Asymptotically conical Ricci-flat Kähler metrics with cone singularities

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Declaration of Originality

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Abstract

The main result proved in this thesis is an existence theorem for asymptotically conical Ricci-flat Kähler metrics on $\mathbb{C}^2$ with cone singularities along a smooth complex curve. These metrics are expected to arise as blow-up limits of non-collapsed sequences of Kähler-Einstein metrics with cone singularities.

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Contents

1 Introduction

The first three subsections in this introduction are meant to explain the words in the title, to provide background and to motivate the topic. The fourth subsection states the main result of the thesis and outlines the strategy of proof.

1.1 Metrics with cone singularities

We introduce the concept of a Kähler metric with cone singularities along a divisor. There is a model metric $g_{(\beta)}$ and several ways one can proceed to give distinct definitions, measuring the deviation from the model in different norms. The one we give is well suited for the development of a linear theory. Our reference is Section 4 in Donaldson's paper [15].

Fix $0 < \beta < 1$. On $\mathbb{R}^2 \setminus \{0\}$ with polar coordinates (ρ, θ) consider the metric

$$g_\beta = d\rho^2 + \beta^2 \rho^2 d\theta^2. \tag{1.1}$$

This is the metric of a cone of total angle $2\pi\beta$. The apex of the cone is located at 0 and g_β is singular at this point. We give two geometric pictures of g_β:

- Take a wedge of angle $2\pi\beta$ in the Euclidean plane and use a rotation to identify the edges.
- Take a coordinate axis in $\mathbb{R}^3$ and a ray which starts at 0 and makes an angle ψ with the axis. We ask that $0 < \psi < \pi/2$ and $\sin\psi = \beta$. Consider the surface of revolution obtained by rotating the ray around the axis.

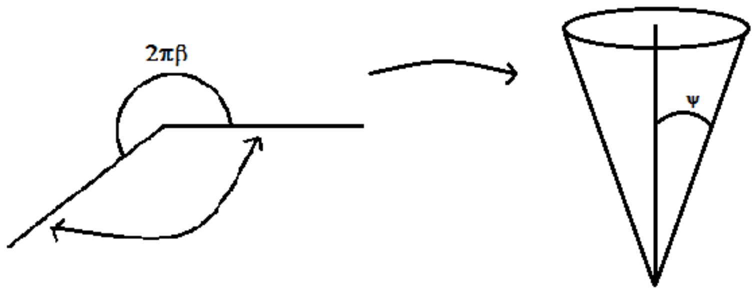

Figure 1: The holonomy of g_β along a simple loop around the apex is an anti-clockwise rotation of angle $2\pi(1-\beta)$ in the sense of the metric g_β.

The metric g_β induces a complex structure on the punctured plane, given by an anti-clockwise rotation of angle $\pi/2$. A basic fact is that we can change coordinates so that this complex structure extends smoothly to the origin. Indeed, set

$$z_1 = \rho^{1/\beta} e^{i\theta} \tag{1.2}$$

to get

$$g_\beta = \beta^2 |z_1|^{2\beta-2} |dz_1|^2. \tag{1.3}$$

Denote by $\mathbb{C}_\beta$ the complex plane endowed with the singular metric 1.3.

We are concerned with metrics which are modeled, in transverse directions to a divisor, by g_β. To begin with we take the product of $\mathbb{C}_\beta$ with $\mathbb{C}^{n-1}$. If $(z_1, \ldots, z_n)$ are standard complex coordinates on $\mathbb{C}^n$, what we get is the model metric

$$g_{(\beta)} = \beta^2 |z_1|^{2\beta-2} |dz_1|^2 + \sum_{j=2}^{n} |dz_j|^2; \tag{1.4}$$

with a singularity along $D = \{z_1 = 0\}$. Set $\{v_1, \ldots, v_n\}$ to be the vectors

$$v_1 = |z_1|^{1-\beta} \frac{\partial}{\partial z_1}, \quad v_j = \frac{\partial}{\partial z_j} \text{ for } j = 2, \ldots n. \tag{1.5}$$

Note that, with respect to $g_{(\beta)}$, these vectors are orthogonal and their length is constant. We move on and consider the situation of a complex manifold X of complex dimension n and a smooth divisor $D \subset X$. Let g be a smooth Kähler metric on $X \setminus D$ and let $p \in D$. Take $(z_1, \ldots, z_n)$ to be complex coordinates centered at p such that $D = \{z_1 = 0\}$. In the complement of D we have smooth functions $g_{i\bar{j}}$ given by $g_{i\bar{j}} = g(v_i, \overline{v}_j)$.

Definition 1 *We say that g has cone angle $2\pi\beta$ along D if for every $p\in D$ and holomorphic coordinates as above the functions $g_{i\bar{j}}$ admit a Hölder continuous extension to D. We also require the matrix $(g_{i\bar{j}}(p))$ to be positive definite and that $g_{1\bar{j}}=0$ when $j\geq 2$ and $z_1=0$.*

We make two remarks on this definition

- It requires a simple computation to check that the definition, in particular the vanishing condition on $g_{1\bar{j}}$ for $j\geq 2$, is independent of the choice of coordinates. In a coordinate chart as above we can define a symmetric tensor h by means of the equation

$$g = g_{(\beta)} + h.$$

  We get a Hermitian matrix $(h_{i\bar{j}})$, where $h_{i\bar{j}} = h(v_i,\overline{v}_j)$. At the point $p\in D$ we can rescale z_1 and perform a linear transformation in the $z_2,\dots,z_n$ variables so that $h_{i\bar{j}}(p)=0$ for all $1\leq i,j\leq n$; but it is interesting to note that- contrary to the case for smooth metrics- this requirement *doesn't* fix the coordinates up to first order. Let $\{\tilde{v}_1,\dots,\tilde{v}_n\}$ be some other frame field of vectors in the complement of D such that, with respect to $g_{(\beta)}$, the length of the vectors $\tilde{v}_1,\dots,\tilde{v}_n$ is constant and the vectors are pairwise orthogonal. For example we can take $\tilde{v}_1 = e^{i\theta}v_1$ with $\theta=\arg(z_1)$ and $\tilde{v}_j = v_j$ for $j\geq 2$. We can define functions $\tilde{g}_{i\bar{j}} = g(\tilde{v}_i,\tilde{v}_j)$. The vanishing condition $g_{1\bar{j}}=0$ for $j\geq 2$ implies that the functions $\tilde{g}_{i\bar{j}}$ extend Hölder continuously to D and hence the definition is independent of the frame field of vectors chosen.

- There are two types of coordinates we can take around D. The first one is given by holomorphic coordinates $z_1,\dots,z_n$ such that $D=\{z_1=0\}$. In the second one we replace the coordinate z_1 with $\rho e^{i\theta}$, by means of 1.2, and leave $z_2,\dots,z_n$ unchanged. We refer to the later as cone coordinates. In other words, there are two relevant differential structures on X in our situation. One is given by the complex manifold structure we started with, the other is given by declaring the cone coordinates to be smooth. The two structures are equivalent by a map modeled on 1.2 in a neighborhood of D. The notion of a function being Hölder continuous (without specifying the exponent) is independent of the coordinates we take. Let f be a function on X and $\alpha\in(0,1)$, we say that $f\in C^{\alpha}$ if it belongs to this space, in the usual sense, in the cone coordinates. We can incorporate the exponent α in Definition 1 by requiring the functions $g_{i\bar{j}}$ to be C^{α}. It is interesting to note that the notion we get is independent of the complex coordinates $z_1,\dots,z_n$ only if we add the restriction that $\alpha\leq\beta^{-1}-1$. Indeed let g be a metric in a domain of $\mathbb{C}^2$, with standard complex coordinates $(\tilde{z}_1,\tilde{z}_2)$, of cone angle $2\pi\beta$ along $D=\{\tilde{z}_1=0\}$. Write $\tilde{g}_{i\bar{j}}$, $1\leq i,j\leq 2$, for the coefficients of g as we defined previously; so that $\tilde{g}_{i\bar{j}}$ are smooth functions on the complement of D which extend Hölder continuously to D . Set $\tilde{z}_1=z_1$ and $\tilde{z}_2 = z_1+z_2$, so that

$$\frac{\partial}{\partial z_1} = \frac{\partial}{\partial \tilde{z}_1} + \frac{\partial}{\partial \tilde{z}_2}, \quad \frac{\partial}{\partial z_2} = \frac{\partial}{\partial \tilde{z}_2}.$$

  In the coordinates (z_1,z_2) we get that

$$g_{1\bar{1}} = \tilde{g}_{1\bar{1}} + |z_1|^{1-\beta}(\tilde{g}_{1\bar{2}}+\tilde{g}_{2\bar{1}}) + |z_1|^{2-2\beta}\tilde{g}_{2\bar{2}}, \quad g_{1\bar{2}} = \tilde{g}_{1\bar{2}} + |z_1|^{1-\beta}\tilde{g}_{2\bar{2}}, \quad g_{2\bar{2}} = \tilde{g}_{2\bar{2}}.$$

  The function $|z_1|^{1-\beta}$ belongs to C^{α} only if $\alpha\leq\beta^{-1}-1$.

Next we provide examples of metrics which satisfy Definition 1. We begin with a local description of a general type of metric with cone singularities. Let F be a smooth positive function and let η be a smooth Kähler form, both defined on a domain in $\mathbb{C}^n$ which contains the origin. Consider the $(1,1)$ form

$$\omega = \eta + i\partial\overline{\partial}(F|z_1|^{2\beta}). \tag{1.6}$$

Straightforward calculation gives us that

$$i\partial\overline{\partial}(F|z_1|^{2\beta}) = |z_1|^{2\beta}i\partial\overline{\partial}F + \beta|z_1|^{2\beta-2}\left(\overline{z}_1 idz_1\wedge\overline{\partial}F + z_1 i\partial F\wedge d\overline{z}_1\right) + \beta^2|z_1|^{2\beta-2}F idz_1\wedge d\overline{z}_1.$$

Let I be the complex structure of $\mathbb{C}^n$ and $g=\omega(.,I.)$. Let $v_1,\dots,v_n$ be as in 1.5. We want to compute $g_{i\bar{j}} = g(v_i,\overline{v}_j)$. We write $\eta=\sum_{i,j=1}^{n}\eta_{i\bar{j}} idz_i\wedge d\overline{z}_j$. Note that the coefficients $\eta_{i\bar{j}}$ are given by the contraction of η with the standard coordinate vectors $\partial/\partial z_i$, $\partial/\partial \overline{z}_j$, while to obtain $g_{i\overline{j}}$ we must contract g with $v_i, \overline{v}_j$. It is easy to check that

$$g_{1\overline{1}} = |z_1|^{2-2\beta}\eta_{1\overline{1}} + |z_1|^2 \frac{\partial^2 F}{\partial z_1 \partial \overline{z}_1} + \beta \left(z_1 \frac{\partial F}{\partial z_1} + \overline{z}_1 \frac{\partial F}{\partial \overline{z}_1} \right) + \beta^2 F;$$

$$g_{1\overline{j}} = |z_1|^{1-\beta}\eta_{1\overline{j}} + |z_1|^{1+\beta} \frac{\partial^2 F}{\partial z_1 \partial \overline{z}_j} + \beta |z_1|^{\beta-1} \left(z_1 \frac{\partial F}{\partial z_j} + \overline{z}_1 \frac{\partial F}{\partial \overline{z}_j} \right) \quad \text{for} j \geq 2;$$

$$g_{j\overline{k}} = \eta_{j\overline{k}} + |z_1|^{2\beta} \frac{\partial^2 F}{\partial z_j \partial \overline{z}_k} \quad \text{for} j, k \geq 2.$$

It is then clear that, in a neighborhood of 0, g defines a Kähler metric with cone angle $2\pi\beta$ along $D = \{z_1 = 0\}$. Indeed this metric is C^α for $\alpha = \beta^{-1} - 1$. There is a useful way of thinking of the metric g: On $\mathbb{C}^{n+1}$ with standard complex coordinates $(z_1, \ldots, z_{n+1})$ consider the $(1,1)$ form

$$\Gamma = \eta + i\partial\overline{\partial}(F|z_{n+1}|^2).$$

This form defines a smooth Kähler metric on $\mathbb{C}^{n+1}$ in a neighborhood of 0. Let us delete a ray in the complex plane corresponding to the z_1 variable and define

$$\Phi(z_1, \ldots, z_n) = (z_1, \ldots, z_n, z_1^\beta),$$

so that $\omega = \Phi^*\Gamma$. The pullback of Γ by Φ is independent of the branch of z_1^β that we take and we can think of the metric g in the complement of D as the restriction of the smooth metric defined by Γ to a smooth complex hypersurface in $\mathbb{C}^{n+1}$. A well-known principle says that the holomorphic sectional curvature of a complex submanifold of a Kähler manifold is less or equal than that of the ambient manifold, see Section 0.5 in Griffiths-Harris [18]. We conclude that we can restrict g to a smaller neighborhood of 0 if necessary so that its sectional curvature is uniformly bounded from above.

Remark 1 *It is easy to see from the expressions of the coefficients $g_{i\overline{j}}$ that in order that ω defines a Kähler metric with cone singularities in a neighborhood of 0, it is enough that η is a closed $(1,1)$ form such that $(\eta_{j\overline{k}})_{2\leq j,k\leq n}$ is a positive matrix at 0. In this more general situation we can not conclude the upper bound on the sectional curvature of g. A good example to have in mind is the following: Let a be a real number with $|a| < 1$. Consider the metric defined in the unit disc of the complex numbers given by*

$$g_a = (a + |z_1|^{2\beta-2})|dz_1|^2.$$

The Gaussian curvature of this metric is equal to

$$K_a = -4(\beta-1)^2 a \frac{|z_1|^{2-4\beta}}{(1 + |z_1|^{2-2\beta}a)^3}.$$

If $1/2 < \beta < 1$, then K_a is unbounded below when $a > 0$ and unbounded above if $a < 0$. Of course we can take the product of g_a with a flat euclidean factor $\mathbb{C}^{n-1}$ to fit this example into our discussion.

The following lemma compiles our results into a global form.

Lemma 1 *Let (X, η) be a smooth compact Kähler manifold and $D \subset X$ a smooth divisor. Let h be a smooth Hermitian metric on the line bundle $[D]$. Let $s \in H^0([D])$ be such that $D = \{s = 0\}$. For $\epsilon > 0$ set $\omega = \eta + \epsilon i\partial\overline{\partial}|s|_h^{2\beta}$. If we take ϵ small enough then the form ω defines a Kähler metric with cone singularities as in Definition 1. The metric ω is, up to quasi-isometry, independent of the choices of η, ϵ, s and h. The bisectional curvature of ω is uniformly bounded from above.*

Proof: Indeed we have shown that $\eta + i\partial\overline{\partial}|s|_h^{2\beta}$ is a Kähler metric in a sufficiently small tubular neighborhood U of D with cone angle $2\pi\beta$ along D. We take $\epsilon > 0$ small enough so that $\eta > -\epsilon i\partial\overline{\partial}|s|_h^{2\beta}$ in the complement of U and we check that ω has the desired properties. □

The statement about the bisectional curvature in Lemma 1 is a little bit technical at this point, but it will be relevant in the next subsection. We don't need to recall the definition of bisectional curvature right now. We just say that on a Kähler manifold a uniform (upper) bound in any of the following three quantities: holomorphic sectional curvature, sectional curvature, bisectional curvature; implies a uniform (upper) bound in the other two quantities. In this direction we mention the following

Conjecture 1 *If there is a 'polyhomogeneous' Kähler metric with cone angle $2\pi\beta$ along D, bounded sectional curvature and $1/2 < \beta < 1$, then there is a holomorphic splitting*

$$TX|_D = TD \oplus \nu_D;$$

where $\nu_D = TX/TD$ is the normal bundle.

In Subsection 6.2 we touch on ideas related to this conjecture. We haven't defined the notion of polyhomogeneity. We simply mention that the reference metric ω in Lemma 1 is polyhomogeneus as well as any metric with cone singularities as in Definition 1 which is Einstein in the complement of D (see Jeffres-Mazzeo-Rubinstein [20]).

We change gears and discuss the foundations of the linear theory for metrics with cone singularities. We introduce the space of $C^{2,\alpha}$ functions. First we work on $\mathbb{C}^n$ with standard complex coordinates $(z_1, \dots, z_n)$ endowed with the model metric $g_{(\beta)}$. In the complement of $D = \{z_1 = 0\}$ there is (up to a factor of $\sqrt{2}$) an orthonormal basis of the $(1,0)$ forms given by $\epsilon_1 = \beta|z_1|^{\beta-1}dz_1$, $\epsilon_2 = dz_2, \dots, \epsilon_n = dz_n$. Let η be a $(1,0)$ form and write $\eta = \sum_{j=1}^n \eta_j \epsilon_j$. We say that η is C^α if the components η_j are C^α for $j = 1, \dots, n$ and $\eta_1 = 0$ when $z_1 = 0$. If η is a $(1,1)$ form we write $\eta = \sum_{i,j} \eta_{i\bar{j}} \epsilon_i \wedge \bar{\epsilon}_j$. We say that η is C^α if the components $\eta_{i\bar{j}}$ are C^α for $i, j = 1, \dots, n$ and $\eta_{1\bar{j}} = \eta_{j\bar{1}} = 0$ when $z_1 = 0$ and $j \geq 2$. These definitions can be compared with Definition 1 and similar remarks apply. A (real) function f is said to be $C^{2,\alpha}$ if $f, \partial f, \partial\overline{\partial} f$ are C^α. Let us point out that -in contrast with the standard $\beta = 1$ case- we are not requiring all the second derivatives to be C^α. The function spaces C^α and $C^{2,\alpha}$ clearly depend on the parameter β, in the literature it is usual to find the notation $C^{\alpha,\beta}$ for the space C^α and $C^{2,\alpha,\beta}$ for $C^{2,\alpha}$. In the setting of a compact complex manifold X, a smooth divisor $D \subset X$ and a fixed parameter $0 < \beta < 1$ it is straightforward, by using a finite collection of charts in which $D = \{z_1 = 0\}$, to define the function spaces C^α, $C^{2,\alpha}$ and to endow them with norms so that they become Banach spaces. Let ω be a C^α Kähler metric with cone angle $2\pi\beta$ along D and write $\triangle_\omega$ for the Laplace operator of the metric. The following result is of fundamental importance:

Theorem 1 *Assume that $\alpha < \beta^{-1} - 1$. Then $\triangle_\omega : C^{2,\alpha} \to C^\alpha$ is a Fredholm operator with zero index.*

In order to illustrate some of the applications of Theorem 1 we consider the functional $\mathcal{F} : U \to C^\alpha$, where U is a neighborhood of 0 in $C^{2,\alpha}$ and $\mathcal{F}(u) = \log(\omega_u^n/\omega^n)$, with $\omega_u = \omega + i\partial\overline{\partial}u$. The derivative at 0 of $\mathcal{F}$ is given, up to a constant multiple, by $\triangle_\omega$. We can use Theorem 1 together with the implicit function theorem, to conclude that for any function f which is sufficiently small in C^α and such that $\int_X e^f \omega^n = \int_X \omega^n$ there exists $u \in C^{2,\alpha}$ such that $\omega_u^n = e^f \omega^n$. Theorem 1 is proved in [15]. First one works with the model metric $g_{(\beta)}$ in $\mathbb{C}^n$. Let $p \notin D = \{z_1 = 0\}$. Denote by Γ_p the Green's function for the Laplacian $\triangle$ of $g_{(\beta)}$ with a single pole at p. One uses separation of variables, together with a check of convergence, to write a series expansion in a neighborhood of 0

$$\Gamma_p = \sum_{j,k \geq 0} a_{j,k}(y) \rho^{2j+k/\beta} \cos(k\theta),$$

where $y = (z_2, \dots, z_n)$, $z_1 = \rho^{1/\beta} e^{i\theta}$ and $a_{j,k}$ are smooth functions. The expression for the coefficients $a_{j,k}$ is explicit in terms of Bessel functions. One then writes $G(x,y) = \Gamma_x(y)$ and differentiates twice (with some care) the integral representation $u(x) = \int G(x,y)\triangle u(y)dy$ to obtain interior Schauder estimates for $\triangle$. In the setting of Theorem 1 one can patch these local estimates together and use standard arguments to obtain a parametrix for the operator $\triangle_\omega$. We will discuss related topics and the content of [15] with more detail in Section 4.

1.2 Kähler-Einstein metrics with cone singularities

Let X be a compact complex manifold, $D \subset X$ a smooth divisor and $0 < \beta < 1$. We are interested in Kähler-Einstein (KE) metrics with cone angle $2\pi\beta$ along D. These are metrics with cone singularities, as in Definition 1, such that the Ricci tensor is a constant multiple of the metric in the complement of D. Among the precedents which motivate this topic we mention the following ones:

- Riemann surfaces with conical singularities and constant Gaussian curvature. This is a classical topic (see [33], [29]). It has strong connections with the study of polyhedral shapes in three dimensional space forms ([32], [27]). The study of constant Gaussian curvature metrics on the Riemann sphere with three cone singularities is essentially equivalent to the study of the hypergeometric equation ([16]).

- Three dimensional hyperbolic metrics with cone singularities along a knot. See [19].
- Anti-self-dual-connections on four-manifolds with cone singularities along an embedded surface, [24]. Holomorphic vector bundles with parabolic structures, [5].

In the context of Kähler geometry we can say that, if we fix the parameter $0 < \beta < 1$, it is expected to find relations between algebraic compactifications of moduli spaces of pairs (X, D) and the metric degenerations of the corresponding KE metrics with cone angle $2\pi\beta$. Understanding the differential geometry of the limits should give us information on the possible singular pairs (W, Δ) which arise in the algebraic compactification. As a prototypical of example we consider the space $\mathcal{M}$ of four distinct unordered points in $\mathbb{CP}^1$ modulo the action of Möbius transformations. It is well-known that $\mathcal{M}$ has the structure of a Riemann surface and that $\mathcal{M} \cong \mathbb{C}$. The Riemann sphere is the only algebraic compactification of $\mathcal{M}$ and it is obtained by adding a single point to the space. On the other hand we can fix $1/2 < \beta < 1$ and consider the space $\mathcal{P}$ of spherical metrics on $\mathbb{CP}^1$ with cone singularities of angle $2\pi\beta$ at four distinct points modulo isometry. It is well-known that to each point of $\mathcal{M}$ there corresponds a unique point in $\mathcal{P}$, and that this map is an homeomorphism if we endow $\mathcal{P}$ with the Gromov-Hausdorff distance. If $\overline{\mathcal{P}}$ is an algebraic compactification of $\mathcal{M}$, then it would follow that there is only one possible limit for any sequence of metrics in $\mathcal{P}$ which degenerates. A little of thought shows that this limit should correspond to a spherical metric with two cone singularities ('american football') of angle $2\pi\gamma$ with $\gamma = 2\beta - 1$. Algebraically this corresponds to two distinct points in $\mathbb{CP}^1$ counted with multiplicity two. If we fix $\beta = 1/2$ (or $0 < \beta < 1/2$) then the discussion involves limits of flat (or hyperbolic) metrics with cone singularities.

Now we state a general existence result for KE metrics with cone singularities, similar to the the well-known Calabi conjecture for smooth metrics. Note that a Kähler form with cone singularities is Hölder continuous in cone coordinates and it is straightforward to see that it represents a de Rham cohomology class. The next theorem summarizes work of [6], [20] and [26] among others.

Theorem 2 *Let X be a compact complex manifold, $D \subset X$ be a smooth divisor and $\beta \in (0, 1)$. Assume that*

- $c_1(X) - (1-\beta)c_1([D]) < 0$. *Then there exists a unique Kähler metric ω_{KE} on X with cone angle $2\pi\beta$ along D such that $Ric(\omega_{KE}) = -\omega_{KE}$ in the complement of D.*
- $c_1(X) - (1-\beta)c_1([D]) = 0$. *Then in any Kähler class on X there exists a unique Kähler metric ω_{KE} with cone angle $2\pi\beta$ along D such that $Ric(\omega_{KE}) = 0$ in the complement of D.*
- *There exists a Kähler-Einstein metric on X with positive scalar curvature, $D \in |\lambda K_X^{-1}|$ with $\lambda \geq 1$ a rational number and $\beta > 1 - \lambda^{-1}$; so that $c_1(X) - (1-\beta)c_1([D]) > 0$. Then there exists a unique Kähler metric ω_{KE} on X with cone angle $2\pi\beta$ along D such that $Ric(\omega_{KE}) = \omega_{KE}$ in the complement of D.*

Theorem 2 requires $c_1(X)$ to be 'more positive than usual'. For example, in the second bullet we require $c_1(X) = (1-\beta)c_1([D])$ rather than $c_1(X) = 0$ for the existence of a Ricci-flat metric. This can be justified heuristically by thinking of KE metrics with cone singularities as having a big lump of positive Ricci curvature concentrated along D. A consequence of Theorem 2 is that every projective manifold has a KE metric with cone singularities of negative scalar curvature. Indeed take any $\beta \in (0, 1)$ and let H be an ample class in X. If we take $m = m(\beta)$ sufficiently large we can guarantee that $c_1(X) - (1-\beta)c_1(mH) < 0$ and, by Bertini's theorem, that there is a smooth divisor $D \in |mH|$. The hypothesis of the first bullet in Theorem 2 are then satisfied. Let us give a sketch of the proof of this first bullet, the main reason being that the techniques we use are relevant to our future work.

Proof: The hypothesis that $c_1(X) - (1-\beta)c_1([D]) < 0$ implies that there is a smooth Kähler form η such that $-(2\pi)^{-1}[\eta] = c_1(X) - (1-\beta)c_1([D])$. Take s to be a holomorphic section of $[D]$ such that $s^{-1}(0) = D$ and let h be a smooth Hermitian metric on $[D]$. Fix $\epsilon > 0$ so that we have the reference metric $\omega = \eta + \epsilon i\partial\overline{\partial}|s|_h^{2\beta}$, as in Lemma 1. We claim that there is a C^α function f on X, smooth in the complement of D, such that $\mathrm{Ric}(\omega) = -\omega + i\partial\overline{\partial}f$. Indeed, the cohomology condition on η implies that there is a smooth function F on X with $i\partial\overline{\partial}F = \eta + \mathrm{Ric}(\eta) + (1-\beta)i\partial\overline{\partial}\log|s|_h^2$. We use that $\mathrm{Ric}(\omega) - \mathrm{Ric}(\eta) = i\partial\overline{\partial}\log(\eta^n/\omega^n)$ to obtain

$$\mathrm{Ric}(\omega) = \mathrm{Ric}(\eta) + i\partial\overline{\partial}\log\left(\frac{\eta^n}{\omega^n}\right) = i\partial\overline{\partial}F - \eta - i\partial\overline{\partial}\log\left(\frac{|s|_h^{2-2\beta}\omega^n}{\eta^n}\right) = -\omega + i\partial\overline{\partial}f;$$

where

$$f = F + \epsilon|s|_h^{2\beta} - \log\left(\frac{|s|_h^{2-2\beta}\omega^n}{\eta^n}\right).$$

It is easy to check that f is a smooth function in the complement of D which extends as a C^α function to X, as we claimed.

We want to find $u \in C^{2,\alpha}$ a solution of

$$(\omega + i\partial\overline{\partial}u)^n = e^{f+u}\omega^n. \tag{1.7}$$

It is easy to argue that if we set $\omega_{KE} = \omega + i\partial\overline{\partial}u$, then ω_{KE} defines a Kähler metric with cone angle $2\pi\beta$ along D and $\mathrm{Ric}(\omega_{KE}) = -\omega_{KE}$ in the complement of D. In order to solve equation 1.7 we use the Aubin-Yau continuity method. A novel feature is that the path we use *doesn't* start with the reference metric ω, as we shall explain.

Consider the functional $\mathcal{F} : U \to C^\alpha$, where U is a neighborhood of 0 in $C^{2,\alpha}$ and $\mathcal{F}(\tilde{u}) = \log(\omega_{\tilde{u}}^n/\omega^n) - \tilde{u}$, with $\omega_{\tilde{u}} = \omega + i\partial\overline{\partial}\tilde{u}$. It is clear that $\mathcal{F}(0) = 0$ and that the derivative at 0 is given by $D_0\mathcal{F}(\tilde{u}) = \triangle_\omega\tilde{u} - \tilde{u}$, with $\triangle_\omega$ the (negative definite or 'analyst') Laplacian. Integration by parts shows that $D_0\mathcal{F}$ has no kernel, so that the implicit function theorem together with Theorem 1 implies that there is $\epsilon > 0$ such that for every $h \in C^\alpha$ with $\|h\|_\alpha < \epsilon$ there is $\tilde{u} \in C^{2,\alpha}$ such that $\mathcal{F}(\tilde{u}) = h$. Recall that in a compact manifold for any $C^{\tilde{\alpha}}$ function f and $\alpha < \tilde{\alpha}$ there is a sequence of smooth functions which converges to f in the C^α norm. It is then easy to argue that there is a smooth function f_0 (in the complex coordinates) such that $\|f - f_0\|_\alpha < \epsilon$. We call $h = f - f_0$ and take $\tilde{u} \in C^{2,\alpha}$ with $\mathcal{F}(\tilde{u}) = h$, so that $\omega_0 = \omega + i\partial\overline{\partial}\tilde{u}$ satisfies $\omega_0^n = e^{h+\tilde{u}}\omega^n$. The improvement is that now we have $\mathrm{Ric}(\omega_0) = -\omega_0 + i\partial\overline{\partial}f_0$, with f_0 a *smooth* function.

In order to solve equation 1.7 it is enough to find $u_1 \in C^{2,\alpha}$ such that $(\omega_0 + i\partial\overline{\partial}u_1)^n = e^{f_0+u_1}\omega_0^n$; because then $u = \tilde{u} + u_1$ is the solution that we want. We use the Aubin-Yau continuity path

$$(\omega_0 + i\partial\overline{\partial}u_t)^n = e^{tf_0+u_t}\omega_0^n \tag{1.8}$$

and consider the set

$$T = \{t \in [0,1] \text{ such that there exists } u_t \in C^{2,\alpha} \text{ a solution of 1.8}\}.$$

If we denote $\omega_t = \omega_0 + i\partial\overline{\partial}u_t$, then $\mathrm{Ric}(\omega_t) = -\omega_t + (1-t)i\partial\overline{\partial}f_0$. We start the continuity path at $t = 0$ with $u_0 = 0$. The goal is to show that T is open and closed.

