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Multi-Games and Bayesian Nash Equilibriums

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Declaration

I hereby certify that all material in this dissertation that has not been carried out by me under the guidance of my supervisor Professor Abbas Edalat has been properly acknowledged.

Ali Ghoroghi

Abstract

A growing awareness of the prominent role the environment plays in multi-agent systems has led to gradual acceptance of its importance by the multi-agent system community in general. Within this line of research, we propose a new class of games, called Multi-Games. A Multi-game is one in which a given number of players play a fixed finite number of basic games simultaneously. The basic games in a multi-game can be regarded as different environments for the players, and, in particular, we submit that multi-games can be used to model investment in multiple national and continental markets within a global economy. Furthermore, when the players' weights for different games in the multi-game are classed as private information or as types with given conditional probability distributions, we obtain a particular class of Bayesian games.

The main contribution of this thesis is to illustrate how, for the class of so-called completely pure regular multi-games with finite sets of types, the Nash equilibria of the basic games can be used to compute a Bayesian Nash equilibrium in multi-games, with complexity independent of the number of types. Following the presentation of the main results, the thesis presents two algorithms that allow us to establish whether we have a Bayesian Nash equilibrium which can be determined with lower computational complexity.

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Lastly, and most importantly, thanks to Professor Peter McBurney and Dr Alessandra Russo for taking the time to read this thesis and for examining me.

Dedication

This thesis is lovingly dedicated to my parents,
Mrs. Fakhri Hodaee and Mr. Fazlollah Ghoroghi
who bore me, raised me, supported me, and loved me;
and to my wife Afrouz, son Arash and daughter Ana
for their support, encouragement and quiet patience.

‘Play the game for more than you can afford to lose... only then will you learn the game.’

Winston Churchill

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The Mathematical Notations

$-i$	set of all players except player *i*.
$\arg\max_{x\in X} f(x)$	set of all *x* where function *f* attains its maximum in the set *X*.
$\sigma = (\sigma_1, \ldots, \sigma_N)$	the (mixed) strategy profile of the *N* players.
$\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_N)$	mixed strategy profile of players other than player i.
Θ	set of vectors of types in game with incomplete information.
θ	a type profile; $(\theta_1, \ldots, \theta_n)$.
Θ_i	player *i*'s set of types in game with incomplete information.
θ_i	player *i*'s type in game with incomplete information.
Θ_{-i}	the set of all possible types of players other than player i.
θ_{-i}	a profile of types of players except *i*.
I	a finite set of players expressed as $I = \{1, 2, \ldots, N\} (N \geq 2)$.
i	player, $i \in I$.
J	a finite set of games expressed as $J = \{1, 2, \ldots, M\} (M \geq 1)$.
j	game, $j \in J$.
M	the number of games.
N	the number of players.
p	common prior in a game with incomplete information.

$s = (s_1, \ldots, s_N)$ the strategy profile which is a list consisting of one strategy for each of the N players; the "outcome" of the game.

S the set of all possible strategy profiles.

S_i the strategy space (or strategy set) of player i.

$s_i(\theta)$ a (pure) strategy of player i with type θ.

$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N)$ the strategy profile of the other N - 1 players.

u payoff function in a strategic-form game.

$u_i(\sigma_i, \sigma_{-i})$ the *expected payoff* to player i as a function of the mixed strategy profile played by the n players in the game.

$u_i(s_1, \ldots, s_N)$ the payoff to player i as a function of the strategy profile played by the N players in the game. Payoffs should be thought of as utilities of the outcomes.

$\mathcal{M}(S_i)$ the set of possible mixed strategies available to player i.

Chapter 1

Introduction

In the opening chapter of this thesis, we briefly talk about game theory, multi-agent systems and then introduce a new class of games, called multi-games, in which a given number of players play a fixed finite number of basic games simultaneously. We then expand on these in the following chapters and explain game theory in more detail as well as discuss in more depth the multi-games. We will then explain the motivation and background to the work, before discussing the contribution of this research respectively. In Section 1.5, we discuss related work. In the last section of this chapter, we provide a guide to the thesis.

1.1 Game Theory

Game theory is the academic study of conflict and cooperation between two or more independent, self-interested and rational players (agents). Rational players aim towards very specific goals when making any choice. Players may be groups or individuals or a combination of these.

Game theory furnishes a framework for communication between players and a formulation and analysis of their strategies [TS01, OR94]. In other words, game theory refers to the study of decision-making between people when decisions are reliant upon each other. This forms part of a highly academic sector within applied mathematics, even though the term might sound somewhat playful.

Game theory allows us to effectively analyse, create, plan, and gain insight into situations in which a strategy might be used [Per12, MSZ13]. Game theory is the most comprehensive interaction theory to date [Sho08].

In 1838, Antoine Cournot conducted one of the earliest formal game-theoretic analyses. Game theory was originally introduced to model the behaviour of rational agents whereby players make independent decisions in order to maximise their utility or payoff in an economy [NM44]. Further enhancements to the field were undertaken in 1950 by John Nash, who demonstrated that finite games always have an equilibrium point, at which all players can choose actions that are best for them given their opponents' choices [Nas50]. Since 1950, game theory has been applied in economic theories, sociology, psychology, philosophy, biology, military applications, politics and networks [MSZ13]. Recently, game theory has become a common and pervasive occurrence in computer science. Artificial intelligence, e-commerce, and networking are just a few computer science fields in which game theory has become an inherent component. Game theory also applies to the internet, which requires the development of systems that encompass more than one entity and are associated with specific information and interests.

"Game theory is a sort of umbrella or 'unified field' theory for the rational side of social science, where 'social' is interpreted broadly, to include human as well as non-human players (computers, animals, plants)."(Yisrael Aumann 1987)

The algorithmic game theory has emerged as a result of the integration of the computational approach with game theoretical models [NRTV07]. The algorithmic game theory and multi-agent systems are both important research areas in Artificial Intelligence.

1.2 Multi-Agent System and Multi-Games

A multi-agent system is formed of a number of independent entities known as agents, that engage in mutual interaction in order to pursue individual interests (Competition), achieve a common objective (Collaboration) and negotiate with each other [SL09].

Traditional artificial intelligence has been concerned with developing models of particular facets of individual agents; by contrast, multi-agent system addresses the interplay among different agents which exhibit several characteristics, including autonomy of action (Autonomous), response to phenomena in their surrounding environment (Reactive), own initiative as catalyst for action (Proactive), and interaction with other agents (Social) [Woo09, WJ95].

Multi-agent systems research employs many techniques to model and investigate different facets of agents, for example classical logic, non-monotonic logic reasoning and even machine learning. Recently, multi-agent systems research has been aided by the tools and techniques of game theory, particularly when applied to problems such as negotiations [Woo09]. Indeed, due to the solid mathematical basis of game theory, in modelling interactions among self interested agents, it has been a predominant theoretical tool in use for analysis of multi-agent systems [Woo09].

Because of the interactive relationship between the agents in a multi-agent system, the actions of one agent may have repercussions for the others as well. Game theory can be applied to model this interplay and it has been used as a very popular technique in multi-agent systems. Due to the general acceptance that most agents in multi-agent systems are self interested, there has been a great increase in interest in the application of game theory and its models to multi-agent systems, particularly when involving self-interested agents or players. Game theoretic work initially entered the multi-agent system's literature as a result of Jeffrey Rosenchein and colleagues' work [PW02]. Although multi-agent systems comprise the greater part of game-theoretic work, the scope of multi-agent systems is significantly broad, incorporating non-game-theoretic subjects like models of software engineering and logical reasoning regarding the perspectives and goals of other agents, sharing of tasks, argumentation, distributed sensing, and coordination between multiple agents [EL10, LS08, Woo09, SL09].

A growing awareness of the prominent role the environment plays in multi-agent systems has led to gradual acceptance of its importance by the multi-agent system community in general [WHHS09]. Suppose a multi-agent system in which agents interact together simultaneously within many environments. A concept is needed to act as a model, which can be applied to this multi-agent system as a representation of the players or agents interacting together simultaneously in many different environments.

For the purpose of this research, we introduce a new class of games, called multi-games which can be used to model economic, human or technological behaviour, where each agent can allocate its resources in varying proportions to play in a number of different environments simultaneously, each representing a basic game in its own right. Each agent can have the same set of strategies for the different basic games. The payoff for each agent in a multi-game is assumed to be the convex linear combination of payoffs obtained for the basic games, weighted by the proportions allocated to them (A convex combination is defined as a linear combination $\theta_1 x_1 + \cdots + \theta_N x_N$ of the vectors $x_1, x_2, \ldots, x_N$ where $\theta_i \in [0, 1]$, $\sum_{i=1}^{N} \theta_i = 1$).

In the multi-games, each basic game is the environment in which the agents, or players interact in. In particular, we submit that multi-games can be used to model investment in multiple national and continental markets within a global economy, where agents have to interact in different environments at the same time, though it can also be applied to any multi-agent system where a number of agents are interacting simultaneously within a number of environments.

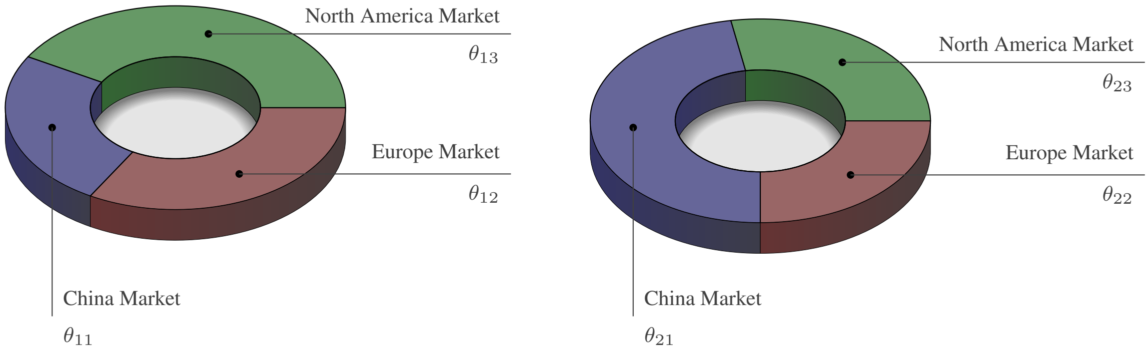
In other words, the purpose of using this model is to add a new dimension to the description of a range of situations, achieved through the employment of game theoretic models. This is done by linearly combining the payoff matrices of various games and linking them through the use of a type for each agent, which represents the amount of investment that an agent is willing to commit in that particular game.

Next, we provide an example of multi-games in the case of investment in a global economy with different national or continental markets.

Example 1.1. *(Battle in Smart phones Market)*

Consider two multinational smart-phone producer companies which can invest in the national economies of the US, EU and China as the world's largest market for smart-phones with different cost of investment, advertising, rates of profit, labour value, interest rates, etc. Suppose that they need to decide in what ratio to divide their funds for investment in the three regions and, in addition, whether to enter into a particular venture or not. We thus have three games, $\mathcal{G}_1$ for US, $\mathcal{G}_2$ for EU and $\mathcal{G}_3$ for China, one for each national economy, each with two players and two strategies for entering (E) and not entering (E'). Let u_{ij} denote the payoff function for player i in $\mathcal{G}_j$ (with $i = 1, 2$ and $j = 1, 2, 3$).

Suppose that the first player invests θ_{11}, θ_{12} and $\theta_{13} = 1 - \theta_{11} - \theta_{12}$ fractions of its funds in US, EU and China respectively, and assume that θ_{21}, θ_{22} and $\theta_{23} = 1 - \theta_{21} - \theta_{22}$ are the corresponding fractions for the second player, see Figure 1.1.



(a) Fractions for Producer 1, $\theta_{11} + \theta_{12} + \theta_{13} = 1$ (b) Fractions for Producer 2, $\theta_{21} + \theta_{22} + \theta_{23} = 1$

Figure 1.1: Illustration of an example of the fractions of two multinational companies' funds in US, EU and China.

Thus, the payoff to the first player for the strategy profiles (X_1, Y_1) in $\mathcal{G}_1$, (X_2, Y_2) in $\mathcal{G}_2$ and (X_3, Y_3) in $\mathcal{G}_3$, with $X_j, Y_j \in \{E, E'\}$ for $j = 1, 2, 3$, would be

$$\theta_{11}u_{11}(X_1, Y_1) + \theta_{12}u_{12}(X_2, Y_2) + \theta_{13}u_{13}(X_3, Y_3),$$

whereas the payoff for the second player for the same strategy profile would be

$$\theta_{21}u_{21}(X_1, Y_1) + \theta_{22}u_{22}(X_2, Y_2) + \theta_{23}u_{23}(X_3, Y_3).$$

1.3 Motivation and Background

The structure of a multi-game is based on a model named double game. In 2010, Edalat introduced a new game framework called double game, which combines a standard dilemma and a social game. Consequently, research has been done by Ounsley on the double game utilising the concept of Nash equilibrium and by Ghoroghi on an application of the double game [Oun10, Gho10].

The idea of multi-games originated following research for an application of double games and we generalised the idea of a double game to multi-games following the unpublished paper [EGS12]. In this research, we generalised definitions of double game and proved some new theorems. We then present two algorithms for both double and multi games that allow us to establish whether we have a Bayesian Nash equilibrium which can be determined with lower computational complexity. The algorithm for multi-games determines this with a lower computational complexity which is polynomial in M for a given N. Multi-games are more realistic in real-world socio-economic contexts and the study of multi-games may exhibit interesting results in comparison to double games.

Furthermore the idea of changing types in a multi-game seemed more appealing. Therefore, we pursued this line of research into how changes in the types affect and induce changes in the Nash equilibrium set for multi-games. Changes in types might result from changes in information, preferences, or might result from errors in identifying the types of the other player. A player might receive new information and s/he might then reconsider the type s/he had originally chosen. Obviously a change affecting the types of one player might produce a new game with a new Nash equilibrium.

On the other hand, in order to be considered useful in predicting its economic behaviour, an equilibrium strategy for players must lend itself to effective computation. The main problem is that the computational complexity of obtaining the Nash equilibrium becomes greater with the increase in the number of features. This means that the players cannot derive any advantages even with the identification of equilibrium. Such an equilibrium strategy for players can only be useful for determining or predicting economic behaviour if it can be efficiently computed. Clearly, if the computation of a Nash equilibrium is unfeasible because of its high complexity, then its existence, despite having theoretical significance, has no value in practice. It is thus useful to have models in game theory for which the computation of a Nash equilibrium can be more efficiently done than in general. Therefore, we pursued this line of research attempting to reduce the computational complexity in order to establish whether we have a Bayesian Nash equilibrium which can be determined.

1.4 Contributions of The Thesis

- This research introduces multi-games as a new form of game. In multi-games, a given number of players divide up their resources according to different weights over a given number of games, which are then played out simultaneously. All players play at the same time but each can use the same set of strategies for the games. Players use a particular assortment of weights, one for each of the games. Combined, these signify the percentage of the players' investment in each of the games. The convex combination of the payoff a player acquires in the games, along with the assigned weights, makes up a player's total payoff. Each of the games can be thought of as being an alternate environment for the players. It is our suggestion that investments in various continental and national markets in a worldwide economy can be represented using multi-games.

- A class of Bayesian games is achieved when the players' weights for certain multi-games involve private information or types that have certain conditional probability distributions. We show that for the class of so-called completely pure regular multi-games and with finite sets of types, the Nash equilibria of the basic games can be used to compute a Bayesian Nash equilibrium in multi-games with complexity independent of the number of types. We developed two algorithms in order to establish whether we have a Bayesian Nash equilibrium which can be determined with lower computational complexity.

1.5 Related Work

Our construction of the multi-games is a novel approach in the game theory field. At first glance, an N-player multi-game may seem similar to poly-matrix games [Yan68], but is dissimilar in structure. This is discussed further in Section 3.2. Also, we show that the double game, as an instance of multi-games, provides a generalisation of the altruistic extension in [CKKS11] which can be considered as a double game with the first game identified as the original game and the second game as a symmetric altruistic game in Chapter 5.

Hypergame theory which can be thought of as a linked set of games [BD79], is used to reason on two or more perspectives of a competitive situation. Both enemy capability and possible intent can be recorded in a parsimonious notation called the Hypergame Normal Form. Hypergame theory abandons the assumption of perfect knowledge where one player can perceive options for himself. Thus each player makes a rational decision according to its own perception of the game. Therefore in a Hypergame it may seem as if two games are played, these are each player's subjective game both leading to actions in the real game which gives payoffs [Van00, VL02].

Drama theory, developed by [How71, Bry03] follows in the path of hypergames in allowing the differing perceptions by each player. It also allows players to change their preferences during the game. This can be used to simulate responses during the course of a game which results in change following change until the game is over or certain actions become necessary. A drama unfolds through episodes in which characters interact. An episode goes through phases of scene-setting, build-up, climax and decision. Finally comes the action that sets up the next episode. This has been used for defence, political and commercial relations since the early 90s.

Another similar field is concurrent games, which are a form of game semantics [AM99]. This was designed to overcome the problems in sequential forms of game semantics for linear logic. Game semantics is used to model interactions between a system and its environment, so one player in the game represents the system as a proponent and the other the environment as the opponent. This sequential format however has very limitative consequences. Abramsky created a new form called concurrent games which allowed the players to act "in a distributed, asynchronous fashion", taking notice of each other only when they choose to. These games no longer follow the normal format of logical games.

Playing simultaneous games to test different strategies is also related to ideas in evolutionary game theory. Nash doctoral dissertation contains a seminal idea that equilibrium of a game can be understood either as the rational behaviour of a fixed group of individuals, or as the not necessarily rational, but average behaviour of a population of individuals. John Maynard Smith revived the idea in a biological setting where the players are competing species that possess different strategies [Smi82]. Each individual in a habitat competes in interspecies scenarios which can be thought of as two player games. This comes from an assumption that all members of the same species are irrational and can be generalised as one player [You11]. In each game however there are different payoffs for each player depending on the opponent species and the variation within each species which leads to evolution. This links evolutionary game theory to two player games with varying payoffs as a large population of individuals who are recurrently and uniformly randomly matched in pairs, play a finite and symmetric two-player game.

1.6 A Guide to the Chapters

In the next chapter, we will talk about the fundamentals of game theory. In chapter 3, we will introduce a formal definition of the multi-game in addition to some examples of multi-games.

In chapter 4, for convenience and ease of presentation, we restrict ourselves to the class of N-player double game in which each player has the same set of strategies in the two basic games. We later define the class of *pure regular* double game, in which for pairs of extreme types there are 2^N pure Nash equilibrium in which the strategy of each player only depends on its own type. Similarly, for a double game with ℓ_i types for each player $i \in I = \{1, \ldots, N\}$, we define the notion of a *completely pure regular* double game where there are $\ell_1 \times \cdots \times \ell_i \times \cdots \times \ell_N$ pure Nash equilibrium for all possible pairs of types of N players in which the strategy of each player only depends on its own type. We then derive an algorithm for establishing that a double game is completely pure regular with complexity independent of the number of the types and actions. We also show that a pure Bayesian equilibrium for a completely pure regular double game can be obtained directly from this algorithm, thus reducing the complexity of computation.

Chapter 5 will present an example where we apply this framework to obtain a double game extension of the Prisoner's Dilemma in order to model pro-social behaviour. In this double game of Prisoner's Dilemma, the first game is the classical Prisoner's Dilemma and the second game captures the social or moral gain for cooperation for each player. Furthermore, we consider the double game for the Prisoner's Dilemma where the social (altruistic) coefficient of each player forms a finite discrete set of incomplete information or types thus giving rise to a Bayesian game.

In chapter 6, we introduce N-player multi-games with M games and define the class of pure regular multi-game and a completely pure regular multi-game. We then present an algorithm that allows us establish whether we have a Bayesian Nash equilibrium which can be determined with lower computational complexity.

Chapter 7 will introduce the computer program which has been developed on the basis of the proposed algorithms and we will discuss the results of analysing these algorithms for a various number of types, players and games.

In Chapter 8, we will discuss some attempts to compare various strategies in a round-robin tournament of the double game for Prisoner's Dilemma, in which the players can change their social coefficients incrementally from one round to the next. In the last chapter, chapter 9, we will conclude the discussion and highlight future works.

Chapter 2

Background

In this chapter, we briefly talk about the fundamentals of game theory besides some examples to clarify them. The end of this chapter contains a short overview of Robinson-Goforth topology of 2×2 Games.

2.1 Game Theory Definitions

Within the decision theory, a game refers to a strategic interaction [OR94] and so game theory is “the formal study of decision-making where several players must make choices that potentially affect the interests of the other players” [TS01]. Game theory contains a number of important key words:

- *Utility*; Under utility theory, the participants’ decisions or predilections influence the values assigned to certain results. It has been proposed by Von Neumann and Morgenster [NM44] that the results can be given substitute numbers in order to ensure that a rational individual making a decision will consistently be able to choose the optimal expected utility.
- *Rationality*; The idea is that each agent is rational under game theory. The objective of each agent is to achieve the optimal expected value of his/her *payoff* [NM44].

In terms of player i’s payoff, we are referring to one of the two following situations:

- Nature and every other player have selected their strategies and the game has finished. Utility for player i is then received; or
- Player *i*'s expected utility, obtained by her/him as a function of the strategies s/he and the other players selected.

Overall, games can be categorised as shown in Figure 2.1:

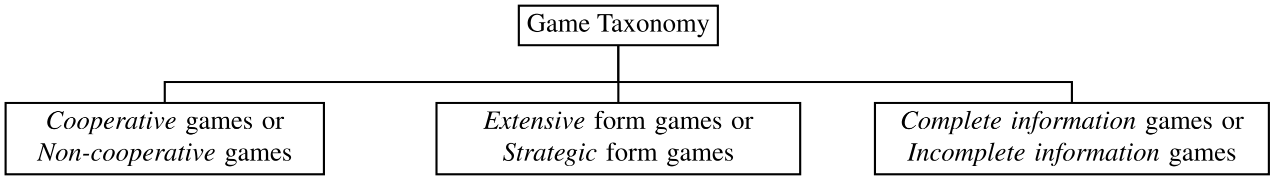


Figure 2.1: Illustration of important game categories.

Coalitional or *cooperative*, this type of game forms payoffs that are achievable for potential teams if they choose to work together. Games are examined in terms of the relationships between participants under cooperative game theory. On the other hand, *Non-cooperative* game theory selects payoffs based on the evaluation of strategic choices regarding the sequence of participants' decisions and actions. Under the non-cooperative game theory, two fundamental approaches are utilised: *extensive form* game, and *strategic form* or *normal form* game.

Extensive form games can be illustrated through game trees. Figure 2.2 represents an example of an extensive game with two players. Each player has an action set comprised of two actions. Player 1 can choose z or u and player 2 can select between v or w.

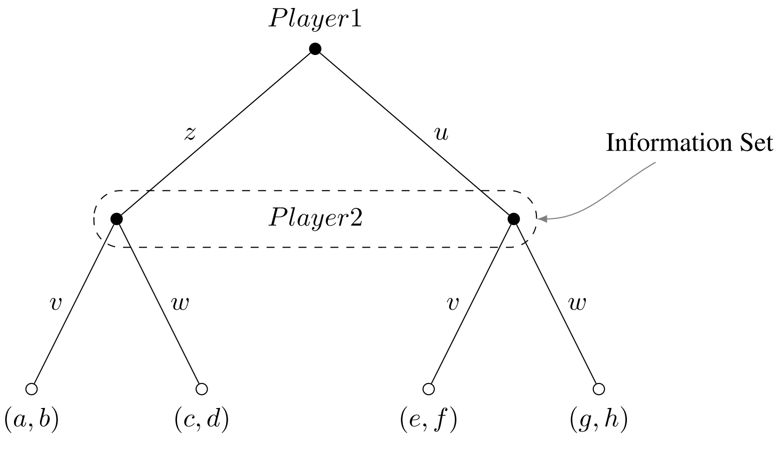


Figure 2.2: Illustration of a game tree.

Definition 2.1. *[NGNP09] (Extensive Form Game). An extensive form game $\mathcal{G}$ is defined as a tuple $\langle I, (A_i)_{i\in I}, \mathcal{H}, P, (u_i)_{i\in I}\rangle$, where $I = \{1, 2, \ldots, N\}$ is a finite set of players; $A_1, A_2, \ldots, A_N$ are the action sets of player $i \in I$, respectively; $\mathcal{H}$ is the set of all terminal nodes (terminal histories). $P : t \rightarrow I$ where $t \notin \mathcal{H}$ is a mapping that associates node t to player i and $u_i : \mathcal{H} \rightarrow \mathbb{R}$ as a function of the terminal node reached, called the utility functions (payoff functions).*

An information set illustrates every possible move that could have been made during the game up to a given node, based on what that player has seen.

Definition 2.2. *[NGNP09] (Strategic Form Game). A strategic form game or normal (static) form game $\mathcal{G}$ is defined as a tuple $\langle I, (S_i)_{i\in I}, (u_i)_{i\in I}\rangle$, where $I = \{1, 2, \ldots, N\}$ is a finite set of players; $S_1, S_2, \ldots, S_N$ are the strategy sets of the players $i \in I$, respectively; and $u_i : S_1 \times S_2 \times \cdots \times S_N \rightarrow \mathbb{R}$ are mappings called the utility functions.*

Figure 2.3 represents an example of 2×2 game, The four outcomes for each table are represented by the four cells of the matrix. Player one controls the choice between z and u, and player 2, between v and w. The players choices then determine the outcome of the game. The corresponding cell to the choices contains each player's payoff, represented by two numbers, for players 1 and 2 respectively.

		Player 2	
		v	w
Player 1	z	$(3, 6)$	$(0, 7)$
	u	$(0, 4)$	$(3, 5)$

Figure 2.3: Payoff matrix: standard notation for the strategic form.

All games can be categorised as follows:

- *Complete information* games; A game can be thought of as a complete information game when each player has the same level of knowledge (or more) as the players who have already acted in each move.
- *Incomplete information* games. When the player has less knowledge that those who have already acted, it is considered to be incomplete information.

The following two game types fall under the category of complete information games:

- *Perfect information*: Here, the players know everything that has happened previously in the game (i.e. all other players' actions). All the players also hold full payoff knowledge of their own and other players' payoffs.

- *Imperfect information*: Here, players do not know everything, but some things that have happened previously in the game, and they do not know of every action taken previously by the other players. However, all players know every payoff that could be achieved.

As many factors are influenced by the relationship between players' planned actions and expectations, we usually use the word "strategy" (which can be thought of as a function or mapping) instead of "action". In the case of strategy we refer to *pure* strategy in order to distinguish them from *mixed* strategies, which are randomizations over pure strategies. Given a player *i* with S_i as the set of pure strategies, a mixed strategy σ_i for player *i* is a probability distribution over S_i. That is, $\sigma_i : S_i \rightarrow [0, 1]$ assigns to each pure strategy $s_i \in S_i$, a probability $\sigma_i(s_i)$ such that $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$. The probability of a pure strategy profile (combination) $(s_1, \ldots, s_N)$ is $\sigma(s_1, \ldots, s_N) = \prod_{i \in I} \sigma_i(s_i)$ [NGNP09].

All players are assumed to be rational and therefore they choose the strategies which are desirable for them, with respect to what the other players do. Thus, each player's predicted strategy must be that player's best response to the predicted strategies of the other players.

Definition 2.3. *[Ras06](Best response) The best response for player $i \in I = \{1, 2, \ldots, N\}$ in a normal form game $\mathcal{G}$ to the strategies s_{-i} chosen by the other players is strategy $s_i \in BR_i(s_{-i})$ if there exists another strategy s'_i of player i such that*

$$BR_i(s_{-i}) = \{s_i \in S_i | u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \, \forall s'_i \in S_i\} = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i})$$

for each $s_i \in S_i$. $BR_i(s_{-i})$ is a set and $BR_i : S_{-i} \rightarrow S_i$.

A strategy is dominated if it is not the best response strategy whatever the strategy choice of the opposition. Predictions are easy when there are *dominant strategies*. A dominant strategy for a player is one that produces the highest payoff of any strategy available for every possible action by the other players [HV95]. Now we present some definitions regarding dominance in game theory. Definitions 2.4 to 2.7 are taken from [Ras06].

Definition 2.4. *Strategy* s_i *dominates another possible strategy* s'_i *of player* i *if* $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$, $\forall\ s_{-i} \in S_{-i}$ *and* $u_i(s_i, s'_{-i}) > u_i(s'_i, s'_{-i})$ *for some* $s'_{-i} \in S_{-i}$.

Definition 2.5. *(Dominant strategy) Strategy* s_i *is dominant if* s_i *dominates another possible strategy* s'_i *of player i,* $\forall s'_i \neq s_i$.

Definition 2.6. *Strategy* s_i *strictly dominates another possible strategy* s'_i *of player* i *if* $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i}), \forall\ s_{-i} \in S_{-i}$.

Definition 2.7. *(Strictly dominant strategy) Strategy* s_i *is strictly dominant if* s_i *strictly dominates another possible strategy* s'_i *of player i,* $\forall s'_i \neq s_i$.

Dominant strategies do not always exist, and then we can turn to notions of equilibrium. Nash equilibrium appears when the only criteria is that every decision made by a player is the best response to the other players' best response strategies. The notion of Nash equilibrium has become the key concept in game theory since John Nash's celebrated proof of the existence of a mixed Nash equilibrium for all finite games in 1950 [Nas50]. In the following sections, we discuss the notion of Nash equilibrium in pure strategies and mixed strategies.

2.2 Nash Equilibrium in Pure Strategies

Definition 2.8. *[NGNP09] Consider* $\mathcal{G} = \langle I, (S_i)_{i \in I}, (u_i)_{i \in I} \rangle$*, a strategic form game, where* $I = \{1, 2, \ldots, N\}$ *is a finite set of players. The strategy profile (vector)* $s = (s_1, s_2, \ldots, s_N)$ *can be recognised as a pure strategy Nash equilibrium of* $\mathcal{G}$ *if*

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$$

for all $s'_i \in S_i$ *and for all players i. In other words, pure strategy profile* $s = (s_1, s_2, \ldots, s_N)$ *is a Nash equilibrium if each* s_i *is a best response to* s_{-i}.

Definition 2.9. *[Ras06] (ϵ-Nash). Fix* $\epsilon > 0$*. A strategy profile* $s = (s_1, \ldots, s_N)$ *is an* ϵ*-Nash equilibrium if, for all players i and for all strategies* $s'_i \neq s_i, u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) - \epsilon$.

2.3 Nash Equilibrium in Mixed Strategies

A strategic form game, may not involve a Nash equilibrium but instead requires that players choose pure strategies with given probabilities. It also requires that players act rationally. Mixed strategy is a probability distribution over pure strategies.

