 = 0, \tag{5.2.1}$$

where $\mathcal{V}$ is some random variable, independent of the Brownian motion W and $\sigma(\mathcal{V}) \subseteq \mathcal{F}_0$. The process $(Z_s)_{s\geq 0}$, in finance, clearly corresponds to the logarithm of the underlying stock price. This is of course a simple example of stochastic differential equations with random coefficients, existence and uniqueness of which were studied by Kohatsu-Higa, León and Nualart [110], see also Alòs, León and Nualart [2]. In the case where $\mathcal{V}$ is a discrete random variable, this model reduces to the mixture of distributions, analysed, in the Gaussian case by Brigo and Mercurio [32, 33]. In a stochastic volatility model where the instantaneous variance process $(V_t)_{t\geq 0}$ is uncorrelated with the asset price process, the mixing result [135] implies that European options with maturity τ are the same as those evaluated from the SDE (5.2.1) with $\mathcal{V} = \tau^{-1}\int_0^\tau V_s \mathrm{d}s$. As τ tends to zero, the distribution of $\mathcal{V}$ approaches a Dirac Delta centred at the initial variance V_0. Asymptotics of the implied volatility are well-known and weaknesses of classical stochastic volatility models are well-documented [71]. Although such models fit into our framework, we will not consider them further in this paper. Define pathwise the process M by $M_s := -\frac{1}{2}s + W_s$ and let $(\mathcal{T}_s)_{s\geq 0}$ be given by $\mathcal{T}_s := s\mathcal{V}$. Then $\mathcal{T}$ is an independent increasing time-change process and $Z = M_{\mathcal{T}}$. In this way our model can be thought of as a random time change. Let now N be a Lévy process such that $(\mathrm{e}^{N_s})_{s\geq 0}$ is a $(\mathcal{F}_s)_{s\geq 0}$-adapted martingale; define $\mathcal{V} := \tau^{-1}\int_0^\tau V_s \mathrm{d}s$ where V is a positive and independent process, then $(\mathrm{e}^{N_{\mathcal{T}_s}})_{s\geq 0}$ is a classical time-changed exponential Lévy process, and pricing vanilla options is now standard [44, Section 15.5]. However, note that here, as the maturity τ tends to zero, $\mathcal{V}$ converges in distribution to a Dirac Delta, in which case asymptotics are well-known [145].

The model (5.2.1) is also related to the Uncertain Volatility Model of Avellaneda and Parás [6] (see also [50, 96, 121]), in which the Black-Scholes volatility is allowed to evolve randomly within two bounds. In this UVM framework, sub-and super-hedging strategies (corresponding to best and worst case scenarios) are usually derived via the Black-Scholes-Barenblatt equation, and Fouque and Ren [67] recently provided approximation results when the two bounds become close to each other. One can also, at least formally, look at (5.2.1) from the perspective of fractional stochastic volatility models, first proposed by Comte et al. in [43], and later developed and revived in [42, 74, 78]. In these models, standard stochastic volatility models are generalised by replacing the Brownian motion driving the instantaneous volatility by a fractional Brownian motion. This preserves the martingale property of the stock price process, and allows, in the case of short memory (Hurst parameter H between 0 and 1/2) for short-maturity steep skew of the implied volatility smile. However, the Mandelbrot-van Ness representation [123] of the fractional Brownian motion reads

$$W_t^H := \int_0^t \frac{\mathrm{d}W_s}{(t-s)^\gamma} + \int_{-\infty}^0 \left(\frac{1}{(t-s)^\gamma} - \frac{1}{(-s)^\gamma} \right) \mathrm{d}W_s,$$

for all $t \geq 0$, where $\gamma := 1/2 - H$. This representation in particular indicates that, at time zero, the instantaneous variance, being driven by a fractional Brownian motion, incorporates some randomness (through the second integral). Finally, we agree that, at first sight, randomising the variance may sound unconventional. As mentioned in the introduction, we see this model as a building block for more involved models, in particular stochastic volatility with random initial variance, the full study of which is the purpose of ongoing research. After all, market data only provides us with an initial value of the stock price, and the initial level of the variance is unknown, usually let as a parameter to calibrate. In this sense, it becomes fairly natural to leave the latter random.

The framework constituted by the stochastic differential equation (5.2.1) is a simple case of a diffusion in random environment. We refer the interested reader to the seminal paper by Papanicolaou and Varadhan [133], the monographs by Komorowski et al. [111], by Sznitman [143], and the lectures notes by Bolthausen and Sznitman [26] and by Zeitouni [148]. We recall here briefly this framework, and link it to our framework. The classical set-up (say in $\mathbb{R}^d$) is that of a given probability space $(\widetilde{\Omega}, \mathcal{A}, \mathbb{Q})$ describing the random environment and a group of transformations $(\tau_x)_{x\in\mathbb{R}^d}$, jointly measurable in $x \in \mathbb{R}^d$ and $\omega \in \widetilde{\Omega}$ (the transformation essentially indicates a translation of the environment in the x-direction). Consider now two functions $b, a : \widetilde{\Omega} \to \mathbb{R}^d$ and define

$$b(x,\omega) := b(\tau_x(\omega)) \qquad \text{and} \qquad a(x,\omega) := a(\tau_x(\omega)), \quad \text{for all } x \in \mathbb{R}^d, \omega \in \widetilde{\Omega}.$$

For each $x \in \mathbb{R}^d$ and $\omega \in \widetilde{\Omega}$, we let $\mathbb{Q}_{x,\omega}$ denote the unique solution to the martingale problem starting at x and associated to the differential operator $\mathcal{L}^\omega := \frac{1}{2}a(\cdot,\omega)\Delta + b(\cdot,\omega)\nabla$. The probability law $\mathbb{Q}_{x,\omega}$ is called the *quenched law*, and one can define the solution to the corresponding stochastic differential equation $\mathbb{Q}_{x,\omega}$-almost surely. The *annealed law* is the semi-product $\mathbb{Q}_x := \mathbb{Q} \times \mathbb{Q}_{x,\omega}$, and corresponds to averaging over the random environment. Most of the results in the literature, using the method of the environment viewed from the particle, however, impose Lipschitz continuity on the drift $b()$ and the diffusion coefficient $a()$, and uniform ellipticity of $a()$. These conditions clearly do not hold for (5.2.1), where $b(\cdot,\omega) \equiv -\frac{1}{2}\omega$ and $a(\cdot,\omega) \equiv \omega$, since ω takes values in $[0,\infty)$. We shall leave more precise details and applications of random environment to future research. As far as we are aware, this framework has not been applied yet in mathematical finance, the closest being the recent publication by Spiliopoulos [140], who proves quenched (almost sure with respect to the environment) large deviations for a multi-scale diffusion (in a certain regime), assuming stationarity and ergodicity of the random environment.

5.2.1.1 Cumulant generating function

In [62, 63, 95], the authors used the theory of large deviations, and in particular the Gärtner-Ellis theorem (Theorem 1.2.3), to prove small-and large-maturity behaviours of the implied volatility in the Heston model and more generally (in [95]) for affine stochastic volatility models. This approach relies solely on the knowledge of the cumulant generating function of the underlying stock price, and its rescaled limiting behaviour. For any $\tau \geq 0$, let $\Lambda^Z(u,\tau) := \log \mathbb{E}(\mathrm{e}^{uZ_\tau})$ denote the cumulant generating function of Z_τ, defined on the effective domain $\mathcal{D}_\tau^Z := \{u \in \mathbb{R} : |\Lambda^Z(u,\tau)| < \infty\}$; similarly denote $\Lambda^{\mathcal{V}}(u) \equiv \log \mathbb{E}(\mathrm{e}^{u\mathcal{V}})$, whenever it is well defined. A direct application of the tower property for expectations yields

$$\Lambda^Z(u,\tau) = \Lambda^{\mathcal{V}}\left(\frac{u(u-1)\tau}{2}\right), \qquad \text{for all } u \in \mathcal{D}_\tau^Z. \tag{5.2.2}$$

Unfortunately, the cumulant generating function of $\mathcal{V}$ is not available in closed-form in general. In Section 5.2.3 below, we shall see some examples where such a closed-form solution is available, and where direct computations are therefore possible. We note in passing that this simple representation allows, at least in principle, for straightforward (numerical) computations of the slopes of the wings of the implied volatility using Roger Lee's Moment Formula [116] (see also Section 5.2.3.2). The latter are indeed given directly by the boundaries (in $\mathbb{R}$) of the effective domain of $\Lambda^{\mathcal{V}}$. Note further that the model (5.2.1) could be seen as a time-changed Brownian motion (with drift); the representation (5.2.2) clearly rules out the case where Z is a simple exponential Lévy process (in which case $\Lambda^Z(u,\tau)$ would be linear in τ). In view of Roger Lee's formula, this also implies that, contrary to the Lévy case, the slopes of the implied volatility wings are not constant in time in our model.

5.2.2 CEV randomisation

As mentioned above, this paper is a first step towards the introduction of 'random environment' into the realm of mathematical finance, and we believe that, seeing it 'at work' through a specific, yet non-trivial, example, will speak for its potential prowess. We assume from now on that $\mathcal{V}$ corresponds to the distribution of the random variable generated, at some time t, by the solution to the CEV stochastic differential equation $\mathrm{d}Y_u = \xi Y_u^p \mathrm{d}B_u$, $Y_0 = y_0 > 0$ where $p \in \mathbb{R}$, $\xi > 0$ and B is a standard Brownian motion, independent of W. The CEV process [30, 103] is the unique strong solution to this stochastic differential equation, up to the stopping time $\tau_0^Y := \inf_{u>0}\{Y_u = 0\}$. The behaviour of the process after τ_0^Y depends on the value of p, and shall be discussed below. We let $\Gamma(n;x) := \Gamma(n)^{-1}\int_0^x t^{n-1}\mathrm{e}^{-t}\mathrm{d}t$ denote the normalised lower incomplete Gamma function, and $\mathrm{m}_t := \mathbb{P}(Y_t = 0) = \mathbb{P}(\mathcal{V} = 0)$ represent the mass at the origin. Straightforward computations show that, whenever the origin is an absorbing boundary, the density $\zeta_p(y) \equiv \mathbb{P}(Y_t \in \mathrm{d}y)/\mathrm{d}y$ is norm decreasing and (η defined in (5.2.4))

$$\mathrm{m}_t = 1 - \Gamma\left(-\eta; \frac{y_0^{2(1-p)}}{2\xi^2(1-p)^2 t}\right) > 0; \tag{5.2.3}$$

otherwise $\mathrm{m}_t = 0$ and the density ζ_p is norm preserving. When $p \in [1/2, 1)$, the origin is naturally absorbing. When $p \geq 1$, the process Y never hits zero $\mathbb{P}$-almost surely. Finally, when $p < 1/2$, the origin is an attainable boundary, and can be chosen to be either absorbing or reflecting. Absorption is compulsory if Y is required to be a martingale [94, Chapter III, Lemma 3.6]. Here it is only used as a building block for the instantaneous variance, and such a requirement is therefore not needed, so that both cases (absorption and reflection) will be treated. Define the constants

$$\eta := \frac{1}{2(p-1)}, \qquad \vartheta := \log(y_0) - \frac{\xi^2 t}{2}, \tag{5.2.4}$$

and the function φ_η by

$$\varphi_\eta(y) := \frac{y_0^{1/2} y^{1/2-2p}}{|1-p|\xi^2 t} \exp\left(-\frac{y^{2(1-p)} + y_0^{2(1-p)}}{2\xi^2 t(1-p)^2}\right) \mathrm{I}_\eta\left(\frac{(y_0 y)^{1-p}}{(1-p)^2\xi^2 t}\right),$$

where I_η is the modified Bessel function of the first kind of order η [1, Section 9.6]. The CEV density, $\zeta_p(y) := \mathbb{P}(Y_t \in \mathrm{d}y)/\mathrm{d}y$, reads

$$\zeta_p(y) = \begin{cases} \varphi_{-\eta}(y), & \text{if } p \in [1/2, 1) \text{ or } p < \frac{1}{2} \text{ with absorption,} \\ \varphi_\eta(y), & \text{if } p > 1 \text{ or } p < \frac{1}{2} \text{ with reflection,} \\ \dfrac{1}{y\xi\sqrt{2\pi t}} \exp\left(-\dfrac{(\log(y) - \vartheta)^2}{2\xi^2 t}\right), & \text{if } p = 1, \end{cases} \tag{5.2.5}$$

valid for $y \in (0, \infty)$. When $p \geq 1$, the density ζ_p converges to zero around the origin, implying that paths are being pushed away from the origin. On the other hand ζ_p diverges to infinity at the origin when $p < 1/2$, so that the paths have a propensity towards the vicinity of the origin.

5.2.3 The cumulant generating function approach

In the literature on implied volatility asymptotics, the cumulant generating function of the stock price has proved to be an extremely useful tool to obtain sharp estimates. This is obviously the case for the wings of the smile (small and large strikes) via Roger Lee's formula, mentioned in Section 5.2.1.1, but also to describe short-and large-maturity asymptotics, as developed for instance in [95] or in Chapters 2, 3 and 4, via the use of (a refined version of) the Gärtner-Ellis theorem (Theorem 1.2.3). As shown in Section 5.2.1.1, the cumulant generating function of a stock price satisfying (5.2.1) is fully determined by that of the random variable $\mathcal{V}$. However, even though the density of the latter is known in closed-form (see Equation (5.2.5)), the cumulant generating function is not so for general values of p. In the cases $p = 0$ (with either reflecting or absorbing boundary) and $p = 1/2$, a closed-form expression is available and direct computations are possible.

5.2.3.1 Computation of the cumulant generating function

Let us denote by $\Lambda^{\mathcal{V}}_{0,r}$, $\Lambda^{\mathcal{V}}_{0,a}$ and $\Lambda^{\mathcal{V}}_{1/2}$ the cumulant generating function (cgf) of the random variable $\mathcal{V}$ when $p = 0$ (the subscript 'r' and 'a' denote the behaviour at the origin) and $p = 1/2$. Straightforward computations yield

$$\begin{aligned}
\Lambda^{\mathcal{V}}_{0,a}(u) &= \log\left(\mathrm{m}_t + \frac{1}{2}\exp\left(\frac{(u\xi^2 t - 2y_0)u}{2}\right)\left\{\mathrm{e}^{2uy_0}\mathcal{E}\left(\frac{u\xi^2 t + y_0}{\xi\sqrt{2t}}\right) + \mathrm{e}^{2uy_0} - 1\right.\right.\\
&\qquad \left.\left. -\mathcal{E}\left(\frac{u\xi^2 t - y_0}{\xi\sqrt{2t}}\right)\right\}\right),\\
\Lambda^{\mathcal{V}}_{0,r}(u) &= \log\left(\frac{1}{2}\exp\left(\frac{(u\xi^2 t - 2y_0)u}{2}\right)\left\{\mathrm{e}^{2uy_0}\mathcal{E}\left(\frac{u\xi^2 t + y_0}{\xi\sqrt{2t}}\right) + \mathrm{e}^{2uy_0} + 1\right.\right.\\
&\qquad \left.\left. +\mathcal{E}\left(\frac{u\xi^2 t - y_0}{\xi\sqrt{2t}}\right)\right\}\right),\\
\Lambda^{\mathcal{V}}_{1/2}(u) &= \frac{2y_0 u}{2 - u\xi^2 t},
\end{aligned} \tag{5.2.6}$$

where $\mathcal{E}(z) \equiv \frac{2}{\sqrt{\pi}}\int_0^z \exp(-x^2)\mathrm{d}x$ is the error function. Note that when $p = 1/2$ and $p = 0$ in the absorption case, one needs to take into account the mass at zero in (5.2.3) when computing these expectations.

