THE VERTEX GROUPS OF CONNECTED TREE PRODUCTS OF GROUPOIDS & HNN GROUPOIDS

THESIS SUBMITTED FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

(MATHEMATICS)

John Bruce Tough

April 1975

Imperial College, London

ABSTRACT

We define the term 'tree product of groupoids'. Then, using the theory of groupoids and defining a particular graph construction which we call a 'regular representative system', we prove that the vertex group of any connected tree product of groupoids is an HNN group with base-part some tree product of groups. For special connected tree products of groupoids we obtain a similar characterisation theorem without needing a 'regular representative system'. Also we define the term 'HNN groupoid', and prove that the vertex group of any connected HNN groupoid is an HNN group with base-part some tree product of groups. As an application of these results we characterise the subgroups of any tree product of groups, and the subgroups of any HNN group.

ACKNOWLEDGMENTS

I wish to thank Dr. B. Baumslag for his supervision and encouragement.

Also I wish to thank the Science Research Council for their grant.

TABLE OF CONTENTS

TABLE OF CONTENTS (CONT.)

INTRODUCTION

Hanna Neumann [8] has described the subgroups of any generalised product of groups as generalised products of groups, but her method is complicated and involves transfinite induction.

Karrass & Solitar [5] define a particular kind of generalised product of groups which they call a 'tree product of groups'. In particular a free product of two groups with an amalgamation is a special case of a tree product of groups. Then they show that if G is any free product of two groups with an amalgamation and H is any subgroup of G then H is a Higman, Neumann, Neumann group (HNN group) with base-part some tree product of groups. Their method does not use transfinite induction, and consists of defining a 'compatible regular extended Schreier system (cress) for G mod H' ([5] page 239), and then using a cress to construct a 'Kurosch rewriting process for G mod H' ([7] page 230). This produces a presentation for H, and the result follows from a detailed investigation of this presentation. However, they are unable to use the method to characterise the subgroups of an arbitrary tree product of groups.

Also Cohen [2] uses Serre's theory of groups acting on graphs to obtain a similar result to that of Karrass & Solitar, but again it is difficult to see how to generalise Cohen's method.

Our aim, here, is to describe the subgroups of any tree product of groups. Our method uses the theory of groupoids as described by Higgins [4]. To be more precise we define what we mean by a 'tree product of groupoids'. Then, using a graph construction which we call a 'regular representative system', we show that the vertex group of any connected tree product of groupoids is an HNN group with base-part some tree product of groups (theorem 3). From this result, and using a result of Higgins (proposition 8), it will follow that any subgroup of any tree product of groups is an HNN group with base-part some tree product of groups (theorem 7).

Further we define what we mean by an 'HNN groupoid'. Then we shall see that the vertex group of any connected HNN groupoid is an HNN group with base-part some tree product of groups (theorem 6). From theorem 6, and again using proposition 8, it will follow that any subgroup of any HNN group is an HNN group with base-part some tree product of groups. Similar results to this have been obtained by Karrass & Solitar [6] & Cohen [2].

Now we give a note on the convention we adopt in our work.

All groups and groupoids we consider will be multiplicative, and all maps will be written on the right. Any reference to other authors is denoted by using square brackets, for example (Higgins [4] page 31).

All definitions are underlined.

Chapter 1

PRELIMINARIES

In this chapter we give some basic definitions and results taken from group theory, graph theory and groupoid theory.

We begin in section 1 by defining the terms: graph, groupoid, graph homomorphism and groupoid homomorphism. Then we describe the notions of a 'path in a graph' and a 'connected graph'. Using these notions we define the terms 'free groupoid on a graph', and a special kind of graph called a 'tree'. Next we describe what is meant by a 'level-function on a tree induced by a vertex'. Finally we define the term 'quotient groupoid'. The definition of a 'level-function on a tree induced by a vertex' is due to Karrass & Solitar ([5] page 231). All the other definitions and results given in section 1 are due to Higgins [4] .

Next in section 2 we give the definition of a 'presentation for a groupoid'. We follow the definition given by Higgins ([3] page 10). As a special case we obtain the definition of a 'presentation for a group', and this definition agrees with the usual definition of a presentation for a group (see for example [7] page 7).

Then in section 3 we use the notion of a 'presentation for a groupoid' to define the term 'tree product of groupoids'. This is an obvious generalisation of Karrass & Solitar's definition of a 'tree product of groups' ([5] page 228). Also we give a result on tree products of groupoids which follows easily from a result on tree products of groups due to Karrass & Solitar ([5] page 231).

And in section 4 we give the well-known definition of an 'HNN group' (see, for example, [5] page 237). In addition we give a basic property of HNN groups (see [5] page 238).

Finally in section 5 we give three results which are of basic importance to our approach to the problem of characterising the vertex group of any connected tree product of groupoids.

1.1 On graphs & groupoids

1.1.1 Definition of a graph, groupoid, graph homomorphism and groupoid homomorphism

A directed graph consists of (1) a non-empty set of vertices I say, (2) a set of edges G say, and (3) an incidence map from G into the cartesian product IxI. For each edge g of G, if the image of g under the incidence map is (i,j) then we call i,j the initial, terminal vertex of g respectively. Also we call i and j the vertices of g. If i = j then g is a point, otherwise g is an arrow.

All graphs we consider will be directed, and so we omit 'directed' for convenience.

Any graph which contains no edges is called an empty graph (it consists simply of a set of vertices), and any graph which contains only points is called a discrete graph.

We sometimes call a graph with vertex set I, say, an I-graph.

For any pair of vertices i and j of any graph G, we write G_{ij} for the set of edges in G with initial vertex i and terminal vertex j.

A <u>groupoid</u> is a graph G together with,

(1) a law of partial multiplication: for any vertices i, j and k of G and any $g \in G_{ij}$, $h \in G_{jk}$ then the product gh is defined in G and belongs to G_{ik},

(2) associativity: for any vertices i,j,k and l of G and any $f \in G_{ij}$, $g \in G_{jk}$ and $h \in G_{kl}$ then $(fg)h = f(gh)$,

(3) a set of identities: for each vertex i of G there exists an element of G_{ii}, written e_i, such that for any vertices j and k of G and any $g \in G_{ij}$ and $h \in G_{ki}$ then $e_i g = g$ and $he_i = h$,

(4) an inverse law: for any vertices i and j of G and any $g \in G_{ij}$ there exists an element of G_{ji}, written g^{-1}, such that $gg^{-1} = e_i$ and $g^{-1}g = e_j$.

It is easy to see that for any groupoid G and any vertex i of G, then G_{ii} is a group, which we call the <u>vertex group of G at i</u>.

A <u>graph homomorphism</u> Θ: $G \rightarrow H$ is a pair of maps, one mapping the vertex set of G into the vertex set of H and called the <u>vertex map of Θ</u>, and the other mapping the set of edges of G into the set of edges of H and called the <u>edge map of Θ</u>, such that for each edge g of G the initial, terminal vertex of the image of g under the edge map of Θ coincides with the image of the initial, terminal vertex of g under the vertex map of Θ, respectively.

A groupoid homomorphism is just a graph homomorphism which preserves products and identity elements (and so also inverses).

A (graph) groupoid homomorphism is called a <u>(graph) groupoid surjection</u> if both its vertex map and edge map are surjections. A (graph) groupoid homomorphism whose vertex and edge maps are both injections is called a <u>(graph) groupoid injection</u>. A (graph) groupoid homomorphism satisfying both of these conditions is called a <u>(graph) groupoid isomorphism</u>.

Consider any (graph) groupoid homomorphism $\Theta : G \to H$. Let I be any set and suppose that G and H have vertex set I. If the vertex map of Θ is the identity map on I, then we call Θ a <u>(graph) groupoid I-homomorphism</u>. The definition of a <u>(graph) groupoid I-surjection, -injection and -isomorphism</u> follow in an obvious way.

A <u>subgraph</u> H of a graph G is a graph whose vertices, edges are contained in the set of vertices, edges of G respectively, and whose incidence map is simply the restriction of the incidence map of G.

Similarly a <u>subgroupoid</u> H of a groupoid G is a subgraph of G which contains the identity element e_i of G for each vertex i of H, and which is closed under multiplication and inverse.

Let G be any graph, and G_α ($\alpha \in A$) any collection of subgraphs of G. The <u>graph-union of G_α ($\alpha \in A$)</u>, written $\bigcup_{\alpha \in A} G_\alpha$, is that subgraph of G with vertex set the union of the vertex sets of the G_α and edge set the union of the edge sets of the G_α . Suppose that the intersection of the vertex sets of the G_α is non-empty. Then the graph-intersection of $G_\alpha (\alpha \in A)$, written $\bigcap_{\alpha \in A} G_\alpha$, is that subgraph of G with vertex set the intersection of the vertex sets of the G_α and edge set the intersection of the edge sets of the G_α . Let G_α and G_β be any subgraphs of G with common vertex sets. Then the graph-difference of G_α and G_β is that subgraph of G with vertex set the same as G_α (& G_β) and edge set consisting of the edges of G_α not belonging to G_β .

Let G be any groupoid, and H be any subgraph of G. By the subgroupoid of G generated by H we mean the graph-intersection of all the subgroupoids of G which contain H.

1.1.2 Paths & components

Let [n] denote the graph $0 \longrightarrow 1 \longrightarrow 2 \cdots n-1 \longrightarrow n$ with n + 1 vertices and n edges joining them in sequence ($n \geq 0$). If X is any graph and i,j are any vertices of X we define a directed path in X of length n from i to j to be a graph homomorphism, p: $[n] \longrightarrow X$ say, whose vertex map takes o to i and n to j. In particular, for each vertex i of X, there is one directed path in X of length 0 from i to i, which we denote by $\emptyset_i$, and which we call the empty path at the vertex i. Equivalently we may consider a directed path in X of length n ($n \geq 0$) to be a sequence of edges of X, $(x_1, \ldots, x_n)$ say, such that for each $1 < r \leq n$ the terminal vertex of x_{r-1}

coincides with the initial vertex of x_r. If $p = (x_1,\ldots,x_n)$ and $q = (y_1,\ldots,y_m)$ are directed paths in X from i to j and from j to k, say, respectively, then $pq = (x_1,\ldots,x_n, y_1,\ldots y_m)$ is a directed path in X from i to k. Clearly this multiplication of directed paths in X is associative.

Now we come to the notion of a 'path in X'.

For each edge x of X let us introduce the symbol $\bar{x}$, and let us define the initial, terminal vertex of $\bar{x}$ to be the terminal, initial vertex of x respectively. Let $\bar{X}$ denote the set of elements $\bar{x}$ as x ranges through X. Then, clearly, $\bar{X}$ is a graph with the same vertex set as X and with no edge in common with X. We define a <u>path in X</u> to be a directed path in $X \cup \bar{X}$ (by $X \cup \bar{X}$ we mean the graph with vertex set the same as X (& $\bar{X}$) and with edge set the union of the edge sets of X and $\bar{X}$). Then we see that for each edge x of X there are two paths in X of length 1, namely x and $\bar{x}$. However we still have only one path in X of length 0 at each vertex of X.

Let p be any path in X from i to j say. Then generalising some terminology given in (1.1.1) we call i and j the <u>vertices of p</u>. Also we call i the <u>initial vertex of p</u> and j the <u>terminal vertex of p</u>. Let us make the convention that for each edge x of X the symbol $\bar{\bar{x}}$ is to be identified with x. Then if $p = (y_1,\ldots,y_n)$ is a path in X from i to j, we have that $(\bar{y}_n,\ldots,\bar{y}_1)$ is a path in X from j to i, which we denote by $\bar{p}$.

A graph X is <u>connected</u> if there is at least one path in X from i to j for each pair of vertices i and j of X. A maximal connected subgraph of X is called a <u>connected component of X</u> or simply a <u>component</u> of X.

Similarly a maximal connected subgroupoid of a groupoid G is called a <u>(connected) component of G</u>.

It is easy to see that components of (graphs) groupoids are themselves (graphs) groupoids.

Let X be any graph and Y any subgraph of X. We say that Y <u>spans X</u> if for each pair of vertices i and j of X such that there is a path in X from i to j then there is also a path in Y from i to j.

Now we give the definition of a 'free groupoid'.

Let X be any graph and $p = (y_1, \ldots, y_n)$ be any path in X. If for some $1 \leqslant r < n$ $\bar{y}_{r+1} = y_r$ or $y_{r+1} = \bar{y}_r$ then $(y_1, \ldots, y_{r-1}, y_{r+2}, \ldots, y_n)$ is also a path in X which we call a <u>simple reduction of p</u>. Let us write $p \sim q$ if there exists a finite sequence of paths in X $(p=)\ p_0, p_1, \ldots, p_m (=q)$ $(m \geqslant 0)$ such that for each $1 \leqslant r \leqslant m$ p_r is a simple reduction of p_{r-1} or vice versa. This is an equivalence relation on the paths in X, and we write [p] for the equivalence class containing p. Since equivalent paths have the same initial, terminal vertex we can assign these as initial, terminal vertex of the equivalence class containing them. Then the set of equivalence classes of paths in X acquires the structure of a graph with vertex set the same as X. In fact this graph is a groupoid with multiplication as follows: if p and q are two paths in X such that the terminal vertex of p coincides with the initial vertex of q then $[p][q] = [pq]$. It is easy to see that this groupoid has identity elements $[\emptyset_i]$ where i ranges through the vertices of X, and the inverse is given by $[p]^{-1} = [\bar{p}]$. We call this groupoid the <u>free groupoid on X</u>.

We can describe free groupoids in another way as follows.

Let X be any graph and let $p = (y_1, \ldots, y_n)$ be any path in X. We call p <u>reduced</u> if for each $1 \leqslant r < n$ $y_r \neq \bar{y}_{r+1}$ (that is p has no simple reduction). Clearly any path in X is equivalent to an unique reduced path in X. We can give the set of reduced paths in X the structure of a groupoid as follows: if p and q are reduced paths in X from i to j and from j to k, say, respectively, then their product is defined to be the reduced path in X obtained from pq by successive simple reductions. It is not difficult to see that this multiplication is associative, and then it is clear that this groupoid is the free groupoid on X.

Let X be any I-graph and let F(X) be the free groupoid on X. Then it is easy to see that the inclusion map from X into F(X) is a graph I-homomorphism, and X generates F(X).

A result of Higgins ([3] page 14) tells us that the vertex group of any connected free groupoid is a free group.

Now we give another result due to Higgins ([4] page 35).

Proposition 1

Let G be any groupoid and X be any subgraph of G. Then G is the free groupoid on X iff each element of G is either an identity element or is uniquely expressed as a product $x_1^{\epsilon_1} \ldots x_n^{\epsilon_n}$ $(n \geqslant 1)$ where each x_r is an edge of X and $\epsilon_r = \pm 1$, and if for some $1 \leqslant r < n$ $x_r = x_{r+1}$ then $\epsilon_r = \epsilon_{r+1}$.

1.1.3 Trees

Let X be any graph, and $p = (y_1, \ldots, y_n)$ be any path in X. We call p closed if the initial and terminal vertex of p coincide. If there are no non-empty closed and reduced paths in X, then we call X circuit-free. A connected circuit-free graph is called a tree.

We have the following well-known result (see Higgins [4] page 40),

Proposition 2

(1) Every circuit-free subgraph of a graph X is contained in a maximal circuit-free subgraph of X.

(2) A circuit-free subgraph of X is maximal (among all circuit-free subgraphs) iff it spans X.

Corollary

Every connected graph is spanned by a tree.

Let X be any graph. A tree of X is a tree which is also a subgraph of X. Clearly, if X is connected, then any maximal tree of X has the same vertex set as X. Then the corollary says that every connected graph contains a maximal tree.

Let T be any tree and i be any vertex of T. For each vertex j of T let l(j) denote the length of that unique reduced path in T from i to j. Then the map l from the vertex set of T into the set of non-negative integers is called the level-function on T induced by i.

This definition is due to Karrass & Solitar ([5] page 231). Also we have the following result due to Karrass & Solitar ([5] page 231).

Proposition 3

For each vertex j of T other than i, there exists an unique vertex k of T such that $l(k) < l(j)$ and k,j are the vertices of an arrow of T.

We call the vertex k in this proposition the predecessor of j with respect to l, and it is easy to see that $l(k) = l(j) - 1$. For any vertex j of T we call l(j) the l-level of j.

1.1.4 Quotient groupoids

A subgroupoid N of a groupoid G is a normal subgroupoid of G if (1) G and N have common vertex set, and (2) for each $n \in N_{ii}$ and $g \in G_{ij}$ then $g^{-1}ng$ belongs to N_{jj}.

For any groupoid homomorphism, $\theta : G \longrightarrow H$ say, we define the kernel of θ, written $\ker\theta$, to be the graph of edges of G which map to identity edges of H under θ. Then $\ker\theta$ is a normal subgroupoid of G.

Let G be any groupoid, and H be any subgraph of G. By the normal subgroupoid of G generated by H we mean the graph-intersection of all the normal subgroupoids of G which contain H.

Let N be any normal subgroupoid of G. The components of N define a partition on the vertex set of G, and we write $\bar{i}$ for the class containing i, and $\bar{I}$ for the set of classes. Also, N defines an equivalence relation on the edges of G as follows: $g \equiv h(\text{mod } N)$ iff $g = n_1hn_2$ for some n_1, n_2 belonging to N. Two equivalent edges of G must have their initial vertices in the same component of N, and similarly for their terminal vertices, so each class $\bar{g}$ of edges can be assigned an unique initial, terminal vertex in $\bar{I}$. This assignment gives the set of equivalence classes of G, written G/N, the structure of an $\bar{I}$-graph. We now define a partial multiplication in G/N as follows: the product $\bar{g}\bar{h}$ is defined iff there exist $g_1 \in \bar{g}$ and $h_1 \in \bar{h}$ such that g_1h_1 is defined in G, and then $\bar{g}\bar{h} = \overline{g_1h_1}$. It is easy to check that this multiplication is well-defined. Moreover, with this multiplication, G/N becomes an $\bar{I}$-groupoid, with identity elements given by the components of N and inverses given by $\bar{g}^{-1} = \overline{g^{-1}}$ as g ranges through G. We call G/N a <u>quotient groupoid</u>. Note that the vertex map $i \to \bar{i}$ as i ranges through I, and the edge map $g \to \bar{g}$ as g ranges through G, constitute a groupoid surjection. Note also that if N is discrete then G/N is an I-groupoid, and the groupoid homomorphism just given is a groupoid I-surjection.

