One-way quantum computation with four-dimensional photonic qudits

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(Dated: February 19, 2013)

We consider the possibility of performing linear optical quantum computation making use of extra photonic degrees of freedom. In particular we focus on the case where we use photons as quadbits, 4-dimensional photonic qudits. The basic 2-quadbit cluster state is a hyper-entangled state across polarization and two spatial mode degrees of freedom. We examine the non-deterministic methods whereby such states can be created from single photons and/or Bell pairs, and then give some mechanisms for performing higher-dimensional fusion gates.

PACS numbers:

I. INTRODUCTION

Optical quantum computation is a strong candidate for a scalable quantum computer. Photons have low decoherence rates, and high fidelity optical components are readily available. In this article we focus on the linear optical quantum computation (LOQC) paradigm, for which the resource overheads of the original LOQC proposal [1] have been greatly reduced by making use [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] of the one-way quantum computation model [14, 15].

Significant hurdles to practical LOQC remain, however. At present the primary obstacle is a deterministic source of photons. Much progress has been made along these lines [16, 17], but it is clear that there is still a long way to go. Particularly exciting is the possibility of creating "on-demand" entangled pairs of photons [18, 19], which obviate the need for initially creating such entangled pairs from single photons [20]. The investigations of this paper are based around an assumption that at some time in the near future efficient deterministic sources of either single photons or entangled photon pairs will become available.

It is not always obvious how to compare the resource requirements of various different proposals for implementing LOQC within the cluster state paradigm (e.g. how many single photon sources, memory units and feed-forward steps is an entangled pair source "worth"?). Since the primary difficulties for LOQC relate to sources and detectors, it is clear that schemes which reduce the number of photons actually used in an implementation are desirable [31]. A travelling photonic wavepacket is in principle a multi-mode creature, and thus can be treated as a d-dimensional quantum system (a "qu*d*it"). There is a d-dimensional version of cluster state computing [21, 22], and one purpose of this paper is to explore procedures whereby such d-dimensional clusters can be created. The second motive is to examine some basic "initial state" resource tradeoffs, such as: "how many Bell pairs does it take to make a hyper-entangled state".

For concreteness we focus on the *quadbit* case - specifically, we treat a single photon as a four-dimensional quantum system; using the two polarization states of two different spatial modes to encode the four levels.

II. QUADBIT CLUSTER STATES

A. General quadbit cluster states

In this section we review the features of quadbit cluster states we shall make use of - a pedagogical overview of the higher-dimensional cluster state computing can be found in [22].

We label the computational basis states $\{|\bar{0}\rangle, |\bar{1}\rangle, |\bar{2}\rangle, |\bar{3}\rangle\}$ (use of the overbar is to prevent confusion with 0 and 1 photon Fock states). In terms of these we can define the quadbit version of a Hadamard rotation, which rotates the computational basis state $|\bar{i}\rangle$ to $|+_i\rangle$ $(i = 0, 1, 2, 3)$, where

$$|+_i\rangle \;=\; \frac{1}{2}(|\bar{0}\rangle + \mathrm{e}^{\mathrm{i}\frac{i\pi}{2}}|\bar{1}\rangle + \mathrm{e}^{\mathrm{i}\,i\pi}|\bar{2}\rangle + \mathrm{e}^{\mathrm{i}\frac{3i\pi}{2}}|\bar{3}\rangle), \qquad (1)$$

A 2-quadbit cluster state $|QdC_2\rangle$ is then given by the superposition

$$|QdC_2\rangle = \frac{1}{2}\sum_{i=0}^{3}|\bar{i}\rangle|+_i\rangle,$$

which should be compared with the equivalent 2-qubit cluster state $|C_2\rangle = (|0\rangle|+\rangle + |1\rangle|-\rangle)/\sqrt{2}$. In the case of qubits a two-qubit (non-destructive) parity gate operation would fuse [13] two 2-qubit clusters into the state $|C_3\rangle = (|+\rangle|00\rangle|+\rangle + |-\rangle|11\rangle|-\rangle)/\sqrt{2}$, and repeated such fusion operations allows for the growth of arbitrary cluster states (the redundant encoding of the central qubit is easily removed by a measurement in the $|\pm\rangle$ basis, yielding the 3-qubit cluster state as claimed). Similarly, in the quadbit cluster case arbitrary quadbit clusters can be grown using a quadbit fusion operation. Applied to two 2-quadbit clusters such a fusion would achieve the state $|QdC_3\rangle = \sum_{i=0}^{3}|+_i\rangle|\overline{ii}\rangle|+_i\rangle/2$.

B. Optical quadbit cluster states

We define a quadbit single photon quantum state in two polarization/spatial modes as follows:

$$|\bar{0}\rangle \equiv |H\rangle_1 , \ |\bar{1}\rangle \equiv |V\rangle_1 , \ |\bar{2}\rangle \equiv |H\rangle_2 , \ |\bar{3}\rangle \equiv |V\rangle_2 , \quad (2)$$

where $H(V)$ denotes horizontal (vertical) polarization, and the subindex 1(2) denotes spatial mode $k_1(k_2)$.

Consider now a so-called hyper-entangled state (HES) [23], which is a two-photon state entangled in both polarization and spatial modes. Two-photon HES's can be generated by spontaneous parametric down-conversion [24]. As with generation of single photons, such a mechanism of HES production is not scalable. In Section III A, we will consider scalable production of HES's given deterministic single photon sources or entangled pairs. It is possible to represent an HES as product of Bell states, with a virtual tensor product structure between the spatial and polarization modes, for example:

$$|\Phi^+_{\mathrm{HES}}\rangle = \frac{1}{2}(|H\rangle|H\rangle + |V\rangle|V\rangle) \otimes (|1\rangle|3\rangle + |2\rangle|4\rangle). \quad (3)$$

(We will always use the $\otimes$ symbol to refer to this virtual tensor product of spatial modes and polarizations). Using the identification in Eq. (2), we see that $|\Phi^+_{\mathrm{HES}}\rangle$ is equal to

$$\begin{aligned} |\Phi^+_{\mathrm{HES}}\rangle &= \frac{1}{2}(|H\rangle_1|H\rangle_3 + |V\rangle_1|V\rangle_3 + |H\rangle_2|H\rangle_4 + |V\rangle_2|V\rangle_4) \\ &= \frac{1}{2}(|\bar{0}\rangle|\bar{0}\rangle + |\bar{1}\rangle|\bar{1}\rangle + |\bar{2}\rangle|\bar{2}\rangle + |\bar{3}\rangle|\bar{3}\rangle). \end{aligned}$$

As any single mode unitary operation can be implemented with linear optics [25], a simple circuit can be constructed which rotates the quadbit in modes 3 and 4 to yield the optical 2-quadbit cluster state $|QdC_2\rangle$ defined above.

