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**EXISTENCE OF GLOBAL WEAK SOLUTIONS TO  
COMPRESSIBLE ISENTROPIC FINITELY EXTENSIBLE  
BEAD-SPRING CHAIN MODELS FOR DILUTE POLYMERS**

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We prove the existence of global-in-time weak solutions to a general class of models that arise from the kinetic theory of dilute solutions of nonhomogeneous polymeric liquids, where the polymer molecules are idealized as bead-spring chains with finitely extensible nonlinear elastic (FENE) type spring potentials. The class of models under consideration involves the unsteady, compressible, isentropic, isothermal Navier–Stokes system in a bounded domain  $\Omega$  in  $\mathbb{R}^d$ ,  $d = 2$  or  $3$ , for the density  $\rho$ , the velocity  $\underline{u}$  and the pressure  $p$  of the fluid, with an equation of state of the form  $p(\rho) = c_p \rho^\gamma$ , where  $c_p$  is a positive constant and  $\gamma > \frac{3}{2}$ . The right-hand side of the Navier–Stokes momentum equation includes an elastic extra-stress tensor, which is the sum of the classical Kramers expression and a quadratic interaction term. The elastic extra-stress tensor stems from the random movement of the polymer chains and is defined through the associated probability density function that satisfies a Fokker–Planck-type parabolic equation, a crucial feature of which is the presence of a centre-of-mass diffusion term.

*Keywords:* Kinetic polymer models; FENE chain; compressible Navier–Stokes–Fokker–Planck system; nonhomogeneous dilute polymer; variable density.

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## 1. Introduction

This paper establishes the existence of global-in-time weak solutions to a large class of bead-spring chain models with finitely extensible nonlinear elastic (FENE) type spring potentials, — a system of nonlinear partial differential equations that arises from the kinetic theory of dilute polymer solutions. The solvent is a compressible, isentropic, isothermal Newtonian fluid confined to a bounded Lipschitz domain  $\Omega \subset \mathbb{R}^d$ ,  $d = 2$  or  $3$ , with boundary  $\partial\Omega$ . For the sake of simplicity of presentation we shall suppose that  $\Omega$  has a ‘solid boundary’  $\partial\Omega$ , and the velocity field  $\underline{u}$  will be therefore

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assumed to satisfy the no-slip boundary condition  $\underline{u} = \underline{0}$  on  $\partial\Omega$ . The equations of continuity and balance of linear momentum have the form of the compressible Navier–Stokes equations (cf. Lions<sup>23</sup>, Feireisl<sup>18</sup>, Novotný & Straškraba<sup>24</sup>, or Feireisl & Novotný<sup>19</sup>) in which the *elastic extra-stress* tensor  $\underline{\underline{\tau}}$  (i.e., the polymeric part of the Cauchy stress tensor) appears as a source term in the conservation of momentum equation:

Given  $T \in \mathbb{R}_{>0}$ , find  $\rho : (\underline{x}, t) \in \Omega \times [0, T] \mapsto \rho(\underline{x}, t) \in \mathbb{R}$  and  $\underline{u} : (\underline{x}, t) \in \bar{\Omega} \times [0, T] \mapsto \underline{u}(\underline{x}, t) \in \mathbb{R}^d$  such that

$$\frac{\partial \rho}{\partial t} + \nabla_{\underline{x}} \cdot (\underline{u} \rho) = 0 \quad \text{in } \Omega \times (0, T], \quad (1.1a)$$

$$\rho(\underline{x}, 0) = \rho_0(\underline{x}) \quad \forall \underline{x} \in \Omega, \quad (1.1b)$$

$$\frac{\partial(\rho \underline{u})}{\partial t} + \nabla_{\underline{x}} \cdot (\rho \underline{u} \otimes \underline{u}) - \nabla_{\underline{x}} \cdot \underline{\underline{S}}(\underline{u}, \rho) + \nabla_{\underline{x}} p(\rho) = \rho \underline{f} + \nabla_{\underline{x}} \cdot \underline{\underline{\tau}} \quad \text{in } \Omega \times (0, T], \quad (1.1c)$$

$$\underline{u} = \underline{0} \quad \text{on } \partial\Omega \times (0, T], \quad (1.1d)$$

$$(\rho \underline{u})(\underline{x}, 0) = (\rho_0 \underline{u}_0)(\underline{x}) \quad \forall \underline{x} \in \Omega. \quad (1.1e)$$

It is assumed that each of the equations above has been written in its nondimensional form;  $\rho$  denotes a nondimensional solvent density,  $\underline{u}$  is a nondimensional solvent velocity, defined as the velocity field scaled by the characteristic flow speed  $U_0$ . Here  $\underline{\underline{S}}(\underline{u}, \rho)$  is the Newtonian part of the viscous stress tensor defined by

$$\underline{\underline{S}}(\underline{u}, \rho) := \mu^S(\rho) \left[ \underline{\underline{D}}(\underline{u}) - \frac{1}{d} (\nabla_{\underline{x}} \cdot \underline{u}) \underline{\underline{I}} \right] + \mu^B(\rho) (\nabla_{\underline{x}} \cdot \underline{u}) \underline{\underline{I}}, \quad (1.2)$$

where  $\underline{\underline{I}}$  is the  $d \times d$  identity tensor,  $\underline{\underline{D}}(\underline{u}) := \frac{1}{2} (\nabla_{\underline{x}} \underline{u} + (\nabla_{\underline{x}} \underline{u})^T)$  is the rate of strain tensor, with  $(\nabla_{\underline{x}} \underline{u})(\underline{x}, t) \in \mathbb{R}^{d \times d}$  and  $(\nabla_{\underline{x}} \underline{u})_{ij} = \frac{\partial u_i}{\partial x_j}$ . The shear viscosity,  $\mu^S(\cdot) \in \mathbb{R}_{>0}$ , and the bulk viscosity,  $\mu^B(\cdot) \in \mathbb{R}_{\geq 0}$ , of the solvent are both scaled and, generally, density-dependent. In addition,  $p$  is the nondimensional pressure satisfying the isentropic equation of state

$$p(\rho) = c_p \rho^\gamma, \quad (1.3)$$

where  $c_p \in \mathbb{R}_{>0}$  and  $\gamma > \frac{3}{2}$ . For the compressible Navier–Stokes equations at least, it is known that this condition on  $\gamma$  can be replaced by  $\gamma > \frac{d}{2}$  for  $d = 2, 3$ . Although the analysis presented herein covers both  $d = 2$  and  $d = 3$ , our main interest is in the physically relevant case of  $d = 3$ . For the sake of streamlining the exposition we have therefore assumed that  $\gamma > \frac{3}{2}$  for both  $d = 2$  and  $d = 3$ , instead of  $\gamma > \frac{d}{2}$ .

**Remark 1.1.** Our analysis applies, without alterations, to some other familiar *monotonic* equations of state, such as the (Kirkwood-modified) Tait equation of state  $p(\rho) = A_0 (\rho/\rho_*)^\gamma - A_1$ , where  $\gamma > \frac{3}{2}$ ,  $A_0$  and  $A_1$  are constants,  $A_0 - A_1 = p_*$  is the equilibrium reference pressure, and  $\rho_*$  is the equilibrium reference density.

On the right-hand side of (1.1c),  $\underline{f}$  is the nondimensional density of body forces and  $\underline{\underline{\tau}}$  denotes the elastic extra-stress tensor. In a *bead-spring chain model*, consisting of  $K + 1$  beads coupled with  $K$  elastic springs to represent a polymer chain,  $\underline{\underline{\tau}}$  is

defined by a version of the *Kramers expression* depending on the probability density function  $\psi$  of the (random) conformation vector  $\underline{q} := (\underline{q}_1^T, \dots, \underline{q}_K^T)^T \in \mathbb{R}^{Kd}$  of the chain (see equation (1.11) below), with  $\underline{q}_i$  representing the  $d$ -component conformation/orientation vector of the  $i$ th spring. The Kolmogorov equation satisfied by  $\psi$  is a second-order parabolic equation, the Fokker–Planck equation, whose transport coefficients depend on the velocity field  $\underline{u}$ , and the hydrodynamic drag coefficient appearing in the Fokker–Planck equation is, generally, a nonlinear function of the density  $\rho$ . The domain  $D$  of admissible conformation vectors  $D \subset \mathbb{R}^{Kd}$  is a  $K$ -fold Cartesian product  $D_1 \times \dots \times D_K$  of bounded open  $d$ -dimensional balls  $D_i$  centred at the origin  $\mathbf{0} \in \mathbb{R}^d$ ,  $i = 1, \dots, K$ .

Let  $\mathcal{O}_i := [0, \frac{b_i}{2})$  denote the image of  $D_i$  under the mapping  $\underline{q}_i \in D_i \mapsto \frac{1}{2}|\underline{q}_i|^2$ , and consider the *spring potential*  $U_i \in C^1(\mathcal{O}_i; \mathbb{R}_{\geq 0})$ ,  $i = 1, \dots, K$ . We shall suppose that  $U_i(0) = 0$  and that  $U_i$  is unbounded on  $\mathcal{O}_i$  for each  $i = 1, \dots, K$ . The elastic spring-force  $\underline{F}_i : D_i \subseteq \mathbb{R}^d \rightarrow \mathbb{R}^d$  of the  $i$ th spring in the chain is defined by

$$\underline{F}_i(\underline{q}_i) := U_i'(\frac{1}{2}|\underline{q}_i|^2) \underline{q}_i, \quad i = 1, \dots, K, \quad (1.4)$$

and the partial Maxwellian  $M_i$ , associated with the spring potential  $U_i$ , is defined by

$$M_i(\underline{q}_i) := \frac{1}{\mathcal{Z}_i} e^{-U_i(\frac{1}{2}|\underline{q}_i|^2)}, \quad \mathcal{Z}_i := \int_{D_i} e^{-U_i(\frac{1}{2}|\underline{q}_i|^2)} d\underline{q}_i, \quad i = 1, \dots, K.$$

The (total) Maxwellian in the model is then

$$M(\underline{q}) := \prod_{i=1}^K M_i(\underline{q}_i) \quad \forall \underline{q} := (\underline{q}_1^T, \dots, \underline{q}_K^T)^T \in D := \prod_{i=1}^K D_i. \quad (1.5)$$

Observe that, for  $i = 1, \dots, K$ ,

$$M(\underline{q}) \nabla_{\underline{q}_i} [M(\underline{q})]^{-1} = -[M(\underline{q})]^{-1} \nabla_{\underline{q}_i} M(\underline{q}) = \nabla_{\underline{q}_i} (U_i(\frac{1}{2}|\underline{q}_i|^2)) = U_i'(\frac{1}{2}|\underline{q}_i|^2) \underline{q}_i, \quad (1.6a)$$

and, by definition,

$$\int_D M(\underline{q}) d\underline{q} = 1. \quad (1.6b)$$

The associated bead-spring chain model is referred to as a FENE (finitely extensible nonlinear elastic) model; in the case of  $K = 1$ , the corresponding model is called a FENE dumbbell model.

We shall assume that  $D_i = B(0, b_i^{\frac{1}{2}})$  with  $b_i > 0$  for  $i = 1, \dots, K$ , and that for  $i = 1, \dots, K$  there exist constants  $c_{ij} > 0$ ,  $j = 1, 2, 3, 4$ , and  $\theta_i > 1$  such that the spring potential  $U_i \in C^1[0, \frac{b_i}{2})$  and the associated partial Maxwellian  $M_i$  satisfy

$$c_{i1} [\text{dist}(\underline{q}_i, \partial D_i)]^{\theta_i} \leq M_i(\underline{q}_i) \leq c_{i2} [\text{dist}(\underline{q}_i, \partial D_i)]^{\theta_i} \quad \forall \underline{q}_i \in D_i, \quad (1.7a)$$

$$c_{i3} \leq \text{dist}(\underline{q}_i, \partial D_i) U_i'(\frac{1}{2}|\underline{q}_i|^2) \leq c_{i4} \quad \forall \underline{q}_i \in D_i. \quad (1.7b)$$

It follows from (1.7a,b) that (if  $\theta_i > 1$ , as has been assumed here),

$$\int_{D_i} \left[ 1 + [U_i(\frac{1}{2}|\underline{q}_i|^2)]^2 + [U_i'(\frac{1}{2}|\underline{q}_i|^2)]^2 \right] M_i(\underline{q}_i) d\underline{q}_i < \infty, \quad i = 1, \dots, K. \quad (1.8)$$

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**Example 1.1.** In the classical FENE dumbbell model,  $K = 1$  and the spring force is given by  $\underline{F}(\underline{q}) = (1 - |\underline{q}|^2/b)^{-1} \underline{q}$ ,  $\underline{q} \in D = B(0, b^{\frac{1}{2}})$ , corresponding to  $U(s) = -\frac{b}{2} \log(1 - \frac{2s}{b})$ ,  $s \in \mathcal{O} = [0, \frac{b}{2})$ ,  $b > 2$ . More generally, in a FENE bead-spring chain, one considers  $K + 1$  beads linearly coupled with  $K$  springs, each with a FENE spring potential. Direct calculations show that the partial Maxwellians  $M_i$  and the elastic potentials  $U_i$ ,  $i = 1, \dots, K$ , of the FENE bead spring chain satisfy the conditions (1.7a,b) with  $\theta_i := \frac{b_i}{2}$ , provided that  $b_i > 2$ ,  $i = 1, \dots, K$ . Thus, (1.8) also holds when  $b_i > 2$ ,  $i = 1, \dots, K$ .

The governing equations of the general nonhomogeneous bead-spring chain models with centre-of-mass diffusion considered here are (1.1a–e), where the extra-stress tensor  $\underline{\tau}$  is defined by

$$\underline{\tau}(\psi)(\underline{x}, t) := \underline{\tau}_1(\psi)(\underline{x}, t) - \left( \int_{D \times D} \gamma(\underline{q}, \underline{q}') \psi(\underline{x}, \underline{q}, t) \psi(\underline{x}, \underline{q}', t) \, d\underline{q} \, d\underline{q}' \right) \underline{I}, \quad (1.9)$$

$\gamma : D \times D \rightarrow \mathbb{R}_{\geq 0}$  is a smooth, time-independent,  $\underline{x}$ -independent and  $\psi$ -independent interaction kernel, which we shall henceforth consider to be  $\gamma(\underline{q}, \underline{q}') \equiv \mathfrak{z}$ , where  $\mathfrak{z} \in \mathbb{R}_{> 0}$ ; thus,

$$\underline{\tau}(\psi) := \underline{\tau}_1(\psi) - \mathfrak{z} \left( \int_D \psi \, d\underline{q} \right)^2 \underline{I}. \quad (1.10)$$

Here,  $\underline{\tau}_1(\psi)$  is the *Kramers expression*; that is,

$$\underline{\tau}_1(\psi) := k \left[ \left( \sum_{i=1}^K C_i(\psi) \right) - (K + 1) \int_D \psi \, d\underline{q} \right] \underline{I}, \quad (1.11)$$

where  $k \in \mathbb{R}_{> 0}$ , with the first term in the square brackets being due to the  $K$  springs and the second to the  $K + 1$  beads in the bead-spring chain representing the polymer molecule. Further,

$$C_i(\psi)(\underline{x}, t) := \int_D \psi(\underline{x}, \underline{q}, t) U_i'(\frac{1}{2} |\underline{q}_i|^2) \underline{q}_i \underline{q}_i^T \, d\underline{q}, \quad i = 1, \dots, K. \quad (1.12)$$

The probability density function  $\psi$  satisfies the Fokker–Planck (forward Kolmogorov) equation

$$\begin{aligned} \frac{\partial \psi}{\partial t} + \underline{\nabla}_x \cdot (\underline{u} \psi) + \sum_{i=1}^K \underline{\nabla}_{\underline{q}_i} \cdot \left( \underline{\sigma}(\underline{u}) \underline{q}_i \psi \right) &= \varepsilon \Delta_x \left( \frac{\psi}{\zeta(\rho)} \right) \\ + \frac{1}{4\lambda} \sum_{i=1}^K \sum_{j=1}^K A_{ij} \underline{\nabla}_{\underline{q}_i} \cdot \left( M \underline{\nabla}_{\underline{q}_j} \left( \frac{\psi}{\zeta(\rho) M} \right) \right) &\quad \text{in } \Omega \times D \times (0, T], \end{aligned} \quad (1.13)$$

with  $\underline{\sigma}(\underline{v}) \equiv \underline{\nabla}_x \underline{v}$  and a, generally, density-dependent scaled drag coefficient  $\zeta(\cdot) \in \mathbb{R}_{> 0}$ . A concise derivation of the Fokker–Planck equation (1.13) can be found in Section 1 of Barrett and Süli<sup>7</sup>. Let  $\partial \overline{D}_i := D_1 \times \dots \times D_{i-1} \times \partial D_i \times D_{i+1} \times \dots \times D_K$ .

We impose the following boundary and initial conditions on solutions of (1.13):

$$\left[ \frac{1}{4\lambda} \sum_{j=1}^K A_{ij} M \nabla_{q_j} \left( \frac{\psi}{\zeta(\rho) M} \right) - \sigma(u) q_i \psi \right] \cdot \frac{q_i}{|q_i|} = 0$$

on  $\Omega \times \partial \bar{D}_i \times (0, T]$ , for  $i = 1, \dots, K$ , (1.14a)

$$\varepsilon \nabla_x \left( \frac{\psi}{\zeta(\rho)} \right) \cdot \underline{n} = 0 \quad \text{on } \partial \Omega \times D \times (0, T], \quad (1.14b)$$

$$\psi(\cdot, \cdot, 0) = \psi_0(\cdot, \cdot) \geq 0 \quad \text{on } \Omega \times D, \quad (1.14c)$$

where  $q_i$  is normal to  $\partial D_i$ , as  $D_i$  is a bounded ball centred at the origin, and  $\underline{n}$  is normal to  $\partial \Omega$ .

The nondimensional constant  $k > 0$  featuring in (1.11) is a constant multiple of the product of the Boltzmann constant  $k_B$  and the absolute temperature  $T$ . In (1.13),  $\varepsilon > 0$  is the centre-of-mass diffusion coefficient defined as  $\varepsilon := (\ell_0/L_0)^2/(4(K+1)\lambda)$  with  $L_0$  a characteristic length-scale of the solvent flow,  $\ell_0 := \sqrt{k_B T/\mathbb{H}}$  signifying the characteristic microscopic length-scale and  $\lambda := \frac{\zeta_0 U_0}{4\mathbb{H}L_0}$ , where  $\zeta_0 > 0$  is a characteristic drag coefficient and  $\mathbb{H} > 0$  is a spring-constant. The nondimensional parameter  $\lambda \in \mathbb{R}_{>0}$ , called the Deborah number (and usually denoted by  $\text{De}$ ), characterizes the elastic relaxation property of the fluid, and  $\underline{\mathbb{A}} = (A_{ij})_{i,j=1}^K$  is the symmetric positive definite *Rouse matrix*, or connectivity matrix; for example,  $\underline{\mathbb{A}} = \text{tridiag}[-1, 2, -1]$  in the case of a (topologically) linear chain. Concerning these scalings and notational conventions, we remark that the factor  $\frac{1}{4\lambda}$  in equation (1.13) above appears as a factor  $\frac{1}{2\lambda}$  in the Fokker–Planck equation in our earlier papers<sup>5,6,8,10</sup>. Two remarks are in order concerning (1.10).

**Remark 1.2.** The analysis in the present paper applies to a general class of extra stress tensors (1.10), with Kramers type expressions of the form

$$\underline{\tau}_1(\psi) := k \left[ \left( \sum_{i=1}^K \underline{\mathbb{C}}_i(\psi) \right) - \mathfrak{k} \int_D \psi \, d\underline{q} \, \underline{\mathbb{I}} \right], \quad (1.15)$$

with  $\mathfrak{k} \in \mathbb{R}$  and  $\mathfrak{z} > 0$ . Since the actual value of  $\mathfrak{k}$  is of no particular relevance in our analysis, we took  $\mathfrak{k} = K + 1$  in the second term in (1.15), yielding (1.11), as this choice simplifies the expressions that arise in the course of the proof. As will be shown below, if  $\mathfrak{k} \geq K + 1$  and  $\mathfrak{z} \geq 0$  a formal energy identity holds, and if  $\mathfrak{k} < K + 1$  and  $\mathfrak{z} > 0$ , one can still prove a formal energy inequality. In contrast with this, in the case of *incompressible* flows,  $\mathfrak{k}$  and  $\mathfrak{z}$  are of no relevance in the analysis and can be any two real numbers; indeed, in Barrett & Süli<sup>6,8,9</sup>,  $\mathfrak{k} = K$  and  $\mathfrak{z} = 0$  were the values used in (1.15) and (1.10), respectively, resulting in  $\underline{\tau}(\psi)$  that is the classical Kramers expression for the extra stress tensor.

**Remark 1.3.** Our second remark concerns the quadratic modification to  $\underline{\tau}_1(\psi)$  appearing as the second term in (1.10). By defining the *polymer number density*

$$\varrho(\underline{x}, t) := \int_D \psi(\underline{x}, \underline{q}, t) \, d\underline{q}, \quad (\underline{x}, t) \in \Omega \times [0, T], \quad (1.16)$$

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formally integrating (1.13) over  $D$  and using the boundary condition (1.14a), we deduce that

$$\frac{\partial \varrho}{\partial t} + \nabla_x \cdot (\underline{u} \varrho) = \varepsilon \Delta_x \left( \frac{\varrho}{\zeta(\rho)} \right) \quad \text{on } \Omega \times (0, T], \quad (1.17a)$$

together with the boundary and initial conditions (which result from integrating (1.14b,c) over  $D$ )

$$\varepsilon \nabla_x \cdot \left( \frac{\varrho}{\zeta(\rho)} \right) \cdot \underline{n} = 0 \quad \text{on } \partial\Omega \times (0, T] \quad \text{and} \quad \varrho(\underline{x}, 0) = \int_D \psi_0(\underline{x}, \underline{q}) \, d\underline{q} \quad \text{for } \underline{x} \in \Omega. \quad (1.17b)$$

In order to avoid potential confusion concerning our notational conventions, we draw the reader's attention to the fact that, here and throughout the rest of the paper, the symbol  $\rho$  signifies the density of the solvent, while the symbol  $\varrho$  denotes the polymer number density, as defined in (1.16).

If  $\nabla_x \cdot \underline{u} \equiv 0$  and  $\varrho(\cdot, 0)$  is constant, and either  $\varepsilon = 0$  or  $\zeta(\rho)$  is independent of  $\rho$  (and therefore identically equal to a constant), then  $\varrho(\underline{x}, t)$  is constant ( $\equiv \varrho(\cdot, 0)$ ), for all  $(\underline{x}, t) \in \Omega \times (0, T]$ , and hence

$$\int_D \psi(\underline{x}, \underline{q}, t) \, d\underline{q} = \int_D \psi_0(\underline{x}, \underline{q}) \, d\underline{q} \in \mathbb{R}_{>0} \quad \text{for all } (\underline{x}, t) \in \Omega \times (0, T];$$

in other words, the polymer number density is constant. This conservation property then guarantees complete control of  $\varrho$  (when  $\mathfrak{z} = 0$ , and a fortiori, for  $\mathfrak{z} > 0$ ) in terms of the initial probability density function  $\psi_0$  in the course of the weak compactness argument upon which the proof of existence of weak solutions rests (cf. Barrett & Süli<sup>6</sup> for the analysis in the case of  $\varepsilon > 0$ , constant  $\zeta$  and  $\mathfrak{z} = 0$ ). In particular, the time derivative  $\frac{\partial \psi}{\partial t}$  can be bounded in a sufficiently strong norm to enable the application of an Aubin–Lions–Simon type compactness theorem that ensures strong convergence of the sequence of approximations to the probability density function  $\psi$ .

In the present paper, in order to focus on the essential new difficulty — the lack of the divergence-free property of  $\underline{u}$  — we shall suppose that the drag coefficient in the Fokker–Planck equation is identically equal to a constant, which we shall henceforth, without loss of generality, assume to be equal to 1, i.e.,  $\zeta(\rho) \equiv 1$ . Setting  $\mathfrak{z} = 0$  in this context results in the loss of a bound on the  $L^1(0, T; \mathfrak{X}')$  norm of the time derivative of the probability density function  $\psi$ , for any reasonable choice of the function space  $\mathfrak{X}$ . Failure to control  $\frac{\partial \psi}{\partial t}$  or a time-difference of  $\psi$  in even such a weak sense brings into question the meaningfulness of the model in the case of compressible flows for solutions of as low a degree of regularity as is guaranteed by the formal energy bound in the case of  $\mathfrak{z} = 0$ , see Remarks 4.1 and 4.2 below. Motivated by the papers of Constantin<sup>14</sup>, Constantin et al.<sup>15</sup> and Bae & Trivisa<sup>3</sup>, we have therefore included the quadratic term in (1.10), with  $\mathfrak{z} > 0$ . As we shall show later on, inclusion of the quadratic term into (1.10) does not destroy energy balance thanks to the fact that the polymer number density function  $\varrho$  satisfies the initial-boundary-value problem (1.17a,b), and has the beneficial effect of guaranteeing

$L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$  norm control for  $\varrho$ , rendering the time derivative of the probability density function finite in the norm of  $L^2(0, T; \mathfrak{X}')$ , with a suitable choice of  $\mathfrak{X}$  as a Maxwellian-weighted Sobolev space of sufficiently high order. This then enables the application of Dubinskii's extension of the Aubin–Lions–Simon compactness theorem (cf. Dubinskii<sup>17</sup> and Barrett & Süli<sup>11</sup>).

**Definition 1.1.** The collection of equations and structural hypotheses (1.1a–e)–(1.14a–c) together with the assumption that the Rouse matrix  $A$  is symmetric and positive definite (as is always the case, by definition,) will be referred to throughout the paper as model (P), or as the *compressible FENE-type bead-spring chain model with centre-of-mass diffusion*. It will also be assumed throughout the paper that the shear viscosity,  $\mu^S \in \mathbb{R}_{>0}$ , the bulk viscosity,  $\mu^B \in \mathbb{R}_{\geq 0}$ , and the drag coefficient,  $\zeta \in \mathbb{R}_{>0}$ , are independent of the density  $\rho$ . For the ease of exposition we shall set  $\zeta \equiv 1$ .

For a survey of macroscopic models of *compressible* viscoelastic flow, the reader is referred to the paper by Bollada & Phillips<sup>13</sup>. Closer to the subject of the present paper, Bae & Trivisa<sup>3</sup> have established the existence of global weak solutions to Doi's rod-model in three-dimensional bounded domains. The model concerns suspensions of rod-like molecules in compressible fluids and involves the coupling of a Fokker–Planck type equation with the compressible Navier–Stokes system. Also, Jiang, Jiang & Wang<sup>21</sup> have studied the existence of global weak solutions to the equations of compressible flow of nematic liquid crystals in two dimensions.

Despite their importance, we shall, for the sake of simplicity, neglect all thermal effects and will focus instead on mechanical properties of the fluid in the isothermal setting. Since the argument is long and technical, we give a brief overview of the main steps of the proof.

At the heart of the proof is a formal energy identity, which we shall state under the assumption that  $\underline{u}$ ,  $\rho$ ,  $\psi$  and  $\varrho$  are sufficiently smooth, and, at least for our purposes in this introductory section,  $\rho$  is nonnegative, and  $\psi$  and  $\varrho$  are positive. Instead of (1.11), used in the rest of the paper, we shall adopt for the moment the more general formula (1.15), in order to explain the admissible range of  $\mathfrak{k}$  alluded to in Remark 1.2 as well as our reasons for choosing  $\mathfrak{k} = K + 1$  in (1.11). The formal energy identity satisfied by  $\underline{u}$ ,  $\rho$ ,  $\psi$  and  $\varrho$ , upon letting

$$P(\rho) := \frac{p(\rho)}{\gamma - 1} \quad \text{and} \quad \mathcal{F}(s) := s(\log s - 1) + 1 \text{ for } s > 0 \text{ and } \mathcal{F}(0) := 1,$$

is (cf. Barrett & Süli<sup>12</sup>):

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left[ \frac{1}{2} \rho |\underline{u}|^2 + P(\rho) + \mathfrak{z} \varrho^2 + k(\mathfrak{k} - (K + 1)) \mathcal{F}(\varrho) + k \int_D M \mathcal{F} \left( \frac{\psi}{M} \right) d\mathfrak{q} \right] d\mathfrak{x} \\ & + \mu^S \int_{\Omega} \left| D(\underline{u}) - \frac{1}{d} (\nabla_x \cdot \underline{u}) I \right|^2 d\mathfrak{x} + \mu^B \int_{\Omega} |\nabla_x \cdot \underline{u}|^2 d\mathfrak{x} \\ & + 2\varepsilon \mathfrak{z} \int_{\Omega} |\nabla_x \varrho|^2 d\mathfrak{x} + 4\varepsilon k(\mathfrak{k} - (K + 1)) \int_{\Omega} |\nabla_x \sqrt{\varrho}|^2 d\mathfrak{x} \end{aligned}$$

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$$\begin{aligned}
 & + 4\varepsilon k \int_{\Omega \times D} M \left| \nabla_x \sqrt{\frac{\psi}{M}} \right|^2 d\tilde{q} d\tilde{x} \\
 & + \frac{k}{\lambda} \sum_{i=1}^K \sum_{j=1}^K A_{ij} \int_{\Omega \times D} M \nabla_{q_j} \sqrt{\frac{\psi}{M}} \cdot \nabla_{q_i} \sqrt{\frac{\psi}{M}} d\tilde{q} d\tilde{x} = \int_{\Omega} \rho \tilde{f} \cdot \underline{u} d\tilde{x}. \quad (1.18)
 \end{aligned}$$

The integral over  $\Omega$  of the expression in the square brackets in the first line of (1.18) is the total energy, and the sum of the terms on the second, third and fourth line of (1.18) is the dissipation of the total energy; recall that the Rouse matrix  $A = (A_{ij})_{i,j=1}^K$  is, by definition, symmetric and positive definite. While, as an energy identity, (1.18) is meaningful for all  $\mathfrak{z} \geq 0$  and all  $\mathfrak{k} \geq K + 1$ , it will transpire in the course of the proof that  $\mathfrak{z} > 0$  is necessary in order to ensure control of  $\frac{\partial \psi}{\partial t}$  or of a time-difference of  $\psi$  (cf. Remarks 1.3 and 4.2). Once  $\mathfrak{z}$  has been chosen to be positive, the two terms in (1.18) that include the factor  $(\mathfrak{k} - (K + 1))$  are of lower order and contribute no additional information; we have therefore, for the sake of simplicity, set  $\mathfrak{k} = K + 1$  in (1.15), yielding (1.11). If  $\mathfrak{k} < K + 1$  and  $\mathfrak{z} > 0$ , then upon moving the two terms containing the factors  $(\mathfrak{k} - (K + 1))$  from the left-hand side of (1.18) to the right, Gronwall's inequality yields a formal energy *inequality*; since the ultimate outcome of the analysis is no different from the one with  $\mathfrak{k} = K + 1$  and  $\mathfrak{z} > 0$ , we shall not discuss this case further.

The main idea of the proof is to construct a sequence of approximating solutions, whose existence one can prove, and then pass to the limit. Available results in the literature concerning the existence of weak solutions to the compressible Navier–Stokes equations are based on considering sequences of approximating problems that are defined by spatial Galerkin discretization, using bases of smooth functions. Our construction of a sequence of approximating problems for the compressible Navier–Stokes–Fokker–Planck system will be based on discretizing the problem with respect to the temporal variable. For a similar approach, in the case of a coupled compressible Navier–Stokes–Cahn–Hilliard system, we refer the reader to the work of Abels and Feireisl<sup>1</sup> and to Remark 3.1 below, which explains the aspects in which our temporal approximation differs from the one in Abels & Feireisl<sup>1</sup>. The inclusion of the Fokker–Planck equation into the analysis is nontrivial, the main hurdle being to ensure that the presence of the extra stress term  $\underline{\tau}$  (cf. (1.9)) on the right-hand side of the Navier–Stokes momentum equation does not destroy the Lions–Feireisl compactness argument for the compressible Navier–Stokes equations. We have only been able to achieve this for  $\mathfrak{z} > 0$ . As the same requirement on  $\mathfrak{z}$  has been found to be necessary in the, related, Doi model for suspensions of rod-like molecules in a compressible fluid, considered by Bae and Trivisa<sup>3</sup>, we are confident that the condition  $\mathfrak{z} > 0$  is not a byproduct of our time-discrete approach to the proof of existence of weak solutions.

The proof of the central theorem in the paper, Theorem 6.1, stating the existence of global bounded-energy weak solutions to problem (P), consists of six steps, which are outlined below.



*Step 1.* Following the approach in Barrett & Süli<sup>4,6,7,8,9</sup> we observe that if  $\frac{\psi}{M}$  is bounded above then, for  $L \in \mathbb{R}_{>0}$  sufficiently large, the third term in (1.13), referred to as the *drag term* is equal to

$$\sum_{i=1}^K \nabla_{q_i} \cdot \left( \underline{\sigma}(u) \underline{q}_i M \beta^L \left( \frac{\psi}{M} \right) \right)$$

(recall that, by hypothesis,  $\zeta \equiv 1$ ), where  $\beta^L \in C(\mathbb{R})$  is a cut-off function defined as

$$\beta^L(s) := \min\{s, L\}. \quad (1.19)$$

It then follows that, for  $L \gg 1$ , any solution  $\psi$  of (1.13), such that  $\frac{\psi}{M}$  is bounded above by  $L$ , also satisfies

$$\begin{aligned} \frac{\partial \psi}{\partial t} + \nabla_x \cdot (u \psi) + \sum_{i=1}^K \nabla_{q_i} \cdot \left( \underline{\sigma}(u) \underline{q}_i M \beta^L \left( \frac{\psi}{M} \right) \right) \\ = \varepsilon \Delta_x \psi + \frac{1}{4\lambda} \sum_{i=1}^K \sum_{j=1}^K A_{ij} \nabla_{q_i} \cdot \left( M \nabla_{q_j} \left( \frac{\psi}{M} \right) \right) \quad \text{in } \Omega \times D \times (0, T]. \end{aligned} \quad (1.20)$$

We impose the following boundary and initial conditions:

$$\left[ \frac{1}{4\lambda} \sum_{j=1}^K A_{ij} M \nabla_{q_j} \left( \frac{\psi}{M} \right) - \underline{\sigma}(u) \underline{q}_i M \beta^L \left( \frac{\psi}{M} \right) \right] \cdot \frac{\underline{q}_i}{|\underline{q}_i|} = 0$$

on  $\Omega \times \partial \bar{D}_i \times (0, T]$ , for  $i = 1, \dots, K$ , (1.21a)

$$\varepsilon \nabla_x \psi \cdot \underline{n} = 0 \quad \text{on } \partial \Omega \times D \times (0, T], \quad (1.21b)$$

$$\psi(\cdot, \cdot, 0) = M(\cdot) \beta^L(\psi_0(\cdot, \cdot)/M(\cdot)) \geq 0 \quad \text{on } \Omega \times D. \quad (1.21c)$$

The model with cut-off parameter  $L > 1$  is further regularized, by introducing a dissipation term of the form  $-\alpha \Delta \rho$ , with  $\alpha > 0$ , into the continuity equation (1.1a) and supplementing the resulting parabolic equation with a homogeneous Neumann boundary condition on  $\partial \Omega \times (0, T]$ . In addition, the equation of state (1.3) is replaced by a regularized equation of state,  $p_\kappa(\rho) = p(\rho) + \kappa(\rho^4 + \rho^\Gamma)$ , where  $\kappa \in \mathbb{R}_{>0}$  and  $\Gamma = \max\{\gamma, 8\}$ . The resulting problem is denoted by  $(P_{\kappa, \alpha, L})$ .

*Step 2.* Ideally, one would like to pass to the limits  $\kappa \rightarrow 0_+$ ,  $\alpha \rightarrow 0_+$ ,  $L \rightarrow +\infty$  to deduce the existence of solutions to (P). Unfortunately, such a direct attack at the problem is fraught with technical difficulties. Instead, we shall first (semi)discretize the problem  $(P_{\kappa, \alpha, L})$  with respect to  $t$ , with step size  $\Delta t$ . This then results in a time-discrete version  $(P_{\kappa, \alpha, L}^{\Delta t})$  of  $(P_{\kappa, \alpha, L})$ .

*Step 3.* By using Schauder's fixed point theorem, we will show in Section 3 the existence of solutions to  $(P_{\kappa, \alpha, L}^{\Delta t})$ . In the course of the proof, for technical reasons, a further cut-off, now from below, with a cut-off parameter  $\delta \in (0, 1)$ , is required.

In addition, a fourth-order hyperviscosity term is added to the Navier–Stokes momentum equation (1.1c). We shall let  $\delta$  pass to 0 to complete the proof of existence of solutions to  $(P_{\kappa,\alpha,L}^{\Delta t})$  in the limit of  $\delta \rightarrow 0_+$  in Section 3; cf. Lemma 3.3.

*Step 4.* In Section 4 we then go on to derive bounds on the sequence of solutions to problem  $(P_{\kappa,\alpha,L}^{\Delta t})$ ; in particular, we develop various bounds on the sequence of weak solutions to  $(P_{\kappa,\alpha,L}^{\Delta t})$  that are uniform in the time step  $\Delta t$  and the cut-off parameter  $L$ , and thus permit the extraction of weakly convergent subsequences, as  $L \rightarrow +\infty$  and  $\Delta t \rightarrow 0_+$ , with  $\Delta t = o(L^{-1})$ , when  $L \rightarrow +\infty$ . The weakly convergent subsequences will then be shown to converge strongly in suitable norms. This will allow us to pass to the limit as  $L \rightarrow +\infty$ , with  $\Delta t = o(L^{-1})$ . The main result of Section 4 is Theorem 4.1, which summarizes the outcome of this limiting process.

*Step 5.* Section 5 is concerned with passage to the limit  $\alpha \rightarrow 0_+$  with the parabolic regularization parameter that was introduced into the continuity equation in Step 1. The main result of Section 5 is Theorem 5.1, which summarizes the outcome of this limiting process.

*Step 6.* Finally, in Section 6 we pass to the limit  $\kappa \rightarrow 0_+$  with the regularization parameter that was introduced into the equation of state in Step 1, which then leads to our main result, Theorem 6.1, stating the existence of global, bounded-energy, weak solutions to problem (P).

## 2. The polymer model $(P_{\kappa,\alpha,L})$

Let  $\Omega \subset \mathbb{R}^d$  be a bounded open set with a Lipschitz-continuous boundary  $\partial\Omega$ , and suppose that the set  $D := D_1 \times \cdots \times D_K$  of admissible conformation vectors  $\underline{q} := (q_1^T, \dots, q_K^T)^T$  in (1.13) is such that  $D_i$ ,  $i = 1, \dots, K$ , is an open ball in  $\mathbb{R}^d$ ,  $d = 2$  or  $3$ , centred at the origin, with boundary  $\partial D_i$  and radius  $\sqrt{b_i}$ ,  $b_i > 2$ ; let

$$\partial D := \bigcup_{i=1}^K \partial \bar{D}_i, \quad \text{where } \partial \bar{D}_i := D_1 \times \cdots \times D_{i-1} \times \partial D_i \times D_{i+1} \times \cdots \times D_K. \quad (2.1)$$

Collecting (1.1a–e), (1.10), (1.11), (1.12), (1.20) and (1.21a–c), we then consider the following regularized initial-boundary-value problem, dependent on the following given regularization parameters  $\kappa > 0$ ,  $\alpha > 0$  and  $L > 1$ . As has been already emphasized in the Introduction, the centre-of-mass diffusion coefficient  $\varepsilon > 0$  is a physical parameter and is regarded as being fixed throughout.

$(P_{\kappa,\alpha,L})$  Find  $\rho_{\kappa,\alpha,L} : (\underline{x}, t) \in \Omega \times [0, T] \mapsto \rho_{\kappa,\alpha,L}(\underline{x}, t) \in \mathbb{R}$  and  $\underline{u}_{\kappa,\alpha,L} : (\underline{x}, t) \in \bar{\Omega} \times [0, T] \mapsto \underline{u}_{\kappa,\alpha,L}(\underline{x}, t) \in \mathbb{R}^d$  such that

$$\frac{\partial \rho_{\kappa,\alpha,L}}{\partial t} + \tilde{\nabla}_{\underline{x}} \cdot (\underline{u}_{\kappa,\alpha,L} \rho_{\kappa,\alpha,L}) - \alpha \Delta_{\underline{x}} \rho_{\kappa,\alpha,L} = 0 \quad \text{in } \Omega \times (0, T], \quad (2.2a)$$

$$\alpha \tilde{\nabla}_{\underline{x}} \rho_{\kappa,\alpha,L} \cdot \tilde{n} = 0 \quad \text{on } \partial\Omega \times (0, T], \quad (2.2b)$$

$$\rho_{\kappa,\alpha,L}(\underline{x}, 0) = \rho^0(\underline{x}) \quad \forall \underline{x} \in \Omega, \quad (2.2c)$$

$$\begin{aligned} \frac{\partial(\rho_{\kappa,\alpha,L} u_{\kappa,\alpha,L})}{\partial t} - \frac{\alpha}{2} (\Delta_x \rho_{\kappa,\alpha,L}) u_{\kappa,\alpha,L} + \nabla_x \cdot (\rho_{\kappa,\alpha,L} u_{\kappa,\alpha,L} \otimes u_{\kappa,\alpha,L}) \\ - \nabla_x \cdot \mathcal{S}(u_{\kappa,\alpha,L}, \rho_{\kappa,\alpha,L}) + \nabla_x p_{\kappa}(\rho_{\kappa,\alpha,L}) \\ = \rho_{\kappa,\alpha,L} f + \nabla_x \cdot \mathcal{T}(\psi_{\kappa,\alpha,L}) \quad \text{in } \Omega \times (0, T], \end{aligned} \quad (2.2d)$$

$$u_{\kappa,\alpha,L} = 0 \quad \text{on } \partial\Omega \times (0, T], \quad (2.2e)$$

$$(\rho_{\kappa,\alpha,L} u_{\kappa,\alpha,L})(x, 0) = (\rho^0 u_0)(x) \quad \forall x \in \Omega, \quad (2.2f)$$

where  $\psi_{\kappa,\alpha,L} : (x, q, t) \in \bar{\Omega} \times \bar{D} \times [0, T] \mapsto \psi_{\kappa,\alpha,L}(x, q, t) \in \mathbb{R}$ . Here  $\mathcal{S}(\cdot, \cdot)$  and  $\mathcal{T}(\cdot, \cdot)$  are given by (1.2) and (1.10), and  $\rho^0(\alpha)$  is a regularization of  $\rho_0$ , see (3.15) below. In addition,  $p_{\kappa}(\cdot)$  is a regularization of  $p(\cdot)$ , (1.3), defined by

$$p_{\kappa}(s) := p(s) + \kappa (s^4 + s^{\Gamma}), \quad \text{where } \kappa \in \mathbb{R}_{>0} \text{ and } \Gamma = \max\{\gamma, 8\}. \quad (2.3)$$

The Fokker–Planck equation with microscopic cut-off satisfied by  $\psi_{\kappa,\alpha,L}$  is:

$$\begin{aligned} \frac{\partial \psi_{\kappa,\alpha,L}}{\partial t} + \nabla_x \cdot \left( u_{\kappa,\alpha,L} M \beta^L \left( \frac{\psi_{\kappa,\alpha,L}}{M} \right) \right) + \sum_{i=1}^K \nabla_{q_i} \cdot \left[ \sigma(u_{\kappa,\alpha,L}) q_i M \beta^L \left( \frac{\psi_{\kappa,\alpha,L}}{M} \right) \right] \\ = \varepsilon \Delta_x \psi_{\kappa,\alpha,L} + \frac{1}{4\lambda} \sum_{i=1}^K \sum_{j=1}^K A_{ij} \nabla_{q_i} \cdot \left( M \nabla_{q_j} \left( \frac{\psi_{\kappa,\alpha,L}}{M} \right) \right) \quad \text{in } \Omega \times D \times (0, T]. \end{aligned} \quad (2.4)$$

Here, for a given  $L > 1$ ,  $\beta^L \in C(\mathbb{R})$  is defined by (1.19),  $\sigma(v) \equiv \nabla_x v$ , and

$$A \in \mathbb{R}^{K \times K} \text{ is symmetric positive definite with smallest eigenvalue } a_0 \in \mathbb{R}_{>0}. \quad (2.5)$$

We impose the following boundary and initial conditions:

$$\begin{aligned} \left[ \frac{1}{4\lambda} \sum_{j=1}^K A_{ij} M \nabla_{q_j} \left( \frac{\psi_{\kappa,\alpha,L}}{M} \right) - \sigma(u_{\kappa,\alpha,L}) q_i M \beta^L \left( \frac{\psi_{\kappa,\alpha,L}}{M} \right) \right] \cdot \frac{q_i}{|q_i|} = 0 \\ \text{on } \Omega \times \partial \bar{D}_i \times (0, T], \quad i = 1, \dots, K, \end{aligned} \quad (2.6a)$$

$$\varepsilon \nabla_x \psi_{\kappa,\alpha,L} \cdot \tilde{n} = 0 \quad \text{on } \partial\Omega \times D \times (0, T], \quad (2.6b)$$

$$\psi_{\kappa,\alpha,L}(\cdot, \cdot, 0) = M(\cdot) \beta^L(\psi_0(\cdot, \cdot)/M(\cdot)) \geq 0 \quad \text{on } \Omega \times D, \quad (2.6c)$$

where  $\tilde{n}$  is the unit outward normal to  $\partial\Omega$ . The boundary conditions for  $\psi_{\kappa,\alpha,L}$  on  $\partial\Omega \times D \times (0, T]$  and  $\Omega \times \partial D \times (0, T]$  have been chosen so as to ensure that

$$\int_{\Omega \times D} \psi_{\kappa,\alpha,L}(x, q, t) dq dx = \int_{\Omega \times D} \psi_{\kappa,\alpha,L}(x, q, 0) dq dx \quad \forall t \in (0, T]. \quad (2.7)$$

Henceforth, we shall write

$$\hat{\psi}_{\kappa,\alpha,L} = \frac{\psi_{\kappa,\alpha,L}}{M}, \quad \hat{\psi}_0 = \frac{\psi_0}{M}.$$

Thus, for example, (2.6c) in terms of this compact notation becomes:  $\hat{\psi}_{\kappa,\alpha,L}(\cdot, \cdot, 0) = \beta^L(\hat{\psi}_0(\cdot, \cdot))$  on  $\Omega \times D$ .

### 3. Existence of a solution to $(\mathbf{P}_{\kappa,\alpha,L}^{\Delta t})$ , a discrete-in-time approximation of $(\mathbf{P}_{\kappa,\alpha,L})$

For later purposes, we recall the following Lebesgue interpolation result (which is a simple consequence of the Riesz–Thorin theorem) and the Gagliardo–Nirenberg inequality. Let  $1 \leq r \leq v \leq s < \infty$ ; then, for any bounded Lipschitz domain  $\mathcal{O}$ ,

$$\|\eta\|_{L^v(\mathcal{O})} \leq \|\eta\|_{L^r(\mathcal{O})}^{1-\vartheta} \|\eta\|_{L^s(\mathcal{O})}^{\vartheta} \quad \forall \eta \in L^s(\mathcal{O}), \quad (3.1)$$

where  $\vartheta = \frac{s(v-r)}{v(s-r)}$ . Let  $r \in [2, \infty)$  if  $d = 2$ , and  $r \in [2, 6]$  if  $d = 3$  and  $\vartheta = d \left(\frac{1}{2} - \frac{1}{r}\right)$ . Then, there is a constant  $C = C(\Omega, r, d)$ , such that

$$\|\eta\|_{L^r(\Omega)} \leq C \|\eta\|_{L^2(\Omega)}^{1-\vartheta} \|\eta\|_{H^1(\Omega)}^{\vartheta} \quad \forall \eta \in H^1(\Omega). \quad (3.2)$$

We note also the generalized Korn’s inequality

$$\begin{aligned} \int_{\Omega} \left[ |D(\tilde{w})|^2 - \frac{1}{d} |\nabla_x \cdot \tilde{w}|^2 \right] dx &= \int_{\Omega} \left[ |D(\tilde{w}) - \frac{1}{d} (\nabla_x \cdot \tilde{w}) I|^2 dx \right. \\ &\geq c_0 \|\tilde{w}\|_{H^1(\Omega)}^2 \quad \forall \tilde{w} \in H_0^1(\Omega), \end{aligned} \quad (3.3)$$

where  $c_0 > 0$ , see Dain<sup>16</sup>. We remark that the notation  $|\cdot|$  will be used to signify one of the following. When applied to a real number  $x$ ,  $|x|$  will denote the absolute value of the number  $x$ ; when applied to a vector  $v$ ,  $|v|$  will stand for the Euclidean norm of the vector  $v$ ; and, when applied to a square matrix  $\tilde{A}$ ,  $|\tilde{A}|$  will signify the Frobenius norm,  $[\text{tr}(\tilde{A}^T \tilde{A})]^{\frac{1}{2}}$ , of the matrix  $\tilde{A}$ , where, for a square matrix  $\tilde{B}$ ,  $\text{tr}(\tilde{B})$  denotes the trace of  $\tilde{B}$ .

Let  $\mathcal{F} \in C(\mathbb{R}_{>0})$  be defined by  $\mathcal{F}(s) := s(\log s - 1) + 1$ ,  $s > 0$ . As  $\lim_{s \rightarrow 0^+} \mathcal{F}(s) = 1$ , the function  $\mathcal{F}$  can be considered to be defined and continuous on  $[0, \infty)$ , where it is a nonnegative, strictly convex function with  $\mathcal{F}(1) = 0$ .

We assume the following:

$$\begin{aligned} \partial\Omega &\in C^{2,\theta}, \quad \theta \in (0, 1); \quad \rho_0 \in L_{\geq 0}^{\infty}(\Omega); \quad u_0 \in L^2(\Omega); \\ \psi_0 &\geq 0 \text{ a.e. on } \Omega \times D \text{ with } \mathcal{F}(\hat{\psi}_0) \in L_M^1(\Omega \times D) \text{ and } \int_D \psi_0(\cdot, q) dq \in L_{\geq 0}^{\infty}(\Omega); \\ \mu^S &\in \mathbb{R}_{>0}, \quad \mu^B \in \mathbb{R}_{\geq 0}; \quad \text{the Rouse matrix } \tilde{A} \in \mathbb{R}^{K \times K} \text{ satisfies (2.5);} \\ p, p_{\kappa} &\in C^1(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0}) \text{ are defined by (1.3) and (2.3);} \\ \tilde{f} &\in L^2(0, T; L^{\infty}(\Omega)) \text{ and } D_i = B(0, b_i^{\frac{1}{2}}), \quad \theta_i > 1, \quad i = 1, \dots, K, \text{ in (1.7a,b).} \end{aligned} \quad (3.4)$$

We introduce  $P, P_{\kappa} \in C^1(\mathbb{R}_{\geq 0}, \mathbb{R}_{\geq 0})$ , for  $\kappa > 0$ , such that

$$\begin{aligned} s P'_{(\kappa)}(s) - P_{(\kappa)}(s) &= p_{(\kappa)}(s) \quad \text{and} \quad P_{(\kappa)}(0) = P'_{(\kappa)}(0) = 0 \\ \Rightarrow \quad P(s) &= \frac{p(s)}{\gamma - 1} = \frac{c_p}{\gamma - 1} s^{\gamma} \quad \text{and} \quad P_{\kappa}(s) = P(s) + \kappa \left( \frac{s^4}{3} + \frac{s^{\Gamma}}{\Gamma - 1} \right). \end{aligned} \quad (3.5)$$

Here, and throughout, the subscript “ $(\cdot)$ ” means: *with and without the subscript “ $\cdot$ ”*. We adopt a similar notation for superscripts.

In (3.4),  $L_M^r(\Omega \times D)$ , for  $r \in [1, \infty)$ , denotes the Maxwellian-weighted  $L^r$  space over  $\Omega \times D$  with norm

$$\|\varphi\|_{L_M^r(\Omega \times D)} := \left\{ \int_{\Omega \times D} M |\varphi|^r \, d\tilde{q} \, d\tilde{x} \right\}^{\frac{1}{r}}.$$

Similarly, we introduce  $L_M^r(D)$ , the Maxwellian-weighted  $L^r$  space over  $D$ . Letting

$$\|\varphi\|_{H_M^1(\Omega \times D)} := \left\{ \int_{\Omega \times D} M \left[ |\varphi|^2 + |\nabla_x \varphi|^2 + |\nabla_q \varphi|^2 \right] \, d\tilde{q} \, d\tilde{x} \right\}^{\frac{1}{2}}, \quad (3.6)$$

we then set

$$X \equiv H_M^1(\Omega \times D) := \left\{ \varphi \in L_{\text{loc}}^1(\Omega \times D) : \|\varphi\|_{H_M^1(\Omega \times D)} < \infty \right\}. \quad (3.7)$$

It is shown in Appendix C of Barrett & Süli<sup>5</sup> (with the set  $X$  denoted by  $\widehat{X}$  there) that

$$C^\infty(\overline{\Omega \times D}) \text{ is dense in } X. \quad (3.8)$$

We have from Sobolev embedding that

$$H^1(\Omega; L_M^2(D)) \hookrightarrow L^s(\Omega; L_M^2(D)), \quad (3.9)$$

where  $s \in [1, \infty)$  if  $d = 2$  or  $s \in [1, 6]$  if  $d = 3$ . In addition, we note that the embeddings

$$H_M^1(D) \hookrightarrow L_M^2(D), \quad (3.10a)$$

$$H_M^1(\Omega \times D) \equiv L^2(\Omega; H_M^1(D)) \cap H^1(\Omega; L_M^2(D)) \hookrightarrow L_M^2(\Omega \times D) \equiv L^2(\Omega; L_M^2(D)) \quad (3.10b)$$

are compact if  $\theta_i \geq 1$ ,  $i = 1, \dots, K$ , in (1.7a,b); see Appendix D of Barrett & Süli<sup>5</sup>.

We recall the Aubin–Lions–Simon compactness theorem, see, e.g., Simon<sup>25</sup>. Let  $\mathfrak{X}_0$ ,  $\mathfrak{X}$  and  $\mathfrak{X}_1$  be Banach spaces with a compact embedding  $\mathfrak{X}_0 \hookrightarrow \mathfrak{X}$  and a continuous embedding  $\mathfrak{X} \hookrightarrow \mathfrak{X}_1$ . Then, for  $\varsigma_i \in [1, \infty)$ ,  $i = 0, 1$ , the embedding

$$\left\{ \eta \in L^{\varsigma_0}(0, T; \mathfrak{X}_0) : \frac{\partial \eta}{\partial t} \in L^{\varsigma_1}(0, T; \mathfrak{X}_1) \right\} \hookrightarrow L^{\varsigma_0}(0, T; \mathfrak{X}) \quad (3.11)$$

is compact.

Let  $\mathfrak{X}$  be a Banach space. We shall denote by  $C_w([0, T]; \mathfrak{X})$  the set of all functions  $\eta \in L^\infty(0, T; \mathfrak{X})$  such that  $t \in [0, T] \mapsto \langle \varphi, \eta(t) \rangle_{\mathfrak{X}} \in \mathbb{R}$  is continuous on  $[0, T]$  for all  $\varphi \in \mathfrak{X}'$ , the dual space of  $\mathfrak{X}$ . Here, and throughout,  $\langle \cdot, \cdot \rangle_{\mathfrak{X}}$  denotes the duality pairing between  $\mathfrak{X}'$  and  $\mathfrak{X}$ . Whenever  $\mathfrak{X}$  has a predual,  $\mathfrak{E}$ , say, (viz.  $\mathfrak{E}' = \mathfrak{X}$ ), we shall denote by  $C_{w^*}([0, T]; \mathfrak{X})$  the set of all functions  $\eta \in L^\infty(0, T; \mathfrak{X})$  such that  $t \in [0, T] \mapsto \langle \eta(t), \zeta \rangle_{\mathfrak{E}} \in \mathbb{R}$  is continuous on  $[0, T]$  for all  $\zeta \in \mathfrak{E}$ .

We note from Lemma 3.1(a) in Barrett & Süli<sup>12</sup> and Lemma 6.2 in Novotný & Straškraba<sup>24</sup> (or Lemma E.1 in Appendix E in Barrett & Süli<sup>12</sup>) that if  $\{\eta_n\}_{n \in \mathbb{N}}$  is such that

$$\|\eta_n\|_{L^\infty(0,T;L^r(\Omega))} + \left\| \frac{\partial \eta_n}{\partial t} \right\|_{L^\varsigma(0,T;W_0^{1,\nu}(\Omega)')} \leq C, \quad r, \varsigma, \nu \in (1, \infty), \quad (3.12a)$$

then there exists a subsequence (not indicated) of  $\{\eta_n\}_{n \in \mathbb{N}}$  and a function  $\eta \in C_w([0, T]; L^r(\Omega))$  such that

$$\eta_n \rightarrow \eta \quad \text{in } C_w([0, T]; L^r(\Omega)). \quad (3.12b)$$

Throughout we will assume that (3.4) hold, so that (1.8) and (3.10a,b) hold. We note for future reference that (1.12) and (1.8) yield that, for  $\varphi \in L_M^2(\Omega \times D)$  and  $i = 1, \dots, K$ ,

$$\int_{\Omega} |C_i(M\varphi)|^2 dx \underset{\approx}{\sim} \int_{\Omega} \left| \int_D M \varphi U_i' q_i q_i^T dq \right|^2 dx \leq C \left( \int_{\Omega \times D} M |\varphi|^2 dq dx \right), \quad (3.13)$$

where  $C$  is a positive constant.

We state a simple integration-by-parts formula.

