A BOLTZMANN MODEL FOR ROD ALIGNMENT AND SCHOOLING FISH

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ABSTRACT. We consider a Boltzmann model introduced by Bertin, Droz and Grégoire as a binary interaction model of the Vicsek alignment interaction. This model considers particles lying on the circle. Pairs of particles interact by trying to reach their mid-point (on the circle) up to some noise. We study the equilibria of this Boltzmann model and we rigorously show the existence of a pitchfork bifurcation when a parameter measuring the inverse of the noise intensity crosses a critical threshold. The analysis is carried over rigorously when there are only finitely many non-zero Fourier modes of the noise distribution. In this case, we can show that the critical exponent of the bifurcation is exactly $1/2$. In the case of an infinite number of non-zero Fourier modes, a similar behavior can be formally obtained thanks to a method relying on integer partitions first proposed by Ben-Naïm and Krapivsky.

Keywords: kinetic equation; binary interaction; mid-point rule; equilibria; pitchfork bifurcation; integer partition; swarms.

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1. INTRODUCTION

This paper is concerned with the study of some interaction mechanisms between large collections of agents subject to social interaction. Specifically, we consider a Boltzmann model introduced in [9] as a binary interaction counterpart of the Vicsek alignment interaction [42]. The goal of the present work is to study the equilibria of this Boltzmann model and to rigorously show that this model exhibits pitchfork bifurcations (or second order phase transitions).

Systems of self-propelled particles interacting through local alignment have triggered considerable literature since the seminal work of Vicsek and co-authors [42]. Indeed, this simple model exhibits all the universal features of collective systems observed in nature and in particular, the emergence of symmetry-breaking phase transitions from disorder to globally aligned phases. We refer for instance to [1, 16, 22, 23, 30, 32] for the study of these phase transitions. A recent review on this ever-growing literature can be found in [43]. The overwhelming majority of references rely on Individual-Based Models (IBM) or particle models [5, 15, 16, 17, 20, 21, 33, 35, 36, 37], mostly with applications to animal collective behavior from bacterias to mammals [2, 19, 31]. When the number of agents becomes very large, kinetic models [6, 10, 11, 28, 34, 38] or hydrodynamic models [3, 4, 10, 27, 26, 24, 29, 39, 40, 41] are more efficient and have received an increasing attention in the literature.

The present work is concerned with a kinetic, Boltzmann-like model which has been proposed as a kinetic version of the Vicsek particle model in [8, 9, 10]. This model shows strong similarity with a model proposed by Ben-Naïm and Krapivsky in [7]. A zero-noise version of this model has been studied in [25]; it is shown that generically, Dirac deltas are the stable equilibria of this model. Here, we study the noisy version of this model and show that peaked equilibria (i.e. noisy versions of the Dirac deltas) emerge when the noise intensity becomes smaller than a critical value, and that, at the same time, uniform equilibria become unstable. Our rigorous proof is limited to the case where the noise has a finite number of Fourier coefficients, leaving the case of generic noises open. However, some formal results can be found by adapting the method of integer partitions by Ben-Naïm and Krapivsky [7].

The main concern of this paper is the following Boltzmann equation:

$$\begin{aligned}\partial_t f(t,x_1) \quad = \quad & \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(t,x_1')f(t,x_2')g(x_1-\hat{x}_{12}')\,\beta(|\sin(x_2'-\hat{x}_{12}')|)\,\frac{dx_1'}{2\pi}\,\frac{dx_2'}{2\pi} \\ & -f(t,x_1)\int_{-\pi}^{\pi} f(t,x_2)\,\beta(|\sin(x_2-\hat{x}_{12})|)\,\frac{dx_2}{2\pi}. \end{aligned} \tag{1}$$

Here, $\hat{x}_{12} = \mathrm{Arg}\{\frac{e^{ix_1}+e^{ix_2}}{|e^{ix_1}+e^{ix_2}|}\}$ is the argument (modulo 2π) of the midpoint on the smallest arc on the unit circle between e^{ix_1} and e^{ix_2}, $\hat{x}_{12}' = \mathrm{Arg}\{\frac{e^{ix_1'}+e^{ix_2'}}{|e^{ix_1'}+e^{ix_2'}|}\}$. The quantity $2|\sin(x_2-\hat{x}_{12})|$ is the euclidean distance in $\mathbb{R}^2$ between x_1 and x_2. As usual in kinetic theory, the collision rate between two particles is a function β of this distance. The unknown f is a probability density on the circle $\mathbb{S}^1 \approx \mathbb{R}/(2\pi\mathbb{Z})$, giving *e.g.* the distribution of directions in a fish school, and g is a given probability density modeling the noise in the model. The first term at the right-hand side (the gain term) expresses the rate at which particles acquire the velocity x_1 as a result of collisions of two particles of velocities x_1' and x_2'. The post-collision velocity x_1 of particle 1 is distributed around the "mid-point" (in the sense above) $\hat{x}_{12}'$ of the two pre-collisional velocities

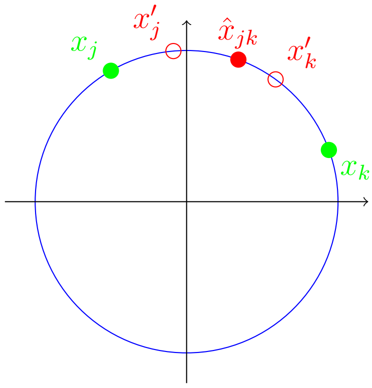

FIGURE 1. The jump process in the BDG model

x'_1 and x'_2 according to the probability distribution g. The loss term (the second term) is found in a similar way reversing the roles of the pre- and post-collisional velocities. In our case β is just a constant (to mimic "Maxwellian molecules" in gas dynamics) or if one takes a collision rate proportional to the relative velocities of the particles as usual in kinetic theory, $\beta(x)$ is proportional to x. A space-dependent version of this equation was first formulated by E. Bertin, M. Droz and G. Grégoire in [9] as a model for swarm dynamics inspired by the so-called Vicsek model [42] (see also e.g. [8, 10]). Here, we consider only the distribution of velocities, so there are no spatial derivatives in our equation. We also do not include a "self-diffusion" term, similar to the one in (2) below. Thus, the model we consider provides a clean and clear setting in which to investigate the competing effects of the allignment mechanism and the strength of the noise which facilitates our rigorous investigation of the critical penomena associated to this competition.

A rigorous derivation of equation (1) as a limit as $N \to \infty$ of an N-particle system was carried out in [13, 14], where a general *propagation of chaos* result is obtained for *pair interaction driven N-particle systems*. These are defined as Markov jump processes in an N-fold product space $\mathbb{T}^N = (\mathbb{S}^1)^N$, where jumps almost surely only involve two coordinates. The jumps are triggered by a Poisson clock with rate proportional to N, and the outcome of a jump is independent of the clock. A jump involves first a choice of a pair (j,k) from the set $1 \leq j < k \leq N$, and then a transition $x \mapsto x'$, independent of (j,k):

$$x = (x_1, ..., x_j, ..., x_k, ..., x_N) \mapsto (x_1, ..., x'_j, ..., x'_k, ..., x_N) = x' \,.$$

The jump process behind equation (1) is defined in the N-dimensional torus, represented by coordinates $x_j \in [-\pi, \pi[$. The jumps take a pair (x_j, x_k) to

$$(x'_j, x'_k) = (\hat{x}_{jk} + X_j, \hat{x}_{jk} + X_k) \qquad \text{mod} \qquad 2\pi \times 2\pi \,,$$

where X_j and X_k are independent and equally distributed angles (see Figure 1). Of course this is not well defined on the set $x_j = -x_k$, but that is a set of measure zero, and at least if the distribution of x_j has a density, this case may be neglected.

An interesting feature of this process is that, although propagation of chaos holds, as required for the derivation of equation (1), this equation has strongly peaked solutions, which implies certain dependence between two particles distributed according to the density f. We will expand on this statement below, where the formal calculations in going from an N-particle system to the kinetic equation are repeated.

The main new results in this paper concern equation (1). First, it is easy to see that the uniform density, $f(x) = 1/2\pi$ is a stationary equilibrium, and that the (linearized) stability of this equilibrium depends on the first moment γ_1 of the noise distribution g. The moment γ_1 indicates how peaked g is (the larger γ_1, the more strongly peaked g is). Second, in the Maxwellian case, we explicitly construct non-uniform stationary solutions when the noise distribution g has a finite number of non-zero Fourier coefficients. We rigorously prove the existence of a pitchfork bifurcation (or second-order phase transition) when γ_1 crosses a critical value $\gamma_c = \pi/4$. For $\gamma_1 \leq \gamma_c$, the uniform stationary distributions is stable. For $\gamma_1 > \gamma_c$ and close to it, there exists another class of equilibria which are stable while the uniform stationary distribution becomes unstable. Additionally, we can prove that the associated critical exponent is $1/2$ when considering the first moment of the stationary solution as an order parameter. This trnasition and critcal exponent had been predicted in [9, 10], and our proof bears out their conclusions.