1.2 <u>Presentations for groups & groupoids</u>

Throughout this section let X be any I-graph, and G be any I-groupoid.

Let θ: $X \to G$ be any graph I-homomorphism, and let R be any discrete subgraph of F(X) (the free groupoid on X). Clearly θ extends to an unique groupoid I-homomorphism, θ': $F(X) \to G$ say. Then we say that <u>R holds in G under θ</u> if θ' maps each element of R to an identity element of G.

Now let θ : $X \to G$ be any graph I-homomorphism, and let R be any discrete subgraph of F(X) which holds in G under θ. Then we call the triple $\langle X, R, \theta \rangle$ an <u>I-presentation for G</u> if $X\theta$ generates G, and for each graph I-homomorphism ψ: $X \to H$ such that $X\psi$ generates H and R holds in H under ψ, there exists a unique groupoid I-homomorphism ϕ: $G \to H$ say such that $\theta\phi = \psi$.

This definition is taken from Higgins ([3] page 10).

If $\langle X,R,\Theta\rangle$ is any I-presentation for G then we call X the <u>generator graph</u> of the I-presentation, and we call R the <u>relator graph</u> of the I-presentation.

In the case that X is a subgraph of G and Θ is the inclusion map, then we abbreviate the notation $\langle X,R,\Theta\rangle$ to $\langle X,R\rangle$. Most of the presentations we consider will be of this kind.

Now we give a result due to Higgins ([3] page 10), Proposition 4

Let $\langle X,R,\theta\rangle$ be any I-presentation for G, and let N denote the normal subgroupoid of F(X) (the free groupoid on X) generated by R. Then G is groupoid I-isomorphic to the quotient groupoid F(X)/N (that is there is a groupoid I-isomorphism from G to F(X)/N).

Every I-groupoid G has an I-presentation. For choose any I-subgraph of G, X say, which generates G, and let N denote the kernel of the unique groupoid I-surjection from F(X) onto G extending the inclusion map from X into G. Then we have that G and F(X)/N are groupoid I-isomorphic, and it is not difficult to see that $\langle X,N\rangle$ is an I-presentation for G.

Suppose now that G is a group (that is a groupoid with a single vertex). Then the above definition gives us a presentation for G. This definition of a presentation for a group agrees with the usual definition of a presentation for a group (see, for example, [7] page 7).

If G is any group, then by the standard presentation for G, we mean that presentation for G with generators all of the elements of G and relators all expressions fgh^{-1} where f, g, and h range through G and $fg = h$ in G.

1.3 Tree products of groups & groupoids

Throughout this section let G_α ($\alpha \in A$) be any collection of I-groupoids which have mutually disjoint edge sets (for some vertex set I).

Suppose we are given a set of groupoid I-isomorphisms, Θ say, and that for each element θ of Θ the domain of θ is an I-subgroupoid of one of the G_α ($\alpha \in A$) and the range of θ is also an I-subgroupoid of one of the G_α ($\alpha \in A$).

Consider any $\theta \in \Theta$. Let us define the initial, terminal vertex of θ to be that groupoid among the G_α ($\alpha \in A$) which contains the domain, range of θ respectively. Then, clearly, with this definition Θ becomes a graph with vertex set the set of groupoids $\{G_\alpha : \alpha \in A\}$. If Θ is a tree, then we call Θ a tree of groupoids G_α ($\alpha \in A$). (Sometimes we shall abbreviate the phrase 'G_α is a vertex of Θ' to 'α is a vertex of Θ'. Under this abuse of definition we sometimes consider the vertex set of Θ to be A).

Now let Θ be any tree of groupoids G_α ($\alpha \in A$). For each $\alpha \in A$ choose any I-presentation for G_α, $\langle X_\alpha, R_\alpha \rangle$ say. Let G denote the I-groupoid which has the I-presentation with generator graph the union of the X_α ($\alpha \in A$), and relator graph the union of the R_α ($\alpha \in A$) together with the graph with points, $u(u\theta)^{-1}$ where u ranges through the domain of θ and θ ranges through Θ. (Here, for each $\theta \in \Theta$ with initial, terminal vertex G_α, G_β say respectively, then we suppose that in the point $u(u\theta)^{-1}$ u is written as a path in X_α and $u\theta$ is written as a path in X_β).

Then we call G a <u>tree product of groupoids G_α ($\alpha \in A$)</u>, or more precisely the <u>tree product of $\mathcal{C}\Theta$</u>.

Tree products of groupoids are special cases of generalised products of groupoids (see Higgins [3] page 15 for the definition of a generalised product of groupoids). It is straightforward to see that tree products of groupoids are independent of the particular presentations used in their definition (see, again, Higgins [3] page 15).

Now we give a short-hand notation for describing trees of groupoids. So let $\mathcal{C}\Theta$ be any tree of groupoids G_α ($\alpha \in A$), and choose any element 0 say of A, and let 1 denote the level-function on $\mathcal{C}\Theta$ induced by the vertex 0. (Note that we are here considering $\mathcal{C}\Theta$ to have vertex set A). Consider any $\theta \in \mathcal{C}\Theta$, and suppose the initial, terminal vertex of θ is G_α, G_β say respectively. If we denote the domain, range of θ by U_α, V_α respectively, then of course we have U_α is a subgroupoid of G_α and V_α is a subgroupoid of G_β. Also it is clear that either G_β is the predecessor of G_α with respect to 1 or vice versa. Without loss of generality we suppose that G_β is the predecessor of G_α with respect to 1. Then let us write θ_α for θ . If we use this convention for each edge of $\mathcal{C}\Theta$, then we can express $\mathcal{C}\Theta$ as $\mathcal{C}\Theta = \{\theta_\alpha : U_\alpha \longrightarrow V_\alpha , \alpha \in A\text{-}0\}$.

In this case it is clear that we can describe the tree product of $\mathcal{C}\Theta$ as that I-groupoid which has an I-presentation with generator graph the union of the X_α ($\alpha \in A$), and relator graph the union of the R_α ($\alpha \in A$) together with the graph with points, $u(u\theta_\alpha)^{-1}$ where u ranges through U_α and α ranges through $A\text{-}o$.

In the case that each G_α is a group, then the above two definitions give us a tree of groups G_α $(\alpha \in A)$ and a tree product of groups G_α $(\alpha \in A)$. These definitions agree with those given by Karrass & Solitar in [5] page 228 .

We close this section with a result on tree products of groupoids, which is a straightforward generalisation of a result on tree products of groups due to Karrass & Solitar ([5] page 232).

Proposition 5

Let G be any tree product of I-groupoids G_α $(\alpha \in A)$. Then for each element α of A the map $G_\alpha \longrightarrow G$ given by $g \to g$ (as g ranges through G_α) is a groupoid I-injection.

1.4 On HNN Groups

Let G be any group, and suppose we have a set of group isomorphisms $\{\theta_\alpha : U_\alpha \longrightarrow V_\alpha, \alpha \in A\}$ where for each $\alpha \in A$ U_α and V_α are subgroups of G. For each $\alpha \in A$ let us introduce the symbol t_α. Choose any presentation for G $\langle X, R \rangle$ say. Let H be that group which has a presentation with generators X together with the elements t_α $(\alpha \in A)$, and relators R together with the expressions, $t_\alpha u t_\alpha^{-1} (u\theta_\alpha)^{-1}$ as u ranges through U_α and α ranges through A .

Then we call H the HNN group with base-part G and free-part generated by the elements t_α $(\alpha \in A)$. Also for each $\alpha \in A$ we call θ_α the group isomorphism associated with the generator t_α.

It is straightforward to see that HNN groups are independent of the particular presentations used to define them (see, for example, [5] page 237).

We close this section with the following well-known result (again see [5] page 238).

Proposition 6

Let H be an HNN group with base-part G and free-part generated by $\{t_\alpha : \alpha \in A\}$. Then G is naturally embedded in H, and $\{t_\alpha : \alpha \in A\}$ freely generate a free subgroup of H (that is the free-part of H is a free group freely generated by $\{t_\alpha : \alpha \in A\}$).

1.5 Three basic results

In this final section we give three basic results involving some of the definitions we have discussed in the earlier sections. We shall use these results in our proofs of theorems 3, 5 and 6. The first result is due to Higgins ([3] page 13).

To begin with we need two definitions.

Let G be any graph or groupoid. By a set of representative vertices for G we mean a subset of the vertex set of G which contains precisely one vertex from each component of G (this unique vertex is called the representative vertex for the component).

Now let G be any connected I-groupoid with an I-presentation $\langle X,R\rangle$ say. Let F(X) be the free groupoid on X, and choose any maximal tree T of F(X). Also choose any vertex i of G. Consider any element r of R with vertices j say. Then by the <u>conjugation of r by T in i</u> we mean the path $t_j r t_j^{-1}$ where t_j is that unique reduced path in T from i to j.

<u>Proposition 7</u>

Let G be any I-groupoid. Choose any maximal circuit-free subgraph X of G, and any set of representative vertices J for G. For each representative vertex j let $\langle G_{jj}, R_{jj}\rangle$ be the standard presentation for the group G_{jj}. Then G has an I-presentation with generator graph $X \cup (\bigcup_{j\in J} G_{jj})$ and relator graph $\bigcup_{j\in J} R_{jj}$.

<u>Lemma 1</u>

Let $\Theta = \{\theta_\alpha : U_\alpha \longrightarrow V_\alpha, \alpha\in A\text{-}o\}$ be any tree of I-groupoids G_α ($\alpha\in A$), and let G be the tree product of Θ. For each $\alpha\in A$ choose an I-presentation for G_α $\langle X_\alpha, R_\alpha\rangle$, and for each $\alpha\in A\text{-}o$ choose any maximal circuit-free subgraph Z_α of U_α, and any set of representative vertices J_α for U_α.

Then G has an I-presentation with generator graph $\bigcup_{\alpha\in A} X_\alpha$, and relator graph $\bigcup_{\alpha\in A} R_\alpha$ together with the graph of points, $u(u\theta_\alpha)^{-1}$ where u ranges through $(U_\alpha)_{jj}$ and j ranges through J_α and α ranges through $A\text{-}o$, and the graph of points, $z(z\theta_\alpha)^{-1}$ where z ranges through Z_α and α ranges through $A\text{-}o$.

Proof

Let ⟨X,R⟩ denote the I-presentation described in the lemma.

Consider any $\alpha \in A\text{-}o$ and any $u \in U_\alpha$. Then to prove the lemma it suffices to show that the point $u(u\theta_\alpha)^{-1}$ belongs to the normal subgroupoid of F(X) (the free groupoid on X) generated by R.

To begin with, we have that u belongs to some component U'_α of U_α. Since Z_α is a maximal circuit-free subgraph of U_α, it follows that some component Z'_α of Z_α is a maximal tree of U'_α. Let j denote the representative vertex for U'_α. Then we can express u as $p^{-1}u_1q$, where u_1 belongs to $(U_\alpha)_{jj}$ and p,q is that unique reduced path in Z'_α from j to the initial vertex of u and from j to the terminal vertex of u, respectively. Then it is easy to see that $u(u\theta_\alpha)^{-1}$ can be expressed as a product of conjugates of the expressions $p(p\theta_\alpha)^{-1}$, $q(q\theta_\alpha)^{-1}$ and $u_1(u_1\theta_\alpha)^{-1}$. And so it follows that $u(u\theta_\alpha)^{-1}$ belongs to the normal subgroupoid of F(X) generated by R.

And so the lemma is proved.

Lemma 2

Let G be any connected I-groupoid with an I-presentation $\langle X, R \rangle$. Let F(X) be the free groupoid on X. Choose any maximal tree of F(X), and any vertex i of G. For each relator r let r' be the conjugation of r in i using this tree. Choose any set of free generators W for the free group $F(X)_{ii}$. For each relator r let r'' be the expression r' rewritten in terms of the set of free generators W.

Then (1) $\langle X, \{r' : r \in R\} \rangle$ is an I-presentation for G, and

(2) $\langle W, \{r'' : r \in R\} \rangle$ is a presentation for the vertex group of G at i.

Proof

First let N(R) and $N(\{r' : r \in R\})$ denote the normal subgroupoid of F(X) generated by R and $\{r' : r \in R\}$ respectively.

Clearly we have $N(R) = N(\{r' : r \in R\})$.

Then, by the remarks following proposition 4, we have that $\langle X, \{r' : r \in R\} \rangle$ is an I-presentation for G.

Now it is easy to see that the vertex group of the quotient groupoid $F(X)/N(\{r' : r \in R\})$ at the vertex i is the factor group of the free group $F(X)_{ii}$ modulo the group $N(\{r' : r \in R\})_{ii}$.

And so it follows that $\langle W, \{r'' : r \in R\} \rangle$ is a presentation for the vertex group of G at the vertex i.

And so the lemma is proved.

Chapter 2

CHARACTERISATION OF THE VERTEX GROUP OF ANY CONNECTED TREE PRODUCT OF GROUPOIDS

Throughout this chapter and the next, suppose we are given any collection G_α ($\alpha \in A$) of I-groupoids (for some vertex set I) whose edge sets are mutually disjoint, and any tree $\mathcal{\Theta}$ of I-groupoids, G_α ($\alpha \in A$), and let G be the tree product of $\mathcal{\Theta}$ and suppose that G is connected. Suppose we have chosen any element of I, which we call the 'origin', and any element of A, which we denote by 0. Further let λ denote the level-function on $\mathcal{\Theta}$ induced by 0. Finally we suppose that $\mathcal{\Theta}$ is expressed as $\mathcal{\Theta} = \{\theta_\alpha : U_\alpha \rightarrow V_\alpha\ ,\ \alpha \in A - o\}$ (see (1.3)).

Our object is to obtain a presentation for the vertex group of G at the origin, from which we hope to deduce the structure of the vertex group of G at the origin.

We begin, here, by describing the general procedure we shall use to obtain presentations for the vertex group of G at the origin.

Our starting-point is to choose any 'representative system' $\{Q_\alpha : \alpha \in A\}$ say. Of course we have not defined what we mean by a representative system, but at present it suffices to know that associated with $\{Q_\alpha : \alpha \in A\}$ we have for each $\alpha \in A$, a maximal circuit-free subgraph of G_α, X_α say, and a set of representative vertices for G_α.

Then for each $\alpha \in \Lambda$, using proposition 7 with the circuit-free graph X_α together with the given set of representative vertices for G_α, we construct an I-presentation for G_α.

Next for each $\alpha \in \Lambda - o$ we choose any maximal circuit-free subgraph of U_α and a set of representative vertices for U_α.

Now we use lemma 1 with the given circuit-free subgraph of U_α and the given set of representative vertices for U_α (as α ranges through $\Lambda - o$) together with the given I-presentation for G_α (as α ranges through Λ), to obtain an I-presentation for G.

We shall see that this I-presentation for G has the form $\langle X \cup Y, R \rangle$ where X is the graph-union of the X_α, and Y is some discrete graph. Note that $X \cup Y$ is a connected I-graph.

Now we choose any maximal tree of $F(X \cup Y)$ (the free groupoid on $X \cup Y$), and for each relator r let us write r' for the conjugation of r in the origin using this tree. Then from the first part of lemma 2 we have another I-presentation for G, $\langle X \cup Y, \{r' : r \in R\} \rangle$, and of course each relator in this I-presentation has vertices the origin.

Next using the representative system $\{Q_\alpha : \alpha \in \Lambda\}$ we describe a method for obtaining a set of free generators, W say, for the vertex group of $F(X \cup Y)$ at the origin.

For each relator r in R let us write r'' for the relator r' rewritten in terms of the free generators W. Then by the second part of lemma 2 we obtain a presentation for the vertex group of G at the origin $\langle W, \{r'' : r \in R\} \rangle$.

Finally from this presentation we shall try to deduce the structure of the vertex group of G at the origin.

In section 1 we define what we mean by a 'representative system', and we show how we can use representative systems to give us sets of free generators for the vertex group of F(XUY) at the origin (theorem 1 and corollary).

In section 2 we define a particular kind of representative system which we call a 'regular representative system', and we prove the existence of a regular representative system (theorem 2).

In section 3 we prove our main theorem (theorem 3). That is we shall prove that the vertex group of G at the origin is an HNN group with base-part some tree product of groups. We prove the theorem by choosing a regular representative system and following through the procedure outlined above.

Finally a word on terminology. Throughout this chapter and the next we shall abbreviate 'predecessor of α with respect to λ' to simply 'predecessor of α'.

2.1 <u>On representative systems</u>

To define a representative system we first need to choose for each $\alpha \in A$ a maximal circuit-free subgraph of G_α and a set of representative vertices for G_α which contains the origin.

For each $\alpha \in A$ choose any maximal circuit-free subgraph X_α of G_α, and let X be the graph-union of all the X_α-then X is a connected I-graph. Also, for each $\alpha \in A$, and each component of G_α which does not contain the origin we define the representative vertex for that component to be any vertex i of the component such that there exists a non-empty reduced path in X from the origin to i which does not end in an element of $X_\alpha^{\pm 1}$.

Now consider any element α of A.

For each non-origin representative vertex i for G_α, choose any non-empty reduced path in X from the origin to i which does not end in an element of $X_\alpha^{\pm 1}$. Consider the graph whose edges are all these chosen paths. Clearly this graph is a tree with vertex set the set of representative vertices for G_α. Let us write Q_α for the graph-union of this tree and the circuit-free graph X_α. Obviously Q_α is a maximal tree of F(X) (the free groupoid on X). For each vertex i let us write $q_{(\alpha, i)}$ for that unique reduced path in Q_α from the origin to i. In particular, then, if i denotes the origin we have that $q_{(\alpha, i)} = \emptyset_i$ (the empty path at the origin).

In this way we construct each Q_α.

Consider any non-origin representative vertex i for G_o. Then, of course, the non-empty path $q_{(o,i)}$ ends in an element of $X_\alpha^{\pm 1}$ for some $\alpha \in A - o$. In this case let us call i an <u>α-vertex for G_o</u>.

Then we call the set of trees $\{Q_\alpha : \alpha \in A\}$ a <u>representative system</u> if for each element α of $A - o$, with predecessor β say, then,

(1) for each representative vertex i for G_α we have $q_{(\alpha, i)} = q_{(\beta, i)}$, and

(2) for each α-vertex i for G_o we have $q_{(\alpha, i)} = q_{(o, i)}$.

