Geophysical Journal International

Geophys. J. Int. (2015) **203,** 1172–1178 doi: 10.1093/gji/ggv346
GJI Seismology

Generalized seismic wavelets

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Accepted 2015 August 14. Received 2015 August 13; in original form 2015 July 6

SUMMARY

The Ricker wavelet, which is often employed in seismic analysis, has a symmetrical form. Seismic wavelets observed from field data, however, are commonly asymmetric with respect to the time variation. In order to better represent seismic signals, asymmetrical wavelets are defined systematically as fractional derivatives of a Gaussian function in which the Ricker wavelet becomes just a special case with the integer derivative of order 2. The fractional value and a reference frequency are two key parameters in the generalization. Frequency characteristics, such as the central frequency, the bandwidth, the mean frequency and the deviation, may be expressed analytically in closed forms. In practice, once the statistical properties (the mean frequency and deviation) are numerically evaluated from the discrete Fourier spectra of seismic data, these analytical expressions can be used to uniquely determine the fractional value and the reference frequency, and subsequently to derive various frequency quantities needed for the wavelet analysis. It is demonstrated that field seismic signals, recorded at various depths in a vertical borehole, can be closely approximated by generalized wavelets, defined in terms of fractional values and reference frequencies.

Key words: Time-series analysis; Numerical solutions; Computational seismology; Wave propagation.

INTRODUCTION

The Ricker wavelet is a well-known symmetrical waveform in the time domain (Ricker 1953). In order to better represent practically observed non-Ricker forms of seismic signals (Hosken 1988), the symmetric Ricker wavelet is generalized to be asymmetrical.

While the Ricker wavelet is the second derivative of a Gaussian function, generalization is achieved by modifying the derivative order from the integer '2' to a fractional value. For mathematical convenience, the base function is the same Gaussian function, rather than other alternative forms. Therefore, generalized wavelets are systematically defined by fractional derivatives of a Gaussian function.

Generalized wavelets have similar Fourier spectra because they are derived from the same Gaussian function. Their spectra differ from each other only in a frequency-related factor $(\mathrm{i}\omega)^u$, where ω is the angular frequency and u is the fractional order of the time derivative. Although there are possibly different definitions for field seismic wavelets, the current paper provides a systematic definition of non-Ricker wavelets and meanwhile can use the Ricker wavelet as a benchmark. This paper will prove that the power spectrum of a generalized wavelet is close to a Gaussian distribution, and thus the mean frequency is approximately equal to the central frequency. The degree of similarity between a generalized wavelet spectrum and a Gaussian distribution depends upon the fractional value.

For a generalized wavelet with a variable fractional value u, the central frequency and the frequency band can be expressed analytically, using a special function, the Lambert W function (Lambert 1758, 1772; Euler 1779; Corless *et al.* 1996; Banwell & Jayakumar 2000; Valluri *et al.* 2000; Packel & Yuen 2004; Shafee 2007; Wang 2015a,b). The mean frequency and its deviation can also be derived analytically in terms of the Gamma function. Thus, the analytical relationships between the theoretical properties (the central frequency and the bandwidth) and statistical parameters (the mean frequency and its deviation) may be established.

While these frequency analyses are inspired by previous studies (Wang 2015a,b) on the Ricker wavelet, which is just a special case with $u = 2$, the systematic definition of generalized wavelets and the non-trivial development of frequency relationships will certainly set up a solid foundation for field seismic signal analysis. In practice, given a discrete Fourier spectrum, the mean frequency and the standard deviation can be evaluated numerically. Once these two quantities are measured from field seismic data, the analytical expressions mentioned above can be used to derive the fractional value, the reference frequency and other frequency parameters for wavelet and spectral analysis.

WAVELETS DEFINED IN THE FREQUENCY DOMAIN

According to Ricker (1943, 1944), a wavelet function (of the displacement, velocity or acceleration type) may be expressed as a polynomial of various derivatives of a potential function. Therefore, we set the potential function in (negative) Gaussian:

$$g(\tau) = -\sqrt{\pi}\omega_0 \exp\left(-\frac{\omega_0^2}{4}(\tau - \tau_0)^2\right), \quad (1)$$

where τ is a time variable (in seconds), τ_0 is the time position of the symmetrical centre and ω_0 is a reference frequency (in radians per second). This ω_0 parameter is inversely proportional to the deviation of the Gaussian distribution and should physically reflect the viscoelastic property of the subsurface media. Setting the potential function as a negative Gaussian will produce a conventional polarity of the wavelets.

