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Generalized hidden symmetries and the Kerr-Sen black hole

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Abstract: We elaborate on basic properties of generalized Killing-Yano tensors which naturally extend Killing-Yano symmetry in the presence of skew-symmetric torsion. In particular, we discuss their relationship to Killing tensors and the separability of various field equations. We further demonstrate that the Kerr-Sen black hole spacetime of heterotic string theory, as well as its generalization to all dimensions, possesses a generalized closed conformal Killing-Yano 2-form with respect to a torsion identified with the 3-form occuring naturally in the theory. Such a 2-form is responsible for complete integrability of geodesic motion as well as for separability of the scalar and Dirac equations in these spacetimes.

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Contents

1 Introduction

Killing-Yano symmetry [1] is a fundamental hidden symmetry which plays a crucial rôle in higher-dimensional rotating black hole spacetimes with spherical horizon topology [2–4], in the case that the supporting matter consists of a cosmological constant alone. Such black holes are uniquely characterized by the existence of this symmetry [5] and derive from it many remarkable properties, such as complete integrability of geodesic motion [6–8], separability of the scalar [9, 10] and Dirac [11, 12] perturbations, and the special algebraic type of the Weyl tensor [13, 14]. This symmetry may be also related to a recent progress in the study of gravitational perturbations in these spacetimes [15–17]. Unfortunately, the demonstrated uniqueness [18–20] prevents these results being extended for black holes of more general theories with additional matter content, such as the black holes of various supergravities or string theory. These black holes are usually much more complicated and the presence of matter tends to spoil many of the elegant characteristics of their vacuum brethren. For example, the Weyl tensor is no longer guaranteed to be algebraically special. On the other hand, one may hope that for at least some of these black holes one can define an appropriate generalization of the Killing-Yano symmetry and infer some of the black hole properties from it.

One possible generalization is an extension of the Killing-Yano symmetry in the presence of skew-symmetric torsion. This generalization was first introduced by Bochner and Yano [21] from the mathematical point of view and recently rediscovered in [22–24] as a hidden symmetry of the Chong-Cvetic-Lü-Pope rotating black hole of $D = 5$ minimal gauged supergravity [25]. More specifically, it was shown that the Chong-Cvetic-Lü-Pope black hole admits a 'generalized Killing-Yano tensor' if one identifies the torsion 3-form with the dual of the Maxwell field, $\boldsymbol{T} = *\boldsymbol{F}/\sqrt{3}$. This identification is rather natural as no additional field is introduced into the theory and the torsion is 'T-harmonic' due to the Maxwell equations. Moreover, the discovered generalized Killing-Yano tensor shares almost identical properties with its vacuum cousin; it gives rise to all isometries of the spacetime [26] and implies separability of the Hamilton-Jacobi, Klein-Gordon, and Dirac equations in this background [24, 27]. Importantly, it was also shown that the Chong-Cvetic-Lü-Pope black hole is the unique solution of minimal gauged supergravity admitting a generalized Killing-Yano tensor with T-harmonic torsion [28]. The relationship between the existence of generalized Killing-Yano symmetries and separability of the Dirac equation was investigated in [29].

These results give rise to the natural question of whether there are some other physically interesting spacetimes which admit Killing-Yano tensors with skew symmetric torsion, or whether the above example is unique, relying on the simplicity of minimal gauged supergravity. It is the purpose of this paper to present a family of spacetimes admitting generalized Killing-Yano symmetry, and hence to show that such symmetry is more widely applicable.

It is well known, that pseudo-Riemannian manifolds with skew symmetric torsion occur naturally in superstring theories, where the torsion may be identified (up to a factor) with a 3-form field strength occurring in the theory [30, 31]. Black hole spacetimes of such a theory are natural candidates to admit generalized Killing-Yano symmetries. We shall consider an effective field theory describing the low-energy heterotic string theory and demonstrate that the generalized Killing-Yano symmetry appears naturally for the Kerr-Sen solution [32], as well as for its higher-dimensional generalizations found by Cvetic and Youm [33] and Chow [34]. The torsion we identify in both instances is the 3-form field strength $\boldsymbol{H}$. We shall also demonstrate that this symmetry, in common with the vacuum and minimal supergravity cases, is responsible for the complete integrability of geodesic motion and separability of suitable scalar and Dirac equations in these spacetimes.

The paper is organized as follows. The general properties of Killing-Yano tensors with an arbitrary torsion 3-form are studied and compared to the vacuum case in section 2. The existence of generalized Killing-Yano symmetries and the corresponding implications for the separability of various field equations are demonstrated for the four-dimensional Kerr-Sen black hole in section 3 and for the 'charged Kerr-NUT' spacetimes in all dimensions in section 4. Section 5 is devoted to conclusions.

2 Generalized Killing-Yano symmetries

2.1 Definition

Throughout this paper, M^D will be a D-dimensional spacetime, equipped with the metric

$$\boldsymbol{g} = g_{ab}\boldsymbol{d}x^a\boldsymbol{d}x^b\,. \tag{2.1}$$

Where we need to distinguish between even and odd dimensions, we will set $D = 2n+\varepsilon$, with $\varepsilon = 0,1$ respectively. Henceforth, $\{\boldsymbol{X}_a\}$ will be an orthonormal basis for TM, $\boldsymbol{g}(\boldsymbol{X}_a, \boldsymbol{X}_b) = \eta_{ab}$, with dual basis $\{\boldsymbol{e}^a\}$ for T^*M with $\boldsymbol{g}^{-1}(\boldsymbol{e}^a, \boldsymbol{e}^b) = \eta^{ab}$. We additionally define

$$\boldsymbol{X}^a = \eta^{ab}\boldsymbol{X}_b\,, \qquad \boldsymbol{e}_{\boldsymbol{a}} = \eta_{ab}\boldsymbol{e}^b. \tag{2.2}$$

Note that $\boldsymbol{X}$ will always be a vector regardless of index position, while $\boldsymbol{e}$ is always a 1-form. In order to state some of our formulae succinctly it will be convenient to make use of the n-fold contracted wedge product introduced in a recent paper [29]. This is defined for any p-form $\boldsymbol{\alpha}$ and q-form $\boldsymbol{\beta}$ inductively by

$$\boldsymbol{\alpha} \underset{0}{\wedge} \boldsymbol{\beta} = \boldsymbol{\alpha}\wedge\boldsymbol{\beta}\,, \qquad \boldsymbol{\alpha}\underset{n}{\wedge}\boldsymbol{\beta} = \boldsymbol{X}^a\lrcorner\,\boldsymbol{\alpha} \underset{n-1}{\wedge} \boldsymbol{X}_a\lrcorner\,\boldsymbol{\beta}\,, \tag{2.3}$$

where the 'hook' operator $\lrcorner$ corresponds to the inner derivative.[1]

We wish to consider a connection ∇^T which has the same geodesics as the Levi-Civita connection and which preserves the metric:

$$\nabla^T_{\dot\gamma}\dot\gamma = 0\,, \qquad \nabla^T_X\boldsymbol{g} = 0\,. \tag{2.4}$$

Such a connection has totally anti-symmetric torsion which may be identified with a 3-form, $\boldsymbol{T}$, after lowering indices with the metric. The connection $\boldsymbol{\nabla}^T$ acts on a vector field $\boldsymbol{Y}$ as

$$\nabla^T_X\boldsymbol{Y} = \nabla_X\boldsymbol{Y} + \frac{1}{2}\boldsymbol{T}(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{X}_a)\boldsymbol{X}^a\,, \tag{2.5}$$

where $\boldsymbol{X},\boldsymbol{Y}$ are vector fields and $\boldsymbol{\nabla}$ is the Levi-Civita connection. As usual, we introduce connection 1-forms $(\boldsymbol{\omega}^T)^a{}_b$ by

$$\nabla^T_{X_b}\boldsymbol{X}_a = (\boldsymbol{\omega}^T)^c{}_a(\boldsymbol{X}_b)\,\boldsymbol{X}_c\,. \tag{2.6}$$

[1] In components the contracted wedge product takes the following form:

$$(\alpha\underset{n}{\wedge}\beta)_{c_1\dots c_{p+q-2n}} = \frac{(p+q-2n)!}{(p-n)!(q-n)!}\,\alpha^{a_1\dots a_n}{}_{[c_1\dots c_{p-n}}\beta_{|a_1\dots a_n|c_{p-n+1}\dots c_{p+q-2n}]}\,.$$

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Comparing (2.5) and (2.6) we have

$$\boldsymbol{\omega}^T_{ab} = \boldsymbol{\omega}_{ab} - \frac{1}{2} T_{abc} \boldsymbol{e}^c\,, \tag{2.7}$$

where $\boldsymbol{\omega}_{ab}$ is the Levi-Civita connection 1-form which obeys the Cartan relations

$$\boldsymbol{\omega}_{ab} = -\boldsymbol{\omega}_{ba}\,, \quad \boldsymbol{d}\boldsymbol{e}^a + \boldsymbol{\omega}^a{}_b \wedge \boldsymbol{e}^b = 0\,. \tag{2.8}$$

The 1-form $\boldsymbol{\omega}^T_{ab}$ satisfies the Cartan relations with torsion

$$\boldsymbol{\omega}^T_{ab} = -\boldsymbol{\omega}^T_{ba}\,, \quad \boldsymbol{d}\boldsymbol{e}^a + (\boldsymbol{\omega}^T)^a{}_b \wedge \boldsymbol{e}^b = \boldsymbol{T}^a\,, \tag{2.9}$$

where $\boldsymbol{T}_a(\boldsymbol{X},\boldsymbol{Y}) = \boldsymbol{T}(\boldsymbol{X}_a,\boldsymbol{X},\boldsymbol{Y})$.

The connection (2.5) induces a connection on forms given by

$$\nabla^T_X \boldsymbol{\Psi} = \nabla_X \boldsymbol{\Psi} + \frac{1}{2} (\boldsymbol{X} \lrcorner \boldsymbol{T}) \underset{1}{\wedge} \boldsymbol{\Psi}\,, \tag{2.10}$$

for a p-form $\boldsymbol{\Psi}$. We additionally define two differential operators related to the exterior derivative and its dual

$$\boldsymbol{d}^T \boldsymbol{\Psi} = \boldsymbol{e}^a \wedge \nabla^T_{X_a} \boldsymbol{\Psi} = \boldsymbol{d}\boldsymbol{\Psi} - \boldsymbol{T} \underset{1}{\wedge} \boldsymbol{\Psi}\,, \tag{2.11}$$

$$\boldsymbol{\delta}^T \boldsymbol{\Psi} = -\boldsymbol{X}^a \lrcorner \nabla^T_{X_a} \boldsymbol{\Psi} = \boldsymbol{\delta}\boldsymbol{\Psi} - \frac{1}{2} \boldsymbol{T} \underset{2}{\wedge} \boldsymbol{\Psi}\,. \tag{2.12}$$

These respectively raise and lower the degree of the form.

Definition. A *generalized conformal Killing-Yano* (GCKY) tensor $\boldsymbol{k}$ [23] is a p-form satisfying for any vector field $\boldsymbol{X}$

$$\nabla^T_X \boldsymbol{k} - \frac{1}{p+1} \boldsymbol{X} \lrcorner \boldsymbol{d}^T \boldsymbol{k} + \frac{1}{D-p+1} \boldsymbol{X}^\flat \wedge \boldsymbol{\delta}^T \boldsymbol{k} = 0\,. \tag{2.13}$$

In analogy with Killing-Yano tensors defined with respect to the Levi-Civita connection, a GCKY tensor $\boldsymbol{f}$ obeying $\boldsymbol{\delta}^T \boldsymbol{f} = 0$ is called a *generalized Killing-Yano* (GKY) tensor, and a GCKY $\boldsymbol{h}$ obeying $\boldsymbol{d}^T \boldsymbol{h} = 0$ a *generalized closed conformal Killing-Yano* (GCCKY) tensor.

2.2 Basic properties

Lemma 1. *GCKY tensors possess the following basic properties:*

1. *A GCKY 1-form is equal to a conformal Killing 1-form.*
2. *The Hodge star $*$ maps GCKY p-forms into GCKY $(D-p)$-forms. In particular, the Hodge star of a GCCKY p-form is a GKY $(D-p)$-form and vice versa.*
3. *GCCKY tensors form a (graded) algebra with respect to a wedge product, i.e., when $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ is a GCCKY p-form and q-form, respectively, then $\boldsymbol{h}_3 = \boldsymbol{h}_1 \wedge \boldsymbol{h}_2$ is a GCCKY $(p+q)$-form.*

4. *Let $\boldsymbol{k}$ be a GCKY p-form for a metric $\boldsymbol{g}$ and a torsion 3-form $\boldsymbol{T}$. Then, $\tilde{\boldsymbol{k}} = \Omega^{p+1}\boldsymbol{k}$ is a GCKY p-form for the metric $\tilde{\boldsymbol{g}} = \Omega^2\boldsymbol{g}$ and the torsion $\tilde{\boldsymbol{T}} = \Omega^2\boldsymbol{T}$.*

5. *Let $\boldsymbol{\xi}$ be a conformal Killing vector, $L_\xi\boldsymbol{g} = 2f\boldsymbol{g}$, for some function f, and $\boldsymbol{k}$ a GCKY p-form with torsion $\boldsymbol{T}$, obeying $L_\xi\boldsymbol{T} = 2f\boldsymbol{T}$. Then $\tilde{\boldsymbol{k}} = L_\xi\boldsymbol{k} - (p+1)f\boldsymbol{k}$ is a GCKY p-form with $\boldsymbol{T}$.*

Proof. The properties 1.-3. were proved in [23]. Let us prove the remaining two properties. To prove 4., we note that the left hand side of (2.13) may be re-written in the form

$$\begin{aligned}\nabla_X\boldsymbol{k} - \frac{1}{p+1}\boldsymbol{X}\lrcorner\,\boldsymbol{dk} + \frac{1}{D-p+1}\boldsymbol{X}^\flat\wedge\boldsymbol{\delta k}+&\\ +\frac{1}{2}(\boldsymbol{X}\lrcorner\,\boldsymbol{T})\underset{1}{\wedge}\boldsymbol{k} + \frac{1}{p+1}\boldsymbol{X}\lrcorner\,(\boldsymbol{T}\underset{1}{\wedge}\boldsymbol{k}) + \frac{1}{2(D-p+1)}\boldsymbol{X}^\flat\wedge(\boldsymbol{T}\underset{2}{\wedge}\boldsymbol{k}) &= 0\,. \end{aligned}\tag{2.14}$$

The first line is the standard conformal Killing-Yano operator, which transforms homogeneously under a conformal transformation $\boldsymbol{g}\to\Omega^2\boldsymbol{g}$ provided $\boldsymbol{k}\to\Omega^{p+1}\boldsymbol{k}$ (see, e.g., [35]). Note that a n-fold contracted wedge product introduces a conformal factor of Ω^{-2n}. The remaining terms may then be seen to transform homogeneously with the correct weight to make (2.13) conformally invariant provided $\boldsymbol{T}\to\Omega^2\boldsymbol{T}$.