Definition 2.10. *[NGNP09] Let* $\mathcal{G} = \langle I, (S_i)_{i\in I}, (u_i)_{i\in I}\rangle$*, be strategic form game with finite set of strategies of each player* $i \in I$ *where* $I = \{1, 2\ , \ldots, N\}$*. Given a player* i *with* S_i *as the set of pure strategies, a mixed strategy* σ_i *for player* i *is a probability distribution over* S_i*,* $\sigma_i : S_i \rightarrow [0, 1]$ *and* $u_i(\sigma_1, \ldots, \sigma_N) = \sum_{(s_1,\ldots,s_N)\in S} \sigma(s_1, \ldots, s_N)u_i(s_1, \ldots, s_N)$*. A mixed strategy profile* $\sigma = (\sigma_1, \ldots, \sigma_N)$ *is a Nash equilibrium if* $\forall i \in I$*,*

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \mathcal{M}(S_i)$ *where* $\mathcal{M}(S_i)$ *is the set of possible mixed strategies available to player i.*

Theorem 2.1. *[Nas50](Nash's Theorem) Let* $\mathcal{G} = \langle I, (S_i)_{i\in I}, (u_i)_{i\in I}\rangle$ *be a finite strategic form game where* $I = \{1, 2\ , \ldots, N\}$ *is finite and* S_i *is finite for each player* i*. Then* $\mathcal{G}$ *has at least one mixed strategy Nash equilibrium.*

2.4 Bayesian Game

Private information for each player refers to confidential data that is not known by any other member [NGNP09]. The player's private information is what defines the *type* of the player. Each player can be of several types where a type is to be thought of as a full description of the player's beliefs. This can include beliefs regarding game information (the 'state of nature'), beliefs held regarding the 'state of nature' viewpoints of the other players, and so on. In 1968, John Harsanyi [Har68] proposed *Bayesian form games* to represent incomplete information games. Harsanyi suggested that a method for solving games with incomplete information is by transforming it into a game with imperfect information, in which a probability distribution for each unknown value, referred to as a type, is provided. The Harsanyi's transformation essentially entails the following stages:

- Within the strategic decision scenario, all asymmetric information should be represented in terms of the way in which profiles of actions influence utility payoffs.
- Transform the above model on the realisation of random variables where the ex-ante probability distribution before the vector of types is chosen is *common knowledge* among all the players. Aumann [Aum76] defined: “A fact is common knowledge among the players if every player knows it and every player knows that every player knows it, and so on”.

Definition 2.11. *[FT91] (Bayesian game) A Bayesian form game $\mathcal{G}$ is defined as a tuple $\mathcal{G} = \langle I, \Theta, S, p, u\rangle$, $I = \{1, 2, \ldots, N\}$ and $\Theta = \times_{i \in I}\Theta_i$, where Θ_i is the type space (the set of types) of player i. $S = \times_{i \in I}S_i$ where S_i is the set of available strategies to player i. S is called “states of the world” . $p : \Theta \rightarrow [0, 1]$ and p_i is a (discrete) probability function specifying i’s belief about the type of other players given his own type. $u = (u_1, \ldots, u_n)$, where $u_i : S \times \Theta \rightarrow \mathbb{R}$ is the utility function (payoff utility) for player i.*

Bayesian Nash equilibrium is also the basis of games with incomplete information as shown by Harsanyi in 1960’s [Har95]. A Bayesian Nash equilibrium is a Nash equilibrium in a Bayesian normal form game [AH02].

Definition 2.12. *[FT91] A Bayesian Nash equilibrium in a game $\mathcal{G}$ of incomplete information with a finite number of types θ_i for each player $i \in I$, a prior distribution p, and pure strategy spaces S_i is the Nash equilibrium of the “expanded game” in which each player i’s space of pure strategies is the set $S_i^{\Theta_i}$ of strategy maps from Θ_i to S_i i.e. $s_i(.) : \Theta_i \rightarrow S_i$. A pure strategy for player i is a function $s_i(.) : \Theta_i \rightarrow S_i$ that specifies a pure action $s_i(\theta_i)$, which is what i will choose when his type is θ_i. Given a strategy profile $s(.)$ and an $s'_i(.) \in S_i^{\Theta_i}$, let $(s'_i(.), s_{-i}(.))$ denote the profile where player i plays $s'_i(.)$ and the other players follow $s(.)$, and let*

$$\big(s'_i(\theta_i), s_{-i}(\theta_{-i})\big) = \big(s_1(\theta_1), \ldots, s_{i-1}(\theta_{i-1}), s'_i(\theta_i), s_{i+1}(\theta_{i+1}), \ldots, s_N(\theta_N)\big)$$

denote the value of this profile at $\theta = (\theta_i, \theta_{-i})$. Then, strategy profile $s(.)$ is a pure Bayesian equilib-

rium of $\mathcal{G}$ *if and only if, for all* $i \in I$ *and all* $\theta_i \in \Theta_i$ *such that* $p(\theta_i) > 0$,

$$s_i(.) \in \arg\max_{s'_i(.) \in S_i^{\Theta_i}} \sum_{\theta_i} \sum_{\theta_{-i}} p(\theta_i, \theta_{-i}) u_i\big(s'_i(\theta_i), s_{-i}(\theta_{-i}), (\theta_i, \theta_{-i})\big)$$

or

$$s_i(\theta_i) \in \arg\max_{s'_i \in S_i} \sum_{\theta_{-i}} p(\theta_{-i}|\theta_i) u_i\big(s'_i, s_{-i}(\theta_{-i}), (\theta_i, \theta_{-i})\big) \tag{2.1}$$

for all $s'_i \in S_i$. *Given* $p(\theta_1, \ldots, \theta_N)$, *Bayes rule can be used to measure conditional distribution* $p(\theta_{-i}|\theta_i)$, *referring to player i's belief regarding the other players' type distribution.*

$$p_i(\theta_{-i}|\theta_i) = \frac{p(\theta_{-i}, \theta_i)}{p(\theta_i)} = \frac{p(\theta_{-i}, \theta_i)}{\sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i}, \theta_i)}.$$

Note that actions and strategies in the Bayesian game context are used in different ways. A strategy for a player i is defined as a mapping from Θ_i to S_i where a strategy s_i of a player i specifies a pure action for each type of player i. The notation $s_i(.)$ refers to the pure action of player i corresponding to an arbitrary type from his type set.

There is also a definition for mixed Nash equilibrium in Bayesian games, which holds for the general case of continuous type θ_i for player $i \in I$ where $I = \{1, 2\ , \ldots, N\}$ is finite.

Definition 2.13. *[OR94] In a game* $\mathcal{G}$ *of incomplete information with a finite number of types* θ_i *for each player* $i \in I$, *and pure strategy spaces* S_i, *we denote the set of mixed strategies over the strategy set* σ *by* $\mathcal{M}(S_i)$. *A mixed strategy is a function* $\sigma_i : \Theta_i \to \mathcal{M}(S_i)$ *that specifies a lottery* $\sigma_i(\theta_i)$ *for each of i's possible types* $\theta_i \in \Theta_i$. *A strategy profile* $\sigma \in \mathcal{M}(S_i)$ *is a mixed strategy Bayesian equilibrium of* $\mathcal{G}$ *if and only if, for all* $i \in I$ *and all* $\theta_i \in \Theta_i$, *such that* $p(\theta_i) > 0$,

$$\sigma_i(\theta_i) \in \arg\max_{\sigma'_i \in \mathcal{M}(S_i)} \sum_{\theta_{-i}} p(\theta_{-i}|\theta_i) u_i\big(\sigma'_i, \sigma_{-i}(\theta_{-i}), (\theta_i, \theta_{-i})\big)$$

for all $\sigma'_{\theta_i} \in \mathcal{M}(S_i)$.

Theorem 2.2. *Every finite game of incomplete information possesses at least one Bayesian Nash equilibrium [JR06].*

Every finite Bayesian game of incomplete information admits a mixed Bayesian Nash equilibrium. We provide three examples of an incomplete information game, followed by the identification of the games' Bayesian Nash equilibria and mixed strategy Bayesian Nash equilibria.

Example 2.1. Consider a game with two players. Each player has a strategy set with two actions. Player 1 knows that player 2 has two possible types (the world has two possible states).

$$\begin{cases} I = \{1, 2\}, \\ S_1 = \{z, u\}, \\ S_2 = \{v, w\}, \\ \Theta_1 = \{\theta_1^1\}, \\ \Theta_2 = \{\theta_2^1, \theta_2^2\}, \\ p(\theta_2^1|\theta_1^1) = 0.7,\ p(\theta_2^2|\theta_1^1) = 0.3,\ p(\theta_1^1|\theta_2^1) = p(\theta_1^1|\theta_2^2) = 1. \end{cases}$$

$(u_i)_{i \in I}$ are given in Figure 2.4.

		Player 2	
		v	w
Player 1	z	$(3, 5)$	$(0, 4)$
	u	$(0, 7)$	$(3, 6)$

(a) θ_2^1

		Player 2	
		v	w
Player 1	z	$(3, 6)$	$(0, 7)$
	u	$(0, 4)$	$(3, 5)$

(b) θ_2^2

Figure 2.4: A variant of a game in which player 1 knows that player 2 has two possible types, and player 1 has only one type.

Now we use the Nash equilibrium concept in an expanded game, where each of player 2's different types has a different strategy. Playing v is a dominant strategy for type θ_2^1 of player 2 and playing w is a dominant strategy for type θ_2^2 of player 2. Player 1's expected utility by playing z is $0.7 \times 3 + 0.3 \times 0 = 2.1$ and by playing u is $0.7 \times 0 + 0.3 \times 3 = 0.9$, thus the Bayesian Nash equilibrium is pure strategies:
$(\overset{\theta_1^1}{\overset{\downarrow}{z}}, \overset{\theta_2^1}{\overset{\downarrow}{v}}\overset{\theta_2^2}{\overset{\downarrow}{w}})$

Example 2.2. Here, we assume that each player has two possible types.

$$\begin{cases} \Theta_1 = \{\theta_1^1, \theta_1^2\}, \\ \Theta_2 = \{\theta_2^1, \theta_2^2\}, \\ p(\theta_2^1|\theta_1^1) = 0.5,\ p(\theta_2^2|\theta_1^1) = 0.5,\ p(\theta_2^1|\theta_1^2) = 0.5,\ p(\theta_2^2|\theta_1^2) = 0.5, \\ p(\theta_1^1|\theta_2^1) = \frac{2}{3},\ p(\theta_1^2|\theta_2^1) = \frac{1}{3},\ p(\theta_1^1|\theta_2^2) = \frac{2}{3},\ p(\theta_1^2|\theta_2^2) = \frac{1}{3}. \end{cases}$$

$(u_i)_{i \in I = \{1,2\}}$ are given in Figure 2.5.

		Player 2	
		v	w
Player 1	z	$(1, 2)$	$(0, 1)$
	u	$(0, 0)$	$(1, 1)$

(a) θ_1^1, θ_2^1

		Player 2	
		v	w
Player 1	z	$(1, 0)$	$(0, 1)$
	u	$(0, 1)$	$(1, 0)$

(b) θ_1^1, θ_2^2

		Player 2	
		v	w
Player 1	z	$(0, 2)$	$(1, 1)$
	u	$(1, 0)$	$(0, 1)$

(c) θ_1^2, θ_2^1

		Player 2	
		v	w
Player 1	z	$(0, 0)$	$(1, 1)$
	u	$(1, 1)$	$(0, 0)$

(d) θ_1^2, θ_2^2

Figure 2.5: A variant of a game in which each player is unsure of the other player's preferences.

Here, Bayesian Nash equilibria are pure strategies:(zz, vw), (zu, vw), (uz, wv) and (uu, wv). Figure 2.6 shows the expected payoffs for types θ_1^1 and θ_2^1 of player 1, and for types θ_1^2 and θ_2^2 of player 2.

		Player 2			
		vv	vw	wv	ww
Player 1	zz	$(\frac{2}{3}, 1)$	$(\frac{1}{2}, \frac{3}{2})$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{3}, 1)$
	zu	$(1, \frac{5}{6})$	$(\frac{1}{2}, 1)$	$(\frac{1}{2}, \frac{2}{3})$	$(0, \frac{5}{6})$
	uz	$(0, \frac{2}{3})$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{5}{6})$	$(1, \frac{2}{3})$
	uu	$(\frac{1}{3}, \frac{1}{2})$	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, 1)$	$(\frac{2}{3}, \frac{1}{2})$

Figure 2.6: Illustration of expected payoffs.

Example 2.3. We present an example of a Bayesian game, and then obtain its possible mixed strategy

Bayesian Nash equilibria. Consider a game with two players. Here, with probability $\theta_2^1 = \frac{1}{3}$ nature decides payoff matrix as given in Figure 2.7 (a), and with probability $\theta_2^2 = \frac{2}{3}$ nature selects payoff matrix as given in Figure 2.7 (b). Player 1 knows the choice of nature but player 2 does not. In Figure 2.7, the probabilities of playing each strategy are shown i.e. q is the probability player 2 plays v and p is the probability player 1 plays z with payoff matrix (a) and p' is the probability player 1 plays z with payoff matrix (b)[1].

			Player 2	
			q	$(1-q)$
			v	w
Player 1	p	z	$(1,1)$	$(0,0)$
	$1-p$	u	$(0,0)$	$(0,0)$

(a) First payoff matrix - $\theta_2^1 = \frac{1}{3}$

			Player 2	
			q	$(1-q)$
			v	w
Player 1	p'	z	$(0,0)$	$(0,0)$
	$1-p'$	u	$(0,0)$	$(2,2)$

(b) Second payoff matrix - $\theta_2^2 = \frac{2}{3}$

Figure 2.7: Payoff matrix representation for an example of mixed strategy Bayesian Nash equilibrium.

Player 1 would play z for the first payoff matrix if $1q + 0(1-q) > 0$, the results can be summarized as:

$$\begin{cases} q > 0 \Longrightarrow p = 1 & (2.2) \\ q = 0 \Longrightarrow p \in [0,1] & (2.3) \end{cases}$$

and for the second payoff matrix, player 1 would play u if $2(1-q) > 0$;

$$\begin{cases} q < 1 \Longrightarrow p' = 0 & (2.4) \\ q = 1 \Longrightarrow p' \in [0,1]. & (2.5) \end{cases}$$

Player 2 would play v if:

$$\frac{1}{3}[1p + 0(1-p)] + \frac{2}{3}[0p' + 0(1-p')] > \frac{1}{3}[0p + 0(1-p)] + \frac{2}{3}[0p' + 2(1-p')] \Longrightarrow p > 4(1-p').$$

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[1] The idea of this example has been borrowed from Guillermo Ordoñez, Notes on Bayesian Games, ECON 201B - Game Theory, UCLA, February 1, 2006.

Assessing player 2's best response, the results can be summarized as:

$$\begin{cases} p = 4(1-p') \implies q \in [0,1] & (2.6) \\ p > 4(1-p') \implies q = 1 & (2.7) \\ p < 4(1-p') \implies q = 0. & (2.8) \end{cases}$$

We have three cases:

- $q = 0$, then from 2.3 and 2.4 we have $p \in [0,1]$ and $p' = 0$.

  From 2.8 when $p' = 0$, then $p < 4$ which always holds that $q = 0$. Therefore, there are many Bayesian Nash equilibria in which player 2 chooses w and if the first payoff matrix occurs, player 1 chooses $pz + (1-p)u$, where $p \in [0,1]$, and if the second payoff matrix occurs, player 1 chooses u.

- $q = 1$, then from 2.5 and 2.2 we have $p' \in [0,1]$ and $p = 1$.

  From 2.7 when $p = 1$, then p' should be $p' > \frac{3}{4}$ that $q = 1$. Then for $q = 1$, $p' \in (\frac{3}{4}, 1]$ and $p = 1$, there are many Bayesian Nash equilibria in which player 2 chooses v and also player 1 chooses z if the first payoff matrix occurs and $p'z + (1-p')u$, where $p' \in (\frac{3}{4}, 1]$ if the second payoff matrix occurs.

- $q \in (0,1)$, then from 2.2 and 2.4 we have $p' = 0$ and $p = 1$.

  From 2.6, it should be the case that $p = 4(1-p')$, but it is impossible for $p = 4(1-p')$ to hold when both $p' = 0$ and $p = 1$. Thus this case is not a Bayesian Nash equilibrium.

2.5 Iterated Game

In reality, one discord scenario results in a consequent discord scenario. This differs from the one-shot scenarios that occur in the game context. Therefore, it is important that an individual considers the difference between the game context and reality when considering using game theory in reality. Therefore, individuals must bear the consequences that decisions have on further discord scenarios that occur later down the line rather than only taking into account the payoffs that will be received directly after a decision has been taken. This consideration represents the repeated game. According to Taylor [Tay87], repeated games are a type of game in which the stage game (basic game) takes place over a number of rounds.

Repeated games are also known as iterated games. All of these terms refer to an extensive form game, in which there are multiple recurrences of a stage game. In repeated games with an identical group of players, the players' behaviour is significantly different to players' behaviour in one-shot games. There are two kinds of repeated games, as follows:

- *Finitely repeated games*, in which all players are aware of the quantity of repetitions that will be carried out.
- *Infinitely repeated games*, in which the game has no known stopping point. Here, the players behave in a way that lends itself to a never-ending game, or a game that only ceases with a certain level of probability.

Incomplete information can be presented in repeated games, as was first illustrated by Aumann and Maschler [Sor96]. When the game is repeated with only partial information known to all of the players, this indicates an incomplete information repeated game. In games, it is presumed that payoff functions and all potential strategies are known to all players. However, every potential strategy that could be followed is not fully known to each player in reality. Additionally, since players are not aware of certain related elements, there is no way for players to know in advance what payoff will be given if any particular action is carried out by the players. Repeated games with incomplete information pose a potential problem for the player, since s/he might share her/his own private information in the process of maximising its payoffs.

Nash standard existence can be used when a game is either played once or is going to be repeated finitely, as well as when there are a finite number of types and actions. For the infinitely repeated game with a limit of average payoff, there is a proof of existence of Nash equilibrium with lack of information on one side [SST02].

One of the most famous experiments for the repeated game was conducted by Axelrod [Axe84]. Axelrod studied the results of two computer-based Prisoner's Dilemma tournaments. Researchers in a wide range of fields submitted strategies, which were then placed in a round-robin competition to determine the best strategy. Ultimately, a Tit-for-Tat strategy, introduced by Anatol Rapoport achieved the best score. According to this strategy, the player first cooperates and then matches the other player's move.

As this research concerns the most famous game, the Prisoner's Dilemma, the next section will contain further explanation about this game.

2.6 Prisoner's Dilemma

The Prisoner's Dilemma is a fundamental problem of game theory that attempts to mathematically analyse the behaviour of individuals in a strategic situation, in which the success of each individual does not depend entirely on one's choice, but on the opponent's as well [Axe84]. Essentially, it is an abstract formulation of some common situations in which what is best for each person individually leads to mutual defection, whereas everyone would have been better off with mutual cooperation [Axe84]. It has provided a tool for experimental studies in various disciplines such as economics, social psychology, evolutionary biology and fields that are involved with the modelling of social processes, such as behaviour in decision making [Axe84, AH81].

Each of the players competing in the Prisoner's Dilemma has the choice to cooperate (C) or defect (D) and the payoff values gained by the combination of the aforementioned actions are T, R, P and S. Prisoner's Dilemma is a non-zero-sum game where one player's gain (or loss) does not necessarily result in the other players' loss (or gain). The payoff matrix of the Prisoner's Dilemma is presented in Figure 2.8.

		Player 2	
		C	D
Player 1	C	(R, R)	(S, T)
	D	(T, S)	(P, P)

Figure 2.8: Payoff matrix representation for Prisoner's Dilemma.

The values of T, R, P and S satisfy the following two inequalities:

$$T > R > P > S \text{ and } R > \frac{(T + S)}{2}$$

The first equation specifies the order of the payoffs and defines the dilemma, since the best a player can do is get T (i.e. the temptation to defect payoff when the other player cooperates), the worst a player can do is get S (i.e. the sucker's payoff for cooperating while the other player defects), and, in ordering the other two outcomes, R (i.e. the reward payoff for mutual cooperation), is assumed to be better than P (i.e. the punishment payoff for mutual defection). The second equation ensures that, in the repeated game, the players cannot get out of the dilemma by taking turns in exploiting each other. This means that an even chance of exploiting and being exploited is not as good an outcome for a player as mutual cooperation. Therefore, it is assumed that R is greater than the average of T and S [Axe84]. Finally, a special case of the Prisoner's Dilemma occurs when the apparent advantage of defecting over cooperating is not dependent on the opponent's choice, and the disadvantage of the opponent defecting over cooperating is not dependent on one's choice, as can be illustrated in the following equations [Axe84]:

$$T + S = P + R$$

The Prisoner's Dilemma is considered a standard method for modelling social dilemmas [Ost07, Shu70] and has also been used to model conditional altruistic cooperation, which has also been tested by real monetary payoffs [Gin09]. In the 1980's, Axelrod organised two international round-robin tournaments in which strategies for the repeated Prisoner's Dilemma competed with each other- [Axe80a, Axe80b]. In the competition, Tit-for-Tat, i.e., cooperate on the first move and then reciprocate the opponent's last move, proved to be robust and became the overall winner of the tournaments [Axe80a, Axe80b]. Axelrod then promoted Tit-for-Tat, and the four associated characteristics of (i) be nice, (ii) reciprocate, (iii) don't be envious, (iv) don't be too clever, as the way reciprocal altruism has evolved [Axe84].

Algorithmic game theory focusses on algorithmic features of games such as computational complexity. In the next section, we will briefly discuss computational complexity in game theory.

2.7 Complexity

Instead of restricting analysis to abridged abstractions, game theory enables the detailed modelling of actual settings. This is all the more important as assessing the feasibility of a solution concept is vital. Game theory can be defined as strategic decision-making analysis. In the context of a game, a strategy represents a detailed plan outlining all the steps and moves needed to play the game. In the case of a game in normal form, one fundamental problem of algorithmic game theory is the finding of a Nash equilibrium. According to Nash's result [Nas50, Nas51], at least one Nash equilibrium is exhibited by each strategic game. The Nash equilibrium is useful in formulating a game strategy and planning every possible move. One of game theory's complexities is the result of all potential moves. The more strategies that are possibly employed, the greater the computational complexity will be in obtaining the Nash equilibrium, and therefore the number of strategies can be considered as an indicator of complexity. Lemke and Howson [LH64] formulated a now well-known algorithm for the calculation of a Nash equilibrium in two player non-zero sum games which revealed that the complexity is no more than exponential. The computational complexity is unidentified not only in the general case of two player non-zero sum games, but also in the case of symmetric two player games and the pure strategy Bayesian Nash equilibria [NGNP09].

There are two main categories of computational complexity, namely, those associated with a polynomial-time algorithm and those characterised by NP-hardness. However, since Nash's theorem specifies that the equilibrium of each game is may be mixed, it is impossible to implement the notion of NP-hardness to every scenario. It has become clear in a number of papers that computation of a Nash equilibrium or even an approximate ϵ-Nash equilibrium is in general a computationally hard problem [DGP06, EY10]. It has been proved that finding a Nash equilibrium is complete for the complexity class PPAD (Polynomial Parity Arguments on Directed graphs) [DGP06]. In the case of Bayesian games, it is proven that determining whether a pure strategy Bayesian Nash equilibrium exists is NP-complete [CS08]. To the best of our knowledge there is not a considerable number of research articles on computational complexity in obtaining the Nash equilibrium in the games with incomplete information.

Recalling the idea of changing types within multi-games, changing payoffs and then the Nash equilibrium of a game has also been considered in [RG05]. A summary of their work is as follows:

Robinson-Goforth Topology of 2×2 Games

Robinson and Goforth introduced a classification of the $2{\times}2$ games with a topological structure [RG05]. They showed that 2×2 games can be transformed by swaps in adjoining payoff ranks. They presented a numbering system to recognize and prioritize 144 strict ordinal games. Rapoport and Guyer [RG78] were the first to note that, with respect to 2×2 games, there are 576 ways to arrange two sequences of four distinct numbers in a bi-matrix (A bi-matrix is a matrix showing payoffs for both players in a single cell). We let

$$\begin{pmatrix} R & S \\ T & P \end{pmatrix}$$

be a matrix payoff to determine a 2×2 game. Rank ordering of the four payoff values R, S, T, P determine the characteristics of the game. Figure 2.9 shows that each 2×2 game is characterized by the ranking of the payoffs S, R, T and P with $R = 1$, $P = 0$. The ranking ordering partitions the (S, T) plane, which displays 12 symmetrical 2×2 games. Considering that we have assumed the values of R and P to be 1 and 0 respectively, the diagram shows which game we would have considering varying values of T and S to create regions of different inequalities. For example, where $S < 0$ and $T > 1$, we get $T > R > P > S$ which corresponds to the top left region and the Prisoner's Dilemma game. Not all regions have been named [Hau02].

Here, we present an example of transformation according to Robinson and Goforth's classification. Figure 2.10 shows a transformation of Prisoner's Dilemma to Alibi Game[2]. Let the payoff most preferred by player 2 be designated with ordinal 4 in the first game. If the payoff 3 becomes more and more attractive to player 2, it will eventually be preferred to the outcome with ordinal 4. When this switch in preference occurs, the effect on the payoff matrix is to exchange the positions of the 3 and 4 in the matrix for player 2. Therefore, we have a new game.

Caveat: In this research, we use the word "payoff" to mean expected payoff and actual payoff, even

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[2] Alibi game was introduced by Robinson and Goforth in 2004 as an asymmetric variant of the classic Prisoner's Dilemma.

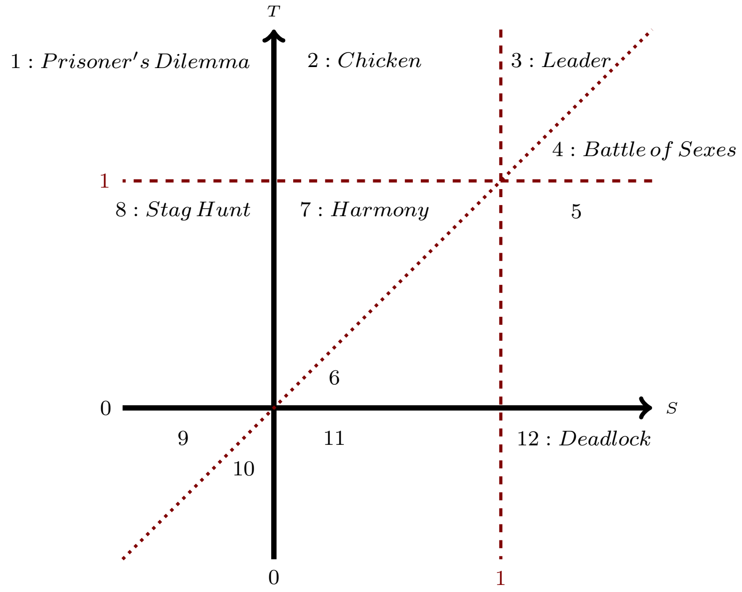


Figure 2.9: Partitioning the (S, T) plane, which displays 12 symmetrical 2×2 games.

		Player 2	
		C	D
Player 1	C	$(3, \mathbf{3})$	$(1, \mathbf{4})$
	D	$(4, 1)$	$(2, 2)$

(a) Prisoner's Dilemma

		Player 2	
		C	D
Player 1	C	$(3, \mathbf{4})$	$(1, \mathbf{3})$
	D	$(4, 1)$	$(2, 2)$

(b) Alibi Game

Figure 2.10: Exchanges the ordinal values for the two payoffs for player 2.

though the two definitions vary. Also we focus principally on non-cooperative strategic form games with incomplete information, which involve only a finite number of rational players, and which give each player only a finite number of actions to choose from.

Chapter 3

Multi-Games

In this chapter, we define the multi-games. We will then discuss the difference between a poly-matrix game and a multi-game, before going on to illustrate some examples of multi-games in the case of double games.

3.1 Formal Definition of Multi-Games

A multi-game is defined as follows. Consider M finite N-player games $\mathcal{G}_{j\in J}$ where $J = \{1, 2, ..., M\}$ with the strategy set S_{ij} and payoff matrix u_{ij} for player $i \in I$ where $I = \{1, 2, ..., N\}$ is finite, in the game $\mathcal{G}_j$. Assume that each player i is equipped with a set of M weights θ_{ij} with $\sum_{j=1}^{M} \theta_{ij} = 1$. We define the multi-games N-player game $\mathcal{G}$ with M basic games $\mathcal{G}_j$ as the finite strategy game with players $i \in I$ each having the strategy set $\prod_{j\in J} S_{ij}$ and payoff

$$u_i\left(\prod_{j\in J} s_{ij}\right) = \sum_{j\in J} \theta_{ij} u_{ij}(s_{ij})$$

for strategy profile $s_{ij} \in S_{ij}$ and possibly incomplete information (types) θ_{ij} for $j \in J$. We say the multi-game is *uniform* if for each player i, the set S_{ij} is independent of the game $\mathcal{G}_j$, i.e., we can write $S_{ij} = S_i$, and $s_{ij} = s_{ij'}$ for $j, j' \in J$. We assume that the values 0 and 1 will always be included in the set of possible types for each player, which we call the *extreme* types. Consider that the number of games can be different for the players, so players assign the weight 0 to some of the games.

Here, we explain the case of the multi-game with two players and M games by using matrices of payoffs. Assume that the strategy set for player i consists of actions $s_i \in S_i$ and we denote the weights for players 1 and 2 respectively by θ_{1j} and θ_{2j}, with $j \in J$ where $J = \{1, 2, ..., M\}$. If the payoff matrix for the basic game $\mathcal{G}_j$ is given as in Figure 3.1, then the payoff matrix for the multi-game $\mathcal{G}$ will be given as in Figure 3.2.

		Player 2	
		s_{21}	s_{22}
Player 1	s_{11}	(a_{1j}, a_{2j})	(b_{1j}, b_{2j})
	s_{12}	(c_{1j}, c_{2j})	(d_{1j}, d_{2j})

Figure 3.1: Payoff matrix of the basic games.

	s_{21}	s_{22}
s_{11}	$\sum_{j=1}^{M} \theta_{1j}a_{1j}, \sum_{j=1}^{M} \theta_{2j}a_{2j}$	$\sum_{j=1}^{M} \theta_{1j}b_{1j}, \sum_{j=1}^{M} \theta_{2j}b_{2j}$
s_{12}	$\sum_{j=1}^{M} \theta_{1j}c_{1j}, \sum_{j=1}^{M} \theta_{2j}c_{2j}$	$\sum_{j=1}^{M} \theta_{1j}d_{1j}, \sum_{j=1}^{M} \theta_{2j}d_{2j}$

Figure 3.2: Payoff matrix for the multi-game.

If $\theta_{1j} = 1$, this means that the first player invests totally in the game $\mathcal{G}_j$ whereas when $\theta_{1j} = 0$, the first player does not invest anything in the game $\mathcal{G}_j$. Similarly for the second player with weight θ_{2j}.

At first glance, an N-player multi-game may seem similar to poly-matrix games, therefore, in the next section we briefly explain about poly-matrix games.

3.2 Poly-Matrix Games

A poly-matrix game [Yan68] is an N-player non-zero sum, non-cooperative game, where the utility of each player is the sum of utilities influenced by her/his interactions with each of the $N - 1$ other players. In this game, each player plays a 2-player game with each other player, and her/his strategies are the same in each of these games; the utilities are then added.