An equation very similar to (1) is studied by Ben-Naim and Krapivsky in [7] as a model for rod alignment:

$$\text{(2)} \qquad \frac{\partial}{\partial t} f(x,v) \;=\; D\frac{\partial^2}{\partial x^2} f(x,t) + \int_{-\pi}^{\pi} f(x+y/2,t) f(x-y/2,t)\,\frac{dy}{2\pi} - f(x,t)\,.$$

While in equation (1) all particles remain fixed between the pair interactions, the model of Ben-Naim and Krapivsky assumes that each particle follows a Brownian motion between the jumps. On the other hand, contrary to equation (1), the jumps in equation (2) imply perfect alignment. More considerations about this model will be found in Section 3, and in particular in Section 6, where the analysis in [7] is studied in more detail. Their analysis also uses the Fourier series expansion of the stationary solution, and semi explicit expressions for the Fourier coefficients are obtained by expanding these coefficients as a power series of the first coefficient, a_1. We adapt their method to our case, and at the same time we try to clarify some technical points of the method. The result is formal in the sense that we do not prove convergence of any of the series appearing in the work, but it does provide new insights in the behavior of the model.

The layout of the paper is as follows. In Section 2, we review the simple case where the model is posed on the real line (instead of the circle). In this case, an explicit formula for the equilibria can be found in Fourier-transformed variables. Going back to the model posed on the circle in Section 3, we show that the Fourier coefficients of the distribution function satisfy a fully-coupled nonlinear dynamical system. The linearization of this system about an isotropic equilibrium is studied in Section 4. We show that the isotropic equilibrium is unstable for noise intensities below a certain threshold and that the instability only appears in the first Fourier coefficient, suggesting that the first Fourier mode acts as an order parameter for this symmetry-breaking phase transition. In Section 5, we rigorously prove the emergence of the phase transition and determine the critical exponent in the case where the noise probability has only finitely many non-zero Fourier modes. Indeed, in such a circumstance, any equilibrium solution has also finitely many non-zero Fourier coefficients, and finding such an equilibium can be rigorously accomplished using the Implicit Function Theorem. We also show that the critical exponent of the phase transition is equal to $1/2$. It is interesting to contrast this result with that of [23] where all critical exponents between $1/4$ and 1 were found for the Vicsek dynamics. Removing the assumption of finitely many modes, only formal calculations can be performed at present. The work of Ben-Naïm and Krapivsky [7] suggests that the critical exponent $1/2$ persists. In Section 6, we relate their integer partition method to our approach. Finally, conclusions and perspectives are drawn in Section 7.

2. THE MODEL ON THE REAL LINE

In order to get a preliminary sense of the behavior of the model, it is useful to investigate the more simple case where $x \in \mathbb{R}$. In this case, the Boltzmann equation is given by:

$$\partial_t f(t,x_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,x_1')f(t,x_2')g(x_1-\hat{x}_{12}')\,\beta(|x_2'-\hat{x}_{12}'|)\,dx_1'\,dx_2' \\ -f(t,x_1)\int_{-\infty}^{\infty} f(t,x_2)\,\beta(|x_2-\hat{x}_{12}|)\,dx_2\,.$$

where now, $\hat{x}_{12}=(x_1+x_2)/2$ and $x_2-\hat{x}_{12}=(x_2-x_1)/2$. This corresponds to pair interactions given by

$$(x_j,x_k) \mapsto \left(\frac{x_j+x_k}{2}+X_1, \frac{x_j+x_k}{2}+X_2\right) \tag{3}$$

where X_1 and X_2 are two independent, identically distributed random variables. The process is then similar to models considered in models of trade [18] and is interesting in the present context mostly because it permits rather explicit calculations. A very similar model was also obtained [7] as a limit of nearly aligned rods. Another related model that takes into acocunt spatial heterogeneities is investigated in interesting recent work [12].

By a simple change of variables $x_2'=x_1'+y$, and using the fact that we look for f being a probability distribution, the Boltzmann equation in the Maxwellian case simplifies to:

$$\partial_t f(t,x) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t,x')f(t,x'+y)g(x-x'-\frac{y}{2})\,dxdy - f(t,x)\,.$$

We note that this can be written equivalently as

$$\partial_t f = (2(f*f)(2\cdot))*g - f\,.$$

(Here, for any function h on the line, $2h(2\cdot)$ denotes the function rescaled function taking the value $2h(2x)$ at x. If h is the probability density of a random variable X, then $2h(2\cdot)$ is the probability density of $X/2$.) Therefore, equilibria are solutions of the fixed-point equation:

$$f = (2(f*f)(2\cdot))*g\,, \tag{4}$$

which expresses that the distribution of $\frac{x_1+x_2}{2}+X$ when x_1 and x_2 are i.i.d. with density f and X is a random variable of density g must be equal to f itself.

Theorem 1. *We suppose that $g \in \mathcal{P}_2 \cap L^1(\mathbb{R}) \cap C^0(\mathbb{R})$ where $\mathcal{P}_2$ is the space of probability measures of $\mathbb{R}$ with bounded second moments. Additionally, we suppose that g has zero mean. The solutions in $\mathcal{P}_2 \cap L^1(\mathbb{R})$ of (4) are given by translations by an arbitrary real number of a probability $f \in \mathcal{P}_2 \cap L^1(\mathbb{R})$ whose Fourier transform $\hat{f}(\xi)$ has the expression:*

$$\hat{f}(\xi) = \prod_{j=0}^{\infty} \hat{g}(\xi/2^j)^{2^j}.$$

Proof. We define

$$\hat{g}_n(\xi) = \prod_{j=0}^{n-1} \hat{g}(\xi/2^j)^{2^j}.$$

We note that $\hat{g}_n$ is the Fourier transform of g_n which satisfies the recursion for $n \geq 1$:

$$g_n = g * (2g_{n-1}(2\cdot)) * (2g_{n-1}(2\cdot)). \tag{5}$$

and $g_0 = g$. Now, by recursion, g_n is a probability density. Indeed, supposing that g_{n-1} is a probability density, we obtain g_n as the convolution of three probability densities. Now, we write, uniformly on any compact set for ξ: $\hat{g}(\xi) = 1 - \frac{1}{2}\gamma_2\xi^2 + o(\xi^2)$, where $\gamma_2 = \int_{\mathbb{R}} g(x)\, x^2\, dx$ is the second moment of g. Then, uniformly for ξ in any bounded interval and $n \in \mathbb{N}$, we get:

$$\begin{aligned} \log \hat{g}_n &= \sum_{j=0}^{n-1} 2^j \log\left(1 - \frac{1}{2}\gamma_2(\xi/2^j)^2 + o((\xi/2^j)^2)\right) \\ &= -\frac{1}{2}\gamma_2\xi^2 \sum_{j=0}^{n-1} 2^{-j} + O(\xi^2)\,. \end{aligned}$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} \log \hat{g}_n(\xi) = -\gamma_2\xi^2 + O(\xi^2)\,,$$

uniformly for ξ in any compact set of $\mathbb{R}$. Hence, this defines $\hat{g}_\infty(\xi)$ as a continuous function of ξ which by Levi's continuity theorem, is the Fourier transform of a probability measure g_∞. Now, taking $n \to \infty$ in (5), we get

$$g_\infty = g * (2g_\infty(2\cdot)) * (2g_\infty(2\cdot)). \tag{6}$$

which expresses g_∞ as the convolution of a continuous function g with a measure $(2g_\infty(2\cdot)) * (2g_\infty(2\cdot))$. Therefore, g_∞ is a continuous function and consequently an element of $L^1(\mathbb{R})$. Finally, by a simple change of variables, (6) is nothing but Eq. (4) with $f = g_\infty$. Therefore, g_∞ is a solution of (4).