Theorem 1 implies that T is open. The fact that T is closed follows from the following a priori estimate: There is a constant C, independent of $t \in T$, such that $\|u_t\|_{2,\alpha} \leq C$. The proof of this estimate is divided into three steps:

- C^0 *estimate*. This is an application of the maximum principle. If u_t attains its maximum at $p \in X \setminus D$ then 1.8 implies that $tf_0(p) + u_t(p) \leq 0$, so that $\sup u_t \leq \max\{-\inf f_0, 0\}$. If the maximum is attained at $p \in D$ then one considers $\tilde{u}_t = u_t + \delta|s|_h^\epsilon$ for a suitable choice of δ and ϵ positive and small. The function $\tilde{u}_t$ attains its maximum outside D, one gets a uniform upper bound on the supremal of $\tilde{u}_t$ which indeed implies a uniform upper bound on $\sup u_t$. Similarly one gets a uniform lower bound on $\inf u_t$. As a result $\|u_t\|_0 \leq C$.
- C^2 *estimate*. The technique is the maximum principle again. Since the reference metric ω has bisectional curvature bounded by above, there is a constant C_3 such that $\mathrm{Bisec}(\omega) \leq C_3$. In the complement of D, equation 1.8 gives us $\mathrm{Ric}(\omega_t) = -\omega_t + (1-t)i\partial\overline{\partial}f_0$. Since f_0 is smooth there is a constant $C_2 > 0$ such that $i\partial\overline{\partial}f_0 \geq -C_2\omega$. Set $C_1 = 1$ so that $\mathrm{Ric}(\omega_t) \geq -C_1\omega_t - C_2\omega$. Write $\omega_t = \omega + i\partial\overline{\partial}\tilde{u}_t$ and $A = C_2 + 2C_3 + 1$. The Chern-Lu inequality tells us that

$$\triangle_{\omega_t}(\mathrm{tr}_{\omega_t}\omega - A\tilde{u}_t) \geq -C_1 - An + \mathrm{tr}_{\omega_t}\omega. \tag{1.9}$$

Note that $\tilde{u}_t = \tilde{u} + u_t$, so the previous bullet gives us a uniform bound on $\|\tilde{u}_t\|_0$. We use 1.9 and the maximum principle (as in the previous item) to get the uniform bound $\mathrm{tr}_{\omega_t}\omega \leq C$. This bound together with the equation 1.8 imply that $C^{-1}\omega \leq \omega_t \leq C\omega$.

- $C^{2,\alpha}$ *estimate*. This is a local result. We want to appeal to the 'interior Schauder estimates for the complex Monge-Ampere operator'. In the case that $\beta = 1$ (no cone singularities) there is a large literature on this topic; we mention, among others, the work of Caffarelli and Safanov for the real Monge-Ampere operator. More recently, Chen-Wang ([10]) gave a new proof of these estimates by means of a 'blow-up' argument, similar in spirit to Leon Simon's proof of the Schauder estimates for the Laplace operator. This technique works in the setting of metrics with cone singularities. Our previous C^2 estimate together with Theorem 1.7 in [10] gives us that $\|u\|_{2,\alpha} \leq C$. Alternatively we can refer to Evans-Krillov theory and its analogue for metrics with cone singularities, see [20].

□

Another important result concerning KE metrics with cone singularities is the regularity theory for such a class of metrics. More precisely, the result we want to refer to says that these metrics are 'polyhomogeneous'. The basic reference for this is [20]. We proceed to the statement of the theorem. Let $p \in D$ and $(z_1, \ldots, z_n)$ holomorphic coordinates centered at p in which $D = \{z_1 = 0\}$. We write $z_1 = \rho^{1/\beta} e^{i\theta}$ and denote by $y = (z_2, \ldots, z_n)$ the other coordinate functions.

Theorem 3 *Let ω_{KE} be a Kähler-Einstein metric on X with cone angle $2\pi\beta$ along D and $\beta \in (1/2, 1)$. Then for every $p \in D$ we can find holomorphic coordinates $(z_1, \ldots, z_n)$ as above such that $\omega_{KE} = i\partial\overline{\partial}\phi$, with*

$$\phi = a_0(y) + (a_{01}(y)\cos(\theta) + a_{10}(y)\sin(\theta))\rho^{1/\beta} + a_2(y)\rho^2 + O(\rho^{2+\epsilon}). \tag{1.10}$$

Where a_0, a_{01}, a_{10}, a_2 are smooth functions of y and $\epsilon = \epsilon(\beta) > 0$.

When $\beta \in (0, 1/2]$ the same statement holds if we replace $1/\beta$ with 2 in the expansion 1.10. The proof of Theorem 3 uses tools from the 'Edge Calculus'.

As shown in the paper of Song-Wang [30], Theorem 3 implies that the norm of the Riemann curvature tensor of a KE metric with cone angle $2\pi\beta$ is bounded by $\rho^{1/\beta-2}$. The energy of such a metric g is defined to be

$$E(g) = \frac{1}{8\pi^2}\int_X |\mathrm{Rm}(g)|^2 = \frac{1}{8\pi^2}\lim_{\epsilon\to 0}\int_{X\setminus U_\epsilon} |\mathrm{Rm}(g)|^2,$$

where U_ϵ is a tubular neighborhood of D of radius ϵ, $\mathrm{Rm}(g)$ denotes the Riemann curvature tensor of g and we integrate using the volume form defined by g. It follows that $E(g)$ is finite by comparison with the integral $\int_0^1 \rho^{2/\beta-3} d\rho < \infty$. There is a topological formula for the energy which can be compared with the Chern-Weil formulae in [24] for connections with cone singularities.

Theorem 4 *[30]. The energy of a Kähler-Einstein metric of cone angle $2\pi\beta$ along D is finite and can be expressed in terms of $c_1(X), c_2(X), \beta, c_1([D])$ and the cohomology class of the Kähler form.*

Next we state a compactness theorem. Let X be a smooth Fano manifold, $D_i \subset X$ smooth divisors with $D_i \in |\lambda K_X^{-1}|$ for some fixed rational number $\lambda \geq 1$. Fix $1 - \lambda^{-1} < \beta < 1$. Assume that there exist KE metrics g_i on X with cone angle $2\pi\beta$ along D_i, we normalize so that $\mathrm{Ric}(g_i) = \mu g_i$, with $\mu = 1 - (1-\beta)\lambda$. It is well-known that under these conditions there exists, taking a subsequence if necessary, a Gromov-Hausdorff limit W of the sequence g_i. Indeed, one can approximate the metrics g_i by smooth metrics with a uniform lower bound on the Ricci curvature and a uniform upper bound on the diameter; then one can appeal to the standard Gromov's compactness theorem (see [7]). The set W is initially a metric space (W, d). A major theorem asserts that W is indeed homeomorphic to a normal projective variety. The normalization condition on the metrics g_i allow us to think of their respective Kähler forms as the curvatures of correponding (singular) Hermitian metrics on K_X^{-1}. A consequence is that for any pair of natural numbers i, m there is a map $T_i : X \to \mathbb{CP}^N$ with $N = N(m)$, defined up to the action of $U(N+1)$, given by an orthonormal (w.r.t. g_i) basis of $H^0(-mK_X)$. -More precisely, the maps T_i are defined using the smooth metrics which approximate the sequence g_i-. Kodaira's Theorem tells us that for m large enough the T_i are indeed well-defined maps in the whole of X and indeed these are embeddings. The theorem we want to quote reads as follows:

Theorem 5 *[8] There is a $\mathbb{Q}$-Fano variety W and a Weil divisor $\Delta \subset W$ such that:*

- *The pair $(W, (1-\beta)\Delta)$ is KLT*
- *There is a weak conical KE metric for the triple (W, Δ, β) which induces the distance d on the Gromov-Hausdorff limit (W, d).*
- *There is $m \in \mathbb{N}$ with the property that, up to a subsequence, we have embeddings $T_i : X \to \mathbb{CP}^N$ and $T : W \to \mathbb{CP}^N$ such that $T_i(X)$ converges to $T(W)$ as algebraic varieties and $T_i(D_i) \to T(\Delta)$ as algebraic cycles.*

We refer to [8] for the definitions of the terms in the statement of the theorem. The proof of this compactness result uses ideas from convergence theory of Riemannian manifolds and the Hörmander technique in complex analysis. Theorem 4 is not used in the proof of Theorem 5. On the other hand one might expect that -as in the case of smooth metrics- the bound on the energy should give us, at least in the case of two complex dimensions, more information on the differential-geometric structure of the limits (W, Δ).

Finally we mention the celebrated work of Chen-Donaldson-Sun which establishes the existence of Kähler-Einstein metrics with positive scalar curvature on K-stable Fano manifolds ([7], [8], [9]). KE metrics with cone singularities play a key role in this work. It would take us a long digression to explain the meaning of K-stability, so we will limit ourselves to say that this is an algebraic concept (i.e. that it makes sense for varieties defined over some other fields rather than $\mathbb{C}$), whose definition is motivated with ideas coming from Geometric Invariant Theory. The strategy to prove the existence of a KE metric on a K-stable Fano manifold is a variant of the continuity method, which resembles the 'opening of an umbrella'. First one fixes a natural number $\lambda \geq 2$ and a smooth divisor $D \in |-\lambda K_X|$. They consider KE metrics in the cohomology class $2\pi c_1(X)$ with cone singularities along D of cone angle $2\pi\beta$ along D and then they let $\beta \to 1$. It is a consequence of Theorem 1 and the fact that there are no holomorphic vector fields on X tangent to D that the set of β for which there is a KE metric with cone angle $2\pi\beta$ along D is open. As a starting point in the continuity path one can take $\beta_0 = 1 - \lambda^{-1}$, so that there is a Ricci-flat metric ω_{β_0} in $2\pi c_1(X)$ with cone angle $2\pi\beta_0$ along D. The hard work is to derive two compactness theorems on sequences $(X_i, D_i, \omega_{\beta_i})$ of KE metrics with cone angle $2\pi\beta_i$ along D_i. (In practice, for the application we are describing, the pair (X_i, D_i) is independent of i). The first compactness theorem concerns the case that $\lim_{i\to\infty} \beta_i < 1$; the second one regards the case when $\lim_{i\to\infty} \beta_i = 1$. These compactness theorems are formulated in such a way that one can conclude that either there is a smooth KE metric on X or there is an algebraic variety W with a divisor $\Delta \subset W$ which one can use to contradict the definition of K-stability of X. In the simplest case when $X = \mathbb{CP}^1$ and $\lambda = 2$ we start with the flat metric of a regular tetrahedron and deform through spherical metrics with cone singularities to finally get the round metric. It is interesting to see what happens when X does not admit a smooth KE metric. For example we consider the case when X is the blowup of $\mathbb{CP}^2$ at $q_1 = [1,0,0]$ and $q_2 = [0,1,0]$. Let $D \subset X$ be the proper transform of a smooth cubic $C \subset \mathbb{CP}^2$ which passes through the points q_1, q_2 and meets the line at infinity at a third distinct point q_3. We would have to digress into a discussion on toric geometry and Futaki invariants to justify the following speculations, so we will simply state them. Consider the one parameter subgroup of biholomorphisms m_λ of X induced by the action on $\mathbb{CP}^2$ given by $[u, v, w] \to [\lambda u, \lambda v, w]$. Let $\Delta = \lim_{\lambda\to 0} m_\lambda(D) \subset X$. Then $\Delta \subset X$ is a singular curve which is the proper transform of three lines in $\mathbb{CP}^2$ meeting at $\tilde{p} = [0,0,1]$ -the lines $\overline{\tilde{p}q_1}$, $\overline{\tilde{p}q_2}$ and a third one-. There is a critical angle $\beta_0 = 21/25$. For any $\beta \in (0, \beta_0)$ there should be a KE metric g_β on X in the class $2\pi c_1(X)$ with cone angle $2\pi\beta$ along D and $\mathrm{Ric}(g_\beta) = \beta g_\beta$. There should be a KE metric g_{β_0} on X with cone angle $2\pi\beta_0$ along Δ in a suitable sense, so that $g_\beta \to g_{\beta_0}$ as $\beta \to \beta_0$. Let p be the point in X which projects to $\tilde{p}$, so that Δ is singular at p. Consider the re-scaled metrics $(X, R_\beta g_\beta, p)$, with $R_\beta = |\mathrm{Rm}(g_\beta)|(p)$. We expect that $R_\beta \to \infty$ as $\beta \to \beta_0$ and that the rescaled metrics converge to a Ricci-flat metric on $\mathbb{C}^2$ with cone angle $2\pi\beta_0$ along a smooth cubic with three different asymptotic lines. The main result of this thesis proves the existence of these model Ricci-flat metrics on $\mathbb{C}^2$.

1.3 Asymptotically conical Ricci-flat Kähler metrics

We give a brief review of material on asymptotically conical Ricci-flat Kähler metrics. This subsection is merely expository and has the purpose of providing context. There is a large literature on the topic we want to discuss. The sequence of articles of Conlon-Hein [12], [13] and [14], gives a good panorama of what is known in this area to date and provides adequate references.

Let $(L, \overline{g})$ be a compact Riemannian manifold. On $(0,\infty)\times L$ consider the metric

$$g_c = dr^2 + r^2\overline{g},$$

where r is the coordinate on $(0,\infty)$. We say that g_c is a Riemannian cone with link $(L,\overline{g})$. The function r is then characterized as the intrinsic distance to the apex of the cone in the metric completion. Let μ be a negative number. A Riemannian manifold (M, g) is called asymptotically conical ('AC'), asymptotic to g_c at rate μ, if the following condition holds: There exists a diffeomorphism $\Phi : (R,\infty)\times L \to M \setminus K$, for some $R > 0$ and $K \subset M$ is compact, such that $|\nabla^j (\Phi^* g - g_c)|_{g_c} = O(r^{\mu-j})$ for all $j \geq 0$. We have denoted by ∇ the Levi-Civita connection of g_c. When g_c is a quotient of the euclidean flat space by a finite subgroup of the orthogonal matrices which acts freely on the unit sphere, the corresponding AC metrics are called 'asymptotically locally euclidean' or ALE.

In the case that g_c admits a parallel complex structure I_c, i.e. that it is Kähler, the pair $(L,\overline{g})$ inherits ths structure of a so-called a Sasaki manifold. We have a Kähler form $\omega_c = g_c(I_c.,.)$ and it is not hard to check that $\omega_c = (i/2)\partial\overline{\partial}r^2$. We use the embedding of L into the cone -given by setting $r = 1$- to think of $I_c\frac{\partial}{\partial r}$ as a vector field on L, known as the Reeb vector field. There are two types of Sasaki manifolds. If the Reeb vector field generates an S^1 action then $(L,\overline{g})$ is said to be quasiregular; otherwise it is called irregular. The term regular is used for the quasiregular ones in which the S^1 action is free. Of particular interest is the case when g_c is Ricci-flat. Moreover, we assume that we have a holomorphic volume form Ω_c and that $\omega_c^n = c_n\Omega_c \wedge \overline{\Omega}_c$, where n is the complex dimension, $c_n = 1$ if n is even and $c_n = i$ if n is odd. It is not hard to prove that this can only happen if $(L,\overline{g})$ is an Einstein manifold with positive scalar curvature. In the case of a regular Ricci-flat Kähler cone, the Kähler quotient of g_c by the free S^1 action at $r = 1$, is a Kähler-Einstein metric of positive scalar curvature. The inverse proccess is known as the 'Calabi ansatz'; it produces a Kähler Ricci-flat cone metric in complex dimension n out of a KE metric with positive scalar curvature in complex dimension $n-1$.

Consider now the case of a Kähler manifold (M, g, ω, I) of complex dimension n with a non-vanishing holomorphic volume form Ω. Let g_c be a Ricci-flat Kähler cone metric as in the previous paragraph. Let μ be a negative number. We say that (M, g) is an asymptotically conical Ricci-flat Kähler metric, asymptotic to g_c at rate μ, if the following two conditions hold:

- $\omega^n = c_n\Omega \wedge \overline{\Omega}$.
- There is a diffeomorphism $\Phi : (R,\infty)\times L \to M \setminus K$, for some $R > 0$ and $K \subset M$ compact, such that $|\nabla^j (\Phi^* g - g_c)|_{g_c} = O(r^{\mu-j})$, $|\nabla^j (\Phi^*\omega - \omega_c)|_{g_c} = O(r^{\mu-j})$ and $|\nabla^j (\Phi^* I - I_c)|_{g_c} = O(r^{\mu-j})$ for all $j \geq 0$.

The prototype of an AC Ricci-flat Kähler metric is the Eguchi-Hanson metric on $T^*\mathbb{CP}^1$, asymptotic to $\mathbb{C}^2/\pm 1$. The metric has cohomogeneity one, is explicit and admits different descriptions: via an ODE, as a Kähler quotient of a linear representation or via the Gibbons-Hawking ansatz. The most general existence result for AC Ricci-flat Kähler metrics, to the author's knowledge, is Theorem 2.1 in [12]. Roughly speaking it says that given a complex manifold (M, I) with a holomorphic volume form Ω, a Ricci-flat Kähler cone metric g_c as before and a diffeomorphism $\Phi : (R,\infty)\times L \to M \setminus K$ such that $|\nabla^j (\Phi^*\Omega - \Omega_c)|_{g_c} = O(r^{\nu-j})$ for some $\nu < 0$ and all $j \geq 0$; then every Kähler class in $H^2_{dR}(M)$ which satisfies a mild technical condition has an AC Ricci-flat Kähler metric. The proof of this result is PDE based and goes along the lines of Joyce's work on the Calabi conjecture for ALE manifolds, see Chapter 8 in [21].

A good reason for studying AC Ricci-flat Kähler metrics is that these can arise as bubbles when a non-collapsed sequence of Kähler-Einstein metrics (X_i, ω_i) develops an isolated singularity. For simplicity let us assume that the underlying topological space is fixed and that $\mathrm{Ric}(\omega_i) = \omega_i$, so that the non-collapsed condition is automatically satisfied. It follows from Gromov's compactness theorem that, up to a subsequence, the sequence converges to a metric space (X, d). Under the hypothesis we are working with, some deep results of Cheeger and Colding imply that there exists a metric tangent cone to X at x_∞ for any $x_\infty \in X$. Suppose that the tangent cone at x_∞ has a smooth link $(L,\overline{g})$, so that we have a Ricci-flat Kähler cone metric g_c as the ones we have described before. Moreover, assume that x_∞ is an isolated singularity in X, so that there exists a neighborhood of x_∞ in which all points distinct from x_∞ have the flat space $\mathbb{C}^n$ as a tangent cone. Then, at a heuristic level and true under some additional technical hypothesis, there is a sequence of points $x_i \in X_i$ with $x_i \to x_\infty$ such that if we let $\lambda_i = |\mathrm{Rm}(\omega_i)|(x_i)$; then $(X_i, \lambda_i \omega_i, x_i) \to (M, \omega, p)$, where (M, ω) is an AC Ricci-flat Kähler metric, asymptotic to the tangent cone of X at x_∞. We refer to Theorem 5.1 in Bando-Kasue-Nakajima [3] for a precise result along these lines in the situation in which ALE spaces arise as bubbles. A good example that fits in this context is provided by Ricci-flat metrics on a Kummer K3 surface which degenerate to the flat orbifold $T^4/\pm 1$. The Eguchi-Hanson metric arises as the blow up limit of the metrics at the singular points. We discuss convergence theory with more detail in the case of two complex dimensions and related results of Anderson [1] in Section 6. The Bishop-Gromov volume monotonicity theorem underpins many aspects of the theory.

There should be a parallel to the theory of AC Ricci-flat Kähler metrics in the setting of metrics with cone singularities. Our main result is an existence theorem which provides interesting examples. We restrict to two complex dimensions. This restriction is irrelevant in our analytic work but it simplifies the constructions in Sections 2 and 3.

1.4 Content of the thesis

We work on $\mathbb{C}^2$ with standard complex coordinates z, w. Let $P = P(z, w)$ be a degree d (≥ 2) polynomial such that $C = \{P = 0\}$ is a smooth complex curve. We restrict to the case in which C has d different asymptotic lines. Write

$$P = P_d + Q,$$

where P_d is the homogeneous degree d part of P. Our restriction means that the zero locus of P_d consists of d distinct complex lines $L_1, \ldots, L_d$. See Figure 2.

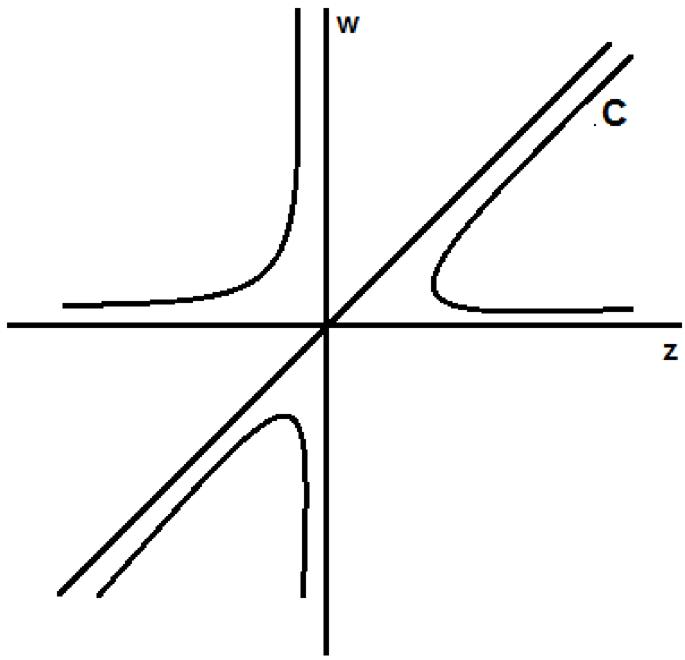

Figure 2: For example we can consider the curve $C = \{zw(z-w) = 1\}$. As a topological space C is a torus with three points removed.

We fix a number β such that

$$\frac{d-2}{d} < \beta < 1. \tag{1.11}$$

By results of Troyanov [33] and Luo-Tian [27], under condition 1.11 there is a (unique up to scale) compatible metric g on $\mathbb{CP}^1$ with constant positive Gaussian curvature and cone angle $2\pi\beta$ at the points corresponding to the lines $L_1, \dots, L_d$. Write $L = \cup_{k=1}^{d} L_k$. In Section 2 we construct a flat Kähler metric g_F on $\mathbb{C}^2 \setminus L$. The starting point in the construction of the flat metric g_F is the existence of the spherical metric g. The link between these two metrics is given by means of the Hopf bundle. The metric g_F is singular along L. More precisely, around each point $0 \neq p \in L$ we can find holomorphic coordinates (z_1, z_2) centered at p in which g_F agrees with the model metric $g_{(\beta)} = \beta^2 |z_1|^{2\beta-2} |dz_1|^2 + |dz_2|^2$. The property of g_F that we shall exploit the most is the one of being a metric cone, with its apex at 0.

We denote by $\mathcal{D}$ the set of all diffeomorphisms H of $\mathbb{C}^2$ for which there exists a compact set K such that $H(C \setminus K) \subset L$ and are asymptotic to the identity in the following sense: there are constants A_j such that $|H(x) - x| \leq A_0$, $|DH(x) - Id| \leq A_1 |x|^{-1}$ and $|D^\alpha H(x)| \leq A_j |x|^{-j}$ for all $x \in \mathbb{C}^2$ and $j = |\alpha| \geq 2$. It is elementary to prove that $\mathcal{D}$ is not empty, see Subsection 3.1.

Our main theorem states the existence of a Ricci-flat Kähler metric g_{RF} on $\mathbb{C}^2 \setminus C$ asymptotic to g_F. Write ω_{RF} for the Kähler form associated to g_{RF}. Let r be the intrinsic distance in g_F to 0 and set $\Omega = (\sqrt{2})^{-1} dz \wedge dw$. Our main result is the following

THEOREM 1 *There is a Kähler metric g_{RF} on $\mathbb{C}^2$ with cone angle $2\pi\beta$ along C and $H \in \mathcal{D}$ such that*

-
$$\omega_{RF}^2 = |P|^{2\beta-2} \Omega \wedge \overline{\Omega} \tag{1.12}$$

-
$$|(H^{-1})^* g_{RF} - g_F|_{g_F} \leq A r^{\gamma} \tag{1.13}$$

outside a compact set, for some constants $A > 0$ and $\gamma < 0$.

Equation 1.12 implies that g_{RF} is Ricci-flat. Indeed, the Ricci form of g_{RF} is given by $-i\partial\overline{\partial} \log \det(g_{RF})$. This is zero since, up to a constant factor, $\det(g_{RF})$ is equal to $|P|^{2\beta-2}$ and P is holomorphic non-vanishing in the complement of the curve. The asymptotic behavior 1.13 indeed holds in a stronger C^α sense, as will be clear from the proof of the Theorem. In the case that $d = 2$ we can assume that $C = \{zw = 1\}$. Then g_{RF} is invariant under the S^1 action $e^{i\theta}(z, w) = (e^{i\theta} z, e^{-i\theta} w)$ and it agrees with the metric described, by means of the Gibbons-Hawking ansatz, in Section 5 of [15]. When $d = 3$ and $\beta = 1/2$ the metrics are quotients of Kronheimer's D_4 gravitational instantons by an involution. See [22].

We briefly give some context for THEOREM 1, along the lines of Sections 5 and 6 of [15]. Consider a compact complex surface X and a family of smooth curves C_ϵ which converge as $\epsilon \to 0$ to a curve C_0 singular at $p \in X$. Assume the singularity is modeled on $\{P_d = 0\}$. Let β be fixed as in 1.11 and suppose that we have Kähler metrics ω_ϵ with cone angle $2\pi\beta$ along C_ϵ and $\mathrm{Ric}(\omega_\epsilon) = \omega_\epsilon$, say, on the complement of the curves. We would expect that (under favorable conditions) after re-scaling ω_ϵ around small balls centered at p we will get a metric g_{RF} in the limit, of the kind given by our THEOREM 1. We say more about this conjectural picture in Section 6.

We will now outline the strategy we follow to prove THEOREM 1. In a few words we can say that our approach is PDE based and goes along the lines of Yau's proof of the Calabi conjecture. The work of Yau has been extended to the context of metrics with cone singularities by Brendle [6], Jeffres-Mazzeo-Rubinstein [20] and to the context of ALE metrics by Joyce [21]. Our work is a mixture of [6], [20] and [21].

In Section 3 we construct $H \in \mathcal{D}$ and a reference metric ω which has cone angle $2\pi\beta$ along C and is asymptotic to ω_F. In Subsection 3.3 we construct another metric, ω_B, which is quasi-isometric to ω. We prove that ω_B has bisectional curvature bounded from above. This follows the lines of Appendix A in Jeffres-Mazzeo-Rubinstein [20]. Later on we will use this bound to get our C^2 estimate. We finish Section 3 by proving a Sobolev inequality for the reference metric ω.

Our analytic work begins in Section 4, where we develop the linear theory we need. First we review some foundational material from [15]. We state the interior Schauder estimates (Theorem 7), which are of fundamental importance in our analysis. Having the interior estimates at hand, in subsections 4.2 and 4.3 we set up a theory of 'weighted Hölder spaces'. Our main references in doing this are Pacard [28] and Bartnik [4], see also Chapter 8 in [31]. The main result of Section 4 is Proposition 5, which establishes good mapping properties for the Laplacian acting in our weighted spaces. This parallels known results in the case of asymptotically conical smooth metrics as stated in Theorem 2.11 of Conlon-Hein [12]. In subsection 4.4, as an application of Proposition 5 and the implicit function theorem, we show the existence of a metric ω_0 asymptotic to ω_F such that

$$\omega_0^2 = e^{-f_0}|P|^{2\beta-2}\Omega \wedge \overline{\Omega}, \tag{1.14}$$

with f_0 a smooth function of compact support. What will be important for us, apart from the fast decay of f_0, is that 1.14 implies that ω_0 has bounded Ricci curvature. The bound $\mathrm{Ric}(\omega_0) \geq -B\omega_0$ for some $B > 0$ is what we will use to derive the C^2 estimate.

To prove the theorem it is enough to show that there exists $u \in C^{2,\alpha}_\delta$ (our notation for the weighted Hölder spaces) such that

$$(\omega_0 + i\partial\overline{\partial}u)^2 = e^{f_0}\omega_0^2.$$

We set $\omega_{RF} = \omega_0 + i\partial\overline{\partial}u$ to be our solution. Standard elliptic regularity theory implies that u is smooth on the complement of the curve. The positivity of ω_{RF} follows from the equation, the decay of $\partial\overline{\partial}u$ and the conectedness of $\mathbb{C}^2 \setminus C$. In order to solve the equation we use the continuity method and consider the set

$$T = \{t \in [0,1] : \exists\, u_t \in C^{2,\alpha}_\delta \quad \text{solving} \quad (\omega_0 + i\partial\overline{\partial}u_t)^2 = e^{tf_0}\omega_0^2\}. \tag{1.15}$$

We want to prove that $1 \in T$. Proposition 5 implies that T is open and $0 \in T$ trivially ($u_0 = 0$). The closedness of T follows from the a priori estimate $\|u_t\|_{2,\alpha,\delta} \leq C$ for some constant $C > 0$ independent of $t \in T$. This is the content of Proposition 7. We prove this proposition into several steps. First we estimate the C^0 norm of u, to do this we use the Sobolev inequality (for the metric ω_0) and then we run a Moser iteration following Chapter 8 of Joyce [21]. To estimate the C^2 norm of u we use the maximum principle and the Chern-Lu inequality (in a slightly different way than in [20]). Here it is crucial that we have an upper bound on the bisectional curvature of ω_B and a lower bound on the Ricci curvature of $\omega_t = \omega_0 + i\partial\overline{\partial}u_t$ in the form of $\mathrm{Ric}(\omega_t) \geq -A\omega_B$, for some $A > 0$. This bound holds for ω_0 by 1.14 and it holds for ω_t since along the continuity path 1.15

$$\mathrm{Ric}(\omega_t) = (1-t)\mathrm{Ric}(\omega_0). \tag{1.16}$$

The C^2 estimate gives us the unform bound $C^{-1}\omega \leq \omega_t \leq C\omega$. Then we can apply the interior $C^{2,\alpha}$ estimate given by Theorem 1.7 of Chen-Wang [10] . Finally we proceed to the weighted estimates. We start by proving a bound on $\|u_t\|_{0,\mu}$ for some $\delta < \mu < 0$. The technique is again Moser iteration and we follow [21]. Finally the bound on $\|u_t\|_{2,\alpha,\delta}$ follows from the linear theory developed.