Consider attempting to fuse two 2-quadbit clusters, the first in modes (1,2;3,4) (as in Eq.(3)), the second in modes (5,6;7,8). The procedure required to fuse the quadbit in spatial modes 1,2 with that in 5,6 is a gate which (when successful) performs a projective measurement of the form:

$$|HH\rangle_{1\,5}\langle HH| + |VV\rangle_{1\,5}\langle VV| + |HH\rangle_{2\,6}\langle HH| + |VV\rangle_{2\,6}\langle VV|.$$

That is, a successful measurement should reveal "the photons were in corresponding spatial modes with the same polarization", but should not reveal in which spatial modes and with what polarization. In section IV we will show that such a fusion is possible, although we have only found methods of doing it that make use of ancillary systems, and for which the success probability strongly depends on the nature of the ancillas available.

III. GENERATION OF QUADBIT CLUSTER STATES

Before discussing possible fusion mechanisms, we turn to examining some "initial state resource tradeoffs". This is because, as in the case of single photons, parametric downconversion is not a suitable source for scalable LOQC. Therefore we may well need to generate deterministic HES's from a deterministic source of either single photons, Bell pairs or GHZ states. Whether the constructions we give are optimal (or even close to being so) we cannot determine. Thus the procedures we present should be seen as simply giving upper bounds on the resources required. Also, because the most efficient fusion gate we will present for quadbit clusters destroys the photons involved (much like Type-II fusion for qubits) we will need to look at mechanisms for generating an initial resource of 3 and 4 quadbit cluster states.

Basic notation for the figures, and a brief outline of the operation of the fundamental optical components is set out in Appendix A.

A. General procedure for HES generation

The general circuit we present (Fig.1) is built from two copies of a sub-circuit we label J_1, and we first explain the operation of this circuit.

The circuit J_1 consists of three beam splitters (BSs) with two vacuum inputs. Consider the case where a Bell state $(|H\rangle_1|H\rangle_2 + |V\rangle_1|V\rangle_2)/\sqrt{2}$ is input into J_1. The first BS creates a bunched two-photon state in modes 1 and 2, and then two vacuum inputs are applied from modes $1'$ and $2'$ with two regular BSs. After the circuit J_1, the state of two photons in mode 1, $1'$, 2, and $2'$ is equal to

$$\begin{aligned} |M\rangle_{121'2'} = \frac{1}{4}(|H\rangle|H\rangle + |V\rangle|V\rangle) \\ \otimes \sum_{j=1}^{2} \left[\mathrm{e}^{\mathrm{i}j\pi}(|j\rangle|j\rangle + |j'\rangle|j'\rangle + \sqrt{2}|j\rangle|j'\rangle)\right]. \quad (4) \end{aligned}$$

It is a combination of four states of bunched photon pairs in a spatial mode ($|j\rangle|j\rangle$ and $|j'\rangle|j'\rangle$) and two anti-bunched states in two different spatial modes ($|j\rangle|j'\rangle$).

We turn now to the full circuit J_2 depicted in Figure 1 (b). At the centre of the circuit is a source S_2 which can be either single photons, Bell pairs or a 4-photon GHZ state. This source is then fed into two copies of the J_1 gate, the outputs of which impinge on 50:50 beam splitters as shown. It is easiest to begin with the case that the source consists of two Bell pairs.

The initial state of the two Bell pairs $|\Phi^+\rangle_{1\,2}|\Phi^+\rangle_{3\,4}$ is

$$\frac{1}{2}(|H\rangle_1|H\rangle_2 + |V\rangle_1|V\rangle_2)(|H\rangle_3|H\rangle_4 + |V\rangle_3|V\rangle_4). (5)$$

According to Eq. (4), the state after the two J_1 circuits

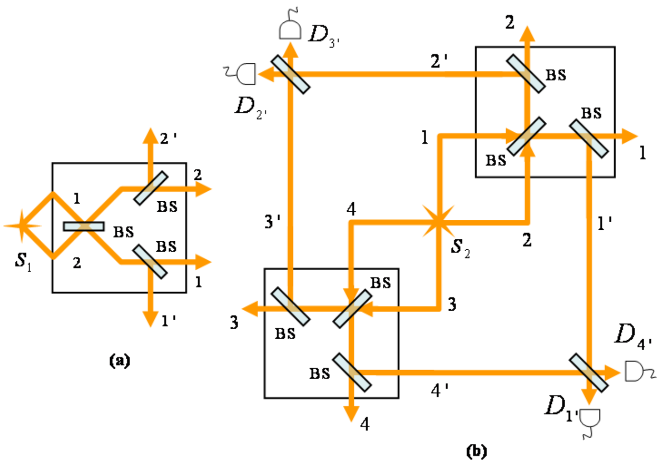

FIG. 1: (color online) (a) Circuit J_1 (b) Circuit J_2 for a hyper-entangled state from four entangled photons

is equal to

$$
\begin{aligned}
&|M\rangle_{1\,2\,1'\,2'}|M\rangle_{3\,4\,3'\,4'} \\
&\quad = \frac{1}{16}(|H\rangle|H\rangle + |V\rangle|V\rangle)(|H\rangle|H\rangle + |V\rangle|V\rangle) \\
&\qquad \otimes \sum_{j=1}^{2}\left[\mathrm{e}^{\mathrm{i}j\pi}(|j\rangle|j\rangle + |j'\rangle|j'\rangle + \sqrt{2}|j\rangle|j'\rangle)\right] \\
&\qquad \sum_{k=3}^{4}\left[\mathrm{e}^{\mathrm{i}k\pi}(|k\rangle|k\rangle + |k'\rangle|k'\rangle + \sqrt{2}|k\rangle|k'\rangle)\right]. \quad (6)
\end{aligned}
$$

At the end of the J_1 circuits, two BSs are applied in modes $1', 4'$ and $2', 3'$, after which detectors are located. Successful operation occurs when two identically polarized photons are detected in modes $1', 4'$ or $2', 3'$ respectively, and the success probability of the detection pattern is 1/16. To see how this works, note that it is the components of the state in Eq. (6) which consist of two bunched photons ($|j'\rangle|j'\rangle$ and $|k'\rangle|k'\rangle$) that can yield successful detection : the anti-bunched photonic states ($|j\rangle|j'\rangle$ and $|k\rangle|k'\rangle$) result in destructive interference. For example, if we detect two horizontal photons in modes $1'$ and $4'$ but nothing in modes $2'$ and $3'$, the outcome state is

$$
\begin{aligned}
|\psi'_{\rm HES}\rangle \;=\; & \frac{1}{2\sqrt{2}}(|H\rangle|H\rangle + |V\rangle|V\rangle) \\
& \otimes(|1\rangle|1\rangle - |2\rangle|2\rangle + |3\rangle|3\rangle - |4\rangle|4\rangle). \qquad (7)
\end{aligned}
$$

This state is, up to a linear optical transformation (in this case two BSs in mode 1 and 2 and mode 3 and 4), a hyper-entangled state.