Remark 2. *The equilibrium distribution g_∞ has a second moment that is twice that of g. Figure 2 shows the solution to equation (4) in the case where $g(x) = \frac{1}{2}1_{[-1,1]}$, where $1_{[-1,1]}$ is the indicator function of the interval $[-1,1]$. When g is a centered Gaussian, then f is also a Gaussian with twice its variance. Indeed, since the convolution of two centered Gaussians is a centered Gaussian whose variance is the sum of the variances of the factors, it follows from (5) that each g_n is Gaussian, and then the limiting variance can be read off from (6).*

Remark 3. *A model where the pair interacts more weakly can be obtained by replacing Equation (3) with*

$$(x_j, x_k) \;\mapsto\; (\lambda x_j + (1-\lambda)x_k + X_1, (1-\lambda)x_j + \lambda x_k + X_2)\ .$$

One can then proceed in the same way by taking the Fourier transform to get

$$\hat{f}(\xi) \;=\; \hat{f}(\lambda\xi)\hat{f}((1-\lambda)\xi)\hat{g}(\xi)\,,$$

and as in the case of $\lambda = 1/2$ obtain a solution

$$\hat{f}(\xi) \;=\; \prod_{k=0}^{\infty}\prod_{j=0}^{k} \hat{g}\left(\lambda^j(1-\lambda)^{k-j}\xi\right)^{\binom{k}{j}}\ .$$

In this case the variance of f can be expressed in terms of the variance of g as

$$\mathit{Var}[f] \;=\; \frac{1}{2\lambda(1-\lambda)}\mathit{Var}[g]$$

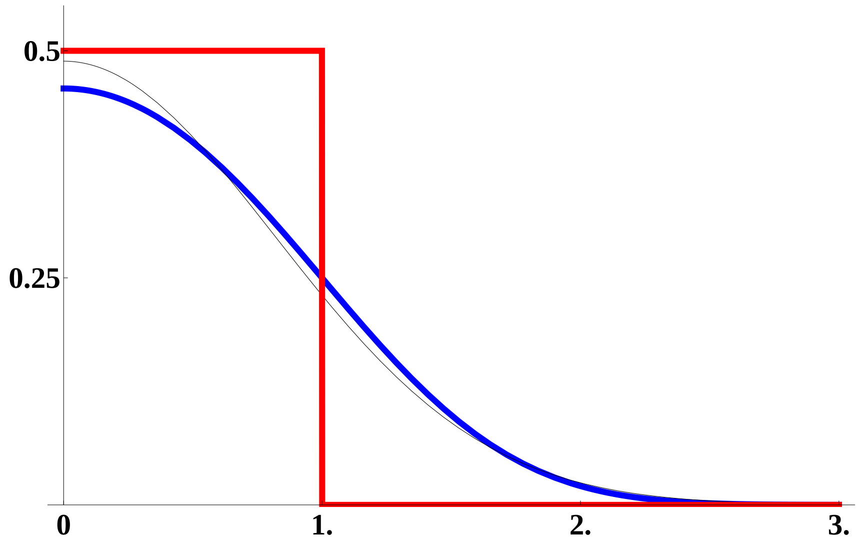

FIGURE 2. A solution f to equation (4) (the blue, thick curve) with $g(x) = \frac{1}{2}1_{[-1,1]}$ (red, thick curve) compared with the Gaussian function with the same variance (the thin curve).

Now, we are going to apply the same method to the original model posed on the circle. But we will see that the difficulties are considerably bigger.

3. FOURIER SERIES EXPANSION OF THE MODEL ON THE CIRCLE

Now, we are back to model (1) posed on the circle. We first remark that, by the change of variables $x_2' = x_1' + y$, $y \in]-\pi, \pi]$, we have $\hat{x}_{12}' = x_1' + y/2$, $x_2' - \hat{x}_{12}' = y/2$, so that the model can be written:

$$\partial_t f(t,x) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \Big(f(t,x')f(t,x'+y)g(x-x'-\frac{y}{2}) - f(t,x)f(t,x+y)\Big)\tilde{\beta}(y)\frac{dx'}{2\pi}\frac{dy}{2\pi}, \quad (7)$$

where either $\tilde{\beta}$ is constant (in fact, we take $\tilde{\beta} = 1$) independent of the velocities of the interacting pair, corresponding to Maxwellian molecules in gas dynamics, or else $\tilde{\beta} = |\sin(y/2)|$, corrsponding to hard-sphere collisions in gas dynamics. Below, we refer to these two choices for $\tilde{\beta}$ as the Maxwellian case and the hard-sphere case, respectively.

Multiplying with a test function ϕ, integrating over $[-\pi, \pi]$, and performing a change of variables gives the following weak form of the equation,

$$\frac{d}{dt}\int_{S^1} f(t,x)\phi(x)\frac{dx}{2\pi} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(t,x)f(t,x+y)g(z)\tilde{\beta}(y)\left(\phi(x+y/2+z)-\phi(x)\right)\frac{dx}{2\pi}\frac{dy}{2\pi}\frac{dz}{2\pi}. \quad (8)$$

Note that formally the system conserves mass:

$$\int_{-\pi}^{\pi} f(x,t)\,dx = \text{Constant.}$$

We may therefore require that $f(x,t)\,dx$ is a probability, i.e. take this constant equal to unity. This means that our equation describes the evolution of the probability density for the velocities. Since the equation is non-linear, multiplying f by a constant has an effect, but since the non-linearity is homogeneous of degree 2, the effect can be absorbed into the time scale (and for the same reason there is no loss of generality is setting $\tilde\beta = 1$ in the Maxwellian case).

Because all functions are periodic, it is natural to consider to rewrite the system in terms of the Fourier series. Introducing

$$\begin{aligned} f(x) &= \sum_{k=-\infty}^{\infty} a_k e^{ikx} \qquad\qquad a_k = \int_{-\pi}^{\pi} f(x) e^{-ikx}\,\frac{dx}{2\pi}\,. \\ \gamma_k &= (2\pi)^{-1}\int_{-\pi}^{\pi} g(z)e^{-ikz}dx, \qquad \Gamma(u) = (2\pi)^{-1}\int_{-\pi}^{\pi}\tilde\beta(y)e^{iuy}dy, \end{aligned}$$

we have the following:

Proposition 4. *Suppose that g is even and let $a_k(t)$ be the Fourier coefficients of a solution of Eq. (7) which is an even probability density. Then, $a_0 = 1$ and a_k for $k \neq 0$ satisfy $a_{-k} = a_k$ and solve the following system:*

$$\begin{aligned} \frac{d}{dt}a_k(t) &= (\,2\gamma_k\Gamma(k/2) - \Gamma(0) - \Gamma(k)\,)\,a_k(t) + \\ & \sum_{n=1}^{k-1}(\gamma_k\Gamma(n-k/2) - \Gamma(n))\,a_n(t)a_{k-n}(t) + \\ & \sum_{n=k+1}^{\infty}(2\gamma_k\Gamma(n-k/2) - \Gamma(n) - \Gamma(n-k))\,a_n(t)a_{n-k}(t) \end{aligned} \tag{9}$$

The function $\Gamma(u)$, which is to be evaluated only on half-integer points, is

$$\Gamma(u) = \frac{\sin(\pi u)}{\pi u} = \begin{cases} 1 & \textit{when} \quad u = 0 \\ 0 & \textit{when} \quad u \in \mathbb{Z}\setminus\{0\} \\ \frac{2(-1)^\ell}{\pi(2\ell+1)} & \textit{when} \quad u = \ell + 1/2 \end{cases} \tag{10}$$

in the Maxwellian case, when $\tilde\beta(1) \equiv 1$; and

$$\Gamma(u) = \frac{2 - 4u\sin(\pi u)}{\pi - 4\pi u^2} = \begin{cases} 2/(\pi(1-4u^2)) & \textit{when} \quad u \in \mathbb{Z} \\ 1/\pi & \textit{when} \quad u = \pm 1/2 \\ \frac{2(-1)^\ell\ell + (-1)^\ell - 1}{2\pi\ell^2 + 2\pi\ell} & \textit{when} \quad u = \ell + 1/2, \ell \neq 0, -1 \end{cases},$$

in the hard-sphere case, when $\tilde\beta(y) = |\sin(y/2)|$.

Proof. Taking $\phi(x) = e^{-ikz}$ in (8), we get (with $a_k = a_k(t)$) for $k \neq 0$

$$\begin{aligned}
\frac{d}{dt} a_k &= \sum_n \sum_m a_m a_n \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{imx} e^{in(x+y)} g(z)\tilde{\beta}(y) \left(e^{-ik(x+y/2+z)} - e^{-ikx}\right) \frac{dx}{2\pi}\frac{dy}{2\pi}\frac{dz}{2\pi} \\
&= \sum_n \sum_m a_m a_n \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(z)\tilde{\beta}(y) \left(e^{i((m+n-k)x+(n-k/2)y-kz)} - \right. \\
&\qquad\qquad \left. e^{i(m+n-k)x+ny)}\right) \frac{dx}{2\pi}\frac{dy}{2\pi}\frac{dz}{2\pi} \\
&= \sum_n a_{k-n} a_n \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(z)\tilde{\beta}(y) \left(e^{i((n-k/2)y-kz)} - e^{i(ny)}\right) \frac{dy}{2\pi}\frac{dz}{2\pi}
\end{aligned}$$

which leads to

$$\begin{aligned}
\frac{d}{dt} a_k(t) &= \sum_n a_{k-n}(t) a_n(t) \left(\gamma_k \Gamma(n-k/2) - \Gamma(n)\right) \\
&= \sum_{i+j=k} a_i(t) a_j(t) \left(\gamma_k \Gamma((j-i)/2) - \Gamma(j)\right) . \qquad (11)
\end{aligned}$$

Using that $\gamma_{-k} = \gamma_k$ and $a_{-k} = a_k$, we get (9).