**Lemma 3.1.** *Let  $\varphi \in H_M^1(D)$  and suppose that  $B \in \mathbb{R}^{d \times d}$ ; then,*

$$\int_D M \sum_{i=1}^K (B q_i) \cdot \nabla_{q_i} \varphi dq = \int_D M \varphi \left[ \left( \sum_{i=1}^K U_i'(\frac{1}{2}|q_i|^2) q_i q_i^T \right) - K I \right] : B dq. \quad (3.14)$$

**Proof.** By Theorem C.1 in Appendix C of Barrett & Süli<sup>5</sup>, the set  $C^\infty(\overline{D})$  is dense in  $H_M^1(D)$ ; hence, for any  $\widehat{\varphi} \in H_M^1(D)$  there exists a sequence  $\{\widehat{\varphi}_n\}_{n \geq 0} \subset C^\infty(\overline{D})$  converging to  $\widehat{\varphi}$  in  $H_M^1(D)$ . As  $M \in C^1(\overline{D})$  and vanishes on  $\partial D$ , the same is true of each of the functions  $M\widehat{\varphi}_n$ ,  $n \geq 1$ . By replacing  $\widehat{\varphi}$  by  $\widehat{\varphi}_n$  on both sides of (3.14), the resulting identity is easily verified by using the classical divergence theorem for smooth functions, noting (1.6a) and that  $M\widehat{\varphi}_n$  vanishes on  $\partial D$ . Then, (3.14) itself follows by letting  $n \rightarrow \infty$ , recalling the definition of the norm in  $H_M^1(D)$  and hypothesis (1.8).  $\square$

We now formulate our discrete-in-time approximation of problem  $(P_{\kappa, \alpha, L})$  for fixed parameters  $\kappa$ ,  $\alpha \in (0, 1]$  and  $L > 1$ . For any  $T > 0$  and  $N \geq 1$ , let  $N \Delta t = T$  and  $t_n = n \Delta t$ ,  $n = 0, \dots, N$ . To prove existence of a solution under minimal smoothness requirements on the initial data, recall (3.4), we regularize the initial data in terms of the parameters  $\alpha$ ,  $\Delta t$  and  $L$ . Specifically, we shall assign to  $\rho_0 \in L_{\geq 0}^\infty(\Omega)$  the function  $\rho^0 = \rho^0(\alpha) \in H^1(\Omega)$ , appearing in (2.2c,f), defined as the unique solution of the problem:

$$\int_{\Omega} \left[ \rho^0 \eta + \alpha \nabla_x \rho^0 \cdot \nabla_x \eta \right] dx = \int_{\Omega} \rho_0 \eta dx \quad \forall \eta \in H^1(\Omega). \quad (3.15)$$

Hence,

$$\rho^0(\cdot) \in [0, \|\rho_0\|_{L^\infty(\Omega)}], \quad (3.16a)$$

$$\text{and } \rho^0 \rightarrow \rho_0 \text{ weakly-}^* \text{ in } L^\infty(\Omega), \text{ strongly in } L^2(\Omega), \text{ as } \alpha \rightarrow 0_+. \quad (3.16b)$$

Therefore, by (3.1) and (3.16a,b), also

$$\rho^0 \rightarrow \rho_0 \text{ strongly in } L^p(\Omega), \text{ as } \alpha \rightarrow 0_+, \quad p \in [1, \infty). \quad (3.16c)$$

Similarly, we assign to  $u_0 \in \underline{L}^2(\Omega)$  the function  $u^0 = u^0(\alpha, \Delta t) \in \underline{H}_0^1(\Omega)$ , defined as the unique solution of the problem:

$$\int_{\Omega} \left[ \rho^0 \underline{u}^0 \cdot \underline{v} + \Delta t \underline{\nabla}_x \underline{u}^0 : \underline{\nabla}_x \underline{v} \right] dx = \int_{\Omega} \rho^0 u_0 \cdot v dx \quad \forall v \in \underline{H}_0^1(\Omega). \quad (3.17)$$

Hence, it follows from (3.17) and (3.16a) that there exists a  $C \in \mathbb{R}_{>0}$ , independent of  $\Delta t$ ,  $L$ ,  $\alpha$  and  $\kappa$ , such that

$$\int_{\Omega} \left[ \rho^0 |\underline{u}^0|^2 + \Delta t |\underline{\nabla}_x \underline{u}^0|^2 \right] dx \leq \int_{\Omega} \rho^0 |u_0|^2 dx \leq C, \quad (3.18a)$$

$$\text{and } \int_{\Omega} \rho^0 (u^0 - u_0) \cdot v dx \rightarrow 0 \quad \forall v \in \underline{L}^2(\Omega), \text{ as } \Delta t \rightarrow 0_+. \quad (3.18b)$$

Analogously, we shall assign a certain ‘smoothed’ initial datum,  $\widehat{\psi}^0 = \widehat{\psi}^0(L, \Delta t) \in H_M^1(\Omega \times D)$ , to the given initial datum  $\widehat{\psi}_0 = \frac{\psi_0}{M}$  such that

$$\int_{\Omega \times D} M \left[ \widehat{\psi}^0 \varphi + \Delta t \left( \underline{\nabla}_x \widehat{\psi}^0 \cdot \underline{\nabla}_x \varphi + \underline{\nabla}_q \widehat{\psi}^0 \cdot \underline{\nabla}_q \varphi \right) \right] dq dx = \int_{\Omega \times D} M \beta^L(\widehat{\psi}_0) \varphi dq dx \quad \forall \varphi \in H_M^1(\Omega \times D). \quad (3.19)$$

For  $r \in [1, \infty)$ , let

$$Z_r := \{ \varphi \in L_M^r(\Omega \times D) : \varphi \geq 0 \text{ a.e. on } \Omega \times D \}. \quad (3.20)$$

It is proved in the Appendix of Barrett & Süli<sup>10</sup> that there exists a unique  $\widehat{\psi}^0 \in H_M^1(\Omega \times D)$  satisfying (3.19); furthermore,  $\widehat{\psi}^0 \in Z_2$ ,

$$\begin{aligned} \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}^0) dq dx + 4 \Delta t \int_{\Omega \times D} M \left[ |\underline{\nabla}_x \sqrt{\widehat{\psi}^0}|^2 + |\underline{\nabla}_q \sqrt{\widehat{\psi}^0}|^2 \right] dq dx \\ \leq \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_0) dq dx \end{aligned} \quad (3.21a)$$

and

$$\widehat{\psi}^0 = \beta^L(\widehat{\psi}^0) \rightarrow \widehat{\psi}_0 \text{ weakly in } L_M^1(\Omega \times D), \text{ as } L \rightarrow \infty, \quad \Delta t \rightarrow 0_+. \quad (3.21b)$$

Finally, by choosing  $\varphi(x, q) = \varrho^0(x) \otimes 1(q)$  in (3.19), where

$$\varrho^0(x) := \int_D M(q) \widehat{\psi}^0(x, q) dq, \quad x \in \Omega,$$

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yields, on noting (1.19) and (3.4), that

$$\begin{aligned} & \frac{1}{2} \left[ \|\varrho^0\|_{L^2(\Omega)}^2 + \int_{\Omega} \left( \int_D M(\widehat{\psi}^0 - \beta^L(\widehat{\psi}_0)) \, dq \right)^2 dx \right] + \Delta t \|\nabla_x \varrho^0\|_{L^2(\Omega)}^2 \\ &= \frac{1}{2} \int_{\Omega} \left( \int_D M \beta^L(\widehat{\psi}_0) \, dq \right)^2 dx \\ &\leq \frac{1}{2} \int_{\Omega} \left( \int_D M \widehat{\psi}_0 \, dq \right)^2 dx \leq C. \end{aligned} \quad (3.22)$$

Next, we define

$$V := \{w \in H_0^1(\Omega) : w \in L^\infty(\Omega), \nabla_x \cdot w \in L^\infty(\Omega)\} \quad (3.23)$$

and

$$\begin{aligned} Y^n := & L^2(t_{n-1}, t_n; H^1(\Omega)) \cap H^1(t_{n-1}, t_n; H^1(\Omega)') \\ & \cap L^\infty(t_{n-1}, t_n; L^{\frac{F}{2}}(\Omega)) \cap L_{\geq 0}^{\frac{4F}{3}}(\Omega \times (t_{n-1}, t_n)). \end{aligned} \quad (3.24)$$

We recall also that, for all  $v, w \in H^1(\Omega)$ ,

$$(v \otimes v) : \nabla_x w = [(v \cdot \nabla_x)w] \cdot v = -[(v \cdot \nabla_x)v] \cdot w + (v \cdot \nabla_x)(v \cdot w) \quad \text{a.e. in } \Omega. \quad (3.25)$$

Noting the above, our discrete-in-time approximation of  $(P_{\kappa, \alpha, L})$  is then defined as follows.

**( $\mathbf{P}_{\kappa, \alpha, L}^{\Delta t}$ )** Let  $N \in \mathbb{N}_{\geq 1}$  and set  $\Delta t := T/N$ ; let us further suppose that  $\rho_{\kappa, \alpha, L}^0 := \rho^0 \in L_{\geq 0}^\infty(\Omega)$ ,  $u_{\kappa, \alpha, L}^0 := u^0 \in H_0^1(\Omega)$  and  $\widehat{\psi}_{\kappa, \alpha, L}^0 := \widehat{\psi}^0 \in Z_2$ . For  $n = 1, \dots, N$ , and given  $(\rho_{\kappa, \alpha, L}^{n-1}, u_{\kappa, \alpha, L}^{n-1}, \widehat{\psi}_{\kappa, \alpha, L}^{n-1}) \in L_{\geq 0}^\Gamma(\Omega) \times H_0^1(\Omega) \times Z_2$ , find

$$\begin{aligned} \rho_{\kappa, \alpha, L}^{[\Delta t], n} \in Y^n \quad \text{with} \quad \rho_{\kappa, \alpha, L}^n(\cdot) := \rho_{\kappa, \alpha, L}^{[\Delta t], n}(\cdot, t_n) \in L_{\geq 0}^\Gamma(\Omega), \quad u_{\kappa, \alpha, L}^n \in H_0^1(\Omega) \\ \text{and} \quad \widehat{\psi}_{\kappa, \alpha, L}^n \in X \cap Z_2, \end{aligned} \quad (3.26)$$

such that  $\rho_{\kappa, \alpha, L}^{[\Delta t], n}(\cdot, t_{n-1}) = \rho_{\kappa, \alpha, L}^{n-1}(\cdot)$ ,

$$\begin{aligned} \int_{t_{n-1}}^{t_n} \left[ \left\langle \frac{\partial \rho_{\kappa, \alpha, L}^{[\Delta t], n}}{\partial t}, \eta \right\rangle_{H^1(\Omega)} + \int_{\Omega} \left( \alpha \nabla_x \rho_{\kappa, \alpha, L}^{[\Delta t], n} - \rho_{\kappa, \alpha, L}^{[\Delta t], n} u_{\kappa, \alpha, L}^n \right) \cdot \nabla_x \eta \, dx \right] dt = 0 \\ \forall \eta \in L^2(t_{n-1}, t_n; H^1(\Omega)), \end{aligned} \quad (3.27a)$$



$$\begin{aligned}
& \int_{\Omega} \left[ \frac{\rho_{\kappa,\alpha,L}^n u_{\kappa,\alpha,L}^n - \rho_{\kappa,\alpha,L}^{n-1} u_{\kappa,\alpha,L}^{n-1}}{\Delta t} - \frac{1}{2} \frac{\rho_{\kappa,\alpha,L}^n - \rho_{\kappa,\alpha,L}^{n-1}}{\Delta t} u_{\kappa,\alpha,L}^n \right] \cdot w \, dx \\
& + \frac{1}{2} \int_{\Omega} \rho_{\kappa,\alpha,L}^{n-1} \left[ [(u_{\kappa,\alpha,L}^{n-1} \cdot \nabla_x) u_{\kappa,\alpha,L}^n] \cdot w - [(u_{\kappa,\alpha,L}^{n-1} \cdot \nabla_x) w] \cdot u_{\kappa,\alpha,L}^n \right] dx \\
& + \int_{\Omega} S(u_{\kappa,\alpha,L}^n) : \nabla_x w \, dx - \int_{\Omega} \left( \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} p_{\kappa}(\rho_{\kappa,\alpha,L}^{[\Delta t],n}) dt \right) \nabla_x \cdot w \, dx \\
& = \int_{\Omega} \rho_{\kappa,\alpha,L}^n f^n \cdot w \, dx - \int_{\Omega} \tau_1(M \widehat{\psi}_{\kappa,\alpha,L}^n) : \nabla_x w \, dx \\
& \quad - 2\mathfrak{z} \int_{\Omega} \left( \int_D M \beta^L(\widehat{\psi}_{\kappa,\alpha,L}^n) dq \right) \nabla_x \left( \int_D M \widehat{\psi}_{\kappa,\alpha,L}^n dq \right) \cdot w \, dx \quad \forall w \in V,
\end{aligned} \tag{3.27b}$$

$$\begin{aligned}
& \int_{\Omega \times D} M \frac{\widehat{\psi}_{\kappa,\alpha,L}^n - \widehat{\psi}_{\kappa,\alpha,L}^{n-1}}{\Delta t} \varphi \, dq \, dx \\
& + \sum_{i=1}^K \int_{\Omega \times D} M \left[ \frac{1}{4\lambda} \sum_{j=1}^K A_{ij} \nabla_{q_j} \widehat{\psi}_{\kappa,\alpha,L}^n - [\sigma(u_{\kappa,\alpha,L}^n) q_i] \beta^L(\widehat{\psi}_{\kappa,\alpha,L}^n) \right] \cdot \nabla_{q_i} \varphi \, dq \, dx \\
& + \int_{\Omega \times D} M \left[ \varepsilon \nabla_x \widehat{\psi}_{\kappa,\alpha,L}^n - \beta^L(\widehat{\psi}_{\kappa,\alpha,L}^n) u_{\kappa,\alpha,L}^n \right] \cdot \nabla_x \varphi \, dq \, dx = 0 \quad \forall \varphi \in X;
\end{aligned} \tag{3.27c}$$

where, for  $t \in (t_{n-1}, t_n]$  and  $n = 1, \dots, N$ ,

$$f^{\{\Delta t\}}(\cdot, t) = f^n(\cdot) := \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} f(\cdot, t) dt \in L^\infty(\Omega). \tag{3.28}$$

It follows from (3.4) and (3.28) that

$$\int_{t_{n-1}}^{t_n} \|f^{\{\Delta t\}}\|_{L^\infty(\Omega)}^2 dt \leq \int_{t_{n-1}}^{t_n} \|f\|_{L^\infty(\Omega)}^2 dt, \quad n = 1, \dots, N, \tag{3.29a}$$

$$f^{\{\Delta t\}} \rightarrow f \quad \text{strongly in } L^2(0, T; L^r(\Omega)), \quad \text{as } \Delta t \rightarrow 0_+, \tag{3.29b}$$

where  $r \in [1, \infty]$ .

**Remark 3.1.** A possible alternative to our temporal approximation scheme (3.27a) for (2.2a), which is the weak formulation of a parabolic initial boundary-value problem posed over the time slab  $\Omega \times [t_{n-1}, t_n]$ ,  $n = 1, \dots, N$ , would have been to proceed as in the work of Abels and Feireisl<sup>1</sup> and approximate (2.2a) by an implicit finite difference scheme with respect to  $t$ . That would have avoided the use of  $\rho_{\kappa,\alpha,L}^{[\Delta t],n}$  here, but would have had the disadvantage, from the point of view of constructive considerations in numerical analysis at least, that nonnegativity of  $\rho_{\kappa,\alpha,L}^n$  will have been guaranteed for  $\Delta t \leq \Delta t_0$  only, where  $\Delta t_0 \in (0, T]$  is sufficiently small, with the value of  $\Delta t_0$  not being easily quantifiable in terms of the data and its independence of  $\kappa$ ,  $\alpha$  and  $L$  being less than obvious. In contrast with that, our  $\rho_{\kappa,\alpha,L}^{[\Delta t],n}$ , and thereby

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also  $\rho_{\kappa,\alpha,L}^n$ , will be shown to be nonnegative for all  $\Delta t = \frac{T}{N}$  and all  $n = 1, \dots, N$ , regardless of the choice of  $\kappa, \alpha, L$  and  $N$ .

We rewrite (3.27a) as

$$\int_{t_{n-1}}^{t_n} \left[ \left\langle \frac{\partial \rho_{\kappa,\alpha,L}^{[\Delta t],n}}{\partial t}, \eta \right\rangle_{H^1(\Omega)} + c(u_{\kappa,\alpha,L}^n)(\rho_{\kappa,\alpha,L}^{[\Delta t],n}, \eta) \right] dt = 0 \quad \forall \eta \in L^2(t_{n-1}, t_n; H^1(\Omega)), \quad (3.30)$$

where, for all  $v \in \underline{H}_0^1(\Omega)$  and  $\eta_i \in H^1(\Omega)$ ,  $i = 1, 2$ ,

$$c(v)(\eta_1, \eta_2) := \int_{\Omega} \left( \alpha \nabla_x \eta_1 - \eta_1 v \right) \cdot \nabla_x \eta_2 \, dx. \quad (3.31)$$

Similarly, on noting (1.2) and (1.11), we rewrite (3.27b) as

$$b(\rho_{\kappa,\alpha,L}^n)(u_{\kappa,\alpha,L}^n, w) = \ell_b(\rho_{\kappa,\alpha,L}^{[\Delta t],n}, \widehat{\psi}_{\kappa,\alpha,L}^n)(w) \quad \forall w \in \underline{V}; \quad (3.32)$$

where, for all  $\eta \in L_{\geq 0}^2(\Omega)$  and  $w_i \in \underline{H}_0^1(\Omega)$ ,  $i = 1, 2$ ,

$$\begin{aligned} b(\eta)(w_1, w_2) &:= \frac{1}{2} \int_{\Omega} (\eta + \rho_{\kappa,\alpha,L}^{n-1}) w_1 \cdot w_2 \, dx + \Delta t \mu^S \int_{\Omega} D(w_1) : D(w_2) \, dx \\ &\quad + \Delta t \left( \mu^B - \frac{\mu^S}{d} \right) \int_{\Omega} (\nabla_x \cdot w_1) (\nabla_x \cdot w_2) \, dx \\ &\quad + \frac{1}{2} \int_{\Omega} \rho_{\kappa,\alpha,L}^{n-1} \left[ [(u_{\kappa,\alpha,L}^{n-1} \cdot \nabla_x) w_1] \cdot w_2 - [(u_{\kappa,\alpha,L}^{n-1} \cdot \nabla_x) w_2] \cdot w_1 \right] \, dx \end{aligned} \quad (3.33a)$$

and, for all  $\eta \in Y^n$  with  $\eta(\cdot, t_n) \in L_{\geq 0}^2(\Omega)$ ,  $\varphi \in X$  and  $w \in \underline{V}$ ,

$$\begin{aligned} &\ell_b(\eta, \varphi)(w) \\ &:= \int_{\Omega} \left[ \rho_{\kappa,\alpha,L}^{n-1} u_{\kappa,\alpha,L}^{n-1} \cdot w + \Delta t \eta(\cdot, t_n) f^n \cdot w - \Delta t k \sum_{i=1}^K C_i(M \varphi) : \nabla_x w \right] dx \\ &\quad + \int_{\Omega} \left( \int_{t_{n-1}}^{t_n} p_{\kappa}(\eta) dt + \Delta t k (K+1) \int_D M \varphi dq \right) \nabla_x \cdot w \, dx \\ &\quad - 2 \Delta t \mathfrak{z} \int_{\Omega} \left( \int_D M \beta^L(\varphi) dq \right) \nabla_x \left( \int_D M \varphi dq \right) \cdot w \, dx. \end{aligned} \quad (3.33b)$$

It follows for fixed  $u_{\kappa,\alpha,L}^{n-1} \in \underline{H}_0^1(\Omega)$ ,  $\rho_{\kappa,\alpha,L}^{n-1} \in L_{\geq 0}^{\Gamma}(\Omega)$  and  $\eta \in L_{\geq 0}^2(\Omega)$ , and the generalized Korn's inequality, (3.3), that the nonsymmetric bilinear functional  $b(\eta)(\cdot, \cdot)$  is a nonsymmetric continuous coercive bilinear functional on  $\underline{H}_0^1(\Omega) \times \underline{H}_0^1(\Omega)$ . In addition, for fixed  $u_{\kappa,\alpha,L}^{n-1} \in \underline{H}_0^1(\Omega)$ ,  $\rho_{\kappa,\alpha,L}^{n-1} \in L_{\geq 0}^{\Gamma}(\Omega)$ ,  $\eta \in Y^n$  with  $\eta(\cdot, t_n) \in L_{\geq 0}^2(\Omega)$  and  $\varphi \in X$ , it follows, on recalling (3.13), (2.3) and (3.24), that  $\ell_b(\eta, \varphi)(\cdot)$  is a continuous linear functional on  $\underline{V}$ .

It is also convenient to rewrite (3.27c) as

$$a(\widehat{\psi}_{\kappa,\alpha,L}^n, \varphi) = \ell_a(u_{\kappa,\alpha,L}^n, \beta^L(\widehat{\psi}_{\kappa,\alpha,L}^n))(\varphi) \quad \forall \varphi \in X, \quad (3.34)$$

where, for all  $\varphi_i \in X$ ,  $i = 1, 2$ ,

$$\begin{aligned} & a(\varphi_1, \varphi_2) \\ & := \int_{\Omega \times D} M \left[ \varphi_1 \varphi_2 + \Delta t \varepsilon \nabla_x \varphi_1 \cdot \nabla_x \varphi_2 + \frac{\Delta t}{4\lambda} \sum_{i=1}^K \sum_{j=1}^K A_{ij} \nabla_{q_j} \varphi_1 \cdot \nabla_{q_i} \varphi_2 \right] dq dx \end{aligned} \quad (3.35a)$$

and, for all  $v \in \underline{H}^1(\Omega)$ ,  $\xi \in L^\infty(\Omega \times D)$  and  $\varphi \in X$ ,

$$\ell_a(v, \xi)(\varphi) := \int_{\Omega \times D} M \left[ \widehat{\psi}_{\kappa, \alpha, L}^{n-1} \varphi + \Delta t \xi \left( \sum_{i=1}^K [\sigma(v) q_i] \cdot \nabla_{q_i} \varphi + v \cdot \nabla_x \varphi \right) \right] dq dx. \quad (3.35b)$$

Clearly,  $a(\cdot, \cdot)$  is a symmetric continuous coercive bilinear functional on  $X \times X$ . In addition, it is easily deduced for fixed  $v \in \underline{H}^1(\Omega)$  and  $\xi \in L^\infty(\Omega \times D)$ , on noting (3.2), that  $\ell_a(v, \xi)(\cdot)$  is a continuous linear functional on  $X$ .

In order to prove existence of a solution, (3.26), to  $(P_{\kappa, \alpha, L}^{\Delta t}, \rho_{\kappa, \alpha, L}^{[\Delta t], n}(\cdot, t_{n-1}) = \rho_{\kappa, \alpha, L}^{n-1}(\cdot))$  and (3.27a-c), which is equivalent to (3.30), (3.32) and (3.34), we require two convex regularizations of the entropy function

$$\mathcal{F} : s \in \mathbb{R}_{\geq 0} \mapsto \mathcal{F}(s) = s(\log s - 1) + 1 \in \mathbb{R}_{\geq 0},$$

denoted by  $\mathcal{F}^L$  and  $\mathcal{F}_\delta^L$ .

For any  $L > 1$ , we define  $\mathcal{F}^L \in C(\mathbb{R}_{\geq 0}) \cap C_{\text{loc}}^{2,1}(\mathbb{R}_{>0})$  by

$$\mathcal{F}^L(s) := \begin{cases} s(\log s - 1) + 1, & 0 \leq s \leq L, \\ \frac{s^2 - L^2}{2L} + s(\log L - 1) + 1, & L \leq s. \end{cases} \quad (3.36)$$

Note that

$$[\mathcal{F}^L]'(s) = \begin{cases} \log s, & 0 < s \leq L, \\ \frac{s}{L} + \log L - 1, & L \leq s, \end{cases} \quad (3.37a)$$

and

$$[\mathcal{F}^L]''(s) = \begin{cases} \frac{1}{s}, & 0 < s \leq L, \\ \frac{1}{L}, & L \leq s. \end{cases} \quad (3.37b)$$

Hence, on noting the definition (1.19) of  $\beta^L$ , we have that

$$\beta^L(s) = \min\{s, L\} = ([\mathcal{F}^L]''(s))^{-1}, \quad s \in \mathbb{R}_{\geq 0}, \quad (3.38a)$$

with the convention  $\frac{1}{\infty} := 0$  when  $s = 0$ , and

$$[\mathcal{F}^L]''(s) \geq \mathcal{F}''(s) = \frac{1}{s}, \quad s \in \mathbb{R}_{>0}. \quad (3.38b)$$

We shall also require the following inequality, relating  $\mathcal{F}^L$  to  $\mathcal{F}$ :

$$\mathcal{F}^L(s) \geq \mathcal{F}(s), \quad s \in \mathbb{R}_{\geq 0}. \quad (3.39)$$

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For  $0 \leq s \leq 1$ , (3.39) trivially holds, with equality. For  $s \geq 1$ , it follows from (3.38b), with  $s$  replaced by a dummy variable  $\sigma$ , after integrating twice over  $\sigma \in [1, s]$ , and noting that  $[\mathcal{F}^L]'(1) = \mathcal{F}'(1)$  and  $\mathcal{F}^L(1) = \mathcal{F}(1)$ .

For  $L > 1$  and  $\delta \in (0, 1)$ , the function  $\mathcal{F}_\delta^L \in C^{2,1}(\mathbb{R})$  is defined by

$$\mathcal{F}_\delta^L(s) := \begin{cases} \frac{s^2 - \delta^2}{2\delta} + s(\log \delta - 1) + 1, & s \leq \delta, \\ \mathcal{F}^L(s), & \delta \leq s. \end{cases} \quad (3.40)$$

Hence,

$$[\mathcal{F}_\delta^L]'(s) = \begin{cases} \frac{s}{\delta} + \log \delta - 1, & s \leq \delta, \\ [\mathcal{F}^L]'(s), & \delta \leq s, \end{cases} \quad (3.41a)$$

$$[\mathcal{F}_\delta^L]''(s) = \begin{cases} \frac{1}{\delta}, & s \leq \delta, \\ [\mathcal{F}^L]''(s), & \delta \leq s. \end{cases} \quad (3.41b)$$

We note that

$$\mathcal{F}_\delta^L(s) \leq \mathcal{F}^L(s) \quad \forall s \geq 0, \quad (3.42a)$$

$$\mathcal{F}_\delta^L(s) \geq \begin{cases} \frac{s^2}{2\delta}, & s \leq 0, \\ \frac{s^2}{4L} - C(L), & s \geq 0; \end{cases} \quad (3.42b)$$

and that  $[\mathcal{F}_\delta^L]''(s)$  is bounded below by  $\frac{1}{L}$  for all  $s \in \mathbb{R}$ . Finally, we set

$$\beta_\delta^L(s) := ([\mathcal{F}_\delta^L]'')^{-1}(s) = \max\{\beta^L(s), \delta\}, \quad (3.43)$$

and observe that  $\beta_\delta^L(s)$  is bounded above by  $L$  and bounded below by  $\delta$  for all  $s \in \mathbb{R}$ . Note also that both  $\beta^L$  and  $\beta_\delta^L$  are Lipschitz continuous on  $\mathbb{R}$ , with Lipschitz constants equal to 1.

In addition, we regularize the bilinear functional  $b(\eta)(\cdot, \cdot)$ , (3.33a), on  $X \times X$ , by introducing the Banach space

$$\mathcal{V} := \{\tilde{w} \in H^2(\Omega) \cap H_0^1(\Omega) : \nabla_x \cdot \tilde{w} \in H^2(\Omega)\}, \quad (3.44)$$

which is compactly embedded in  $\mathcal{V}$ , (3.23). We then define, for  $\delta \in \mathbb{R}_{>0}$ ,  $\eta \in L_{\geq 0}^2(\Omega)$  and  $w_i \in \mathcal{V}$ ,  $i = 1, 2$ ,

$$\begin{aligned} b_\delta(\eta)(w_1, w_2) &:= b(\eta)(w_1, w_2) \\ &+ \Delta t \delta \sum_{|\lambda|=2} \int_{\Omega} \left[ \frac{\partial^{|\lambda|} \tilde{w}_1}{\partial x_1^{\lambda_1} \cdots \partial x_d^{\lambda_d}} \cdot \frac{\partial^{|\lambda|} \tilde{w}_2}{\partial x_1^{\lambda_1} \cdots \partial x_d^{\lambda_d}} + \frac{\partial^{|\lambda|} (\nabla_x \cdot w_1)}{\partial x_1^{\lambda_1} \cdots \partial x_d^{\lambda_d}} \frac{\partial^{|\lambda|} (\nabla_x \cdot w_2)}{\partial x_1^{\lambda_1} \cdots \partial x_d^{\lambda_d}} \right] dx. \end{aligned} \quad (3.45)$$

It follows that  $b_\delta(\eta)(\cdot, \cdot)$  is a nonsymmetric continuous coercive bilinear functional on  $\mathcal{V} \times \mathcal{V}$  for fixed  $\eta \in L_{\geq 0}^2(\Omega)$  and  $\delta \in (0, 1)$ . We also replace  $\ell_b(\eta, \varphi)(\cdot)$  by  $\ell_{b,\delta}(\eta, \varphi)(\cdot)$ ,

where  $\beta^L(\varphi)$  in  $\ell_b(\eta, \varphi)(\cdot)$  is replaced by  $\beta_\delta^L(\varphi)$ ; that is, for  $\delta \in (0, 1)$ ,

$$\begin{aligned} \ell_{b,\delta}(\eta, \varphi)(w) &:= \ell_b(\eta, \varphi)(w) \\ &+ 2 \Delta t \mathfrak{z} \int_{\Omega} \left( \int_D M [\beta^L(\varphi) - \beta_\delta^L(\varphi)] \, dq \right) \nabla_x \left( \int_D M \varphi \, dq \right) \cdot w \, dx. \end{aligned} \quad (3.46)$$

As for fixed  $\eta \in Y^n$  with  $\eta(\cdot, t_n) \in L_{\geq 0}^2(\Omega)$  and  $\varphi \in X$ ,  $\ell_b(\eta, \varphi)(\cdot)$  is a continuous linear functional on  $\mathcal{V}$ , it follows that  $\ell_{b,\delta}(\eta, \varphi)(\cdot)$ , for  $\delta \in (0, 1)$ , is a continuous linear functional on  $\mathcal{V}$ .

Next, we introduce

$$\Upsilon^n := L^2(t_{n-1}, t_n; H^1(\Omega)) \cap H^1(t_{n-1}, t_n; H^1(\Omega)') \cap L_{\geq 0}^\infty(\Omega \times (t_{n-1}, t_n)). \quad (3.47)$$

We note that  $\Upsilon^n \hookrightarrow C([t_{n-1}, t_n]; L_{\geq 0}^2(\Omega))$  and  $\Upsilon^n \subset Y^n$ , (cf. (3.24)). Finally, we regularize the initial datum for the problem posed in the time-slab  $\Omega \times [t_{n-1}, t_n]$  by setting

$$\rho_{\kappa,\alpha,L,\delta}^{n-1} = \beta^{\delta^{-1}}(\rho_{\kappa,\alpha,L}^{n-1}), \quad (3.48)$$

where  $\beta^{\delta^{-1}}$  is given by (1.19) with  $L = \delta^{-1}$ .

We now consider the following regularized version of the coupled system (3.30), (3.32) and (3.34) for a given  $\delta \in (0, 1)$ :

For  $(\rho_{\kappa,\alpha,L}^{n-1}, u_{\kappa,\alpha,L}^{n-1}, \psi_{\kappa,\alpha,L}^{n-1}) \in L_{\geq 0}^\Gamma(\Omega) \times \underline{H}_0^1(\Omega) \times Z_2$ , find  $(\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}, u_{\kappa,\alpha,L,\delta}^n, \widehat{\psi}_{\kappa,\alpha,L,\delta}^n) \in \Upsilon^n \times \mathcal{V} \times X$  such that  $\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}(\cdot, t_{n-1}) = \rho_{\kappa,\alpha,L,\delta}^{n-1}(\cdot)$ ,

$$\begin{aligned} \int_{t_{n-1}}^{t_n} \left[ \left\langle \frac{\partial \rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}}{\partial t}, \eta \right\rangle_{H^1(\Omega)} + c(u_{\kappa,\alpha,L,\delta}^n)(\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}, \eta) \right] dt = 0 \\ \forall \eta \in L^2(t_{n-1}, t_n; H^1(\Omega)), \end{aligned} \quad (3.49a)$$

$$b_\delta(\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}(\cdot, t_n))(u_{\kappa,\alpha,L,\delta}^n, w) = \ell_{b,\delta}(\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}, \widehat{\psi}_{\kappa,\alpha,L,\delta}^n)(w) \quad \forall w \in \mathcal{V}, \quad (3.49b)$$

$$a(\widehat{\psi}_{\kappa,\alpha,L,\delta}^n, \varphi) = \ell_a(u_{\kappa,\alpha,L,\delta}^n, \beta_\delta^L(\widehat{\psi}_{\kappa,\alpha,L,\delta}^n))(\varphi) \quad \forall \varphi \in X. \quad (3.49c)$$

The existence of a solution to (3.49a–c) will be proved by using a fixed-point argument. Given  $(\tilde{u}, \tilde{\psi}) \in \mathcal{V} \times L_M^2(\Omega \times D)$ , let  $(\rho^*, u^*, \psi^*) \in \Upsilon^n \times \mathcal{V} \times X$  be such that  $\rho^*(\cdot, t_{n-1}) = \rho_{\kappa,\alpha,L,\delta}^{n-1}(\cdot)$ ,

$$\int_{t_{n-1}}^{t_n} \left[ \left\langle \frac{\partial \rho^*}{\partial t}, \eta \right\rangle_{H^1(\Omega)} + c(\tilde{u})(\rho^*, \eta) \right] dt = 0 \quad \forall \eta \in L^2(t_{n-1}, t_n; H^1(\Omega)), \quad (3.50a)$$

$$a(\psi^*, \varphi) = \ell_a(\tilde{u}, \beta_\delta^L(\tilde{\psi}))(\varphi) \quad \forall \varphi \in X, \quad (3.50b)$$

$$b_\delta(\rho^*(\cdot, t_n))(u^*, w) = \ell_{b,\delta}(\rho^*, \psi^*)(w) \quad \forall w \in \mathcal{V}. \quad (3.50c)$$

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For fixed  $\underline{v} \in \underline{V}$ , it follows that  $c(\underline{v})(\cdot, \cdot)$ , (3.31), is a nonsymmetric continuous bilinear functional on  $H^1(\Omega) \times H^1(\Omega)$ , and, moreover, for all  $\eta \in H^1(\Omega)$ ,

$$\begin{aligned} c(\underline{v})(\eta, \eta) &= \alpha \|\underline{\nabla}_x \eta\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_{\Omega} (\underline{\nabla}_x \cdot \underline{v}) \eta^2 \, dx \\ &\geq \alpha \|\underline{\nabla}_x \eta\|_{L^2(\Omega)}^2 - \frac{1}{2} \|\underline{\nabla}_x \cdot \underline{v}\|_{L^\infty(\Omega)} \|\eta\|_{L^2(\Omega)}^2. \end{aligned} \quad (3.51)$$

Hence for any fixed  $\tilde{u} \in \underline{V}$ , the existence of a unique weak solution

$$\rho^* \in L^2(t_{n-1}, t_n; H^1(\Omega)) \cap H^1(t_{n-1}, t_n; H^1(\Omega)') \hookrightarrow C([t_{n-1}, t_n]; L^2(\Omega)) \quad (3.52)$$

satisfying  $\rho^*(\cdot, t_{n-1}) = \rho_{\kappa, \alpha, L, \delta}^{n-1}(\cdot)$  and (3.50a) is immediate; see, for example, Wloka<sup>26</sup>, Thm. 26.1. Further, on choosing, for  $s \in (t_{n-1}, t_n]$ ,  $\eta(\cdot, t) = \chi_{[t_{n-1}, s]} e^{-\|\underline{\nabla}_x \cdot \tilde{u}\|_{L^\infty(\Omega)}(t-t_{n-1})} [\rho^*(\cdot, t)]_-$  in (3.50a), where, for a set  $S \subset \mathbb{R}$ ,  $\chi_S$  denotes the characteristic function of  $S$ , and recalling (3.51), we obtain that

$$\begin{aligned} &e^{-\|\underline{\nabla}_x \cdot \tilde{u}\|_{L^\infty(\Omega)}(s-t_{n-1})} \|[\rho^*(\cdot, s)]_-\|_{L^2(\Omega)}^2 \\ &+ 2\alpha \int_{t_{n-1}}^s \int_{\Omega} e^{-\|\underline{\nabla}_x \cdot \tilde{u}\|_{L^\infty(\Omega)}(t-t_{n-1})} |\underline{\nabla}_x [\rho^*(x, t)]_-|^2 \, dx \, dt \leq 0, \quad s \in (t_{n-1}, t_n]. \end{aligned} \quad (3.53)$$

Next, we define

$$R(t) := e^{\|\underline{\nabla}_x \cdot \tilde{u}\|_{L^\infty(\Omega)}(t-t_{n-1})} \|\rho_{\kappa, \alpha, L, \delta}^{n-1}\|_{L^\infty(\Omega)}, \quad t \in [t_{n-1}, t_n]; \quad (3.54)$$

hence,

$$\begin{aligned} &\int_{t_{n-1}}^{t_n} \left[ \left\langle \frac{\partial(\rho^* - R)}{\partial t}, \eta \right\rangle_{H^1(\Omega)} + c(\tilde{u})(\rho^* - R, \eta) \right] dt \\ &= - \int_{t_{n-1}}^{t_n} R \int_{\Omega} (\underline{\nabla}_x \cdot \tilde{u} + \|\underline{\nabla}_x \cdot \tilde{u}\|_{L^\infty(\Omega)}) \eta \, dx \, dt \quad \forall \eta \in L^2(t_{n-1}, t_n; H^1(\Omega)). \end{aligned} \quad (3.55)$$

Then, similarly to (3.53), on choosing

$$\eta(\cdot, t) = \chi_{[t_{n-1}, s]} e^{-\|\underline{\nabla}_x \cdot \tilde{u}\|_{L^\infty(\Omega)}(t-t_{n-1})} [\rho^*(\cdot, t) - R(t)]_+$$

in (3.55) for  $s \in (t_{n-1}, t_n]$ , we obtain that

$$\begin{aligned} &e^{-\|\underline{\nabla}_x \cdot \tilde{u}\|_{L^\infty(\Omega)}(s-t_{n-1})} \|[\rho^*(\cdot, s) - R(s)]_+\|_{L^2(\Omega)}^2 \\ &+ 2\alpha \int_{t_{n-1}}^s \int_{\Omega} e^{-\|\underline{\nabla}_x \cdot \tilde{u}\|_{L^\infty(\Omega)}(t-t_{n-1})} |\underline{\nabla}_x [\rho^*(x, t) - R(t)]_+|^2 \, dx \, dt \leq 0, \end{aligned} \quad (3.56)$$

On noting (3.52), extending (3.53) and (3.56) from the interval  $(t_{n-1}, t_n]$  to  $[t_{n-1}, t_n]$  by letting  $s \rightarrow t_{n-1}$  in (3.53) and (3.56), and combining the resulting inequalities, we deduce that

$$\rho^*(\cdot, t) \in [0, R(t)] \quad \text{for } t \in [t_{n-1}, t_n], \quad \text{and so } \rho^* \in \Upsilon^n. \quad (3.57)$$

As  $a(\cdot, \cdot)$  is a symmetric continuous coercive bilinear functional on  $X \times X$  and  $\ell_a(\underline{v}, \xi)(\cdot)$  is a continuous linear functional on  $X$  for fixed  $\underline{v} \in \underline{H}^1(\Omega)$  and  $\xi \in L^\infty(\Omega \times D)$ , the Lax–Milgram theorem yields the existence of a unique solution  $\psi^* \in X$  to (3.50b). Similarly, for  $\delta \in (0, 1)$ , as  $b_\delta(\eta)(\cdot, \cdot)$  is a nonsymmetric continuous coercive bilinear functional on  $\mathcal{Y} \times \mathcal{Y}$  for fixed  $\eta \in L^2_{\geq 0}(\Omega)$ , and  $\ell_{b,\delta}(\eta, \varphi)(\cdot)$  is a continuous linear functional on  $\mathcal{Y}$  for fixed  $\eta \in \Upsilon^n$  with  $\eta(\cdot, t_n) \in L^2_{\geq 0}(\Omega)$  and  $\varphi \in X$ , the Lax–Milgram theorem yields the existence of a unique solution  $\underline{u}^* \in \mathcal{Y}$  to (3.50c). Therefore the overall procedure (3.50a–c) that, for  $\rho_{\kappa,\alpha,L}^{n-1} \in L^\Gamma_{\geq 0}(\Omega)$  fixed, maps  $(\tilde{u}, \tilde{\psi}) \in \mathcal{V} \times L^2_M(\Omega \times D)$  to  $(\rho^*, \underline{u}^*, \psi^*) \in \Upsilon^n \times \mathcal{Y} \times X$ , with  $\rho^*(\cdot, t_{n-1}) = \rho_{\kappa,\alpha,L}^{n-1}$ , is well defined.

**Lemma 3.2.** *Let  $\mathcal{T} : \mathcal{V} \times L^2_M(\Omega \times D) \rightarrow \mathcal{Y} \times X$  denote the nonlinear map that takes the functions  $(\tilde{u}, \tilde{\psi})$  to  $(\underline{u}^*, \psi^*) = \mathcal{T}(\tilde{u}, \tilde{\psi})$  via the procedure (3.50a–c). Then, the mapping  $\mathcal{T}$  has a fixed point. Hence, there exists a solution  $(\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}, \underline{u}_{\kappa,\alpha,L,\delta}^n, \tilde{\psi}_{\kappa,\alpha,L,\delta}^n) \in \Upsilon^n \times \mathcal{Y} \times X$  to (3.49a–c).*

**Proof.** Clearly, a fixed point of  $\mathcal{T}$  yields a solution of (3.49a–c). In order to show that  $\mathcal{T}$  has a fixed point, we apply Schauder’s fixed-point theorem; that is, we need to show that: (i)  $\mathcal{T} : \mathcal{V} \times L^2_M(\Omega \times D) \rightarrow \mathcal{V} \times L^2_M(\Omega \times D)$  is continuous; (ii)  $\mathcal{T}$  is compact; and (iii) there exists a  $C_* \in \mathbb{R}_{>0}$  such that

$$\|\tilde{u}\|_{H^1(\Omega)} + \|\tilde{u}\|_{L^\infty(\Omega)} + \|\nabla_x \cdot \tilde{u}\|_{L^\infty(\Omega)} + \|\tilde{\psi}\|_{L^2_M(\Omega \times D)} \leq C_* \quad (3.58)$$

for every  $(\tilde{u}, \tilde{\psi}) \in \mathcal{V} \times L^2_M(\Omega \times D)$  and  $\varkappa \in (0, 1]$  satisfying  $(\tilde{u}, \tilde{\psi}) = \varkappa \mathcal{T}(\tilde{u}, \tilde{\psi})$ .

(i) Let  $\{\tilde{u}^{(m)}, \tilde{\psi}^{(m)}\}_{m \in \mathbb{N}}$  be such that, as  $m \rightarrow \infty$ ,

$$\begin{aligned} \tilde{u}^{(m)} &\rightarrow \tilde{u} \quad \text{strongly in } \tilde{H}_0^1(\Omega), & \tilde{u}^{(m)} &\rightarrow \tilde{u} \quad \text{strongly in } \tilde{L}^\infty(\Omega), \\ \nabla_x \cdot \tilde{u}^{(m)} &\rightarrow \nabla_x \cdot \tilde{u} \quad \text{strongly in } L^\infty(\Omega), & \tilde{\psi}^{(m)} &\rightarrow \tilde{\psi} \quad \text{strongly in } L^2_M(\Omega \times D). \end{aligned} \quad (3.59)$$

It follows immediately from (3.59), (3.43), (1.7a) and (3.1) that

$$M^{\frac{1}{2}} \beta_\delta^L(\tilde{\psi}^{(m)}) \rightarrow M^{\frac{1}{2}} \beta_\delta^L(\tilde{\psi}) \quad \text{strongly in } L^r(\Omega \times D) \quad \text{as } m \rightarrow \infty \quad (3.60)$$

for all  $r \in [1, \infty)$ . In order to prove that  $\mathcal{T} : \mathcal{V} \times L^2_M(\Omega \times D) \rightarrow \mathcal{V} \times L^2_M(\Omega \times D)$  is continuous, we need to show that  $(\underline{v}^{(m)}, \xi^{(m)}) := \mathcal{T}(\tilde{u}^{(m)}, \tilde{\psi}^{(m)})$  is such that, as  $m \rightarrow \infty$ ,

$$\begin{aligned} \underline{v}^{(m)} &\rightarrow \underline{v} \quad \text{strongly in } \tilde{H}_0^1(\Omega), & \underline{v}^{(m)} &\rightarrow \underline{v} \quad \text{strongly in } \tilde{L}^\infty(\Omega), \\ \nabla_x \cdot \underline{v}^{(m)} &\rightarrow \nabla_x \cdot \underline{v} \quad \text{strongly in } L^\infty(\Omega), & \xi^{(m)} &\rightarrow \xi \quad \text{strongly in } L^2_M(\Omega \times D), \end{aligned} \quad (3.61)$$

where  $(\underline{v}, \xi) := \mathcal{T}(\tilde{u}, \tilde{\psi})$ . We have from the definition of  $\mathcal{T}$ , recall (3.50a–c), that, for all  $m \in \mathbb{N}$ ,  $(\underline{v}^{(m)}, \xi^{(m)}) \in \mathcal{Y} \times X$  is the unique solution to

$$a(\xi^{(m)}, \varphi) = \ell_a(\tilde{u}^{(m)}, \beta_\delta^L(\tilde{\psi}^{(m)}))(\varphi) \quad \forall \varphi \in X, \quad (3.62a)$$

$$b_\delta(\tilde{\rho}^{(m)}(\cdot, t_n))(\underline{v}^{(m)}, w) = \ell_{b,\delta}(\tilde{\rho}^{(m)}, \xi^{(m)})(w) \quad \forall w \in \mathcal{Y}, \quad (3.62b)$$

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where  $\tilde{\rho}^{(m)} \in \Upsilon^n$  is the unique solution to  $\tilde{\rho}^{(m)}(\cdot, t_{n-1}) = \rho_{\kappa, \alpha, L, \delta}^{n-1}(\cdot)$  and

$$\int_{t_{n-1}}^{t_n} \left[ \left\langle \frac{\partial \tilde{\rho}^{(m)}}{\partial t}, \eta \right\rangle_{H^1(\Omega)} + c(\tilde{u}^{(m)})(\tilde{\rho}^{(m)}, \eta) \right] dt = 0 \quad \forall \eta \in L^2(t_{n-1}, t_n; H^1(\Omega)). \quad (3.63)$$

It follows from (3.59) that  $\|\nabla_x \cdot \tilde{u}^{(m)}\|_{L^\infty(\Omega)} \leq C_*$  for all  $m \in \mathbb{N}$ . By choosing, for  $s \in (t_{n-1}, t_n]$ , the function  $\eta(\cdot, t) = \chi_{[t_{n-1}, s]} e^{-C_*(t-t_{n-1})} \tilde{\rho}^{(m)}(\cdot, t)$  in (3.63) yields, on noting (3.51), the first two bounds in

$$\begin{aligned} & \|\tilde{\rho}^{(m)}\|_{C([t_{n-1}, t_n]; L^2(\Omega))}^2 + \alpha \|\tilde{\rho}^{(m)}\|_{L^2(t_{n-1}, t_n; H^1(\Omega))}^2 \\ & + \|\tilde{\rho}^{(m)}\|_{L^\infty(t_{n-1}, t_n; L^\infty(\Omega))}^2 + \left\| \frac{\partial \tilde{\rho}^{(m)}}{\partial t} \right\|_{L^2(t_{n-1}, t_n; H^1(\Omega)')}^2 \leq C, \end{aligned} \quad (3.64)$$

where, here and below,  $C$  is independent of  $m$ . The third bound in (3.64) follows from applying the bound (3.57) to (3.63) with  $\tilde{u}$  in (3.54) replaced by  $\tilde{u}^{(m)}$  and noting (3.59). The fourth bound in (3.64) follows immediately from the first two bounds in (3.64), (3.59) and (3.63). Choosing  $\varphi = \xi^{(m)}$  in (3.62a) yields, on noting (3.35a,b) and (3.59), that

$$\|\xi^{(m)}\|_{H_M^1(\Omega \times D)}^2 \leq C. \quad (3.65)$$

Choosing  $w = v^{(m)}$  in (3.62b) yields, on noting (3.45), (3.46), (3.33a,b),  $\tilde{\rho}^{(m)}(\cdot, t_n) \in L_{\geq 0}^2(\Omega)$ ,  $\rho_{\kappa, \alpha, L}^{n-1} \in L_{\geq 0}^\Gamma(\Omega)$ , (3.3), (1.6b), (3.13), (3.65), (2.3) and (3.64), that

$$\|v^{(m)}\|_{H^2(\Omega)}^2 + \|\nabla_x \cdot v^{(m)}\|_{H^2(\Omega)}^2 \leq C. \quad (3.66)$$

It follows from (3.64), (3.65), (3.66), (3.11), (3.43), (1.7a), (3.1), (3.13) and the compactness of  $\mathcal{V}$  in  $V$  and  $X \equiv H_M^1(\Omega \times D)$  in  $L_M^2(\Omega \times D)$  that there exists a subsequence  $\{(\tilde{\rho}^{(m_k)}, v^{(m_k)}, \xi^{(m_k)})\}_{m_k \in \mathbb{N}}$  and functions  $(\tilde{\rho}, v, \xi) \in \Upsilon^n \times \mathcal{V} \times X$  such that, as  $m_k \rightarrow \infty$ , for any  $r \in [1, \infty)$ ,

$$\begin{aligned} \tilde{\rho}^{(m_k)} & \rightarrow \tilde{\rho} & \text{weakly in } L^2(t_{n-1}, t_n; H^1(\Omega)), \\ & & \text{strongly in } L^r(t_{n-1}, t_n; L^r(\Omega)), \end{aligned} \quad (3.67a)$$

$$\frac{\partial \tilde{\rho}^{(m_k)}}{\partial t} \rightarrow \frac{\partial \tilde{\rho}}{\partial t} \quad \text{weakly in } L^2(t_{n-1}, t_n; H^1(\Omega)'), \quad (3.67b)$$

$$\tilde{\rho}^{(m_k)}(\cdot, t_n) \rightarrow \tilde{\rho}(\cdot, t_n) \quad \text{weakly in } L^2(\Omega), \quad (3.67c)$$

$$\begin{aligned} v^{(m_k)} & \rightarrow v & \text{weakly in } H^2(\Omega), \\ & & \text{strongly in } H_0^1(\Omega) \cap L^\infty(\Omega), \end{aligned} \quad (3.67d)$$

$$\begin{aligned} \nabla_x \cdot v^{(m_k)} & \rightarrow \nabla_x \cdot v & \text{weakly in } H^2(\Omega), \\ & & \text{strongly in } L^\infty(\Omega), \end{aligned} \quad (3.67e)$$



$$\begin{aligned} \xi^{(m_k)} &\rightarrow \xi && \text{weakly in } H_M^1(\Omega \times D), \\ &&& \text{strongly in } L_M^2(\Omega \times D), \end{aligned} \quad (3.67f)$$

$$M^{\frac{1}{2}} \beta_\delta^L(\xi^{(m_k)}) \rightarrow M^{\frac{1}{2}} \beta_\delta^L(\xi) \quad \text{strongly in } L^r(\Omega \times D), \quad (3.67g)$$

$$C_i(\xi^{(m_k)}) \rightarrow C_i(\xi) \quad \text{strongly in } L^2(\Omega), \quad i = 1, \dots, K. \quad (3.67h)$$

We deduce from (3.63), (3.67a,b) and (3.59) that  $\tilde{\rho} \in \Upsilon^n$  is the unique solution to  $\tilde{\rho}(\cdot, t_{n-1}) = \rho_{\kappa, \alpha, L, \delta}^{n-1}(\cdot)$  and

$$\int_{t_{n-1}}^{t_n} \left[ \left\langle \frac{\partial \tilde{\rho}}{\partial t}, \eta \right\rangle_{H^1(\Omega)} + c(\tilde{u})(\tilde{\rho}, \eta) \right] dt = 0 \quad \forall \eta \in L^2(t_{n-1}, t_n; H^1(\Omega)). \quad (3.68)$$

Choosing  $\eta = 1$  in (3.68), on noting (3.31) and (3.48), yields that

$$\int_{\Omega} \tilde{\rho}(\cdot, t_n) dx = \int_{\Omega} \rho_{\kappa, \alpha, L, \delta}^{n-1} dx \leq \int_{\Omega} \rho_{\kappa, \alpha, L}^{n-1} dx. \quad (3.69)$$

It follows from (3.62a), (3.35a,b), (3.67f), (3.59) and (3.60) that  $\xi, \tilde{\psi} \in X$  and  $\tilde{u} \in \mathcal{V}$  satisfy

$$a(\xi, \varphi) = \ell_a(\tilde{u}, \beta_\delta^L(\tilde{\psi}))(\varphi) \quad \forall \varphi \in C^\infty(\overline{\Omega \times D}). \quad (3.70)$$

Then, noting that  $a(\cdot, \cdot)$  is a continuous bilinear functional on  $X \times X$ , that  $\ell_a(\tilde{u}, \beta_\delta^L(\tilde{\psi}))(\cdot)$  is a continuous linear functional on  $X$ , and recalling (3.8), we deduce that  $\xi \in X$  is the unique solution of (3.70) for all  $\varphi \in X$ . It further follows from (3.62b), (3.45), (3.46), (3.33a,b) and (3.67a,c,d,e,f,g,h) that  $\varrho \in \mathcal{V}$  is the unique solution to

$$b_\delta(\tilde{\rho}(\cdot, t_n))(\varrho, w) = \ell_{b, \delta}(\tilde{\rho}, \xi)(w) \quad \forall w \in \mathcal{V}. \quad (3.71)$$

Combining (3.70) with  $\varphi \in X$  with (3.71) and (3.68), we have that  $(\varrho, \xi) = \mathcal{T}(\tilde{u}, \tilde{\psi}) \in \mathcal{V} \times X$ . As  $(\varrho, \xi)$  is unique for fixed  $(\tilde{u}, \tilde{\psi})$ , the whole sequence converges in (3.67a-h), and so (3.61) holds. Therefore the mapping  $\mathcal{T} : \mathcal{V} \times L_M^2(\Omega \times D) \rightarrow \mathcal{V} \times L_M^2(\Omega \times D)$  is continuous.

(ii) Since the embeddings  $\mathcal{V} \hookrightarrow \mathcal{V}$  and  $X \hookrightarrow L_M^2(\Omega \times D)$  are compact, we directly deduce that the mapping  $\mathcal{T} : \mathcal{V} \times L_M^2(\Omega \times D) \rightarrow \mathcal{V} \times L_M^2(\Omega \times D)$  is compact. It therefore remains to show that (iii) holds.

(iii) Let us suppose that  $(\tilde{u}, \tilde{\psi}) = \varkappa \mathcal{T}(\tilde{u}, \tilde{\psi})$ ; then,  $(\tilde{\rho}, \tilde{u}, \tilde{\psi}) \in \Upsilon^n \times \mathcal{V} \times X$  satisfies  $\tilde{\rho}(\cdot, t_{n-1}) = \rho_{\kappa, \alpha, L, \delta}^{n-1}(\cdot)$  and

$$\int_{t_{n-1}}^{t_n} \left[ \left\langle \frac{\partial \tilde{\rho}}{\partial t}, \eta \right\rangle_{H^1(\Omega)} + c(\tilde{u})(\tilde{\rho}, \eta) \right] dt = 0 \quad \forall \eta \in L^2(t_{n-1}, t_n; H^1(\Omega)), \quad (3.72a)$$

$$b_\delta(\tilde{\rho}(\cdot, t_n))(\tilde{u}, w) = \varkappa \ell_{b, \delta}(\tilde{\rho}, \tilde{\psi})(w) \quad \forall w \in \mathcal{V}, \quad (3.72b)$$

$$a(\tilde{\psi}, \varphi) = \varkappa \ell_a(\tilde{u}, \beta_\delta^L(\tilde{\psi}))(\varphi) \quad \forall \varphi \in X. \quad (3.72c)$$

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Choosing  $w = \tilde{u}$  in (3.72b) yields, as  $\varkappa \in (0, 1]$ , that

$$\begin{aligned}
& \frac{\varkappa}{2} \int_{\Omega} \left[ \tilde{\rho}(\cdot, t_n) |\tilde{u}|^2 + \rho_{\kappa, \alpha, L}^{n-1} |\tilde{u} - u_{\kappa, \alpha, L}^{n-1}|^2 - \rho_{\kappa, \alpha, L}^{n-1} |u_{\kappa, \alpha, L}^{n-1}|^2 \right] dx \\
& + \Delta t \mu^S \int_{\Omega} |D(\tilde{u})|^2 dx + \Delta t \left( \mu^B - \frac{\mu^S}{d} \right) \int_{\Omega} |\nabla_x \cdot \tilde{u}|^2 dx \\
& + \Delta t \delta \sum_{|\lambda|=2} \int_{\Omega} \left[ \left| \frac{\partial^{|\lambda|} \tilde{u}}{\partial x_1^{\lambda_1} \dots \partial x_d^{\lambda_d}} \right|^2 + \left| \frac{\partial^{|\lambda|} (\nabla_x \cdot \tilde{u})}{\partial x_1^{\lambda_1} \dots \partial x_d^{\lambda_d}} \right|^2 \right] dx \\
& \leq \varkappa \Delta t \left[ \int_{\Omega} \tilde{\rho}(\cdot, t_n) f^n \cdot \tilde{u} dx - k \sum_{i=1}^K \int_{\Omega} C_i(M \tilde{\psi}) : \nabla_x \tilde{u} dx \right] \\
& + \varkappa \int_{\Omega} \left( \int_{t_{n-1}}^{t_n} p_{\kappa}(\tilde{\rho}) dt + \Delta t k (K+1) \int_D M \tilde{\psi} dq \right) \nabla_x \cdot \tilde{u} dx \\
& - 2 \varkappa \Delta t \mathfrak{z} \int_{\Omega} \left( \int_D M \beta_{\delta}^L(\tilde{\psi}) dq \right) \nabla_x \left( \int_D M \tilde{\psi} dq \right) \cdot \tilde{u} dx. \quad (3.73)
\end{aligned}$$

On recalling (3.5), we choose  $\eta(\cdot, t) = \chi_{[t_{n-1}, s]} P'_{\kappa}(\tilde{\rho}(\cdot, t) + \varsigma)$  in (3.72a), for any  $s \in (t_{n-1}, t_n]$  and any fixed  $\varsigma \in \mathbb{R}_{>0}$ , to obtain, on noting (3.31), that

$$\begin{aligned}
& \int_{\Omega} P_{\kappa}(\tilde{\rho}(\cdot, s)) dx + \alpha \kappa \int_{t_{n-1}}^s \int_{\Omega} (4\tilde{\rho}^2 + \Gamma \tilde{\rho}^{\Gamma-2}) |\nabla_x \tilde{\rho}|^2 dx dt \\
& \leq \int_{\Omega} P_{\kappa}(\tilde{\rho}(\cdot, s) + \varsigma) dx + \alpha \int_{t_{n-1}}^s \int_{\Omega} P''_{\kappa}(\tilde{\rho} + \varsigma) |\nabla_x \tilde{\rho}|^2 dx dt \\
& = \int_{\Omega} P_{\kappa}(\rho_{\kappa, \alpha, L, \delta}^{n-1} + \varsigma) dx + \int_{t_{n-1}}^s \int_{\Omega} \tilde{\rho} \tilde{u} \cdot \nabla_x P'_{\kappa}(\tilde{\rho} + \varsigma) dx dt \\
& = \int_{\Omega} P_{\kappa}(\rho_{\kappa, \alpha, L, \delta}^{n-1} + \varsigma) dx + \int_{\Omega} \left( \int_{t_{n-1}}^s [P_{\kappa}(\tilde{\rho} + \varsigma) - \tilde{\rho} P'_{\kappa}(\tilde{\rho} + \varsigma)] dt \right) \nabla_x \cdot \tilde{u} dx. \quad (3.74)
\end{aligned}$$

As  $\rho_{\kappa, \alpha, L, \delta}^{n-1} \in L^{\infty}_{\geq 0}(\Omega)$ ,  $\tilde{\rho} \in L^{\infty}(t_{n-1}, t_n; L^{\infty}_{\geq 0}(\Omega))$  and  $\tilde{u} \in \mathcal{Y}$ , one can pass to the limit  $\varsigma \rightarrow 0_+$  in (3.74) using Lebesgue's dominated convergence theorem to obtain, for any  $s \in (t_{n-1}, t_n]$ , that

$$\begin{aligned}
& \int_{\Omega} P_{\kappa}(\tilde{\rho}(\cdot, s)) dx + \alpha \kappa \int_{t_{n-1}}^s \left[ \|\nabla_x(\tilde{\rho}^2)\|_{L^2(\Omega)}^2 + \frac{4}{\Gamma} \|\nabla_x(\tilde{\rho}^{\frac{\Gamma}{2}})\|_{L^2(\Omega)}^2 \right] dt \\
& \leq \int_{\Omega} P_{\kappa}(\rho_{\kappa, \alpha, L, \delta}^{n-1}) dx + \int_{\Omega} \left( \int_{t_{n-1}}^s [P_{\kappa}(\tilde{\rho}) - \tilde{\rho} P'_{\kappa}(\tilde{\rho})] dt \right) \nabla_x \cdot \tilde{u} dx \\
& \leq \int_{\Omega} P_{\kappa}(\rho_{\kappa, \alpha, L}^{n-1}) dx - \int_{\Omega} \left( \int_{t_{n-1}}^s p_{\kappa}(\tilde{\rho}) dt \right) \nabla_x \cdot \tilde{u} dx, \quad (3.75)
\end{aligned}$$

where we have noted (3.48) and (3.5) for the final inequality. We remark that we needed to choose  $P'_{\kappa}(\tilde{\rho}(\cdot, t) + \varsigma)$ , as opposed to  $P'_{\kappa}(\tilde{\rho}(\cdot, t))$ , in the testing procedure

since  $P_\kappa''(\tilde{\rho})$ , that would appear in (3.74), may not be well-defined for  $\gamma \in (\frac{3}{2}, 2)$ , as we only know that  $\tilde{\rho}$  is nonnegative as opposed to being strictly positive.