It is interesting to note that the failure outcomes can still yield photons in useful states. In particular the failure outcome where only the vacuum is detected leaves all the photons still in two Bell pairs ; this occurs with probability 1/16, and obviously the gate can then simply be repeated. This suggests the overall success probability is essentially 1/8. Some of the detection patterns, while not yielding an HES do still leave two of the photons in Bell pair, which could be recycled.

We are also able to use for the source a four-qubit GHZ state of the form $(|HHHH\rangle_{1234} + |VVVV\rangle_{1234})/\sqrt{2}$ rather than two Bell pairs; this yields a higher success probability. This also has the advantage that in this case we need not assume the four detectors are polarization sensitive : they need only count numbers of photons at the output of the primed modes. Upon successful detection, when two photons are detected in any two spatial modes, the state in modes 1 to 4 becomes a HES with a success probability 3/16, which is higher than the case of two Bell pairs. Interestingly, no photon detection yields a 4-photon entangled state such as $(|\Phi^+\rangle_{12}|\Phi^+\rangle_{34} + |\Psi^+\rangle_{12}|\Psi^+\rangle_{34})/\sqrt{2}$.

Finally, if we wish to create a HES ballistically from single photons, then we can replace the two Bell pairs input at the source S_2 by two copies of the circuit for generating a Bell pair from 4 single photons (Figure 8 in Appendix C). In this case we find that the success probability is $1/16^3$.

B. Generating larger quadbit cluster states

1. 3 quadbit cluster state

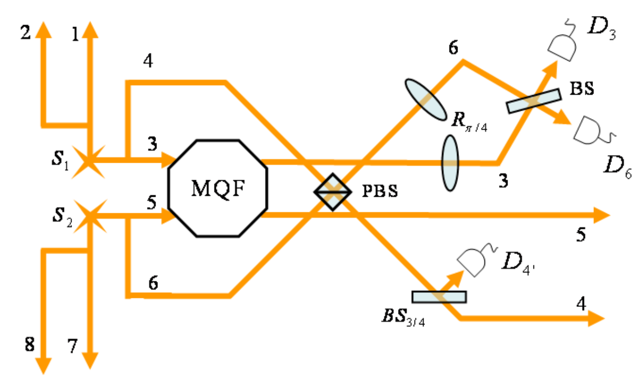

FIG. 2: (color online) Circuit K_1 for a 3 quadbit cluster state from two HESs

To create a 3 quadbit cluster state, we use the "modified quantum filter" (MQF) scheme we present in Appendix B 2. This circuit implements a parity gate between the input photons in a manner which does not destroy the input photons when it is successful, and moreover is unaffected by situations wherein one of the input modes is empty.

Our circuit for generating a 3 quadbit cluster from two HES's is depicted in Figure 2. S_1 and S_2 are sources of initial HESs each in $|\Phi^+_{\mathrm{HES}}\rangle$. Note that there is *one* photon spread across spatial modes (3,4) and one photon spread across spatial modes (5,6) - the circuit is a two-photon gate, and only one photon will be detected - this is reminiscent of fusing together two Bell pairs by Type-I fusion to create a 3 qubit GHZ (cluster) state, and in fact this gate does act as a Type-I fusion gates for quadbits. After a successful operation in modes 3 and 5 of the MQF, the outcome state is equal to

$$\frac{\sqrt{2}}{6}\big[|H\rangle_1|H\rangle_3|H\rangle_5|H\rangle_7+|V\rangle_1|V\rangle_3|V\rangle_5|V\rangle_7 \\ +2(|H\rangle_2|H\rangle_4+|V\rangle_2|V\rangle_4)(|H\rangle_6|H\rangle_8+|V\rangle_6|V\rangle_8)\big], \tag{8}$$

(the measurement operator for operation of the MQF's is presented in Eq. (B5) in Section B 2). Note that in Eq. (8), the first two terms contain a photon in mode 3 and the other terms also have a photon in mode 6 (these are the modes which will be detected). After a polarizing beam splitter (PBS) between modes 4 and 6, two $R_{\pi/4}$s, and a BS in mode 3 and 6, detection of a photon in either mode 3 or 6 results in successful gate operation. The outcome state results from only four terms in Eq. (8), such as $|H\rangle_1|H\rangle_3|H\rangle_5|H\rangle_7$, $|V\rangle_1|V\rangle_3|V\rangle_5|V\rangle_7$, $|H\rangle_2|H\rangle_4|H\rangle_6|H\rangle_8$, and $|V\rangle_2|V\rangle_4|V\rangle_6|V\rangle_8$. The extra beam splitter ($\mathrm{BS}_{3/4}$) with vacuum input in mode 4 balances amplitudes in the final state. For example, after a successful detection in the MQF, the detection of a vertical photon in mode 3 and vacuum in modes 6 and 4 yields a final state

$$\begin{aligned}|QdC_3'\rangle &= \frac{1}{2}\big(|H\rangle_1|H\rangle_5|H\rangle_7-|V\rangle_1|V\rangle_5|V\rangle_7 \\ &\quad +|H\rangle_2|H\rangle_4|H\rangle_8-|V\rangle_2|V\rangle_4|V\rangle_8\big), \\ &= \frac{1}{2}\big(|\bar{0}\rangle|\bar{0}\rangle|\bar{0}\rangle-|\bar{1}\rangle|\bar{1}\rangle|\bar{1}\rangle+|\bar{2}\rangle|\bar{2}\rangle|\bar{2}\rangle-|\bar{3}\rangle|\bar{3}\rangle|\bar{3}\rangle\big),\end{aligned} \tag{9}$$

where the set $\{|\bar{0}\rangle,|\bar{1}\rangle,|\bar{2}\rangle,|\bar{3}\rangle\}$ is defined by $\{|H\rangle_1, |V\rangle_1, |H\rangle_2, |V\rangle_2\}$, $\{|H\rangle_5, |V\rangle_5, |H\rangle_4, |V\rangle_4\}$, and $\{|H\rangle_7, |V\rangle_7, |H\rangle_8, |V\rangle_8\}$.