The number of players is $N \geq 2$, each player $i \in I = \{1, ..., N\}$ has a finite set of pure strategies $S_i = \{s_i^1, ..., s_i^{t_i}\}$ where $|S_i| = t_i$. For player i and each other player m, if player i chooses pure strategy s_i^k and player $m \in I, m \neq i$ chooses pure strategy s_m^l, according to [How72, ABH06], it is possible to assign a partial payoff $a_{i \neq m}^{im}(s_i^k, s_m^l)$ for player i. If $(s_1, ..., s_N)$ is the vector of pure strategies chosen by players $1, ..., N$, for any pure strategies $(s_1, ..., s_N)$, the total payoff for player $i \in I$ is

$$A_i(s_1, ..., s_N) = \sum_{i \neq m} a^{im}(s_i^k, s_m^l).$$

Let $t_i \times t_m$ matrix $A_{im} = (a_{im}^{kl})$ denote the matrix of partial payoffs to player *i* resulting from the choices of pure strategies made by her/him and player *m*. Therefore, player *i*'s payoff with respect to player *m*'s decisions does not depend on any other players' strategies. In a poly-matrix game each player *i* attempts to maximize her/his own total payoff by choosing a mixed strategy vector X_i over her/his set of pure strategies such that $(X_i)^T = (x_i^1, ..., x_i^{t_i})$. If $X = (X_1, ..., X_N)$ is a set of mixed strategies for the N players, then the expected payoff to player *i* is

$$E_i(X) = (X_i)^T \sum_{m \neq i} A_{im} X_m = \sum_{m \neq i} \sum_{k=1}^{t_i} \sum_{l=1}^{t_m} \alpha_{im}^{kl} x_i^k x_m^l.$$

According to [ABH06], the mixed strategies $X^* = (X_1^*, ..., X_i^*, ..., X_N^*)$ can be recognised as a Nash equilibrium of the poly-matrix game if and only if for any other *N*-tuple $X = (X_1^*, ..., X_{i-1}^*, X_i, X_{i+1}^*, ..., X_N^*)$,

$$(X_i^*)^T \sum_{m \neq i} A_{im} X_m^* \geq (X_i)^T \sum_{m \neq i} A_{im} X_m^*, \text{ for } i \in N.$$

A poly-matrix game has at least one Nash equilibrium [Nas50]. The equilibria of a poly-matrix was studied by Yanovskaya [Yan68] and the problem of computing an equilibrium for a poly-matrix game has been considered by Howson [How72], Quintas [Qui89] and Eaves [Cur73]. Howson and Rosenthal [HR74] showed the equivalence of a Bayesian Nash equilibrium of 2-player games with incomplete information and a Nash equilibrium of *N*-player poly-matrix games. Now, we provide a simple example of a poly-matrix game.

Example 3.1. Consider a game with three players. Each player has a strategy set with two actions.

$$\begin{cases} I = \{1, 2\} \\ S_1 = \{z, u\} \\ S_2 = \{v, w\} \\ S_3 = \{x, y\} \end{cases}$$

Figures 3.3(a), (b) and (c) are payoff matrices for each pair of players. $(u_i)_{i \in I}$ are given in Figure 3.3(d).

Player 2

		v	w
Player 1	z	(4,2)	(0,3)
	u	(3,5)	(2,4)

(a)

Player 3

		x	y
Player 1	z	(4,2)	(2,1)
	u	(3,1)	(2,3)

(b)

Player 3

		x	y
Player 2	v	(2,4)	(2,1)
	w	(1,1)	(3,2)

(c)

Player 2, Player 3

		vx	vy	wx	wy
Player 1	z	(8,4,6)	(6,4,2)	(4,4,3)	(2,6,3)
	u	(8,7,5)	(5,7,4)	(5,5,2)	(4,7,5)

(d)

Figure 3.3: Payoff matrices representation for an example of poly-matrix game.

$(\frac{1}{2}, \frac{1}{2})(1, 0)(0, 1)$ is a mixed strategy Nash equilibrium of the poly-matrix game.

In an N-player multi-game, the payoff for player $i \in I = \{1, 2, ..., N\}$, relative to the decision of player $m \in I$, $m \neq i$ is dependent on other players' choices. While in a poly-matrix game, the partial payoff to player *i*, when player *i* chooses her/his strategy in connection with player *m*'s decision is not correlated with any choice of strategies made by the other players.

3.3 Some Applications of Multi-Games

Here, we explain some applications of multi-games through the use of some examples. In Chapter 5, we will apply the framework of multi-games to obtain a double game extension of the Prisoner's Dilemma in order to model pro-social behaviour. We will then discuss its Bayesian Nash equilibrium.

Now we present an example of a multi-game through applying transactional analysis.

Example 3.2. *(Transactional analysis)*

Assume that we aim to develop a model of irrational social conflict between two hostile groups in psychological but not material war against each other. We combine ideas from transactional analysis (a model of human interaction) using a double game in which each player is characterized by a moral type that measures the moral consciousness of the player.

In "irrational" social conflicts, two rival groups fight each other not for any material gain but for purely psychological reasons based on the antagonistic identities of the two groups. The question is: How can we model such psychological war in the double game?

Sigmund Freud has established theories on human personality. Freud's theories on modelling the structure of personality have helped to develop what is called transactional analysis, a popularization of classical psychoanalysis by the American psychologist Eric Berne, who maintained Freud's notion of personality [Ber96].

Berne describes the social interactions between people and the way they enter social games without even being aware of this complicated process. Berne defines games as a process of interconnected transactions that lead to a specific result for both parties in the game. Transactional analysis is fundamentally defined as a method of studying interaction amongst individuals.

Berne introduced three ego states. Ego states represent recurring sequences of individual human behaviour, emotions, and cognition. The ego states are Parent, Adult and Child as described in the following:

- *Parent*: Berne stated that the maximum amount of retention of events in a child's mind is that of the parent because parents are the most vital people in a child's life, hence the first ego state is referred to as the parent.

- *Child*: As compared to parents, children record the sentiments or the feelings associated with the events in their mind.
- *Adult*: It is known as the last ego stage. When a child is about one year old, s/he begins to display signs of adult behaviour. For instance, s/he thinks that s/he can hold a cup of tea in her/his hand. This behaviour develops in the child so that s/he can learn to differentiate between the child and parent's behaviour. Berne intrinsically defines adult's behaviour as filtering information or data based on past experience.

Transactional analysis, as a theory, is associated with personality and explains the psychological structure of individuals and how personality is formed in individuals. Transactional analysis is also a communication theory, which can be used for examining different types of behaviour between individuals. Transactional analysis can be used to explain the growth of maturity in a child because the first signs of age development in a child begin from childhood.

Interactions occurring between two or more people involve a variety of stimuli and responses that form the core of the person's behaviour. Figure 3.4 provides sample transactions between the parent, adult, or child of one person and the parent, adult, or child of another (represented by P, A and C respectively).

Now we define a game with two players and consider all payoffs that could result for one of the players, in a conflict where each player takes one of the defined ego states.

Let $p_i^{XX'}$ be the payoff value gained by player i when s/he plays with ego state X and the other player plays with ego state X', $(X, X' \in \{A, C, P\})$. The payoff matrix is given in Figure 3.5.

In the full transactional model, each player has a choice of nine types of transactions corresponding to the nine pairs of ego states the two players can adopt. In order to simplify the illustration and explanation of this example, the parent ego state has been disregarded among the nine pairs of transactions.

Parent to Parent (PP)	Shall we stop mutual condemnation for a while?
Parent to Adult (PA)	Let me tell you how bad mutual condemnation is.
Parent to Child (PC)	Never condemn others.
Adult to Parent (AP)	What shall I do to stop mutual condemnation?
Adult to Adult (AA)	Let's stop mutual condemnation
Adult to Child (AC)	You are condemning me.
Child to Parent (CP)	Please don't condemn me.
Child to Adult (CA)	Why do you condemn me?
Child to Child (CC)	You condemn me so I condemn you.

Figure 3.4: Some examples of transactions.

		Player 2		
		A	C	P
Player 1	A	(p_1^{AA}, p_2^{AA})	(p_1^{AC}, p_2^{AC})	(p_1^{AP}, p_2^{AP})
	C	(p_1^{CA}, p_2^{CA})	(p_1^{CC}, p_2^{CC})	(p_1^{CP}, p_2^{CP})
	P	(p_1^{PA}, p_2^{PA})	(p_1^{PC}, p_2^{PC})	(p_1^{PP}, p_2^{PP})

Figure 3.5: Payoff matrix when players play with their particular ego states.

We assume that the strategy taken by each of the players comes from a pair of ego states; one belonging to the player, and the other to the opponent. For instance, parents to parents PP; means when a player chooses this strategy, s/he uses her/his parent ego state and addresses it to the opponent's parent ego state. Similarly, for child to adult CA, a player chooses a child ego state and addresses it to the opponent's adult ego state.

It is assumed that the first game consisting of actions AA and CC is a Prisoner's Dilemma, with AA corresponding to "cooperation" and CC to "defection". The consequence of this assumption in the model is a symmetric series of payoff values for the payoff matrix. Figure 3.6 shows a self-evaluated game in which strategies consist of four transactions belonging to the combinations of adult and child ego states as well as their related summarized payoff values.

It is assumed that the values in each row and each column of the payoff matrix is either monotonically

		Player 2			
		AA	AC	CA	CC
Player 1	AA	(a, a)	(f, e)	(h, g)	(c, b)
	AC	(e, f)	(l, l)	(s, r)	(u, t)
	CA	(g, h)	(r, s)	(v, v)	(z, y)
	CC	(b, c)	(t, u)	(y, z)	(d, d)

Figure 3.6: Payoff matrix due when the player strategy taken results from a pair of ego states.

increasing or monotonically decreasing as required by the conditions of the Prisoner's Dilemma.

$$c < d < a < b \qquad c < h < f < a \qquad u < s < l < e \qquad z < v < r < g \qquad d < y < t < b$$

$$a < e < g < b \qquad f < l < r < t \qquad h < s < v < y \qquad c < u < z < d$$

Another step towards simplifying the problem at this stage is through a reduction of the variety of payoff parameters. This is achieved by replacing some payoff parameters with a linear combination of the others without losing their monotonic order. By applying the following assignments only four parameters remain;

$$e = r = l = \frac{(a+b)}{2} \qquad g = t = b \qquad f = a \qquad h = \frac{(a+c)}{2} \qquad v = s = \frac{(a+d)}{2}$$

$$u = \frac{(c+d)}{2} \qquad y = \frac{(b+d)}{2} \qquad z = d$$

Therefore, the matrix payoff would resemble Figure 3.7.

		Player 2			
		AA	AC	CA	CC
Player 1	AA	(a, a)	$(a, \frac{a+b}{2})$	$(\frac{a+c}{2}, b)$	(c, b)
	AC	$(\frac{a+b}{2}, a)$	$(\frac{a+b}{2}, \frac{a+b}{2})$	$(\frac{a+d}{2}, \frac{a+b}{2})$	$(\frac{c+d}{2}, b)$
	CA	$(b, \frac{a+c}{2})$	$(\frac{a+b}{2}, \frac{a+d}{2})$	$(\frac{a+d}{2}, \frac{a+d}{2})$	$(d, \frac{b+d}{2})$
	CC	(b, c)	$(b, \frac{c+d}{2})$	$(\frac{b+d}{2}, d)$	(d, d)

Figure 3.7: Payoff matrix of the basic game for the example of transactional analysis.

It is desirable to enrich the payoff model through the addition of another consideration regarding the source of payoff values after the two players have taken a strategy. So far it has been considered that in essence only the players themselves can give payoff values to the result of strategies, taken by both themselves and their opponents. The role of a 'third party' in this evaluation has been disregarded.

At this stage, it is intended to involve a third party's role in the evaluation. The players' values have an effect on their decision making. Their values, beliefs and attitudes are developed throughout the course of their lives, family and friends. Specifically social expectations and the experiences they have had can all contribute to how they make a decision. Here, we introduce a game called society-evaluated that represents social judgement or expectations.

In general, members of society are assumed to regard strategies taken from an adult oriented ego state as being more constructive than from a child ego state. Under this assumption, we consider a moral game, in which the highest moral payoff value is given for strategies with the most number of adult ego state elements in a player's strategy. Figure 3.8 accommodates these payoffs for the moral game with *m* as a positive value. In assigning moral payoffs in conjunction with the properties of the first matrix, it has been assumed that $c < d < a < m < b$.

		Player 2			
		AA	AC	CA	CC
Player 1	AA	(m, m)	$(\frac{m}{2}, m)$	$(\frac{-m}{2}, m)$	$(-m, m)$
	AC	$(m, \frac{m}{2})$	$(\frac{m}{2}, \frac{m}{2})$	$(\frac{-m}{2}, \frac{m}{2})$	$(-m, \frac{m}{2})$
	CA	$(m, \frac{-m}{2})$	$(\frac{m}{2}, \frac{-m}{2})$	$(\frac{-m}{2}, \frac{-m}{2})$	$(-m, \frac{-m}{2})$
	CC	$(m, -m)$	$(\frac{m}{2}, -m)$	$(\frac{-m}{2}, -m)$	$(-m, -m)$

Figure 3.8: Payoff matrix of the moral game for the example of transactional analysis.

The last task is to combine the payoff values of the first game and the second game. This is an instance of the double game in which each player is characterized by her/his type for each game. The payoff values of the double game are given in Figure 3.9.

A player may choose to value social judgement over self evaluation; thus behaving in a way that will encourage society to have admiration or approval for the player. However, a player may also decide that her/his personal beliefs and values are more important than social expectations, resulting in the player acting on personal expectations.

		Player 2			
		AA	AC	CA	CC
Player 1	AA	$((1-\theta_1)a+\theta_1 m,$ $(1-\theta_2)a+\theta_2 m)$	$((1-\theta_1)a+\theta_1\frac{m}{2},$ $(1-\theta_2)\frac{a+b}{2}+\theta_2 m)$	$((1-\theta_1)\frac{a+c}{2}-\theta_1\frac{m}{2},$ $(1-\theta_2)b-\theta_2 m)$	$((1-\theta_1)c-\theta_1 m,$ $(1-\theta_2)b+\theta_2 m)$
	AC	$((1-\theta_1)\frac{a+b}{2}+\theta_1 m,$ $(1-\theta_2)a+\theta_2\frac{m}{2})$	$((1-\theta_1)\frac{a+b}{2}+\theta_1\frac{m}{2},$ $(1-\theta_2)\frac{a+b}{2}+\theta_2\frac{m}{2})$	$((1-\theta_1)\frac{a+d}{2}-\theta_1\frac{m}{2},$ $(1-\theta_2)\frac{a+b}{2}+\theta_2\frac{m}{2})$	$((1-\theta_1)\frac{c+d}{2}-\theta_1 m,$ $(1-\theta_2)b+\theta_2\frac{m}{2})$
	AC	$((1-\theta_1)b+\theta_1 m,$ $(1-\theta_2)\frac{a+c}{2}-\theta_2\frac{m}{2})$	$((1-\theta_1)\frac{a+b}{2}+\theta_1\frac{m}{2},$ $(1-\theta_2)\frac{a+d}{2}-\theta_2\frac{m}{2})$	$((1-\theta_1)\frac{a+d}{2}-\theta_1\frac{m}{2},$ $(1-\theta_2)\frac{a+d}{2}-\theta_2\frac{m}{2})$	$((1-\theta_1)d-\theta_1 m,$ $(1-\theta_2)\frac{b+d}{2}-\theta_2\frac{m}{2})$
	CC	$((1-\theta_1)b+\theta_1 m,$ $(1-\theta_2)c-\theta_2 m)$	$((1-\theta_1)b+\theta_1\frac{m}{2},$ $(1-\theta_2)\frac{c+d}{2}-\theta_2 m)$	$((1-\theta_1)\frac{b+d}{2}-\theta_1\frac{m}{2},$ $(1-\theta_2)d-\theta_2 m)$	$((1-\theta_1)d-\theta_1 m,$ $(1-\theta_2)d-\theta_2 m)$

Figure 3.9: Payoff matrix of the double game for the example of transactional analysis.

If the values of types a, b, c, d and m were known, then Nash equilibrium would present the best strategies for both players, for given values of θ_1 and θ_2. In other words, different types lead to different Nash equilibria for the two players. Figure 3.10 shows the different regions of θ_1 and θ_2 where different pure Nash equilibria appear for assumed values $a = 0$, $b = -1$, $c = 1$, $d = \frac{-1}{2}$ and $m = \frac{1}{2}$ (shown in different colours).

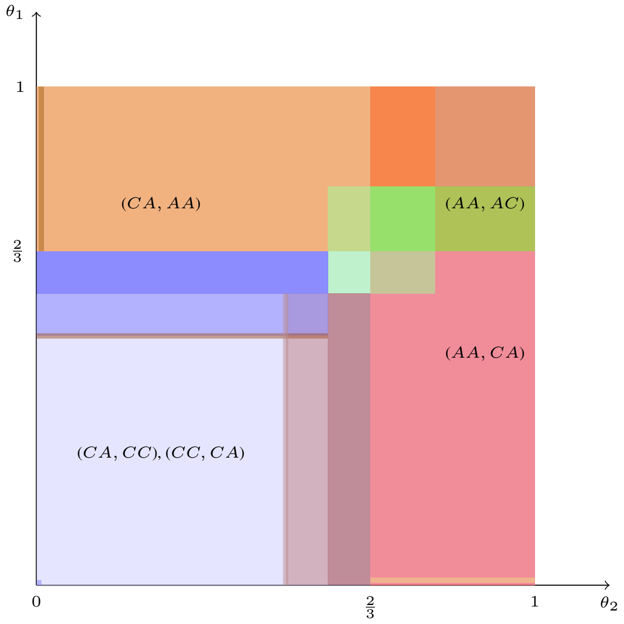


Figure 3.10: Variation of Nash equilibrium in the different regions of θ_1 and θ_2 for the example of transactional analysis.

Now, we present an example that uses gamification in a double game.

Example 3.3. *(Gamification)*

Recent developments in e-commerce have focused on gamification and the impact this can have on human's behaviour. Gamification can be defined as the integration of motivational games or game-like activities into the corporate domain. The effectiveness of the approach has been subject to extensive debate amongst theorists and business owners. Gamification has been incorporated into various aspects of business structure, including HR, skills training, health and safety, customer engagement and research and development [Wer14]. Gamification is used to manipulate end-user behaviour by providing them with extrinsic and intrinsic rewards. Motivation that is prompted by a desire for tangible rewards is defined as extrinsic whereas motivation that is prompted by an inherent passion or interest is defined as intrinsic.

We can use the idea of multi-games in the gamified systems with two games, where one game is a serious game and another game is an entertainment game [AD14]. Here we present a simple example of the double game with gamification in the context of Group Project Assessment in which challenges, points and levels can all create rewards which a player may or may not find appealing to strive towards.

In [Pit00] suggested the application of game theory into a common-practice method of assessment within academic environments - group projects. In a typical group project, a team of two students or more, are given a task to be carried out jointly. The task could be a range of group activities such as carrying out experiments or making presentations. The arrangements of these groups could be random, self selected or, more commonly, selected so that the groups are of equal ability and contain a range of students with varying capability. Generally, the students can either be given the same mark, or obtain marks based on their own relative contribution. These however introduce some problems such as lack of cooperation, time wasting, and marking difficulties.

In a group with a bright student, it would seem logical that the highest achieving student did all or most of the work. This way both the low achieving students in the group would get a good mark, and the bright student would also ensure the work was not affected by the others. This discourages cooperation and team work within students. To battle this, many academics introduce marks for either teamwork or relative contribution. There are however few ways of knowing if the group worked as a team. Asking them could lead to students pretending they worked as a team. Marks for relative effort may also result in players being shunned out of the group in order to maximise contribution or lying about contribution, both of which go against the purpose of a group project.

In the serious game, we introduce a group project where groups of three students have to write a report over six weeks on the assigned topic. Each group consists of a bright, average and poor student, who are all trying to get the largest payoff possible (best grades). The final overall mark for the six week project is given to each student as their final grade for the project.

We assume that if a bright student puts in a good effort, he will gain a marks for the overall project and b marks if he puts in a weak effort. For the average student, we assume $\frac{a}{2}$ and $\frac{b}{2}$ for good and weak effort respectively. However, we assume in the case of the poor student that a good effort, meaning he takes responsibility for a large part of the project, will negatively affect the quality of the work as the others could have done it better, therefore resulting in $\frac{-a}{4}$. However, if he puts in a weak effort, he will not affect the quality of the work, contributing 0 marks to the project. In order to ensure that no player scores a negative mark in the project, we assume $a, b \geq 0$ and $a \leq 6b$. Therefore, the matrix payoff would resemble Figure 3.11. The Nash equilibrium for this game is achieved when both the bright student and average student choose the good effort strategy and the poor student chooses the weak effort strategy.

		Average student				
		Good effort	Weak effort	Good effort	Weak effort	
Bright student	Good effort	$(\frac{5a}{4}, \frac{5a}{4}, \frac{5a}{4})$	$(\frac{3a+2b}{4}, \frac{3a+2b}{4}, \frac{3a+2b}{4})$			Good effort
	Weak effort	$(\frac{4b+a}{4}, \frac{4b+a}{4}, \frac{4b+a}{4})$	$(\frac{6b-a}{4}, \frac{6b-a}{4}, \frac{6b-a}{4})$			
	Good effort			$(\frac{6a}{4}, \frac{6a}{4}, \frac{6a}{4})$	$(\frac{4a+2b}{4}, \frac{4a+2b}{4}, \frac{4a+2b}{4})$	Weak effort
	Weak effort			$(\frac{4b+2a}{4}, \frac{4b+2a}{4}, \frac{4b+2a}{4})$	$(\frac{6b}{4}, \frac{6b}{4}, \frac{6b}{4})$	Poor student

Figure 3.11: Payoff matrix representation for serious game.

For our second game, we look to gamification. Challenges are the most integral feature of a gamified group project as tasks are assigned to each player and rewards offered if they are successful. This increases motivation as students perceive each challenge as a further step towards completion of the task. Point systems are also important as points are awarded based on the quality of the completed task. To ensure that a gamified group project is effective, it must offer rewards that appeal to the students, as different players will be motivated by different factors. A gamified group project must also consider the social aspect of participation by providing leader-boards to monitor progress and facilitate an increased level of commitment and competition. The success of each student depends on the quality of her/his decisions. The majority of individuals require recognition or acknowledgement for their achievements and they can achieve this by performing well or being committed in a series of tasks. If their effort is recognised, their level of motivation will naturally increase.

Similar to Example 3.3 we can say that a gamified group project can be considered to be a double game comprised of a group project task and a designed gamification. Similarly to how social judgement and expectations influenced a player's decisions in the moral judgement game, students will make decisions based on how s/he regards the teacher's approval. In this example, it is a sense of personal achievement, teacher recognition, glory or self-satisfaction that can affect a player's decision-making approach.

For our entertainment game, we propose that the teacher takes a test every week for the six weeks of the project on the topic of the project and records the hours each student attends after school sessions to complete the group project. This data is then turned into a rank for a leader-board containing every student in the group. The top half of the class are put in the ‘ positive’ category and the lower half in the ‘negative’ category. This would act as motivation for the players to attend more sessions and work on the project more in order to receive both recognition and praise as well as a sense of achievement in beating others.

Here, we give values to the self-satisfaction given to students based on whether they fall in the positive or negative category based on their rank which takes into account their test results and after class attendance. It is assumed that if a player chooses the good effort strategy, s/he is guaranteed to be in the positive category and if s/he chooses a weak strategy, s/he is guaranteed to be in the negative category. For a bright student, if s/he falls in the positive category because s/he chose the good effort strategy, s/he will have a $\frac{m}{2}$ payoff value when the poor student is also in that category, as s/he is expecting themselves to do better and will not have a large payoff. If s/he falls into the negative category by choosing the weak effort strategy s/he will have a $-m$ payoff value because s/he will be dissatisfied by the result when the poor student has done better, and s/he has performed below expectations. The same applies to the average student in this case, with the same payoff values.

Furthermore, for both the average and bright students, if they are the only student in their group with the good effort strategy, their payoff will be m as they will have the satisfaction of being the only student form the group in the positive category. For the poor student, her/his payoff will be m when s/he has chosen the good effort strategy regardless of the other students and 0 if s/he has chosen the weak effort strategy. Figure 3.12 accommodates these payoffs for the entertainment game with *m* as a positive value.

		Average student					
		Good effort	Weak effort	Good effort	Weak effort		
Bright student	Good effort	$(\frac{m}{2}, \frac{m}{2}, m)$	$(\frac{m}{2}, -m, m)$			Good effort	Poor student
	Weak effort	$(-m, \frac{m}{2}, m)$	$(-m, -m, m)$				
	Good effort			$(m, m, 0)$	$(m, \frac{-m}{2}, 0)$	Weak effort	
	Weak effort			$(-m, m, 0)$	$(-m, \frac{-m}{2}, 0)$		

Figure 3.12: Payoff matrix representation for entertainment game.

Here, a player may choose to regard highly the sense of achievement, reward or competition that comes with gamification, rather than not give much importance to them. If the rewarding aspects of gamification offered in a game appeal to them, they may choose to pursue the task further than they would if they did not care for the rewards or status. In the double game, a player may choose to change her/his type so that s/he finds the competitiveness of a leader-board ranking effort to be an exciting prospect, and therefore makes decisions to put in more effort than s/he initially had. A player may also choose to regard the initial serious game more highly because of her/his personal interest in the mark obtained for it. The payoff values of the double game are given in Figure 3.13 for assumed values $a = 4$, $b = 2$ and $m = 2$.

		Average student					
		Good	Weak	Good	Weak		
Bright student	Good	$(5 - 4\theta_1, 5 - 4\theta_2, 5 - 3\theta_3)$	$(4 - 3\theta_1, 4 - 6\theta_2, 4 - 2\theta_3)$			Good	Poor student
	Weak	$(3 - 5\theta_1, 3 - 2\theta_2, 3 - \theta_3)$	$(2 - 4\theta_1, 2 - 2\theta_2, 2)$				
	Good			$(6 - 4\theta_1, 6 - 4\theta_1, 6 - 6\theta_3)$	$(5 - 3\theta_1, 5 - 6\theta_2, 5 - 5\theta_3)$	Weak	
	Weak			$(4 - 6\theta_1, 4 - 2\theta_2, 4 - 4\theta_3)$	$(3 - 5\theta_1, 3 - 4\theta_2, 3 - 3\theta_3)$		

Figure 3.13: Payoff matrix representation for the double game in the gamification example where $a = 4$, $b = 2$ and $m = 2$.

Different types lead to different Nash equilibria for the two players. Figure 3.14 shows the pure Nash equilibria for the gamification example where types θ_1, θ_2 and θ_3 have values of $0,\ 1,$ and $\frac{1}{2}$. On the figure, the Nash equilibria for different values of θ_1, θ_2 and θ_3 are shown. For example, a value of 1 for θ_1, $\frac{1}{2}$ for θ_2 and 0 for θ_3 correspond to GGW. This means that players one and two should have a good effort strategy and player three should have a weak effort strategy.

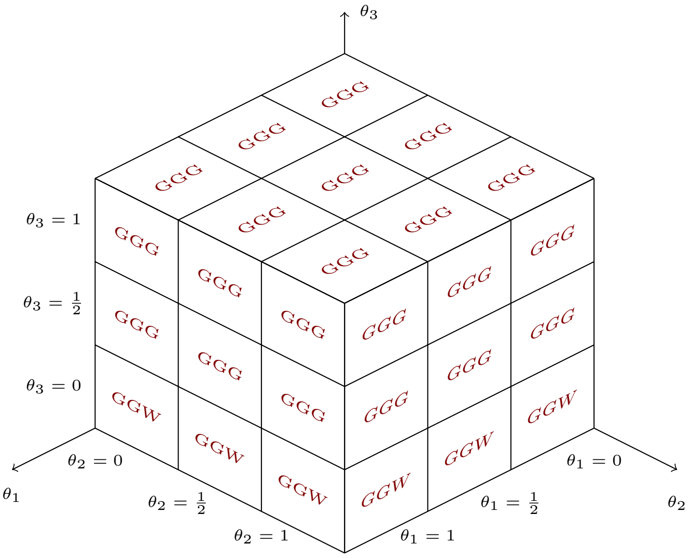


Figure 3.14: The pure regular strategies for the types with assumed values for Example 3.3.

Now if the types are all private information and can each take only a finite number of values between 0 and 1, then the double game is reduced to a Bayesian game with a finite set of types for the three players and we can look for a Bayesian Nash equilibrium.

Chapter 4

Double Games with *N* Players

> In this chapter, we introduce the N-player double game, in which each player has the same set of strategies in two basic games. We later define the class of *pure regular* double game in which for pairs of extreme types there are 2^N pure Nash equilibrium in which the strategy of each player only depends on its own type. Similarly, we define the notion of a *completely pure regular* double game where there are pure Nash equilibrium for all possible pairs of types for the N players, and in which the strategy of each player only depends on its own type. We then derive a test for establishing that a double game is completely pure regular with computational complexity independent of the number of types and actions. We also show that a pure Bayesian equilibrium for a completely pure regular double game can be obtained directly from this test, thus reducing the complexity of the computation.

We suppose that the multi-game is uniform with $M = 2$. We now have two basic games $\mathcal{G}_1$ and $\mathcal{G}_2$ and N players. For finite sets of types, we assume that the weights θ_{ij} for each player $i \in I$ and each game $j \in J$ are selected from finite discrete sets $I = \{1, 2, \ldots, N\}$ and $J = \{1, 2\}$, which would denote the types of each player i for a given game j. As $M = 2$, we can use only θ_i instead of θ_{ij} ($\theta_{11} = \theta_1, \theta_{12} = 1 - \theta_1, \theta_{21} = \theta_2$ and $\theta_{22} = 1 - \theta_2$). When these values represent private information, we have a Bayesian game.

4.1 Coherent Pairs of Nash Equilibrium

In the finite discrete case, the finite set of types for each player is given by a set of increasing values, say θ_i^k $(1 \leq k \leq \ell_i)$ where ℓ_i is the number of types for player $i \in I = \{1, 2, \ldots, N\}$, and each type is restricted to its unit interval. We assume that in the discrete case, we always have 0 and 1 as types for each player, i.e., $\theta_i^1 = 0$ and $\theta_i^{\ell_i} = 1$, which we call the *extreme* types. We put $\theta_i^- := 0$ and $\theta_i^+ := 1$. Let $\mathcal{G}^{(\theta_1,\ldots,\theta_N)}$ denote double game $\mathcal{G}$ with the types taking the specific values $\theta_1, \ldots, \theta_N$. In addition, in the discrete case, we let $\mathcal{G}^{k_1 k_2 \cdots k_N}$ denote double game $\mathcal{G}$ with the types $\theta_1^{k_1}, \theta_2^{k_2}, \ldots, \theta_N^{k_N}$ selected for the N players respectively. We refer to a Nash equilibrium for $\mathcal{G}^{(\theta_1,\ldots,\theta_N)}$ as a *local* Nash equilibrium for double game $\mathcal{G}$. In the continuous case, let $\mathcal{G}^{\theta_1 \theta_2 \cdots \theta_N}$ denote double game $\mathcal{G}$ with the types $\theta_1, \theta_2, \ldots, \theta_N$ selected for the N players respectively.