2 Flat metrics

We continue with the notation from the Introduction. Let $L_k = \{l_k = 0\}$ with l_k linear functions of z, w for $k = 1, \ldots, d$. We set $L = \cup_{k=1}^d L_k = \{P_d = l_1 \ldots l_d = 0\}$, where P_d is the homogeneous degree d part of P. Recall that $\Omega = (\sqrt{2})^{-1}dz \wedge dw$. The main result of this section is the following

Proposition 1 *There exists a Kähler metric g_F on $\mathbb{C}^2 \setminus L$ with Kähler form ω_F such that*

$$\omega_F^2 = |P_d|^{2\beta-2}\Omega \wedge \overline{\Omega}. \tag{2.1}$$

The metric is a cone with apex at 0, is invariant under the S^1 action $e^{it}(z,w) = (e^{it}z, e^{it}w)$ and

$$\omega_F = \frac{i}{2}\partial\overline{\partial}r^2, \tag{2.2}$$

where $r = r(z,w)$ is the intrinsic distance to 0.

As we said this result is a consequence of the fact that under the condition 1.11 there is a compatible metric g on $\mathbb{CP}^1$ with constant positive Gaussian curvature and cone angle $2\pi\beta$ at the points corresponding to the lines L. It will turn out at the end that g is a Kähler quotient of g_F by the S^1 action. Note that $(d-2)/d < \beta$ is a necessary condition for the existence of such a metric g. In fact, Gauss-Bonnet tells us that

$$2 + d\beta - d = \frac{1}{2\pi} \int_{\mathbb{CP}^1} K_g dV_g. \tag{2.3}$$

Finally we state some other properties of the metrics given by Proposition 1 that follow from the proof of it

- For every $p \notin L$ we can find holomorphic coordinates (z_1, z_2) around p such that the metric is given by $|dz_1|^2 + |dz_2|^2$. Because of this we refer to these metrics as 'Flat metrics'.
- For every $0 \neq p \in L$ we can find holomorphic coordinates (z_1, z_2) on a neighborhood U around p such that $U \cap L = \{z_1 = 0\}$ and the metric g_F agrees with the model $g_{(\beta)} = \beta^2 |z_1|^{2\beta-2}|dz_1|^2 + |dz_2|^2$.
- Let $\lambda > 0$ and denote $m_\lambda(z, w) = (\lambda z, \lambda w)$. Write $c = 2 + d\beta - d$. The neighborhoods in the previous two items can be taken to be invariant under m_λ and

  $$r^2 \circ m_\lambda = \lambda^c r^2$$

  for all $\lambda > 0$. Note that 1.11 means that $0 < c < 2$.

2.1 Spherical metrics with cone singularities on $\mathbb{CP}^1$

This subsection is a review of well-known material. Our main references are [33] and [27]. Our purpose is to state, without a proof, Theorem 6 below.

First we give a local model for a spherical metric with a cone singularity. Let W be a wedge in the two-sphere of radius 1 defined by two geodesics that intersect with angle $\pi\beta$. Consider the model metric given by identifying two copies of W isometrically along their boundary. The expression of this metric in geodesic coordinates (ρ, θ) is

$$d\rho^2 + \beta^2 \sin^2(\rho) d\theta^2. \tag{2.4}$$

2.4 induces a complex structure on a punctured neighborhood of the origin given by an anti-clockwise rotation of angle $\pi/2$. The fact is that we can change coordinates so that this complex structure extends smoothly to 0. Indeed, if we write $\eta = (\tan(\rho/2))^{1/\beta} e^{i\theta}$ our model metric takes the following form

$$4\beta^2 \frac{|\eta|^{2\beta-2}}{(1+|\eta|^{2\beta})^2} |d\eta|^2. \tag{2.5}$$

Let $L_1, \ldots, L_d \in \mathbb{CP}^1$ be d distinct points. We want to define the notion of a spherical metric g with a cone singularity of angle $2\pi\beta$ at the given points. There are two equivalent points of view

- g is a metric on the two-sphere minus d points which is locally isometric to the round sphere of radius 1 . Around each of the singular points there are polar coordinates (ρ, θ) such that g is given by 2.4. The metric g endows the punctured sphere with the complex structure of $\mathbb{CP}^1 \setminus \{L_1, \ldots, L_d\}$.
- g is a compatible metric on $\mathbb{CP}^1 \setminus \{L_1, \ldots, L_d\}$ of constant Gaussian curvature equal to 1 . Around each singular point we can find a complex coordinate η in which g is given by 2.5.

The content of the Gauss-Bonnet theorem in this setting reads as follows

Lemma 2 *Let g be a spherical metric with cone angle $2\pi\beta$ at d distinct points. Then the total volume of g is*

$$Vol(g) = 2\pi(2 + d\beta - d). \tag{2.6}$$

Proof: Denote the standard round metric of radius 1 by g_0 and write $g = e^{\phi} g_0$. Let ω and ω_0 be the corresponding area forms. Since $\mathrm{Vol}(g_0) = 4\pi$, we need to show that

$$\frac{1}{2\pi} \int_{\mathbb{CP}^1} \omega - \omega_0 = d\beta - d.$$

The function ϕ is smooth away of the d given points and the fact that the Gaussian curvature of both metrics is 1 implies that $\omega = \omega_0 - i\partial\overline{\partial}\phi$. Let p be a singular point and C_ϵ be a circle that shrinks to p as $\epsilon \to 0$. By Stokes' theorem it is enough to prove that

$$\frac{i}{2\pi} \lim_{\epsilon \to 0} \int_{C_\epsilon} \overline{\partial}\phi = \beta - 1.$$

This is an easy computation if we use coordinates in which g is given by 2.5.

□

The main result we want to recall is the following

Theorem 6 *([33], [27]). Let $L_1, \dots, L_d \in \mathbb{CP}^1$ be d distinct points, $d \geq 2$. If $(d-2)/d < \beta < 1$ then there exists a unique spherical metric with cone angle $2\pi\beta$ at the given points.*

We mention that when $d = 3$ and $1/3 < \beta < 1$, the metric g is given by doubling the spherical equilateral triangle T with interior angles equal to $\beta\pi$. If G is a conformal equivalence between the upper half plane and T, then g will be the pull back by G of the smooth constant curvature metric on T, extended to $\mathbb{C}$ by requiring the conjugation map to be an isometry. The construction of such a map G is related to the study of the hypergeometric equation. See Chapter 15 in [16]. The techniques used to establish Theorem 6 are not used in this thesis. We give a brief sketch of the two different proofs of 6 given in our references.

- Luo-Tian [27]. First they prove the uniqueness of a spherical metric with cone angle $2\pi\beta$ at $L_1, \dots, L_d \in \mathbb{CP}^1$. If g_1 and g_2 are two such metrics, then $g_2 = e^{\psi} g_1$ for some function ψ. They show that the gradient of ψ (w.r.t. any of the metrics) is an holomorphic vector field which vanishes at the given points. The condition that $\beta < 1$ is needed. If $d \geq 3$ it follows that $g_1 = g_2$.

  To prove existence they define $\mathcal{P}_d$ to be the space of all boundaries of labeled d-vertex convex polytopes in the round S^3 with total angle of $2\pi\beta$ at the vertices, modulo the ambient isometries. The space $\mathcal{P}_d$ is endowed with the Hausdorff topology. The condition that $(d-2)/d < \beta$ ensures that $\mathcal{P}_d$ is not empty. Let $\mathcal{M}_d$ be the space of labeled d points in $\mathbb{CP}^1$ modulo the action of Möbius transformations. The manifolds $\mathcal{P}_d$ and $\mathcal{M}_d$ have the same dimension.

  Each element of $\mathcal{P}_d$ represents a spherical metric on $\mathbb{CP}^1$ with cone angle $2\pi\beta$ at d distinct points. This is the metric induced by the round metric on S^3. There is a natural map $\Pi : \mathcal{P}_d \to \mathcal{M}_d$ obtained by recording the complex structure given by the metric. The uniqueness they proved implies that Π is injective. They use an elementary argument to show that Π is proper. Since $\mathcal{M}_d$ is connected, it follows that Π is an homeomorphism. This concludes the proof of 6.

- Troyanov [33]. This only deals with the existence part. He starts with an arbitrary metric g_0 which has cone angle $2\pi\beta$ at the points $L_1, \dots, L_d \in \mathbb{CP}^1$. The goal is to find a continuous function ϕ, smooth in the complement of the d given points, that solves the equation

  $$\triangle_0 \phi = K_0 - e^{2\phi}. \tag{2.7}$$

  Where $\triangle_0$ is the Laplacian of g_0 and K_0 is the Gaussian curvature of g_0. Then $g = e^{2\phi} g_0$ is the desired spherical metric. In order to solve 2.7, Troyanov uses the variational method. The hypothesis $(d-2)/d < \beta < 1$ implies that the appropriate functional is bounded below. The relevant technical point is to establish the Trudinger inequality and compute the value of the Trudinger constant.

2.2 Spherical metrics with cone singularities on the 3-sphere

This subsection contains the first step towards the proof of Proposition 1. We construct metrics on the 3-sphere with cone singularities of angle $2\pi\beta$ transverse to the Hopf circles corresponding to L. First we describe a local model for the singularities.

Write $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ and take polar coordinates $(r_1, \theta_1), (r_2, \theta_2)$ on each factor. Consider the product of a standard cone of total angle $2\pi\beta$ with an Euclidean plane

$$g_{(\beta)} = dr_1^2 + \beta^2 r_1^2 d\theta_1^2 + dr_2^2 + r_2^2 d\theta_2^2. \tag{2.8}$$

We want to write $g_{(\beta)}$ as a Riemannian cone. Recall that if (L, g_L) is a compact Riemannian manifold then the Riemannian cone (with link L) is the space $(0,\infty)\times L$ with the metric $dr^2 + r^2 g_L$. The coordinate r is then characterized as the intrinsic distance to the apex. In our situation L is the 3-sphere and we allow g_L to have mild singularities. More generally we could speak about metric cones. The general fact is that the product of two metric cones is a metric cone. In our case this amounts to checking that if we define $r \in (0,\infty)$ and $\rho \in (0, \pi/2)$ by

$$r_1 = r\sin\rho, \qquad r_2 = r\cos\rho;$$

then we get $g_{(\beta)} = dr^2 + r^2 \overline{g}_{(\beta)}$, where

$$\overline{g}_{(\beta)} = d\rho^2 + \beta^2 \sin^2(\rho) d\theta_1^2 + \cos^2(\rho) d\theta_2^2. \tag{2.9}$$

We think of $\overline{g}_{(\beta)}$ as a metric on the 3-sphere with a cone singularity of angle $2\pi\beta$ transverse to the circle given by the intersection of $\{0\}\times\mathbb{R}^2$ with the unit sphere. Alternatively we can also say that $\overline{g}_{(\beta)}$ is the restriction of $g_{(\beta)}$ to the unit sphere.

Let $S^3 = \{|z|^2 + |w|^2 = 1\} \subset \mathbb{C}^2$ equipped with the S^1-action $e^{it}(z,w) = (e^{it}z, e^{it}w)$ and let $H : S^3 \to \mathbb{CP}^1$ be the Hopf bundle. Denote by g the compatible metric on $\mathbb{CP}^1$ with constant curvature $K_g = 4$ and cone angle $2\pi\beta$ at the points corresponding to L. (Note that this is $1/4$ times the spherical metrics we considered in Subsection 2.1.)

Lemma 3 *There is an S^1-invariant metric $\overline{g}$ on $S^3 \setminus L$ such that*

- *$H : (S^3 \setminus L, \overline{g}) \to (\mathbb{CP}^1 \setminus L, g)$ is a riemannian submersion.*
- *$\overline{g}$ is locally isometric to the round 3-sphere of radius 1.*
- *Each $p \in L$ has a neighborhood in which $\overline{g}$ agrees with $\overline{g}_{(\beta)}$.*

Proof: First we write g in complex coordinates. W.l.o.g. we assume that $L_j = \{z = a_j w\}$ with $a_j \in \mathbb{C}$ for $j = 1, \dots, d-1$ and $L_d = \{w = 0\}$. Set $\xi = z/w$, then $g = e^{2\phi}|d\xi|^2$ with ϕ a function of ξ. Consider the function

$$u = \phi - (\beta - 1)\sum_{j=1}^{d-1} \log|\xi - a_j|.$$

The point of defining u in this way is that around each a_j there is a complex coordinate η centered at a_j in which

$$g = \beta^2 \frac{|\eta|^{2\beta-2}}{(1+|\eta|^{2\beta})^2}|d\eta|^2,$$

so that $\phi = \log\beta + (\beta-1)\log|\eta| - \log(1+|\eta|^{2\beta})$. It is easy to check from here that u is a continuous function on $\mathbb{C}$ and that

$$\lim_{\xi\to a_j} |\xi - a_j| \frac{\partial u}{\partial \xi} = 0$$

for $j = 1, \dots, d-1$. On $\mathbb{C}\setminus\{a_1, \dots, a_{d-1}\}$ define the real 1-form

$$\alpha_0 = \frac{i}{c}(\partial u - \overline{\partial} u), \tag{2.10}$$

where $c = 2 + d\beta - d$. It follows that, for $j = 1, \dots, d-1$,

$$\lim_{\epsilon \to 0} \int_{C_\epsilon(a_j)} \alpha_0 = 0, \tag{2.11}$$

where $C_\epsilon(a_j) = \{|\xi - a_j| = \epsilon\}$. On the other hand

$$d\alpha_0 = -\frac{2i}{c}\partial\overline{\partial}u = \frac{1}{c}K_g dV_g, \tag{2.12}$$

so 2.3 gives us that

$$\frac{1}{2\pi}\int_{\mathbb{C}} d\alpha_0 = 1. \tag{2.13}$$

On the trivial S^1-bundle $\mathbb{C} \setminus \{a_1, \dots, a_{d-1}\} \times S^1$ with coordinates (ξ, e^{it}) consider the connection $\alpha = dt + \alpha_0$ and the metric

$$\overline{g} = g + \frac{c^2}{4}\alpha^2. \tag{2.14}$$

Let $p = (\xi_0, e^{it_0}) \in \mathbb{C} \setminus \{a_1, \dots, a_{d-1}\} \times S^1$, we want to prove that $\overline{g}$ is isometric to the round sphere of radius 1 in a neighborhood of p. There are polar coordinates (ρ, θ) around ξ_0 in which

$$g = d\rho^2 + \frac{\sin^2(2\rho)}{4}d\theta^2.$$

In these coordinates $d\alpha_0 = (1/c)K_g dV_g = (2/c)\sin(2\rho)d\rho d\theta$. Doing a change of gauge if necessary we can assume that $\alpha_0 = (2/c)\sin^2(\rho)d\theta$. It follows that $(c/2)\alpha = (c/2)dt + \sin^2(\rho)d\theta$. If we assume $t_0 \in (-\pi, \pi)$, say, and define $\overline{t} = (c/2)t$ we get

$$\overline{g} = d\rho^2 + \frac{\sin^2(2\rho)}{4}d\theta^2 + (d\overline{t} + \sin^2(\rho)d\theta)^2.$$

This can be recognized as the round sphere of radius 1. We use the map

$$(\xi, e^{it}) \to \left(z = \frac{\xi}{\sqrt{1+|\xi|^2}}e^{it}, \ \ w = \frac{1}{\sqrt{1+|\xi|^2}}e^{it}\right)$$

to think of $\overline{g}$ as a metric on $S^3 \setminus L$. The S^1 invariance and the first item of the lemma are clear from the definition 2.14 of $\overline{g}$. We already checked the second item so let's prove the last one.

Assume first that $p \in L_j$ for some $1 \leq j \leq d-1$. Write $p = (a_j, e^{it_0})$. There are polar coordinates (ρ, θ) around a_j in which

$$g = d\rho^2 + \beta^2\frac{\sin^2(2\rho)}{4}d\theta^2.$$

In these coordinates $d\alpha_0 = (1/c)K_g dV_g = (2/c)\beta\sin(2\rho)d\rho d\theta$. It follows from 2.11 that we can perform a change of gauge so that $\alpha_0 = (2/c)\beta\sin^2(\rho)d\theta$. It follows that $(c/2)\alpha = (c/2)dt + \beta\sin^2(\rho)d\theta$. If we assume $t_0 \in (-\pi, \pi)$, say, and define $\overline{t} = (c/2)t$ we have that

$$\overline{g} = d\rho^2 + \beta^2\frac{\sin^2(2\rho)}{4}d\theta^2 + (d\overline{t} + \beta\sin^2(\rho)d\theta)^2.$$

Write $\theta_2 = \overline{t}$, $\theta_1 = \theta + \beta^{-1}\overline{t}$ to get $\overline{g} = d\rho^2 + \beta^2\sin^2(\rho)d\theta_1^2 + \cos^2(\rho)d\theta_2^2$. This matches with the expression 2.9 of the metric $\overline{g}_{(\beta)}$. Finally consider the case of $p \in L_d = \{w = 0\}$. In the coordinates

$$(\eta, e^{is}) \to \left(z = \frac{1}{\sqrt{1+|\eta|^2}}e^{is}, \ \ w = \frac{\eta}{\sqrt{1+|\eta|^2}}e^{is}\right)$$

we have $p = (0, e^{is_0})$. These coordinates are related to (ξ, e^{it}) via $\eta = 1/\xi$ and $e^{is} = (\xi/|\xi|)e^{it}$. So that $\alpha = dt + \alpha_0 = ds + \beta_0$ with $\beta_0 = d(\arg\eta) + \alpha_0$. Now $\lim_{\epsilon\to 0}\int_{|\eta|=\epsilon}\alpha_0 = -\lim_{N\to\infty}\int_{|\xi|=N}\alpha_0$. It follows from 2.11, 2.13 and Stokes' theorem that $\lim_{N\to\infty}\int_{|\xi|=N}\alpha_0 = 2\pi$. As a result $\lim_{\epsilon\to 0}\int_{|\eta|=\epsilon}\beta_0 = 0$. From here we can proceed as before, finding polar coordinates in which $g = d\rho^2 + \beta^2\frac{\sin^2(2\rho)}{4}d\theta^2$ and changing gauge so that $\beta_0 = (2/c)\beta\sin^2(\rho)d\theta$.

□

We have set

$$c = 2 + d\beta - d. \tag{2.15}$$

This number will appear frequently in the following sections.

Remark 2 *The proof above gives us that the fibers of H have constant length πc. Since $Vol(g) = (\pi/2)c$ we have $Vol(\overline{g}) = (\pi^2/2)c^2$.*

In a coordinate free way we can say that the metric of Lemma 3 is given by 2.14 . Where α is the unique connection, up to gauge equivalence, on the Hopf bundle with $d\alpha = c^{-1}H^*(K_g dV_g)$ which satisfies the following condition:

- If $p \in \mathbb{CP}^1$ is a point in L and γ_ϵ is a loop that shrinks to p as $\epsilon \to 0$, then the holonomy of α along γ_ϵ goes to the identity as $\epsilon \to 0$.

2.3 Proof of Proposition 1

Having Lemma 3 at hand we prove Proposition 1 in this subsection. We set g_F to be the Riemannian cone with $(S^3, \overline{g})$ as a link and we check that it has the desired properties.

On $(0,\infty)\times\mathbb{C}\setminus\{a_1,\dots,a_{d-1}\}\times S^1$ with coordinates (r,ξ,e^{it}) define

$$g_F = dr^2 + r^2\overline{g}. \tag{2.16}$$

We use the notation introduced in Lemma 3 and we write $\xi = x + iy$. Consider the almost-complex structure given by

$$I\frac{\tilde\partial}{\partial x} = \frac{\tilde\partial}{\partial y}, \qquad I\frac{\partial}{\partial r} = \frac{2}{cr}\frac{\partial}{\partial t}$$

where

$$\frac{\tilde\partial}{\partial x} = \frac{\partial}{\partial x} - \alpha\left(\frac{\partial}{\partial x}\right)\frac{\partial}{\partial t}, \qquad \frac{\tilde\partial}{\partial y} = \frac{\partial}{\partial y} - \alpha\left(\frac{\partial}{\partial y}\right)\frac{\partial}{\partial t}$$

are the horizontal lifts of $\partial/\partial x$ and $\partial/\partial y$. Finally set $\omega_F = g_F(I.,.)$.

Claim 1 $\big((0,\infty)\times\mathbb{C}\setminus\{a_1,\dots,a_{d-1}\}\times S^1, g_F, I\big)$ *is a Kähler manifold. I.e. $d\omega_F = 0$ and I is integrable. Moreover,*

$$\omega_F = \frac{i}{2}\partial\overline{\partial}r^2. \tag{2.17}$$

Proof: We compute in the coframe $\{dx, dy, dr, \alpha\}$ where

$$\omega_F = r^2e^{2\phi}dx\wedge dy + \frac{cr}{2}dr\wedge\alpha,$$

so that $d\omega_F = 2re^{2\phi}drdxdy - (cr/2)(4/c)e^{2\phi}drdxdy = 0$. The integrability of I amounts to check that

$$\left[\frac{\tilde\partial}{\partial x} + i\frac{\tilde\partial}{\partial y}, \frac{\partial}{\partial r} + i\frac{2}{cr}\frac{\partial}{\partial t}\right] = 0.$$

Finally $dId(r^2) = d(2rIdr) = -cd(r^2\alpha) = -2crdr\wedge\alpha - 4r^2e^{2\phi}dx\wedge dy$. Using that $2i\partial\overline{\partial} = -dId$ we deduce 2.17

□

Claim 2 *Set A to be the constant $(c/2)^{1/c}$. The functions*

$$z = \xi w, \quad w = Ar^{2/c}e^{u/c}e^{it} \tag{2.18}$$

give a biholomorphism between $(0,\infty) \times \mathbb{C} \setminus \{a_1, \dots, a_{d-1}\} \times S^1$ with the complex structure I and $\mathbb{C}^2 \setminus L$. If we write $\Omega = (\sqrt{2})^{-1}dzdw$, then

$$\omega_F^2 = |P_d|^{2\beta-2}\Omega \wedge \overline{\Omega}. \tag{2.19}$$

Proof: It is easy to see that the pair (z,w) defines a diffeomorphism between the corresponding spaces. The Cauchy-Riemann equations for a function h to be holomorphic with respect to I are given by

$$\frac{\partial h}{\partial r} + i\frac{2}{cr}\frac{\partial h}{\partial t} = 0, \qquad \frac{\partial h}{\partial x} + i\frac{\partial h}{\partial y} = \alpha\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\frac{\partial h}{\partial t}.$$

If we ask h to have weight 1 with respect to the circle action the equations become

$$\frac{\partial h}{\partial r} = \frac{2}{cr}h, \qquad \frac{\partial h}{\partial \overline{\xi}} = i\alpha\left(\frac{\partial}{\partial \overline{\xi}}\right)h = \frac{1}{c}\frac{\partial u}{\partial \overline{\xi}}h.$$

From here we see that see that z and w are holomorphic.

Now we compute the volume form of g_F in the complex coordinates z, w. First define a basis $\{\tau_1, \tau_2\}$ of the $(1,0)$ forms

$$\tau_1 = dr + i\frac{cr}{2}\alpha, \quad \tau_2 = e^{\phi}rd\xi. \tag{2.20}$$

Up to a factor of $\sqrt{2}$ this is an orthonormal basis for the $(1,0)$ forms in $\mathbb{C}^2 \setminus L$, i.e.

$$\omega_F = (i/2)\tau_1\overline{\tau_1} + (i/2)\tau_2\overline{\tau_2}.$$

Define a two by two matrix (a_{ij}) by means of

$$dz = a_{11}\tau_1 + a_{12}\tau_2, \quad dw = a_{21}\tau_1 + a_{22}\tau_2.$$

From here we get

$$\Omega \wedge \overline{\Omega} = |\det(a_{ij})|^2\omega_F^2.$$

Since $z = \xi w$ we have that $a_{11} = \xi a_{21}$ and $a_{12} = \xi a_{22} + we^{-\phi}r^{-1}$. It follows that $\det(a_{ij}) = -we^{-\phi}r^{-1}a_{21}$. We can easily compute, from the formula given for w, that $a_{21} = (2/c)r^{-1}w$. We put these things together to get

$$\omega_F^2 = (c^2/4)|w|^{-4}r^4e^{2\phi}\Omega \wedge \overline{\Omega}.$$

Now we use that $r^4 = (4/c^2)|w|^{2c}e^{-2u}$, $\phi - u = (\beta - 1)\sum_{j=1}^{d-1}\log|(z/w) - a_j|$ and $2c - 4 = 2d\beta - 2d = 2\beta - 2 + (d-1)(2\beta - 2)$ to conclude that

$$\omega_F^2 = |z - a_1w|^{2\beta-2}\dots|z - a_{d-1}w|^{2\beta-2}|w|^{2\beta-2}\Omega \wedge \overline{\Omega}.$$

This is formula 2.19.

□

The proof of Proposition 1 is now complete. Note that we have two natural systems of coordinates: the complex coordinates (z,w) and the spherical coordinates (r,θ), where θ denotes a point in the 3-sphere. For $\lambda > 0$ define $D_\lambda(r,\theta) = (\lambda r, \theta)$ and $m_\lambda(z,w) = (\lambda z, \lambda w)$. Equation 2.18 gives that $D_\lambda = m_{\lambda^{2/c}}$ and Equation 2.17 implies that $m_\lambda^*\omega_F = \lambda^c\omega_F$. In our derivation, the number c comes from the Gauss-Bonnet formula 2.3. Alternatively, assume that $m_\lambda^*\omega_F = \lambda^{\tilde{c}}\omega_F$ for some $\tilde{c} > 0$. If we pull back the equation 2.19 by m_λ we get that $\tilde{c}$ must agree with c.

We summarize the results of this subsection in the form of a recipe which allows us to go from the metric g_F on $\mathbb{C}^2$ in Proposition 1 to the corresponding g on $\mathbb{CP}^1$ in Theorem 6 and vice versa. From 2.18 we get

$$r^2 = \frac{2}{c}|w|^c e^{-u}. \tag{2.21}$$

We recall that

$$u = \phi - (\beta - 1)\sum_{j=1}^{d-1} \log|\xi - a_j| \quad , \quad g = e^{2\phi}|d\xi|^2. \tag{2.22}$$

Where ϕ a function of $\xi = z/w$. We are writing the lines as $L_j = \{z = a_j w\}$ with $a_j \in \mathbb{C}$ for $j = 1, \ldots, d-1$ and $L_d = \{w = 0\}$. 2.21 together with 2.22 allow us to write g_F explicitly in terms of g and vice-versa.

2.4 Explicit examples and quotients

In this subsection we write explicit expressions for the metrics g_F in the particular cases of $d = 2$, $0 < \beta < 1$ and $d = 3$, $\beta = 1/2$. We take this as a chance to test the equations 2.21 and 2.22.

Let us begin with the case $d = 2$ and $0 < \beta < 1$. It turns out that the local model for a spherical metric with angle $2\pi\beta$ at 0 given by

$$g = \beta^2 \frac{|\xi|^{2\beta}}{(1+|\xi|^{2\beta})^2}|d\xi|^2, \tag{2.23}$$

defines globally a metric on $\mathbb{CP}^1$ with angle $2\pi\beta$ at 0 and ∞. This space is also known as the 'rugby ball' since it is obtained by removing a spherical wedge of angle $2\pi(1-\beta)$ delimited by two geodesics joining antipodal points and gluing the sides. We use our formula 2.21 to get $r^2 = \beta^{-2}(|z|^{2\beta} + |w|^{2\beta})$, so that $g_F = |z|^{2\beta-2}|dz|^2 + |w|^{2\beta-2}|dw|^2$. Up to a constant factor this is the space $\mathbb{C}_\beta \times \mathbb{C}_\beta$. When $\beta = 1/k$ with k a natural number, the metric g_F is a global quotient of the euclidean metric. Indeed, let $A_k \subset SU(2)$ be the cyclic group generated by $(x, y) \to (e^{2\pi i/k}x, e^{-2\pi i/k}y)$. The functions $z = x^k$, $w = y^k$ and $t = xy$ are invariant under the action of A_k and give a complex isomorphism

$$\mathbb{C}^2/A_k \cong \{zw = t^k\} \subset \mathbb{C}^3.$$

Consider the group $G \subset U(2)$ generated by A_k and $(x, y) \to (e^{2\pi i/k}x, y)$. Then $A_k \subset G$ is normal and $G/A_k \cong \mathbb{Z}_k$ acts on $\mathbb{C}^2/A_k$ via $(z, w, t) \to (z, w, e^{2\pi i/k}t)$. The functions z, w give a complex isomorphism $\mathbb{C}^2/G \cong \mathbb{C}^2$. We can push forward the euclidean metric since $G \subset U(2)$. If we take care of the normalization of the volume form we get the same expression for g_F as the one we derived before. Note that the map $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$, $\Phi(x, y) = (x^k, y^k)$, maps complex lines to complex lines. $\Phi(\{x = \eta y\}) = \{x = \eta^k y\}$, therefore Φ induces a map F on $\mathbb{CP}^1$ given by $\xi = F(\eta) = \eta^k$. We can pullback the metric 2.23 (with $\beta = 1/k$) under F to get $F^*g = (1+|\eta|^2)^{-2}|d\eta|^2$, the round metric on the sphere of radius 1/2.