When the generalized quadbit Hadamard operation and a phase shift are employed on a vertical photon in mode 5 and 4, the outcome state is equivalent to a 3-quadbit cluster state $|QdC_3\rangle$ in Section II A. Therefore, we obtain a three-quadbit cluster state in modes 1,2,4,5,7, and 8 with success probability 1/256.

2. *4 quadbit cluster state*

A slight modification of the circuit in the previous subsection can easily build a 4-quadbit cluster state. We start from the intermediate state in Eq. (8) (see Figure 3). Because the state does not contain an input of vacuum states in mode 4 and 6, the original QF can be used (see Section B 2). When the original QF is successfully applied in modes 4 and 6 to the outcome in Eq. (8), the final state is equal to

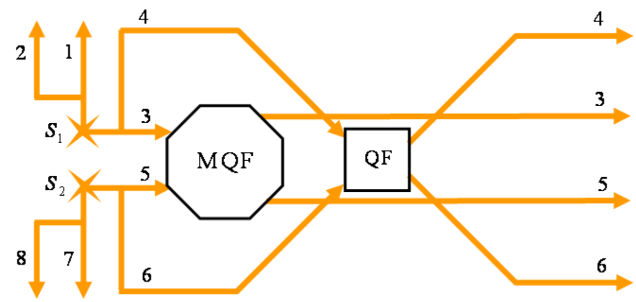

FIG. 3: (color online) Circuit K_2 for a 4-quadbit cluster state from 2 hyper-entangled pairs

$$\begin{aligned}|\mathrm{QdC}_4'\rangle &= \frac{1}{2}\big(|H\rangle_1|H\rangle_3|H\rangle_5|H\rangle_7+|V\rangle_1|V\rangle_3|V\rangle_5|V\rangle_7 \\ &\quad +|H\rangle_2|H\rangle_4|H\rangle_6|H\rangle_8+|V\rangle_2|V\rangle_4|V\rangle_6|V\rangle_8\big).\end{aligned} \tag{10}$$

This is equivalent to

$$|\mathrm{QdC}_4\rangle = \frac{1}{2}\sum_{d=0}^{3}|+_d\rangle|\bar{d}\rangle|+_d\rangle|+_d\rangle, \tag{11}$$

up to a local operation on the second photon. Note this is a 4-quadbit state of "star" form - i.e a central quadbit with three leaves, and thus is useful for creating quadbit clusters with nontrivial topology.

From the resource point of view, two hyper-entangled states and six single photons (four horizontal and two vertical photons) are used to create such a 4 quadbit cluster with success probability 1/1024.

IV. FUSING QUADBIT CLUSTER STATES

In order to perform optical quadbit one-way quantum computation, we require a procedure for building large multi-quadbit cluster states. The Type-I style gate (of section III B 1) could be used; however its success probability is very low.

In Figure 4, we present a Type-II-like fusion gate between two quadbit cluster states. The total circuit is comprised of two sub-circuits we label T_3, consisting of two four-port interferometers. The operation of the T_3 gate is discussed in Appendix B 1. The basic effect of gate T_3 is to destroy the spatial mode information carried by the photons while leaving their polarization information in fact.

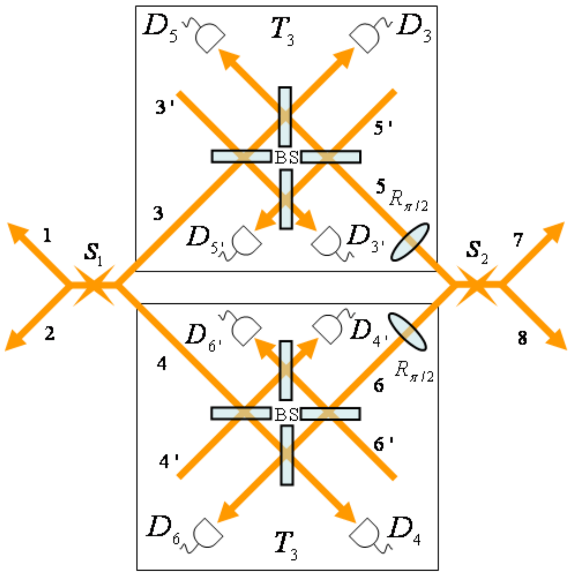

FIG. 4: (color online) Circuit K_3 of a Type2-like fusion gate on two hyper-entangled pairs

As shown in Figure 4, the initial state is prepared in $|\Phi^+_{\text{HES}}\rangle_{1\,2\,3\,4}|\Phi^+_{\text{HES}}\rangle_{5\,6\,7\,8}$. What we desire of this gate is that when it succeeds it tells us "the photons were either in modes 3 and 5 or they were in modes 4 and 6, and their polarization was the same". However it should not reveal in which pair of spatial modes they were, and with what polarization.

After two $R_{\pi/2}$s in mode 5 and 6, the intermediate state is

$$\frac{1}{4}(|H\rangle_1|H\rangle_3 + |V\rangle_1|V\rangle_3 + |H\rangle_2|H\rangle_4 + |V\rangle_2|V\rangle_4)$$
$$(|V\rangle_5|H\rangle_7 + |H\rangle_5|V\rangle_7 + |V\rangle_6|H\rangle_8 + |H\rangle_6|V\rangle_8). \quad (12)$$

Based on the discussion in Appendix B 1, if the upper T_3 gate is implemented (without extra photons in modes $3'$, $5'$) and a successful detection occurs (i.e. a single horizontal and a single vertical photon are detected in two of the modes 3, 5, $3'$, and $5'$), it generates the Bell state in modes 1 and 7:

$$(|H\rangle_1|H\rangle_7 \pm |V\rangle_1|V\rangle_7)/\sqrt{2}. \quad (13)$$

Note that the parts of the input state with amplitude in modes 4, 6 are then wiped out.

On the other hand, if the lower T_3 gate detects one horizontal and one vertical photon, originating from modes 4 and 6, it makes a Bell state in mode 2 and 8

$$(|H\rangle_2|H\rangle_8 \pm |V\rangle_2|V\rangle_8)/\sqrt{2}. \quad (14)$$

and amplitude for modes 3 and 5 is wiped out.