Now, suppose we have a family of diffeomorphisms $\phi_t : M\to M$ with $t\in(-\epsilon,\epsilon)$ such that $\phi_\cdot : (-\epsilon,\epsilon)\times M\to M$ is smooth and $\phi_0 = id_M$. We suppose further that this family of diffeomorphisms are conformal transformations of (M,g,T), i.e.,

$$\boldsymbol{g} = (\phi_t)_*\left(\Omega(t,x)^2\boldsymbol{g}\right), \qquad \boldsymbol{T} = (\phi_t)_*\left(\Omega(t,x)^2\boldsymbol{T}\right), \tag{2.15}$$

for some smooth, non-zero function $\Omega : (-\epsilon,\epsilon)\times M\to\mathbb{R}$, where clearly $\Omega(0,x) = 1$. Here $(\phi_t)_* : T^{\star n}_{\phi_t(x)}M\to T^{\star n}_x M$ is the pull-back operator. We may differentiate the first equation with respect to t and evaluate the result at $t=0$ to find

$$0 = 2\Omega\dot{\Omega}\Big|_{t=0}(\phi_0)_*\boldsymbol{g} + \Omega^2\,\frac{d}{dt}\,(\phi_t)_*\boldsymbol{g}\Big|_{t=0} \tag{2.16}$$

and hence

$$\mathcal{L}_\xi\boldsymbol{g} = 2f\boldsymbol{g}\,, \tag{2.17}$$

where we define $f(x) = -\dot{\Omega}(0,x)$, and $\xi = (d\phi_t/dt)_{t=0}$. Similarly, the transformation properties of $\boldsymbol{T}$ imply

$$\mathcal{L}_\xi\boldsymbol{T} = 2f\boldsymbol{T}. \tag{2.18}$$

Now suppose that $\boldsymbol{k}$ is a GCKY p-form of the metric $\boldsymbol{g}$ with torsion $\boldsymbol{T}$. By the conformal invariance of (2.13), $\Omega(t,x)^{p+1}\boldsymbol{k}$ is a GCKY p-form of the metric $\Omega(t,x)^2\boldsymbol{g}$ with torsion $\Omega(t,x)^2\boldsymbol{T}$. Pulling this back by the diffeomorphism ϕ_t, we deduce that

$$\boldsymbol{k}(t) = (\phi_t)_*\left(\Omega(t,x)^{p+1}\boldsymbol{h}\right) \tag{2.19}$$

is a GCKY p-form of the metric $\boldsymbol{g}$ with torsion $\boldsymbol{T}$, *for all values of* t. In particular so is $\tilde{\boldsymbol{k}} = \dot{\boldsymbol{k}}(0)$, as solutions of (2.13) form a vector space. Differentiating (2.19) with respect to t and setting $t=0$, we find that

$$\tilde{\boldsymbol{k}} = L_\xi\boldsymbol{k} - (p+1)f\boldsymbol{k} \tag{2.20}$$

is a GCKY p-form.

This supposed that we started with a family of conformal diffeomorphisms, from which we constructed ξ. Locally however, we may start with ξ obeying (2.17), (2.18) and find a family of conformal diffeomorphisms for $t \in (-\epsilon, \epsilon)$ such that $\dot{\phi}|_{t=0} = \xi$.

□

2.3 GCKY forms and Killing tensors

In this section we prove that generalized (conformal) Killing-Yano tensors of arbitrary rank give rise to (conformal) Killing tensors. Conformal Killing tensors are associated with conserved quantities along null geodesics which are of higher order in geodesics' momenta. Let γ be an (affine parametrized) null geodesic with tangent vector $l^a = dx^a/ds$,

$$l^2 = \boldsymbol{l} \cdot \boldsymbol{l} = 0\,, \quad \nabla^T_l \boldsymbol{l} = \nabla_l \boldsymbol{l} = 0\,. \tag{2.21}$$

The requirement that the quantity

$$C = K_{a_1 \dots a_r} l^{a_1} \dots l^{a_r} \tag{2.22}$$

is preserved along γ, that is $\nabla_l C = 0$, is taken as a definition for $\boldsymbol{K}$ to be a conformal Killing tensor [36]. That is, a *conformal Killing tensor* $\boldsymbol{K}$ of rank r is a symmetric tensor which obeys

$$K_{a_1 a_2 \dots a_r} = K_{(a_1 a_2 \dots a_r)}\,, \quad \nabla_{(b} K_{a_1 a_2 \dots a_r)} = g_{(b a_1} \tilde{K}_{a_2 \dots a_r)}\,. \tag{2.23}$$

The tensor $\tilde{\boldsymbol{K}}$ is determined by tracing both sides of equation (2.23). If $\tilde{\boldsymbol{K}}$ vanishes, the tensor $\boldsymbol{K}$ is a *Killing tensor* [37]. In this case the quantity C is preserved also along timelike (spacelike) geodesics γ with $u^a = dx^a/d\tau$,

$$\nabla^T_u \boldsymbol{u} = \nabla_u \boldsymbol{u} = 0\,. \tag{2.24}$$

Let us now show how these objects follow from the existence of GCKY tensors. For a GKY p-form $\boldsymbol{k}$ and timelike geodesic $\boldsymbol{u}$ let us define a $p-1$ form $\boldsymbol{w}_{(k)}$,

$$\boldsymbol{w}_{(k)} = \boldsymbol{u} \lrcorner\, \boldsymbol{k}\,. \tag{2.25}$$

Using the GKY equation, one can easily show that such a form is 'torsion' parallel transported,

$$\nabla^T_u \boldsymbol{w}_{(k)} = \boldsymbol{u} \lrcorner\, \nabla^T_u \boldsymbol{k} = \frac{1}{p+1} \boldsymbol{u} \lrcorner\, \boldsymbol{u} \lrcorner\, \boldsymbol{d}^T \boldsymbol{k} = 0\,. \tag{2.26}$$

Lemma 2. *Let $\boldsymbol{h}$ and $\boldsymbol{k}$ be two GKY tensors of rank p. Then*

$$K_{ab} = h_{(a|c_1 \dots c_{p-1}|} k_{b)}{}^{c_1 \dots c_{p-1}} \tag{2.27}$$

is a Killing tensor of rank 2.

Proof. We construct $\boldsymbol{w}_{(k)} = \boldsymbol{u} \lrcorner\, \boldsymbol{k}$ and $\boldsymbol{w}_{(h)} = \boldsymbol{u} \lrcorner\, \boldsymbol{h}$, which automatically satisfy $\nabla^T_u \boldsymbol{w}_{(k)} = 0$ and $\nabla^T_u \boldsymbol{w}_{(h)} = 0$. Hence, any product of $\boldsymbol{w}$'s, $\boldsymbol{l}$'s, and the metric $\boldsymbol{g}$ is torsion parallel propagated along γ. In particular, we find

$$\nabla^T_u C = \nabla_u C = 0\,, \qquad C = \boldsymbol{w}_{(k)} \cdot \boldsymbol{w}_{(h)} = u^a u^b K_{ab}\,. \tag{2.28}$$

This is of the form (2.22) and hence K_{ab} is a Killing tensor. □

Now, let us consider a GCKY p-form $\boldsymbol{k}$ and a null geodesic $\boldsymbol{l}$. Then the following p-form $\boldsymbol{F}_{(k)}$:

$$\boldsymbol{F}_{(k)} = \boldsymbol{l}^\flat \wedge (\boldsymbol{l} \lrcorner\, \boldsymbol{k})\,, \tag{2.29}$$

obeys

$$\nabla^T_l \boldsymbol{F}_{(k)} = \boldsymbol{l}^\flat \wedge (\boldsymbol{l} \lrcorner\, \nabla^T_l \boldsymbol{k}) = \frac{1}{D-p+1} \boldsymbol{l}^\flat \wedge [\boldsymbol{l} \lrcorner\, (\boldsymbol{l}^\flat \wedge \boldsymbol{\delta}^T \boldsymbol{k})] = 0\,. \tag{2.30}$$

Lemma 3. *Let* $\boldsymbol{h}$ *and* $\boldsymbol{k}$ *be two GCKY tensors of rank* p*. Then*

$$Q_{ab} = h_{(a|c_1 \ldots c_{p-1}|}k_{b)}{}^{c_1 \ldots c_{p-1}} \tag{2.31}$$

is a conformal Killing tensor of rank 2.

Proof. Let $\boldsymbol{F}_{(k)}$ and $\boldsymbol{F}_{(h)}$ be 'torsion' parallel transported forms along γ constructed from $\boldsymbol{k}$ and $\boldsymbol{h}$ as above, $\boldsymbol{F}_{(k)} = \boldsymbol{l}^\flat \wedge (\boldsymbol{l} \lrcorner\, \boldsymbol{k})\,,\ \boldsymbol{F}_{(h)} = \boldsymbol{l}^\flat \wedge (\boldsymbol{l} \lrcorner\, \boldsymbol{h})\,.$ Then, any product of $\boldsymbol{F}$'s, $\boldsymbol{l}$'s, and the metric $\boldsymbol{g}$ is also torsion parallel propagated along γ. In particular, this is true for the product

$$(\boldsymbol{F}_{(k)} \cdot \boldsymbol{F}_{(h)})_{ab} = F_{(k)a}{}^{c_1 \ldots c_{p-1}} F_{(h)bc_1 \ldots c_{p-1}} = l_a l_b C\,, \tag{2.32}$$

where $C = l^a l^b Q_{ab}$. Hence we have

$$\nabla^T_l \left[(\boldsymbol{F}_{(k)} \cdot \boldsymbol{F}_{(h)})_{ab}\right] = l_a l_b \nabla^T_l C = l_a l_b \nabla_l C = 0\,. \tag{2.33}$$

This means that $\nabla_l C = 0$, and, comparing with (2.22), we realize that Q_{ab} is a conformal Killing tensor. □

Remark. It is obvious from the proofs that both lemmas are valid for 'standard' (conformal) Killing-Yano tensors as well.

One might ask whether these results admit a converse, in particular: can all Killing tensors be written as a product of GKY tensors with respect to a suitable torsion? This problem is already complicated in four dimensions in the absence of torsion where it was first considered by Collinson [38] and finally addressed by Ferrando and Sáez [39]. In that case, algebraic and differential conditions are imposed upon the Killing tensor. It is obvious that introducing arbitrary torsion adds extra degrees of freedom which may be exploited. Naïve counting arguments suggest that even with these additional degrees of freedom it is not possible to write every Killing tensor as a product of GKY tensors, however, we have been unable to exhibit an explicit counter-example.

Let us finally comment on whether a Killing tensor K_{ab} constructed from a GKY p-form induces symmetries of the scalar wave operator $\Box = g^{ab}\nabla_a\nabla_b$. It was demonstrated by Carter [40] that a commutator of the symmetry operator $\hat{K} = \nabla_a K^{ab} \nabla_b$ with the scalar wave operator reads

$$\left[\Box, \nabla_a K^{ab} \nabla_b\right] = \frac{4}{3} \nabla_a \left(K_c{}^{[a} R^{b]c}\right) \nabla_b\,. \tag{2.34}$$

The expression on the r.h.s. automatically vanishes whenever the Killing tensor is a square of a Killing-Yano tensor [41] of arbitrary rank. One can easily show that this is no longer true in the presence of torsion and in general torsion anomalies appear on the r.h.s. In other words, GKY p-forms do not in general produce symmetry operators for the Klein-Gordon equation.

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2.4 Spinning particles and the Dirac equation

A key property of Killing-Yano tensors in the absence of torsion (and other matter fields) is that they are intimately related to an enhanced worldline supersymmetry of spinning particles in the semiclassical approximation [42]. Even more interestingly, such property remains true at the 'quantum' level. This is reflected by the fact that Killing-Yano tensors give rise to symmetry operators for the Dirac equation [35, 43] and no anomalies appear in the transition. This is no longer true in the presence of matter fields. For example, for the Maxwell field an anomaly appears already at the spinning particle level and destroys the supersymmetry unless the electromagnetic field obeys some additional restrictions [44]. Similarly, it was recently shown that, up to an explicit anomaly, GCKY tensors correspond to an enhanced worldline supersymmetry of spinning particles [23, 45, 46] in the presence of torsion and provide symmetry operators for the (torsion modified) Dirac operator [29]. Let us briefly recapitulate these results.

Since the torsion field naturally couples to particle's spin, it is not very surprising that the appropriate Dirac operator picks up a torsion correction. It was argued in [29] that in the presence of torsion the natural Dirac operator to consider is

$$\mathcal{D} = \gamma^a \nabla_a - \frac{1}{24} T_{abc} \gamma^{abc} \,. \tag{2.35}$$

In the context of the supergravities we will later work with, this is also a natural Dirac operator to consider as it gives the equation of motion for the linearized gaugino field χ in string frame. See for example the Lagrangian given in section 4 of [47]. After a conformal rescaling to string frame together with a rescaling of χ, we find that the new spinor obeys (2.35).

It was further shown in [29] that given a GCKY tensor $\boldsymbol{k}$ and provided that the corresponding anomaly terms[2]

$$\boldsymbol{A}_{(cl)}(\boldsymbol{k}) = \frac{\boldsymbol{d}(\boldsymbol{d}^T \boldsymbol{k})}{p+1} - \frac{\boldsymbol{T} \wedge \boldsymbol{\delta}^T \boldsymbol{k}}{D-p+1} - \frac{1}{2} \boldsymbol{dT} \underset{1}{\wedge} \boldsymbol{k} \,, \tag{2.36}$$

$$\boldsymbol{A}_{(q)}(\boldsymbol{k}) = \frac{\boldsymbol{\delta}(\boldsymbol{\delta}^T \boldsymbol{k})}{D-p+1} - \frac{1}{6(p+1)} \boldsymbol{T} \underset{3}{\wedge} \boldsymbol{d}^T \boldsymbol{k} + \frac{1}{12} \boldsymbol{dT} \underset{3}{\wedge} \boldsymbol{k} \,, \tag{2.37}$$

vanish, one can construct an operator L_k which (on-shell) commutes with $\mathcal{D}$, $[\mathcal{D}, L_k] = 0$. Such an operator provides an on-shell symmetry operator for a massless Dirac equation. When $\boldsymbol{k}$ is in addition $\boldsymbol{d}^T$-closed or $\boldsymbol{\delta}^T$-coclosed the operator L_k may be modified to produce off-shell (anti)-commuting operators M_k or K_k.