5.2.3.2 Roger Lee's wing formula

In [116], Roger Lee provided a precise link between the slope of the total implied variance in the wings and the boundaries of the domain of the cumulant generating function of the stock price. More precisely, for any $\tau \geq 0$, let $u_+(\tau)$ and $u_-(\tau)$ be defined as

$$u_+(\tau) := \sup\{u \geq 1 : |\Lambda^Z(u,\tau)| < \infty\} \qquad \text{and} \qquad u_-(\tau) := \sup\{u \geq 0 : |\Lambda^Z(-u,\tau)| < \infty\}.$$

The implied volatility $\sigma_\tau(k)$ then satisfies

$$\limsup_{k\uparrow\infty} \frac{\sigma_\tau(k)^2\tau}{k} = \psi(u_+(\tau) - 1) =: \beta_+(\tau) \qquad \text{and} \qquad \limsup_{k\downarrow-\infty} \frac{\sigma_\tau(k)^2\tau}{|k|} = \psi(u_-(\tau))) =: \beta_-(\tau),$$

where the function ψ is defined by $\psi(u) = 2 - 4\left(\sqrt{u(u+1)} - u\right)$. Combining (5.2.6) and (5.2.2) yields a closed-form expression for the cumulant generating function of the stock price when $p \in \{0, 1/2\}$. It is clear that, when $p = 0$, $u_\pm(\tau) = \pm\infty$ for any $\tau \geq 0$, and hence the slopes of the left and right wings are equal to zero (the total variance flattens for small and large strikes). In the case where $p = 1/2$, explosion will occur as soon as $\left(\frac{1}{2}u(u-1)\tau\xi^2 t - 2\right) = 0$, so that

$$u_\pm(\tau) = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{16}{\xi^2 t\tau}}, \quad \text{and} \quad \beta_-(\tau) = \beta_+(\tau) = \frac{2}{\xi\sqrt{t\tau}}\left(\sqrt{\xi^2 t\tau + 16} - 4\right), \quad \text{for all } \tau > 0.$$

The left and right slopes are the same, but the product $\xi^2 t$ can be directly calibrated on the observed wings. Note that $\beta_\pm(\tau)$ is concave and increasing from 0 to 2 as the product $\xi^2 t$ ranges from zero to infinity. As τ tends to infinity, $\beta_\pm(\tau)$ converges to 2, so that the implied volatility smile does not 'flatten out', as is usually the case for Itô diffusions or affine stochastic volatility models (see for instance [95]). In Section 5.2.5 below, we make this more precise by investigating the large-time behaviour of the implied volatility using the density of the CEV-distributed variance.

5.2.3.3 Small-time asymptotics

In order to study the small-maturity behaviour of the implied volatility, one could, whenever the moment generating function of the stock price is available in closed form (e.g. in the case $p \in \{0, 1/2\}$), apply the methodology developed in [62]. The latter is based on the Gärtner-Ellis theorem, which, essentially, consists of finding a smooth convex pointwise limit (as τ tends to zero) of some rescaled version of the cumulant generating function. In the case where $p = 1/2$, it is easy to show that

$$\lim_{\tau\downarrow 0} \tau^{1/2} \log \Lambda^Z\left(\frac{u}{\sqrt{\tau}}, \tau\right) = \begin{cases} 0, & \text{if } u \in \left(-\frac{2}{\xi\sqrt{t}}, \frac{2}{\xi\sqrt{t}}\right), \\ +\infty, & \text{otherwise.} \end{cases} \tag{5.2.7}$$

The nature of this limiting behaviour falls outside the scope of the Gärtner-Ellis theorem, which requires strict convexity of the limiting rescaled cumulant generating function. It is easy to see that any other rescaling would yield even more degenerate behaviour. One could adapt the proof of the Gärtner-Ellis theorem, as was done in Chapter 3 for the small-maturity behaviour of the forward implied volatility smile in the Heston model (see also [47] and references therein for more examples of this kind). In the case (5.2.7), we are exactly as in the framework of Chapter 3, in which the small-maturity smile (squared) indeed explodes as $\tau^{-1/2}$, precisely the same explosion as the one in (5.1.1). Unfortunately, as we mentioned above, the cumulant generating function of the stock price is not available in general, and this approach is hence not amenable here.

5.2.3.4 Large-time asymptotics

The analysis above, based on the cumulant generating function of the stock price, can be carried over to study the large-time behaviour of the implied volatility. In the case $p = 1/2$, computations are fully explicit, and the following pointwise limit follows from simple straightforward manipulations:

$$\lim_{\tau\uparrow\infty} \tau^{-1}\Lambda^Z(u,\tau) = \begin{cases} 0, & \text{if } u \in [0,1], \\ +\infty, & \text{otherwise.} \end{cases}$$

The nature of this asymptotic behaviour, again, falls outside the scope of standard large deviations analysis, and tedious work, in the spirit of Chapter 3, would be needed to pursue this route.

5.2.4 Small-time behaviour of option prices and implied volatility

For any $k \in \mathbb{R}^*$, $T > 0$, and $p > 1$, the quantity (BS defined in (1.0.2))

$$\mathrm{J}^p(k) := \begin{cases} \int_0^\infty \mathrm{BS}\left(k, \frac{y}{T}, T\right) y^{-p}\mathrm{d}y, & \text{if } k > 0, \\ \int_0^\infty \left(\mathrm{e}^k - 1 + \mathrm{BS}\left(k, \frac{y}{T}, T\right)\right) y^{-p}\mathrm{d}y, & \text{if } k < 0, \end{cases} \tag{5.2.8}$$

is clearly independent of T and is well defined. Indeed, consider the case $k > 0$. Since the stock price is a martingale starting at one, Call options are always bounded above by one, and hence $\mathrm{J}^p(k) \leq \int_0^1 \mathrm{BS}(k, y/T, T)y^{-p}\mathrm{d}y + \int_1^\infty y^{-p}\mathrm{d}y$. The second integral is finite since $p > 1$. When $k > 0$, the asymptotic behaviour

$$\mathrm{BS}\left(k, \frac{y}{T}, T\right) \sim \exp\left(-\frac{k^2}{2y} + \frac{k}{2}\right) \frac{y^{3/2}}{k^2\sqrt{2\pi}}$$

holds as y tends to zero, so that $\lim_{y\downarrow 0} \mathrm{BS}(k, y/T, T)y^{-p} = 0$, and hence the integral is finite. A similar analysis holds when $k < 0$ (using put-call symmetry). Define now the following constants:

$$\beta_p := \frac{1}{3-2p}, \qquad \overline{y}_p := \left(\frac{k^2\xi^2 t(1-p)}{2}\right)^{\beta_p}, \qquad y^* := \frac{k^2\xi^2 t}{2}, \tag{5.2.9}$$

the first two only when $p < 1$, and note that $\beta_p \in (0, 1)$; define further the following functions from $\mathbb{R}_+^*$ to $\mathbb{R}$:

$$\begin{cases} f_0(y) := \frac{k^2}{2y} + \frac{y^{2(1-p)}}{2\xi^2 t(1-p)^2}, & f_1(y) := \frac{(yy_0)^{(1-p)}}{\xi^2 t(1-p)^2}, \\ g_0(y) := \frac{k^2}{2y} + \frac{\log(y)}{\xi^2 t}, & g_1(y) := \frac{\log(y)}{\xi^2 t}, \end{cases} \tag{5.2.10}$$

as well as the following ones, parameterised by p:

	$p<1$	$p=1$	$p>1$
$c_1(t,p)$	$f_0(\overline{y}_p)$	$1/(2\xi^2 t)$	0
$c_2(t,p)$	$f_1(\overline{y}_p)$	$1/(2\xi^2 t)$	0
$c_3(t,p)$	$\frac{6-5p}{6-4p}$	$g_0(y^*) - \frac{\vartheta}{\xi^2 t}$	$2p-1$
$c_4(t,p)$	0	$g_1(y^*) - \frac{\vartheta}{\xi^2 t} - 2$	0
$c_5(t,p)$	$\frac{y_0^{\frac{p}{2}} \overline{y}_p^{\frac{3}{2}(1-p)} \mathrm{e}^{\frac{k}{2} - \frac{y_0^{2(1-p)}}{2\xi^2 t(1-p)^2} + \frac{f_1'(\overline{y}_p)^2}{2f_0''(\overline{y}_p)}}}{k^2 \xi \sqrt{2\pi f_0''(\overline{y}_p) t}}$	$\frac{\exp\left(\frac{k}{2} - \frac{\vartheta^2}{2\xi^2 t} + \frac{\vartheta \log(y^*)}{\xi^2 t}\right)}{4\sqrt{\pi}\|k\|^{-1}\xi^{-3}t^{-3/2}}$	$\frac{2(p-1)\mathrm{e}^{-\frac{y_0^{2(p-1)}}{2\xi^2 t(1-p)^2}} \mathrm{J}^{2p}(k)}{(2(1-p)^2\xi^2 t)^{\eta+1}\Gamma(\eta+1)}$
$h_1(\tau,p)$	$\tau^{\beta_p - 1}$	$(\log(\tau) + \log\|\log(\tau)\|)^2$	0
$h_2(\tau,p)$	$\tau^{(\beta_p-1)/2}$	$\frac{(\log\|\log(\tau)\|)^2}{\|\log(\tau)\|}$	0
$\mathcal{R}(\tau,p)$	$\mathcal{O}\left(\tau^{(1-\beta_p)/2}\right)$	$\mathcal{O}\left(\frac{1}{\|\log(\tau)\|}\right)$	$\mathcal{O}(\tau^{p-1})$

Table 5.1: List of constants and functions

The following theorem (proved in Section 5.3.1) is the central result of this chapter (although its equivalent below, in terms of implied volatility, is more informative for practical purposes):

Theorem 5.2.1. *The following expansion holds for all $k \in \mathbb{R}^*$ as τ tends to zero:*

$$\mathbb{E}\left(\mathrm{e}^{Z_\tau} - \mathrm{e}^k\right)^+ = (1-\mathrm{e}^k)^+ + \mathrm{e}^{-c_1(t,p)h_1(\tau,p)+c_2(t,p)h_2(\tau,p)} \tau^{c_3(t,p)} |\log(\tau)|^{c_4(t,p)} c_5(t,p) \left[1 + \mathcal{R}(\tau,p)\right].$$

Remark 5.2.2.

(i) Whenever $p \leq 1$, c_1 and c_2 are strictly positive; the function c_5 is always strictly positive; when $p<1$, c_3 is strictly positive; when $p=1$, the functions c_3 and c_4 can take positive and negative values;

(ii) Whenever $p \leq 1$, $h_2(\tau,p) \leq h_1(\tau,p)$ for τ small enough, so that the leading order is provided by h_1;

(iii) In the lognormal case $p=1$, $h_1(\tau,1) \sim (\log \tau)^2$ as τ tends to zero, so that the exponential decay of option prices is governed at leading order by $\exp(-c_1(t,1)(\log\tau)^2)$.

Using Theorem 5.2.1 and small-maturity asymptotics for the Black-Scholes model in Lemma 3.3.4, it is straightforward to translate option price asymptotics into asymptotics of the implied volatility:

Theorem 5.2.3. *For any $k \in \mathbb{R}^*$, the small-maturity implied volatility smile behaves as follows:*

$$\sigma_\tau^2(k) \sim \begin{cases} (1-\beta_p)\left(\dfrac{k^2\xi^2 t(1-p)}{2\tau}\right)^{\beta_p}, & \text{if } p < 1,\\ \dfrac{k^2\xi^2 t}{\tau\log(\tau)^2}, & \text{if } p = 1,\\ \dfrac{k^2}{2(2p-1)\tau|\log(\tau)|}, & \text{if } p > 1. \end{cases}$$

This theorem only presents the leading-order asymptotic behaviour of the implied volatility as the maturity becomes small. One could in principle (following [69] or [38]) derive higher-order terms, but these additional computations would impact the clarity of this singular behaviour. In the at-the-money $k = 0$ case, the implied volatility converges to a constant:

Lemma 5.2.4. *The at-the-money implied volatility $\sigma_\tau(0)$ converges to $\mathbb{E}(\sqrt{\mathcal{V}})$ as τ tends to zero.*

The proof of the lemma follows steps analogous to Lemma 3.4.3, and we omit the details here. Note that, from Theorem 5.2.3, as p approaches 1 from below, the rate of explosion approaches τ^{-1}. When p tends to 1 from above, the explosion rate is $1/(\tau|\log\tau|)$ instead. So there is a 'discontinuity' at $p = 1$ and the actual rate of explosion is less than both these limits. As an immediate consequence of Theorem 5.2.1 we have the following corollary. Define the following functions:

$$h^*(\tau,p) := \begin{cases} \tau^{1-\beta_p}, & \text{if } p < 1,\\ \left|\log(\tau)^{-1}\right|, & \text{if } p > 1,\\ \log(\tau)^{-2}, & \text{if } p = 1, \end{cases} \qquad \text{and} \qquad \Lambda_p^*(k) := \begin{cases} c_1(t,p), & \text{if } p \le 1,\\ 2p-1, & \text{if } p > 1, \end{cases}$$

where $c_1(t,p)$ is defined in Table 5.1, and depends on k (through $\overline{y}_p$).