We essentially desire both of these T_3 gates to be able to succeed simultaneously and indistinguishably. In order to attain this, extra photons are injected into the spatial modes $3'$, $4'$, $5'$, and $6'$. We will consider various possible initial states for these ancillary photons. The basic idea is that indistiguishable events occur if two photons in different polarizations are detected in both the upper and lower T_3 gates simultaneously. These events can arise from either the ancillary photons or the 'actual inputs' - and our lack of knowledge about which possibility occurs gives an amplitude for both T_3 gates working. The success probability relies on the input state of the extra two photons, and we discuss several possibilities.

The first case is that two single photons are injected in mode $3'$, $4'$, $5'$, and $6'$ in the state

$$|\text{Ex}_1\rangle = \frac{1}{4}(|H\rangle_{3'} + |V\rangle_{3'} + |H\rangle_{4'} + |V\rangle_{4'})$$
$$(-|H\rangle_{5'} + |V\rangle_{5'} - |H\rangle_{6'} + |V\rangle_{6'}), \quad (15)$$

where each photon is a superposed state in two spatial modes in both polarizations.

When we detect two different polarized photons in the upper T_3 gate and two different polarized photons in the lower one, we do not know whether the four photons detected in both T_3 gates come from hyper-entangled states or the extra input photons. For example, the upper T_3 gate succeeds upon detection of a horizontal photon in mode 3 and a vertical photon in mode 5. The photons could come from any two modes out of modes 3, 5, $3'$, and $5'$. According to Eq. (15), the detection works on various input states like $|H\rangle_3|V\rangle_5$, $|V\rangle_3|H\rangle_5$,$|H\rangle_{3'}|V\rangle_{5'}$, and $|V\rangle_{3'}|H\rangle_{5'}$. If the detected photons were $|H\rangle_3|V\rangle_5$, $|V\rangle_3|H\rangle_5$, the remaining state from Eq. (12) is equal to the state in Eq. (13). However, if the detected photons were $|H\rangle_{3'}|V\rangle_{5'}$, and $|V\rangle_{3'}|H\rangle_{5'}$, the lower circuit could be activated by $|H\rangle_4|V\rangle_6$, $|V\rangle_4|H\rangle_6$ and the remaining state equals Eq.(14). The same logic can be applied the other way around between the upper and lower circuits. Thus, for the successful cases (two different polarized photons detected in each T_3 gate), the final state is equivalent to

$$\frac{1}{2}(|H\rangle_1|H\rangle_7 + |V\rangle_1|V\rangle_7 + |H\rangle_2|H\rangle_8 + |V\rangle_2|V\rangle_8), \quad (16)$$

which is a superposition state of Eq. (13) and Eq. (14). For this case, the total success probability is 1/64.

We now consider injecting a Bell pair in mode $3'$, $4'$, $5'$, and $6'$ instead of two single photons such as

$$|\text{Ex}_2\rangle = \frac{1}{2\sqrt{2}}\big[(|H\rangle_{3'} + |V\rangle_{3'})(-|H\rangle_{5'} + |V\rangle_{5'})$$
$$+ (|H\rangle_{4'} + |V\rangle_{4'})(-|H\rangle_{6'} + |V\rangle_{6'})\big] \quad (17)$$

it can readily be seen that the same indistinguishability of T_3 gate operations occurs - in this case with total success probability 1/32.

Finally, the most efficient state to use is the ancillary input a HES

$$|\text{Ex}_3\rangle = \frac{1}{2}(|H\rangle_{3'}|V\rangle_{5'} + |V\rangle_{3'}|H\rangle_{5'}$$
$$+ |H\rangle_{4'}|V\rangle_{6'} + |V\rangle_{4'}|H\rangle_{6'}). \quad (18)$$

In this case the total success probability is 1/16.

Interestingly, even in some failure cases, we still have a chance to have remanent entanglement between two photons in mode 1 (or 2) and 7 (or 8) of Figure 4. Without the help of extra photons in the primed modes, the success probability is 1/2 to generate a Bell pair from two HESs through this circuit. If we use two extra photons, the possibility of obtaining some entanglement between 1 (or 2) and 7 (or 8) becomes higher than 1/2. This could possibly be useful for some hybrid qubit/quadbit cluster states computing schemes. For example, we imagine a modified qubit cluster state possessing a HES at the edge and fuse two copies of this state on the HES side in circuit K_3. With an extra HES in the primed modes, a HES or a Bell pair is generated among mode 1, 2, 7, and 8 with overall probability 3/4.

We see therefore that we can use this Type2-like circuit to create a Bell pair from HESs. As shown in Figure 4, we prepare two HESs with no extra photon. With probability 3/4 we achieve a Bell state. Although this seems perverse - destroying two HES's to create a Bell pair, it raises an interesting possibility of attaching systems which have the form of a HES at the end of the (qubit) cluster state. If we perform this fusion gate on two such photons, it appears that we could fuse the larger qubit cluster state with the probability 3/4.

V. SUMMARY OF SOME RESOURCE TRADEOFFS

A. Difficulties of quantifying tradeoffs

A Bell pair can be created from 4 single photons with probability 1/4 (see Appendix C for a proof - previously published results [20] suggested the success probability was 3/16). Such creation is *ballistic* - the single photons are fired in, and (up to some local linear transformation) the desired Bell state is created 1 in 4 times. We could say that a Bell pair is "worth" 16 single photons on average - this indicates how much easier things will be if we have a deterministic source of Bell pairs. Now a trivial extension of this ballistic scheme can create a 3-photon GHZ state from 6 single photons with probability 1/32, and can create (ballistically) a 4-photon GHZ state with probability 1/128 (see Table I). From this we might conclude a 3-photon GHZ state is worth 96 photons. However we can also create a 3-photon GHZ state by using a Type-I gate [13] and fusing two Bell pairs. The Type-I gate succeeds with probability 1/2, and each Bell pair is worth 16 photons, so this indicates the GHZ state is only worth 64 single photons. The difference, of course, is that with the latter technique we would have to store the Bell pair, once created, in order for it to be available to combine with the second Bell pair. While the ability to postselect on successfully generated states (and then store them) lies at the heart of why it is we can turn exponentially decreasing probabilities into efficient methods for creating large entangled states, such storage is likely to present practical problems. (It is worth noting that the percolation techniques of [3] ameliorate many of these issues).