Assume that there is a uniform N-player double game $\mathcal{G}$ with basic games $\mathcal{G}_1$ and $\mathcal{G}_2$. Given a player $i \in I = \{1, 2, \ldots, N\}$, we denote as usual the strategy set of the opponent of i by S_{-i}. Also we let $\Theta = \{(\theta_1, \ldots, \theta_N) | \theta_i \in \Theta_i\}$, where Θ_i is the set of types of player i. Given a player i, we denote the set of types for players other than player i by Θ_{-i}. Given players i and m where $m \neq i$ and $m \in I$, let $\Theta^e_{-(i,m)}$ be the set of extreme types for players other than player i and player m.

Definition 4.1. *A double game $\mathcal{G}$ has a* coherent *set of* pure Nash equilibria *with a player $i \in I = \{1, \ldots, N\}$ having a given type $\theta_i = \theta_i^*$ if there is an action $s \in S_i$ for which there exist $s_p^{e_p} \in S_p$, for all $p \neq i$, $p \in I$ and all $e_p \in \{+, -\}$ such that the 2^{N-1} strategy profiles $(s_1^{e_1}, \ldots, s_{i-1}^{e_{i-1}}, s, s_{i+1}^{e_{i+1}}, \ldots, s_N^{e_N})$, are pure Nash equilibria for $\mathcal{G}$ with type $\theta_i = \theta_i^*$ for player i and with extreme types $(\theta_1^{e_1}, \ldots, \theta_{i-1}^{e_{i-1}}, \theta_{i+1}^{e_{i+1}}, \ldots, \theta_N^{e_N}) \in \Theta_{-i}$ for the other players.*

Assume that double game $\mathcal{G}$ has a coherent set of pure Nash equilibria $(s_1^{e_1}, \ldots, s_{i-1}^{e_{i-1}}, s, s_{i+1}^{e_{i+1}}, \ldots, s_N^{e_N})$ with a player $i \in I = \{1, \ldots, N\}$ having a given type $\theta_i = \theta_i^*$ where $s \in S_i$ and $s_p^{e_p} \in S_p, p \neq i, p \in I$ and $e_p \in \{+, -\}$. We take any $\theta_m \in \{\theta_m^-, \theta_m^+\}$, where $m \neq i, m \in I$ and fix the extreme types of the other players. The pair of profiles $(s_1^{e_1}, \ldots, s_{i-1}^{e_{i-1}}, s, s_{i+1}^{e_{i+1}}, \ldots, s_{m-1}^{e_{m-1}}, s_m^-, s_{m+1}^{e_{m+1}}, \ldots, s_N^{e_N}), (s_1^{e_1}, \ldots, s_{i-1}^{e_{i-1}}, s, s_{i+1}^{e_{i+1}}, \ldots, s_{m-1}^{e_{m-1}}, s_m^+, s_{m+1}^{e_{m+1}}, \ldots, s_N^{e_N})$ where $s_m^-, s_m^+ \in S_m$, is called the *coherent pair* of pure Nash equilibrium for game $\mathcal{G}$ with player i having a given type $\theta_i = \theta_i^*$ and player m having type θ_m.

For example, consider a double game $\mathcal{G}$ with three players. Assume that there are actions $s_1 \in S_1$ and $s_2^-, s_2^+ \in S_2$ and $s_3^- \in S_3$. Suppose $\mathcal{G}$ has a coherent pair of Nash equilibria $\big((s_1, s_2^-, s_3^-), (s_1, s_2^+, s_3^-)\big)$ when we fix the extreme type θ_3^- for player 3 and we fix type $\theta_1 = \theta_1^*$ for player 1. Figure 4.1 shows the coherent pair of Nash equilibria $\big((s_1, s_2^-, s_3^-), (s_1, s_2^+, s_3^-)\big)$ for $\mathcal{G}$.

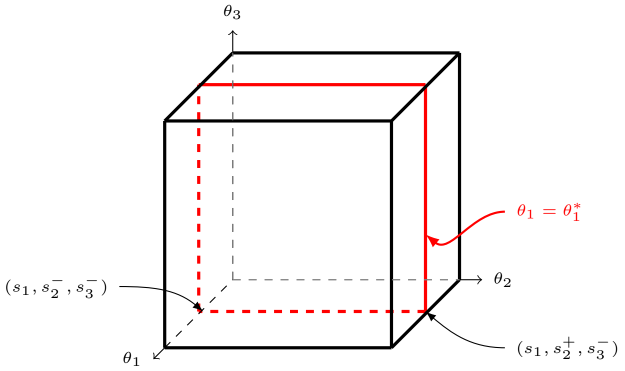


Figure 4.1: Illustration of a coherent pair in a double game with three players.

The following example shows a double game $\mathcal{G}$ which has a coherent pair $\big((z, v), (z, w)\big)$ of pure Nash equilibrium, with the first player having a given type θ_1. This example shows that if one player has more than two actions, then for the different parameter value of θ_2, we may not have a pure Nash equilibrium. Alternatively we may have a pure Nash equilibrium which, in the first player's action, is not equal to the first action of the coherent pair of pure Nash equilibrium for $\mathcal{G}$ where the first player has a given type θ_1. Therefore, when considering the double game, we restrict ourselves to having only two actions for each player.

Example 4.1. Consider a double game $\mathcal{G}$. We present an example for the case that each player has a strategy set: $S_1 = \{z, u\}$ and $S_2 = \{v, w, y\}$, with the payoff matrices of two basic games given in Figure 4.2. The pair of profiles $\big((z, v), (z, y)\big)$ is a coherent pair of pure Nash equilibrium for $\mathcal{G}$ with player 1 having type $\theta_1 = 0$ and player 2 having extreme types $\theta_2 = 0$ and $\theta_2 = 1$.

		Player 2		
		v	w	y
Player 1	z	$(3, 2)$	$(2, 2)$	$(2, 1)$
	u	$(2, 1)$	$(3, 2)$	$(1, 3)$

(a) Game 1

		Player 2		
		v	w	y
Player 1	z	$(3, 1)$	$(2, 2)$	$(2, 3)$
	u	$(2, 6)$	$(3, 4)$	$(1, 1)$

(b) Game 2

Figure 4.2: Payoff matrices representation for Example 4.1 (1).

The payoff matrix of double game $\mathcal{G}$ for $\theta_1 = 0$ and $\theta_2 = \frac{1}{4}$ is given in Figure 4.3 and strategy profile (u, w) is a pure Nash equilibrium that is not coherent with strategy profiles (z, v) and (z, y).

		Player 2		
		v	w	y
Player 1	z	$(3, \frac{7}{4})$	$(2, \frac{8}{4})$	$(2, \frac{6}{4})$
	u	$(2, \frac{9}{4})$	$(3, \frac{10}{4})$	$(1, \frac{10}{4})$

Figure 4.3: Payoff matrix of the double game in Example 4.1 (1).

Now we assume the payoff matrices of two basic games are given in Figure 4.4. Strategy profiles (z, v) and (z, y) are a coherent pair, with player 1 having type $\theta_1 = 0$ and player 2 having extreme types $\theta_2 = 0$ and $\theta_2 = 1$.

		Player 2		
		v	w	y
Player 1	z	$(3, 2)$	$(2, 2)$	$(2, 1)$
	u	$(2, 1)$	$(3, 1)$	$(1, 3)$

(a) Game 1

		Player 2		
		v	w	y
Player 1	z	$(3, 1)$	$(2, 2)$	$(2, 3)$
	u	$(2, 6)$	$(3, 4)$	$(1, 1)$

(b) Game 2

Figure 4.4: Payoff matrices representation for Example 4.1 (2).

The payoff matrix of $\mathcal{G}$ for $\theta_1 = 0$ and $\theta_2 = \frac{1}{4}$ is given in Figure 4.5 and $\mathcal{G}$ fails to have the pure Nash equilibrium.

		Player 2		
		v	w	y
Player 1	z	$(3, \frac{7}{4})$	$(2, \frac{8}{4})$	$(2, \frac{6}{4})$
	u	$(2, \frac{9}{4})$	$(3, \frac{7}{4})$	$(1, \frac{10}{4})$

Figure 4.5: Payoff matrix of the double game in Example 4.1 (2).

Therefore, for the purpose of this research, we restrict ourselves to having only two actions for each player in the case of a double game.

The following lemma and proof are by Abbas Edalat.

Lemma 4.1. *If the double game with two players has a coherent pair* $\big((z, v), (z, w)\big)$ *of pure Nash equilibrium with the first player having type* $\theta_1 = \theta_1^*$*, then there exists an integer* $\mathcal{S}$ *with* $1 \leq \mathcal{S} \leq \ell_2$ *such that* $\mathcal{G}^{k_1 k_2}$ *has* (z, v) *as a pure Nash equilibrium for* $1 \leq k_2 < \mathcal{S}$ *and has* (z, w) *as a pure Nash equilibrium for* $\mathcal{S} < k_2 \leq \ell_2$*.*

Proof. Consider a double game $\mathcal{G}$. Each player has a strategy set with two actions: $S_1 = \{z, u\}$ and $S_2 = \{v, w\}$, with the payoff matrices of two basic games given in Figure 4.6.

Player 2

		v	w
Player 1	z	(a_1, a_2)	(b_1, b_2)
	u	(c_1, c_2)	(d_1, d_2)

(a) Game 1

Player 2

		v	w
Player 1	z	(e_1, e_2)	(f_1, f_2)
	u	(g_1, g_2)	(h_1, h_2)

(b) Game 2

Figure 4.6: Payoff matrix representation for Lemma 4.1.

The payoff matrix for $\mathcal{G}$ is given in Figure 4.7.

	v	w
z	$(1-\theta_1)a_1 + \theta_1 e_1, (1-\theta_2)a_2 + \theta_2 e_2$	$(1-\theta_1)b_1 + \theta_1 f_1, (1-\theta_2)b_2 + \theta_2 f_2$
u	$(1-\theta_1)c_1 + \theta_1 g_1, (1-\theta_2)c_2 + \theta_2 g_2$	$(1-\theta_1)d_1 + \theta_1 h_1, (1-\theta_2)d_2 + \theta_2 h_2$

Figure 4.7: Payoff matrix representation of the double game for Lemma 4.1.

Suppose that for $\theta_2 = 0$ and $\theta_2 = 1$ and for a given $\theta_1 = \theta_1^*$ we obtain the pair of coherent Nash equilibria (z, v) and (z, w) thus;

$$\begin{cases} u_2(z, v; \theta_1^*, 0) \geq u_2(z, w; \theta_1^*, 0), & (4.1) \\ u_2(z, w; \theta_1^*, 1) \geq u_2(z, v; \theta_1^*, 1). & (4.2) \end{cases}$$

Using Figure 4.7 and Equations 4.1 and 4.2, we have

$$\begin{cases} (1-\theta_2)a_2 + \theta_2 e_2 \geq (1-\theta_2)b_2 + \theta_2 f_2, & (4.3) \\ (1-\theta_2)b_2 + \theta_2 f_2 \geq (1-\theta_2)a_2 + \theta_2 e_2. & (4.4) \end{cases}$$

By rearranging the above inequalities, we can see

$$\begin{cases} \theta_2(b_2 + e_2 - a_2 - f_2) + a_2 - b_2 \geq 0, \\ \theta_2(b_2 + e_2 - a_2 - f_2) + a_2 - b_2 \leq 0. \end{cases}$$

We let

$$L_{\theta_2} = \theta_2(b_2 + e_2 - a_2 - f_2) + a_2 - b_2.$$

The line L_{θ_2} as a linear function of θ_2 divides the space of pure Nash equilibrium for $\mathcal{G}$ with player i having a given type $\theta_1 = \theta_1^*$ into two regions. One region of pure Nash equilibrium (z, v) and another region of pure Nash equilibrium (z, w) which satisfy Inequalities 4.3 and 4.4. Figure 4.8 shows the line L_{θ_2}.

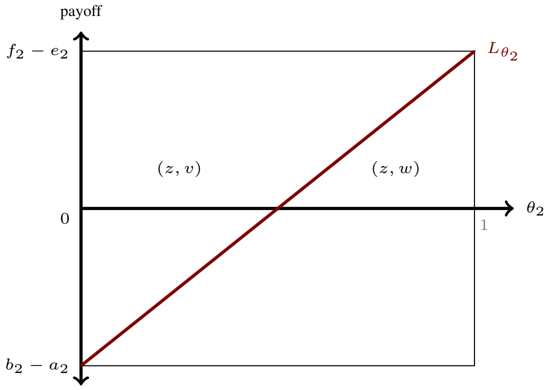


Figure 4.8: Division of the space of pure Nash equilibrium for type θ_1^* into two regions by the line L_{θ_2} as a linear function of θ_2.

Hence there exists an integer $\mathcal{S}$ with $1 \leq \mathcal{S} \leq \ell_2$ such that $\mathcal{G}^{k_1 k_2}$ has (z, v) as a pure Nash equilibrium for $1 \leq k_2 < \mathcal{S}$ and has (z, w) as a pure Nash equilibrium for $\mathcal{S} < k_2 \leq \ell_2$. ☐

We call $\mathcal{S}$ given in Lemma 4.1, the *type changing point*.

The following example shows a double game with two players, such that the first basic game fails to have a pure Nash equilibrium and the second basic game has two pure Nash equilibria.

Example 4.2. In this example, each player has a strategy set with two actions: $S_1 = \{z, u\}$ and $S_2 = \{v, w\}$, with the payoff matrices of two basic games given in Figure 4.9. The first basic game is similar to a Matching Pennies game that has no pure Nash equilibrium but it has a unique mixed Nash equilibrium.

		Player 2	
		v	w
Player 1	z	$(1, 0)$	$(0, 1)$
	u	$(0, 1)$	$(1, 0)$

(a) Game 1

		Player 2	
		v	w
Player 1	z	$(3, 1)$	$(3, 0.5)$
	u	$(3, 0.5)$	$(3, 1)$

(b) Game 2

Figure 4.9: Payoff matrices representation for Example 4.2.

	v	w
z	$(1-\theta_1)+3\theta_1, 0+\theta_2$	$0+3\theta_1, (1-\theta_2)+0.5\theta_2$
u	$0+3\theta_1, (1-\theta_2)+0.5\theta_2$	$(1-\theta_1)+3\theta_1, 0+\theta_2$

Figure 4.10: Payoff matrix representation of the double game in Example 4.2.

The payoff matrix of double game $\mathcal{G}$ is given in Figure 4.10.

We observe there are pure Nash equilibria for certain values of θ_1 and θ_2 for $\mathcal{G}$. For example, the profile (z, v) is a pure Nash equilibrium if $\theta_1 \leq 1$ and $\theta_2 \geq \frac{2}{3}$. Figure 4.11 shows the different regions of θ_1 and θ_2 where different pure Nash equilibria appear. Uncoloured zones represent the regions where there is no pure Nash equilibrium.

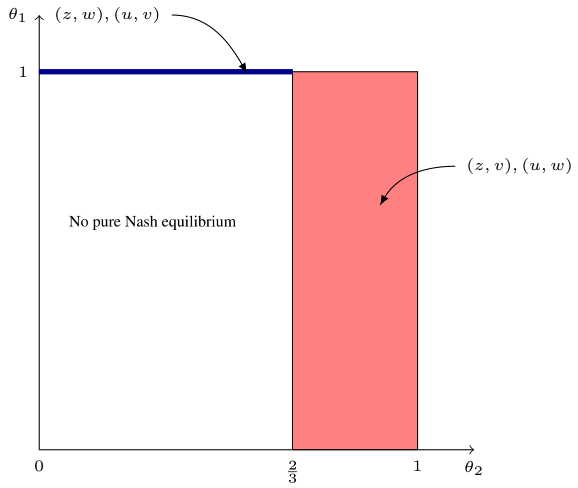


Figure 4.11: Illustration of Nash equilibrium in the different regions of θ_1 and θ_2 for Example 4.2.

Next, we examine how information about the set of local pure Nash equilibrium for a double game with various types of *N* players, can be used to deduce the Bayesian Nash equilibrium for the double game.

4.2 Pure Regular Double Game

We consider a double game to be pure regular if it has a set of 2^N pairs of pure Nash equilibrium for all extreme types, for which the strategy of each player only depends on its own type. Here is the exact definition.

Definition 4.2. *A double game with a finite set of types for each player* $i \in I = \{1, \dots, N\}$*, is* pure regular *if there are actions* $s_i \in S_i$ *for* $i \in I$ *such that the strategy profiles* $(s_1, \dots, s_i, \dots, s_N)$ *are pure Nash equilibria for the double game with player* i *having extreme types* $\theta_i^{e_i}$, $e_i \in \{-, +\}$.

Figure 4.12 shows pure regularity in a double game $\mathcal{G}$ with three players while each player has a strategy set: $S_1 = \{v, x\}$, $S_2 = \{z, w\}$ and $S_3 = \{u, y\}$. Here, there are eight pure strategy profiles $(v, u, z), (v, u, w), (v, y, z), (v, y, w), (x, u, z)$, $(x, u, w), (x, y, z), (x, y, w)$ for $\mathcal{G}$ with each player utilising its extreme types.

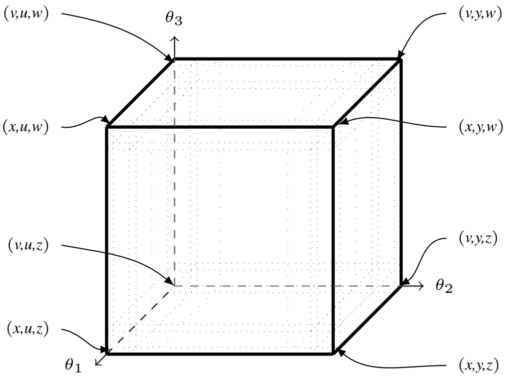


Figure 4.12: Illustration of eight pure regular Nash equilibria in a double game with three players.

For a double game with a finite set of types for each player, we can go further, as follows.

4.3 Complete Pure Regular Double Game

Definition 4.3. *We say a double game with a finite set of types for each player* $i \in I = \{1, \dots, N\}$ *given by* θ_i^k *(*$1 \leq k \leq \ell_i$*) is* completely pure regular *if there are pure strategies* $s_k \in S_i$ *for* $(1 \leq k \leq \ell_i)$ *such that the strategy profiles* $(s_{k_1}, \dots, s_{k_i}, \dots, s_{k_N})$ *are pure Nash equilibria for the game* $\mathcal{G}^{k_1 \cdots k_i \cdots k_N}$ *for* $(1 \leq k_i \leq \ell_i)$*.*

It is clear that a completely pure regular double game is pure regular and thus our terminology is consistent. Let $s_i(.) : \Theta_i \to S_i$ be a function that specifies a pure strategy $s_i(\theta_i)$ for player $i \in I = \{1, \ldots, N\}$ where Θ_i is set of types for player i and $\theta_i \in \Theta_i$. We will prove that for a completely pure regular double game, the strategy profile $\big(s_1(.), s_2(.), \ldots, s_N(.)\big)$ is a pure Bayesian strategy profile, in which each player $i \in I = \{1, \ldots, N\}$ takes strategy $s_i(.)$ where $s_i(.)$ refers to the pure strategy of player i corresponding to a type from Θ_i.

Theorem 4.1. *If the double game is completely pure regular, then for all prior distributions p, such that $p(\theta_i) > 0$ for all $\theta_i \in \Theta_i$, $\forall i \in I$ for the N players' types, the Bayesian pure strategy profile $\big(s_1(.), s_2(.), \ldots, s_N(.)\big)$ is a pure Bayesian Nash equilibrium.*

Proof. Since the finite types θ_i for each player i are considered to be private information and can each take only a finite number of values between 0 and 1, the double game is reduced to a Bayesian game with a finite set of types for the N players. Let the double game $\mathcal{G}$ be a Bayesian game with a finite number of types for each player $i \in I = \{1, \ldots, N\}$, and $s_i(.) : \Theta_i \to S_i$. Based on the assumption that $\mathcal{G}$ is completely pure regular, thus;

$$s_i(\theta_i) \in \arg\max_{s'_i \in S_i} u_i\big(s'_i, s_{-i}(\theta_{-i}), (\theta_i, \theta_{-i})\big) \tag{4.5}$$

for all $s'_i \in S_i$. Since $\sum_{\theta_{-i}} p(\theta_{-i}|\theta_i) = 1$, we can rewrite 4.5 as follows;

$$s_i(\theta_i) \in \arg\max_{s'_i \in S_i} u_i\big(s'_i, s_{-i}(\theta_{-i}), (\theta_i, \theta_{-i})\big). \sum_{\theta_{-i}} p(\theta_{-i}|\theta_i). \tag{4.6}$$

We know that utility $u_i\big(s'_i, s_{-i}(\theta_{-i}), (\theta_i, \theta_{-i})\big)$ in a multi-game is independent of θ_{-i}, therefore, we have;

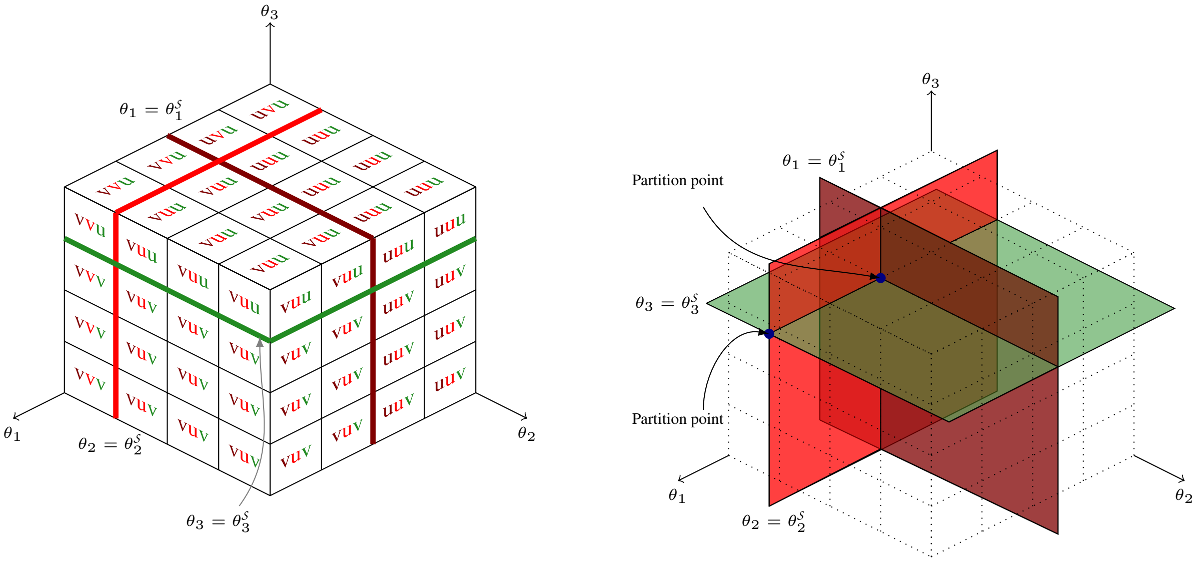
$$s_i(\theta_i) \in \arg\max_{s'_i \in S_i} \sum_{\theta_{-i}} p(\theta_{-i}|\theta_i) u_i\big(s'_i, s_{-i}(\theta_{-i}), (\theta_i, \theta_{-i})\big). \tag{4.7}$$

for all $s'_i \in S_i$. Recall Definition 2.12, the strategy profile $s_i(.)$ in 4.7 is a pure Bayesian equilibrium for game $\mathcal{G}$ for all $i \in I$ and all $\theta_i \in \Theta_i$ such that $p(\theta_i) > 0$. Therefore, the Bayesian pure strategy profile $\big(s_1(.), \ldots, s_i(.), \ldots, s_N(.)\big)$ is a pure Bayesian Nash equilibrium. □

4.4 Separatrix Hyperplane

In an N-player double game, we introduce a function $p_{im} : \Theta^e_{-(i,m)} \times \Theta_i \longmapsto \Theta_m$ where $i, m \in I = \{1, 2, \ldots, N\}$ and $i \neq m$. This function has two arguments. The first one is the set of extreme types for players other than player i and player m and all of their types are fixed at their extreme types, while the second argument is the set of types for player i. The function p_{im} returns the type changing point for player m given in Lemma 4.1.

If p_{im} does not depend on either of its two arguments, then there is a *separatrix hyperplane* $\theta_m = \theta^S_m$ such that $\theta_m = \theta^S_m$ is always independent of $\theta_i \in \Theta_i$ and $\theta \in \Theta^e_{-(i,m)}$. The hyperplane $\theta_m = \theta^S_m$ is called a *constant type changing hyperplane*. The point of intersection of the constant type changing hyperplanes is called the *partition point* $P \in Q = [0, 1]^N$. Figure 4.13 shows constant type changing hyperplanes and some partition points in a 3-player double game $\mathcal{G}$ while each player has a strategy set with two actions: $\{u, v\}$.



(a) (b)

Figure 4.13: Illustration of the type changing hyperplanes and partition points in a 3-player double game.

Theorem 4.2. *An N-player double game is completely pure regular if and only if for all types* θ_i^k *(*$1 \leq k \leq \ell_i$*) for each player* $i \in I = \{1, 2, \ldots, N\}$*, the value of the type changing point* p_{im} *of each player* $m \in I$*,* $m \neq i$*, is constant.*

Proof. ($\Rightarrow$) Now we assume that the double game is completely pure regular. Fix $m \in I = \{1, 2, \ldots, N\}$; take any extreme types of m, $\theta_m \in \{\theta_m^-, \theta_m^+\}$. Also for any player $i \in I$, where $i \neq m$, we take $\theta_i \in \{\theta_i^-, \theta_i^+\}$. Let $s_i(.) : \Theta_i \to S_i$ where Θ_i is a set of types for player i and $\theta_i \in \Theta_i$ such that $\big(s_1(\theta_1), \ldots, s_N(\theta_N)\big)$ is a pure Nash equilibrium. Recall Section 4.4 that $\Theta^e_{-(i,m)}$ is a set of extreme types for players other than player i and player m and all of their types are fixed at the their extreme types. As the double game is completely pure regular, $p_{im} : \Theta^e_{-(i,m)} \times \Theta_i \longmapsto \Theta_m$ is constant, otherwise $s_m(\theta_m)$ depends on θ_i or $\theta^e_{-(i,m)}$, which contradicts the complete pure regularity for the double game. Therefore, the value of type changing point p_{im} for each player m, is constant for all types θ_i^k $(1 \leq k \leq \ell_i)$ of each player i. $\forall m\ \forall i \neq m$

$$p_{im}\big(\theta_{-(i,m)}, \theta_i\big) = \theta_m^S$$

($\Leftarrow$) Now we assume that $\forall m\ \forall i \neq m,\ p_{im}\big(\theta_{-(i,m)}, \theta_i\big) = \theta_m^S$. Thus p_{im} is constant and the strategies of each player $m \in I$ only depends on its own type; therefore, the double game is completely pure regular. ☐

Corollary 4.1. *The double game is completely pure regular if and only if there is a partition point $P \in Q = [0,1]^N$ such that the sub-hypercubes generated by P using hyperplanes through P parallel to the coordinate planes partition Q into regions of constant Nash equilibrium.*

In a double game, the constant type changing hyperplanes partition the space of pure Nash equilibrium into several blocks of pure Nash equilibrium for each player $i \in I = \{1, \ldots, N\}$ for the given type θ_i. We let q_i denote the number of blocks containing constant Nash equilibrium for player i for each θ_i. For each player i, as we restrict ourselves to a uniform double game with only two actions for each player then we have $q_i \leq 2$, thus $\prod_{i=1}^{N} q_i \leq 2^N$. Figure 4.14 shows an example of the number of blocks containing constant Nash equilibrium in a double game $\mathcal{G}$ with two players.

In an N-player double game, for each player $i \in I = \{1, 2, \ldots, N\}$, there exists two N-1 player games at the extreme types θ_i^- and θ_i^+ that are called *opposite faces* for player i.

Corollary 4.2. *A 2-player double game is completely pure regular if and only if the partition point of each face is the same as the partition point of its opposite face.*

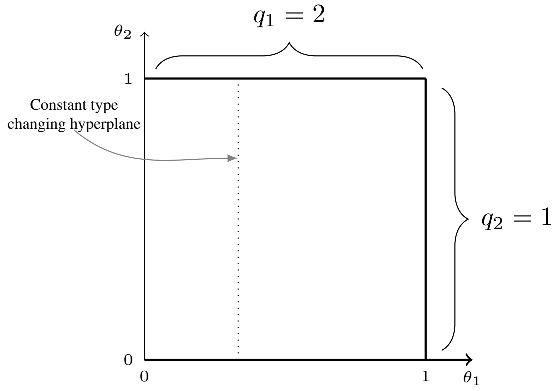


Figure 4.14: An example of the number of blocks containing constant Nash equilibrium in a double game $\mathcal{G}$ with two players.

If the partition points at two opposite faces are the same, we call them *matched* partition points.

Theorem 4.3. *In any N-player double game with finite sets of types, if all 2- player sub-games are completely pure regular then any k-dimensional sub-game, ($2 \leq k \leq N$) is completely pure regular.*

Proof. Assume all 2-player sub-games are completely pure regular. Consider a k-dimensional sub-game with k players. Let $B \subset I = \{1, 2, \dots, N\}$, $|B| = k$. Take $m \in B$; take any extreme type of m, $\theta_m \in \{\theta_m^-, \theta_m^+\}$. Also for any player $i \in B$, where $i \neq m$, we take $\theta_i \in \{\theta_i^-, \theta_i^+\}$. Recall Section 4.4 that $\Theta^e_{-(i,m)}$ is the set of extreme types of players other than player i and player m. Using Corollary 4.2, there are matched partition points for each 2-player game. Thus have $\forall m \, \forall i \neq m \, \forall x \in \Theta^e_{-(i,m)}$

$$p_{im}(x, \theta_i^+) = p_{im}(x, \theta_i^-) = \theta_m^S. \tag{4.8}$$

Let $A \subseteq I = \{1, \dots, N\}$. Let Θ^e_{-A} be the set of extreme types of players other than subset A,

$$\text{If } \theta \in \Theta^e_{-A} \text{ then } |\theta| = N - |A|.$$

Let

$$\theta_t : \Theta^e_{-A} := \{\theta_t : \theta \,|\, \theta \in \Theta^e_{-A}\}$$

where $\theta_t : \theta$ is the concatenation of θ_t and the list θ. Then

$$\text{if } \theta \in \theta_t : \Theta^e_{-A} \text{ then } |\theta| = N - |A| + 1.$$

Since $\theta_t^- : \Theta^e_{-(i,m,t)} \subset \Theta^e_{-(i,m)}$ and $\theta_t^+ : \Theta^e_{-(i,m,t)} \subset \Theta^e_{-(i,m)}$, and by Equation 4.8, then

$$\forall m, i, t \in B\ , t \neq i \neq m,\ \ , t \neq m\ , \forall y \in \theta_t^+ : \Theta^e_{-(i,m,t)}\ , \forall z \in \theta_t^- : \Theta^e_{-(i,m,t)},$$

$$p_{im}(y, \theta_i^+) = p_{im}(z, \theta_i^+) = \theta_m^S,$$

$$p_{im}(y, \theta_i^-) = p_{im}(z, \theta_i^-) = \theta_m^S,$$

which shows in each k- dimensional sub-game, the value of the type changing point of each player is constant. Therefore, any $k-$dimensional sub-game, ($2 \leq k \leq N$) is completely pure regular.