Corollary 5.2.5. *For any $p \in \mathbb{R}$, the sequence $(Z_\tau)_{\tau\ge 0}$ satisfies a large deviations principle with speed $h^*(\tau,p)$ and rate function Λ_p^* as τ tends to zero. Furthermore, the rate function is good only when $p < 1$.*

Proof. The proof of Theorem 5.2.1 holds with only minor modifications for digital options, which are equivalent to probabilities of the form $\mathbb{P}(Z_\tau \le k)$ or $\mathbb{P}(Z_\tau \ge k)$. For $p \in (-\infty, 1]$, one can then show that

$$\lim_{\tau\downarrow 0} h^*(\tau,p)\log\mathbb{P}(Z_\tau \le k) = -\inf\left\{\Lambda_p^*(x) : x \le k\right\}.$$

The infimum is null whenever $k > 0$ and $p < 1$, and $\Lambda_1^*(x) \equiv 1/(2\xi^2 t)$ is constant. Consider now an open interval $(a,b) \subset \mathbb{R}$. Since $(a,b) = (-\infty,b) \setminus (-\infty,a]$, then by continuity and convexity of Λ_p^*, we obtain

$$\lim_{\tau\downarrow 0} h^*(\tau,p)\log\mathbb{P}(Z_\tau \in (a,b)) = -\inf_{x\in(a,b)} \Lambda_p^*(x).$$

Since any Borel set of the real line can be written as a (countable) union / intersection of open intervals, the corollary follows from the definition of the large deviations principle [48, Section 1.2]. When $p \in (1,\infty)$, the only non-trivial choice of speed is $|(\log\tau)^{-1}|$, in which case

$\lim_{\tau\downarrow 0}|(\log\tau)^{-1}|\log\mathbb{P}(Z_\tau \le k) = -(2p-1)$. Clearly, the constant function is a rate function (the level sets, either the empty set or the real line, being closed in $\mathbb{R}$), and the corollary follows. $\square$

Remark 5.2.6. In the case $p = 1/2$, as discussed in Section 5.2.3.4, the cumulant generating function of Z is available in closed-form. However, the large deviations principle does not follow from the Gärtner-Ellis theorem, since the pointwise rescaled limit of the cgf is degenerate (in the sense of (5.2.7)).

5.2.4.1 Small-maturity at-the-money skew and convexity

The goal of this section is to compute asymptotics for the at-the-money skew and convexity of the smile as the maturity becomes small. These quantities are useful to traders who actually observe them (or approximations thereof) on real data. We define the left and right derivatives by $\partial_k^-\sigma_\tau^2(0) := \lim_{k\uparrow 0}\partial_k\sigma_\tau^2(k)|_{k=0}$ and $\partial_k^+\sigma_\tau^2(0) := \lim_{k\downarrow 0}\partial_k\sigma_\tau^2(k)|_{k=0}$, and similarly $\partial_{kk}^-\sigma_\tau^2(0) := \lim_{k\uparrow 0}\partial_{kk}\sigma_\tau^2(k)|_{k=0}$ and $\partial_{kk}^+\sigma_\tau^2(0) := \lim_{k\downarrow 0}\partial_{kk}\sigma_\tau^2(k)|_{k=0}$. The following lemma describes this short-maturity behaviour in the general case where $\mathcal{V}$ is any random variable supported on $[0,\infty)$.

Lemma 5.2.7. *Consider* (5.2.1) *and assume that* $\mathbb{E}(\mathcal{V}^{n/2}) < \infty$ *for* $n = -1, 1, 3$, *and* $\mathrm{m}_t := \mathbb{P}(\mathcal{V} = 0) < 1$. *As* τ *tends to zero,*

$$\begin{aligned}
\partial_k^-\sigma_\tau^2(0) &\sim -\frac{\mathbb{E}(\sqrt{\mathcal{V}})}{48}\left(\mathbb{E}(\mathcal{V}^{3/2}) - \mathbb{E}(\sqrt{\mathcal{V}})^3\right)\tau - \mathrm{m}_t\frac{\mathbb{E}(\sqrt{\mathcal{V}})\sqrt{\pi}}{\sqrt{2\tau}},\\
\partial_k^+\sigma_\tau^2(0) &\sim -\frac{\mathbb{E}(\sqrt{\mathcal{V}})}{48}\left(\mathbb{E}(\mathcal{V}^{3/2}) - \mathbb{E}(\sqrt{\mathcal{V}})^3\right)\tau + \mathrm{m}_t\frac{\mathbb{E}(\sqrt{\mathcal{V}})\sqrt{\pi}}{\sqrt{2\tau}},\\
\partial_{kk}^-\sigma_\tau^2(0) &\sim \partial_{kk}^+\sigma_\tau^2(0) \sim \frac{\mathbb{E}(\sqrt{\mathcal{V}})}{\tau}\left(\mathbb{E}\left(\mathcal{V}^{-1/2}\right) - \mathbb{E}(\sqrt{\mathcal{V}})^{-1}\left(1 - \frac{\mathrm{m}_t^2\sqrt{\pi}}{8}\right)\right).
\end{aligned}$$

Jensen's inequality and the fact that the support of $\mathcal{V}$ is in $\mathbb{R}_+$ imply that both $\mathbb{E}(\mathcal{V}^{3/2})-\mathbb{E}(\sqrt{\mathcal{V}})^3$ and $\mathbb{E}\left(\mathcal{V}^{-1/2}\right) - \mathbb{E}(\sqrt{\mathcal{V}})^{-1}$ are strictly positive. The small-maturity at-the-money skew is always negative for small m_t. Note that this in particular means that the smile generated by (5.2.1) is not necessarily symmetric. When $\mathrm{m}_t > 0$, the at-the-money left skew explodes to $-\infty$ and the at-the-money right skew explodes to $+\infty$. Furthermore, the small-maturity at-the-money convexity tends to infinity. In the CEV case, however, the moments are not available in closed-form in general.

Proof. We first focus on the at-the-money skew. By definition $C(k,\tau) = \mathrm{BS}(k, \sigma_\tau^2(k), \tau)$ and therefore

$$\partial_k C(k,\tau) = \partial_k\mathrm{BS}(k,\sigma_\tau^2(k),\tau) + \partial_k\sigma_\tau^2(k)\partial_w\mathrm{BS}(k,\sigma_\tau^2(k),\tau),$$

where $\partial_w\mathrm{BS}$ is the partial derivative with respect to the second argument. Also by (5.3.1), an immediate application of Leibniz's integral rule yields

$$\partial_k C(k,\tau) = \int_0^\infty \partial_k\mathrm{BS}(k,y,\tau)\zeta_p(y)\mathrm{d}y + \mathrm{m}_t\partial_k\left(1-\mathrm{e}^k\right)^+, \tag{5.2.11}$$

We first assume that $\mathrm{m}_t = 0$. The at-the-money skew is then given by

$$\partial_k \sigma_\tau^2(k)|_{k=0} = \left(\partial_w \mathrm{BS}(k, \sigma_\tau^2(k), \tau)|_{k=0}\right)^{-1} \left(\int_0^\infty \partial_k \mathrm{BS}(k, y, \tau)|_{k=0} \zeta_p(y)\mathrm{d}y - \partial_k \mathrm{BS}(k, \sigma_\tau^2(k), \tau)|_{k=0}\right).$$

Recall now from Lemma 5.2.4 that $\sigma_\tau(0) = \mathbb{E}(\sqrt{\mathcal{V}}) + o(1)$. Straightforward computations then yield

$$\begin{cases} \partial_k \mathrm{BS}(k, \sigma_\tau^2(k), \tau)|_{k=0} & = -\mathcal{N}\left(\frac{\sigma_\tau(0)\sqrt{\tau}}{2}\right) \\ & = -\frac{1}{2} + \frac{\sigma_\tau(0)\sqrt{\tau}}{2\sqrt{2\pi}} - \frac{\sigma_\tau^3(0)\tau^{3/2}}{48\sqrt{2\pi}} + \mathcal{O}\left(\sigma_\tau^5(0)\tau^{5/2}\right), \\ \partial_k \mathrm{BS}(k, y, \tau)|_{k=0} & = -\mathcal{N}\left(\frac{\sqrt{y}\sqrt{\tau}}{2}\right) \\ & = -\frac{1}{2} + \frac{\sqrt{y}\sqrt{\tau}}{2\sqrt{2\pi}} - \frac{y^{3/2}\tau^{3/2}}{48\sqrt{2\pi}} + \mathcal{O}\left(\tau^{5/2}\right), \\ \partial_w \mathrm{BS}(k, \sigma_\tau^2(k), \tau)|_{k=0} & = \frac{\sqrt{\tau}}{\sigma_\tau(0)\sqrt{2\pi}} \exp\left(-\frac{\sigma_\tau^2(0)\tau}{8}\right). \end{cases} \tag{5.2.12}$$

as τ tends to zero. Hence

$$\partial_k \sigma_\tau^2(k)|_{k=0} = \exp\left(\frac{\sigma_\tau^2(0)\tau}{8}\right) \frac{\sigma_\tau(0)}{2} \left(\mathbb{E}(\sqrt{\mathcal{V}}) - \sigma_\tau(0) - \frac{(\mathbb{E}(\mathcal{V}^{3/2}) - \sigma_\tau(0)^3)\tau}{24} + \mathcal{O}\left(\sigma_\tau(0)^3\tau^5\right)\right),$$

and so, as τ tends to zero,

$$\partial_k \sigma_\tau^2(k)|_{k=0} \sim -\frac{\mathbb{E}(\sqrt{\mathcal{V}})}{48} \left(\mathbb{E}(\mathcal{V}^{3/2}) - \mathbb{E}(\sqrt{\mathcal{V}})^3\right)\tau, \tag{5.2.13}$$

The small-maturity convexity follows similar arguments, which we only outline:

$$\begin{aligned} \partial_{kk} C(k, \tau) = & \partial_{kk} \mathrm{BS}(k, \sigma_\tau^2(k), \tau) + 2\partial_k \sigma_\tau^2(k) \partial_{wk} \mathrm{BS}(k, \sigma_\tau^2(k), \tau) \\ & + \left(\partial_k \sigma_\tau^2(k)\right)^2 \partial_{ww} \mathrm{BS}(k, \sigma_\tau^2(k), \tau) + \partial_{kk} \sigma_\tau^2(k) \partial_w \mathrm{BS}(k, \sigma_\tau^2(k), \tau), \end{aligned} \tag{5.2.14}$$

and

$$\partial_{kk} C(k, \tau) = \int_0^\infty \partial_{kk} \mathrm{BS}(k, y, \tau) \zeta_p(y)\mathrm{d}y + \mathrm{m}_t \partial_{kk}\left(1 - \mathrm{e}^k\right)^+. \tag{5.2.15}$$

Likewise, we first consider the case $\mathrm{m}_t = 0$. Straightforward computations yield

$$\begin{cases} \partial_{kk} \mathrm{BS}(k, \sigma_\tau^2(k), \tau)|_{k=0} & = \frac{\exp\left(-\frac{\sigma_\tau^2(0)\tau}{8}\right)}{\sigma_\tau(0)\sqrt{\tau}\sqrt{2\pi}} - \mathcal{N}\left(\frac{\sigma_\tau(0)\sqrt{\tau}}{2}\right) \\ & = \frac{1}{\sqrt{2\pi}\sigma_\tau(0)\sqrt{\tau}} - \frac{1}{2} + \mathcal{O}\left(\sigma_\tau(0)\sqrt{\tau}\right), \\ \partial_{kw} \mathrm{BS}(k, \sigma_\tau^2(k), \tau)|_{k=0} & = \exp\left(-\frac{\sigma_\tau^2(0)\tau}{8}\right) \frac{\sqrt{\tau}}{4\sigma_\tau(0)\sqrt{2\pi}}, \\ \partial_{ww} \mathrm{BS}(k, \sigma_\tau^2(k), \tau)|_{k=0} & = -\frac{\tau^2 \mathrm{e}^{-\frac{\tau\sigma_\tau^2(0)}{8}}(\tau\sigma_\tau^2(0) + 4)}{16\sqrt{2\pi}(\tau\sigma_\tau^2(0))^{3/2}} \\ & = -\frac{\sqrt{\tau}}{4\sqrt{2\pi}\sigma_\tau^3(0)} + \frac{3\tau^{5/2}\sigma_\tau(0)}{512\sqrt{2\pi}} + \mathcal{O}\left(\frac{\tau^{3/2}}{\sigma_\tau(0)}\right). \end{cases} \tag{5.2.16}$$

Using (5.2.14) and (5.2.15) in conjunction with (5.2.12),(5.2.16) and (5.2.13), we obtain $\partial_{kk}\sigma_\tau^2(0) \sim \frac{1}{\tau}\mathbb{E}(\sqrt{\mathcal{V}})\left(\mathbb{E}\left(\mathcal{V}^{-1/2}\right) - \mathbb{E}(\sqrt{\mathcal{V}})^{-1}\right)$. When $\mathrm{m}_t > 0$, we need to take right and left derivatives in (5.2.11) and (5.2.15) to account for the atomic term. Since $\partial_k^-\left(1 - \mathrm{e}^k\right)^+|_{k=0} = \partial_{kk}^-\left(1 - \mathrm{e}^k\right)^+|_{k=0} = -1$ and $\partial_k^+\left(1 - \mathrm{e}^k\right)^+|_{k=0} = \partial_{kk}^+\left(1 - \mathrm{e}^k\right)^+|_{k=0} = 0$, the lemma follows immediately. □

5.2.5 Large-time behaviour of option prices and implied volatility

In this section we compute the large-time behaviour of option prices and implied volatility. The proofs are given in Section 5.3.2. It turns out that asymptotics are degenerate in the sense that option prices decay algebraically to their intrinsic values. The structure of the asymptotic depends on the value of p and whether the origin is reflecting or absorbing:

Theorem 5.2.8. *Define the following quantity:*

$$\mathfrak{M}(\eta) := \frac{2^{3-6p-\eta}\Gamma\left(\frac{1}{2}-2p\right)}{\sqrt{\pi}\Gamma(1+\eta)|1-p|^{2\eta+1}(\xi^2 t)^{\eta+1}} \exp\left(-\frac{y_0^{2(1-p)}}{2\xi^2 t(1-p)^2}\right),$$

with η given in (5.2.4). *The following expansions hold for all $k \in \mathbb{R}$ as τ tends to infinity:*

(i) if $p < 3/4$ and the origin is absorbing then

$$\mathbb{E}\left(\mathrm{e}^{Z_\tau} - \mathrm{e}^k\right)^+ = 1 - \mathrm{m}_t + \mathrm{m}_t(1-\mathrm{e}^k)^+ - 8\mathrm{e}^{k/2} y_0 \left(\frac{1}{2} - 2p\right)\mathfrak{M}(-\eta)\frac{1+\mathcal{O}\left(\tau^{-1}\right)}{\tau^{2-2p}};$$