Resource counting is made even messier by the following observation: Sometimes we may require the use of an ancillary entangled state within some larger ballistic circuit (a Bell pair say). One may think that we could replace this Bell pair by 4 single photons (as in Fig. 8 in Appendix C) to obtain a ballistic single photon scheme, and only take a hit of 1/4 in the overall success probability of the larger circuit. However the ballistic scheme presumes the ideal state is produced "up to easily implementable linear optical transformations" - and it is generally a smaller set of detection outcomes which yield the desired state for input into the larger circuit.

The final feature that makes resource counting difficult is the nature of failure outcomes: sometimes failed gates acting on suitably large input states still leave some of the systems in useful resource states. The potential for recycling (which also requires quantum memory) often greatly complicates the question of optimizing resource counting [5].

B. Resources for quadbit cluster states

As shown in Table I, A various combination of resources can be used to create any desired state. Without an entangled source (i.e. only single photon sources) one can generate a Bell pair from 4 single photons, a 3-photon GHZ state from 6 photons, and 4-photon GHZ state from 8 photons. However, using entangled sources, the desired many-photon state can be built with much higher probabilities.

In terms of quadbit cluster states, the counterpart of a Bell pair for qubit is a HES. So, to build a HES requires a source such as 8 single photons, two Bell pairs, or one

Resource						Output	Probability
SP	BP	3GHZ	4GHZ	HES	3QdC		
4						BP	1/4
6						3GHZ	1/32
8						4GHZ	1/128
	2					3GHZ	1/2
	1	1				4GHZ	1/2
8						HES	1/4096
	2					HES	1/16
			1			HES	3/16
4				2		3QdC	1/256
6				2		4QdC	1/1024
6				1	1	4QdC	1/256
2					2	4QdC	1/64
	1				2	4QdC	1/32
				1	2	4QdC	1/16

TABLE I: Resource costs for multi-quadbit cluster states (SP = single photon, BP = Bell pair, HES = hyper-entangled state, and QdC = quadbit cluster)

4-photon GHZ state. Based on the circuit J_2 the optimal probability is 3/16, obtained when using a 4-photon GHZ state.

The bottom of the table shows various ways of building multi-quadbit cluster states by proposed methods with the help of extra photons. We only have one method to build 3 quadbit cluster stated using circuit K_1, while several possible methods are available to create a 4 quadbit cluster state through circuits K_2 and K_3. For the 4 quadbit cluster state, the success probability without a 3 quadbit cluster state is 1/1024 (using circuit K_2).

VI. CONCLUSION

We have initiated the study of building higher dimensional cluster states of photons. Although we have presented several "modules" within our constructions that we expect to be of generic use for LOQC using higher dimensional photonic states, it is unclear to us whether the procedures we have outlined are close to the best possible. If they are, then there seems to be limited advantage in using higher dimensional cluster states built up from single photons from a strict resource counting perspective. It is possible, however, that in the future deterministic sources of hyper-entanglement become available.

Very recently, qubit one-way quantum computation using a hyper-entangled state (HES) has been demonstrated [26, 27]. In these papers, a four-qubit cluster state is created from a HES generated by a spontaneous parametric down conversion. They assume that photon's polarization and its spatial modes are defined as a qubit respectively and destroying a single photon performs two single qubit measurements simultaneously. Note that this is quite different to our proposal, where we use a single photon as a higher-dimensional quantum unit. An equivalence between these schemes arises for a 2-quadbit HES and a 4-qubit linear cluster state because $2+2 = 2\cdot 2$ [26, 27].

Acknowledgments

We acknowledge useful discussions with Jens Eisert, David Gross and Konrad Kieling and thank the support of the US Army Research Office (W911NF-05-0397) and the UK EPSRC. This work was supported in part by the UK Engineering and Physical Sciences Research Council through their Quantum Information Processing Interdisciplinary Research Centre, and by the European Union through their networks SCALA and CONQUEST. J.J. was also supported by the Overseas Research Student Award Program.

APPENDIX A: BASIC OPTICAL TOOLS

We use the convention that a regular beam splitter (Fig. 5 (a)) which we denote BS_{r^2} acts upon two spatial modes 1,2 according to

$$\begin{aligned} |H(V)\rangle_1 &\rightarrow t|H(V)\rangle_1 + r|H(V)\rangle_2, \\ |H(V)\rangle_2 &\rightarrow -r|H(V)\rangle_1 + t|H(V)\rangle_2, \end{aligned} \tag{A1}$$

where $t^2 + r^2 = 1$. A 50:50 BS ($t = r = 1/\sqrt{2}$) acts according to $|H(V)\rangle_1 \rightarrow (|H(V)\rangle_1 + |H(V)\rangle_2)/\sqrt{2}$ and $|H(V)\rangle_2 \rightarrow (-|H(V)\rangle_1 + |H(V)\rangle_2)/\sqrt{2}$. On a single photon this is a Hadamard gate in the spatial mode. However, when two horizontal (vertical) photons are injected in modes 1 and 2 respectively, the total photonic state shows destructive interference between modes 1 and 2 - the two photons bunch into a single mode together. More explicitly, if two polarized photons $|H\rangle_1|H\rangle_2$ are injected onto a 50:50 BS, the total state is equal to $\frac{1}{\sqrt{2}}|H\rangle|H\rangle\otimes(-|1\rangle|1\rangle+|2\rangle|2\rangle)$ where $|H^2\rangle_i$ denote two horizontal photons in mode i. This state could be called an entangled state in spatial mode. In terms of the quadbit encoding Eq. 2, the regular BS induces unitary rotations between state $|\bar{0}\rangle$,$|\bar{2}\rangle$ and between $|\bar{1}\rangle$,$|\bar{3}\rangle$.

In Fig. 5 (b), a polarization rotator (R_θ) in mode 1 is depicted - its action is taken to be

$$\begin{aligned} |H\rangle_1 &\rightarrow \cos\theta|H\rangle_1 + \sin\theta|V\rangle_1\,, \\ |V\rangle_1 &\rightarrow -\sin\theta|H\rangle_1 + \cos\theta|V\rangle_1, \end{aligned} \tag{A2}$$

where angle θ is the amount of the rotating axis of linear polarizer. A $\pi/4$ polarization rotator $R_{\pi/4}$ acts such that $|H\rangle_1 \rightarrow |+\rangle_1 = (|H\rangle_1 + |V\rangle_1)/\sqrt{2}$ and $|V\rangle_1 \rightarrow |-\rangle_1 = (-|H\rangle_1 + |V\rangle_1)/\sqrt{2}$. On a single photon this is a Hadamard gate in polarization; on two differently polarized photons in the same spatial mode $|H\rangle_1|V\rangle_1$ are transformed by a rotation $R_{\pi/4}$ into the state $\frac{1}{\sqrt{2}}(-|H\rangle|H\rangle + |V\rangle|V\rangle) \otimes |1\rangle|1\rangle$, again because of destructive interference. This photonic state has entanglement in polarization (although the two photons are located in a single spatial mode together). In terms of the quadbit encoding Eq. 2, the polarization rotator induces unitary rotations between state $|\bar{0}\rangle$,$|\bar{1}\rangle$ and between $|\bar{2}\rangle$,$|\bar{3}\rangle$.