Consider a completely pure regular double game $\mathcal{G}$ with three players, where each player has a strategy set with two actions: $\{s, u\}$. Figure 4.15 (a) shows the completely pure regular 2-player sub-game and matched partition points where

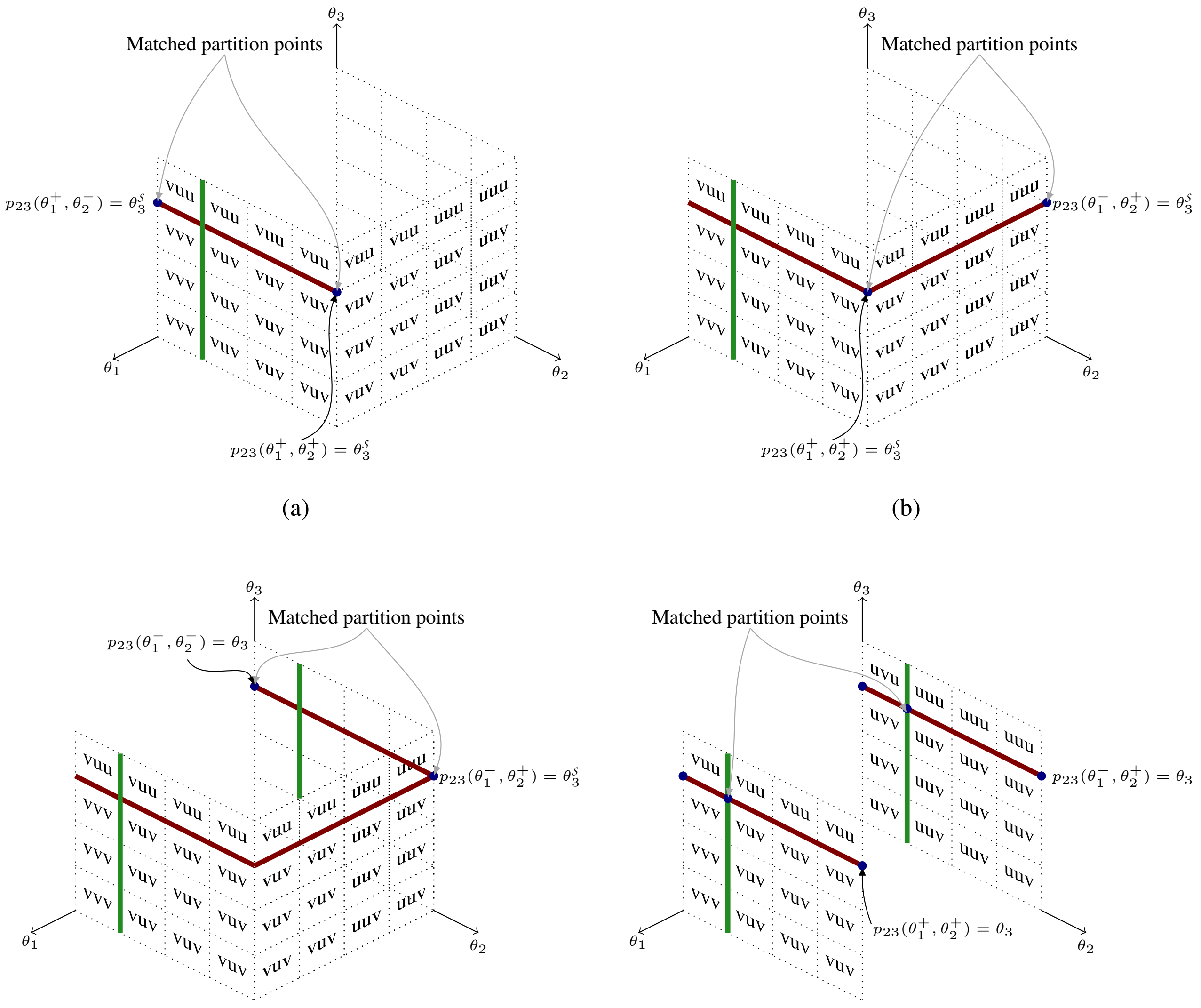
$$p_{23}(\theta_1^+, \theta_2^+) = p_{23}(\theta_1^+, \theta_2^-) = \theta_3^S$$

Therefore, type changing point $p_{23}(\theta_1^+, \theta_2^+)$ is matched with type changing point $p_{23}(\theta_1^-, \theta_2^+)$ $\big($Figure-4.15 (b)$\big)$ and also type changing point $p_{23}(\theta_1^-, \theta_2^+)$ is matched with type changing point $p_{23}(\theta_1^-, \theta_2^-)$ $\big($Figure 4.15 (c)$\big)$. Figure 4.15 (d) shows that the partition point of a 2-player sub-game matches with the partition point of its opposite 2-player. Similarly, we can prove that the value of the type changing point of each player is constant. Therefore, $3-$dimensional sub-game is completely pure regular.

□

Corollary 4.3. *An N-player double game is completely pure regular, if and only if, for each player* $i \in I = \{1, 2, \ldots, N\}$*, the two N-1 player games at the extreme types* θ_i^- *and* θ_i^+ *are completely pure regular with the matched partition points.*

Figure 4.16 (a) represents a double game $\mathcal{G}$ with three players while each player has a strategy set with two actions: $\{u, v\}$. The double game is completely pure regular and constant type changing hyperplanes partition the space of pure Nash equilibrium into 8 blocks of pure Nash equilibrium with constant pure Nash equilibrium in each. Figure 4.16 (b) represents an example of a double game $\mathcal{G}$



(a)

(b)

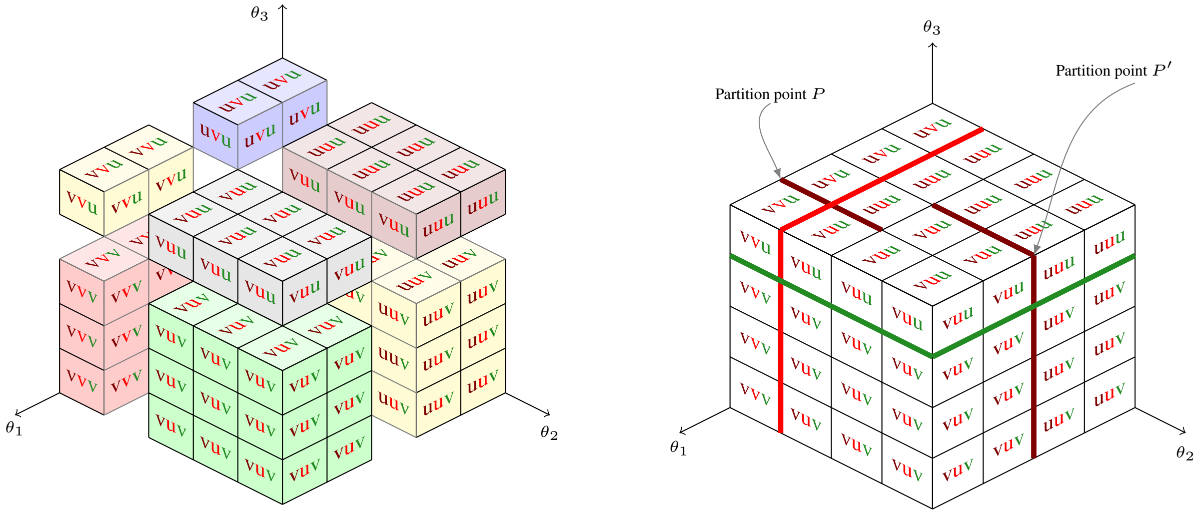
(c)

(d)

Figure 4.15: The opposite faces and matched partition points in a 3-player double game.

with two unmatched partition points on the two opposite faces in a double game $\mathcal{G}$ for three players.

Now we present an efficient algorithm for the double game in order to establish whether we have a Bayesian Nash equilibrium that can be determined with lower computational complexity. We can determine if a double game with finite types for the N players is completely pure regular by using Algorithm 1. The basic idea is a recursive check to see if the opposite faces for each player $i \in I = \{1, \ldots, N\}$ are pure regular and if the partition point of each face are matched with the partition points of its opposite face. At the lowest recursion level, we have a double game with two players and we test whether the double game is pure regular and if the partition points of each face are matched with the partition point of its opposite face. Thus, we can only check each 2-player game.



(a) (b)

Figure 4.16: Illustration of the blocks of pure Nash equilibrium with constant pure Nash equilibrium (a) and the unmatched partition points (b), in a double game G for three players.

Algorithm 1: Algorithm to test for the property of being complete pure regular in $\mathcal{G}_N$ with N players

---

Input: An N-player double game $\mathcal{G}_N$.
Output: $\mathcal{G}_N$ is completely pure regular or not.

```
$CPR(\mathcal{G})$: $\mathcal{G}$ is completely pure regular.
$PR(\mathcal{G})$: The opposite faces for players are pure regular in $\mathcal{G}$.
$M(\mathcal{G})$: In $\mathcal{G}$, the partition point of each face matches with the partition
        point of its opposite face.
    begin
        $\mathcal{G} := \mathcal{G}_N$
        if $CPR(\mathcal{G}) = PR(\mathcal{G}) \wedge M(\mathcal{G})$.
            $PR(\mathcal{G}) = \bigwedge_{\substack{t \in [\mathcal{G}] \\ e \in \{+,-\}}} PR(\mathcal{G}_{-t}^{\theta_t^e})$.      Check recursively
            $M(\mathcal{G}) = \bigwedge_{\substack{t \in [\mathcal{G}] \\ e \in \{+,-\}}} M(\mathcal{G}_{-t}^{\theta_t^e})$.      Check recursively
        then
            $\mathcal{G}_N$ is Completely Pure Regular
        else
            $\mathcal{G}_N$ is not Completely Pure Regular
    end
```

Corollary 4.4. *Given any double game with a finite number of types for N players, we can decide if it is completely pure regular, in which case a Bayesian pure Nash equilibrium is obtained $O(2^N)$.*

Proof. The number of k-dimensional hypercubes on the boundary of an N-cube is $2^{N-k}\binom{N}{k}$. In the double game, for a 2-player game, we have $k = 2$ and so the number of 2-player games is

$\frac{1}{2!}.N(N-1)2^{N-2}$. Thus, for Algorithm 1, the computational complexity is $O\big(2^N\big)$. □

Therefore, for the class of completely pure regular double games, the Nash equilibria of the basic games can be used to compute a Bayesian Nash equilibrium of the double game with respect to the number of players.

Chapter 5

An Application: A Double Game for Prisoner's Dilemma

> In this chapter, we explain a double game extension of the Prisoner's Dilemma to model pro-social behaviour. In this double game for Prisoner's dilemma, the first game is the classical Prisoner's dilemma and the second game captures the social or moral gain for cooperation for each player. We furthermore consider the double game for the Prisoner's dilemma where the social (altruistic) coefficient of each player forms a finite discrete set of incomplete information or types, thus giving rise to a Bayesian game. We prove that this double game is in fact pure regular and determine its Bayesian equilibrium when it is completely pure regular.

Game theory has been an important tool for addressing problems regarding the origins of conventions, fairness, and pro-social behaviour in general. In an overwhelming number of situations, people do not seem to behave in their self-interest, but rather behave pro-socially, contrary to what classical game theory suggests (that people always act in their own self-interest).

We now review the concept of pro-social behaviour and moral gain in Prisoner's Dilemma. Recall Section 2.6, the Prisoner's Dilemma is considered a standard method for modelling social dilemmas and has also been used to model conditional altruistic cooperation, which has also been tested by real monetary payoffs. However, when confronted with the choice to cooperate or defect, human beings not only consider their material score, but also the social and moral payoffs of any decision they make. This means that the material payoffs presented in the Prisoner's Dilemma cannot provide a complete picture of the decision making process human beings follow. In fact, according to some researchers, human social evolution has a moral direction which has extended our moral campus in the course of thousands of years of increasing communication and complexity in human social organisation [Wri01, Wri10]. Moreover, there are individual and temporal variations in pro-social attitudes of human beings with some making decisions more based on self-interest than others. A more adequate model of human behaviour should take into account these aspects of social evolution as well. The same applies to economic decisions by corporations or governments, in which actions taken can have significant social and environmental implications, which are not incorporated in the material gains they produce. In [She94], it was proposed that a *coefficient of morality* be introduced to the Prisoner's Dilemma and the payoff values of the players be accordingly changed. The so-called *altruistic extension* of any finite strategic game was defined in [CKKS11], which endows each player with an altruistic level in the unit interval which provides the weight of the pro-social attitude of the player. This modification aims to reflect real-life situations and dilemmas more accurately by taking into account both material and moral/social gains. Thus, for each player, the payoff is a weighted, linear combination of the payoffs for the Prisoner's Dilemma and the social game. In essence, it is a convex combination of the payoffs occurring from both material and social dilemmas.

We show that the double game, as an instance of multi-games, provides a generalisation of the altruistic extension in [CKKS11] which can be considered as a double game with the first game identified as the original game and the second game as a symmetric altruistic game. In a general double game, the social or altruistic game is allowed to be non-symmetric, which means that in general the altruistic payoffs for the different players may be different even for the same strategy profile.

5.1 The Social Game

The social game encourages cooperation and discourages defection, as cooperating is usually considered to be the ethical and moral choice to make when interacting with others in social dilemmas. This can be done in different ways corresponding to different types of payoff matrices. Here, we will restrict to the case that the social game encourages cooperation and discourages defection for each player, independently of the action chosen by the other player.

We present the normal form and the mathematical formulation of the social game as follows. Assume that the competing participants in the social game are player 1 and player 2. Each of them has the choice to select between "C" and "D ". When they have both made their choice, the payoffs assigned to them are calculated according to Figure 5.1, where M_1, M_2 and M_1', M_2' satisfy:

$$M_1 > M_1', \quad M_2 > M_2'.$$

When $M_1 = M_2$ and $M_1' = M_2'$, we will have a symmetric social game and our framework reduces to the altruistic extension in [CKKS11].

		Player 2	
		C	D
Player 1	C	(M_1, M_2)	(M_1, M_2')
	D	(M_1', M_2)	(M_1', M_2')

Figure 5.1: Payoff matrix representation of Social game.

Thus, in the social game we treat in this chapter, the players are individually and independently rewarded for cooperating and punished for defecting. This can be interpreted in the following way. Cooperation by an individual, independent of the action of the opponent, is socially rewarded by inducing a good conscience, whereas defection is punished by creating a guilty one. The values of M_1, M_2 and M_1', M_2' are assumed to be socially determined to correspond to the average moral norm in the given society and are considered to have evolved in the course of increasing complexity, communication and moral growth in human history.

5.2 The Double Game Extension of the Prisoner's Dilemma

Although the payoffs for the social game are determined by the social context of the game, there is still individual variation in pro-social behaviour of the players. We assume each player has a social coefficient taking values between 0 and 1, which reflects how pro-social they are in practice in each round of the game. In our particular social game, the social coefficient of a player signifies how much the player cares about the morality or the social aspect of their action. The payoffs of the double game for each player are then the weighted sum or convex combination of the payoffs of the Prisoner's Dilemma (Figure 2.8) and social game (Figure 5.1) using the player's social coefficient as represented in Figure 5.2, where θ_1 and θ_2 (with $0 \leq \theta_1, \theta_2 \leq 1$) are the social coefficients of players 1 and 2, respectively. Note that the two players can still play the standard version of the Prisoner's Dilemma by selecting their social coefficients to be equal to 0, in which case the double game reduces to the Prisoner's Dilemma.

	C	D
C	(1 - θ_1)R + $\theta_1 M_1$, (1 - θ_2)R + $\theta_2 M_2$	(1 - θ_1)S + $\theta_1 M_1$, (1 - θ_2)T + $\theta_2 S$
D	(1 - θ_1)T + $\theta_1 S$, (1 - θ_2)S + $\theta_2 M_2$	(1 - θ_1)P + $\theta_1 S$, (1 - θ_2)P + $\theta_2 S$

Figure 5.2: Payoff matrix representation of the double game in Prisoner's Dilemma example.

In addition to the inequalities satisfied in the payoffs for Prisoner's Dilemma and social game, we stipulate the two new inequalities below that connect the payoff values from both the Prisoner's Dilemma and the social game:

$$\begin{cases} M_1,\ M_2 > \dfrac{(R+P)}{2}, \\ T > R > M_1 \geq M_2 > P > S \ \text{ or } \ T > R > M_2 \geq M_1 > P > S. \end{cases}$$

First, we argue that M_1 and M_2 should be less than T, but greater than P. The former should hold, otherwise if M_1 and M_2 are equal to or greater than T, then, there is no dilemma as to what the best strategy is (one should select the highest possible social coefficient and always choose "C " in order to achieve the highest available payoff), and the social game loses its meaning. On the other hand, the latter should hold, because, if M_1 and M_2 are equal to or less than P, then, cooperation is discouraged, since one would have no incentive to select a high social coefficient and choose "C ". In addition, M_1 and M_2 should be strictly less than R, as we would like to encourage cooperation in the social game by assigning it a payoff value that is somewhat less than the payoff value obtained through mutual cooperation in the Prisoner's Dilemma. This, we believe, reflects more accurately real-life situations, as, in general, the decisions based on moral incentives do not bring high material benefits. Finally, we assume that M_1 and M_2 should be greater than the average of R and P, so that the dilemma of whether to cooperate or defect becomes more intense.

Then, we argue that M_1' and M_2' should be equal to S, so as to discourage defection with a high social coefficient, which would be self-contradictory, and, to punish, in a sense, defection, since M_1' and M_2' are the payoff values for defection in the social game, which, by its definition, should not give a high value to defection.

The selection of the social coefficient reveals, in part, the strategy that one will follow in a given game. To illustrate this with an example, note that the choice of social coefficient equal to 1 implies cooperation, since defection would give a payoff of 0, and, similarly, the choice of social coefficient equal to 0 most probably implies defection, since cooperation in that case would give a payoff of 0, unless it is mutual, in which case it would be beneficial. On the other hand, selecting a social coefficient between 0 and 1 leaves room for more complex and sophisticated strategies. Finally, as we will see later on in Chapter 8, in the implementation of the double game, certain restrictions are imposed on how much a player can increase or decrease the social coefficient in a single round. This is done, since, in general, humans do not change their moral values radically in a short amount of time.

5.3 Double Game with Complete Information

If we assume that the players know each other's social coefficients prior to every game, the double game becomes a game with complete information and all payoffs are known to both players. In this section we focus on the analysis of Nash equilibrium in this type of double game. It is well known that the Nash equilibrium for the Prisoner's Dilemma is mutual defection, represented by (D, D). However, from the perspective of the social game, the best response of any player is to cooperate, as this always leads to a better score as compared to defecting. As a result, if the players make their decisions with no concern for their opponents' behaviour, it leads to a Nash equilibrium of mutual cooperation, represented by (C, C). However, this simplicity cannot be incorporated in the double game, due to the inclusion of the social coefficient, which alters the reward for all outcomes.

In accordance with the equilibrium, we must find out how the social coefficients of the two players alter the potential payoffs for four possible outcomes of the game. The payoffs for each possible outcome change along with the variation in the social coefficients θ_1 and θ_2 of players 1 and 2, respectively, as shown in Figure 5.7. At this stage, we consider the payoff equations for player 1. We note that, by symmetry, a similar analysis can be conducted for player 2. Furthermore, without loss of generality, we assume the equality $M_1' = M_2' = S$, since the social game punishes defection.

For θ_1 let us label the three crossing points of the payoff equations as $\theta_1 = a_1$ for $u_1(D, D)$ = $u_1(C, D)$, $\theta_1 = b_1$ for $u_1(D, C)$ = $u_1(C, C)$ and $\theta_1 = c_1$ for $u_1(D, C)$ = $u_1(C, D)$. By equating the equations for each payoff, we find the values of the crossing points (similarly for θ_2) to be:

$$
\begin{aligned}
a_1 &= \frac{P - S}{M_1 + P - 2S}, & a_2 &= \frac{P - S}{M_2 + P - 2S}, \\
b_1 &= \frac{T - R}{T - S + M_1 - R}, & b_2 &= \frac{T - R}{T - S + M_2 - R}, \\
c_1 &= \frac{T - S}{M_1 + T - 2S}, & c_2 &= \frac{T - S}{M_2 + T - 2S}.
\end{aligned}
$$

We have the three following cases:

$$
\begin{cases}
a_1 < b_1 < c_1, \ a_2 < b_2 < c_2 \text{ if } P - S < T - R, \\
b_1 < a_1 < c_1, \ b_2 < a_2 < c_2 \text{ if } P - S > T - R, \\
a_1 = b_1 < c_1, \ a_2 = b_2 < c_2 \text{ if } P - S = T - R.
\end{cases}
$$

We can obtain the Nash equilibrium points for $M_1, M_2 > R$ and $M_1, M_2 < R$ and $M_1 = M_2 = R$; note that for all values of θ_1 and θ_2 that are greater than b_1 or b_2, the equilibria are equal. To illustrate the method, we will compute below the Nash equilibrium for the two generic cases of $a_1 < b_1, \ a_2 < b_2$ and $b_1 < a_1, \ b_2 < a_2$ when $T > R > M_1 > M_2 > P > M_1' = M_2' = S$. We used ideas of Ounsley [Oun10] to write the two next sections.

5.3.1 Case: $a_1 < b_1$ and $a_2 < b_2$

Figure 5.3 shows the variation in the payoffs resulting from each outcome of the double game with different values of θ_1. We can describe the order of preference for all values of θ_1 lying between 0 and 1 by using the values of a_1, b_1 and c_1 and the functions shown in Figure 5.6, and then we can obtain the equilibria for different social coefficients.

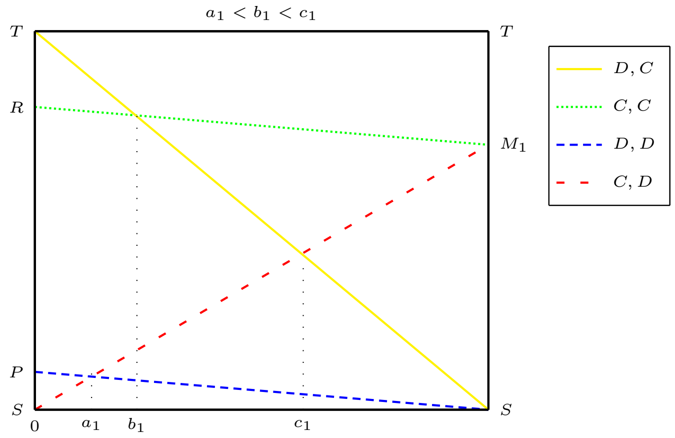


Figure 5.3: Change of payoffs $T > R > M_1 > M_2 > P > M_1' = M_2' = S$.

Figure 5.4 shows the preference ordering of player 1 for the variation of θ_1.

Type θ_1	Preference ordering of player 1
$0 \leq \theta_1 < a_1$	$(D, C) > (C, C) > (D, D) > (C, D)$
$\theta_1 = a_1$	$(D, D) = (C, D)$
$a_1 < \theta_1 < b_1$	$(D, C) > (C, C) > (C, D) > (D, D)$
$\theta_1 = b_1$	$(D, C) = (C, C)$
$b_1 < \theta_1 < c_1$	$(C, C) > (D, C) > (C, D) > (D, D)$
$\theta_1 = c_1$	$(D, C) = (C, D)$
$c_1 < \theta_1 \leq 1$	$(C, C) > (C, D) > (D, C) > (D, D)$

Figure 5.4: Preference ordering of player 1 for the variation of θ_1.

Since the double game is a symmetric game, the same inequalities also exist for player 2, with the social coefficient for player 2 being θ_2 instead of θ_1, and each outcome being replaced with its mirror point. Figure 5.5 shows the preference ordering of player 1 for the variation of θ_2. For instance while (D, C) is shown to be the most preferable for player 1 in Figure 5.3, (C, D) would take its place for player 2. The pair (θ_1, θ_2) is a point of unit square $[0, 1] \times [0, 1]$.

Type θ_2	Preference ordering of player 2
$0 \leq \theta_2 < a_1$	$(C,D) > (C,C) > (D,D) > (D,C)$
$\theta_2 = a_1$	$(D,D) = (D,C)$
$a_1 < \theta_2 < b_1$	$(C,D) > (C,C) > (D,C) > (D,D)$
$\theta_2 = b_1$	$(C,D) = (C,C)$
$b_1 < \theta_2 < c_1$	$(C,C) > (C,D) > (D,C) > (D,D)$
$\theta_2 = c_1$	$(C,D) = (D,C)$
$c_1 < \theta_2 \leq 1$	$(C,C) > (D,C) > (C,D) > (D,D)$

Figure 5.5: Preference ordering of player 2 for the variation of θ_2.

The equilibria for different social coefficients in the case of $a_1 < b_1$ are given in Figure 5.6 for the generic sub-rectangles.

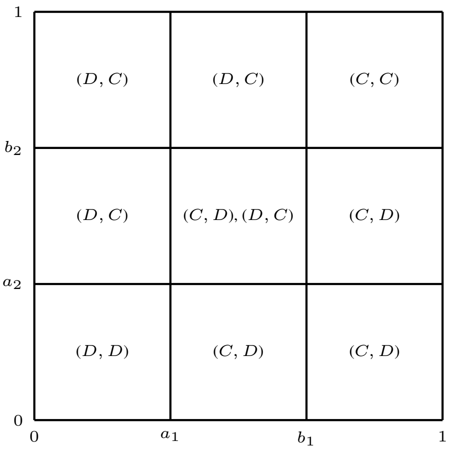


Figure 5.6: The set of Nash equilibria for each of the 9 generic regions ($a_1 < b_1,\ a_2 < b_2$).

The equilibria for different social coefficients in the case of $a_1 < b_1$ are given are presented in Figure 5.7, which includes the boundary points of these 9 regions. Note that on any boundary point of 2 or 4 generic regions, the set of equilibria is precisely the union of equilibria in the neighbouring generic regions.

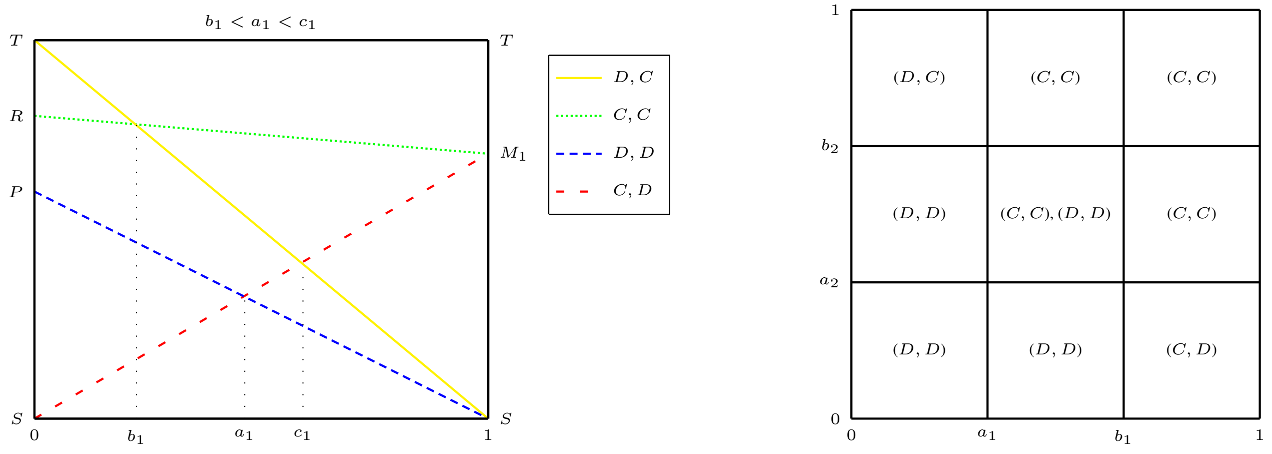
5.3.2 Case: $b_1 < a_1$ and $b_2 < a_2$

Figure 5.8 (a) illustrates the change in payoffs resulting from each outcome of the double game with different values of θ_1 in the case of $b_1 < a_1$. Figure 5.8 (b) provides the set of equilibria for each of

$b_2 < \theta_2 \leq 1$	(D, C)	(D, C)	(D, C)	$(C, C), (D, C)$	(C, C)
$\theta_2 = b_2$	(D, C)	$(C, D), (D, C)$	$(C, D), (D, C)$	$(C, C), (C, D), (D, C)$	$(C, C), (C, D)$
$a_2 < \theta_2 < b_2$	(D, C)	$(C, D), (D, C)$	$(C, D), (D, C)$	$(C, D), (D, C)$	(C, D)
$\theta_2 = a_2$	$(D, D), (D, C)$	$(D, D), (D, C), (C, D)$	$(C, D), (D, C)$	$(C, D), (D, C)$	(C, D)
$0 \leq \theta_2 < a_2$	(D, D)	$(D, D), (C, D)$	(C, D)	(C, D)	(C, D)
	$0 \leq \theta_1 < a_1$	$\theta_1 = a_1$	$a_1 < \theta_1 < b_1$	$\theta_1 = b_1$	$b_1 < \theta_1 \leq 1$

Figure 5.7: Nash equilibria for different social coefficients for $a_1 < b_1$ and $a_2 < b_2$.

the 9 generic regions.



(D, C)	(C, C)	(C, C)
(D, D)	$(C, C), (D, D)$	(C, C)
(D, D)	(D, D)	(C, D)

(a) (b)

Figure 5.8: (a) Change of payoffs (The variation of payoffs ($T > R > M_1 > M_2 > P > M_1' = M_2' = S$)). (b) The set of Nash equilibria for each of the 9 generic regions ($b_1 < a_1$, $b_2 < a_2$).

Figure 5.9 presents the equilibria for all possible social coefficients.

$a_2 < \theta_2 \leq 1$	(D, C)	(D, C)	(C, C)	(C, C)	(C, C)
$\theta_2 = a_2$	(D, C)	$(D, C), (D, D), (C, C)$	$(D, D), (C, C)$	$(D, D), (C, C)$	(C, C)
$b_2 < \theta_2 < a_2$	(D, D)	$(D, D), (C, C)$	$(D, D), (C, C)$	$(D, D), (C, C)$	(C, C)
$\theta_2 = b_2$	(D, D)	$(D, D), (C, C)$	$(D, D), (C, C)$	$(C, D), (D, D), (C, C)$	(C, D)
$0 \leq \theta_2 < b_2$	(D, D)	(D, D)	(D, D)	(C, D)	(C, D)
	$0 \leq \theta_1 < b_1$	$\theta_1 = b_1$	$b_1 < \theta_1 < a_1$	$\theta_1 = a_1$	$a_1 < \theta_1 \leq 1$

Figure 5.9: Nash equilibria for different social coefficients for $b_1 < a_1$ and $b_2 < a_2$.

Now we consider the double game for the Prisoner's Dilemma where the social (altruistic) coefficient of each player forms a finite discrete set of incomplete information or types thus giving rise to a Bayesian game.