Now consider the case when $d = 3$ and $\beta = 1/2$. We take our lines to be $L_1 = \{z = 0\}$, $L_2 = \{z = w\}$ and $L_3 = \{w = 0\}$. Let $D_4 \subset SU(2)$ be the subgroup generated by $(x, y) \to (ix, -iy)$ and $(x, y) \to (-y, x)$. The polynomials $z = (x^2+y^2)^2$, $w = (x^2-y^2)^2$ and $t = 2(x^5y - y^5x)$ are invariant under the action and give the complex isomorphism

$$\mathbb{C}^2/D_4 \cong \{zw(z-w) = t^2\}.$$

Let $G \subset U(2)$ be the subgroup generated by D_4 and $(x, y) \to (y, x)$. Then $D_4 \subset G$ is normal and $K = G/D_4 \cong \mathbb{Z}_2$ acts on $\mathbb{C}^2/D_4$ as $(z, w, t) \to (z, w, -t)$. The functions z, w give an isomorphism of complex manifolds $\mathbb{C}^2/G \cong \mathbb{C}^2$. We can push forward the euclidean metric $\omega_{euc} = (i/2)\partial\overline{\partial}(|x|^2 + |y|^2)$ to obtain a flat Kähler metric with cone angle $\beta = 1/2$ along L. From the formulas for z, w we have that $|z| + |w| = 2|x|^4 + 2|y|^4$ and $|z - w| = 4|x|^2|y|^2$ so that $2(|x|^2 + |y|^2)^2 = |z| + |w| + |z - w|$. From here we get that

$$r^2 = a\left(|z| + |w| + |z-w|\right)^{1/2}, \tag{2.24}$$

where $a = 8\sqrt{2}$ is determined by the normalization condition 2.1. We can now use the equations 2.21 2.22 to get

$$g = \frac{1}{8}\frac{1}{|\xi||\xi-1| + |\xi|^2|\xi-1| + |\xi||\xi-1|^2}|d\xi|^2$$

Indeed, the map $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ given by $\Phi(x,y) = \big((x^2+y^2)^2, (x^2-y^2)^2\big)$ maps lines to lines and induces $F : \mathbb{CP}^1 \to \mathbb{CP}^1$ given by

$$\xi = F(\eta) = \frac{(\eta^2+1)^2}{(\eta^2-1)^2}. \tag{2.25}$$

Then one can check that $F^*g = (1+|\eta|^2)^{-2}|d\eta|^2$ (the smooth metric with constant curvature 4). The map F has degree 4 and has six critical points at $0, \pm 1, \pm i, \infty$. It maps the spherical triangle $T = \{|\eta| \leq 1,\ 0 \leq \arg(\eta) \leq \pi/2\}$ to the upper half plane $H = \{\mathrm{Im}(\xi) \geq 0\}$. Then we recognize g as the metric obtained by gluing two copies of T along the boundary.

2.5 A different approach

We mention another approach to Proposition 1. This fits our work into the setting of the so called 'Calabi ansatz' (see page 11 of LeBrun [25]). In this subsection we view a Kähler metric as the curvature form of a Hermitian metric on a complex line bundle.

We think of $\mathbb{C}^2$ as the total space of $\mathcal{O}_{\mathbb{CP}^1}(-1)$ with the zero section collapsed at 0. The bundle projection is given by $\Pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1$, $\Pi(z,w) = [z : w]$. We can then identify (smooth) Hermitian metrics on $\mathcal{O}_{\mathbb{CP}^1}(-1)$ with (smooth) functions $h : \mathbb{C}^2 \to \mathbb{R}_{\geq 0}$ such that $h(\lambda p) = |\lambda|^2 h(p)$ for all $\lambda \in \mathbb{C}$, $p \in \mathbb{C}^2$ and $h(p) = 0$ only when $p = 0$. The first basic fact we need is that an area form ω in $\mathbb{CP}^1$ induces a Hermitian metric h_ω. We use coordinates $\xi = z/w$, $\eta = w/z$ on $\mathbb{CP}^1$. Write $\omega = e^{2\phi}(i/2)d\xi d\overline{\xi}$ with $\phi = \phi(\xi)$ on $U = \Pi(\{w \neq 0\})$ and $\omega = e^{2\psi}(i/2)d\eta d\overline{\eta}$ with $\psi = \psi(\eta)$ on $V = \Pi(\{z \neq 0\})$. Then h_ω is given by

$$h_\omega = |w|^2 e^{-\phi}, \text{ if } w \neq 0; \quad h_\omega = |z|^2 e^{-\psi}, \text{ if } z \neq 0. \tag{2.26}$$

The second basic fact is that a Hermitian metric h gives a 2-form ω_h on $\mathbb{CP}^1$ by means of

$$\omega_h = i\partial\overline{\partial}\log h(\xi, 1) \text{ on } U, \text{ and } \omega_h = i\partial\overline{\partial}\log h(1,\eta) \text{ on } V. \tag{2.27}$$

We also mention that h induces Hermitian metrics on the other complex line bundles over $\mathbb{CP}^1$. If we regard P_d as a section of $\mathcal{O}_{\mathbb{CP}^1}(d)$, then we have $|P_d|_h^2 = h(\xi,1)^{-d}|\xi - a_1|^2 \ldots |\xi - a_{d-1}|^2$ on U and a corresponding expression on V.

One can then rephrase the existence of the spherical metric with cone singularities g on $\mathbb{CP}^1$ by saying that there is a Hermitian metric h, continuous on $\mathbb{C}^2$ and smooth outside L such that

$$h = |P_d|_h^{\beta-1} h_{\omega_h} \tag{2.28}$$

Where by $|P_d|_h$ we mean $|P_d|_h \circ \Pi$. Here we could be more precise and instead of saying that h is merely continuous we could give a local model for h around points of L. From 2.28 one gets that ω_h has constant Gaussian curvature equal to $c = 2 + d\beta - d$ outside L and one can also argue that $(2\pi)^{-1}\int_{\mathbb{CP}^1}\omega_h = 1$. The potential for ω_F is then given by $r^2 = a h^{c/2}$ for some constant $a > 0$ determined by 2.1.

3 Reference Metrics

Recall thet we denote by $\mathcal{D}$ the set of diffeomorphisms H of $\mathbb{C}^2$ which, outside a compact set, map the curve C to the asymptotic lines L and are asymptotic to the identity in the following sense: $H(x) = x + h(x)$, with $D^\alpha h(x) = O(|x|^{-|\alpha|})$ for any multi-index α. The main result of this section is the following

Proposition 2 *There exist $H \in \mathcal{D}$ and Kähler metrics ω, ω_B on $\mathbb{C}^2$ with cone singularities of angle $2\pi\beta$ along C such that*

- $|(H^{-1})^*\omega - \omega_F|_{g_F} = O(r^{-2/c})$
- *Bisec(ω_B) $\leq Q_1$*
- $Q_2^{-1}\omega_B \leq \omega \leq Q_2\omega_B$

for some positive constants Q_1, Q_2.

We review the definition of a metric having cone singularities in Subsection 4.1. The statement about the singularities will follow from the fact that around points of C one can write the metrics as (smth) $+i\partial\overline{\partial}(F|z_1|^{2\beta})$ with F a smooth positive function and where (smth) denotes a smooth $(1,1)$ form positive in the direction tangent to C. (See Remark 1 in the Introduction and Lemma 12 in this Section). The metric ω is isometric to the flat metric ω_F in a neighborhood of C at infinity. In the notation of the next subsection this neighborhood is $U_{\delta/2,2R}$.

3.1 A diffeomorphism

Let $C=\{P=0\}$. The homogeneous degree d part of P is $P_d=l_1\dots l_d$. We write $l_j=z-a_jw$, for $j=1,\dots,d-1$ and $l_d=w$. W.l.o.g. let us assume that $a_j\neq 0$ for all $j=1,\dots,d-1$. First we look at the piece of C which is asymptotic to $L_d=\{w=0\}$.

Lemma 4 *There exist $R,\delta>0$ and $\Phi=\Phi(z):\{|z|>R\}\to\mathbb{C}$ bounded holomorphic, which depend only on P, such that*

$$C\cap U_{d,\delta,R}=\{(z,\Phi(z))\},$$

where $U_{d,\delta,R}=\{|w|<\delta|z|,\quad |z|>R\}$.

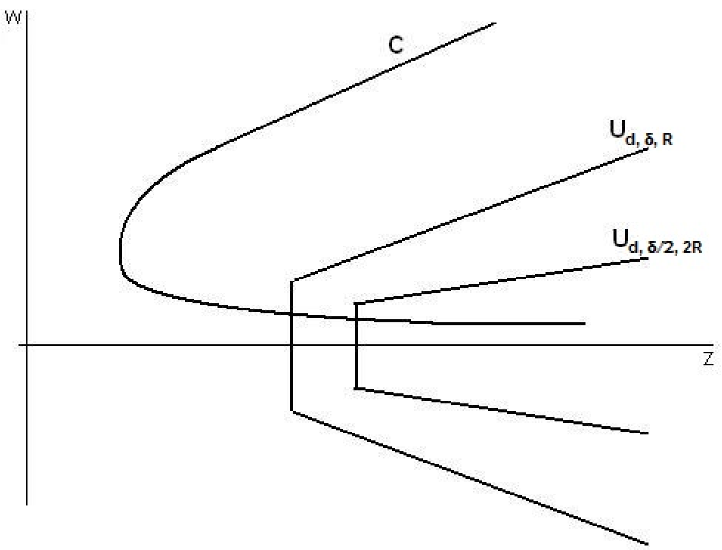

Figure 3: In the region $U_{d,\delta,R}$ the curve can be written as a graph.

Proof: For $j=1,\dots,d-1$ let S_j be an orthogonal linear transformation that takes L_d to L_j. Write $U_{j;\delta,R}=S_j(U_{d;\delta,R})$ and $U_{\delta,R}=\cup_{j=1}^d U_{j;\delta,R}$. Taking δ small enough we can assume that the sets $U_{j,\delta,R}$ are pairwise disjoint. Write

$$P=P_d+Q \tag{3.1}$$

with Q a polynomial of degree $d-1$. On the complement of $U_{\delta,R}$ we have that $|P_d(x)|\geq C_1|x|^d$ for some $C_1>0$. Since $\deg(Q)=d-1$ we can find $C_2>0$ such that $|Q(x)|\leq C_2|x|^{d-1}$. It follows that for R big enough

$$C\cap\{|z|>R\}\subset U_{\delta,R}. \tag{3.2}$$

For each z with $|z|>R$ we write

$$P(z,w)=P_z(w)=a(w-h_1(z))\dots(w-h_d(z)). \tag{3.3}$$

With $a=(-1)^{d-1}a_1\dots a_{d-1}\neq 0$ and $h_j:\{|z|>R\}\to\mathbb{C}$ holomorphic. It follows from 3.2 that for each j, $\{(z,h_j(z)),\ |z|>R\}\subset U_{i,\delta,R}$ for some $i=i(j)$. In particular this implies that there is a constant $A>0$ such that

$$|h_j(z)|\leq A|z| \tag{3.4}$$

for $j = 1, \ldots, d$. We want to show that we can label the functions h_j in a way such that $i(j) = j$. First we note that if $i(j_0) = d$ then h_{j_0} is bounded. Indeed $|l_1 \ldots l_{d-1}(x)| \geq c|x|^{d-1}$ for some $c > 0$ and all $x \in U_{d,\delta,R}$, so that $|h_{j_0}(z)| = |Q|/|l_1 \ldots l_{d-1}| \leq C_2/c$. From 3.3 we get that the coefficient in front of w in the polynomial $P_z(w)$ is given by

$$(-1)^{d-1} a \sum_{j=1}^{d} \Pi_{i \neq j} h_i(z). \tag{3.5}$$

On the other hand 3.1 and $P_d = w(z - a_1 w) \ldots (z - a_{d-1} w)$, imply that 3.5 is a polynomial of degree $d - 1$ in z (with leading term z^{d-1}) . If we had $i(j_0) = i(j_1) = d$ for some $j_0 \neq j_1$ then h_{j_0} and h_{j_1} would be bounded. This together with the bound 3.4 would imply that the absolute value of 3.5 would be bounded by a constant times $|z|^{d-2}$, contradicting 3.5 being a degree $d - 1$ polynomial.

Changing coordinates we can argue the same way for the other asymptotic lines. We conclude that the map $j \to i(j)$ is injective and we can perform the desired labeling. The lemma follows by setting $\Phi = h_d$. In fact $h_j(z) = (1/a_j)z + \phi_j(z)$ with ϕ_j bounded for $j = 1, \ldots d - 1$ so that 3.3 gives

$$P(z, w) = (l_1 + \phi_1) \ldots (l_{d-1} + \phi_{d-1})(w - \Phi) \tag{3.6}$$

□

Lemma 5 *Let $\delta > 0$ be small enough and $R > 0$ big enough, then there exists a diffeomorphism $H \in \mathcal{D}$ such that H is holomorphic in $U_{\delta/2,2R}$ and H is the identity outside $U_{\delta,R}$.*

Proof: Let $\chi = \chi(t)$ be a smooth cut-off function with $\chi(t) = 1$ for $t \leq 1$ and $\chi(t) = 0$ for $t \geq 2$. We first define H in the region asymptotic to L_d. Let

$$h(z, w) = \chi\left(\frac{2|w|}{\delta|z|}\right)(1 - \chi)(R^{-1}|z|). \tag{3.7}$$

It follows that $h = 1$ on $U_{d,\delta/2,2R}$, $h = 0$ outside $U_{d,\delta,R}$ and $|D^\alpha h(x)| \leq C_{|\alpha|}|x|^{-|\alpha|}$ for any multi-index α. We set

$$H_d(z, w) = (z, w - h\Phi). \tag{3.8}$$

Since Φ is a bounded holomorphic function of z and in the region $U_{d,\delta,R}$ we have $|z| \geq c|(z, w)|$ for some $c > 0$, we conclude that there are constants A_j such that $|H_d(x) - x| \leq A_0$, $|DH_d(x) - Id| \leq A_1|x|^{-1}$ and $|D^\alpha H_d(x)| \leq A_j|x|^{-j}$ for all $x \in \mathbb{C}^2$ and $j = |\alpha| \geq 2$. We proceed similarly for the other asymptotic regions, and in an obvious notation we set

$$H = H_1 \circ \ldots \circ H_d. \tag{3.9}$$

□

From now on we fix $\delta, R > 0$ and H.

3.2 Construction of ω

We start by deriving some consequences of

$$r^2 \circ m_\lambda = \lambda^c r^2 \tag{3.10}$$

for all $\lambda > 0$ and $c = 2 + d\beta - d$. First of all we get that $m_\lambda^* \omega_F = \lambda^c \omega_F$. Since ω_F is positive we can find $a > 0$ such that $\omega_F \geq a\omega_{euc}$ on the euclidean unit sphere, the scaling property then gives

$$\omega_F(p) \geq a|p|^{c-2}\omega_{euc}. \tag{3.11}$$

For every $p \in \mathbb{C}^2$. ($|p|$ denotes the euclidean norm). On the other hand, from the continuity of r one gets

$$b^{-1}|p|^c \leq r^2(p) \leq b|p|^c \tag{3.12}$$

for some $b > 0$. Differentiating equation 3.10 on $\mathbb{C}^2 \setminus L$ we get that $D^\alpha r \circ m_\lambda = \lambda^{c-|\alpha|} D^\alpha r$ for any multi-index α. For $\epsilon > 0$ denote $U_\epsilon = U_{\epsilon,0}$, with the notation as in the previous subsection. From the smoothness of r on the complement of L it follows that

$$|D^\alpha r^2(p)| \leq A|p|^{c-|\alpha|} \tag{3.13}$$

on $\mathbb{C}^2 \setminus U_\epsilon$, where the constant A depends on ϵ and $|\alpha|$. It follows from 3.13 and 3.11 that in the complement of U_ϵ there exist $a_\epsilon > 0$ such that

$$a_\epsilon |p|^{c-2} \omega_{euc} \leq \omega_F(p) \leq a_\epsilon^{-1} |p|^{c-2} \omega_{euc}. \tag{3.14}$$

Let us denote by I the complex structure of $\mathbb{C}^2$ and let G be the inverse of H.

Lemma 6

$$|G^* I - I|_{g_F} = O(r^{-2/c}). \tag{3.15}$$

Proof: First we note that $|G^* I - I|_{g_{euc}} = O(|p|^{-1})$, since $G^* I - I$ is basically given by $\overline{\partial} G$. From 3.12 we can replace $O(|p|^{-1})$ with $O(r^{-2/c})$. Secondly, there exist $\epsilon > 0$ such that G is holomorphic in $U_{2\epsilon}$. (More precisely this is true outside a compact set). So $G^* I = I$ in $U_{2\epsilon}$. In a vector space with an inner product the norm of an endomorphism doesn't change if we multiply the inner product by a positive constant. Hence $|G^* I - I|_{|p|^{c-2} g_{euc}} = O(|p|^{-1})$. Finally 3.14 gives the lemma.

□

We move on and define

$$\eta = \frac{i}{2} \partial \overline{\partial} (r^2 \circ H). \tag{3.16}$$

Lemma 7 *There exists a compact K such that $\eta > 0$ outside K. Moreover,*

$$|G^* \eta - \omega_F|_{g_F} = O(r^{-2/c}).$$

Proof: Denote $H(z, w) = (u, v)$, so that $r^2 = r^2(u, v)$. Write $U = U_{\delta,R}$ and $U' = U_{\delta/2,2R}$, the subsets introduced in the previous subsection. We remove compact sets whenever necessary. Note that $G^* \eta = \omega_F$ in $H(U')$, clearly we can pick $\epsilon > 0$ such that $U_\epsilon \subset H(U')$. In $\mathbb{C}^2 \setminus H(U')$ we are then able to use the bounds 3.13. Set $p_0 = (u_0, v_0) = H(x_0)$ with $x_0 = (z_0, w_0) \notin U'$. First we compute $\eta(x_0)$

$$\frac{\partial}{\partial z}(r^2 \circ H) = \frac{\partial r^2}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial r^2}{\partial \overline{u}}\frac{\partial \overline{u}}{\partial z} + \frac{\partial r^2}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial r^2}{\partial \overline{v}}\frac{\partial \overline{v}}{\partial z},$$

$$\frac{\partial^2}{\partial z \partial \overline{z}}(r^2 \circ H) = \frac{\partial^2 r^2}{\partial^2 u}\frac{\partial u}{\partial \overline{z}}\frac{\partial u}{\partial z} + \frac{\partial^2 r^2}{\partial u \partial \overline{u}}\frac{\partial \overline{u}}{\partial \overline{z}}\frac{\partial u}{\partial z} + \frac{\partial^2 r^2}{\partial u \partial v}\frac{\partial v}{\partial \overline{z}}\frac{\partial u}{\partial z} + \frac{\partial^2 r^2}{\partial u \partial \overline{v}}\frac{\partial \overline{v}}{\partial \overline{z}}\frac{\partial u}{\partial z} + \frac{\partial r^2}{\partial u}\frac{\partial^2 u}{\partial z \partial \overline{z}}$$

$$+(\ldots),$$

where $(\ldots)$ consists of 15 terms that the reader can figure out. The second term is equal to

$$\frac{\partial^2 r^2}{\partial u \partial \overline{u}}(p_0)\left(1 + O(|x|^{-1})\right).$$

The first, third and fourth terms can be bounded by $A|x|^{c-2}|x|^{-1}$ and the fifth by $A|x|^{c-1}|x|^{-2}$ for some constant $A > 0$. It is easy to see that the remaining 15 terms can be bounded by $A|x|^{c-2}|x|^{-1}$ (the ones which contain second derivatives of r^2) or $A|x|^{c-1}|x|^{-2}$ (the ones which contain second derivatives of H). We conclude that we can bound all this terms by a constant times $|x|^{c-3}$. We argue similarly for the other derivatives in $\partial\overline{\partial}(r^2 \circ H)$ to conclude that

$$G^* \eta(p_0) = \omega_F(p_0) + O(|x|^{c-3}) dz d\overline{z} + O(|x|^{c-3}) dz d\overline{w} + O(|x|^{c-3}) dw d\overline{z} + O(|x|^{c-3}) dw d\overline{w}$$

Note that $dz d\overline{z} = du d\overline{u} + \nu$ where ν is a 2-form with $|\nu|_{euc} = O(|p|^{-1})$. From 3.11 we get $|du d\overline{u}|_{g_F} = O(|x|^{2-c})$. We argue equally for the other terms to conclude that

$$|G^*\eta - \omega_F|_{g_F}(p_0) = O(|p_0|^{-1}). \tag{3.17}$$

3.12 then gives the result.

□

Remark 3 *As we already said, $G^*(\eta) = \omega_F$ on a region $U_{\delta',R'}$ for some $\delta', R' > 0$. In the complement of this region one can extend 3.17 to*

$$|\nabla^i(G^*\eta - \omega_F)|_{g_F}(p_0) = O(r^{-(2/c)-i}), \tag{3.18}$$

where ∇ is the Levi-Civita connection of g_F.

We continue with the construction of ω. Let h be a cut-off function with $h = 1$ on B_N (the euclidean ball of radius N, say) and $h = 0$ on B^c_{N+1} where N is large enough so that $C \cap B^c_N \subset U'$ and $\eta > 0$ outside B_N . Consider

$$\omega' = \frac{i}{2}\partial\overline{\partial}\left(h|P|^{2\beta} + (1-h)(r^2 \circ H)\right). \tag{3.19}$$

Note that $\omega' = \eta > 0$ on B^c_{N+1}. On the other hand

$$\omega' = \frac{i}{2}\partial\overline{\partial}|P|^{2\beta} = \beta^2|P|^{2\beta-2}\frac{i}{2}\partial P \wedge \overline{\partial P} \geq 0$$

on B_N. Finally consider the annulus $B_{N+1} \setminus B_N$

Claim 3 *There is $a > 0$ such that $\omega' \geq -a\omega_{euc}$ on $B_{N+1} \setminus B_N$.*

Proof: Indeed, for $x \in C \cap (B_{N+1} \setminus B_N)$ we can find holomorphic coordinates (z_1, z_2) such that $C = \{z_1 = 0\}$ and $r^2 \circ H = |z_1|^{2\beta} + |z_2|^2$. In this coordinates $P = fz_1$ for some non-vanishing holomorphic f. Then we have $2\omega' = i\partial\overline{\partial}\left(h(|f|^{2\beta}|z_1|^{2\beta}) + (1-h)(|z_1|^{2\beta} + |z_2|^2)\right) =$ (smooth) $+i\partial\overline{\partial}u$, where $u = |z_1|^{2\beta}(h|f|^{2\beta} + 1 - h)$. On a smaller neighborhood we can assume $|f|^{2\beta} \geq \epsilon > 0$ so that $i\partial\overline{\partial}u = iu\partial\log u \wedge \overline{\partial}\log u + ui\partial\overline{\partial}\log u \geq ui\partial\overline{\partial}\log F$ where $F = h|f|^{2\beta} + 1 - h$. Note that F is smooth and $F \geq \min\{\epsilon, 1\}$ to conclude the claim. □

Lemma 8 *There is a Kähler metric ω on $\mathbb{C}^2$ with cone singularities of angle $2\pi\beta$ along C such that $\omega = \eta$ outside a compact set.*

Proof: Let $\chi = \chi(t)$ be a smooth cut-off function with $\chi(t) = 1$ for $t \leq 1$ and $\chi(t) = 0$ for $t \geq 2$. For $L > 0$ and $x \in \mathbb{C}^2$ let $\chi_L(x) = \chi(L^{-1}|x|)$. Set $\phi = \log(1 + |z|^2 + |w|^2)$ and define

$$\omega_L = \omega' + iK\partial\overline{\partial}(\chi_L\phi) \tag{3.20}$$

with $K > 0$ such that $Ki\partial\overline{\partial}\phi + \omega' > 0$ and $L > N + 2$. If L is big enough we can assume that on the annulus on $B_{2L} \setminus B_L$, $\omega' = \eta$. Recall that $|\eta|_{euc} \geq C_1|x|^{c-2}$ on the other hand, on $B_{2L} \setminus B_L$ we can bound $|\partial\overline{\partial}(\chi_L\phi)|_{euc} \leq C_2|x|^{-2}\log|x|$ (with C_2 independent of L). Taking L large we get that ω_L is positive everywhere. Fix such a large L and define $\omega = \omega_L$. The statement about the cone singularities follows from Lemma 12.

□

For reference in the future we say something about the volume form of ω. Define a function f in $\mathbb{C}^2$ by means of the equation

$$\omega^2 = e^f|P|^{2\beta-2}\Omega \wedge \overline{\Omega}. \tag{3.21}$$

Lemma 9 *Outside a compact set f is a smooth function with*

$$|D^\alpha f(x)| \leq A_{|\alpha|}|x|^{-1-|\alpha|}. \tag{3.22}$$

Proof: Consider first the complement of $U_{\delta,R}$, where H is the identity and $\eta = \omega_F$. Compare 2.1 and 3.21 to obtain

$$e^f = |P|^{2-2\beta}|P_d|^{2\beta-2} = \left|1 + \frac{Q}{P_d}\right|^{2-2\beta}.$$

In the complement of $U_{\delta,R}$ we have constants $b_{|\alpha|}$ such that

$$|D^\alpha(Q/P_d)|(x) \leq b_{|\alpha|}|x|^{-1-|\alpha|}.$$

3.22 then follows from $f = (2-2\beta)\log|1+Q/P_d|$. Secondly we consider the region $U_{\delta/2,2R}$, where H is holomorphic and $\eta = H^*\omega_F$. We see that $e^f = |P/(P_d \circ H)|^{2-2\beta}$. We focus in $U_{d,\delta/2,2R}$ and use 3.6 to get $P/(P_d \circ H) = (1+\psi_1(z))\dots(1+\psi_{d-1}(z))$ where $\psi_j(z)$ are holomorphic with $|\psi_j(z)| \leq A|z|^{-1}$ for some $A > 0$. Note that in $U_{d,\delta/2,2R}$ we have $|z| \geq a|(z,w)|$ for some $a > 0$. As before we get 3.22. Finally consider the region $U_{\delta,R} \setminus U_{\delta/2,2R}$. By Lemma 7 we can write $\eta = H^*\omega_F + \xi$ where ξ is a 2-form with $|\xi|_{g_F} = O(|x|^{-1})$. We conclude that $\eta^2 = \left(1 + O(|x|^{-1})\right) H^*\omega_F^2$ and we can proceed as before.

□

At this point we have proved the first item in Proposition 2. We call ω our reference metric. In the future we will need a metric with bisectional curvature bounded from above. The author was not able to prove that ω has this property. To remedy this we introduce another metric, ω_B. This is the content of the next subsection.

3.3 Upper bound on Bisec(ω_B)

First we define ω_B. Fix $0 < \delta < c$. Note that the function $p \to |p|^\delta$ is plurisubharmonic in $\mathbb{C}^2$. In fact,

$$a^{-1}|p|^{\delta-2}\omega_{euc} \leq i\partial\overline{\partial}|p|^\delta \leq a|p|^{\delta-2}\omega_{euc}$$

for some $a > 0$. The diffeomorphism H is asymptotic to the identity, so that there is $K > 0$ such that, outside a ball of radius K, $a^{-1}|p|^{\delta-2}\omega_{euc} \leq i\partial\overline{\partial}|H|^\delta \leq a|p|^{\delta-2}\omega_{euc}$. In the construction of ω (Lemma 8) we take $L >> K$. Let $\psi = \psi(t)$ be a smooth convex function of one real variable which is equal to the identity for large values of t and is constant when $t \leq K$. Define $h = \psi \circ |H|^\delta$, then h is smooth and $\nu = i\partial\overline{\partial}h = \psi'' i\partial|H|^\delta \wedge \overline{\partial}|H|^\delta + \psi' i\partial\overline{\partial}|H|^\delta$, since the first term is non-negative we have that $\nu \geq 0$ in all of $\mathbb{C}^2$. Moreover, outside a compact set there is $a > 0$ such that

$$a^{-1}|p|^{\delta-2}\omega_{euc} \leq \nu(p) \leq a|p|^{\delta-2}\omega_{euc}.$$

We define ω_B as

$$\omega_B = \omega + \Lambda\nu, \tag{3.23}$$

where $\Lambda > 0$ will be specified later on. From the definition it follows that

$$Q_2^{-1}\omega \leq \omega_B \leq Q_2\omega \tag{3.24}$$

for some $Q_2 > 0$. The goal is to prove the following

Lemma 10

$$Bisec(\omega_B) \leq Q_1.$$

Let us start by recalling the definition of bisectional curvature. Let ω be a Kähler metric on an open subset U of $\mathbb{C}^2$. For $x \in U$ and $v, w \in T_x^{1,0}\mathbb{C}^2$ with $|v|_\omega = |w|_\omega = 1$ we set

$$\mathrm{Bisec}_\omega(v,w) = R(v, \overline{v}, w, \overline{w}),$$

where R is the Riemann curvature tensor of ω. Recall that if (z_1, z_2) are holomorphic coordinates around x in which $\omega = \sum_{i,j=1}^2 g_{i\overline{j}} i dz_i d\overline{z_j}$ and $v = v_1\partial/\partial z_1 + v_2\partial/\partial z_1$, $w = w_1\partial/\partial z_1 + w_2\partial/\partial z_2$ then

$$\mathrm{Bisec}_\omega(v,w) = \sum_{i,j,k,l=1}^2 R_{i\overline{j}k\overline{l}} v_i \overline{v_j} w_k \overline{w_l},$$

where

$$R_{i\bar{j}k\bar{l}} = -g_{i\bar{j},k\bar{l}} + \sum_{s,t=1}^{2} g^{s\bar{t}} g_{i\bar{t},k} g_{s\bar{j},\bar{l}}.$$

Indexes after the comma indicate differentiation and $(g^{i\bar{j}})$ denotes the inverse transpose of the positive Hermitian matrix $(g_{i\bar{j}})$, the index i being for the rows and j for the columns.

In Appendix A of [20] it is shown that if η is a smooth Kähler form in the unit ball $B_1 \subset \mathbb{C}^2$, say, and F is a smooth positive function such that

$$\omega = \eta + i\partial\overline{\partial}(F|z_1|^{2\beta}) \tag{3.25}$$

is Kähler on $B_1 \setminus \{z_1 = 0\}$. Then there exist a number C such that $\mathrm{Bisec}(\omega) \leq C$ on $B_{1/2} \setminus \{z_1 = 0\}$, say. We choose $\Lambda > 0$ in 3.23 such that ω_B can be written in the form 3.25 around the points of the curve. Then [20] gives us an upper bound on $\mathrm{Bisec}(\omega_B)$ on compacts sets. In order to extend this bound to $\mathbb{C}^2$ we use the 'asymptotically conical' behavior of ω_B.

At points $x \in C$ where $\chi_L(x) = 0$ the metric ω in Lemma 8 can't be written in the form 3.25. (Compare with the one in Lemma 12). The author hasn't been able to get an upper bound on $\mathrm{Bisec}(\omega)$ around such points.