For a.e.  $\underline{x} \in \Omega$ , let

$$\tilde{\varrho}(\underline{x}) := \int_D M(\underline{q}) \tilde{\psi}(\underline{x}, \underline{q}) \, d\underline{q} \quad \text{and} \quad \varrho_{\kappa, \alpha, L}^{n-1}(\underline{x}) := \int_D M(\underline{q}) \widehat{\psi}_{\kappa, \alpha, L}^{n-1}(\underline{x}, \underline{q}) \, d\underline{q}. \quad (3.76)$$

Choosing  $\varphi(\underline{x}, \underline{q}) = \tilde{\varrho}(\underline{x}) \otimes 1(\underline{q})$  in (3.72c) yields that

$$\begin{aligned} & \frac{\varkappa}{2} \left[ \|\tilde{\varrho}\|_{L^2(\Omega)}^2 + \|\tilde{\varrho} - \varrho_{\kappa, \alpha, L}^{n-1}\|_{L^2(\Omega)}^2 \right] + \Delta t \varepsilon \|\nabla_x \tilde{\varrho}\|_{L^2(\Omega)}^2 \\ & = \frac{\varkappa}{2} \|\varrho_{\kappa, \alpha, L}^{n-1}\|_{L^2(\Omega)}^2 + \varkappa \Delta t \int_\Omega \left( \int_D M \beta_\delta^L(\tilde{\psi}) \, d\underline{q} \right) \tilde{u} \cdot \nabla_x \tilde{\varrho} \, dx. \end{aligned} \quad (3.77)$$

Combining (3.73), (3.75) for  $s = t_n$  and (3.77) yields, on noting (3.3), (3.4) and (3.76), that, for all  $\varkappa \in (0, 1]$ ,

$$\begin{aligned} & \frac{\varkappa}{2} \int_\Omega \left[ \tilde{\rho}(\cdot, t_n) |\tilde{u}|^2 + \rho_{\kappa, \alpha, L}^{n-1} |\tilde{u} - u_{\kappa, \alpha, L}^{n-1}|^2 \right] \, dx + \varkappa \int_\Omega P_\kappa(\tilde{\rho}(\cdot, t_n)) \, dx \\ & + \varkappa \alpha \kappa \int_{t_{n-1}}^{t_n} \left[ \|\nabla_x(\tilde{\rho}^2)\|_{L^2(\Omega)}^2 + \frac{4}{\Gamma} \|\nabla_x(\tilde{\rho}^{\frac{\Gamma}{2}})\|_{L^2(\Omega)}^2 \right] \, dt + \Delta t \mu^S c_0 \|\tilde{u}\|_{H^1(\Omega)}^2 \\ & + \Delta t \delta \sum_{|\lambda|=2} \int_\Omega \left[ \left| \frac{\partial^{|\lambda|} \tilde{u}}{\partial x_1^{\lambda_1} \dots \partial x_d^{\lambda_d}} \right|^2 + \left| \frac{\partial^{|\lambda|} (\nabla_x \cdot \tilde{u})}{\partial x_1^{\lambda_1} \dots \partial x_d^{\lambda_d}} \right|^2 \right] \, dx \\ & + \varkappa \mathfrak{z} \left[ \|\tilde{\varrho}\|_{L^2(\Omega)}^2 + \|\tilde{\varrho} - \varrho_{\kappa, \alpha, L}^{n-1}\|_{L^2(\Omega)}^2 \right] + 2 \Delta t \mathfrak{z} \varepsilon \|\nabla_x \tilde{\varrho}\|_{L^2(\Omega)}^2 \\ & \leq \frac{\varkappa}{2} \int_\Omega \rho_{\kappa, \alpha, L}^{n-1} |u_{\kappa, \alpha, L}^{n-1}|^2 \, dx + \varkappa \int_\Omega P_\kappa(\rho_{\kappa, \alpha, L}^{n-1}) \, dx \\ & + \varkappa \Delta t \int_\Omega \tilde{\rho}(\cdot, t_n) f^n \cdot \tilde{u} \, dx + \varkappa \mathfrak{z} \|\varrho_{\kappa, \alpha, L}^{n-1}\|_{L^2(\Omega)}^2 \\ & + \varkappa k \Delta t \left[ (K+1) \int_\Omega \tilde{\varrho} \nabla_x \cdot \tilde{u} \, dx - \sum_{i=1}^K \int_\Omega C_i(M \tilde{\psi}) : \nabla_x \tilde{u} \, dx \right]. \end{aligned} \quad (3.78)$$

Choosing  $\varphi = [\mathcal{F}_\delta^L]'(\tilde{\psi})$  in (3.72c) and noting (3.43) implies that

$$\begin{aligned} & \int_{\Omega \times D} M \left( \mathcal{F}_\delta^L(\tilde{\psi}) - \mathcal{F}_\delta^L(\varkappa \widehat{\psi}_{\kappa, \alpha, L}^{n-1}) + \frac{1}{2L} |\tilde{\psi} - \varkappa \widehat{\psi}_{\kappa, \alpha, L}^{n-1}|^2 \right) \, d\underline{q} \, d\underline{x} \\ & + \frac{\Delta t}{4\lambda} \sum_{i=1}^K \sum_{j=1}^K A_{ij} \int_{\Omega \times D} M \nabla_{q_j} \tilde{\psi} \cdot \nabla_{q_i} ([\mathcal{F}_\delta^L]'(\tilde{\psi})) \, d\underline{q} \, d\underline{x} \\ & + \Delta t \varepsilon \int_{\Omega \times D} M \nabla_x \tilde{\psi} \cdot \nabla_x ([\mathcal{F}_\delta^L]'(\tilde{\psi})) \, d\underline{q} \, d\underline{x} \end{aligned}$$

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$$\begin{aligned}
 &\leq \varkappa \Delta t \int_{\Omega \times D} M \left[ \left( \sum_{i=1}^K \sigma(\tilde{u}) q_i \right) \cdot \nabla_{q_i} \tilde{\psi} + \tilde{u} \cdot \nabla_x \tilde{\psi} \right] dq dx \\
 &= \varkappa \Delta t \int_{\Omega} \left[ \sum_{i=1}^K C_i(M \tilde{\psi}) : \sigma(\tilde{u}) - (K+1) (\nabla_x \cdot \tilde{u}) \int_D M \tilde{\psi} dq \right] dx, \quad (3.79)
 \end{aligned}$$

where in the transition to the final line we applied (3.14) with  $\underline{B} = \underline{\sigma}(\tilde{u})$  (on account of it being independent of the variable  $q$ ), and recalled (1.12). Combining (3.78) and (3.79), and noting (2.5) and (3.43), yields, for all  $\varkappa \in (0, 1]$  and  $\varsigma \in \mathbb{R}_{>0}$ , that

$$\begin{aligned}
 &\frac{\varkappa}{2} \int_{\Omega} \left[ \tilde{\rho}(\cdot, t_n) |\tilde{u}|^2 + \rho_{\kappa, \alpha, L}^{n-1} |\tilde{u} - u_{\kappa, \alpha, L}^{n-1}|^2 \right] dx + \varkappa \int_{\Omega} P_{\kappa}(\tilde{\rho}(\cdot, t_n)) dx \\
 &\quad + \varkappa \alpha \kappa \int_{t_{n-1}}^{t_n} \left[ \|\nabla_x(\tilde{\rho}^2)\|_{L^2(\Omega)}^2 + \frac{4}{\Gamma} \|\nabla_x(\tilde{\rho}^{\frac{\Gamma}{2}})\|_{L^2(\Omega)}^2 \right] dt \\
 &\quad + k \int_{\Omega \times D} M \left( \mathcal{F}_{\delta}^L(\tilde{\psi}) + \frac{1}{2L} |\tilde{\psi} - \varkappa \hat{\psi}_{\kappa, \alpha, L}^{n-1}|^2 \right) dq dx + \Delta t \mu^S c_0 \|\tilde{u}\|_{H^1(\Omega)}^2 \\
 &\quad + \Delta t \delta \sum_{|\lambda|=2} \int_{\Omega} \left[ \left| \frac{\partial^{|\lambda|} \tilde{u}}{\partial x_1^{\lambda_1} \cdots \partial x_d^{\lambda_d}} \right|^2 + \left| \frac{\partial^{|\lambda|} (\nabla_x \cdot \tilde{u})}{\partial x_1^{\lambda_1} \cdots \partial x_d^{\lambda_d}} \right|^2 \right] dx \\
 &\quad + \frac{\Delta t k a_0}{4 \lambda L} \sum_{i=1}^K \int_{\Omega \times D} M |\nabla_{q_i} \tilde{\psi}|^2 dq dx + \frac{\Delta t k \varepsilon}{L} \int_{\Omega \times D} M |\nabla_x \tilde{\psi}|^2 dq dx \\
 &\quad + \varkappa \mathfrak{J} \left[ \|\tilde{\varrho}\|_{L^2(\Omega)}^2 + \|\tilde{\varrho} - \varrho_{\kappa, \alpha, L}^{n-1}\|_{L^2(\Omega)}^2 \right] + 2 \Delta t \mathfrak{J} \varepsilon \|\nabla_x \tilde{\varrho}\|_{L^2(\Omega)}^2 \\
 &\leq \frac{\varkappa}{2} \int_{\Omega} \rho_{\kappa, \alpha, L}^{n-1} |u_{\kappa, \alpha, L}^{n-1}|^2 dx + \varkappa \int_{\Omega} P_{\kappa}(\rho_{\kappa, \alpha, L}^{n-1}) dx + k \int_{\Omega \times D} M \mathcal{F}_{\delta}^L(\varkappa \hat{\psi}_{\kappa, \alpha, L}^{n-1}) dq dx \\
 &\quad + \varkappa \mathfrak{J} \|\varrho_{\kappa, \alpha, L}^{n-1}\|_{L^2(\Omega)}^2 + \varkappa \Delta t \int_{\Omega} \tilde{\rho}(\cdot, t_n) f^n \cdot \tilde{u} dx \\
 &\leq \frac{\varkappa}{2} \int_{\Omega} \rho_{\kappa, \alpha, L}^{n-1} |u_{\kappa, \alpha, L}^{n-1}|^2 dx + \varkappa \int_{\Omega} P_{\kappa}(\rho_{\kappa, \alpha, L}^{n-1}) dx + k \int_{\Omega \times D} M \mathcal{F}_{\delta}^L(\varkappa \hat{\psi}_{\kappa, \alpha, L}^{n-1}) dq dx \\
 &\quad + \varkappa \mathfrak{J} \|\varrho_{\kappa, \alpha, L}^{n-1}\|_{L^2(\Omega)}^2 + \frac{\varkappa \Delta t}{2} \left[ \varsigma \int_{\Omega} \tilde{\rho}(\cdot, t_n) |\tilde{u}|^2 dx + \frac{1}{\varsigma} \|f^n\|_{L^\infty(\Omega)}^2 \int_{\Omega} \rho_{\kappa, \alpha, L}^{n-1} dx \right], \quad (3.80)
 \end{aligned}$$

where, in deriving the final inequality, we have noted (3.69). It is easy to see that  $\mathcal{F}_{\delta}^L(s)$  is nonnegative for all  $s \in \mathbb{R}$ , with  $\mathcal{F}_{\delta}^L(1) = 0$ . Furthermore, for any  $\varkappa \in (0, 1]$ ,  $\mathcal{F}_{\delta}^L(\varkappa s) \leq \mathcal{F}_{\delta}^L(s)$  if  $s < 0$  or  $1 \leq \varkappa s$ , and also  $\mathcal{F}_{\delta}^L(\varkappa s) \leq \mathcal{F}_{\delta}^L(0) \leq 1$  if  $0 \leq \varkappa s \leq 1$ . Thus we deduce that

$$\mathcal{F}_{\delta}^L(\varkappa s) \leq \mathcal{F}_{\delta}^L(s) + 1 \quad \forall s \in \mathbb{R}, \quad \forall \varkappa \in (0, 1]. \quad (3.81)$$

Hence, the bounds (3.80) and (3.81), on noting (3.42b) and, from (3.43) and (1.19), that  $\beta_{\delta}^L(\cdot) \leq L$ , give rise, for  $\varsigma$  sufficiently small, to the desired bound (3.58) with  $C_*$  dependent only on  $\delta, L, \Delta t, M, k, \mu^S, c_0, a_0, f, \rho_{\kappa, \alpha, L}^{n-1}, u_{\kappa, \alpha, L}^{n-1}$  and  $\hat{\psi}_{\kappa, \alpha, L}^{n-1}$ .

Therefore (iii) holds, and so  $\mathcal{T}$  has a fixed point, proving existence of a solution to (3.49a-c).  $\square$

Similarly to (3.77), choosing  $\varphi(\underline{x}, \underline{q}) = \varrho_{\kappa, \alpha, L, \delta}^n(\underline{x}) \otimes 1(\underline{q})$  in (3.49c), where  $\varrho_{\kappa, \alpha, L, \delta}^n(\underline{x}) := \int_D M(\underline{q}) \widehat{\psi}_{\kappa, \alpha, L, \delta}^n(\underline{x}, \underline{q}) \, d\underline{q} \, d\underline{x}$ , yields that

$$\begin{aligned} & \frac{1}{2} \left[ \|\varrho_{\kappa, \alpha, L, \delta}^n\|_{L^2(\Omega)}^2 + \|\varrho_{\kappa, \alpha, L, \delta}^n - \varrho_{\kappa, \alpha, L}^{n-1}\|_{L^2(\Omega)}^2 \right] + \Delta t \varepsilon \|\nabla_x \varrho_{\kappa, \alpha, L, \delta}^n\|_{L^2(\Omega)}^2 \\ &= \frac{1}{2} \|\varrho_{\kappa, \alpha, L}^{n-1}\|_{L^2(\Omega)}^2 \\ & \quad + \Delta t \int_{\Omega} \left( \int_D M \beta_{\delta}^L(\varrho_{\kappa, \alpha, L, \delta}^n) \, d\underline{q} \right) \varrho_{\kappa, \alpha, L, \delta}^n \cdot \nabla_x \varrho_{\kappa, \alpha, L, \delta}^n \, d\underline{x}. \end{aligned} \quad (3.82)$$

Choosing  $w = \varrho_{\kappa, \alpha, L, \delta}^n$  in (3.49b) and  $\eta = P'_{\kappa}(\rho_{\kappa, \alpha, L, \delta}^{[\Delta t], n})$  in (3.49a), and combining with (3.82), yields, similarly to (3.78), that

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \left[ \rho_{\kappa, \alpha, L, \delta}^{[\Delta t], n}(\cdot, t_n) |\varrho_{\kappa, \alpha, L, \delta}^n|^2 + \rho_{\kappa, \alpha, L}^{n-1} |u_{\kappa, \alpha, L, \delta}^n - u_{\kappa, \alpha, L}^{n-1}|^2 \right] d\underline{x} \\ & \quad + \int_{\Omega} P_{\kappa}(\rho_{\kappa, \alpha, L, \delta}^{[\Delta t], n}(\cdot, t_n)) d\underline{x} \\ & \quad + \alpha \kappa \int_{t_{n-1}}^{t_n} \left[ \|\nabla_x [(\rho_{\kappa, \alpha, L, \delta}^{[\Delta t], n})^2]\|_{L^2(\Omega)}^2 + \frac{4}{\Gamma} \|\nabla_x [(\rho_{\kappa, \alpha, L, \delta}^{[\Delta t], n})^{\frac{\Gamma}{2}}]\|_{L^2(\Omega)}^2 \right] dt \\ & \quad + \Delta t \mu^S c_0 \|\varrho_{\kappa, \alpha, L, \delta}^n\|_{H^1(\Omega)}^2 \\ & \quad + \Delta t \delta \sum_{|\lambda|=2} \int_{\Omega} \left[ \left| \frac{\partial^{|\lambda|} \varrho_{\kappa, \alpha, L, \delta}^n}{\partial x_1^{\lambda_1} \dots \partial x_d^{\lambda_d}} \right|^2 + \left| \frac{\partial^{|\lambda|} (\nabla_x \cdot \varrho_{\kappa, \alpha, L, \delta}^n)}{\partial x_1^{\lambda_1} \dots \partial x_d^{\lambda_d}} \right|^2 \right] d\underline{x} \\ & \quad + \mathfrak{z} \left[ \|\varrho_{\kappa, \alpha, L, \delta}^n\|_{L^2(\Omega)}^2 + \|\varrho_{\kappa, \alpha, L, \delta}^n - \varrho_{\kappa, \alpha, L}^{n-1}\|_{L^2(\Omega)}^2 \right] + 2 \Delta t \mathfrak{z} \varepsilon \|\nabla_x \varrho_{\kappa, \alpha, L, \delta}^n\|_{L^2(\Omega)}^2 \\ & \leq \frac{1}{2} \int_{\Omega} \rho_{\kappa, \alpha, L}^{n-1} |u_{\kappa, \alpha, L}^{n-1}|^2 d\underline{x} \\ & \quad + \int_{\Omega} P_{\kappa}(\rho_{\kappa, \alpha, L}^{n-1}) d\underline{x} \\ & \quad + \Delta t \int_{\Omega} \rho_{\kappa, \alpha, L, \delta}^{[\Delta t], n}(\cdot, t_n) f^n \cdot \varrho_{\kappa, \alpha, L, \delta}^n d\underline{x} + \mathfrak{z} \|\varrho_{\kappa, \alpha, L}^{n-1}\|_{L^2(\Omega)}^2 \\ & \quad + k \Delta t (K+1) \int_{\Omega} \varrho_{\kappa, \alpha, L, \delta}^n \nabla_x \cdot \varrho_{\kappa, \alpha, L, \delta}^n d\underline{x} \\ & \quad - k \Delta t \sum_{i=1}^K \int_{\Omega} C_i(M \widehat{\psi}_{\kappa, \alpha, L, \delta}^n) : \nabla_x \varrho_{\kappa, \alpha, L, \delta}^n d\underline{x}. \end{aligned} \quad (3.83)$$

Choosing  $\varphi = [\mathcal{F}_{\delta}^L]'(\widehat{\psi}_{\kappa, \alpha, L, \delta}^n)$  in (3.49c), combining with (3.83) and noting (3.42a), yields, similarly to (3.80), that, for  $\varsigma \in \mathbb{R}_{>0}$  sufficiently small, the solution

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$(\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}, u_{\kappa,\alpha,L,\delta}^n, \widehat{\psi}_{\kappa,\alpha,L,\delta}^n) \in \Upsilon^n \times \mathcal{Y} \times X$  of (3.49a-c) satisfies

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega} \left[ \rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}(\cdot, t_n) |u_{\kappa,\alpha,L,\delta}^n|^2 + \rho_{\kappa,\alpha,L}^{n-1} |u_{\kappa,\alpha,L,\delta}^n - u_{\kappa,\alpha,L}^{n-1}|^2 \right] dx \\
& + \int_{\Omega} P_{\kappa}(\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}(\cdot, t_n)) dx \\
& + \alpha \kappa \int_{t_{n-1}}^{t_n} \left[ \|\widetilde{\nabla}_x [(\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n})^2]\|_{L^2(\Omega)}^2 + \frac{4}{\Gamma} \|\widetilde{\nabla}_x [(\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n})^{\frac{\Gamma}{2}}]\|_{L^2(\Omega)}^2 \right] dt \\
& + k \int_{\Omega \times D} M \left( \mathcal{F}_{\delta}^L(\widehat{\psi}_{\kappa,\alpha,L,\delta}^n) + \frac{1}{2L} |\widehat{\psi}_{\kappa,\alpha,L,\delta}^n - \widehat{\psi}_{\kappa,\alpha,L}^{n-1}|^2 \right) dq dx \\
& + \Delta t \mu^S c_0 \|u_{\kappa,\alpha,L,\delta}^n\|_{H^1(\Omega)}^2 \\
& + \Delta t \delta \sum_{|\lambda|=2} \int_{\Omega} \left[ \left| \frac{\partial^{|\lambda|} u_{\kappa,\alpha,L,\delta}^n}{\partial x_1^{\lambda_1} \dots \partial x_d^{\lambda_d}} \right|^2 + \left| \frac{\partial^{|\lambda|} (\nabla_x \cdot u_{\kappa,\alpha,L,\delta}^n)}{\partial x_1^{\lambda_1} \dots \partial x_d^{\lambda_d}} \right|^2 \right] dx \\
& + \frac{\Delta t k a_0}{4 \lambda L} \sum_{i=1}^K \int_{\Omega \times D} M |\nabla_{q_i} \widehat{\psi}_{\kappa,\alpha,L,\delta}^n|^2 dq dx \\
& + \frac{\Delta t k \varepsilon}{L} \int_{\Omega \times D} M |\nabla_x \widehat{\psi}_{\kappa,\alpha,L,\delta}^n|^2 dq dx \\
& + \mathfrak{z} \left[ \|\varrho_{\kappa,\alpha,L,\delta}^n\|_{L^2(\Omega)}^2 + \|\varrho_{\kappa,\alpha,L,\delta}^n - \varrho_{\kappa,\alpha,L}^{n-1}\|_{L^2(\Omega)}^2 \right] + 2 \Delta t \mathfrak{z} \varepsilon \|\nabla_x \varrho_{\kappa,\alpha,L,\delta}^n\|_{L^2(\Omega)}^2 \\
& \leq \frac{1}{2} \int_{\Omega} \rho_{\kappa,\alpha,L}^{n-1} |u_{\kappa,\alpha,L}^{n-1}|^2 dx + \int_{\Omega} P_{\kappa}(\rho_{\kappa,\alpha,L,\delta}^{n-1}) dx + k \int_{\Omega \times D} M \mathcal{F}_{\delta}^L(\widehat{\psi}_{\kappa,\alpha,L}^{n-1}) dq dx \\
& + \mathfrak{z} \|\varrho_{\kappa,\alpha,L}^{n-1}\|_{L^2(\Omega)}^2 + \Delta t \int_{\Omega} \rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}(\cdot, t_n) f^n \cdot u_{\kappa,\alpha,L,\delta}^n dx \\
& \leq \frac{1}{2} \int_{\Omega} \rho_{\kappa,\alpha,L}^{n-1} |u_{\kappa,\alpha,L}^{n-1}|^2 dx + \int_{\Omega} P_{\kappa}(\rho_{\kappa,\alpha,L}^{n-1}) dx + k \int_{\Omega \times D} M \mathcal{F}^L(\widehat{\psi}_{\kappa,\alpha,L}^{n-1}) dq dx \\
& + \mathfrak{z} \|\varrho_{\kappa,\alpha,L}^{n-1}\|_{L^2(\Omega)}^2 \\
& + \frac{\Delta t}{2} \left[ \varsigma \int_{\Omega} \rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}(\cdot, t_n) |u_{\kappa,\alpha,L,\delta}^n|^2 dx + \frac{1}{\varsigma} \|f^n\|_{L^{\infty}(\Omega)}^2 \int_{\Omega} \rho_{\kappa,\alpha,L}^{n-1} dx \right] \leq C,
\end{aligned} \tag{3.84}$$

where  $C$  is independent of  $\delta$  and  $\Delta t$ .

On choosing, for any  $s \in (t_{n-1}, t_n]$ ,  $\eta(\cdot, t) = \chi_{[t_{n-1}, s]} [\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}(\cdot, t)]^{\vartheta-1}$ , for  $\vartheta = 2$  and  $\frac{\Gamma}{2}$ , in (3.49a), we obtain, on noting (3.31), (3.48) and (3.84), that

$$\begin{aligned}
& \frac{1}{\vartheta} \|\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}(\cdot, s)\|_{L^{\vartheta}(\Omega)}^{\vartheta} + \frac{4\alpha(\vartheta-1)}{\vartheta^2} \int_{t_{n-1}}^s \|\widetilde{\nabla}_x [(\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n})^{\frac{\vartheta}{2}}]\|_{L^2(\Omega)}^2 dt \\
& = \frac{1}{\vartheta} \left[ \|\rho_{\kappa,\alpha,L,\delta}^{n-1}\|_{L^{\vartheta}(\Omega)}^{\vartheta} + (\vartheta-1) \int_{t_{n-1}}^s \int_{\Omega} u_{\kappa,\alpha,L,\delta}^n \cdot \widetilde{\nabla}_x [(\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n})^{\vartheta}] dx dt \right]
\end{aligned}$$

$$\begin{aligned} &\leq \frac{1}{\vartheta} \left[ \|\rho_{\kappa,\alpha,L}^{n-1}\|_{L^\vartheta(\Omega)}^\vartheta + \Delta t \|u_{\kappa,\alpha,L,\delta}^n\|_{L^2(\Omega)}^2 \right. \\ &\quad \left. + \frac{(\vartheta-1)^2}{4} \int_{t_{n-1}}^{t_n} \|\nabla_x [(\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n})^\vartheta]\|_{L^2(\Omega)}^2 dt \right] \leq C, \end{aligned} \quad (3.85)$$

where  $C$  is independent of  $\delta$  and  $\Delta t$ . On denoting by  $\bar{f}$   $\eta$ , the mean value of the function  $\eta$  over  $\Omega$ , it follows from a Poincaré inequality, (3.84) and (3.85) for  $\vartheta = \frac{\Gamma}{2}$  that

$$\begin{aligned} \|\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}\|_{L^\Gamma(t_{n-1},t_n;L^\Gamma(\Omega))}^\Gamma &= \|(\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n})^{\frac{\Gamma}{2}}\|_{L^2(t_{n-1},t_n;L^2(\Omega))}^2 \\ &\leq 2\|(I-\bar{f})(\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n})^{\frac{\Gamma}{2}}\|_{L^2(t_{n-1},t_n;L^2(\Omega))}^2 + 2\|\bar{f}(\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n})^{\frac{\Gamma}{2}}\|_{L^2(t_{n-1},t_n;L^2(\Omega))}^2 \\ &\leq C\|\nabla_x [(\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n})^{\frac{\Gamma}{2}}]\|_{L^2(t_{n-1},t_n;L^2(\Omega))}^2 + C\Delta t\|\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}\|_{L^\infty(t_{n-1},t_n;L^{\frac{\Gamma}{2}}(\Omega))}^\Gamma \leq C. \end{aligned} \quad (3.86)$$

Next, we obtain from (3.31), (3.2), (3.84) and (3.85) for  $\vartheta = \frac{\Gamma}{2}$ , on recalling that  $\Gamma \geq 8$ , that

$$\begin{aligned} &\left| \int_{t_{n-1}}^{t_n} c(u_{\kappa,\alpha,L,\delta}^n)(\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}, \eta) dt \right| \leq \alpha \|\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}\|_{L^2(t_{n-1},t_n;H^1(\Omega))} \|\eta\|_{L^2(t_{n-1},t_n;H^1(\Omega))} \\ &\quad + \left| \int_{t_{n-1}}^{t_n} \|\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}\|_{L^3(\Omega)} \|u_{\kappa,\alpha,L,\delta}^n\|_{L^6(\Omega)} \|\nabla_x \eta\|_{L^2(\Omega)} dt \right| \\ &\leq C \|\eta\|_{L^2(t_{n-1},t_n;H^1(\Omega))} + C \left| \int_{t_{n-1}}^{t_n} \|u_{\kappa,\alpha,L,\delta}^n\|_{H^1(\Omega)} \|\eta\|_{H^1(\Omega)} dt \right| \\ &\leq C \left[ 1 + \left( \Delta t \|u_{\kappa,\alpha,L,\delta}^n\|_{H^1(\Omega)}^2 \right)^{\frac{1}{2}} \right] \|\eta\|_{L^2(t_{n-1},t_n;H^1(\Omega))} \\ &\leq C \|\eta\|_{L^2(t_{n-1},t_n;H^1(\Omega))} \quad \forall \eta \in L^2(t_{n-1},t_n;H^1(\Omega)). \end{aligned} \quad (3.87)$$

Hence, we deduce from (3.85) for  $\vartheta = 2$  and  $\frac{\Gamma}{2}$ , (3.87), on noting (3.49a), (3.84), (3.5) and (3.86) that  $\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n} \in \Upsilon^n$  is such that

$$\begin{aligned} &\|\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}\|_{L^\infty(t_{n-1},t_n;L^{\frac{\Gamma}{2}}(\Omega))} + \|\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}\|_{L^2(t_{n-1},t_n;H^1(\Omega))}^2 + \|\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}\|_{H^1(t_{n-1},t_n;H^1(\Omega)')}^2 \\ &\quad + \|\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}(\cdot, t_n)\|_{L^\Gamma(\Omega)}^\Gamma + \|(\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n})^{\frac{\Gamma}{2}}\|_{L^2(t_{n-1},t_n;H^1(\Omega))}^2 \leq C, \end{aligned} \quad (3.88)$$

where  $C$  is independent of  $\delta$  and  $\Delta t$ . Furthermore, we deduce from (3.2) and the last bound in (3.88) that

$$\begin{aligned} \|\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}\|_{L^\Gamma(t_{n-1},t_n;L^{3\Gamma}(\Omega))}^\Gamma &= \|(\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n})^{\frac{\Gamma}{2}}\|_{L^2(t_{n-1},t_n;L^6(\Omega))}^2 \\ &\leq C \|(\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n})^{\frac{\Gamma}{2}}\|_{L^2(t_{n-1},t_n;H^1(\Omega))}^2 \leq C. \end{aligned} \quad (3.89)$$

Finally, it follows from (3.1) with  $v = \frac{4\Gamma}{3}$ ,  $r = \frac{\Gamma}{2}$  and  $s = 3\Gamma$  yielding  $\vartheta = \frac{3}{4}$ , the

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first bound in (3.88) and (3.89) that

$$\begin{aligned} \|\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}\|_{L^{\frac{4\Gamma}{3}}(t_{n-1},t_n;L^{\frac{4\Gamma}{3}}(\Omega))} &\leq \|\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}\|_{L^\infty(t_{n-1},t_n;L^{\frac{\Gamma}{2}}(\Omega))}^{\frac{\Gamma}{3}} \|\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}\|_{L^\Gamma(t_{n-1},t_n;L^{3\Gamma}(\Omega))} \\ &\leq C, \end{aligned} \quad (3.90)$$

where  $C$  is independent of  $\delta$  and  $\Delta t$ .

As the bounds (3.84), (3.89) and (3.90) are independent of  $\delta$ , we are now ready to pass to the limit  $\delta \rightarrow 0_+$  in (3.49a–c), to deduce the existence of a solution  $\{(\rho_{\kappa,\alpha,L}^{[\Delta t],n}, u_{\kappa,\alpha,L}^n, \widehat{\psi}_{\kappa,\alpha,L}^n)\}_{n=1}^N$  to  $(P_{\kappa,\alpha,L}^{\Delta t})$ .

**Lemma 3.3.** *There exists a subsequence (not indicated) of the sequence of functions  $\{(\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}, u_{\kappa,\alpha,L,\delta}^n, \widehat{\psi}_{\kappa,\alpha,L,\delta}^n)\}_{\delta>0}$ , and functions  $\rho_{\kappa,\alpha,L}^{[\Delta t],n} \in Y^n$  with  $\rho_{\kappa,\alpha,L}^n(\cdot) = \rho_{\kappa,\alpha,L}^{[\Delta t],n}(\cdot, t_n) \in L_{\geq 0}^\Gamma(\Omega)$ ,  $u_{\kappa,\alpha,L}^n \in \underline{H}_0^1(\Omega)$  and  $\widehat{\psi}_{\kappa,\alpha,L}^n \in X \cap Z_2$ ,  $n = 1, \dots, N$ , with*

$$\varrho_{\kappa,\alpha,L}^n(\cdot) := \int_D M(q) \widehat{\psi}_{\kappa,\alpha,L}^n(\cdot, q) \, dq \in H^1(\Omega), \quad n = 1, \dots, N, \quad (3.91)$$

such that, as  $\delta \rightarrow 0_+$ ,

$$\begin{aligned} \rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n} &\rightarrow \rho_{\kappa,\alpha,L}^{[\Delta t],n} && \text{weakly in } L^2(t_{n-1}, t_n; H^1(\Omega)), \\ & && \text{weakly in } H^1(t_{n-1}, t_n; H^1(\Omega)'), \end{aligned} \quad (3.92a)$$

$$\begin{aligned} \rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n} &\rightarrow \rho_{\kappa,\alpha,L}^{[\Delta t],n} && \text{strongly in } L^2(t_{n-1}, t_n; L^r(\Omega)), \\ & && \text{strongly in } L^v(\Omega \times (t_{n-1}, t_n)), \end{aligned} \quad (3.92b)$$

$$\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n}(\cdot, t_n) \rightarrow \rho_{\kappa,\alpha,L}^n(\cdot) \quad \text{weakly in } L^\Gamma(\Omega), \quad (3.92c)$$

$$(\rho_{\kappa,\alpha,L,\delta}^{[\Delta t],n})^\vartheta \rightarrow (\rho_{\kappa,\alpha,L}^{[\Delta t],n})^\vartheta \quad \text{weakly in } L^2(t_{n-1}, t_n; H^1(\Omega)), \quad \vartheta = 2, \frac{\Gamma}{2}, \quad (3.92d)$$

and

$$\begin{aligned} \underline{u}_{\kappa,\alpha,L,\delta}^n &\rightarrow \underline{u}_{\kappa,\alpha,L}^n && \text{weakly in } \underline{H}_0^1(\Omega), \\ & && \text{strongly in } \underline{L}^r(\Omega), \end{aligned} \quad (3.93a)$$

$$\delta \frac{\partial^{|\lambda|} \underline{u}_{\kappa,\alpha,L,\delta}^n}{\partial x_1^{\lambda_1} \cdots \partial x_d^{\lambda_d}} \rightarrow 0 \quad \text{strongly in } \underline{L}^2(\Omega), \quad \forall |\lambda| = 2, \quad (3.93b)$$

$$\delta \frac{\partial^{|\lambda|} (\nabla_x \cdot \underline{u}_{\kappa,\alpha,L,\delta}^n)}{\partial x_1^{\lambda_1} \cdots \partial x_d^{\lambda_d}} \rightarrow 0 \quad \text{strongly in } L^2(\Omega), \quad \forall |\lambda| = 2, \quad (3.93c)$$



where  $r \in [1, \infty)$  if  $d = 2$  and  $r \in [1, 6)$  if  $d = 3$ , and  $v \in [1, \frac{4\Gamma}{3})$ ; and

$$M^{\frac{1}{2}} \nabla_{\sim q} \widehat{\psi}_{\kappa, \alpha, L, \delta}^n \rightharpoonup M^{\frac{1}{2}} \nabla_{\sim q} \widehat{\psi}_{\kappa, \alpha, L}^n \quad \text{weakly in } \widetilde{L}^2(\Omega \times D), \quad (3.94a)$$

$$M^{\frac{1}{2}} \nabla_{\sim x} \widehat{\psi}_{\kappa, \alpha, L, \delta}^n \rightharpoonup M^{\frac{1}{2}} \nabla_{\sim x} \widehat{\psi}_{\kappa, \alpha, L}^n \quad \text{weakly in } \widetilde{L}^2(\Omega \times D), \quad (3.94b)$$

$$M^{\frac{1}{2}} \widehat{\psi}_{\kappa, \alpha, L, \delta}^n \rightarrow M^{\frac{1}{2}} \widehat{\psi}_{\kappa, \alpha, L}^n \quad \text{strongly in } L^2(\Omega \times D), \quad (3.94c)$$

$$M^{\frac{1}{2}} \beta_{\delta}^L(\widehat{\psi}_{\kappa, \alpha, L, \delta}^n) \rightarrow M^{\frac{1}{2}} \beta^L(\widehat{\psi}_{\kappa, \alpha, L}^n) \quad \text{strongly in } L^s(\Omega \times D), \quad (3.94d)$$

$$C_i(M \widehat{\psi}_{\kappa, \alpha, L, \delta}^n) \rightharpoonup C_i(M \widehat{\psi}_{\kappa, \alpha, L}^n) \quad \text{strongly in } \widetilde{L}^2(\Omega), \quad i = 1, \dots, K, \quad (3.94e)$$

$$\begin{aligned} \varrho_{\kappa, \alpha, L, \delta}^n &\rightharpoonup \varrho_{\kappa, \alpha, L}^n && \text{weakly in } H^1(\Omega), \\ &&& \text{strongly in } L^r(\Omega), \end{aligned} \quad (3.94f)$$

where  $s \in [1, \infty)$ . Furthermore,  $(\rho_{\kappa, \alpha, L}^{[\Delta t], n}, \underline{u}_{\kappa, \alpha, L}^n, \widehat{\psi}_{\kappa, \alpha, L}^n)$  solves (3.27a-c) for  $n = 1, \dots, N$ . Hence, there exists a solution  $\{(\rho_{\kappa, \alpha, L}^{[\Delta t], n}, \underline{u}_{\kappa, \alpha, L}^n, \widehat{\psi}_{\kappa, \alpha, L}^n)\}_{n=1}^N$  to  $(P_{\kappa, \alpha, L}^{\Delta t})$ .

**Proof.** The weak convergence results (3.92a,c,d) follow immediately from (3.88). The strong convergence results (3.92b) follow from (3.92a), (3.11), (3.90) and the interpolation result (3.1). Hence  $\rho_{\kappa, \alpha, L}^{[\Delta t], n} \in Y^n$  with  $\rho_{\kappa, \alpha, L}^n(\cdot) = \rho_{\kappa, \alpha, L}^{[\Delta t], n}(\cdot, t_n) \in L_{\geq 0}^{\Gamma}(\Omega)$  as  $\rho_{\kappa, \alpha, L, \delta}^{[\Delta t], n} \in \Upsilon^n$ . The weak convergence result (3.93a) and the strong convergence results (3.93b,c) follow immediately from (3.84), and hence  $\underline{u}_{\kappa, \alpha, L}^n \in \underline{H}_0^1(\Omega)$  as  $\underline{u}_{\kappa, \alpha, L, \delta}^n \in \mathcal{V}$ . The strong convergence result (3.93a) follows as  $\underline{H}^1(\Omega)$  is compactly embedded in  $\underline{L}^r(\Omega)$  for the stated values of  $r$ .

The weak convergence results (3.94a,b) follow from (3.84); the strong convergence result (3.94c) and the fact that  $\widehat{\psi}_{\kappa, \alpha, L}^n \geq 0$  a.e. on  $\Omega \times D$  follow from the fourth bound in (3.84), (3.42b) and (3.10b). Hence  $\widehat{\psi}_{\kappa, \alpha, L}^n \in X \cap Z_2$ . The desired results (3.94d,e) follow from (3.94c), (3.43), (1.12) and (3.13). See the proof of Lemma 3.3 in Barrett & Süli<sup>6</sup> for details of the results (3.94a-e). Finally, (3.94f) follows from (3.84) and (3.94c).

It follows from (3.92a-c), (3.93a-c), (3.94a-f), (3.31), (3.45), (3.46), (3.33a,b), (3.35a,b) and (3.8) that we may pass to the limit  $\delta \rightarrow 0_+$  in (3.49a-c) to obtain that  $(\rho_{\kappa, \alpha, L}^{[\Delta t], n}, \underline{u}_{\kappa, \alpha, L}^n, \widehat{\psi}_{\kappa, \alpha, L}^n)$  solves (3.30), (3.32), and (3.34); that is, (3.27a-c).

Finally, as  $(\rho_{\kappa, \alpha, L}^0, \underline{u}_{\kappa, \alpha, L}^0, \widehat{\psi}_{\kappa, \alpha, L}^0) \in L_{\geq 0}^{\Gamma}(\Omega) \times \underline{H}_0^1(\Omega) \times Z_2$ , performing the above existence proof at each time level  $t_n$ ,  $n = 1, \dots, N$ , yields a solution  $\{(\rho_{\kappa, \alpha, L}^{[\Delta t], n}, \underline{u}_{\kappa, \alpha, L}^n, \widehat{\psi}_{\kappa, \alpha, L}^n)\}_{n=1}^N$  to  $(P_{\kappa, \alpha, L}^{\Delta t})$  with  $\rho_{\kappa, \alpha, L}^n(\cdot) = \rho_{\kappa, \alpha, L}^{[\Delta t], n}(\cdot, t_n)$ ,  $n = 1, \dots, N$ , by noting that  $\rho_{\kappa, \alpha, L}^{[\Delta t], n}$  thus constructed is an element of  $C([t_{n-1}, t_n]; L^2(\Omega))$ ,  $n = 1, \dots, N$ .  $\square$

#### 4. Existence of a solution to $(P_{\kappa, \alpha})$

Next, we derive bounds on the solution of  $(P_{\kappa, \alpha, L}^{\Delta t})$ , independent of  $\Delta t$  and  $L$ . Our starting point is Lemma 3.3, concerning the existence of a solution to the

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problem  $(P_{\kappa,\alpha,L}^{\Delta t})$ . The model  $(P_{\kappa,\alpha,L}^{\Delta t})$  includes ‘microscopic cut-off’ in the drag and convective terms of the Fokker–Planck equation, where  $L > 1$  is a (fixed, but otherwise arbitrary,) cut-off parameter. Our next objective is to pass to the limits  $L \rightarrow \infty$  and  $\Delta t \rightarrow 0_+$  in the model  $(P_{\kappa,\alpha,L}^{\Delta t})$ , with  $L$  and  $\Delta t$  linked by the condition  $\Delta t = o(L^{-1})$ , as  $L \rightarrow \infty$ . To that end, we need to develop various bounds on sequences of weak solutions of  $(P_{\kappa,\alpha,L}^{\Delta t})$  that are uniform in the time step  $\Delta t$  and the cut-off parameter  $L$ , and thus permit the extraction of weakly convergent subsequences, as  $L \rightarrow \infty$ , through the use of a weak compactness argument. The derivation of such bounds, based on the use of the relative entropy associated with the Maxwellian  $M$ , is our main task in this section.

We define

$$\begin{aligned} \rho_{\kappa,\alpha,L}^{[\Delta t]} &:= \rho_{\kappa,\alpha,L}^{[\Delta t],n}, & t \in [t_{n-1}, t_n], \quad n = 1, \dots, N, \\ &\Rightarrow \rho_{\kappa,\alpha,L}^{[\Delta t]}(\cdot, t_n) = \rho_{\kappa,\alpha,L}^n(\cdot), & n = 0, \dots, N, \end{aligned} \quad (4.1a)$$

$$\rho_{\kappa,\alpha,L}^{\{\Delta t\}}(\cdot, t) := \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \rho_{\kappa,\alpha,L}^{[\Delta t]}(\cdot, s) ds, \quad t \in (t_{n-1}, t_n], \quad n = 1, \dots, N. \quad (4.1b)$$

Further, we define the pressure variable

$$p_{\kappa,\alpha,L}^{\{\Delta t\}}(\cdot, t) := \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} p_{\kappa}(\rho_{\kappa,\alpha,L}^{[\Delta t]}(\cdot, s)) ds, \quad t \in (t_{n-1}, t_n], \quad n = 1, \dots, N, \quad (4.2)$$

and the momentum variable

$$\tilde{m}_{\kappa,\alpha,L}^n := \rho_{\kappa,\alpha,L}^n \tilde{u}_{\kappa,\alpha,L}^n, \quad n = 0, \dots, N. \quad (4.3)$$

We then introduce the following definitions:

$$\begin{aligned} \tilde{u}_{\kappa,\alpha,L}^{\Delta t}(\cdot, t) &:= \frac{t - t_{n-1}}{\Delta t} \tilde{u}_{\kappa,\alpha,L}^n(\cdot) + \frac{t_n - t}{\Delta t} \tilde{u}_{\kappa,\alpha,L}^{n-1}(\cdot), \\ &t \in [t_{n-1}, t_n], \quad n = 1, \dots, N, \end{aligned} \quad (4.4a)$$

$$\begin{aligned} \tilde{u}_{\kappa,\alpha,L}^{\Delta t,+}(\cdot, t) &:= \tilde{u}_{\kappa,\alpha,L}^n(\cdot), & \tilde{u}_{\kappa,\alpha,L}^{\Delta t,-}(\cdot, t) &:= \tilde{u}_{\kappa,\alpha,L}^{n-1}(\cdot), \\ &t \in (t_{n-1}, t_n], \quad n = 1, \dots, N. \end{aligned} \quad (4.4b)$$

We shall adopt  $\tilde{u}_{\kappa,\alpha,L}^{\Delta t(\pm)}$  as a collective symbol for  $\tilde{u}_{\kappa,\alpha,L}^{\Delta t}$ ,  $\tilde{u}_{\kappa,\alpha,L}^{\Delta t,\pm}$ . The corresponding notations  $\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t}$ ,  $\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,\pm}$  and  $\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t(\pm)}$ ;  $\rho_{\kappa,\alpha,L}^{\Delta t}$ ,  $\rho_{\kappa,\alpha,L}^{\Delta t,\pm}$  and  $\rho_{\kappa,\alpha,L}^{\Delta t(\pm)}$ ;  $\tilde{m}_{\kappa,\alpha,L}^{\Delta t}$ ,  $\tilde{m}_{\kappa,\alpha,L}^{\Delta t,\pm}$  and  $\tilde{m}_{\kappa,\alpha,L}^{\Delta t(\pm)}$ , and  $\varrho_{\kappa,\alpha,L}^{\Delta t}$ ,  $\varrho_{\kappa,\alpha,L}^{\Delta t,\pm}$  and  $\varrho_{\kappa,\alpha,L}^{\Delta t(\pm)}$  are defined analogously. The notation  $\rho_{\kappa,\alpha,L}^{\Delta t}$  signifying the piecewise linear interpolant of  $\rho_{\kappa,\alpha,L}^{[\Delta t]}$  with respect to the variable  $t$  is not to be confused with  $\rho_{\kappa,\alpha,L}^{[\Delta t]}$  itself, which denotes the function defined piecewise, over the union of time slabs  $\Omega \times [t_{n-1}, t_n]$ ,  $n = 1, \dots, N$ , solving (3.27a) subject to the initial condition  $\rho_{\kappa,\alpha,L}^{[\Delta t]}(\cdot, t_{n-1}) = \rho_{\kappa,\alpha,L}^{n-1}(\cdot)$ ,  $n = 1, \dots, N$ , with  $\rho_{\kappa,\alpha,L}^0 := \rho^0$ . Using the above notation, (3.27a–c) summed for  $n = 1, \dots, N$  can be restated in the form: find  $(\rho_{\kappa,\alpha,L}^{\Delta t}(\cdot, t), \tilde{u}_{\kappa,\alpha,L}^{\Delta t}(\cdot, t), \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t}(\cdot, t)) \in H^1(\Omega) \cap L_{\geq 0}^{\frac{4F}{3}}(\Omega) \times H_0^1(\Omega) \times (X \cap Z_2)$ ,

with  $\frac{\partial \rho_{\kappa,\alpha,L}^{[\Delta t]}}{\partial t}(\cdot, t) \in H^1(\Omega)'$ , a.e.  $t \in (0, T)$ , such that  $\tilde{m}_{\kappa,\alpha,L}^{\Delta t}$  is defined via (4.3) and

$$\int_0^T \left\langle \frac{\partial \rho_{\kappa,\alpha,L}^{[\Delta t]}}{\partial t}, \eta \right\rangle_{H^1(\Omega)} dt + \int_0^T \int_{\Omega} \left( \alpha \tilde{\nabla}_x \rho_{\kappa,\alpha,L}^{[\Delta t]} - \rho_{\kappa,\alpha,L}^{[\Delta t]} \tilde{u}_{\kappa,\alpha,L}^{\Delta t,+} \right) \cdot \tilde{\nabla}_x \eta \, dx \, dt = 0 \quad \forall \eta \in L^2(0, T; H^1(\Omega)), \quad (4.5a)$$

$$\begin{aligned} & \int_0^T \int_{\Omega} \left[ \frac{\partial \tilde{m}_{\kappa,\alpha,L}^{\Delta t}}{\partial t} - \frac{1}{2} \frac{\partial \rho_{\kappa,\alpha,L}^{\Delta t}}{\partial t} \tilde{u}_{\kappa,\alpha,L}^{\Delta t,+} \right] \cdot \tilde{w} \, dx \, dt + \int_0^T \int_{\Omega} S(\tilde{u}_{\kappa,\alpha,L}^{\Delta t,+}) : \tilde{\nabla}_x \tilde{w} \, dx \, dt \\ & + \frac{1}{2} \int_0^T \int_{\Omega} \left[ [(\tilde{m}_{\kappa,\alpha,L}^{\Delta t,-} \cdot \tilde{\nabla}_x) \tilde{u}_{\kappa,\alpha,L}^{\Delta t,+}] \cdot \tilde{w} - [(\tilde{m}_{\kappa,\alpha,L}^{\Delta t,-} \cdot \tilde{\nabla}_x) \tilde{w}] \cdot \tilde{u}_{\kappa,\alpha,L}^{\Delta t,+} \right] \, dx \, dt \\ & - \int_0^T \int_{\Omega} p_{\kappa,\alpha,L}^{\{\Delta t\}} \tilde{\nabla}_x \cdot \tilde{w} \, dx \, dt \\ & = \int_0^T \int_{\Omega} \left[ \rho_{\kappa,\alpha,L}^{\Delta t,+} f^{\{\Delta t\}} \cdot \tilde{w} - \tau_1(M \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}) : \tilde{\nabla}_x \tilde{w} \right] \, dx \, dt \\ & - 2 \mathfrak{z} \int_0^T \int_{\Omega} \left( \int_D M \beta^L(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}) \, dq \right) \tilde{\nabla}_x \varrho_{\kappa,\alpha,L}^{\Delta t,+} \cdot \tilde{w} \, dx \, dt \\ & \quad \forall \tilde{w} \in L^2(0, T; V), \end{aligned} \quad (4.5b)$$

$$\begin{aligned} & \int_0^T \int_{\Omega \times D} M \frac{\partial \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t}}{\partial t} \varphi \, dq \, dx \, dt \\ & + \frac{1}{4\lambda} \sum_{i=1}^K \sum_{j=1}^K A_{ij} \int_0^T \int_{\Omega \times D} M \tilde{\nabla}_{q_j} \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} \cdot \tilde{\nabla}_{q_i} \varphi \, dq \, dx \, dt \\ & + \int_0^T \int_{\Omega \times D} M \left[ \varepsilon \tilde{\nabla}_x \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} - \tilde{u}_{\kappa,\alpha,L}^{\Delta t,+} \beta^L(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}) \right] \cdot \tilde{\nabla}_x \varphi \, dq \, dx \, dt \\ & - \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \left[ \sigma(\tilde{u}_{\kappa,\alpha,L}^{\Delta t,+}) q_i \right] \beta^L(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}) \cdot \tilde{\nabla}_{q_i} \varphi \, dq \, dx \, dt = 0 \\ & \quad \forall \varphi \in L^2(0, T; X); \end{aligned} \quad (4.5c)$$

subject to the initial conditions  $\rho_{\kappa,\alpha,L}^{\Delta t}(0) = \rho^0 \in L_{\geq 0}^{\infty}(\Omega)$ ,  $u_{\kappa,\alpha,L}^{\Delta t}(0) = u^0 \in \underline{H}_0^1(\Omega)$  and  $\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t}(0) = \widehat{\psi}^0 \in X \cap Z_2$ , where we recall (3.17) and (3.19). We emphasize that (4.5a–c) is an equivalent restatement of problem  $(P_{\kappa,\alpha,L}^{\Delta t})$ , for which existence of a solution has been established (cf. Lemma 3.3). We are now ready to embark on the derivation of the required bounds, uniform in the time step  $\Delta t$  and the cut-off parameter  $L$ , on norms of  $\rho_{\kappa,\alpha,L}^{[\Delta t]}(t) \in H_0^1(\Omega) \cap L_{\geq 0}^{\frac{4F}{3}}(\Omega)$ ,  $u_{\kappa,\alpha,L}^{\Delta t,+}(t) \in \underline{H}_0^1(\Omega)$ ,  $\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}(t) \in X \cap Z_2$  and  $\varrho_{\kappa,\alpha,L}^{\Delta t,+}(t) \in H^1(\Omega)$ ,  $t \in (0, T]$ .