Remark 5. *Eq. (9) for the Maxwellian case can be simplified and gives:*

$$\begin{aligned}
\frac{d}{dt} a_k(t) &= \left(\, 2\gamma_k \Gamma(k/2) - 1 \,\right) a_k(t) + \\
&\quad \sum_{n=1}^{k-1} \gamma_k \Gamma(n-k/2) a_n(t) a_{k-n}(t) + \\
&\quad \sum_{n=k+1}^{\infty} 2\gamma_k \Gamma(n-k/2) a_n(t) a_{n-k}(t)
\end{aligned}$$

Remark 6. *For comparison, we note that the Fourier coefficients of solutions to equation (2) satisfy*

$$\frac{d}{dt} a_k(t) = -(1+Dk^2) a_k(t) + \sum_{i+j=k} \Gamma((i-j)/2) a_j(t) a_i(t) \,,$$

with Γ as in equation (10) (see [7]*). The only essential difference with equation (11) is that the diffusion term manifests itself as a multiplier Dk^2 of a_k (and moreover that (11) includes the possibility of non-Maxwellian interactions).*

4. THE LINEARIZED EQUATION

It is easy to verify that $f(x) \equiv 1$ is a solution, which corresponds to $a_0 = 1, a_k = 0$, $(k \neq 0)$. If f is a solution, then any translation of f, i.e. $x \mapsto f(x+s)$) is also a solution. Expressed in terms of the Fourier coefficients, this means that if $(a_k)_{k\in\mathbb{Z}}$ is a solution, then so is $(a_k e^{iks})_{k\in\mathbb{Z}}$.

To investigate the stability of the uniform density, let $f(x,t) = 1+\varepsilon F(x,t)$, and let $b_k(t),\ k \in \mathbb{Z}$ be the Fourier coefficients of $F(x,t)$. Then $b_0 = 0$, and for $k \neq 0$,

$$\frac{d}{dt} b_k(t) = b_k(t) \left(2\gamma_k \Gamma(k/2) - \Gamma(0) - \Gamma(k)\right) .$$

Hence the linearized stability may be determined by analyzing separately the sign of $\mathrm{Re}\lambda_k$ where

$$\lambda_k = (2\gamma_k \Gamma(k/2) - \Gamma(0) - \Gamma(k)). \tag{12}$$

Indeed, if $\mathrm{Re}\lambda_k \leq 0, \forall k \in \mathbb{Z}$, the system is stable, and it is unstable otherwise. Note that $\lambda_0 = 0$ and $\lambda_k \in \mathbb{R}, \forall k \in \mathbb{Z}$ in our case.

Remark 7. *The uniform density is also stationary for the model in* [7], *where its stability is analyzed in very much the same way, giving an explicit expression involving the only parameter in the model, the diffusion coefficient D.*

We assume that g is even. In both the Maxwellian and hard-sphere case, we have the:

Theorem 8. *We have $\lambda_k \leq 0, \forall k \in \mathbb{Z}, |k| \geq 2$, meaning that the linearized stability depends only on the sign of $\lambda_1 = \lambda_{-1}$:*

$$\text{the system is stable} \iff \lambda_1 \leq 0$$

Proof. In the Maxwellian case, we have

$$2\Gamma(k/2) - \Gamma(0) - \Gamma(k) \;=\; \frac{4\sin\left(\frac{k\pi}{2}\right)}{k\pi} - 1.$$

It is easily seen that the right-hand side is negative when $|k| \geq 2$. Hence it is only λ_1 that may become positive, and therefore the condition for stability of the uniform solution is that $\gamma_1 \leq \frac{\pi}{4}$.

In the hard-sphere case, we find that $2\Gamma(1/2) - \Gamma(0) - \Gamma(1) = 2/(3\pi)$, and that for $k > 1$,

$$2\Gamma(k/2) - \Gamma(0) - \Gamma(k) \;=\; -\frac{4\left(2k^4 - 4\sin\left(\frac{k\pi}{2}\right)k^3 + k^2 + \sin\left(\frac{k\pi}{2}\right)k\right)}{(k^2-1)(4k^2-1)\pi}$$

Because Γ is an even function, it is enough to consider $k \geq 2$, and in that case the numerator is larger than

$$\begin{aligned} 4\left(2k^4 - 4\sin\left(\frac{k\pi}{2}\right)k^3 + k^2 + \sin\left(\frac{k\pi}{2}\right)k\right) \;&\geq\; 4\left(2k^4 - 4k^3 + k^2 - k\right) \\ &\geq\; 4(k^2 - k) > 0 \end{aligned}$$

and hence we may deduce that $\lambda_k < 0$ for $|k| > 1$ also in this case. If γ_k changes sign the calculation is more complicated, but the result is the same: it is only the first Fourier modes of the solution f that may cause instability of the uniform stationary states.

For concreteness, we now consider a family of distributions $g(y)$ defined as the periodization of $\frac{1}{\tau}\rho(\frac{y}{\tau})$, where ρ is a given even probability density on $\mathbb{R}$:

$$g_\tau(y) \;=\; 2\pi \sum_{j=-\infty}^{\infty} \frac{1}{\tau}\rho(\frac{y - 2\pi j}{\tau}).$$

Then

$$\gamma_k(\tau) \;=\; \int_{-\pi}^{\pi} e^{-iky} 2\pi \sum_{j=-\infty}^{\infty} \frac{1}{\tau}\rho(\frac{y-2\pi j}{\tau})\,\frac{dy}{2\pi} \;=\; \int_{-\infty}^{\infty} e^{-i\tau k y}\rho(y)\,dy \;=\; \hat{\rho}(\tau k)\,.$$

An example is $\rho(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ which gives $\hat{\rho}(\tau k) = e^{-(\tau k)^2/2}$. When τ is small, the noise is small, and when τ is large, the noise is also very large, and g_τ converges to the uniform distribution when $\tau \to \infty$. Therefore, $\gamma_1(\tau)$ is a continuous function of τ with $\gamma_1(0) = 1$ and $\gamma_1(\tau) \to 0$ as $\tau \to \infty$. Then $\lambda_1 = \lambda_1(\tau) \le 0$ for τ large and $\lambda_1 > 0$ for τ small. This shows that the system is linearly stable for large values of τ and unstable for small ones.

5. AN EXPLICIT EXAMPLE WITH BIFURCATION

The calculation here is restricted to the Maxwellian case, and we only look for even solutions, expressed as a Fourier cosine series. Hence we wish to solve

$$\begin{aligned} a_k &= 2\gamma_k\Gamma(k/2)a_k + \gamma_k\sum_{n=1}^{k-1}\Gamma(n-k/2)a_na_{k-n} + \\ &\quad 2\gamma_k\sum_{n=k+1}^{\infty}\Gamma(n-k/2)a_na_{n-k} \end{aligned} \tag{13}$$

for $k \ge 1$. Note that γ_k is a factor for all terms in the right hand side, implying that if g only has finitely many terms in the Fourier series, only the corresponding terms are nonzero in f.

We will conisder g as a member of a parameterized faimily of noise distributions g_λ so that changing the parameter corresponds to changing the strength of the noise. The Fourier coefficents of g_λ will depend on λ. In the first model we consider below (based on the Fejér kernel), $\gamma_1(\lambda)$ is monotone – even linear – in λ, and we may therefore regard λ as a function of γ_1. That is, we may take the first Fourrier coefficent as the parameter for this family of noise distributions. As γ_1 is varied, the other Fourier coefficent vary along with it to keep the shape of the distribution consistent with our chosen one parameter family.

So, here we make the following assumption on the noise distribution:

Assumption 9. *We assume that $g = g_{\gamma_1}$ is a family of noise distributions with a finite number of non-zero Fourier coefficients: for some $N < \infty$,*

$$g_{\gamma_1}(x) = 1 + 2\gamma_1\,\cos x + 2\sum_{k=2}^{N}\gamma_k(\gamma_1)\,\cos kx, \quad \forall x \in]-\pi,\pi].$$

with C^2 functions $\gamma_1 \in [0,1] \mapsto \gamma_k(\gamma_1) \in [-1,1]$ and with γ_2 such that

$$\gamma_2(\gamma_1) > 0.$$

Note that g is a probability measure as soon as $g \ge 0$. We can now state the following

Theorem 10. *Consider a one-parameter family of noise functions g_{γ_1} satisfying Hypothesis 9. Then:*

(i) *The uniform distribution, with Fourier coefficients $a_0 = 1, a_k = 0 \ \ (k \ge 1)$ is stationary. It is stable for $\gamma_1 < \pi/4$ and unstable for $\gamma_1 > \pi/4$.*

(ii) *In an interval $\frac{\pi}{4} < \gamma_1 < \gamma_{max}$ there is another invariant solution to the dynamic problem, with Fourier coefficients $a_0 = 1$,*
$a_1 = \sqrt{\frac{12(\gamma_1-\pi/4)}{\pi\gamma_2(\pi/4)}} + \mathcal{O}((\gamma_1-\pi/4)^{3/2}), ..., a_k = 0 \ \ (k > N)$.

(iii) *This solution is linearly stable with a leading eigenvalue* $\lambda(\gamma_1) = 1 - \frac{8}{\pi}(\gamma_1 - \pi/4) + \mathcal{O}((\gamma_1 - \pi/4)^{3/2})$.

Before proving this theorem, we give a few comments. One is tempted to think that the same result would hold for any noise distribution, at least provided its Fourier coefficients decay sufficiently fast, but to prove that rigorously requires an additional estimate showing that γ_{max} does not converge to $\pi/4$ when the number of coefficients increases.