In this section, generalized wavelets are defined in the frequency domain. The Fourier transform of the Gaussian function (1) is

$$G(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\tau)\exp(-\mathrm{i}\omega\tau)\,\mathrm{d}\tau$$
$$= -\exp\left(-\frac{\omega^2}{\omega_0^2}\right)\exp\left(-\mathrm{i}\omega\tau_0\right). \quad (2)$$

For any fractional or integer order derivative of the Gaussian function, with respect to the time τ, the frequency spectrum may be expressed as $G(\omega)$ multiplied by a frequency factor $(\mathrm{i}\omega)^u$:

$$(\mathrm{i}\omega)^u G(\omega) = \omega^u \exp\left(-\frac{\omega^2}{\omega_0^2}\right)\exp\left(-\mathrm{i}\omega\tau_0 + \mathrm{i}\pi\left(1 + \frac{u}{2}\right)\right), \quad (3)$$

where u is the order of a time-domain derivative. A normalized spectrum $\Phi^{(u)}(\omega)$ is set, by multiplying a factor of $\omega_0^{-u}(u/2)^{-u/2}\exp(u/2)$ to spectrum (3), as

$$\Phi^{(u)}(\omega) = \left(\frac{u}{2}\right)^{-u/2}\frac{\omega^u}{\omega_0^u}\exp\left(-\frac{\omega^2}{\omega_0^2} + \frac{u}{2}\right)$$
$$\times \exp\left(-\mathrm{i}\omega\tau_0 + \mathrm{i}\pi\left(1 + \frac{u}{2}\right)\right). \quad (4)$$

Performing an inverse Fourier transform on $\Phi^{(u)}(\omega)$, one can generate a time-domain wavelet $\phi^{(u)}(\tau)$. When u is an integer, the inverse Fourier transform can be derived analytically. But, for any fractional value of u, the inverse Fourier transform needs to be calculated numerically.

Fig. 1 displays a series of wavelets $\phi^{(u)}(\tau)$ (in solid curves), defined with a sample reference frequency $\omega_0 = 60\pi$ rad s^{-1}, equivalent to the ordinary frequency of 30 Hz. The fractional value u varies between 0.4 and 2.2. These wavelets centred at time $\tau_0 = 0.05$ s are overlaid by approximations (the dashed curves) obtained using time-domain fractional derivatives, described in the following section.

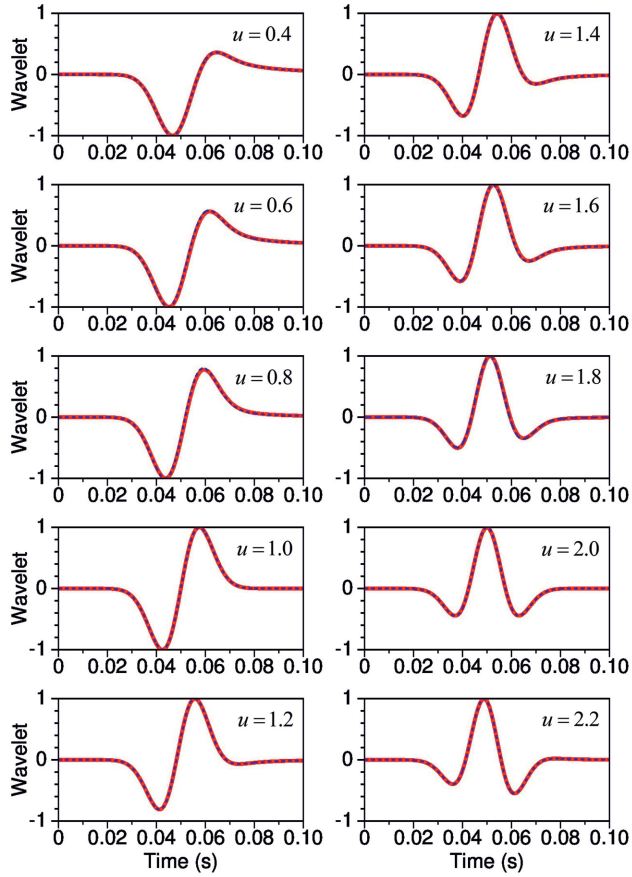

Figure 1. Generalized wavelets defined by fractional derivatives of a Gaussian function. Dashed curves are the approximations obtained by the time-domain fractional derivatives and are overlaid on solid curves which are the accurate waveforms obtained by a Fourier transform method. The fraction u varies from 0.4 to 2.2. When $u = 2.0$, the second derivative, it is the Ricker wavelet.

WAVELETS DEFINED BY FRACTIONAL DERIVATIVES

Since wavelets are commonly presented in the time domain, this section presents generalized wavelets as time-domain fractional derivatives of a Gaussian function.