Lemma 4. *Let $\boldsymbol{f}$ be a GKY p-form for which $\boldsymbol{A}_{(cl)}(\boldsymbol{f}) = 0 = \boldsymbol{A}_{(q)}(\boldsymbol{f})$. Then, the following operator:*

$$\begin{aligned} K_f = {} & f^a{}_{b_1 \dots b_{p-1}} \gamma^{b_1 \dots b_{p-1}} \nabla_a + \frac{1}{2(p+1)^2} (df)_{b_1 \dots b_{p+1}} \gamma^{b_1 \dots b_{p+1}} + \frac{1-p}{8(p+1)} T^a{}_{b_1 b_2} f_{a b_3 \dots b_{p+1}} \gamma^{b_1 \dots b_{p+1}} \\ & - \frac{p-1}{4} T^{ab}{}_{b_1} f_{a b b_2 \dots b_{p-1}} \gamma^{b_1 \dots b_{p-1}} + \frac{(p-1)(p-2)}{24} T^{abc} f_{abc b_1 \dots b_{p-3}} \gamma^{b_1 \dots b_{p-3}} \end{aligned} \tag{2.38}$$

[2] It is only the first condition, $\boldsymbol{A}_{(cl)} = 0$, which emerges from the classical spinning particle approximation. Correspondingly, we call $\boldsymbol{A}_{(cl)}$ a *classical anomaly* and $\boldsymbol{A}_{(q)}$ (which appears only at the operator level) a '*quantum anomaly*'.

graded anti-commutes with the Dirac operator $\mathcal{D}$, $\{\mathcal{D}, K_f\}_+ \equiv \mathcal{D}K_f + (-1)^p K_f \mathcal{D} = 0$. In particular, when p is odd, K_f is a symmetry operator for the massive Dirac equation $(\mathcal{D}+m)\psi = 0$.

The first two terms in (2.38) correspond to the symmetry operator for the 'classical' Dirac operator in the absence of torsion [35, 43]. The third term is a 'leading' torsion correction, present already at the classical spinning particle level [23]. The last two terms are 'quantum corrections' due to the presence of torsion. Similarly, one has

Lemma 5. *Let $\boldsymbol{h}$ be a GCCKY p-form for which $\boldsymbol{A}_{(cl)}(\boldsymbol{h}) = 0 = \boldsymbol{A}_{(q)}(\boldsymbol{h})$. Then, the following operator:*

$$M_h = h_{b_1\dots b_p}\gamma^{ab_1\dots b_p}\nabla_a - \frac{p(D-p)}{2(D-p+1)}(\delta h)_{b_1\dots b_{p-1}}\gamma^{b_1\dots b_{p-1}} - \frac{1}{24}T_{b_1b_2b_3}h_{b_4\dots b_{p+3}}\gamma^{b_1\dots b_{p+3}} + \frac{p}{4}T^a{}_{b_1b_2}h_{ab_3\dots b_{p+1}}\gamma^{b_1\dots b_{p+1}} + \frac{p(p-1)(D-p-1)}{8(D-p+1)}T^{ab}{}_{b_1}h_{abb_2\dots b_{p-1}}\gamma^{b_1\dots b_{p-1}}\,, \tag{2.39}$$

graded commutes with the Dirac operator $\mathcal{D}$, $[\mathcal{D}, M_h]_- \equiv \mathcal{D}M_h - (-1)^p M_h\mathcal{D} = 0$.

Since GCCKY tensors form an algebra with respect to the wedge product, it is natural to ask whether the anomalies respect this algebra. It was shown in [29] that provided the classical anomaly vanishes for $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$, both GCCKY tensors, then it also vanishes for $\boldsymbol{h}_1 \wedge \boldsymbol{h}_2$.

Note that we assert that if the anomalies vanish then L_k is a symmetry operator. In the case that the anomalies do not vanish, it may still be possible to modify L_k to give a symmetry operator. This is in fact the case in the 5-dimensional minimal supergravity case considered in [23]. Although the anomalies do not vanish for the GKY tensor exhibited, a symmetry operator may nevertheless be constructed by making use of properties of the torsion in this special case.

2.5 GCCKY 2-form

Let us consider a *non-degenerate*[3] GCCKY 2-form $\boldsymbol{h}$

$$\nabla^T_X \boldsymbol{h} = \boldsymbol{X}^\flat \wedge \boldsymbol{\xi}\,, \qquad \boldsymbol{\xi} = -\frac{1}{D-1}\boldsymbol{\delta}^T\boldsymbol{h}\,. \tag{2.40}$$

In the absence of torsion such an object [called the *principal conformal Killing-Yano* (PCKY) tensor] implies the existence of towers of explicit and hidden symmetries and determines uniquely (up to $[D/2]$ functions of one variable) the canonical form of the metric [7, 18, 19]. In this subsection we shall see that in the presence of torsion, the GCCKY 2-form $\boldsymbol{h}$ is in general a much weaker structure. Our presentation closely follows the review [49] while we stress some important differences.

[3] By non-degenerate we mean that the skew symmetric matrix h_{ab} has the maximal possible rank and that its eigenvalues are functionally independent in some spacetime domain. The degenerate case without torsion has been studied in [20, 48]

2.5.1 Canonical basis

For a non-degenerate $\boldsymbol{h}$ one can introduce a Darboux basis in which

$$\boldsymbol{g} = \delta_{ab}\boldsymbol{e}^a\boldsymbol{e}^b = \sum_{\mu=1}^{n}(\boldsymbol{e}^\mu\boldsymbol{e}^\mu + \boldsymbol{e}^{\hat{\mu}}\boldsymbol{e}^{\hat{\mu}}) + \varepsilon\boldsymbol{e}^0\boldsymbol{e}^0\,, \tag{2.41}$$

$$\boldsymbol{h} = \sum_{\mu=1}^{n} x_\mu \boldsymbol{e}^\mu \wedge \boldsymbol{e}^{\hat{\mu}}\,, \tag{2.42}$$

where x_μ are the 'eigenvalues' of $\boldsymbol{h}$. We refer to $\{\boldsymbol{e}\}$ as the *canonical basis* associated with the GCCKY 2-form $\boldsymbol{h}$. This basis is fixed uniquely up to 2D rotations in each of the 'GKY 2-planes' $\boldsymbol{e}^\mu \wedge \boldsymbol{e}^{\hat{\mu}}$. This freedom can be exploited, for example, to simplify the canonical form of the torsion 3-form.

2.5.2 Towers of hidden symmetries

According to the property 3 in lemma 1, the GCCKY 2-form generates a tower of GCCKY tensors

$$\boldsymbol{h}^{(j)} \equiv \boldsymbol{h}^{\wedge j} = \underbrace{\boldsymbol{h} \wedge \ldots \wedge \boldsymbol{h}}_{\text{total of } j \text{ factors}}\,. \tag{2.43}$$

Because $\boldsymbol{h}$ is non-degenerate, one has a set of n non-vanishing GCCKY $(2j)$-forms $\boldsymbol{h}^{(j)}$, $\boldsymbol{h}^{(1)} = \boldsymbol{h}$. In an odd number of spacetime dimensions $\boldsymbol{h}^{(n)}$ is dual to a Killing vector $\boldsymbol{\eta} = *\boldsymbol{h}^{(n)}$, whereas in even dimensions it is proportional to the totally antisymmetric tensor. Contrary to a torsion-less case, forms $\boldsymbol{h}^{(j)}$ do not necessary admit a potential, as in general $(\boldsymbol{d}^T)^2 \neq 0 \neq \mathbf{d}^T\mathbf{d}$. On the other hand, each $\boldsymbol{h}^{(j)}$ still gives rise to a GKY $(D-2j)$-form

$$\boldsymbol{f}^{(j)} \equiv *\boldsymbol{h}^{(j)}\,, \tag{2.44}$$

which in its turn generates a Killing tensor $\boldsymbol{K}^{(j)}$ by lemma 2,

$$K^{(j)}_{ab} \equiv \frac{1}{(D-2j-1)!(j!)^2} f^{(j)}{}_{ac_1\ldots c_{D-2j-1}} f^{(j)}{}_b{}^{c_1\ldots c_{D-2j-1}}\,. \tag{2.45}$$

The choice of the coefficient in the definition (2.45) gives the Killing tensor an elegant form in the canonical basis [see eq. (2.46) below]. Including the metric $\boldsymbol{g}$, which is a trivial Killing tensor, as the zeroth element, $\boldsymbol{K}^{(0)} = \boldsymbol{g}$, we obtain a tower of n irreducible Killing tensors. The explicit form of this tower in the canonical basis is

$$\boldsymbol{K}^{(j)} = \sum_{\mu=1}^{n} A^{(j)}_\mu (\boldsymbol{e}^\mu\boldsymbol{e}^\mu + \boldsymbol{e}^{\hat{\mu}}\boldsymbol{e}^{\hat{\mu}}) + \varepsilon A^{(j)}\boldsymbol{e}^0\boldsymbol{e}^0\,, \qquad j = 0,\ldots,n-1, \tag{2.46}$$

$$A^{(j)} = \sum_{\nu_1<\cdots<\nu_j} x^2_{\nu_1}\ldots x^2_{\nu_j}\,, \qquad A^{(j)}_\mu = \sum_{\substack{\nu_1<\cdots<\nu_j\\ \nu_i\neq\mu}} x^2_{\nu_1}\ldots x^2_{\nu_j}\,. \tag{2.47}$$

Similar to the torsion-less case one also finds the recursive relation

$$\boldsymbol{K}^{(j)} = A^{(j)}\boldsymbol{g} - \boldsymbol{Q}\cdot\boldsymbol{K}^{(j-1)}\,, \quad \boldsymbol{K}^{(0)} = \boldsymbol{g}\,, \tag{2.48}$$

where $\boldsymbol{Q} = \boldsymbol{h}^2$ is a conformal Killing tensor. Additionally $\boldsymbol{K}^{(i)} \cdot \boldsymbol{K}^{(j)} = \boldsymbol{K}^{(j)} \cdot \boldsymbol{K}^{(i)}$, which means that $\boldsymbol{K}^{(j)}$'s have common eigenvectors [8].[4]

A powerful property of $\boldsymbol{K}^{(j)}$'s in the absence of torsion is that they Schouten-Nijenhuis commute [7, 8]. This means that the corresponding integrals of motion for geodesic trajectories (characterized by velocity $\boldsymbol{u}$)

$$\kappa_j = K^{(j)}_{ab} u^a u^b \,, \tag{2.49}$$

are *in involution*, i.e., they mutually Poisson commute, $\{\kappa_i, \kappa_j\} = 0$. Contrary to this, in the presence of torsion one rather finds

$$\big[K^{(j)}, K^{(l)}\big]^T_{abc} \equiv K^{(j)\ e}{}_{(a}\, \nabla^T_{|e|} K^{(l)}_{bc)} - K^{(l)\ e}{}_{(a}\, \nabla^T_{|e|} K^{(j)}_{bc)} = 0 \,, \tag{2.50}$$

which means that κ_j's are generally not in involution, unless the torsion $\boldsymbol{T}$ obeys some additional conditions. Similarly, in the absence of torsion the tensors $\boldsymbol{K}^{(j)}$ automatically give rise to symmetry operators for the Klein-Gordon equation and forms $\boldsymbol{f}^{(j)}$ produce symmetry operators for the Dirac equation. None of these statements remain generally true in the presence of torsion (see previous subsections).

The most striking difference between the principal conformal Killing-Yano tensor and a non-degenerate GCCKY 2-form is that the first one generates 'naturally' a tower of $n+\varepsilon$ Killing fields whereas the latter does not. More specifically, one can show that in the absence of torsion $\boldsymbol{\xi}$, given by (2.40), is a (primary) Killing vector and that additional Killing vectors are constructed as $\boldsymbol{\xi}^{(j)} = \boldsymbol{K}^{(j)} \cdot \boldsymbol{\xi}$ and $\boldsymbol{\eta}$ (in an odd number of dimensions). When the torsion is present, neither $\boldsymbol{\delta}^T \boldsymbol{h}$ nor $\boldsymbol{\delta h}$ are in general Killing vectors and the whole construction breaks down already in the first step; except in odd dimensions one still has at least one Killing field $\boldsymbol{\eta}$ derived from $\boldsymbol{h}$. It is a very interesting open question whether (and if so how) the existence of a non-degenerate GCCKY 2-form $\boldsymbol{h}$ implies the existence of $n+\varepsilon$ isometries. If such construction exists, one can upgrade n natural coordinates x_μ by adding Killing coordinates to form a complete *canonical basis* as in the case without torsion. Such a result would open a possibility for constructing a 'torsion canonical metric'. The Kerr-Sen black hole spacetime (and more generally the charged Kerr-NUT metrics) studied in the following two sections provide an example of geometries with a non-degenerate GCCKY 2-form and $n+\varepsilon$ isometries.

3 Kerr-Sen black hole

The Kerr-Sen black hole [32] is a solution of the low-energy string theory effective action, which in string frame reads

$$S = -\int d^4x \sqrt{-g}\, e^{-\Phi} \left(-R + \frac{1}{12} H_{abc} H^{abc} - g^{ab} \partial_a \Phi \partial_b \Phi + \frac{1}{8} F_{ab} F^{ab} \right) . \tag{3.1}$$

[4]As an alternative to the above construction, the tower of Killing tensors $\boldsymbol{K}^{(j)}$ can be generated with the help of a generating function $W(\beta) \equiv \det\big(I + \sqrt{\beta} w^{-1} F\big)$ [6, 7]. Here, $w = u^a u_a$, $\boldsymbol{u}$ being the geodesic velocity vector, and $\boldsymbol{F}$ is a 'torsion-parallel-propagated' 2-form, $\nabla^T_c F_{ab} = 0$, which is a projection of $\boldsymbol{h}$ along the geodesic, $F_{ab} = P^c_a h_{cd} P^d_b$, $P^b_a = \delta^b_a - w^{-1} u^b u_a$.