(ii) if $p < 1/4$ and the origin is reflecting then

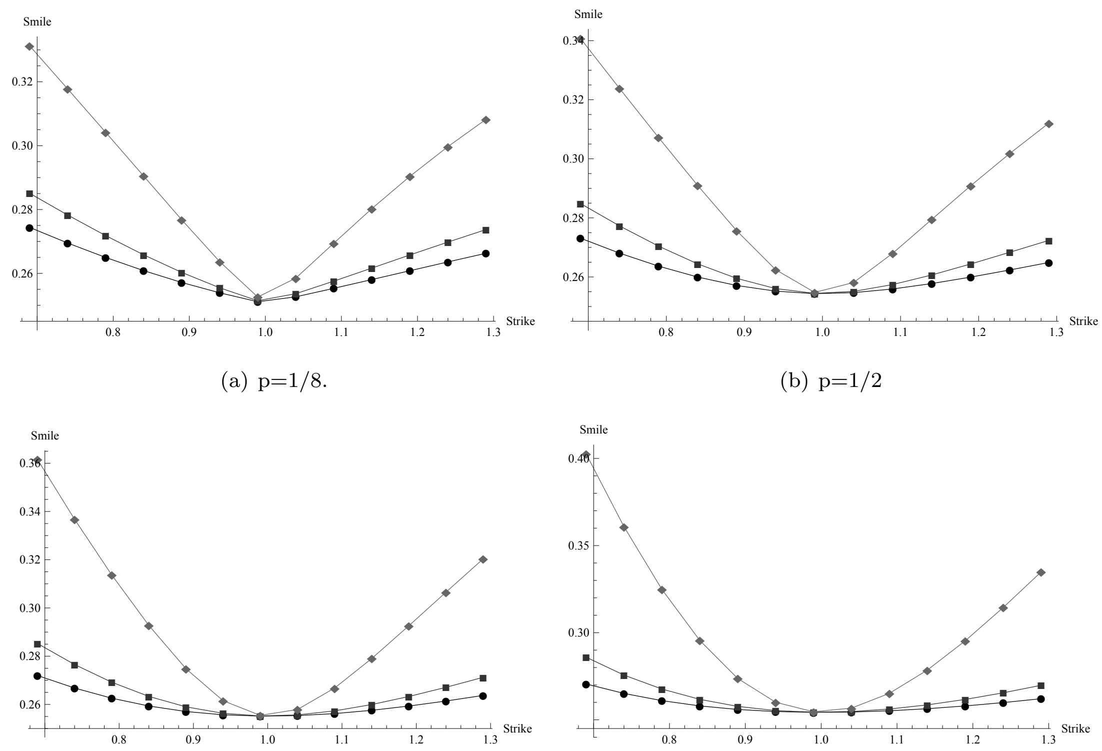
$$\mathbb{E}\left(\mathrm{e}^{Z_\tau} - \mathrm{e}^k\right)^+ = 1 - \mathrm{e}^{k/2}\mathfrak{M}(\eta)\frac{1+\mathcal{O}\left(\tau^{-1}\right)}{\tau^{1-2p}}.$$

For other values of p, asymptotics are more difficult to derive and we leave this for future research. The asymptotic behaviour of option prices is fundamentally different to Black-Scholes asymptotics (Lemma D.0.11) and it is not clear that one can deduce asymptotics for the implied volatility. For example, the intrinsic values do not necessarily match as τ tends to infinity because of the mass at the origin. The one exception is when the origin is reflecting, in which case the implied volatility tends to zero. This result follows directly from the comparison of Theorem 5.2.8 and Lemma D.0.11.

Theorem 5.2.9. *If $p < 1/4$ and the origin is reflecting, then for all $k \in \mathbb{R}$ as τ tends to infinity:*

$$\sigma_\tau^2(k) \sim \frac{8(1-2p)\log\tau}{\tau}.$$

Although, we have provided the large-time asymptotics in this section, it is not our intention to use this model for options with large expiries. Our intention (as mentioned in Section 5.1) is to use these models as building blocks for more complicated models (such as stochastic volatility models where the initial variance is sampled from a continuous distribution) so that we are able to better match steep small-maturity observed smiles. In these types of more sophisticated models, the large-time behaviour is governed more from the chosen stochastic volatility model rather than the choice of distribution for the initial variance (see Chapters 2 and 4 for examples), especially if the variance process possesses some ergodic properties. This also suggests to use this class of models to introduce two different time scales: one to match the small-time smile (the distribution for the initial variance) and one to match the medium to large-time smile (the chosen stochastic volatility model).



(a) p=1/8.

(b) p=1/2

(c) p=1

(d) p=3/2

Figure 5.1: Here we plot $K \mapsto \sigma_\tau(\log K)$ for maturities of 1 (circles), 1/2 (squares) and 1/12 (diamonds) for increasing values of the CEV power p. The smile is obtained by numerically solving for the option price using (5.3.1) and then using a simple root search to solve for the implied volatility. Parameters of the model are given in the text.

5.2.6 Numerics

We calculate option prices using the representation (5.3.1) and a global adaptive Gauss-Kronrod quadrature scheme. We then compute the smile σ_τ with a simple root-finding algorithm. In Figure 5.1 we plot the smile for different maturities and values for the CEV power p. The model parameters are $y_0 = 0.07$, $\xi = 0.2 y_0^{1/2-p}$ and $t = 1/2$. Note here that we set ξ to be a different value for each p. This is done so that the models are comparable: ξ is then given in the same units and the quadratic variation of the CEV variance dynamics are approximately matched for different values of p. The graphs highlight the steepness of the smiles as the maturity gets smaller and the role of p in the shape of the small-maturity smile. Out-of-the money volatilities (for $K \notin [0.9, 1.1]$) explode at a quicker rate as p increases (this can be seen from Theorem 5.2.3). The volatility for strikes close to at-the-money $K \in [0.9, 1.1]$ appears to be less explosive as one increases p. This can be explained from the strike dependence of the coefficients of the asymptotic in Theorem 5.2.3 and is discussed further in Section 5.2.7.1.

5.2.7 Application to forward smile asymptotics

We now show how our model (5.2.1) and the asymptotics derived above for the implied volatility can be directly translated into asymptotics of the forward implied volatility in stochastic volatility models. Suppose now that the log stock price process X satisfies the following SDE:

$$\begin{aligned} \mathrm{d}X_s &= -\frac{1}{2}Y_s\mathrm{d}s + \sqrt{Y_s}\mathrm{d}W_s, & X_0 = 0,\\ \mathrm{d}Y_s &= \xi_s Y_s^p \mathrm{d}B_s, & Y_0 = y_0 > 0,\\ \mathrm{d}\langle W, B\rangle_s &= \rho \mathrm{d}s, \end{aligned} \tag{5.2.17}$$

with $p \in \mathbb{R}$, $|\rho| < 1$ and $(W_s)_{s\geq 0}$ and $(B_s)_{s\geq 0}$ are two standard Brownian motions. Fix the forward-start date $t > 0$ and set

$$\xi_u := \begin{cases} \xi, & \text{if } 0 \leq u \leq t,\\ \bar{\xi}, & \text{if } u > t, \end{cases} \tag{5.2.18}$$

where $\xi > 0$ and $\bar{\xi} \geq 0$. This includes the Heston model and 3/2 model with zero mean reversion ($p = 1/2$ and $p = 3/2$ respectively) as well as the SABR model ($p = 1$). Here we impose the condition that if the variance hits the origin, it is either absorbed or reflected (see Section 5.2.1 for further details). Consider the CEV process for the variance: $\mathrm{d}Y_u = \xi Y_u^p \mathrm{d}B_u$, $Y_0 = y_0$, where $p \in \mathbb{R}$ and B is a standard Brownian motion. Let $\mathrm{CEV}(t, \xi, p)$ be the distribution such that $\mathrm{Law}(Y_t) = \mathrm{Law}(\mathcal{V}) = \mathrm{CEV}(t, \xi, p)$. Then the following lemma holds (an application of Lemma 1.4.9):

Lemma 5.2.10. *In the model* (5.2.17) *the forward price process* $X^{(t)}_{\cdot}$ *(defined in* (1.0.3)*) solves the following system of SDEs:*

$$\begin{aligned} \mathrm{d}X_\tau^{(t)} &= -\frac{1}{2}Y_\tau^{(t)}\mathrm{d}\tau + \sqrt{Y_\tau^{(t)}}\mathrm{d}W_\tau, & X_0^{(t)} = 0,\\ \mathrm{d}Y_\tau^{(t)} &= \bar{\xi}\left(Y_\tau^{(t)}\right)^p \mathrm{d}B_\tau, & Y_0^{(t)} \sim \mathrm{CEV}(t, \xi, p),\\ \mathrm{d}\langle W, B\rangle_\tau &= \rho \mathrm{d}\tau, \end{aligned} \tag{5.2.19}$$

where $Y_0^{(t)}$ *is independent to the Brownian motions* $(W_\tau)_{\tau\geq 0}$ *and* $(B_\tau)_{\tau\geq 0}$.

If we set $\bar{\xi} = 0$, then $X^{(t)}_{\cdot} = Z$ and the following corollary provides forward smile asymptotics:

Corollary 5.2.11. *When* $\bar{\xi} = 0$ *in* (5.2.18)*, Theorem 5.2.1, Theorem 5.2.3 and Lemma 5.2.4 hold with* $Z = X^{(t)}_{\cdot}$ *and* $\sigma_\tau = \sigma_{t,\tau}$.

Remark 5.2.12.

(i) This result explicitly links the shape and fatness of the right tail of the variance distribution at the forward-start date and the asymptotic form and explosion rate of the small-maturity forward smile. Take for example $p > 1$: the density of the variance in the right wing is dominated by the polynomial y^{-2p} and the exponential dependence on y is irrelevant. So the smaller p in this case, the fatter the right tail and hence the larger the coefficient of the expansion. This also explains the algebraic (not exponential) small-maturity dependence for forward-start option prices.

(ii) The asymptotics in the $p > 1$ case are extreme and the algebraic dependence on τ is similar to small-maturity exponential Lévy models. This extreme nature is related to the fatness of the right tail of the variance distribution: for example, the 3/2 model ($p = 3/2$) allows for the occurrence of extreme paths with periods of very high instantaneous volatility (see [54, Figure 3]).

(iii) The asymptotics in Theorems 5.2.1 and 5.2.3 remain the same (at this order) regardless of whether the variance process is absorbing or reflecting at zero when $p \in (-\infty, 1/2)$. Intuitively this is because absorption or reflection primarily influences the left tail whereas small-maturity forward smile asymptotics are influenced by the shape of the right tail of the variance distribution.

5.2.7.1 Conjecture

When $p = 1/2$ in Corollary 5.2.11, the asymptotics are the same as in Theorem 3.4.1 for the Heston model. This confirms that the key quantity determining the small-maturity forward smile explosion rate is the variance distribution at the forward-start date. The dynamics of the stock price are actually irrelevant at this order. This leads us to the following conjecture:

Conjecture 5.2.13. The leading-order small-maturity forward smile asymptotics generated from (5.2.17) are equivalent to those given in Corollary 5.2.11.

Practitioners have stated [10, 36] that the Heston model ($p = 1/2$) produces small-maturity forward smiles that are too convex and 'U-shaped' and inconsistent with observations. Furthermore, it has been empirically stated [10] that SABR or lognormal based models for the variance ($p = 1$) produce less convex or 'U-shaped' small-maturity forward smiles. Our results provide theoretical insight into this effect. We observed in Section 5.2.6 and Figure 5.1 that the explosion effect was more stable for strikes close to the money as one increased p. The strike dependence of the asymptotic implied volatility in Theorem 5.2.3 is given by $K \mapsto \sqrt{|\log K|}$ for $p = 1/2$ and $K \mapsto |\log K|$ for $p = 1$. It is clear that the shape of the forward implied volatility is more stable and less U-shaped in the lognormal $p = 1$ case.

5.3 Proofs

5.3.1 Proof of Theorem 5.2.1

Let $C(k,\tau) := \mathbb{E}(\mathrm{e}^{X_\tau} - \mathrm{e}^k)^+$. This function clearly depends on the parameter t, but we omit this dependence in the notations. The tower property implies

$$C(k,\tau) = \int_0^\infty \mathrm{BS}(k,y,\tau)\zeta_p(y)\mathrm{d}y + \mathrm{m}_t\left(1-\mathrm{e}^k\right)^+, \tag{5.3.1}$$

where BS is defined in (1.0.2), ζ_p is density of $\mathcal{V}$ given in (5.2.5) and m_t is the mass at the origin (5.2.3). Our goal is to understand the asymptotics of this integral as τ tends to zero. We break the proof of Theorem 5.2.1 into three parts: in Section 5.3.1.1 we prove the case $p > 1$, in Section 5.3.1.2 we prove the case $p \in (-\infty, 1)$ and in Section 5.3.1.3 we prove the case $p = 1$. We only prove the result for $k > 0$, the arguments being completely analogous when $k < 0$. The key insight is that one has to re-scale the variance in terms of the maturity τ before asymptotics can be computed. The nature of the re-scaling depends critically on the CEV power p and fundamentally different asymptotics result in each case. Note that for $k > 0$, $\left(1-\mathrm{e}^k\right)^+ = 0$, so that the atomic term in (5.3.1) is irrelevant for the analysis. When $k < 0$, asymptotics follow directly using Put-Call symmetry.

5.3.1.1 Case: $p > 1$

In Lemma 5.3.1 we prove a bound on the CEV density. This is sufficient to allow us to prove asymptotics for option prices in Lemma 5.3.2 after rescaling the variance by τ. This rescaling is critical because it is the only one making $\mathrm{BS}(k, y/\tau, \tau)$ independent of τ. Let

$$\chi(\tau,p) := \frac{\tau^{2p}}{|1-p|\xi^2 t\Gamma(1+|\eta|)\Big(2(1-p)^2\xi^2 t\Big)^{|\eta|}} \exp\left(-\frac{y_0^{2(1-p)}}{2\xi^2 t(1-p)^2}\right),$$

and we have the following lemma:

Lemma 5.3.1. *The following bounds hold for the CEV density for all $y,\tau > 0$ when $p > 1$:*

$$\frac{\chi(\tau,p)}{y^{2p}}\left\{1-\frac{1}{2\xi^2 t(1-p)^2}\left(\frac{\tau}{y}\right)^{2p-2}\right\} \le \zeta_p\left(\frac{y}{\tau}\right),$$

$$\frac{\chi(\tau,p)}{y^{2p}}\left\{1+\exp\left(\frac{y_0^{2-2p}}{2(p-1)^2 t\xi^2}\right)\left[\frac{1}{2\xi^2 t(1-p)^2}\left(\frac{\tau}{y}\right)^{2p-2} + \frac{1}{\xi^2 t(1-p)^2}\left(\frac{\tau}{yy_0}\right)^{p-1}\right]\right\} \ge \zeta_p\left(\frac{y}{\tau}\right).$$

Proof. From [120] we know that for $x > 0$ and $\nu > -1/2$:

$$\frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu \le I_\nu(x) \le \frac{\cosh(x)}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu. \tag{5.3.2}$$

Also since $\cosh(x) < \mathrm{e}^x$ holds for $x > 0$, the expression for the CEV density in (5.2.5) implies that for $p > 1$,

$$\frac{\chi(\tau,p)}{y^{2p}}\exp\left(-\frac{1}{2\xi^2 t(1-p)^2}\left(\frac{\tau}{y}\right)^{2p-2}\right) \le \zeta_p\left(\frac{y}{\tau}\right) \le \frac{\chi(\tau,p)}{y^{2p}}\mathrm{e}^{m(y,\tau)},$$

where

$$m(y,\tau) := -\frac{1}{2\xi^2 t(1-p)^2}\left(\frac{\tau}{y}\right)^{2p-2} + \frac{1}{\xi^2 t(1-p)^2}\left(\frac{\tau}{yy_0}\right)^{p-1}.$$