For a derivative of order u, which can be either a positive fraction or an integer, of the Gaussian function $g(\tau)$, the following definition of Caputo (1967) is used:

$$g^{(u)}(\tau) = \frac{1}{\Gamma(m-u)}\int_0^{\tau}(\tau - \xi)^{m-u-1} g^{(m)}(\xi)\,\mathrm{d}\xi, \quad (5)$$

where $g^{(u)}(\tau) \equiv \mathrm{d}^u g(\tau)/\mathrm{d}\tau^u$, m is an integer, $m - 1 \le u < m$, $g^{(m)}(\tau)$ is the mth integer-order derivative and $\Gamma(s)$ is the Gamma function. Eq. (5) can be understood as a Laplace convolution of two causal functions:

$$g^{(u)}(\tau) = g^{(m)}(\tau) * h(\tau), \quad (6)$$

where $g^{(m)}(\tau) = 0$, for $\tau < 0$, and

$$h(\tau) = \begin{cases} \dfrac{\tau^{m-u-1}}{\Gamma(m-u)}, & \tau > 0, \\ 0, & \tau \le 0. \end{cases} \quad (7)$$

The convolution expression (6) is equivalent to a Fourier transform domain multiplication between frequency spectra of $g^{(m)}(\tau)$ and $h(\tau)$. The Fourier transform of $h(\tau)$ is the factor $(\mathrm{i}\omega)^u$ in eq. (3).

For fractional derivatives, there are various definitions, such as conventional Riemann–Liouville integral (Podlubny 2002) and its modification (Jumarie 2006). These definitions form the integer-order derivative $\mathrm{d}^m/\mathrm{d}\tau^m$ outside the integral, rather than being inside of the integral. They are theoretically equivalent, based on the Leibniz integral rule. Caputo's definition is used here because any integer-order derivative $g^{(m)}(\tau)$ can be easily expressed in an analytical form, and fractional derivatives can be written explicitly as follows.

For $0 \le u < 1$, where $m = 1$:

$$g^{(u)}(\tau) = \frac{\sqrt{\pi}\omega_0^3}{2\Gamma(1-u)} \int_0^\tau (\tau-\xi)^{-u}(\xi-\tau_0) \times \exp\left(-\frac{\omega_0^2}{4}(\xi-\tau_0)^2\right) \mathrm{d}\xi. \quad (8)$$

For $1 \le u < 2$, where $m = 2$:

$$g^{(u)}(\tau) = \frac{\sqrt{\pi}\omega_0^3}{2\Gamma(2-u)} \int_0^\tau (\tau-\xi)^{1-u}\left(1-\frac{\omega_0^2}{2}(\xi-\tau_0)^2\right) \times \exp\left(-\frac{\omega_0^2}{4}(\xi-\tau_0)^2\right) \mathrm{d}\xi. \quad (9)$$

For $2 \le u < 3$, where $m = 3$:

$$g^{(u)}(\tau) = \frac{3\sqrt{\pi}\omega_0^5}{4\Gamma(3-u)} \int_0^\tau (\tau-\xi)^{2-u}(\xi-\tau_0) \times \left(\frac{\omega_0^2}{6}(\xi-\tau_0)^2-1\right) \exp\left(-\frac{\omega_0^2}{4}(\xi-\tau_0)^2\right) \mathrm{d}\xi. \quad (10)$$

Considering special cases with the integer values $u = 0, 1, 2$, the corresponding m values are $m = 1, 2, 3$, eqs (8)–(10) will be

$$g^{(u)}(\tau) = g^{(m-1)}(\xi)\big|_0^\tau = g^{(m-1)}(\tau) - g^{(m-1)}(0). \quad (11)$$

What one expected here is $g^{(u)}(\tau) = g^{(m-1)}(\tau)$, for $u = 0, 1, 2$. This expectation is fulfilled if $g^{(m-1)}(0)$ is zero-valued. Hence, a mathematical condition for Caputo's integral (5) is

$$g^{(m-1)}(\tau) = 0, \quad \text{for} \quad \tau \le 0. \quad (12)$$

This condition elicits the importance of the time-shift τ_0 in the Gaussian function (1): The time-shift will make the above condition be satisfied by Caputo's definition (5), and explicitly by three eqs (8–10), in which the integrals are implemented using the following Simpson's 3/8 rule in each subinterval:

$$\int_a^b f(\xi)\mathrm{d}\xi \approx \frac{(b-a)}{8}\left(f(a)+3f\left(\frac{2a+b}{3}\right) +3f\left(\frac{a+2b}{3}\right)+f(b)\right). \quad (13)$$

A wavelet is defined by the normalized derivative $\phi^{(u)}(\tau) = \overline{g^{(u)}(\tau)}$, where the overbar denotes the normalization. Fractional derivatives are approximations due to the integral singularity when $\xi \to \tau$. However, Fig. 1 indicates that these approximated wavelets (dashed curves) have some negligible errors, only when $u = 0.8$ and 1.8, if they are compared to the accurate waveforms (solid curves), obtained by a Fourier transform method, presented in the previous section.