Note that for each element α of A we have associated with $\{Q_\alpha : \alpha \in A\}$ some maximal circuit-free subgraph of G_α, namely X_α, and some set of representative vertices for G_α

containing the origin, namely the set of representative vertices chosen in the construction of $\{Q_\alpha : \alpha \in A\}$.

Now in the following lemma we give two elementary properties of representative systems.

So let $\{Q_\alpha : \alpha \in A\}$ be any representative system and for each element α of A let X_α be that maximal circuit-free subgraph of G_α associated with $\{Q_\alpha : \alpha \in A\}$.

Then,

Lemma 3

Consider any element α of A-o. Then the set of representative vertices for G_α has empty intersection with the set of α-vertices for G_o.

Further suppose β is the predecessor of α. Then for each vertex i we have $q_{(\alpha,i)} = q_{(\beta,j)}p$ for some reduced path p in X_α and some representative vertex j for G_α.

Proof

(1) Consider any non-origin vertex i.

If i is a representative vertex for G_α, then from the definition of Q_α we have that the non-empty path $q_{(\alpha,i)}$ does not end in an element of $X_\alpha^{\pm 1}$.

Now suppose i is an α- vertex for G_o.

This means that $q_{(o,i)}$ ends in an element of $X_\alpha^{\pm 1}$. Also by the definition of a representative system we have that $q_{(\alpha,i)} = q_{(o,i)}$, and so $q_{(\alpha,i)}$ ends in an element of $X_\alpha^{\pm 1}$.

And so we cannot have that i is both a representative vertex for G_α and an α-vertex for G_o.

This proves the first part of the lemma.

(2) Now consider any vertex i.

If i is a representative vertex for G_α then from the definition of $q_{(\alpha,i)}$ we have $q_{(\alpha,i)} = q_{(\beta,i)}$. And so, in this case, we have $q_{(\alpha,i)} = q_{(\beta,j)}p$ with i = j and p the empty path at the vertex i.

On the other hand, if i is not a representative vertex for G_α, then let j denote the representative vertex for that component of G_α which contains i. Also let p denote that unique non-empty reduced path in X_α from j to i. Then again from the definition of $q_{(\alpha,i)}$ we have $q_{(\alpha,i)} = q_{(\alpha,j)}p$. And then, since j is a representative vertex for G_α, we have $q_{(\alpha,j)} = q_{(\beta,j)}$. Thus, $q_{(\alpha,i)} = q_{(\beta,j)}p$.

This proves the second part of the lemma.

Corollary

Consider any element α of Λ-o of λ-level m, and let α_1 denote the predecessor of α, and for each $1 < r \leqslant m$ let α_r denote the predecessor of α_{r-1}.

Then for each vertex i we have $q_{(\alpha,i)} = q_{(o,j)}p_m \cdots p_1 p$ where p is some reduced path in X_α, and for each $1 \leqslant r \leqslant m$ p_r is some reduced path in X_{α_r}, and j is some representative vertex for G_o.

Now we show how we can use representative systems to give us sets of free generators for the vertex groups of connected free groupoids.

Theorem 1

Let $\{Q_\alpha : \alpha \in \Lambda\}$ be any representative system. For each $\alpha \in \Lambda$ let X_α denote the maximal circuit-free subgraph of G_α associated with $\{Q_\alpha : \alpha \in \Lambda\}$, and let X denote the graph-union of all the X_α.

Then the vertex group of F(X) (the free groupoid on X) at the origin is freely generated by the elements, $q_{(\alpha,i)} q^{-1}_{(\beta,i)}$ where β is the predecessor of α, and i ranges through those elements of I other than representative vertices for G_α and α-vertices for G_o, and α ranges through Λ-o.

Proof

Consider any element α of $A-o$, with predecessor β say, and any element i of I which is neither a representative vertex for G_α nor an α-vertex for G_o. Let us write $f_{(\alpha,i)}$ for the element $q_{(\alpha,i)}q^{-1}_{(\beta,i)}$.

Let F denote the set of all these elements.

Then we must prove that the vertex group of F(X) at the origin is freely generated by the set of elements F.

The proof of the theorem is based on the following result due to Higgins ([3] page 14).

Consider any maximal tree T of X, and for each element i of I let t_i be that unique reduced path in T from the origin to i. Then the vertex group of F(X) at the origin is freely generated by the elements, $t_j x t_i^{-1}$ where j,i is the initial, terminal vertex of x respectively, and x ranges through X - T.

It is not difficult to obtain the following generalisation of this result,

Lemma 4

The vertex group of F(X) at the origin is freely generated by the elements $q_{(o,j)}yq^{-1}_{(o,i)}$ where j is the representative vertex for that component of G_α which contains i, and y is that unique non-empty reduced path in X_α from j to i, and i ranges through those elements of I other than representative vertices for G_α and α-vertices for G_o, and α ranges through $A-o$.

For convenience let us now introduce some notation.

Consider any element α of A-o and any vertex i, other than a representative vertex for G_α or an α-vertex for G_o. Let j be the representative vertex for that component of G_α which contains i, and let y be that unique non-empty reduced path in X_α from j to i. We write $w_{(\alpha,i)}$ for the element $q_{(o,j)} y\, q^{-1}_{(o,i)}$.

Let us write W for the set of all these elements.

Then the lemma says that W is a set of free generators for the vertex group of F(X) at the origin.

To show that F is a set of free generators for the vertex group of F(X) at the origin, we shall investigate how each element of F is expressed as a product of elements (or their inverses) of W.

To do this, we have,

<u>Lemma 5</u>

Consider any element α of A-o and any vertex i. Then $q_{(\alpha,i)} q^{-1}_{(o,i)}$ is expressed as a product of elements of the form $w^{\epsilon}_{(\beta,j)}$ where $\epsilon = \pm 1$ and j ranges through those elements of I other than representative vertices for G_β or β-vertices for G_o, and β ranges through those elements of A-o such that $\lambda(\beta) < \lambda(\alpha)$.

Proof

We shall prove the lemma by induction on the λ-level of α.

Suppose first that α has λ-level 1.

If i is either a representative vertex for G_α or an α-vertex for G_o, then obviously $q_{(\alpha,i)} = q_{(o,i)}$ and so the lemma follows trivially in this case.

On the other hand, if i is neither a representative vertex for G_α nor an α-vertex for G_o, then we have $q_{(\alpha,i)} = q_{(\alpha,j)}y$ where j is the representative vertex for that component of G_α which contains i, and y is that unique non-empty reduced path in X_α from j to i. Also we have $q_{(\alpha,j)} = q_{(o,j)}$ since the predecessor of α is o. And so $q_{(\alpha,i)}q_{(o,i)}^{-1} = q_{(o,j)}yq_{(o,i)}^{-1} = w_{(\alpha,i)}$. Thus the lemma holds in this case.

Now choose any $n > 1$ and suppose the lemma holds for each α of λ-level $< n$. Suppose α has λ-level n, and let β denote the predecessor of α. Then of course β has λ-level n-1.

If i is an α-vertex for G_o then $q_{(\alpha,i)} = q_{(o,i)}$.

If i is a representative vertex for G_α then we have $q_{(\alpha,i)} = q_{(\beta,i)}$ and so $q_{(\alpha,i)}q_{(o,i)}^{-1} = q_{(\beta,i)}q_{(o,i)}^{-1}$. Then the lemma holds by our induction hypothesis.

Finally suppose i is neither a representative vertex for G_α nor an α-vertex for G_o. Let j denote the representative vertex for that component of G_α which contains i, and let y be that unique non-empty reduced path in X_α from j to i. Then we have $q_{(\alpha,i)} = q_{(\alpha,j)}y$. Also, since β is the predecessor of α, we have $q_{(\alpha,j)} = q_{(\beta,j)}$. Thus

$q_{(\alpha,i)}q_{(o,i)}^{-1} = (q_{(\beta,j)}q_{(o,j)}^{-1})\ (q_{(o,j)}yq_{(o,i)}^{-1}) = (q_{(\beta,j)}q_{(o,j)}^{-1})\ w_{(\alpha,i)}$. Then, again by our induction hypothesis, we see that the lemma holds in this case.

Thus the lemma holds for $\lambda(\alpha) = n$.

And so the lemma is proved, by induction on the λ-level of α.

Now consider any element α of A-o, with predecessor β say. Choose any vertex i which is neither a representative vertex for G_α nor an α-vertex for G_o. Let j denote the representative vertex for that component of G_α which contains i, and let y be that unique non-empty reduced path in X_α from j to i. Then, clearly,

$$f_{(\alpha,i)} = q_{(\alpha,i)}q_{(\beta,i)}^{-1} = (q_{(\beta,j)}q_{(o,j)}^{-1})(q_{(o,j)}yq_{(o,i)}^{-1})(q_{(o,i)}q_{(\beta,i)}^{-1}) = uw_{(\alpha,i)}v,$$

where by lemma 5 we have that u and v can be expressed as a product of elements of the kind $w_{(\gamma,k)}$ where k ranges through those vertices other than representative vertices for G_γ or γ-vertices for G_o, and γ ranges through those elements of A-o such that $\lambda(\gamma) < \lambda(\beta)$.

From this it is easy to show, by induction on the λ-level of α, that F is a set of free generators for the vertex group of F(X) at the origin.

And so the theorem is proved.

Corollary

Suppose the hypotheses of the theorem hold. Also suppose we have a set of discrete graphs $\{Y_\alpha : \alpha \in \Lambda\}$ whose edge sets are mutually disjoint, where for each element α of Λ the vertex set of Y_α is the set of representative vertices for G_α. Let Y denote the graph-union of the Y_α and let F(XUY) be the free groupoid on XUY.

Then the vertex group of F(XUY) at the origin is freely generated by the elements given in the theorem, together with the elements $q_{(\alpha,i)} y \, q_{(\alpha,i)}^{-1}$ where y ranges through $(Y_\alpha)_{ii}$ and i ranges through the representative vertices for G_α and α ranges through Λ.

In the appendix we give an example of a connected tree product of groupoids. In this example we choose a representative system and then follow through the general procedure given in the introduction to this chapter, and so obtain a presentation for the vertex group of this connected tree product of groupoids. However, we shall see that we cannot characterise this vertex group precisely using this presentation.

Further the example indicates the kind of condition we must impose upon the representative systems we use before we can obtain useful presentations for the vertex group of G.

The condition is that the representative systems be 'regular'.

And so, in the next section, we define what we mean by a 'regular representative system'.

2.2 On regular representative systems.

Let $\{Q_\alpha : \alpha \in A\}$ be any representative system. Recall, then, that for each element α of A-o with predecessor β say then,

(1) for each representative vertex i for G_α $q_{(\alpha,i)} = q_{(\beta,i)}$, and,

(2) for each α-vertex i for G_o $q_{(\alpha,i)} = q_{(o,i)}$.

If, in addition, the following condition holds,

(3) for each element α of A-o and each α-vertex i for G_o then $q_{(\alpha,i)} = q_{(\beta,i)}$ for all those elements β of A such that $\lambda(\beta) < \lambda(\alpha)$, then we call $\{Q_\alpha : \alpha \in A\}$ a regular representative system.

In the next lemma we give two basic properties of regular representative systems. So suppose $\{Q_\alpha : \alpha \in A\}$ is any regular representative system, and for each element α of A-o , let I_α denote the union of the set of representative vertices for G_α and the set of α-vertices for G_o.

Lemma 6

Consider any element α of A-o . Then for each α-vertex i for G_o we have that i is a representative vertex for each G_β such that $\lambda(\beta) < \lambda(\alpha)$.

Further we have that distinct elements of I_α belong to distinct components of U_α .

Proof

(1) Let i be any α-vertex for G_o, and let β be any element of A such that $\lambda(\beta) < \lambda(\alpha)$. Then, since $\{Q_\alpha : \alpha \in A\}$ is regular, we have $q_{(\alpha,i)} = q_{(\beta,i)}$. Further, since i is not a representative vertex for G_α, it follows that $q_{(\alpha,i)}$ ends in an element of $X_\alpha^{\pm 1}$. And so, from the definition of the path $q_{(\beta,i)}$, it follows that i is a representative vertex for G_α.

This proves the first part of the lemma.

(2) Let β be the predecessor of α. Then we have that U_α is a subgroupoid of G_α and V_α is a subgroupoid of G_β and Θ_α is a groupoid I-isomorphism from U_α to V_α.

Consider any distinct elements i and j of I_α, and suppose these vertices belong to the same component of U_α. Then, of course, they also belong to the same component of G_α, and the same component of G_β.

First, suppose that neither i nor j is a representative vertex for G_α. Then both vertices are α-vertices for G_o. And so, from the first part of the lemma, we have that i and j are representative vertices for G_β. This is a contradiction since i and j belong to the same component of G_β.

Next, suppose that one of the vertices, i say, is a representative vertex for G_α. Then, from the definition of $q_{(\alpha,i)}$, we have $q_{(\alpha,i)} = q_{(\beta,i)}$. Also, since j is an α-vertex for G_o, we have $q_{(\alpha,j)} = q_{(\beta,j)}$. Further it is clear that $q_{(\alpha,j)} = q_{(\alpha,i)}p$ and $q_{(\beta,i)} = q_{(\beta,j)}q$ for some non-empty reduced path p,q in G_α, G_β respectively. From these equations we obtain $q_{(\alpha,j)} = q_{(\alpha,j)}qp$, a contradiction.

This proves the second part of the lemma.

Theorem 2

There exists a regular representative system.

Proof

To begin with, for each element α of Λ, choose any maximal circuit-free subgraph X_α of G_α, and let X be the graph-union of the X_α. Then X is a connected I-graph.

We shall prove the theorem in four steps, as follows.

First, we construct a very particular maximal tree T of X containing X_o.

Second, using the level-function 1 on T induced by the origin (of I), we choose for each element α of Λ a particular set of representative vertices for G_α.

Third, using for each element α of Λ, the maximal circuit-free subgraph X_α of G_α and the set of representative vertices for G_α chosen in step 2, we construct a representative system $\{Q_\alpha : \alpha \in \Lambda\}$.

Fourth, we show that $\{Q_\alpha : \alpha \in \Lambda\}$ is regular, and to help us we give three lemmas.

Step 1 Construction of the tree T.

To help us to construct T we shall first describe a sequence of graphs C_r $(r \geq 1)$ say. So for each $r \geq 1$, let C_r be the graph-union of all those X_α where α has λ-level r.

Now choose any circuit-free subgraph T_1 of C_1 such that the graph-union $X_o \cup T_1$ is a maximal circuit-free subgraph of

$X_o \cup C_1$. It is easy to see that such a T_1 can be chosen.

Next choose any circuit-free subgraph T_2 of C_2 such that the graph-union $X_o \cup T_1 \cup T_2$ is a maximal circuit-free subgraph of $X_o \cup C_1 \cup C_2$.

We continue in this way and so obtain a sequence of graphs $T_r (r \geqslant 1)$ such that for each $r \geqslant 1$ T_r is a circuit-free subgraph of C_r and $X_o \cup T_1 \cup .. \cup T_r$ is a maximal circuit-free subgraph of $X_o \cup C_1 \cup .. \cup C_r$.

Let T denote the graph-union of X_o together with all the T_r.

Then we have,

<u>Lemma 7</u>

T is a maximal tree of X.

<u>Proof</u>

To prove the lemma we show that T is a connected circuit-free I-graph. Then, from proposition 2, it follows that T is a maximal tree of X.

First, then, we show that T is a connected I-graph.

Consider any distinct vertices i and j.

Since X is a connected I-graph there exists a non-empty path in X from i to j. Then, of course, this path is of the form $x_1^{\epsilon_1} x_2^{\epsilon_2} \dots x_m^{\epsilon_m}$ for some $m \geqslant 1$ where for each $1 \leqslant r \leqslant m$ $\epsilon_r = \pm 1$ and x_r is some element of X. Let n denote the maximum of the λ-levels of those elements of Λ, α, such that X_α contains some x_r $(1 \leqslant r \leqslant m)$.

Then, clearly, the vertices i and j belong to the same connected component of $X_o UC_1 U..UC_n$.

Now, since $X_o UT_1 U...UT_n$ is a maximal circuit-free subgraph of $X_o UC_1 U..UC_n$, it follows that the vertices i and j belong to the same connected component of $X_o UT_1 U..UT_n$. This means that there exists a path in $X_o UT_1 U...UT_n$ with vertices i and j.

Thus we have a path in T from i to j.

Therefore T is a connected I-graph.

Now we show that T is circuit-free.

Consider any non-empty closed and reduced path in T, $x_1^{\epsilon_1} x_2^{\epsilon_2} ... x_m^{\epsilon_m}$ for some $m \geqslant 1$ where for each $1 \leqslant r \leqslant m$ $\epsilon_r = \pm 1$ and x_r is some element of T.

Again let n denote the maximum of the λ-levels of those elements α of Λ such that X_α contains some $x_r (1 \leqslant r \leqslant m)$.

Then, for each $1 \leqslant r \leqslant m$, it is easy to see that x_r belongs to $X_o UT_1 U...UT_n$.

Thus the given path is a non-empty closed and reduced path in $X_o UT_1 U...UT_n$. But this is a contradiction since $X_o UT_1 U...UT_n$ is circuit-free.

This proves that T is circuit-free.

And so the lemma is proved.

For each element i of I let us write t_i for that unique reduced path in T from the origin to i.

It is very easy to see that for each connected component of X_o which does not contain the origin, there exists an unique vertex of the component i such that t_i ends in an element of $X_{\alpha}^{\pm 1}$ for some element α of $\mathcal{A}$-o.

Step 2 Choosing a set of representative vertices for each G_{α}.

As already mentioned we shall use the level-function l to help us to choose a particular set of representative vertices for each G_{α}.

So consider any element α of $\mathcal{A}$. For each component of G_{α} choose any vertex of the component of minimal l-level, and let this vertex be the representative vertex for the component. In this way we choose a set of representative vertices for G_{α}, and note that the origin belongs to this set of vertices.

And so we choose a set of representative vertices for each G_{α}.

Note, then, that the representative vertices for G_o are uniquely determined. Indeed the set of non-origin representative vertices for G_o is precisely the set of non-origin elements i of I such that t_i ends in an element of $X_{\alpha}^{\pm 1}$ for some element α of $\mathcal{A}$-o.