We illustrate the theorem by showing numerical calculations using the family of noise distributions obtained as a convex combination of a Fejér kernel and of the uniform distribution.

$$g_\lambda(x) = (1-\lambda) + \lambda \frac{1}{N}\left(\frac{\sin(Nx/2)}{x/2}\right)^2 .$$

For such a noise distribution, we have $\gamma_k = \lambda(N-k)/N$ for $1 \le k < N$. Therefore, this family can be put in the framework of Hypothesis 9 if we link λ to γ_1 by $\lambda = \frac{N}{N-1}\gamma_1$. In the numerical simulations, we use $N = 9$. Fig. 3 shows the Fourier coefficient a_1 as a function of the parameter γ_1. This figure exhibits a typical pitchfork bifurcation pattern. The order parameter a_1 is identically zero as long as γ_1 is less than the critical value $\gamma_{1c} = \pi/4$ and the associated uniform equilibrium is stable. When γ_1 becomes larger than the critical value γ_{1c} a second branch of non-uniform equilibria starts. This branch is stable while the branch of uniform equilibria becomes unstable. In fact the non-uniform equilibria forms a continuum, because the system is rotationally invariant, and therefore, if f is a non-isotropic equilibrium, then any $f(e^{i\theta_0}x)$ with $\theta_0 \in]0, 2\pi[$ is another equilibrium. This feature is represented by the lower branch in the diagram. In physical terms, the system exhibits a symmetry-breaking second-order phase transition as γ_1 crosses γ_{1c}. From the point (ii) of the theorem, it appears that the critical exponent is $1/2$, *i.e.* the order parameter behaves like $a_1 \sim (\gamma_1 - \gamma_{1c})^{1/2}$ when $\gamma_1 \overset{>}{\to} \gamma_{1c}$. Fig. 4 shows the noise function g and the corresponding stationary solution f when $\gamma_1 = \pi/4 + 0.1$.

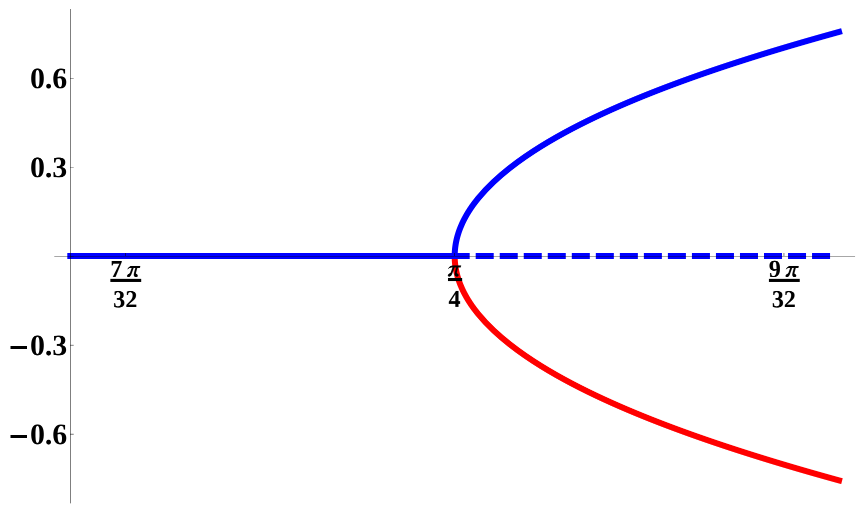

FIGURE 3. The stationary solution $\bar{a}_1$ plotted as a function of γ_1. The noise function is a parameterized Fejér kernel of order 9.

Proof of Theorem 10. The first statement, (i), is an immediate consequence of the analysis of the linearized system in Section 4.

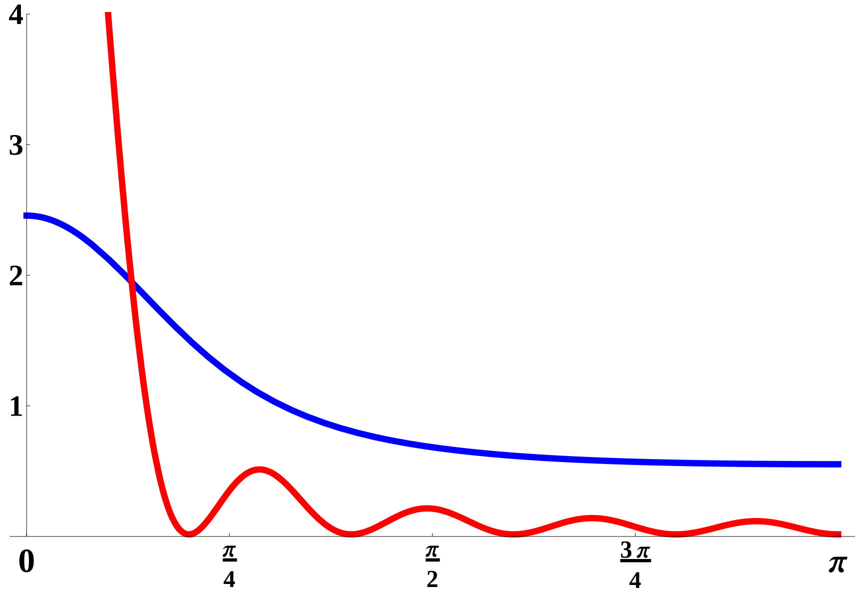

FIGURE 4. The parameterized Fejér kernel of order 9 with $\gamma_1 = \pi/4 + 0.1$ (red), and the corresponding solution $f(x)$ (blue)

To prove (ii) and (iii) we first note that in the Maxwellian case, $\Gamma(n - k/2) = 0$ when k is even and different from $2n$. Therefore, if $k \neq 0$ is even, there is only one non-zero term in the right hand side (13) and we get:

$$a_k = \gamma_k {a_{k/2}}^2 , \quad \forall k \neq 0, \quad k \text{ even}.$$

We now concentrate on the case of k odd. First, after a minor reformulation,

$$a_1 = 2\gamma_1\Gamma(1/2)a_1 + 2\gamma_1\Gamma(3/2)a_2a_1 + 2\gamma_1\sum_{n=3}^{N}\Gamma(n-1/2)a_na_{n-1} ,$$

$$a_3 = 2\gamma_3\Gamma(3/2)a_3 + 2\gamma_3\Gamma(1/2)a_2a_1 + 2\gamma_3\Gamma(5/2)a_4a_1 + 2\gamma_3\sum_{n=5}^{N}\Gamma(n-3/2)a_na_{n-3}$$

$$\vdots$$

$$a_k = 2\gamma_k\Gamma(k/2)a_k + 2\gamma_k\Gamma(1-k/2)a_{k-1}a_1 + 2\gamma_k\sum_{n=2}^{(k-1)/2}\Gamma(n-k/2)a_na_{k-n} +$$
$$+ 2\gamma_k\Gamma(1+k/2)a_{k+1}a_1 + 2\gamma_k\sum_{n=k+2}^{N}\Gamma(n-k/2)a_na_{n-k}$$

Because k is odd, either n or $n-k$ is even. So all terms contain a factor of the form a_pa_q, where p is odd and $q \geq 2$ is even. Above we have separated all terms that contain a factor a_1. We write q in factorized form as

$$q = \omega(q)2^{m(q)} \equiv 2\omega(q)\eta(q)$$

with $\omega(q)$ containing all odd factors of q. With this notation,

$$
\begin{aligned}
a_q &= \gamma_q a^2_{\omega(q)2^{m(q)-1}} = \gamma_q \gamma^2_{\omega(q)2^{m(q)-1}} a^{2^2}_{\omega(q)2^{m(q)-2}} = \dots \\
&= \gamma_q \prod_{j=1}^{m(q)-1} \gamma^{2^j}_{\omega(q)2^{m(q)-j}} a^{2^{m(q)}}_{\omega(q)} \equiv \tilde{\gamma}_q a^{2\eta(q)}_{\omega(q)} \,.
\end{aligned} \tag{14}
$$

If $a_1 \neq 0$, we may write $a_p = a_1 \tilde{a}_p$ for all p odd (this obviously holds also for $p = 1$, with $\tilde{a}_1 = 1$), and then

$$
\frac{a_q a_p}{a_1} = \tilde{\gamma}_q \gamma_2^{-\eta(q)} a_2^{\eta(q)} \tilde{a}^{2\eta(q)}_{\omega(q)} \tilde{a}_p \,. \tag{15}
$$

Inserting these expressions in the equation for a_1 we get, after dividing through by a_1, and using $\Gamma(x) = \frac{\sin(\pi x)}{\pi x}$,

$$
0 = \left(\frac{4}{\pi} \gamma_1 - 1 \right) - \frac{4}{3\pi} \gamma_1 a_2 + \gamma_1 R_2 \equiv F_2(\gamma_1, a_2, \tilde{a}_3, \tilde{a}_5, \dots) \,,
$$

where R_2 is a sum of terms of the form (15) with $p \geq 3$ and $q \geq 2$, *i.e.* monomials in a_2 and $\tilde{a}_p, p = 3, 5, 7 \dots$ of degree at least two. Similarly the equation for a_3 becomes

$$
0 = \frac{4}{\pi} \gamma_3 a_2 - \left(\frac{4}{3\pi} \gamma_3 + 1 \right) \tilde{a}_3 + \gamma_3 R_3 \equiv F_3(\gamma_1, a_2, \tilde{a}_3, \tilde{a}_5, \dots) \,,
$$

where again R_3 is a sum of monomials of order at least two. And the remaining equations are of the form

$$
0 = (2\Gamma(k/2)\gamma_k - 1)\, \tilde{a}_k + \gamma_k R_k \equiv F_k(\gamma_1, a_2, \tilde{a}_3, \tilde{a}_5, \dots) \,,
$$

with R_k as before. We have replaced all γ_k by γ_1 owing to the parametrization of γ_k by γ_1. As written here, the functions F_k depend only on one coefficient, γ_1. Here we also note that $\gamma_k = 0$ implies that $\tilde{a}_k = 0$, and hence restricting the analysis to noise functions with only finitely many non-zero coefficients, the system of equations $(F_k = 0)_{k=2,3,5,\dots}$ is reduced to a system of polynomial equations for the unknowns $(a_2, \tilde{a}_3, \dots, \tilde{a}_N)$, with a right-hand side being a function of γ_1.