FREQUENCY CHARACTERISTICS

For any real-valued wavelet $\phi^{(u)}(\tau)$, the frequency spectrum $\Phi^{(u)}(\omega)$ needs to be considered only for $\omega \ge 0$. The amplitude spectrum is

$$|\Phi^{(u)}(\omega)| = \left(\frac{u}{2}\right)^{-u/2} \frac{\omega^u}{\omega_0^u} \exp\left(-\frac{\omega^2}{\omega_0^2}+\frac{u}{2}\right). \quad (14)$$

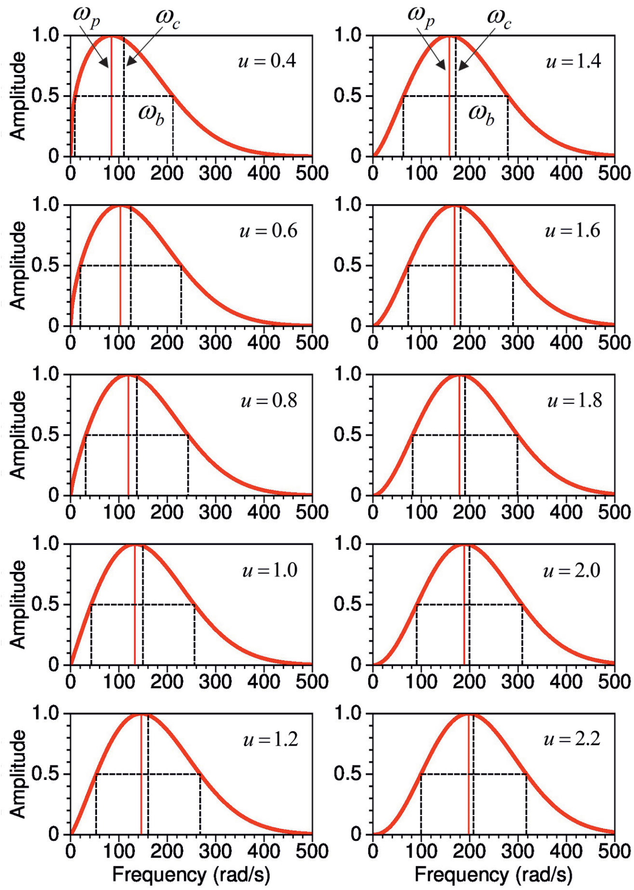

Figure 2. The amplitude spectra of various wavelets defined by fractional derivatives of a Gaussian function. A solid vertical line indicates the peak frequency ω_p and dashed vertical lines indicate the central frequency ω_c and the half-bandwidth ω_b.

Fig. 2 displays the amplitude spectra $|\Phi^{(u)}(\omega)|$ with different u values ($0.4 \le u \le 2.2$). The peak frequency is

$$\omega_p = \omega_0\sqrt{\frac{u}{2}}. \quad (15)$$

For example, the peak frequencies are $\omega_p = (\frac{1}{\sqrt{2}}\omega_0, \frac{\sqrt{3}}{2}\omega_0, \omega_0)$ for the first, one-and-a-half and second derivatives, respectively. This proves that only the Ricker wavelet with $u = 2$ has its peak frequency equal to ω_0. When $u < 2$, the peak frequency is generally smaller than the reference frequency.

The peak amplitude is unity. To determine the frequency band, an equation is set up so that the amplitude spectrum equals to one half,

$$\left(\frac{\omega}{\omega_0}\sqrt{\frac{2}{u}}\right)^u \exp\left(-\frac{\omega^2}{\omega_0^2}+\frac{u}{2}\right) = \frac{1}{2}. \quad (16)$$

Manipulating this equation leads to the following form:

$$-\frac{2}{u}\frac{\omega^2}{\omega_0^2}\exp\left(-\frac{2}{u}\frac{\omega^2}{\omega_0^2}\right) = -\frac{1}{2^{2/u}e}, \quad (17)$$

where e is Euler's number, $e = \exp(1)$. Eq. (17) is an inverse exponential equation,

$$z \exp z = x, \quad (18)$$

Table 1. Numerical values of the Lambert $W(x)$ function, where $x = -(2^{2/u}e)^{-1}$, corresponding to various u values.

u	x	$W_{-1}(x)$	$W_0(x)$
0.4	−0.011496	−6.307471	−0.011631
0.5	−0.022992	−5.472285	−0.023540
0.6	−0.036498	−4.899655	−0.037908
0.7	−0.050771	−4.480056	−0.053565
0.8	−0.065033	−4.157871	− 0.069729
0.9	−0.078841	−3.901753	−0.085914
1.0	−0.091970	−3.692635	−0.101828
1.1	−0.104323	−3.518224	−0.117307
1.2	−0.115875	−3.370224	−0.132260
1.3	−0.126643	−3.242825	−0.146646
1.4	−0.136667	−3.131827	−0.160452
1.5	−0.145993	−3.034117	−0.173685
1.6	−0.154674	−2.947335	−0.186360
1.7	−0.162762	−2.869661	−0.198500
1.8	−0.170305	−2.799662	−0.210129
1.9	−0.177350	−2.736198	−0.221274
2.0	−0.183940	−2.678347	−0.231961
2.1	−0.190112	−2.625357	−0.242216
2.2	−0.195903	−2.576607	−0.252065

with a solution $z = W(x)$, where $W(x)$ is the Lambert W function (Corless *et al.* 1996; Wang 2015a). Hence the solution to this equation is

$$W\left(-\frac{1}{2^{2/u}e}\right) = -\frac{2}{u}\frac{\omega^2}{\omega_0^2}. \tag{19}$$