Also it is easy to see that if i is any non-origin vertex and t_i ends in an element of $X_{\alpha}^{\pm 1}$ for some element α of $\mathcal{A}$

then i is not a representative vertex for G_α.

Step 3 Construction of the representative system $\{Q_\alpha : \alpha \in A\}$

Here we shall use, for each element α of A, the maximal circuit-free subgraph X_α of G_α and the set of representative vertices for G_α chosen in step 2, to construct the representative system $\{Q_\alpha : \alpha \in A\}$.

We shall construct $\{Q_\alpha : \alpha \in A\}$ by induction on the λ-level of elements of A.

So, first, we construct Q_o.

To be precise, for each non-origin representative vertex i for G_o, we shall define a non-empty reduced path in X from the origin to i, which we shall write $q_{(o,i)}$, and then we shall write Q_o for the graph whose edges are all these paths together with the edges of X_o.

We shall define the paths, $q_{(o,i)}$ as i ranges through the non-origin representative vertices for G_o, by induction on the 1-level of the representative vertices for G_o.

To begin with, then, consider any representative vertex i for G_o of 1-level 1. In this case t_i is a path of length 1. From the remarks given in step 2 we have that t_i belongs to $X_\alpha^{\pm 1}$ for some element α of A-o. Then we define $q_{(o,i)}$ to be t_i.

Now choose any n>1 and suppose that we have defined the paths, $q_{(o,i)}$ as i ranges through those representative vertices for G_o of 1-level $<$ n.

Consider any representative vertex i for G_o of l-level n.

Again from the remarks in step 2 we have that t_i ends in an element of $X^{\pm}_{\alpha}$ for some element α of Λ-o. Suppose α has λ-level m (for some $m \geqslant 1$). Let j denote the representative vertex for that component of G_{α} which contains i, and let q be that unique reduced path in X_{α} from j to i. Note that, from the definition of the representative vertices for G_{α}, $l(j) < n$.

Now let α_1 denote the predecessor of α, and let j_1 be the representative vertex for that component of G_{α_1}, which contains j, and let q_1 be that unique reduced path in X_{α_1} from j_1 to j. Again we observe that $l(j_1) < n$.

Continuing in this way we obtain a set of elements of Λ $\{\alpha_r : 1 \leqslant r \leqslant m\}$ and a set of vertices $\{j_r : 1 \leqslant r \leqslant m\}$ such that for each $1 < r \leqslant m$ α_r is the predecessor of α_{r-1} and j_r is the representative vertex for that component of G_{α_r} which contains j_{r-1}.

It is easy to see that $l(j_r) < n$ for each $1 \leqslant r \leqslant m$.

For each $1 < r \leqslant m$ let us write q_r for that unique reduced path in X_{α_r} from j_r to j_{r-1}.

In particular we have $\alpha_m = o$ and $l(j_m) < n$.

Thus, from our induction hypothesis, the path $q_{(o,j_m)}$ has already been defined. (If j_m is the origin of I then we define $q_{(o,j_m)}$ to be the empty path at the origin).

In this case we define the path $q_{(o,i)}$ to be $q_{(o,j_m)} q_m \dots q_1 q$.

It is not difficult to see that this path is reduced.

And so, by induction , we have defined the reduced paths $q_{(o,i)}$ as i ranges through the non-origin representative vertices for G_o.

Note that for each non-origin representative vertex i for G_o then $q_{(o,i)}$ and t_i end in elements (perhaps different) of the same $X_\alpha^{\pm 1}$ for some element α of A-o.

Then we write Q_o for the graph whose edges are the edges of X_o together with the edges $q_{(o,i)}$ as i ranges through the non-origin representative vertices for G_o.

Now choose any $n \geq 1$ and suppose that for each element α of A of λ-level$< n$, we have defined Q_α .

Consider any element α of A of λ-level n.

Let β denote the predecessor of α. Then of course β has λ-level n-1 and so by induction Q_β has already been defined.

Then for each non-origin representative vertex i for G_α we define $q_{(\alpha,i)}$ to be $q_{(\beta,i)}$, that is that unique reduced path in Q_β from the origin to i.

Then we write Q_α for the graph whose edges are the edges of X_α together with the edges $q_{(\alpha,i)}$ as i ranges through the non-origin representative vertices for G_α.

And so by induction we have defined $\{Q_\alpha : \alpha \in A\}$. It is immediate from its construction that $\{Q_\alpha : \alpha \in A\}$ is a representative system.

Recall that for each non-origin representative vertex i for G_o then $q_{(o,i)}$ and t_i end in elements of the same $X_\alpha^{\pm 1}$ for some element α of A-o. From this it is clear that for any element α of A-o then the α-vertices for G_o are precisely those non-origin representative vertices i for G_o such that the end of t_i belongs to $X_\alpha^{\pm 1}$.

Step 4 $\{Q_\alpha : \alpha \in A\}$ is regular

Let us recall here that the representative system $\{Q_\alpha : \alpha \in A\}$ is regular if for each element α of A-o and each α-vertex i for G_o then $q_{(\alpha,i)} = q_{(\beta,i)}$ for each element β of A such that $\lambda(\beta) < \lambda(\alpha)$.

As we have mentioned, to help us see that $\{Q_\alpha : \alpha \in A\}$ is regular, we now give three lemmas.

<u>Lemma 8</u>

Consider any element α of A-o and any α-vertex i for G_o. If α has λ-level m, then the origin and i belong to different components of $X_o U C_1 U..UC_{m-1}$.

<u>Proof</u>

To begin with, since i is an α-vertex for G_o, we have from the remark given at the end of step 3 that t_i ends in an element of $X_\alpha^{\pm 1}$.

Suppose that the origin and i belong to the same component of $X_o U C_1 U..U C_{m-1}$. Then, since $X_o U T_1 U..U T_{m-1}$ is a maximal circuit-free subgraph of $X_o U C_1 U..U C_{m-1}$, it follows that the origin and i belong to the same component of $X_o U T_1 U..U T_{m-1}$. This means there exists a path p in $X_o U T_1 U..U T_{m-1}$ from i to the origin.

Then it is clear that the reduction of the path $t_i p$ is a non-empty closed and reduced path in T.

Of course this is a contradiction.

This proves the lemma.

In exactly the same way we prove,

Lemma 9

Consider any elements α, β of Λ-o and any distinct α-vertex i, β-vertex j for G_o. If $m=\min\{\lambda(\alpha), \lambda(\beta)\}$ then i and j belong to different components of $X_o U C_1 U..$ $U C_{m-1}$.

Lemma 10

Choose any $m > o$ and any component of $X_0 U C_1 U..U C_{m-1}$ and any vertex i of this component of minimal l-level. Then either i is the origin or i is an α-vertex for G_o for some element α of Λ of λ-level $\geqslant$ m.

Proof

Suppose i is not the origin, and that t_i ends in an element of $X_\alpha^{\pm 1}$ for some element α of Λ of λ-level $<$ m.

Let j denote the initial vertex of the terminal element of t_i.

Then, since j has lesser 1-level than i, it follows that j does not belong to the given component of $X_o U C_1 U \ldots U C_{m-1}$.

On the other hand, since the terminal element of t_i belongs to $X_\alpha^{\pm 1}$ for some element α of A of λ-level $<$ m, it follows that j does belong to the given component of $X_o U C_1 U..U C_{m-1}$.

This contradiction means that t_i ends in an element of $X_\alpha^{\pm 1}$ for some element α of A of λ-level $\geq$ m, and this means that i is an α-vertex for G_o.

Thus the lemma is proved.

Now from these three lemmas it is easy to see that $\{Q_\alpha : \alpha \in A\}$ is regular.

For consider any element α of A-0 and any α-vertex i for G_o.

Let m denote the λ-level of α.

Consider that component of $X_o U C_1 U..U C_{m-1}$ which contains i. Then from lemmas 8, 9 and 10 it is clear that i is that unique vertex of this component of minimal 1-level.

Now choose any element β of A of lesser λ-level than α.

Consider that component of X_β which contains i. Obviously this component of X_β is contained in the given component of $X_o U C_1 U..U C_{m-1}$, and so it follows that i is that unique vertex of this component of X_β of minimal 1-level.

Then, from the definition of the set of representative vertices for G_{β}, it follows that i is a representative vertex for G_{β}.

And so it is clear that $q_{(\alpha, i)} = q_{(\beta, i)}$ for each element β of Λ of lesser λ-level than α.

But this is precisely the condition required for $\{Q_{\alpha} : \alpha \in \Lambda\}$ to be regular.

And so the theorem is proved.

2.3 The main theorem

In this section we prove our main theorem, that is we characterise the vertex group of G at the origin as an HNN group with base-part some tree product of groups (theorem 3).

Our method of proof will be to choose any regular representative system, and then follow through the general procedure given in the introduction to this chapter, to obtain a presentation for the vertex group of G at the origin, and from this presentation we shall see that this group has the structure just described.

First though we make a simple observation.

Consider any regular representative system $\{Q_{\alpha} : \alpha \in \Lambda\}$. Choose any element α of $\Lambda - 0$ and let β be the precedessor of α. Also choose any vertex i, and let j,k be the representative vertex for that component of G_{α}, G_{β} respectively which contains

i. Then for any element u of $(U_\alpha)_{ii}$ we have that $q_{(\alpha,i)}uq^{-1}_{(\alpha,i)}$ is an element of the group $q_{(\alpha,j)}(G_\alpha)_{jj}$ $q^{-1}_{(\alpha,j)}$. To see this we have $q_{(\alpha,i)} = q_{(\alpha,j)}p$ for some reduced path p in G_α. So we can express $q_{(\alpha,i)}uq^{-1}_{(\alpha,i)}$ as $q_{(\alpha,j)}(q^{-1}_{(\alpha,j)}q_{(\alpha,i)}uq^{-1}_{(\alpha,i)}q_{(\alpha,j)})q^{-1}_{(\alpha,j)}$ = $q_{(\alpha,j)}(pup^{-1})q^{-1}_{(\alpha,j)}$ and of course we have that pup^{-1} belongs to $(G_\alpha)_{jj}$. Similarly we have that $q_{(\beta,i)}(u\Theta_\alpha)$ $q^{-1}_{(\beta,i)}$ is an element of the group $q_{(\beta,k)}(G_\beta)_{kk}q^{-1}_{(\beta,k)}$. Further, in the case that i is a representative vertex for G_α or an α-vertex for G_o we have $q_{(\alpha,i)} = q_{(\beta,i)}$ and so $q_{(\beta,i)}(u\Theta_\alpha)q^{-1}_{(\beta,i)} = q_{(\alpha,i)}(u\Theta_\alpha)q^{-1}_{(\alpha,i)}$.

Theorem 3

Let $\{Q_\alpha : \alpha \in A\}$ be any regular representative system. For each element α of A-o let I_α be the set of representative vertices for G_α and α-vertices for G_o, and choose any set of vertices K_α such that $I_\alpha \cap K_\alpha$ is empty and $I_\alpha \cup K_\alpha$ is a set of representative vertices for U_α. Again for each element α of A-o and each element i of I_α, let $\sigma_{(\alpha,i)}$ denote the group isomorphism given by $q_{(\alpha,i)}uq^{-1}_{(\alpha,i)} \longrightarrow q_{(\alpha,i)}u\Theta_\alpha q^{-1}_{(\alpha,i)}$ as u ranges through $(U_\alpha)_{ii}$. Let Σ be the set of all these group isomorphisms.

Then Σ is a tree of groups $q_{(\alpha,i)}(G_\alpha)_{ii}q^{-1}_{(\alpha,i)}$ as i ranges through the representative vertices for G_α and α ranges through A.

Further the vertex group of G at the origin is the HNN group with base-part the tree product of ζ and free-part generated by the elements $q_{(\alpha,i)}q^{-1}_{(\beta,i)}$ where β is the predecessor of α and i ranges through K_α and α ranges through $\mathcal{A}-0$.

Finally for each element α of $\mathcal{A}-0$, with predecessor β say, and each element i of K_α, then the group isomorphism associated with the generator $q_{(\alpha,i)}q^{-1}_{(\beta,i)}$ is given by $q_{(\beta,i)}u\theta_\alpha q^{-1}_{(\beta,i)} \longrightarrow q_{(\alpha,i)}uq^{-1}_{(\alpha,i)}$ as u ranges through $(U_\alpha)_{ii}$.

<u>Proof</u>

We begin immediately with the following,

<u>Lemma 11</u>

ζ is a tree of groups $q_{(\alpha,i)}(G_\alpha)_{ii}q^{-1}_{(\alpha,i)}$ as i ranges through the representative vertices for G_α and α ranges through $\mathcal{A}$.

<u>Proof</u>

First we show that the set of group isomorphisms ζ can be considered as a graph with vertices the groups $q_{(\alpha,i)}(G_\alpha)_{ii}$ $q^{-1}_{(\alpha,i)}$ as i ranges through the representative vertices for G_α and α ranges through $\mathcal{A}$.

So consider any element σ of Σ. Then we have $\sigma = \sigma_{(\alpha,i)}$ for some element i of I_α and some element α of A-o. Let β denote the predecessor of α. Then $q_{(\alpha,i)} = q_{(\beta,i)}$ and $\sigma_{(\alpha,i)}$ has domain $q_{(\alpha,i)}(U_\alpha)_{ii}$ $q^{-1}_{(\alpha,i)}$ and range $q_{(\alpha,i)}(V_\alpha)_{ii}q^{-1}_{(\alpha,i)} = q_{(\beta,i)}(V_\alpha)_{ii}$ $q^{-1}_{(\beta,i)}$.

Suppose first that i is a representative vertex for G_α.

Let j denote the representative vertex for that component of G_β which contains i. Then the range of $\sigma_{(\alpha,i)}$ is a subgroup of $q_{(\beta,j)}(G_\beta)_{jj}q^{-1}_{(\beta,j)}$.

In this case we define the initial, terminal vertex of $\sigma_{(\alpha,i)}$ to be the group $q_{(\alpha,i)}(G_\alpha)_{ii}q^{-1}_{(\alpha,i)}$, $q_{(\beta,j)}(G_\beta)_{jj}$ $q^{-1}_{(\beta,j)}$ respectively.

Now suppose that i is an α-vertex for G_o.

Then of course i is a representative vertex for G_β. Let l denote the representative vertex for that component of G_α which contains i. Then the domain of $\sigma_{(\alpha,i)}$ is a subgroup of $q_{(\alpha,l)}(G_\alpha)_{ll}q^{-1}_{(\alpha,l)}$.

In this case we define the initial, terminal vertex of $\sigma_{(\alpha,i)}$ to be the group $q_{(\alpha,l)}(G_\alpha)_{ll}q^{-1}_{(\alpha,l)}$, $q_{(\beta,i)}(G_\beta)_{ii}$ $q^{-1}_{(\beta,i)}$ respectively.

And so it is easy to see that Σ is a graph with vertices the groups $q_{(\alpha,i)}(G_\alpha)_{ii}q^{-1}_{(\alpha,i)}$ as i ranges through the representative vertices for G_α and α ranges through A.

Now we show that Σ is in fact a tree.

To begin with let Σ_1 denote that subgraph of Σ consisting of the group isomorphisms $\sigma_{(\alpha, i)}$ as i ranges through the representative vertices for G_α and α ranges through $A - o$.

Then it is easy to see that Σ_1 is circuit-free. Also it is easy to see that for each component of Σ_1 there exists a unique representative vertex i for G_o such that the group $q_{(o,i)}(G_o)_{ii}q^{-1}_{(o,i)}$ is a vertex of this component.

Now put $\Sigma_2 = \Sigma - \Sigma_1$.

We shall construct a graph, which we denote by $\bar{\Sigma}_2$, with vertices the connected components of Σ_1. To do this consider any edge σ of Σ_2. Then we introduce the symbol $\bar{\sigma}$ and we define the initial vertex of $\bar{\sigma}$ to be that component of Σ_1 which contains the initial vertex of σ and we define the terminal vertex of $\bar{\sigma}$ to be that component of Σ_1 which contains the terminal vertex of σ. Then we write $\bar{\Sigma}_2$ for the graph consisting of the elements, $\bar{\sigma}$ as σ ranges through Σ_2. It is clear that the vertices of $\bar{\Sigma}_2$ are the components of Σ_1.

Also it is straightforward to see that Σ is a tree iff $\bar{\Sigma}_2$ is a tree.

To show that $\bar{\Sigma}_2$ is a tree, we use the following result due to Karrass & Solitar ([5] page 231).

Consider any graph and choose any vertex of the graph, and call this vertex the 'start'. Suppose we associate to each vertex of the graph, some non-negative integer, such that the non-negative integer associated to the 'start' is 0. Also suppose that the non-negative integer associated to the initial vertex of each edge of the graph is less than the non-negative integer associated to the terminal vertex of that edge. Finally suppose that each non-'start' vertex of the graph is the terminal vertex of a unique edge. Then the graph is a tree.

Lemma 12

$\bar{\xi}_2$ is a tree.

Proof

To begin with let us call that component of ξ_1 which contains the vertex group of G_o at the origin, the 'start'.

Now consider any component of ξ_1. Let i denote that unique representative vertex for G_o such that the group $q_{(o,i)}(G_o)_{ii}q^{-1}_{(o,i)}$ is a vertex of the given component. Let us count the number of non-origin representative vertices j for G_o such that $q_{(o,j)}$ is an initial segment of $q_{(o,i)}$(that is $q_{(o,i)} = q_{(o,j)}p$ for some reduced path p). This is the non-negative integer we shall associate to the given component of ξ_1.

Clearly the non-negative integer associated to the 'start' is 0.

Next consider any edge $\bar{\sigma}$ of $\bar{\xi}_2$.

We shall show that the non-negative integer associated to the initial vertex of $\bar{\sigma}$ is less than that associated to the terminal vertex of $\bar{\sigma}$.

We have $\sigma = \sigma_{(\alpha, i)}$ for some element α of A-o and some α-vertex i for G_o. Then of course we have that i is a representative vertex for each G_β where β has lesser λ-level than α. And so it follows that the terminal vertex of $\bar{\sigma}$ is that component of ξ_1 which contains the vertex $q_{(o,i)}(G_o)_{ii}q^{-1}_{(o,i)}$. Let j be that unique representative vertex for G_o such that the group $q_{(o,j)}(G_o)_{jj}q^{-1}_{(o,j)}$ belongs to the initial vertex of $\bar{\sigma}$. Then it is easy to see that $q_{(o,j)}$ is some proper initial segment of $q_{(o,i)}$.