5.4 Double Game with Incomplete Information

We now assume that the players do not know each other's social coefficients prior to any game, which means that they do not know the full values of the payoff matrix. Thus, the game has two-sided incomplete information. We assume that the social coefficient of each player has a finite number of possible values and that the probability distribution of the social coefficient is common knowledge between the two players. We also assume that, at the start of the game, each player is aware of the value of its own social coefficient (private information) or type, but not the value of the social coefficient of the opponent. In that way, each player can rely on a probabilistic inference to predict the opponent's actions. From Figures 5.7 and 5.9, we see immediately that for extreme types $\theta_1, \theta_2 = 0, 1$, we have the pure Nash Equilibrium (D, D), (C, D), (D, C) and (C, C) and it follows immediately that in both cases of $a_1 < b_1$, $a_2 < b_2$, and $b_1 < a_1$, $b_2 < a_2$, we have pure regular games.

We present two specific examples with finite sets of types for two players. Assume that

$$T > R > M > P > M' = S \text{ and } a_1 < b_1,\ a_2 < b_2$$

Example 5.1. We choose the four discrete values, or types,

$$\theta_1^1 = 0,\ \theta_1^2 = a_1,\ \theta_1^3 = b_1,\ \theta_1^4 = 1,$$
$$\theta_2^1 = 0,\ \theta_2^2 = a_2,\ \theta_2^3 = b_2,\ \theta_2^4 = 1.$$

Note that θ_1^2, θ_2^2, θ_1^3, θ_2^3 give the values at the boundaries of the three generic regions in the unit square with each other. Figure 5.10 gives the set of Nash equilibria for all possible pairs (θ_1^m, θ_2^n) $(0 \le m, n \le 4)$.

θ_2^4	(D, C)	(D, C)	(C, C), (D, C)	(C, C)
θ_2^3	(D, C)	(C, D), (D, C)	(C, D), (D, C), (C, C)	(C, C), (C, D)
θ_2^2	(D, D), (D, C)	(D, D), (D, C), (C, D)	(C, D), (D, C)	(C, D)
θ_2^1	(D, D)	(D, D), (C, D)	(C, D)	(C, D)
	θ_1^1	θ_1^2	θ_1^3	θ_1^4

Figure 5.10: Nash equilibria for different social coefficients for $a_1 < b_1$ and $a_2 < b_2$ with four types per player.

From this figure, we see that for pairs of types where there is a choice of pure Nash Equilibrium, we can choose a pure Nash Equilibrium such that we obtain Figure 5.11.

θ_2^4	(D, C)	(D, C)	(C, C)	(C, C)
θ_2^3	(D, C)	(D, C)	(C, C)	(C, C)
θ_2^2	(D, D)	(D, D)	(C, D)	(C, D)
θ_2^1	(D, D)	(D, D)	(C, D)	(C, D)
	θ_1^1	θ_1^2	θ_1^3	θ_1^4

Figure 5.11: Nash equilibria chosen for different social coefficients from Figure 5.10.

From Figure 5.11, we see that the double game is completely pure regular with $(DDCC, DDCC)$ as a pure Bayesian Nash Equilibrium.

Example 5.2. We take 5 discrete values, or types for each player as follows,

$$\theta_1^1 = 0,\ \theta_1^2 = a_1,\ \theta_1^3 = \frac{a_1 + b_1}{2},\ \theta_1^4 = b_1,\ \theta_1^5 = 1,$$
$$\theta_2^1 = 0,\ \theta_2^2 = a_2,\ \theta_2^3 = \frac{a_2 + b_2}{2},\ \theta_2^4 = b_2,\ \theta_2^5 = 1.$$

Figure 5.12 gives the set of Nash equilibria for all possible pairs (θ_1^i, θ_2^j) $(0 \leq i, j \leq 5)$.

θ_2^5	(D, C)	(D, C)	(D, C)	$(C, C), (D, C)$	(C, C)
θ_2^4	(D, C)	$(C, D), (D, C)$	$(C, D), (D, C)$	$(C, D), (D, C), (C, C)$	$(C, C), (C, D)$
θ_2^3	(D, C)	$(C, D), (D, C)$	$(C, D), (D, C)$	$(C, D), (D, C)$	(C, D)
θ_2^2	$(D, D), (D, C)$	$(D, D), (D, C), (C, D)$	$(C, D), (D, C)$	$(C, D), (D, C)$	(C, D)
θ_2^1	(D, D)	$(D, D), (C, D)$	(C, D)	(C, D)	(C, D)
	θ_1^1	θ_1^2	θ_1^3	θ_1^4	θ_1^5

Figure 5.12: Nash equilibria for different social coefficients for $a_1 < b_1, a_2 < b_2$ with five types per player.

From this figure, we can see that the double game is not completely pure regular.

Chapter 6

Multi-Games with N Players

> In this chapter, we extend the concept of multi-games to consider games with N players and M games. Each player has the same set of strategies in the M basic games. We later define the class of pure regular multi-game where, for pairs of extreme types there are pure Nash equilibria in which the strategy of each player only depends on its own type. Similarly, we define the notion of a completely pure regular multi-game where there are pure Nash equilibria for all possible pairs of types for the N players, in which the strategy of each player only depends on its own type. We then go on to derive a test for establishing whether multi-games are completely pure regular and show that a pure Bayesian equilibrium for a completely pure regular multi-game can be obtained directly from this test, thus reducing the complexity of computation.

We suppose our multi-game with M basic games is uniform. For each type of player $i \in I = \{1, \ldots, N\}$, there are M points $\theta_{i1}, \theta_{i2}, \ldots, \theta_{iM}$ in the Euclidean space, $\mathbb{R}^M$. In other words, for all $j \in J = \{1, \ldots, M\}$ and a player i, we have a set of all possible types $\Theta_i = \{(\theta_{i1}, \theta_{i2}, \ldots, \theta_{iM}) : \theta_{ij} \geq 0, \sum_{j=1}^{M} \theta_{ij} = 1\}$. Each type $\bar{\theta}_i = (\theta_{i1}, \theta_{i2}, \ldots, \theta_{iM})$ for player i is equipped with a set of M weights θ_{ij} with $\sum_{j=1}^{M} \theta_{ij} = 1$. The standard $(M-1)$-simplex is the subset of $\mathbb{R}^M$ given by

$$\Delta_{\theta_{ij}}^{M-1} = \{(\theta_{i1}, \theta_{i2}, \ldots, \theta_{iM}) \in \mathbb{R}^M \mid \sum_{j=1}^{M} \theta_{ij} = 1 \wedge \theta_{ij} \geq 0 \ \forall j\}.$$

The simplex $\Delta_{\theta_{ij}}^{M-1}$ lies in the affine hyperplane H of dimension M-1 that is called the *extreme face* for player $i \in I = \{1, \ldots, N\}$, where $j \in J = \{1, \ldots, M\}$. Figure 6.1 shows the hyperplane H of dimension 2 for the set Θ_2 for player 2 where $M = 3$.

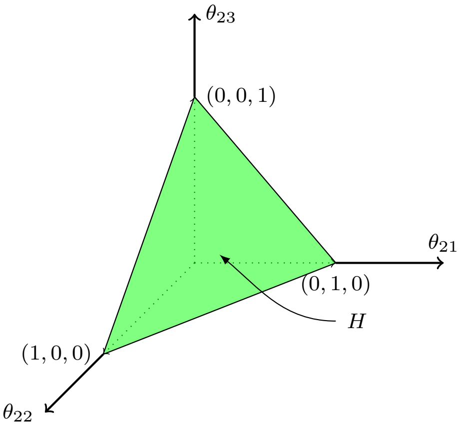


Figure 6.1: Illustration of a simplex of Θ_2 in a multi-game with three games.

In the finite discrete case, the finite set of types for each player is given by a set of types, say $\bar{\theta}_i^k = (\theta_{i1}^k, \theta_{i2}^k, \ldots, \theta_{iM}^k)$, $k = \{1, \ldots, M, \ldots, \ell_i\}$ where ℓ_i is the number of types for player $i \in I = \{1, \ldots, N\}$ and M is the number of games, and each type is restricted to its unit interval. For every game $j \in J = \{1, \ldots, M\}$, each player has an extreme type consisting of one component with value 1 and the rest with value 0;

$$\theta_{in}^j = \delta_{jn} = \begin{cases} 1, & j = n, \\ 0, & j \neq n. \end{cases}$$

We let game $\mathcal{G}^{(\theta_1, \ldots, \theta_i, \ldots, \theta_N)}$ where $\bar{\theta}_i = (\theta_{i1}, \theta_{i2}, \ldots, \theta_{iM})$, denote the multi-game $\mathcal{G}$ with the types taking the specific types $\theta_1, \ldots, \theta_i, \ldots, \theta_N$. In the discrete case, additionally we let game $\mathcal{G}^{k_1 k_2 \cdots k_N}$ denote the multi-game $\mathcal{G}$ with types $\theta_1^{k_1}, \theta_2^{k_2}, \ldots, \theta_N^{k_N}$ selected for N players respectively. We refer to a Nash equilibrium for game $\mathcal{G}^{(\theta_1, \ldots, \theta_N)}$ as a local Nash equilibrium for the multi-game $\mathcal{G}$. In the continuous case, we let game $\mathcal{G}^{\theta_1 \theta_2 \cdots \theta_N}$ denote the multi-game $\mathcal{G}$ with types $\theta_1, \theta_2, \ldots, \theta_N$ selected for the N players respectively.

Assume that we have a uniform N-player multi-game $\mathcal{G}$ with basic games $\mathcal{G}_1$ to $\mathcal{G}_M$. Given a player $i \in I = \{1, \ldots, N\}$, we denote, as usual, the strategy set of the opponent of i by S_{-i}. We denote the set of types for players other than player i by Θ_{-i}.

Next, we explain how information about the set of local pure Nash equilibria for multi-games with M games for various types of N players, can be used to deduce the Bayesian Nash equilibrium for the multi-games.

6.1 Pure Regular Multi-Games

We say a multi-game is pure regular if it has a set of M^N pure Nash equilibria for all players for their extreme types, and that the strategy of each player $i \in I = \{1, \ldots, N\}$ only depends on its own extreme type, which we re-label as $s_{ij} \in S_i$ where $j \in J = \{1, \ldots, M\}$ is the game for the extreme type.

Definition 6.1. *We say a multi-game with a finite set of types for each player* $i \in I = \{1, \ldots, N\}$*, is* pure regular *if there are actions* $s_{ij_i} \in S_i$ *for* $i \in I$ *and* $j_i \in J = \{1, \ldots, M\}$ *such that the strategy profiles* $(s_{1j_1}, \ldots, s_{ij_i}, \ldots, s_{Nj_N})$ *are pure Nash equilibria for the multi-game with player* i *having extreme types* $\theta_i^{j_i}$ *with respect to game* j_i*.*

In other words, we can think of pure regularity as an $M.N$ tuple of actions $(s_{11}, \ldots, s_{1M}; s_{21}, \ldots, s_{2M}; \ldots; s_{i1}, \ldots, s_{iM}; \ldots; s_{N1}, \ldots, s_{NM})$ in which within each block $(s_{i1}, \ldots, s_{iM})$ for a player $i \in I = \{1, \ldots, N\}$ there is a strategy corresponding to a particular game $j_i \in J = \{1, \ldots, M\}$. If we choose one strategy from each block then we have a strategy profile that is a pure Nash equilibrium for the multi-game with player i having extreme types $\theta_i^{j_i}$ with respect to game j_i.

Figure 6.2 shows a pure regular multi-game $\mathcal{G}$ with three players and three games, each player has a strategy set: $S_1 = \{s_{11}, s_{12}, s_{13}\}$, $S_2 = \{s_{21}, s_{22}, s_{23}\}$ and $S_3 = \{s_{31}, s_{32}, s_{33}\}$. We suppose that $\mathcal{G}$ has a set of $M^N = 3^3 = 27$ pure Nash equilibrium with three players having extreme types, for which the strategy of each player only depends on its own type. We say that the $3^3 = 27$ strategy profiles induce pure regularity.

Figure 6.3 shows simplices for a pure regular multi-game $\mathcal{G}$ with two players and three games, each player has a strategy set: $S_1 = \{s_{11}, s_{12}, s_{13}\}$ and $S_2 = \{s_{21}, s_{22}, s_{23}\}$. We set the same colours for the same Nash equilibria.

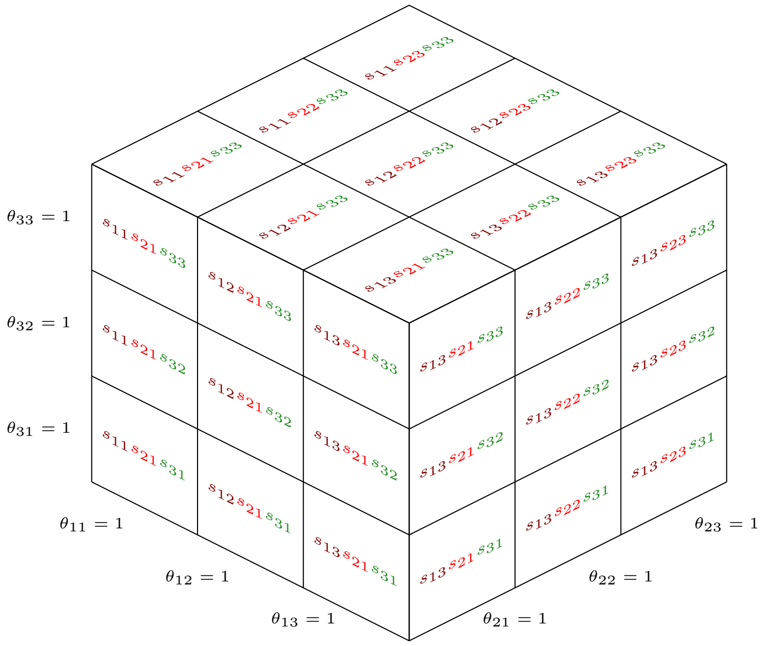


Figure 6.2: The pure regular strategies for all extreme types within a pure regular multi-game of three games and three players.

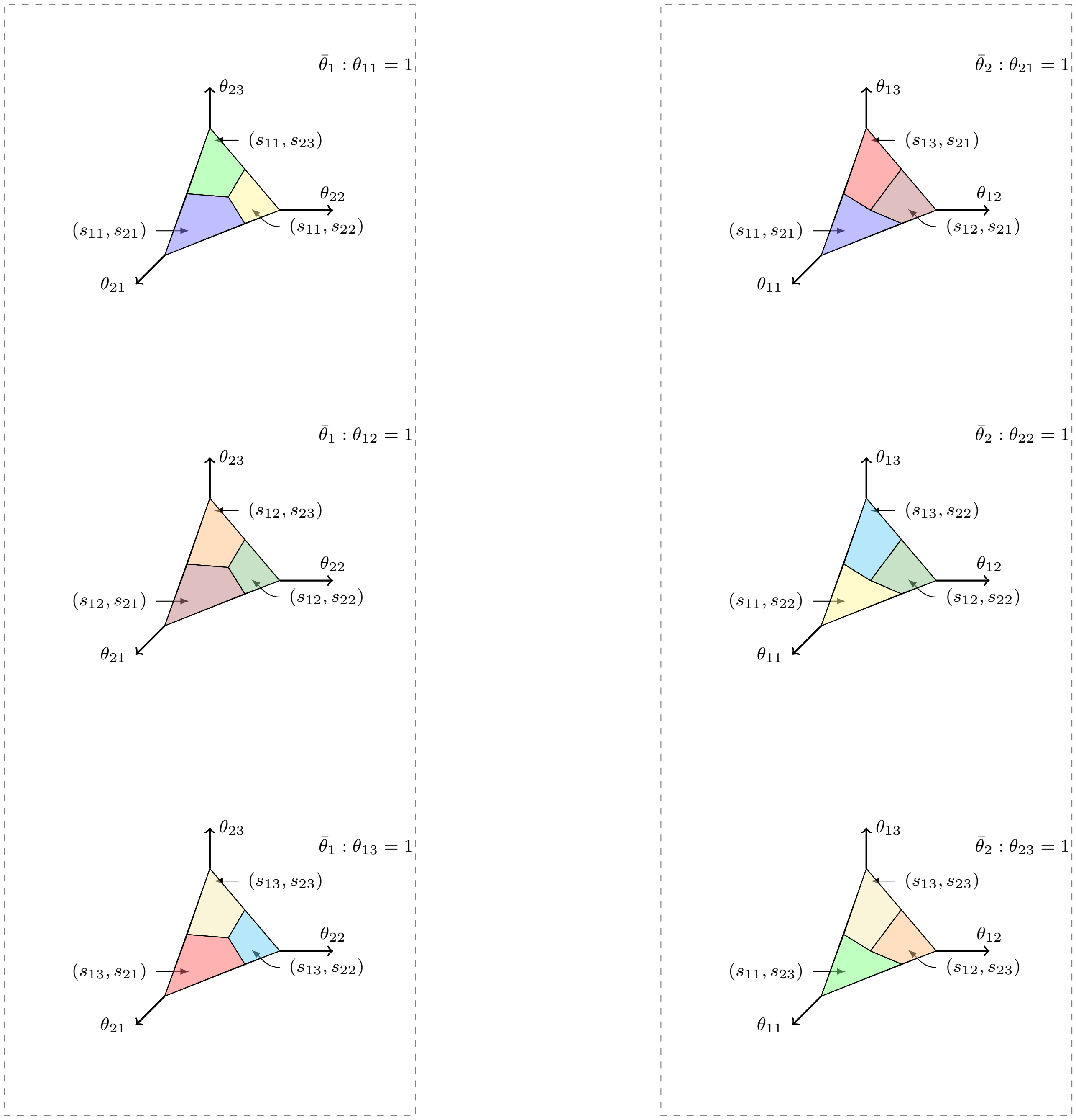
6.2 Completely Pure Regular Multi-Games

We say a multi-game is completely pure regular if it has a set of pure Nash equilibrium for all types θ_i^k, for which the strategy of each player $i \in I = \{1, \ldots, N\}$ only depends on its own type. In other words, the optimal response of each player only depends on its type. For each player i, there are l_i types and each type is a vector with M components.

Definition 6.2. *We say a multi-game with a finite set of types for each player* $i \in I = \{1, \ldots, N\}$ *given by* θ_i^k $(1 \leq k \leq \ell_i)$ *is* completely pure regular *if there are pure strategies* $s_k \in S_i$ *for* $(1 \leq k \leq \ell_i)$ *such that the strategy profiles* $(s_{k_1}, \ldots, s_{k_i}, \ldots, s_{k_N})$ *are pure Nash equilibria for the game* $\mathcal{G}^{k_1 \cdots k_i \cdots k_N}$ *for* $(1 \leq k_i \leq \ell_i)$.

It is clear that a completely pure regular multi-game is pure regular and thus our terminology is consistent.

The following example shows a multi-game $\mathcal{G}$ with two players and three games such that $\mathcal{G}$ is pure regular but $\mathcal{G}$ is not complete pure regular. Therefore, pure regularity in a multi-game is not a sufficient condition to guarantee that a multi-game is completely pure regular.



(a) (b)

Figure 6.3: The extreme faces in a pure regular multi-game with three games and two players.

Example 6.1. Consider a multi-game $\mathcal{G}$ with three games and two players. Each player has a strategy set: $S_1 = \{s_{11}, s_{12}, s_{13}\}$ and $S_2 = \{s_{21}, s_{22}, s_{23}\}$, with the payoff matrices of the three basic games given in Figure 6.4.

Figure 6.5(a) shows the pure Nash equilibria for $\mathcal{G}$ with player 1 and player 2 having their extreme types. As shown in the figure, the strategy of each player only depends on its own extreme type, thus

Player 2

Player 1

	s_{21}	s_{22}	s_{23}
s_{11}	$(3, 3)$	$(3, 2)$	$(3, 2)$
s_{12}	$(2, 3)$	$(2, 2.5)$	$(2, 2)$
s_{13}	$(2, 3)$	$(2, 2)$	$(2, 1)$

(a) Game 1

Player 2

Player 1

	s_{21}	s_{22}	s_{23}
s_{11}	$(2, 2)$	$(2, 3)$	$(2, 2)$
s_{12}	$(3, 2)$	$(3, 3)$	$(3, 1)$
s_{13}	$(2, 2)$	$(2, 3)$	$(2, 1)$

(b) Game 2

Player 2

Player 1

	s_{21}	s_{22}	s_{23}
s_{11}	$(2, 2)$	$(2, 2)$	$(1, 3)$
s_{12}	$(2, 2)$	$(2, 2)$	$(2, 3)$
s_{13}	$(3, 2)$	$(3, 1)$	$(3, 3)$

(c) Game 3

Figure 6.4: Payoff matrices representation for Example 6.1 (1).

$\mathcal{G}$ is pure regular. Figure 6.5(b) shows that for the given set of type $\bar{\theta}_2 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ for player 2 and extreme types for players 1, there are pure Nash equilibria but with different strategies for player 2, thus $\mathcal{G}$ is not complete pure regular.

	$\bar{\theta}_2 : \theta_{21} = 1$	$\bar{\theta}_2 : \theta_{22} = 1$	$\bar{\theta}_2 : \theta_{23} = 1$
$\bar{\theta}_1 : \theta_{11} = 1$	(s_{11}, s_{21})	(s_{11}, s_{22})	(s_{11}, s_{23})
$\bar{\theta}_1 : \theta_{12} = 1$	(s_{12}, s_{21})	(s_{12}, s_{22})	(s_{12}, s_{23})
$\bar{\theta}_1 : \theta_{13} = 1$	(s_{13}, s_{21})	(s_{13}, s_{22})	(s_{13}, s_{23})

(a)

	$\bar{\theta}_2 : (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$
$\bar{\theta}_1 : \theta_{11} = 1$	(s_{11}, s_{23})
$\bar{\theta}_1 : \theta_{12} = 1$	(s_{12}, s_{22})
$\bar{\theta}_1 : \theta_{13} = 1$	(s_{13}, s_{21})

(b)

Figure 6.5: Illustration of pure Nash equilibria for a variety of extreme types in Example 6.1 (1).

Figure 6.6 illustrates vector $\bar{\theta}_2 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ for extreme types for player 1. The green zone shows a subset of Θ_2 that $\mathcal{G}$, with respect to extreme types for player 1 and this subset, has a Nash equilibrium for, while player 2's strategy is s_{23}. Similarly, the blue zone and yellow zone are corresponding to pure Nash equilibria in which strategies of player 2 are s_{22} and s_{21} respectively.

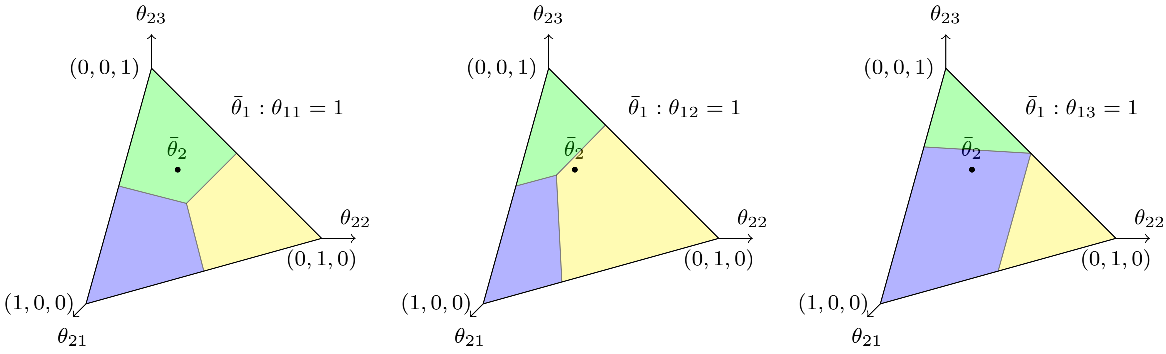


Figure 6.6: Illustration of different zones of pure Nash equilibrium for types of player 2.

Now, assume the payoff matrices of the three basic games are given as in Figure 6.7.

Figure 6.8(a) represents the pure Nash equilibria for $\mathcal{G}$ with player 1 and player 2 having extreme

		Player 2		
		s_{21}	s_{22}	s_{23}
Player 1	s_{11}	$(3,3)$	$(3,2)$	$(3,2)$
	s_{12}	$(2,3)$	$(2,2)$	$(2,2)$
	s_{13}	$(2,3)$	$(2,2)$	$(2,2)$

(a) Game 1

		Player 2		
		s_{21}	s_{22}	s_{23}
Player 1	s_{11}	$(2,2)$	$(2,3)$	$(2,2)$
	s_{12}	$(3,2)$	$(3,3)$	$(3,2)$
	s_{13}	$(2,2)$	$(2,3)$	$(2,2)$

(b) Game 2

		Player 2		
		s_{21}	s_{22}	s_{23}
Player 1	s_{11}	$(2,2)$	$(2,2)$	$(2,3)$
	s_{12}	$(2,2)$	$(2,2)$	$(2,3)$
	s_{13}	$(3,2)$	$(3,2)$	$(3,3)$

(c) Game 3

Figure 6.7: Payoff matrices representation for Example 6.1 (2).

types. The strategy of each player only depends on its own extreme type as shown in the figure, thus $\mathcal{G}$ is pure regular. Figure 6.8(b) shows that for a given set of types $\bar{\theta}_2 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ for player 2 and extreme types for player 1, there are pure Nash equilibria but with similar strategies for player 2, thus $\mathcal{G}$ is complete pure regular for type $\bar{\theta}_2 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$.

	$\bar{\theta}_2 : \theta_{21} = 1$	$\bar{\theta}_2 : \theta_{22} = 1$	$\bar{\theta}_2 : \theta_{23} = 1$
$\bar{\theta}_1 : \theta_{11} = 1$	(s_{11}, s_{21})	(s_{11}, s_{22})	(s_{11}, s_{23})
$\bar{\theta}_1 : \theta_{12} = 1$	(s_{12}, s_{21})	(s_{12}, s_{22})	(s_{12}, s_{23})
$\bar{\theta}_1 : \theta_{13} = 1$	(s_{13}, s_{21})	(s_{13}, s_{22})	(s_{13}, s_{23})

(a)

	$\bar{\theta}_2 : (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$
$\bar{\theta}_1 : \theta_{11} = 1$	(s_{11}, s_{23})
$\bar{\theta}_1 : \theta_{12} = 1$	(s_{12}, s_{23})
$\bar{\theta}_1 : \theta_{13} = 1$	(s_{13}, s_{23})

(b)

Figure 6.8: Illustration of pure Nash equilibria for a variety of extreme types in Example 6.1 (2).

In the following sections, we introduce an algorithm in order to establish whether we have a Bayesian Nash equilibrium which can be determined with lower computational complexity. With respect to the number of games M, this algorithm is applicable while the set of types Θ_i for each player $i \in I = \{1, \dots, N\}$ is split into at most M disjoint sets. Therefore, for applying the algorithm we restrict ourselves to having only at most M possible actions for each player to guarantee the set of types Θ_i for each player i is split into at most M disjoint sets. The following example represents a multi-game with two players and three games. One of the players can choose more than three actions (more than the number of games), which results in the set of types Θ_1 for player 1 is split into 4 disjoint sets, more than the algorithm would be able to use to give a suitable result.

Example 6.2. Consider a multi-game $\mathcal{G}$ with three games and two players. Player 1 has the strategy set $S_1 = \{s_{11}, s_{12}, s_{13}, s_{14}\}$ and player 2 has the strategy set $S_2 = \{s_{21}, s_{22}, s_{23}\}$, with the payoff matrices of the three basic games given in Figure 6.9.

		Player 2		
		s_{21}	s_{22}	s_{23}
Player 1	s_{11}	(3, 2.5)	$(3, 2)$	$(3, 2)$
	s_{12}	(1.5, 2.5)	(1, 2.3)	$(2, 2)$
	s_{13}	(2.5, 3.1)	$(2, 3)$	$(2, 2)$
	s_{14}	(2.9, 3.1)	(2.5, 3)	$(2, 2)$

(a) Game 1

		Player 2		
		s_{21}	s_{22}	s_{23}
Player 1	s_{11}	$(2, 2)$	$(1, 3)$	$(2, 2)$
	s_{12}	(3.5, 2)	$(3, 3)$	$(3, 2)$
	s_{13}	$(3, 2)$	$(2, 3)$	$(2, 2)$
	s_{14}	$(3, 2)$	$(2, 3)$	$(2, 2)$

(b) Game 2

		Player 2		
		s_{21}	s_{22}	s_{23}
Player 1	s_{11}	$(2, 2)$	$(2, 2)$	$(2, 3)$
	s_{12}	$(2, 2)$	$(2, 2)$	$(2, 3)$
	s_{13}	$(3, 2)$	$(3, 2)$	$(3, 3)$
	s_{14}	(2.5, 2)	$(2, 2)$	$(2, 3)$

(c) Game 3

Figure 6.9: Payoff matrices representation for Example 6.2.

Figure 6.10 (a) shows the pure Nash equilibria for the extreme types of player 1 and player 2. The strategy of each player only depends on its own extreme type as shown in the figure, thus $\mathcal{G}$ is pure regular. Strategy profile (s_{14}, s_{21}) is a pure Nash equilibrium for $\mathcal{G}$ with player 1 and player 2 having types $\bar{\theta}_1 = (\frac{1}{2}, \frac{1}{2}, 0)$ and $\bar{\theta}_2 = (1, 0, 0)$ respectively.

	$\bar{\theta}_2 : \theta_{21} = 1$	$\bar{\theta}_2 : \theta_{22} = 1$	$\bar{\theta}_2 : \theta_{23} = 1$
$\bar{\theta}_1 : \theta_{11} = 1$	(s_{11}, s_{21})	(s_{11}, s_{22})	(s_{11}, s_{23})
$\bar{\theta}_1 : \theta_{12} = 1$	(s_{12}, s_{21})	(s_{12}, s_{22})	(s_{12}, s_{23})
$\bar{\theta}_1 : \theta_{13} = 1$	(s_{13}, s_{21})	(s_{13}, s_{22})	(s_{13}, s_{23})

(a)

	$\bar{\theta}_2 : \theta_{21} = 1$
$\bar{\theta}_1 : \theta_{11} = 1$	(s_{11}, s_{21})
$\bar{\theta}_1 : \theta_{12} = 1$	(s_{12}, s_{21})
$\bar{\theta}_1 : \theta_{13} = 1$	(s_{13}, s_{21})
$\bar{\theta}_1 = (\frac{1}{2}, \frac{1}{2}, 0)$	(s_{14}, s_{21})

(b)

Figure 6.10: Illustration of pure Nash equilibria for a variety of extreme types in Example 6.2.

Figure 6.11 shows four different zones of pure Nash equilibrium for $\mathcal{G}$ with player 1 and player 2 having type $\theta_1 \in \Theta_1$ and $\bar{\theta}_2 = (1, 0, 0)$ respectively and the set of types Θ_1 is split into 4 disjoint sets.

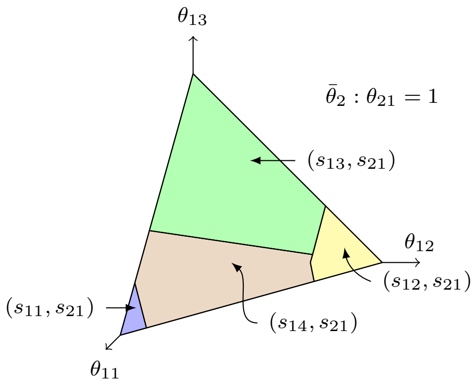


Figure 6.11: Illustration of a simplex with four different zones of pure Nash equilibria.