For $x < 0$, the $W(x)$ function has two branches, $W_{-1}(x) \le -1$ and $W_0(x) \ge -1$. Then, the frequency band $[\omega_{\ell 1}, \omega_{\ell 2}]$ may be analytically defined as

$$\omega_{\ell 1} = \omega_0\sqrt{-\frac{u}{2}W_0\left(-\frac{1}{2^{2/u}e}\right)},$$

$$\omega_{\ell 2} = \omega_0\sqrt{-\frac{u}{2}W_{-1}\left(-\frac{1}{2^{2/u}e}\right)}. \tag{20}$$

Correspondingly, the central frequency, the geometric centre of the frequency band, is

$$\omega_c = \frac{\omega_0}{2}\left(\sqrt{-\frac{u}{2}W_{-1}\left(-\frac{1}{2^{2/u}e}\right)} + \sqrt{-\frac{u}{2}W_0\left(-\frac{1}{2^{2/u}e}\right)}\right), \tag{21}$$

and the half-bandwidth is

$$\omega_b = \frac{\omega_0}{2}\left(\sqrt{-\frac{u}{2}W_{-1}\left(-\frac{1}{2^{2/u}e}\right)} - \sqrt{-\frac{u}{2}W_0\left(-\frac{1}{2^{2/u}e}\right)}\right). \tag{22}$$

For practical application, Table 1 lists numeric values of the Lambert W function: $W_{-1}(x)$ and $W_0(x)$, where $x = -(2^{2/u}e)^{-1} < 0$, corresponding to a series of u values, $0.4 \le u \le 2.2$.

STATISTICAL PROPERTIES

The statistical properties of the discrete Fourier spectrum of a field seismic signal can be described by the mean frequency and the standard deviation. This section derives analytical expressions for these two frequency parameters.

The mean frequency and its deviation may be evaluated from the power spectrum $|\Phi^{(u)}(\omega)|^2$ by using (Berkhout 1984; Cohen & Lee 1989)

$$\omega_m = \frac{\int_0^\infty \omega|\Phi^{(u)}(\omega)|^2\,\mathrm{d}\omega}{\int_0^\infty |\Phi^{(u)}(\omega)|^2\,\mathrm{d}\omega},$$

$$\omega_\sigma = \left(\frac{\int_0^\infty (\omega-\omega_m)^2|\Phi^{(u)}(\omega)|^2\,\mathrm{d}\omega}{\int_0^\infty |\Phi^{(u)}(\omega)|^2\,\mathrm{d}\omega}\right)^{1/2}. \tag{23}$$

The three definite integrals are

$$\int_0^\infty |\Phi^{(u)}(\omega)|^2\,\mathrm{d}\omega = \frac{\omega_0 e^u}{2\sqrt{2}u^u}\Gamma\left(u+\frac{1}{2}\right), \tag{24}$$

$$\int_0^\infty \omega|\Phi^{(u)}(\omega)|^2\,\mathrm{d}\omega = \frac{\omega_0^2 e^u}{4u^{u-1}}\Gamma(u), \tag{25}$$

$$\int_0^\infty (\omega-\omega_m)^2|\Phi^{(u)}(\omega)|^2\mathrm{d}\omega = \frac{\omega_0^3 e^u}{4u^u}\left\{\Gamma\left(u+\frac{1}{2}\right)\times\left(\frac{1}{\sqrt{2}}\left(u+\frac{1}{2}\right)+\frac{\sqrt{2}\omega_m^2}{\omega_0^2}\right) - \frac{2u\omega_m}{\omega_0}\Gamma(u)\right\}. \tag{26}$$

Then, the analytical expression for the mean frequency is

$$\omega_m = \frac{\omega_0}{\sqrt{2}}\frac{u\Gamma(u)}{\Gamma(u+\frac{1}{2})}, \tag{27}$$

and for its deviation is

$$\omega_\sigma = \frac{\omega_0}{\sqrt{2}}\sqrt{\frac{1}{2}+u-\left(\frac{u\Gamma(u)}{\Gamma(u+\frac{1}{2})}\right)^2}. \tag{28}$$

For $u = 2$, where $\Gamma(u) = 1$, $\Gamma(u+\frac{1}{2}) = \frac{3}{4}\sqrt{\pi}$, the statistic quantities of the Ricker spectrum are (Wang 2015b)

$$\omega_m = \frac{4}{3}\sqrt{\frac{2}{\pi}}\omega_0, \quad \omega_\sigma = \omega_0\sqrt{\frac{5}{4}-\frac{32}{9\pi}}. \tag{29}$$

Fig. 3 displays the power spectra, with annotations of the mean frequencies and the deviations, of wavelets defined by different u values. It reveals that the mean frequency ω_m is close to the central frequency ω_c. This observation was found first in the case $u = 2$, when the mean frequency was evaluated from the power spectrum of a Ricker wavelet (Wang 2015b).