Here, g_{ab} stands for the metric in string frame, Φ is the dilaton, $\boldsymbol{F} = \boldsymbol{dA}$ is the Maxwell field, and there is additionally a 3-form $\boldsymbol{H} = \boldsymbol{dB} - \frac{1}{4}\boldsymbol{A}\wedge\boldsymbol{dA}$ where $\boldsymbol{B}$ is an antisymmetric tensor field. The solution can be obtained by applying the Hassan-Sen transformation [50] to the Kerr geometry [51]. Its geometry (especially in Einstein frame) is the subject of study of many papers. For example, some algebraic properties of this solution were studied in [52], separability of the Hamilton-Jacobi equation in both (string and Einstein) frames was proved in [53, 54], separability of the charged scalar in Einstein frame was demonstrated in [55] and the Killing tensor underlying these results was constructed in [56]. Although the Einstein frame metric, $\boldsymbol{g}_E = e^{-\Phi}\boldsymbol{g}$, is very similar to the Kerr geometry and consequently inherits some of its properties, we shall see that from the point of view of hidden symmetries it is the string frame which is more fundamental.[5] Namely, we shall demonstrate that the string frame metric $\boldsymbol{g}$ possesses a GCCKY 2-form with respect to a natural torsion identified with the 3-form $\boldsymbol{H}$ occurring in the theory. For this reason, in our study we mainly concentrate on the string frame.

3.1 Kerr-Sen black hole in string frame

3.1.1 Metric and fields

In Boyer-Lindquist coordinates the string frame Kerr-Sen black hole solution reads [32, 55]

$$\begin{aligned}
ds^2 &= e^{\Phi}\left\{-\frac{\Delta}{\rho_b^2}\left(dt - a\sin^2\theta d\varphi\right)^2 + \frac{\sin^2\theta}{\rho_b^2}\left[adt - (r^2+2br+a^2)d\varphi\right]^2 + \frac{\rho_b^2}{\Delta}dr^2 + \rho_b^2 d\theta^2\right\},\\
\boldsymbol{H} &= -\frac{2ba}{\rho_b^4}\,\boldsymbol{d}t\wedge\boldsymbol{d}\varphi\wedge\left[\left(r^2 - a^2\cos^2\theta\right)\sin^2\theta\boldsymbol{d}r - r\Delta\sin 2\theta\boldsymbol{d}\theta\right],\\
\boldsymbol{A} &= -\frac{Qr}{\rho_b^2}\left(\boldsymbol{d}t - a\sin^2\theta\boldsymbol{d}\varphi\right),\\
\Phi &= 2\ln\left(\frac{\rho}{\rho_b}\right),
\end{aligned} \tag{3.2}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta\,,\qquad \rho_b^2 = \rho^2 + 2br\,,\qquad \Delta = r^2 - 2(M-b)r + a^2\,. \tag{3.3}$$

The solution describes a black hole with mass M, charge Q, angular momentum $J = Ma$, and magnetic dipole momentum $\mu = Qa$. When the twist parameter $b = Q^2/2M$ is set to zero, the solution reduces to the Kerr geometry, which can be easily recognized in the brackets.

Let us choose the following basis of 1-forms:

$$\begin{aligned}
\boldsymbol{e}^0 &= \frac{\rho\sqrt{\Delta}}{\rho_b^2}\left(\boldsymbol{d}t - a\sin^2\theta\boldsymbol{d}\varphi\right), & \boldsymbol{e}^1 &= \frac{\rho}{\sqrt{\Delta}}\boldsymbol{d}r\,,\\
\boldsymbol{e}^2 &= \frac{\rho\sin\theta}{\rho_b^2}\left[adt - (r^2+2br+a^2)d\varphi\right], & \boldsymbol{e}^3 &= \rho\boldsymbol{d}\theta\,,
\end{aligned} \tag{3.4}$$

[5]This will be especially true for higher-dimensional generalizations of the Kerr-Sen geometry discussed in the next section.

and define the functions

$$T_0^{\pm} = -\frac{2a\sin\theta}{\rho}\left(-\frac{r+b}{\rho_b^2} \pm \frac{r}{\rho^2}\right), \quad T_1^{\pm} = -\frac{2a\cos\theta\sqrt{\Delta}}{\rho}\left(-\frac{1}{\rho_b^2} \pm \frac{1}{\rho^2}\right). \tag{3.5}$$

Then, the metric and the fields take the form

$$\boldsymbol{g} = -\boldsymbol{e}^{0^2} + \boldsymbol{e}^{1^2} + \boldsymbol{e}^{2^2} + \boldsymbol{e}^{3^2}, \quad \boldsymbol{H} = T_0^+\boldsymbol{e}^{012} + T_1^+\boldsymbol{e}^{023}, \quad \boldsymbol{A} = -\frac{Qr}{\rho\sqrt{\Delta}}\boldsymbol{e}^0, \tag{3.6}$$

and the inverse metric is given by

$$\begin{aligned}
\boldsymbol{X}_0 &= \frac{1}{\rho\sqrt{\Delta}}\left[(r^2+2br+a^2)\boldsymbol{\partial}_t + a\boldsymbol{\partial}_\varphi\right], & \boldsymbol{X}_1 &= \frac{\sqrt{\Delta}}{\rho}\boldsymbol{\partial}_r, \\
\boldsymbol{X}_2 &= -\frac{1}{\rho\sin\theta}\left(a\sin^2\theta\boldsymbol{\partial}_t + \boldsymbol{\partial}_\varphi\right), & \boldsymbol{X}_3 &= \frac{1}{\rho}\boldsymbol{\partial}_\theta.
\end{aligned} \tag{3.7}$$

In order to separate the Dirac equation, we shall also need the spin connection. This can be obtained from the Cartan's equation $\boldsymbol{d}\boldsymbol{e}^a + \boldsymbol{\omega}^a{}_b \wedge \boldsymbol{e}^b = 0$ and is given as follows:

$$\begin{aligned}
\boldsymbol{\omega}_{01} &= -A\boldsymbol{e}^0 - B\boldsymbol{e}^2, & \boldsymbol{\omega}_{02} &= -B\boldsymbol{e}^1 + C\boldsymbol{e}^3, & \boldsymbol{\omega}_{03} &= -D\boldsymbol{e}^0 - C\boldsymbol{e}^2, \\
\boldsymbol{\omega}_{12} &= B\boldsymbol{e}^0 - E\boldsymbol{e}^2, & \boldsymbol{\omega}_{13} &= D\boldsymbol{e}^1 - E\boldsymbol{e}^3, & \boldsymbol{\omega}_{23} &= -C\boldsymbol{e}^0 - F\boldsymbol{e}^2,
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
A &= \frac{\rho_b^2}{\rho^2}\frac{d}{dr}\left(\frac{\rho\sqrt{\Delta}}{\rho_b^2}\right), & B &= \frac{a(r+b)\sin\theta}{\rho\rho_b^2}, & C &= \frac{a\cos\theta\sqrt{\Delta}}{\rho\rho_b^2}, \\
D &= -\frac{a^2\sin\theta\cos\theta}{\rho^3}, & E &= \frac{r\sqrt{\Delta}}{\rho^3}, & F &= -\frac{1}{\sin\theta}\frac{\rho_b^2}{\rho^2}\frac{d}{d\theta}\left(\frac{\rho\sin\theta}{\rho_b^2}\right).
\end{aligned} \tag{3.9}$$

3.1.2 Hidden symmetries

Besides two obvious isometries $\boldsymbol{\partial}_t$ and $\boldsymbol{\partial}_\varphi$, the Kerr-Sen geometry admits an irreducible Killing tensor

$$\boldsymbol{K} = a^2\cos^2\theta(\boldsymbol{e}^0\boldsymbol{e}^0 - \boldsymbol{e}^1\boldsymbol{e}^1) + r^2(\boldsymbol{e}^2\boldsymbol{e}^2 + \boldsymbol{e}^3\boldsymbol{e}^3). \tag{3.10}$$

Such a tensor is responsible for separability of (charged) Hamilton-Jacobi equation and hence for the complete integrability of the motion of (charged) particles. Moreover, the metric possesses two GCCKY 2-forms. The first one naturally generalizes the closed conformal Killing-Yano 2-form of the Kerr geometry with respect to the torsion identified with 3-form $\boldsymbol{H}$. More specifically, if we identify

$$\boldsymbol{T}_+ = \boldsymbol{H}, \tag{3.11}$$

where the 3-form $\boldsymbol{H}$ is given by (3.6), then one can explicitly verify that

$$\boldsymbol{h}_+ = r\boldsymbol{e}^0 \wedge \boldsymbol{e}^1 + a\cos\theta\boldsymbol{e}^2 \wedge \boldsymbol{e}^3 \tag{3.12}$$

is a GCCKY 2-form obeying (2.40). This is true with no restriction on the function $\Delta = \Delta(r)$. Contrary to the Kerr case, one does not simply recover the isometries from the divergence of $\boldsymbol{h}_+$; one has

$$\boldsymbol{\xi}_+ = -\frac{1}{3}\boldsymbol{\delta}^{T_+}(\boldsymbol{h}_+) = -\frac{\sqrt{\Delta}}{\rho}\,\boldsymbol{e}^0 + \frac{a\sin\theta}{\rho}\,\boldsymbol{e}^2\,, \qquad \boldsymbol{\xi}_+^\sharp = e^{-\Phi}\boldsymbol{\partial}_t\,. \tag{3.13}$$

One can easily check that in the limit $b = 0$, the torsion $\boldsymbol{T}_+$ vanishes and one recovers the standard form of the Kerr geometry and its corresponding PCKY tensor, as found by Floyd and Penrose [57, 58]. Let us also observe that if one identifies $x_1 = r$ and $x_2 = a\cos\theta$, one recovers a canonical basis for $\boldsymbol{h}_+$. This leads to a transformation to the 'canonical coordinates' and the Carter-Plebanski-like form [59, 60] of the Kerr-Sen geometry (see also section 4.1).

There is yet another GCCKY 2-form in the Kerr-Sen geometry,

$$\boldsymbol{h}_- = r\boldsymbol{e}^0\wedge\boldsymbol{e}^1 - a\cos\theta\boldsymbol{e}^2\wedge\boldsymbol{e}^3\,, \qquad \boldsymbol{\xi}_- = -\frac{1}{3}\boldsymbol{\delta}^{T_-}(\boldsymbol{h}_-) = -\frac{\sqrt{\Delta}}{\rho}\,\boldsymbol{e}^0 - \frac{a\sin\theta}{\rho}\,\boldsymbol{e}^2\,, \tag{3.14}$$

with respect to a different torsion $\boldsymbol{T}_-$ given by

$$\boldsymbol{T}_- = T_0^-\boldsymbol{e}^{012} + T_1^-\boldsymbol{e}^{023}\,. \tag{3.15}$$

Such a torsion is rather peculiar. It remains non-trivial in the limit of the Kerr geometry where one has

$$\boldsymbol{T} \equiv (\boldsymbol{T}_-)|_{b=0} = \frac{4a}{\rho^3}\Big(r\sin\theta\boldsymbol{e}^{012} + \cos\theta\sqrt{\Delta}\boldsymbol{e}^{023}\Big)|_{b=0}\,, \tag{3.16}$$

and

$$\boldsymbol{\delta T} = 0 \quad\Leftrightarrow\quad \boldsymbol{T} = *\boldsymbol{d}\alpha\,, \quad \alpha = 4\arctan\left(\frac{a\cos\theta}{r}\right). \tag{3.17}$$

Although such a torsion seems unfamiliar and cannot be related to fields occurring naturally in the theory, it is 'encoded' in the geometry of the spacetime. Whether it is of some physical interest remains an open question.

Let us finally remark that contrary to the GCCKY tensor in the Chong-Cvetic-Lü-Pope black hole spacetime of minimal supergravity [22–24], neither the 2-form $\boldsymbol{h}_+$ nor $\boldsymbol{h}_-$ are closed and hence neither can be generated from a potential. We shall see in section 3.1.5 that both these 2-forms give rise to symmetry operators of appropriately modified Dirac equations in the Kerr-Sen black hole background.

3.1.3 Motion of charged particles

Motion of test particles in the string frame of the Kerr-Sen black hole background is studied in detail in [54]; it is completely integrable due to the Killing tensor (3.10). Let us here, for completeness, demonstrate that the same remains true also for charged particles. The motion of a particle with charge e is governed by the minimally coupled Hamilton-Jacobi equation

$$\frac{\partial S}{\partial\lambda} + g^{ab}(\partial_a S + eA_a)(\partial_b S + eA_b) = 0\,. \tag{3.18}$$

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Using (3.7) and (3.6), this equation takes the following explicit form

$$\frac{\partial S}{\partial \lambda} - \frac{1}{\rho^2 \Delta}\Big[(r^2+2br+a^2)\partial_t S + a\partial_\varphi S - eQr\Big]^2 + \frac{\Delta}{\rho^2}(\partial_r S)^2 + \\ + \frac{1}{\rho^2 \sin^2\theta}(a\sin^2\theta\partial_t S + \partial_\varphi S)^2 + \frac{1}{\rho^2}(\partial_\theta S)^2 = 0\,. \tag{3.19}$$

It allows a separation of variables $S = -\lambda\kappa_0 - Et + L\varphi + R(r) + \Theta(\theta)$, where the functions $R(r)$ and $\Theta(\theta)$ obey the ordinary differential equations

$$\begin{aligned} &R'^2 - \frac{W_r^2}{\Delta^2} - \frac{V_r}{\Delta} = 0\,, \quad W_r = -E(r^2+2br+a^2) + aL - eQr\,, \quad V_r = \kappa + \kappa_0 r^2\,,\\ &\Theta'^2 + \frac{W_\theta^2}{\sin^2\theta} - V_\theta = 0\,, \quad W_\theta = -aE\sin^2\theta + L\,, \qquad V_\theta = -\kappa + \kappa_0 a^2\cos^2\theta. \end{aligned} \tag{3.20}$$

Identifying $p_a = \partial_a S + eA_a$, we find the particle's momentum $\boldsymbol{p}$ (obeying $\dot{p}_a = -eF_{ab}p^b$)

$$\boldsymbol{p} = -\left(E + \frac{eQr}{\rho_b^2}\right)\boldsymbol{d}t + \left(L + \frac{aeQr\sin^2\theta}{\rho_b^2}\right)\boldsymbol{d}\varphi + \sigma_r\sqrt{\frac{W_r^2}{\Delta^2} + \frac{V_r}{\Delta}}\,\boldsymbol{d}r + \sigma_\theta\sqrt{V_\theta - \frac{W_\theta^2}{\sin^2\theta}}\,\boldsymbol{d}\theta\,. \tag{3.21}$$

Here, $\sigma_r, \sigma_\theta = \pm$ are independent signs, parameters E and L are separation constants corresponding to the Killing fields $\boldsymbol{\partial}_t$ and $\boldsymbol{\partial}_\varphi$, κ_0 is the normalization of the momentum, and κ denotes a separation constant associated with the Killing tensor (3.10).