**4.1.  $L$ ,  $\Delta t$ -independent bounds on the spatial derivatives of  $u_{\kappa,\alpha,L}^{\Delta t}$  and  $\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t}$**

We note that it is *not* possible to pass to the limit  $\delta \rightarrow 0_+$  in (3.84) to obtain strong enough  $L$ -independent bounds due to the fourth, seventh and eighth of the ten terms on the left-hand side. Similarly, it is *not* possible to pass to the limit in these terms even before we use the bound  $[\mathcal{F}_\delta^L]''(\cdot) \geq \frac{1}{L}$ ; recall its use in (3.79) to obtain (3.80), and hence (3.84). However, it is a simple matter to pass to the limit  $\delta \rightarrow 0_+$  in (3.83). Noting (3.92c,d), (3.93a), (3.94e,f) and the convexity of  $P_\kappa(\cdot)$ , we may pass to the limit  $\delta \rightarrow 0_+$  in (3.83) to obtain for  $n = 1, \dots, N$  that

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega} \left[ \rho_{\kappa,\alpha,L}^n |u_{\kappa,\alpha,L}^n|^2 + \rho_{\kappa,\alpha,L}^{n-1} |u_{\kappa,\alpha,L}^n - u_{\kappa,\alpha,L}^{n-1}|^2 \right] dx + \int_{\Omega} P_\kappa(\rho_{\kappa,\alpha,L}^n) dx \\
& + \alpha \kappa \int_{t_{n-1}}^{t_n} \left[ \|\nabla_x [(\rho_{\kappa,\alpha,L}^{\Delta t})^n]^2\|_{L^2(\Omega)}^2 + \frac{4}{\Gamma} \|\nabla_x [(\rho_{\kappa,\alpha,L}^{\Delta t})^{\frac{\Gamma}{2}}]^2\|_{L^2(\Omega)}^2 \right] dt \\
& + \Delta t \mu^S c_0 \|u_{\kappa,\alpha,L}^n\|_{H^1(\Omega)}^2 \\
& + \mathfrak{z} \left[ \|\varrho_{\kappa,\alpha,L}^n\|_{L^2(\Omega)}^2 + \|\varrho_{\kappa,\alpha,L}^n - \varrho_{\kappa,\alpha,L}^{n-1}\|_{L^2(\Omega)}^2 \right] + 2 \Delta t \mathfrak{z} \varepsilon \|\nabla_x \varrho_{\kappa,\alpha,L}^n\|_{L^2(\Omega)}^2 \\
& \leq \frac{1}{2} \int_{\Omega} \rho_{\kappa,\alpha,L}^{n-1} |u_{\kappa,\alpha,L}^{n-1}|^2 dx + \int_{\Omega} P_\kappa(\rho_{\kappa,\alpha,L}^{n-1}) dx + \Delta t \int_{\Omega} \rho_{\kappa,\alpha,L}^n f^n \cdot u_{\kappa,\alpha,L}^n dx \\
& + \mathfrak{z} \|\varrho_{\kappa,\alpha,L}^{n-1}\|_{L^2(\Omega)}^2 + k \Delta t (K+1) \int_{\Omega} \varrho_{\kappa,\alpha,L}^n \nabla_x \cdot u_{\kappa,\alpha,L}^n dx \\
& - k \Delta t \sum_{i=1}^K \int_{\Omega} C_i(M \widehat{\psi}_{\kappa,\alpha,L}^n) : \nabla_x u_{\kappa,\alpha,L}^n dx. \tag{4.6}
\end{aligned}$$

Summing the above over  $n$ , and adopting the notation (4.1a), (4.4a,b) and (3.22),

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega} \rho_{\kappa,\alpha,L}^{\Delta t,+}(t_n) |u_{\kappa,\alpha,L}^{\Delta t,+}(t_n)|^2 dx + \frac{1}{2\Delta t} \int_0^{t_n} \int_{\Omega} \rho_{\kappa,\alpha,L}^{\Delta t,-} |u_{\kappa,\alpha,L}^{\Delta t,+} - u_{\kappa,\alpha,L}^{\Delta t,-}|^2 dx dt \\
& + \int_{\Omega} P_\kappa(\rho_{\kappa,\alpha,L}^{\Delta t,+}(t_n)) dx + \mu^S c_0 \int_0^{t_n} \|u_{\kappa,\alpha,L}^{\Delta t,+}\|_{H^1(\Omega)}^2 dt \\
& + \alpha \kappa \int_0^{t_n} \left[ \|\nabla_x [(\rho_{\kappa,\alpha,L}^{\Delta t})^2]\|_{L^2(\Omega)}^2 + \frac{4}{\Gamma} \|\nabla_x [(\rho_{\kappa,\alpha,L}^{\Delta t})^{\frac{\Gamma}{2}}]\|_{L^2(\Omega)}^2 \right] dt \\
& + \mathfrak{z} \|\varrho_{\kappa,\alpha,L}^{\Delta t,+}(t_n)\|_{L^2(\Omega)}^2 \\
& + \mathfrak{z} \int_0^{t_n} \left[ \|\varrho_{\kappa,\alpha,L}^{\Delta t,+} - \varrho_{\kappa,\alpha,L}^{\Delta t,-}\|_{L^2(\Omega)}^2 + 2\varepsilon \|\nabla_x \varrho_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^2(\Omega)}^2 \right] dt \\
& \leq \frac{1}{2} \int_{\Omega} \rho^0 |u^0|^2 dx + \int_{\Omega} P_\kappa(\rho^0) dx + \mathfrak{z} \|\varrho^0\|_{L^2(\Omega)}^2 \\
& + \int_0^{t_n} \int_{\Omega} \rho_{\kappa,\alpha,L}^{\Delta t,+} f \cdot u_{\kappa,\alpha,L}^{\Delta t,+} dx dt \\
& + k (K+1) \int_0^{t_n} \int_{\Omega} \varrho_{\kappa,\alpha,L}^{\Delta t,+} \nabla_x \cdot u_{\kappa,\alpha,L}^{\Delta t,+} dx dt
\end{aligned}$$

$$-k \sum_{i=1}^K \int_0^{t_n} \int_{\Omega \approx} C_i(M \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}) : \nabla_x \underset{\approx}{u}_{\kappa,\alpha,L}^{\Delta t,+} \underset{\approx}{dx} \underset{\approx}{dt}, \quad n = 1, \dots, N. \quad (4.7)$$

We now require a suitable  $\widehat{\psi}_{\kappa,\alpha,L}^n$  analogue of (3.79). The appropriate choice of test function in (3.27c) for this purpose is  $\varphi = [\mathcal{F}^L]'(\widehat{\psi}_{\kappa,\alpha,L}^n)$ . While Lemma 3.3 guarantees that  $\widehat{\psi}_{\kappa,\alpha,L}^n$  belongs to  $Z_2$ , and is therefore nonnegative a.e. on  $\Omega \times D$ , there is unfortunately no reason why  $\widehat{\psi}_{\kappa,\alpha,L}^n$  should be strictly positive on  $\Omega \times D$ , and therefore the expression  $[\mathcal{F}^L]'(\widehat{\psi}_{\kappa,\alpha,L}^n)$  may in general be undefined. Similarly to (3.74), we shall circumvent this problem by choosing  $\varphi = [\mathcal{F}^L]'(\widehat{\psi}_{\kappa,\alpha,L}^n + \varsigma)$  in (3.27c), which leads, for any fixed  $\varsigma \in \mathbb{R}_{>0}$ , to

$$\begin{aligned} 0 &= \int_{\Omega \times D} M \frac{\widehat{\psi}_{\kappa,\alpha,L}^n - \widehat{\psi}_{\kappa,\alpha,L}^{n-1}}{\Delta t} [[\mathcal{F}^L]'(\widehat{\psi}_{\kappa,\alpha,L}^n + \varsigma)] \underset{\approx}{dq} \underset{\approx}{dx} \\ &\quad - \int_{\Omega \times D} M \underset{\approx}{u}_{\kappa,\alpha,L}^n \cdot \left( \nabla_x [[\mathcal{F}^L]'(\widehat{\psi}_{\kappa,\alpha,L}^n + \varsigma)] \right) \beta^L(\widehat{\psi}_{\kappa,\alpha,L}^n) \underset{\approx}{dq} \underset{\approx}{dx} \\ &\quad + \frac{1}{4\lambda} \sum_{i=1}^K \sum_{j=1}^K \int_{\Omega \times D} A_{ij} M \nabla_{q_j} \widehat{\psi}_{\kappa,\alpha,L}^n \cdot \nabla_{q_i} [[\mathcal{F}^L]'(\widehat{\psi}_{\kappa,\alpha,L}^n + \varsigma)] \underset{\approx}{dq} \underset{\approx}{dx} \\ &\quad + \varepsilon \int_{\Omega \times D} M \nabla_x \widehat{\psi}_{\kappa,\alpha,L}^n \cdot \nabla_x [[\mathcal{F}^L]'(\widehat{\psi}_{\kappa,\alpha,L}^n + \varsigma)] \underset{\approx}{dq} \underset{\approx}{dx} \\ &\quad - \sum_{i=1}^K \int_{\Omega \times D} M \beta^L(\widehat{\psi}_{\kappa,\alpha,L}^n) [\sigma(\underset{\approx}{u}_{\kappa,\alpha,L}^n) \underset{\approx}{q}_i] \cdot \nabla_{q_i} [[\mathcal{F}^L]'(\widehat{\psi}_{\kappa,\alpha,L}^n + \varsigma)] \underset{\approx}{dq} \underset{\approx}{dx} \\ &\quad =: \sum_{i=1}^5 \mathbf{T}_i. \end{aligned} \quad (4.8)$$

It follows from (3.38a) that

$$\begin{aligned} \mathbf{T}_1 &\geq \frac{1}{\Delta t} \int_{\Omega \times D} M \left[ \mathcal{F}^L(\widehat{\psi}_{\kappa,\alpha,L}^n + \varsigma) - \mathcal{F}^L(\widehat{\psi}_{\kappa,\alpha,L}^{n-1} + \varsigma) \right] \underset{\approx}{dq} \underset{\approx}{dx} \\ &\quad + \frac{1}{2\Delta t L} \int_{\Omega \times D} M (\widehat{\psi}_{\kappa,\alpha,L}^n - \widehat{\psi}_{\kappa,\alpha,L}^{n-1})^2 \underset{\approx}{dq} \underset{\approx}{dx}. \end{aligned} \quad (4.9)$$

In addition, it follows from (3.38a) that

$$\begin{aligned} \mathbf{T}_2 &= - \int_{\Omega \times D} M \frac{\beta^L(\widehat{\psi}_{\kappa,\alpha,L}^n)}{\beta^L(\widehat{\psi}_{\kappa,\alpha,L}^n + \varsigma)} \underset{\approx}{u}_{\kappa,\alpha,L}^n \cdot \nabla_x \widehat{\psi}_{\kappa,\alpha,L}^n \underset{\approx}{dq} \underset{\approx}{dx} \\ &= \int_{\Omega} \left( \int_D M \widehat{\psi}_{\kappa,\alpha,L}^n \underset{\approx}{dq} \right) \nabla_x \cdot \underset{\approx}{u}_{\kappa,\alpha,L}^n \underset{\approx}{dx} \\ &\quad + \int_{\Omega \times D} M \left[ 1 - \frac{\beta^L(\widehat{\psi}_{\kappa,\alpha,L}^n)}{\beta^L(\widehat{\psi}_{\kappa,\alpha,L}^n + \varsigma)} \right] \underset{\approx}{u}_{\kappa,\alpha,L}^n \cdot \nabla_x \widehat{\psi}_{\kappa,\alpha,L}^n \underset{\approx}{dq} \underset{\approx}{dx}. \end{aligned} \quad (4.10)$$

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Thanks to (2.5), we have that

$$\mathsf{T}_3 \geq \frac{a_0}{4\lambda} \int_{\Omega \times D} M [[\mathcal{F}^L]''(\widehat{\psi}_{\kappa,\alpha,L}^n + \varsigma)] |\nabla_{\widetilde{q}} \widehat{\psi}_{\kappa,\alpha,L}^n|^2 \, d\widetilde{q} \, dx, \quad (4.11a)$$

$$\mathsf{T}_4 \geq \varepsilon \int_{\Omega \times D} M [[\mathcal{F}^L]''(\widehat{\psi}_{\kappa,\alpha,L}^n + \varsigma)] |\nabla_x \widehat{\psi}_{\kappa,\alpha,L}^n|^2 \, d\widetilde{q} \, dx. \quad (4.11b)$$

It is tempting to bound  $[\mathcal{F}^L]''(\widehat{\psi}_{\kappa,\alpha,L}^n + \varsigma)$  below further by  $(\widehat{\psi}_{\kappa,\alpha,L}^n + \varsigma)^{-1}$  using (3.38b). We have refrained from doing so as the precise form of (4.11b) will be required to absorb the extraneous term that the process of shifting  $\widehat{\psi}_{\kappa,\alpha,L}^n$  by the addition of  $\varsigma > 0$  generates in the last term in (4.10). Similarly, (4.11a) is required for the last line in (4.12) below. Finally, it follows from (3.38a) and (1.6a) that

$$\begin{aligned} \mathsf{T}_5 &= - \sum_{i=1}^K \int_{\Omega} \left[ \int_D M [(\nabla_x u_{\kappa,\alpha,L}^n) q_i] \cdot \nabla_{q_i} \widehat{\psi}_{\kappa,\alpha,L}^n \, d\widetilde{q} \right] dx \\ &\quad + \int_{\Omega \times D} M \left[ 1 - \frac{\beta^L(\widehat{\psi}_{\kappa,\alpha,L}^n)}{\beta^L(\widehat{\psi}_{\kappa,\alpha,L}^n + \varsigma)} \right] \sum_{i=1}^K [(\nabla_x u_{\kappa,\alpha,L}^n) q_i] \cdot \nabla_{q_i} \widehat{\psi}_{\kappa,\alpha,L}^n \, d\widetilde{q} \, dx \\ &= - \int_{\Omega \times D} M \sum_{i=1}^K U'(\tfrac{1}{2}|q_i|^2) \widehat{\psi}_{\kappa,\alpha,L}^n (q_i q_i^T) : \nabla_x u_{\kappa,\alpha,L}^n \, d\widetilde{q} \, dx \\ &\quad + K \int_{\Omega} \left( \int_D M \widehat{\psi}_{\kappa,\alpha,L}^n \, d\widetilde{q} \right) \nabla_x \cdot u_{\kappa,\alpha,L}^n \, dx \\ &\quad + \int_{\Omega \times D} M \left[ 1 - \frac{\beta^L(\widehat{\psi}_{\kappa,\alpha,L}^n)}{\beta^L(\widehat{\psi}_{\kappa,\alpha,L}^n + \varsigma)} \right] \sum_{i=1}^K [(\nabla_x u_{\kappa,\alpha,L}^n) q_i] \cdot \nabla_{q_i} \widehat{\psi}_{\kappa,\alpha,L}^n \, d\widetilde{q} \, dx. \end{aligned} \quad (4.12)$$

Substituting (4.9)–(4.12) into (4.8), multiplying by  $\Delta t$ , summing over  $n$  and adopting the notation (4.4a,b) yields, for  $n = 1, \dots, N$ , that

$$\begin{aligned} &\int_{\Omega \times D} M \mathcal{F}^L(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}(t_n) + \varsigma) \, d\widetilde{q} \, dx \\ &\quad + \frac{1}{2\Delta t L} \int_0^{t_n} \int_{\Omega \times D} M (\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} - \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,-})^2 \, d\widetilde{q} \, dx \, dt \\ &\quad + \int_0^{t_n} \int_{\Omega \times D} M [[\mathcal{F}^L]''(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} + \varsigma)] \left[ \frac{a_0}{4\lambda} |\nabla_{\widetilde{q}} \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}|^2 + \varepsilon |\nabla_x \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}|^2 \right] \, d\widetilde{q} \, dx \, dt \\ &\leq \int_{\Omega \times D} M \mathcal{F}^L(\beta^L(\widehat{\psi}^0) + \varsigma) \, d\widetilde{q} \, dx \\ &\quad - (K+1) \int_0^{t_n} \int_{\Omega} \left( \int_D M \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} \, d\widetilde{q} \right) \nabla_x \cdot u_{\kappa,\alpha,L}^{\Delta t,+} \, dx \, dt \\ &\quad + \int_0^{t_n} \int_{\Omega \times D} M \sum_{i=1}^K U'_i(\tfrac{1}{2}|q|^2) \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} (q_i q_i^T) : \nabla_x u_{\kappa,\alpha,L}^{\Delta t,+} \, d\widetilde{q} \, dx \, dt \end{aligned}$$

$$\begin{aligned}
& - \int_0^{t_n} \int_{\Omega \times D} M \left[ 1 - \frac{\beta^L(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+})}{\beta^L(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} + \varsigma)} \right] u_{\kappa,\alpha,L}^{\Delta t,+} \cdot \nabla_x \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} \, dq \, dx \, dt \\
& - \int_0^{t_n} \int_{\Omega \times D} M \left[ 1 - \frac{\beta^L(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+})}{\beta^L(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} + \varsigma)} \right] \sum_{i=1}^K \left[ (\nabla_x u_{\kappa,\alpha,L}^{\Delta t,+}) q_i \right] \cdot \nabla_{q_i} \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} \, dq \, dx \, dt,
\end{aligned} \tag{4.13}$$

where we have noted (3.21b). The denominator in the prefactor of the second integral on the left-hand side motivates us to link  $\Delta t$  to  $L$  so that  $\Delta t L = o(1)$ , as  $\Delta t \rightarrow 0_+$  (or, equivalently,  $\Delta t = o(L^{-1})$ , as  $L \rightarrow \infty$ ), in order to drive the integral multiplied by the prefactor to 0 in the limit of  $\Delta t \rightarrow 0_+$ , once the product of the two has been bounded above by a constant, independent of  $\Delta t$  and  $L$ .

Comparing (4.13) with (4.7), and noting (1.12), we see that after multiplying (4.13) by  $k$  and adding the resulting inequality to (4.7) the last two terms on the right-hand side of (4.7) are cancelled by  $k$  times the second and third terms on the right-hand side of (4.13). Hence, for  $n = 1, \dots, N$ , we deduce that

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega} \rho_{\kappa,\alpha,L}^{\Delta t,+}(t_n) |u_{\kappa,\alpha,L}^{\Delta t,+}(t_n)|^2 \, dx + \frac{1}{2\Delta t} \int_0^{t_n} \int_{\Omega} \rho_{\kappa,\alpha,L}^{\Delta t,-} |u_{\kappa,\alpha,L}^{\Delta t,+} - u_{\kappa,\alpha,L}^{\Delta t,-}|^2 \, dx \, dt \\
& + \int_{\Omega} P_{\kappa}(\rho_{\kappa,\alpha,L}^{\Delta t,+}(t_n)) \, dx + k \int_{\Omega \times D} M \mathcal{F}^L(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}(t_n) + \varsigma) \, dq \, dx \\
& + \alpha \kappa \int_0^{t_n} \left[ \|\nabla_x [(\rho_{\kappa,\alpha,L}^{\Delta t,+})^2]\|_{L^2(\Omega)}^2 + \frac{4}{\Gamma} \|\nabla_x [(\rho_{\kappa,\alpha,L}^{\Delta t,+})^{\frac{\Gamma}{2}}]\|_{L^2(\Omega)}^2 \right] \, dt \\
& + \mu^S c_0 \int_0^{t_n} \|u_{\kappa,\alpha,L}^{\Delta t,+}\|_{H^1(\Omega)}^2 \, dt + \frac{k}{2\Delta t L} \int_0^{t_n} \int_{\Omega \times D} M (\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} - \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,-})^2 \, dq \, dx \, dt \\
& + k \int_0^{t_n} \int_{\Omega \times D} M [[\mathcal{F}^L]''(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} + \varsigma)] \left[ \frac{\alpha_0}{4\lambda} |\nabla_q \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}|^2 + \varepsilon |\nabla_x \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}|^2 \right] \, dq \, dx \, dt \\
& + \mathfrak{z} \|\varrho_{\kappa,\alpha,L}^{\Delta t,+}(t_n)\|_{L^2(\Omega)}^2 + \mathfrak{z} \int_0^{t_n} \left[ \|\varrho_{\kappa,\alpha,L}^{\Delta t,+} - \varrho_{\kappa,\alpha,L}^{\Delta t,-}\|_{L^2(\Omega)}^2 + 2\varepsilon \|\nabla_x \varrho_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^2(\Omega)}^2 \right] \, dt \\
& \leq \frac{1}{2} \int_{\Omega} \rho^0 |u^0|^2 \, dx + \int_{\Omega} P_{\kappa}(\rho^0) \, dx + \mathfrak{z} \|\varrho^0\|_{L^2(\Omega)}^2 \\
& + \int_0^{t_n} \int_{\Omega} \rho_{\kappa,\alpha,L}^{\Delta t,+} f \cdot u_{\kappa,\alpha,L}^{\Delta t,+} \, dx \, dt + k \int_{\Omega \times D} M \mathcal{F}^L(\beta^L(\widehat{\psi}^0) + \varsigma) \, dq \, dx \\
& - k \int_0^{t_n} \int_{\Omega \times D} M \left[ 1 - \frac{\beta^L(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+})}{\beta^L(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} + \varsigma)} \right] u_{\kappa,\alpha,L}^{\Delta t,+} \cdot \nabla_x \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} \, dq \, dx \, dt \\
& - k \int_0^{t_n} \int_{\Omega \times D} M \left[ 1 - \frac{\beta^L(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+})}{\beta^L(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} + \varsigma)} \right] \sum_{i=1}^K \left[ (\nabla_x u_{\kappa,\alpha,L}^{\Delta t,+}) q_i \right] \cdot \nabla_{q_i} \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} \, dq \, dx \, dt.
\end{aligned} \tag{4.14}$$

Similarly to (3.69), we have on choosing  $\eta = 1$  in (3.27a) that

$$\int_{\Omega} \rho_{\kappa,\alpha,L}^n \, dx = \int_{\Omega} \rho_{\kappa,\alpha,L}^{n-1} \, dx = \int_{\Omega} \rho^0 \, dx, \quad n = 1, \dots, N. \tag{4.15}$$

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Noting (4.15) and (4.4b), we have that

$$\begin{aligned} & \left| \int_0^{t_n} \int_{\Omega} \rho_{\kappa,\alpha,L}^{\Delta t,+} f \cdot u_{\kappa,\alpha,L}^{\Delta t,+} dx dt \right| \\ & \leq \frac{1}{2} \left[ \int_0^{t_n} \int_{\Omega} \rho_{\kappa,\alpha,L}^{\Delta t,+} |u_{\kappa,\alpha,L}^{\Delta t,+}|^2 dx dt + \int_0^{t_n} \|f\|_{L^\infty(\Omega)}^2 dt \int_{\Omega} \rho^0 dx \right]. \end{aligned} \quad (4.16)$$

Next we recall from Barrett & Süli<sup>6</sup> the bound

$$\int_{\Omega \times D} M \mathcal{F}^L(\beta^L(\widehat{\psi}^0) + \varsigma) dq dx \leq \frac{3\varsigma}{2} |\Omega| + \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}^0 + \varsigma) dq dx. \quad (4.17)$$

Let  $\underline{b} := (b_1, \dots, b_K)$ , recall (3.4), and  $b := |\underline{b}|_1 := b_1 + \dots + b_K$ ; then we can bound the magnitude of the last term on the right-hand side of (4.14), on noting (3.38a) and (1.6b), by

$$\begin{aligned} & \frac{k a_0}{8\lambda} \left( \int_0^{t_n} \int_{\Omega \times D} M[[\mathcal{F}^L]''(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} + \varsigma)] |\nabla_q \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}|^2 dq dx dt \right) \\ & + \varsigma \frac{2k \lambda b}{a_0} \left( \int_0^{t_n} \int_{\Omega} |\nabla_x u_{\kappa,\alpha,L}^{\Delta t,+}|^2 dx dt \right), \end{aligned} \quad (4.18)$$

see Barrett & Süli<sup>8</sup> for the details. Similarly, the second to last term on the right-hand side of (4.14) can be bounded by

$$\begin{aligned} & \frac{k \varepsilon}{2} \left( \int_0^{t_n} \int_{\Omega \times D} M[[\mathcal{F}^L]''(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} + \varsigma)] |\nabla_x \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}|^2 dq dx dt \right) \\ & + \varsigma \frac{k}{2\varepsilon} \left( \int_0^{t_n} \int_{\Omega} |u_{\kappa,\alpha,L}^{\Delta t,+}|^2 dx dt \right). \end{aligned} \quad (4.19)$$

Noting (4.14)–(4.19), and using (3.38b) to bound the expression  $[\mathcal{F}^L]''(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} + \varsigma)$  from below by

$$\mathcal{F}''(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} + \varsigma) = (\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} + \varsigma)^{-1}$$

and (3.39) to bound  $\mathcal{F}^L(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} + \varsigma)$  by  $\mathcal{F}(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} + \varsigma)$  from below yields, for  $n = 1, \dots, N$ , that

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \rho_{\kappa,\alpha,L}^{\Delta t,+}(t_n) |u_{\kappa,\alpha,L}^{\Delta t,+}(t_n)|^2 dx + \frac{1}{2\Delta t} \int_0^{t_n} \int_{\Omega} \rho_{\kappa,\alpha,L}^{\Delta t,-} |u_{\kappa,\alpha,L}^{\Delta t,+} - u_{\kappa,\alpha,L}^{\Delta t,-}|^2 dx dt \\ & + \int_{\Omega} P_{\kappa}(\rho_{\kappa,\alpha,L}^{\Delta t,+}(t_n)) dx + k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}(t_n) + \varsigma) dq dx \\ & + \alpha \kappa \int_0^{t_n} \left[ \|\nabla_x [(\rho_{\kappa,\alpha,L}^{\Delta t})^2]\|_{L^2(\Omega)}^2 + \frac{4}{\Gamma} \|\nabla_x [(\rho_{\kappa,\alpha,L}^{\Delta t})^{\frac{\Gamma}{2}}]\|_{L^2(\Omega)}^2 \right] dt \end{aligned}$$



$$\begin{aligned}
& + \mu^S c_0 \int_0^{t_n} \|u_{\kappa,\alpha,L}^{\Delta t,+}\|_{H^1(\Omega)}^2 dt \\
& + \frac{k}{2\Delta t L} \int_0^{t_n} \int_{\Omega \times D} M (\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} - \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,-})^2 dq dx dt \\
& + \frac{k}{2} \int_0^{t_n} \int_{\Omega \times D} \frac{M}{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} + \varsigma} \left[ \frac{a_0}{4\lambda} |\nabla_q \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}|^2 + \varepsilon |\nabla_x \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}|^2 \right] dq dx dt \\
& + \mathfrak{z} \|\varrho_{\kappa,\alpha,L}^{\Delta t,+}(t_n)\|_{L^2(\Omega)}^2 + \mathfrak{z} \int_0^{t_n} \left[ \|\varrho_{\kappa,\alpha,L}^{\Delta t,+} - \varrho_{\kappa,\alpha,L}^{\Delta t,-}\|_{L^2(\Omega)}^2 + 2\varepsilon \|\nabla_x \varrho_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^2(\Omega)}^2 \right] dt \\
& \leq \frac{1}{2} \int_{\Omega} \rho^0 |u^0|^2 dx + \int_{\Omega} P_{\kappa}(\rho^0) dx + \mathfrak{z} \|\varrho^0\|_{L^2(\Omega)}^2 \\
& + \frac{1}{2} \int_0^{t_n} \|f\|_{L^\infty(\Omega)}^2 dt \int_{\Omega} \rho^0 dx + k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}^0 + \varsigma) dq dx + \frac{3k\varsigma}{2} |\Omega| \\
& + 2k\varsigma \max \left\{ \frac{\lambda b}{a_0}, \frac{1}{4\varepsilon} \right\} \int_0^{t_n} \|u_{\kappa,\alpha,L}^{\Delta t,+}\|_{H^1(\Omega)}^2 dt + \frac{1}{2} \int_0^{t_n} \int_{\Omega} \rho_{\kappa,\alpha,L}^{\Delta t,+} |u_{\kappa,\alpha,L}^{\Delta t,+}|^2 dx dt.
\end{aligned} \tag{4.20}$$

Passing to the limit  $\varsigma \rightarrow 0_+$  in (4.20), and then applying a discrete Gronwall inequality yields, for  $n = 1, \dots, N$ , that

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega} \rho_{\kappa,\alpha,L}^{\Delta t,+}(t_n) |u_{\kappa,\alpha,L}^{\Delta t,+}(t_n)|^2 dx + \frac{1}{2\Delta t} \int_0^{t_n} \int_{\Omega} \rho_{\kappa,\alpha,L}^{\Delta t,-} |u_{\kappa,\alpha,L}^{\Delta t,+} - u_{\kappa,\alpha,L}^{\Delta t,-}|^2 dx dt \\
& + \int_{\Omega} P_{\kappa}(\rho_{\kappa,\alpha,L}^{\Delta t,+}(t_n)) dx + k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}(t_n)) dq dx \\
& + \alpha \kappa \int_0^{t_n} \left[ \|\nabla_x [(\rho_{\kappa,\alpha,L}^{\Delta t,+})^{\frac{1}{2}}]\|_{L^2(\Omega)}^2 + \frac{4}{\Gamma} \|\nabla_x [(\rho_{\kappa,\alpha,L}^{\Delta t,+})^{\frac{\Gamma}{2}}]\|_{L^2(\Omega)}^2 \right] dt \\
& + \mu^S c_0 \int_0^{t_n} \|u_{\kappa,\alpha,L}^{\Delta t,+}\|_{H^1(\Omega)}^2 dt + \frac{k}{2\Delta t L} \int_0^{t_n} \int_{\Omega \times D} M (\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} - \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,-})^2 dq dx dt \\
& + k \int_0^{t_n} \int_{\Omega \times D} M \left[ \frac{a_0}{2\lambda} \left| \nabla_q \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}} \right|^2 + 2\varepsilon \left| \nabla_x \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}} \right|^2 \right] dq dx dt \\
& + \mathfrak{z} \|\varrho_{\kappa,\alpha,L}^{\Delta t,+}(t_n)\|_{L^2(\Omega)}^2 + \mathfrak{z} \int_0^{t_n} \left[ \|\varrho_{\kappa,\alpha,L}^{\Delta t,+} - \varrho_{\kappa,\alpha,L}^{\Delta t,-}\|_{L^2(\Omega)}^2 + 2\varepsilon \|\nabla_x \varrho_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^2(\Omega)}^2 \right] dt \\
& \leq e^{t_n} \left[ \frac{1}{2} \int_{\Omega} \rho^0 |u^0|^2 dx + \int_{\Omega} P_{\kappa}(\rho^0) dx + k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}^0) dq dx \right. \\
& \quad \left. + \mathfrak{z} \|\varrho^0\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_0^{t_n} \|f\|_{L^\infty(\Omega)}^2 dt \int_{\Omega} \rho^0 dx \right] \\
& \leq e^{t_n} \left[ \frac{1}{2} \int_{\Omega} \rho^0 |u^0|^2 dx + \int_{\Omega} P_{\kappa}(\rho^0) dx + k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}^0) dq dx \right. \\
& \quad \left. + \mathfrak{z} \int_{\Omega} \left( \int_D M \widehat{\psi}^0 dq \right)^2 dx + \frac{1}{2} \int_0^{t_n} \|f\|_{L^\infty(\Omega)}^2 dt \int_{\Omega} \rho^0 dx \right] \leq C,
\end{aligned} \tag{4.21}$$

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where  $C$  is a positive constant, independent of the parameters  $\Delta t$ ,  $L$ ,  $\alpha$  and  $\kappa$ . Here, we have noted (3.18a), (3.21a) and (3.22) for the penultimate inequality in (4.21), and (3.16a) for the final inequality.

Next we bound the extra stress term (1.12). As we do not have a bound on  $\|M^{\frac{1}{2}}\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^2(\Omega\times D)}$  in (4.21), we will need a weaker bound than (3.13). First, we deduce from (1.12), (1.6a) and as  $M = 0$  on  $\partial D$  that

$$C_i(M\varphi) \approx - \int_D (\nabla_{q_i} M) q_i^T \varphi \, dq = \int_D M (\nabla_{q_i} \varphi) q_i^T \, dq + \left( \int_D M \varphi \, dq \right) I. \quad (4.22)$$

Hence, for  $r \in [1, 2)$ , on noting that  $\nabla_{q_i} \varphi = \nabla_{q_i} (\sqrt{\varphi})^2 = 2\sqrt{\varphi} \nabla_{q_i} \sqrt{\varphi}$  for any sufficiently smooth nonnegative function  $\varphi$ , we have that

$$\begin{aligned} & \int_{\Omega} \|C_i(M\varphi)\|_{L^r(\Omega)} \\ & \leq C \left[ \int_{\Omega} \left( \int_D M \varphi \, dq \right)^{\frac{r}{2}} \left( \int_D M |\nabla_{q_i} \sqrt{\varphi}|^2 \, dq \right)^{\frac{r}{2}} \, dx + \int_{\Omega} \left( \int_D M \varphi \, dq \right)^r \, dx \right]^{\frac{1}{r}} \\ & \leq C \left[ \|\nabla_{q_i} \sqrt{\varphi}\|_{L_M^2(\Omega\times D)} \left\| \int_D M \varphi \, dq \right\|_{L^{\frac{r}{2-r}}(\Omega)}^{\frac{1}{2}} + \left\| \int_D M \varphi \, dq \right\|_{L^r(\Omega)} \right]. \quad (4.23) \end{aligned}$$

Therefore, for  $r \in [1, 2)$  and  $s \in [1, 2]$ , it follows that, for any such function  $\varphi$ ,

$$\begin{aligned} & \|C_i(M\varphi)\|_{L^s(0,T;L^r(\Omega))} \\ & \leq C \left[ \|\nabla_{q_i} \sqrt{\varphi}\|_{L^2(0,T;L_M^2(\Omega\times D))} \left\| \int_D M \varphi \, dq \right\|_{L^v(0,T;L^{\frac{r}{2-r}}(\Omega))}^{\frac{1}{2}} \right. \\ & \quad \left. + \left\| \int_D M \varphi \, dq \right\|_{L^s(0,T;L^r(\Omega))} \right], \quad (4.24) \end{aligned}$$

where  $v = \frac{s}{2-s}$  if  $s \in [1, 2)$  and  $v = \infty$  if  $s = 2$ . We deduce from (4.24) and (4.21) that, for  $i = 1, \dots, K$ ,

$$\|C_i(M\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+})\|_{L^s(0,T;L^r(\Omega))} \leq C \quad \text{if} \quad \|\varrho_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^v(0,T;L^{\frac{r}{2-r}}(\Omega))} \leq C, \quad (4.25)$$

where  $r \in [1, 2)$ ,  $s \in [1, 2]$  and  $v = \frac{s}{2-s}$  if  $s \in [1, 2)$  and  $v = \infty$  if  $s = 2$ .

It follows from (4.21) and (3.2) that

$$\|\varrho_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^{\frac{2}{\vartheta}}(0,T;L^v(\Omega))} \leq C \|\varrho_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^\infty(0,T;L^2(\Omega))}^{1-\vartheta} \|\varrho_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^2(0,T;H^1(\Omega))}^{\vartheta} \leq C, \quad (4.26)$$

where  $\vartheta = \frac{(v-2)d}{2v}$ , and  $v \in (2, \infty)$  if  $d = 2$  and  $v \in (2, 6]$  if  $d = 3$ . For example, we have that

$$\|\varrho_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^\infty(0,T;L^2(\Omega))} + \|\varrho_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^{\frac{2(d+2)}{d}}(\Omega_T)} + \|\varrho_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^4(0,T;L^{\frac{2d}{d-1}}(\Omega))} \leq C, \quad (4.27)$$

and hence we deduce from (4.25) and (1.11) that

$$\|C_i(M \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+})\|_{L^2(0,T;L^{\frac{4}{3}}(\Omega))} + \|C_i(M \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+})\|_{L^{\frac{4(d+2)}{3d+4}}(\Omega_T)} \leq C, \quad i = 1, \dots, K, \quad (4.28a)$$

$$\|\tau_1(M \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+})\|_{L^2(0,T;L^{\frac{4}{3}}(\Omega))} + \|\tau_1(M \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+})\|_{L^{\frac{4(d+2)}{3d+4}}(\Omega_T)} \leq C, \quad (4.28b)$$

where  $C$  is independent of  $\Delta t$ ,  $L$ ,  $\alpha$  and  $\kappa$ .

**Remark 4.1.** We note from (4.21) and (4.26) that if  $\mathfrak{z} = 0$ , then the bounds (4.27) and (4.28a,b) no longer hold. In this case, we have only the following weaker bounds.

Similarly to (4.15), we have on choosing  $\varphi = 1$  in (3.27c), and noting (4.4b) and (3.21b), that, for a.a.  $t \in (0, T)$ ,

$$\int_{\Omega} \varrho_{\kappa,\alpha,L}^{\Delta t,+} dx = \int_{\Omega \times D} M \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} dq dx = \int_{\Omega \times D} M \widehat{\psi}^0 dq dx \leq C. \quad (4.29)$$

Next we deduce from (4.21) and (4.29) that

$$\begin{aligned} \|\nabla_x \varrho_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^2(0,T;L^1(\Omega))}^2 &= 4 \left\| \int_D M \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}} \nabla_x \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}} dq \right\|_{L^2(0,T;L^1(\Omega))}^2 \\ &\leq 4 \|\varrho_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^\infty(0,T;L^1(\Omega))} \left\| \nabla_x \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}} \right\|_{L^2(0,T;L_M^2(\Omega \times D))}^2 \leq C. \end{aligned} \quad (4.30)$$

It follows from Sobolev embedding, (4.30) and (4.29) that

$$\|\varrho_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^2(0,T;L^{\frac{d}{d-1}}(\Omega))} \leq C \|\varrho_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^2(0,T;W^{1,1}(\Omega))} \leq C. \quad (4.31)$$

Therefore, we obtain from (4.25), (4.31) and (1.11) that

$$\|\tau_1(M \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+})\|_{L^{\frac{4}{3}}(0,T;L^{\frac{2d}{2d-1}}(\Omega))} \leq C, \quad (4.32)$$

where  $C$  is independent of  $\Delta t$ ,  $L$ ,  $\alpha$  and  $\kappa$ .

#### 4.2. $L$ , $\Delta t$ -independent bounds on the spatial and temporal derivatives of $\rho_{\kappa,\alpha,L}^{[\Delta t]}$

In addition to the bounds on  $\rho_{\kappa,\alpha,L}^{[\Delta t]}$  and  $\rho_{\kappa,\alpha,L}^{\Delta t,+}$  in (4.21), we establish further relevant bounds here. Similarly to (3.85), on choosing, for any  $s \in (0, T]$ ,  $\eta(\cdot, t) = \chi_{[0,s]} [\rho_{\kappa,\alpha,L}^{[\Delta t]}(\cdot, t)]^{\vartheta-1}$ , for  $\vartheta = 2$  and  $\frac{\Gamma}{2}$ , in (4.5a), we obtain, on noting (4.21), that

$$\begin{aligned} &\frac{1}{\vartheta} \|\rho_{\kappa,\alpha,L}^{[\Delta t]}(\cdot, s)\|_{L^\vartheta(\Omega)}^\vartheta + \frac{4\alpha(\vartheta-1)}{\vartheta^2} \int_0^s \|\nabla_x [(\rho_{\kappa,\alpha,L}^{[\Delta t]})^{\frac{\vartheta}{2}}]\|_{L^2(\Omega)}^2 dt \\ &= \frac{1}{\vartheta} \left[ \|\rho^0\|_{L^\vartheta(\Omega)}^\vartheta + (\vartheta-1) \int_0^s \int_{\Omega} u_{\kappa,\alpha,L}^{\Delta t,+} \cdot \nabla_x [(\rho_{\kappa,\alpha,L}^{[\Delta t]})^\vartheta] dx dt \right] \\ &\leq \frac{1}{\vartheta} \left[ \|\rho^0\|_{L^\vartheta(\Omega)}^\vartheta + \int_0^s \|u_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^2(\Omega)}^2 dt + \frac{(\vartheta-1)^2}{4} \int_0^s \|\nabla_x [(\rho_{\kappa,\alpha,L}^{[\Delta t]})^\vartheta]\|_{L^2(\Omega)}^2 dt \right] \\ &\leq C, \end{aligned} \quad (4.33)$$

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where  $C$  is independent of  $\Delta t$  and  $L$ . Similarly to (3.86), it follows from a Poincaré inequality, (4.21) and (4.33) for  $\vartheta = \frac{\Gamma}{2}$  that

$$\begin{aligned} \|\rho_{\kappa,\alpha,L}^{[\Delta t]}\|_{L^\Gamma(0,T;L^\Gamma(\Omega))} &= \|(\rho_{\kappa,\alpha,L}^{[\Delta t]})^{\frac{\Gamma}{2}}\|_{L^2(0,T;L^2(\Omega))}^2 \\ &\leq C \left[ \|\nabla_x [(\rho_{\kappa,\alpha,L}^{[\Delta t]})^{\frac{\Gamma}{2}}]\|_{L^2(0,T;L^2(\Omega))}^2 + \|\rho_{\kappa,\alpha,L}^{[\Delta t]}\|_{L^\infty(0,T;L^{\frac{\Gamma}{2}}(\Omega))}^\Gamma \right] \leq C. \end{aligned} \quad (4.34)$$

Similarly to (3.87), we obtain from (3.31), (3.2), (4.21) and (4.33) for  $\vartheta = \frac{\Gamma}{2}$  that

$$\begin{aligned} \left| \int_0^T c(u_{\kappa,\alpha,L}^{\Delta t,+})(\rho_{\kappa,\alpha,L}^{[\Delta t]}, \eta) \, dt \right| &\leq C \|\eta\|_{L^2(0,T;H^1(\Omega))} \\ &\quad + C \left| \int_0^T \|u_{\kappa,\alpha,L}^{\Delta t,+}\|_{H^1(\Omega)} \|\eta\|_{H^1(\Omega)} \, dt \right| \\ &\leq C \left[ 1 + \|u_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^2(0,T;H^1(\Omega))} \right] \|\eta\|_{L^2(0,T;H^1(\Omega))} \\ &\leq C \|\eta\|_{L^2(0,T;H^1(\Omega))} \quad \forall \eta \in L^2(0,T;H^1(\Omega)). \end{aligned} \quad (4.35)$$

Hence, similarly to (3.88), we deduce from (4.33) for  $\vartheta = 2$  and  $\frac{\Gamma}{2}$ , (4.35), on noting (4.5a) and (3.31), (4.21) and (4.34) that

$$\begin{aligned} \|\rho_{\kappa,\alpha,L}^{[\Delta t]}\|_{L^\infty(0,T;L^{\frac{\Gamma}{2}}(\Omega))} + \|\rho_{\kappa,\alpha,L}^{[\Delta t]}\|_{L^2(0,T;H^1(\Omega))}^2 \\ + \|\rho_{\kappa,\alpha,L}^{[\Delta t]}\|_{H^1(0,T;H^1(\Omega)')}^2 + \|(\rho_{\kappa,\alpha,L}^{[\Delta t]})^{\frac{\Gamma}{2}}\|_{L^2(0,T;H^1(\Omega))}^2 \leq C, \end{aligned} \quad (4.36)$$

where  $C$  is independent of  $\Delta t$  and  $L$ . Similarly to (3.89), we deduce from (3.2) and the last bound in (4.36) that

$$\|\rho_{\kappa,\alpha,L}^{[\Delta t]}\|_{L^\Gamma(0,T;L^{3\Gamma}(\Omega))} = \|(\rho_{\kappa,\alpha,L}^{[\Delta t]})^{\frac{\Gamma}{2}}\|_{L^2(0,T;L^6(\Omega))}^2 \leq C \|(\rho_{\kappa,\alpha,L}^{[\Delta t]})^{\frac{\Gamma}{2}}\|_{L^2(0,T;H^1(\Omega))}^2 \leq C. \quad (4.37)$$

Next, similarly to (3.90), it follows from (3.1), the first bound in (4.36) and (4.37) that

$$\|\rho_{\kappa,\alpha,L}^{[\Delta t]}\|_{L^{\frac{4\Gamma}{3}}(0,T;L^{\frac{4\Gamma}{3}}(\Omega))} \leq \|\rho_{\kappa,\alpha,L}^{[\Delta t]}\|_{L^\infty(0,T;L^{\frac{\Gamma}{2}}(\Omega))}^{\frac{\Gamma}{3}} \|\rho_{\kappa,\alpha,L}^{[\Delta t]}\|_{L^\Gamma(0,T;L^{3\Gamma}(\Omega))} \leq C, \quad (4.38)$$

where  $C$  is independent of  $\Delta t$  and  $L$ . Finally, it follows from (4.2), (2.3) and (4.38) that, with  $C$  independent of  $\Delta t$  and  $L$ ,

$$\|\rho_{\kappa,\alpha,L}^{\{\Delta t\}}\|_{L^{\frac{4}{3}}(\Omega_T)} \leq \|p_\kappa(\rho_{\kappa,\alpha,L}^{[\Delta t]})\|_{L^{\frac{4}{3}}(\Omega_T)} \leq C \|\rho_{\kappa,\alpha,L}^{[\Delta t]}\|_{L^{\frac{4\Gamma}{3}}(\Omega_T)} \leq C. \quad (4.39)$$

### 4.3. *Passing to the limit $\Delta t \rightarrow 0_+$ ( $L \rightarrow \infty$ ) in the continuity equation (4.5a)*

As noted after (4.13), we shall assume that

$$\Delta t = o(L^{-1}) \quad \text{as } \Delta t \rightarrow 0_+ \text{ (} L \rightarrow \infty \text{)}. \quad (4.40)$$

Requiring, for example, that  $0 < \Delta t \leq C_0/(L \log L)$ ,  $L > 1$ , with an arbitrary (but fixed) constant  $C_0$  will suffice to ensure that (4.40) holds. We have the following convergence results.

**Lemma 4.1.** *There exists a subsequence (not indicated) of the sequence of functions  $\{(\rho_{\kappa,\alpha,L}^{[\Delta t]}, u_{\kappa,\alpha,L}^{\Delta t,+}, \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+})\}_{\Delta t > 0}$ , and functions*

$$\rho_{\kappa,\alpha} \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^1(\Omega)') \cap C_w([0, T]; L^{\frac{\Gamma}{2}}(\Omega)) \cap L^{\frac{4\Gamma}{3}}_{\geq 0}(\Omega_T) \quad (4.41)$$

with  $\rho_{\kappa,\alpha}(\cdot, 0) = \rho^0(\cdot)$  and  $u_{\kappa,\alpha} \in L^2(0, T; \widetilde{H}_0^1(\Omega))$  such that, as  $\Delta t \rightarrow 0_+$  ( $L \rightarrow \infty$ ),

$$\begin{aligned} \rho_{\kappa,\alpha,L}^{[\Delta t]} &\rightarrow \rho_{\kappa,\alpha} \quad \text{weakly in } L^2(0, T; H^1(\Omega)), \\ &\quad \text{weakly in } H^1(0, T; H^1(\Omega)'), \end{aligned} \quad (4.42a)$$

$$\begin{aligned} \rho_{\kappa,\alpha,L}^{[\Delta t]} &\rightarrow \rho_{\kappa,\alpha} \quad \text{in } C_w([0, T]; L^{\frac{\Gamma}{2}}(\Omega)), \\ &\quad \text{weakly in } L^{\frac{4\Gamma}{3}}(\Omega_T), \end{aligned} \quad (4.42b)$$

$$\left(\rho_{\kappa,\alpha,L}^{[\Delta t]}\right)^{\frac{\Gamma}{2}} \rightarrow \rho_{\kappa,\alpha}^{\frac{\Gamma}{2}} \quad \text{weakly in } L^2(0, T; H^1(\Omega)), \quad (4.42c)$$

$$\begin{aligned} \rho_{\kappa,\alpha,L}^{[\Delta t]} &\rightarrow \rho_{\kappa,\alpha} \quad \text{strongly in } L^2(0, T; L^r(\Omega)), \\ &\quad \text{strongly in } L^v(\Omega_T), \end{aligned} \quad (4.42d)$$

where  $r \in [1, \infty)$  if  $d = 2$  and  $r \in [1, 6)$  if  $d = 3$ , and  $v \in [1, \frac{4\Gamma}{3})$ ;

$$p_{\kappa}(\rho_{\kappa,\alpha,L}^{[\Delta t]}) \rightarrow p_{\kappa}(\rho_{\kappa,\alpha}) \quad \text{strongly in } L^s(\Omega_T), \quad (4.43a)$$

$$p_{\kappa,\alpha,L}^{\{\Delta t\}} \rightarrow p_{\kappa}(\rho_{\kappa,\alpha}) \quad \text{weakly in } L^s(\Omega_T), \quad (4.43b)$$

where  $s \in (1, \frac{4}{3})$ ; and

$$u_{\kappa,\alpha,L}^{\Delta t,+} \rightarrow u_{\kappa,\alpha} \quad \text{weakly in } L^2(0, T; \widetilde{H}_0^1(\Omega)). \quad (4.44)$$

Moreover, we have that

$$\begin{aligned} \int_0^T \left\langle \frac{\partial \rho_{\kappa,\alpha}}{\partial t}, \eta \right\rangle_{H^1(\Omega)} dt + \int_0^T \int_{\Omega} \left( \alpha \nabla_x \rho_{\kappa,\alpha} - \rho_{\kappa,\alpha} u_{\kappa,\alpha} \right) \cdot \nabla_x \eta \, dx \, dt = 0 \\ \forall \eta \in L^2(0, T; H^1(\Omega)). \end{aligned} \quad (4.45)$$

**Proof.** The convergence results (4.42a,b) follow immediately from (4.36), (3.12a,b) and (4.38). The strong convergence results (4.42d) follow from (4.42a), (3.11), (4.38) and the interpolation result (3.1). The weak convergence result (4.42c) is then a consequence of (4.36) and (4.42d). Therefore, we have the desired result (4.41). As  $L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^1(\Omega)') \hookrightarrow C([0, T]; L^2(\Omega))$ , we obtain that  $\rho_{\kappa,\alpha}(\cdot, 0) = \rho^0(\cdot)$ .

Next, it follows from (2.3), for  $s \in [1, \frac{4}{3}]$ , that

$$\begin{aligned} &\|p_{\kappa}(\rho_{\kappa,\alpha}) - p_{\kappa}(\rho_{\kappa,\alpha,L}^{[\Delta t]})\|_{L^s(\Omega_T)} \\ &\leq C \left[ \|\rho_{\kappa,\alpha}\|_{L^{s\Gamma}(\Omega_T)}^{\Gamma-1} + \|\rho_{\kappa,\alpha,L}^{[\Delta t]}\|_{L^{s\Gamma}(\Omega_T)}^{\Gamma-1} \right] \|\rho_{\kappa,\alpha} - \rho_{\kappa,\alpha,L}^{[\Delta t]}\|_{L^{s\Gamma}(\Omega_T)}. \end{aligned} \quad (4.46)$$

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Hence, the result (4.43a) follows from (4.46) and (4.42d).

It follows from (4.2) and (4.43a) that, for  $s \in (1, \frac{4}{3})$ ,

$$\int_{\Omega_T} p_{\kappa,\alpha,L}^{\{\Delta t\}} \eta \, dx \, dt = \int_{\Omega_T} p_{\kappa}(\rho_{\kappa,\alpha,L}^{[\Delta t]}) \eta^{\{\Delta t\}} \, dx \, dt \quad \forall \eta \in L^{\frac{s}{s-1}}(\Omega_T), \quad (4.47)$$

where

$$\eta^{\{\Delta t\}}(\cdot, t) := \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \eta(\cdot, t') \, dt', \quad t \in (t_{n-1}, t_n], \quad n = 1, \dots, N. \quad (4.48)$$

We note that

$$\lim_{\Delta t \rightarrow 0_+} \|\eta - \eta^{\{\Delta t\}}\|_{L^r(\Omega_T)} = 0 \quad \forall \eta \in L^r(\Omega_T), \quad r \in [1, \infty). \quad (4.49)$$

Therefore, the desired result (4.43b) follows from (4.47), (4.43a) and (4.49). Finally, the weak convergence result (4.44) follows immediately from (4.21).

It is now a simple matter to pass to the limit  $\Delta t \rightarrow 0_+$  ( $L \rightarrow \infty$ ) for the subsequence in (4.5a), on noting (4.42a-c) and (4.44), to obtain (4.45).  $\square$

In order to pass to the limit  $\Delta t \rightarrow 0_+$  ( $L \rightarrow \infty$ ) in the momentum equation (4.5b), we will need a strong convergence result for  $\nabla_x \rho_{\kappa,\alpha,L}^{[\Delta t]}$ . First, it follows from (4.4b), (3.18a), and (4.21) that

$$\begin{aligned} \|\nabla_x u_{\kappa,\alpha,L}^{\Delta t,-}\|_{L^2(0,T;L^2(\Omega))}^2 &= \Delta t \|\nabla_x u_{\kappa,\alpha,L}^0\|^2 + \int_{\Delta t}^T \|\nabla_x u_{\kappa,\alpha,L}^{\Delta t,-}\|^2 \, dt \\ &\leq \int_{\Omega} \rho^0 |u_0|^2 \, dx + \int_0^{T-\Delta t} \|\nabla_x u_{\kappa,\alpha,L}^{\Delta t,+}\|^2 \, dt \leq C; \end{aligned} \quad (4.50)$$

hence, we obtain from (3.2), a Poincaré inequality, (4.21), (4.50) and (4.4a,b) that

$$\|u_{\kappa,\alpha,L}^{\Delta t(\pm)}\|_{L^2(0,T;L^6(\Omega))} \leq \|u_{\kappa,\alpha,L}^{\Delta t(\pm)}\|_{L^2(0,T;H^1(\Omega))} \leq \|\nabla_x u_{\kappa,\alpha,L}^{\Delta t(\pm)}\|_{L^2(0,T;L^2(\Omega))} \leq C, \quad (4.51)$$

where  $C$  is independent of  $\Delta t$ ,  $L$ ,  $\alpha$  and  $\kappa$ .

**Lemma 4.2.** *There exists a  $C \in \mathbb{R}_{>0}$ , independent of  $\Delta t$  and  $L$ , such that*

$$\begin{aligned} &\|\rho_{\kappa,\alpha,L}^{\Delta t(\pm)}\|_{L^\infty(0,T;L^\Gamma(\Omega))} + \left\| \sqrt{\rho_{\kappa,\alpha,L}^{\Delta t,\pm}} u_{\kappa,\alpha,L}^{\Delta t,\pm} \right\|_{L^\infty(0,T;L^2(\Omega))} \\ &\quad + \|\rho_{\kappa,\alpha,L}^{\Delta t,\pm} u_{\kappa,\alpha,L}^{\Delta t,\pm}\|_{L^\infty(0,T;L^{\frac{2\Gamma}{\Gamma+1}}(\Omega))} + \left\| \sqrt{\rho_{\kappa,\alpha,L}^{\Delta t,\pm}} u_{\kappa,\alpha,L}^{\Delta t,\pm} \right\|_{L^2(0,T;L^{\frac{6\Gamma}{\Gamma+3}}(\Omega))} \leq C, \end{aligned} \quad (4.52a)$$

$$\begin{aligned} &\|\rho_{\kappa,\alpha,L}^{[\Delta t]} u_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^\infty(0,T;L^1(\Omega))} + \|\rho_{\kappa,\alpha,L}^{[\Delta t]} u_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^2(0,T;L^{\frac{6\Gamma}{\Gamma+12}}(\Omega))} \\ &\quad + \left\| \sqrt{\rho_{\kappa,\alpha,L}^{[\Delta t]}} u_{\kappa,\alpha,L}^{\Delta t,+} \right\|_{L^2(0,T;L^{\frac{6\Gamma}{\Gamma+6}}(\Omega))} + \|\rho_{\kappa,\alpha,L}^{[\Delta t]} u_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^v(\Omega_T)} \leq C, \end{aligned} \quad (4.52b)$$

$$\|\nabla_x \rho_{\kappa,\alpha,L}^{[\Delta t]}\|_{L^v(\Omega_T)} \leq C, \quad (4.52c)$$

where  $v = \frac{8\Gamma-12}{3\Gamma} \geq \frac{13}{6}$  as  $\Gamma \geq 8$ .

Hence, in addition to (4.41),  $\rho_{\kappa,\alpha} \in L^\infty(0, T; L^\Gamma(\Omega))$  and for a further subsequence of the subsequence of Lemma 4.1, it follows that, as  $\Delta t \rightarrow 0_+$  ( $L \rightarrow \infty$ ),

$$\begin{aligned} \widetilde{\nabla}_x \rho_{\kappa,\alpha,L}^{[\Delta t]} &\rightharpoonup \widetilde{\nabla}_x \rho_{\kappa,\alpha} \quad \text{weakly in } L^v(\Omega_T), \\ &\text{strongly in } L^2(\Omega_T), \end{aligned} \quad (4.53a)$$

$$\begin{aligned} \rho_{\kappa,\alpha,L}^{\Delta t,+} &\rightharpoonup \rho_{\kappa,\alpha} \quad \text{weakly-}\star \text{ in } L^\infty(0, T; L^\Gamma(\Omega)), \\ &\text{strongly in } L^2(0, T; H^1(\Omega)'), \end{aligned} \quad (4.53b)$$

and, for any nonnegative  $\eta \in C[0, T]$ ,

$$\int_0^T \left( \int_\Omega P_\kappa(\rho_{\kappa,\alpha}) \, dx \right) \eta \, dt \leq \liminf_{\Delta t \rightarrow 0_+ (L \rightarrow \infty)} \int_0^T \left( \int_\Omega P_\kappa(\rho_{\kappa,\alpha,L}^{\Delta t,+}) \, dx \right) \eta \, dt. \quad (4.53c)$$

**Proof.** The first two bounds in (4.52a) follow immediately from (4.4a,b), (4.21), (3.5), (3.16a) and (3.18a). The third bound in (4.52a) follows immediately from the first two on noting that

$$\|\rho_{\kappa,\alpha,L}^{\Delta t,\pm} u_{\kappa,\alpha,L}^{\Delta t,\pm}\|_{L^{\frac{2\Gamma}{\Gamma+1}}(\Omega)} \leq \left\| \sqrt{\rho_{\kappa,\alpha,L}^{\Delta t,\pm}} \right\|_{L^{2\Gamma}(\Omega)} \left\| \sqrt{\rho_{\kappa,\alpha,L}^{\Delta t,\pm}} u_{\kappa,\alpha,L}^{\Delta t,\pm} \right\|_{L^2(\Omega)}. \quad (4.54)$$

It follows from the first bound in (4.52a), (3.2) and (4.51) that

$$\begin{aligned} &\left\| \sqrt{\rho_{\kappa,\alpha,L}^{\Delta t,\pm}} u_{\kappa,\alpha,L}^{\Delta t,\pm} \right\|_{L^2(0, T; L^{\frac{6\Gamma}{\Gamma+3}}(\Omega))} \\ &\leq \left\| \sqrt{\rho_{\kappa,\alpha,L}^{\Delta t,\pm}} \right\|_{L^\infty(0, T; L^{2\Gamma}(\Omega))} \|u_{\kappa,\alpha,L}^{\Delta t,\pm}\|_{L^2(0, T; L^6(\Omega))} \leq C, \end{aligned} \quad (4.55)$$

and hence the fourth bound in (4.52a).

Next we note, for any  $\eta \in L^2(0, T; H^1(\Omega))$  and for a.a.  $s \in (t_{n-1}, t_n]$ , that

$$\begin{aligned} &\left| \int_\Omega \left[ \rho_{\kappa,\alpha,L}^{[\Delta t]}(x, s) - \rho_{\kappa,\alpha,L}^{\Delta t,+}(x, s) \right] \eta(x, s) \, dx \right| = \left| \int_s^{t_n} \int_\Omega \eta(x, s) \frac{\partial \rho_{\kappa,\alpha,L}^{[\Delta t]}(x, t)}{\partial t} \, dx \, dt \right| \\ &\leq \|\eta(\cdot, s)\|_{H^1(\Omega)} \int_{t_{n-1}}^{t_n} \left\| \frac{\partial \rho_{\kappa,\alpha,L}^{[\Delta t]}}{\partial t} \right\|_{H^1(\Omega)'} \, dt. \end{aligned} \quad (4.56)$$

It follows from (4.56) with  $\eta = |u_{\kappa,\alpha,L}^{\Delta t,+}|$ , (4.52a), (4.36) and (4.21) that

$$\begin{aligned} &\|\rho_{\kappa,\alpha,L}^{[\Delta t]} u_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^\infty(0, T; L^1(\Omega))} \leq \|\rho_{\kappa,\alpha,L}^{\Delta t,+} u_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^\infty(0, T; L^1(\Omega))} \\ &+ \frac{1}{2} \left[ \left\| \frac{\partial \rho_{\kappa,\alpha,L}^{[\Delta t]}}{\partial t} \right\|_{L^2(0, T; H^1(\Omega)')}^2 + \|u_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^2(0, T; H^1(\Omega))}^2 \right] \leq C, \end{aligned} \quad (4.57)$$

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and hence the first desired bound in (4.52b). Similarly to (4.55), it follows from (4.36), (3.2) and (4.51) that

$$\|\rho_{\kappa,\alpha,L}^{[\Delta t]} u_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^2(0,T;L^{\frac{6\Gamma}{\Gamma+12}}(\Omega))} \leq \|\rho_{\kappa,\alpha,L}^{[\Delta t]}\|_{L^\infty(0,T;L^{\frac{\Gamma}{2}}(\Omega))} \|u_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^2(0,T;L^6(\Omega))}, \quad (4.58a)$$

$$\left\| \sqrt{\rho_{\kappa,\alpha,L}^{[\Delta t]}} u_{\kappa,\alpha,L}^{\Delta t,+} \right\|_{L^2(0,T;L^{\frac{6\Gamma}{\Gamma+6}}(\Omega))} \leq \left\| \sqrt{\rho_{\kappa,\alpha,L}^{[\Delta t]}} \right\|_{L^\infty(0,T;L^\Gamma(\Omega))} \|u_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^2(0,T;L^6(\Omega))}, \quad (4.58b)$$

and hence the second and third bounds in (4.52b). It follows from (3.1) with  $v = \frac{8\Gamma-12}{3\Gamma}$ ,  $r = 1$  and  $s = \frac{6\Gamma}{\Gamma+12}$  that  $v\vartheta = 2$  (with  $\vartheta \in (0, 1)$  for  $\Gamma \geq 8$ ) and so

$$\|\rho_{\kappa,\alpha,L}^{[\Delta t]} u_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^v(\Omega_T)} \leq \|\rho_{\kappa,\alpha,L}^{[\Delta t]} u_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^\infty(0,T;L^1(\Omega))}^{1-\vartheta} \|\rho_{\kappa,\alpha,L}^{[\Delta t]} u_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^2(0,T;L^s(\Omega))}^{\vartheta}. \quad (4.59)$$

Thus, (4.59) and the first two bounds in (4.52b) yield the fourth bound in (4.52b). On noting this bound and recalling from (3.4) that  $\partial\Omega \in C^{2,\theta}$ ,  $\theta \in (0, 1)$ , and  $\rho^0 \in L^\infty(\Omega)$  satisfying (3.16a), we can now apply the parabolic regularity result, Lemma 7.38 in Novotný & Straškraba<sup>24</sup> (or Lemma G.2 in Appendix G in Barrett & Süli<sup>12</sup>), to (4.5a) to obtain that the solution  $\rho_{\kappa,\alpha,L}^{[\Delta t]}$  satisfies the bound (4.52c).