For fixed $\tau > 0$, note that $m(\cdot,\tau) : \mathbb{R}_+ \mapsto \mathbb{R}_+$ takes a maximum positive value at $y = y_0\tau$ with $m(y_0\tau,\tau) = y_0^{2-2p}/(2(p-1)^2 t\xi^2)$. When $m > 0$ Taylor's Theorem with remainder yields $\mathrm{e}^{m(y,\tau)} = 1 + \mathrm{e}^{\gamma} m(y,\tau)$ for some $\gamma \in (0, m(y,\tau))$, and hence $\mathrm{e}^{m(y,\tau)} \le 1 + \mathrm{e}^{m(y_0\tau,\tau)} m(y,\tau)$. If $m < 0$ then $\mathrm{e}^{m(y,\tau)} \le 1 + |m(y,\tau)| \le 1 + \mathrm{e}^{m(y_0\tau,\tau)}|m(y,\tau)|$. The result for the upper bound then follows by the triangle inequality for $|m(y,\tau)|$. The lower bound simply follows from the inequality $1 - x \le \mathrm{e}^{-x}$, valid for $x > 0$, and

$$1 - \frac{1}{2\xi^2 t(1-p)^2}\left(\frac{\tau}{y}\right)^{2p-2} \le \exp\left(-\frac{1}{2\xi^2 t(1-p)^2}\left(\frac{\tau}{y}\right)^{2p-2}\right).$$

□

Lemma 5.3.2. *When $p > 1$, Theorem 5.2.1 holds.*

Proof. The substitution $y \to y/\tau$ and (5.3.1) imply that the option price reads

$$C(k,\tau) = \int_0^\infty \mathrm{BS}(k,y,\tau)\zeta_p(y)\mathrm{d}y = \tau^{-1}\int_0^\infty \mathrm{BS}(k,y/\tau,\tau)\zeta_p(y/\tau)\mathrm{d}y.$$

Using Lemma 5.3.1 and Definition (5.2.8), we obtain the following bounds:

$$\frac{\chi(\tau,p)}{\tau}\left[\mathrm{J}^{2p}(k) - \frac{\tau^{2p-2}}{2\xi^2 t(1-p)^2}\mathrm{J}^{4p-2}(k)\right] \le C(k,\tau),$$

$$\frac{\chi(\tau,p)}{\tau}\left[\mathrm{J}^{2p}(k) + \exp\left(\frac{y_0^{2-2p}}{2(p-1)^2 t\xi^2}\right)\left(\frac{\tau^{2p-2}}{2\xi^2 t(1-p)^2}\mathrm{J}^{4p-2}(k) + \frac{\tau^{p-1}}{\xi^2 t(1-p)^2 y_0^{p-1}}\mathrm{J}^{3p-1}(k)\right)\right] \ge C(k,\tau).$$

Hence for $\tau < 1$:

$$\left|\frac{C(k,\tau)\tau}{\chi(\tau,p)\mathrm{J}^{2p}(k)} - 1\right| \le \exp\left(\frac{y_0^{2-2p}}{2(p-1)^2 t\xi^2}\right)\left(\frac{\mathrm{J}^{4p-2}(k)}{2\xi^2 t(1-p)^2\mathrm{J}^{2p}(k)} + \frac{\mathrm{J}^{3p-1}(k)}{\xi^2 t(1-p)^2 y_0^{p-1}\mathrm{J}^{2p}(k)}\right)\tau^{p-1},$$

which proves the lemma since $\mathrm{J}^q(k)$ is strictly positive, finite and independent of τ whenever $q > 1$. □

5.3.1.2 Case: $p < 1$

We use the representation in (5.3.1) and break the domain of the integral up into a compact part and an infinite (tail) one. We prove in Lemma 5.3.4 that the tail integral is exponentially subdominant (compared to the compact part) and derive asymptotics for the integral in Lemma 5.3.5. This allows us to apply the Laplace method to the integral. We start with the following bound for the modified Bessel function of the first kind and then prove a tail estimate in Lemma 5.3.4.

Lemma 5.3.3. *The following bound holds for all $x > 0$ and $\nu > -3/2$:*

$$\mathrm{I}_\nu(x) < \frac{\nu+2}{\Gamma(\nu+2)}\left(\frac{x}{2}\right)^\nu \mathrm{e}^{2x}.$$

Proof. Let $x > 0$. Using (5.3.2) and the fact that that $\cosh(x) < \mathrm{e}^x$, we obtain

$$\frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu \le \mathrm{I}_\nu(x) \le \frac{\mathrm{e}^x}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu, \tag{5.3.3}$$

whenever $\nu > -1/2$. From [139, Theorem 7, page 522], for $\nu \ge -2$, the inequality $\mathrm{I}_\nu(x) < \mathrm{I}_{\nu+1}(x)^2/\mathrm{I}_{\nu+2}(x)$ holds, and hence combining it with the bounds in (5.3.3) we can write

$$\mathrm{I}_\nu(x) < \frac{\Gamma(\nu+3)}{(\Gamma(\nu+2))^2}\left(\frac{x}{2}\right)^\nu \mathrm{e}^{2x},$$

when $\nu > -3/2$. The lemma then follows from the trivial identity $\Gamma(\nu+3) = (\nu+2)\Gamma(\nu+2)$. □

Lemma 5.3.4. *Let $L > 1$ and $p < 1$. Then the following tail estimate holds as τ tends to zero:*

$$\int_L^\infty \mathrm{BS}\left(k, \frac{y}{\tau^{\beta_p}}, \tau\right)\zeta_p\left(\frac{y}{\tau^{\beta_p}}\right)\mathrm{d}y = \mathcal{O}\left(\exp\left(-\frac{1}{4\xi^2 t(1-p)}\left[\frac{L^{1-p}}{\tau^{(1-\beta_p)/2}} - y_0^{1-p}\right]^2\right)\right).$$

Proof. Lemma 5.3.3 and the density in (5.2.5) imply

$$\zeta_p\left(\frac{y}{\tau^{\beta_p}}\right) \le \frac{b_0}{\tau^{-2p\beta_p}} y^{-2p}\exp\left(-\frac{1}{2\xi^2 t(1-p)^2}\left\{\frac{y^{1-p}}{\tau^{\beta_p(1-p)}} - y_0^{1-p}\right\}^2 + \frac{(yy_0)^{1-p}}{\tau^{\beta_p(1-p)}\xi^2 t(1-p)^2}\right),$$

where the constant b_0 is given by

$$\frac{(\eta+2)}{|1-p|\xi^2 t\Gamma(\eta+2)\left(2(1-p)^2\xi^2 t\right)^\eta}, \qquad \text{resp.} \qquad \frac{(|\eta|+2)}{|1-p|\xi^2 t\Gamma(|\eta|+2)\left(2(1-p)^2\xi^2 t\right)^{|\eta|}},$$

if the origin is reflecting (resp. absorbing) when $p < 1/2$; the exact value of b_0 is however irrelevant for the analysis. Set now $L > 1$. Using this upper bound and the no-arbitrage inequality $\mathrm{BS}(\cdot) < 1$, we find

$$\begin{aligned}
&\int_L^\infty \mathrm{BS}\left(k, \frac{y}{\tau^{\beta_p}}, \tau\right)\zeta_p\left(\frac{y}{\tau^{\beta_p}}\right)\mathrm{d}y \le \int_L^\infty \zeta_p\left(\frac{y}{\tau^{\beta_p}}\right)\mathrm{d}y \\
&\le \frac{b_0}{\tau^{-2p\beta_p}}\int_L^\infty y^{-2p}\exp\left(-\frac{1}{2\xi^2 t(1-p)^2}\left\{\frac{y^{1-p}}{\tau^{\beta_p(1-p)}} - y_0^{1-p}\right\}^2 + \frac{(yy_0)^{1-p}}{\tau^{\beta_p(1-p)}\xi^2 t(1-p)^2}\right)\mathrm{d}y \\
&\le \frac{b_0}{\tau^{-2p\beta_p}}\int_L^\infty y^{1-2p}\exp\left(-\frac{1}{2\xi^2 t(1-p)^2}\left\{\frac{y^{1-p}}{\tau^{\beta_p(1-p)}} - y_0^{1-p}\right\}^2 + \frac{(yy_0)^{1-p}}{\tau^{\beta_p(1-p)}\xi^2 t(1-p)^2}\right)\mathrm{d}y,
\end{aligned}$$

where the last line follows since $y^{1-2p} > y^{-2p}$. Setting $q = \left(y^{1-p}/\tau^{\beta_p(1-p)} - y_0^{1-p}\right)/(\xi\sqrt{t}(1-p))$ yields

$$\begin{aligned}
&\int_L^\infty y^{1-2p}\exp\left(-\frac{\left(\frac{y^{1-p}}{\tau^{\beta_p(1-p)}} - y_0^{1-p}\right)^2}{2\xi^2 t(1-p)^2} + \frac{(yy_0)^{1-p}}{\tau^{\beta_p(1-p)}\xi^2 t(1-p)^2}\right)\mathrm{d}y \\
&= \frac{\xi\sqrt{t}(1-p)}{\tau^{2\beta_p(p-1)}}\left[\xi\sqrt{t}(1-p)\int_{L_\tau}^\infty q\mathrm{e}^{-\frac{q^2}{2}+\frac{y_0^{1-p}q}{\xi\sqrt{t}(1-p)}}\mathrm{d}q + y_0^{1-p}\int_{L_\tau}^\infty \mathrm{e}^{-\frac{q^2}{2}+\frac{y_0^{1-p}q}{\xi\sqrt{t}(1-p)}}\mathrm{d}q\right],
\end{aligned} \tag{5.3.4}$$

with $L_\tau := \left(L^{1-p}/\tau^{\beta_p(1-p)} - y_0^{1-p}\right)/(\xi\sqrt{t}(1-p)) > 0$ for small enough τ since $L > 1$ and $p \in (-\infty, 1)$. Set now (we always choose the positive root)

$$\tau^* := \left(\frac{L^{1-p}}{5y_0^{1-p}}\right)^{(\beta_p(1-p))^{-1}},$$

so that, for $\tau < \tau^*$ we have $L_\tau > 4y_0^{1-p}/(\xi\sqrt{t}(1-p))$ and hence for $q > L_\tau$:

$$\frac{y_0^{1-p}q}{\xi\sqrt{t}(1-p} \leq \frac{q^2}{4}.$$

In particular, for the integrals in (5.3.4) we have the following bounds for $\tau < \tau^*$:

$$\int_{L_\tau}^\infty q\exp\left(-\frac{q^2}{2} + \frac{y_0^{1-p}q}{\xi\sqrt{t}(1-p)}\right)\mathrm{d}q \leq \int_{L_\tau}^\infty q\exp\left(-\frac{q^2}{4}\right)\mathrm{d}q,$$
$$\int_{L_\tau}^\infty \exp\left(-\frac{q^2}{2} + \frac{y_0^{1-p}q}{\xi\sqrt{t}(1-p)}\right)\mathrm{d}q \leq \int_{L_\tau}^\infty \exp\left(-\frac{q^2}{4}\right)\mathrm{d}q.$$

For the first integral we simply obtain $\int_{L_\tau}^\infty q\exp(-q^2/4)\mathrm{d}q = 2\exp(-L_\tau^2/4)$. For the second integral we use the upper bound for the complementary normal distribution function [147, Section 14.8] to write $\int_{L_\tau}^\infty \mathrm{e}^{-q^2/4}\mathrm{d}q \leq 4L_\tau^{-1}\mathrm{e}^{-L_\tau^2/4}$. The lemma then follows from noting that $1-\beta_p = 2\beta_p(1-p)$. □

Lemma 5.3.5. *When $p < 1$, Theorem 5.2.1 holds.*

Proof. Let $\widetilde{\tau} := \tau^{\beta_p}$, with β_p defined in (5.2.9). Applying the substitution $y \to y/\widetilde{\tau}$ to (5.3.1) yields

$$\begin{aligned}C(k,\tau) &= \int_0^\infty \mathrm{BS}(k,y,\tau)\zeta_p(y)\mathrm{d}y = \frac{1}{\widetilde{\tau}}\int_0^\infty \mathrm{BS}\left(k,\frac{y}{\widetilde{\tau}},\tau\right)\zeta_p\left(\frac{y}{\widetilde{\tau}}\right)\mathrm{d}y\\ &= \frac{1}{\widetilde{\tau}}\int_0^L \mathrm{BS}\left(k,\frac{y}{\widetilde{\tau}},\tau\right)\zeta_p\left(\frac{y}{\widetilde{\tau}}\right)\mathrm{d}y + \frac{1}{\widetilde{\tau}}\int_L^\infty \mathrm{BS}\left(k,\frac{y}{\widetilde{\tau}},\tau\right)\zeta_p\left(\frac{y}{\widetilde{\tau}}\right)\mathrm{d}y,\end{aligned}$$

for some $L > 0$ to be chosen later. We start with the first integral. Using the asymptotics for the modified Bessel function of the first kind (1.5.6) as τ tends to zero, we obtain

$$\zeta_p\left(\frac{y}{\widetilde{\tau}}\right) = \frac{\tau^{3p\beta_p/2}y_0^{p/2}\mathrm{e}^{-\frac{y_0^{2(1-p)}}{2\xi^2t(1-p)^2}}}{\xi y^{3p/2}\sqrt{2\pi t}}\mathrm{e}^{-\frac{1}{\tau^{2\beta_p(1-p)}}\frac{y^{2(1-p)}}{2\xi^2t(1-p)^2}+\frac{1}{\tau^{\beta_p(1-p)}}\frac{(yy_0)^{(1-p)}}{\xi^2t(1-p)^2}}\left[1+\mathcal{O}\left(\tau^{(1-p)\beta_p}\right)\right].$$

Note that this expansion does not depend on the sign of η and so the same asymptotics hold regardless of whether the origin is reflecting or absorbing. In the Black-Scholes model, Call option prices satisfy (Lemma D.0.10):

$$\mathrm{BS}\left(k,\frac{y}{\widetilde{\tau}},\tau\right) = \frac{y^{3/2}}{k^2\sqrt{2\pi}}\left(\frac{\tau}{\widetilde{\tau}}\right)^{3/2}\exp\left(-\frac{k^2}{2y}\frac{\widetilde{\tau}}{\tau}+\frac{k}{2}\right)\left(1+\mathcal{O}\left(\frac{\tau}{\widetilde{\tau}}\right)\right),$$

as τ tends to zero. Using the identity $1-\beta_p = 2\beta_p(1-p)$ we then compute

$$\begin{aligned}&\frac{1}{\tau^{\beta_p}}\int_0^L \mathrm{BS}\left(k,\frac{y}{\tau^{\beta_p}},\tau\right)\zeta_p\left(\frac{y}{\tau^{\beta_p}}\right)\mathrm{d}y\\ &\quad= \frac{\tau^{\beta_p(4-3p)/2}y_0^{p/2}\mathrm{e}^{-\frac{y_0^{2(1-p)}}{2\xi^2t(1-p)^2}+\frac{k}{2}}}{2\pi k^2\xi\sqrt{t}}\int_0^L y^{\frac{3}{2}(1-p)}\mathrm{e}^{-\frac{f_0(y)}{\tau^{1-\beta_p}}+\frac{f_1(y)}{\tau^{(1-\beta_p)/2}}}\mathrm{d}y\left[1+\mathcal{O}\left(\tau^{(1-\beta_p)/2}\right)\right],\end{aligned}$$

where f_0, f_1 are defined in (5.2.10). Solving the equation $f_0'(y) = 0$ gives $y = \overline{y}_p$ with $\overline{y}_p$ defined in (5.2.9) and we always choose the positive root and set $L > \overline{y}_p$.