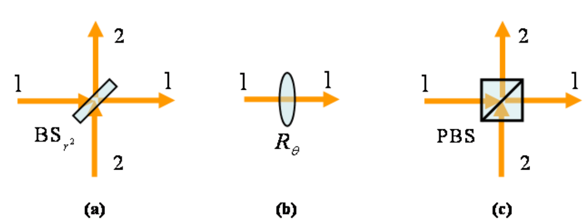

FIG. 5: (color online) (a) Beam splitter BS_{r^2} (b) Polarization rotator R_θ (c) Polarizing beam splitter (PBS)

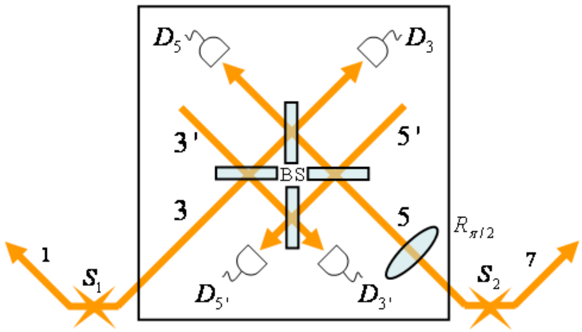

FIG. 6: (color online) The Type3 fusion gate T_3 creates indistinguishability of photons in mode 3 and 5

In Fig.5(c) a polarizing beam-splitter (PBS) is depicted. It acts such that a vertical photon is reflected into a different spatial mode ($1 \to 2$ and $2 \to 1$), while a horizontal photon passes through a PBS. In terms of the quadbit encoding Eq. 2, the PBS induces a swap operation between $|\bar{1}\rangle$ and $|\bar{3}\rangle$.

APPENDIX B: ADVANCED OPTICAL TOOLS

1. Gate T_3

Here we present a gate similar to a qubit Type-II gate, which is used to create a Type-II-like gate for quadbits. In Figure 6, we begin with two Bell-pairs $|\psi^+\rangle_{13}|\psi^+\rangle_{57}$ and act the gate on modes 3 and 5. The state after a $R_{\pi/2}$ in mode 5 is equal to

$$\frac{1}{2}(|H\rangle_1|H\rangle_3+|V\rangle_1|V\rangle_3)(|V\rangle_5|H\rangle_7+|H\rangle_5|V\rangle_7). \quad \text{(B1)}$$

A four-port interferometer is then introduced [28]. This interferometer essentially erases spatial mode information . Its operation on a single photon input into any of the modes 3, 5, $3'$, $5'$ is:

$$\begin{aligned} |P\rangle_3 &\to \big(|P\rangle_3+|P\rangle_5+|P\rangle_{3'}+|P\rangle_{5'}\big)/2, \\ |P\rangle_5 &\to \big(-|P\rangle_3+|P\rangle_5-|P\rangle_{3'}+|P\rangle_{5'}\big)/2, \\ |P\rangle_{3'} &\to \big(-|P\rangle_3-|P\rangle_5+|P\rangle_{3'}+|P\rangle_{5'}\big)/2, \\ |P\rangle_{5'} &\to \big(|P\rangle_3-|P\rangle_5-|P\rangle_{3'}+|P\rangle_{5'}\big)/2, \end{aligned} \quad \text{(B2)}$$

where P denotes either horizontal or vertical polarization (H, V). In the case that inputs $3'$ and $5'$ are vacuum, as in our example here, we need only consider here the first and second transformations in Eq. (B2). Note, however, that the possibility of extra input photons in these modes will be used in Section IV to create indistinguishability in polarization.

To continue with our example, when we detect a horizontal photon in mode 3 and a vertical photon in mode 5, the original photons in Eq.(B1) could be either $|V\rangle_3$ $|H\rangle_5$ or $|H\rangle_3$ $|V\rangle_5$. When a horizontal and a vertical photon are detected in any mode out of 3, 5, $3'$, and $5'$, the final state is

$$\frac{1}{\sqrt{2}}(|H\rangle_1|H\rangle_7 \pm |V\rangle_1|V\rangle_7). \quad \text{(B3)}$$

The failure cases (the detection of two horizontal or two vertical photons) yield a product state. That is, in the failure cases, the detection of two horizontal (vertical) photons comes only from $|H\rangle_1|V\rangle_7$ ($|V\rangle_1|H\rangle_7$). The failure outcome gives us the polarization information of the other photons.

The overall effect of the gate is to remove spatial mode information in a way that does not destroy polarization information. This gate T_3 has success probability 1/2 and it destroys both input photons.

2. Quantum filter (QF)

The original qubit QF [29] depicted in Figure 7 is useful for generating multi-qubit entangled states deterministically. Two photons in modes i and j are injected into the QF; successful detection occurs if and only if their polarizations are the same. The most interesting feature is that the injected photons survive in the output modes after the QF. Consider, as shown in Figure 7 (a), when two input Bell-pairs (as in Eq. (5)) are passed through a QF set up in mode 2 and 3. (The $\mathrm{BS}_{3/4}$ is required to balance coefficients in the two different polarization modes). The output state is simply the fusion of the initial Bell states, i.e. the GHZ state $(|HHHH\rangle_{1234}+|VVVV\rangle_{1234})/\sqrt{2}$.

In cases where only a single photon is input there is a fundamental asymmetry between horizontal and vertical input photons in the workings of the QF. For example, the QF is activated (a success detection is achieved) when a vertical photon in mode i is input along with vacuum in mode j. Taking into account vacuum inputs, the action of the QF circuit when successful can be represented by a measurement operator

$$\begin{aligned} \hat{S}_{ij} &= \frac{1}{4}(|HH\rangle_{ij}\langle HH|+|VV\rangle_{ij}\langle VV|)+\frac{1}{4}|\text{ø}V\rangle_{ij}\langle \text{ø}V| \\ &\quad +\frac{1}{2}(|V\text{ø}\rangle_{ij}\langle V\text{ø}|+|\text{ø}\,\text{ø}\rangle_{ij}\langle \text{ø}\,\text{ø}|). \end{aligned} \quad \text{(B4)}$$

This asymmetry between H, V is highly detrimental to our purposes, as is the fact that the filter will trigger success when only single photon is input.