The first desired result in (4.53a) follows immediately from (4.52c). Next we obtain from (4.33) for  $\vartheta = 2$  that, for any  $s \in (0, T]$ ,

$$\begin{aligned} & \frac{1}{2} \|\rho_{\kappa,\alpha,L}^{[\Delta t]}(\cdot, s)\|_{L^2(\Omega)}^2 + \alpha \int_0^s \|\nabla_x \rho_{\kappa,\alpha,L}^{[\Delta t]}\|_{L^2(\Omega)}^2 dt \\ &= \frac{1}{2} \left[ \|\rho^0\|_{L^2(\Omega)}^2 - \int_0^s \int_\Omega (\nabla_x \cdot u_{\kappa,\alpha,L}^{\Delta t,+}) (\rho_{\kappa,\alpha,L}^{[\Delta t]})^2 dx dt \right]. \end{aligned} \quad (4.60)$$

Integrating (4.60) over  $s \in (0, T)$ , and performing integration by parts, yields that

$$\begin{aligned} & \frac{1}{2} \|\rho_{\kappa,\alpha,L}^{[\Delta t]}\|_{L^2(\Omega_T)}^2 + \alpha \int_0^T (T-t) \|\nabla_x \rho_{\kappa,\alpha,L}^{[\Delta t]}\|_{L^2(\Omega)}^2 dt \\ &= \frac{1}{2} \left[ T \|\rho^0\|_{L^2(\Omega)}^2 - \int_0^T (T-t) \int_\Omega (\nabla_x \cdot u_{\kappa,\alpha,L}^{\Delta t,+}) (\rho_{\kappa,\alpha,L}^{[\Delta t]})^2 dx dt \right]. \end{aligned} \quad (4.61)$$

Similarly, on choosing for any  $s \in (0, T]$ ,  $\eta(\cdot, t) = \chi_{[0,s]} \rho_{\kappa,\alpha}(\cdot, t)$  in (4.45), and integrating over  $s \in (0, T)$  yields that

$$\begin{aligned} & \frac{1}{2} \|\rho_{\kappa,\alpha}\|_{L^2(\Omega_T)}^2 + \alpha \int_0^T (T-t) \|\nabla_x \rho_{\kappa,\alpha}\|_{L^2(\Omega)}^2 dt \\ &= \frac{1}{2} \left[ T \|\rho^0\|_{L^2(\Omega)}^2 - \int_0^T (T-t) \int_\Omega (\nabla_x \cdot u_{\kappa,\alpha}) (\rho_{\kappa,\alpha})^2 dx dt \right]. \end{aligned} \quad (4.62)$$

We deduce from (4.61), (4.62), (4.42d) (with  $v = 4$ ) and (4.44) that

$$\lim_{\Delta t \rightarrow 0_+} \int_0^T (T-t) \|\nabla_x \rho_{\kappa,\alpha,L}^{[\Delta t]}\|_{L^2(\Omega)}^2 dt = \int_0^T (T-t) \|\nabla_x \rho_{\kappa,\alpha}\|_{L^2(\Omega)}^2 dt. \quad (4.63)$$



By applying the elementary identity  $|a - b|^2 = |a|^2 - |b|^2 - 2(a - b) \cdot b$  with  $\underline{a} = \nabla_x \rho_{\kappa, \alpha, L}^{[\Delta t]}$  and  $\underline{b} = \nabla_x \rho_{\kappa, \alpha, L}$ , it follows from (4.63) and (4.42a) that

$$\lim_{\Delta t \rightarrow 0_+} \int_0^T (T - t) \|\nabla_x (\rho_{\kappa, \alpha, L} - \rho_{\kappa, \alpha, L}^{[\Delta t]})\|_{L^2(\Omega)}^2 dt = 0, \quad (4.64a)$$

and hence, for a.a.  $t \in (0, T)$ ,

$$\|\nabla_x (\rho_{\kappa, \alpha, L} - \rho_{\kappa, \alpha, L}^{[\Delta t]})(\cdot, t)\|_{L^2(\Omega)}^2 \rightarrow 0, \quad \text{as } \Delta t \rightarrow 0_+ \text{ (} L \rightarrow \infty \text{)}. \quad (4.64b)$$

Therefore, we obtain the second desired result (4.53a) from (4.64b), (4.52c) and Vitali's convergence theorem. The details of the argument are as follows. With  $v \geq \frac{13}{6} > 2$ , the bound (4.52c) implies that  $|\nabla_x \rho_{\kappa, \alpha, L}^{[\Delta t]}|^2$  is equi-integrable in  $L^1(\Omega_T)$ , i.e.,  $\nabla_x \rho_{\kappa, \alpha, L}^{[\Delta t]}$  is 2-equi-integrable. Further, thanks to (4.64a), a subsequence of  $\nabla_x \rho_{\kappa, \alpha, L}^{[\Delta t]}$  is a.e. convergent on  $\Omega_T$  (cf. Theorem 2.20 (iii) in Fonseca & Leoni<sup>20</sup>), and thus by Egoroff's theorem (cf. Theorem 2.22 in Fonseca & Leoni<sup>20</sup>) it also converges in measure. Hence, by Vitali's convergence theorem (cf. Theorem 2.24 in Fonseca & Leoni<sup>20</sup>, with  $p = 2$ .) we have strong convergence of the subsequence (not indicated).

The first stated convergence result in (4.53b) follows directly from the first bound in (4.52a). Next, it follows from (4.56) and (4.36) that

$$\|\rho_{\kappa, \alpha, L}^{[\Delta t]} - \rho_{\kappa, \alpha, L}^{\Delta t, +}\|_{L^2(0, T; H^1(\Omega)')}^2 \leq (\Delta t)^2 \left\| \frac{\partial \rho_{\kappa, \alpha, L}^{[\Delta t]}}{\partial t} \right\|_{L^2(0, T; H^1(\Omega)')}^2 \leq C (\Delta t)^2. \quad (4.65)$$

Hence the desired convergence results (4.53b) follow immediately from (4.65) and (4.42d) with  $r > \frac{2d}{d+2}$  (to ensure that  $L^r(\Omega) \subset H^1(\Omega)'$ ).

Finally, it follows for any nonnegative  $\eta \in C[0, T]$ , on noting the convexity of  $P_\kappa(\cdot)$ , that

$$\begin{aligned} & \int_0^T \left( \int_\Omega P_\kappa(\rho_{\kappa, \alpha, L}^{\Delta t, +}) dx \right) \eta dt \\ & \geq \int_0^T \left( \int_\Omega \left[ P_\kappa(\rho_{\kappa, \alpha}) + P'_\kappa(\rho_{\kappa, \alpha}) (\rho_{\kappa, \alpha, L}^{\Delta t, +} - \rho_{\kappa, \alpha}) \right] dx \right) \eta dt. \end{aligned} \quad (4.66)$$

This yields the desired result (4.53c) on noting (4.53b) and that  $P'_\kappa(\rho_{\kappa, \alpha}) \in L^1(0, T; L^{\frac{r}{r-1}}(\Omega))$  by (4.41).  $\square$

The following result is also required to pass to the limit  $\Delta t \rightarrow 0_+$  ( $L \rightarrow \infty$ ) in the momentum equation (4.5b).

**Lemma 4.3.** *There exists a  $C \in \mathbb{R}_{>0}$ , independent of  $\Delta t$  and  $L$ , such that*

$$\|\nabla_x \cdot (\rho_{\kappa, \alpha, L}^{[\Delta t]} u_{\kappa, \alpha, L}^{\Delta t, +})\|_{L^s(\Omega_T)} + \left\| \frac{\partial \rho_{\kappa, \alpha, L}^{[\Delta t]}}{\partial t} \right\|_{L^s(\Omega_T)} + \|\Delta_x \rho_{\kappa, \alpha, L}^{[\Delta t]}\|_{L^s(\Omega_T)} \leq C, \quad (4.67)$$

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where  $s = \frac{8\Gamma-12}{7\Gamma-6} \geq \frac{26}{25}$  as  $\Gamma \geq 8$ . In addition, we have that

$$\lim_{\Delta t \rightarrow 0_+} \lim_{(L \rightarrow \infty)} \|\rho_{\kappa,\alpha,L}^{[\Delta t]} u_{\kappa,\alpha,L}^{\Delta t,+} - \rho_{\kappa,\alpha,L}^{\Delta t,-} u_{\kappa,\alpha,L}^{\Delta t,-}\|_{L^2(\Omega_T)} = 0, \quad (4.68a)$$

$$\lim_{\Delta t \rightarrow 0_+} \lim_{(L \rightarrow \infty)} \|\rho_{\kappa,\alpha,L}^{\Delta t,+} u_{\kappa,\alpha,L}^{\Delta t,+} - \rho_{\kappa,\alpha,L}^{\Delta t,-} u_{\kappa,\alpha,L}^{\Delta t,-}\|_{L^2(\Omega_T)} = 0. \quad (4.68b)$$

**Proof.** It follows from (4.34) and (4.52c) that

$$\|\rho_{\kappa,\alpha,L}^{[\Delta t]}\|_{L^{\frac{8\Gamma-12}{3\Gamma}}(\Omega_T)} + \|\nabla_x \rho_{\kappa,\alpha,L}^{[\Delta t]}\|_{L^{\frac{8\Gamma-12}{3\Gamma}}(\Omega_T)} \leq C, \quad (4.69)$$

where we have noted that  $\frac{8\Gamma-12}{3\Gamma} < \Gamma$ , with  $\Gamma \geq 8$ . Hence, (4.69) and (4.51) yield that

$$\|\nabla_x \rho_{\kappa,\alpha,L}^{[\Delta t]} \nabla_x u_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^s(\Omega_T)} \leq \|\nabla_x \rho_{\kappa,\alpha,L}^{[\Delta t]}\|_{L^{\frac{8\Gamma-12}{3\Gamma}}(\Omega_T)} \|\nabla_x u_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^2(\Omega_T)} \leq C, \quad (4.70)$$

where  $s = \frac{8\Gamma-12}{7\Gamma-6}$ . From the first bound in (4.69) and (4.70), we obtain the first bound in (4.67). As  $\partial\Omega \in C^{2,\theta}$ ,  $\theta \in (0, 1)$ , it follows from (3.15), (3.16a) and elliptic regularity that, for all  $r \in [1, \infty)$ ,

$$\|\rho^0\|_{W^{2,r}(\Omega)} \leq C(\alpha) \|\rho_0 - \rho^0\|_{L^\infty(\Omega)} \leq C(\alpha) \quad \text{and} \quad \nabla_x \rho^0 \cdot \tilde{n} = 0 \text{ on } \partial\Omega. \quad (4.71)$$

On noting (4.71) and the first bound in (4.67), we can now apply to (4.5a) the parabolic regularity result, Lemma 7.37 in Novotný & Straškraba<sup>24</sup> (or Lemma G.1 in Appendix G in Barrett & Süli<sup>12</sup>), to obtain that the solution  $\rho_{\kappa,\alpha,L}^{[\Delta t]}$  satisfies the last two bounds in (4.67).

Next, we note that

$$\begin{aligned} \|\rho_{\kappa,\alpha,L}^{[\Delta t]} u_{\kappa,\alpha,L}^{\Delta t,+} - \rho_{\kappa,\alpha,L}^{\Delta t,-} u_{\kappa,\alpha,L}^{\Delta t,-}\|_{L^2(\Omega_T)}^2 &= \int_0^T \int_\Omega |\rho_{\kappa,\alpha,L}^{[\Delta t]} u_{\kappa,\alpha,L}^{\Delta t,+} - \rho_{\kappa,\alpha,L}^{\Delta t,-} u_{\kappa,\alpha,L}^{\Delta t,-}|^2 dx dt \\ &= \int_0^T \int_\Omega \left| \sqrt{\rho_{\kappa,\alpha,L}^{[\Delta t]}} \rho_{\kappa,\alpha,L}^{\Delta t,-} (u_{\kappa,\alpha,L}^{\Delta t,+} - u_{\kappa,\alpha,L}^{\Delta t,-}) \right. \\ &\quad \left. + \left( \sqrt{\rho_{\kappa,\alpha,L}^{[\Delta t]}} - \sqrt{\rho_{\kappa,\alpha,L}^{\Delta t,-}} \right) \left( \sqrt{\rho_{\kappa,\alpha,L}^{[\Delta t]}} u_{\kappa,\alpha,L}^{\Delta t,+} + \sqrt{\rho_{\kappa,\alpha,L}^{\Delta t,-}} u_{\kappa,\alpha,L}^{\Delta t,-} \right) \right|^2 dx dt \\ &\leq 2 \int_0^T \int_\Omega \rho_{\kappa,\alpha,L}^{[\Delta t]} \rho_{\kappa,\alpha,L}^{\Delta t,-} |u_{\kappa,\alpha,L}^{\Delta t,+} - u_{\kappa,\alpha,L}^{\Delta t,-}|^2 dx dt \\ &\quad + 2 \int_0^T \int_\Omega \left| \sqrt{\rho_{\kappa,\alpha,L}^{[\Delta t]}} - \sqrt{\rho_{\kappa,\alpha,L}^{\Delta t,-}} \right|^2 \left| \sqrt{\rho_{\kappa,\alpha,L}^{[\Delta t]}} u_{\kappa,\alpha,L}^{\Delta t,+} + \sqrt{\rho_{\kappa,\alpha,L}^{\Delta t,-}} u_{\kappa,\alpha,L}^{\Delta t,-} \right|^2 dx dt \\ &\leq 2 \int_0^T \int_\Omega \rho_{\kappa,\alpha,L}^{[\Delta t]} \rho_{\kappa,\alpha,L}^{\Delta t,-} |u_{\kappa,\alpha,L}^{\Delta t,+} - u_{\kappa,\alpha,L}^{\Delta t,-}|^2 dx dt \\ &\quad + 2 \int_0^T \int_\Omega \left| \rho_{\kappa,\alpha,L}^{[\Delta t]} - \rho_{\kappa,\alpha,L}^{\Delta t,-} \right| \left| \sqrt{\rho_{\kappa,\alpha,L}^{[\Delta t]}} u_{\kappa,\alpha,L}^{\Delta t,+} + \sqrt{\rho_{\kappa,\alpha,L}^{\Delta t,-}} u_{\kappa,\alpha,L}^{\Delta t,-} \right|^2 dx dt. \end{aligned} \quad (4.72)$$

It follows from (4.36) and (4.21) that

$$\begin{aligned} & \left\| \sqrt{\rho_{\kappa,\alpha,L}^{[\Delta t]}} \sqrt{\rho_{\kappa,\alpha,L}^{\Delta t,-}} \left( u_{\kappa,\alpha,L}^{\Delta t,+} - u_{\kappa,\alpha,L}^{\Delta t,-} \right) \right\|_{L^2(0,T;L^{\frac{2\Gamma}{\Gamma+2}}(\Omega))} \\ & \leq \left\| \sqrt{\rho_{\kappa,\alpha,L}^{[\Delta t]}} \right\|_{L^\infty(0,T;L^\Gamma(\Omega))} \left\| \sqrt{\rho_{\kappa,\alpha,L}^{\Delta t,-}} \left( u_{\kappa,\alpha,L}^{\Delta t,+} - u_{\kappa,\alpha,L}^{\Delta t,-} \right) \right\|_{L^2(\Omega_T)} \\ & \leq C (\Delta t)^{\frac{1}{2}}. \end{aligned} \quad (4.73)$$

Similarly, it follows from (4.51), (4.36) and (4.52a) and that

$$\begin{aligned} & \left\| \sqrt{\rho_{\kappa,\alpha,L}^{[\Delta t]}} \sqrt{\rho_{\kappa,\alpha,L}^{\Delta t,-}} \left( u_{\kappa,\alpha,L}^{\Delta t,+} - u_{\kappa,\alpha,L}^{\Delta t,-} \right) \right\|_{L^2(0,T;L^{\frac{6\Gamma}{\Gamma+9}}(\Omega))} \\ & \leq \left\| \sqrt{\rho_{\kappa,\alpha,L}^{[\Delta t]}} \rho_{\kappa,\alpha,L}^{\Delta t,-} \right\|_{L^\infty(0,T;L^{\frac{2\Gamma}{3}}(\Omega))} \left\| u_{\kappa,\alpha,L}^{\Delta t,+} - u_{\kappa,\alpha,L}^{\Delta t,-} \right\|_{L^2(0,T;L^6(\Omega))} \\ & \leq C \left\| \sqrt{\rho_{\kappa,\alpha,L}^{[\Delta t]}} \right\|_{L^\infty(0,T;L^\Gamma(\Omega))} \left\| \sqrt{\rho_{\kappa,\alpha,L}^{\Delta t,-}} \right\|_{L^\infty(0,T;L^{2\Gamma}(\Omega))} \leq C. \end{aligned} \quad (4.74)$$

We deduce from (4.73), (4.74) and (3.1) as  $\frac{6\Gamma}{\Gamma+9} > 2 > \frac{2\Gamma}{\Gamma+2}$  that

$$\lim_{\Delta t \rightarrow 0_+} \lim_{(L \rightarrow \infty)} \left\| \sqrt{\rho_{\kappa,\alpha,L}^{[\Delta t]}} \sqrt{\rho_{\kappa,\alpha,L}^{\Delta t,-}} \left( u_{\kappa,\alpha,L}^{\Delta t,+} - u_{\kappa,\alpha,L}^{\Delta t,-} \right) \right\|_{L^2(\Omega_T)} = 0, \quad (4.75)$$

and so the first term on the right-hand side of (4.72) converges to zero, as  $\Delta t \rightarrow 0_+$  ( $L \rightarrow \infty$ ).

Next, we deal with the second term on the right-hand side of (4.72). It follows from (4.67) that

$$\|\rho_{\kappa,\alpha,L}^{[\Delta t]} - \rho_{\kappa,\alpha,L}^{\Delta t,-}\|_{L^s(\Omega_T)} \leq \Delta t \left\| \frac{\partial \rho_{\kappa,\alpha,L}^{[\Delta t]}}{\partial t} \right\|_{L^s(\Omega_T)} \leq C \Delta t, \quad (4.76)$$

and from (4.36) and (4.52a) that

$$\|\rho_{\kappa,\alpha,L}^{[\Delta t]} - \rho_{\kappa,\alpha,L}^{\Delta t,-}\|_{L^\infty(0,T;L^{\frac{\Gamma}{2}}(\Omega))} \leq C. \quad (4.77)$$

Hence, the bounds (4.76) and (4.77) yield, on noting (3.1) and as  $\Gamma \geq 8$ , that

$$\lim_{\Delta t \rightarrow 0_+} \lim_{(L \rightarrow \infty)} \|\rho_{\kappa,\alpha,L}^{[\Delta t]} - \rho_{\kappa,\alpha,L}^{\Delta t,-}\|_{L^v(0,T;L^r(\Omega))} = 0, \quad \text{for any } v \in [1, \infty), r \in [1, 4). \quad (4.78)$$

As  $\frac{6\Gamma}{\Gamma+3} \geq \frac{48}{11} > 4$  for  $\Gamma \geq 8$ , it follows from (3.1) and (4.52a) that

$$\begin{aligned} & \left\| \sqrt{\rho_{\kappa,\alpha,L}^{\Delta t,\pm}} u_{\kappa,\alpha,L}^{\Delta t,\pm} \right\|_{L^3(0,T;L^3(\Omega))} \\ & \leq \left\| \sqrt{\rho_{\kappa,\alpha,L}^{\Delta t,\pm}} u_{\kappa,\alpha,L}^{\Delta t,\pm} \right\|_{L^\infty(0,T;L^2(\Omega))}^{\frac{1}{3}} \left\| \sqrt{\rho_{\kappa,\alpha,L}^{\Delta t,\pm}} u_{\kappa,\alpha,L}^{\Delta t,\pm} \right\|_{L^2(0,T;L^4(\Omega))}^{\frac{2}{3}} \leq C. \end{aligned} \quad (4.79)$$

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In addition, (4.52b) and (4.51) yield that

$$\begin{aligned} & \left\| \sqrt{\rho_{\kappa,\alpha,L}^{[\Delta t]}} u_{\kappa,\alpha,L}^{\Delta t,+} \right\|_{L^4(0,T;L^{\frac{12}{7}}(\Omega))} \\ & \leq \left\| \rho_{\kappa,\alpha,L}^{[\Delta t]} u_{\kappa,\alpha,L}^{\Delta t,+} \right\|_{L^\infty(0,T;L^1(\Omega))}^{\frac{1}{2}} \left\| u_{\kappa,\alpha,L}^{\Delta t,+} \right\|_{L^2(0,T;L^6(\Omega))}^{\frac{1}{2}} \leq C. \end{aligned} \quad (4.80)$$

As  $\frac{6\Gamma}{\Gamma+6} \geq \frac{24}{7}$  for  $\Gamma \geq 8$ , it then follows from (3.1), (4.80) and (4.52b) that

$$\begin{aligned} & \left\| \sqrt{\rho_{\kappa,\alpha,L}^{[\Delta t]}} u_{\kappa,\alpha,L}^{\Delta t,+} \right\|_{L^{\frac{28}{13}}(0,T;L^3(\Omega))} \\ & \leq \left( \int_0^T \left\| \sqrt{\rho_{\kappa,\alpha,L}^{[\Delta t]}} u_{\kappa,\alpha,L}^{\Delta t,+} \right\|_{L^{\frac{12}{7}}(\Omega)}^{\frac{4}{13}} \left\| \sqrt{\rho_{\kappa,\alpha,L}^{[\Delta t]}} u_{\kappa,\alpha,L}^{\Delta t,+} \right\|_{L^{\frac{24}{7}}(\Omega)}^{\frac{24}{13}} dt \right)^{\frac{13}{8}} \\ & \leq \left\| \sqrt{\rho_{\kappa,\alpha,L}^{[\Delta t]}} u_{\kappa,\alpha,L}^{\Delta t,+} \right\|_{L^4(0,T;L^{\frac{12}{7}}(\Omega))}^{\frac{1}{7}} \left\| \sqrt{\rho_{\kappa,\alpha,L}^{[\Delta t]}} u_{\kappa,\alpha,L}^{\Delta t,+} \right\|_{L^2(0,T;L^{\frac{24}{7}}(\Omega))}^{\frac{6}{7}} \leq C. \end{aligned} \quad (4.81)$$

Combining (4.78), (4.79) and (4.81) yields that the second term on the right-hand side of (4.72) converges to zero, as  $\Delta t \rightarrow 0_+$  ( $L \rightarrow \infty$ ). Therefore, we have the desired result (4.68a).

We now adapt the argument above to prove the desired result (4.68b). The bounds (4.72)–(4.75) remain true with  $\rho_{\kappa,\alpha,L}^{[\Delta t]}$  replaced by  $\rho_{\kappa,\alpha,L}^{\Delta t,+}$ . Similarly, (4.76) remains true with  $\rho_{\kappa,\alpha,L}^{[\Delta t]}$  on the left-hand side of the inequality replaced by  $\rho_{\kappa,\alpha,L}^{\Delta t,+}$ . Hence, the bounds (4.77) and (4.78) remain true with  $\rho_{\kappa,\alpha,L}^{[\Delta t]}$  replaced by  $\rho_{\kappa,\alpha,L}^{\Delta t,+}$ . Therefore, on combining all these modified bounds with (4.79), we obtain the desired result (4.68b).  $\square$

#### 4.4. $L$ , $\Delta t$ -independent bounds on the time-derivatives of $\tilde{m}_{\kappa,\alpha,L}^{\Delta t}$ and $\hat{\psi}_{\kappa,\alpha,L}^{\Delta t}$

On noting from (4.4a,b) that

$$\tilde{m}_{\kappa,\alpha,L}^{\Delta t} = \frac{t - t_{n-1}}{\Delta t} \tilde{m}_{\kappa,\alpha,L}^{\Delta t,+} + \frac{t_n - t}{\Delta t} \tilde{m}_{\kappa,\alpha,L}^{\Delta t,-}, \quad t \in (t_{n-1}, t_n], \quad n = 1, \dots, N, \quad (4.82)$$

an elementary calculation yields, for any  $s \in [1, \infty]$ , that

$$\int_0^T \|\tilde{m}_{\kappa,\alpha,L}^{\Delta t}\|_{L^s(\Omega)}^2 dt \leq \frac{1}{2} \int_0^T \left( \|\tilde{m}_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^s(\Omega)}^2 + \|\tilde{m}_{\kappa,\alpha,L}^{\Delta t,-}\|_{L^s(\Omega)}^2 \right) dt. \quad (4.83)$$

In order to pass to the limit  $\Delta t \rightarrow 0_+$  ( $L \rightarrow \infty$ ) in the momentum equation (4.5b), we require the following result.

**Lemma 4.4.** *There exists a  $C \in \mathbb{R}_{>0}$ , independent of  $\Delta t$  and  $L$ , such that*

$$\|\tilde{m}_{\kappa,\alpha,L}^{\Delta t(\pm)}\|_{L^\infty(0,T;L^{\frac{2\Gamma}{\Gamma+1}}(\Omega))} + \|\tilde{m}_{\kappa,\alpha,L}^{\Delta t(\pm)}\|_{L^2(0,T;L^{\frac{6\Gamma}{\Gamma+6}}(\Omega))} + \|\tilde{m}_{\kappa,\alpha,L}^{\Delta t(\pm)}\|_{L^v(\Omega_T)} \leq C, \quad (4.84a)$$

$$\|\tilde{m}_{\kappa,\alpha,L}^{\Delta t(\pm)} \otimes u_{\kappa,\alpha,L}^{\Delta t(\pm)}\|_{L^2(0,T;L^{\frac{6\Gamma}{4\Gamma+3}}(\Omega))} \leq C, \quad (4.84b)$$

where  $v = \frac{10\Gamma-6}{3(\Gamma+1)} \geq \frac{74}{27}$  as  $\Gamma \geq 8$ .

**Proof.** The first bound in (4.84a) for  $\tilde{m}_{\kappa,\alpha,L}^{\Delta t,\pm}$  follows immediately from the third bound in (4.52a). The corresponding bound for  $\tilde{m}_{\kappa,\alpha,L}^{\Delta t}$  is a direct consequence of (4.82). Similarly to (4.58a), it follows from (4.52a) and (4.51) that

$$\|\rho_{\kappa,\alpha,L}^{\Delta t,\pm} \tilde{u}_{\kappa,\alpha,L}^{\Delta t,\pm}\|_{L^2(0,T;L^{\frac{6\Gamma}{\Gamma+6}}(\Omega))} \leq \|\rho_{\kappa,\alpha,L}^{\Delta t,\pm}\|_{L^\infty(0,T;L^\Gamma(\Omega))} \|\tilde{u}_{\kappa,\alpha,L}^{\Delta t,\pm}\|_{L^2(0,T;L^6(\Omega))} \leq C, \quad (4.85)$$

and hence the second bound in (4.84a) for  $\tilde{m}_{\kappa,\alpha,L}^{\Delta t,\pm}$ . The corresponding bound for  $\tilde{m}_{\kappa,\alpha,L}^{\Delta t}$  is then a direct consequence of (4.83). Similarly to (4.59), it follows from (3.1) with  $v = \frac{10\Gamma-6}{3(\Gamma+1)}$ ,  $r = \frac{2\Gamma}{\Gamma+1}$  and  $s = \frac{6\Gamma}{\Gamma+6}$  that  $v\vartheta = 2$  (with  $\vartheta \in (0, 1)$ ) thanks to  $\Gamma \geq 8$ , and so

$$\|\tilde{m}_{\kappa,\alpha,L}^{\Delta t(\pm)}\|_{L^v(\Omega_T)} \leq \|\tilde{m}_{\kappa,\alpha,L}^{\Delta t(\pm)}\|_{L^\infty(0,T;L^r(\Omega))}^{1-\vartheta} \|\tilde{m}_{\kappa,\alpha,L}^{\Delta t(\pm)}\|_{L^2(0,T;L^s(\Omega))}^\vartheta. \quad (4.86)$$

Hence (4.86) and the first two bounds in (4.84a) yield the third bound in (4.84a). Finally, combining the first bound in (4.84a) and (4.51) yields (4.84b).  $\square$

Next, we need to bound the time-derivative of  $\tilde{m}_{\kappa,\alpha,L}^{\Delta t}$  independently of  $\Delta t$  and  $L$ . It follows from (4.39) that we will need to choose at least  $w \in L^4(0, T; W_0^{1,4}(\Omega))$  in (4.5b). We now rewrite the time-derivative of  $\rho_{\kappa,\alpha,L}^{\Delta t}$  in (4.5b) using (4.5a). Adopting the notation (4.48), we have for any  $w \in L^4(0, T; W_0^{1,4}(\Omega))$  that  $\tilde{u}_{\kappa,\alpha,L}^{\Delta t,+} \cdot w^{\{\Delta t\}} \in L^2(0, T; H^1(\Omega))$ , and so (4.5b) yields that

$$\begin{aligned} - \int_0^T \int_\Omega \frac{\partial \rho_{\kappa,\alpha,L}^{\Delta t}}{\partial t} \tilde{u}_{\kappa,\alpha,L}^{\Delta t,+} \cdot w \, dx \, dt &= - \int_0^T \left\langle \frac{\partial \rho_{\kappa,\alpha,L}^{\Delta t}}{\partial t}, \tilde{u}_{\kappa,\alpha,L}^{\Delta t,+} \cdot w^{\{\Delta t\}} \right\rangle_{H^1(\Omega)} dt \\ &= \int_0^T \int_\Omega \left( \alpha \nabla_x \rho_{\kappa,\alpha,L}^{\Delta t} - \rho_{\kappa,\alpha,L}^{\Delta t} \tilde{u}_{\kappa,\alpha,L}^{\Delta t,+} \right) \cdot \nabla_x (\tilde{u}_{\kappa,\alpha,L}^{\Delta t,+} \cdot w^{\{\Delta t\}}) \, dx \, dt \\ &= \alpha \int_0^T \int_\Omega \nabla_x \rho_{\kappa,\alpha,L}^{\Delta t} \cdot \nabla_x (\tilde{u}_{\kappa,\alpha,L}^{\Delta t,+} \cdot w^{\{\Delta t\}}) \, dx \, dt \\ &\quad - \int_0^T \int_\Omega \rho_{\kappa,\alpha,L}^{\Delta t} \tilde{u}_{\kappa,\alpha,L}^{\Delta t,+} \cdot \nabla_x (\tilde{u}_{\kappa,\alpha,L}^{\Delta t,+} \cdot w) \, dx \, dt \\ &\quad + \int_0^T \int_\Omega \rho_{\kappa,\alpha,L}^{\Delta t} \tilde{u}_{\kappa,\alpha,L}^{\Delta t,+} \cdot \nabla_x (\tilde{u}_{\kappa,\alpha,L}^{\Delta t,+} \cdot (w - w^{\{\Delta t\}})) \, dx \, dt. \end{aligned} \quad (4.87)$$

Next we note that, for all  $w \in L^4(0, T; W_0^{1,4}(\Omega))$ ,

$$\begin{aligned} &\int_0^T \int_\Omega \rho_{\kappa,\alpha,L}^{\Delta t} \tilde{u}_{\kappa,\alpha,L}^{\Delta t,+} \cdot \nabla_x (\tilde{u}_{\kappa,\alpha,L}^{\Delta t,+} \cdot w) \, dx \, dt \\ &= \int_0^T \int_\Omega \tilde{m}_{\kappa,\alpha,L}^{\Delta t,-} \cdot \nabla_x (\tilde{u}_{\kappa,\alpha,L}^{\Delta t,+} \cdot w) \, dx \, dt \\ &\quad + \int_0^T \int_\Omega \left( \rho_{\kappa,\alpha,L}^{\Delta t} \tilde{u}_{\kappa,\alpha,L}^{\Delta t,+} - \rho_{\kappa,\alpha,L}^{\Delta t,-} \tilde{u}_{\kappa,\alpha,L}^{\Delta t,-} \right) \cdot \nabla_x (\tilde{u}_{\kappa,\alpha,L}^{\Delta t,+} \cdot w) \, dx \, dt. \end{aligned} \quad (4.88)$$

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Therefore, on combining (4.5b), (4.87) and (4.88), one can rewrite (4.5b) as

$$\begin{aligned}
& \int_0^T \int_{\Omega} \left[ \frac{\partial m_{\kappa,\alpha,L}^{\Delta t}}{\partial t} \cdot w + \frac{\alpha}{2} \nabla_x \rho_{\kappa,\alpha,L}^{[\Delta t]} \cdot \nabla_x (u_{\kappa,\alpha,L}^{\Delta t,+} \cdot w^{\{\Delta t\}}) \right. \\
& \quad \left. - [(m_{\kappa,\alpha,L}^{\Delta t,-} \cdot \nabla_x) w] \cdot u_{\kappa,\alpha,L}^{\Delta t,+} \right] dx dt \\
& + \int_0^T \int_{\Omega} S(u_{\kappa,\alpha,L}^{\Delta t,+}) : \nabla_x w dx dt - \int_0^T \int_{\Omega} p_{\kappa,\alpha,L}^{\{\Delta t\}} \nabla_x \cdot w dx dt \\
& - \frac{1}{2} \int_0^T \int_{\Omega} \left( \rho_{\kappa,\alpha,L}^{[\Delta t]} u_{\kappa,\alpha,L}^{\Delta t,+} - \rho_{\kappa,\alpha,L}^{\Delta t,-} u_{\kappa,\alpha,L}^{\Delta t,-} \right) \cdot \nabla_x (u_{\kappa,\alpha,L}^{\Delta t,+} \cdot w) dx dt \\
& + \frac{1}{2} \int_0^T \int_{\Omega} \rho_{\kappa,\alpha,L}^{[\Delta t]} u_{\kappa,\alpha,L}^{\Delta t,+} \cdot \nabla_x (u_{\kappa,\alpha,L}^{\Delta t,+} \cdot (w - w^{\{\Delta t\}})) dx dt \\
& = \int_0^T \int_{\Omega} \left[ \rho_{\kappa,\alpha,L}^{\Delta t,+} f^{\{\Delta t\}} \cdot w - \tau_1(M \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}) : \nabla_x w \right] dx dt \\
& \quad - 2\mathfrak{z} \int_0^T \int_{\Omega} \left( \int_D M \beta^L(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}) dq \right) \nabla_x \rho_{\kappa,\alpha,L}^{\Delta t,+} \cdot w dx dt \\
& \qquad \qquad \qquad \forall w \in L^4(0, T; W_0^{1,4}(\Omega)). \quad (4.89)
\end{aligned}$$

We have the following result.

**Lemma 4.5.** *There exists a  $C \in \mathbb{R}_{>0}$ , independent of  $\Delta t$  and  $L$ , such that*

$$\left\| \frac{\partial m_{\kappa,\alpha,L}^{\Delta t}}{\partial t} \right\|_{L^s(0,T;W_0^{1,4}(\Omega)')} \leq C, \quad (4.90)$$

where  $s = \frac{8\Gamma-12}{7\Gamma-6} \geq \frac{26}{25}$  as  $\Gamma \geq 8$ .

**Proof.** Let  $v = \frac{8\Gamma-12}{3\Gamma} \geq \frac{13}{6}$  as in Lemma 4.2. Then for  $s' = \frac{s}{s-1} = \frac{8\Gamma-12}{\Gamma-6} > 8$ , we have that

$$\frac{1}{s'} + \frac{1}{v} + \frac{1}{2} = 1. \quad (4.91)$$

It follows from (4.89), (4.91),  $W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$ ,  $H^1(\Omega) \hookrightarrow L^6(\Omega)$ , (4.52a-c), (4.51), (4.84a), on noting that  $\frac{10\Gamma-6}{3(\Gamma+1)} \geq \frac{8\Gamma-12}{7\Gamma-6}$ , (4.39), (4.28b), (1.19), (4.21) and (3.29a) that, for any  $w \in L^{s'}(0, T; W_0^{1,4}(\Omega))$ ,

$$\begin{aligned}
& \left| \int_0^T \int_{\Omega} \frac{\partial m_{\kappa,\alpha,L}^{\Delta t}}{\partial t} \cdot w dx dt \right| \\
& \leq C \|\nabla_x \rho_{\kappa,\alpha,L}^{[\Delta t]}\|_{L^v(\Omega_T)} \|u_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^2(0,T;H^1(\Omega))} \|w^{\{\Delta t\}}\|_{L^{s'}(0,T;W_0^{1,4}(\Omega))}
\end{aligned}$$

$$\begin{aligned}
& + C \left[ \|\rho_{\kappa,\alpha,L}^{[\Delta t]} \tilde{u}_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^v(\Omega_T)} + \|\tilde{m}_{\kappa,\alpha,L}^{\Delta t,-}\|_{L^v(\Omega_T)} + 1 \right] \\
& \quad \times \|\tilde{u}_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^2(0,T;H^1(\Omega))} \|\tilde{w}\|_{L^{s'}(0,T;W^{1,4}(\Omega))} \\
& + C \|\rho_{\kappa,\alpha,L}^{[\Delta t]} \tilde{u}_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^v(\Omega_T)} \|\tilde{u}_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^2(0,T;H^1(\Omega))} \|\tilde{w}^{\{\Delta t\}}\|_{L^{s'}(0,T;W^{1,4}(\Omega))} \\
& + \|\tilde{p}_{\kappa,\alpha,L}^{\{\Delta t\}}\|_{L^{\frac{4}{3}}(\Omega_T)} \|\tilde{w}\|_{L^4(0,T;W^{1,4}(\Omega))} \\
& + C \|\tau_1(M \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+})\|_{L^2(0,T;L^{\frac{4}{3}}(\Omega))} \|\tilde{w}\|_{L^2(0,T;W^{1,4}(\Omega))} \\
& + \|\varrho_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^\infty(0,T;L^2(\Omega))} \|\varrho_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^2(0,T;H^1(\Omega))} \|\tilde{w}\|_{L^2(0,T;L^\infty(\Omega))} \\
& + \|\rho_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^\infty(0,T;L^2(\Omega))} \|\tilde{f}^{\{\Delta t\}}\|_{L^2(0,T;L^\infty(\Omega))} \|\tilde{w}\|_{L^2(\Omega_T)} \\
& \leq C \left[ \|\tilde{w}\|_{L^{s'}(0,T;W^{1,4}(\Omega))} + \|\tilde{w}^{\{\Delta t\}}\|_{L^{s'}(0,T;W^{1,4}(\Omega))} \right]. \tag{4.92}
\end{aligned}$$

The desired result (4.90) then follows immediately from (4.92) on noting, as  $s' \geq 4$ , that, for any  $w \in L^{s'}(0,T;W_0^{1,4}(\Omega))$ ,

$$\begin{aligned}
\|\tilde{w}^{\{\Delta t\}}\|_{L^{s'}(0,T;W^{1,4}(\Omega))}^{s'} & = \Delta t \sum_{n=1}^N \left\| \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \tilde{w} \, dt \right\|_{W^{1,4}(\Omega)}^{s'} \\
& \leq C \Delta t \sum_{n=1}^N \left( \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \|\tilde{w}\|_{W^{1,4}(\Omega)}^4 \, dt \right)^{\frac{s'}{4}} \leq C \|\tilde{w}\|_{L^{s'}(0,T;W^{1,4}(\Omega))}^{s'}. \tag{4.93}
\end{aligned}$$

That completes the proof.  $\square$

Next, we bound the time derivative of  $\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t}$ .

**Lemma 4.6.** *There exists a  $C \in \mathbb{R}_{>0}$ , independent of  $\Delta t$  and  $L$ , such that*

$$\left\| M \frac{\partial \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t}}{\partial t} \right\|_{L^2(0,T;H^s(\Omega \times D)')} \leq C, \tag{4.94}$$

where  $s > 1 + \frac{1}{2}(K+1)d$ .

**Proof.** It follows from (4.5c), (4.21), (1.19) and (4.27) that, for any  $\varphi \in L^2(0,T;W^{1,\infty}(\Omega \times D))$ ,

$$\begin{aligned}
& \left| \int_0^T \int_{\Omega \times D} M \frac{\partial \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t}}{\partial t} \varphi \, dq \, dx \, dt \right| \\
& \leq 2\varepsilon \left| \int_0^T \int_{\Omega \times D} M \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}} \nabla_x \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}} \cdot \nabla_x \varphi \, dq \, dx \, dt \right| \\
& \quad + \frac{1}{2\lambda} \left| \sum_{i=1}^K \sum_{j=1}^K A_{ij} \int_0^T \int_{\Omega \times D} M \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}} \nabla_{q_j} \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}} \cdot \nabla_{q_i} \varphi \, dq \, dx \, dt \right|
\end{aligned}$$

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$$\begin{aligned}
& + \left| \int_0^T \int_{\Omega \times D} M u_{\kappa, \alpha, L}^{\Delta t, +} \beta^L(\widehat{\psi}_{\kappa, \alpha, L}^{\Delta t, +}) \cdot \nabla_x \varphi \, dq \, dx \, dt \right| \\
& + \left| \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \left[ \sigma(u_{\kappa, \alpha, L}^{\Delta t, +}) q_i \right] \beta^L(\widehat{\psi}_{\kappa, \alpha, L}^{\Delta t, +}) \cdot \nabla_{q_i} \varphi \, dq \, dx \, dt \right| \\
& \leq C \max \left\{ 1, \|\varrho_{\kappa, \alpha, L}^{\Delta t, +}\|_{L^\infty(0, T; L^2(\Omega))} \right\} \left[ \|\nabla_x \sqrt{\widehat{\psi}_{\kappa, \alpha, L}^{\Delta t, +}}\|_{L^2(0, T; L_M^2(\Omega \times D))} \right. \\
& \quad \left. + \|\nabla_q \sqrt{\widehat{\psi}_{\kappa, \alpha, L}^{\Delta t, +}}\|_{L^2(0, T; L_M^2(\Omega \times D))} + \|u_{\kappa, \alpha, L}^{\Delta t, +}\|_{L^2(0, T; H^1(\Omega))} \right] \|\varphi\|_{L^2(0, T; W^{1, \infty}(\Omega \times D))} \\
& \leq C \|\varphi\|_{L^2(0, T; W^{1, \infty}(\Omega \times D))}. \tag{4.95}
\end{aligned}$$

The desired result (4.94) then follows on noting that  $H^s(\Omega \times D) \hookrightarrow W^{1, \infty}(\Omega \times D)$  for the stated bound on  $s$ .  $\square$

**Remark 4.2.** We note that allowing  $\mathfrak{z} = 0$  would impact on the proof of Lemma 4.5. As is already clear from the formal energy inequality (1.18), by setting  $\mathfrak{z} = 0$  one loses control over the  $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$  norm of  $\varrho_{\kappa, \alpha, L}^{\Delta t, +}$ ; instead, one can only control weaker norms of  $\varrho_{\kappa, \alpha, L}^{\Delta t, +}$ , leading to (4.32) in place of (4.28b). While this weaker control is not sufficient to prove (4.90) as stated, one can prove a weaker result by replacing the  $W^{1, 4}(\Omega)$  norm on the test function  $w$  by the  $W^{1, 2d}(\Omega)$  norm throughout the proof. Admitting  $\mathfrak{z} = 0$  in Lemma 4.6, on the other hand, results in unsurmountable difficulties: the proof of the lemma cannot be completed without an  $L^r(0, T; L^2(\Omega))$  norm bound on  $\varrho_{\kappa, \alpha, L}^{\Delta t, +}$ , with  $r > 2$  at least; in particular, we are unable to prove (4.94), or a weaker result on the time-difference of  $\widehat{\psi}_{\kappa, \alpha, L}^{\Delta t}$ , when  $\mathfrak{z} = 0$ . Remark 4.1, and equation (4.31) in particular, indicate that an  $L^r(0, T; L^2(\Omega))$  norm bound on  $\varrho_{\kappa, \alpha, L}^{\Delta t, +}$ , with  $r > 2$ , is unlikely to hold without requiring  $\mathfrak{z} > 0$ , regardless of the choice of  $\mathfrak{k}$  in (1.15).

#### 4.5. *Passing to the limit $\Delta t \rightarrow 0_+$ ( $L \rightarrow \infty$ ) in the momentum equation (4.5b) and the Fokker–Planck equation (4.5c)*

We are now ready to pass to the limit  $\Delta t \rightarrow 0_+$  ( $L \rightarrow \infty$ ) in the momentum equation (4.5b) and the Fokker–Planck equation (4.5c). We have the following convergence results.

**Lemma 4.7.** *We have that*

$$\widetilde{m}_{\kappa, \alpha} := \rho_{\kappa, \alpha} u_{\kappa, \alpha} \in L^v(\Omega_T) \cap W^{1, s}(0, T; W_0^{1, 4}(\Omega)') \cap C_w([0, T]; L^{\frac{2\Gamma}{\Gamma+1}}(\Omega)), \tag{4.96}$$

where  $v = \frac{10\Gamma-6}{3(\Gamma+1)}$  and  $s = \frac{8\Gamma-12}{7\Gamma-6}$ , with  $\Gamma \geq 8$ . In addition, for a further subsequence



of the subsequences of Lemmas 4.1 and 4.2 it follows that, as  $\Delta t \rightarrow 0_+$  ( $L \rightarrow \infty$ ),

$$\tilde{m}_{\kappa,\alpha,L}^{\Delta t(\pm)} \rightarrow \tilde{m}_{\kappa,\alpha} \quad \text{weakly in } \tilde{L}^v(\Omega_T), \quad (4.97a)$$

$$\tilde{m}_{\kappa,\alpha,L}^{\Delta t} \rightarrow \tilde{m}_{\kappa,\alpha} \quad \text{weakly in } W^{1,s}(0,T; \tilde{W}_0^{1,4}(\Omega)'), \quad (4.97b)$$

$$\tilde{m}_{\kappa,\alpha,L}^{\Delta t(\pm)} \rightarrow \tilde{m}_{\kappa,\alpha} \quad \text{strongly in } L^2(0,T; H^1(\Omega)'), \quad (4.97c)$$

$$\tilde{m}_{\kappa,\alpha,L}^{\Delta t(\pm)} \otimes \tilde{u}_{\kappa,\alpha,L}^{\Delta t,+} \rightarrow \tilde{m}_{\kappa,\alpha} \otimes \tilde{u}_{\kappa,\alpha} \quad \text{weakly in } L^2(0,T; L^{\frac{6\Gamma}{4\Gamma+3}}(\Omega)), \quad (4.97d)$$

$$\tilde{m}_{\kappa,\alpha,L}^{\Delta t(\pm)} \rightarrow \tilde{m}_{\kappa,\alpha} \quad \text{in } C_w([0,T]; L^{\frac{2\Gamma}{\Gamma+1}}(\Omega)). \quad (4.97e)$$

**Proof.** The weak convergence result (4.97a) for some limit function  $\tilde{m}_{\kappa,\alpha} \in \tilde{L}^v(\Omega_T)$ , which is the common limit of  $\tilde{m}_{\kappa,\alpha,L}^{\Delta t}$ ,  $\tilde{m}_{\kappa,\alpha,L}^{\Delta t,+}$  and  $\tilde{m}_{\kappa,\alpha,L}^{\Delta t,-}$ , follows immediately from (4.84a), (4.68b), (4.3) and (4.82). The weak convergence result (4.97b) and the strong convergence result (4.97c) for  $\tilde{m}_{\kappa,\alpha,L}^{\Delta t}$  follow immediately for  $\tilde{m}_{\kappa,\alpha} \in \tilde{L}^v(\Omega_T) \cap W^{1,s}(0,T; \tilde{W}_0^{1,4}(\Omega)')$  from (4.84a), (4.90) and (3.11), on noting that  $L^v(\Omega)$  is compactly embedded in  $H^1(\Omega)'$ , which is in turn continuously embedded in  $W^{1,4}(\Omega)'$ . The corresponding result (4.97c) for  $\tilde{m}_{\kappa,\alpha,L}^{\Delta t(\pm)}$  then follows from (4.97c) for  $\tilde{m}_{\kappa,\alpha,L}^{\Delta t}$ , and (4.68b). It follows from (4.53b), (4.44) and (4.97c) for  $\tilde{m}_{\kappa,\alpha,L}^{\Delta t,+}$  that  $\tilde{m}_{\kappa,\alpha} = \rho_{\kappa,\alpha} \tilde{u}_{\kappa,\alpha}$ , and hence (4.96). The result (4.97d) follows immediately from (4.84b), (4.97c) and (4.44). Finally (4.97e) follows from the bound on the first term in (4.84a), (4.90) and (3.12a,b).  $\square$

Next, noting (4.4a,b), a simple calculation yields that [see (6.32)–(6.34) in Barrett & Süli<sup>5</sup> for details]:

$$\begin{aligned} & \int_0^T \int_{\Omega \times D} M |\tilde{\nabla}_x \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t}}|^2 \, dq \, dx \, dt \\ & \leq 2 \int_0^T \int_{\Omega \times D} M \left[ |\tilde{\nabla}_x \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}}|^2 + |\tilde{\nabla}_x \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,-}}|^2 \right] \, dq \, dx \, dt, \end{aligned} \quad (4.98)$$

and an analogous result with  $\tilde{\nabla}_x$  replaced by  $\tilde{\nabla}_q$ . Then, the bound (4.21), on noting (3.21a), (4.98) and the convexity of  $\mathcal{F}$ , imply the existence of a  $C \in \mathbb{R}_{>0}$ , independent of  $\Delta t$  and  $L$ , such that:

$$\begin{aligned} & \text{ess. sup}_{t \in [0,T]} \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t(\pm)}(t)) \, dq \, dx \\ & + \frac{1}{\Delta t L} \int_0^T \int_{\Omega \times D} M (\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} - \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,-})^2 \, dq \, dx \, dt \\ & + \int_0^T \int_{\Omega \times D} M |\tilde{\nabla}_x \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t(\pm)}}|^2 \, dq \, dx \, dt + \int_0^T \int_{\Omega \times D} M |\tilde{\nabla}_q \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t(\pm)}}|^2 \, dq \, dx \, dt \\ & \leq C. \end{aligned} \quad (4.99)$$

**Lemma 4.8.** *For a further subsequence of the subsequences of Lemmas 4.1, 4.2*

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and 4.7, there exists a function

$$\widehat{\psi}_{\kappa,\alpha} \in L^v(0, T; Z_1) \cap H^1(0, T; M^{-1}(H^s(\Omega \times D))'), \quad (4.100a)$$

where  $v \in [1, \infty)$  and  $s > 1 + \frac{1}{2}(K+1)d$ , with finite relative entropy and Fisher information, i.e.,

$$\mathcal{F}(\widehat{\psi}_{\kappa,\alpha}) \in L^\infty(0, T; L_M^1(\Omega \times D)) \quad \text{and} \quad \sqrt{\widehat{\psi}_{\kappa,\alpha}} \in L^2(0, T; H_M^1(\Omega \times D)), \quad (4.100b)$$

such that, as  $\Delta t \rightarrow 0_+$  ( $L \rightarrow \infty$ ),

$$M^{\frac{1}{2}} \underset{\sim}{\nabla}_x \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t(\pm)}} \rightarrow M^{\frac{1}{2}} \underset{\sim}{\nabla}_x \sqrt{\widehat{\psi}_{\kappa,\alpha}} \quad \text{weakly in } L^2(0, T; L^2(\Omega \times D)), \quad (4.101a)$$

$$M^{\frac{1}{2}} \underset{\sim}{\nabla}_q \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t(\pm)}} \rightarrow M^{\frac{1}{2}} \underset{\sim}{\nabla}_q \sqrt{\widehat{\psi}_{\kappa,\alpha}} \quad \text{weakly in } L^2(0, T; L^2(\Omega \times D)), \quad (4.101b)$$

$$M \frac{\partial \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t}}{\partial t} \rightarrow M \frac{\partial \widehat{\psi}_{\kappa,\alpha}}{\partial t} \quad \text{weakly in } L^2(0, T; H^s(\Omega \times D)'), \quad (4.101c)$$

$$\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t(\pm)} \rightarrow \widehat{\psi}_{\kappa,\alpha} \quad \text{strongly in } L^v(0, T; L_M^1(\Omega \times D)), \quad (4.101d)$$

$$\beta^L(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t(\pm)}) \rightarrow \widehat{\psi}_{\kappa,\alpha} \quad \text{strongly in } L^v(0, T; L_M^1(\Omega \times D)), \quad (4.101e)$$

$$\tau(M \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}) \rightarrow \tau(M \widehat{\psi}_{\kappa,\alpha}) \quad \text{strongly in } L^r(\Omega_T), \quad (4.101f)$$

where  $r \in [1, \frac{4(d+2)}{3d+4})$ ; and, for a.a.  $t \in (0, T)$ ,

$$\begin{aligned} & \int_{\Omega \times D} M(q) \mathcal{F}(\widehat{\psi}_{\kappa,\alpha}(x, q, t)) \, dq \, dx \\ & \leq \liminf_{\Delta t \rightarrow 0_+ (L \rightarrow \infty)} \int_{\Omega \times D} M(q) \mathcal{F}(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}(x, q, t)) \, dq \, dx. \end{aligned} \quad (4.101g)$$

In addition, we have that

$$\varrho_{\kappa,\alpha} := \int_D M \widehat{\psi}_{\kappa,\alpha} \, dq \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad (4.102)$$

and, as  $\Delta t \rightarrow 0_+$  ( $L \rightarrow \infty$ ),

$$\begin{aligned} \varrho_{\kappa,\alpha,L}^{\Delta t,+} & \rightarrow \varrho_{\kappa,\alpha} \quad \text{weakly-}\star \text{ in } L^\infty(0, T; L^2(\Omega)), \\ & \text{weakly in } L^2(0, T; H^1(\Omega)), \end{aligned} \quad (4.103a)$$

$$\varrho_{\kappa,\alpha,L}^{\Delta t,+} \int_D M \beta^L(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}) \, dq \rightarrow \varrho_{\kappa,\alpha} \quad \text{strongly in } L^{\frac{5\varsigma}{3(\varsigma-1)}}(0, T; L^\varsigma(\Omega)), \quad (4.103b)$$

for any  $\varsigma \in (1, 6)$ .

**Proof.** In order to prove the strong convergence result (4.101d), we will apply Dubinskii's compactness theorem (Theorem 3.1 in Barrett & Süli<sup>12</sup>) with  $\mathfrak{X} = L_M^1(\Omega \times D)$ ,  $\mathfrak{X}_1 = M^{-1} H^s(\Omega \times D)'$  and

$$\mathfrak{M} = \left\{ \varphi \in Z_1 : \int_{\Omega \times D} M \left[ |\underset{\sim}{\nabla}_q \sqrt{\varphi}|^2 + |\underset{\sim}{\nabla}_x \sqrt{\varphi}|^2 \right] \, dq \, dx < \infty \right\}. \quad (4.104)$$

See Section 5 in Barrett & Süli<sup>6</sup> for the proof of the compactness of the embedding  $\mathfrak{M} \hookrightarrow \mathfrak{X}$ , and the continuity of the embedding  $\mathfrak{X} \hookrightarrow \mathfrak{X}_1$ . Hence, the desired result (4.101d) for  $\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t}$  and  $v = 1$  follows from Dubinskiĭ's compactness theorem (cf. Theorem 3.1 in Barrett & Süli<sup>12</sup>). The desired result (4.101d) for  $\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,\pm}$  and  $v = 1$  then follows from (4.101d) for  $\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t}$  and  $v = 1$ , (4.82) with  $m_{\kappa,\alpha,L}^{\Delta t(\pm)}$  replaced by  $\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t(\pm)}$ , the second bound in (4.99) and (4.40). The desired result (4.101d) for  $v \in (1, \infty)$  then follows from (4.101d) for  $v = 1$ , the first bound in (4.99) (note that  $\mathcal{F}(s) \geq [s - e + 1]_+$  for all  $s \geq 0$ , since  $\mathcal{F}(s) \geq 0$  and, by convexity of  $\mathcal{F}$ ,  $\mathcal{F}(s) \geq \mathcal{F}(e) + (s - e)\mathcal{F}'(e) = s - e + 1$ ) and an interpolation result, see Lemma 5.1 in Barrett & Süli<sup>6</sup>. The weak convergence result (4.101c) follows immediately from (4.94). The weak convergence results (4.101a,b) follow immediately from the last two terms in (4.99), on noting an argument similar to that in the proof of Lemma 3.3 in Barrett & Süli<sup>6</sup> in order to identify the limit. The result (4.101e) follows from (4.101d) and the Lipschitz continuity of  $\beta^L$ , see (5.8) in Barrett & Süli<sup>9</sup> for details. The result (4.101g) follows from (4.101d) and Fatou's lemma, see (6.46) in Barrett & Süli<sup>6</sup> for details. In addition, the convergence results (4.101a–d,g) yield the desired results (4.100a,b).

The results (4.103a) for some limit function  $\varrho_{\kappa,\alpha}$  follow immediately from the bounds on  $\varrho_{\kappa,\alpha,L}^{\Delta t,+}$  in (4.21). The fact that  $\varrho_{\kappa,\alpha} = \int_D M \widehat{\psi}_{\kappa,\alpha} dq$  follows from (3.91), (4.4b) and (4.101d), and hence the desired result (4.102). The strong convergence results (4.103b) follow from noting the embedding  $H^1(\Omega) \hookrightarrow L^6(\Omega)$ , (1.19), (4.101d,e) and (3.1).