From this it follows that the non-negative integer associated to the initial vertex of $\bar{\sigma}$ is less than that associated to the terminal vertex of $\bar{\sigma}$.

It remains to show, then, that each non-'start' component of ξ_1 is the terminal vertex of a unique element of $\bar{\xi}_2$.

So consider any non-'start' component of ξ_1.

Let i be that unique representative vertex for G_o such that the group $q_{(o,i)}(G_o)_{ii}q^{-1}_{(o,i)}$ is a vertex of the given component.

Obviously i is not the origin, and so we have that i is an α-vertex for G_o for some element α of A-o.

Thus $\sigma_{(\alpha,i)}$ belongs to ξ_2, and the given component of ξ_1 is the terminal vertex of $\bar{\sigma}_{(\alpha,i)}$.

Suppose the given component of ξ_1 is the terminal vertex of some other edge $\bar{\sigma}_{(\beta,j)}$ of $\bar{\xi}_2$, for some element β of A-o and some β-vertex j for G_o. Then we must have i = j, and so it follows that $\alpha = \beta$.

Thus we have shown that each non-'start' component of ξ_1 is the terminal vertex of a unique edge of $\bar{\xi}_2$.

And so we have shown that $\bar{\xi}_2$ satisfies the conditions given just before lemma 12.

Therefore $\bar{\xi}_2$ is a tree, and so lemma 12 is proved.

And so, also, lemma 11 is proved.

Now we show that the vertex group of G at the origin is the HNN group described in the statement of the theorem.

As we stated in the beginning of this section, we prove this result using the regular representative system $\{Q_\alpha : \alpha \in A\}$ and following through the procedure given in the introduction to this chapter. This will give us a presentation for the vertex group of G at the origin, and from this presentation we shall see that the vertex group of G at the origin has the structure described.

First though for each element α of A let X_α be the maximal circuit-free subgraph of G_α associated with $\{Q_\alpha : \alpha \in A\}$ and let L_α be the set of representative vertices for G_α associated with $\{Q_\alpha : \alpha \in A\}$. Then for each $\alpha \in A$-o we have that I_α is the union of L_α and the set of α-vertices for G_o.

So, now, let us follow through the given procedure.

Step 1 An I-presentation for each G_α.

Consider any element α of A.

We shall use proposition 7 to give us an I-presentation for G_α. Recall, then, that we must choose some maximal circuit-free subgraph of G_α, and some set of representative vertices for G_α. The maximal circuit-free subgraph of G_α we choose is X_α, and the set of representative vertices for G_α we choose is L_α.

Now for each element i of L_α let $\langle (G_\alpha)_{ii}, (R_\alpha)_{ii} \rangle$ be the standard presentation for the group $(G_\alpha)_{ii}$.

Then we obtain an I-presentation for G_α,

$$\langle X_\alpha U(\underset{i \in L_\alpha}{U} (G_\alpha)_{ii}), \underset{i \in L_\alpha}{U} (R_\alpha)_{ii} \rangle .$$

Step 2 An I-presentation for G.

For each element α of A-o, choose any maximal circuit-free subgraph Z_α of U_α.

We now use lemma 1 to give us an I-presentation for G. To do this we must choose for each element α of A, some I-presentation for G_α, and for each element α of A-o, some set of representative vertices for U_α and some maximal circuit-free subgraph of U_α. Here for each element α of A the I-presentation for G_α we choose is that given in step 1, and for each element α of A-o the set of representative vertices for U_α we choose is $I_\alpha UK_\alpha$ and the maximal circuit-free subgraph of U_α we choose is Z_α.

Then we obtain an I-presentation for G with generator graph the union of the generator graphs given in step 1, and with relator graph the union of the relator graphs given in step 1 together with the graphs $\{u(u\theta_\alpha)^{-1} : u \in (U_\alpha)_{ii},\ i \in I_\alpha \cup K_\alpha,\ \alpha \in A\text{-}o\}$ and $\{z(z\theta_\alpha)^{-1} : z \in Z_\alpha,\ \alpha \in A\text{-}o\}$.

Now, to save us from repeating long expressions for graphs of generators and relators, let us introduce some short-hand notation.

So let us denote the relator graph $\{z(z\theta_\alpha)^{-1} : z \in Z_\alpha,\ \alpha \in A\text{-}o\}$ by $\{z(z\theta_\alpha)^{-1} : A\text{-}o\}$ and the relator graph $\{u(u\theta_\alpha)^{-1} : u \in (U_\alpha)_{ii},\ i \in I_\alpha \cup K_\alpha,\ \alpha \in A\text{-}o\}$ by $\{u(u\theta_\alpha)^{-1} : I_\alpha \cup K_\alpha,\ A\text{-}o\}$. Similarly let us denote the graphs $\bigcup_{\alpha \in A} X_\alpha$, $\bigcup_{\alpha \in A} (\bigcup_{i \in L_\alpha} (G_\alpha)_{ii})$, and $\bigcup_{\alpha \in A} (\bigcup_{i \in L_\alpha} (R_\alpha)_{ii})$ by $\{x : A\}$, $\{g : L_\alpha, A\}$, and $\{r : L_\alpha, A\}$ respectively.

Sometimes we shall use obvious generalisations of this notation. For example by $\{u(u\theta_\alpha)^{-1} : I_\alpha, A\text{-}o\}$ we mean $\{u(u\theta_\alpha)^{-1} : u \in (U_\alpha)_{ii},\ i \in I_\alpha;\ \alpha \in A\text{-}o\}$.

With this notation the I-presentation we have for G becomes,

$$< \{x : A\} \cup \{g : L_\alpha, A\},\ \{r : L_\alpha, A\} \cup \{z(z\theta_\alpha)^{-1} : A\text{-}o\} \cup \{u(u\theta_\alpha)^{-1} : I_\alpha \cup K_\alpha, A\text{-}o\} >.$$

Now choose any maximal tree of $F(\{x : A\} \cup \{g : L_\alpha, A\})$ (the free groupoid on $\{x : A\} \cup \{g : L_\alpha, A\}$), and for each relator s in the given I-presentation for G let s' denote the conjugation of s in the origin using this tree (see the introduction to section 5 of chapter 1).

Then by the first part of lemma 2 we have another I-presentation for G,

$$< \{x:A\} \cup \{g:L_\alpha, A\},\ \{r':L_\alpha, A\} \cup \{(z(z\theta_\alpha)^{-1})':A-o\} \cup \{(u(u\theta_\alpha)^{-1})':I_\alpha \cup K_\alpha, A-o\} >.$$

Note then that each relator in this I-presentation has vertices the origin.

Step 3 A set of free generators for the vertex group of $F(\{x:A\} \cup \{g:L_\alpha, A\})$ at the origin.

Now, using the regular representative system $\{Q_\alpha : \alpha \in A\}$ in theorem 1 and its corollary (with $Y = \{g:L_\alpha, A\}$), we see that the vertex group of $F(\{x:A\} \cup \{g:L_\alpha, A\})$ at the origin is freely generated by the elements $q_{(\alpha,i)} g q^{-1}_{(\alpha,i)}$ $(g \in (G_\alpha)_{ii}, i \in L_\alpha, \alpha \in A)$ and $q_{(\alpha,i)} q^{-1}_{(\beta,i)}$ (where β is the predecessor of α, $i \in I-I_\alpha, \alpha \in A-o$).

Let us write $\{q_{(\alpha,i)} g q^{-1}_{(\alpha,i)} : L_\alpha, A\}$ for the set of elements $\{q_{(\alpha,i)} g q^{-1}_{(\alpha,i)} : g \in (G_\alpha)_{ii}, i \in L_\alpha, \alpha \in A\}$, and $\{q_{(\alpha\ i)} q^{-1}_{(\beta,i)} : I-I_\alpha, A-o\}$ for the set of elements $\{q_{(\alpha,i)} q^{-1}_{(\beta,i)} : \beta$ the predecessor of α, $i \in I-I_\alpha, \alpha \in A-o\}$.

Step 4 A presentation for the vertex group of G at the origin.

For each relator s' in the I-presentation for G given at the end of step 2 let us write s'' for s' rewritten in terms of the free generators given in step 3.

Then using the second part of lemma 2 we obtain a presentation for the vertex group of G at the origin,

$$< \{q_{(\alpha,i)} g q^{-1}_{(\alpha,i)} : L_\alpha, A\} \cup \{q_{(\alpha,i)} q^{-1}_{(\beta,i)} : I - I_\alpha, A\text{-}o\},$$
$$\{r'' : L_\alpha, A\} \cup \{(z(z\theta_\alpha)^{-1})'' : A\text{-}o\}$$
$$\cup \{(u(u\theta_\alpha)^{-1})'' : I_\alpha \cup K_\alpha, A\text{-}o\} >.$$

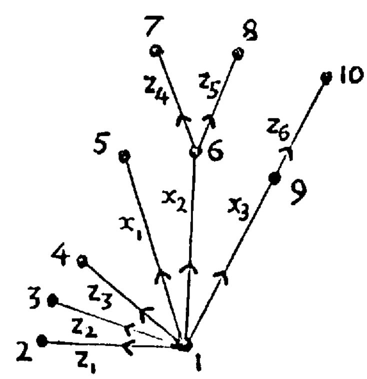
Our aim now is to deduce from this presentation that the vertex group of G at the origin is the HNN group described in the statement of the theorem.

In order to simplify the computational work which follows we make the following three conditions.

For each element α of A-o we suppose that X_α contains Z_α. Also we suppose that both the initial vertex of each edge of Z_α and the terminal vertex of each edge of $X_\alpha - Z_\alpha$ belongs to $I_\alpha \cup K_\alpha$ (the set of representative vertices for U_α). And finally we suppose that the initial vertex of each edge of $X_\alpha - Z_\alpha$ belongs to L_α (the set of representative vertices for G_α).

Since X_α and Z_α are subgraphs of the groupoid G_α it is not difficult to see that X_α and Z_α can be chosen to satisfy these conditions.

To make clear the meaning of these conditions let us give an example. So consider any element α of A-o, and let G'_α be any component of G_α, and let U'_α be that subgroupoid of U_α belonging to G'_α. Also let X'_α be that component of X_α belonging to G'_α, and let Z'_α be that subgraph of Z_α belonging to G'_α. Suppose G'_α has vertex set $\{1\text{-}10\}$ with representative vertex 1, and suppose the components of U'_α have vertex sets $\{1\text{-}4\}$, $\{5\}$, $\{6\text{-}8\}$, and $\{9,10\}$ with representative vertices 1,5,6 and 9 respectively. Then, in accordance with the above conditions, X_α and Z_α are typically of the form shown in the following figure,



$$X'_\alpha - Z'_\alpha = \{x_{1-3}\}$$

$$Z'_\alpha = \{z_{1-6}\}$$

FIG.

Now from the above three conditions we have the following simple properties.

Consider any element α of $\Lambda\text{-}0$. Then for any vertex i which is not a representative vertex for G_α we see that $q_{(\alpha,i)}$ ends in an element of X_α. Also consider any element z of Z_α with initial, terminal vertex j and i say respectively. Then $q_{(\alpha,i)}$ ends in z and i belongs to $I-(I_\alpha UK_\alpha)$ and j belongs to $I_\alpha UK_\alpha$.

We shall see that these properties simplify the computational work which follows.

And so we now describe the forms taken by the relators occurring in the presentation given in step 4.

(a) It is easy to see that any relator in $\{r'' : L_\alpha, A\}$, when reduced, is of the form $(q_{(\alpha,i)} f q^{-1}_{(\alpha,i)})(q_{(\alpha,i)} g q^{-1}_{(\alpha,i)})(q_{(\alpha,i)} h q^{-1}_{(\alpha,i)})^{-1}$ for some element α of A, and some representative vertex i for G_α, and some elements f, g and h of $(G_\alpha)_{ii}$ where fg = h in $(G_\alpha)_{ii}$.

Now consider any element α of A-o, with predecessor β say, and any vertex i, and any element u of $(U_\alpha)_{ii}$.

We discuss the two cases $i \in I_\alpha$ and $i \in K_\alpha$ separately.

(b) Suppose i belongs to I_α.

Note that $q_{(\alpha,i)} = q_{(\beta,i)}$ in this case.

(b.1) First let us assume that i is a representative vertex for G_α.

Let j denote the representative vertex for that component of G_β which contains i, and let p denote that unique reduced path in X_β from j to i.

Then of course $q_{(\beta,i)} = q_{(\beta,j)} p$ and $p u \Theta_\alpha p^{-1} = g$ for some element g of $(G_\beta)_{jj}$.

In this case the relator $u(u\Theta_\alpha)^{-1}$ is written $u(p^{-1}gp)^{-1}$, and it follows that the relator $(u(u\Theta_\alpha)^{-1})''$, when reduced, is written $(q_{(\alpha,i)} u q^{-1}_{(\alpha,i)})(q_{(\beta,j)} g q^{-1}_{(\beta,j)})^{-1}$.

And so we have that the set of relators $(u(u\Theta_\alpha)^{-1})''$, as u ranges through $(U_\alpha)_{ii}$, when reduced, is expressed as $(q_{(\alpha,i)} u q^{-1}_{(\alpha,i)})(q_{(\beta,i)} u\Theta_\alpha q^{-1}_{(\beta,i)})^{-1}$, as u ranges through $(U_\alpha)_{ii}$.

Further we note that $\sigma_{(\alpha,i)}$ takes $q_{(\alpha,i)} u q^{-1}_{(\alpha,i)}$ to $q_{(\alpha,i)} u\Theta_\alpha q^{-1}_{(\alpha,i)}$ for each u in $(U_\alpha)_{ii}$.

(b.2) Second let us assume that i is an α-vertex for G_o.

Then i is a representative vertex for G_β .

This time let j denote the representative vertex for that component of G_α which contains i, and let p denote that unique reduced path in X_α from j to i.

Then $q_{(\alpha,i)} = q_{(\alpha,j)}p$ and $pup^{-1} = g$ for some element g of $(G_\alpha)_{jj}$.

In this case the relator $u(u\theta_\alpha)^{-1}$ is written $(p^{-1}gp)(u\theta_\alpha)^{-1}$, and it follows that the relator $(u(u\theta_\alpha)^{-1})''$, when reduced, is written $(q_{(\alpha,j)}gq^{-1}_{(\alpha,j)})(q_{(\beta,i)}u\theta_\alpha q^{-1}_{(\beta,i)})^{-1}$.

Here again we see that the set of relators, $(u(u\theta_\alpha)^{-1})''$ as u ranges through $(U_\alpha)_{ii}$, when reduced, is expressed as $(q_{(\alpha,i)}uq^{-1}_{(\alpha,i)})(q_{(\beta,i)}u\theta_\alpha q^{-1}_{(\beta,i)})^{-1}$, as u ranges through $(U_\alpha)_{ii}$.

And we note that $\sigma_{(\alpha,i)}$ takes $q_{(\alpha,i)}uq^{-1}_{(\alpha,i)}$ to $q_{(\alpha,i)}u\theta_\alpha q^{-1}_{(\alpha,i)}$ for each u in $(U_\alpha)_{ii}$.

(c) Suppose i belongs to K_α.

Then $q_{(\alpha,i)}q^{-1}_{(\beta,i)}$ is an element of $\{q_{(\alpha,i)}q^{-1}_{(\beta,i)}: K_\alpha, \Lambda\text{-}o\}$.

Now, let j,k be the representative vertex for that component of G_α, G_β respectively which contains i, and let p be that reduced path in X_α from j to i, and let q be that reduced path in X_β from k to i.

Then $q_{(\alpha,i)} = q_{(\alpha,j)}p$ and $pup^{-1} = g$ for some element g of $(G_\alpha)_{jj}$.

Also $q_{(\beta,i)} = q_{(\beta,k)}q$ and $qu\theta_\alpha q^{-1} = h$ for some element h of $(G_\beta)_{kk}$.

In this case we have that the relator $(u(u\theta_\alpha)^{-1}$ is written $(p^{-1}gp)(q^{-1}hq)^{-1}$, and it follows that the relator $(u(u\theta_\alpha)^{-1})''$, when reduced, is written $(q_{(\alpha,j)}gq^{-1}_{(\alpha,j)})$ $(q_{(\alpha,i)}q^{-1}_{(\beta,i)})(q_{(\beta,k)}hq^{-1}_{(\beta,k)})^{-1}$ $(q_{(\alpha,i)}q^{-1}_{(\beta,i)})^{-1}$.

Then we see that the set of relators, $(u(u\theta_\alpha)^{-1})''$ as u ranges through $(U_\alpha)_{ii}$, when reduced, is expressed as $(q_{(\alpha,i)}uq^{-1}_{(\alpha,i)})(q_{(\alpha,i)}q^{-1}_{(\beta,i)})(q_{(\beta,i)}u\theta_\alpha q^{-1}_{(\beta,i)})^{-1}$ $(q_{(\alpha,i)}q^{-1}_{(\beta,i)})^{-1}$ as u ranges through $(U_\alpha)_{ii}$.

Finally we consider any relator in $\{(z(z\theta_\alpha)^{-1})'' : A\text{-}o\}$.

(d) Suppose that i belongs to $I-(I_\alpha \cup K_\alpha)$.

Then we have that $q_{(\alpha,i)}$ ends in some element z of Z_α.

Let j be the initial vertex of z. Then j belongs to $I_\alpha \cup K_\alpha$.

Let k be the representative vertex for that component of G_β which contains i (and j), and let p,q be that reduced path in X_β from k to j and from k to i respectively.

Then $q_{(\beta,j)} = q_{(\beta,k)}p$ and $q_{(\beta,i)} = q_{(\beta,k)}q$, and $pz\theta_\alpha q^{-1} = g$ for some element g of $(G_\beta)_{kk}$.

And so the relator $z(z\theta_\alpha)^{-1}$ is written as $z(p^{-1}gq)^{-1}$.

We discuss the two cases $j \in I_\alpha$ and $j \in K_\alpha$ separately.

So first assume that j belongs to I_α.

Then $q_{(\alpha,j)} = q_{(\beta,j)}$ and it follows that the relator $(z(z\theta_\alpha)^{-1})''$, when reduced, is written $(q_{(\alpha,i)}q^{-1}_{(\beta,i)})$ $(q_{(\beta,k)}gq^{-1}_{(\beta,k)})^{-1}$.