We observe that at the critical value of the parameter, $\gamma_1 = \pi/4$, the right-hand side as a function of γ_1 vanishes. Hence, the polynomial system has no degree zero term and is solved by $a_2 = \tilde{a}_3 = \tilde{a}_3 = \dots = \tilde{a}_n = 0$. The implicit function theorem then implies that for a sufficiently small interval around $\gamma_1 = \pi/4$, there is a solution $a_2(\gamma_1), \tilde{a}_3(\gamma_1), \dots, \tilde{a}_3(\gamma_1)$ if the

Jacobian

$$J = \begin{pmatrix} \frac{\partial F_2}{\partial a_2} & \frac{\partial F_2}{\partial \tilde{a}_3} & \cdots & \frac{\partial F_2}{\partial \tilde{a}_N} \\ \\ \frac{\partial F_3}{\partial a_2} & \frac{\partial F_3}{\partial \tilde{a}_3} & \cdots & \frac{\partial F_3}{\partial \tilde{a}_N} \\ \vdots & \vdots & & \vdots \\ \frac{\partial F_N}{\partial a_2} & \frac{\partial F_N}{\partial \tilde{a}_3} & \cdots & \frac{\partial F_N}{\partial \tilde{a}_N} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{4\gamma_1}{3\pi} + \gamma_1 \frac{\partial R_2}{\partial a_2} & \gamma_1 \frac{\partial R_2}{\partial \tilde{a}_3} & \cdots & \gamma_1 \frac{\partial R_2}{\partial \tilde{a}_N} \\ \\ \frac{4\gamma_3}{\pi} + \gamma_3 \frac{\partial R_3}{\partial a_2} & -\left(\frac{4\gamma_3}{3\pi} + 1\right) + \gamma_3 \frac{\partial R_3}{\partial \tilde{a}_3} & \cdots & \gamma_3 \frac{\partial R_3}{\partial \tilde{a}_N} \\ \vdots & \vdots & & \vdots \\ \gamma_N \frac{\partial R_N}{\partial a_2} & \gamma_N \frac{\partial R_N}{\partial \tilde{a}_3} & \cdots & 2(\gamma_N \mathrm{sinc}(\pi N/2) - 1) + \gamma_N \frac{\partial R_N}{\partial \tilde{a}_N} \end{pmatrix}$$

is invertible at $\gamma_1 = \frac{\pi}{4}, a_2 = \tilde{a}_3 = ... = \tilde{a}_N = 0$. Because all the R_k are polynomials of degree greater than two, we find that at the critical point

$$J = \begin{pmatrix} -\frac{1}{3} & 0 & \cdots & 0 \\ \\ \frac{4\gamma_3}{\pi} & -\left(\frac{4\gamma_3}{3\pi} + 1\right) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 2(\gamma_N \mathrm{sinc}(\pi N/2) - 1) \end{pmatrix}$$

Moreover, since, as seen before, the R_k's are sums of monomials in $(a_2, \tilde{a}_3, \ldots, \tilde{a}_N)$ of degree at least two, and thanks to the assumption that $\gamma_k(\gamma_1)$ is C^1, we have:

$$\left(\frac{\partial F_2}{\partial \gamma_1}\right)_{\gamma_1 = \frac{\pi}{4}, a_2 = \tilde{a}_3 = \ldots = \tilde{a}_N = 0} = \frac{4}{\pi}$$

$$\left(\frac{\partial F_k}{\partial \gamma_1}\right)_{\gamma_1 = \frac{\pi}{4}, a_2 = \tilde{a}_3 = \ldots = \tilde{a}_N = 0} = 0 \qquad k = 3, 5, ..., N\,.$$

The implicit function theorem then implies that sufficiently near $\gamma = \pi/4$, the polynomial system can be solved, and that the solutions $a_2, \tilde{a}_3, \tilde{a}_5, ..., \tilde{a}_N$ are differentiable functions of γ_1, with

$$\frac{d}{d\gamma_1} \begin{pmatrix} a_2 \\ \tilde{a}_3 \\ \vdots \\ \tilde{a}_N \end{pmatrix} = J^{-1} \frac{d}{d\gamma_1} \begin{pmatrix} F_2 \\ F_3 \\ \vdots \\ F_N \end{pmatrix} ,$$

where all derivatives in the right hand side are to be evaluated at the critical point. Computing the inverse of the Jacobian, we find easily that $a_2'(\pi/4) = 12/\pi$, and with a little more effort that $\tilde{a}_3'(\pi/4) = \frac{144\gamma_3}{4\pi\gamma_3+3\pi^2}$, and then that $\tilde{a}_k'(\pi(4) = 0$ for $k > 3$. Hence

$$
\begin{aligned}
a_2(\gamma_1) &= \frac{12}{\pi}\left(\gamma_1 - \frac{\pi}{4}\right) + \mathcal{O}\left(\left(\gamma_1 - \frac{\pi}{4}\right)^2\right) , \\
\tilde{a}_3(\gamma_1) &= \frac{144\gamma_3}{4\pi\gamma_3 + 3\pi^2}\left(\gamma_1 - \frac{\pi}{4}\right) + \mathcal{O}\left(\left(\gamma_1 - \frac{\pi}{4}\right)^2\right) , \\
\tilde{a}_k(\gamma_1) &= \mathcal{O}\left(\left(\gamma_1 - \frac{\pi}{4}\right)^2\right) , \qquad\qquad k = 5, 7, 9, ...
\end{aligned}
$$

The Fourier coefficients $a_1, ..., a_N$ of a stationary solution may now be computed directly from $a_2(\gamma_1), \tilde{a}_3(\gamma_1), ..., \tilde{a}_N(\gamma_1)$ using $a_p = a_1\tilde{a}_p$ and Eq. (14). Because $a_2 = \gamma_2 a_1^2$ and $\gamma_2 > 0$, and because we expect all coefficients $a_1, ..., a_N$ to be real, only $\gamma_1 \geq \pi/4$ yields an admissible solution. All coefficients are continuous functions of γ_1, and therefore when $\gamma_1 - \pi/4$ is sufficiently small, the Fourier cosine series with these coefficients is non-negative. Interestingly the behavior of a_2 near the critical point is completely independent of the other coefficients of the noise function than γ_2.

The uniform distribution, with $a_k = 0, \ k = 1, 2, 3....$ is always a stationary solution, and the linearized analysis from Section 4 showed that this solution is stable for $\gamma_1 < \pi/4$ and unstable for $\gamma_1 > \pi/4$. The analysis in this section shows that in an interval $\pi/4 < \gamma_1 < \gamma_{max}$ there is a new invariant solution defined by the coefficients $\bar{a}_1(\gamma_1), ..., \bar{a}_N(\gamma_1)$ defined as above. It now remains to prove that this new solution is linearly stable. Setting $\mathbf{a}(t) = (a_1(t), a_2(t), ..., a_N(t))^{tr}$, we may write Eq. (9) as

$$
\frac{d}{dt}\mathbf{a}(t) = Q(\gamma_1 : \mathbf{a}(t)) - \mathbf{a}(t) ,
$$

where $Q(\gamma_1 : \mathbf{a})$ is a vector whose k-th element is given by the right hand side of Eq. (13). To prove linear stability of the stationary distributions $\bar{\mathbf{a}}(\gamma_1)$ computed from above amounts to proving that the eigenvalues of the Jacobian matrix

$$
\frac{\partial}{\partial \mathbf{a}} Q(\gamma_1 : \bar{\mathbf{a}}(\gamma_1)) = \left(\frac{\partial}{\partial a_j} Q_k(\gamma_1 : \bar{\mathbf{a}}(\gamma_1))\right)_{j,k=1}^{N}
$$

all lie inside the unit circle. The characteristic polynomial is

$$
p(\gamma_1, \lambda) = \det\left(\frac{\partial}{\partial \mathbf{a}} Q(\gamma_1 : \bar{\mathbf{a}}(\gamma_1)) - \lambda I\right) .
$$