Theorem 6.1. *If a multi-game is completely pure regular, then for all conditional probability distributions for the types of the N players, the Bayesian pure strategy $(s_1(.), s_2(.), \dots, s_N(.))$, is a pure Bayesian Nash equilibrium.*

Proof. The proof is similar to proof Theorem 4.1. □

6.3 Separatrix Hyperplane

In a pure regular multi-game $\mathcal{G}$, assume all extreme types for the players other than player $i \in I = \{1, \dots, N\}$ are fixed. Let $u_i(s_{1j_1}, \dots, s_{ij_i}, \dots, s_{Nj_N}) = \bar{\theta}_i.\bar{a}_{ij_1\cdots j_i\cdots j_N}$ be the utility payoff for player i with type $\bar{\theta}_i$. For a given type $\bar{\theta}_{i\in I}$, strategy profile $(s_{1j_1}, \dots, s_{ij_i}, \dots, s_{Nj_N})$ is a Nash equilibrium if

$$\bar{\theta}_i.\bar{a}_{ij_1\cdots j_i\cdots j_N} \geq \bar{\theta}_i.\bar{a}_{ij_1\cdots k_i\cdots j_N}, \text{ where } j_i, k_i \in J = \{1, \dots, M\} \text{ and } j_i \neq k_i.$$

We let;

$$P_{i;j_1\cdots j_i\cdots j_N;j_1\cdots k_i\cdots j_N}(\bar{\theta}_i) = \bar{\theta}_i.(\bar{a}_{ij_1\cdots j_i\cdots j_N} - \bar{a}_{ij_1\cdots k_i\cdots j_N}) \geq 0,$$

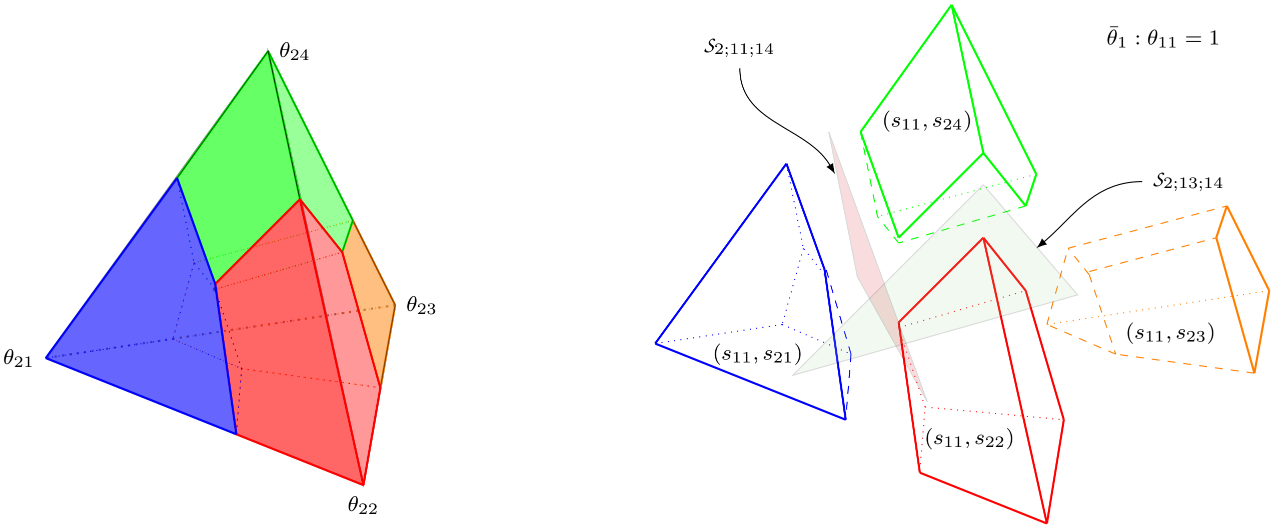
$$\mathcal{S}_{i;j_1\cdots j_i\cdots j_N;j_1\cdots k_i\cdots j_N} = \{\bar{\theta}_i | P_{i;j_1\cdots j_i\cdots j_N;j_1\cdots k_i\cdots j_N}(\bar{\theta}_i) = 0\}.$$

The hyperplane $\mathcal{S}_{i;j_1\cdots j_i\cdots j_N;j_1\cdots k_i\cdots j_N}$ is a separatrix hyperplane in $\mathcal{G}$. This separates the space of the types for player $i \in I = \{1, \dots, N\}$ where $j_i, k_i \in J = \{1, \dots, M\},\ j_i \neq k_i$, and the extreme types of the other players are fixed.

Like in a double game a multi-game is completely pure regular if the set of types is partitioned into polytopes with constant Nash Equilibria. Figure 6.12 shows an example of a simplex of a multi-game $\mathcal{G}$ with four games and two players, Each player has a strategy set: $S_1 = \{s_{11}, s_{12}, s_{13}, , s_{14}\}$ and $S_2 = \{s_{21}, s_{22}, s_{23}, , s_{24}\}$, separated by six separatrix hyperplanes while $\bar{\theta}_1 = (1, 0, 0)$.

The following example shows a multi-game $\mathcal{G}$ in which the separatrix hyperplanes do not intersect each other on the extreme face.

Example 6.3. Consider a multi-game $\mathcal{G}$ with three games and two players. Player 1 has the strategy set $S_1 = \{s_{11}, s_{12}, s_{13}\}$ and player 2 has the strategy set $S_2 = \{s_{21}, s_{22}, s_{23}\}$, with the payoff matrices



(a) (b)

Figure 6.12: Illustration of separatrix hyperplanes in a multi-game where $M = 4$ and $N = 2$.

of the three basic games given in Figure 6.13.

		Player 2		
		s_{21}	s_{22}	s_{23}
Player 1	s_{11}	(3, 2.5)	(3, 2)	(3, 2)
	s_{12}	(1.5, 2.5)	(1, 2.3)	(2, 2)
	s_{13}	(2.5, 3.1)	(2, 3)	(2, 2)

(a) Game 1

	Player 2		
	s_{21}	s_{22}	s_{23}
s_{11}	(2, 2)	(1, 3)	(2, 2)
s_{12}	(3.5, 2)	(3, 3)	(3, 2)
s_{13}	(3, 2)	(2, 3)	(2, 2)

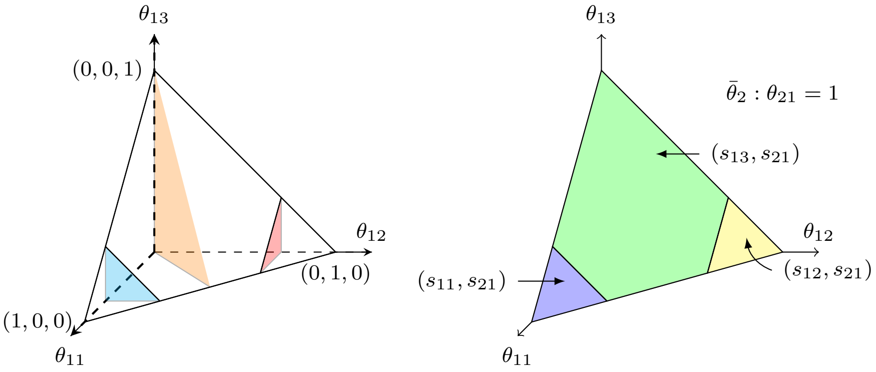
(b) Game 2

	Player 2		
	s_{21}	s_{22}	s_{23}
s_{11}	(2, 2)	(2, 2)	(2, 3)
s_{12}	(2, 2)	(2, 2)	(2, 3)
s_{13}	(3, 2)	(3, 2)	(3, 3)

(c) Game 3

Figure 6.13: Payoff matrices representation for three games.

Figure 6.14 shows an extreme face for player 1 separated by three separatrix hyperplanes while player 2 has chosen type $\bar{\theta}_2 = (1, 0, 0)$. Note the separatrix hyperplanes do not intersect each other on the extreme face in this particular example.



(a) (b)

Figure 6.14: Illustration of a simplex while the separatrix hyperplanes do not intersect each other on the extreme face.

Now, we present a test for establishing that a multi-game is completely pure regular.

6.4 Boundary Test

Consider a multi-game $\mathcal{G}$ with M games and N players. Let;

$$s_i(.) : \Theta_i \to S_i,\ i \in I = \{1, \ldots, N\},$$

$$s_i : \theta_i \mapsto s_{ij} \text{ if } \theta_i \in \Theta_{ij},\ j \in J = \{1, \ldots, M\},$$

$$s_i(\theta_i) = s_{ij}.$$

where Θ_i is a set of types for each player i, $\Theta_{ij} \subseteq \Theta_i$. Let $(\bar{\theta}_1, \ldots, \bar{\theta}_N)$ be a set of types for the N-player multi-game $\mathcal{G}$. For a pure regular $\mathcal{G}$, the boundary test is defined as follows:

- For each type $\bar{\theta}_i$, $i \in I = \{1, \ldots, N\}$, we test if there is a constant pure strategy s_{ij} such that the strategy profiles $(s_{1j}, \ldots, s_{ij}, \ldots, s_{Nj})$ where $j \in J = \{1, \ldots, M\}$, are pure Nash equilibria for all extreme types for all players other than player $i \in I$ with respect to game $j \in J$.

A multi-game $\mathcal{G}$ satisfies the boundary test if it is completely pure regular on the boundary i.e. when all types except for a single player are extreme.

Theorem 6.2. *A multi-game $\mathcal{G}$ is completely pure regular if, and only if, $\mathcal{G}$ is pure regular, and satisfies the boundary test.*

Proof. ($\Rightarrow$) The working assumption is that the multi-game $\mathcal{G}$ is completely pure regular. Bearing in mind Definition 6.1 and Definition 6.2, if we restrict the types to extreme types, then $\mathcal{G}$ is pure regular. As $\mathcal{G}$ is completely pure regular, if we restrict the types to boundary types then $\mathcal{G}$ satisfies the boundary test.

($\Leftarrow$) Let $(\bar{\theta}_1, \ldots, \bar{\theta}_N)$ be a set of types for the N-player multi-game $\mathcal{G}$. If the test succeeds then for all extreme types for the players other than player $i \in I = \{1, \ldots, N\}$ and for each type $\bar{\theta}_{i \in I}$,

$$P_{i;j_1 \cdots j_i \cdots j_N; j_1 \cdots k_i \cdots j_N}(\bar{\theta}_i) = \bar{\theta}_i.(\bar{a}_{ij_1 \cdots j_i \cdots j_N} - \bar{a}_{ij_1 \cdots k_i \cdots j_N}) \geq 0,$$

where $\bar{\theta}_i.a_{ij_1\cdots j_i\cdots j_N}$ is the utility payoff for player i with type $\bar{\theta}_i$ and $j_i, k_i \in J = \{1, \ldots, M\}, j_i \neq k_i$. We observe that $\bar{\theta}_i.(\bar{a}_{ij_1\cdots j_i\cdots j_N} - \bar{a}_{ij_1\cdots k_i\cdots j_N}) \geq 0$ is independent of θ_{-i}. Therefore, each $\bar{\theta}_i$ is in one of the disjoint sets Θ_{ij}. In addition the set of types Θ_i for each player i is split by separatrix hyperplanes into at most M disjoint sets

$$\Theta_i = \bigcup_{j=1}^{M} \Theta_{ij},\ \Theta_{ij} \cap \Theta_{ij'} = \emptyset,\ j \neq j',\ j' \in J,\ i \in I.$$

Let $s_i(.) : \Theta_i \to S_i$, $s_i : \bar{\theta}_i \mapsto s_{ij_i}$ if $\bar{\theta}_i \in \Theta_{ij_i}$. Based on our assumption, for each type $\bar{\theta}_{i\in I} \in \Theta_{ij_i}$, $s(\bar{\theta}_i) = s_{ij_i}$ such that;

$$P_{i;j_1\cdots j_i\cdots j_N;j_1\cdots k_i\cdots j_N}(\bar{\theta}_i) = \bar{\theta}_i.(\bar{a}_{ij_1\cdots j_i\cdots j_N} - \bar{a}_{ij_1\cdots k_i\cdots j_N}) \geq 0.$$

Therefore, the strategy profile $(s_{1j_1}, \ldots, s_{ij_i}, \ldots, s_{Nj_N})$ is a Nash equilibrium for $\mathcal{G}^{\bar{\theta}_1\cdots\bar{\theta}_N}$ such that s_{ij_i} only depends on player i's type. Thus $\mathcal{G}$ is completely pure regular. □

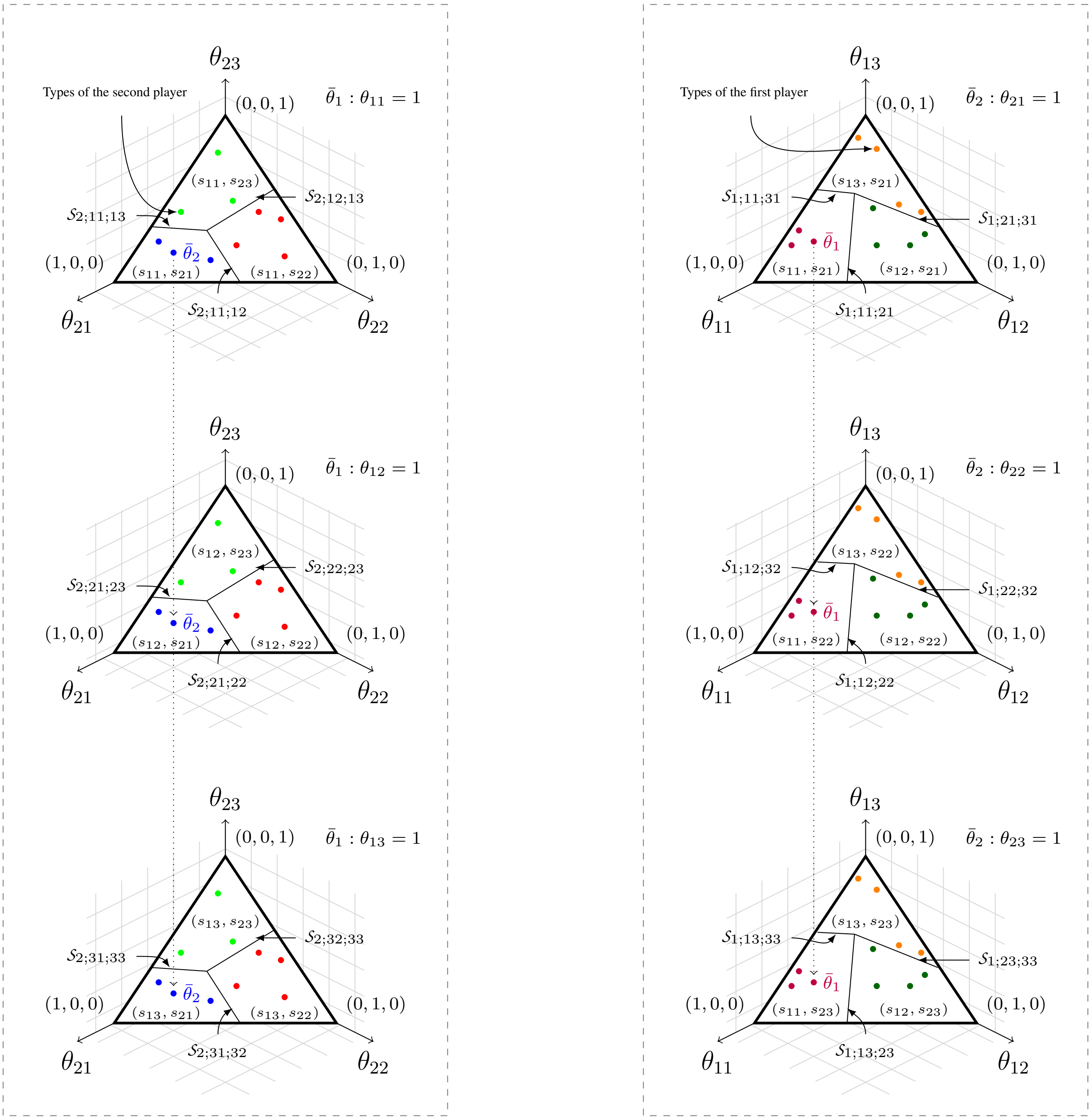
Figure 6.15 represents disjoint sets of types for a multi-game $\mathcal{G}$ with three games and two players, at the boundary types of two players.

By applying Theorem 6.2, Algorithm 2 can be employed to establish whether a multi-game $\mathcal{G}$ with finite types for the N players and M games is completely pure regular.

Algorithm 2: Algorithm to test for the property of being complete pure regular in multi-game $\mathcal{G}$ with *N* players and *M* games.

Input: An *N*-player multi-game $\mathcal{G}$ with *M* games.
Output: Multi-game $\mathcal{G}$ is complete pure regular or not.

```
CPR(G): Game G is completely pure regular.
PR(G): Game G is pure regular.
    begin
        if $CPR(\mathcal{G}) = PR(\mathcal{G})$ and the boundary test succeeds.
        then
            $\mathcal{G}$ is completely pure regular
        else
            $\mathcal{G}$ is not completely pure regular
    end
```



(a) (b)

Figure 6.15: Disjoint sets of types for a multi-game $\mathcal{G}$ with three games and two players at boundary types.

Corollary 6.1. *Given any multi-game with a finite number of types for N players and M games, we can decide if it is completely pure regular with computational complexity $O\big(NM^{N-1}\big)$.*

Proof. In a multi-game, the number of extreme faces when considering all combinations of extreme types is NM^{N-1}. Since Algorithm 2 checks the extreme faces, thus the computational complexity is $O\big(NM^{N-1}\big)$. □

For a given number of players, the complexity of this algorithm is polynomial in M, the number of strategies. For the class of completely pure regular multi-games, the Nash equilibria of the basic games can be used to compute a Bayesian Nash equilibrium of the multi-games with respect to the number of players and games. Let M^{ℓ_i} denotes the number strategies for player i in the expanded game therefore the number of strategies is M^{ℓ} where $\ell = \max_{1 \le i \le N} \ell_i$. If finding Nash equilibrium is "hard" in terms of the number of strategies, then the classical complexity is $O(2^{M^{\ell}})$ which is large even if ℓ and M are small compared to $O(NM^{N-1})$ which is polynomial in M for a given N.

Figure 6.16 and Figure 6.17 represent the extreme faces of the completely pure regular 3- player multi-game with three games, each player has a strategy set: $S_1 = \{s_{11}, s_{12}, s_{13}\}$, $S_2 = \{s_{21}, s_{22}, s_{23}\}$ and $S_3 = \{s_{31}, s_{32}, s_{33}\}$. We set the same colours for the same Nash equilibria. Algorithm 2 checks 27 extreme faces for this game.

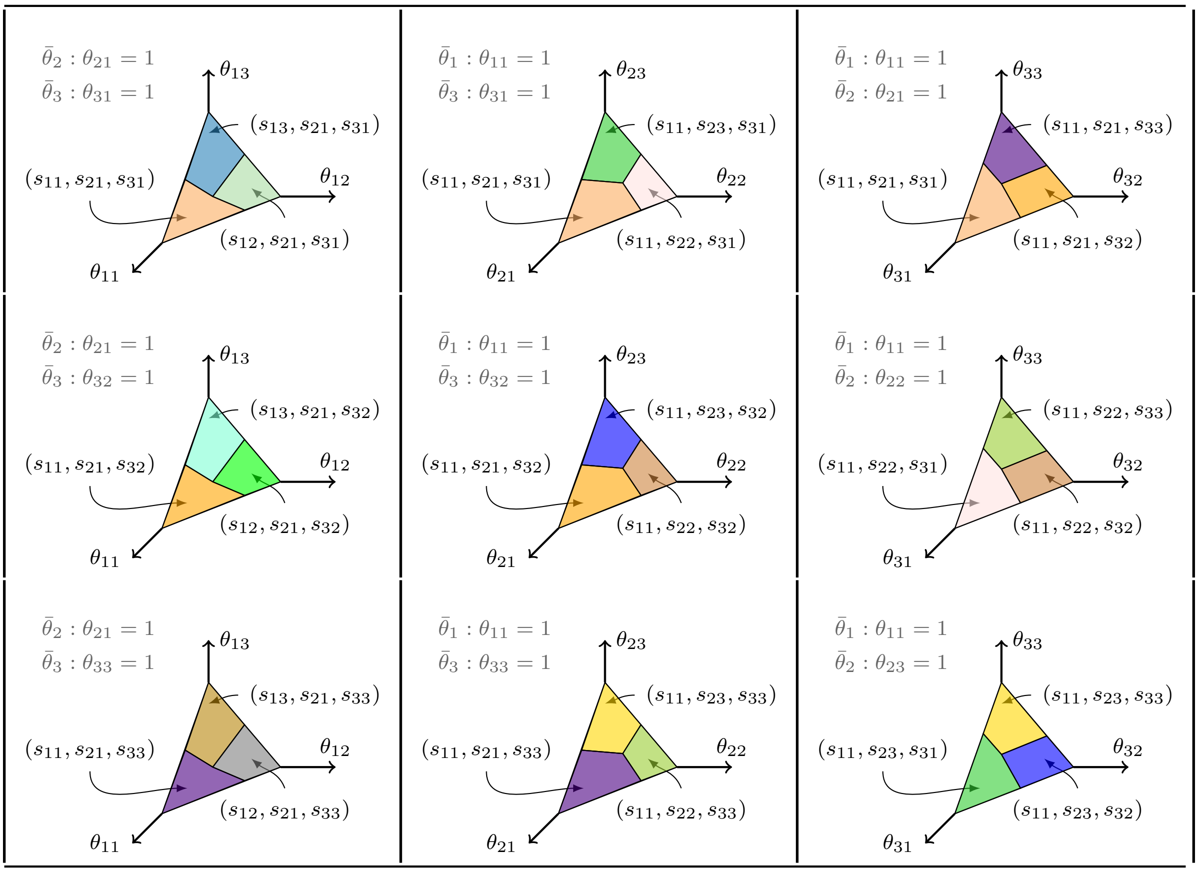


Figure 6.16: The extreme faces of the completely pure regular 3- player multi-game with three games (1).

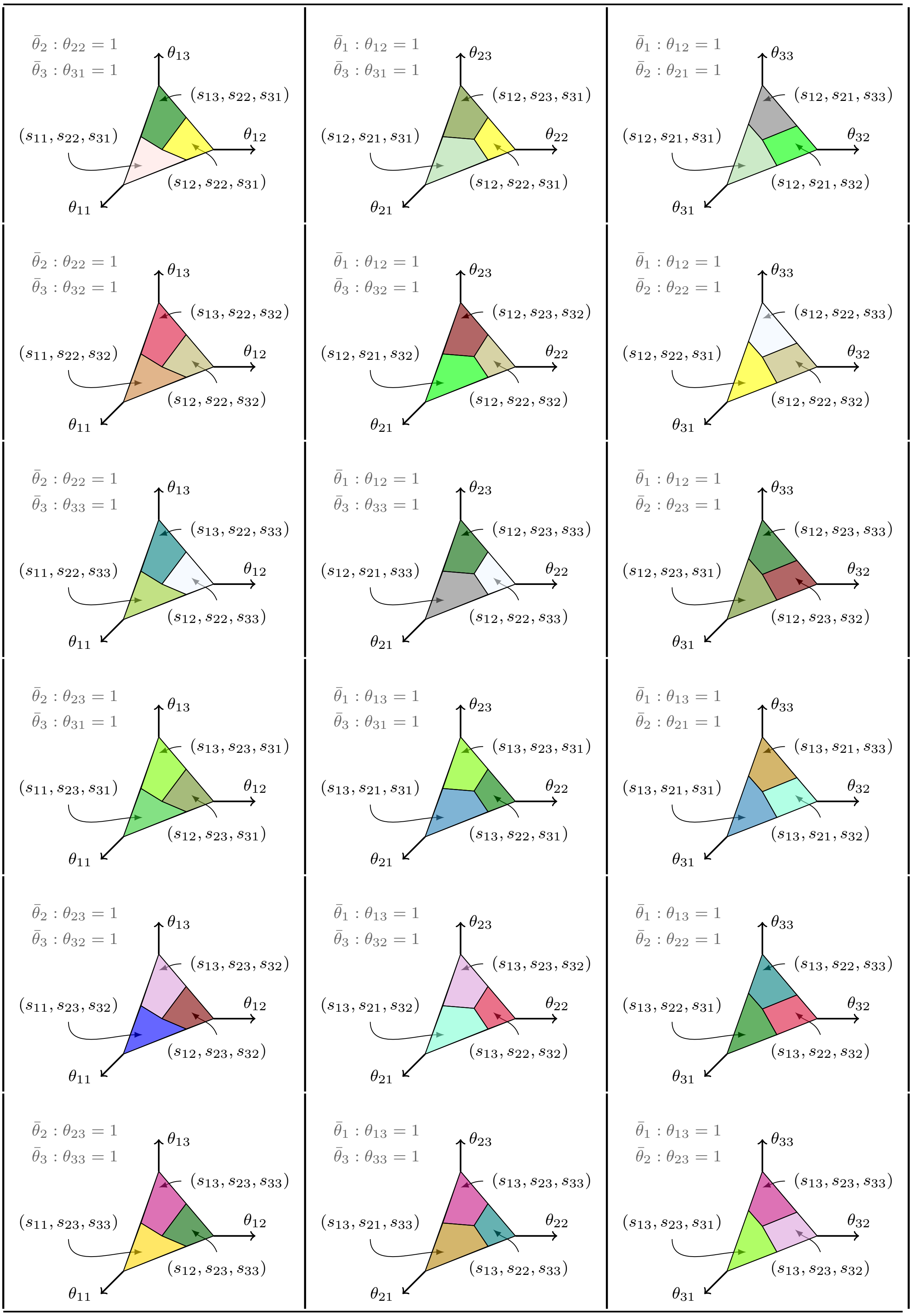


Figure 6.17: The extreme faces of the completely pure regular 3- player multi-game with three games (2).

Chapter 7

Simulation of Algorithms

> A computer program is developed on the basis of the proposed Algorithm 1 and Algorithm 2 in order to establish the existence of a Bayesian Nash equilibrium that can be determined with lower computational complexity. For validation and verification of the computer experiments, mathematical models were manually analysed according to the coded algorithms and methodology. The results sufficiently support the results of the software. The program was written in C# computer language.

What follows is more information about the computer program.

7.1 C# Implementation of Algorithm 1 and Algorithm 2

The implementation of Algorithm 1 and Algorithm 2 was an important aspect to this research, as it allowed the theoretical work to be put into perspective and provided an experimental test-bed for the different results. It was created using C#, while Excel and Matlab were used to test the results that assisted in the analysis of the algorithms. C# was the natural choice for implementation, due to its object-oriented characteristics. These helped to increase the level of organization and software engineering features in the code. The source code consists of two base classes, namely `Form` and `Controller`. The program consists of approximately 2,000 lines of code. The simulation code is available on `https://github.com/alighoroghi/thesis`.

7.2 Inputs and Outputs

Here we explain the data that the user has to enter into the system.

1. The number of games can be between two and ten games.
2. The number of players can be between two and ten players.
3. The number of strategies per player can vary for each player but it can be between two and the number of games being played.
4. For the number of types of players, we assume that every player has a set of extreme types (an extreme type for each game) plus a user determined number of non-extreme types. The value of non-extreme types can be randomly generated or chosen by the user.
5. Payoffs can be generated randomly or by the user.
6. The software indicates whether the multi-game is pure regular.
7. The software indicates whether the multi-game is completely pure regular. If yes, it shows the Bayesian Nash equilibrium.
8. For more than one run of the simulation, users determine the number of games, the number of players and the number of runs.

7.3 Implementation

This software contains two classes; `Form` and `Controller`. The `Form` class provides a graphical user interface for taking the user inputs and displaying the results, while the `Controller` class processes the user's inputs and computes the results. Figure 7.1 shows a UML diagram for the C# implementation of the algorithms.

Getting the user inputs:

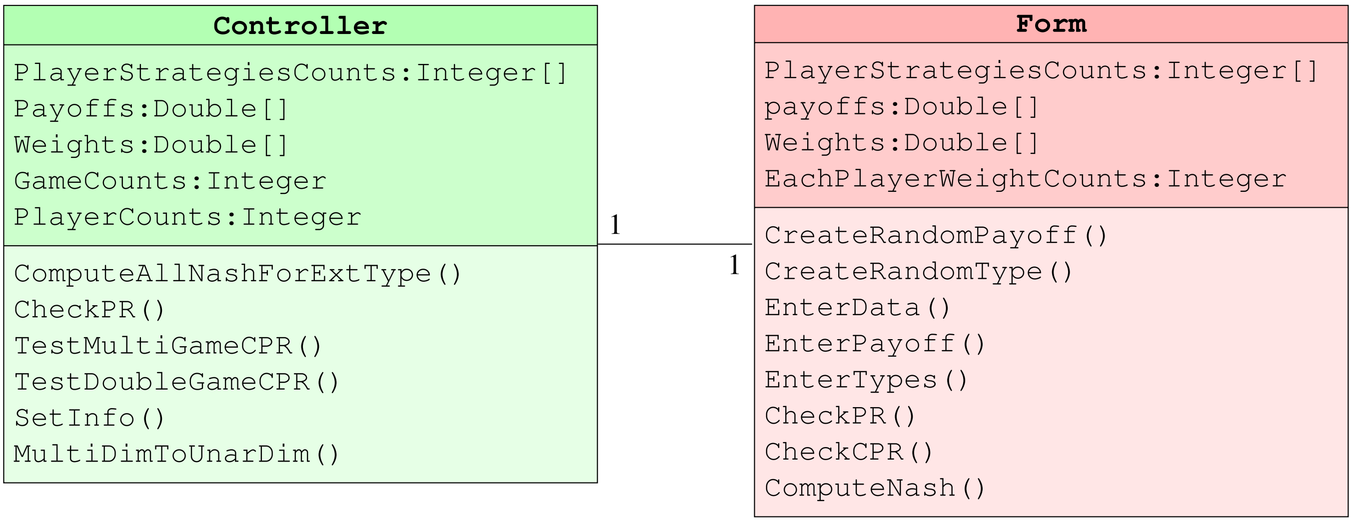


Figure 7.1: UML Diagram for the C# implementation of the algorithms.

- At first the user should input the number of games and players on the input tab and click on the `EnterData` button to store these values.
- The `EnterPayoff` function stores each players' payoffs.
- The `EnterTypes` function sets the non-extreme types (each player has six types).

Computing the results:

After setting the required values on the form the user should click on the `Compute Nash` button. This results in calling up the function `ComputeNash` in the `Form` class, which updates the variable values on the `Controller` class and calls up the `ComputeAllNashForExtType`. In this function all Nash equilibria are calculated based on all combinations of player's types. Then the `Controller` class determines if the game is pure regular and stores the result in the `ISPureRegular` variable.

Displaying the results:

- The function `CheckPR` in the `Form` class checks the `Controller`'s conclusion stored in `ISPureRegular` and notifies the user of whether the game was pure regular or not.
- The function `CheckCPR` in the `Form` class first checks if `ISPureRegular` is true and in that case calls up the `CheckCPR` function in the `Controller` class, which then determines

whether the game is completely pure regular. The result is stored in the `ISCompletePure-Regular` variable, and based on the result the `Form` class notifies the user of whether the game was completely pure regular or not. To avoid the complexity resulting from a large number of games or players, the inputs are formatted as lists using the function `MultiDimToUnarDim`.