3.1.4 Separability of the charged scalar field equation

Given that the Hamilton-Jacobi equation for the motion of charged particles separates for the Sen black hole in string frame, it is natural to consider whether the equations for a charged scalar field separate. The matter content of the theory is determined by (3.1), and contains a scalar field in the form of the dilaton Φ. In order to remain within this model, one should consider perturbations of the fields appearing in the action. Since all the fields have non-trivial background values, the linear perturbations of the fields couple to one another, so that one may not consistently consider a linearised perturbation of one field in isolation. In order to circumvent this problem, we introduce a new charged scalar field which vanishes in the background and consider perturbations of this. One might hope that analysing such a test field may give some insight into the dynamics of perturbations in the background, while remaining tractable.

A reasonable guess for the appropriate field equation of a charged scalar field in this background would be the minimally coupled Klein-Gordon equation. This does not separate in string frame. Considering the action (3.1), one concludes that the naïve Klein-Gordon equation is not the most natural equation for a charged scalar field in this background. We instead consider a field ψ whose equations of motion derive from the action

$$S = \int d^4x\sqrt{-g}\,\frac{e^{-\Phi}}{2}\left(g^{ab}\mathcal{D}_a\psi\mathcal{D}_b\psi + m^2\psi^2\right). \tag{3.22}$$

We introduce here the gauge covariant derivative $\mathcal{D}_a = \nabla_a + ieA_a$, where $\boldsymbol{A}$ is the background 1-form field given in section 3.1.1. In the case $m = 0$, this field obeys the standard charged massless Klein-Gordon equation when we transform to Einstein frame. In that frame the charged Klein-Gordon equation has been separated by Wu and Cai [55].

For the particular $\boldsymbol{A}$, Φ of the Kerr-Sen background, we find that $\nabla_a A^a = A^a \nabla_a \Phi = 0$, so the equations which arise upon varying S may be written as

$$\nabla^a \nabla_a \psi - \nabla^a \Phi \nabla_a \psi + 2ieA^a \nabla_a \psi + e^2 A^2 \psi - m^2 \psi = 0. \tag{3.23}$$

A short calculation with the line element given above shows that $\sqrt{-g} = (\rho^4/\rho_b^2)\sin\theta$. Making use of the expression $\nabla_a \nabla^a \psi = (-g)^{-1/2}\partial_a(\sqrt{-g}g^{ab}\partial_b \psi)$ we find that (3.23) separates multiplicatively with the ansatz

$$\psi = R(r)\Theta(\theta)e^{-i\omega t + ih\varphi}\,. \tag{3.24}$$

The resulting ordinary differential equations for the functions $R(r), \Theta(\theta)$ are

$$\begin{aligned} \frac{1}{R}\frac{d}{dr}\left(\Delta \frac{dR}{dr}\right) + \frac{U_r^2}{\Delta} - r^2 m^2 - \kappa = 0\,,\\ \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) - \frac{U_\theta^2}{\sin^2\theta} - a^2 m^2 \cos^2\theta - \kappa = 0\,, \end{aligned} \tag{3.25}$$

where the potential functions U_r, U_θ are given by

$$U_r = ah - \omega(r^2 + 2br + a^2) - eQr, \qquad U_\theta = h - a\omega \sin^2\theta, \tag{3.26}$$

and κ is a separation constant, related to the Killing tensor (3.10).

3.1.5 Separability of the Dirac equation

The torsion modified Dirac equation for a particle carrying charge e reads

$$[\gamma^a (D_\pm)_a + m]\psi_\pm = 0\,, \qquad (D_\pm)_a = X_a + \frac{1}{4}\gamma^b\gamma^c\omega_{bc}(X_a) - \frac{1}{24}\gamma^b\gamma^c(T_\pm)_{abc} + ieA_a\,. \tag{3.27}$$

Using the connection (3.8) and the inverse basis (3.7) we find its explicit form

$$\begin{aligned} \Biggl\{ \frac{\gamma^0}{\rho\sqrt{\Delta}}\Bigl[(r^2{+}2br{+}a^2)\partial_t{+}a\partial_\varphi{-}ieQr\Bigr]{+}\gamma^1\left(E{+}\frac{A}{2}{+}\frac{\sqrt{\Delta}}{\rho}\partial_r\right){-}\frac{\gamma^2}{\rho\sin\theta}(a\sin^2\theta\partial_t{+}\partial_\varphi)\\ +\gamma^3\left(D - \frac{F}{2} + \frac{1}{\rho}\partial_\theta\right) + \frac{\gamma^{012}}{4}(2B - T_0^\pm) + \frac{\gamma^{023}}{4}(2C - T_1^\pm) + m\Biggr\}\psi_\pm = 0\,. \end{aligned} \tag{3.28}$$

We use the following representation of gamma matrices $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$:

$$\gamma^0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} \sigma^1 & 0 \\ 0 & -\sigma^1 \end{pmatrix}, \tag{3.29}$$

where σ^i are Pauli matrices.

To keep track of various signs, we first consider the case of $\boldsymbol{T}_+$. Separation of the Dirac equation can be achieved with the ansatz

$$\psi_+ = \frac{\rho_b}{\rho}\begin{pmatrix} (r + ia\cos\theta)^{-1/2} R_+ S_+ \\ (r - ia\cos\theta)^{-1/2} R_+ S_- \\ (r - ia\cos\theta)^{-1/2} R_- S_+ \\ (r + ia\cos\theta)^{-1/2} R_- S_- \end{pmatrix} e^{i(h\varphi - \omega t)}\,, \tag{3.30}$$

with functions $R_\pm = R_\pm(r)$ and $S_\pm = S_\pm(\theta)$. Inserting this ansatz in (3.28), we obtain eight equations with four separation constants. The consistency of these equations implies that only one of the separation constants is independent, we denote it by κ. Finally one obtains the following four coupled first order ordinary differential equations for $R_\pm$ and $S_\pm$:

$$\begin{aligned}\frac{dR_\pm}{dr} + R_\pm\frac{\Delta' \pm 4iU_r}{4\Delta} + R_\mp\frac{mr\mp\kappa}{\sqrt{\Delta}} &= 0\,,\\ \frac{dS_\pm}{d\theta} + S_\pm\frac{\cos\theta \pm 2U_\theta}{2\sin\theta} + S_\mp\big(\pm ima\cos\theta - \kappa\big) &= 0\,,\end{aligned} \tag{3.31}$$

where U_r and U_θ are given by eq. (3.26) of the previous section.

Similarly, for $\boldsymbol{T}_-$ one has the separation ansatz

$$\psi_- = \frac{\rho_b}{\rho}\begin{pmatrix}(r-ia\cos\theta)^{-1/2}R_+S_+\\(r+ia\cos\theta)^{-1/2}R_+S_-\\(r+ia\cos\theta)^{-1/2}R_-S_+\\(r-ia\cos\theta)^{-1/2}R_-S_-\end{pmatrix}e^{i(h\varphi-\omega t)}\,, \tag{3.32}$$

and the functions $R_\pm$ and $S_\pm$ satisfy the following coupled ODEs:

$$\begin{aligned}\frac{dR_\pm}{dr} + R_\pm\frac{\Delta' \pm 4iU_r}{4\Delta} + R_\mp\frac{mr\mp\kappa}{\sqrt{\Delta}} &= 0\,,\\ \frac{dS_\pm}{d\theta} + S_\pm\frac{\cos\theta \pm 2U_\theta}{2\sin\theta} + S_\mp\big(\mp ima\cos\theta - \kappa\big) &= 0\,.\end{aligned} \tag{3.33}$$

This result is new and non-trivial even in the limit where the Kerr geometry is recovered.

In both cases, separability can be justified by general theory of section 2.4. Indeed, one can easily show that the anomalies (2.36) and (2.37) reduce to

$$(\boldsymbol{A}_\pm)_{(cl)} = \boldsymbol{T}_\pm \wedge \boldsymbol{\xi}_\pm\,,\quad (\boldsymbol{A}_\pm)_{(q)} = -\boldsymbol{\delta}(\boldsymbol{\xi}_\pm)\,, \tag{3.34}$$

and in both considered cases these vanish. The corresponding symmetry operators that commute with the Dirac operator (3.27) are therefore given by lemma 5 and read

$$M_\pm = (h_\pm)_{bc}\gamma^{abc}\nabla_a - \frac{2}{3}(\delta h_\pm)_a\gamma^a + \frac{1}{2}(T_\pm)^a{}_{bc}(h_\pm)_{ad}\gamma^{bcd} + \frac{1}{12}(T_\pm)^{ab}{}_c(h_\pm)_{ab}\gamma^c\,. \tag{3.35}$$

It can be explicitly verified that these commute with the Dirac operator. As expected, the demonstrated separability is underpinned by the existence of GCCKY tensors.

3.2 Kerr-Sen black hole in Einstein frame

Let us now briefly consider the Kerr-Sen geometry in Einstein frame, $\boldsymbol{g}_E = e^{-\Phi}\boldsymbol{g}$. Introducing an ortonormal basis of 1-forms

$$\begin{aligned}\boldsymbol{e}^0_E &= \frac{\sqrt{\Delta}}{\rho_b}\left(\boldsymbol{d}t - a\sin^2\theta\boldsymbol{d}\varphi\right), & \boldsymbol{e}^1_E &= \frac{\rho_b}{\sqrt{\Delta}}\boldsymbol{d}r\,,\\ \boldsymbol{e}^2_E &= \frac{\sin\theta}{\rho_b}\Big[a dt - (r^2+2br+a^2)d\varphi\Big]\,, & \boldsymbol{e}^3_E &= \rho_b\boldsymbol{d}\theta\,,\end{aligned} \tag{3.36}$$

the metric reads

$$\boldsymbol{g}_E = -\boldsymbol{e}_E^{0\,2} + \boldsymbol{e}_E^{1\,2} + \boldsymbol{e}_E^{2\,2} + \boldsymbol{e}_E^{3\,2}\,. \tag{3.37}$$

Obviously, the Kerr-Sen metric in Einstein frame is very similar to the Kerr geometry. Consequently it shares some of its miraculous properties. In particular, this is true for separability of various field equations in its background and for the existence of hidden symmetries. Let us first concentrate on hidden symmetries.

The metric possesses an irreducible Killing tensor [56]

$$\boldsymbol{K} = a^2\cos^2\theta\big(\boldsymbol{e}_E^0\boldsymbol{e}_E^0 - \boldsymbol{e}_E^1\boldsymbol{e}_E^1\big) + r(r+2b)\big(\boldsymbol{e}_E^2\boldsymbol{e}_E^2 + \boldsymbol{e}_E^3\boldsymbol{e}_E^3\big)\,, \tag{3.38}$$

which is responsible for complete integrability of motion of charged particles and separability of the charged scalar field [55]. Moreover, the metric admits two GCCKY 2-forms

$$\boldsymbol{h}_\pm = \sqrt{r(r+2b)}\,\boldsymbol{e}_E^0\wedge\boldsymbol{e}_E^1 \pm a\cos\theta\,\boldsymbol{e}_E^2\wedge\boldsymbol{e}_E^3 \tag{3.39}$$

with respect to the torsions

$$\boldsymbol{T}_\pm = T_0^\pm\boldsymbol{e}_E^{012} + T_1^\pm\boldsymbol{e}_E^{023}\,, \tag{3.40}$$

where

$$T_0^\pm = \frac{2a\sin\theta}{\rho_b^3}\left[r+b\mp\sqrt{r(r+2b)}\right]\,,\quad T_1^\pm = \mp\frac{2a\cos\theta\sqrt{\Delta}}{\rho_b^3\sqrt{r(r+2b)}}\left(r+b\mp\sqrt{r(r+2b)}\right)\,. \tag{3.41}$$

One can easily show that for both choices, anomalies (2.36) and (2.37) vanish and operators (3.35) give commuting operators for the corresponding Dirac equations. By calculations analogous to section 3.1.5, one can demonstrate that both Dirac equations separate. The meaning of torsions $\boldsymbol{T}_\pm$ is at the moment unclear. Let us finally mention that as is the case in the Kerr geometry [61], one can prove complete integrability of stationary string configurations in the background (3.37).

4 Higher-dimensional charged Kerr-NUT spacetimes

Let us now consider the following higher-dimensional generalization of the string frame action (3.1):

$$S = \int_{M^D} e^{\phi\sqrt{D/2-1}}\left(*R + \frac{D-2}{2}*\boldsymbol{d}\phi\wedge\boldsymbol{d}\phi - *\boldsymbol{F}\wedge\boldsymbol{F} - \frac{1}{2}*\boldsymbol{H}\wedge\boldsymbol{H}\right)\,, \tag{4.1}$$

where $\boldsymbol{F} = \boldsymbol{dA}$ and $\boldsymbol{H} = \boldsymbol{dB} - \boldsymbol{A}\wedge\boldsymbol{dA}$. The system consists of a metric g_{ab}, scalar field ϕ, U(1) potential $\boldsymbol{A}$, and 2-form potential $\boldsymbol{B}$. This kind of action gives a bosonic part of supergravity such as heterotic supergravity compactified on a torus. The corresponding equations of motion are

$$\begin{aligned} R_{ab} - \frac{1}{2}Rg_{ab} =& \sqrt{\frac{D-2}{2}}\nabla_a\nabla_b\phi - \sqrt{\frac{D-2}{2}}\,g_{ab}\nabla^2\phi - \frac{D-2}{4}\,g_{ab}(\nabla\phi)^2 \\ &+ \left(F_a{}^cF_{bc} - \frac{1}{4}g_{ab}F^2\right) + \frac{1}{4}\left(H_a{}^{cd}H_{bcd} - \frac{1}{6}g_{ab}H^2\right)\,, \end{aligned}$$

$$
\begin{aligned}
&\boldsymbol{d}\Big(e^{\phi\sqrt{D/2-1}}{*\boldsymbol{F}}\Big) + (-1)^D e^{\phi\sqrt{D/2-1}}{*\boldsymbol{H}}\wedge\boldsymbol{F} = 0\,, \qquad \boldsymbol{d}\Big(e^{\phi\sqrt{D/2-1}}{*\boldsymbol{H}}\Big) = 0\,,\\
&R - \frac{D-2}{2}(\nabla\phi)^2 - \sqrt{2(D-2)}\nabla^2\phi - \frac{1}{2}F^2 - \frac{1}{12}H^2 = 0.
\end{aligned}
\tag{4.2}
$$

Alternatively, one could consider an Einstein frame metric, $\boldsymbol{g}_E = e^{\phi\sqrt{2/(D-2)}}\boldsymbol{g}$. After this transformation, the action (4.1) is equivalent to the action (2.1) in [34] in the case when two scalar and two U(1) charges are set equal.