Finally, we need to prove (4.101f). Similarly to (4.25)–(4.28a,b), we deduce from (4.100b), (4.102), (3.2) and (4.24) that

$$\|\varrho_{\kappa,\alpha}\|_{L^{\frac{2(d+2)}{d}}(\Omega_T)} + \|\tau_1(M \widehat{\psi}_{\kappa,\alpha})\|_{L^{\frac{4(d+2)}{3d+4}}(\Omega_T)} \leq C. \quad (4.105)$$

On recalling (1.11) and (4.22), we have that

$$\tau_1(M \varphi) \underset{\approx}{=} k \int_D M \left( \sum_{i=1}^K \nabla_{q_i} \varphi \otimes q_i \right) dq - k \left( \int_D M \varphi dq \right) I. \quad (4.106)$$

Let  $D_0 \subset \overline{D_0} \subset D$  be an arbitrary Lipschitz subdomain of  $D$ , then (4.106) yields that

$$\begin{aligned} & \int_{\Omega_T} \left| \tau_1(M \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}) - \tau_1(M \widehat{\psi}_{\kappa,\alpha}) \right| dx dt \leq C [T |\Omega \setminus \Omega_0|]^{\frac{d+4}{4(d+2)}} \\ & \leq k |b|_{\frac{1}{2}} \int_0^T \int_{\Omega} \int_{D \setminus D_0} M \sum_{i=1}^K \left( |\nabla_{q_i} \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}| + |\nabla_{q_i} \widehat{\psi}_{\kappa,\alpha}| \right) dq dx dt \\ & \quad + k \int_0^T \int_{\Omega} \left| \int_{D_0} M \sum_{i=1}^K q_i \otimes \nabla_{q_i} (\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} - \widehat{\psi}_{\kappa,\alpha}) dq \right| dx dt \\ & \quad + k d^{\frac{1}{2}} \int_0^T \int_{\Omega \times D} M |\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} - \widehat{\psi}_{\kappa,\alpha}| dq dx dt =: \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3, \end{aligned} \quad (4.107)$$

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where we have recalled (3.4). Further, we deduce from (4.101b,d) that

$$M \nabla_{\sim q_i} \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} = 2M \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}} \nabla_{\sim q_i} \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}} \rightarrow 2M \sqrt{\widehat{\psi}_{\kappa,\alpha}} \nabla_{\sim q_i} \sqrt{\widehat{\psi}_{\kappa,\alpha}} = M \nabla_{\sim q_i} \widehat{\psi}_{\kappa,\alpha},$$

for  $i = 1, \dots, K$ ,

weakly in  $L^1(0, T; \underline{L}^1(\Omega \times D)) = \underline{L}^1(\Omega_T \times D)$  as  $\Delta t \rightarrow 0_+$  ( $L \rightarrow \infty$ ). By the Dunford–Pettis theorem the sequence  $\{M \nabla_{\sim q} \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}\}_{\Delta t > 0}$  is therefore equi-integrable in  $\underline{L}^1(\Omega_T \times D)$ ; hence, for any  $\delta > 0$  there exists a  $\delta_0 = \delta_0(\delta)$  such that for any set  $D_0 \subset D$  with  $T|\Omega||D \setminus D_0| < \delta_0$ ,

$$k |b|_1^{\frac{1}{2}} \int_0^T \int_{\Omega} \int_{D \setminus D_0} M \sum_{i=1}^K \left( |\nabla_{\sim q} \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}| + |\nabla_{\sim q} \widehat{\psi}_{\kappa,\alpha}| \right) dq \, d\mathbf{x} \, dt < \frac{\delta}{3}.$$

We therefore select  $D_0 \subset \overline{D_0} \subset D$  to be a Lipschitz subdomain of  $D$  such that  $T|\Omega||D \setminus D_0| < \delta_0$ , which implies that  $0 < \mathbf{T}_1 < \frac{\delta}{3}$ ; that, now, fixes our choice of  $D_0$ .

Next, we bound  $\mathbf{T}_2$ . By performing partial integration over  $D_0$ , we have that

$$\begin{aligned} \mathbf{T}_2 &= k \int_0^T \int_{\Omega} \left| \int_{D_0} M \sum_{i=1}^K q_i \otimes \nabla_{\sim q_i} (\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} - \widehat{\psi}_{\kappa,\alpha}) dq \right| dx \, dt \\ &\leq k \int_0^T \int_{\Omega} \left| - \int_{D_0} \sum_{i=1}^K \left( \nabla_{\sim q_i} M \otimes q_i \right) (\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} - \widehat{\psi}_{\kappa,\alpha}) dq \right| dx \, dt \\ &\quad + k \int_0^T \int_{\Omega} \left| -K \left[ \int_{D_0} M (\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} - \widehat{\psi}_{\kappa,\alpha}) dq \right] I \right| dx \, dt \\ &\quad + k \int_0^T \int_{\Omega} \left| \int_{\partial D_0} \sum_{i=1}^K M (n_i \otimes q_i) (\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} - \widehat{\psi}_{\kappa,\alpha}) d\sigma(q) \right| dx \, dt, \end{aligned}$$

where the  $d$ -component column vector  $n_i$  is the  $i$ th component of the  $Kd$ -component unit outward (column) normal vector  $\mathbf{n} = (n_1^T, \dots, n_K^T)^T$  to the boundary  $\partial D_0$  of  $D_0$ . As the closure of the Lipschitz subdomain  $D_0$  is a strict subset of the open set  $D$ , we have, on noting (1.6a) and (1.7a,b), that  $\sup_{q \in D_0} \left( \frac{1}{M(q)} |\nabla_{\sim q} M(q)| \right) \leq C(\delta_0) < \infty$ . Hence,

$$\begin{aligned} \mathbf{T}_2 &\leq k \int_0^T \int_{\Omega} \int_{D_0} \left[ |b|_1^{\frac{1}{2}} |\nabla_{\sim q} M| + K d^{\frac{1}{2}} M \right] |\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} - \widehat{\psi}_{\kappa,\alpha}| dq \, dx \, dt \\ &\quad + k |b|_1^{\frac{1}{2}} \int_0^T \int_{\Omega} \int_{\partial D_0} M |\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} - \widehat{\psi}_{\kappa,\alpha}| d\sigma(q) \, dx \, dt \\ &\leq k \left( |b|_1^{\frac{1}{2}} C(\delta_0) + K d^{\frac{1}{2}} \right) \int_0^T \int_{\Omega} \int_{D_0} M |\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} - \widehat{\psi}_{\kappa,\alpha}| dq \, dx \, dt \\ &\quad + k |b|_1^{\frac{1}{2}} \|M\|_{L^\infty(D)} \int_0^T \int_{\Omega} \int_{\partial D_0} |\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} - \widehat{\psi}_{\kappa,\alpha}| d\sigma(q) \, dx \, dt \\ &=: \mathbf{T}_{21} + \mathbf{T}_{22}. \end{aligned} \tag{4.108}$$

Thus, thanks to (4.101d) with  $v = 1$ , there exists a  $\Delta t_0$  such that for all  $\Delta t \leq \Delta t_0$ , we have that  $0 < \mathsf{T}_{21} < \frac{\delta}{6}$  and  $0 < \mathsf{T}_3 < \frac{\delta}{3}$ .

Finally, we shall show that, for  $\Delta t_0$  sufficiently small, also  $0 < \mathsf{T}_{22} < \frac{\delta}{6}$ . In the process of doing so we shall repeatedly use the following result. As the closure of  $D_0$  is a compact subset of  $D$ , we have from (1.7a) that

$$\sup_{\tilde{q} \in D_0} [M(\tilde{q})]^{-1} \leq C(D_0) < \infty. \quad (4.109)$$

We begin by noting that (4.99) and (4.109) imply that  $\{\sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}}\}_{\Delta t > 0}$  is a bounded sequence in  $L^2(0, T; H^1(\Omega \times D_0))$ ; hence, by Sobolev embedding, it is also a bounded sequence in  $L^2(0, T; L^{\frac{2((K+1)d}{(K+1)d-2}}(\Omega \times D_0))$ . Further, by (4.99) and (4.109) we have that  $\{\sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}}\}_{\Delta t > 0}$  is a bounded sequence in the function space  $L^\infty(0, T; L^2(\Omega \times D_0))$ . It then follows from (3.1) that  $\{\sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}}\}_{\Delta t > 0}$  is a bounded sequence in the space  $L^{\frac{2((K+1)d+2)}{(K+1)d}}(0, T; L^{\frac{2((K+1)d+2)}{(K+1)d}}(\Omega \times D_0))$ ; thus,

$$\int_0^T \int_{\Omega \times D_0} |\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}|^{\frac{(K+1)d+2}{(K+1)d}} \, d\tilde{q} \, d\tilde{x} \, dt \leq C(D_0), \quad (4.110)$$

where the constant  $C(D_0)$  is independent of  $\Delta t$  and  $L$ . Now, for any  $s \in (1, 2)$ , we have by Hölder's inequality, (4.99), (4.109) and the inequality  $(a^{\frac{s}{2}} + b^{\frac{s}{2}}) \leq 2^{1-\frac{s}{2}}(a+b)^{\frac{s}{2}}$  with  $a, b \geq 0$ , which follows from the concavity of the function  $x \in [0, \infty) \mapsto x^{\frac{s}{2}} \in [0, \infty)$ , that

$$\begin{aligned} & \int_0^T \int_{\Omega \times D_0} \left( |\nabla_x \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}|^s + |\nabla_q \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}|^s \right) \, d\tilde{q} \, d\tilde{x} \, dt \\ &= 2^s \int_0^T \int_{\Omega \times D_0} |\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}|^{\frac{s}{2}} \left( \left| \nabla_x \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}} \right|^s + \left| \nabla_q \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}} \right|^s \right) \, d\tilde{q} \, d\tilde{x} \, dt \\ &\leq 2^{\frac{s}{2}+1} \left( \int_0^T \int_{\Omega \times D_0} |\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}|^{\frac{s}{2-s}} \, d\tilde{q} \, d\tilde{x} \, dt \right)^{\frac{2-s}{2}} \\ &\quad \times \left( \int_0^T \int_{\Omega \times D_0} \left| \nabla_x \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}} \right|^2 + \left| \nabla_q \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}} \right|^2 \, d\tilde{q} \, d\tilde{x} \, dt \right)^{\frac{s}{2}} \\ &\leq C \left( \int_0^T \int_{\Omega \times D_0} |\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}|^{\frac{s}{2-s}} \, d\tilde{q} \, d\tilde{x} \, dt \right)^{\frac{2-s}{2}}. \end{aligned}$$

Comparing this with (4.110) indicates that  $s \in (1, 2)$  should be chosen so that

$$\frac{s}{2-s} \leq \frac{(K+1)d+2}{(K+1)d}.$$

The largest such  $s$  is  $s = \frac{(K+1)d+2}{(K+1)d+1}$ ; using this value of  $s$ , we then deduce on noting (4.110) that

$$\|\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}\|_{L^s(0,T;W^{1,s}(\Omega \times D_0))} \leq C(D_0). \quad (4.111)$$

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Note further that, thanks to (4.109) and (4.101d), we have, for any  $v \in [1, \infty)$ , that

$$\|\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} - \widehat{\psi}_{\kappa,\alpha}\|_{L^v(0,T;L^1(\Omega \times D_0))} \rightarrow 0, \quad \text{as } \Delta t \rightarrow 0_+ \ (L \rightarrow \infty). \quad (4.112)$$

We shall now use (4.111) and (4.112) to show that  $\mathbf{T}_{22}$  converges to 0 as  $\Delta t \rightarrow 0_+$  ( $L \rightarrow \infty$ ). To this end, we shall make use of the following sharp trace inequality, established recently by Auchmuty (cf. Theorem 6.3 inequality (6.3) in Auchmuty<sup>2</sup>): suppose that  $\mathcal{O}$  is a bounded Lipschitz domain and let  $r \in (1, 2)$ ; then, the following inequality holds for all  $\varphi \in W^{1,r}(\mathcal{O})$ :

$$\int_{\partial\mathcal{O}} |\varphi|^{2-\frac{1}{r}} d\sigma \leq \frac{|\partial\mathcal{O}|}{|\mathcal{O}|} \|\varphi\|_{L^{2-\frac{1}{r}}(\mathcal{O})}^{2-\frac{1}{r}} + \left(2 - \frac{1}{r}\right) k_{\mathcal{O}} \|\varphi\|_{L^1(\mathcal{O})}^{1-\frac{1}{r}} \|\nabla\varphi\|_{L^r(\mathcal{O})}, \quad (4.113)$$

where  $k_{\mathcal{O}}$  is a positive constant, which depends on  $\mathcal{O}$  only. We deduce from (4.113) and (3.1) for any  $r \in (1, 2)$  that

$$\int_{\partial\mathcal{O}} |\varphi|^{2-\frac{1}{r}} d\sigma \leq C(\mathcal{O}, r) \|\varphi\|_{L^1(\mathcal{O})}^{1-\frac{1}{r}} \|\varphi\|_{W^{1,r}(\mathcal{O})} \quad \forall \varphi \in W^{1,r}(\mathcal{O}). \quad (4.114)$$

We apply (4.114) with  $\mathcal{O} = D_0$ , integrate the resulting inequality over  $(0, T) \times \Omega$  and apply Hölder's inequality; this yields for any  $r \in (1, 2)$  and for all  $\varphi \in L^r(0, T; W^{1,r}(\Omega \times D_0))$  that

$$\begin{aligned} & \int_0^T \int_{\Omega \times \partial D_0} |\varphi|^{2-\frac{1}{r}} d\sigma(q) dx dt \\ & \leq C(D_0, r) \|\varphi\|_{L^1(0,T;L^1(\Omega \times D_0))}^{1-\frac{1}{r}} \|\varphi\|_{L^r(0,T;W^{1,r}(\Omega \times D_0))}. \end{aligned} \quad (4.115)$$

Motivated by the bound (4.111), we fix

$$r = s = \frac{(K+1)d+2}{(K+1)d+1} \in (1, 2)$$

in (4.115). It follows from (4.115), (4.111) and (4.112) that

$$\int_0^T \int_{\Omega \times \partial D_0} |\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} - \widehat{\psi}_{\kappa,\alpha}|^{2-\frac{1}{s}} d\sigma(q) dx dt \rightarrow 0, \quad \text{as } \Delta t \rightarrow 0_+ \ (L \rightarrow \infty). \quad (4.116)$$

Since  $2 - \frac{1}{s} > 1$ , it follows from (4.116) that  $\mathbf{T}_{22}$  converges to 0, as  $\Delta t \rightarrow 0_+$  ( $L \rightarrow \infty$ ). We thus deduce that there exists a  $\Delta t_0$  such that for all  $\Delta t \leq \Delta t_0$ , we have that  $0 < \mathbf{T}_{22} < \frac{\delta}{6}$ . Finally, by recalling the inequalities (4.107) and (4.108) and the bounds on  $\mathbf{T}_1$ ,  $\mathbf{T}_{21}$ ,  $\mathbf{T}_{22}$  and  $\mathbf{T}_3$ , it then follows that for each  $\delta > 0$  there exists a  $\Delta t_0$  such that for all  $\Delta t \leq \Delta t_0$ , we have that

$$\int_{\Omega_T} |\tau_1(M\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}) - \tau_1(M\widehat{\psi}_{\kappa,\alpha})| dx dt < \delta.$$

Thus we have proved that

$$\tau_1(M\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}) \rightarrow \tau_1(M\widehat{\psi}_{\kappa,\alpha}) \quad \text{strongly in } L^1(\Omega_T), \quad \text{as } \Delta t \rightarrow 0_+ \ (L \rightarrow \infty).$$

This, together with (4.28b), (4.105) and (3.1), implies that, as  $\Delta t \rightarrow 0_+$  ( $L \rightarrow \infty$ ),

$$\tau_1(M\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}) \rightharpoonup \tau_1(M\widehat{\psi}_{\kappa,\alpha}) \quad \text{strongly in } L^r(\Omega_T) \text{ for all } r \in \left[1, \frac{4(d+2)}{3d+4}\right). \quad (4.117)$$

We note further that, according to (4.103b) with  $\varsigma = 2$ ,  $\varrho_{\kappa,\alpha,L}^{\Delta t,+} \rightarrow \varrho_{\kappa,\alpha}$  strongly in  $L^{\frac{10}{3}}(0,T;L^2(\Omega))$  as  $\Delta t \rightarrow 0_+$  ( $L \rightarrow \infty$ ); therefore  $(\varrho_{\kappa,\alpha,L}^{\Delta t,+})^2 \rightarrow \varrho_{\kappa,\alpha}^2$  strongly in  $L^{\frac{5}{3}}(0,T;L^1(\Omega))$ , and thus strongly in  $L^1(0,T;L^1(\Omega)) = L^1(\Omega_T)$ , as  $\Delta t \rightarrow 0_+$  ( $L \rightarrow \infty$ ). Also, by (4.27), since  $\frac{8(d+2)}{3d+4} < \frac{2(d+2)}{d}$  for  $d \in \{2,3\}$ , we have that  $\{(\varrho_{\kappa,\alpha,L}^{\Delta t,+})^2\}_{\Delta t > 0}$  is a bounded sequence in  $L^{\frac{4(d+2)}{3d+4}}(\Omega_T)$ ; consequently from (4.105) and (3.1),  $(\varrho_{\kappa,\alpha,L}^{\Delta t,+})^2 \rightarrow (\varrho_{\kappa,\alpha})^2$  strongly in  $L^r(\Omega_T)$  for all  $r \in \left[1, \frac{4(d+2)}{3d+4}\right)$ . Combining this with (4.117) we deduce (4.101f) thanks to (1.10).  $\square$

We are now ready to pass to the limit with  $\Delta t \rightarrow 0_+$  ( $L \rightarrow \infty$ ) in (4.5a–c) to prove the existence of a weak solution to the regularized problem  $(P_{\kappa,\alpha})$ .

**Theorem 4.1.** *The triple  $(\rho_{\kappa,\alpha}, \underline{u}_{\kappa,\alpha}, \widehat{\psi}_{\kappa,\alpha})$ , defined as in Lemmas 4.1 and 4.8, is a global weak solution to problem  $(P_{\kappa,\alpha})$ , in the sense that*

$$\int_0^T \left\langle \frac{\partial \rho_{\kappa,\alpha}}{\partial t}, \eta \right\rangle_{H^1(\Omega)} dt + \int_0^T \int_{\Omega} \left( \alpha \nabla_x \rho_{\kappa,\alpha} - \rho_{\kappa,\alpha} \underline{u}_{\kappa,\alpha} \right) \cdot \nabla_x \eta \, dx \, dt = 0 \quad \forall \eta \in L^2(0,T;H^1(\Omega)), \quad (4.118a)$$

with  $\rho_{\kappa,\alpha}(\cdot, 0) = \rho^0(\cdot)$ ,

$$\begin{aligned} & \int_0^T \left\langle \frac{\partial(\rho_{\kappa,\alpha} \underline{u}_{\kappa,\alpha})}{\partial t}, \underline{w} \right\rangle_{W_0^{1,4}(\Omega)} dt + \frac{\alpha}{2} \int_0^T \int_{\Omega} \nabla_x \rho_{\kappa,\alpha} \cdot \nabla_x (\underline{u}_{\kappa,\alpha} \cdot \underline{w}) \, dx \, dt \\ & + \int_0^T \int_{\Omega} \left[ S(\underline{u}_{\kappa,\alpha}) - \rho_{\kappa,\alpha} \underline{u}_{\kappa,\alpha} \otimes \underline{u}_{\kappa,\alpha} - p_{\kappa}(\rho_{\kappa,\alpha}) I \right] : \nabla_x \underline{w} \, dx \, dt \\ & = \int_0^T \int_{\Omega} \left[ \rho_{\kappa,\alpha} \underline{f} \cdot \underline{w} - \left( \tau_1(M\widehat{\psi}_{\kappa,\alpha}) - \mathfrak{z} \varrho_{\kappa,\alpha}^2 I \right) : \nabla_x \underline{w} \right] dx \, dt \end{aligned} \quad \forall \underline{w} \in L^r(0,T;W_0^{1,4}(\Omega)), \quad (4.118b)$$

with  $(\rho_{\kappa,\alpha} \underline{u}_{\kappa,\alpha})(\cdot, 0) = (\rho^0 \underline{u}_0)(\cdot)$  and  $r = \frac{8\Gamma-12}{\Gamma-6}$ ,  $\Gamma \geq 8$ , and

$$\begin{aligned} & \int_0^T \left\langle M \frac{\partial \widehat{\psi}_{\kappa,\alpha}}{\partial t}, \varphi \right\rangle_{H^s(\Omega \times D)} dt \\ & + \frac{1}{4\lambda} \sum_{i=1}^K \sum_{j=1}^K A_{ij} \int_0^T \int_{\Omega \times D} M \nabla_{q_j} \widehat{\psi}_{\kappa,\alpha} \cdot \nabla_{q_i} \varphi \, dq \, dx \, dt \\ & + \int_0^T \int_{\Omega \times D} M \left[ \varepsilon \nabla_x \widehat{\psi}_{\kappa,\alpha} - \underline{u}_{\kappa,\alpha} \widehat{\psi}_{\kappa,\alpha} \right] \cdot \nabla_x \varphi \, dq \, dx \, dt \end{aligned}$$

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$$- \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \left[ \underset{\approx}{\sigma}(u_{\kappa,\alpha}) \underset{\approx}{q}_i \right] \underset{\approx}{\widehat{\psi}}_{\kappa,\alpha} \cdot \underset{\approx}{\nabla}_{q_i} \underset{\approx}{\varphi} \underset{\approx}{dq} \underset{\approx}{dx} \underset{\approx}{dt} = 0$$

$$\forall \varphi \in L^2(0, T; H^s(\Omega \times D)), \quad (4.118c)$$

with  $\widehat{\psi}_{\kappa,\alpha}(\cdot, 0) = \widehat{\psi}_0(\cdot)$  and  $s > 1 + \frac{1}{2}(K+1)d$ .

In addition, the weak solution  $(\rho_{\kappa,\alpha}, \underline{u}_{\kappa,\alpha}, \widehat{\psi}_{\kappa,\alpha})$  satisfies, for a.a.  $t' \in (0, T)$ , the following energy inequality:

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \rho_{\kappa,\alpha}(t') |\underline{u}_{\kappa,\alpha}(t')|^2 \underset{\approx}{dx} + \int_{\Omega} P_{\kappa}(\rho_{\kappa,\alpha}(t')) \underset{\approx}{dx} + k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_{\kappa,\alpha}(t')) \underset{\approx}{dq} \underset{\approx}{dx} \\ & + \alpha \kappa \int_0^{t'} \left[ \|\nabla_x(\rho_{\kappa,\alpha}^2)\|_{L^2(\Omega)}^2 + \frac{4}{\Gamma} \|\nabla_x(\rho_{\kappa,\alpha}^{\frac{\Gamma}{2}})\|_{L^2(\Omega)}^2 \right] dt + \mu^S c_0 \int_0^{t'} \|\underline{u}_{\kappa,\alpha}\|_{H^1(\Omega)}^2 dt \\ & + k \int_0^{t'} \int_{\Omega \times D} M \left[ \frac{a_0}{2\lambda} \left| \nabla_q \sqrt{\widehat{\psi}_{\kappa,\alpha}} \right|^2 + 2\varepsilon \left| \nabla_x \sqrt{\widehat{\psi}_{\kappa,\alpha}} \right|^2 \right] \underset{\approx}{dq} \underset{\approx}{dx} dt \\ & + \mathfrak{z} \|\varrho_{\kappa,\alpha}(t')\|_{L^2(\Omega)}^2 + 2\mathfrak{z} \varepsilon \int_0^{t'} \|\nabla_x \varrho_{\kappa,\alpha}\|_{L^2(\Omega)}^2 dt \\ & \leq e^{t'} \left[ \frac{1}{2} \int_{\Omega} \rho^0 |\underline{u}_0|^2 \underset{\approx}{dx} + \int_{\Omega} P_{\kappa}(\rho^0) \underset{\approx}{dx} + k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_0) \underset{\approx}{dq} \underset{\approx}{dx} \right. \\ & \quad \left. + \mathfrak{z} \int_{\Omega} \left( \int_D M \widehat{\psi}_0 \underset{\approx}{dq} \right)^2 \underset{\approx}{dx} + \frac{1}{2} \int_0^{t'} \|f\|_{L^\infty(\Omega)}^2 dt \int_{\Omega} \rho^0 \underset{\approx}{dx} \right] \leq C, \end{aligned} \quad (4.119)$$

where  $C \in \mathbb{R}_{>0}$  is independent of  $\alpha$  and  $\kappa$ .

**Proof.** The limit equation (4.118a) has already been established in Lemma 4.1, see (4.45).

We now pass to the limit  $\Delta t \rightarrow 0_+$  ( $L \rightarrow \infty$ ), subject to (4.40), for the subsequence of  $\{(\rho_{\kappa,\alpha,L}^{[\Delta t]}, \underline{u}_{\kappa,\alpha,L}^{\Delta t,+}, \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+})\}_{\Delta t > 0}$  of Lemma 4.8 in (4.89) initially for a fixed test function  $w \in C_0^\infty(\Omega_T)$ . We consider first the five terms on the left-hand side of (4.89). On noting (4.97b,d), (4.53a), (4.44), (4.52c) and (4.49), we obtain the first two terms on the left-hand side of (4.118b) and the  $\rho_{\kappa,\alpha} \underline{u}_{\kappa,\alpha} \otimes \underline{u}_{\kappa,\alpha}$  term from the first term on the left-hand side of (4.89). The second and third terms on the left-hand side of (4.89) give rise to the remaining terms on the left-hand side of (4.118b), on noting (4.44) and (4.43b). The fourth and fifth terms on the left-hand side of (4.89) converge to zero, on noting (4.68a), (4.51), (4.52b) and (4.49).

We now consider the two terms on the right-hand side of (4.89). The first term gives rise to the  $\underline{f}$  and  $\underline{\tau}_1$  contributions on the right-hand side of (4.118b), on noting (4.53b), (3.29b) and (4.101f). The second term on the right-hand side of (4.89) converges to the  $\varrho_{\kappa,\alpha}^2$  term on the right-hand side of (4.118b), on noting (4.103a,b) and performing integration by parts. Therefore, we have obtained (4.118b) for any  $w \in C_0^\infty(\Omega_T)$ . The desired result (4.118b) for any  $w \in L^r(0, T; W_0^{1,4}(\Omega))$  then follows from the denseness of  $C_0^\infty(\Omega_T)$  in  $L^r(0, T; W_0^{1,4}(\Omega))$ , (4.96), (4.53a,b), (4.44),



(4.91), (4.97d), (4.41), (3.5), (3.4), (4.101f), and finally (4.102), which with (3.2) yields, similarly to (4.26) and (4.27), that  $\varrho_{\kappa,\alpha} \in L^4(0, T; L^{\frac{2d}{d-1}}(\Omega))$ .

Similarly, we now pass to the limit  $\Delta t \rightarrow 0_+$  ( $L \rightarrow \infty$ ) for the subsequence in (4.5c) initially for a fixed test function  $\varphi \in C([0, T]; C^\infty(\overline{\Omega \times D}))$ . The first term of (4.5c) converges to the first term of (4.118c), on noting (4.101c). For the second term of (4.5c), we note that

$$\begin{aligned} & \int_0^T \int_{\Omega \times D} M \nabla_{q_i} \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} \cdot \nabla_{q_j} \varphi \, dq \, dx \, dt \\ &= 2 \int_0^T \int_{\Omega \times D} M \left( \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}} - \sqrt{\widehat{\psi}_{\kappa,\alpha}} \right) \nabla_{q_i} \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}} \cdot \nabla_{q_j} \varphi \, dq \, dx \, dt \\ & \quad + 2 \int_0^T \int_{\Omega \times D} M \sqrt{\widehat{\psi}_{\kappa,\alpha}} \nabla_{q_i} \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}} \cdot \nabla_{q_j} \varphi \, dq \, dx \, dt =: T_1 + T_2. \end{aligned} \quad (4.120)$$

Next, on noting (4.99) and that  $|\sqrt{c_1} - \sqrt{c_2}| \leq \sqrt{|c_1 - c_2|}$  for all  $c_1, c_2 \in \mathbb{R}_{\geq 0}$ , we have that

$$\begin{aligned} |T_1| &\leq C \left\| \sqrt{\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}} - \sqrt{\widehat{\psi}_{\kappa,\alpha}} \right\|_{L^2(0,T;L^2_M(\Omega \times D))} \|\nabla_{q_j} \varphi\|_{L^\infty(0,T;L^\infty(\Omega \times D))} \\ &\leq C \|\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+} - \widehat{\psi}_{\kappa,\alpha}\|_{L^1(0,T;L^1_M(\Omega \times D))}^{\frac{1}{2}} \|\nabla_{q_j} \varphi\|_{L^\infty(0,T;L^\infty(\Omega \times D))}, \end{aligned} \quad (4.121)$$

and so (4.101d) yields that  $T_1$  converges to zero as  $\Delta t \rightarrow 0_+$ . Similarly, as  $M^{\frac{1}{2}} \sqrt{\widehat{\psi}_{\kappa,\alpha}} \nabla_{q_j} \varphi \in L^2(0, T; L^2(\Omega \times D))$ , it follows from (4.101b) that, as  $\Delta t \rightarrow 0_+$ ,

$$\begin{aligned} T_2 &\rightarrow 2 \int_0^T \int_{\Omega \times D} M \sqrt{\widehat{\psi}_{\kappa,\alpha}} \nabla_{q_i} \sqrt{\widehat{\psi}_{\kappa,\alpha}} \cdot \nabla_{q_j} \varphi \, dq \, dx \, dt \\ &= \int_0^T \int_{\Omega \times D} M \nabla_{q_i} \widehat{\psi}_{\kappa,\alpha} \cdot \nabla_{q_j} \varphi \, dq \, dx \, dt. \end{aligned} \quad (4.122)$$

Hence the second term in (4.5c) converges to the second term in (4.118c). For the fourth term in (4.5c), we note that

$$\begin{aligned} & \int_0^T \int_{\Omega \times D} M \left[ \sigma(\widehat{u}_{\kappa,\alpha,L}^{\Delta t,+}) q_i \right] \beta^L(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}) \cdot \nabla_{q_i} \varphi \, dq \, dx \, dt \\ &= \int_0^T \int_{\Omega \times D} M \left[ \nabla_x \widehat{u}_{\kappa,\alpha,L}^{\Delta t,+} q_i \right] \left( \beta^L(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}) - \widehat{\psi}_{\kappa,\alpha} \right) \cdot \nabla_{q_i} \varphi \, dq \, dx \, dt \\ & \quad + \int_0^T \int_{\Omega \times D} M \left[ \nabla_x \widehat{u}_{\kappa,\alpha,L}^{\Delta t,+} q_i \right] \widehat{\psi}_{\kappa,\alpha} \cdot \nabla_{q_i} \varphi \, dq \, dx \, dt =: T_3 + T_4. \end{aligned} \quad (4.123)$$

Next, on noting (4.51), (3.1), (1.19), (4.29) and (4.102), we have that

$$\begin{aligned} |T_3| &\leq C \left\| \int_D M |\beta^L(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}) - \widehat{\psi}_{\kappa,\alpha}| \, dq \right\|_{L^2(0,T;L^2(\Omega))} \|\nabla_{q_i} \varphi\|_{L^\infty(0,T;L^\infty(\Omega \times D))} \\ &\leq C \|\beta^L(\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t,+}) - \widehat{\psi}_{\kappa,\alpha}\|_{L^2(0,T;L^1_M(\Omega \times D))}^{\frac{2}{5}} \\ & \quad \times \|\varrho_{\kappa,\alpha,L}^{\Delta t,+} + \varrho_{\kappa,\alpha}\|_{L^2(0,T;L^6(\Omega))}^{\frac{3}{5}} \|\nabla_{q_i} \varphi\|_{L^\infty(0,T;L^\infty(\Omega))}, \end{aligned} \quad (4.124)$$

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and so (4.101e) and (4.26) (with  $v = 6$  and  $\vartheta = \frac{d}{3}$ ,  $d = 2, 3$ ) yield that  $\mathbf{T}_3$  converges to zero as  $\Delta t \rightarrow 0_+$ . Similarly, as (4.102) yields that  $\int_D M \widehat{\psi}_{\kappa,\alpha} \underline{q}_i \otimes \nabla_{q_i} \varphi \, d\underline{q} \in L^2(\Omega_T)$ , it follows from (4.44) that, as  $\Delta t \rightarrow 0_+$ ,

$$\mathbf{T}_4 \rightarrow \int_0^T \int_{\Omega \times D} M [(\nabla_x u_{\kappa,\alpha}) \underline{q}_i] \widehat{\psi}_{\kappa,\alpha} \cdot \nabla_{q_i} \varphi \, d\underline{q} \, dx \, dt. \quad (4.125)$$

Hence the last term in (4.5c) converges to the last term in (4.118c). Similarly to the second and last terms, the third term in (4.5c) converges to the third term in (4.118c). Therefore, we have obtained (4.118c) for any  $\varphi \in C([0, T]; C^\infty(\overline{\Omega \times D}))$ . The desired result (4.118c), for any  $\varphi \in L^2(0, T; H^s(\Omega \times D))$ , then follows from the denseness of the function space  $C([0, T]; C^\infty(\overline{\Omega \times D}))$  in  $L^2(0, T; H^s(\Omega \times D))$ ,  $H^s(\Omega \times D) \hookrightarrow W^{1,\infty}(\Omega \times D)$ , (4.100a,b), (4.102) and (4.44).

Next we shall verify the attainment of the respective initial data by  $\rho_{\kappa,\alpha} \underline{u}_{\kappa,\alpha}$  and  $\widehat{\psi}_{\kappa,\alpha}$ . We have already established that  $\rho_{\kappa,\alpha} \underline{u}_{\kappa,\alpha} \in C_w([0, T]; L^{\frac{2\Gamma}{\Gamma+1}}(\Omega))$ , see (4.96). That  $\widehat{\psi}_{\kappa,\alpha} \in C_w([0, T]; L_M^1(\Omega \times D))$  follows from  $\mathcal{F}(\psi_{\kappa,\alpha}) \in L^\infty(0, T; L_M^1(\Omega \times D))$  and  $\widehat{\psi}_{\kappa,\alpha} \in H^1(0, T; M^{-1}(H^s(\Omega \times D)))'$  (cf. (4.100b) and (4.94)) with  $s > 1 + \frac{1}{2}(K+1)d$ , by Lemma 3.1(b) in Barrett & Süli<sup>12</sup> on taking  $\mathfrak{X} := L_M^\Phi(\Omega \times D)$ , the Maxwellian weighted Orlicz space with Young's function  $\Phi(r) = \mathcal{F}(1 + |r|)$  (cf. Kufner, John & Fučík<sup>22</sup>, Sec. 3.18.2) whose separable predual  $\mathfrak{E} := E_M^\Psi(\Omega \times D)$  has Young's function  $\Psi(r) = \exp|r| - |r| - 1$ , and  $\mathfrak{Y} := M^{-1}(H^s(\Omega \times D))'$  whose predual with respect to the duality pairing  $\langle M \cdot, \cdot \rangle_{H^s(\Omega \times D)}$  is  $\mathfrak{F} := H^s(\Omega \times D)$ , with  $s > 1 + \frac{1}{2}(K+1)d$ , and noting the embedding  $C_{w^*}([0, T]; L_M^\Phi(\Omega \times D)) \hookrightarrow C_w([0, T]; L_M^1(\Omega \times D))$ . The last embedding and that  $\mathfrak{F} \hookrightarrow \mathfrak{E}$  are proved by adapting Def. 3.6.1. and Thm. 3.2.3 in Kufner, John & Fučík<sup>22</sup> to the measure  $M(\underline{q}) \, d\underline{q} \, dx$  to show that  $L^\infty(\Omega \times D) \hookrightarrow L_M^\Xi(\Omega \times D)$  for any Young's function  $\Xi$ , and then adapting Theorem 3.17.7 *ibid.* to deduce that  $\mathfrak{F} \hookrightarrow L^\infty(\Omega \times D) \hookrightarrow E_M^\Psi(\Omega \times D) = \mathfrak{E}$ .

We are now ready to prove that  $\rho_{\kappa,\alpha} \underline{u}_{\kappa,\alpha}$  and  $\widehat{\psi}_{\kappa,\alpha}$  satisfy the initial conditions  $(\rho_{\kappa,\alpha} \underline{u}_{\kappa,\alpha})(\cdot, 0) = (\rho^0 \underline{u}_0)(\cdot)$  and  $\widehat{\psi}_{\kappa,\alpha}(\cdot, \cdot, 0) = \widehat{\psi}_0(\cdot, \cdot)$  in the sense of  $C_w([0, T]; L^{\frac{2\Gamma}{\Gamma+1}}(\Omega))$  and  $C_w([0, T]; L_M^1(\Omega \times D))$ , respectively. The desired result for  $\rho_{\kappa,\alpha} \underline{u}_{\kappa,\alpha}$  follows immediately from (4.97e) and (3.18b). We now consider  $\widehat{\psi}_{\kappa,\alpha}$ . According to (4.94), there exists a  $C \in \mathbb{R}_{>0}$ , independent of  $\Delta t$  and  $L$ , such that, for all  $\varphi \in L^2(0, T; H^s(\Omega \times D))$ ,

$$\left| \int_0^T \int_{\Omega \times D} M \frac{\partial \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t}}{\partial t} \varphi \, d\underline{q} \, dx \, dt \right| \leq C \|\varphi\|_{L^2(0,T;H^s(\Omega \times D))},$$

where  $s > 1 + \frac{1}{2}(K+1)d$ . Choosing, in particular  $\varphi(\underline{x}, \underline{q}, t) = \phi(\underline{x}, \underline{q}) \left(1 - \frac{t}{\delta}\right)_+$ , with  $\phi \in H^s(\Omega \times D)$ ,  $0 < \delta < T$ , integrating by parts with respect to  $t$  and using that  $\widehat{\psi}_{\kappa,\alpha,L}^{\Delta t}(\cdot, \cdot, 0) = \widehat{\psi}^0(\cdot, \cdot)$ , we have that, for all  $\phi \in H^s(\Omega \times D)$ ,

$$\left| \frac{1}{\delta} \int_0^\delta \int_{\Omega \times D} M \widehat{\psi}_{\kappa,\alpha,L}^{\Delta t} \phi \, d\underline{q} \, dx \, dt - \int_{\Omega \times D} M \widehat{\psi}^0 \phi \, d\underline{q} \, dx \right| \leq C \delta^{\frac{1}{2}} \|\phi\|_{H^s(\Omega \times D)}.$$

For  $\delta \in (0, T)$  and  $\phi$  fixed, we now pass to the limit  $\Delta t \rightarrow 0_+$  ( $L \rightarrow \infty$ ) in this inequality using (4.101d) and (3.21b) to deduce that, for all  $\phi \in H^s(\Omega \times D)$ ,

$$\left| \frac{1}{\delta} \int_0^\delta \int_{\Omega \times D} M \widehat{\psi}_{\kappa, \alpha} \phi \, dq \, dx \, dt - \int_{\Omega \times D} M \widehat{\psi}_0 \phi \, dq \, dx \right| \leq C \delta^{\frac{1}{2}} \|\phi\|_{H^s(\Omega \times D)},$$

where we have recalled that  $H^s(\Omega \times D) \hookrightarrow W^{1, \infty}(\Omega \times D)$ . Thus, noting the weak continuity result  $\widehat{\psi}_{\kappa, \alpha} \in C_w([0, T]; L^1_M(\Omega \times D))$  established above, it follows on passing to the limit  $\delta \rightarrow 0_+$  that

$$\int_{\Omega \times D} M \widehat{\psi}_{\kappa, \alpha}(0) \phi \, dq \, dx = \int_{\Omega \times D} M \widehat{\psi}_0 \phi \, dq \, dx \quad \forall \phi \in H^s(\Omega \times D).$$

Hence, we have  $\widehat{\psi}_{\kappa, \alpha}(\cdot, \cdot, 0) = \widehat{\psi}_0$  in  $L^1_M(\Omega \times D)$ .

It remains to prove the inequality (4.119). For  $t' \in (0, T]$  fixed, let  $n = n(t', \Delta t)$  be a positive integer such that  $0 \leq (n-1)\Delta t < t' \leq n\Delta t \leq T$ . It follows from (4.21) and (4.4b), on noting that the interval  $(0, t']$  is contained in  $(0, t_n]$ , that

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} \rho_{\kappa, \alpha, L}^{\Delta t, +}(t') |u_{\kappa, \alpha, L}^{\Delta t, +}(t')|^2 \, dx + \int_{\Omega} P_{\kappa}(\rho_{\kappa, \alpha, L}^{\Delta t, +}(t')) \, dx \\ & + k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_{\kappa, \alpha, L}^{\Delta t, +}(t')) \, dq \, dx \\ & + \alpha \kappa \int_0^{t'} \left[ \|\nabla_x [(\rho_{\kappa, \alpha, L}^{\Delta t})^2]\|_{L^2(\Omega)}^2 + \frac{4}{\Gamma} \|\nabla_x [(\rho_{\kappa, \alpha, L}^{\Delta t})^{\frac{\Gamma}{2}}]\|_{L^2(\Omega)}^2 \right] dt \\ & + \mu^S c_0 \int_0^{t'} \|u_{\kappa, \alpha, L}^{\Delta t, +}\|_{H^1(\Omega)}^2 dt \\ & + k \int_0^{t'} \int_{\Omega \times D} M \left[ \frac{a_0}{2\lambda} \left| \nabla_q \sqrt{\widehat{\psi}_{\kappa, \alpha, L}^{\Delta t, +}} \right|^2 + 2\varepsilon \left| \nabla_x \sqrt{\widehat{\psi}_{\kappa, \alpha, L}^{\Delta t, +}} \right|^2 \right] dq \, dx \, dt \\ & + \mathfrak{z} \|\varrho_{\kappa, \alpha, L}^{\Delta t, +}(t')\|_{L^2(\Omega)}^2 + 2\mathfrak{z} \varepsilon \int_0^{t'} \|\nabla_x \varrho_{\kappa, \alpha, L}^{\Delta t, +}\|_{L^2(\Omega)}^2 dt \\ & \leq e^{t_n} \left[ \frac{1}{2} \int_{\Omega} \rho^0 |u_0|^2 \, dx + \int_{\Omega} P_{\kappa}(\rho^0) \, dx + k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_0) \, dq \, dx \right. \\ & \quad \left. + \mathfrak{z} \int_{\Omega} \left( \int_D M \widehat{\psi}_0 \, dq \right)^2 \, dx + \frac{1}{2} \int_0^{t_n} \|f\|_{L^\infty(\Omega)}^2 dt \int_{\Omega} \rho^0 \, dx \right] \leq C, \quad (4.126) \end{aligned}$$

where  $C \in \mathbb{R}_{>0}$  is independent of  $\alpha$  and  $\kappa$ . Clearly  $n = n(t', \Delta t) \geq \frac{t'}{\Delta t} \rightarrow \infty$  as  $\Delta t \rightarrow 0_+$ . Since  $t' \in (t_{n-1}, t_n]$  and  $t_n - t_{n-1} = \Delta t$ , we deduce that as  $\Delta t \rightarrow 0_+$  (and, hence,  $n = n(t', \Delta t) \rightarrow \infty$ ) both  $t_{n-1}$  and  $t_n$  converge to  $t'$ ; hence

$$e^{t_n} \rightarrow e^{t'} \quad \text{and} \quad \int_0^{t_n} \|f\|_{L^\infty(\Omega)}^2 dt \rightarrow \int_0^{t'} \|f\|_{L^\infty(\Omega)}^2 dt, \quad \text{as } \Delta t \rightarrow 0_+. \quad (4.127)$$

We multiply (4.126) by any nonnegative  $\eta \in C_0^\infty(0, T)$ , integrate over  $(0, T)$ , and pass to the limit  $\Delta t \rightarrow 0_+$  (and  $L \rightarrow \infty$ ) in the resulting inequality. It then follows from (4.97d), (4.53c), (4.101g); weak lower-semicontinuity, via the weak convergence

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results (4.42a,c), (4.44), (4.101a,b) and (4.103a); and (4.127), that we obtain the inequality (4.119) multiplied by  $\eta$  and integrated over  $(0, T)$ . The desired result (4.119) then follows from the well-known variant of du Bois-Reymond's lemma according to which, if  $\phi \in L^1(0, T)$ , then

$$\int_0^T \phi \eta dt \geq 0 \quad \forall \eta \in C_0^\infty(0, T) \text{ with } \eta \geq 0 \text{ on } (0, T) \quad \Rightarrow \quad \phi \geq 0 \quad \text{a.e. in } (0, T). \quad (4.128)$$

That completes the proof of the theorem.  $\square$

### 5. Existence of a solution to $(\mathbf{P}_\kappa)$

It follows from the bounds on  $\varrho_{\kappa, \alpha}$  in (4.119), similarly to (4.26) and (4.27), that

$$\|\varrho_{\kappa, \alpha}\|_{L^\infty(0, T; L^2(\Omega))} + \|\varrho_{\kappa, \alpha}\|_{L^{\frac{2(d+2)}{d}}(\Omega_T)} + \|\varrho_{\kappa, \alpha}\|_{L^4(0, T; L^{\frac{2d}{d-1}}(\Omega))} \leq C, \quad (5.1)$$

where throughout this section  $C$  is a generic positive constant, independent of  $\alpha$ . Hence, we deduce from (5.1), (4.24) and (4.119), similarly to (4.28b), that

$$\|\tau_1(M \widehat{\psi}_{\kappa, \alpha})\|_{L^2(0, T; L^{\frac{4}{3}}(\Omega))} + \|\tau_1(M \widehat{\psi}_{\kappa, \alpha})\|_{L^{\frac{4(d+2)}{3d+4}}(\Omega_T)} \leq C. \quad (5.2)$$

Similarly to (4.94), it follows from (5.1) and (4.119) that

$$\left\| M \frac{\partial \widehat{\psi}_{\kappa, \alpha}}{\partial t} \right\|_{L^2(0, T; H^s(\Omega \times D)')} \leq C, \quad (5.3)$$

where  $s > 1 + \frac{1}{2}(K+1)d$ . We have the following analogue of Lemma 4.8.

**Lemma 5.1.** *There exist functions*

$$\underset{\sim}{u}_\kappa \in L^2(0, T; H_0^1(\Omega)) \quad \text{and} \quad \widehat{\psi}_\kappa \in L^v(0, T; Z_1) \cap H^1(0, T; M^{-1}(H^s(\Omega \times D))'), \quad (5.4a)$$

where  $v \in [1, \infty)$  and  $s > 1 + \frac{1}{2}(K+1)d$ , with finite relative entropy and Fisher information,

$$\mathcal{F}(\widehat{\psi}_\kappa) \in L^\infty(0, T; L_M^1(\Omega \times D)) \quad \text{and} \quad \sqrt{\widehat{\psi}_\kappa} \in L^2(0, T; H_M^1(\Omega \times D)), \quad (5.4b)$$

and a subsequence of  $\{(\rho_{\kappa, \alpha}, \underset{\sim}{u}_{\kappa, \alpha}, \widehat{\psi}_{\kappa, \alpha})\}_{\alpha > 0}$  such that, as  $\alpha \rightarrow 0_+$ ,

$$\underset{\sim}{u}_{\kappa, \alpha} \rightarrow \underset{\sim}{u}_\kappa \quad \text{weakly in } L^2(0, T; H_0^1(\Omega)), \quad (5.5)$$

and

$$M^{\frac{1}{2}} \nabla_x \sqrt{\widehat{\psi}_{\kappa,\alpha}} \rightharpoonup M^{\frac{1}{2}} \nabla_x \sqrt{\widehat{\psi}_{\kappa}} \quad \text{weakly in } L^2(0, T; L^2(\Omega \times D)), \quad (5.6a)$$

$$M^{\frac{1}{2}} \nabla_q \sqrt{\widehat{\psi}_{\kappa,\alpha}} \rightharpoonup M^{\frac{1}{2}} \nabla_q \sqrt{\widehat{\psi}_{\kappa}} \quad \text{weakly in } L^2(0, T; L^2(\Omega \times D)), \quad (5.6b)$$

$$M \frac{\partial \widehat{\psi}_{\kappa,\alpha}}{\partial t} \rightharpoonup M \frac{\partial \widehat{\psi}_{\kappa}}{\partial t} \quad \text{weakly in } L^2(0, T; H^s(\Omega \times D)'), \quad (5.6c)$$

$$\widehat{\psi}_{\kappa,\alpha} \rightarrow \widehat{\psi}_{\kappa} \quad \text{strongly in } L^v(0, T; L_M^1(\Omega \times D)), \quad (5.6d)$$

$$\tau(M \widehat{\psi}_{\kappa,\alpha}) \rightharpoonup \tau(M \widehat{\psi}_{\kappa}) \quad \text{strongly in } L^r(\Omega_T), \quad (5.6e)$$

where  $r \in [1, \frac{4(d+2)}{3d+4})$ , and, for a.a.  $t \in (0, T)$ ,

$$\int_{\Omega \times D} M(q) \mathcal{F}(\widehat{\psi}_{\kappa,\alpha}) \, dq \, dx \leq \liminf_{\alpha \rightarrow 0_+} \int_{\Omega \times D} M(q) \mathcal{F}(\widehat{\psi}_{\kappa,\alpha}) \, dq \, dx. \quad (5.6f)$$

In addition, we have that

$$\varrho_{\kappa} := \int_D M \widehat{\psi}_{\kappa} \, dq \in L^{\infty}(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad (5.7)$$

and, as  $\alpha \rightarrow 0_+$ ,

$$\varrho_{\kappa,\alpha} \rightarrow \varrho_{\kappa} \quad \begin{array}{l} \text{weakly-}\star \text{ in } L^{\infty}(0, T; L^2(\Omega)), \\ \text{weakly in } L^2(0, T; H^1(\Omega)), \end{array} \quad (5.8a)$$

$$\varrho_{\kappa,\alpha} \rightarrow \varrho_{\kappa} \quad \text{strongly in } L^{\frac{5\varsigma}{\varsigma-1}}(0, T; L^{\varsigma}(\Omega)), \quad (5.8b)$$

for any  $\varsigma \in (1, 6)$ .

**Proof.** The convergence result (5.5) and the first result in (5.4a) follow immediately from the bound on  $u_{\kappa,\alpha}$  in (4.119). The remainder of the results follow from the bounds on  $\widehat{\psi}_{\kappa,\alpha}$  and  $\varrho_{\kappa,\alpha}$  in (4.119) in the same way as those of Lemma 4.8.  $\square$

We have the following analogue of Lemmas 4.2 and 4.4.

**Lemma 5.2.** *Let  $\Gamma \geq 8$ ; then, there exists a  $C \in \mathbb{R}_{>0}$ , independent of  $\alpha$ , such that*

$$\begin{aligned} & \|\rho_{\kappa,\alpha}\|_{L^{\infty}(0,T;L^{\Gamma}(\Omega))} + \|u_{\kappa,\alpha}\|_{L^2(0,T;H^1(\Omega))} + \left\| \sqrt{\rho_{\kappa,\alpha}} u_{\kappa,\alpha} \right\|_{L^{\infty}(0,T;L^2(\Omega))} \\ & + \|\rho_{\kappa,\alpha} u_{\kappa,\alpha}\|_{L^{\infty}(0,T;L^{\frac{2\Gamma}{\Gamma+1}}(\Omega))} + \|\rho_{\kappa,\alpha} u_{\kappa,\alpha}\|_{L^2(0,T;L^{\frac{6\Gamma}{\Gamma+6}}(\Omega))} \\ & + \|\rho_{\kappa,\alpha} u_{\kappa,\alpha}\|_{L^{\frac{10\Gamma-6}{3(\Gamma+1)}}(\Omega_T)} + \left\| \rho_{\kappa,\alpha} |u_{\kappa,\alpha}|^2 \right\|_{L^2(0,T;L^{\frac{6\Gamma}{4\Gamma+3}}(\Omega))} \leq C, \end{aligned} \quad (5.9a)$$

$$\begin{aligned} & \sqrt{\alpha} \|\nabla_x \rho_{\kappa,\alpha}\|_{L^2(\Omega_T)} + \alpha \|\nabla_x \rho_{\kappa,\alpha}\|_{L^{\frac{10\Gamma-6}{3(\Gamma+1)}}(\Omega_T)} + \alpha \|(\nabla_x \rho_{\kappa,\alpha} \cdot \nabla_x) u_{\kappa,\alpha}\|_{L^{\frac{5\Gamma-3}{4\Gamma}}(\Omega_T)} \\ & + \alpha \|\nabla_x \rho_{\kappa,\alpha} \otimes u_{\kappa,\alpha}\|_{L^{\frac{5\Gamma-3}{4\Gamma}}(\Omega_T)} \leq C, \end{aligned} \quad (5.9b)$$

$$\left\| \frac{\partial \rho_{\kappa,\alpha}}{\partial t} \right\|_{L^2(0,T;H^1(\Omega)')} \leq C. \quad (5.9c)$$

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Hence, there exists a function  $\rho_\kappa \in C_w([0, T]; L^{\frac{\Gamma}{\geq 0}}(\Omega)) \cap H^1(0, T; H^1(\Omega)')$ , and for a further subsequence of the subsequence of Lemma 5.1, it follows that, as  $\alpha \rightarrow 0_+$ ,

$$\begin{aligned} \rho_{\kappa, \alpha} &\rightarrow \rho_\kappa && \text{in } C_w([0, T]; L^\Gamma(\Omega)), \\ &&& \text{weakly in } H^1(0, T; H^1(\Omega)'), \end{aligned} \quad (5.10a)$$

$$\rho_{\kappa, \alpha} \rightarrow \rho_\kappa \quad \text{strongly in } L^2(0, T; H^1(\Omega)'), \quad (5.10b)$$

$$\alpha \nabla_x \rho_{\kappa, \alpha} \rightarrow 0 \quad \text{strongly in } L^r(\Omega_T), \quad r \in [1, \frac{10\Gamma-6}{3(\Gamma+1)}), \quad (5.10c)$$

$$\alpha (\nabla_x \rho_{\kappa, \alpha} \cdot \nabla_x) u_{\kappa, \alpha} \rightarrow 0 \quad \text{weakly in } L^{\frac{5\Gamma-3}{4\Gamma}}(\Omega_T), \quad (5.10d)$$

$$\begin{aligned} \alpha \nabla_x \rho_{\kappa, \alpha} \otimes u_{\kappa, \alpha} &\rightarrow 0 \quad \text{weakly in } L^{\frac{5\Gamma-3}{4\Gamma}}(\Omega_T), \\ &\text{strongly in } L^1(0, T; L^{\frac{3}{2}}(\Omega)), \end{aligned} \quad (5.10e)$$

and, for any nonnegative  $\eta \in C[0, T]$ ,

$$\int_0^T \left( \int_\Omega P_\kappa(\rho_\kappa) \, dx \right) \eta \, dt \leq \liminf_{\alpha \rightarrow 0_+} \int_0^T \left( \int_\Omega P_\kappa(\rho_{\kappa, \alpha}) \, dx \right) \eta \, dt. \quad (5.10f)$$

**Proof.** The first three bounds in (5.9a) follow directly from (4.119). The last four bounds in (5.9a) follow, similarly to (4.84a,b), from the first two bounds in (5.9a). On recalling (4.41), we choose  $\eta = \rho_{\kappa, \alpha}$  in (4.45) to obtain, on noting (5.9a), that

$$\begin{aligned} &\frac{1}{2} \|\rho_{\kappa, \alpha}(\cdot, T)\|_{L^2(\Omega)}^2 + \alpha \|\nabla_x \rho_{\kappa, \alpha}\|_{L^2(\Omega_T)}^2 \\ &= \frac{1}{2} \left[ \|\rho^0\|_{L^2(\Omega)}^2 - \int_0^T \int_\Omega (\nabla_x \cdot u_{\kappa, \alpha}) \rho_{\kappa, \alpha}^2 \, dx \right] \\ &\leq C \left[ 1 + \|u_{\kappa, \alpha}\|_{L^2(0, T; H^1(\Omega))} \|\rho_{\kappa, \alpha}\|_{L^4(0, T; L^4(\Omega))}^2 \right] \leq C. \end{aligned} \quad (5.11)$$

Hence the first bound (5.9b). On noting the sixth bound in (5.9a) and recalling from (3.4) that  $\partial\Omega \in C^{2, \theta}$ ,  $\theta \in (0, 1)$ , and  $\rho^0 \in L^\infty(\Omega)$  satisfying (3.16a), we can now apply the parabolic regularity result, Lemma 7.38 in Novotný & Straškraba<sup>24</sup> (or Lemma G.2 in Appendix G in Barrett & Süli<sup>12</sup>), to (4.118a) to obtain that the solution  $\rho_{\kappa, \alpha}$  satisfies the second bound in (5.9b). The third and fourth bounds in (5.9b) follow from the second bounds in (5.9a,b), similarly to (4.70). The bound (5.9c) follows immediately from (4.118a), the fifth bound in (5.9a) and the first bound in (5.9b).

The convergence results (5.10a,b) follow immediately from the first bound in (5.9a,c), (3.12a,b) and (3.11). The first two bounds in (5.9b) and (3.1) yield the desired result (5.10c). The convergence result (5.10d) and the first result in (5.10e) follow from the final two bounds in (5.9b), the second bound in (5.9a) and (5.10c). The second result in (5.10e) also follows from the second bound in (5.9a) and (5.10c)

with  $r = 2$  on noting that, for all  $\eta_1 \in L^2(\Omega_T)$  and  $\eta_2 \in L^2(0, T; H^1(\Omega))$ ,

$$\|\eta_1 \eta_2\|_{L^1(0, T; L^{\frac{3}{2}}(\Omega))} \leq \|\eta_1\|_{L^2(\Omega_T)} \|\eta_2\|_{L^2(0, T; L^6(\Omega))} \leq C \|\eta_1\|_{L^2(\Omega_T)} \|\eta_2\|_{L^2(0, T; H^1(\Omega))}. \quad (5.12)$$

Finally, the result (5.10f) follows, similarly to (4.66), from (5.10a) and the convexity of  $P_\kappa$ .  $\square$

Next, we set  $v = \frac{10\Gamma-6}{3(\Gamma+1)} \geq \frac{74}{27}$ , which appears in Lemma 5.2, and  $s' = \frac{5\Gamma-3}{\Gamma-3} \geq 5$  so that (4.91) holds. Then, similarly to (4.92), it follows from (4.118b), (4.91),  $W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$ ,  $H^1(\Omega) \hookrightarrow L^6(\Omega)$ , (5.9a,b), (5.1), on noting that  $\frac{2d}{d-1} > \frac{8}{3}$ , (5.2), and (3.4) that, for any  $w \in L^{s'}(0, T; \tilde{W}_0^{1,4}(\Omega))$ ,

$$\begin{aligned} & \left| \int_0^T \left\langle \frac{\partial(\rho_{\kappa,\alpha} \tilde{u}_{\kappa,\alpha})}{\partial t}, \tilde{w} \right\rangle_{\tilde{W}_0^{1,4}(\Omega)} dt - \int_0^T \int_\Omega p_\kappa(\rho_{\kappa,\alpha}) \tilde{\nabla}_x \cdot \tilde{w} dx dt \right| \\ & \leq C \left[ \alpha \|\tilde{\nabla}_x \rho_{\kappa,\alpha}\|_{L^v(\Omega_T)} + \|\rho_{\kappa,\alpha} \tilde{u}_{\kappa,\alpha}\|_{L^v(\Omega_T)} + 1 \right] \\ & \quad \times \|\tilde{u}_{\kappa,\alpha}\|_{L^2(0, T; H^1(\Omega))} \|w\|_{L^{s'}(0, T; W^{1,4}(\Omega))} \\ & + C \left[ \|\mathcal{T}_1(M \hat{\psi}_{\kappa,\alpha})\|_{L^2(0, T; L^{\frac{4}{3}}(\Omega))} + \|\varrho_{\kappa,\alpha}\|_{L^4(0, T; L^{\frac{2d}{d-1}}(\Omega))}^2 \right] \|w\|_{L^2(0, T; W^{1,4}(\Omega))} \\ & + \|\rho_{\kappa,\alpha}\|_{L^\infty(0, T; L^2(\Omega))} \|f\|_{L^2(0, T; L^\infty(\Omega))} \|w\|_{L^2(\Omega_T)} \leq C \|w\|_{L^{s'}(0, T; W^{1,4}(\Omega))}. \quad (5.13) \end{aligned}$$

For  $r, s \in (1, \infty)$ , let

$$\begin{aligned} L_0^r(\Omega) & := \{\zeta \in L^r(\Omega) : \int_\Omega \zeta dx = 0\}, \quad E^{r,s}(\Omega) := \{w \in L^r(\Omega) : \tilde{\nabla}_x \cdot w \in L^s(\Omega)\} \\ \text{and } E_0^{r,s}(\Omega) & := \{w \in E^{r,s}(\Omega) : w \cdot \tilde{n} = 0 \text{ on } \partial\Omega\}. \quad (5.14) \end{aligned}$$

The equality  $w \cdot \tilde{n} = 0$  on  $\partial\Omega$  should be understood in the sense of traces of Sobolev functions, with equality in  $W^{1-\frac{s}{v}, v'}(\partial\Omega)'$ , where  $\frac{1}{v} + \frac{1}{v'} = 1$  and  $v = \min\{r, s\}$ ; cf. Lemma 3.10 in Novotný & Straškraba<sup>24</sup>.