To prove Lemma 10 it suffices to bound from above $\mathrm{Bisec}(\omega_F + G^*\nu)$ in a region U_{δ_0,R_0} for some $\delta_0, R_0 > 0$. Note that outside a compact set $G^*\nu = i\partial\overline{\partial}|p|^{\delta}$. Let $0 \neq q \in L$ and B a neighborhood of q where there exist coordinates (ξ_1, ξ_2) which map B to the unit ball in $\mathbb{C}^2$ in which $\omega_F = |\xi_1|^{2\beta-2} i d\xi_1 d\overline{\xi}_1 + i d\xi_2 d\overline{\xi}_2$. We might also assume that $|q| \geq 2$ and that B is contained in the euclidean ball of radius half the euclidean distance from q to 0. Let $m_\lambda : B \to \lambda B$ for $\lambda \geq 1$ be the multiplication by λ in $\mathbb{C}^2$. We simplify notation and write ν for $G^*\nu$. Then

$$m_\lambda^*(\omega_F + \nu) = \lambda^c(\omega_F + \lambda^{\delta-c}\nu).$$

We will show that we have an upper bound for the bisectional curvature of $\omega_F + \lambda^{-c} m_\lambda^* \nu$ on $B_{1/2}$ which is independent of $\lambda \geq 1$. By a covering argument this gives the desired bound on U_{δ_0,R_0} and hence proves Lemma 10.

Write $\nu_{i\bar{j}} = \nu(\frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \overline{\xi}_j})$. Let $Q > 0$ be such that

$$Q^{-1}(\delta_{ij}) \leq (\nu_{i\bar{j}}) \leq Q(\delta_{ij}); \quad |\nu_{i\bar{j},k}| \leq Q; \quad |\nu_{i\bar{j},k\bar{l}}| \leq Q \tag{3.26}$$

on $B_{1/2}$. Write $\omega = \omega_{(\beta)} + \epsilon\nu$ with ν a smooth Kähler form in the unit ball in $\mathbb{C}^2$, $0 < \epsilon < 1$,

$$\omega_{(\beta)} = |\xi_1|^{2\beta-2} i d\xi_1 d\overline{\xi}_1 + i d\xi_2 d\overline{\xi}_2$$

and

$$\nu = \sum_{i,j=1}^{2} \nu_{i\bar{j}} i d\xi_i d\overline{\xi}_j.$$

The desired bound then follows from the following

Lemma 11 *There is a constant C, independent of $\epsilon > 0$, such that $\mathrm{Bisec}(\omega) \leq C$ on $B_{1/2}$. In fact C depends only on Q, where $Q > 0$ is such that, on $B_{1/2}$, $Q^{-1}\omega_{euc} \leq \nu \leq Q\omega_{euc}$ and $|\nu_{i\bar{j},k}|, |\nu_{i\bar{j},k\bar{l}}| \leq Q$ for any i, j, k, l.*

Proof: This follows the lines of Appendix A in [20]. Write

$$\omega = |\xi_1|^{2\beta-2} i d\xi_1 d\overline{\xi}_1 + \sum_{i,j=1}^{2} \tilde{g}_{i\bar{j}} i d\xi_i d\overline{\xi}_j,$$

so that

$$\tilde{g}_{1\bar{1}} = \epsilon\nu_{1\bar{1}}, \quad \tilde{g}_{1\bar{2}} = \epsilon\nu_{1\bar{2}} \quad \tilde{g}_{2\bar{2}} = 1 + \epsilon\nu_{2\bar{2}}.$$

Let $x = (x_1, x_2) \in B_{1/2} \setminus \{\xi_1 = 0\}$. Define new coordinates (z_1, z_2) around x via

$$\xi_1 = z_1$$
$$\xi_2 = z_2 + \frac{a}{2}(z_1 - x_1)^2 + b(z_1 - x_1)(z_2 - x_2) + \frac{c}{2}(z_2 - x_2)^2,$$

where

$$a = -(\tilde{g}_{2\bar{2}}(x))^{-1}\tilde{g}_{1\bar{2},1}(x), \quad b = -(\tilde{g}_{2\bar{2}}(x))^{-1}\tilde{g}_{1\bar{2},2}(x), \quad c = -(\tilde{g}_{2\bar{2}}(x))^{-1}\tilde{g}_{2\bar{2},2}(x).$$

In this new coordinates we have

$$\omega = |z_1|^{2\beta-2} i dz_1 d\bar{z}_1 + \sum_{i,j} \hat{g}_{i\bar{j}} i dz_i d\bar{z}_j.$$

Claim 4 $\hat{g}_{i\bar{j},k}(x) = 0$ *when* $j \neq 1$.

Indeed, write $d\xi_2 = A dz_1 + B dz_2$, with $A = a(z_1 - x_1) + b(z_2 - x_2)$ and $B = 1 + b(z_1 - x_1) + c(z_2 - x_2)$. A straightforward computation gives

$$\hat{g}_{1\bar{2}} = \tilde{g}_{1\bar{2}}\overline{B} + \tilde{g}_{2\bar{2}}A\overline{B}, \quad \hat{g}_{2\bar{2}} = |B|^2\tilde{g}_{2\bar{2}}.$$

From here we get

$$\hat{g}_{1\bar{2},1}(x) = \tilde{g}_{1\bar{2},1}(x) + \tilde{g}_{2\bar{2}}(x)a, \quad \hat{g}_{1\bar{2},2}(x) = \tilde{g}_{1\bar{2},2}(x) + \tilde{g}_{2\bar{2}}(x)b, \quad \hat{g}_{2\bar{2},2}(x) = \tilde{g}_{2\bar{2},2}(x) + \tilde{g}_{2\bar{2}}(x)c.$$

Our choice of a, b, c implies that these three numbers are zero. The Kähler condition $\hat{g}_{i\bar{j},k} = \hat{g}_{k\bar{j},i}$ implies that $\hat{g}_{2\bar{2},1}(x) = \hat{g}_{1\bar{2},2}(x) = 0$ and the claim follows.

We compute the bisectional curvature of ω at x using the coordinates (z_1, z_2). Let $v = v_1\partial/\partial z_1 + v_2\partial/\partial z_1$ and $w = w_1\partial/\partial z_1 + w_2\partial/\partial z_2 \in T_x^{1,0}\mathbb{C}^2$ with $|v|_\omega = |w|_\omega = 1$. Note that this implies that $|v_1|, |w_1| \leq C|z_1|^{1-\beta}$ and $|v_2|, |w_2| \leq C$. Write $\omega = \sum_{i,j=1}^2 g_{i\bar{j}} i dz_i d\overline{z_j}$. So that $g_{i\bar{j}} = \hat{g}_{i\bar{j}}$ when $(i,j) \neq (1,1)$ and $g_{1\bar{1}} = |z_1|^{2\beta-2} + \hat{g}_{1\bar{1}}$. Write $\text{Bisec}_\omega(v, w) = T_1 + T_2$, where

$$T_1 = -\sum_{i,j,k,l} g_{i\bar{j},k\bar{l}}(x) v_i \overline{v}_j w_k \overline{w}_l$$

and

$$T_2 = \sum_{s,t,i,j,k,l=1}^{2} g^{s\bar{t}}(x) g_{i\bar{t},k}(x) g_{s\bar{j},\bar{l}}(x) v_i \overline{v}_j w_k \overline{w}_l.$$

Claim 5

$$T_1 \leq C - (\beta - 1)^2|z_1|^{2\beta-4}|v_1|^2|w_1|^2.$$

In fact $g_{1\bar{1},1\bar{1}} = (\beta - 1)^2|z_1|^{2\beta-4} + \hat{g}_{1\bar{1},1\bar{1}}$, and we have

$$\hat{g}_{1\bar{1}} = \tilde{g}_{1\bar{1}} + A\tilde{g}_{2\bar{1}} + \overline{A}\tilde{g}_{1\bar{2}} + |A|^2\tilde{g}_{2\bar{2}}.$$

From here we compute

$$\hat{g}_{1\bar{1},1\bar{1}}(x) = \tilde{g}_{1\bar{1},1\bar{1}}(x) + a\tilde{g}_{2\bar{1},\bar{1}}(x) + \overline{a}\tilde{g}_{1\bar{2},1}(x) + |a|^2\tilde{g}_{2\bar{2}}.$$

Since the differential at x of the change of coordinates between (ξ_1, ξ_2) and (z_1, z_2) is the identity, we have that

$$\tilde{g}_{i\bar{j},k}(x) = \frac{\partial \tilde{g}_{i\bar{j}}}{\partial \xi_k}(x) = \frac{\partial \nu_{i\bar{j}}}{\partial \xi_k}(x), \quad \tilde{g}_{i\bar{j},k\bar{l}}(x) = \frac{\partial^2 \tilde{g}_{i\bar{j}}}{\partial \bar{\xi}_l \partial \xi_k}(x) = \frac{\partial^2 \nu_{i\bar{j}}}{\partial \bar{\xi}_l \partial \xi_k}(x).$$

From this fact, and $|a| = |-(\tilde{g}_{2\bar{2}}(x))^{-1}\tilde{g}_{1\bar{2},1}(x)| \leq |\tilde{g}_{1\bar{2},1}(x)|$ we get that $|\hat{g}_{1\bar{1},1\bar{1}}(x)| \leq C$. Similarly, when $(i,j,k,l) \neq (1,1,1,1)$ we have $|g_{i\bar{j},k\bar{l}}(x)| = |\hat{g}_{i\bar{j},k\bar{l}}(x)| \leq C$, and the claim follows.

Claim 6

$$T_2 \leq C + (\beta - 1)^2|z_1|^{2\beta-4}|v_1|^2|w_1|^2.$$

Define a non-negative bilinear Hermitian form on tensors $a = [a_{i\bar{j}k}]$ satisfying $a_{i\bar{j}k} = a_{k\bar{j}i}$ by

$$\langle [a_{i\bar{j}k}], [b_{p\bar{q}r}] \rangle = \sum g^{q\bar{j}}(x)(w_i a_{i\bar{j}k} v_k)\overline{(w_p b_{p\bar{q}r})v_r}.$$

Then

$$T_2 = \|D + E\|^2$$

with $D_{ijk} = \hat{g}_{i\bar{j},k}$ and $E_{ijk} = (\beta - 1)|z_1|^{2\beta-4}\overline{z_1}$ if $(ijk) = (111)$ and $E_{ijk} = 0$ otherwise. We first estimate

$$\|E\|^2 = (\beta - 1)^2 |z_1|^{4\beta-6} g^{1\bar{1}}(x)|v_1|^2|w_1|^2,$$

where $g^{1\bar{1}} = \det(g)^{-1} g_{2\bar{2}}$.

$$\det(g) = (|z_1|^{2\beta-2} + \hat{g}_{1\bar{1}})\hat{g}_{2\bar{2}} - |\hat{g}_{1\bar{2}}|^2 = \hat{g}_{2\bar{2}}|z_1|^{2\beta-2}\left(1 + (\hat{g}_{2\bar{2}})^{-1}\det(\hat{g})|z_1|^{2-2\beta}\right).$$

Unwinding notation we have that at the point x, $\hat{g}_{2\bar{2}} = 1 + \epsilon\nu_{2\bar{2}}(x)$ and $\det(\hat{g})(x) = \epsilon\nu_{1\bar{1}}(x) + \epsilon^2\det(\nu)(x)$. We conclude that $(\hat{g}_{2\bar{2}})^{-1}\det(\hat{g}) \geq Q^{-1}\epsilon$, so

$$g^{1\bar{1}}(x) \leq (1 + \delta)^{-1}|z_1|^{2-2\beta}$$

with $\delta = Q^{-1}\epsilon|z_1|^{2-2\beta}$. We get

$$\|E\|^2 \leq (1 + \delta)^{-1}(\beta - 1)^2|z_1|^{2\beta-4}|v_1|^2|w_1|^2.$$

Next we do a trick

$$\|T_2\|^2 \leq (1 + \delta^{-1})\|D\|^2 + (1 + \delta)\|E\|^2.$$

The claim (and the lemma) will follow if we can bound $\epsilon^{-1}|z_1|^{2\beta-2}\|D\|^2$.

$$\|D\|^2 = \sum_{s,t,i,j,k,l=1}^{2} g^{s\bar{t}}(x)\hat{g}_{i\bar{t},k}(x)\hat{g}_{s\bar{j},\bar{l}}(x)v_i\overline{v}_j w_k\overline{w}_l = \sum_{i,j,k,l=1}^{2} g^{1\bar{1}}(x)\hat{g}_{i\bar{1},k}(x)\hat{g}_{1\bar{j},\bar{l}}(x)v_i\overline{v}_j w_k\overline{w}_l.$$

(The second equality follows from the first claim.) Since $g^{1\bar{1}}(x) \leq |z_1|^{2-2\beta}$ and $|\hat{g}_{i\bar{j},k}(x)| \leq C\epsilon$, the estimate follows.

□

3.4 The Sobolev inequality

Recall that the Sobolev inequality says that there exists a constant C such that for every smooth function ϕ in $\mathbb{R}^4$ with compact support, we have

$$\left(\int_{\mathbb{R}^4} |\phi|^4\right)^{1/2} \leq C\int_{\mathbb{R}^4} |\nabla\phi|^2. \tag{3.27}$$

Let g be a Riemannian metric on $\mathbb{R}^4$. We can ask about inequality 3.27, for some constant C, if we replace the euclidean measure by the measure defined by g and the euclidean gradient by the gradient with respect to g. It is clear that 3.27 holds if g is quasi-isometric to the euclidean metric, i.e. $\Lambda^{-1}g \leq g_{euc} \leq \Lambda g$ for some constant $\Lambda > 0$. Consider now the case of the Flat metric g_F on $\mathbb{C}^2 \setminus L$ given by Proposition 1. We claim that there exists a diffeomorphism Φ of $\mathbb{C}^2 \setminus L$ such that $\Lambda^{-1}g_{euc} \leq \Phi^* g_F \leq \Lambda g_{euc}$ for some $\Lambda > 0$. This is indeed clear from the construction of g_F : The spherical metric with cone singularities on $\mathbb{CP}^1$ is, up to a diffeomorphism, quasi-isometric to the round metric and the same is true for the singular metric $\overline{g}$ on the three-sphere; finally, the euclidean metric on $\mathbb{R}^4$ is the cone over the round three sphere of radius one. We conclude that the Sobolev inequality 3.27 holds for g_F. Here we remark that we have established the inequality for compactly supported functions ϕ which are smooth in cone coordinates. This means that ϕ is a smooth function of $\rho e^{i\theta}$, z_2; whenever $z_1 = \rho^{1/\beta}e^{i\theta}$, z_2 are holomorphic coordinates around a point of $L \setminus \{0\}$ such that $L = \{z_1 = 0\}$. Consider now our reference metric ω, given by Lemma 8.

We claim that there exists a diffeomorphism Ψ of $\mathbb{C}^2 \setminus C$ such that $\Lambda^{-1}\omega_{euc} \leq \Psi^*\omega \leq \Lambda\omega_{euc}$ for some $\Lambda > 0$. Indeed $\Phi \circ H$ will do the job outside a compact set. It is easy to patch this with a diffeomorphism supported in a tubular neighborhood of the curve, modeled on $\rho e^{i\theta} \to \rho^{1/\beta} e^{i\theta}$ in transverse directions to C. The claim follows. As a result of the discussion we have the following

Proposition 3 *There exists a constant C such that the Sobolev inequality 3.27 holds for the reference metric ω and all functions ϕ with compact support, smooth in cone coordinates.*

4 Linear analysis

In this section we define Banach spaces of continuous functions on $\mathbb{C}^2$ in which the Laplacian of the reference metric ω acts as a Fredholm operator. The main result is Proposition 5. Our references are Pacard's notes [28], Bartnik's article [4] and Chapter 8 in Szekelyhidi's book [31]. To set up the corresponding linear theory for ALE metrics, Joyce [21] invokes the explicit expression of the Green's function of the euclidean space. We avoid those arguments by means of some others conventional methods used in the study of AC manifolds. In a few words we can say that, provided we know the interior Schauder estimates (Theorem 7), the standard techniques used to establish the linear theory for AC manifolds work in our setting.

4.1 Interior Schauder estimates

This subsection is a review of material in [15]. The main point is to state, without a proof, Theorem 7. We also recall the definition of a metric with a cone singularity.

Consider the singular metric $g_{(\beta)} = \beta^2|z_1|^{2\beta-2}|dz_1|^2 + |dz_2|^2$ on $\mathbb{C}^2$. We want to define Hölder continuous $(1,0)$ and $(1,1)$ forms. Note that under the map $z_1 = r_1^{1/\beta}e^{i\theta_1}$ we have $g_{(\beta)} = dr_1^2 + \beta^2 r_1^2 d\theta_1^2 + |dz_2|^2$. Set $\epsilon = dr_1 + i\beta r_1 d\theta_1$. A $(1,0)$ form η is called C^α if $\eta = f_1\epsilon + f_2 dw$ with f_1, f_2 C^α functions in the usual sense in the cone coordinates $(r_1e^{i\theta_1}, z_2)$. It is also required that $f_1 = 0$ on $\{z_1 = 0\}$. If we change ϵ by $\tilde{\epsilon} = e^{i\theta}\epsilon = \beta|z_1|^{\beta-1}dz_1$, say, in the definition; then the vanishing condition implies that we get the same space. In order to define C^α $(1,1)$ forms we use the basis $\{\epsilon\overline{\epsilon}, \epsilon d\overline{w}, dw\overline{\epsilon}, dwd\overline{w}\}$, as above we ask the components to be C^α functions and we require the components corresponding to $\epsilon d\overline{w}, dw\overline{\epsilon}$ to vanish on the singular set. Finally we set $C^{2,\alpha}$ to be the space of C^α functions u such that $\partial u, \partial\overline{\partial}u$ are C^α. We define the C^α norm of a function $\|f\|_\alpha$ as the sum of its C^0 norm $\|f\|_0$ and its C^α semi-norm $[f]_\alpha$; in the cone coordinates this last semi-norm agrees with the standard

$$[f]_\alpha = \sup_{x,y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}.$$

To define the $C^{2,\alpha}$ norm of a function f we simply add $\|f\|_\alpha$, the C^α norm of the components of ∂f in the basis $\{\epsilon, dw\}$ and the C^α norm of the components of $i\partial\overline{\partial}f$ in the basis $\{\epsilon\overline{\epsilon}, \epsilon d\overline{w}, dw\overline{\epsilon}, dwd\overline{w}\}$.

We are interested in the equation $\triangle u = f$, where $\triangle$ is the Laplace operator of $g_{(\beta)}$. We define L_1^2 on domains of $\mathbb{C}^2$ by means of the usual norm $\|u\|_{L_1^2} = \int |\nabla u|^2 + \int u^2$. In the coordinates $(r_1e^{i\theta_1}, z_2)$, $\beta^2 g_{euc} \leq g_{(\beta)} \leq (1+\beta^2)g_{euc}$, , so that L_1^2 coincides with the standard Sobolev space in these coordinates. Let u be a function that is locally in L_1^2. We say that u is a weak solution of $\triangle u = f$ if

$$\int \langle \nabla u, \nabla \phi \rangle = -\int f\phi$$

for all smooth compactly supported ϕ.

Theorem 7 *[15]. Fix $\alpha < \beta^{-1} - 1$, then there exists a constant C such that if u is a weak solution of $\triangle u = f$ on B_2 and $f \in C^\alpha(B_2)$ then $u \in C^{2,\alpha}(B_1)$ and*

$$\|u\|_{C^{2,\alpha}(B_1)} \leq C\left(\|f\|_{C^\alpha(B_2)} + \|u\|_{C^0(B_2)}\right). \tag{4.1}$$

We mention 3 differences between Theorem 7 and the standard Schauder estimates

- We don't have estimates for all the second derivatives of u. (E.g. $\partial^2 u/\partial r_1^2$).

- If $\triangle u \in C^\alpha$; then the component of ∂u corresponding to ϵ needs to vanish along the singular set.
- The estimates require $\alpha < \beta^{-1} - 1$.

There is a geometric explanation for the second point. The metric $g_{(\beta)}$ has non trivial holonomy. If γ is a simple loop around the singular set then parallel transport along γ is a rotation of angle $2\pi(1-\beta)$ on the $\mathbb{C}_\beta$ factor and it is the identity on the $\mathbb{C}$ factor. If u is a function with bounded Hessian then the component of its gradient along the $\mathbb{C}_\beta$ factor must vanish; because parallel transport of the gradient of u along small loops around the singular set that shrink to a point must leave the gradient unchanged in the limit.

More technically, the three points can be explained by the fact that if p is a point outside the singular set and $\Gamma_p = G(.,p)$, where G is the Green's function for $\triangle$; then around points of $\{z_1 = 0\}$ one can write a convergent series expansion

$$\Gamma_p = \sum_{j,k\geq 0} a_{j,k}(z_2) r_1^{(k/\beta)+2j} \cos(k\theta_1) \tag{4.2}$$

with $a_{j,k}$ smooth functions. The proof of Proposition 7 given in [15] uses classical methods. The expression 4.2 is proved by separation of variables and a check of convergence. The coefficients $a_{j,k}$ are given in terms of Bessel functions. If u is a function with compact support such that $\triangle u = f$, then

$$u(x) = \int G(x,y) f(y) dy. \tag{4.3}$$

To show the estimate 4.1 one has to differentiate 4.3 twice. The proof then follows the one of the standard Scahuder estimates. There are modifications due to the fact that $\triangle$ is not translation invariant.

We move on to give the definition of a metric with cone singularities. But first let η be a $(1,1)$ form on B_2 with $\|\eta\|_{C^\alpha(B_2)} \leq \epsilon$. Assume that η has support contained in B_1 and consider the operator $Lu = \triangle u + \langle \partial\overline{\partial} u, \eta \rangle$. If $\epsilon < 1/(2C)$ we can use 4.1 to get the estimate

$$\|u\|_{C^{2,\alpha}(B_1)} \leq 2C \left(\|Lu\|_{C^\alpha(B_2)} + \|u\|_{C^0(B_2)} \right) \tag{4.4}$$

for all functions $u \in C^{2,\alpha}(B_2)$. Now let C be our smooth curve in $\mathbb{C}^2$ and let ω be a (smooth) Kähler metric in the complement of C. We say that ω is a metric with cone singularities along C of angle $2\pi\beta$ if around each $p \in C$ we can find holomorphic coordinates (z_1, z_2) such that

$$\omega = \omega_{(\beta)} + \eta \tag{4.5}$$

with $\eta \in C^\alpha$ and $\eta(p) = 0$. More precisely, $\eta(p) = 0$ means that the coefficients of η in the basis $\{\epsilon\overline{\epsilon}, \epsilon d\overline{w}, dw\overline{\epsilon}, dwd\overline{w}\}$ vanish at p. This agrees with the Definition 1 given in the Introduction, as follows from the first remark after Definition 1.

Given our curve C and a bounded open subset U of $\mathbb{C}^2$ we can define the space $C^{2,\alpha}(U)$ by taking a finite cover of U with coordinates in which $C = \{z_1 = 0\}$. Let $p \in C$ and write ω as in 4.5. After a dilation and multiplying by a cut-off function we can assume that in a smaller neighborhood of p we have $\triangle_\omega = L$ with L as in 4.4. From here we get that

$$\|u\|_{C^{2,\alpha}(U)} \leq C \left(\|\triangle_\omega u\|_{C^\alpha(V)} + \|u\|_{C^0(V)} \right) \tag{4.6}$$

for all $u \in C^{2,\alpha}(V)$. In 4.6 we assume that U is compactly contained in V. The constant C depends on ω, U, V.

Finally let us say that the metrics we have constructed in the previous section have cone singularities in the sense of 4.5 because of the following

Lemma 12 *Let ω be a Kähler metric on $\mathbb{C}^2 \setminus C$ such that around each $p \in C$ we can find holomorphic coordinates (z_1, z_2) such that*

$$\omega = \Omega + i\partial\overline{\partial}(F|z_1|^{2\beta})$$

with Ω a smooth $(1,1)$ form such that $\Omega(\partial/\partial z_2, \partial/\partial \overline{z}_2)(p) > 0$ and F a smooth positive function. Then ω has cone singularities in the sense of 4.5.

Proof: This follows from the computation

$$i\partial\overline{\partial}(F|z_1|^{2\beta}) = |z_1|^{2\beta}i\partial\overline{\partial}F + \beta|z_1|^{2\beta-2}\left(\overline{z}_1 idz_1\overline{\partial}F + z_1\partial F d\overline{z}_1\right) + \beta^2 F|z_1|^{2\beta-2}idz_1 d\overline{z}_1.$$

Set $\tilde{z_1} = az_1$, $\tilde{z_2} = bz_2$ with $a = F(p)^{1/2}$ and $b = \left(\Omega(\partial/\partial z_2, \partial/\partial\overline{z}_2)(p)\right)^{1/2}$ to get 4.5

□

4.2 Weighted Hölder spaces

Now we begin to work in a direction adapted to our needs. We introduce weights to the previous Hölder spaces. In this subsection we work with the flat metrics g_F from Section 2. The property we shall exploit the most is the one of being a metric cone. If γ is the weight parameter in our space of functions, then a function in the space is bounded by r^γ. In particular, if $\gamma < 0$, we allow our functions to blow up at the apex of the cone.

Let g_F be the flat metric. Write $B_R = \{r < R\}$ for the metric ball of radius R around the origin. Consider the annulus $A_1 = B_2 \setminus \overline{B_1}$ and the bigger one $\tilde{A}_1 = B_4 \setminus \overline{B_{1/2}}$. We know that around each $p \in L \cap A_1$ we can find coordinates (z_1, z_2) in which $g_F = g_{(\beta)}$ and that g_F is locally isometric to the euclidean metric outside L. We fix a finite cover of A_1 by such coordinates and define the spaces $C^\alpha(A_1)$ and $C^{2,\alpha}(A_1)$ in the obvious way . Alternatively (in more intrinsic terms) we can define the space C^α functions in any domain by considering the distance induced by g_F and applying the standard definition. To measure the $C^{2,\alpha}$ norm of a function we can take an orthonormal basis for the $(1,0)$ forms $\{\tau_1, \tau_2\}$, for example by applying Gram-Schmidt to $\{dz, dw\}$ over $A_1 \setminus L$, and sum the C^α norm of the components of ∂u and $\partial\overline{\partial}u$ with respect to τ_i and $\tau_i\overline{\tau_j}$ respectively. It follows from the first bullet after Definition 1 that this is independent of the choice of orthonormal basis $\{\tau_1, \tau_2\}$. One can replace A_1 with $\tilde{A}_1$ in the above discussion without any change. It follows from the standard Schauder estimates and Theorem 7 that there is a constant C such that for every $u \in C^{2,\alpha}(\tilde{A}_1)$

$$\|u\|_{C^{2,\alpha}(A_1)} \leq C\left(\|f\|_{C^\alpha(\tilde{A}_1)} + \|u\|_{C^0(\tilde{A}_1)}\right), \tag{4.7}$$

where $\triangle u = f$ is the Laplacian of u with respect to g_F.

Let $\gamma \in \mathbb{R}$, we want to define the space C^α_γ. For $\lambda > 0$, denote $A_\lambda = B_{2\lambda} \setminus B_\lambda$. In other words $A_\lambda = D_\lambda(A_1)$ where D_λ is the map given in spherical coordinates by $D_\lambda(r, \theta) = (\lambda r, \theta)$. Note that in complex coordinates $D_\lambda(z, w) = (\lambda^{2/c}z, \lambda^{2/c}w)$. Let f be a continuous function on $\mathbb{C}^2 \setminus \{0\}$. Define $f_{\lambda,\gamma} = \lambda^{-\gamma}.(f \circ D_\lambda)$ and think of it as a function on A_1. Finally we set

$$\|f\|_{\alpha,\gamma} = \sup_{\lambda>0}\|f_{\lambda,\gamma}\|_{C^\alpha(A_1)}. \tag{4.8}$$

It follows that if $f \in C^\alpha_\gamma$ (the space of functions in $\mathbb{C}^2 \setminus \{0\}$ for which the above norm is finite), then $|f(x)| \leq Ar(x)^\gamma$ for some constant A. In fact, if we let $\|f\|_{0,\gamma} = \sup_{\lambda>0}\|f_{\lambda,\gamma}\|_{C^0(A_1)}$ we clearly have $\|f\|_{0,\gamma} \leq \|f\|_{\alpha,\gamma}$ and $\|f\|_{0,\gamma}$ is easily seen to be equivalent to $\sup_x r(x)^{-\gamma}|f(x)|$. It is clear that if we use $\tilde{A}_1$ instead we would get an equivalent norm, i.e, there exist a constant C such that

$$\sup_{\lambda>0}\|f_{\lambda,\gamma}\|_{C^\alpha(\tilde{A}_1)} \leq C\|f\|_{\alpha,\gamma}.$$

Having said what is the space $C^{2,\alpha}$ on A_1 we can define the space $C^{2,\alpha}_\delta$ to be the space of functions u on $\mathbb{C}^2 \setminus \{0\}$ for which

$$\|u\|_{2,\alpha,\delta} = \sup_{\lambda>0}\|u_{\lambda,\delta}\|_{C^{2,\alpha}(A_1)} \tag{4.9}$$

is finite. As above δ is any fixed real number.

With these definitions we claim that $\triangle$ defines a bounded operator from $C^{2,\alpha}_\delta$ to $C^\alpha_{\delta-2}$. Indeed, from the expression

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\triangle_{\overline{g}}, \tag{4.10}$$

we get that $\triangle u_\lambda = \lambda^2(\triangle u)_\lambda$. We denote $u_\lambda = u \circ D_\lambda$. Now take $u \in C^{2,\alpha}_\delta$, write $\triangle u = f$ and let $\lambda > 0$. Then

$$f_{\lambda,\delta-2} = \lambda^{-\delta+2}(\triangle u)_\lambda = \lambda^{-\delta}\triangle u_\lambda$$

and our claim follows from the fact that $\triangle : C^{2,\alpha}(A_1) \to C^\alpha(A_1)$ is a bounded operator.