4.1 Metric and fields

The most general known 'charged Kerr-NUT' solution of the theory (4.1) in all dimensions was obtained by Chow [34]. It generalizes the equal charge solution of Cvetic and Youm [33] by including NUT parameters which in its turn is a $D \geq 4$ generalization of the Kerr-Sen solution studied in the previous section. One can also understand Chow's solution as a generalization of the Kerr-NUT solution of Chen, Lü, and Pope [4] by including matter fields ϕ, $\boldsymbol{A}$ and $\boldsymbol{B}$. The metric and the fields are given by[6]

$$
\begin{aligned}
\boldsymbol{g} &= \sum_{\mu=1}^{n}\frac{\boldsymbol{d}x_\mu^2}{Q_\mu} + \sum_{\mu=1}^{n}Q_\mu\Big(\boldsymbol{\mathcal{A}}_\mu - \sum_{\nu=1}^{n}\frac{2N_\nu s^2}{HU_\nu}\boldsymbol{\mathcal{A}}_\nu\Big)^2 + \varepsilon S\Big(\boldsymbol{\mathcal{A}} - \sum_{\nu=1}^{n}\frac{2N_\nu s^2}{HU_\nu}\boldsymbol{\mathcal{A}}_\nu\Big)^2\,,\\
\phi &= \sqrt{\frac{2}{D-2}}\ln H\,,\quad \boldsymbol{A} = \sum_{\nu=1}^{n}\frac{2N_\nu sc}{HU_\nu}\boldsymbol{\mathcal{A}}_\nu\,,\\
\boldsymbol{B} &= \Big(\sum_{k=0}^{n-1}(-1)^k c_{n-k-1}\boldsymbol{d}\psi_k + \varepsilon\,\tilde{c}\boldsymbol{d}\psi_n\Big)\wedge\Big(\sum_{\nu=1}^{n}\frac{2N_\nu s^2}{HU_\nu}\boldsymbol{\mathcal{A}}_\nu\Big)\,,
\end{aligned}
\tag{4.3}
$$

where we have defined the following 1-forms:

$$
\boldsymbol{\mathcal{A}}_\mu = \sum_{k=0}^{n-1}A_\mu^{(k)}\boldsymbol{d}\psi_k\,,\qquad \boldsymbol{\mathcal{A}} = \sum_{k=0}^{n}A^{(k)}\boldsymbol{d}\psi_k\,,
\tag{4.4}
$$

and the following functions:

$$
\begin{aligned}
&H = 1 + \sum_{\mu=1}^{n}\frac{2N_\mu s^2}{U_\mu}\,,\quad N_\mu = m_\mu x_\mu^{1-\varepsilon}\,,\qquad S = \frac{\tilde{c}}{A^{(n)}}\,,\\
&Q_\mu = \frac{X_\mu}{U_\mu}\,,\qquad U_\mu = \prod_{\substack{\nu=1\\ \nu\neq\mu}}^{n}(x_\mu^2 - x_\nu^2)\,,\quad X_\mu = \sum_{k=0}^{n-1}c_k x_\mu^{2k} + 2N_\mu + \varepsilon\,\frac{(-1)^n\tilde{c}}{x_\mu^2}\,,\\
&A_\mu^{(k)} = \sum_{\substack{1\leq\nu_1<\cdots<\nu_k\leq n\\ \nu_i\neq\mu}} x_{\nu_1}^2\cdots x_{\nu_k}^2\,,\quad A^{(k)} = \sum_{1\leq\nu_1<\cdots<\nu_k\leq n} x_{\nu_1}^2\cdots x_{\nu_k}^2\,,\quad A_\mu^{(0)} = A^{(0)} = 1\,.
\end{aligned}
\tag{4.5}
$$

[6] As usual for these kind of solutions (see, e.g., [4, 5]), we work with an analytically continued metric, where one of the x_μ correspond to the Wick rotated radial coordinate r and (in even dimensions) the corresponding parameter m_μ is imaginary mass. The advantage of this continuation is that the radial and longitudinal coordinates appear on exactly the same footing and the metric takes an extremely symmetric form. Let us stress that working in this continuation affects neither the existence of hidden symmetries nor separability of the field equations studied below.

We have introduced $s = \sinh\delta$, $c = \cosh\delta$, $c_{n-1} = -1$, and m_μ ($\mu = 1,\dots,n$), c_k ($k = 0,\dots,n-2$), $\tilde{c}$, and δ are arbitrary constants. We have verified directly that these fields satisfy the equations of motion (4.2).

Let us remark that the Kerr-NUT solution [4] is recovered for $\delta = 0$. On the other hand, when $D = 4$ and the NUT parameter m_2 is set to zero, one recovers the Kerr-Sen solution of the previous section if the fields are rescaled as $\phi \to -\Phi$ and $\boldsymbol{A} \to \boldsymbol{A}/2$, and the following transformation of coordinates, and parameters is performed:

$$\begin{aligned} &x_1 \to ir\,, && x_2 \to a\cos\theta\,, && \psi_0 \to t - a\varphi\,, && \psi_1 \to \varphi/a\,, \\ &c_0 \to a^2\,, && 2m_1 s^2 \to 2ib\,, && im_1 \to b - M\,. \end{aligned} \tag{4.6}$$

Let us also introduce the orthonormal basis

$$\boldsymbol{e}^\mu = \frac{\boldsymbol{d}x_\mu}{\sqrt{Q_\mu}}\,,\quad \boldsymbol{e}^{\hat\mu} = \sqrt{Q_\mu}\Big(\boldsymbol{\mathcal{A}}_\mu - \sum_{\nu=1}^{n}\frac{2N_\nu s^2}{HU_\nu}\boldsymbol{\mathcal{A}}_\nu\Big)\,,\quad \boldsymbol{e}^0 = \sqrt{S}\Big(\boldsymbol{\mathcal{A}} - \sum_{\nu=1}^{n}\frac{2N_\nu s^2}{HU_\nu}\boldsymbol{\mathcal{A}}_\nu\Big)\,, \tag{4.7}$$

in which the metric and the field strengths are written as

$$\begin{aligned} \boldsymbol{g} &= \sum_{\mu=1}^{n}(\boldsymbol{e}^\mu\boldsymbol{e}^\mu + \boldsymbol{e}^{\hat\mu}\boldsymbol{e}^{\hat\mu}) + \varepsilon\boldsymbol{e}^0\boldsymbol{e}^0\,, \\ \boldsymbol{F} &= \frac{c}{s}\sum_{\rho=1}^{n} H_\rho\,\boldsymbol{e}^\rho\wedge\boldsymbol{e}^{\hat\rho}\,, \\ \boldsymbol{H} &= -\Big(\sum_{\mu=1}^{n}\sqrt{Q_\mu}\,\boldsymbol{e}^{\hat\mu} + \varepsilon\sqrt{S}\,\boldsymbol{e}^0\Big)\wedge\Big(\sum_{\rho=1}^{n}H_\rho\,\boldsymbol{e}^\rho\wedge\boldsymbol{e}^{\hat\rho}\Big)\,, \end{aligned} \tag{4.8}$$

where we have denoted $H_\mu = \partial_\mu \ln H$. The inverse frame is given by

$$\begin{aligned} \boldsymbol{X}_\mu &= \sqrt{Q_\mu}\,\boldsymbol{\partial}_{x_\mu}\,, \\ \boldsymbol{X}_{\hat\mu} &= \sum_{k=0}^{n-1}\frac{(-1)^k x_\mu^{2(n-k-1)}}{U_\mu\sqrt{Q_\mu}}\,\boldsymbol{\partial}_{\psi_k} + \frac{2N_\mu s^2}{U_\mu\sqrt{Q_\mu}}\,\boldsymbol{\partial}_{\psi_0} + \frac{\varepsilon(-1)^n x_\mu^{-2}}{U_\mu\sqrt{Q_\mu}}\,\boldsymbol{\partial}_{\psi_n}\,, \\ \boldsymbol{X}_0 &= \frac{1}{\sqrt{S}A^{(n)}}\boldsymbol{\partial}_{\psi_n}\,, \end{aligned} \tag{4.9}$$

and the spin connection is calculated to be [34][7]

$$\begin{aligned} \boldsymbol{\omega}^\mu{}_\nu &= -\frac{x_\nu\sqrt{Q_\nu}}{x_\mu^2 - x_\nu^2}\,\boldsymbol{e}^\mu - \frac{x_\mu\sqrt{Q_\mu}}{x_\mu^2 - x_\nu^2}\,\boldsymbol{e}^\nu \quad (\text{for } \mu\neq\nu)\,, \\ \boldsymbol{\omega}^\mu{}_{\hat\mu} &= -H\partial_\mu\Big(\frac{\sqrt{Q_\mu}}{H}\Big)\,\boldsymbol{e}^{\hat\mu} + \sum_{\rho\neq\mu}\frac{\sqrt{Q_\rho}}{2}\partial_\mu\ln(HU_\rho)\,\boldsymbol{e}^{\hat\rho} + \varepsilon\sqrt{S}\Big(\frac{1}{x_\mu} + \frac{1}{2}\partial_\mu\ln H\Big)\,\boldsymbol{e}^0 \\ \boldsymbol{\omega}^\mu{}_{\hat\nu} &= \frac{\sqrt{Q_\nu}}{2}\partial_\mu\ln(HU_\nu)\,\boldsymbol{e}^{\hat\mu} - \frac{x_\mu\sqrt{Q_\mu}}{x_\mu^2 - x_\nu^2}\,\boldsymbol{e}^{\hat\nu} \quad (\text{for } \mu\neq\nu)\,, \end{aligned}$$

[7] Note that there is a typo in the second line in (3.18) of [34].

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$$
\begin{aligned}
\omega^{\hat\mu}{}_{\hat\nu} &= -\frac{\sqrt{Q_\nu}}{2}\partial_\mu \ln(HU_\nu)\, e^\mu + \frac{\sqrt{Q_\mu}}{2}\partial_\nu \ln(HU_\mu)\, e^\nu \quad (\text{for } \mu\neq\nu)\,,\\
\omega^{\mu}{}_{0} &= \sqrt{S}\Big(\frac{1}{x_\mu}+\frac{1}{2}\partial_\mu \ln H\Big)\, e^{\hat\mu} - \frac{\sqrt{Q_\mu}}{x_\mu}\, e^0\,,\\
\omega^{\hat\mu}{}_{0} &= -\sqrt{S}\Big(\frac{1}{x_\mu}+\frac{1}{2}\partial_\mu \ln H\Big)\, e^{\mu}\,.
\end{aligned} \tag{4.10}
$$

4.2 Hidden symmetries

The metric (4.3) possesses $n+\varepsilon$ obvious isometries $\boldsymbol{\partial}_{\psi_k}$ $(k=0,\ldots,n-1+\varepsilon)$. Our claim is that in addition to these Killing vectors, the metric possesses a GCCKY 2-form $\boldsymbol{h}$,

$$
\boldsymbol{h} = \sum_{\mu=1}^{n} x_\mu\, \boldsymbol{e}^\mu \wedge \boldsymbol{e}^{\hat\mu}\,, \tag{4.11}
$$

which represents a natural generalization of the PCKY tensor of the Kerr-NUT spacetime [5] with respect to the following torsion:

$$
\boldsymbol{T} = -\sum_{\mu=1}^{n}\sum_{\substack{\nu=1\\ \nu\neq\mu}}^{n} \sqrt{Q_\mu} H_\nu\, \boldsymbol{e}^{\hat\mu\nu\hat\nu} - \varepsilon \sum_{\mu=1}^{n}\sqrt{S} H_\mu\, \boldsymbol{e}^{0\mu\hat\mu} + \varepsilon \sum_{\mu=1}^{n} \frac{f}{x_\mu}\, \boldsymbol{e}^{0\mu\hat\mu}\,, \tag{4.12}
$$

where f is an arbitrary function. Notice that this torsion is unique in an even number of spacetime dimensions; the non-uniqueness in odd dimensions, expressed by function f, follows from the fact that the 2-form $\boldsymbol{h}$ is necessary degenerate in odd dimensions. Using the orthonormal basis (4.7) and the connection (4.10) one can verify that

$$
\begin{aligned}
\nabla^T_{X_\mu}\boldsymbol{h} &= \sum_{\nu=1}^{n} \sqrt{Q_\nu}\, \boldsymbol{e}^\mu \wedge \boldsymbol{e}^{\hat\nu} + \varepsilon\sqrt{S}\, \boldsymbol{e}^\mu \wedge \boldsymbol{e}^0 + \frac{\varepsilon}{2} f\, \boldsymbol{e}^\mu \wedge \boldsymbol{e}^0\,,\\
\nabla^T_{X_{\hat\mu}}\boldsymbol{h} &= \sum_{\substack{\nu=1\\ \nu\neq\mu}}^{n} \sqrt{Q_\nu}\, \boldsymbol{e}^{\hat\mu} \wedge \boldsymbol{e}^{\hat\nu} + \varepsilon\sqrt{S}\, \boldsymbol{e}^{\hat\mu} \wedge \boldsymbol{e}^0 + \frac{\varepsilon}{2} f\, \boldsymbol{e}^{\hat\mu} \wedge \boldsymbol{e}^0\,,\\
\nabla^T_{X_0}\boldsymbol{h} &= -\sum_{\rho=1}^{n} \sqrt{Q_\rho}\, \boldsymbol{e}^{\hat\rho} \wedge \boldsymbol{e}^0\,.
\end{aligned} \tag{4.13}
$$

and also

$$
\boldsymbol{\xi} = -\frac{1}{D-1}\,\boldsymbol{\delta}^T\boldsymbol{h} = \sum_{\mu=1}^{n} \sqrt{Q_\mu}\, \boldsymbol{e}^{\hat\mu} + \varepsilon\left(\sqrt{S}+\frac{f}{2}\right)\boldsymbol{e}^0\,. \tag{4.14}
$$

It is then easy to prove that $\boldsymbol{h}$ obeys (2.40) for any vector field $\boldsymbol{X}$ and hence it is a GCCKY 2-form.