Let $I(\tau) := \int_0^L y^{\frac{3}{2}(1-p)} \exp\left(-\frac{f_0(y)}{\tau^{1-\beta_p}} + \frac{f_1(y)}{\tau^{(1-\beta_p)/2}}\right) \mathrm{d}y$. Then for some $\varepsilon > 0$ small enough, as τ tends to zero:

$$
\begin{aligned}
I(\tau) &\sim \mathrm{e}^{-\frac{f_0(\overline{y}_p)}{\tau^{1-\beta_p}} + \frac{f_1(\overline{y}_p)}{\tau^{(1-\beta_p)/2}} + \frac{f_1'(\overline{y}_p)^2}{2f_0''(\overline{y}_p)}} \overline{y}_p^{\frac{3}{2}(1-p)} \int_{\overline{y}_p-\varepsilon}^{\overline{y}_p+\varepsilon} \exp\left(-\frac{1}{2}\left[\frac{\sqrt{f_0''(\overline{y}_p)}(y-\overline{y}_p)}{\tau^{(1-\beta_p)/2}} - \frac{f_1'(\overline{y}_p)}{\sqrt{f_0''(\overline{y}_p)}}\right]^2\right) \mathrm{d}y \\
&\sim \mathrm{e}^{-\frac{f_0(\overline{y}_p)}{\tau^{1-\beta_p}} + \frac{f_1(\overline{y}_p)}{\tau^{(1-\beta_p)/2}} + \frac{f_1'(\overline{y}_p)^2}{2f_0''(\overline{y}_p)}} \overline{y}_p^{\frac{3}{2}(1-p)} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left[\frac{\sqrt{f_0''(\overline{y}_p)}(y-\overline{y}_p)}{\tau^{(1-\beta_p)/2}} - \frac{f_1'(\overline{y}_p)}{\sqrt{f_0''(\overline{y}_p)}}\right]^2\right) \mathrm{d}y \\
&= \exp\left(-\frac{f_0(\overline{y}_p)}{\tau^{1-\beta_p}} + \frac{f_1(\overline{y}_p)}{\tau^{(1-\beta_p)/2}} + \frac{f_1'(\overline{y}_p)^2}{2f_0''(\overline{y}_p)}\right) \tau^{(1-\beta_p)/2} \overline{y}_p^{\frac{3}{2}(1-p)} \sqrt{\frac{2\pi}{f_0''(\overline{y}_p)}}.
\end{aligned}
$$

The $\sim$ approximations here are exactly of the same type as in [75], and we refer the interested reader to this paper for details. It follows that as τ tends to zero:

$$
\frac{1}{\tau^{\beta_p}} \int_0^L \mathrm{BS}\left(k, \frac{y}{\tau^{\beta_p}}, \tau\right) \zeta_p\left(\frac{y}{\beta_p}\right) \mathrm{d}y = \exp\left(-\frac{c_1(t,p)}{\tau^{1-\beta_p}} + \frac{c_2(t,p)}{\tau^{(1-\beta_p)/2}}\right) c_5(t,p) \tau^{c_3(t,p)} \left[1 + \mathcal{O}\left(\tau^{\frac{1-\beta_p}{2}}\right)\right],
$$

with the functions c_1, c_2, c_3 and c_5 given in Table 5.1. From Lemma 5.3.4 we know that

$$
\frac{1}{\tau^{\beta_p}} \int_L^\infty \mathrm{BS}\left(k, \frac{y}{\tau^{\beta_p}}, \tau\right) \zeta_p(y/\beta_p) \mathrm{d}y = \mathcal{O}\left(\exp\left(-\frac{1}{2\xi^2 t(1-p)}\left(\frac{L^{1-p}}{\tau^{(1-\beta_p)/2}} - y_0^{1-p}\right)^2\right)\right).
$$

Choosing $L > \max\left(1, \left(2\xi^2 t(1-p) f_0(\overline{y}_p)\right)^{1/(2-2p)}, \overline{y}_p\right)$ makes this tail term exponentially subdominant to $\tau^{-\beta_p} \int_0^L \mathrm{BS}(k, y/\tau^{\beta_p}, \tau) \zeta_p(y/\beta_p) \mathrm{d}y$, which completes the proof of the lemma. □

5.3.1.3 Case: $p = 1$

We now consider the lognormal case $p = 1$. The proof is similar to Section 5.3.1.2, but we need to re-scale the variance by $\tau|\log(\tau)|$. We prove a tail estimate in Lemma 5.3.6 and derive asymptotics for option prices in Lemma 5.3.7.

Lemma 5.3.6. *The following tail estimate holds for* $p = 1$ *and* $L > 0$ *as* τ *tends to zero (*ϑ *defined in* (5.2.4)*):*

$$
\int_L^\infty \mathrm{BS}\left(k, \frac{y}{\tau|\log(\tau)|}, \tau\right) \zeta_1\left(\frac{y}{\tau|\log(\tau)|}\right) \mathrm{d}y = \mathcal{O}\left(\exp\left\{-\frac{1}{2\xi^2 t}\left[\log\left(\frac{L}{\tau|\log(\tau)|}\right) - \vartheta\right]^2\right\}\right).
$$

Proof. By no-arbitrage arguments, the Call price is always bounded above by one, so that

$$
\int_L^\infty \mathrm{BS}\left(k, \frac{y}{\tau|\log(\tau)|}, \tau\right) \zeta_1\left(\frac{y}{\tau|\log(\tau)|}\right) \mathrm{d}y \le \int_L^\infty \zeta_1\left(\frac{y}{\tau|\log(\tau)|}\right) \mathrm{d}y.
$$

With the substitution $q = \frac{1}{\xi\sqrt{t}}[\log(y/(\tau|\log(\tau)|)) - \vartheta]$, the lemma follows from the bound for the complementary Gaussian distribution function [147, Section 14.8]. □

Lemma 5.3.7. *Let $p = 1$. The following expansion holds for option prices as τ tends to zero:*

$$C(k,\tau) = c_5(t,1)\exp\Big(-c_1(t,1)h_1(\tau,p)+c_2(t,1)h_2(\tau,p)\Big)\tau^{c_3(t,1)}|\log(\tau)|^{c_4(t,1)}\left(1+\mathcal{O}\left(\frac{1}{|\log(\tau)|}\right)\right),$$

with the functions $c_1, c_2, ..., c_5$, h_1 and h_2 given in Table 5.1.

Proof. Let $\widetilde{\tau} := \tau|\log(\tau)|$. With the substitution $y \to y/\widetilde{\tau}$ and using (5.3.1), the option price is given by

$$\begin{aligned}C(k,\tau) &= \int_0^\infty \mathrm{BS}(k,y,\tau)\zeta_1(y)\mathrm{d}y = \frac{1}{\widetilde{\tau}}\int_0^\infty \mathrm{BS}\left(k,\frac{y}{\widetilde{\tau}},\tau\right)\zeta_1\left(\frac{y}{\widetilde{\tau}}\right)\mathrm{d}y\\ &= \frac{1}{\widetilde{\tau}}\left\{\int_0^L \mathrm{BS}\left(k,\frac{y}{\widetilde{\tau}},\tau\right)\zeta_1\left(\frac{y}{\widetilde{\tau}}\right)\mathrm{d}y + \int_L^\infty \mathrm{BS}\left(k,\frac{y}{\widetilde{\tau}},\tau\right)\zeta_1\left(\frac{y}{\widetilde{\tau}}\right)\mathrm{d}y\right\} =: \underline{C}(k,\tau)+\overline{C}(k,\tau),\end{aligned}$$

for some $L > 0$. Consider the first term. Using Lemma D.0.10 with $\widetilde{\tau} = \tau|\log(\tau)|$, we have, as τ tends to zero,

$$\mathrm{BS}\left(k,\frac{y}{\tau|\log(\tau)|},\tau\right) = \exp\left(-\frac{k^2|\log(\tau)|}{2y}+\frac{k}{2}\right)\frac{y^{3/2}}{k^2|\log(\tau)|^{3/2}\sqrt{2\pi}}\left[1+\mathcal{O}\left(\frac{1}{|\log(\tau)|}\right)\right].$$

Therefore

$$\begin{aligned}\underline{C}(k,\tau) &= \frac{\mathrm{e}^{k/2}\left(1+\mathcal{O}\left(\frac{1}{|\log(\tau)|}\right)\right)}{|\log(\tau)|^{3/2}\xi k^2 2\pi\sqrt{t}}\int_0^L \exp\left(-\frac{k^2|\log(\tau)|}{2y}-\frac{\left(\log\left(\frac{y}{\tau|\log(\tau)|}\right)-\vartheta\right)^2}{2\xi^2 t}\right)y^{1/2}\mathrm{d}y\\ &= \exp\left(\frac{k}{2}-\frac{(\log(\tau)+\log|\log(\tau)|)^2+\vartheta^2}{2\xi^2 t}-\frac{\vartheta(\log(\tau)+\log|\log(\tau)|)}{\xi^2 t}\right)\frac{I_1(\tau)\left[1+\mathcal{O}\left(\frac{1}{|\log(\tau)|}\right)\right]}{\xi k^2 2\pi\sqrt{t}|\log(\tau)|^{3/2}},\end{aligned}$$

where $I_1(\tau) := \int_0^L g_2(y)\exp\left(-g_0(y)|\log\tau| + g_1(y)\log|\log(\tau)|\right)\mathrm{d}y$ with g_0 and g_1 defined in (5.2.10) and

$$g_2(y) := \sqrt{y}\exp\left(\frac{\vartheta\log(y)}{\xi^2 t}\right).$$

The dominant contribution from the integrand is the $|\log(\tau)|$ term; the minimum of g_0 is attained at y^* given in (5.2.9), and $g_0''(y^*) = 4/(\xi^6 t^3 k^4) > 0$. Set

$$\begin{aligned}I_0(\tau) &:= \int_{-\infty}^\infty \exp\left(-\frac{1}{2}\left((y-y^*)\sqrt{|\log(\tau)|g_0''(y^*)}-\frac{g'(y^*)\log|\log(\tau)|}{\sqrt{|\log(\tau)|g_0''(y^*)}}\right)^2\right)\mathrm{d}y\\ &= \sqrt{\frac{2\pi}{g_0''(y^*)|\log(\tau)|}}.\end{aligned}$$

Then for some $\varepsilon > 0$ as τ tends to zero, with $L > y^*$,

$$\begin{aligned}I_1(\tau) &\sim \int_{y^*-\epsilon}^{y^*+\epsilon} g_2(y)\exp\Big\{-g_0(y)|\log(\tau)|+g_1(y)\log|\log(\tau)|\Big\}\mathrm{d}y\\ &\sim g_2(y^*)\mathrm{e}^{-g_0(y^*)|\log(\tau)|+g_1(y^*)\log|\log(\tau)|}\int_{y^*-\epsilon}^{y^*+\epsilon}\mathrm{e}^{-\frac{1}{2}g_0''(y^*)(y-y^*)^2|\log(\tau)|+g_1'(y^*)(y-y^*)\log|\log(\tau)|}\mathrm{d}y\\ &\sim g_2(y^*)\exp\left(-g_0(y^*)|\log(\tau)|+g_1(y^*)\log|\log(\tau)|+\frac{(g_1'(y^*)\log|\log(\tau)|)^2}{2g_0''(y^*)|\log(\tau)|}\right)I_0(\tau)\\ &= g_2(y^*)\exp\left(-g_0(y^*)|\log(\tau)|+g_1(y^*)\log|\log(\tau)|+\frac{(g_1'(y^*)\log|\log(\tau)|)^2}{2g_0''(y^*)|\log(\tau)|}\right)\sqrt{\frac{2\pi}{g_0''(y^*)|\log(\tau)|}}.\end{aligned}$$

where again the $\sim$ approximations here are exactly of the same type as in [75], and we refer the interested reader to this paper for details. Therefore as τ tends to zero:

$$\underline{C}(k,\tau) = c_5(t,1)\exp\Big(-c_1(t,1)h_1(\tau,1)+c_2(t,1)h_2(\tau,1)\Big)\tau^{c_3(t,1)}|\log(\tau)|^{c_4(t,1)}\left[1+\mathcal{O}\left(\frac{1}{|\log(\tau)|}\right)\right],$$

with the functions $c_1, c_2, ..., c_5$, h_1 and h_2 given in Table 5.1. For ease of computation we note that

$$c_5(t,1) = \frac{\sqrt{y^*}\exp\left(\frac{k}{2}-\frac{\vartheta^2}{2\xi^2 t}+\frac{\vartheta\log(y^*)}{\xi^2 t}\right)}{k^2\xi\sqrt{2\pi t}\sqrt{g_0''(y^*)}} = \frac{|k|\xi^3 t^{3/2}\exp\left(\frac{k}{2}-\frac{\vartheta^2}{2\xi^2 t}+\frac{\vartheta\log(y^*)}{\xi^2 t}\right)}{4\sqrt{\pi}}.$$

Now by Lemma 5.3.6,

$$\begin{aligned}\overline{C}(k,\tau) &= \frac{1}{\tau|\log(\tau)|}\int_L^\infty \mathrm{BS}\left(k,\frac{y}{\tau|\log(\tau)|},\tau\right)\zeta_1\left(\frac{y}{\tau|\log(\tau)|}\right)\mathrm{d}y \\ &= \frac{1}{\tau|\log(\tau)|}\mathcal{O}\left(\exp\left\{-\frac{1}{2\xi^2 t}\left[\log\left(\frac{L}{\tau|\log(\tau)|}\right)-\vartheta\right]^2\right\}\right).\end{aligned}$$

Since for some $B>0$ we have that

$$\exp\left(-\frac{1}{2\xi^2 t}\left[\log\left(\frac{L}{\tau|\log(\tau)|}\right)-\vartheta\right]^2\right) \le B\left(\tau|\log(\tau)|\right)^{\frac{1}{\xi^2 t}(\log(L)-\vartheta)}\exp\left(-\frac{1}{2\xi^2 t}h_1(\tau,1)\right),$$

choosing L such that $\log(L)>\vartheta$ yields

$$\mathcal{O}\left(\exp\left\{-\frac{1}{2\xi^2 t}\left[\log\left(\frac{L}{\tau|\log(\tau)|}\right)-\vartheta\right]^2\right\}\right) = \mathcal{O}\left(\exp\left(-\frac{1}{2\xi^2 t}h_1(\tau,1)\right)\right).$$