Note that $q_{(\alpha,i)}q^{-1}_{(\beta,i)}$ belongs to $\{q_{(\alpha,i)}q^{-1}_{(\beta,i)}: I-(I_\alpha \cup K_\alpha), A-o\}$.

Second let us assume that j belongs to K_α.

Then it follows that the relator $(z(z\theta_\alpha)^{-1})''$, when reduced, is written $(q_{(\alpha,j)}q^{-1}_{(\beta,j)})^{-1}(q_{(\alpha,i)}q^{-1}_{(\beta,i)})(q_{(\beta,k)}gq^{-1}_{(\beta,k)})^{-1}$

And note here that $q_{(\alpha,i)}q^{-1}_{(\beta,i)}$ belongs to $\{q_{(\alpha,i)}q^{-1}_{(\beta,i)}: I-(I_\alpha \cup K_\alpha), A-o\}$, whereas $q_{(\alpha,j)}q^{-1}_{(\beta,j)}$ belongs to $\{q_{(\alpha,i)}q^{-1}_{(\beta,i)}: K_\alpha, A-o\}$.

Now from these remarks we observe the following.

(1) Consider any element α of $A-o$, with predecessor β say, and any element i of $I-(I_\alpha \cup K_\alpha)$. We have that $q_{(\alpha,i)}q^{-1}_{(\beta,i)}$ is an element of $\{q_{(\alpha,i)}q^{-1}_{(\beta,i)}: I-(I_\alpha \cup K_\alpha), A-o\}$, and $q_{(\alpha,i)}$ ends in some element z of Z_α.

Then from (a), (b), (c) and (d), we see that among the relators in the presentation given in step 4 the only occurrence of the generator $q_{(\alpha,i)}q^{-1}_{(\beta,i)}$ (or its inverse) is in the relator $(z(z\theta_\alpha)^{-1})''$.

And so we can omit the set of generators $\{q_{(\alpha,i)}q^{-1}_{(\beta,i)}: I-(I_\alpha \cup K_\alpha), A-o\}$ and the set of relators

$\{(z(z\Theta_\alpha)^{-1})'' : A\text{-}o\}$ from the presentation given in step 4.

That is we have a presentation for the vertex group of G at the origin,

$$< \{q_{(\alpha,i)}q^{-1}_{(\beta,i)} : K_\alpha, A\text{-}o\} \cup \{q_{(\alpha,i)}gq^{-1}_{(\alpha,i)} : L_\alpha, A\},$$
$$\{r'' : L_\alpha, A\} \cup \{(u(u\Theta_\alpha)^{-1})'' : I_\alpha UK_\alpha, A\text{-}o\} >$$

(2) From (a), (b) and the construction of the tree ξ given in lemma 11, it follows that the tree product of ξ has a presentation,

$$< \{q_{(\alpha,i)}gq^{-1}_{(\alpha,i)} : L_\alpha, A\},$$
$$\{r'' : L_\alpha, A\} \cup \{(u(u\Theta_\alpha)^{-1})'' : I_\alpha, A\text{-}o\} >.$$

(3) Consider any element α of $A\text{-}o$, with predecessor β say, and any element i of K_α. We have $q_{(\alpha,i)}q^{-1}_{(\beta,i)}$ belongs to $\{q_{(\alpha,i)}q^{-1}_{(\beta,i)} : K_\alpha, A\text{-}o\}$.

Then from (a), (b), (c) and (d) we see that among the relators in the presentation given in (1) the only occurrences of the generator $q_{(\alpha,i)}q^{-1}_{(\beta,i)}$ (or its inverse) are in the relators $(u(u\Theta_\alpha)^{-1})''$ as u ranges through $(U_\alpha)_{ii}$.

And so from (c), (1) and (2) we see that the vertex group of G at the origin is precisely the group described in the theorem.

Thus the theorem is proved in the case that each X_α and Z_α $(\alpha \in A\text{-}o)$ satisfies the conditions following step 4.

In the general case the computations become more intricate, but it is fairly straightforward to see that the theorem is still true.

This completes the proof of the theorem.

Chapter 3

THE VERTEX GROUPS OF TWO IMPORTANT KINDS OF CONNECTED TREE PRODUCTS OF GROUPOIDS

In this chapter we suppose that G satisfies one of two conditions, the first being that G_o is connected, and the second being that for each element α of $A - o$ U_α is discrete.

In the first case we shall see that we can characterise the vertex group of G at the origin as an HNN group whose base-part is some tree product of groups, using simply a representative system (theorem 4). This theorem is a straightforward special case of theorem 3.

In the second case we shall obtain a similar characterisation of the vertex group of G at the origin, without needing even a representative system (theorem 5). The basic point of interest in the proof of theorem 5 is that for each element α of A we choose a particular I-presentation for G_α unlike the usual kind of I-presentation described in proposition 7.

Theorem 4

Suppose that G_o is connected, and choose any representative system $\{Q_\alpha : \alpha \in A\}$. For each $\alpha \in A - o$ choose any set of representative vertices for U_α containing the set of representative vertices for G_α. Again for each $\alpha \in A - o$ and each representative vertex i for G_α, let $\sigma_{(\alpha, i)}$ denote the group isomorphism given by $q_{(\alpha,i)} u q^{-1}_{(\alpha,i)} \rightarrow q_{(\alpha,i)} u\theta_\alpha q^{-1}_{(\alpha,i)}$ as u ranges through $(U_\alpha)_{ii}$. Let Σ be the set of all these group isomorphisms.

Then Σ is a tree of groups $q_{(\alpha,i)}(G_\alpha)_{ii} q^{-1}_{(\alpha,i)}$ as i ranges through the representative vertices for G_α and α ranges through A.

Further the vertex group of G at the origin is the HNN group with base-part the tree product of Σ and free-part generated by the elements $q_{(\alpha,i)} q^{-1}_{(\beta,i)}$ where β is the predecessor of α and i ranges through the representative vertices for U_α other than representative vertices for G_α and α ranges through $A - o$.

Finally for each $\alpha \in A - o$, with predecessor β say, and each representative vertex i for U_α other than a representative vertex for G_α, then the group isomorphism associated with $q_{(\alpha,i)} q^{-1}_{(\beta,i)}$ is given by $q_{(\beta,i)} u\theta_\alpha q^{-1}_{(\beta,i)} \longrightarrow q_{(\alpha,i)} u q^{-1}_{(\alpha,i)}$ as u ranges through $(U_\alpha)_{ii}$.

Now for the remainder of this chapter suppose we have chosen, for each element α of $\mathcal{A}$, some maximal circuit-free subgraph X_α of G_α. Let X be the graph-union of the X_α, and suppose we have also chosen some maximal tree T of X. For each vertex i let us write t_i for that unique reduced path in T from the origin to i.

Before we give the next theorem we give a definition, and make some simple observations.

For each element α of $\mathcal{A}$ we define the <u>α-part of I</u> to consist of those vertices i such that t_i does not end in an element of $X_\alpha^{\pm 1}$.

Now consider any element α of $\mathcal{A}$, and any vertex i which does not belong to the α-part of I. Of course this means that t_i ends in an element of $X_\alpha^{\pm 1}$. Let $(i \neq) j$ denote that vertex such that t_j is that largest initial segment of t_i which does not end in an element of $X_\alpha^{\pm 1}$. Then it follows that j belongs to the α-part of I. Also we have that $t_j^{-1}t_i$ is a path in X_α. Now consider any element g of $(G_\alpha)_{ii}$. Writing $t_i g t_i^{-1}$ as $t_j(t_j^{-1}t_i g t_i^{-1}t_j)t_j^{-1}$ we see that $t_i g t_i^{-1}$ belongs to the group $t_j(G_\alpha)_{jj}t_j^{-1}$.

Theorem 5

Suppose for each $\alpha \in A\text{-}o$ U_α is discrete. For each $\alpha \in A\text{-}o$ and each vertex i let $\sigma_{(\alpha,i)}$ denote the group isomorphism given by $t_i u t_i^{-1} \longrightarrow t_i u\theta_\alpha t_i^{-1}$ as u ranges through $(U_\alpha)_{ii}$. Let Σ be the set of all these group isomorphisms.

Then Σ is a tree of groups $t_i(G_\alpha)_{ii}t_i^{-1}$ as i ranges through the α-part of I and α ranges through A.

Further the vertex group of G at the origin is the HNN group with base-part the tree product of Σ and free-part generated by the elements $t_j x t_i^{-1}$ where j,i is the initial, terminal vertex of x respectively and x ranges through X-T.

Finally consider any edge x of X-T and suppose x belongs to X_α and has initial, terminal vertex j,i respectively. Then the group isomorphism associated with $t_j x t_i^{-1}$ is given by $t_i g t_i^{-1} \longrightarrow t_j(xgx^{-1})t_j^{-1}$ as g ranges through $(G_\alpha)_{ii}$.

Proof

First we make two observations.

(1) For each element α of A the α-part of I contains the origin and forms a set of representative vertices for $X_\alpha \cap T$, considered as an I-graph.

(2) Consider any non-origin vertex i. Then there exists a unique element α of A such that i does not belong to the α-part of I. That is the end of t_i belongs to $X_\alpha^{\pm 1}$.

Now we show that Σ is a tree of groups.

Lemma 13

Σ is a tree of groups $t_i(G_\alpha)_{ii}t_i^{-1}$ where i ranges through the α-part of I and α ranges through A .

Proof

To prove the lemma we shall show that, for each vertex i, the set of group isomorphisms, $\sigma_{(\alpha,i)}$ as α ranges through A-o , constitute a tree Σ_i say.

Then we shall see that the union of the trees $\Sigma_i (i \in I)$ is also a tree, and contains the same set of group isomorphisms as Σ . Then it will follow that Σ is a tree.

So let us construct the trees Σ_i.

First suppose i is the origin. We associate two vertices to each group isomorphism, $\sigma_{(\alpha,i)}$ as α ranges through A-o , as follows.

So consider any element α of A-o , with predecessor β say. Then the group isomorphism $\sigma_{(\alpha,i)}$ has domain $(U_\alpha)_{ii}$, a subgroup of $(G_\alpha)_{ii}$, and range $(V_\alpha)_{ii}$, a subgroup of $(G_\beta)_{ii}$. In this case we define the vertices of $\sigma_{(\alpha,i)}$ to be the groups $(G_\alpha)_{ii}$ and $(G_\beta)_{ii}$ (it does not matter which of these groups we define to be the initial vertex of $\sigma_{(\alpha,i)}$ and which the terminal vertex of $\sigma_{(\alpha,i)}$). Observe that since i is the origin we have i belongs to both the α-part and the β-part of I.

In this way we define the vertices of each group isomorphism $\sigma_{(\alpha,i)}$ as α ranges through A-o .

Then it is clear that the set of group isomorphisms, $\sigma_{(\alpha,i)}$ as α ranges through A-o , constitutes a tree of groups $(G_\alpha)_{ii}$

as α ranges through A. And this is the tree we denote by ξ_i.

Now suppose that i is a non-origin vertex.

Then, from the remarks preceding the theorem, we have that there exists a unique element γ of A such that i does not belong to the γ-part of I. This means that t_i ends in an element of $X_\gamma^{\pm 1}$. Then let $(i \neq)j$ denote that vertex such that t_j is that largest initial segment of t_i which does not end in an element of $X_\gamma^{\pm 1}$. Recall then that j belongs to the γ-part of I and $t_j^{-1}t_i$ is a path in X_γ.

We associate two vertices to each group isomorphism, $\sigma_{(\alpha,i)}$ as α ranges through A-o, as follows.

So consider any element α of A-o with predecessor β say. Note then that the group isomorphism $\sigma_{(\alpha,i)}$ has domain $t_i(U_\alpha)_{ii}t_i^{-1}$ and range $t_i(V_\alpha)_{ii}t_i^{-1}$.

We deal with the three cases: $\alpha \neq \gamma \neq \beta$, $\alpha = \gamma$, and $\gamma = \beta$ separately.

First then suppose $\alpha \neq \gamma \neq \beta$. In this case we have that i belongs to both the α-part and the β-part of I. Then we define the vertices of $\sigma_{(\alpha,i)}$ to be the groups $t_i(G_\alpha)_{ii}t_i^{-1}$ and $t_i(G_\beta)_{ii}t_i^{-1}$.

Next suppose $\alpha = \gamma$. In this case we have that i belongs to the β-part of I. Observe also that the domain of $\sigma_{(\alpha,i)}$ can be expressed as $t_j(t_j^{-1}t_i(U_\alpha)_{ii}t_i^{-1}t_j)t_j^{-1}$ which is a subgroup of $t_j(G_\alpha)_{jj}t_j^{-1}$. Then we define the vertices of $\sigma_{(\alpha,i)}$ to be the groups $t_j(G_\alpha)_{jj}t_j^{-1}$ and $t_i(G_\beta)_{ii}t_i^{-1}$.

Finally suppose $\gamma = \beta$. In this case we have that i belongs to the α-part of I. Observe, this time, that the range of

$\sigma_{(\alpha,i)}$ can be expressed at $t_j(t_j^{-1}t_i(V_\alpha)_{ii}t_i^{-1}t_j)t_j^{-1}$ which is a subgroup of $t_j(G_\beta)_{jj}t_j^{-1}$. Then we define the vertices of $\sigma_{(\alpha,i)}$ to be the groups $t_i(G_\alpha)_{ii}t_i^{-1}$ and $t_j(G_\beta)_{jj}t_j^{-1}$.

In this way we define the vertices of each group isomorphism $\sigma_{(\alpha,i)}$ as α ranges through A-o .

Then it is easy to see that the set of group isomorphisms, $\sigma_{(\alpha,i)}$ as α ranges through A-o , constitute a tree of groups $t_i(G_\alpha)_{ii}t_i^{-1}$ as α ranges through A-γ together with the group $t_j(G_\gamma)_{jj}t_j^{-1}$. And this is the tree we denote by Σ_i.

Now it is not difficult to see that Σ is the union of the $\Sigma_i(i \in I)$, and that Σ is a tree of groups $t_i(G_\alpha)_{ii}t_i^{-1}$ as i ranges through the α-part of I and α ranges through A .

Thus the lemma is proved.

Now to characterise the vertex group of G at the origin. The steps of the proof are as follows.

To begin with we choose a particular I-presentation for each G_α (and we shall see that these I-presentations are unlike those usually considered).

Then, using these I-presentations in lemma 1, we obtain an I-presentation for G, $\langle Y,S \rangle$ say.

Next we choose any maximal tree of F(Y) (the free groupoid on Y), and using this tree we form the conjugation in the origin of each relator in S, and so obtain another I-presentation for G, $\langle Y,S' \rangle$ say, by the first part of lemma 2. Recall then that each relator in S' has vertices the origin.

Then we use the result of Higgins given in theorem 1 to obtain a set of free generators W for the vertex group of F(Y) at the origin.

And so, rewriting each relator in S' in terms of the set of free generators W, we obtain a presentation for the vertex group of G at the origin $\langle W, S'' \rangle$ say, by the second part of lemma 2.

Finally we describe the forms taken by the relators in S'', and so deduce that the vertex group of G at the origin is as described in the statement of the theorem.

Step 1 An I-presentation for each G_α.

Choose any element α of A.

We use the following lemma to give us a particular I-presentation for G_α. The proof of the lemma is quite straightforward and so is omitted.

Lemma 14

Choose any subgraph Y_α of X_α, together with any set of representative vertices I_α for Y_α considered as an I-graph. For each element i of I_α let $\langle (G_\alpha)_{ii}, (R_\alpha)_{ii} \rangle$ be the standard presentation for $(G_\alpha)_{ii}$.

Consider any edge x of $X_\alpha - Y_\alpha$ with initial and terminal vertex j,i respectively. Let k,l denote the representative vertex for that component of Y_α which contains j,i respectively. Also let us write R_x for the graph of points $h(xgx^{-1})^{-1}$ as g ranges through $(G_\alpha)_{ii}$ and $xgx^{-1} = h$ in $(G_\alpha)_{jj}$, where the elements g are written in terms of $Y_\alpha \cup (G_\alpha)_{ll}$ and the elements h are written in terms of $Y_\alpha \cup (G_\alpha)_{kk}$.

Then G_α has an I-presentation with generator graph X_α together with the graphs $(G_\alpha)_{ii}$ $(i \in I_\alpha)$, and relator graph the union of the graphs $(R_\alpha)_{ii}$ $(i \in I_\alpha)$ together with the graphs $R_x (x \in X_\alpha - Y_\alpha)$.

Recall now that the α-part of I is a set of representative vertices for $X_\alpha \cap T$ considered as an I-graph. And so, using lemma 14 with $Y_\alpha = X_\alpha \cap T$ and I_α the α-part of I, we obtain an I-presentation for G_α with generator graph X_α together with the graphs $(G_\alpha)_{ii}$ (as i ranges through the α-part of I), and relator graph the union of the graphs $(R_\alpha)_{ii}$ (as i ranges through the α-part of I) together with the graphs R_x (as x ranges through $X_\alpha - (X_\alpha \cap T) = X_\alpha - T$).

In this way we choose an I-presentation for each G_α.

Step 2 An I-presentation for G

Here we use lemma 1, together with the I-presentation for each G_α given in step 1, to obtain an I-presentation for G.

To do this, we need, for each $\alpha \in A\text{-}o$, some set of representative vertices for U_α and some maximal circuit-free subgraph of U_α. Of course, since each U_α ($\alpha \in A\text{-}o$) is discrete, it follows that the only set of representative vertices for U_α is I itself and the only maximal circuit-free subgraph of U_α is the empty I-graph.

And so we obtain an I-presentation for G with generator graph the union of the generator graphs of the I-presentations for the G_α given in step 1, and with relator graph the union of the relator graphs of the I-presentations for the G_α given in step 1 together with the points $u(u\theta_\alpha)^{-1}$ as u ranges through $(U_\alpha)_{ii}$ and i ranges through I and α ranges through $A\text{-}o$.

Of course, in this I-presentation for G, each relator of the form $u(u\theta_\alpha)^{-1}$ can be written in many ways in terms of the given generators of G. We adopt the following rule for writing such relators.