7.4 Results of Simulations

Algorithm 1 and Algorithm 2 are simulated by generating 10,000 games with random payoffs and non-extreme types. The number of players, games and strategies were varied. All the computational experiments where run on a Windows machine with an Intel(R) Core(TM) i7, 2.00GHz CPU and 8GB of RAM.

To evaluate the results, we compute the number of double games or multi-games which have pure Nash equilibria when all players have extreme types ($\#$ of NE-ET). Similarly, we also compute the number of pure regular games ($\#$ of PRs) and completely pure regular games ($\#$ of CPRs).

First we discuss the results of the simulation for the double game. Figure 7.2 shows the results obtained from the simulation. A percentage for ($\#$ of NE-ET), ($\#$ of PRs) and ($\#$ of CPRs) were calculated for a varying number of types for each player (2 to 30 types) and compared to each other while the number of players are two, then three, four and five.

Figure 7.2 (a) shows that 70% of the two-player simulated double games had pure Nash equilibrium when each player chose one of his extreme types. Results also show that 51% of the games were pure regular, 5% more than the number of games that were completely pure regular. This means that on average, 73% of games with pure Nash equilibrium for extreme types are pure regular, which also contributes to 66% of the completely pure regular games. Figure 7.2 (b) shows a 3-player double game. We can see from the games simulated that only 32% had pure Nash equilibrium for extreme types, almost half as many as that of a two player double game. Similarly, the percentage of pure regular games and completely pure regular games also dropped with results showing that only 20% of games simulated were pure regular and 19%, completely pure regular. Figure 7.2(c) continues this trend, showing that as the number of players in the double game increases, there is a decrease in the number of games with pure Nash equilibrium for extreme types. Only 15% of the 4-player double games had pure Nash equilibrium for extreme types, while even less were pure regular games 10% and completely pure regular games 9%. Figure 7.2 (d) also follows the trend of a decline in ($\#$ of NE-ET), ($\#$ of PRs) and ($\#$ of CPRs) when the number of players increases to five.

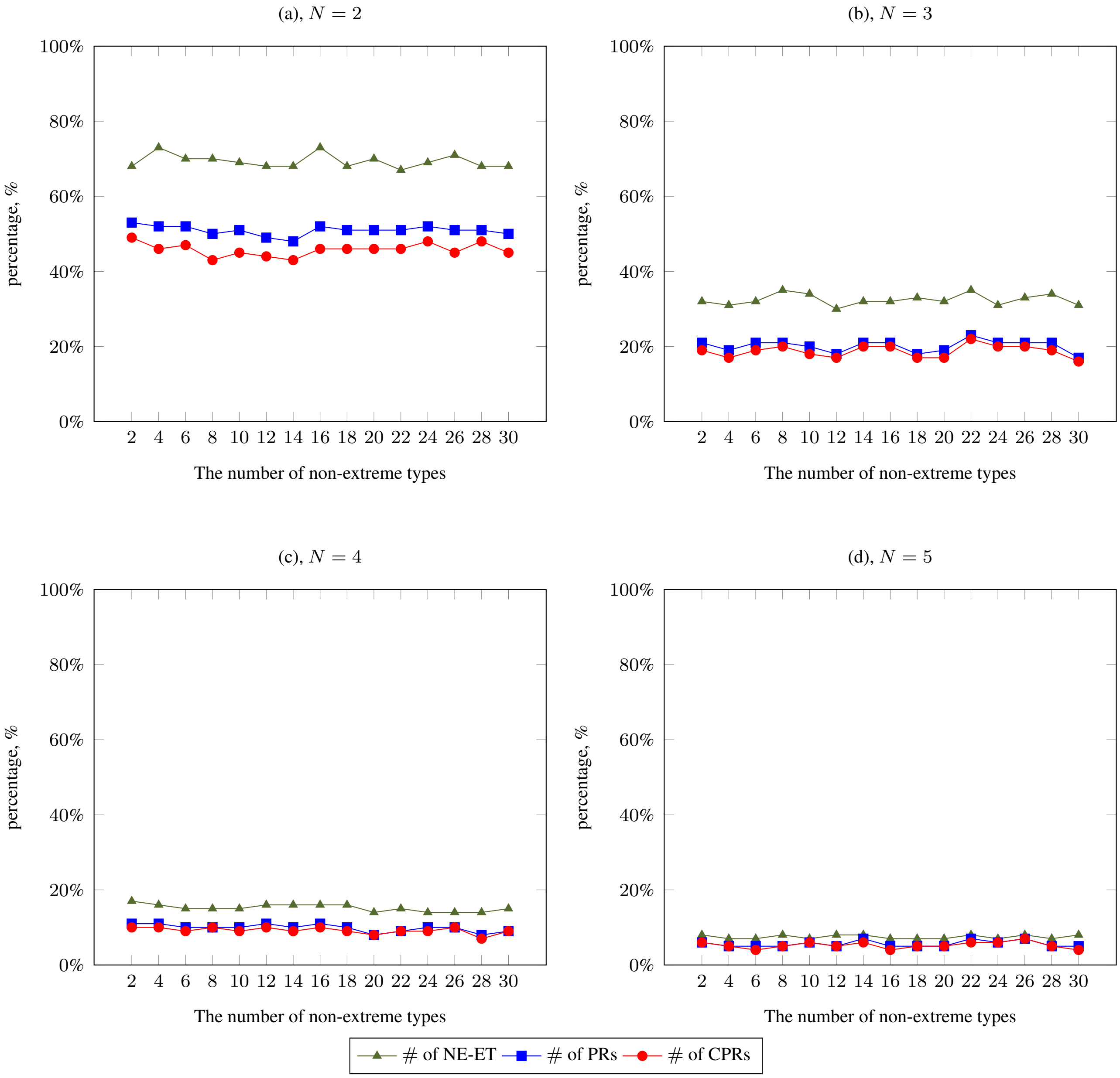


Figure 7.2: The simulation results for various numbers of players and types in the double game.

Figure 7.3 shows that about 69% of the two-player double games have pure Nash equilibrium when the players select their extreme types. The percentage of games that are pure regular is about 51% and the percentage of games that are completely pure regular is about 45%. Pooling the data according to the number of players and the number of double games suggests that there is a concerning decrease in (# of NE-ET), (# of PRs) and (# of CPRs) when the number of players increases.

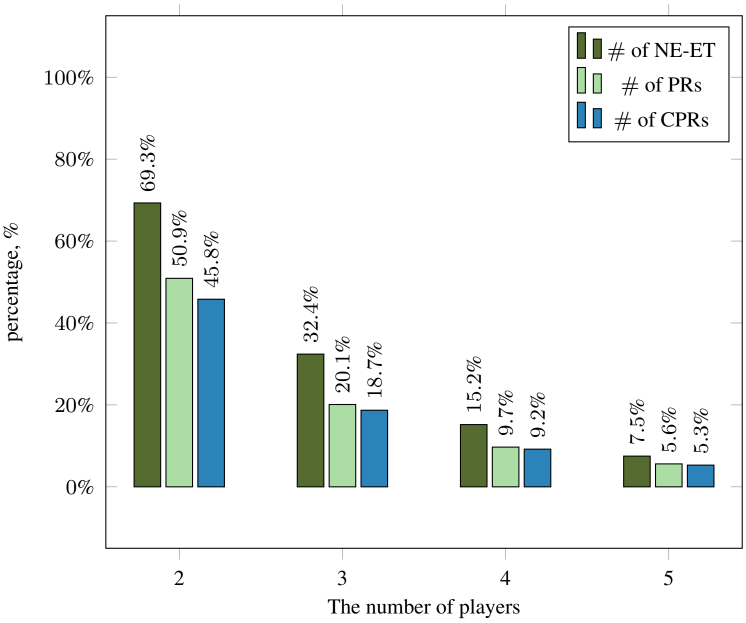


Figure 7.3: The simulation results for various numbers of players in the double game.

Now, we consider multi-games consisting of three games. Figure 7.4 (a, b) illustrates the percentages for (# of NE-ET), (# of PRs) and (# of CPRs) that were calculated for the varying number of types for each player. These were compared to each other while the number of players were two and three.

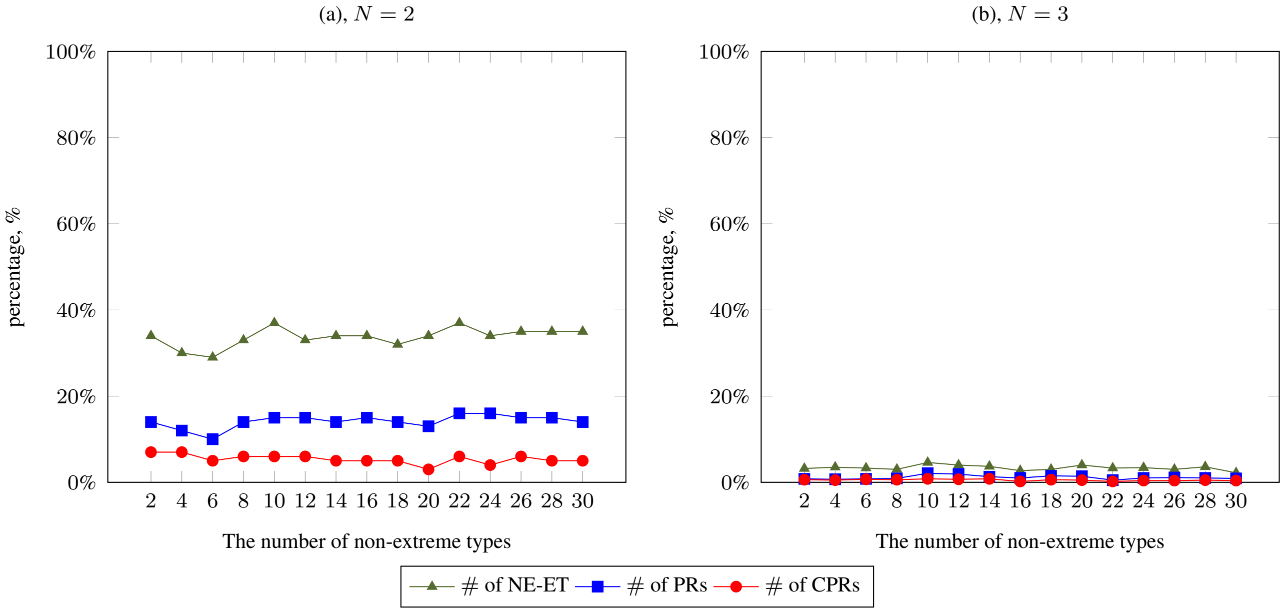


Figure 7.4: The simulation results for various numbers of players and types in the 3-game multi-game.

Figure 7.5 shows that about 33% percent of the multi-games with two players and three games have pure Nash equilibrium when the players utilise their extreme types. The percentage of pure regular games is about 14% while the percentage of completely pure regular games is about 5%. The percentages for (# of NE-ET), (# of PRs) and (# of CPRs) have dramatically decreased as the number of players increases.

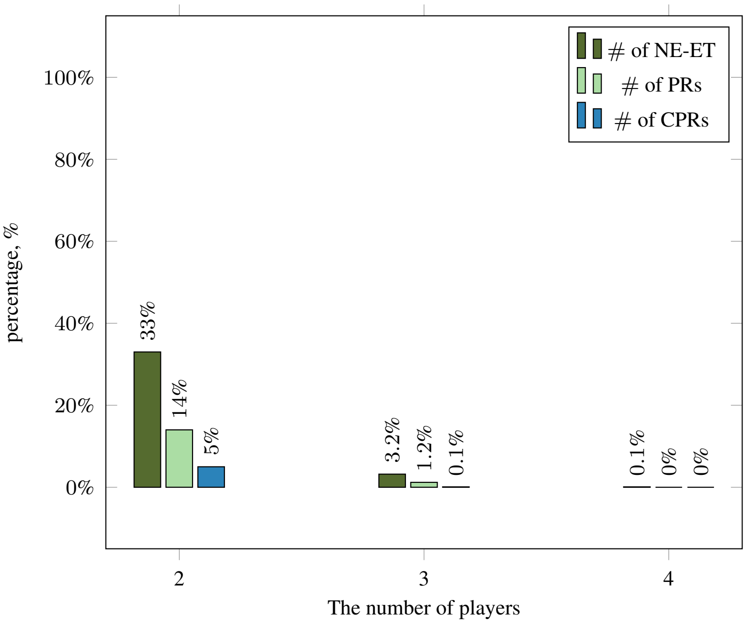


Figure 7.5: The simulation results for various numbers of players in the 3-game multi-game.

Figure 7.6 shows the results obtained from the simulation of varying numbers of types for each player, in the case of a multi-game consisting of four games and 2 players.

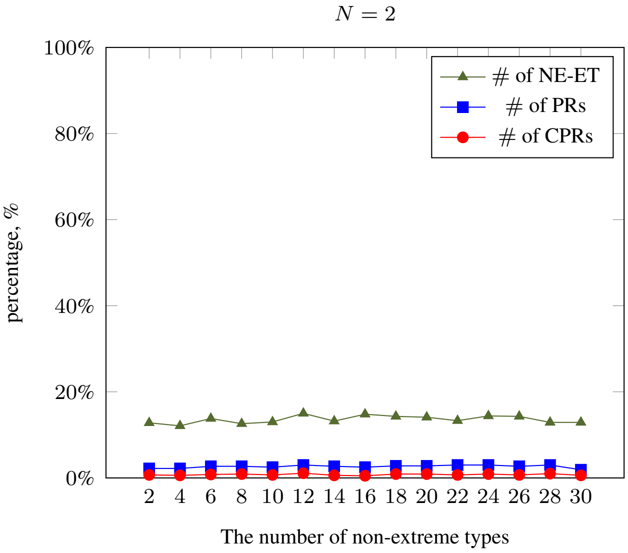


Figure 7.6: The simulation results for various number of types in the multi-game with four games and two players.

Figure 7.7 (a, b) represents the results obtained from the simulation of varying numbers of games containing two players and three players and two non-extreme types for each player. The results of the computer simulation indicate that both Algorithm 1 and Algorithm 2 can perform more efficiently when the double games and multi games have fewer players, compared to when they have to accommodate a larger number. Results show a decline in the number of games with pure Nash equilibrium for extreme types and consequently pure regular games and completely pure regular games as the number of players rises. Similarly, as the number of games increase, the algorithms perform less effectively.

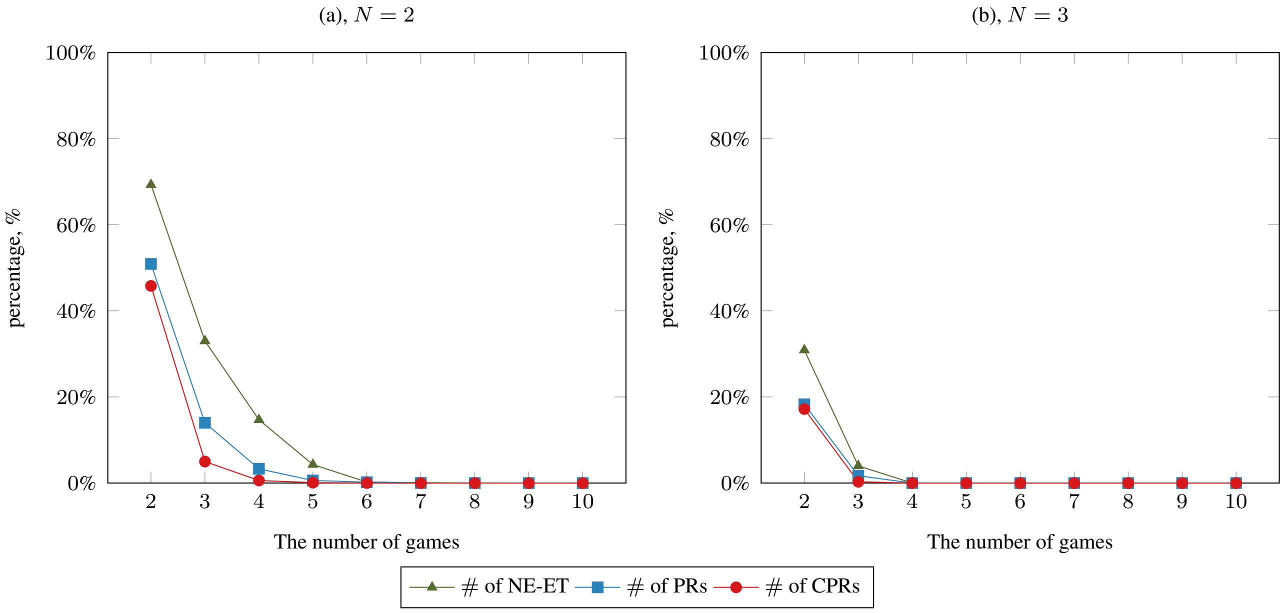


Figure 7.7: The simulation results for various numbers of games in the multi-game for $N = 2$ and $N = 3$ and two non-extreme types for each player.

It is also evident that the number of non-extreme types does not affect the number of games with pure Nash equilibrium for extreme types, as well as the number of pure regular and completely pure regular games.

Chapter 8

Iterated Multi-Game

> The multi-games become more interesting when we have repeated interactions and the players come to compete over a number of rounds. In this chapter, we introduce some attempts to apply repeated double games with incomplete information in order to analyze the effectiveness of strategies within iterated double games.

We let double game $\mathcal{G}$ (the stage game) be played a finite number of times. The following attempts used the style of a round robin tournament. A round robin tournament involves a group of players who play each other in turn at a number of rounds of the game. Each new round gives the player a better insight into her/his opponent and allows them to make decisions on the basis of her/his previous experience. We assumed that in the repeated double game, players are able to change their type for each round following repeated interactions in the round robin tournament, and so this model seems more rational and akin to real life.

8.1 Iterated Double Game for the Transactional Analysis Example

Recall example 3.2, where we discussed the transactional analysis, some strategies were designed for the iterated double game according to [Gho10]. The structure of the tournament was round-robin, and, thus, all the strategies in the iterated double game competed against all the other strategies and against themselves once. It was assumed that each player plays in response to actions that were chosen by her/him and the opponent before. The results of the simulation of 13 strategies showed that the Tit-for-Tat strategy had a higher average expected value of payoffs for variation of θ_i for player $i \in \{1, 2\}$ in 200 iterations.

In the Tit-for-Tat strategy, it is assumed that the attractiveness of the action AA is greater than AC, while AC is more attractive than CA, and CA is more attractive than CC. At stage $t = 0$, player i plays AA. From stage $t = 1$ to $t = T$, player i looks at the previous actions of both players; if both actions are the same, player i plays the same action; if the attractiveness of the opponent's action is less than player i's action, s/he plays an action with equal or greater attractiveness than the other player's last action; otherwise, s/he repeats the opponent's last action. For example, if player 1 chose CA and player 2 chose CC in the previous stage, player 1 plays AA, AC, or CA, all with a $\frac{1}{3}$ probability. If player 1 chose CC and player 2 chose CA in the previous stage, given that player 2 chose an action with higher attractiveness, player 1 then chooses CA in the current stage (the same as player 2's previous action).

8.2 Iterated Double Game with Morality Aspect

Recall the example in Section 5.4, two attempts were made to apply a repeated double game with a morality aspect.

8.2.1 Iterated Double Game for Altruistic - the First attempt

A computer tournament of the double game was made to operate as a framework, testing the validity of the theoretical results by comparing the performance of various competing strategies in [EGS12]. The structure of the tournament was a round-robin. Each game between any two strategies consisted of 200 rounds and the total score of a strategy within a game was the sum of the payoffs acquired from all the rounds. It was assumed that the numerical values for the payoff values are the following:

$$T = 5,\ R = 3,\ M = 2.5,\ P = 1,\ M' = 0 \text{ and } S = 0$$

A set of assumptions were made for the tournament:

- Given the aforementioned values for the payoffs, a strategy could score between 0 and 1000 in a stage game.
- A score of 0 can be obtained through the player either using a strategy that has a social coefficient equal to 0 and cooperating throughout the entire game, while the opponent only defects, or a strategy that has a social coefficient equal to 1 and defecting throughout the entire game, irrespective of what the opponent does.
- On the other hand, a score of 1000 can only be obtained through a strategy that has a social coefficient equal to 0 and defecting throughout the entire game, while the opponent only cooperates.
- A score of 200 can be obtained through two strategies that have social coefficients equal to 0 and mutual defection throughout the entire game.
- A score of 500 can be obtained through a strategy that has a social coefficient equal to 1 and cooperation throughout the entire game, irrespective of what the opponent does.
- A score of 600 can be obtained through two strategies that have social coefficients equal to 0 and mutual cooperation throughout the entire game.

A significant aspect of the tournament was the selection of a social coefficient according to the strategy. It was assumed that the social coefficient was part of a discrete set of five distinct values $0,\ a,\ \frac{a+b}{2},\ b,\ 1.$

The strategies were allowed to change their social coefficients within the game and adapt them to the environment they faced. However, since it is difficult to quantify exactly by how much human beings change their social values, the strategies were allowed to change their social coefficients stepwise, and, as a result, they could either increase them or decrease them by one value at any round. This was done to avoid having strategies changing their social coefficients from a value of 0 to that of 1 in a single round, since, it is believed that, only under extreme and unprecedented circumstances would such a sudden change occur in one's social values.

The strategies participating in the tournament varied in several ways, such as the choice made for the first round of the game and the initial social coefficient. Some strategies take into account the decisions that the opponent has made up to the point of consideration in the game, some use probabilistic estimations and even randomness in making their decisions, some have already made up their mind and follow rules that do not change according to the flow of the game. In essence, a strategy consists of an algorithm and so it operates according to certain instructions, changes the social coefficient and provides the decision of whether to cooperate or defect.

The initial social coefficient of a strategy shows its intentions, since a low social coefficient usually implies proneness to defection, a high social coefficient on the other hand implies cooperation. Varying initial social coefficients across tournaments means changing initial conditions, and, as a result, dynamic environments. Certain strategies have complex ways for dealing with their opponents' initial behaviour, and, so, what they may infer from it, may, in some cases, pre-determine the rest of the course of the game. In addition, most strategies have algorithms that modify their social coefficient in almost every round and adapt to the environment that has been developed from their opponents' actions. Then, they can respond effectively to both cooperative and defective behaviours and not be restricted by their choice of initial social coefficient.

As mentioned in the theoretical part of the analysis of the double game in Section 5.4, different kinds of behaviour are observed when θ_1 and θ_2 change.

With $0 \leq \theta_1 \leq a$ and $0 \leq \theta_2 \leq a$, the Nash equilibrium is provided by (D, D). For such social coefficients, we expect to see defective behaviours, as strategies try to recognise their opponents' intentions and see whether they can get away with defection, or if they will face retaliatory behaviour.

With $a < \theta_1 < b$ and $a < \theta_2 < b$, the Nash equilibria are provided by (C, D) and (D, C). For such social coefficients, the player who defects first has an advantage and dominates by gaining from an opponent's cooperation. However, with the increase of the social coefficient, cooperating can be beneficial, since a player can gain the reward from the social game. In that way, a lot of strategies change their behaviour at this stage and employ a cooperative approach to the game.

With $b \leq \theta_1 \leq 1$ and $b \leq \theta_2 \leq 1$, the Nash equilibrium is provided by (C, C). For such social coefficients, strategies with cooperative behaviour can gain the social rewards and not suffer the social punishments of the social game.

The following strategies were the main strategies used in [EGS12]:

SEG: SEG is based on two parts; deciding whether to cooperate or defect and altering the social coefficient based on some pre-defined conditions. For the former, it behaves as the Nash equilibria indicate, thus, its decision of whether to cooperate or defect depends only on the theoretical work and the results drawn from it. For the latter, it changes its social coefficient according to the following conditions:

- If SEG chooses C and its opponent chooses C in the previous round, it does not change its social coefficient.
- If SEG chooses C and its opponent chooses D in the previous round, it increases its social coefficient.
- If SEG chooses D and its opponent chooses C in the previous round, it decreases its social coefficient.
- If SEG chooses D and its opponent chooses D in the previous round, it increases its social coefficient.

ALLC: This strategy has an initial social coefficient equal to 0; it constantly chooses C and never changes its social coefficient.

ALLD: This strategy has an initial social coefficient equal to 0; it constantly chooses D and never changes its social coefficient.

Tit-for-Tat: During the first round, player $i \in \{1, 2\}$ cooperates and randomly chooses a moral coefficient; during subsequent rounds, if the opponent cooperated in the previous round then player i will increase her/his moral coefficient, and if her/his opponent defected then s/he will decrease her/his moral coefficient in the current round.

Positive-people strategy: During the first round, the player cooperates and randomly chooses a moral coefficient; during subsequent rounds, if the opponent cooperated in the previous round then s/he will increase her/his moral coefficient.

Negative-people strategy: During the first round, the player cooperates and randomly chooses a moral coefficient; during subsequent rounds, if her/his opponent defected then s/he will decrease her moral coefficient in the current round.

Nonsense-people strategy: During the first round, the player cooperates and randomly chooses a moral coefficient; during subsequent rounds, regardless of whether the opponent defected or cooperated in the previous round, s/he will not change her/his moral coefficient in the current round.

The results of the tournament showed that the winning strategy was SEG, as its average and cumulative scores were much higher than those of any other participating strategy. Its algorithm is a mixture of the results of theoretical work and some conditions on how to alter its social coefficient, so as to adapt to the course of action of any game. It works on the principle of adjusting its social coefficient based not only on its opponent's behaviour, but its own as well. If its opponent defected and it either cooperated or defected in the previous round, it increases its social coefficient to avoid the disastrous cycles of mutual defection that would be caused by a low social coefficient. It should also be mentioned that its initial social coefficient was 0, the result of which was that in the first round it defects.

8.2.2 Iterated Double Game for Altruistic - the Second Attempt

The most recent study about the repeated double game was conducted by Boyd [Boy12] using the framework of a tournament. The participants in the study included computer science, game theory, and machine-learning researchers as well as university students of varying educational levels. The aim of the study was to create a model for recording the effectiveness of strategies within iterated double games.

In Boyd's study, participants were offered a type of financial reward in order to encourage them to be willing to document and share their strategies, as well as to encourage them to create strategies as well as they could. The reward offered to participants was in the form of a reasonably large donation to a popular charity. The amount donated to charity was in line with the amount that would have been paid in cash.

The game was comprised of 100-200 rounds. The player with the highest-scoring strategy was declared the winning player, as follows:

$$\big((1-\beta) \times \text{Material score}\big) + (\beta \times \text{Social score})$$

where β represents the global social coefficient that reflects the social tendency; therefore reflecting the importance of the social score to overall society. β was set to $\frac{1}{2}$ to ensure that players were motivated by the monetary prize as well by the thought of benefiting others.

In the tournament it was decided to track the players' materially-motivated scores and socially-motivated scores as two different scoring groups, rather than tracking the overall score of each player. The players' scores in each of these groups were documented in aggregate as each round was played. Each player's present state was reflected through his aggregate scores in each of the motivation groups, along with θ. Because this method retained the significance of the separate scores, thus indicating the weight of each player's social and material motivations, this was believed to be a more effective method for calculating the dual game used in the study. In this tournament, the payoff amounts were as follows:

$$T = 5, R = 3, M_1 = M_2 = 2.5, P = 1, M_1' = M_2' = S = 0$$

The tournament also had a number of further restrictions:

- The social coefficient of every participant was represented by a fixed, distinct number from the set $\{0, 0.2, 0.4, 0.6, 0.8, 1\}$.
- Because people rarely alter their ethical beliefs to a great degree in a small time period, the social coefficient of an agent was only able to rise by 0.2 per round.
- Players were not allowed to try to collaborate immediately following a drop in their social coefficient after having defected; nor were they allowed to try to defect immediately following a collaboration which increased their social coefficient.

Gao [Gao12] used reinforcement learning theory to propose the most effective strategy as the winning strategy out of almost two-dozen possible strategies.

A Reinforcement Learning Based Strategy for the Double Game Prisoner's Dilemma

Founded on a reinforcement learning rule, Gao's strategy [Gao12] is highly appropriate in situations where the other players in repeated games are not known to the player. This is because there is no obligation to provide a framework of a player's environment in Gao's strategy.

The theory of reinforcement learning stems from the concept that when moves are associated with desirable outcomes, the likelihood that a player will perform those moves will increase. Similarly, a move that is associated with unwanted outcomes is less likely to be performed. Reinforcement learning entails two processes: firstly, the updating of the social coefficient; and secondly, the player deciding on an action.

Therefore, the social coefficient will initially be updated according to the Prisoner's Dilemma double game payoff matrix, along with the previous actions taken by each of the players. Once this has taken place, reinforcement learning is used to modify the other player's actions during the stage in which the player decides on an action.

Chapter 9

Conclusion and Future Work

9.1 Summary

Game theory can be used to analyze as well as predict human behaviour in several strategic situations. The main contribution of this work is to build appropriate mathematical methods using game theory to simulate the behaviour of opponents, and to understand its dominant factors in order to build more realistic models. We address these issues through the introduction of a novel model in game theory called multi-games. In multi-games, a given number of players divide up their resources according to different weights for a given number of games, which are then played out simultaneously. All players play at the same time but each can use the same set of strategies for the games. Players use a particular assortment of weights, one for each of the games. Combined, these signify the percentage of the players' investment in each of the games. The convex combination of the payoff a player acquired in the games, along with the assigned weights, makes up a player's total payoff. Within multi-games, basic games can be thought of as alternative environments for the players. It can be argued that investments in the global economy (in terms of various markets) can be modelled as multi-games. A certain class of Bayesian games is achieved when players' weights for certain multi-games involve private information or types that have certain conditional probability distributions.

We have shown that for the class of so-called completely pure regular multi-games with finite sets of types, the Nash equilibria of basic games can be used to compute a Bayesian Nash equilibrium for multi-games, with complexity independent of the number of the types and actions. Following the presentation of the main results, this thesis presented two algorithms for the purpose of establishing whether we have a Bayesian Nash equilibrium that can be determined with lower computational complexity.

9.2 Future Work

In this section we identify three possible future directions or work.

Although, throughout this work, some of the results have been proved for pure Bayesian Nash equilibria, future work leading on from our research will mainly be focused on extending the multi-games to compute mixed strategy Bayesian Nash equilibrium in N-player multi-games.

Another challenging question is if we can reduce the complexity of computing a Bayesian Nash equilibrium for a pure regular (but not completely pure regular) multi-game.

Furthermore, we will consider a particular class of the multi-games in which the payoff for player $i \in I = \{1, ..., N\}$ not only depends on θ_i but also depends on the other players' types. This framework can be used for modelling economic, human or technological behaviour in scenarios where each player can allocate their resources in varying proportions in order to play in a number of different environments. However, the payoff for each player is assumed to be the convex linear combination of payoffs obtained for the basic games, weighted by allocated proportions to their own and other players' weights. In other words, the purpose of using this model is to add a new dimension to multi-games by linearly combining the payoff matrices of various games and linking them through the use of all players' types for each player. This then represents the amount of investment that a player is willing to commit in that particular game when considering their own and other players' weights.

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