Let us give an equivalent norm in $C^{2,\alpha}_\delta$ which will make evident the fact that if u belongs to this space then $|\partial\overline{\partial}u|_{g_F} = O(r^{\delta-2})$. In order to do this we note that on $\mathbb{C}^2 \setminus L$ we have an (up to a factor of $\sqrt{2}$) orthonormal basis (w.r.t. g_F) of the $(1,0)$ forms given by $\{\tau_1, \tau_2\}$ (see 2.20) such that $D^*_\lambda \tau_i = \lambda\tau_i$. Given a function u we write $\partial u = \sum_i u_i\tau_i$ and $\partial\overline{\partial}u = \sum_{i,j} u_{i\overline{j}}\tau_i\overline{\tau_j}$. We claim that

$$\|u\|_{2,\alpha,\delta} = \|u\|_{0,\delta} + \sum_i \|u_i\|_{\alpha,\delta-1} + \sum_{i,j}\|u_{i\overline{j}}\|_{\alpha,\delta-2} \tag{4.11}$$

defines an equivalent norm as the previous one. (Our claim justifies the abuse of notation since 4.11 is not exactly equal to 4.9). Since $\triangle u = u_{1\overline{1}} + u_{2\overline{2}}$ we see again that $\triangle : C^{2,\alpha}_\delta \to C^\alpha_{\delta-2}$ is a bounded map. We compute $\|u_{\lambda,\delta}\|_{C^{2,\alpha}(A_1)}$ using the basis $\{\tau_1,\tau_2\}$. Since D_λ is holomorphic we have that $\partial u_\lambda = D^*_\lambda \partial u = \lambda\sum_i (u_i)_\lambda \tau_i$ and that $\partial\overline{\partial}u_\lambda = D^*_\lambda\partial\overline{\partial}u = \lambda^2\sum_{i,j}(u_{i\overline{j}})_\lambda\tau_i\overline{\tau_j}$. Our claim then follows from

$$\|u_{\lambda,\delta}\|_{C^{2,\alpha}(A_1)} = \|\lambda^{-\delta}u_\lambda\|_{C^0(A_1)} + \sum_i \|\lambda^{-\delta+1}(u_i)_\lambda\|_{C^\alpha(A_1)} + \sum_{i,j}\|\lambda^{-\delta+2}(u_{i\overline{j}})_\lambda\|_{C^\alpha(A_1)}.$$

In arguments in which the Hölder exponent α is not crucially needed we will say that a function is in C^2 if the components $u_{i\overline{j}}$ are continuous. Similarly we can give a definition of C^2_δ.

We are now ready to state our first main estimate

Lemma 13 *Let $\alpha < \beta^{-1} - 1$ and $\delta \in \mathbb{R}$. Then there is a constant $C = C(\alpha,\delta)$ such that for every $u \in C^{2,\alpha}_\delta$ with $\triangle u = f$*

$$\|u\|_{2,\alpha,\delta} \le C\left(\|f\|_{\alpha,\delta-2} + \|u\|_{0,\delta}\right).$$

Proof: Write $\delta = \gamma + 2$. Let $\lambda > 0$ we apply the interior estimate 4.7 to $u_{\lambda,\delta} = \lambda^{-\delta}u_\lambda$ to get

$$\|u_{\lambda,\delta}\|_{C^{2,\alpha}(A_1)} \le C\left(\|\lambda^{-\delta+2}f_\lambda\|_{C^\alpha(\tilde{A}_1)} + \|\lambda^{-\delta}u_\lambda\|_{C^0(\tilde{A}_1)}\right).$$

Note that the first term on the r.h.s. is bounded by $\|f\|_{\alpha,\gamma}$ and the second term is bounded by $\|u\|_{0,\gamma+2}$. □

Remark 4 *In fact we have proved that if u is locally in $C^{2,\alpha}$, $\triangle u \in C^\alpha_{\delta-2}$ and $\|u\|_{0,\delta}$ is finite, then $u \in C^{2,\alpha}_\delta$ and the above estimate holds.*

Our next goal is to bound $\|u\|_{0,\delta}$ in terms of $\|f\|_{\alpha,\delta-2}$. It turns out that this is true, except when δ belongs to the discrete set of 'Indicial Roots'. In order to explain what is this set we digress a little and discuss some basics of spectral theory for $\triangle_{\overline{g}}$, the Laplacian of the singular metric on the 3-sphere.

First we note that on $(S^3, \overline{g})$ there is an obvious definition of the spaces L^2 and L^2_1. Since there is a diffeomorphism χ of $S^3 \setminus L$ such that $\chi^*\overline{g}$ is quasi-isometric to a smooth metric on S^3 we see that L^2 and L^2_1 correspond under χ to the usual spaces. In particular we have that $L^2_1 \subset L^2$ is compact. If we write the norms as $\|f\|^2_{L^2} = \int f^2$ and $\|u\|^2_{L^2_1} = \int u^2 + \int|\nabla u|^2$ we see that $f \in L^2$ defines a bounded linear functional T on L^2_1 by $T(\phi) = \int f\phi$. If u is such that $T = \langle u, -\rangle_{L^2_1}$ then u is said to be a weak solution of $-\triangle_{\overline{g}}u + u = f$. The map $K(f) = u$ is a bounded linear map between L^2 and L^2_1, composing this map with the compact inclusion we have a map $K : L^2 \to L^2$ which is compact and self-adjoint. It follows from the spectral theorem that we can find an orthonormal basis $\{\phi_i\}_{i\ge0}$ of L^2 such that $K(\phi_i) = s_i\phi_i$ and $s_i \to 0$. Unwinding the definitions we get that $\triangle_{\overline{g}}\phi_i = -\lambda_i\phi_i$ with $0 = \lambda_0 \le \lambda_1 \le \lambda_2 \le \ldots$ and $\lambda_i = (1 - s_i)/s_i \to \infty$. For each λ_i define $\delta^\pm_i$ to be the solutions of $\{s(s+2) = \lambda_i\}$ with δ^+_i non-negative and δ^-_i non-positive (in fact ≤ -2). The set of Indicial Roots is set to be $I = \{\delta^\pm_i, i \ge 0\}$. With this definition we can state the following

Lemma 14 *Let $u \in C^2_\delta$ be such that $\triangle u = 0$ and $\delta \notin I$. Then $u = 0$.*

Proof: Write $u(r,\theta) = \sum_{i=0}^{\infty} u_i(r)\phi_i(\theta)$, where $u_i(r) = \int_{S^3} u(r,.)\phi_i$. It follows from Hölder's inequality that if $|u| \leq Cr^\delta$ then $|u_i(r)| \leq C(\mathrm{Vol}(\overline{g}))^{1/2} r^\delta$. On the other hand the equation $\triangle u = 0$ implies

$$u_i'' + \frac{3}{r}u_i' - \frac{\lambda_i}{r^2}u_i = 0,$$

so that $u_i = Ar^{\delta_i^+} + Br^{\delta_i^-}$ for some constants A and B. Since $\delta \neq \delta_i^{\pm}$ we get that $u_i = 0$.

□

Proposition 4 *Let $\alpha < \beta^{-1} - 1$ and $\delta \in \mathbb{R} \setminus I$. Then there exist $C = C(\alpha, \delta)$ such that*

$$\|u\|_{2,\alpha,\delta} \leq C\|f\|_{\alpha,\delta-2} \tag{4.12}$$

for every $u \in C^{2,\alpha}_\delta$ with $\triangle u = f$.

Proof: If the result was not true then we would be able to take a sequence $\{u_k\}$ with $\|u_k\|_{2,\alpha,\delta} = 1$, $\triangle u_k = f_k$ and $\|f_k\|_{\alpha,\delta-2} \to 0$. It follows from Lemma 13 that $\|u_k\|_{0,\delta} \geq 2\epsilon$ for some $\epsilon > 0$. Hence we can find x_k such that $r(x_k)^{-\delta}|u_k(x_k)| \geq \epsilon$. Consider the sequence $\tilde{u_k} = (u_k)_{L_k,\delta}$ where $L_k = r(x_k)$. Write $x_k = (r(x_k), \theta_k)$, then $|\tilde{u_k}(\tilde{x_k})| \geq \epsilon$ with $\tilde{x_k} = (1, \theta_k)$. On the other hand $\tilde{f_k} = \triangle \tilde{u_k} = L_k^{-\delta+2}(f_k)_{L_k} = (f_k)_{L_k,\gamma}$. (Where $\gamma = \delta - 2$). The key point is that $\|u\|_{2,\alpha,\delta} = \|u_{L,\delta}\|_{2,\alpha,\delta}$ and $\|f\|_{\alpha,\gamma} = \|f_{L,\gamma}\|_{\alpha,\gamma}$ for any $L > 0$ and f, g any functions. So that $\|\tilde{u_k}\|_{2,\alpha,\delta} = 1$ and $\|\tilde{f_k}\|_{\alpha,\delta-2} \to 0$. Let $K_n = \overline{B_n} \setminus B_{1/n}$ for n an integer ≥ 2. Arzela-Ascoli and the bound $\|\tilde{u_k}\|_{2,\alpha,\delta} = 1$ imply that we can take a subsequence $\tilde{u_k}^{(n)}$ which converges in $C^2(K_n)$ to some function u_n such that $\triangle u_n = 0$. The diagonal subsequence $\tilde{u_n}^{(n)}$ converges to a function u in $\mathbb{C}^2 \setminus \{0\}$ which is in C^2_δ and $\triangle u = 0$. Since $|\tilde{u_k}(\tilde{x_k})| \geq \epsilon$ we see that $u \neq 0$, but this contradicts Lemma 14

□

In practice we will only use the estimate 4.12 for functions u with support outside B_1. For these functions we can give another equivalent definition of the norms 4.8 and 4.9. Slightly abusing notation let us set

$$\|f\|_{\alpha,\gamma} = \|f\|_{0,\gamma} + [f]_{\alpha,\gamma-\alpha} \tag{4.13}$$

for functions f with $\mathrm{supp}(f) \subset B_1^c$, where

$$[f]_{\alpha,\gamma-\alpha} = \sup_{x,y} \min\{r(x), r(y)\}^{-\gamma+\alpha} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}$$

when $\gamma < 0$. (If $\gamma > 0$ we replace $\min\{r(x), r(y)\}$ by $\max\{r(x), r(y)\}$.)

Claim 7 *4.8 and 4.13 define equivalent norms*

Proof: We prove that 4.13 is bounded by a constant times 4.8. Consider the case of $\gamma < 0$. Take $x, y \in \mathbb{C}^2$ with $r(x) \leq r(y)$ such that

$$(1/2)[f]_{\alpha,\gamma-\alpha} \leq r(x)^{-\gamma+\alpha} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}.$$

Assume first that $r(y) \geq (5/4)r(x)$, say. Then $d(x, y) \geq d(y, 0) - d(x, 0) \geq (1/4)r(x)$, so that

$$(1/2)[f]_{\alpha,\gamma-\alpha} \leq r(x)^{-\gamma}|f(x)| + r(x)^{-\gamma}|f(y)|.$$

When $\gamma < 0$, $r(x)^{-\gamma}|f(y)| \leq r(y)^{-\gamma}|f(y)|$ and this last term is bounded by 4.8. When $r(y) \leq (5/4)r(x)$ we write $x = (r(x), \theta)$ and $y = (r(y), \psi)$. Let $\tilde{x} = (3/2, \theta)$ and $\tilde{y} = (\frac{3r(y)}{2r(x)}, \psi)$. Set $\lambda = (2/3)r(x)$ so that $D_\lambda(\tilde{x}) = x$ and $D_\lambda(\tilde{y}) = y$. Note that $\tilde{x}, \tilde{y} \in A_1$ $(r(\tilde{y}) \leq 15/8 < 2)$, so that 4.8 gives us a bound for

$$\lambda^{-\gamma} \frac{|f(x) - f(y)|}{d(\tilde{x}, \tilde{y})^\alpha} = (2/3)^{-\gamma} r(x)^{-\gamma+\alpha} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}.$$

From this we get that 4.13 is bounded by a constant times 4.8. The reverse inequality follows similarly.
□

Putting 4.13 and 4.11 together we get that the norm 4.9 we defined is equivalent (for functions u with $\mathrm{supp}(u) \subset B_1^c$) to the commonly used [21] [12].

Finally let us point out that $(-2,0) \cap I = \phi$ independently of $\overline{g}$. In fact, for this range one can give an alternative proof of 4.12 which does not use the spectrum of $\triangle_{\overline{g}}$.

Lemma 15 *Let $u \in C^2$ with supp(u) $\subset B_1^c$. Assume $\triangle u = f \in C^0_{\delta-2}$ for some $\delta \in (-2,0)$ and that $u \in C^0_\mu$ for some $\mu < 0$. Then*

$$\|u\|_{0,\delta} \leq c_\delta \|f\|_{0,\delta-2}$$

with $c_\delta = -(\delta+2)^{-1}\delta^{-1}$.

Proof: From 4.10 we have that $\triangle r^\delta = (\delta+2)\delta r^{\delta-2} = -Q_\delta r^{\delta-2}$ with $Q_\delta = -(\delta+2)\delta > 0$. On $U_R = B_R \setminus B_1$ consider the function $h = u - Ar^\delta - m_R$ where $m_R = \sup_{\partial B_R} u$ and $A = \|f\|_{0,\delta-2}/Q_\delta$. Then

$$\triangle h = f + \|f\|_{0,\delta-2} r^{\delta-2} \geq 0.$$

$h \leq 0$ on ∂B_1 since u has support outside B_1. By our choice of m_R, $h \leq 0$ on ∂B_R . The maximum principle implies that $h \leq 0$ in U_R, i.e. for every $x \in U_R$ we have that

$$u(x) \leq (\|f\|_{0,\delta-2}/Q_\delta) r(x)^\delta + m_R.$$

Since $u \in C^0_\mu$ for some $\mu < 0$ we get that $\lim_{R\to\infty} m_R = 0$. We let $R \to \infty$ and get the desired upper bound on u. The lower bound, and hence the lemma, follows by applying the upper bound to $-u$. $\square$

Let us explain how one can use the maximum principle in the context of metrics with cone singularities. Let $A = B_{R_2} \setminus B_{R_1} \subset \mathbb{C}^2 \setminus \{0\}$. Let $h \in C^2(A)$ be such that $\triangle h \geq 0$ and $h|_{\partial A} \leq 0$. We claim that $h \leq 0$ on A, if this was not the case we can find $p \in A$ such that $h(p) = \sup_A h = 2m > 0$. If $p \notin L$ this would contradict the usual maximum principle. Then $p \in L$. Let $\epsilon < \beta$ and δ be small enough such that $\delta |P_d|^{2\epsilon} \leq m$ on ∂A. Consider the function $H = h + \delta |P_d|^{2\epsilon}$. By our choices H has a local maximum at some point $q \in A$. Since $i\partial\overline{\partial}|P_d|^{2\epsilon} \geq 0$ we still have $\triangle H \geq 0$. Since $\epsilon < \beta$ and h is a C^1 function, we have that $q \notin L$, contradicting the usual maximum principle. In fact this argument can be adapted to other situations. For example the same holds if h is C^α, smooth outside L with $\triangle h \geq 0$ (one then needs to take $\epsilon < \alpha\beta$).

4.3 Main result

In this subsection we study the mapping properties (between weighted spaces) of the Laplacian of a metric ω with cone singularities along C asymptotic to ω_F. We fix ω given by Lemma 8.

We want to define our weighted Hölder spaces. The notation is the one of Subsection 3.1. Fix N large enough such that $C \cap B_N^c \subset U_{2R,\delta/2}$. Let χ be a smooth function equal to 1 on B_{N+1}^c which vanishes on B_N. For a function $u : \mathbb{C}^2 \to \mathbb{R}$ we write $u_\infty = \chi u \circ G$. We change notation and introduce a $'$ on the norms of the previous subsection. The space $C^{2,\alpha}_\delta$ (C^α_γ) is defined to be the set of functions u (f) such that the norm

$$\|u\|_{2,\alpha,\delta} = \|u\|_{C^{2,\alpha}(B_{N+1})} + \|u_\infty\|'_{2,\alpha,\delta} \tag{4.14}$$

$$\|f\|_{\alpha,\gamma} = \|f\|_{C^{\alpha}(B_{N+1})} + \|f_\infty\|'_{\alpha,\gamma} \tag{4.15}$$

is finite. The fact is that these are Banach spaces.

Write $\triangle$ for the Laplacian of ω. We apply the estimates of the previous two subsections to get the following

Corollary 1 *Let $\delta \notin I$ and $\alpha < \beta^{-1} - 1$. Then there exist a compact set K and a constant C such that for all $u \in C^{2,\alpha}_\delta$ with $\triangle u = f$ we have*

$$\|u\|_{2,\alpha,\delta} \leq C \left(\|u\|_{C^0(K)} + \|f\|_{\alpha,\delta-2} \right). \tag{4.16}$$

Proof: The key point is that if $v \in (C^{2,\alpha}_{\delta})'$ with support on B_L^c,

$$\|\triangle_{G^*g} v - \triangle_F v\|'_{\alpha,\delta-2} \leq c_L \|v\|'_{2,\alpha,\delta} \tag{4.17}$$

with $c_L \to 0$ as $L \to \infty$, where g is the metric corresponding to ω and $\triangle_F$ is the Laplacian of the flat metric. Since $G^*g = g_F$ in a region $U_{\delta',R'}$ and $|G^*g - g_F|_{g_F} = O(r^{\mu})$ for some $\mu < 0$ with derivatives on the complement of $U_{\delta',R'}$, 4.17 holds. The corollary then follows from 4.12 and the interior estimates.

□

Lemma 16 $\triangle : C^{2,\alpha}_{\delta} \to C^{\alpha}_{\delta-2}$ *has finite dimensional kernel for any* δ *and closed image when* $\delta \notin I$.

Proof: Let us start by proving the statement about the kernel. Assume first that $\delta \notin I$ and let $u_k \in C^{2,\alpha}_{\delta}$ with $\triangle u_k = 0$ and $\|u_k\|_{2,\alpha,\delta} = 1$. By Arzela-Ascoli we can take a subsequence which converges in $C^0(K)$ to some function. We apply the estimate 4.16 to conclude that the subsequence is Cauchy in $C^{2,\alpha}_{\delta}$ and hence $\ker(\triangle)$ is finite dimensional. In the case that $\delta \in I$ just take $\tilde{\delta} > \delta$, $\tilde{\delta} \notin I$ and note that $C^{2,\alpha}_{\delta} \subset C^{2,\alpha}_{\tilde{\delta}}$.

To prove that the image is closed let us write $C^{2,\alpha}_{\delta} = V \oplus \ker(\triangle)$ for some closed subspace V. We claim that there exist a constant C such that $\|u\|_{2,\alpha,\delta} \leq C\|f\|_{\alpha,\delta-2}$ for every $u \in V$. If this was not true then we would get a sequence such that $\|u_k\|_{2,\alpha,\delta} = 1$ and $\|f_k\|_{\alpha,\delta-2} \to 0$. It follows from Arzela-Ascoli and 4.16 that, after taking a subsequence, we can assume that u_k converges in $C^{2,\alpha}_{\delta}$ to some function u with $\triangle u = 0$. Since $u \in V$ then $u = 0$ and this contradicts $\|u_k\|_{2,\alpha,\delta} = 1$. Finally let $f_k = \triangle u_k$ with $f_k \to f$ in $C^{\alpha}_{\delta-2}$. We can assume that $u_k \in V$. The estimate we just proved implies that $\{u_k\}$ is Cauchy and converges to some $u \in C^{2,\alpha}_{\delta}$ with $\triangle u = f$.

□

Let $\mathcal{H}$ be the the completion of the space of compactly supported functions ϕ, smooth in the cone coordinates, under the Dirichlet norm $\int |\nabla \phi|^2$. (In a more precise notation we should write $\int_{\mathbb{C}^2} |\nabla^{\omega}\phi|\omega^2$). We recall the content of Subsection 3.4 in the form of the following

Lemma 17 *(Sobolev inequality.) There exists* C *such that*

$$\left(\int |\phi|^4\right)^{1/2} \leq C \int |\nabla \phi|^2 \tag{4.18}$$

for every $\phi \in \mathcal{H}$.

Proof: This follows since we can find a diffeomorphism of $\mathbb{C}^2 \setminus C$ under which ω is quasi-isometric to the euclidean metric. □

Let $f \in L^{4/3}$. It follows from 4.18 that $T_f(\phi) = \int f\phi$ defines a bounded functional on $\mathcal{H}$. A weak solution of $\triangle u = f$ is a function $u \in \mathcal{H}$ such that $-\int \langle \nabla u, \nabla \phi \rangle = \int f\phi$ for every $\phi \in \mathcal{H}$. It follows from Theorem 7 that if f is locally in C^{α} then u is locally in $C^{2,\alpha}$.

Lemma 18 *Let* $f \in C^{\alpha}_c$ *and* $u \in \mathcal{H}$ *be a weak solution of* $\triangle u = f$. *Then* $u \in C^{2,\alpha}_{\delta}$ *for any* $\delta > -2$

Proof: Take $\psi = \psi(t)$ to be a smooth non-decreasing function of one real variable with $\psi(t) = t$ when $t \geq 2$ and $\psi(t) = 1$ when $t \leq 1$. Define $\rho = \psi \circ r$ and let

$$\|u\|^2_{L^2_{\delta}} = \int |u|^2 \rho^{-2\delta}\rho^{-4}.$$

Since $u \in \mathcal{H}$ we get that $\int |u|^4$ is finite (in fact it is bounded by $\|f\|_{L^{4/3}}$). From Hölder's inequality we have that

$$\|u\|^2_{L^2_{\delta}} \leq \left(\int |u|^4\right)^{1/2} \left(\int \rho^{-4(\delta+2)}\right)^{1/2}.$$

If $\delta > -1$ we conclude that $\|u\|_{L^2_{\delta}}$ is finite.

In the interior Schauder estimates one can replace the C^0 norm in the r.h.s by the L^2 norm. Using the interior estimates in this form one gets that if u is locally in $C^{2,\alpha}$ and $\|u\|_{L^2_{\delta}}$ is finite, then $u \in C^{2,\alpha}_{\delta}$ and

$$\|u\|_{2,\alpha,\delta} \leq C\left(\|f\|_{\alpha,\delta-2} + \|u\|_{L^2_\delta}\right).$$

Hence $u \in C^0_\delta$ for any $\delta > -1$. One can then use Lemma 15 to show that in fact this is true for any $\delta > -2$.

□

Proposition 5 *$\triangle : C^{2,\alpha}_\delta \to C^\alpha_{\delta-2}$ is an isomorphism when $\delta \in (-2, 0)$ and is surjective when $\delta \in (0, 2) \setminus I$.*

Proof: The fact that $\triangle$ is injective when $\delta < 0$ follows from the maximum principle or by integration by parts. The key is to prove that the map is onto. By lemma 16 it is enough to prove that the image is dense. We know from lemma 18 that the space of C^α functions with compact support is contained in the image, one detail is that this space is not dense in $C^\alpha_{\delta-2}$. But this can be overcome as follows: Take $f \in C^\alpha_{\delta-2}$ and $\delta < \tilde{\delta} < 2$ with $[\delta, \tilde{\delta}] \cap I = \phi$. Let h_n be a sequence of smooth cut-off functions with $h_n = 1$ on B_n and $h_n = 0$ on B^c_{n+1}. The sequence of functions $f_n = h_n f \to f$ in $C^\alpha_{\tilde{\delta}-2}$ so that we can find $u \in C^{2,\alpha}_{\tilde{\delta}}$ with $\triangle u = f$. It follows from the proof of lemma 16 that we can take $u \in C^{2,\alpha}_{\delta'}$ for any $\delta' \in (\delta, \tilde{\delta}]$ and with $\|u\|_{2,\alpha,\delta'} \leq C\|f\|_{\alpha,\delta}$ with C independent of δ'. By taking the limit as $\delta' \to \delta$ we get that $u \in C^{2,\alpha}_\delta$

□

Remark 5 *Let $\omega_u = \omega + i\partial\overline{\partial}u$ be a Kähler metric on $\mathbb{C}^2 \setminus C$ with $u \in C^{2,\alpha}_\delta$ for some $\delta < 2$. Then Proposition 5 holds for the Laplacian of ω_u.*

Finally we mention some properties of these weighted spaces that will be useful to us later.

- Multiplication gives a bounded map
$$C^\alpha_{\gamma_1} \times C^\alpha_{\gamma_2} \to C^\alpha_{\gamma_1+\gamma_2}$$
- Let $\{f_j\}_{j=1}^\infty \subset C^\alpha_\gamma$ with $\|f_j\|_{\alpha,\gamma} \leq C$ for some constant C. Then, after taking a subsequence, we can assume that $f_j \to f$ uniformly in compact subsets to some function f. Moreover $f \in C^\alpha_\gamma$ and $\|f\|_{\alpha,\gamma} \leq C$.
- Let $f \in C^{\tilde{\alpha}}_{\tilde{\gamma}}$ and $\alpha < \tilde{\alpha}$, $\tilde{\gamma} < \gamma$. Then for every $\epsilon > 0$ we can find $h \in C^\infty_c$ such that $\|f - h\|_{\alpha,\gamma} < \epsilon$.

4.4 Application

In this subsection we use Proposition 5 and the implicit function theorem to prove the existence of a metric ω_0 with bounded Ricci curvature. In fact ω_0 is Ricci-flat outside a compact set. It is not hard to see that the metrics ω and ω_B constructed in Section 3 have unbounded Ricci curvature. One can easily adapt the proof of Proposition 6 to show that in the general setting of a compact Kähler manifold with a smooth divisor $D \subset X$ there are metrics with cone singularities along D and bounded Ricci curvature.

Proposition 6 *There exists $u_0 \in C^{2,\alpha}_\delta$ for some $\delta < 2$ such that $\omega_0 = \omega + i\partial\overline{\partial}u_0$ is a Kähler form on $\mathbb{C}^2 \setminus C$ with*
$$\omega_0^2 = e^{-f_0}|P|^{2\beta-2}\Omega \wedge \overline{\Omega}$$
and $f_0 \in C^\infty_c$.

Proof: Write
$$\omega^2 = e^{-f}|P|^{2\beta-2}\Omega \wedge \overline{\Omega}.$$
We claim that there exists $0 < \tilde{\alpha} < \beta^{-1} - 1$ and $\tilde{\gamma} < 0$ such that $f \in C^{\tilde{\alpha}}_{\tilde{\gamma}}$. The fact that $f \in C^{\tilde{\alpha}}$ on compacts subsets follows from the expression 4.5. Lemma 9 then proves the claim. (We can take any $\tilde{\gamma} > -2/c$.) Let $0 < \alpha < \tilde{\alpha}$ and $\tilde{\gamma} < \gamma < 0$ such that $\delta = \gamma + 2 \notin I$. Then there exist $\{h_j\}_{j=1}^\infty \subset C^\infty_c$ such that $\lim_{j\to\infty} \|f - h_j\|_{\alpha,\gamma} = 0$. We can assume that $\gamma \notin I$.

Consider the bounded map $\mathcal{F} : U \subset C^{2,\alpha}_\delta \to C^\alpha_{\delta-2}$ defined in a neighborhood of 0 and given by

$$\mathcal{F}(u) = \log \frac{(\omega + i\partial\overline{\partial}u)^2}{\omega^2}.$$

So that $\mathcal{F}(0) = 0$ and $D\mathcal{F}|_0 = \triangle$. By 5 and the implicit function theorem, we can solve $\mathcal{F}(u_0) = f - h_N$ for some $N >> 1$. We get that

$$(\omega + i\partial\overline{\partial}u_0)^2 = e^{f-h_N}\omega^2$$

and the proposition is proved with $f_0 = h_N$.

□

In Proposition 6 the function f_0 is smooth with respect to the complex coordinates. One can use the same proof, with the obvious modification, to get a Kähler metric $\tilde{\omega}_0$ such that $\tilde{\omega}_0^2 = e^{-\tilde{f}_0}|P|^{2\beta-2}\Omega \wedge \overline{\Omega}$, with $\tilde{f}_0$ a compactly supported function, smooth in the cone coordinates. What will be relevant for us in the next section is that $i\partial\overline{\partial}f_0$ is bounded with respect to ω_0; the metric $\tilde{\omega}_0$ would do the job as well.

5 A priori estimates for the Monge-Ampere equation

Let ω_0 be given by Proposition 6. Recall that

$$\omega_0^2 = e^{-f_0}|P|^{2\beta-2}\Omega \wedge \overline{\Omega},$$

with $f_0 \in C_c^\infty$. Fix $0 < \alpha < \beta^{-1} - 1$ and $-2 < \delta < 0$. The main result of this section is the following

Proposition 7 *There exists a constant C independent of $t \in [0,1]$ such that if $u_t \in C^{2,\alpha}_\delta$ solves*

$$(\omega_0 + i\partial\overline{\partial}u_t)^2 = e^{tf_0}\omega_0^2,$$

then $\|u_t\|_{2,\alpha,\delta} \leq C$.

In the next subsections we derive a priori estimates on different norms of $\|u_t\|$ which can be stated in the same form as Proposition 7. To avoid repetition we only state the estimate proved. We simplify notation and write $f = tf_0$ and $u = u_t$. We hope that this simplified notation doesn't cause any confusion at the end. The set up for this section is then a smooth function with compact support f and $u \in C^{2,\alpha}_\delta$, a solution of

$$(\omega_0 + i\partial\overline{\partial}u)^2 = e^f\omega_0^2. \tag{5.1}$$

We denote by ω_u the corresponding Kähler form $\omega_0 + i\partial\overline{\partial}u$.

5.1 C^0 estimate

The goal is to prove the following

Proposition 8 $\|u\|_0 \leq C$.