Using the two statistical parameters ω_m and ω_σ, an equivalent Gaussian function can be constructed. Fig. 3 above also shows how the wavelet power spectra (solid curves) are close to the Gaussian distribution (dashed curves).

Frequency characteristics can be presented as the reference frequency ω_0 multiplied by fraction-dependent coefficients:

$$\begin{bmatrix}\omega_p\\ \omega_c\\ \omega_b\\ \omega_m\\ \omega_\sigma\end{bmatrix} = \omega_0\begin{bmatrix}\alpha_p(u)\\ \alpha_c(u)\\ \alpha_b(u)\\ \alpha_m(u)\\ \alpha_\sigma(u)\end{bmatrix}. \tag{30}$$

These frequency coefficients $(\alpha_p, \alpha_c, \alpha_b, \alpha_m, \alpha_\sigma)$ can be found from eqs (15), (21), (22), (27) and (28), respectively. Fig. 4 compares

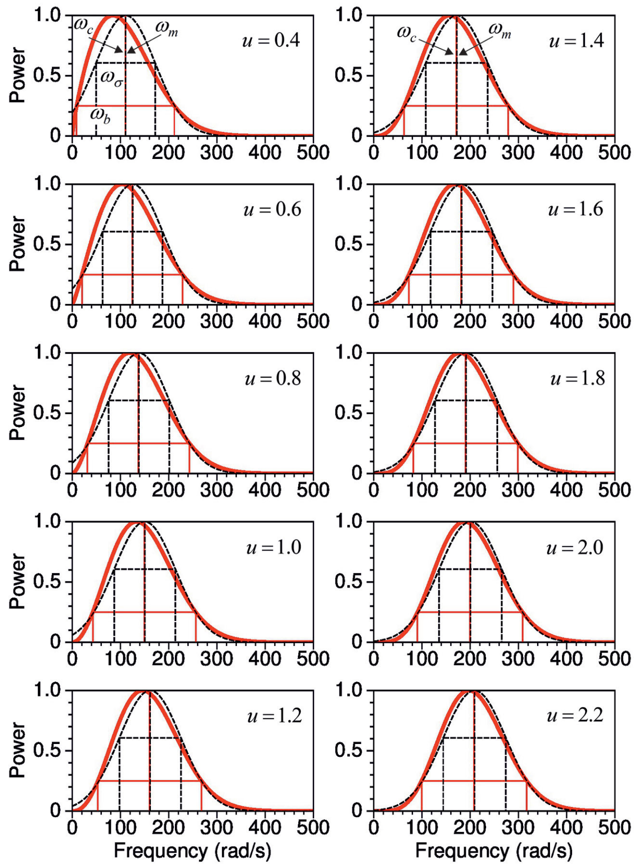

Figure 3. The power spectra of various wavelets defined by fractional derivatives of a Gaussian function. Solid vertical lines indicate the central frequencies ω_c and the half-bandwidths ω_b. Dashed vertical lines indicate the mean frequencies ω_m and the deviations ω_σ. Dashed curves are the equivalent Gaussian distributions defined by $(\omega_m, \omega_\sigma)$.

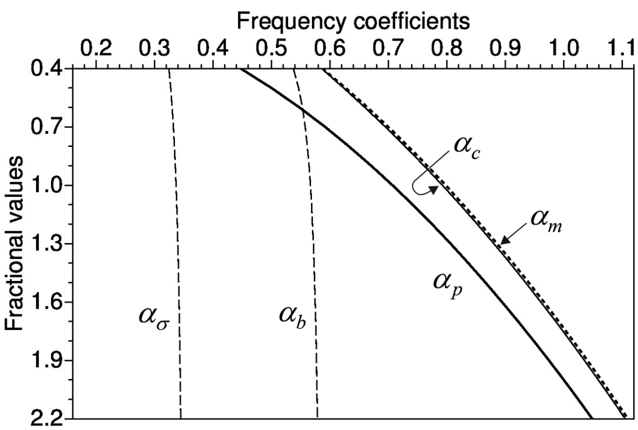

Figure 4. Frequency coefficients, α_p, α_c, α_b, α_m, α_σ, versus the fractional value of u. Multiplying these coefficients to a reference frequency will produce the peak frequency, the central frequency, the half-bandwidth, the mean frequency and the deviation, respectively.

these frequency coefficients with respect to the fractional value of u. These coefficients are actually the scaled frequency quantities. Thus, the illustration of Fig. 4 is an excellent summary of the relationships among those frequencies:

(1) The central frequency ω_c is approximately equal to the mean frequency ω_m;

(2) the peak frequency ω_p is less than the central frequency ω_c and the mean frequency ω_m;

(3) the half-bandwidth ω_b is wider than the standard deviation ω_σ; and

(4) variations in the half-bandwidth ω_b and the deviation ω_σ are both relatively small, along with the fraction.