At $\gamma_1 = \pi/4, a_1 = ... = a_N = 0$, $\frac{\partial}{\partial \mathbf{a}} Q(\gamma_1 : \bar{\mathbf{a}}(\gamma_1))$ is a diagonal matrix whose diagonal entries are the coefficients $\lambda_k + 1$ as determined by (12). They are explicitly given here by:

$$
1, \quad 0, \quad -\frac{4}{3\pi/2}\gamma_3, \quad 0, \quad \frac{4}{5\pi/2}\gamma_5, ...
$$

They all lie inside the unit circle except the first one. They are continuous functions of γ_1 Therefore, as γ_1 is moved around the critical value $\pi/4$ by a small amount, they all stay within the unit circle, except perhpas the first one, which are going to study now. We note that, $\lambda = 1$ is a simple eigenvalue at this point:

$$
p(\frac{\pi}{4}, 1) = 0 .
$$

We will now again use the implicit function theorem to show that there is a function $\lambda(\gamma_1)$ such that $\lambda(\pi/4) = 1$, $p(\gamma_1, \lambda(\gamma_1)) = 0$, and

$$\lambda'(\frac{\pi}{4}) = -\left(\frac{\partial p(\gamma_1,\lambda)}{\partial \lambda}\right)^{-1}_{\gamma_1=\frac{\pi}{4},\lambda=1}\left(\frac{\partial p(\gamma_1,\lambda)}{\partial \gamma_1}\right)_{\gamma_1=\frac{\pi}{4},\lambda=1} = -\frac{8}{\pi} \tag{16}$$

This implies that for $\gamma_1 > \pi/4$, sufficiently small, $|\lambda(\gamma_1)| < 1$, and that $\bar{\mathbf{a}}(\gamma_1)$ is a stable (hyperbolic) fixed point for the system in Eq. (9) in the Maxwellian case and with N non-zero noise coefficients γ_k.

To obtain (16) we write $\frac{\partial}{\partial \mathbf{a}}Q(\gamma_1 : \bar{\mathbf{a}}(\gamma_1)) - \lambda I$ in more detail. Explicitly for 5 non-zero coefficients γ_k, this matrix is equal to:

$$\begin{pmatrix}
-\frac{4a_2\gamma_1}{3\pi}+\frac{4\gamma_1}{\pi}-\lambda & 2\left(\frac{2a_3}{5\pi}-\frac{2a_1}{3\pi}\right)\gamma_1 & 2\left(\frac{2a_2}{5\pi}-\frac{2a_4}{7\pi}\right)\gamma_1 & 2\left(\frac{2a_5}{9\pi}-\frac{2a_3}{7\pi}\right)\gamma_1 & \frac{4a_4\gamma_1}{9\pi} \\
2a_1\gamma_2 & -\lambda & 0 & 0 & 0 \\
\frac{4a_2\gamma_3}{\pi}+\frac{4a_4\gamma_3}{5\pi} & \frac{4a_1\gamma_3}{\pi}-\frac{4a_5\gamma_3}{7\pi} & -\frac{4\gamma_3}{3\pi}-\lambda & \frac{4a_1\gamma_3}{5\pi} & -\frac{4a_2\gamma_3}{7\pi} \\
0 & 2a_2\gamma_4 & 0 & -\lambda & 0 \\
-\frac{4a_4\gamma_5}{3\pi} & \frac{4a_3\gamma_5}{\pi} & \frac{4a_2\gamma_5}{\pi} & -\frac{4a_1\gamma_5}{3\pi} & \frac{4\gamma_5}{5\pi}-\lambda
\end{pmatrix}$$

Substituting γ_1 with $\pi/4 + \tau$ and λ with $1 + \mu$ we find, retaining only the lowest order terms in each coefficient and only coefficients of order one or less in τ and μ,

$$\begin{pmatrix}
-\mu & -\frac{4a_1\tau}{3\pi}-\frac{a_1}{3} & \frac{12\tau}{5\pi} & 0 & 0 \\
2a_1\bar{\gamma}_2 & -1 & 0 & 0 & 0 \\
\frac{48\tau\bar{\gamma}_3}{\pi^2} & \frac{4a_1\bar{\gamma}_3}{\pi} & -\frac{4\bar{\gamma}_3}{3\pi}-1 & \frac{4a_1\bar{\gamma}_3}{5\pi} & -\frac{48\tau\bar{\gamma}_3}{7\pi^2} \\
0 & \frac{24\tau\bar{\gamma}_4}{\pi} & 0 & -1 & 0 \\
0 & 0 & \frac{48\tau\bar{\gamma}_5}{\pi^2} & -\frac{4a_1\bar{\gamma}_5}{3\pi} & \frac{4\bar{\gamma}_5}{5\pi}-1\,.
\end{pmatrix}$$

In this expression $\bar{\gamma}_k = \gamma_k(\pi/4)$. It is easy to see that this matrix has essentially the same form for any number of non-zero coefficients γ_k, a five-diagonal matrix where the diagonal elements except the first one are of order $\mathcal{O}(1)$ and all other elements are $\mathcal{O}(\mu + \tau^{1/2})$ (because $\bar{a}_1 \sim \bar{a}_2^{1/2} = \mathcal{O}(\tau^{1/2})$). Hence, expanding the determinant, we find, after some computation, that

$$\begin{aligned}
p(\frac{\pi}{4}+\tau, 1+\mu) &= C_N(\mu + \frac{2}{3}\bar{\gamma}_2 a_1^2) + \mathcal{O}(\mu^2 + \tau^{3/2}) \\
&= C_N(\mu + \frac{2}{3}a_2) + \mathcal{O}(\mu^2 + \tau^{3/2}) \\
&= C_N(\mu + \frac{8}{\pi}\tau) + \mathcal{O}(\mu^2 + \tau^{3/2})
\end{aligned}$$

where C_N is the product of the diagonal elements from row three and below. And we conclude, as stated in eq. (16) that

$$-\frac{\partial p}{\partial \gamma_1} / \frac{\partial p}{\partial \lambda} = -8/\pi \, ,$$

when evaluated at the critical point $\gamma_1 = \pi/4, \lambda = 1$. Again we note that this is independent of the Fourier coefficients of the noise function.

6. THE METHOD OF PARTITIONS OF INTEGERS BY BEN-NAIM AND KRAPIVSKY

In this section we adapt a method of Ben-Naim and Krapivsky [7] to the construction of invariant densities for our equation in the Maxwellian case. We no longer require Hypothesis 9, but on the other hand, we shall not control the convergence of infinite sums, and our conclusions are therefore formal. Nonetheless, as in [7], the method provides another view of the phase transition studied here.

With γ_k defined as above and $\Gamma(u) = \sin(\pi u)/(\pi u)$, we let

$$G_{i,j} \quad = \quad \frac{\gamma_{i+j}}{1 - 2\gamma_{i+j}\Gamma\left(\frac{i+j}{2}\right)} \Gamma\left(\frac{i-j}{2}\right) \, ,$$

which is defined for $i, j \in \mathbb{Z}$. Clearly

$$G_{i,j} = G_{j,i}, \quad G_{i,j} = G_{-i,-j}, \qquad \text{and} \qquad G_{j,j} = \gamma_{2j} \, .$$

Also

$$G_{i,j} = 0 \quad \text{when} \quad (i-j) \neq 0 \quad \text{is even} \, , \tag{17}$$

whereas for $j - i$ odd, $G_{i,j}$ satisfies

$$|G_{i,j}| \leq \frac{|\gamma_{i+j}|}{1 - 4|\gamma_{i+j}|/(\pi|i+j|)} \frac{2}{\pi|i-j|} \, .$$

Because we only look for even solutions, $a_j = a_{-j}$, equation (9) may now be written

$$a_k \quad = \quad \sum_{j=1}^{k-1} G_{k-j,j} a_{k-j} a_j + 2 \sum_{j=1}^{\infty} G_{k+j,-j} a_{k+j} a_j \, . \tag{18}$$

It follows from (17) and (18) that when k is a power of two, one can express a_k in terms of $a_1 = a_{-1}$. Hence with $k = 2^m$,

$$a_{2^m} \quad = \quad \gamma_{2^m} \left(a_{2^{m-1}}\right)^2 \, ,$$

and iterating gives

$$a_{2^m} = \prod_{j=0}^{m-1} \left(\gamma_{2^{m-j}}\right)^{2^j} a_1^{2^m} \, . \tag{19}$$

One might hope that it is possible to express *every* a_k as, if not a polynomial in a_1, at least as a power series in a_1. The strategy in [7] provides such an expression, and a_1 itself is considered an *order parameter* and denoted R: for $k \geq 2$,

$$a_k \quad = \quad \sum_{n=0}^{\infty} p_{k,n} R^{|k|+2n} \, , \tag{20}$$

where the the coefficients $p_{k,n}$ are a sum of various products of $G_{i,j}$ computed using a generalized integer partition of k as a sum of $k+n$ terms of $+1$ and n terms of -1. The formula corresponding to (20) in [7] is written with k instead of $|k|$ in the exponent of R, and this leads to the erroneous formula (15) in their paper. We will now derive a correct replacement of their formula (15) adapted to our case.