So consider any element α of $A\text{-}o$, with predecessor β say, and any vertex i, and any element u of $(U_\alpha)_{ii}$. Let j,k be the representative vertex for that component of $X_\alpha \cap T, X_\beta \cap T$ which contains i respectively. Then in the relator $u(u\theta_\alpha)^{-1}$ we write u in terms of $(X_\alpha \cap T)U(G_\alpha)_{jj}$ and we write $u\theta_\alpha$ in terms of $(X_\beta \cap T)U(G_\beta)_{kk}$.

Now let us introduce a short-hand notation similar to that given in theorem 3, to describe the generator and relator graphs we consider.

So let us denote the generator graph $U(\bigcup_{\alpha\in A}\bigcup_{i\in I_\alpha}(G_\alpha)_{ii})$ by $\{g:I_\alpha, A\}$, and the relator graph $\bigcup_{\alpha\in A}(\bigcup_{i\in I_\alpha}(R_\alpha)_{ii})$ by $\{r:I_\alpha, A\}$. Also we denote the relator graph $\bigcup_{x\in X-T} R_x$ by $\{r:X-T\}$, and the relator graph $\{u(u\theta_\alpha)^{-1}: u\in(U_\alpha)_{ii}, i\in I, \alpha\in A-o\}$ by $\{u(u\theta_\alpha)^{-1}:I, A-o\}$.

Then in this notation the I-presentation for G we obtain in step 2 becomes,

$$\langle\, X\cup\{g:I_\alpha, A\},\ \{r:I_\alpha, A\}\cup\{r:X-T\}\cup\{u(u\theta_\alpha)^{-1}:I, A-o\}\,\rangle.$$

Step 3 A second I-presentation for G

Choose any maximal tree of $F(X\cup\{g:I_\alpha, A\})$ (the free groupoid on $X\cup\{g:I_\alpha, A\}$), and using this tree let us form the conjugation in the origin of each relator in the I-presentation for G given in step 2.

Then by the first part of lemma 2 we obtain a second I-presentation for G,

$$\langle\, X\cup\{g:I_\alpha, A\},\ \{r':I_\alpha, A\}\cup\{r':X-T\}\cup\{(u(u\theta_\alpha)^{-1})':I, A-o\}\,\rangle.$$

Note that each relator in this I-presentation has vertices the origin.

Step 4 A set of free generators for the vertex group of $F(X \cup \{g : I_\alpha, \mathcal{A}\})$ at the origin

Using the maximal tree T of $X \cup \{g : I_\alpha, \mathcal{A}\}$, and Higgins' result given in theorem 1 we obtain that the vertex group of $F(X \cup \{g : I_\alpha, \mathcal{A}\})$ at the origin is freely generated by the elements $t_i g t_i^{-1}$ (as g ranges through $(G_\alpha)_{ii}$ and i ranges through I_α and α ranges through $\mathcal{A}$) together with the elements $t_j x t_i^{-1}$ (where j,i is the initial, terminal vertex of x respectively and x ranges through X-T).

Let us denote the set of elements $t_i g t_i^{-1}$ (as g ranges through $(G_\alpha)_{ii}$ and i ranges through I_α and α ranges through $\mathcal{A}$) by $\{t_i g t_i^{-1} : I_\alpha, \mathcal{A}\}$, and the set of elements $t_j x t_i^{-1}$ (where j,i is the initial, terminal vertex of x respectively and x ranges through X-T) by $\{t_j x t_i^{-1} : X-T\}$.

Step 5 A presentation for the vertex group of G at the origin

Now let us rewrite each relator in the I-presentation for G given in step 3 in terms of the set of free generators given in step 4.

Then by the second part of lemma 2 we have the following presentation for the vertex group of G at the origin,

$$\langle \{t_i g t_i^{-1} : I_\alpha, \mathcal{A}\} \cup \{t_j x t_i^{-1} : X-T\},\ \{r'' : I_\alpha, \mathcal{A}\} \cup \{r'' : X-T\} \cup \{(u(u\theta_\alpha)^{-1})'' : I, \mathcal{A}\text{-}o\} \rangle .$$

Step 6 Investigation of the forms taken by the relators in the presentation given in step 5.

Here, finally, we describe the forms taken by the relators in the presentation given in step 5. From this description we shall see that the vertex group of G at the origin has the structure given in the statement of the theorem.

(a) Consider any relator in $\{r'' : I_\alpha, A\}$.

Clearly this relator, when reduced, is of the form $(t_i f t_i^{-1})(t_i g t_i^{-1})(t_i h t_i^{-1})^{-1}$ for some $\alpha \in A$ and some $i \in I_\alpha$ and some f, g and h belonging to $(G_\alpha)_{ii}$ where fg = h in $(G_\alpha)_{ii}$.

(b) Now consider any relator in $\{(u(u\theta_\alpha)^{-1})'' : I, A-o\}$.

So choose any element α of $A-o$, with predecessor β say, and any vertex i, and any element u of $(U_\alpha)_{ii}$.

Let j be the representative vertex for that component of $X_\alpha \cap T$ which contains i, and let p be that reduced path in $X_\alpha \cap T$ from j to i.

Also let k be the representative vertex for that component of $X_\beta \cap T$ which contains i, and let q be that reduced path in $X_\beta \cap T$ from k to i.

Then $pup^{-1} = g$ for some element g of $(G_\alpha)_{jj}$, and $qu\theta_\alpha q^{-1} = h$ for some element h of $(G_\beta)_{kk}$.

In this case we have that the relator $u(u\theta_\alpha)^{-1}$ is written $(p^{-1}gp)(q^{-1}hq)^{-1}$, and it follows that the relator $(u(u\theta_\alpha)^{-1})''$, when reduced, is written $(t_j g t_j^{-1})(t_k h t_k^{-1})^{-1}$.

And so we see that the set of relators, $(u(u\theta_\alpha)^{-1})''$ as u ranges through $(U_\alpha)_{ii}$, when reduced, is expressed as $(t_i u t_i^{-1})(t_i u\theta_\alpha t_i^{-1})^{-1}$ as u ranges through $(U_\alpha)_{ii}$.

Further we note that $\sigma_{(\alpha,i)}$ takes $t_i u t_i^{-1}$ to $t_i u\theta_\alpha t_i^{-1}$ for each u in $(U_\alpha)_{ii}$.

From (a), (b) and the construction of the tree ξ given in lemma 14, we obtain that the tree product of ξ has a presentation,

$$\langle \{t_i g t_i^{-1} : I_\alpha, A\}, \{r'' : I_\alpha, A\} \cup \{(u(u\theta_\alpha)^{-1})'' : I, A\text{-}o\} \rangle.$$

(c) Next consider any relator in $\{r'' : X-T\}$.

So choose any element α of A , and any element x of X_α -T with initial, terminal vertex j,i say respectively, and any element g of $(G_\alpha)_{ii}$, and suppose $xgx^{-1} = h$ in $(G_\alpha)_{jj}$.

We discuss the relator $(h(xgx^{-1})^{-1})''$.

Let k,l denote the representative vertex for that component of $X_\alpha \cap T$ which contains j, i respectively, and let p,q be that reduced path in $X_\alpha \cap T$ from k to j and from l to i respectively.

Then $php^{-1} = h_1$ for some element h_1 of $(G_\alpha)_{kk}$, and $qgq^{-1} = g_1$ for some element g_1 of $(G_\alpha)_{ll}$.

In this case the relator $(h(xgx^{-1})^{-1})$ is written $(p^{-1}h_1 p)(x(q^{-1}g_1 q)x^{-1})^{-1}$, and it follows that the relator $(h(xgx^{-1})^{-1})''$, when reduced, is written $(t_k h_1 t_k^{-1})(t_j x t_i^{-1})$ $(t_l g_1 t_l^{-1})^{-1}(t_j x t_i^{-1})^{-1}$.

Thus we see that the set of relators, $(h(xgx^{-1})^{-1})''$ as

g ranges through $(G_\alpha)_{ii}$, when reduced, is expressed as $(t_j(xgx^{-1})t_j^{-1})(t_jxt_i^{-1})(t_igt_i^{-1})^{-1}(t_jxt_i^{-1})^{-1}$ as g ranges through $(G_\alpha)_{ii}$.

From these remarks it is straightforward to see that the vertex group of G at the origin is the HNN group with base-part the tree product of $\leq$ and free-part generated by $\{t_jxt_i^{-1}:X-T\}$. Also for each $\alpha\in A$ and each $x\in X_\alpha-T$ with initial, terminal vertex j,i respectively, then we see that the group isomorphism associated with the generator $t_jxt_i^{-1}$ is given by $t_igt_i^{-1}\longrightarrow t_j(xgx^{-1})t_j^{-1}$ as g ranges through $(G_\alpha)_{ii}$.

Thus the theorem is proved.

In closing this chapter we mention that in the proof of theorem 5 it is important that we choose I-presentations for the G_α according to lemma 14. If, as usual, we choose I-presentations for the G_α according to proposition 7, then the method breaks down - for we are then faced with the same kind of problem which appears in the example considered in the appendix.

Chapter 4

DEFINITION OF HNN GROUPOIDS AND CHARACTERISATION OF THE VERTEX GROUP OF ANY CONNECTED HNN GROUPOID

In this chapter we define what we mean by an 'HNN groupoid', and then we show that the vertex group of any connected HNN groupoid is an HNN group with base-part some tree product of groups (theorem 6).

4.1 Definition of an HNN groupoid

Consider any I-groupoid G, for some vertex set I. Let $\{\theta_\alpha : U_\alpha \longrightarrow V_\alpha , \alpha \in A\}$ be any set of groupoid isomorphisms where, for each $\alpha \in A$, U_α and V_α are I-subgroupoids of G. Here we do not require that the θ_α be groupoid I-isomorphisms.

Now for each $\alpha \in A$ and each vertex i, let j denote the image of i under the vertex map of θ_α, and then let us introduce the edge $s_{(\alpha,i)}$ with initial vertex j and terminal vertex i.

Choose any I-presentation $\langle X,R \rangle$ for G.

Let H be the I-groupoid with the I-presentation with generator graph $X \cup \{s_{(\alpha,i)} : \alpha \in A, i \in I\}$, and relator graph R together with the graph of points $s_{(\alpha,i)} u s^{-1}_{(\alpha,j)} (u\theta_\alpha)^{-1}$ where u ranges through $(U_\alpha)_{ij}$ and i,j range through I and α ranges through A.

Then we call H the HNN groupoid with base-groupoid G, groupoid isomorphisms $\{\theta_\alpha : U_\alpha \longrightarrow V_\alpha , \alpha \in A\}$ and related graph $\{s_{(\alpha,i)} : \alpha \in A, i \in I\}$.

It is not difficult to see that HNN groupoids are independent of the particular I-presentation used in their definition.

4.2 Some constructions

Here we describe the constructions we use to prove theorem 6.

Also we give some elementary properties of these constructions.

So let H be any connected HNN I-groupoid with base-groupoid G, groupoid isomorphisms $\{\theta_\alpha : U_\alpha \rightarrow V_\alpha , \alpha \in A\}$, and related graph $\{s_{(\alpha, i)} : \alpha \in A , i \in I\}$.

Choose any element of I, and call this vertex the origin. Our object is to describe the vertex group of H at the origin. To do this we need to choose a maximal circuit-free subgraph X of G, and for each $\alpha \in A$ a set of representative vertices I_α for U_α , and a maximal tree T of H, and a set of representative vertices I_G for G.

To begin with, then, choose any maximal circuit-free subgraph X of G, and for each $\alpha \in A$ choose any set of representative vertices I_α for U_α .

Now for each $\alpha \in A$ put $S_\alpha = \{s_{(\alpha, i)} : i \in I_\alpha\}$, and then write S for the graph-union of the $S_\alpha (\alpha \in A)$. Clearly we have that $X \cup S$ is a connected I-graph. Choose any maximal tree T of $X \cup S$ containing X.

Let l denote the level-function on T induced by the origin. Then for each component of G there exists a unique vertex of the component of minimal l-level - choose this vertex to be the representative vertex for the component. In this way we obtain a set of representative vertices I_G for G which we call the set of representative vertices for G minimal with respect to l.

Finally we make two observations.

First, for each vertex i let t_i be that reduced path in T from the origin to i. Then it is easy to see that the set of non-origin representative vertices for G consists precisely of the set of non-origin vertices i such that t_i ends in an element of $S^{\pm 1}$.

Second, consider any vertex i, and let k denote the representative vertex for that component of G which contains i. Then for any element g of G_{ii} we have that $t_i g t_i^{-1}$ belongs to the group $t_k G_{kk} t_k^{-1}$.

4.3 The theorem

Throughout this section suppose we have the following.

Let H be the connected HNN I-groupoid with base-groupoid G, groupoid isomorphisms $\{\theta_\alpha : U_\alpha \to V_\alpha, \alpha \in A\}$, and related graph $\{s_{(\alpha, i)} : \alpha \in A, i \in I\}$.

Suppose we have chosen any element of I which we call the origin, and any maximal circuit-free subgraph X of G, and for each $\alpha \in A$ any set of representative vertices I_α for U_α.

Put $S_\alpha = \{s_{(\alpha,i)} : i \in I_\alpha\}$ for each $\alpha \in A$, and $S = \bigcup_{\alpha \in A} S_\alpha$, and suppose we have chosen any maximal tree T of XUS containing X. For each vertex i let t_i denote the reduced path in T from the origin to i.

Finally we denote by I_G the set of representative vertices for G minimal with respect to the level-function on T induced by the origin.

Theorem 6

For each $\alpha \in A$ and each $s \in S_\alpha \cap T$ with initial, terminal vertex j,i say respectively let σ_s denote the group isomorphism given by $t_i u t_i^{-1} \longrightarrow t_j u \theta_\alpha t_j^{-1}$ as u ranges through $(U_\alpha)_{ii}$. Let Σ denote the set of all these group isomorphisms.

Then Σ is a tree of groups $t_i G_{ii} t_i^{-1}$ where i ranges through the representative vertices for G.

Further the vertex group of H at the origin is the HNN group with base-part the tree product of Σ and free-part generated by $t_j s t_i^{-1}$ where j,i is the initial, terminal vertex of s respectively and s ranges through S-T.

Finally consider any edge s of S-T and suppose s belongs to S_α and has initial, terminal vertex j,i respectively. Then the group isomorphism associated with $t_j s t_i^{-1}$ is given by $t_i u t_i^{-1} \longrightarrow t_j u \theta_\alpha t_j^{-1}$ as u ranges through $(U_\alpha)_{ii}$.

Proof

We begin by proving that Σ is a tree.

Lemma 15

ξ is a tree of groups $t_i G_{ii} t_i^{-1}$ as i ranges through the representative vertices for G.

Proof

First we show that ξ is a graph with vertices the groups $t_i G_{ii} t_i^{-1}$ as i ranges through the representative vertices for G.

To do this we define an initial and terminal vertex for each group isomorphism in ξ .

So consider any element σ of ξ. Then $\sigma = \sigma_s$ for some edge s of $S \cap T$. Suppose s belongs to S_α and has initial, terminal vertex j,i say respectively. Then σ has domain $t_i (U_\alpha)_{ii} t_i^{-1}$ and range $t_j (V_\alpha)_{jj} t_j^{-1}$. Let k,l be the representative vertex for that component of G which contains j,i respectively (note that at least one of k = j or l = i is true). Then we have that $t_i (U_\alpha)_{ii} t_i^{-1}$ is a subgroup of $t_l G_{ll} t_l^{-1}$ and $t_j (V_\alpha)_{jj} t_j^{-1}$ is a subgroup of $t_k G_{kk} t_k^{-1}$. In this case we define the initial, terminal vertex of σ to be the group $t_l G_{ll} t_l^{-1}$, $t_k G_{kk} t_k^{-1}$ respectively.

If we define the initial, terminal vertex of each element of ξ in this way, we see that ξ acquires the structure of a graph with vertices the groups $t_i G_{ii} t_i^{-1}$ as i ranges through the representative vertices for G.

Now to show that ξ is a tree.

To do this we construct a graph Λ of group isomorphisms, λ_i as i ranges through the non-origin representative vertices for G. We shall see that Λ is a tree, and from this it will follow that ξ is a tree.

We construct the group isomorphisms λ_i as follows.

Consider any group isomorphism σ in ξ. Then $\sigma = \sigma_s$ for some edge s of $S\cap T$. Let us suppose that s belongs to S_α and has initial, terminal vertex j,i say respectively. Of course σ has domain $t_i(U_\alpha)_{ii}t_i^{-1}$ and range $t_j(V_\alpha)_{jj}t_j^{-1}$.

It is easy to see that either t_j is an initial segment of t_i in which case t_i ends in s and i is a non-origin representative vertex for G, or t_i is an initial segment of t_j in which case t_j ends in s^{-1} and j is a non-origin representative vertex for G.

First, suppose that t_j is an initial segment of t_i. Let k be the representative vertex for that component of G which contains j. Then σ has initial vertex $t_iG_{ii}t_i^{-1}$ and terminal vertex $t_kG_{kk}t_k^{-1}$. In this case we define the group isomorphism λ_i to be σ.

Second, suppose that t_i is an initial segment of t_j. This time let k be the representative vertex for that component of G which contains i. Then σ has initial vertex $t_kG_{kk}t_k^{-1}$ and terminal vertex $t_jG_{jj}t_j^{-1}$. In this case we define the group isomorphism λ_j to be σ^{-1}.

Then we write Λ for the graph of group isomorphisms λ_i as i ranges through the non-origin representative vertices for G.

It is easy to see that each edge of ξ is either an edge of Λ or the inverse of an edge of Λ, and vice versa.

Then it follows that ξ is a tree iff Λ is a tree.

It only remains to show that Λ is a tree.

This we do using Karrass & Solitar's result given in theorem 3. For completeness we restate their result here.

Choose any vertex of Λ, which we call the 'start'. Then to each vertex of Λ associate a non-negative integer, such that the non-negative integer associated with the 'start' is 0. Suppose that each non-'start' vertex of Λ is the terminal vertex of a unique edge of Λ. Also suppose that for each edge λ of Λ, the non-negative integer associated with the initial vertex of λ is less than that associated with the terminal vertex of λ. Then Λ is a tree.

This result holds if we replace 'initial vertex' by 'terminal vertex' and vice versa.

To use this result to show that Λ is a tree, we choose the group $t_i G_{ii} t_i^{-1}$ ($=G_{ii}$, where i denotes the origin) to be the 'start' of Λ. Also, for each representative vertex i for G, the non-negative integer we associate with the vertex $t_i G_{ii} t_i^{-1}$ is to be the length of the path t_i.