FIELD SIGNAL ANALYSIS

Once the mean frequency and its deviation are evaluated from field seismic spectra, the fractional value u and the reference frequency ω_0 can be derived.

The fractional value of u can be uniquely determined by the ratio of the standard deviation to the mean frequency, using the following equation:

$$\left(\frac{1}{2u}+1\right)\left(\frac{\Gamma(u+\frac{1}{2})}{\sqrt{u}\,\Gamma(u)}\right)^2-1=\frac{\omega_\sigma^2}{\omega_m^2}, \tag{31}$$

in which the factor related to the ratio of the Gamma functions can be expressed as an asymptotic series (Graham *et al.* 1994),

$$\frac{\Gamma(u+\frac{1}{2})}{\sqrt{u}\,\Gamma(u)}=1-\frac{1}{8u}+\frac{1}{128u^2}+\frac{5}{1024u^3}-\frac{21}{32768u^4}+\cdots. \tag{32}$$

For the nonlinear eq. (31) with a single variable, a simple iterative procedure can find an optimal u value.

Once the fraction u is determined, the reference frequency can be determined based on the sum of ω_m^2 and ω_σ^2, using the following expression:

$$\omega_0=2\sqrt{\frac{\omega_m^2+\omega_\sigma^2}{1+2u}}. \tag{33}$$

This expression suggests to use both parameters ω_m and ω_σ, instead of only ω_m or only ω_σ, for estimating ω_0. It is an effort to minimize any potential bias error in these two values, numerically evaluated from a discrete Fourier spectrum.

Fig. 5 displays a series of field waveforms recorded at different depths in a vertical borehole. These waveforms (solid curves) are extracted from a vertical seismic profiling (VSP) data set generated by a dynamite source shot in a 15 m depth hole. They are the first arrivals, obtained by median filtering, which removes the VSP upgoing wavefield, and cosine-square tapering, which suppresses the downgoing free-surface multiples behind the first arrivals (Wang 2014).

In the frequency domain, the mean frequency ω_m and the standard deviation ω_σ are evaluated from the power spectra. Fig. 5 clearly indicates that the mean frequency ω_m is decreasing gradually along the depth, while the standard deviation ω_σ is almost a constant. Based on ω_m and ω_σ, the fractional value u and the reference frequency ω_0 are also derived. The reference frequency ω_0 has a relatively small variation, as the deviation from the mean of 72.32π is 2.87π. The fractional value u is also decreasing gradually from 1.9 down to 1.2, similar to the variation of the mean frequency ω_m.

The strong covariance between u and ω_m can be explained analytically by making an approximation to eq. (27) as

$$\omega_m\approx\omega_0\sqrt{\frac{u}{2}}\left(1+\frac{1}{8u}\right), \tag{34}$$

if assuming the reference frequency ω_0 to be a constant. The correlation coefficient between u and ω_m is 0.97 in this case (Fig. 6).