Let us note that if we choose $f=0$, the torsion $\boldsymbol{T}$ becomes very natural and can be identified with the 3-form field strength $\boldsymbol{H}$. In that case we also have

$$
\boldsymbol{T} = \boldsymbol{H} = -\frac{s}{c}\,\boldsymbol{F}\wedge\boldsymbol{\xi}\,. \tag{4.15}
$$

In any case, the GCCKY 2-form $\boldsymbol{h}$ gives rise to towers of hidden symmetries as discussed in section 2.5.2. In particular, one obtains the tower of GCCKY $(2j)$-forms $\boldsymbol{h}^{(j)}$, $j = 1, \ldots, n-1$, and the following mutually Schouten commuting rank-2 irreducible Killing tensors [34]:

$$\boldsymbol{K}^{(j)} = \sum_{\mu=1}^{n} A_\mu^{(j)} (\boldsymbol{e}^\mu \boldsymbol{e}^\mu + \boldsymbol{e}^{\hat{\mu}} \boldsymbol{e}^{\hat{\mu}}) + \varepsilon A^{(j)} \boldsymbol{e}^0 \boldsymbol{e}^0 \,, \qquad j = 1, \ldots, n-1 \,. \tag{4.16}$$

Together with the Killing vector fields $\boldsymbol{\partial}_{\psi_k}$ these Killing tensors are responsible for complete integrability of geodesic motion in the charged Kerr-NUT spacetimes, as discussed by Chow [34]. Similarly, the GCCKY tensors $\boldsymbol{h}^{(j)}$ are responsible for separability of the Dirac equation. This requires the vanishing of the anomalies, explicitly demonstrated in appendix A. We shall now demonstrate that these hidden symmetries allow one to separate the scalar and Dirac test fields in the charged Kerr-NUT background. For simplicity, we consider only uncharged fields; the calculations extend results demonstrated in [9, 11].

4.3 Separability of the scalar equation

As in the four-dimensional case, let us consider a new scalar field φ which in string frame obeys the following 'massless' equation:[8]

$$\Box\varphi + \sqrt{\frac{D-2}{2}} \nabla_a \varphi \nabla^a \phi = 0 \,, \tag{4.17}$$

where the background scalar field ϕ is given by (4.3), $\phi = \sqrt{\frac{2}{D-2}} \ln H$. This equation can be written as

$$\Box\varphi + \sum_{\mu=1}^{n} H_\mu Q_\mu \frac{\partial \varphi}{\partial x_\mu} = 0 \,, \tag{4.18}$$

and, using the basis (4.9), it takes the following explicit form:

$$\begin{aligned} \sum_{\mu=1}^{n} \frac{1}{U_\mu} \Bigg\{ X_\mu \frac{\partial^2 \varphi}{\partial x_\mu^2} + X_\mu' \frac{\partial \varphi}{\partial x_\mu} + \frac{1}{X_\mu} \Bigg[\sum_{k=0}^{n-1} (-1)^k x_\mu^{2(n-k-1)} \frac{\partial}{\partial \psi_k} + 2N_\mu s^2 \frac{\partial}{\partial \psi_0} + \varepsilon \frac{(-1)^n}{x_\mu^2} \frac{\partial}{\partial \psi_n} \Bigg]^2 \varphi \\ + \varepsilon \left[\frac{(-1)^{n-1}}{\tilde{c} x_\mu^2} \frac{\partial^2 \varphi}{\partial \psi_n^2} + \frac{X_\mu}{x_\mu} \frac{\partial \varphi}{\partial x_\mu} \right] \Bigg\} = 0 \,. \end{aligned} \tag{4.19}$$

This equation allows a multiplicative separation of variables

$$\varphi = \prod_{\mu=1}^{n} R_\mu(x_\mu) \prod_{k=0}^{n-1+\varepsilon} e^{i p_k \psi_k} \,. \tag{4.20}$$

Indeed, plugging this ansatz into eq. (4.19), it assumes the form

$$\sum_{\mu=1}^{n} \frac{G_\mu}{U_\mu} \varphi = 0 \,, \tag{4.21}$$

[8] This equation is equivalent to the massless Klein-Gordon equation in the Einstein frame $\Box_E \varphi = 0$, which was proved to separate by Chow [34].

where G_μ is a function of x_μ only

$$G_\mu = X_\mu \frac{R''_\mu}{R_\mu} + \left(X'_\mu + \varepsilon \frac{X_\mu}{x_\mu}\right) \frac{R'_\mu}{R_\mu} - \frac{W_\mu^2}{X_\mu} + \varepsilon \frac{(-1)^n p_n^2}{\tilde{c} x_\mu^2}\,, \tag{4.22}$$

and

$$W_\mu = \sum_{k=0}^{n-1} (-1)^k x_\mu^{2(n-k-1)} p_k + 2N_\mu s^2 p_0 + \varepsilon \frac{(-1)^n p_n}{x_\mu^2}\,. \tag{4.23}$$

The general solution of (4.21) is

$$G_\mu = \sum_{j=1}^{n-1} k_j x_\mu^{2(n-1-j)}\,, \tag{4.24}$$

where k_j are arbitrary constants. Hence, the functions R_μ satisfy the ordinary second order differential equations

$$(X_\mu R'_\mu)' + \varepsilon \frac{X_\mu}{x_\mu} R'_\mu - \left(\frac{W_\mu^2}{X_\mu} + \sum_{j=1}^{n-1} k_j x_\mu^{2(n-1-j)} - \varepsilon \frac{(-1)^n p_n^2}{\tilde{c} x_\mu^2} \right) R_\mu = 0\,, \tag{4.25}$$

and we have shown that the scalar field equation (4.17) admits the multiplicative separation of variables (4.20).

4.4 Separability of the Dirac equation

Finally, we demonstrate separability of the torsion modified Dirac equation. For the time being, we will work with the torsion (4.12), including an arbitrary function in odd spacetime dimensions.[9] For simplicity we consider an uncharged field for which the Dirac equation reads

$$\big(\gamma^a D_a + m\big)\Psi = 0\,, \qquad D_a = X_a + \frac{1}{4}\gamma^b\gamma^c \omega_{bc}(X_a) - \frac{1}{24}\gamma^b\gamma^c T_{abc}\,. \tag{4.26}$$

Using the connection (4.10), the inverse basis (4.9) and the torsion (4.12) this equation takes the following explicit form:

$$\begin{aligned}
\Bigg\{ \sum_{\mu=1} \gamma^\mu \sqrt{Q_\mu} & \left[\frac{\partial}{\partial x_\mu} + \frac{X'_\mu}{4X_\mu} + \frac{\varepsilon}{2x_\mu} + \frac{1}{2} \sum_{\nu \neq \mu} \frac{x_\mu}{x_\mu^2 - x_\nu^2} \right] \\
& + \sum_{\mu=1}^{n} \gamma^{\hat{\mu}} \sqrt{Q_\mu} \left[\sum_{k=0}^{n-1+\varepsilon} \frac{(-1)^k x_\mu^{2(n-k-1)}}{X_\mu} \frac{\partial}{\partial \psi_k} + \frac{2N_\mu s^2}{X_\mu} \frac{\partial}{\partial \psi_0} + \frac{1}{2} \sum_{\substack{\nu=1 \\ \nu\neq\mu}}^{n} \frac{x_\nu}{x_\mu^2 - x_\nu^2} (\gamma^\nu \gamma^{\hat{\nu}}) \right] \\
& + \varepsilon\, \gamma^0 \sqrt{S} \left[\frac{1}{c} \frac{\partial}{\partial \psi_n} - \sum_{\mu=1}^{n} \frac{F}{x_\mu} (\gamma^\nu \gamma^{\hat{\nu}}) \right] + m \Bigg\} \tilde{\Psi} \;=\; 0\,,
\end{aligned} \tag{4.27}$$

[9]We will find in appendix A that we must specialize to the case $\boldsymbol{T} = \boldsymbol{H}$, that is, $f = 0$, in order both anomalies (2.36) and (2.37) vanish. However, separability of the Dirac equation can be demonstrated for other choices of f as well.

where we have set $\Psi = \sqrt{H}\widetilde{\Psi}$, using the fact that

$$\left(\frac{\partial}{\partial x_\mu} - \frac{H_\mu}{2}\right)(\sqrt{H}\widetilde{\Psi}) = \sqrt{H}\frac{\partial \widetilde{\Psi}}{\partial x_\mu}\,, \tag{4.28}$$

and we define an arbitrary function F by $F = 1/2 + f(4\sqrt{S})^{-1}$.

Let us use the following representation of γ-matrices: $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$,

$$\begin{aligned}
\gamma^\mu &= \underbrace{\sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3}_{\mu-1} \otimes \sigma_1 \otimes I \otimes \cdots \otimes I\,, \\
\gamma^{\hat\mu} &= \underbrace{\sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3}_{\mu-1} \otimes \sigma_2 \otimes I \otimes \cdots \otimes I\,, \\
\gamma^0 &= \sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3\,,
\end{aligned} \tag{4.29}$$

where I is the 2×2 identity matrix and σ_i are the Pauli matrices. In this representation, we write the 2^n components of the spinor field as $\Psi_{\epsilon_1\epsilon_2\cdots\epsilon_n}$ $(\epsilon_\mu = \pm 1)$, and it follows that

$$\begin{aligned}
(\gamma^\mu\Psi)_{\epsilon_1\epsilon_2\cdots\epsilon_n} &= \left(\prod_{\nu=1}^{\mu-1}\epsilon_\nu\right)\Psi_{\epsilon_1\cdots\epsilon_{\mu-1}(-\epsilon_\mu)\epsilon_{\mu+1}\cdots\epsilon_n}\,, \\
(\gamma^{\hat\mu}\Psi)_{\epsilon_1\epsilon_2\cdots\epsilon_n} &= -i\epsilon_\mu\left(\prod_{\nu=1}^{\mu-1}\epsilon_\nu\right)\Psi_{\epsilon_1\cdots\epsilon_{\mu-1}(-\epsilon_\mu)\epsilon_{\mu+1}\cdots\epsilon_n}\,, \\
(\gamma^0\Psi)_{\epsilon_1\epsilon_2\cdots\epsilon_n} &= \left(\prod_{\rho=1}^{n}\epsilon_\rho\right)\Psi_{\epsilon_1\cdots\epsilon_n}\,.
\end{aligned} \tag{4.30}$$

We consider the separable solution

$$\widetilde{\Psi} = \hat{\Psi}(x)\exp\left(i\sum_{k=0}^{n-1+\varepsilon} p_k\psi_k\right), \tag{4.31}$$

where p_k $(k = 0, \ldots, n-1+\varepsilon)$ are arbitrary constants. Using (4.30), we obtain

$$\begin{aligned}
\Bigg\{\sum_{\mu=1}\sqrt{Q_\mu}\left(\prod_{\rho=1}^{\mu-1}\epsilon_\rho\right)&\left[\frac{\partial}{\partial x_\mu} + \frac{X'_\mu}{4X_\mu} + \frac{\varepsilon}{2x_\mu} + \frac{\epsilon_\mu Y_\mu}{X_\mu} + \frac{1}{2}\sum_{\nu\neq\mu}\frac{1-\epsilon_\mu\epsilon_\nu}{x_\mu + x_\nu}\right] \\
+\,\varepsilon\, i\sqrt{S}\left(\prod_{\rho=1}^{\mu-1}\epsilon_\rho\right)\left[\frac{p_n}{c} - \sum_{\mu=1}^{n}\frac{\epsilon_\mu F}{x_\mu}\right]\Bigg\}&\hat{\Psi}_{\epsilon_1\ldots\epsilon_{\mu-1}(-\epsilon_\mu)\epsilon_{\mu+1}\ldots\epsilon_n} + m\,\hat{\Psi}_{\epsilon_1\ldots\epsilon_n} \;=\; 0\,,
\end{aligned} \tag{4.32}$$

where

$$Y_\mu = \sum_{k=0}^{n-1+\varepsilon}(-1)^k x_\mu^{2(n-k-1)}p_k + 2N_\mu s^2 p_0\,. \tag{4.33}$$

Following further [11] we set[10]

$$\hat{\Psi}_{\epsilon_1\ldots\epsilon_n}(x) = \left(\prod_{1\le\mu<\nu\le n} \frac{1}{\sqrt{x_\mu + \epsilon_\mu\epsilon_\nu x_\nu}} \right) \left(\prod_{\mu=1}^{n} \chi^{(\mu)}_{\epsilon_\mu}(x_\mu) \right) . \tag{4.34}$$

Thereafter, we have the following equation following from (4.32):

$$\sum_{\mu=1}^{n} \frac{P^{(\mu)}_{\epsilon_\mu}(x_\mu)}{\prod_{\substack{\nu=1\\ \nu\neq\mu}}^{n}(\epsilon_\mu x_\mu - \epsilon_\nu x_\nu)} + \frac{\varepsilon\, i\sqrt{c}}{\prod_{\rho=1}^{n}(\epsilon_\rho x_\rho)} \left(-\sum_{\mu=1}^{n} \frac{F}{\epsilon_\mu x_\mu} + \frac{p_n}{c} \right) + m = 0 \,, \tag{4.35}$$

where

$$P^{(\mu)}_{\epsilon_\mu} = (-1)^\mu (\epsilon_\mu)^{n-\mu} \sqrt{(-1)^{\mu-1} X_\mu}\, \frac{1}{\chi^{(\mu)}_{\epsilon_\mu}} \left(\frac{d}{dx_\mu} + \frac{X'_\mu}{4X_\mu} + \frac{\epsilon_\mu Y_\mu}{X_\mu} \right) \chi^{(\mu)}_{-\epsilon_\mu} \,, \tag{4.36}$$

are functions of x_μ only.