We now introduce the Bogovskii operator  $\mathcal{B} : L_0^r(\Omega) \rightarrow \tilde{W}_0^{1,r}(\Omega)$ ,  $r \in (1, \infty)$ , such that

$$\int_\Omega \left( \tilde{\nabla}_x \cdot \mathcal{B}(\zeta) - \zeta \right) \eta dx = 0 \quad \forall \eta \in L^{\frac{r}{r-1}}(\Omega); \quad (5.15)$$

which satisfies

$$\|\mathcal{B}(\zeta)\|_{W^{1,r}(\Omega)} \leq C \|\zeta\|_{L^r(\Omega)} \quad \forall \zeta \in L_0^r(\Omega), \quad (5.16a)$$

$$\|\mathcal{B}(\tilde{\nabla}_x \cdot w)\|_{L^r(\Omega)} \leq C \|w\|_{L^r(\Omega)} \quad \forall w \in E_0^{r,s}(\Omega), \quad (5.16b)$$

see Lemma 3.17 in Novotný & Straškraba<sup>24</sup> (or Lemma C.1 in Appendix C in Barrett & Süli<sup>12</sup>).

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**Lemma 5.3.** *There exists a  $C(\alpha) \in \mathbb{R}_{>0}$  such that*

$$\|\nabla_x \cdot (\rho_{\kappa,\alpha} \underset{\sim}{u}_{\kappa,\alpha})\|_{L^s(\Omega_T)} + \left\| \frac{\partial \rho_{\kappa,\alpha}}{\partial t} \right\|_{L^s(\Omega_T)} + \|\Delta_x \rho_{\kappa,\alpha}\|_{L^s(\Omega_T)} \leq C(\alpha), \quad (5.17)$$

where  $s = \frac{5\Gamma-3}{4\Gamma}$ . In addition, there exists a  $C \in \mathbb{R}_{>0}$ , independent of  $\alpha$ , such that

$$\|\rho_{\kappa,\alpha}\|_{L^{\Gamma+1}(\Omega_T)} \leq C. \quad (5.18)$$

**Proof.** We prove (5.17), similarly to (4.67). The first bound in (5.17) follows from (5.9b). On noting (4.71) and the first bound in (5.17), we can now apply the parabolic regularity result, Lemma 7.37 in Novotný & Straškraba<sup>24</sup> (or Lemma G.1 in Appendix G in Barrett & Süli<sup>12</sup>), to (4.118a) to obtain that the solution  $\rho_{\kappa,\alpha}$  satisfies the last two bounds in (4.67). It follows from (5.17), (5.9a,b) and (4.118a) that

$$\begin{aligned} \frac{\partial \rho_{\kappa,\alpha}}{\partial t} &= \nabla_x \cdot \left( \alpha \nabla_x \rho_{\kappa,\alpha} - \rho_{\kappa,\alpha} \underset{\sim}{u}_{\kappa,\alpha} \right) \in L^s(\Omega_T) \\ \text{with } \left( \alpha \nabla_x \rho_{\kappa,\alpha} - \rho_{\kappa,\alpha} \underset{\sim}{u}_{\kappa,\alpha} \right) \cdot \underset{\sim}{n} &= 0 \quad \text{on } \partial\Omega \times (0, T), \end{aligned} \quad (5.19)$$

and hence, on recalling (5.14) and that  $s = \frac{5\Gamma-3}{4\Gamma} < \frac{10\Gamma-6}{3(\Gamma+1)} = r$ , we have that

$$\alpha \nabla_x \rho_{\kappa,\alpha} - \rho_{\kappa,\alpha} \underset{\sim}{u}_{\kappa,\alpha} \in L^s(0, T; E_0^{r,s}(\Omega)). \quad (5.20)$$

Therefore (5.19), (5.20), (5.16b) and (5.9a,b) yield that

$$\begin{aligned} \left\| \underset{\sim}{\mathcal{B}} \left( \frac{\partial \rho_{\kappa,\alpha}}{\partial t} \right) \right\|_{L^s(0, T; L^r(\Omega))} &= \left\| \underset{\sim}{\mathcal{B}} (\nabla_x \cdot (\alpha \nabla_x \rho_{\kappa,\alpha} - \rho_{\kappa,\alpha} \underset{\sim}{u}_{\kappa,\alpha})) \right\|_{L^s(0, T; L^r(\Omega))} \\ &\leq C \|\alpha \nabla_x \rho_{\kappa,\alpha} - \rho_{\kappa,\alpha} \underset{\sim}{u}_{\kappa,\alpha}\|_{L^s(0, T; L^r(\Omega))} \leq C. \end{aligned} \quad (5.21)$$

On recalling the notation used in (3.86), then, similarly to (4.15), we obtain, on choosing  $\eta = 1$  in (4.118a) and noting (3.16a), that, for all  $t \in [0, T]$ ,

$$0 \leq f \rho_{\kappa,\alpha}(t) = f \rho^0 \leq \|\rho_0\|_{L^\infty(\Omega)}. \quad (5.22)$$

We now choose  $w = \eta \underset{\sim}{\mathcal{B}}((I - f)\rho_{\kappa,\alpha})$  in (5.13), where  $\eta \in C_0^\infty(0, T)$ , to obtain, on



noting (2.3), (1.3), (5.22), (5.16a), (5.21) and (5.9a) that, for  $s' = \frac{5\Gamma-3}{\Gamma-3}$ ,

$$\begin{aligned}
& \left| \int_0^T \eta \int_{\Omega} (c_p \rho_{\kappa,\alpha}^{\gamma+1} + \kappa (\rho_{\kappa,\alpha}^5 + \rho_{\kappa,\alpha}^{\Gamma+1})) \, dx \, dt \right| \\
& \leq C \|\eta\|_{L^\infty(0,T)} \\
& \quad \times \left[ \|f \rho_{\kappa,\alpha}\|_{L^\infty(0,T)} \|p_\kappa(\rho_{\kappa,\alpha})\|_{L^1(\Omega_T)} + \|\mathcal{B}((I-f)\rho_{\kappa,\alpha})\|_{L^{s'}(0,T;W^{1,4}(\Omega))} \right] \\
& + \left| \int_0^T \int_{\Omega} \rho_{\kappa,\alpha} \tilde{u}_{\kappa,\alpha} \cdot \left[ \frac{d\eta}{dt} \mathcal{B}((I-f)\rho_{\kappa,\alpha}) + \eta \mathcal{B}\left(\frac{\partial \rho_{\kappa,\alpha}}{\partial t}\right) \right] \, dx \, dt \right| \\
& \leq C \|\eta\|_{L^\infty(0,T)} \left[ \|\rho_{\kappa,\alpha}\|_{L^\Gamma(\Omega_T)}^\Gamma + \|\rho_{\kappa,\alpha}\|_{L^{s'}(0,T;L^4(\Omega))} \right] \\
& + \|\rho_{\kappa,\alpha} \tilde{u}_{\kappa,\alpha}\|_{L^\infty(0,T;L^{\frac{2\Gamma}{\Gamma+1}}(\Omega))} \left\| \frac{d\eta}{dt} \right\|_{L^1(0,T)} \|\mathcal{B}((I-f)\rho_{\kappa,\alpha})\|_{L^\infty(0,T;L^{\frac{2\Gamma}{\Gamma-1}}(\Omega))} \\
& + \|\rho_{\kappa,\alpha} \tilde{u}_{\kappa,\alpha}\|_{L^\infty(0,T;L^{\frac{2\Gamma}{\Gamma+1}}(\Omega))} \|\eta\|_{L^\infty(0,T)} \left\| \mathcal{B}\left(\frac{\partial \rho_{\kappa,\alpha}}{\partial t}\right) \right\|_{L^1(0,T;L^{\frac{2\Gamma}{\Gamma-1}}(\Omega))} \\
& \leq C \left[ \|\eta\|_{L^\infty(0,T)} + \left\| \frac{d\eta}{dt} \right\|_{L^1(0,T)} \right]. \tag{5.23}
\end{aligned}$$

We now consider (5.23) with  $\eta = \eta_m \in C_0^\infty(0,T)$ ,  $m \in \mathbb{N}$ , where  $\eta_m \in [0,1]$  with  $\eta_m(t) = 1$  for  $t \in [\frac{1}{m}, T - \frac{1}{m}]$  and  $\|\frac{d\eta_m}{dt}\|_{L^\infty(0,T)} \leq 2m$  yielding  $\|\frac{d\eta_m}{dt}\|_{L^1(0,T)} \leq 4$ . As  $\eta_m \rightarrow 1$  pointwise in  $(0,T)$ , as  $m \rightarrow \infty$ , we obtain (5.18).  $\square$

We have the following analogue of Lemmas 4.5 and 4.7.

**Lemma 5.4.** *Let  $\Gamma \geq 8$ ; then, there exists a  $C \in \mathbb{R}_{>0}$ , independent of  $\alpha$ , such that*

$$\left\| \frac{\partial(\rho_{\kappa,\alpha} \tilde{u}_{\kappa,\alpha})}{\partial t} \right\|_{L^{\frac{\Gamma+1}{\Gamma}}(0,T;W_0^{1,\Gamma+1}(\Omega)')} \leq C. \tag{5.24}$$

Hence, for a further subsequence of the subsequence of Lemma 5.2, it follows that, as  $\alpha \rightarrow 0_+$ ,

$$\begin{aligned}
\rho_{\kappa,\alpha} \tilde{u}_{\kappa,\alpha} & \rightarrow \rho_\kappa \tilde{u}_\kappa & \text{weakly in } L^{\frac{10\Gamma-6}{3(\Gamma+1)}}(\Omega_T), \\
& & \text{weakly in } W^{1,\frac{\Gamma+1}{\Gamma}}(0,T;W_0^{1,\Gamma+1}(\Omega)'), \tag{5.25a}
\end{aligned}$$

$$\begin{aligned}
\rho_{\kappa,\alpha} \tilde{u}_{\kappa,\alpha} & \rightarrow \rho_\kappa \tilde{u}_\kappa & \text{in } C_w([0,T];L^{\frac{2\Gamma}{\Gamma+1}}(\Omega)), \\
& & \text{strongly in } L^2(0,T;H^1(\Omega)'), \tag{5.25b}
\end{aligned}$$

$$\rho_{\kappa,\alpha} \tilde{u}_{\kappa,\alpha} \otimes \tilde{u}_{\kappa,\alpha} \rightarrow \rho_\kappa \tilde{u}_\kappa \otimes \tilde{u}_\kappa \quad \text{weakly in } L^2(0,T;L^{\frac{6\Gamma}{4\Gamma+3}}(\Omega)), \tag{5.25c}$$

$$\rho_{\kappa,\alpha} \rightarrow \rho_\kappa \quad \text{weakly in } L^{\Gamma+1}(\Omega_T), \tag{5.25d}$$

$$p_\kappa(\rho_{\kappa,\alpha}) \rightarrow \overline{p_\kappa(\rho_\kappa)} \quad \text{weakly in } L^{\frac{\Gamma+1}{\Gamma}}(\Omega_T), \tag{5.25e}$$

where  $\overline{p_\kappa(\rho_\kappa)} \in L^{\frac{\Gamma+1}{\Gamma}}_{\geq 0}(\Omega_T)$  remains to be identified.

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**Proof.** We deduce from (5.13), (5.18) and as  $4 < s' < \Gamma + 1$  that

$$\begin{aligned} & \left| \int_0^T \left\langle \frac{\partial(\rho_{\kappa,\alpha} u_{\kappa,\alpha})}{\partial t}, w \right\rangle_{W_0^{1,\Gamma+1}(\Omega)} dt \right| \\ & \leq C \|w\|_{L^{\Gamma+1}(0,T;W_0^{1,\Gamma+1}(\Omega))} \quad \forall w \in L^{\Gamma+1}(0,T;W_0^{1,\Gamma+1}(\Omega)), \end{aligned} \quad (5.26)$$

and hence the desired result (5.24).

The results (5.25a–c) follow similarly to (4.97a–e) from (5.9a), (5.24), (3.12a,b), (3.11), (5.10b) and (5.5). The results (5.25d–f) follow immediately from (5.18), (2.3) and (1.3).  $\square$

We have the following analogue of Theorem 4.1.

**Lemma 5.5.** *The triple  $(\rho_\kappa, \underline{u}_\kappa, \widehat{\psi}_\kappa)$ , defined as in Lemmas 5.1 and 5.2, satisfies*

$$\int_0^T \left\langle \frac{\partial \rho_\kappa}{\partial t}, \eta \right\rangle_{H^1(\Omega)} dt - \int_0^T \int_\Omega \rho_\kappa \underline{u}_\kappa \cdot \nabla_x \eta \, dx \, dt = 0 \quad \forall \eta \in L^2(0,T;H^1(\Omega)), \quad (5.27a)$$

with  $\rho_\kappa(\cdot, 0) = \rho_0(\cdot)$ ,

$$\begin{aligned} & \int_0^T \left\langle \frac{\partial(\rho_\kappa \underline{u}_\kappa)}{\partial t}, w \right\rangle_{W_0^{1,\Gamma+1}(\Omega)} dt \\ & + \int_0^T \int_\Omega \left[ S(\underline{u}_\kappa) - \rho_\kappa \underline{u}_\kappa \otimes \underline{u}_\kappa - \overline{p_\kappa(\rho_\kappa)} I \right] : \nabla_x w \, dx \, dt \\ & = \int_0^T \int_\Omega \left[ \rho_\kappa f \cdot w - \left( \tau_1(M \widehat{\psi}_\kappa) - \mathfrak{z} \varrho_\kappa^2 I \right) : \nabla_x w \right] dx \, dt \\ & \quad \forall w \in L^{\Gamma+1}(0,T;W_0^{1,\Gamma+1}(\Omega)), \end{aligned} \quad (5.27b)$$

with  $(\rho_\kappa \underline{u}_\kappa)(\cdot, 0) = (\rho_0 \underline{u}_0)(\cdot)$ , and

$$\begin{aligned} & \int_0^T \left\langle M \frac{\partial \widehat{\psi}_\kappa}{\partial t}, \varphi \right\rangle_{H^s(\Omega \times D)} dt + \frac{1}{4\lambda} \sum_{i=1}^K \sum_{j=1}^K A_{ij} \int_0^T \int_{\Omega \times D} M \nabla_{q_j} \widehat{\psi}_\kappa \cdot \nabla_{q_i} \varphi \, dq \, dx \, dt \\ & + \int_0^T \int_{\Omega \times D} M \left[ \varepsilon \nabla_x \widehat{\psi}_\kappa - \underline{u}_\kappa \widehat{\psi}_\kappa \right] \cdot \nabla_x \varphi \, dq \, dx \, dt \\ & - \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \left[ \sigma(\underline{u}_\kappa) q_i \right] \widehat{\psi}_\kappa \cdot \nabla_{q_i} \varphi \, dq \, dx \, dt = 0 \\ & \quad \forall \varphi \in L^2(0,T;H^s(\Omega \times D)), \end{aligned} \quad (5.27c)$$

with  $\widehat{\psi}_\kappa(\cdot, 0) = \widehat{\psi}_0(\cdot)$  and  $s > 1 + \frac{1}{2}(K+1)d$ .

In addition, the triple  $(\rho_\kappa, \underline{u}_\kappa, \widehat{\psi}_\kappa)$  satisfies, for a.a.  $t' \in (0, T)$ ,

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega} \rho_\kappa(t') |\underline{u}_\kappa(t')|^2 \, dx + \int_{\Omega} P_\kappa(\rho_\kappa(t')) \, dx + k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_\kappa(t')) \, dq \, dx \\
& + \mu^S c_0 \int_0^{t'} \|\underline{u}_\kappa\|_{H^1(\Omega)}^2 \, dt \\
& + k \int_0^{t'} \int_{\Omega \times D} M \left[ \frac{a_0}{2\lambda} \left| \nabla_q \sqrt{\widehat{\psi}_\kappa} \right|^2 + 2\varepsilon \left| \nabla_x \sqrt{\widehat{\psi}_\kappa} \right|^2 \right] \, dq \, dx \, dt \\
& + \mathfrak{z} \|\varrho_\kappa(t')\|_{L^2(\Omega)}^2 + 2\mathfrak{z} \varepsilon \int_0^{t'} \|\nabla_x \varrho_\kappa\|_{L^2(\Omega)}^2 \, dt \\
& \leq e^{t'} \left[ \frac{1}{2} \int_{\Omega} \rho_0 |\underline{u}_0|^2 \, dx + \int_{\Omega} P_\kappa(\rho_0) \, dx + k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_0) \, dq \, dx \right. \\
& \quad \left. + \mathfrak{z} \int_{\Omega} \left( \int_D M \widehat{\psi}_0 \, dq \right)^2 \, dx + \frac{1}{2} \int_0^{t'} \|f\|_{L^\infty(\Omega)}^2 \, dt \int_{\Omega} \rho_0 \, dx \right] \leq C, \quad (5.28)
\end{aligned}$$

where  $C \in \mathbb{R}_{>0}$  is independent of  $\kappa$ .

**Proof.** Passing to the limit  $\alpha \rightarrow 0_+$  for the subsequence of Lemma 5.4 in (4.118a) yields (5.27a) subject to the stated initial condition, on noting (5.10a,c), (5.25a) and (3.16b).

Similarly to the proof of (4.118b), passing to the limit  $\alpha \rightarrow 0_+$  for the subsequence of Lemma 5.4 in (4.118b) yields (5.27b) subject to the stated initial condition, on noting (5.5), (5.10a,d,e), (5.25a-c,e), (5.6e), (5.8b) and (3.16b). Similarly to the proof of (4.118c), passing to the limit  $\alpha \rightarrow 0_+$  for the subsequence of Lemma 5.4 in (4.118c) yields (5.27c) subject to the stated initial condition, on noting (5.5), (5.6a-d), (5.7) and (3.2). Similarly to the proof of (4.119), we deduce (5.28) from (4.119) using the results (5.25c), (5.10f), (5.6a,b,f), (5.5), (5.8a) and (3.16c).  $\square$

Finally, to obtain the complete analogue of Theorem 4.1, we have to identify  $\overline{p_\kappa(\rho_\kappa)}$ , which appears in (5.27b) and (5.25e), by establishing that  $\overline{p_\kappa(\rho_\kappa)} = p_\kappa(\rho_\kappa)$ . Due to the presence of the extra stress term in the momentum equation, we require a modification of the effective viscous flux compactness result, Proposition 7.36 in Novotný & Straškraba<sup>24</sup>. Such results require pseudodifferential operators identified via the Fourier transform  $\mathfrak{F}$ . We briefly recall the key ideas, and refer to Section 4.4.1 in Novotný & Straškraba<sup>24</sup> for the details. With

$$\mathfrak{S}(\mathbb{R}^d) := \left\{ \eta \in C^\infty(\mathbb{R}^d) : \sup_{x \in \mathbb{R}^d} \left| x_1^{s_1} \cdots x_d^{s_d} \frac{\partial^{\lambda_1} \eta}{\partial x_1^{\lambda_1} \cdots \partial x_d^{\lambda_d}} \right| \leq C(|s|, |\lambda|) \quad \forall s, \lambda \in \mathbb{N}^d \right\}, \quad (5.29)$$

the space of smooth rapidly decreasing (complex-valued) functions, we introduce the Fourier transform  $\mathfrak{F} : \mathfrak{S}(\mathbb{R}^d) \rightarrow \mathfrak{S}(\mathbb{R}^d)$ , and its inverse  $\mathfrak{F}^{-1} : \mathfrak{S}(\mathbb{R}^d) \rightarrow \mathfrak{S}(\mathbb{R}^d)$ ,

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defined by

$$[\mathfrak{F}(\eta)](\tilde{y}) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i\tilde{x}\cdot\tilde{y}} \tilde{\eta}(\tilde{x}) d\tilde{x} \quad \text{and} \quad [\mathfrak{F}^{-1}(\eta)](\tilde{x}) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{i\tilde{x}\cdot\tilde{y}} \tilde{\eta}(\tilde{y}) d\tilde{y}. \quad (5.30)$$

These are extended to  $\mathfrak{F}, \mathfrak{F}^{-1} : \mathfrak{S}(\mathbb{R}^d)' \rightarrow \mathfrak{S}(\mathbb{R}^d)'$ , where  $\mathfrak{S}(\mathbb{R}^d)'$ , the dual of  $\mathfrak{S}(\mathbb{R}^d)$ , is the space of tempered distributions, via

$$\begin{aligned} \langle \mathfrak{F}(\eta), \xi \rangle_{\mathfrak{S}(\mathbb{R}^d)} &= \langle \eta, \mathfrak{F}(\xi) \rangle_{\mathfrak{S}(\mathbb{R}^d)} & \forall \xi \in \mathfrak{S}(\mathbb{R}^d) \\ \langle \mathfrak{F}^{-1}(\eta), \xi \rangle_{\mathfrak{S}(\mathbb{R}^d)} &= \langle \eta, \mathfrak{F}^{-1}(\xi) \rangle_{\mathfrak{S}(\mathbb{R}^d)} & \forall \xi \in \mathfrak{S}(\mathbb{R}^d). \end{aligned} \quad (5.31)$$

We now introduce the inverse divergence operator  $\mathcal{A}_j : \mathfrak{S}(\mathbb{R}^d) \rightarrow \mathfrak{S}(\mathbb{R}^d)'$ ,  $j = 1, \dots, d$ , such that

$$\mathcal{A}_j(\eta) = -\mathfrak{F}^{-1} \left( \frac{iy_j}{|y|^2} [\mathfrak{F}(\eta)](\tilde{y}) \right). \quad (5.32)$$

It follows from Theorems 1.55 and 1.57 in Novotný & Straškraba<sup>24</sup> (or Lemmas B.1 and B.2 in Appendix B in Barrett & Süli<sup>12</sup>) and Sobolev embedding that, for  $j = 1, \dots, d$ ,

$$\|\nabla_x \mathcal{A}_j(\eta)\|_{L^r(\mathbb{R}^d)} \leq C(r) \|\eta\|_{L^r(\mathbb{R}^d)} \quad \forall \eta \in \mathfrak{S}(\mathbb{R}^d), \quad r \in (1, \infty), \quad (5.33a)$$

$$\|\mathcal{A}_j(\eta)\|_{L^{\frac{dr}{d-r}}(\mathbb{R}^d)} \leq C(r) \|\eta\|_{L^r(\mathbb{R}^d)} \quad \forall \eta \in \mathfrak{S}(\mathbb{R}^d), \quad r \in (1, d). \quad (5.33b)$$

Hence, we deduce from (5.33a,b) that  $\mathcal{A}_j$  can be extended to  $\mathcal{A}_j : L^r(\mathbb{R}^d) \rightarrow D^{1,r}(\mathbb{R}^d)$  for  $r \in (1, \infty)$ ,  $j = 1, \dots, d$ , where  $D^{1,r}(\mathbb{R}^d)$  is a homogeneous Sobolev space; see Section 1.3.6 in Novotný & Straškraba<sup>24</sup> (or Appendix A in Barrett & Süli<sup>12</sup>). In addition, by duality,  $\mathcal{A}_j$  can be extended to  $\mathcal{A}_j : D^{1,r}(\mathbb{R}^d)' \rightarrow D^{1,r}(\mathbb{R}^d)$  for  $r \in (1, \infty)$ ,  $j = 1, \dots, d$ , see (4.4.4) in Novotný & Straškraba<sup>24</sup>. Moreover, as  $\mathcal{A}_j(\eta)$  is real, for a real-valued function  $\eta$ , and from the Parseval–Plancherel formula we have, for all  $\eta \in L^r(\mathbb{R}^d)$  and  $\xi \in L^{\frac{r}{r-1}}(\mathbb{R}^d)$ ,  $r \in (1, \infty)$ , having compact support that

$$\int_{\mathbb{R}^d} \mathcal{A}_j(\eta) \xi d\tilde{x} = - \int_{\mathbb{R}^d} \eta \mathcal{A}_j(\xi) d\tilde{x}, \quad j = 1, \dots, d. \quad (5.34)$$

Finally, we introduce the so-called Riesz operator  $\mathcal{R}_{kj} : L^r(\mathbb{R}^d) \rightarrow L^r(\mathbb{R}^d)$ ,  $r \in (1, \infty)$ , defined by

$$\mathcal{R}_{kj}(\eta) = \frac{\partial}{\partial x_k} \mathcal{A}_j(\eta), \quad j, k = 1, \dots, d. \quad (5.35)$$

We note for all  $\eta \in L^r(\mathbb{R}^d)$  and  $\xi \in L^{\frac{r}{r-1}}(\mathbb{R}^d)$ ,  $r \in (1, \infty)$ , that

$$\sum_{j=1}^d \mathcal{R}_{jj}(\eta) = \sum_{j=1}^d \frac{\partial}{\partial x_j} \mathcal{A}_j(\eta) = \eta, \quad (5.36a)$$

$$\mathcal{R}_{kj}(\eta) = \mathcal{R}_{jk}(\eta) \quad \text{and} \quad \int_{\mathbb{R}^d} \mathcal{R}_{jk}(\eta) \xi d\tilde{x} = \int_{\mathbb{R}^d} \eta \mathcal{R}_{jk}(\xi) d\tilde{x}, \quad j, k = 1, \dots, d. \quad (5.36b)$$

Below we use the notation  $\mathcal{A}(\cdot)$  and  $\mathcal{R}(\cdot)$  with components  $\mathcal{A}_i(\cdot)$  and  $\mathcal{R}_{ij}(\cdot)$ ,  $i, j = 1, \dots, d$ , respectively. We shall adopt the convention that whenever any of these operators is applied to a function or a distribution that has been defined on  $\Omega$  only, it is tacitly understood that the function or distribution in question has been extended by 0 from  $\Omega$  to the whole of  $\mathbb{R}^d$ .

We now have the following modification of Proposition 7.36 in Novotný & Straškraba<sup>24</sup>, which is adequate for our purposes.

**Lemma 5.6.** *Given  $\{(g_n, \underline{u}_n, \underline{m}_n, p_n, \underline{\tau}_n, f_n, \underline{F}_n)\}_{n \in \mathbb{N}}$ , we assume for any  $\zeta \in C_0^\infty(\Omega)$  that, as  $n \rightarrow \infty$ ,*

$$g_n \rightarrow g \quad \text{in } C_w([0, T]; L^q(\Omega)), \quad \text{weakly } (-\star) \text{ in } L^\omega(\Omega_T), \quad (5.37a)$$

$$\underline{u}_n \rightarrow \underline{u} \quad \text{weakly in } L^2(0, T; H_0^1(\Omega)), \quad (5.37b)$$

$$\underline{m}_n \rightarrow \underline{m} \quad \text{in } C_w([0, T]; L^z(\Omega)), \quad (5.37c)$$

$$p_n \rightarrow p \quad \text{weakly in } L^r(\Omega_T), \quad (5.37d)$$

$$\underline{\tau}_n \rightarrow \underline{\tau} \quad \text{strongly in } L^1(0, T; L^{\frac{q}{q-1}}(\Omega)), \quad (5.37e)$$

$$f_n \rightarrow f \quad \text{weakly in } L^2(0, T; H^1(\Omega)'), \quad (5.37f)$$

$$\mathcal{A}(\zeta f_n) \rightarrow \mathcal{A}(\zeta f) \quad \text{strongly in } L^2(0, T; L^{\frac{z}{z-1}}(\Omega)), \quad (5.37g)$$

$$\underline{F}_n \rightarrow \underline{F} \quad \text{weakly in } L^s(\Omega_T), \quad (5.37h)$$

where  $q \in (d, \infty)$ ,  $r, s \in (1, \infty)$ ,  $\omega \in [\max\{2, \frac{r}{r-1}\}, \infty]$  and  $z \in (\frac{6q}{5q-6}, \infty)$ .

In addition, suppose that

$$\frac{\partial g_n}{\partial t} + \nabla_x \cdot (\underline{u}_n g_n) = f_n \quad \text{in } C_0^\infty(\Omega_T)', \quad (5.38a)$$

$$\begin{aligned} \frac{\partial \underline{m}_n}{\partial t} + \nabla_x \cdot (\underline{m}_n \otimes \underline{u}_n) - \mu \Delta_x \underline{u}_n - (\mu + \lambda) \nabla_x (\nabla_x \cdot \underline{u}_n) + \nabla_x p_n \\ = \underline{F}_n + \nabla_x \cdot \underline{\tau}_n \quad \text{in } C_0^\infty(\Omega_T)'. \end{aligned} \quad (5.38b)$$

Then it follows that, for any  $\zeta \in C_0^\infty(\Omega)$  and  $\eta \in C_0^\infty(0, T)$ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^T \eta \left( \int_\Omega \zeta g_n [p_n - (2\mu + \lambda) \nabla_x \cdot \underline{u}_n] \, dx \right) dt \\ = \int_0^T \eta \left( \int_\Omega \zeta g [p - (2\mu + \lambda) \nabla_x \cdot \underline{u}] \, dx \right) dt. \end{aligned} \quad (5.39)$$

**Proof.** We adapt the proof of Proposition 7.36 in Novotný & Straškraba<sup>24</sup>, by just pointing out the key differences. As  $q > d$ , then  $q^*$ , the Sobolev conjugate of  $q$  in the notation (1.3.64) of Novotný & Straškraba<sup>24</sup>, is such that  $q^* = \infty$ . Hence our restrictions on  $r, s, \omega$  and  $z$  satisfy the restrictions of Proposition 7.36 in Novotný &

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Straškraba<sup>24</sup>. With any  $\tilde{\zeta} \in C_0^\infty(\Omega)$ , it follows from (5.38a) and properties (5.34)–(5.36a,b) of  $\mathcal{A}_j$  and  $\mathcal{R}_{kj}$  that, for  $i = 1, \dots, d$ ,

$$\frac{\partial}{\partial t} \mathcal{A}_i(\tilde{\zeta} g_n) + \sum_{j=1}^d \mathcal{R}_{ij}(\tilde{\zeta} g_n u_n^j) = \mathcal{A}_i(\tilde{\zeta} f_n) + \mathcal{A}_i(g_n u_n \cdot \nabla_x \tilde{\zeta}) \quad \text{in } C_0^\infty(\Omega_T)', \quad (5.40)$$

where we adopt the notation  $u_n^j$  for the  $j^{\text{th}}$  component of  $u_n$ . With any  $\zeta, \tilde{\zeta} \in C_0^\infty(\Omega)$  and  $\eta \in C_0^\infty(0, T)$ , we now consider  $\eta \zeta \mathcal{A}(\tilde{\zeta} g_n)$  as a test function for (5.38b). It follows from (5.37a) and (5.33a,b) that  $\mathcal{A}(\tilde{\zeta} g_n) \in L^\infty(0, T; \mathcal{W}^{1,q}(\Omega)) \cap L^\omega(0, T; \mathcal{W}^{1,\omega}(\Omega))$ , and hence  $\mathcal{A}(\tilde{\zeta} g_n) \in \underline{L}^\infty(\Omega_T)$  as  $q > d$ . Similarly to (5.9a),  $g_n u_n \in L^2(0, T; \underline{L}^{\frac{6q}{q+6}}(\Omega))$  and  $m_n \otimes u_n \in L^2(0, T; \underline{L}^{\frac{6z}{z+6}}(\Omega))$ . As  $z \in (\frac{6q}{5q-6}, \infty)$ , and therefore  $\frac{z}{z-1} \in (1, \frac{6q}{q+6})$  and  $\frac{6z}{z+6} \in (\frac{q}{q-1}, 6)$ , it follows from (5.40), (5.33a,b), (5.35) and (5.37g) that

$$\frac{\partial}{\partial t} \mathcal{A}(\tilde{\zeta} g_n) \in L^2(0, T; \underline{L}^{\frac{z}{z-1}}(\Omega)).$$

Noting the above and (5.37b–e,h), we see that  $\eta \zeta \mathcal{A}(\tilde{\zeta} g_n)$  is a valid test function for (5.38b), and we obtain, on using integration by parts several times and properties (5.34)–(5.36a,b) of  $\mathcal{A}_j$  and  $\mathcal{R}_{kj}$ , that

$$\begin{aligned} & \int_0^T \eta \left( \int_\Omega \zeta \tilde{\zeta} g_n [p_n - (2\mu + \lambda) \nabla_x \cdot u_n] \, dx \right) dt \\ &= \mu \int_0^T \eta \left( \int_\Omega \left( \nabla_x u_n : \mathcal{A}(\tilde{\zeta} g_n) \otimes \nabla_x \zeta - u_n \otimes \nabla_x \zeta : \mathcal{R}(\tilde{\zeta} g_n) + \tilde{\zeta} g_n u_n \cdot \nabla_x \zeta \right) dx \right) dt \\ &+ \int_0^T \eta \left( \int_\Omega \left( \left( \tau_n - m_n \otimes u_n \right) : \mathcal{A}(\tilde{\zeta} g_n) \otimes \nabla_x \zeta \right. \right. \\ &\quad \left. \left. - [p_n - (\mu + \lambda) \nabla_x \cdot u_n] \mathcal{A}(\tilde{\zeta} g_n) \cdot \nabla_x \zeta \right) dx \right) dt \\ &+ \int_0^T \eta \left( \int_\Omega \zeta \left( \left( \tau_n - m_n \otimes u_n \right) : \mathcal{R}(\tilde{\zeta} g_n) - F_n \cdot \mathcal{A}(\tilde{\zeta} g_n) \right) dx \right) dt \\ &- \int_0^T \frac{d\eta}{dt} \left( \int_\Omega \zeta m_n \cdot \mathcal{A}(\tilde{\zeta} g_n) \, dx \right) dt \\ &- \int_0^T \eta \left( \int_\Omega \zeta m_n \cdot \left[ \mathcal{A}(\tilde{\zeta} f_n) + \mathcal{A}(g_n u_n \cdot \nabla_x \tilde{\zeta}) \right] dx \right) dt \\ &+ \int_0^T \eta \left( \int_\Omega \tilde{\zeta} \sum_{i=1}^d \sum_{j=1}^d g_n u_n^j \mathcal{R}_{ij}(\zeta m_n^i) \, dx \right) dt. \end{aligned} \quad (5.41)$$

The equation (5.41) is exactly the same as (7.5.12) in Novotný & Straškraba<sup>24</sup>, except for the extra  $\underline{\tau}_n$  terms and the change of notation. We will therefore just concentrate on the terms involving  $\underline{\tau}_n$ , as the other terms are dealt with as in Novotný & Straškraba<sup>24</sup>.

It follows from (5.37a), (5.33a,b), see equations (7.5.18)–(7.5.20) in Novotný & Straškraba<sup>24</sup> for the details, that

$$\begin{aligned} \mathcal{A}(\tilde{\zeta} g_n) &\rightarrow \mathcal{A}(\tilde{\zeta} g) \quad \text{weakly in } L^\infty(0, T; W^{1,q}(\Omega)), \\ &\text{strongly in } L^v(\Omega_T), \end{aligned} \quad (5.42a)$$

$$\begin{aligned} \mathcal{R}(\tilde{\zeta} g_n) &\rightarrow \mathcal{R}(\tilde{\zeta} g) \quad \text{weakly in } L^\infty(0, T; L^q(\Omega)), \\ &\text{strongly in } L^v(0, T; H^{-1}(\Omega)), \end{aligned} \quad (5.42b)$$

where  $v \in [1, \infty)$ . It follows from (5.37e) and (5.42a,b) that, as  $n \rightarrow \infty$ ,

$$\begin{aligned} &\int_0^T \eta \left( \int_{\Omega} \tau_n : \left[ \mathcal{A}(\tilde{\zeta} g_n) \otimes \nabla_x \zeta + \mathcal{R}(\tilde{\zeta} g_n) \zeta \right] dx \right) dt \\ &\rightarrow \int_0^T \eta \left( \int_{\Omega} \tau : \left[ \mathcal{A}(\tilde{\zeta} g) \otimes \nabla_x \zeta + \mathcal{R}(\tilde{\zeta} g) \zeta \right] dx \right) dt. \end{aligned} \quad (5.43)$$

Combining (5.43) with the convergence, as  $n \rightarrow \infty$ , of other terms in (5.41) as in the proof of Proposition 7.36 in Novotný & Straškraba<sup>24</sup>, which involves the use of the crucial ‘commutator lemma’ (Lemma 4.25 in Novotný & Straškraba<sup>24</sup>, or Lemma D.3 in Appendix D in Barrett & Süli<sup>12</sup>), we obtain

$$\begin{aligned} &\lim_{n \rightarrow \infty} \int_0^T \eta \left( \int_{\Omega} \zeta \tilde{\zeta} g_n [p_n - (2\mu + \lambda) \nabla_x \cdot u_n] dx \right) dt \\ &= \mu \int_0^T \eta \left( \int_{\Omega} \left( \nabla_x u : \mathcal{A}(\tilde{\zeta} g) \otimes \nabla_x \zeta - u \otimes \nabla_x \zeta : \mathcal{R}(\tilde{\zeta} g) + \tilde{\zeta} g u \cdot \nabla_x \zeta \right) dx \right) dt \\ &\quad + \int_0^T \eta \left( \int_{\Omega} \left( (\tau - m \otimes u) : \mathcal{A}(\tilde{\zeta} g) \otimes \nabla_x \zeta \right. \right. \\ &\quad \quad \left. \left. - [p - (\mu + \lambda) \nabla_x \cdot u] \mathcal{A}(\tilde{\zeta} g) \cdot \nabla_x \zeta \right) dx \right) dt \\ &\quad + \int_0^T \eta \left( \int_{\Omega} \zeta \left( (\tau - m \otimes u) : \mathcal{R}(\tilde{\zeta} g) - F \cdot \mathcal{A}(\tilde{\zeta} g) \right) dx \right) dt \\ &\quad - \int_0^T \frac{d\eta}{dt} \left( \int_{\Omega} \zeta m \cdot \mathcal{A}(\tilde{\zeta} g) dx \right) dt \\ &\quad - \int_0^T \eta \left( \int_{\Omega} \zeta m \cdot \left[ \mathcal{A}(\tilde{\zeta} f) + \mathcal{A}(g u \cdot \nabla_x \tilde{\zeta}) \right] dx \right) dt \\ &\quad + \int_0^T \eta \left( \int_{\Omega} \tilde{\zeta} \sum_{i=1}^d \sum_{j=1}^d g u^j \mathcal{R}_{ij}(\zeta m^i) dx \right) dt, \end{aligned} \quad (5.44)$$

which is exactly the same as (7.5.25) in Novotný & Straškraba<sup>24</sup>, except for the extra  $\tau$  terms and the change of notation.

In addition, the equations (5.38a,b) are exactly the same as in (7.5.7)–(7.5.8) in Novotný & Straškraba<sup>24</sup> except for the extra  $\tau_n$  term. One can use (5.37a–h) to

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pass to the limit  $n \rightarrow \infty$  in (5.38a,b) to obtain

$$\frac{\partial g}{\partial t} + \nabla_x \cdot (\tilde{u} g) = f \quad \text{in } C_0^\infty(\Omega_T)', \quad (5.45a)$$

$$\begin{aligned} \frac{\partial m}{\partial t} + \nabla_x \cdot (\tilde{m} \otimes \tilde{u}) - \mu \Delta_x \tilde{u} - (\mu + \lambda) \nabla_x (\nabla_x \cdot \tilde{u}) + \nabla_x p \\ = F + \nabla_x \cdot \tilde{\tau} \quad \text{in } C_0^\infty(\Omega_T)', \end{aligned} \quad (5.45b)$$

see Novotný & Straškraba<sup>24</sup> for details. Clearly, the  $\tilde{\tau}_n$  term in (5.38b) is easily dealt with using (5.37e). Similarly to (5.40), we deduce that  $\eta \zeta \mathcal{A}(\tilde{\zeta} g)$  is a valid test function for (5.45b), for any  $\zeta, \tilde{\zeta} \in C_0^\infty(\Omega)$  and  $\eta \in C_0^\infty(0, T)$ , and we obtain (5.41) without the subscript  $n$ . Combining this with (5.44), we deduce that, for any  $\zeta, \tilde{\zeta} \in C_0^\infty(\Omega)$  and  $\eta \in C_0^\infty(0, T)$ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_0^T \eta \left( \int_\Omega \zeta \tilde{\zeta} g_n [p_n - (2\mu + \lambda) \nabla_x \cdot \tilde{u}_n] \tilde{u}_n \, dx \right) dt \\ = \int_0^T \eta \left( \int_\Omega \zeta \tilde{\zeta} g [p - (2\mu + \lambda) \nabla_x \cdot \tilde{u}] \tilde{u} \, dx \right) dt. \end{aligned} \quad (5.46)$$

Hence we arrive at (5.39) by taking  $\tilde{\zeta} \in C_0^\infty(\Omega)$  such that  $\tilde{\zeta} \equiv 1$  on the support of the function  $\zeta$ .  $\square$

We need also the following variation of Lemma 5.6 for later use in Section 6.

**Corollary 5.1.** *The results of Lemma 5.6 hold with the assumptions (5.37f,g) replaced by*

$$f_n \rightarrow f \quad \text{weakly in } L^2(\Omega_T), \quad \text{as } n \rightarrow \infty. \quad (5.47)$$

**Proof.** One can still pass to the limit  $n \rightarrow \infty$  in (5.38a) to obtain (5.45a) using (5.47) in place of (5.37f,g). We deduce from (5.47), (5.33b) and (5.34) that

$$\mathcal{A}(\zeta f_n) \rightarrow \mathcal{A}(\zeta f) \quad \text{weakly in } L^2(0, T; L^6(\Omega)), \quad \text{as } n \rightarrow \infty. \quad (5.48)$$

As  $\frac{z}{z-1} < \frac{6q}{q+6} < 6$ , (5.48) ensures that one can still conclude from (5.40) that  $\eta \zeta \mathcal{A}(\tilde{\zeta} g_n)$  is a valid test function for (5.38b). Similarly, one can deduce that  $\eta \zeta \mathcal{A}(\tilde{\zeta} g)$  is a valid test function for (5.45b). The only other place where (5.37f,g) are used in the proof of Lemma 5.6 is in dealing with the term involving  $f_n$  on the right-hand side of (5.41); that is, the term

$$- \int_0^T \eta \left( \int_\Omega \zeta \tilde{m}_n \cdot \mathcal{A}(\tilde{\zeta} f_n) \tilde{u}_n \, dx \right) dt = \int_0^T \eta \left( \int_\Omega \tilde{\zeta} f_n \sum_{i=1}^d \mathcal{A}_i(\zeta m_n^i) \tilde{u}_n \, dx \right) dt, \quad (5.49)$$

where we have noted (5.34). Similarly to (5.42a), it follows from (5.37c), (5.33a,b) and Sobolev embedding, as  $z > \frac{6}{5}$ , that

$$\mathcal{A}(\tilde{\zeta} m_n) \rightarrow \mathcal{A}(\tilde{\zeta} m) \quad \text{weakly in } L^\infty(0, T; W^{1,z}(\Omega)), \quad \text{strongly in } L^v(0, T; L^3(\Omega)), \quad (5.50)$$



where  $v \in [1, \infty)$ . Therefore, (5.50), (5.47) and (5.34) imply that we can pass to the limit  $n \rightarrow \infty$  in (5.49) to obtain the term involving  $f$  on the right-hand side of (5.44).  $\square$

In order to identify  $\overline{p_\kappa(\rho_\kappa)}$  in (5.27b) and (5.25e), we now apply Lemma 5.6 with (5.38a,b) being (4.118a,b) so that  $\mu = \frac{\mu^S}{2}$  and  $\lambda = \mu^B - \frac{\mu^S}{d}$ ,  $g_n = \rho_{\kappa,\alpha}$ ,  $u_n = u_{\kappa,\alpha}$ ,  $m_n = \rho_{\kappa,\alpha} u_{\kappa,\alpha}$ ,  $p_n = p_\kappa(\rho_{\kappa,\alpha})$ ,  $\tau_n = \tau_\kappa(M \widehat{\psi}_{\kappa,\alpha}) + \frac{\alpha}{2} (u_{\kappa,\alpha} \otimes \nabla_x \rho_{\kappa,\alpha})$ ,  $f_n = \alpha \Delta_x \rho_{\kappa,\alpha}$  and  $F_n = \rho_{\kappa,\alpha} f - \frac{\alpha}{2} (\nabla_x \rho_{\kappa,\alpha} \cdot \nabla_x) u_{\kappa,\alpha}$ . With  $\{(\rho_{\kappa,\alpha}, u_{\kappa,\alpha}, \widehat{\psi}_{\kappa,\alpha})\}_{\alpha>0}$  being the subsequence (not indicated) of Lemma 5.4, we have that (5.37a–d) hold with  $g = \rho_\kappa$ ,  $u = u_\kappa$ ,  $m = \rho_\kappa u_\kappa$  and  $p = \overline{p_\kappa(\rho_\kappa)}$ , and  $q = \Gamma$ ,  $\omega = \Gamma + 1$ ,  $z = \frac{2\Gamma}{\Gamma+1}$  and  $r = \frac{\Gamma+1}{\Gamma}$  on recalling (5.10a), (5.25b,d,e) and (5.5). We note that  $\omega = \Gamma + 1 = \frac{r}{r-1} > 2$  and, as  $\Gamma \geq 8$ ,  $z = \frac{2\Gamma}{\Gamma+1} \geq \frac{16}{9} > \frac{24}{17} \geq \frac{6\Gamma}{5\Gamma-6} = \frac{6q}{5q-6}$ . Hence, the constraints on  $q, r, \omega$  and  $z$  hold. The results (5.37e,h) hold with  $\tau = \tau_\kappa(M \widehat{\psi}_\kappa)$ ,  $F = \rho_\kappa f$  and  $s = \frac{5\Gamma-3}{4\Gamma}$ , on recalling (5.6e), (5.10d,e) and (5.25d), and noting that  $\frac{4(d+2)}{3d+4} \geq \frac{20}{13} > \frac{8}{7} \geq \frac{q}{q-1}$ . Finally, the results (5.37f,g) hold with  $f = 0$  on recalling (5.10c) and the properties of  $\mathcal{A}$  and  $\mathcal{R}$ , on noting that  $\frac{z}{z-1} = \frac{2\Gamma}{\Gamma-1} < \frac{10\Gamma-6}{3(\Gamma+1)}$ , see (7.9.21) in Novotný & Straškraba<sup>24</sup> for details. Hence, we obtain from (5.39) for the subsequence of Lemma 5.4 that, for all  $\zeta \in C_0^\infty(\Omega)$  and  $\eta \in C_0^\infty(0, T)$ ,

$$\begin{aligned} \lim_{\alpha \rightarrow 0_+} \int_0^T \eta \left( \int_\Omega \zeta \rho_{\kappa,\alpha} [p_\kappa(\rho_{\kappa,\alpha}) - \mu^* \nabla_x \cdot u_{\kappa,\alpha}] dx \right) dt \\ = \int_0^T \eta \left( \int_\Omega \zeta \rho_\kappa [\overline{p_\kappa(\rho_\kappa)} - \mu^* \nabla_x \cdot u_\kappa] dx \right) dt, \end{aligned} \quad (5.51)$$

where  $\mu^* := \frac{(d-1)}{d} \mu^S + \mu^B$ . The first two bounds in (5.9a) yield that

$$\|\rho_{\kappa,\alpha} \nabla_x \cdot u_{\kappa,\alpha}\|_{L^2(0,T;L^{\frac{2\Gamma}{\Gamma+2}}(\Omega))} \leq C, \quad (5.52)$$

and hence there exists a  $\overline{\rho_\kappa \nabla_x \cdot u_\kappa} \in L^2(0, T; L^{\frac{2\Gamma}{\Gamma+2}}(\Omega))$  such that for a subsequence (not indicated)

$$\rho_{\kappa,\alpha} \nabla_x \cdot u_{\kappa,\alpha} \rightharpoonup \overline{\rho_\kappa \nabla_x \cdot u_\kappa} \quad \text{weakly in } L^2(0, T; L^{\frac{2\Gamma}{\Gamma+2}}(\Omega)), \quad \text{as } \alpha \rightarrow 0_+. \quad (5.53)$$

It follows from the monotonicity of  $p_\kappa(\cdot)$  that

$$\begin{aligned} \rho_{\kappa,\alpha} p_\kappa(\rho_{\kappa,\alpha}) &= (\rho_{\kappa,\alpha} - \rho_\kappa) (p_\kappa(\rho_{\kappa,\alpha}) - p_\kappa(\rho_\kappa)) + (\rho_{\kappa,\alpha} - \rho_\kappa) p_\kappa(\rho_\kappa) + \rho_\kappa p_\kappa(\rho_{\kappa,\alpha}) \\ &\geq (\rho_{\kappa,\alpha} - \rho_\kappa) p_\kappa(\rho_\kappa) + \rho_\kappa p_\kappa(\rho_{\kappa,\alpha}) \quad \text{a.e. in } \Omega_T. \end{aligned} \quad (5.54)$$

We deduce from (5.51), (5.54), (5.53) and (5.25d,e) that for all nonnegative  $\zeta \in C_0^\infty(\Omega)$  and  $\eta \in C_0^\infty(0, T)$ ,

$$\begin{aligned} \int_0^T \eta \left( \int_\Omega \zeta [\overline{\rho_\kappa \nabla_x \cdot u_\kappa} - \rho_\kappa \nabla_x \cdot u_\kappa] dx \right) dt \geq 0 \\ \Rightarrow \overline{\rho_\kappa \nabla_x \cdot u_\kappa} \geq \rho_\kappa \nabla_x \cdot u_\kappa \quad \text{a.e. in } \Omega_T, \end{aligned} \quad (5.55)$$

where we have noted (4.128) with  $(0, T)$  replaced by  $\Omega_T$  for the final implication.

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Next, we introduce  $\mathfrak{L}(s) = s \log s$  for  $s \in [0, \infty)$ . On recalling (3.16a,b), we have for a subsequence (not indicated), via Lebesgue's dominated convergence theorem, that

$$\lim_{\alpha \rightarrow 0_+} \int_{\Omega} \mathfrak{L}(\rho^0) \, dx \underset{\sim}{=} \int_{\Omega} \mathfrak{L}(\rho_0) \, dx. \quad (5.56)$$

We can now follow the discussion in Section 7.9.3 in Novotný & Straškraba<sup>24</sup> to deduce that  $\overline{p_{\kappa}(\rho_{\kappa})} = p_{\kappa}(\rho_{\kappa})$ . For the benefit of the reader, we briefly outline the argument. One deduces from (5.27a) as  $\rho_{\kappa} \in C_w([0, T]; L^{\Gamma}_{\geq 0}(\Omega))$ , via renormalization and noting that  $s \mathfrak{L}'(s) - \mathfrak{L}(s) = s$  for  $s \in [0, \infty)$ , that, for any  $t' \in (0, T]$ ,

$$\int_{\Omega} [\mathfrak{L}(\rho_{\kappa})(t') - \mathfrak{L}(\rho_0)] \, dx \underset{\sim}{=} - \int_0^{t'} \int_{\Omega} \rho_{\kappa} \nabla_x \cdot u_{\kappa} \, dx \, dt. \quad (5.57)$$

On noting (5.17), one can choose, similarly to (4.8),  $\eta = \chi_{[0, t']} [\log(\rho_{\kappa, \alpha} + \varsigma) - 1]$ , where  $\varsigma \in \mathbb{R}_{>0}$ , in (4.118a), and on passing to the limit  $\varsigma \rightarrow 0_+$  obtain that, for any  $t' \in (0, T]$ ,

$$\int_{\Omega} [\mathfrak{L}(\rho_{\kappa, \alpha})(t') - \mathfrak{L}(\rho^0)] \, dx \underset{\sim}{\leq} - \int_0^{t'} \int_{\Omega} \rho_{\kappa, \alpha} \nabla_x \cdot u_{\kappa, \alpha} \, dx \, dt. \quad (5.58)$$

Subtracting (5.57) from (5.58), and passing to the limit  $\alpha \rightarrow 0_+$ , one deduces from (5.56), (5.53) and (5.55) that, for any  $t' \in (0, T]$ ,

$$\int_{\Omega} [\overline{\mathfrak{L}(\rho_{\kappa})(t')} - \mathfrak{L}(\rho_{\kappa})(t')] \, dx \underset{\sim}{\leq} \int_0^{t'} \int_{\Omega} [\rho_{\kappa} \nabla_x \cdot u_{\kappa} - \overline{\rho_{\kappa} \nabla_x \cdot u_{\kappa}}] \, dx \, dt \leq 0, \quad (5.59)$$

where, by noting (5.10a),

$$\mathfrak{L}(\rho_{\kappa, \alpha})(t') \rightarrow \overline{\mathfrak{L}(\rho_{\kappa})(t')} \text{ weakly in } L^r(\Omega), \text{ for any } r \in [1, \Gamma), \text{ as } \alpha \rightarrow 0_+. \quad (5.60)$$

As  $\mathfrak{L}(s)$  is continuous and convex for  $s \in [0, \infty)$ , it follows from (5.10a) and (5.60), see e.g. Corollary 3.33 in Novotný & Straškraba<sup>24</sup> (or Lemma D.1 in Appendix D in Barrett & Süli<sup>12</sup>), that  $\overline{\mathfrak{L}(\rho_{\kappa})(t')} \geq \mathfrak{L}(\rho_{\kappa})(t')$  a.e. in  $\Omega$  for any  $t' \in (0, T]$ . Hence, we deduce from (5.59) that  $\overline{\mathfrak{L}(\rho_{\kappa})(t')} = \mathfrak{L}(\rho_{\kappa})(t')$  a.e. in  $\Omega$  for any  $t' \in (0, T]$ . Therefore, on applying Lemma 3.34 in Novotný & Straškraba<sup>24</sup> (or Lemma D.2 in Appendix D in Barrett & Süli<sup>12</sup>), we conclude from the above, (5.60) and (5.10a) that  $\rho_{\kappa, \alpha}(t) \rightarrow \rho_{\kappa}(t)$  strongly in  $L^1(\Omega)$  for any  $t \in (0, T]$ , as  $\alpha \rightarrow 0_+$ . It immediately follows from this, (5.18), (3.1) and (5.25e), on possibly extracting a further subsequence (not indicated), that, as  $\alpha \rightarrow 0_+$ ,

$$\rho_{\kappa, \alpha} \rightarrow \rho_{\kappa} \quad \text{strongly in } L^r(\Omega_T), \quad \text{for any } r \in [1, \Gamma + 1), \quad (5.61a)$$

$$p_{\kappa}(\rho_{\kappa, \alpha}) \rightarrow p_{\kappa}(\rho_{\kappa}) \quad \text{weakly in } L^{\frac{\Gamma+1}{\Gamma}}(\Omega_T), \quad \text{that is, } \overline{p_{\kappa}(\rho_{\kappa})} = p_{\kappa}(\rho_{\kappa}). \quad (5.61b)$$

Finally, we have the following complete analogue of Theorem 4.1.

**Theorem 5.1.** *The triple  $(\rho_{\kappa}, u_{\kappa}, \widehat{\psi}_{\kappa})$ , defined as in Lemmas 5.1 and 5.2, is a global weak solution to problem  $(P_{\kappa})$ , in the sense that (5.27a,c), with their initial*

conditions, hold and

$$\begin{aligned} & \int_0^T \left\langle \frac{\partial(\rho_\kappa u_\kappa)}{\partial t}, w \right\rangle_{W_0^{1,\Gamma+1}(\Omega)} dt \\ & + \int_0^T \int_\Omega \left[ \underset{\approx}{S}(u_\kappa) - \rho_\kappa \underset{\approx}{u}_\kappa \otimes \underset{\approx}{u}_\kappa - p_\kappa(\rho_\kappa) \underset{\approx}{I} \right] : \underset{\approx}{\nabla}_x \underset{\approx}{w} dx dt \\ & = \int_0^T \int_\Omega \left[ \rho_\kappa \underset{\approx}{f} \cdot \underset{\approx}{w} - \left( \tau_1(M \widehat{\psi}_\kappa) - \mathfrak{z} \varrho_\kappa^2 \underset{\approx}{I} \right) : \underset{\approx}{\nabla}_x \underset{\approx}{w} \right] dx dt \\ & \quad \forall w \in L^{\Gamma+1}(0, T; W_0^{1,\Gamma+1}(\Omega)), \end{aligned} \quad (5.62)$$

with  $(\rho_\kappa \underline{u}_\kappa)(\cdot, 0) = (\rho_0 \underline{u}_0)(\cdot)$ . In addition, the weak solution  $(\rho_\kappa, \underline{u}_\kappa, \widehat{\psi}_\kappa)$  satisfies (5.28).