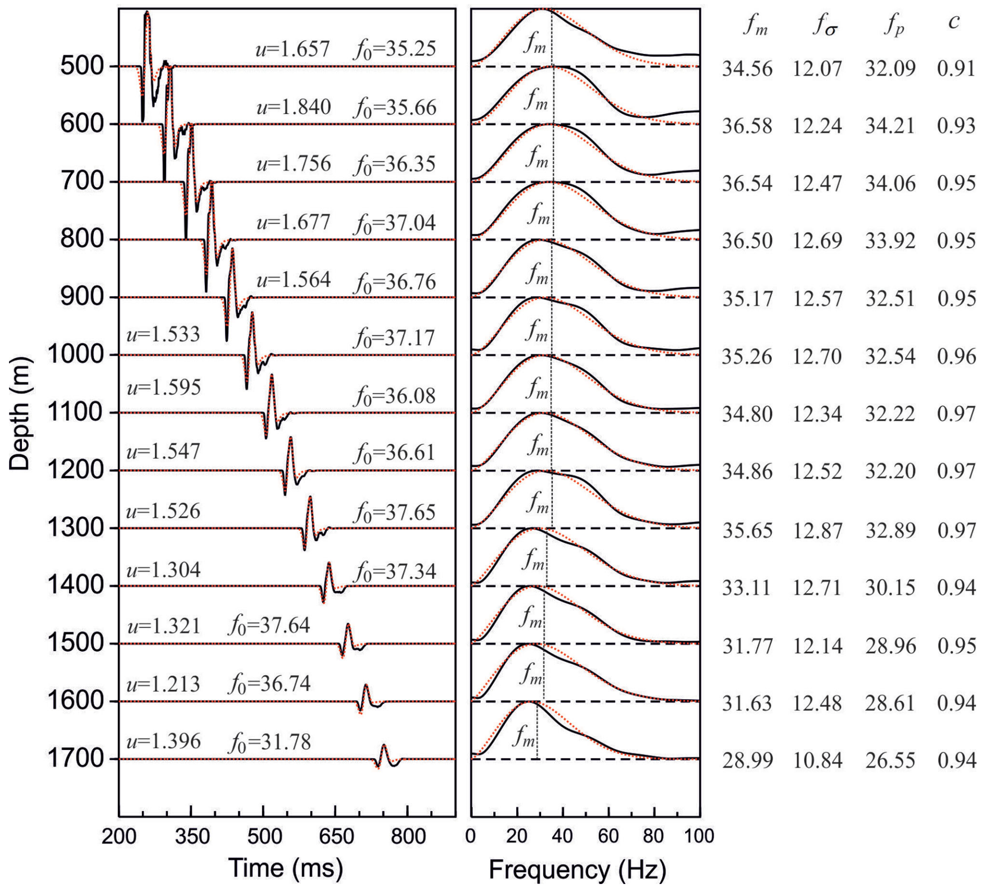

Figure 5. Field signals and spectra (solid curves) of VSP data recorded at various depths, compared to generalized wavelets and spectra (dotted curves). At each depth, the mean frequency f_m and the deviation f_σ are evaluated, and in turn the fractional value u, the reference frequency f_0 and the peak frequency f_p are derived. All of the frequency quantities annotated in this figure are $f = (2\pi)^{-1}\omega$, with units of Hz. The correlation coefficient between a field signal and a generalized wavelet (defined by the fractional value u and the reference frequency) is denoted as c. The average of all c values is 0.95.

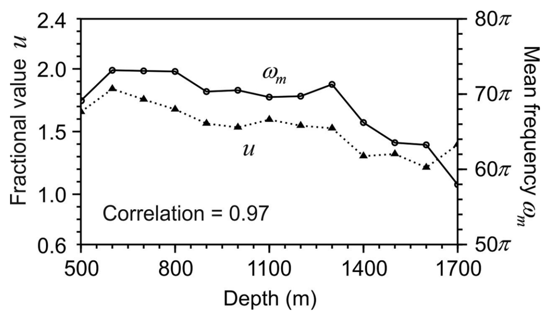

Figure 6. The mean frequency ω_m (solid curve) and the fractional value u (dotted curve) are highly correlated, with a correlation coefficient of 0.97.

The relationship between the peak frequency and the mean frequency can be approximated as

$$\omega_p = \omega_m \frac{\Gamma(u+\frac{1}{2})}{\sqrt{u}\,\Gamma(u)} \approx \omega_m \left(1 - \frac{1}{8u}\right). \tag{35}$$

It suggests that, if u decreases, the difference between ω_p and ω_m is getting larger. This intuitive observation was also clearly presented in Fig. 2, where $\omega_c \approx \omega_m$. In this field data example, the peak frequency ω_p is decreasing monotonically along the depth. This variation would reflect the characteristic of seismic absorption, and thus can potentially be used for Q estimation.

A pair consisting of fractional value u and reference frequency ω_0 defines a generalized wavelet. These field signals are well-fitted by generalized wavelets, as shown in Fig. 5. The correlation coefficients between the field waveforms and the theoretical wavelets have an average value of 0.95. An accurate wavelet is necessary in seismic inversion, either for the reflectivity series or for the velocity variation.

CONCLUSIONS

In order to better represent field seismic signals, asymmetrical wavelets are defined by fractional derivatives of a Gaussian function. The Ricker wavelet is just a special case with an integer derivative of order 2. Since these wavelets and the Ricker wavelet are mathematically derived from the same Gaussian function, their spectral properties are similar to each other and differ in a frequency-related factor $(\mathrm{i}\omega)^u$. This factor is a frequency-domain representation of the fractional derivative with respect to time. Generalized wavelets match field seismic signals with high correlations.

For various wavelets, analytical expressions are found for the central frequency, the bandwidth, the mean frequency and the deviation. The first two frequency characteristics of a wavelet are expressed by the Lambert W function. The last two statistical properties are presented in terms of the Gamma function. In practice, once the mean frequency and its deviation are numerically evaluated from discrete Fourier spectra of field seismic data, the analytical expressions mentioned above can be used to uniquely determine the fractional value of u and the reference frequency ω_0, and can subsequently be used to derive the peak frequency, the central frequency and the bandwidth. These are frequency parameters that are needed practically for wavelet and spectral analyses.

ACKNOWLEDGEMENTS

The author is grateful to the sponsors of the Centre for Reservoir Geophysics, Imperial College London, for supporting this research.

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