6.1. **The recursion formula.** Here we look for an invariant density f whose Fourier coefficients, a_k ($k \geq 2$) are given by a power series in R of the form (20), using, of course, $k = |k|$. For a_1, there is such a representation,

$$p_{1,n} = \delta_{n,0} = \begin{cases} 1 & \text{if} \quad n = 0 \\ 0 & \text{if} \quad n \neq 0 \end{cases} \tag{21}$$

but we will also use a different representation in which $p_{1,0} = 0$. Combining the two expressions gives the equation

$$R = \sum_{n=0}^{\infty} p_{1,n} R^{1+2n} \,,$$

from which the value of R can be determined. Clearly, $R = 0$ is a solution, corresponding to the uniform distribution $f = (2\pi)^{-1}$.

Lemma 11. *For each positive integer k, let $\{p_{k,n}\}$ be a sequence of numbers such that the power series $\sum_{n=0}^{\infty} p_{k,n} z^{k+2n}$ has radius of convergence at least one. For $-1 < R < 1$, define*

$$a_{-k}(R) = a_k(R) = \sum_{n=0}^{\infty} p_{k,n} R^{k+2n} \,.$$

Then the $a_k(R)$ satisfy (18) for all R and all $k \geq 1$ if and only if the numbers $\{p_{k,n}\}$ for $k \geq 1$ and $n \geq 0$ satisfy

$$p_{k,n} = \sum_{j=1}^{k-1} \sum_{\ell=0}^{n} G_{k-j,j} p_{k-j,\ell} p_{j,n-\ell} + 2 \sum_{j=1}^{n} \sum_{\ell=0}^{n-j} G_{k+j,-j} p_{k+j,\ell} p_{j,n-(j+\ell)} \,. \tag{22}$$

Note that for $n = 0$ the second sum is zero.

Proof. Take $k \geq 0$. Substituting (20) into equation (18) gives

$$\begin{aligned} \sum_{n=1}^{\infty} p_{k,n} R^{k+2n} &= \sum_{j=1}^{k-1} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} R^{k+2(\ell+m)} G_{k-j,j}\, p_{k-j,\ell}\, p_{j,m} \\ &\quad + 2 \sum_{j=1}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} R^{k+2(j+\ell+m)} G_{k+j,-j}\, p_{k+j,\ell}\, p_{j,m} \,. \end{aligned} \tag{23}$$

Equating coefficients of like powers of R, we obtain (22). Conversely, if (22) is satisfied for all $k \geq 2$, then (23) is also satisfied for $k \geq 2$. ☐

As the proof of the lemma show, if we could find numbers $p_{k,n}$ such that (22) is satisfied for all $k \geq 1$, then we would construct a family, parameterized by R, of solutions (not necessarily positive) of the invariant measure equation.

This, of course, is more than we expect to find, and so the lemma must be supplemented by two things: (1) A construction of the numbers $p_{k,n}$. (2) A mechanism for selecting a particular value of R.

Following [7], we present a recursive construction of the numbers $p_{k,n}$, and a consistent argument for determining R.

6.2. **The recursion formula.** We need some known values of the $p_{k,n}$ to start the recursive construction. First, notice that when k is a power of two, there is only one non-zero term in the right-hand side of (22), and a simple recursion gives

$$p_{2^m,n} = \prod_{j=0}^{m-1} \left(\gamma_{2^{m-j}}\right)^{2^j} \delta_{n,0} \,, \tag{24}$$

which is consistent with (19).

On the other hand, equation (22) is inconsistent with (21). Indeed, for $k = 1$, the first sum in (22) is zero because the range of summation is empty. Then for $n = 0$ also the second sum is zero, so, $p_{1,0} = 0$. This can be seen already in (23), because there, in the right hand side, the smallest power of R that is present is $R^{1+2(j-m-\ell)}$ with $j = 1$ and $m = \ell = 0$, *i.e.* R^3. However, the coefficient of R^3 is a multiple of $p_{1,0}$, so $p_{1,1} = 0$ as well. Hence the first non-vanishing coefficient for a_1 is $p_{1,2}$.

This discrepancy is the source of the criterion for selecting a particular value of R that yields an invariant density.

To start the recursive determination of the coefficients, note that when $n = 0$, the range in the second sum in (22) is empty. Thus, we have

$$p_{k,0} = \sum_{j=1}^{k-1} G_{k-j,j} p_{k-j,0} p_{j,0} \,.$$

Since as noted above $p_{1,0} = 1$ and $p_{2,0} = \gamma_2$, $p_{3,0}$ is determined and then, recursively, so is $p_{k,0}$ for all k.

Next, we consider $p_{k,n}$ for $k = 1$. Specializing (22) to $k = 1$, we obtain

$$p_{1,n} \;=\; 2\sum_{j=1}^{n}\sum_{\ell=0}^{n-j} G_{1+j,-j}\, p_{1+j,\ell}\, p_{n-(j+\ell)} \,.$$

The first two terms in this sequence are

$$p_{1,2} \;=\; 2G_{3,-2}\, p_{3,0}\, p_{2,0} \,,$$

and

$$p_{1,3} \;=\; 2\left(G_{2,-1} p_{2,0}\, p_{1,2} + G_{3,-2} p_{3,1}\, p_{2,0}\right) \,.$$

Here we have used $p_{1,0} = p_{1,1} = p_{2,1} = 0$, the latter being true because of (24), which reduces to $p_{2,n} = \gamma_2 \delta_{n,0}$ when $k = 2$. All terms in the expression for $p_{1,2}$ have been determined above. To compute $p_{1,3}$, we need $p_{3,1}$. However,

$$p_{3,1} = \sum_{j=1}^{2}\sum_{\ell=0}^{1} G_{k-j,j} p_{k-j,\ell} p_{j,1-\ell} + 2G_{4,-1} p_{4,0} p_{1,0} \,.$$

Since $p_{4,0}$ is known, we have $p_{3,1}$ and hence $p_{1,3}$. So far, we have determined the values of all $p_{k,n}$ for all $k + n \le 4$, and then some. From here it is not hard to see that the values of all of the $p_{k,n}$ are determined. For a discussion of this in terms of integer partitions, see [7]. Though all of the coefficients are determined, it does not seem to be a simple matter to estimate the size of the coefficients in a manner that is useful for proving that they do define power series with even a positive radius of convergence.

6.3. **The consistency condition.** At this stage, we have the coefficients $p_{k,n}$ for all $k \geq 1$ and all $n \geq 0$. The equations (22) are satisfied for all $k \geq 1$, by construction, but not, as we have pointed out, for $k = 1$ by the coefficients given in (21), which corresponds to $a_1(R) = R$ for all $-1 < R < 1$.

Nonetheless, assuming convergence, we have from (19) that $R = a_1$. Using the coefficients derived above, we have

$$a_1(R) = \sum_0^\infty p_{1,n} R^{1+2n} \ ,$$

and the first non-vanishing term in the power series on the right is for $n = 2$, so that $a_1(R) \sim R^5$ at $R = 0$.

Therefore, any value of R giving an invariant measure must satisfy

$$R = a_1(R) \ ,$$

where $a_1(R)$ is the function defined by the power series derived above. Of course, there is always the solution $R = 0$. However, there may be other solutions. In [7], the function $a_1(R)$ is approximately computed numerically and plotted. For noise parameters such that $R = a_1(R)$ has a non-zero solution, they find a non-trivial invariant measure. However, rigorous analysis of this construction, and especially analysis of stability of the invariant measures so constructed, seems difficult, and this has motivated our different treatment. While less general in its scope, due to Hypothesis 9, it does permit rigorous analysis.

7. Conclusion

In this paper, we have studied a Boltzmann model intended to provide a binary interaction description of alignment dynamics which appears in swarming models such as the Vicsek model. In this model, pairs of particles lying on the circle interact by trying to reach their mid-point up to some noise. We have studied the equilibria of this Boltzmann model and, in the case where the noise probability has only a finite number of non-zero Fourier coefficients, rigorously shown the existence of a pitchfork bifurcation as a function of the noise intensity. Such a transition had been predicted, with the correct critical exponent, in [9, 10]. In the case of an infinite number of non-zero Fourier modes, we have adapted a method proposed by Ben-Naïm and Krapivsky to show (at least formally) that a similar behavior can be obtained. In the future, we expect to be able to show the rigorous convergence of the infinite series involved in the Ben-Naïm and Krapivsky argument, and therefore, to give a solid mathematical ground also to this case. Extensions of the model to higher dimensional spheres or other manifolds is also envisionned. Finally, the non-isotropic equilibria found beyond the critical threshold will allow us to develop non-trivial Self-Organized Hydrodynamics, as done earlier in the case of the Vicsek mean-field dynamics [8, 9, 10].

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