**Proof.** The results (5.27a,c) and (5.28) have already been established in Lemma 5.5. Equation (5.62) was established in Lemma 5.5 with  $p_\kappa(\rho_\kappa)$  replaced by  $\bar{p}_\kappa(\rho_\kappa)$ , see (5.27b). The desired result (5.62) then follows immediately from (5.27b) and (5.61b).  $\square$

## 6. Existence of a solution to (P)

It follows from the bounds on  $\varrho_\kappa$  in (5.28), similarly to (4.26) and (4.27), that

$$\|\varrho_\kappa\|_{L^\infty(0,T;L^2(\Omega))} + \|\varrho_\kappa\|_{L^{\frac{2(d+2)}{d}}(\Omega_T)} + \|\varrho_\kappa\|_{L^2(0,T;L^6(\Omega))} + \|\varrho_\kappa\|_{L^4(0,T;L^{\frac{2d}{d-1}}(\Omega))} \leq C, \quad (6.1)$$

where throughout this section  $C$  is a generic positive constant, independent of  $\kappa$ . Hence, we deduce from (6.1), (4.24) and (5.28), similarly to (4.28b), that

$$\|\tau_1(M \widehat{\psi}_\kappa)\|_{L^2(0,T;L^{\frac{4}{3}}(\Omega))} + \|\tau_1(M \widehat{\psi}_\kappa)\|_{L^{\frac{4(d+2)}{3d+4}}(\Omega_T)} + \|\tau_1(M \widehat{\psi}_\kappa)\|_{L^{\frac{4}{3}}(0,T;L^{\frac{12}{7}}(\Omega))} \leq C. \quad (6.2)$$

Similarly to (4.94), it follows from (6.1) and (5.28) that

$$\left\| M \frac{\partial \widehat{\psi}_\kappa}{\partial t} \right\|_{L^2(0,T;H^s(\Omega \times D)')} \leq C, \quad (6.3)$$

where  $s > 1 + \frac{1}{2}(K+1)d$ . We have the following analogue of Lemma 5.1.

**Lemma 6.1.** *There exist functions*

$$\underline{u} \in L^2(0, T; H_0^1(\Omega)) \quad \text{and} \quad \widehat{\psi} \in L^v(0, T; Z_1) \cap H^1(0, T; M^{-1}(H^s(\Omega \times D))'), \quad (6.4a)$$

where  $v \in [1, \infty)$  and  $s > 1 + \frac{1}{2}(K+1)d$ , with finite relative entropy and Fisher information,

$$\mathcal{F}(\widehat{\psi}) \in L^\infty(0, T; L_M^1(\Omega \times D)) \quad \text{and} \quad \sqrt{\widehat{\psi}} \in L^2(0, T; H_M^1(\Omega \times D)), \quad (6.4b)$$

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and a subsequence of  $\{(\rho_\kappa, \underline{u}_\kappa, \widehat{\psi}_\kappa)\}_{\kappa>0}$  such that, as  $\kappa \rightarrow 0_+$ ,

$$\underline{u}_\kappa \rightharpoonup \underline{u} \quad \text{weakly in } L^2(0, T; H_0^1(\Omega)), \quad (6.5)$$

and

$$M^{\frac{1}{2}} \nabla_x \sqrt{\widehat{\psi}_\kappa} \rightharpoonup M^{\frac{1}{2}} \nabla_x \sqrt{\widehat{\psi}} \quad \text{weakly in } L^2(0, T; L^2(\Omega \times D)), \quad (6.6a)$$

$$M^{\frac{1}{2}} \nabla_q \sqrt{\widehat{\psi}_\kappa} \rightharpoonup M^{\frac{1}{2}} \nabla_q \sqrt{\widehat{\psi}} \quad \text{weakly in } L^2(0, T; L^2(\Omega \times D)), \quad (6.6b)$$

$$M \frac{\partial \widehat{\psi}_\kappa}{\partial t} \rightharpoonup M \frac{\partial \widehat{\psi}}{\partial t} \quad \text{weakly in } L^2(0, T; H^s(\Omega \times D)'), \quad (6.6c)$$

$$\widehat{\psi}_\kappa \rightarrow \widehat{\psi} \quad \text{strongly in } L^v(0, T; L_M^1(\Omega \times D)), \quad (6.6d)$$

$$\tau(M \widehat{\psi}_\kappa) \rightarrow \tau(M \widehat{\psi}) \quad \text{strongly in } L^r(\Omega_T), \quad (6.6e)$$

where  $r \in [1, \frac{4(d+2)}{3d+4})$ , and, for a.a.  $t \in (0, T)$ ,

$$\int_{\Omega \times D} M(q) \mathcal{F}(\widehat{\psi}(x, q, t)) \, dq \, dx \leq \liminf_{\kappa \rightarrow 0_+} \int_{\Omega \times D} M(q) \mathcal{F}(\widehat{\psi}_\kappa(x, q, t)) \, dq \, dx. \quad (6.6f)$$

In addition, we have that

$$\varrho := \int_D M \widehat{\psi} \, dq \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad (6.7)$$

and, as  $\kappa \rightarrow 0_+$ ,

$$\begin{aligned} \varrho_\kappa &\rightharpoonup \varrho \quad \text{weakly-}\star \text{ in } L^\infty(0, T; L^2(\Omega)), \\ &\quad \text{weakly in } L^2(0, T; H^1(\Omega)), \end{aligned} \quad (6.8a)$$

$$\varrho_\kappa \rightarrow \varrho \quad \text{strongly in } L^{\frac{5\varsigma}{3(\varsigma-1)}}(0, T; L^\varsigma(\Omega)), \quad (6.8b)$$

for any  $\varsigma \in (1, 6)$ .

**Proof.** The convergence result (6.5) and the first result in (6.4a) follow immediately from the bound on  $\underline{u}_\kappa$  in (5.28). The remainder of the results follow from the bounds on  $\widehat{\psi}_\kappa$  and  $\varrho_\kappa$  in (5.28) in the same way as the results of Lemma 4.8.  $\square$

We have the following analogue of Lemma 5.2.

**Lemma 6.2.** *Let  $\Gamma \geq 8$ ; then, there exists a  $C \in \mathbb{R}_{>0}$ , independent of  $\kappa$ , such that, for any  $\gamma > \frac{3}{2}$  as in (1.3),*

$$\begin{aligned} &\|\rho_\kappa\|_{L^\infty(0, T; L^\gamma(\Omega))} + \|\underline{u}_\kappa\|_{L^2(0, T; H^1(\Omega))} + \kappa^{\frac{1}{\Gamma}} \|\rho_\kappa\|_{L^\infty(0, T; L^\Gamma(\Omega))} \\ &\quad + \left\| \sqrt{\rho_\kappa} \underline{u}_\kappa \right\|_{L^\infty(0, T; L^2(\Omega))} + \|\rho_\kappa \underline{u}_\kappa\|_{L^\infty(0, T; L^{\frac{2\gamma}{\gamma+1}}(\Omega))} \\ &\quad + \|\rho_\kappa \underline{u}_\kappa\|_{L^2(0, T; L^{\frac{6\gamma}{\gamma+6}}(\Omega))} + \left\| \rho_\kappa |\underline{u}_\kappa|^2 \right\|_{L^2(0, T; L^{\frac{6\gamma}{4\gamma+3}}(\Omega))} \leq C, \end{aligned} \quad (6.9a)$$

$$\left\| \frac{\partial \rho_\kappa}{\partial t} \right\|_{L^2(0, T; W^{1,6}(\Omega)')} \leq C. \quad (6.9b)$$

Hence, there exists a function  $\rho \in C_w([0, T]; L_{\geq 0}^\gamma(\Omega)) \cap H^1(0, T; W^{1,6}(\Omega)')$ , and for a further subsequence of the subsequence of Lemma 6.1, it follows that, as  $\kappa \rightarrow 0_+$ ,

$$\begin{aligned} \rho_\kappa &\rightarrow \rho && \text{in } C_w([0, T]; L^\gamma(\Omega)) \\ &&& \text{weakly in } H^1(0, T; W^{1,6}(\Omega)'), \end{aligned} \quad (6.10a)$$

$$\rho_\kappa \rightarrow \rho \quad \text{strongly in } L^2(0, T; H^1(\Omega)'), \quad (6.10b)$$

and, for any nonnegative  $\eta \in C[0, T]$ ,

$$\int_0^T \left( \int_\Omega P(\rho) \, dx \right) \eta \, dt \leq \liminf_{\kappa \rightarrow 0_+} \int_0^T \left( \int_\Omega P(\rho_\kappa) \, dx \right) \eta \, dt. \quad (6.10c)$$

**Proof.** The first four bounds in (6.9a) follow immediately from (5.28). The last three bounds in (6.9a) follow, similarly to (4.84a,b), from the first two bounds in (6.9a). The bound (6.9b) follows immediately from (5.27a), the sixth bound in (6.9a), on noting that  $\frac{6\gamma}{\gamma+6} > \frac{6}{5}$  as  $\gamma > \frac{3}{2}$ . The convergence results (6.10a,b) follow immediately from (6.9a,b), (3.12a,b) and (3.11). The result (6.10c) follows, similarly to (4.66), from (6.10a) and the convexity of  $P$ .  $\square$

Similarly to (5.13), it follows from (5.27b), (6.9a), (6.1), (6.2), (3.4), on noting that  $\gamma > \frac{3}{2}$ , and (3.2) that, for any  $w \in L^\infty(0, T; W_0^{1,r}(\Omega))$  with  $r = \max\{\Gamma + 1, v\}$  and  $v = \max\{\frac{3\gamma}{2\gamma-3}, \frac{12}{5}\}$ ,

$$\begin{aligned} &\left| \int_0^T \left\langle \frac{\partial(\rho_\kappa u_\kappa)}{\partial t}, w \right\rangle_{W_0^{1,\Gamma+1}(\Omega)} \, dt - \int_0^T \int_\Omega p_\kappa(\rho_\kappa) \nabla_x \cdot w \, dx \, dt \right| \\ &\leq C \|\rho_\kappa\|_{L^\infty(0,T;L^\gamma(\Omega))} \|u_\kappa\|_{L^2(0,T;L^6(\Omega))}^2 \|w\|_{L^\infty(0,T;W^{1,\frac{3\gamma}{2\gamma-3}}(\Omega))} \\ &\quad + C \|u_\kappa\|_{L^2(0,T;H^1(\Omega))} \|w\|_{L^2(0,T;H^1(\Omega))} \\ &\quad + C \left[ \|\tau_1(M \widehat{\psi}_\kappa)\|_{L^{\frac{4}{3}}(0,T;L^{\frac{12}{7}}(\Omega))} + \|\varrho_\kappa\|_{L^2(0,T;L^6(\Omega))}^2 \right] \|\nabla_x w\|_{L^\infty(0,T;L^{\frac{12}{5}}(\Omega))} \\ &\quad + \|\rho_\kappa\|_{L^\infty(0,T;L^\gamma(\Omega))} \|f\|_{L^2(0,T;L^\infty(\Omega))} \|w\|_{L^2(0,T;L^3(\Omega))} \\ &\leq C \|w\|_{L^\infty(0,T;W^{1,v}(\Omega))}. \end{aligned} \quad (6.11)$$

We deduce from (6.11) with  $w = \eta v$ , where  $\eta \in C_0^\infty(0, T)$  and  $v \in L^\infty(0, T; W_0^{1,v}(\Omega)) \cap H^1(0, T; L^{\frac{6\gamma}{5\gamma-6}}(\Omega))$  with  $v = \max\{\frac{3\gamma}{2\gamma-3}, \frac{12}{5}\}$ , on noting (6.9a) and (3.2) as  $\frac{2\gamma}{\gamma-1} < 6$  for  $\gamma > \frac{3}{2}$ , that

$$\begin{aligned} &\left| \int_0^T \eta \int_\Omega p_\kappa(\rho_\kappa) \nabla_x \cdot v \, dx \, dt \right| \\ &\leq \left| \int_0^T \int_\Omega \rho_\kappa u_\kappa \cdot \frac{\partial(\eta v)}{\partial t} \, dx \, dt \right| + C \|\eta\|_{L^\infty(0,T)} \|v\|_{L^\infty(I;W^{1,v}(\Omega))} \\ &\leq C \|\rho_\kappa u_\kappa\|_{L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega))} \left\| \frac{d\eta}{dt} \right\|_{L^1(0,T)} \|v\|_{L^\infty(I;L^{\frac{2\gamma}{\gamma-1}}(\Omega))} \end{aligned}$$

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$$\begin{aligned}
& + C \|\eta\|_{L^\infty(0,T)} \left[ \|\rho_\kappa \tilde{u}_\kappa\|_{L^2(0,T;L^{\frac{6\gamma}{\gamma+6}}(\Omega))} \left\| \frac{\partial \tilde{v}}{\partial t} \right\|_{L^2(I;L^{\frac{6\gamma}{5\gamma-6}}(\Omega))} + \|\tilde{v}\|_{L^\infty(I;W^{1,v}(\Omega))} \right] \\
\leq C & \left[ \left\| \frac{d\eta}{dt} \right\|_{L^1(0,T)} + \|\eta\|_{L^\infty(0,T)} \right] \left[ \|\tilde{v}\|_{L^\infty(I;W^{1,v}(\Omega))} + \left\| \frac{\partial \tilde{v}}{\partial t} \right\|_{L^2(I;L^{\frac{6\gamma}{5\gamma-6}}(\Omega))} \right],
\end{aligned} \tag{6.12}$$

where  $I = \text{supp}(\eta) \subset (0, T)$ . With  $v = v(\gamma)$  thus defined, let

$$\vartheta(\gamma) := \frac{\gamma}{v(\gamma)} = \begin{cases} \frac{2\gamma-3}{3} & \text{for } \frac{3}{2} < \gamma \leq 4, \\ \frac{5}{12}\gamma & \text{for } 4 \leq \gamma. \end{cases} \tag{6.13}$$

With  $\vartheta(\gamma) \in \mathbb{R}_{>0}$  defined as above and  $\ell \in \mathbb{N}$ , we now introduce  $\mathbf{b} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  and  $\mathbf{b}_\ell : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  such that

$$\mathbf{b}(s) := s^\vartheta \quad \text{and} \quad \mathbf{b}_\ell(s) := \begin{cases} \mathbf{b}(s) & \text{for } 0 \leq s \leq \ell, \\ \mathbf{b}(\ell) & \text{for } \ell \leq s. \end{cases} \tag{6.14}$$

We note from (6.14), (6.9a) and (3.2) that, for  $v(\gamma) := \max\{\frac{3\gamma}{2\gamma-3}, \frac{12}{5}\}$  as in (6.12) and  $\vartheta(\gamma)$  as in (6.13), we have that

$$\|\mathbf{b}_\ell(\rho_\kappa)\|_{L^\infty(0,T;L^v(\Omega))} \leq \|\rho_\kappa^\vartheta\|_{L^\infty(0,T;L^v(\Omega))} \leq \|\rho_\kappa\|_{L^\infty(0,T;L^\gamma(\Omega))}^\vartheta \leq C, \tag{6.15a}$$

$$\|\mathbf{b}_\ell(\rho_\kappa) \tilde{u}_\kappa\|_{L^2(0,T;L^{\frac{6\gamma}{\gamma+6\vartheta}}(\Omega))} \leq \|\rho_\kappa\|_{L^\infty(0,T;L^\gamma(\Omega))}^\vartheta \|\tilde{u}_\kappa\|_{L^2(0,T;L^6(\Omega))} \leq C, \tag{6.15b}$$

$$\|\mathbf{b}_\ell(\rho_\kappa) \tilde{\nabla}_x \cdot \tilde{u}_\kappa\|_{L^2(0,T;L^{\frac{2\gamma}{\gamma+2\vartheta}}(\Omega))} \leq \|\rho_\kappa\|_{L^\infty(0,T;L^\gamma(\Omega))}^\vartheta \|\tilde{u}_\kappa\|_{L^2(0,T;H^1(\Omega))} \leq C, \tag{6.15c}$$

where  $C \in \mathbb{R}_{>0}$  is independent of  $\kappa, \vartheta$  and  $\ell$ .

As  $\Gamma > 2$ , it follows from (6.9a) and (5.27a), on extending  $\rho_\kappa$  and  $\tilde{u}_\kappa$  from  $\Omega$  to  $\mathbb{R}^d$  by zero, that

$$\frac{\partial \rho_\kappa}{\partial t} + \tilde{\nabla}_x \cdot (\rho_\kappa \tilde{u}_\kappa) = 0 \quad \text{in } C_0^\infty(\mathbb{R}^d \times (0, T))', \tag{6.16}$$

see Lemmas 6.8 in Novotný & Straškraba<sup>24</sup> (or Lemma F.1 in Appendix F in Barrett & Süli<sup>12</sup>). Applying Lemma 6.11 in Novotný & Straškraba<sup>24</sup> (or Lemma F.3 in Appendix F in Barrett & Süli<sup>12</sup>) to (6.16), we have the renormalized equation, for any  $\ell \in \mathbb{N}$ ,

$$\begin{aligned}
\frac{\partial \mathbf{b}_\ell(\rho_\kappa)}{\partial t} + \tilde{\nabla}_x \cdot (\mathbf{b}_\ell(\rho_\kappa) \tilde{u}_\kappa) + (\rho_\kappa (\mathbf{b}_\ell)_+'(\rho_\kappa) - \mathbf{b}_\ell(\rho_\kappa)) \tilde{\nabla}_x \cdot \tilde{u}_\kappa = 0 \\
\text{in } C_0^\infty(\mathbb{R}^d \times (0, T))',
\end{aligned} \tag{6.17}$$

where  $(\mathbf{b}_\ell)_+'(\cdot)$  is the right-derivative of  $\mathbf{b}_\ell(\cdot)$  satisfying

$$(\mathbf{b}_\ell)_+'(s) = \begin{cases} \mathbf{b}'(s) & \text{for } 0 \leq s < \ell, \\ 0 & \text{for } \ell \leq s. \end{cases} \tag{6.18}$$

For any  $\delta \in (0, \frac{T}{2})$ , we now introduce the Friedrichs mollifier, with respect to the time variable,  $\mathcal{S}_\delta : L^1(0, T; L^q(\Omega)) \rightarrow C^\infty(\delta, T - \delta; L^q(\Omega))$ ,  $q \in [1, \infty]$ ,

$$\mathcal{S}_\delta(\eta)(\tilde{x}, t) = \frac{1}{\delta} \int_0^T \omega\left(\frac{t-s}{\delta}\right) \eta(\tilde{x}, s) ds \quad \text{a.e. in } \Omega \times (\delta, T - \delta), \quad (6.19)$$

where  $\omega \in C_0^\infty(\mathbb{R})$ ,  $\omega \geq 0$ ,  $\text{supp}(\omega) \subset (-1, 1)$  and  $\int_{\mathbb{R}} \omega ds = 1$ . It follows from (6.17) and (6.19) that

$$\begin{aligned} \frac{\partial \mathcal{S}_\delta(\mathbf{b}_\ell(\rho_\kappa))}{\partial t} + \tilde{\nabla}_x \cdot \mathcal{S}_\delta(\mathbf{b}_\ell(\rho_\kappa) \tilde{u}_\kappa) + \mathcal{S}_\delta([\rho_\kappa (\mathbf{b}_\ell)'_+(\rho_\kappa) - \mathbf{b}_\ell(\rho_\kappa)] \tilde{\nabla}_x \cdot \tilde{u}_\kappa) = 0 \\ \text{in } C_0^\infty(\mathbb{R}^d \times (\delta, T - \delta))'. \end{aligned} \quad (6.20)$$

In addition, it follows from (6.19), (6.14), (6.9a), (3.2), (6.18) and (6.20) that

$$\begin{aligned} \mathcal{S}_\delta(\mathbf{b}_\ell(\rho_\kappa)) \in C^\infty(\delta, T - \delta; L^\infty(\mathbb{R}^d)), \quad \mathcal{S}_\delta(\mathbf{b}_\ell(\rho_\kappa) \tilde{u}_\kappa) \in C^\infty(\delta, T - \delta; \tilde{L}^6(\mathbb{R}^d)), \\ \mathcal{S}_\delta([\rho_\kappa (\mathbf{b}_\ell)'_+(\rho_\kappa) - \mathbf{b}_\ell(\rho_\kappa)] \tilde{\nabla}_x \cdot \tilde{u}_\kappa), \quad \tilde{\nabla}_x \cdot [\mathcal{S}_\delta(\mathbf{b}_\ell(\rho_\kappa) \tilde{u}_\kappa)] \in C^\infty(\delta, T - \delta; L^2(\mathbb{R}^d)). \end{aligned} \quad (6.21)$$

One can deduce from  $\tilde{u}_\kappa \in L^2(0, T; \tilde{H}_0^1(\Omega))$  and (6.21) that

$$\mathcal{S}_\delta(\mathbf{b}_\ell(\rho_\kappa) \tilde{u}_\kappa) \in C^\infty(\delta, T - \delta; \tilde{E}_0^{6,2}(\Omega)), \quad (6.22)$$

where we recall (5.14). We note from (5.16a), (6.19), (6.14) and (6.15a) that  $\mathcal{B}([\mathcal{S}_\delta(\mathbf{b}_\ell(\rho_\kappa))]) \in L^\infty(\delta, T - \delta; \tilde{W}_0^{1,r}(\Omega))$ ,  $r \in [1, \infty)$ , and, for  $\nu(\gamma) := \max\{\frac{3\gamma}{2\gamma-3}, \frac{12}{5}\}$  as in (6.12), that

$$\begin{aligned} \|\tilde{\mathcal{B}}((I - f)[\mathcal{S}_\delta(\mathbf{b}_\ell(\rho_\kappa))])\|_{L^\infty(\delta, T - \delta; W^{1,\nu}(\Omega))} \leq C \|\mathcal{S}_\delta(\mathbf{b}_\ell(\rho_\kappa))\|_{L^\infty(\delta, T - \delta; L^\nu(\Omega))} \\ \leq C \|\mathbf{b}_\ell(\rho_\kappa)\|_{L^\infty(0, T; L^\nu(\Omega))} \leq C, \end{aligned} \quad (6.23a)$$

and from (6.20), Sobolev embedding, (6.22), (5.16a,b), (6.19), (6.14), (6.18) and (6.15b,c) with  $\vartheta$  as in (6.13) that

$$\begin{aligned} & \left\| \frac{\partial}{\partial t} \tilde{\mathcal{B}}((I - f)[\mathcal{S}_\delta(\mathbf{b}_\ell(\rho_\kappa))]) \right\|_{L^2(\delta, T - \delta; L^{\frac{6\gamma}{\gamma+6\vartheta}}(\Omega))} \\ & \leq \|\tilde{\mathcal{B}}(\tilde{\nabla}_x \cdot [\mathcal{S}_\delta(\mathbf{b}_\ell(\rho_\kappa) \tilde{u}_\kappa)])\|_{L^2(\delta, T - \delta; L^{\frac{6\gamma}{\gamma+6\vartheta}}(\Omega))} \\ & \quad + \|\tilde{\mathcal{B}}((I - f)[\mathcal{S}_\delta([\rho_\kappa (\mathbf{b}_\ell)'_+(\rho_\kappa) - \mathbf{b}_\ell(\rho_\kappa)] \tilde{\nabla}_x \cdot \tilde{u}_\kappa)])\|_{L^2(\delta, T - \delta; L^{\frac{6\gamma}{\gamma+6\vartheta}}(\Omega))} \\ & \leq \|\tilde{\mathcal{B}}(\tilde{\nabla}_x \cdot [\mathcal{S}_\delta(\mathbf{b}_\ell(\rho_\kappa) \tilde{u}_\kappa)])\|_{L^2(\delta, T - \delta; L^{\frac{6\gamma}{\gamma+6\vartheta}}(\Omega))} \\ & \quad + \|\tilde{\mathcal{B}}((I - f)[\mathcal{S}_\delta([\rho_\kappa (\mathbf{b}_\ell)'_+(\rho_\kappa) - \mathbf{b}_\ell(\rho_\kappa)] \tilde{\nabla}_x \cdot \tilde{u}_\kappa)])\|_{L^2(\delta, T - \delta; W^{1, \frac{2\gamma}{\gamma+2\vartheta}}(\Omega))} \\ & \leq C \left[ \|\mathbf{b}_\ell(\rho_\kappa) \tilde{u}_\kappa\|_{L^2(0, T; L^{\frac{6\gamma}{\gamma+6\vartheta}}(\Omega))} + \|[\rho_\kappa (\mathbf{b}_\ell)'_+(\rho_\kappa) - \mathbf{b}_\ell(\rho_\kappa)] \tilde{\nabla}_x \cdot \tilde{u}_\kappa\|_{L^2(0, T; L^{\frac{2\gamma}{\gamma+2\vartheta}}(\Omega))} \right] \\ & \leq C \left[ \|\mathbf{b}_\ell(\rho_\kappa) \tilde{u}_\kappa\|_{L^2(0, T; L^{\frac{6\gamma}{\gamma+6\vartheta}}(\Omega))} + \|\mathbf{b}_\ell(\rho_\kappa) \tilde{\nabla}_x \cdot \tilde{u}_\kappa\|_{L^2(0, T; L^{\frac{2\gamma}{\gamma+2\vartheta}}(\Omega))} \right] \leq C, \end{aligned} \quad (6.23b)$$

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where  $C \in \mathbb{R}_{>0}$  in (6.23a,b) is independent of  $\kappa, \vartheta, \ell$  and  $\delta$ . We now have the following analogue of Lemma 5.3.

**Lemma 6.3.** *With  $\vartheta(\gamma)$  as defined in (6.13), we have that*

$$\|\rho_\kappa\|_{L^{\gamma+\vartheta}(\Omega_T)} + \kappa^{\frac{1}{4+\vartheta}} \|\rho_\kappa\|_{L^{4+\vartheta}(\Omega_T)} + \kappa^{\frac{1}{\Gamma+\vartheta}} \|\rho_\kappa\|_{L^{\Gamma+\vartheta}(\Omega_T)} \leq C. \quad (6.24)$$

**Proof.** For any  $\ell \in \mathbb{N}$  and  $\delta \in (0, \frac{T}{2})$ , we choose

$$\varrho = \mathfrak{B}((I - f)[\mathcal{S}_\delta(\mathbf{b}_\ell(\rho_\kappa))]) \in L^\infty(\delta, T - \delta, W_0^{1,r}(\Omega)),$$

any  $r \in [1, \infty)$ , and  $\eta \in C_0^\infty(0, T)$ , with  $\text{supp}(\eta) \subset (\delta, T - \delta)$ , in (6.12) to obtain, on noting (6.23a,b),  $\frac{6\gamma}{5\gamma-6} \leq \frac{6\gamma}{\gamma+6\vartheta}$  as  $\vartheta \leq \frac{2\gamma}{3} - 1$ , (6.9a) and (2.3), that

$$\begin{aligned} & \left| \int_0^T \eta \int_\Omega p_\kappa(\rho_\kappa) \mathcal{S}_\delta(\mathbf{b}_\ell(\rho_\kappa)) \, dx \, dt \right| \\ & \leq C \left[ \left\| \frac{d\eta}{dt} \right\|_{L^1(0,T)} + \|\eta\|_{L^\infty(0,T)} \right] \left[ 1 + \|p_\kappa(\rho_\kappa)\|_{L^1(\Omega_T)} \|\mathfrak{f} \mathcal{S}_\delta(\mathbf{b}_\ell(\rho_\kappa))\|_{L^\infty(\delta, T-\delta)} \right] \\ & \leq C \left[ \left\| \frac{d\eta}{dt} \right\|_{L^1(0,T)} + \|\eta\|_{L^\infty(0,T)} \right]. \end{aligned} \quad (6.25)$$

We now consider (6.25) with  $\eta = \eta_\ell \in C_0^\infty(0, T)$  with  $\text{supp}(\eta_\ell) \subset (\frac{1}{\ell}, T - \frac{1}{\ell})$ ,  $\ell \in \mathbb{N}$  with  $\ell > \frac{4}{T}$ , where  $\eta_\ell \in [0, 1]$  with  $\eta_\ell(t) = 1$  for  $t \in [\frac{2}{\ell}, T - \frac{2}{\ell}]$  and  $\|\frac{d\eta_\ell}{dt}\|_{L^\infty(0,T)} \leq 2\ell$  yielding  $\|\frac{d\eta_\ell}{dt}\|_{L^1(0,T)} \leq 4$ . For a fixed  $\ell$ , we now let  $\delta \rightarrow 0$  in (6.25) and using the standard convergence properties of mollifiers we obtain that

$$\left| \int_0^T \eta_\ell \int_\Omega p_\kappa(\rho_\kappa) \mathbf{b}_\ell(\rho_\kappa) \, dx \, dt \right| \leq C, \quad (6.26)$$

where  $C \in \mathbb{R}$  is independent of  $\ell$  and  $\kappa$ . Letting  $\ell \rightarrow \infty$  in (6.26), and noting that  $\eta_\ell \rightarrow 1$  pointwise in  $(0, T)$ ,  $\mathbf{b}_\ell(\rho_\kappa) \rightarrow \mathbf{b}(\rho_\kappa) = \rho_\kappa^\vartheta$  pointwise in  $\Omega_T$  and Fatou's lemma, we obtain that

$$\int_0^T \int_\Omega p_\kappa(\rho_\kappa) \rho_\kappa^\vartheta \, dx \, dt \leq C, \quad (6.27)$$

where  $C \in \mathbb{R}$  is independent of  $\kappa$ . Hence the desired result (6.24) follows from (6.27), (2.3) and (1.3).  $\square$

Similarly to (6.11), it follows from (5.27b), (6.9a), (6.1), (6.2) and (3.4), on



noting  $\gamma > \frac{3}{2}$ , that, for any  $w \in L^{\Gamma+1}(0, T; W_0^{1,v}(\Omega))$  with  $v = \max\{\Gamma + 1, \frac{6\gamma}{2\gamma-3}\}$ ,

$$\begin{aligned} & \left| \int_0^T \left\langle \frac{\partial(\rho_\kappa \tilde{u}_\kappa)}{\partial t}, w \right\rangle_{W_0^{1,\Gamma+1}(\Omega)} dt - \int_0^T \int_\Omega p_\kappa(\rho_\kappa) \nabla_x \cdot \tilde{w} dx dt \right| \\ & \leq C \left[ \|\rho_\kappa |\tilde{u}_\kappa|^2\|_{L^2(0,T;L^{\frac{6\gamma}{4\gamma+3}}(\Omega))} + \|\tilde{u}_\kappa\|_{L^2(0,T;H^1(\Omega))} \right] \|\tilde{w}\|_{L^2(0,T;W^{1,\frac{6\gamma}{2\gamma-3}}(\Omega))} \\ & \quad + C \left[ \|\tau_1(M \widehat{\psi}_\kappa)\|_{L^2(0,T;L^{\frac{4}{3}}(\Omega))} + \|\varrho_\kappa\|_{L^4(0,T;L^{\frac{2d}{d-1}}(\Omega))}^2 \right] \|\nabla_x \tilde{w}\|_{L^2(0,T;L^4(\Omega))} \\ & \quad + \|\rho_\kappa\|_{L^\infty(0,T;L^\gamma(\Omega))} \|\tilde{f}\|_{L^2(0,T;L^\infty(\Omega))} \|\tilde{w}\|_{L^2(0,T;L^3(\Omega))} \leq C \|\tilde{w}\|_{L^2(0,T;W^{1,s}(\Omega))}, \end{aligned} \quad (6.28)$$

where  $s = \max\{4, \frac{6\gamma}{2\gamma-3}\}$ . We now have the following analogue of Lemma 5.4.

**Lemma 6.4.** *There exists a  $C \in \mathbb{R}_{>0}$ , independent of  $\kappa$ , such that*

$$\left\| \frac{\partial(\rho_\kappa \tilde{u}_\kappa)}{\partial t} \right\|_{L^{\frac{\Gamma+\vartheta}{\Gamma}}(0,T;W_0^{1,r}(\Omega)')} \leq C, \quad (6.29)$$

where  $\vartheta(\gamma)$  is defined as in (6.13),  $r = \max\{s, \frac{\Gamma+\vartheta}{\vartheta}\}$  and  $s = \max\{4, \frac{6\gamma}{2\gamma-3}\}$ .

Hence, for a further subsequence of the subsequence of Lemma 6.2, it follows that, as  $\kappa \rightarrow 0_+$ ,

$$\rho_\kappa \tilde{u}_\kappa \rightharpoonup \rho u \quad \text{weakly in } W^{1,\frac{\Gamma+\vartheta}{\Gamma}}(0,T;W_0^{1,r}(\Omega)'), \quad (6.30a)$$

$$\begin{aligned} \rho_\kappa \tilde{u}_\kappa &\rightarrow \rho u && \text{in } C_w([0,T];L^{\frac{2\gamma}{\gamma+1}}(\Omega)), \\ &&& \text{strongly in } L^2(0,T;H^1(\Omega)'), \end{aligned} \quad (6.30b)$$

$$\rho_\kappa \tilde{u}_\kappa \rightarrow \rho u \quad \text{weakly in } L^2(0,T;L^{\frac{6\gamma}{\gamma+6}}(\Omega)), \quad (6.30c)$$

$$\rho_\kappa \tilde{u}_\kappa \otimes \tilde{u}_\kappa \rightarrow \rho u \otimes u \quad \text{weakly in } L^2(0,T;L^{\frac{6\gamma}{4\gamma+3}}(\Omega)), \quad (6.30d)$$

$$\rho_\kappa \rightarrow \rho \quad \text{weakly in } L^{\gamma+\vartheta}(\Omega_T), \quad (6.30e)$$

$$\rho_\kappa^\gamma \rightarrow \overline{\rho^\gamma} \quad \text{weakly in } L^{\frac{\gamma+\vartheta}{\gamma}}(\Omega_T), \quad (6.30f)$$

$$\kappa(\rho_\kappa^4 + \rho_\kappa^\Gamma) \rightarrow 0 \quad \text{weakly in } L^{\frac{\Gamma+\vartheta}{\Gamma}}(\Omega_T), \quad (6.30g)$$

where  $\overline{\rho^\gamma} \in L_{>0}^{\frac{\gamma+\vartheta}{\gamma}}(\Omega_T)$  remains to be identified.

**Proof.** It immediately follows from (6.28), (2.3) and (6.24) that, for all functions  $w \in L^{\frac{\Gamma+\vartheta}{\Gamma}}(0,T;W_0^{1,r}(\Omega))$ , we have that

$$\begin{aligned} & \left| \int_0^T \left\langle \frac{\partial(\rho_\kappa \tilde{u}_\kappa)}{\partial t}, w \right\rangle_{W_0^{1,r}(\Omega)} dt \right| \\ & \leq C \|w\|_{L^2(0,T;W^{1,s}(\Omega))} + \|p_\kappa(\rho_\kappa)\|_{L^{\frac{\Gamma+\vartheta}{\Gamma}}(\Omega_T)} \|w\|_{L^{\frac{\Gamma+\vartheta}{\Gamma}}(0,T;W^{1,\frac{\Gamma+\vartheta}{\Gamma}}(\Omega))} \\ & \leq C \|w\|_{L^{\frac{\Gamma+\vartheta}{\Gamma}}(0,T;W^{1,r}(\Omega))}, \end{aligned} \quad (6.31)$$

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where we have noted from (6.13) that  $\frac{\Gamma+\vartheta}{\vartheta} \geq \frac{\gamma+\vartheta}{\vartheta} \geq 2$ . The desired result (6.29) then follows from (6.31).

The results (6.30a–d) follow similarly to (4.97a–e) from (6.9a), (6.29), (3.12a,b), (3.11), (6.10b), and (6.5). The results (6.30e–g) follow immediately from (6.24) and (2.3).  $\square$

We now have the following analogue of Lemma 5.5.

**Lemma 6.5.** *The triple  $(\rho, \underline{u}, \widehat{\psi})$ , defined as in Lemmas 6.1 and 6.2, satisfies*

$$\int_0^T \left\langle \frac{\partial \rho}{\partial t}, \eta \right\rangle_{W^{1,6}(\Omega)} dt - \int_0^T \int_{\Omega} \rho \underline{u} \cdot \nabla_x \eta \, dx \, dt = 0 \quad \forall \eta \in L^2(0, T; W^{1,6}(\Omega)), \quad (6.32a)$$

with  $\rho(\cdot, 0) = \rho_0(\cdot)$ ,

$$\begin{aligned} & \int_0^T \left\langle \frac{\partial(\rho \underline{u})}{\partial t}, w \right\rangle_{W_0^{1,r}(\Omega)} dt + \int_0^T \int_{\Omega} \left[ S(\underline{u}) - \rho \underline{u} \otimes \underline{u} - c_p \overline{\rho^\gamma} I \right] : \nabla_x w \, dx \, dt \\ &= \int_0^T \int_{\Omega} \left[ \rho f \cdot w - \left( \tau_1(M \widehat{\psi}) - \mathfrak{z} \varrho^2 I \right) : \nabla_x w \right] dx \, dt \\ & \quad \forall w \in L^{\frac{\gamma+\vartheta}{\vartheta}}(0, T; W_0^{1,r}(\Omega)), \end{aligned} \quad (6.32b)$$

with  $(\rho \underline{u})(\cdot, 0) = (\rho_0 \underline{u}_0)(\cdot)$ ,  $\vartheta(\gamma)$  defined as in (6.13) and  $r = \max\{4, \frac{6\gamma}{2\gamma-3}\}$ , and

$$\begin{aligned} & \int_0^T \left\langle M \frac{\partial \widehat{\psi}}{\partial t}, \varphi \right\rangle_{H^s(\Omega \times D)} dt + \frac{1}{4\lambda} \sum_{i=1}^K \sum_{j=1}^K A_{ij} \int_0^T \int_{\Omega \times D} M \nabla_{q_j} \widehat{\psi} \cdot \nabla_{q_i} \varphi \, dq \, dx \, dt \\ & + \int_0^T \int_{\Omega \times D} M \left[ \varepsilon \nabla_x \widehat{\psi} - \underline{u} \widehat{\psi} \right] \cdot \nabla_x \varphi \, dq \, dx \, dt \\ & - \int_0^T \int_{\Omega \times D} M \sum_{i=1}^K \left[ \sigma(\underline{u}) q_i \right] \widehat{\psi} \cdot \nabla_{q_i} \varphi \, dq \, dx \, dt = 0 \\ & \quad \forall \varphi \in L^2(0, T; H^s(\Omega \times D)), \end{aligned} \quad (6.32c)$$

with  $\widehat{\psi}(\cdot, 0) = \widehat{\psi}_0(\cdot)$  and  $s > 1 + \frac{1}{2}(K+1)d$ . In addition, the triple  $(\rho, \underline{u}, \widehat{\psi})$  satisfies,

for a.a.  $t' \in (0, T)$ ,

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega} \rho(t') |\underline{u}(t')|^2 \, d\underline{x} + \int_{\Omega} P(\rho(t')) \, d\underline{x} + k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}(t')) \, d\underline{q} \, d\underline{x} \\
& + \mu^S c_0 \int_0^{t'} \|\underline{u}\|_{H^1(\Omega)}^2 \, dt \\
& + k \int_0^{t'} \int_{\Omega \times D} M \left[ \frac{a_0}{2\lambda} \left| \underline{\nabla}_q \sqrt{\widehat{\psi}} \right|^2 + 2\varepsilon \left| \underline{\nabla}_x \sqrt{\widehat{\psi}} \right|^2 \right] \, d\underline{q} \, d\underline{x} \, dt \\
& + \mathfrak{z} \|\varrho(t')\|_{L^2(\Omega)}^2 + 2\mathfrak{z} \varepsilon \int_0^{t'} \|\underline{\nabla}_x \varrho\|_{L^2(\Omega)}^2 \, dt \\
& \leq e^{t'} \left[ \frac{1}{2} \int_{\Omega} \rho_0 |\underline{u}_0|^2 \, d\underline{x} + \int_{\Omega} P(\rho_0) \, d\underline{x} + k \int_{\Omega \times D} M \mathcal{F}(\widehat{\psi}_0) \, d\underline{q} \, d\underline{x} \right. \\
& \quad \left. + \mathfrak{z} \int_{\Omega} \left( \int_D M \widehat{\psi}_0 \, d\underline{q} \right)^2 \, d\underline{x} + \frac{1}{2} \int_0^{t'} \|f\|_{L^\infty(\Omega)}^2 \, dt \int_{\Omega} \rho_0 \, d\underline{x} \right]. \quad (6.33)
\end{aligned}$$

**Proof.** Passing to the limit  $\kappa \rightarrow 0_+$  for the subsequence of Lemma 6.4 in (5.27a) yields (6.32a) subject to the stated initial condition, on noting (6.10a), (6.30c) and that  $\frac{6\gamma}{\gamma+6} > \frac{6}{5}$  as  $\gamma > \frac{3}{2}$ .

Similarly to the proof of (4.118b), passing to the limit  $\kappa \rightarrow 0_+$  for the subsequence of Lemma 6.4 in (5.62) for any  $w \in C_0^\infty(\Omega_T)$  yields (6.32b) for any  $w \in C_0^\infty(\Omega_T)$  subject to the stated initial condition, on noting (6.5), (6.10a), (6.30a,b,d,f,g), (6.6e) and (6.8b). The desired result (6.32b) for any  $w \in L^{\frac{\gamma+\vartheta}{\vartheta}}(0, T; W_0^{1,r}(\Omega))$  then follows from (6.28), (6.30f) and noting from (6.13) that  $r \geq \frac{\gamma+\vartheta}{\vartheta} \geq 2$ . Similarly to the proof of (4.118c), passing to the limit  $\kappa \rightarrow 0_+$  for the subsequence of Lemma 6.4 in (5.27c) yields (6.32c) subject to the stated initial condition, on noting (6.5), (6.6a–d), (6.7) and (3.2). Similarly to the proof of (4.119), we deduce (6.33) from (5.28) using the results (6.30d), (6.10c), (6.6a,b,f), (6.5) and (6.8a).  $\square$

We need to identify  $\overline{\rho^\gamma}$  in (6.32b) and (6.30f). Similarly to (6.14), with  $\ell \in \mathbb{N}$ , we now introduce  $\mathfrak{t} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  and  $\mathfrak{t}_\ell : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  such that

$$\mathfrak{t}(s) := s \quad \text{and} \quad \mathfrak{t}_\ell(s) := \begin{cases} \mathfrak{t}(s) & \text{for } 0 \leq s \leq \ell, \\ \mathfrak{t}(\ell) & \text{for } \ell \leq s. \end{cases} \quad (6.34)$$

Then, similarly to (6.17), we have the renormalized equation, for any  $\ell \in \mathbb{N}$ ,

$$\begin{aligned}
\frac{\partial \mathfrak{t}_\ell(\rho_\kappa)}{\partial t} + \underline{\nabla}_x \cdot (\mathfrak{t}_\ell(\rho_\kappa) \underline{u}_\kappa) + (\rho_\kappa (\mathfrak{t}_\ell)_+(\rho_\kappa) - \mathfrak{t}_\ell(\rho_\kappa)) \underline{\nabla}_x \cdot \underline{u}_\kappa = 0 \\
\text{in } C_0^\infty(\mathbb{R}^d \times (0, T))', \quad (6.35)
\end{aligned}$$

where  $(\mathfrak{t}_\ell)_+(\cdot)$  is defined similarly to (6.18). It follows from (6.34), (5.28) and (6.35)

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that, for any fixed  $\ell \in \mathbb{N}$ ,

$$\begin{aligned} & \|\mathbf{t}_\ell(\rho_\kappa)\|_{L^\infty(\Omega_T)} + \|(\rho_\kappa(\mathbf{t}_\ell)_+(\rho_\kappa) - \mathbf{t}_\ell(\rho_\kappa)) \nabla_x \cdot \mathbf{u}_\kappa\|_{L^2(\Omega_T)} \\ & \quad + \left\| \frac{\partial \mathbf{t}_\ell(\rho_\kappa)}{\partial t} \right\|_{L^2(0,T;H^1(\Omega)')} \leq C(\ell). \end{aligned} \quad (6.36)$$

In order to identify  $\overline{\rho^\gamma}$  in (6.32b) and (6.30f), we now apply Corollary 5.1 with (5.38a,b) being (6.35) and (5.62) so that  $\mu = \frac{\mu^S}{2}$  and  $\lambda = \mu^B - \frac{\mu^S}{d}$ ,  $g_n = \mathbf{t}_\ell(\rho_\kappa)$  for a fixed  $\ell \in \mathbb{N}$ ,  $\mathbf{u}_n = \mathbf{u}_\kappa$ ,  $\mathbf{m}_n = \rho_\kappa \mathbf{u}_\kappa$ ,  $p_n = p_\kappa(\rho_\kappa)$ ,  $\tau_n = \tau_\kappa(M \widehat{\psi}_\kappa)$ ,  $f_n = -(\rho_\kappa(\mathbf{t}_\ell)_+(\rho_\kappa) - \mathbf{t}_\ell(\rho_\kappa)) \nabla_x \cdot \mathbf{u}_\kappa$  and  $F_n = \rho_\kappa f$ . With  $\{(\rho_\kappa, \mathbf{u}_\kappa, \widehat{\psi}_\kappa)\}_{\kappa>0}$  being the subsequence (not indicated) of Lemma 6.4, we have that (5.37a–d) hold with  $g = \mathbf{t}_\ell(\rho)$ ,  $\mathbf{u} = \mathbf{u}$ ,  $\mathbf{m} = \rho \mathbf{u}$  and  $p = c_p \overline{\rho^\gamma}$ , and  $q < \infty$ ,  $\omega = \infty$ ,  $z = \frac{2\gamma}{\gamma+1}$  and  $r = \frac{\Gamma+\vartheta}{\Gamma}$  on recalling (6.36), (3.12a,b), (6.5), (6.30b) and (6.30f,g). We note that  $z = \frac{2\gamma}{\gamma+1} > \frac{6}{5}$  as  $\gamma > \frac{3}{2}$ . Hence, the constraints on  $q, r, \omega$  and  $z$  hold. The results (5.37e,h) hold with  $\tau_\kappa = \tau_\kappa(M \widehat{\psi}_\kappa)$ ,  $F = \rho f$  and  $s = \gamma + \vartheta$ , on recalling (6.6e) and (6.30e), and noting that  $\frac{4(d+2)}{3d+4} \geq \frac{q}{q-1}$ . Finally, the result (5.47) holds with  $f = -(\rho(\mathbf{t}_\ell)_+(\rho) - \mathbf{t}_\ell(\rho)) \nabla_x \cdot \mathbf{u}$  on recalling (6.36). Hence, we obtain from (5.39) for the subsequence of Lemma 6.4 that, for any fixed  $\ell \in \mathbb{N}$  and for all  $\zeta \in C_0^\infty(\Omega)$  and  $\eta \in C_0^\infty(0, T)$ ,

$$\begin{aligned} & \lim_{\kappa \rightarrow 0_+} \int_0^T \eta \left( \int_\Omega \zeta \mathbf{t}_\ell(\rho_\kappa) [p_\kappa(\rho_\kappa) - \mu^* \nabla_x \cdot \mathbf{u}_\kappa] dx \right) dt \\ & \quad = \int_0^T \eta \left( \int_\Omega \zeta \overline{\mathbf{t}_\ell(\rho)} [c_p \overline{\rho^\gamma} - \mu^* \nabla_x \cdot \mathbf{u}] dx \right) dt, \end{aligned} \quad (6.37)$$

where  $\mu^* := \frac{(d-1)}{d} \mu^S + \mu^B$ .

We deduce from (6.37), (6.34), (2.3), (1.3), (6.30f,g) and (6.5) that, for any fixed  $\ell \in \mathbb{N}$ ,

$$c_p [\overline{\mathbf{t}_\ell(\rho) \rho^\gamma} - \overline{\mathbf{t}_\ell(\rho)} \overline{\rho^\gamma}] = \mu^* [\overline{\mathbf{t}_\ell(\rho) \nabla_x \cdot \mathbf{u}} - \overline{\mathbf{t}_\ell(\rho)} \nabla_x \cdot \mathbf{u}] \quad \text{a.e. in } \Omega_T, \quad (6.38)$$

where, as  $\kappa \rightarrow 0_+$ ,

$$\mathbf{t}_\ell(\rho_\kappa) \rho_\kappa^\gamma \rightarrow \overline{\mathbf{t}_\ell(\rho) \rho^\gamma} \quad \text{weakly in } L^{\frac{\gamma+\vartheta}{\gamma}}(\Omega_T), \quad (6.39a)$$

$$\text{and } \mathbf{t}_\ell(\rho_\kappa) \nabla_x \cdot \mathbf{u}_\kappa \rightarrow \overline{\mathbf{t}_\ell(\rho) \nabla_x \cdot \mathbf{u}} \quad \text{weakly in } L^2(\Omega_T). \quad (6.39b)$$

We can now follow the discussion in Sections 7.10.2–7.10.5 in Novotný & Straškraba<sup>24</sup> to deduce that  $\overline{p(\rho)} = p(\rho)$ . For the benefit of the reader, we briefly outline the argument. First, it follows from (6.30f) and (6.39a) that, for any fixed  $\ell \in \mathbb{N}$ ,

$$\begin{aligned} & \int_{\Omega_T} [\overline{\mathbf{t}_\ell(\rho) \rho^\gamma} - \overline{\mathbf{t}_\ell(\rho)} \overline{\rho^\gamma}] dx dt \\ & = \lim_{\kappa \rightarrow 0_+} \left[ \int_{\Omega_T} (\mathbf{t}_\ell(\rho_\kappa) - \mathbf{t}_\ell(\rho)) (\rho_\kappa^\gamma - \rho^\gamma) dx dt + \int_{\Omega_T} (\mathbf{t}_\ell(\rho) - \overline{\mathbf{t}_\ell(\rho)}) (\overline{\rho^\gamma} - \rho^\gamma) dx dt \right] \\ & \geq \limsup_{\kappa \rightarrow 0_+} \int_{\Omega_T} |\mathbf{t}_\ell(\rho_\kappa) - \mathbf{t}_\ell(\rho)|^{\gamma+1} dx dt, \end{aligned} \quad (6.40)$$

where we have noted that the second term on the second line is nonnegative as  $\mathfrak{t}_\ell(s)$  is concave and  $s^\gamma$  is convex for  $s \in [0, \infty)$ .

We now introduce  $\mathfrak{L}_\ell : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ ,  $\ell \in \mathbb{N}$ , such that  $s \mathfrak{L}'_\ell(s) - \mathfrak{L}_\ell(s) = \mathfrak{t}_\ell(s)$  for all  $s \in [0, \infty)$ , so that

$$\mathfrak{L}_\ell(s) := \begin{cases} \mathfrak{L}(s) := s \log s & \text{for } 0 \leq s \leq \ell, \\ s \log \ell + s - \ell & \text{for } \ell \leq s. \end{cases} \quad (6.41)$$

Similarly to (5.57) and (6.17), one deduces, via renormalization, from (6.32a) and (5.27a) that, for any fixed  $\ell \in \mathbb{N}$  and for any  $t' \in (0, T]$ ,

$$\int_{\Omega} [\mathfrak{L}_\ell(\rho)(t') - \mathfrak{L}_\ell(\rho_0)] \, d\tilde{x} = - \int_0^{t'} \int_{\Omega} \mathfrak{t}_\ell(\rho) \nabla_x \cdot \tilde{u} \, d\tilde{x} \, dt, \quad (6.42a)$$

$$\int_{\Omega} [\mathfrak{L}_\ell(\rho_\kappa)(t') - \mathfrak{L}_\ell(\rho_0)] \, d\tilde{x} = - \int_0^{t'} \int_{\Omega} \mathfrak{t}_\ell(\rho_\kappa) \nabla_x \cdot \tilde{u}_\kappa \, d\tilde{x} \, dt. \quad (6.42b)$$

Although establishing (6.42b) is straightforward as  $\rho_\kappa \in C_w([0, T]; L_{\geq 0}^\Gamma(\Omega))$ , proving (6.42a) is not, since  $\rho \in C_w([0, T]; L_{\geq 0}^\gamma(\Omega))$ , and so  $\rho$  may not be in  $L^2(\Omega_T)$  as  $\gamma > \frac{3}{2}$ . Nevertheless, (6.42a) can still be established, see Lemma 7.57 in Novotný & Straškraba<sup>24</sup>. We note that our  $\vartheta(\gamma)$ , recall (6.13), differs from the  $\vartheta(\gamma)$  in Novotný & Straškraba<sup>24</sup> for  $\gamma \geq 4$ , due to the presence of the extra stress term in the momentum equation for our polymer model. However, as  $\rho \in L^2(\Omega_T)$  for  $\gamma \geq 4$ , Lemma 7.57 in Novotný & Straškraba<sup>24</sup> is not required for such  $\gamma$ . Subtracting (6.42a) from (6.42b), and passing to the limit  $\kappa \rightarrow 0_+$ , one deduces from (6.39b) that, for any fixed  $\ell \in \mathbb{N}$  and for any  $t' \in (0, T]$ ,

$$\int_{\Omega} [\overline{\mathfrak{L}_\ell(\rho)(t')} - \mathfrak{L}_\ell(\rho)(t')] \, d\tilde{x} = \int_0^{t'} \int_{\Omega} [\mathfrak{t}_\ell(\rho) \nabla_x \cdot \tilde{u} - \overline{\mathfrak{t}_\ell(\rho) \nabla_x \cdot \tilde{u}}] \, d\tilde{x} \, dt, \quad (6.43)$$

where, on noting (6.10a) and the convexity of  $\mathfrak{L}_\ell$ ,

$$\mathfrak{L}_\ell(\rho_\kappa)(t') \rightarrow \overline{\mathfrak{L}_\ell(\rho)(t')} \geq \mathfrak{L}_\ell(\rho)(t') \quad \text{weakly in } L^\gamma(\Omega), \quad \text{as } \kappa \rightarrow 0_+. \quad (6.44)$$

It follows from (6.40), (6.38), (6.43), (6.44) and (6.4a) that, for any fixed  $\ell \in \mathbb{N}$ ,

$$\begin{aligned} \limsup_{\kappa \rightarrow 0_+} \|\mathfrak{t}_\ell(\rho) - \mathfrak{t}_\ell(\rho_\kappa)\|_{L^{\gamma+1}(\Omega_T)}^{\gamma+1} &\leq \frac{\mu^*}{c_p} \int_{\Omega_T} [\overline{\mathfrak{t}_\ell(\rho) \nabla_x \cdot \tilde{u}} - \overline{\mathfrak{t}_\ell(\rho)} \nabla_x \cdot \tilde{u}] \, d\tilde{x} \, dt \\ &\leq \frac{\mu^*}{c_p} \int_{\Omega_T} [\mathfrak{t}_\ell(\rho) - \overline{\mathfrak{t}_\ell(\rho)}] \nabla_x \cdot \tilde{u} \, d\tilde{x} \, dt \\ &\leq C \|\mathfrak{t}_\ell(\rho) - \overline{\mathfrak{t}_\ell(\rho)}\|_{L^2(\Omega_T)} \\ &\leq C \limsup_{\kappa \rightarrow 0_+} \|\mathfrak{t}_\ell(\rho) - \mathfrak{t}_\ell(\rho_\kappa)\|_{L^2(\Omega_T)} \\ &\leq C \limsup_{\kappa \rightarrow 0_+} \|\mathfrak{t}_\ell(\rho) - \mathfrak{t}_\ell(\rho_\kappa)\|_{L^{\gamma+1}(\Omega_T)}, \end{aligned} \quad (6.45)$$

where  $C \in \mathbb{R}_{>0}$  is independent of  $\ell$  and  $\kappa$ . It follows from (6.24), (6.30e) and (6.34) that, for all  $\ell \in \mathbb{N}$ ,  $\kappa > 0$  and  $r \in [1, \gamma + \vartheta)$ ,

$$\|\rho_\kappa - \mathfrak{t}_\ell(\rho_\kappa)\|_{L^r(\Omega_T)} + \|\rho - \mathfrak{t}_\ell(\rho)\|_{L^r(\Omega_T)} + \|\rho - \overline{\mathfrak{t}_\ell(\rho)}\|_{L^r(\Omega_T)} \leq C \ell^{1 - \frac{\gamma + \vartheta}{r}}, \quad (6.46)$$

where  $C \in \mathbb{R}_{>0}$  is independent of  $\ell$  and  $\kappa$ . It follows from (6.45), (6.46) and (3.1) that

$$\lim_{\ell \rightarrow \infty} \limsup_{\kappa \rightarrow 0_+} \|\mathfrak{t}_\ell(\rho) - \mathfrak{t}_\ell(\rho_\kappa)\|_{L^{\gamma+1}(\Omega_T)} = 0. \quad (6.47)$$

It follows from (6.46), (6.47), (6.24), (3.1) and (6.30f), on extracting a further subsequence (not indicated), that, as  $\kappa \rightarrow 0_+$ ,

$$\rho_\kappa \rightarrow \rho \quad \text{strongly in } L^s(\Omega_T), \quad \text{for any } s \in [1, \gamma + \vartheta(\gamma)], \quad (6.48a)$$

$$\rho_\kappa^\gamma \rightarrow \rho^\gamma \quad \text{weakly in } L^{\frac{\gamma+\vartheta}{\gamma}}(\Omega_T), \quad \text{that is, } \overline{\rho^\gamma} = \rho^\gamma. \quad (6.48b)$$

Finally, we have the analogue of Theorem 5.1.

**Theorem 6.1.** *The triple  $(\rho, \underline{u}, \widehat{\psi})$ , defined as in Lemmas 6.1 and 6.2, is a global weak solution to problem (P), in the sense that (6.32a,c), with their initial conditions, hold and, letting  $r = \max\{4, \frac{6\gamma}{2\gamma-3}\}$ ,*

$$\begin{aligned} & \int_0^T \left\langle \frac{\partial(\rho \underline{u})}{\partial t}, w \right\rangle_{W_0^{1,r}(\Omega)} dt + \int_0^T \int_\Omega \left[ S(\underline{u}) - \rho \underline{u} \otimes \underline{u} - c_p \rho^\gamma I \right] : \nabla_x w \, dx \, dt \\ & = \int_0^T \int_\Omega \left[ \rho f \cdot w - \left( \tau_1(M \widehat{\psi}) - \mathfrak{z} \varrho^2 I \right) : \nabla_x w \right] dx \, dt \\ & \quad \forall w \in L^{\frac{\gamma+\vartheta}{\vartheta}}(0, T; W_0^{1,r}(\Omega)), \end{aligned} \quad (6.49)$$

with  $(\rho \underline{u})(\cdot, 0) = (\rho_0 \underline{u}_0)(\cdot)$ ,  $\vartheta(\gamma)$  defined as in (6.13). In addition, the weak solution  $(\rho, \underline{u}, \widehat{\psi})$  satisfies (6.33).

**Proof.** The results (6.32a,c) and (6.33) have already been established in Lemma 6.5. Equation (6.49) was established in Lemma 6.5 with  $\rho^\gamma$  replaced by  $\overline{\rho^\gamma}$ , see (6.32b). The desired result (6.49) then follows from (6.32b) and (6.48b).  $\square$

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