In order to satisfy (4.35) $P^{(\mu)}_{\epsilon_\mu}$ must assume the form

$$P^{(\mu)}_{\epsilon_\mu}(x_\mu) = Q(\epsilon_\mu x_\mu) \,, \tag{4.37}$$

where (a) in an even dimension ($\varepsilon = 0$) one has

$$Q(y) = -m y^{n-1} + \sum_{j=0}^{n-2} q_j y^j \,, \tag{4.38}$$

whereas (b) in an odd dimension ($\varepsilon = 1$) Q is a solution of

$$\sum_{\mu=1}^{n} \frac{Q(y_\mu)}{\prod_{\substack{\nu=1\\ \nu\neq\mu}}^{n}(y_\mu - y_\nu)} + \frac{i\sqrt{c}}{\prod_{\rho=1}^{n} y_\rho} \left(-\sum_{\mu=1}^{n} \frac{F}{y_\mu} + \frac{p_n}{c} \right) + m = 0 \,. \tag{4.39}$$

In particular, for $F = 1/2$ (which corresponds to the natural torsion $\boldsymbol{T} = \boldsymbol{H}$) we have

$$Q(y) = \sum_{j=-2}^{n-1} q_j y^j \,, \quad q_{n-1} = -m \,, \quad q_{-1} = \frac{i}{\sqrt{c}}(-1)^n p_n \,, \quad q_{-2} = \frac{i}{2}(-1)^{n-1}\sqrt{c} \,. \tag{4.40}$$

In both cases parameters q_j ($j = 0, \ldots, n-2$) are arbitrary.

Let us summarize our result. We have proved that the torsion modified Dirac equation (4.26) in the charged Kerr-NUT spacetime (4.3) allows separation of variables

$$\Psi_{\epsilon_1\ldots\epsilon_n} = \sqrt{H} \left(\prod_{1\le\mu<\nu\le n} \frac{1}{\sqrt{x_\mu + \epsilon_\mu\epsilon_\nu x_\nu}} \right) \left(\prod_{\mu=1}^{n} \chi^{(\mu)}_{\epsilon_\mu}(x_\mu) \right) \exp\left(i \sum_{k=0}^{n-1+\varepsilon} p_k \psi_k \right) , \tag{4.41}$$

[10] Note that there are some mistakes in [11]. In eq. (24) one needs to add a term $1/(2x_\mu)$ and replace the coefficient standing by X'_μ/X_μ by 1/4. In eq. (39) one should have

$$q_{-1} = \frac{i}{\sqrt{c}}(-1)^n N_n \,, \quad q_{-2} = \frac{i}{2}(-1)^{n-1}\sqrt{c} \,.$$

where functions $\chi^{(\mu)}_{\epsilon_\mu}$ satisfy the (coupled) ordinary first order differential equations

$$\left(\frac{d}{dx_\mu} + \frac{1}{4}\frac{X'_\mu}{X_\mu} + \frac{\epsilon_\mu Y_\mu}{X_\mu}\right)\chi^{(\mu)}_{-\epsilon_\mu} - \frac{(-1)^{\mu-1}(\epsilon_\mu)^{n-\mu}Q(\epsilon_\mu x_\mu)}{\sqrt{(-1)^{\mu-1}X_\mu}}\chi^{(\mu)}_{\epsilon_\mu} = 0\,. \tag{4.42}$$

The demonstrated separation is justified by the existence of the GCCKY 2-form $\boldsymbol{h}$. As in the four-dimensional case it is demonstrated in appendix A that for all GCCKY $(2j)$-forms $\boldsymbol{h}^{(j)}$ both anomalies (2.36) and (2.37) vanish and the corresponding operators of lemma 5 provide symmetry operators which commute with the modified Dirac operator.

5 Conclusions

In this paper we have studied an extension of Killing-Yano symmetry in the presence of skew-symmetric torsion. We have demonstrated, that when the torsion is an arbitrary 3-form, one obtains various torsion anomalies and the implications of the existence of the generalized Killing-Yano symmetry are relatively weak. For example, contrary to the vacuum case neither complete integrability of geodesic motion nor separability of test field equations are implied in general. However, in the spacetimes where there is a natural 3-form, obeying the appropriate field equations, these anomalies may disappear and the concept of generalized Killing-Yano symmetry may become very powerful. This is for example the case for the black hole of minimal (gauged) supergravity where the torsion is identified with the dual of Maxwell field [23], or, as demonstrated in this paper, in the case of the Kerr-Sen solution of effective string theory and its higher-dimensional generalizations where the torsion is identified with the 3-form $\boldsymbol{H}$. The latter result also straightforwardly generalizes to $D = 4, 5, 6$ black holes of gauged supergravities [62–65] in the case when two U(1) charges are set equal and other charges vanish. The reason for this is that such solutions can be cast in the form (4.3) while the 'gauging' affects only the explicit form of metric functions $X_\mu = X_\mu(x_\mu)$. The results concerning the existence of hidden symmetries and separability demonstrated in section 4 are, however, valid for arbitrary $X_\mu = X_\mu(x_\mu)$, and so, remain valid for these gauged solutions as well. In all the cases the choice of torsion is very natural and the generalized Killing-Yano symmetry carries non-trivial information about the spacetime; for example it underlines complete integrability of geodesic motion as well as separability of the scalar and Dirac equations.

Let us finally remark that in this paper we have not investigated the separability of Maxwell or gravitational perturbations. In four dimensions there is a direct link between the existence of Killing-Yano tensors and separability of Maxwell equation (see, e.g., [66–68]), which can, perhaps, be extended to the generalized Killing-Yano tensors as well. In higher dimensions such a link, if it exists, remains to be shown. Regarding the separability of gravitational perturbation the connection with Killing-Yano tensors is not obvious even in four dimensions.

It is an interesting question whether the Killing-Yano symmetry and its generalizations can provide new insights into the theory of black holes beyond its many contributions to the vacuum theory.

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A Vanishing of anomalies

In this appendix we wish to justify (at least partially) the separability of the modified Dirac equation for $\boldsymbol{T} = \boldsymbol{H}$ in charged Kerr-NUT spacetimes by proving that for all GC-CKY $(2j)$-forms $\boldsymbol{h}^{(j)}$ both anomalies (2.36) and (2.37) vanish and hence the corresponding operators $M_{h^{(j)}}$, (2.39), give symmetry operators for the Dirac operator (2.35).[11] Recall that from (4.11) the GCCKY 2-form $\boldsymbol{h}^{(1)} = \boldsymbol{h}$ is given by

$$\boldsymbol{h} = \sum_{\mu=1}^{n} x_\mu e^\mu \wedge e^{\hat{\mu}}. \tag{A.1}$$

The torsion 3-form is given by

$$\boldsymbol{T} = -\sum_{\mu\neq\nu} \sqrt{Q_\mu} H_\nu e^{\hat{\mu}} \wedge e^\nu \wedge e^{\hat{\nu}} - \varepsilon \sum_{\mu} \sqrt{S} H_\mu e^0 \wedge e^\mu \wedge e^{\hat{\mu}}, \tag{A.2}$$

where we have taken the arbitrary function f in (4.12) to vanish. From (2.36) and (2.37), we have

$$\begin{aligned} \boldsymbol{A}_{(cl)}(\boldsymbol{h}^{(j)}) &= -\frac{\boldsymbol{T}\wedge\boldsymbol{\delta}^T\boldsymbol{h}^{(j)}}{D-2j+1} - \frac{1}{2}\boldsymbol{dT} \underset{1}{\wedge} \boldsymbol{h}^{(j)}, \\ \boldsymbol{A}_{(q)}(\boldsymbol{h}^{(j)}) &= \frac{\boldsymbol{\delta\delta}^T\boldsymbol{h}^{(j)}}{D-2j+1} + \frac{1}{12}\boldsymbol{dT} \underset{3}{\wedge} \boldsymbol{h}^{(j)}. \end{aligned} \tag{A.3}$$

For $j=1$ it is easy to confirm by using the explicit form of $\boldsymbol{h}$ that

$$\boldsymbol{T}\wedge\boldsymbol{\delta}^T\boldsymbol{h} = \boldsymbol{dT} \underset{1}{\wedge} \boldsymbol{h} = 0, \tag{A.4}$$

$$\boldsymbol{\delta\delta}^T\boldsymbol{h} = \boldsymbol{dT} \underset{3}{\wedge} \boldsymbol{h} = 0, \tag{A.5}$$

which leads to

$$\boldsymbol{A}_{(cl)}(\boldsymbol{h}) = \boldsymbol{A}_{(q)}(\boldsymbol{h}) = 0. \tag{A.6}$$

In general, provided the classical anomaly vanishes for the GCCKY forms $\boldsymbol{h}_1$, $\boldsymbol{h}_2$, then it also vanishes for $\boldsymbol{h}_1 \wedge \boldsymbol{h}_2$. Hence we have $\boldsymbol{A}_{(cl)}(\boldsymbol{h}^{(j)}) = 0$.

[11] In order to justify separability completely, one should additionally prove that all such operators mutually commute. Such a task is rather more difficult.

In order to show $\boldsymbol{A}_{(q)}(\boldsymbol{h}^{(j)})=0$ we prove the following statement

$$\boldsymbol{\delta}\boldsymbol{\delta}^T\boldsymbol{h}^{(j)}=0. \tag{A.7}$$

Using the formula

$$\frac{1}{D-(p_1+p_2)+1}\boldsymbol{\delta}^T(\boldsymbol{h}_1\wedge\boldsymbol{h}_2)=\frac{1}{D-p_1+1}\boldsymbol{\delta}^T\boldsymbol{h}_1\wedge\boldsymbol{h}_2+\frac{(-1)^{p_1}}{D-p_2+1}\boldsymbol{h}_1\wedge\boldsymbol{\delta}^T\boldsymbol{h}_2 \tag{A.8}$$

for GCCKY p_i-forms $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$, we obtain

$$\boldsymbol{\delta}^T\boldsymbol{h}^{(j)}=\frac{(D-2j+1)j}{D-1}\boldsymbol{\delta}^T\boldsymbol{h}\wedge\boldsymbol{h}^{(j-1)}. \tag{A.9}$$

Further we calculate

$$\boldsymbol{\delta}(\boldsymbol{\delta}^T\boldsymbol{h}\wedge\boldsymbol{h}^{(j-1)})=\boldsymbol{\delta}(\boldsymbol{\delta}^T\boldsymbol{h}\wedge\boldsymbol{h}^{(j-2)})\wedge\boldsymbol{h}+\boldsymbol{h}^{(j-2)}\wedge I_1+2(j-2)\boldsymbol{h}^{(j-3)}\wedge\boldsymbol{\delta}^T\boldsymbol{h}\wedge I_2, \tag{A.10}$$

where

$$\begin{aligned}I_1&=\nabla_{X_a}\boldsymbol{\delta}^T\boldsymbol{h}\wedge X_a\lrcorner\,\boldsymbol{h}-X_a\lrcorner\,\boldsymbol{\delta}^T\boldsymbol{h}\wedge\nabla_{X_a}\boldsymbol{h}-\boldsymbol{\delta}^T\boldsymbol{h}\wedge\boldsymbol{\delta}\boldsymbol{h},\\ I_2&=X_a\lrcorner\,\boldsymbol{h}\wedge\nabla_{X_a}\boldsymbol{h}.\end{aligned} \tag{A.11}$$

By a direct calculation we have $I_1=0$ and I_2 is proportional to the factor $\boldsymbol{\delta}^T\boldsymbol{h}$,

$$I_2=\frac{1}{D-1}\boldsymbol{\delta}^T\boldsymbol{h}\wedge\left(\sum_{\mu=1}^{n}(2+x_\mu H_\mu)x_\mu e^\mu\wedge e^{\hat{\mu}}\right), \tag{A.12}$$

so that

$$\boldsymbol{\delta}(\boldsymbol{\delta}^T\boldsymbol{h}\wedge\boldsymbol{h}^{(j-1)})=\boldsymbol{\delta}(\boldsymbol{\delta}^T\boldsymbol{h}\wedge\boldsymbol{h}^{(j-2)})\wedge\boldsymbol{h}. \tag{A.13}$$

By induction, we therefore establish (A.7). Finally, one may show that

$$\boldsymbol{dT}\underset{3}{\wedge}\boldsymbol{h}^{(j)}=0 \tag{A.14}$$

by a direct calculation using $\boldsymbol{dT}=\boldsymbol{dH}=-\boldsymbol{F}\wedge\boldsymbol{F}$. Making use of (A.7) we conclude $\boldsymbol{A}_{(q)}(\boldsymbol{h}^{(j)})=0$.

Next we show that the tower of GKY forms, $\boldsymbol{f}^{(j)}=*\boldsymbol{h}^{(j)}$, also satisfies the anomaly free condition,

$$\boldsymbol{A}_{(cl)}(\boldsymbol{f}^{(j)})=\boldsymbol{A}_{(q)}(\boldsymbol{f}^{(j)})=0, \tag{A.15}$$

where

$$\begin{aligned}\boldsymbol{A}_{(cl)}(\boldsymbol{f}^{(j)})&=\frac{\boldsymbol{d}\boldsymbol{d}^T\boldsymbol{f}^{(j)}}{p+1}-\frac{1}{2}\boldsymbol{dT}\underset{1}{\wedge}\boldsymbol{f}^{(j)},\\ \boldsymbol{A}_{(q)}(\boldsymbol{f}^{(j)})&=-\frac{1}{6(p+1)}\boldsymbol{T}\underset{3}{\wedge}\boldsymbol{d}^T\boldsymbol{f}^{(j)}+\frac{1}{12}\boldsymbol{dT}\underset{3}{\wedge}\boldsymbol{f}^{(j)}.\end{aligned} \tag{A.16}$$

In general, the following result holds for contracted wedge products

$$\boldsymbol{\alpha}\underset{r}{\wedge}*\boldsymbol{\beta}=(-1)^{p(q+r+1)}\frac{r!}{(p-r)!}*(\boldsymbol{\alpha}\underset{p-r}{\wedge}\boldsymbol{\beta}), \tag{A.17}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are p and q forms, respectively. In particular, we have

$$\begin{aligned} \boldsymbol{dT} \underset{3}{\wedge} \boldsymbol{f}^{(j)} &= 3! * (\boldsymbol{dT} \underset{1}{\wedge} \boldsymbol{h}^{(j)}) = 0, \\ \boldsymbol{dT} \underset{1}{\wedge} \boldsymbol{f}^{(j)} &= \frac{1}{3!} * (\boldsymbol{dT} \underset{3}{\wedge} \boldsymbol{h}^{(j)}) = 0. \end{aligned} \tag{A.18}$$

Further we find

$$\begin{aligned} \boldsymbol{dd}^T \boldsymbol{f}^{(j)} &= - * \boldsymbol{\delta\delta}^T \boldsymbol{h}^{(j)} = 0\,, \\ \boldsymbol{T} \underset{3}{\wedge} \boldsymbol{d}^T \boldsymbol{f}^{(j)} &= 3! * (\boldsymbol{\delta}^T \boldsymbol{h}^{(j)} \wedge \boldsymbol{T}) = 0\,. \end{aligned} \tag{A.19}$$

Thus, we conclude the anomaly vanishes for GKY forms.

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