We follow [21] (pages 188-190). The technique is called Moser iteration. Note that $u \in C^0_\delta$ implies that $u \in L^p$ for p large and $\|u\|_0 = \lim_{p\to\infty}\|u\|_{L^p}$. The proof of Proposition 8 begins with

Lemma 19 *Let $p > 2$ with $p\delta + 2 < 0$. Write $\phi = u|u|^{p/2-1}$. Then we have*

$$\int_{\mathbb{C}^2} |\nabla\phi|^2\omega_0^2 \leq p\int_{\mathbb{C}^2} u|u|^{p-2}(1 - e^f)\omega_0^2. \tag{5.2}$$

Proof: This goes along the lines of Joyce's book [21]. We have to check that the relevant integration by parts arguments hold in our context of metrics with cone singularities. Define a 3-form η by

$$\eta = u|u|^{p-2}i\overline{\partial}u \wedge (\omega_0 + \omega_u),$$

where $\omega_u = \omega + i\partial\overline{\partial}u$. This form is C^1 on $\mathbb{C}^2\setminus C$. Fix $R, \epsilon > 0$ and consider the region $U = B_R \setminus \{|P| \leq \epsilon\}$. By Stokes' theorem $\int_U d\eta = \int_{\partial U}\eta$. Use the equation 5.1 to get

$$d\eta = (p-1)|u|^{p-2} i\partial u \wedge \overline{\partial} u \wedge (\omega_0 + \omega_u) + u|u|^{p-2}(e^f - 1)\omega_0^2.$$

For R fixed we let $\epsilon \to 0$. Write $C_\epsilon = \{|P| = \epsilon\} \cap B_R$. Note that $\lim_{\epsilon \to 0} \mathrm{Area}_{g_0}(C_\epsilon) = 0$ and that $|\eta|_{g_0}$ is bounded. We conclude that we can take $U = B_R$ and $\partial U = S_R$. Now note that $\mathrm{Vol}_{g_0}(S_R) \leq CR^3$ and $|\eta|_{g_0} \leq CR^{(p-1)\delta+\delta-1}$ on S_R. The choice $p\delta < -2$ gives $\lim_{R\to\infty} \int_{S_R} \eta = 0$ and we get $\int_{\mathbb{C}^2} d\eta = 0$. The lemma follows from $i\partial u \wedge \overline{\partial} u \wedge \omega_0 = |\nabla u|^2 \omega_0^2$, $|\nabla \phi|^2 = (p^2/4)|u|^{p-2}|\nabla u|^2$ and $i\partial u \wedge \overline{\partial} u \wedge \omega_u = F\omega_0^2$ with $F = |\nabla u|^2_{g_u}(\omega_u^2/\omega_0^2) \geq 0$.

□

Now we prove Proposition 8.
Proof: The Sobolev inequality for the metric ω_0 tells us

$$\left(\int_{\mathbb{C}^2} |\phi|^4 \omega_0^2\right)^{1/2} \leq C \int_{\mathbb{C}^2} |\nabla \phi|^2 \omega_0^2. \tag{5.3}$$

Apply this to $\phi = u|u|^{p/2-1}$ and use 5.2 to get

$$\|u\|^p_{L^{2p}} \leq Cp\|u\|^{p-1}_{L^{p-1}}. \tag{5.4}$$

The next step is to estimate $\|u\|_{L^{p_1}}$ for some $p_1 > 2$. In order to do this we fix some $p_0 > 2$ such that $p_0\delta + 2 < 0$. Use 5.2, 5.3 to get

$$\left(\int |u|^{2p_0}\omega_0^2\right)^{1/2} \leq p_0 \int |1 - e^f||u|^{p_0-1}\omega_0^2.$$

Let $r > 1$ be given by $r(p_0 - 1) = 2p_0$ and q by $r^{-1} + q^{-1} = 1$. Let ρ be a function ≥ 1 that agrees with r outside a compact set, as in the proof of Lemma 18. We replace $|1 - e^f| \leq C\rho^\gamma$. (With $\gamma = \delta - 2$). From the choices it follows that $\|\rho^\gamma\|_{L^q} \leq C$. Hölder's inequality then implies that $\|u\|_{L^{p_1}} \leq C$ with $p_1 = 2p_0$. Using the bound on $\|u\|_{L^{p_1}}$, 5.4 and an induction argument we get a uniform bound (independent of p) on $\|u\|_{L^p}$. Finally $\|u\|_{C^0} = \lim_{p\to\infty} \|u\|_{L^p} \leq C$. □

5.2 C^2 estimate

In this subsection we prove the following

Proposition 9 *$C^{-1}\omega_0 \leq \omega_u \leq C\omega_0$.*

To prove Proposition 9 we use the maximum principle. Our main tool is the Chern-Lu inequality (Lemma 20 below). In Yau's proof of the Calabi conjecture, the constant C in Proposition 9 depends on a lower bound on the bisectional curvature of a reference metric. In our case we don't know of any reference metric with bisectional curvature bounded from below and there might be obstructions to the existence of one. The use of Lemma 20 allows us to overcome this problem. Our methods in this subsection are highly inspired by Jeffres-Mazzeo-Rubinstein [20], although we use the Chern-Lu inequality in a slightly different way than in [20].

Lemma 20 *Let g and $\hat{g}$ be two Kähler metrics on X such that $Ric(g) \geq -Q_2\hat{g}$ and $Bisec(\hat{g}) \leq Q_1$ for some $Q_1, Q_2 > 0$. Set $\phi = tr_g(\hat{g})$. Then*

$$\triangle_g \log \phi \geq -Q\phi, \tag{5.5}$$

where $Q = Q_1 + Q_2$.

Before going to the proof we mention two points:

- Lemma 20 is a particular case of Proposition 7.1 in [20].
- There is a similar formula for $\triangle_{\hat{g}} \log \phi$ if we assume an upper bound on the Ricci curvature of $\hat{g}$ and a lower bound on the bisectional curvature of g. See Chapter 3 in [31].

Proof: Let $x \in X$ and $(z_1, \dots, z_n)$ be holomorphic coordinates around x. The metrics are then given by n by n Hermitian matrices $(g_{i\bar{j}})$ and $(\hat{g}_{i\bar{j}})$. At the point x we require that $(g_{i\bar{j}})$ is diagonal , $(\hat{g}_{i\bar{j}})$ is the identity and all the first derivatives of $(\hat{g}_{i\bar{j}})$ vanish. The function ϕ is then given by

$$\phi = \mathrm{tr}_\omega \hat{\omega} = \sum_{j,k} g^{j\bar{k}} \hat{g}_{j\bar{k}},$$

where $(g^{i\bar{j}})$ is the inverse transpose of $(g_{i\bar{j}})$. First we compute $\triangle_g \phi$. At the point x we have $\triangle_g \phi = \sum_p g^{p\bar{p}} \partial_{\bar{p}} \partial_p \phi$, where

$$\partial_{\bar{p}} \partial_p \phi = \sum_{j,k} (\partial_{\bar{p}} \partial_p g^{j\bar{k}}) \hat{g}_{j\bar{k}} + (\partial_{\bar{p}} \partial_p \hat{g}_{j\bar{k}}) g^{j\bar{k}} = \sum_j \partial_{\bar{p}} \partial_p g^{j\bar{j}} + (\partial_{\bar{p}} \partial_p \hat{g}_{j\bar{j}}) g^{j\bar{j}}.$$

We write $\triangle_\omega \phi = I + II$, with

$$I = \sum_{p,j} g^{p\bar{p}} g^{j\bar{j}} \hat{g}_{j\bar{j},p\bar{p}}, \quad II = \sum_{p,j} g^{p\bar{p}} \partial_{\bar{p}} \partial_p g^{j\bar{j}}.$$

Subindices after the comma indicate differentiation. Since the coordinates are adapted to $\hat{g}$ at x and the bisectional curvature of $\hat{g}$ is bounded from above by Q_1; we have that for every p and j, $-\hat{g}_{j\bar{j},p\bar{p}} \leq Q_1$. It follows that

$$I \geq -Q_1 \sum_{p,j} g^{p\bar{p}} g^{j\bar{j}} = -Q_1 \phi^2. \tag{5.6}$$

Since $(g_{i\bar{j}})$ is diagonal at x, we get

$$II = -\sum_{p,j} g^{p\bar{p}} (g^{j\bar{j}})^2 g_{j\bar{j},p\bar{p}}.$$

Denote by R the Riemann curvature tensor of g, given by

$$R_{i\bar{j}k\bar{l}} = -g_{i\bar{j},k\bar{l}} + \sum_{r,s} g^{r\bar{s}} g_{i\bar{s},k} g_{r\bar{j},\bar{l}}.$$

We conclude that

$$-g_{l\bar{q},p\bar{p}} = R_{l\bar{q}p\bar{p}} - \sum_{r,s} g^{r\bar{s}} g_{l\bar{s},p} g_{r\bar{q},\bar{p}}.$$

We obtain

$$II = \sum_{j,p} g^{p\bar{p}} (g^{j\bar{j}})^2 R_{j\bar{j}p\bar{p}} + P, \tag{5.7}$$

where

$$P = \sum_{p,j,r} (g^{j\bar{j}})^2 g^{r\bar{r}} g^{p\bar{p}} |g_{j\bar{r},p}|^2.$$

Note that $\mathrm{Ric}_{j\bar{j}} = \sum_p g^{p\bar{p}} R_{j\bar{j}p\bar{p}}$ is the Ricci curvature of g. Since $\mathrm{Ric}(g) \geq -Q_2 \hat{g}$, we bound first term in 5.7 by

$$\sum_{j,p} g^{p\bar{p}} (g^{j\bar{j}})^2 R_{j\bar{j}p\bar{p}} = \sum_j (g^{j\bar{j}})^2 \mathrm{Ric}_{j\bar{j}} \geq -Q_2 \phi^2. \tag{5.8}$$

Now we bring in the logarithm to get

$$\triangle_g \log \phi = \frac{\triangle_g \phi}{\phi} - \frac{|\nabla \phi|^2}{\phi^2}.$$

It follows from 5.6, 5.7 and 5.8 that to prove the lemma it is enough to show that $|\nabla \phi|^2 \leq \phi P$. At the point x we have

$$|\nabla\phi|^2 = \sum_p g^{p\overline{p}}|\partial_p\phi|^2, \quad \partial_p\phi = -\sum_j (g^{j\overline{j}})^2 g_{j\overline{j},p}.$$

So that

$$|\nabla\phi|^2 = \sum_{p,j,r} g^{p\overline{p}}(g^{j\overline{j}})^2 g_{j\overline{j},p}(g^{r\overline{r}})^2 g_{r\overline{r},\overline{p}} = \sum_{j,r}(g^{j\overline{j}})^2(g^{r\overline{r}})^2\left(\sum_p g^{p\overline{p}} g_{j\overline{j},p} g_{r\overline{r},\overline{p}}\right).$$

For each j and r fixed, the Cauchy-Schwarz inequality implies that

$$\sum_p g^{p\overline{p}} g_{j\overline{j},p} g_{r\overline{r},\overline{p}} \leq \left(\sum_p g^{p\overline{p}}|g_{j\overline{j},p}|^2\right)^{1/2}\left(\sum_p g^{p\overline{p}}|g_{r\overline{r},\overline{p}}|^2\right)^{1/2}.$$

We use the Cauchy-Schwarz inequality once again to obtain

$$|\nabla\phi|^2 \leq \left(\sum_j (g^{j\overline{j}})^2\left(\sum_p g^{p\overline{p}}|g_{j\overline{j},p}|^2\right)^{1/2}\right)^2 = \left(\sum_j (g^{j\overline{j}})^{1/2}\left(\sum_p (g^{j\overline{j}})^3 g^{p\overline{p}}|g_{j\overline{j},p}|^2\right)^{1/2}\right)^2$$

$$\leq \phi\left(\sum_{j,p}(g^{j\overline{j}})^3 g^{p\overline{p}}|g_{j\overline{j},p}|^2\right) \leq \phi\left(\sum_{j,r,p}(g^{j\overline{j}})^2 g^{r\overline{r}} g^{p\overline{p}}|g_{j\overline{r},p}|^2\right) = \phi P.$$

The proof of the lemma is now complete.

□

We are now ready to prove Proposition 9.

Proof: We set g and $\hat{g}$ to be the Kähler metrics corresponding to ω_u and ω_B, respectively. First we check that the hypothesis of Lemma 20 hold. The upper bound on the bisectional curvature of $\hat{g}$ is given by Lemma 10. Recall that $\omega_u^2 = e^{tf_0}\omega_0^2$, where $\mathrm{Ric}(\omega_0) = i\partial\overline{\partial}f_0$. It follows that $\mathrm{Ric}(\omega_u) = (1-t)\mathrm{Ric}(\omega_0)$. Since f_0 is smooth we clearly have $i\partial\overline{\partial}f_0 \geq -Q_2\omega_B$ for some $Q_2 > 0$. We conclude that the bound $\mathrm{Ric}(g) \geq -Q_2\hat{g}$ holds.

Write $\omega_u = \omega_B + i\partial\overline{\partial}v$. Note that u and v differ by a fixed function. Take the trace w.r.t. ω_u to get $2 = \phi + \triangle_g v$. Consider the function $H = \log\phi - Av$, with $A = Q+1$. We want to show that H is bounded above by a uniform constant. Since $H(y) \to \log 2$ as $y \to \infty$, we can assume that H attains its global maximum at $x \in \mathbb{C}^2$. If $x \notin C$, by Lemma 20 we have

$$0 \geq \triangle_g H(x) \geq -Q\phi - A\triangle_g v = \phi(x) - 2A.$$

Proposition 8 gives us a uniform bound on the C^0 norm of u and hence of v. We conclude that at the point x the function H is bounded from above by a uniform constant. Since x is a maximum point of H the bound holds in all of $\mathbb{C}^2$.

If $x \in C$ we can assume $H(x) \geq \log 2 + 3$ and take $R > 0$ so that $H|_{\partial B_R} \leq \log 2 + 1$. Fix some $0 < \epsilon < \beta$ and consider the function $\tilde{H} = H + (1/N)|P|^{2\epsilon}$, where $N > 0$ is big enough such that $(1/N)|P|^{2\epsilon} \leq 1$ on ∂B_R. By our choices $\max_{y\in\overline{B_R}}\tilde{H} = \tilde{H}(\tilde{x})$ with $\tilde{x} \notin \partial B_R$. Since $H \in C^\alpha$ and $\epsilon < \beta$, we have that $\tilde{x} \notin C$, hence

$$0 \geq \triangle_\omega\tilde{H}(\tilde{x}) = \triangle_\omega H + (1/N)\triangle_\omega|P|^{2\epsilon} \geq \triangle_\omega H(\tilde{x}) \geq \phi(\tilde{x}) - 2A$$

We used that $\triangle_\omega|P|^{2\epsilon} \geq 0$ since $i\partial\overline{\partial}|P|^{2\epsilon} \geq 0$. Note that $H(x) \leq \tilde{H}(x) \leq \tilde{H}(\tilde{x})$ to get the estimate.

We have proved that H is uniformly bounded from above. We use Proposition 8 once again to conclude that $\phi = \mathrm{tr}_g\hat{g} \leq \tilde{C}$. Therefore $\omega_B \leq \tilde{C}\omega_u$. Since the metrics ω_B and ω_0 are fixed there is a fixed constant Λ such that $\Lambda^{-1}\omega_0 \leq \omega_B$, hence $\omega_0 \leq \Lambda\tilde{C}\omega_u$. Finally we use the equation $\omega_u^2 = e^f\omega_0^2$ to get the desired bound $C^{-1}\omega_0 \leq \omega_u \leq C\omega_0$.

□

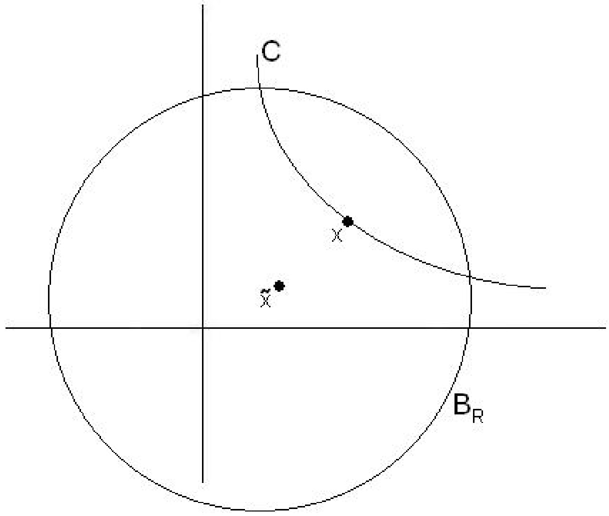

Figure 4: Pushing the maximum outside the curve.

5.3 $C^{2,\alpha}$ estimate

In this subsection we prove the following

Proposition 10 *$\|u\|_{2,\alpha} \leq C$.*

Proposition 10 is a direct consequence of Theorem 8 below. The technique we use is known as 'the blow-up argument'. When $\beta = 1$, the content of Theorem 8 is well-known and is a consequence of the so-called Evans-Krillov theory. The proof we sketch here uses different methods than the ones in that theory and also works in the case of $\beta = 1$. This subsection does not contain any original work and is included here for the sake of completeness. Our references are: [10], Sections 3, 4 and 5 in [11] and the proof of Theorem 2 in [8]. We mention that in Yau's work on the Calabi conjecture, Proposition 10 was proved by means of the maximum principle -a step known as Calabi's third order estimate-; while Theorem 8 is a local statement. Calabi's third order estimate was carried over to the context of metrics with cone singularities, under the assumption that $\beta < 1/2$, in Section 6 of [6].

We work on the space $\mathbb{C}_\beta \times \mathbb{C}^{n-1}$ with complex coordinates $z_1, \ldots, z_n$. If $p \in \mathbb{C}^n$ and $r > 0$, we denote by $B_r(x)$ the metric ball with center at x and radius r -in the distance induced by $g_{(\beta)}$. When $x = 0$ we write $B_r = B_r(0)$.

Theorem 8 *Let $\alpha < \alpha' < \beta^{-1} - 1$ and $\phi \in C^{2,\alpha'}(B_1)$ be such that*

$$K^{-1}\omega_{(\beta)} \leq i\partial\overline{\partial}\phi \leq K\omega_{(\beta)}$$

and

$$\det(\partial\overline{\partial}\phi) = |z_1|^{2\beta-2}e^f.$$

Then there exists a constant C, which depends only only on K and the C^α norm of f in B_1 such that

$$[i\partial\overline{\partial}\phi]_{\alpha, B_{1/4}} \leq C.$$

Proof: This is Theorem 1.7 in [10]. We only sketch the proof. There are three main ingredients:

- Fact 1: Let $\Lambda > 0$ and $\Lambda^{-1} < r < \Lambda$. There exists a constant C which depends only on Λ with the following property: If η is a real closed C^α form on B_r of type $(1,1)$, then there exists a real function $\phi \in C^{2,\alpha}(B_{r/2})$ such that $i\partial\overline{\partial}\phi = \eta$ on $B_{r/2}$ and $\|\phi\|_{2,\alpha,B_{r/4}} \leq C\|\eta\|_{\alpha,B_r}$.

- Fact 2: Let ω_∞ be a C^α Kähler metric on $\mathbb{C}^n$ such that $\omega_\infty^n = \omega_{(\beta)}^n$ and $K^{-1}\omega_{(\beta)} \leq \omega_\infty \leq K\omega_{(\beta)}$ for some $K > 0$. Then there exists a linear transformation L, which preserves $\{z_1 = 0\}$, such that $\omega_\infty = L^*\omega_{(\beta)}$.

- Fact 3: There is a $\delta > 0$ such that Theorem 8 holds when $K = 1 + \delta$.

Fact 1 is proved by analyzing the standard proof of the local $\partial\overline{\partial}$-lemma, see Section 4 in [11]. Fact 2 is proved by means of the maximum principle in Section 5 of [11]; in the case that ω_∞ is known to be a Riemannian cone see Proposition 25 in [8]. Fact 3 is proved in pages 228-229 of [8], it follows from the interior Schauder estimates and the fact that the Laplace operator is the linearization of the Monge-Ampere operator.

Now let us proceed with the proof of Theorem 8. We write $\omega = i\partial\overline{\partial}\phi$. For $q \in B_1$, denote by d_q the distance from q to the boundary of B_1. Define the Hölder radius as the supremal of the $h \in (0, d_q)$ such that $[\omega]_{\alpha, B_h(q)} \leq \delta_0 h^{-\alpha}$, where δ_0 is a small positive number which depends only on K. To prove the theorem it is enough to show that there exists a constant $c_0 > 0$, depending only on K and $\|f\|_{\alpha, B_1}$, such that $h_{\omega,q}/d_q \geq c_0$ for all $q \in B_1$. We argue by contradiction and assume that there are $\omega_k = i\partial\overline{\partial}\phi_k$, $q_k \in B_1$ such that:

$$K^{-1}\omega_{(\beta)} \leq \omega_k \leq K\omega_{(\beta)}, \quad \det(\omega_k) = |z_1|^{2\beta-2}e^{f_k}, \quad \|f_k\|_{\alpha', B_1} \leq 1 \tag{5.9}$$

$$\frac{h_{\omega_k, q_k}}{d_{q_k}} = \epsilon_k \to 0, \quad \frac{h_{\omega_k, q_k}}{d_{q_k}} \leq 2 \inf_{q \in B_1} \frac{h_{\omega_k, q}}{d_q}. \tag{5.10}$$

We rescale and define

$$\hat{z_1} = h_{\omega_k, q_k}^{-1/\beta}(z_1 - z_1(q_k)), \quad z_j = h_{\omega_k, q_k}^{-1}(z_j - z_j(q_k)) \quad \text{for } j = 2, \dots, n.$$

Write this change of coordinates as $\hat{x} = \tilde{\Gamma}_k(x)$. Let Γ_k be the inverse of $\tilde{\Gamma}_k$ and consider $\hat{\omega}_k = h_{\omega_k, q_k}^{-2}\Gamma_k^*\omega_k$, $\hat{f}_k = \Gamma_k^* f_k$. It follows that

$$\det(\hat{\omega}_k) = |\hat{z}_1|^{2\beta-2}e^{\hat{f}_k}, \quad h_{\hat{\omega}_k, 0} = 1. \tag{5.11}$$

Note that $\tilde{\Gamma}_k(B_{d_{q_k}}(q_k)) = B_{1/\epsilon_k}(0)$, so that the ω_k are defined on larger and larger balls. It is not hard to show, by means of the Arzela-Ascoli theorem and a diagonal argument, that 5.9, 5.10 together with Fact 3 and Fact 1; imply that $\hat{\omega}_k$ converges in C^α , up to a subsequence, to a Kähler metric $\hat{\omega}_\infty$ defined on $\mathbb{C}^n$ as the one in Fact 2. It follows that $\hat{\omega}_\infty$ has constant coefficients, so that $h_{\hat{\omega}_\infty, 0} = \infty$. Since $\hat{\omega}_k \to \hat{\omega}_\infty$ in C^α , for k large enough we get that $h_{\hat{\omega}_k, 0} \geq 2$ and this contradicts 5.11.

$\square$

5.4 Weighted estimates

This subsection completes the proof of Proposition 7. Our main results are Proposition 11 and Proposition 12. Our reference is Chapter 8 in Joyce's book [21]. Proposition 11 corresponds to Theorem 8.6.6 in [21] and Proposition 12 to Theorem 8.6.11 in [21].

The first result is a weighted version of Proposition 8. The proof uses Moser iteration and follows the same lines as the one of 8. We fix μ such that $\delta < \mu < 0$.

Proposition 11 $\|u\|_{C^0_\mu} \leq C$.

Let ψ be a smooth convex function of one real variable with $\psi(t) = 1$ for $t \leq 1$ and $\psi(t) = t$ for $t \geq 2$. Recall that r is the intrinsic distance in the flat metric to 0 and define $\rho = \psi \circ r$. In order to prove Proposition 11 we introduce the norm

$$\|u\|^p_{L^p_\mu} = \int_{\mathbb{C}^2} |u|^p \rho^{-p\mu}\rho^{-4}\omega_0^2.$$

Because $u \in C^0_\delta$ and $\delta < \mu$ we have that $u \in C^0_\mu$, $u \in L^p_\mu$ for all $p \geq 1$ and $\|u\|_{C^0_\mu} = \lim_{p\to\infty} \|u\|_{L^p_\mu}$.

Lemma 21 *For $p \geq 2$, $p\mu \leq -2$ we have*

$$\|u\|^p_{L^{2p}_\mu} \leq Cp\left(\|u\|^{p-1}_{L^{p-1}_\mu} + \|u\|^p_{L^p_\mu}\right). \tag{5.12}$$

This lemma corresponds to Proposition 8.6.8 in [21]. 5.12 is a weighted version of inequality 5.4. To prove the lemma we have to use the Sobolev inequality for the metric ω_0 together with an integration by parts argument. We refer to pages 190-192 in [21]. The only work we have to do is to check that the relevant integration by parts hold in our context of metric with cone singularities -as we did in the proof of Proposition 8-; this is straightforward and we omit the details. To prove Proposition 11 we note that if $p_0 = (-4/\mu)$, then $\|u\|_{L^{p_0}_\mu} = \|u\|_{L^{p_0}}$ and we already have a bound on this quantity. Finally, an induction argument using 5.12 gives the desired bound on $\|u\|_{C^0_\mu}$.

We move on to state our second result, Proposition 12. The proof uses the linear theory we developed and follows the lines of pages 193-195 in [21]. We pause for a moment to touch on a technical point: There is a mistake in the definition of the weighted C^α semi-norm given by formula 8.6, page 179 of [21]. The problem is that the semi-norm only compares points which are at distance less than the injectivity radius. This forbids the use of the interior Schauder estimates and scaling arguments needed to establish the linear theory, see the proof of Lemma 13. The arguments in pages 193-195 of [21] deal with this wrong semi-norm; it is not hard to adapt the arguments to prove what we need.

Proposition 12 $\|u\|_{2,\alpha,\delta} \leq C$

Proof: Write $\omega_u^2 = e^f\omega_0^2$ as $i\partial\overline{\partial}u \wedge (2\omega_0 + i\partial\overline{\partial}u) = (e^f - 1)\omega_0^2$. We get

$$\triangle_0 u = (e^f - 1) + \psi, \tag{5.13}$$

with $\psi = u^2_{i\bar{j}}$. We could also have written

$$\triangle u = H(e^f - 1), \tag{5.14}$$

where $\triangle$ is the Laplace operator of the metric $\omega_{u/2} = \omega_0 + i\partial\overline{\partial}(u/2)$ and $H = \omega^2_{u/2}/\omega_0^2$. Since $\omega_{u/2} = (1/2)\omega_0 + (1/2)\omega_u \geq (1/2)\omega_0$ and we have a bound on the $C^{2,\alpha}$ norm of u, we conclude that

$$\|u\|_{C^{2,\alpha}(B_1(x))} \leq C\left(\|\triangle u\|_{C^{2,\alpha}(B_2(x))} + \|u\|_{C^0(B_2(x))}\right), \tag{5.15}$$

with a constant C independent of x. We multiply 5.15 by $\rho(x)^{-\mu}$ to get

$$\|u_{i\bar{j}}\|_{0,\mu} \leq C, \quad \rho(x)^{-\mu}\frac{|u_{i\bar{j}}(x) - u_{i\bar{j}}(y)|}{d(x,y)^\alpha} \leq C \text{ whenever } d(x,y) < 1. \tag{5.16}$$

Take $\mu < \tilde{\mu} < 0$, $\tilde{\mu} = \mu + \alpha$ such that $2\tilde{\mu} < -2$. At this point we impose some restrictions on the choices of δ, μ, $\tilde{\mu}$. We start with $-2 < \delta < -1 - \alpha$, then we take $\delta < \mu < -1 - \alpha$ and $\tilde{\mu} < -1$. We claim that 5.16 implies that $\|u_{i\bar{j}}\|_{\alpha,\tilde{\mu}} \leq C$. In fact one only needs to consider the case of $d(x,y) \geq 1$, let's say that $\rho(x) \leq \rho(y)$ and estimate

$$\rho(x)^{-\tilde{\mu}+\alpha}\frac{|u_{i\bar{j}}(x) - u_{i\bar{j}}(y)|}{d(x,y)^\alpha} \leq \rho(x)^{-\mu}(|u_{i\bar{j}}(x) + |u_{i\bar{j}}(y)|) \leq 2C.$$

We use 5.13 and Proposition 5 to conclude that $\|u\|_{2,\alpha,2+2\tilde{\mu}} \leq C$. Then $\|u_{i\bar{j}}\|_{\alpha,2\tilde{\mu}} \leq C$, so that $\|\psi\|_{\alpha,4\tilde{\mu}} \leq$. Since $4\tilde{\mu} < -4 < \delta - 2$, we can use 5.13 and Proposition 5 again to obtain $\|u\|_{2,\alpha,\delta} \leq C$.

□

5.5 Proof of THEOREM 1

We are ready to give the proof of our main result, THEOREM 1.

Proof: Let ω_0 be the metric given by Proposition 6. Take any $0 < \alpha < \beta^{-1} - 1$, $\delta \in (-2, 0)$ and consider the set

$$T = \{t \in [0,1] : \exists u_t \in C^{2,\alpha}_\delta \quad \text{solving} \quad (\omega_0 + i\partial\overline{\partial}u_t)^2 = e^{tf_0}\omega_0^2\}. \tag{5.17}$$

Clearly $0 \in T$, with $u_0 = 0$. Note that if $t \in T$, then $\omega_t = \omega_0 + i\partial\overline{\partial}u_t$ is a Kähler metric with cone angle $2\pi\beta$ along C. The positivity follows from the equation, the decay of $i\partial\overline{\partial}u_t$ and the connectedness of $\mathbb{C}^2 \setminus C$. Proposition 5 and remark 5, together with the implicit function theorem, imply that the set T is open. Proposition 7 gives us that the set T is closed. We set $\omega_{RF} = \omega_1$. It is easy to check that ω_{RF} has the desired properties. It follows from this proof that we can improve the statement on the asymptotic behavior and we can say that, outside a compact set, $\omega_{RF} - H^*\omega_F \in C^{\alpha}_{\gamma}$ for any $\gamma > \max\{-2/c, -4\}$. $\square$

6 Conjectural picture

6.1 Convergence theory

We review well-known material, for which our reference is Anderson's survey [1]. The energy of a Riemannian manifold is the quantity given by

$$E = \frac{1}{8\pi^2}\int |Rm|^2.$$

It plays a significant role in the study of Einstein metrics on 4-manifolds. If (X^4, g) is closed and Einstein, then E is equal to the Euler characteristic of X; after Chern-Weil. A consequence of this conservation law is that solutions to Einstein's equations can degenerate, in the non-collapsing situation, only by developing orbifold singularities at a finite number of points. The blow-up limits of the solutions are Ricci-flat manifolds asymptotic to a quotient of $\mathbb{R}^4$ by a finite subgroup of $SO(4)$, i.e. Ricci-flat ALE spaces. We put these results together, in the form of the following

Theorem 9 *[1]. Let (X, g_i) be a sequence of Einstein metrics on a smooth four manifold X with Ricci $= \pm 1$ or 0. Assume that the diameter is uniformly bounded from above and volume uniformly bounded from below. Then there exist $\{j\} \subset \{i\}$ a subsequence and a compact Einstein 4-orbifold (X_∞, g_∞) with a finite singular set $S = \{x_1, \dots, x_k\} \subset X_\infty$ (possibly empty) such that*

- *$(X, g_j) \to (X_\infty, g_\infty)$ in the Gromov-Hausdorff sense.*
- *For any $x_a \in S$ there is a sequence $x_{a,j} \in X$ with $\lim_{j\to\infty} x_{a,j} = x_a$ such that if we set $r_j = |Rm_{g_j}|(x_{a,j})$; then $r_j \to \infty$ and*

$$(X, r_j g_j, x_{a,j}) \to (M_a, h_a, x_{a,\infty})$$

  in the pointed Gromov-Hausdorff sense; where (M_a, h_a) is an ALE manifold, possibly with orbifold singularities.
- *Let E be the energy of (X, g_j), which by Chern-Weil is independent of j. Then we have that*

$$E \geq E(g_\infty) + \sum_{a=1}^{k} E(h_a). \tag{6.1}$$

We make some remarks on Theorem 9: The convergence in the first two items can be strengthened to convergence of tensors. The hyperkähler ALE spaces are well understood and classified due to Kronheimer's work [22], [23]. It is possible to get equality in 6.1 if one takes into account 'bubble tree' phenomena.