Hence $\overline{C}(k,\tau)$ is then exponentially subdominant to the compact part since

$$\mathrm{e}^{c_1(t,1)h_1(\tau,1)-c_2(t,1)h_2(\tau,1)}\mathcal{O}\left(\exp\left\{-\frac{1}{2\xi^2 t}\left[\log\left(\frac{L}{\tau|\log(\tau)|}\right)-\vartheta\right]^2\right\}\right) = \mathcal{O}\left(\mathrm{e}^{-c_2(t,1)h_2(\tau,1)}\right),$$

and the result follows. □

5.3.2 Proof of Theorem 5.2.8

The goal of this section is to prove the large-time behaviour of option prices in Theorem 5.2.8. Due to Lemma D.0.11 and the representation (5.3.1) we have the following asymptotics for call option prices as τ tends to infinity:

$$C(k,\tau) = 1-\mathrm{m}_t+\mathrm{m}_t(1-\mathrm{e}^k)^+ + \tau^{-1/2}\mathrm{e}^{k/2}\mathfrak{L}(\tau)(1+\mathcal{O}(\tau^{-1})), \tag{5.3.5}$$

where

$$\mathfrak{L}(\tau) = \int_0^\infty q(z)\mathrm{e}^{-\tau z}\mathrm{d}z, \tag{5.3.6}$$

and we set $q(z)\equiv -8\zeta_p(8z)/\sqrt{\pi z}$. Using asymptotics for the modified Bessel function of the first kind (1.5.11) and the definition of the density in (5.2.5) we obtain the following asymptotics for the density as y tends to zero when $p<1$ and absorption at the origin when $p<1/2$:

$$\zeta_p(y) = \frac{y_0 y^{1-2p}}{|1-p|\xi^2 t\Gamma(|\eta|+1)\left(2(1-p)^2\xi^2 t\right)^{|\eta|}}\exp\left(-\frac{y_0^{2(1-p)}}{2\xi^2 t(1-p)^2}\right)\left(1+\mathcal{O}\left(y^{2(1-p)}\right)\right). \tag{5.3.7}$$

Analogous arguments yield that when $p < 1/2$ and the origin is reflecting, then, as y tends to zero,

$$\zeta_p(y) = \frac{y^{-2p}}{|1-p|\xi^2 t\Gamma(\eta+1)\left(2(1-p)^2\xi^2 t\right)^{\eta}} \exp\left(-\frac{y_0^{2(1-p)}}{2\xi^2 t(1-p)^2}\right)\left(1+\mathcal{O}\left(y^{2(1-p)}\right)\right). \qquad (5.3.8)$$

In order to apply Watson's lemma (Theorem 1.2.4 and Remark 1.2.5) it is sufficient to require that the function q in (5.3.6) satisfies $q(z) = \mathcal{O}(\mathrm{e}^{cz})$ for some $c > 0$ as z tends to infinity. This clearly holds here since $\lim_{z\uparrow\infty}\zeta_p(z) = 0$. We also require (Remark 1.2.5 with $N = 0$) that

$$q(z) = a_0 z^{(\lambda-\mu)/\mu} + \mathcal{O}\left(z^{(1+\lambda-\mu)/\mu}\right), \quad \text{as } z \downarrow 0.$$

When $p \geq 1$, it can be shown that ζ_p is exponentially small, and a different method needs to be used. When $p < 1$ and the density is as in (5.3.7) then in the notation of Theorem 1.2.4 we have $\lambda = 1 - 1/(4(1-p))$ and $\mu = 1/(2(1-p))$. We require these terms to be positive and so $p < 3/4$. Analogously, when $p < 1/2$ and the density is (5.3.8) then $\lambda = 1-3/(4(1-p))$ and $\mu = 1/(2(1-p))$ and we require $p < 1/4$. An application of Watson's Lemma in conjunction with (5.3.5) then yields Theorem 5.2.8.

Appendix A

Verification of Assumption 2.2.1(v)

The tail assumption 2.2.1(v) needs to be verified in order to apply Theorem 2.2.4 in Chapter 2. It is readily satisfied by most models used in practice. Its verification is tedious but straightforward, and we give here an outline for the time-changed exponential Lévy case where the time-change is given by an integrated Feller process (1.3.11), i.e. Proposition 2.3.10(i). Analogous arguments hold for all other models in the chapter.

We recall that the forward cgf is given in (1.3.13) and the limiting cgf and domain (2.3.11),(2.3.12) are given by $\widehat{V} : \widehat{\mathcal{K}}_\infty \ni u \mapsto \frac{\kappa\theta}{\xi^2}\left(\kappa - \sqrt{\kappa^2 - 2\phi(u)\xi^2}\right)$ with $\widehat{\mathcal{K}}_\infty := \left\{u : \phi(u) \leq \kappa^2/(2\xi^2)\right\}$ and ϕ is the Lévy exponent. Straightforward computations yield Assumption 2.2.1(v)(a). For fixed $a \in \widehat{\mathcal{K}}_\infty^0$ denote $L_r : \mathbb{R} \to \mathbb{R}$ by $L_r(z) := \Re(\widehat{V}(\mathtt{i}z + a))$ and $L_i : \mathbb{R} \to \mathbb{R}$ by $L_i(z) := \Im(\widehat{V}(\mathtt{i}z + a))$. Then $\widehat{V}(\mathtt{i}z + a) = L_r(z) + \mathtt{i}L_i(z)$. Similarly we define ϕ_r and ϕ_i such that $\phi(\mathtt{i}z + a) = \phi_r(z) + \mathtt{i}\phi_i(z)$. From [60, Lemma A.1, page 10] we know that ϕ_r has a unique maximum at zero and is bounded way from zero as $|z|$ tends to infinity. Now $L_r(z) := \frac{\kappa^2\theta}{\xi^2} - \frac{\kappa\theta}{\xi^2}\Re\left(\sqrt{\kappa^2 - 2\phi(\mathtt{i}z + a)\xi^2}\right)$ and $\Re\left(\sqrt{\kappa^2 - 2\phi(\mathtt{i}z + a)\xi^2}\right) = \frac{1}{2}\sqrt{2(\kappa^2 - 2\phi_r(z)\xi^2) + 2\sqrt{(\kappa^2 - 2\phi_r(z)\xi^2)^2 + 4\xi^4\phi_i(z)^2}}$. We certainly have

$$\sqrt{2(\kappa^2 - 2\phi_r(z)\xi^2) + 2\sqrt{(\kappa^2 - 2\phi_r(z)\xi^2)^2}} \\ \leq \sqrt{2(\kappa^2 - 2\phi_r(z)\xi^2) + 2\sqrt{(\kappa^2 - 2\phi_r(z)\xi^2)^2 + 4\xi^4\phi_i(z)^2}}, \quad \text{(A.0.1)}$$

with equality only if $\phi_i(z) = 0$. Since ϕ_r has a unique maximum at zero we have $\phi_r(z) < \phi_r(0) \leq \kappa^2/(2\xi^2)$ and further $\sqrt{2(\kappa^2 - 2\phi_r(0)\xi^2)} \leq \sqrt{2(\kappa^2 - 2\phi_r(z)\xi^2) + 2\sqrt{(\kappa^2 - 2\phi_r(z)\xi^2)^2}}$, with the inequality strict for all $z \in \mathbb{R}^*$. Since $\phi_i(0) = 0$ it follows that $u = 0$ is the unique minimum of $\Re\left(\sqrt{\kappa^2 - 2\phi(\mathtt{i}z + a)\xi^2}\right)$. Since ϕ_r is bounded away from $\phi_r(0)$ as $|z|$ tends to infinity there exists a $q^* > 0$ and $M > 0$ such that for $|z| > q^*$ we have that $\phi_r(z) \leq M < \phi_r(0)$. But then for $|z| > q^*$

we certainly have (also using (A.0.1))

$$\frac{1}{2}\Re\left(\sqrt{2(\kappa^2-2\phi(a)\xi^2)}\right)=\frac{1}{2}\sqrt{2(\kappa^2-2\phi_r(0)\xi^2)}<\frac{1}{2}\sqrt{2(\kappa^2-2M\xi^2)}\leq\Re\left(\sqrt{\kappa^2-2\phi(\mathtt{i}z+a)\xi^2}\right).$$

This proves Assumption 2.2.1(v)(b). The proof of Assumption 2.2.1(v)(c) involves tedious but straightforward computations and we only highlight the main steps. Let $a\in\widehat{\mathcal{K}}^0_\infty$ and define $\overline{A}(u,\tau):=A(\phi(u),\tau)-\tau\widehat{V}(u)$ with A given in (1.3.14). From the analysis above we know that the map $z\mapsto\Re d(\phi(\mathtt{i}z+a))$ has a unique minimum at $z=0$. Also we recall that $0<d(\phi(a))$ and straightforward calculations show that $|\gamma(\phi(\mathtt{i}z+a))|<1$ with d and γ given in (1.3.14). Using the triangle and reverse triangle inequality we have for all $z\in\mathbb{R}$ and $\tau>0$ that

$$\begin{aligned}\Re\overline{A}(\phi(\mathtt{i}z+a),\tau)&=\frac{2\kappa\theta}{\xi^2}\log\left|\frac{1-\gamma(\phi(\mathtt{i}z+a))}{1-\gamma(\phi(\mathtt{i}z+a))\mathrm{e}^{-d(\phi(\mathtt{i}z+a))\tau}}\right|\\&\leq\frac{2\kappa\theta}{\xi^2}\log\left(\frac{2}{1-\mathrm{e}^{-d(\phi(a))\tau}}\right).\end{aligned}\tag{A.0.2}$$

Tedious computations also reveal that (B given in (1.3.14)): $\Re B(\phi(\mathtt{i}z+a),\tau)\leq B(\phi(a),\tau)$, for all $z\in\mathbb{R}$ and $\tau>0$. Consider the second and third terms for the forward cgf in (1.3.13). For all $z\in\mathbb{R}$ and $\tau>0$ (using $|y|^{-1}\leq|\Re y|^{-1}$ for all $y\in\mathbb{C}\setminus\{0\}$):

$$\begin{aligned}\Re\log\left(\frac{1}{1-2\beta_tB(\phi(\mathtt{i}z+a),\tau)}\right)&=\log\left|\frac{1}{1-2\beta_tB(\phi(\mathtt{i}z+a),\tau)}\right|\\&\leq\log\left(\frac{1}{1-2\beta_tB(\phi(a),\tau)}\right),\end{aligned}\tag{A.0.3}$$

where we note in the last inequality that $1-2\beta_tB(\phi(a),\tau)>0$. We also compute

$$\Re\left(\frac{B(\phi(\mathtt{i}z+a),\tau)}{1-2\beta_tB(\phi(\mathtt{i}z+a),\tau)}\right)=\frac{\Re B(\phi(\mathtt{i}z+a),\tau)-2\beta_t|B(\phi(\mathtt{i}z+a),\tau)|^2}{1-4\beta_t\Re B(\phi(\mathtt{i}z+a),\tau)+4\beta_t^2|B(\phi(\mathtt{i}z+a),\tau)|^2},$$

and hence using $\Re B(\phi(\mathtt{i}z+a),\tau)\leq|B(\phi(\mathtt{i}z+a),\tau)|$ and that $1-\beta_t\Re B(\phi(\mathtt{i}z+a),\tau)>1/2$ we see that for all $z\in\mathbb{R}$ and $\tau>0$:

$$\Re\left(\frac{B(\phi(\mathtt{i}z+a),\tau)}{1-2\beta_tB(\phi(\mathtt{i}z+a),\tau)}\right)\leq\frac{\Re B(\phi(\mathtt{i}z+a),\tau)}{1-2\beta_t\Re B(\phi(\mathtt{i}z+a),\tau)}\leq\frac{B(\phi(a),\tau)}{1-2\beta_tB(\phi(a),\tau)},\tag{A.0.4}$$

where the last inequality follows since the term in the second inequality is strictly increasing in $\Re B(\phi(\mathtt{i}z+a),\tau)$. Combining (A.0.2), (A.0.3) and (A.0.4) we see that as τ tends to infinity:

$$\Re\left[\tau^{-1}\log\mathbb{E}\left(\mathrm{e}^{(\mathtt{i}z+a)X_\tau^{(t)}}\right)-\widehat{V}(\mathtt{i}z+a)\right]\leq\left[\frac{\widehat{V}(a)v\mathrm{e}^{-\kappa t}}{1-2\beta_t\widehat{V}(a)}+\frac{2\kappa\theta}{\xi^2}\log\left(\frac{2}{1-2\beta_t\widehat{V}(a)}\right)\right]\frac{1}{\tau}+\mathcal{O}\left(\frac{1}{\tau^2}\right),$$

for all $z\in\mathbb{R}$ and where the remainder does not depend on z. This proves Assumption 2.2.1(v)(c).

Appendix B

Properties (i),(ii) and (iii) in Lemma 3.6.3

The purpose of this appendix is to verify properties (i),(ii) and (iii) in Lemma 3.6.3. Denote the cumulant generating function for the random variable Y by

$$\Lambda_Y(u) := \log \mathbb{E}\left(\mathrm{e}^{uY}\right), \text{ for all } u \in \mathcal{D}_Y,$$

where $\mathcal{D}_Y$ is its effective domain. We say that a random variable is degenerate if $\mathcal{D}_Y = \{0\}$ or the random variable is a constant. We now recall the following result [104, Theorem 2.3].

Theorem B.0.8. *Λ_Y is strictly convex on its effective domain if and only if Y is not degenerate.*

The Heston forward cgf $\Lambda_\tau^{(t)}$ in (3.6.3) is therefore strictly convex. From Lemmas 3.2.3 and 3.6.2, for $u \in \mathcal{D}_\Lambda := (-1/\sqrt{\beta_t}, 1/\sqrt{\beta_t})$ and τ small enough, we have

$$\partial_u \Lambda_\tau^{(t)}(u) = \Lambda_0(u)\sqrt{\tau} + \mathcal{O}(\tau), \tag{B.0.1}$$

where we set

$$\Lambda_0(u) := \frac{4\kappa u\left(\theta \mathrm{e}^{2\kappa t}\left(4\kappa - \xi^2 u^2\right) + 2\mathrm{e}^{\kappa t}\left(\theta\xi^2 u^2 - 2\kappa\theta + 2\kappa v\right) - \theta\xi^2 u^2\right)}{\left(\mathrm{e}^{\kappa t}\left(4\kappa - \xi^2 u^2\right) + \xi^2 u^2\right)^2}.$$

The denominator of Λ_0 explodes to infinity at the boundary points, showing that it is steep. This proves (i) for small enough τ.