To obtain a complete quantum filter for both polarizations, a modification is required in the vertical part of the original filter, and this takes two additional ancilla vertical photons. This modified quantum filter (MQF) is shown in Figure 7 (b). The measurement operator for the MQF is

$$\begin{aligned} \hat{S'}_{ij} &= \frac{1}{8}(|HH\rangle_{ij}\langle HH| + |VV\rangle_{ij}\langle VV|) \\ &+\frac{1}{4}|\o\,\o\rangle_{ij}\langle \o\,\o|). \end{aligned} \tag{B5}$$

Note that the $\mathrm{BS}_{3/4}$ with a vacuum input (which essentially just dumped unwanted amplitude) is no longer needed because the MQF balances the outcomes for both polarizations naturally. The success probability of the MQF is 1/128.

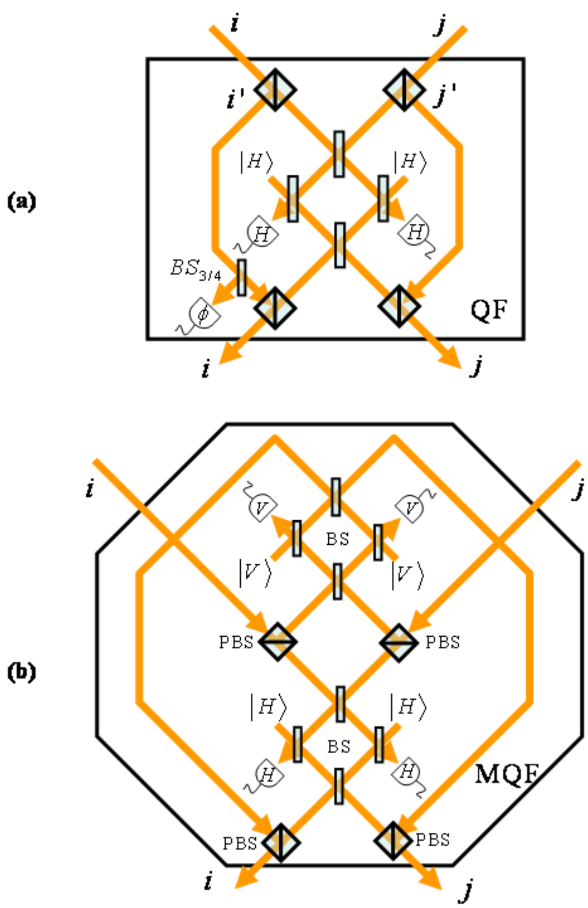

FIG. 7: (color online) Implementation of (a) an original and (b) a modified quantum filter

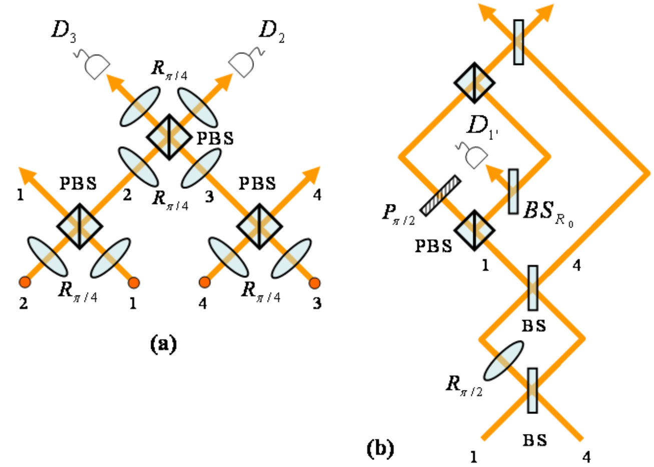

FIG. 8: (color online) (a) The circuit B creates a Bell pair from four single photons in horizontal polarization when single photons are detected at D_2 and D_3 and (b) An extra circuit is required for correcting the case of two-photon detection in circuit B.

APPENDIX C: IMPLEMENTATION OF A BELL PAIR FROM FOUR SINGLE PHOTONS

We present a scheme to build a two-photon Bell state from four single photons. Initially, all photons are prepared with horizontal polarization. As shown in Figure 8(a), each photon passes through a $R_{\pi/4}$ in each mode and two PBSs are applied in modes 1 and 2 as well as 3 and 4. Then, we use a Type-II fusion gate between modes 2 and 3. If two different polarized photons are detected in any mode, we obtain a typical form of a Bell pair with a success probability 3/16, as noted in [13, 20]. However, previously the detection of two photons of the same polarization was regarded as a failure case.

In fact, we can transform this error state to a typical Bell pair probabilistically with a simple circuit in Figure 8 (b). For example, we assume that two horizontal photons are detected in mode 2 and two photons remain in modes 1 and 4. The outcome state contains a mixture of two-photon states in a single spatial mode and a state of one photon in each mode. The first part of the correction circuit consists of two BSs and a $R_{\pi/4}$ (see the bottom of Figure 8 (b)). After the first part, the total state is a bunching state made by two equivalently polarized photons with unbalanced coefficients in each mode, such as:

$$\frac{1}{2\sqrt{3}}(|H^2\rangle_1 - 3|V^2\rangle_1 + |H^2\rangle_4 + |V^2\rangle_4). \tag{C1}$$

In the remainder part of the circuit, the BS_{R_0} ($R_0 = 2/3$)) with a vacuum input plays a key role to balance coefficients in the final state and $P_{\pi/2}$ is a $\pi/2$ phase shifter in mode 1. For successful cases, no photon detection at $D_{1'}$ reduces a coefficient of one photonic state in a certain spatial mode. Finally, the total state is equal to $|\phi^-\rangle_{14}$ with a success probability 1/16. Therefore, the optimal success probability is 1/4 to create a Bell pair linear optically from four single photons.

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[31] In general, within the circuit model, the results of [30] suggest that quantum computation with quadbits can be expected to result in a space saving of at least $O(\log_2 d)$, and a time saving of at least $O(\log_2 d)^2$. The extent to which such savings translate into the cluster state model are largely unexplored - they will depend on optimal decompositions of qudit cluster circuits for general two-qudit unitary operations. Such optimal decompositions are not completely characterized for even the qubit case yet.