Consider now the case of metrics with cone singularities. Let X be a closed complex surface with a Kähler-Einstein metric with cone angle $2\pi\beta$ along a smooth complex curve C. Then, by Atiyah-LeBrun [2] and Song-Wang [30], we know that

$$E = \chi(X) + (\beta - 1)\chi(C). \tag{6.2}$$

This gives us some evidence that there should be a parallel to Theorem 9 in this context. The degeneration of the curves is a new feature of the theory. In order to fix some ideas let $C_\epsilon \subset X$ be a smooth complex curve for each $\epsilon > 0$, with $C_\epsilon \to C_0$ as $\epsilon \to 0$, where C_0 is a curve in X which might be singular.

Fix $0 < \beta < 1$ and assume that there are Kähler-Einstein metrics g_ϵ with cone angle $2\pi\beta$ along C_ϵ, for simplicity also assume that the Ricci curvature is a fixed positive constant. First we focus on the case where the curves C_ϵ develop an isolated singularity. Let $p \in C_0$ and (u, v) complex coordinates centered at p such that $C_0 = \{P_d + (\text{h.o.t.}) = 0\}$, where P_d is a homogeneous degree $d \geq 2$ polynomial and h.o.t. means higher order terms. For the sake of definiteness suppose that in these coordinates $C_\epsilon = \{P_d + (\text{h.o.t.}) = \epsilon Q\}$, where Q is a polynomial such that $Q(0) = 1$. Rescale the coordinates and define $u = \epsilon^{1/d} z$, $v = \epsilon^{1/d} w$. Let $C = \{P_d = 1\}$, a smooth complex curve in $\mathbb{C}^2$. In the (z, w) coordinates we have that $C_\epsilon \to C$. Let g_{RF} be the Ricci-flat metric with cone angle $2\pi\beta$ along C given by THEOREM 1. We would expect that (under favorable conditions) after re-scaling g_ϵ around small balls centered at p we will get the metric g_{RF} in the limit. We give a detailed example of this situation in Subsection 6.3. It would be interesting to extend the previous discussion to the case of convergence with multiplicity. In this case we expect to find Ricci-flat metrics with cone singularities along a smooth complex curve whose asymptotic lines are not necessarily different. We say more on this in Subsection 6.4. To include 'bubble tree' phenomena into the discussion, one should consider the case of a general curve, not necessarily smooth. To take account of degenerations which involve the curves and the ambient surface at the same time one should replace $\mathbb{C}^2$ by a complex surface Z. The general problem would then be to study asymptotically conical Ricci-flat Kahler metrics on a complex surface Z with cone singularities along a complex curve $C \subset Z$.

To finish this subsection we note that there are compactness results for Kähler-Einstein metrics with cone singularities. For example, the ones in Chen-Donaldson-Sun's work [8]. However the main theme in [8] is to endow the limit with an algebraic structure. It should be possible to say much more on the differential geometric strucure of the limits in the case when the curves degenerate to a singular one or converge to C_0 with multiplicity.

6.2 Energy of the metrics

Let (M, h) be a smooth ALE manifold asymptotic to $\mathbb{R}^4/\Gamma$, where Γ is a finite subgroup of $SO(4)$. It is known (see [2]) that the energy of (M, h) is finite and is given by

$$E = \chi(M) - \frac{1}{|\Gamma|}. \tag{6.3}$$

Now let g_{RF} be the metric in THEOREM 1. We have a formula for the energy of g_{RF}, which is a mixture of 6.2 and 6.3:

Proposition 13

$$E = 1 + (\beta - 1)\chi(C) - \frac{Vol(\overline{g})}{2\pi^2}. \tag{6.4}$$

Recall that $\overline{g}$ is the corresponding singular metric on the 3-sphere and we know that $\text{Vol}(\overline{g}) = (\pi^2/2)c^2$ (see Remark 2). The Euler characteristic of C is $\chi(C) = 2 - 2g - d$; where $g = (d-1)(d-2)/2$, by the degree-genus formula. Putting these facts together we obtain a formula for E which only involves d and β. In the case that $d = 2$ we can prove 6.4 by direct computation, using the Gibbons-Hawking description of the metric. Proposition 13 follows immediately if we establish the following two items:

- Let (X, g) be a compact Kähler-Einstein surface with boundary $\partial X = Y$ and cone singularities of angle $2\pi\beta$ along a smooth complex curve $C \subset X$. Let II be the second fundamental form of Y in X and $\hat{\mathcal{R}}$ the restriction of the ambient curvature operator to Y, thought as a symmetric two tensor by means of the three dimensional Hodge operator. Then the energy of g is given by

$$E = \chi(X) + (\beta - 1)\chi(C) - \frac{1}{2\pi^2}\int_Y \left(det(II) + \langle II, \hat{\mathcal{R}}\rangle\right). \tag{6.5}$$

- Let X be a large ball in $(\mathbb{C}^2, g_{RF})$ of radius R. In the limit when $R \to \infty$ we can replace the boundary integral in 6.5 with $\text{Vol}(\overline{g})$.

Consider the flat metric g_F on $\mathbb{C}^2$ and let $Y = S_R$ be the set of points at distance R from the appex. Since the ambient curvature vanishes, the integral appearing in formula 6.5 reduces to $\int_{S_R} \det(II)$. Around each point y of $S_R \setminus L$ there is a neighborhood in $\mathbb{C}^2 \setminus L$ which maps isometrically into $\mathbb{R}^4$ with the Euclidean metric and identifies S_R with the standard round three-sphere of radius R. It follows that the integral is independent of R and $\int_{S_R} \det(II) = \mathrm{Vol}(\overline{g})$. Therefore, in order to establish the second item, what one has to prove is that the convergence of g_{RF} to g_F as $R \to \infty$ is strong and fast enough so that one can replace the integrals. We don't say anymore on this item and move on to discuss the former. From now on we take X to be a large ball in $(\mathbb{C}^2, g_{RF})$.

There are two main steps in the proof of 6.5. First we show that our Ricci-flat metric has cone singularities in a stronger sense; at points of the curve g_{RF} approaches the model $g_{(\beta)}$ with derivatives. In particular this implies that the energy and the integral in formula 6.5 are finite. The second step is to fit 6.5 into a more general setting of connections on the tangent bundle of $X \setminus C$, viewed as a complex rank 2 vector bundle. We write 6.5 as a Chern-Weil type formula. We prove the independence of the formula for an appropriate class of connections, which contains the Levi-Civita connection of g_{RF}. Finally, we construct a model connection in our class for which the relevant Chern-Weil integral can be computed explicitly. We follow closely [17], the approach is similar to the ones in Atiyah-LeBrun [2] and Kronheimer-Mrowka [24].

We use that g_{RF} is Ricci-flat to improve its regularity. We appeal to Theorem 2 in [20]. Let $p \in C$ and z_1, z_2 holomorphic coordinates around p in which $C = \{z_1 = 0\}$. Write $z_1 = \rho^{1/\beta} e^{i\theta}$. In small neighborhood of p the metric g_{RF} has a potential ϕ which has a polyhomogeneous expansion

$$\phi \sim \sum_{j,k} \sum_{l=0}^{N_{jk}} a_{jkl}(\theta, z_2) \rho^{j/\beta + 2k} (\log r)^l .$$

The functions a_{jkl} are smooth and there are no terms of the form $r^a (\log r)^l$ with $l > 0$ and $a \leq 2$. In particular it follows that g_{RF} is locally in C^α for $\alpha = \beta^{-1} - 1$. Denote the normal bundle of the curve by $\nu_C = TX/TC$. The metric g_{RF} induces a Hermitian metric on $TX|_C$, equivalently it gives us the following data:

- A Kähler metric g_C on the curve. Indeed this is simply the restriction of g_{RF} to TC.
- A smooth splitting $TX = TC \oplus \nu_C$ along the curve. In other words, there is a notion of a normal direction to the curve. See Lemma 24.
- A Hermitian metric on ν_C. See Lemma 23.

Let g be a C^α metric with cone angle $2\pi\beta$ along C and $\alpha = \beta^{-1} - 1$. Let $\Pi : P \to X$ denote the $\mathbb{CP}^1$ bundle given by the projectivisation of TX. Let $U = \Pi^{-1}(X \setminus C)$. Since we are assuming that the metric is smooth on the complement of the curve we have a smooth bundle map $\perp : U \to U$, given by taking the orthogonal complement. The Hölder condition then gives us the following

Lemma 22 *The map $\perp$ has a continuous extension $\perp : P \to P$.*

Proof: Let $p \in C$. Take z_1, z_2 to be holomorphic coordinates centered at p such that $C = \{z_1 = 0\}$. The metric g is represented by an Hermitian matrix

$$g\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial \overline{z}_1}\right) = a|z_1|^{2\beta-2}, \quad g\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial \overline{z}_2}\right) = b|z_1|^{\beta-1}, \quad g\left(\frac{\partial}{\partial z_2}, \frac{\partial}{\partial \overline{z}_2}\right) = c.$$

We are assuming that the functions a, b, c are C^α and $b = 0$ when $z_1 = 0$. We can scale each of the coordinate functions and suppose that $a(0) = c(0) = 1$. The condition on the Hölder exponent and the fact that b vanishes when $z_1 = 0$ gives us that the limit

$$\lim_{q \to C} g_q \left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial \overline{z}_2}\right)$$

is finite. The proof of the lemma is now an easy computation, for simplicity we show the case of the directions tangent to the curve

$$\perp \left(\langle \frac{\partial}{\partial z_2} \rangle\right) = \langle \frac{\partial}{\partial z_1} - c^{-1} g\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial \overline{z}_2}\right) \frac{\partial}{\partial z_2} \rangle .$$

□

We use the notation in the proof of Lemma 24. In the coordinate chart z_1, z_2, the line bundle ν_C is trivialized by the section $\left[\frac{\partial}{\partial z_1}\right]$. We set

$$\left|\left[\frac{\partial}{\partial z_1}\right]\right|_h^2 = a^{1/\beta}. \tag{6.6}$$

Lemma 23 *6.6 defines a C^α Hermitian metric on ν_C.*

Proof: Let $\tilde{z}_1, \tilde{z}_2$ be another coordinate system such that $C = \{\tilde{z}_1 = 0\}$. Then $z_1 = f\tilde{z}_1$ for some holomorphic non-vanishing f. Then

$$\left[\frac{\partial}{\partial \tilde{z}_1}\right] = f\left[\frac{\partial}{\partial z_1}\right].$$

Write

$$g\left(\frac{\partial}{\partial \tilde{z}_1}, \frac{\partial}{\partial \overline{\tilde{z}}_1}\right) = \tilde{a}|\tilde{z}_1|^{2\beta-2}.$$

At the curve we have that

$$a|z_1|^{2\beta-2}|dz_1|^2 = a|f|^{2\beta-2}|\tilde{z}_1|^{2\beta-2}|f|^2|d\tilde{z}_1|^2 = \tilde{a}|\tilde{z}_1|^{2\beta-2}|d\tilde{z}_1|^2.$$

Then $\tilde{a} = |f|^{2\beta}a$ and the lemma is proved. □

In the case that g is our Ricci-flat metric, lemmas 24 and 23, indeed provide us with a smooth splitting $TX = TC \oplus \nu_C$ and a smooth Hermitian metric h. We use this data, together with the induced metric g_C, to define around points of the curve what we call adapted coordinates. This concept is parallel to the notion of normal coordinates for smooth Kähler metrics. From now on we assume that $1/2 < \beta < 1$, the discussion when $0 < \beta \leq 1/2$ is easier.

Lemma 24 *Let $p \in C$. Then there exists holomorphic coordinates z_1, z_2 around p such that*

$$|g - g_{(\beta)}| = O(r^{1/\beta}), \quad |\nabla g| = O(r^{1/\beta-1}), \quad |\nabla^2 g| = O(\rho^{1/\beta-2}), \tag{6.7}$$

where $r^2 = |z_1|^{2\beta} + |z_2|^2$ and $\rho = |z_1|^\beta$.

It is a consequence of Lemma 24 that $|\mathrm{Rm}(g)| = O(\rho^{1/\beta-2})$. Since $\beta < 1$ the energy of the metric is finite, by comparison with the integral $\int_0^1 \rho^{2/\beta-4}\rho d\rho$. Note that h gives rise to a connection ∇^v on ν_C. Write s for the local section $\left[\frac{\partial}{\partial z_1}\right]$. In adapted coordinates we have that $|s|_h(p) = 1$, $\nabla^v s(p) = 0$ and z_2 is a standard normal coordinate for g_C at p. We omit the proof of 24.

Now we bring in the Chern-Weil formalism. Let ∇ be a connection on the rank 2 complex vector bundle $T(X \setminus C)$. Let z_1, z_2 be adapted coordinates. Consider the locally defined coordinate $\xi = z_1^\beta$. Write Γ for the Christoffel symbols of ∇ in the coordinates ξ, z_2. Write F for its curvature. We say that ∇ is an adapted connection if for every point $p \in C$ and (z_1, z_2) adapted coordinates centered at p we have that $|\Gamma| = O(r^{1/\beta-1})$ and $|F| = O(\rho^{1/\beta-2})$. The Levi-Civita connection of g is an adapted connection. In our case, since X is a ball in $\mathbb{C}^2$, we have a natural trivialization of the bundle $T(X \setminus C)$. An adapted connection ∇ is represented by a matrix of 1-forms A. We define

$$I(\nabla) = \frac{1}{8\pi^2}\int_X \mathrm{tr}(F_\nabla \wedge F_\nabla) - \frac{1}{8\pi^2}\int_Y dA \wedge A + \frac{2}{3}A \wedge A \wedge A. \tag{6.8}$$

The second term in the r.h.s. of 6.8 can be recognized as a Chern-Simons invariant. It follows from the definition of adapted connection that the integrals are well defined. The standard arguments of Chern-Weil theory apply to give that, in fact, $I(\nabla)$ is independent of the choice of adapted connection. In the case that ∇ is the Levi-Civita connection of a Kähler metric g, it is a well-known fact that $\mathrm{tr}(F_\nabla \wedge F_\nabla) = |\mathrm{Rm}|^2 - |\mathrm{Ric}|^2$. It should be the case that the boundary integrals in formulas 6.5 and 6.8 agree. Therefore, to prove 6.5, it is enough to show that $I(\nabla) = \chi(X) + (\beta - 1)\chi(C)$ for some adapted connection. The construction of such a connection takes place in a tubular neighborhood N of the curve. For each $p \in C$, let V_p be the affine complex line going through p which is orthogonal with respect to

g to the curve. If N is sufficiently small $V_p \cap N$ is biholomorphic to the unit disc $\triangle \subset \mathbb{C}$. We have a map $j : \triangle \to V_p \cap N$, well defined up to rotations. The derivative dj gives us an isomorphism between $\mathbb{C}$ and the fiber of ν_C at p, ν_p. The composite $j \circ (dj)^{-1}$ is well defined. The collection of these maps as p varies over C give rise to a diffeomorphism Ψ from a neighborhood of the zero section of ν_C to N. The map Ψ is holomorphic when we restrict it to the fibers ν_p. Let V be the image under the derivative of Ψ of the tangent spaces to the fibers of ν. Then V is a distribution of complex lines in N. The connection ∇^v induces a distribution $\mathcal{H}$ of horizontal subspaces in ν_D. Since Ψ is not holomorphic, the image under the derivative of $\mathcal{H}$ don't need to be complex subspaces of $\mathbb{C}^2$. However, if N is sufficiently small, this image will lie in a tubular neighborhood of $\mathbb{CP}^1 \subset Gr(2,4)$. Then we can project to get a distribution H of complex lines in N which agrees with TC over C. Over N we have the decomposition $T\mathbb{C}^2 = H \oplus V$. Let α be the 1-form with kernel H and which agrees, after pulled back with Ψ, with the canonical 1-form on the fibers of ν_C. Let P be an endomorphism of $T\mathbb{C}^2$ which is zero outside N and equal to the projection on the V component in a smaller neighborhood of C. We use the splitting $T\mathbb{C}^2 = H \oplus V$, the connection ∇^v and the Levi-Civita connection of g_C to get a smooth connection over N, we can extend it to a smooth connection ∇_0 on TX. Finally we define

$$\nabla = \nabla_0 + (\beta - 1)\alpha \otimes P.$$

It requires some technical work to show that ∇ is an adapted connection and we omit the details. It is easy to check that $I(\nabla) = \chi(X) + (\beta - 1)\chi(C)$. One has to use that $I(\nabla_0) = \chi(X)$ and that $d\alpha$ is, up to a constant factor, a representative of the Poincare dual of C.

6.3 Cubics in $\mathbb{CP}^2$

We illustrate our previous general speculations on convergence theory with an example. In $\mathbb{CP}^2$ with homogeneous coordinates $[x_0, x_1, x_2]$ consider the family of elliptic curves

$$C_\epsilon = \{x_0 x_1 x_2 - \epsilon(x_0^3 + x_1^3 + x_2^3) = 0\}.$$

These curves are smooth when $\epsilon > 0$ and C_0 is the union of three lines. Fix $0 < \beta < 1$ as before. It follows from the third bullet of Theorem 2 in the Introduction, that for each $\epsilon \neq 0$ there exists a Kähler metric g_ϵ on $\mathbb{CP}^2 \setminus C_\epsilon$ with cone angle $2\pi\beta$ along C_ϵ and constant positive Ricci curvature on the complement of the curve, let's say $\mathrm{Ric}(g_\epsilon) = g_\epsilon$. Take a decreasing sequence of positive numbers $\epsilon_j \to 0$. For different values of the parameter ϵ the curves C_ϵ are different complex tori. The metrics g_{ϵ_j} are pairwise non-isometric. Denote by d_ϵ the distance induced by g_ϵ. It follows from Gromov's compactness theorem that there exist a metric space (X, d) such that $(\mathbb{CP}^2, d_{\epsilon_j}) \to (X, d)$ in the Gromov-Hausdorff sense, after taking a subsequence if necessary. In fact, there is a natural candidate for (X, d). The S^1 action $e^{i\theta}(x_0, x_1, x_2) = (e^{i\theta}x_0, e^{i\theta}x_1, e^{i\theta}x_2)$ preserves the metric of $(\mathbb{C}_\beta)^3$. Taking an appropriate Kähler quotient we get a Kähler metric g_0 on $\mathbb{CP}^2$ with cone angle $2\pi\beta$ along C_0 and $\mathrm{Ric}(g_0) = g_0$ on the complement of C_0. When $\beta = 1/k$ the metric g_0 is (up to a constant factor) the push forward of the Fubiny-Study metric under the map $[x_0, x_1, x_2] \to [x_0^k, x_1^k, x_2^k]$. The metric g_0 induces a distance d_0 and our candidate for (X, d) is $(\mathbb{CP}^2, d_0)$.

Formula 6.2 tells us that $E(g_\epsilon) = 3$. On the other hand the energy of the metric g_0 can be computed directly (in the case when $\beta = 1/k$ it is $1/k^2$ times the energy of the Fubini-Study metric) and is given by $E(g_0) = 3\beta^2$. If our conjecture that $(\mathbb{CP}^2, d_{\epsilon_j}) \to (\mathbb{CP}^2, d_0)$ is true, then we are losing an amount of energy equal to

$$E(g_\epsilon) - E(g_0) = 3 - 3\beta^2 = 3(1 - \beta^2). \tag{6.9}$$

Let p denote any of the points $[1,0,0], [0,1,0]$ or $[0,0,1]$ and write $\lambda_j = |\mathrm{Rm}(g_{\epsilon_j})|(p)$. Let g_{RF} be the metric in THEOREM 1 when $C = \{zw = 1\}$ and write $a = |\mathrm{Rm}(g_{RF})|(0)$. We expect that $\lambda_j \to \infty$ and that $(\mathbb{CP}^2, \lambda_j g_{\epsilon_j}, p) \to (\mathbb{C}^2, a g_{RF}, 0)$ in the pointed Gromov-Hausdorff sense. Alternatively, consider the embedding of $\mathbb{C}^2$ into $\mathbb{CP}^2$ given by $(u, v) \to [u, v, 1]$. In these coordinates the point $p = [0,0,1]$ corresponds to 0 and

$$C_\epsilon = \{uv = \epsilon(u^3 + v^3 + 1)\}.$$

Write $u = \sqrt{\epsilon}z$ and $v = \sqrt{\epsilon}w$ so that $C_\epsilon = \{zw = \epsilon^{3/2}z^3 + \epsilon^{3/2}w^3 + 1\}$. Write $(u, v) = F_\epsilon(z, w)$. We can omit the discussion above on convergence of metric spaces and say that we expect that $|\mathrm{Rm}(g_\epsilon)|(p)F_\epsilon^* g_\epsilon \to a g_{RF}$ as $\epsilon \to 0$ in the sense of tensors. Our conjectural formula 6.4 allows us to compute the energy of g_{RF}. In this case $C = \{zw = 1\}$, so that $\chi(C) = 0$. The corresponding metric on the three-sphere has total volume $2\pi^2\beta^2$. We get that $E(g_{RF}) = 1 - \beta^2$. The total amount of energy lost in the convergence of the metrics g_ϵ to the metric g_0 is given by 6.9. This can be explained by the formation of three bubbles with energy $1-\beta^2$, according to our conjectural picture. (Note that $E(ag_{RF}) = E(g_{RF})$ since the energy is scale invariant.)

One can write many other families of smooth cubic curves in $\mathbb{CP}^2$ which degenerate into a singular curve. From the point of view of the metrics, the author believes that the degeneration we have described is the only possible one. Denote by $\mathcal{M}$ the space of all smooth cubics in $\mathbb{CP}^2$ modulo the action of $PSL(3, \mathbb{C})$, it is a classical fact that $\mathcal{M}$ carries a natural structure of a Riemann surface and that $\mathcal{M} \cong \mathbb{C}$, see Chapter 6 in [16]. Fix $0 < \beta < 1$ and let $\mathcal{P}$ be the space of all Kähler-Einstein metrics in $\mathbb{CP}^2$ of normalized volume with cone angle $2\pi\beta$ along a smooth cubic curve, modulo isometry. Endow $\mathcal{P}$ with the Gromov-Hausdorff topology. The theory of existence and uniqueness of such metrics, see [20], establishes an homeomorphism between $\mathcal{M}$ and $\mathcal{P}$. One can use the Gromov-Hausdorff distance to compactify $\mathcal{P}$. It should be the case that this is homeomorphic to an *algebraic* compactification of $\mathcal{M}$. In our case, the Riemann sphere is the only algebraic compactification of $\mathbb{C}$. It is obtained by adding only one point to $\mathcal{M}$, this point should correspond to the metric g_0 with cone singularities along the three lines. It would be interesting to discuss the case of higher degree curves in $\mathbb{CP}^2$, in the next section we say something about the degree four case.

A simpler situation would be to study the behavior of spherical metrics on $\mathbb{CP}^1$ with cone singularities as two of the singular points come together. The case of four points can be related to our discussion in the previous paragraph. We think of $\mathcal{M}$ as the space of four unordered points in the Riemann sphere modulo the action of Möbius transformations. Fix $1/2 < \beta < 1$ and denote by $\tilde{\mathcal{P}}$ the space of spherical metrics on $\mathbb{CP}^1$ with cone angle $2\pi\beta$ at four distinct points, modulo isometry. We can replace $\mathcal{P}$ by $\tilde{\mathcal{P}}$ in our previous discussion without any change. This time, the metric g_0 corresponds to the spherical metric on $\mathbb{CP}^1$ with two cone singularities of angle $2\beta - 1$.

6.4 The case of a general smooth curve in $\mathbb{C}^2$

An interesting project is to extend THEOREM 1 to the case of curves for which the asymptotic lines don't need to be different. Let us consider the example of $C = \{w = z^2\}$. In this case we think that for any $1/2 < \beta < 1$ there should be a Ricci-flat metric with cone angle $2\pi\beta$ along C asymptotic to the cone $\mathbb{C}_\gamma \times \mathbb{C}$, with $\gamma = 2\beta - 1$. A way to work out this relation between β and γ is to cut two disjoint wedge shaped regions of angle $2\pi(1-\beta)$ from the plane, identify the corresponding sides to get a space with two cone singularities of angle $2\pi\beta$ and then let the singular points come together. See Figure 5. In the case that such a metric exists, formula 6.4 allows us to compute its energy

$$E = 1 + (\beta - 1) - \gamma = 1 - \beta. \tag{6.10}$$

We expect to find these metrics in the situation of $C_\epsilon \to 2C_0$. Let us illustrate our speculations with an example, coming from a classical discussion involving Riemann surfaces of genus 3. (See Chapter 12 in [16].) Let Q be a non-degenerate quadratic form in three variables, so $C_0 = \{Q = 0\} \subset \mathbb{CP}^2$ is a smooth conic. Let F be a generic polynomial of degree 4 and let $C_\epsilon = \{Q^2 + \epsilon F\} = 0$. Write $Z = \{F = 0\}$, so that for a typical F the intersection $Z \cap C_0$ consists of 8 distinct points $p_1, \dots, p_8$. For small and non-zero ϵ the curve C_ϵ is smooth and one can think of it as an approximate double cover of C_0, branched over the points $p_1, \dots, p_8$. Fix some $\beta > 1/2$, assume that there exist KE metrics ω_ϵ with cone angle $2\pi\beta$ along C_ϵ and a KE metric ω_0 with cone angle $2\pi\gamma$ along C_0. In this situation we would expect that $\omega_\epsilon \to \omega_0$. We can compute the energy of the metrics using 6.2

$$E(\omega_\epsilon) = 3 + (\beta - 1)\chi(C_\epsilon) = 3 + (\beta - 1)(-4) = 7 - 4\beta$$

$$E(\omega_0) = 3 + (\gamma - 1)\chi(C_0) = 3 + (2\beta - 2)2 = 4\beta - 1.$$

The total amount of energy lost is given by

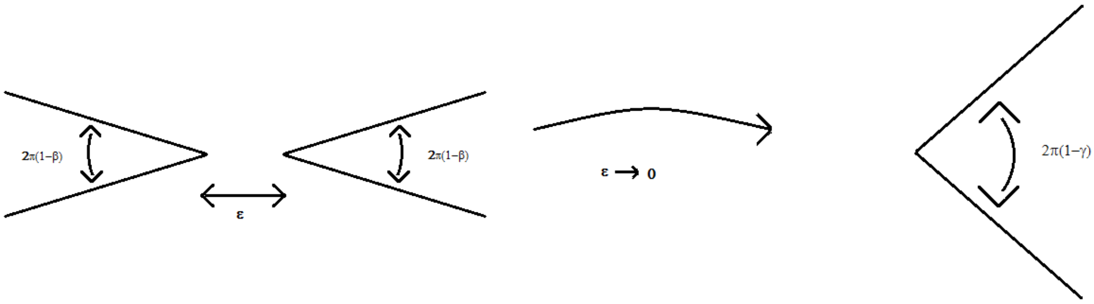

Figure 5: $1-\gamma = 2(1-\beta)$. This picture models the behavior as $\epsilon \to 0$ of the metrics ω_ϵ in a transverse direction to C_0.

$$E(\omega_\epsilon) - E(\omega_0) = 8(1-\beta). \tag{6.11}$$

We expect that re-scaling the metrics ω_ϵ around the points p_i we get a Ricci-flat metric on $\mathbb{C}^2$ in the limit with cone angle $2\pi\beta$ along a parabola, as described above. Then 6.11 can be explained by the formation of eight 'bubbles' with energy given by 6.10.

Consider now the curve $C = \{wz^2 = 1\}$. We can ask for the existence of a Ricci-flat metric with cone angle $2\pi\beta$ transverse to C asymptotic to $\mathbb{C}_\gamma \times \mathbb{C}_\beta$. Let $C_0 \subset X$ be a curve with a normal crossing singularity at p, so that there are complex coordinates (u, v) centered at p in which $C_0 = \{uv + (\text{h.o.t.}) = 0\}$. Assume that the curves C_ϵ converge to C_0 as $\epsilon \to 0$ with multiplicity 2 on the $\{u = 0\}$ axis and with multiplicity 1 on the $\{v = 0\}$ axis. Suppose that there are KE metrics g_ϵ on X with cone angle $2\pi\beta$ along C_ϵ. Under suitable hypothesis one might expect that re-scaling the metrics g_ϵ around small balls centered at p one gets a Ricci-flat metric on $\mathbb{C}^2$ with cone angle $2\pi\beta$ along $C = \{wz^2 = 1\}$ asymptotic to $\mathbb{C}_\gamma \times \mathbb{C}_\beta$.

It is straightforward to state the case of asymptotic lines with higher multiplicity. If $C = \{w = z^n\}$, we set $\frac{n-1}{n} < \beta < 1$ and $\gamma = n\beta - (n-1)$. We ask for the existence of a Ricci-flat metric with cone angle $2\pi\beta$ transverse to C, asymptotic to $\mathbb{C}_\gamma \times \mathbb{C}$. The author believes that there is a general existence theorem for the case of any smooth complex curve, provided that the angle is such that the appropriate flat cone metric exists. The work on the linear theory in Section 4 and the a priori estimates of Section 5 should carry over immediately to this more general case, the asymptotically conical regime is the crucial property we use in those sections. The work one has to do to extend THEOREM 1 to the case of any smooth curve is in the construction of the reference metric. On the other hand, the metrics corresponding to $C = \{w = z^n\}$ should be invariant under the S^1 action $e^{i\theta}(z, w) = (e^{i\theta} z, e^{in\theta} w)$. One can then ask for a more explicit construction, similar in spirit to the Gibbons-Hawking ansatz.

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