We also know that $\Lambda_\tau^{(t)}(0) = \Lambda_\tau^{(t)}(1) = 0$, and so by the strict convexity of $\Lambda_\tau^{(t)}$, we must have $u_\tau^*(0) \in (0, 1)$. By the expansion in (B.0.1) and since $\Lambda_0(0) = 0$ we see that $u_\tau^*(0)$ must converge to zero proving (ii). Finally, (iii) follows from expansion (B.0.1).

Appendix C

Large-maturity Heston cgf expansion

The purpose of this appendix is to extend the expansion in Lemma (2.5.15) to $\Lambda_\tau^{(t)}(\mathtt{i}u+a)$ and show that the remainder is uniform in u. Recall the definition of $\Lambda_\tau^{(t)}$ in (4.6.1) and the Heston forward cgf given in (1.3.7). Also note that when $(u,a) = (0,1)$ below then $\Lambda_\tau^{(t)}(1) = 0$ for all $\tau > 0$ by the martingale property.

Lemma C.0.9. *The following expansion holds (*V*,*H *and* d *given in* (2.3.8) *and* (1.3.6)*) for all* $(u,a) \in \mathbb{R} \times \mathcal{K}_{\mathrm{H}} \setminus \{(0,1)\}$ *as* τ *tends to infinity:*

$$\Lambda_\tau^{(t)}(\mathtt{i}u+a) = V(\mathtt{i}u+a) + \tau^{-1}H(\mathtt{i}u+a) + \mathcal{O}\left(\mathrm{e}^{-d(a)\tau}\right),$$

where the remainder is uniform in u *and* $\mathcal{K}_{\mathrm{H}}$ *is given in Table 2.1.*

Proof. We first consider asymptotics for A in (1.3.8). We write A as

$$A(\mathtt{i}u+a,\tau) = \tau V(\mathtt{i}u+a) - \frac{2\kappa\theta}{\xi^2}\log\left(\frac{1}{1-\gamma(\mathtt{i}u+a)}\right) \\ - \frac{2\kappa\theta}{\xi^2}\log\left(1-\gamma(\mathtt{i}u+a)\mathrm{e}^{-d(\mathtt{i}u+a)\tau}\right). \tag{C.0.1}$$

The last term is the remainder that we want to analyse. Using the Lagrange form of the remainder in Taylor's theorem for small $|x|$ we have that

$$\log\left(1-\gamma(\mathtt{i}u+a)x\right) = -x\frac{\gamma(\mathtt{i}u+a)}{1-\gamma(\mathtt{i}u+a)x^*},$$

for some $x^* \in \{y \in \mathbb{C} : |y| < |x|\}$. Hence we have that

$$\left|\log\left(1-\gamma(\mathtt{i}u+a)\mathrm{e}^{-d(\mathtt{i}u+a)\tau}\right)\right| = \left|\mathrm{e}^{-d(\mathtt{i}u+a)\tau}\frac{\gamma(\mathtt{i}u+a)}{1-\gamma(\mathtt{i}u+a)x^*}\right|,$$

with $x^* \in \{y \in \mathbb{C} : |y| < |\mathrm{e}^{-d(\mathtt{i}u+a)\tau}| = \mathrm{e}^{-\Re d(\mathtt{i}u+a)\tau}\}$. Using $\Re d(\mathtt{i}u+a) \geq d(a)$ (see Appendix A) we have that

$$\left|\log\left(1-\gamma(\mathtt{i}u+a)\mathrm{e}^{-d(\mathtt{i}u+a)\tau}\right)\right| \leq \mathrm{e}^{-d(a)\tau}\left|\frac{\gamma(\mathtt{i}u+a)}{1-\gamma(\mathtt{i}u+a)x^*}\right|$$

with $x^* \in \{y \in \mathbb{C} : |y| < \mathrm{e}^{-d(a)\tau}\}$. Using the reverse triangle inequality and that $|\gamma(\mathtt{i}u+a)| < 1$ (see Appendix A)) we see that

$$\left|\log\left(1-\gamma(\mathtt{i}u+a)\mathrm{e}^{-d(\mathtt{i}u+a)\tau}\right)\right| \leq \mathrm{e}^{-d(a)\tau}\frac{1}{1-\mathrm{e}^{-d(a)\tau}} \leq \mathrm{e}^{-d(a)\tau}\left(1+\mathcal{O}\left(\mathrm{e}^{-d(a)\tau}\right)\right),$$

as τ tends to infinity and where the remainder is uniform in u. Hence using (C.0.1) we have the folowing expansion as τ tends to infinity:

$$A(\mathtt{i}u+a,\tau) = \tau\Re V(\mathtt{i}u+a) - \frac{2\kappa\theta}{\xi^2}\log\left(\frac{1}{1-\gamma(\mathtt{i}u+a)}\right) + \mathcal{O}\left(\mathrm{e}^{-d(a)\tau}\right), \tag{C.0.2}$$

and where the remainder is uniform in u. Analogous arguments yield that (B defined in (1.3.8))

$$B(\mathtt{i}u+a,\tau) = \frac{V(\mathtt{i}u+a)}{\kappa\theta} + \mathcal{O}\left(\mathrm{e}^{-d(a)\tau}\right), \tag{C.0.3}$$

as τ tends to infinity and where the remainder is uniform in u. Consider the second term for the Heston forward cgf given in (1.3.7). Using the expansion for B in (C.0.3) and the Lagrange form of the remainder in Taylor's theorem we find that

$$\begin{aligned}\frac{B(\mathtt{i}u+a,\tau)}{1-2\beta_t B(\mathtt{i}u+a,\tau)} &= \frac{V(\mathtt{i}u+a)/(\kappa\theta)+\mathcal{O}\left(\mathrm{e}^{-d(a)\tau}\right)}{1-2\beta_t\left(V(\mathtt{i}u+a)/(\kappa\theta)+\mathcal{O}\left(\mathrm{e}^{-d(a)\tau}\right)\right)}\\ &= \frac{V(\mathtt{i}u+a)}{\kappa\theta-2\beta_t V(\mathtt{i}u+a)} + \frac{\mathcal{O}\left(\mathrm{e}^{-d(a)\tau}\right)}{\left(1-2\beta_t\left(V(\mathtt{i}u+a)/(\kappa\theta)+x^*\right)\right)^2},\end{aligned}$$

where $\mathcal{O}\left(\mathrm{e}^{-d(a)\tau}\right)$ is uniform in u and $x^* \in \{y \in \mathbb{C} : |y| < \mathrm{e}^{-d(a)\tau}\}$. The last term is our remainder and using $|y|^{-1} \leq |\Re y|^{-1}$ for all $y \in \mathbb{C}\setminus\{0\}$ and $|\Re y| \leq |y|$:

$$\begin{aligned}\left|\frac{1}{\left(1-2\beta_t\left(V(\mathtt{i}u+a)/(\kappa\theta)+x^*\right)\right)^2}\right| &\leq \frac{1}{\left(1-2\beta_t\left(\Re V(\mathtt{i}u+a)/(\kappa\theta)+\mathrm{e}^{-d(a)\tau}\right)\right)^2}\\ &\leq \frac{1}{\left(1-2\beta_t\left(V(a)/(\kappa\theta)+\mathrm{e}^{-d(a)\tau}\right)\right)^2},\end{aligned}$$

where the last line follows since $\Re V(\mathtt{i}u+a) \leq V(a)$ (see Appendix A)). Hence

$$\frac{B(\mathtt{i}u+a,\tau)}{1-2\beta_t B(\mathtt{i}u+a,\tau)} = \frac{V(\mathtt{i}u+a)}{\kappa\theta-2\beta_t V(\mathtt{i}u+a)} + \mathcal{O}\left(\mathrm{e}^{-d(a)\tau}\right), \tag{C.0.4}$$

where the remainder is uniform in u. Analogously for the third term in the Heston forward cgf given in (1.3.7) we find that

$$\log\left(1-2\beta_t B(\mathtt{i}u+a,\tau)\right) = \log\left(1-\frac{2\beta_t V(\mathtt{i}u+a)}{\theta\kappa}\right) + \mathcal{O}\left(\mathrm{e}^{-d(u)\tau}\right), \tag{C.0.5}$$

where the remainder is uniform in u. The result follows after combining (C.0.2), (C.0.4) and (C.0.5).

□

Appendix D

Black-Scholes asymptotics

Lemma D.0.10. *Let $k, y > 0$ and $\widetilde{\tau} : (0,\infty) \to (0,\infty)$ be a continuous function such that $\lim_{\tau\downarrow 0} \frac{\tau}{\widetilde{\tau}(\tau)} = 0$. Then*

$$\mathrm{BS}\left(k, \frac{y}{\widetilde{\tau}(\tau)}, \tau\right) = \frac{y^{3/2}}{k^2\sqrt{2\pi}} \left(\frac{\tau}{\widetilde{\tau}(\tau)}\right)^{3/2} \mathrm{e}^{-\frac{k^2}{2y}\frac{\widetilde{\tau}(\tau)}{\tau}+\frac{k}{2}} \left\{1 - \left(\frac{3}{k^2}+\frac{1}{8}\right)\frac{y\tau}{\widetilde{\tau}(\tau)} + \mathcal{O}\left(\left(\frac{\tau}{\widetilde{\tau}(\tau)}\right)^2\right)\right\},$$

as τ tends to zero, where the function BS *is defined in* (1.0.2).

Proof. Let $k, y > 0$ and set $\tau^*(\tau) \equiv \tau/\widetilde{\tau}(\tau)$. By assumption, $\tau^*(\tau)$ tends to zero as τ approaches zero, and (1.0.2) implies

$$\mathrm{BS}\left(k, \frac{y}{\widetilde{\tau}}, \tau\right) = \mathrm{BS}\left(k, y, \tau^*(\tau)\right) = \mathcal{N}(d_+^*(\tau)) - \mathrm{e}^k\mathcal{N}(d_-^*(\tau)),$$

where we set $d_\pm^*(\tau) := -k/(\sqrt{y\tau^*(\tau)}) \pm \frac{1}{2}\sqrt{y\tau^*(\tau)}$, and $\mathcal{N}$ is the standard normal distribution function. Note that $d_\pm^*$ tends to $-\infty$ as τ tends to zero. The asymptotic expansion $1-\mathcal{N}(z) = (2\pi)^{-1/2}\mathrm{e}^{-z^2/2}\left(z^{-1} - z^{-3} + \mathcal{O}(z^{-5})\right)$, valid for large z ([1, page 932]), yields

$$\begin{aligned}\mathrm{BS}\left(k, \frac{y}{\widetilde{\tau}(\tau)}, \tau\right) &= \mathcal{N}\left(d_+^*(\tau)\right) - \mathrm{e}^k\mathcal{N}\left(d_-^*(\tau)\right) = 1 - \mathcal{N}\left(-d_+^*(\tau)\right) - \mathrm{e}^k(1-\mathcal{N}\left(-d_-^*(\tau)\right)) \\ &= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}d_+^*(\tau)^2/2\right)\left\{\frac{1}{d_-^*(\tau)} - \frac{1}{d_+^*(\tau)} + \frac{1}{d_+^*(\tau)^3} - \frac{1}{d_-^*(\tau)^3} + \mathcal{O}\left(\frac{1}{d_+^*(\tau)^5}\right)\right\},\end{aligned}$$

as τ tends to zero, where we used the identity $\frac{1}{2}d_-^*(\tau)^2 - k = \frac{1}{2}d_+^*(\tau)^2$. The lemma then follows from the following expansions as τ tends to zero:

$$\exp\left(-\frac{1}{2}d_+^*(\tau)^2\right) = \exp\left(-\frac{k^2}{2y\tau^*} + \frac{k}{2}\right)\left(1 - \frac{y}{8}\tau^*(\tau) + \mathcal{O}\left(\tau^*(\tau)^2\right)\right),$$

$$\frac{1}{d_-^*(\tau)} - \frac{1}{d_+^*(\tau)} + \frac{1}{d_+^*(\tau)^3} - \frac{1}{d_-^*(\tau)^3} = \frac{y^{3/2}\tau^*(\tau)^{3/2}}{k^2}\left(1 - \frac{3y}{k^2}\tau^*(\tau) + \mathcal{O}\left(\tau^*(\tau)^2\right)\right).$$

□

Lemma D.0.11. *Let $y > 0$ and $k \in \mathbb{R}$. Then*

$$\mathrm{BS}(k, y, \tau) = 1 - \frac{4}{\sqrt{2\pi\tau y}} \mathrm{e}^{-y\tau/8+k/2} \left(1 + \mathcal{O}(\tau^{-1})\right),$$

as τ tends to infinity, where the function BS *is defined in* (1.0.2).

Proof. Let $y > 0$. Then

$$\mathrm{BS}(k, y, \tau) = \mathcal{N}\left(d_+^*(\tau)\right) - \mathrm{e}^k \mathcal{N}\left(d_-^*(\tau)\right),$$

where we set $d_\pm^*(\tau) := -k/(\sqrt{y\tau}) \pm \frac{1}{2}\sqrt{y\tau}$, and $\mathcal{N}$ is the standard normal distribution function. Hence $d_\pm^*$ tends to $\pm\infty$ as τ tends to infinity. The asymptotic expansion $1 - \mathcal{N}(z) = (2\pi)^{-1/2}\mathrm{e}^{-z^2/2}\left(z^{-1} - z^{-3} + \mathcal{O}(z^{-5})\right)$, valid for large z ([1, page 932]), yields

$$\begin{aligned}
\mathrm{BS}(k, y, \tau) &= \mathcal{N}\left(d_+^*(\tau)\right) - \mathrm{e}^k\left(1 - \mathcal{N}\left(-d_-^*(\tau)\right)\right) \\
&= 1 - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} d_+^*(\tau)^2/2\right) \left\{\frac{1}{d_+^*(\tau)} - \frac{1}{d_-^*(\tau)} + \frac{1}{d_-^*(\tau)^3} - \frac{1}{d_+^*(\tau)^3} + \mathcal{O}\left(\frac{1}{d_+^*(\tau)^5}\right)\right\},
\end{aligned}$$

as τ tends to infinity, where we used the identity $\frac{1}{2}d_-^*(\tau)^2 - k = \frac{1}{2}d_+^*(\tau)^2$. The lemma then follows from the following expansions as τ tends to infinity:

$$\begin{aligned}
\exp\left(-\frac{1}{2}d_+^*(\tau)^2\right) &= \exp\left(-\frac{y\tau}{8} + \frac{k}{2}\right)\left(1 + \mathcal{O}(\tau^{-1})\right), \\
\frac{1}{d_+^*(\tau)} - \frac{1}{d_-^*(\tau)} + \frac{1}{d_-^*(\tau)^3} - \frac{1}{d_+^*(\tau)^3} &= \frac{4}{\sqrt{2\pi\tau y}}\left(1 + \mathcal{O}(\tau^{-1})\right).
\end{aligned}$$

□

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