Then using Karrass & Solitar's result it is quite straightforward to see that Λ is a tree.

And from this we obtain that ξ is a tree.

Thus the lemma is proved.

Now to prove that the vertex group of H at the origin is the HNN group described in the statement of the theorem.

To begin with we obtain an I-presentation for H, using the following easy lemma.

Lemma 16

For each element i of I_G let $\langle G_{ii}, R_{ii}\rangle$ denote the standard presentation for the group G_{ii}.

Then H has an I-presentation with generator graph $XUSU(\bigcup_{i\in I_G} G_{ii})$, and relator graph $\bigcup_{i\in I_G} R_{ii}$ together with the graph of points $s_{(\alpha,i)}us^{-1}_{(\alpha,i)}(u\theta_\alpha)^{-1}$ where u ranges through $(U_\alpha)_{ii}$ and i ranges through I_α and α ranges through A.

For convenience we now introduce some short-hand notation for the graphs of generators and relators in which we are interested.

We denote the graph of generators $\bigcup_{i\in I_G} G_{ii}$ by $\{g:I_G\}$, and the graph of relators $\bigcup_{i\in I_G} R_{ii}$ by $\{r:I_G\}$.

Similarly we write $\{sus^{-1}(u\theta_\alpha)^{-1}:S\}$ for the graph of points $s_{(\alpha,i)}us^{-1}_{(\alpha,i)}(u\theta_\alpha)^{-1}$ where u ranges through $(U_\alpha)_{ii}$ and $s_{(\alpha,i)}$ ranges through S_α and α ranges through A.

Also we shall find it convenient to split up the graph $\{sus^{-1}(u\theta_\alpha)^{-1}:S\}$ as follows. We write $\{sus^{-1}(u\theta_\alpha)^{-1}:S\cap T\}$ for the graph of points $s_{(\alpha,i)}us_{(\alpha,i)}^{-1}(u\theta_\alpha)^{-1}$ where u ranges through $(U_\alpha)_{ii}$ and $s_{(\alpha,i)}$ ranges through $S_\alpha\cap T$ and α ranges through A. And we write $\{sus^{-1}(u\theta_\alpha)^{-1}:S-T\}$ for the graph of points $s_{(\alpha,i)}us^{-1}_{(\alpha,i)}(u\theta_\alpha)^{-1}$ where u ranges through $(U_\alpha)_{ii}$ and $s_{(\alpha,i)}$ ranges through $S_\alpha-T$ and α ranges through A.

Then, with this notation, the I-presentation for H given in lemma 16 can be written,

$$\langle\, XUSU\,\{g:I_G\}\ ,\quad \{r:I_G\}\cup\{sus^{-1}(u\theta_\alpha)^{-1}:\ S\}\,\rangle.$$

For each relator in this I-presentation let us form its conjugation in the origin using the maximal tree T. (Of course, instead of T, we could choose any maximal tree of $F(XUSU\{g:I_G\})$, the free groupoid on $XUSU\{g:I_G\}$).

Then, by the first part of lemma 2, we have another I-presentation for H,

$$\langle\ XUSU\,\{g:I_G\}\ ,\quad \{r':I_G\}\ \cup\ \{(sus^{-1}(u\theta_\alpha)^{-1})':\ S\}\ \rangle.$$

And each relator in this I-presentation has vertices the origin.

Now, using Higgins' result given in theorem 1 with the maximal tree T, we have that the vertex group of $F(XUSU\{g:I_G\})$ at the origin is freely generated by the elements $\{t_i g t_i^{-1} : g\in G_{ii}, i\in I_G\}$ together with the elements $t_j s t_i^{-1}$ (where j,i is the initial, terminal vertex of s respectively and s ranges through S-T).

We abbreviate the set of elements $\{t_i g t_i^{-1} : g\in G_{ii}, i\in I_G\}$ to $\{t_i g t_i^{-1}:I_G\}$, and we write $\{t_j s t_i^{-1}:S-T\}$ for the set of elements $t_j s t_i^{-1}$ (where j,i is the initial, terminal vertex of s respectively and s ranges through S-T).

Next let us rewrite each relator in the second I-presentation given for H in terms of these two sets of free generators.

Then, by the second part of lemma 2, we have a presentation for the vertex group of H at the origin,

$$\langle \{t_j s t_i^{-1} : S-T\} \cup \{t_i g t_i^{-1} : I_G\}, \quad \{r'' : I_G\} \cup \{(sus^{-1}(u\theta_\alpha)^{-1})'' : S\} \rangle.$$

Now the structure of the vertex group of H at the origin follows on investigating the forms taken by the relators in this presentation.

This we now do.

(a) First it is clear that any relator in $\{r'' : I_G\}$, when reduced, is written $(t_i f t_i^{-1})(t_i g t_i^{-1})(t_i h t_i^{-1})^{-1}$ for some representative vertex i for G, and some f,g and h belonging to G_{ii}, where fg = h in G_{ii}.

(b) Next consider any relator in $\{(sus^{-1}(u\theta_\alpha)^{-1})'' : S\}$.

(b.1) First we consider any relator in $\{(sus^{-1}(u\theta_\alpha)^{-1})'' : S \cap T\}$.

So choose any element α of A, and any $s_{(\alpha,i)}$ belonging to $S_\alpha \cap T$ (of course i belongs to I_α), and any element u of $(U_\alpha)_{ii}$.

Let j denote the initial vertex of $s_{(\alpha,i)}$.

Then either t_j is an initial segment of t_i in which case t_i ends in $s_{(\alpha,i)}$ and i is a representative vertex for G, or t_i is an initial segment of t_j in which case t_j ends in $s^{-1}_{(\alpha,i)}$ and j is a representative vertex for G.

To begin with, suppose t_j is an initial segement of t_i.

Then let k denote the representative vertex for that component of G which contains j, and let p be that reduced path in X from k to j.

Then $pu\theta_\alpha p^{-1} = g$ for some element g of G_{kk}.

In this case we have that the relator $s_{(\alpha,i)}us_{(\alpha,i)}^{-1}(u\theta_\alpha)^{-1}$ is written $s_{(\alpha,i)}us_{(\alpha,i)}^{-1}(p^{-1}gp)^{-1}$, and it follows that the relator $(s_{(\alpha,i)}us_{(\alpha,i)}^{-1}(u\theta_\alpha)^{-1})''$, when reduced, is written $(t_iut_i^{-1})(t_kgt_k^{-1})^{-1}$.

And so we see that the set of relators, $(s_{(\alpha,i)}us_{(\alpha,i)}^{-1}(u\theta_\alpha)^{-1})''$ as u ranges through $(U_\alpha)_{ii}$, when reduced, is expressed as $(t_iut_i^{-1})(t_ju\theta_\alpha t_j^{-1})^{-1}$ as u ranges through $(U_\alpha)_{ii}$.

Also we observe that $\sigma_{s_{(\alpha,i)}}$ takes $(t_iut_i^{-1})$ to $(t_ju\theta_\alpha t_j^{-1})$ for each u in $(U_\alpha)_{ii}$.

Now, suppose t_i is an initial segement of t_j.

This time let k denote the representative vertex for that component of G which contains i, and let p be that reduced path in X from k to i.

Then $pup^{-1} = g$ for some element g of G_{kk}.

In this case the relator $s_{(\alpha,i)}us_{(\alpha,i)}^{-1}(u\theta_\alpha)^{-1}$ is written $s_{(\alpha,i)}(p^{-1}gp)s_{(\alpha,i)}^{-1}(u\theta_\alpha)^{-1}$, and it follows that the relator $(s_{(\alpha,i)}us_{(\alpha,i)}^{-1}(u\theta_\alpha)^{-1})''$, when reduced, is written $(t_kgt_k^{-1})(t_ju\theta_\alpha t_j^{-1})$.

Here again, we see that the set of relators, $(s_{(\alpha,i)}us^{-1}_{(\alpha,i)}(u\theta_\alpha)^{-1})''$ as u ranges through $(U_\alpha)_{ii}$, when reduced, is expressed as $(t_iut_i^{-1})(t_ju\theta_\alpha t_j^{-1})^{-1}$ as u ranges through $(U_\alpha)_{ii}$.

And again we observe that $\sigma_{s_{(\alpha,i)}}$ takes $(t_iut_i^{-1})$ to $(t_ju\theta_\alpha t_j^{-1})$ for each u in $(U_\alpha)_{ii}$.

From (a), (b.1), and the construction of the tree ξ given in lemma 15, we obtain that the tree product of ξ has a presentation,

$$\langle \{t_igt_i^{-1}:I_G\}, \{r'':I_G\} \cup \{(sus^{-1}(u\theta_\alpha)^{-1})'':S\cap T\} \rangle$$

(b.2) Finally consider any relator in $\{(sus^{-1}(u\theta_\alpha)^{-1})'':S-T\}$.

So choose any element α of A, and any $s_{(\alpha,i)}$ belonging to S -T, and any element u of $(U_\alpha)_{ii}$.

Let j denote the initial vertex of $s_{(\alpha,i)}$.

Also let k,l denote the representative vertex for that component of G which contains j,i respectively, and let p,q be that reduced path in X from k to j and from l to i respectively.

Then $quq^{-1} = g$ for some element g of G_{ll}, and $pu\theta_\alpha p^{-1} = h$ for some element h of G_{kk}.

Then the relator $s_{(\alpha,i)}us^{-1}_{(\alpha,i)}(u\theta_\alpha)^{-1}$ is written $s_{(\alpha,i)}(q^{-1}gq)s^{-1}_{(\alpha,i)}(p^{-1}hp)^{-1}$, and it follows that the relator $(s_{(\alpha,i)}us^{-1}_{(\alpha,i)}(u\theta_\alpha)^{-1})''$, when reduced, is written $(t_js_{(\alpha,i)}t_i^{-1})(t_lgt_l^{-1})(t_js_{(\alpha,i)}t_i^{-1})^{-1}(t_kht_k^{-1})^{-1}$.

Thus we have that the set of relators, $(s_{(\alpha,i)}us^{-1}_{(\alpha,i)}(u\theta_\alpha)^{-1})''$ as u ranges through $(U_\alpha)_{ii}$, when reduced, is expressed as $(t_j s_{(\alpha,i)} t_i^{-1})(t_i u t_i^{-1})(t_j s_{(\alpha,i)} t_i^{-1})^{-1}(t_j u\theta_\alpha t_j^{-1})^{-1}$ as u ranges through $(U_\alpha)_{ii}$.

From these remarks we see that the vertex group of H at the origin is the HNN group described in the theorem.

Thus the theorem is proved.

We shall see in the next chapter how we can use theorem 6 to help us describe the subgroups of any HNN group.

Chapter 5

APPLICATIONS: THE SUBGROUPS OF TREE PRODUCTS OF GROUPS AND HNN GROUPS

In this chapter we describe the subgroups of any tree product of groups, and the subgroups of any HNN group.

In section 1 we give a basic result of Higgins (proposition 8).

In section 2 we define what we mean by a 'regular representative system for a tree product of groups modulo any one of its subgroups'. We shall see that this definition is a straightforward analog of a 'regular representative system'.

Then, in section 3, we characterise any subgroup H of any tree product of groups G as an HNN group with base-part some tree product of groups. This result follows easily from proposition 8 and theorem 3, using a 'regular representative system for G mod H'.

Finally we observe that we can obtain a similar characterisation of any subgroup of any HNN group, this time using proposition 8 and theorem 6.

5.1 A result of Higgins

Let G be any group and H be any subgroup of G. For any elements a and x of G, if Hax = Hb then x induces a mapping from the right coset Ha of H in G to the right coset Hb. These mappings form a groupoid, which we denote by $\Gamma(G,H)$, under composition of mappings, and the vertices of this groupoid are the right cosets of H in G. Clearly $\Gamma(G,H)$ is connected. Also if x and y induce the same map $Ha \longrightarrow Hb$ then ax = ay and so x = y. So we have a groupoid surjection from $\Gamma(G,H)$ onto G which takes each map of $\Gamma(G,H)$ into that element of G which induces it. And it is clear that the restriction of this groupoid surjection to the vertex group of $\Gamma(G,H)$ at the vertex H is a group isomorphism from this group onto the subgroup H of G.

For any subset K of G let us write $\bar{K}$ for the subgraph of $\Gamma(G,H)$ consisting of all the maps induced by all the elements of K. It is easy to see that if K is a subgroup of G, then $\bar{K}$ is a subgroupoid of $\Gamma(G,H)$.

Now we give a result due to Higgins ([4] page 135).

<u>Proposition 8</u>

Let G be any group and H be any subgroup of G, and let I denote the set of right cosets of H in G. If G has a presentation $\langle X,R \rangle$ then the I-groupoid $\Gamma(G,H)$ has an I-presentation $\langle \bar{X},\bar{R} \rangle$.

And we have two corollaries,

Corollary 1

Let $\mathcal{O} = \{\theta_\alpha : U_\alpha \to V_\alpha, \alpha \in A\text{-}o\}$ be any tree of groups G_α ($\alpha \in A$), and let G be the tree product of $\mathcal{O}$. Then $\Gamma(G,H)$ is the tree product of $\bar{\mathcal{O}} = \{\bar{\theta}_\alpha : \bar{U}_\alpha \to \bar{V}_\alpha, \alpha \in A\text{-}o\}$ where for each $\alpha \in A\text{-}o$ $\bar{\theta}_\alpha$ denotes the groupoid I-isomorphism induced by θ_α.

This corollary follows as a special case of another result of Higgins ([4] page 137).

Corollary 2

Let G be the HNN group with base-part K, free-part generated by $W = \{w_\alpha : \alpha \in A\}$ and for each $\alpha \in A$ let $\theta_\alpha : U_\alpha \to V_\alpha$ be the group isomorphism associated with the generator w_α. Then $\Gamma(G,H)$ is the HNN groupoid with base-groupoid $\bar{K}$, groupoid isomorphisms $\{\bar{\theta}_\alpha : \bar{U}_\alpha \to \bar{V}_\alpha, \alpha \in A\}$, and related graph $\bar{W}$.

To prove this corollary let $\langle Y,S \rangle$ be any presentation for the group K. Then we must show that $\langle \bar{Y},\bar{S} \rangle$ is an I-presentation for the I-groupoid $\bar{K}$. To see this let $\bar{r}$ be any relator in $\bar{K}$, and let r be the element of G which induces $\bar{r}$. Then r is a product of elements of K and is a relator in G. From proposition 6 we have that K is naturally embedded in G, and so r is a relator in K. That is r is a consequence of the relators in S, and so $\bar{r}$ is a consequence of the relators in $\bar{S}$. Thus $\langle \bar{Y},\bar{S} \rangle$ is an I-presentation for $\bar{K}$. Then the corollary follows easily.

5.2 Definition of a 'regular representative system for G mod H'

Throughout this section let $\Theta = \{\theta_\alpha : U_\alpha \rightarrow V_\alpha, \alpha \in A - o\}$ be any tree of groups G_α ($\alpha \in A$), and let G be the tree product of Θ, and let H be any subgroup of G. Also let λ be the level-function on Θ induced by the vertex 0. For each element α of $A - o$ we shall abbreviate 'predecessor of α with respect to λ' to simply 'predecessor of α'.

For each element α of A choose a set of generators for G_α $\{\ldots, x_\alpha, \ldots\}$ say. We call any element of $\{\ldots, x_\alpha, \ldots\} \cup \{\ldots, x_\alpha^{-1}, \ldots\}$ an <u>α-symbol</u>.

Also for each element α of A choose a right coset representative function Q_α for G mod H (see Magnus, Karrass and Solitar [7] page 88). We call each element of Q_α an <u>α-representative</u>.

Let us suppose that the set of right coset representative functions $\{Q_\alpha : \alpha \in A\}$ satisfies the following two conditions,

(1) for each representative q if q = px and x is an α-symbol for some $\alpha \in A$ then both q and p are α-representatives,

(2) for each $\alpha \in A$, when all the α-symbols are completely deleted from the ends of all the α-representatives, then the resulting set of α- representatives constitute a double coset representative function for G mod (H, G_α) (see Magnus, Karrass and Solitar [7] page 239).

We call $\{Q_\alpha : \alpha \in A\}$ a <u>regular representative system for G mod H</u> if the following two conditions are also satisfied,

(3) for each $\alpha \in A-o$, with predecessor β say, then each double coset representative for G mod (H, G_α) is a β-representative,

(4) for each double coset representative q for G mod (H, G_o), if q ends in an α-symbol, for some $\alpha \in A-o$, then q is a β-representative for each $\beta \in A$ of lesser λ-level than α.

The existence of a regular representative system for G mod H follows from the existence of a regular representative system for $\Gamma(G,H)$.

Now, for convenience, we introduce a little notation and terminology connected with any regular representative system $\{Q_\alpha : \alpha \in A\}$ for G mod H.

First, for any representative q and any $\alpha \in A$, we write q^α for the α-representative of the right coset Hq.

Second, consider any $\alpha \in A-o$. Choose any double coset representative function for G mod (H, U_α) in Q_α which contains the double coset representatives for G mod (H, G_α) and the o-representatives in Q_α. Then we call those double coset representatives for G mod (H, U_α) which are neither double coset representatives for G mod (H, G_α) nor o-representatives a <u>complement for U_α</u>.

We close this section by showing how we can use proposition 8 and theorem 3 to describe the structure of the group H.

To begin with, from the first corollary to proposition 8 we have that $\Gamma(G,H)$ is the tree product of $\mathcal{\bar{C}}$ where $\mathcal{\bar{C}}$ is the tree of groupoids $\bar{G}_\alpha$ ($\alpha \in A$) given by $\mathcal{\bar{C}} = \{\bar{\theta}_\alpha : \bar{U}_\alpha \longrightarrow \bar{V}_\alpha, \alpha \in A-o\}$ (see (5.1) for this notation).

Choose any regular representative system $\{Q_\alpha : \alpha \in A\}$ for G mod H. Also choose the vertex H to be the 'origin' of $\Gamma(G, H)$.

For each representative q let us write $\bar{q}$ for that map in $\Gamma(G,H)$ induced by q and with initial vertex H. Then for each $\alpha \in A$ put $\bar{Q}_\alpha = \{\bar{q} : q \in Q_\alpha\}$.

Then it is easy to see that $\{\bar{Q}_\alpha : \alpha \in A\}$ is a regular representative system (see (2.2)).

And so, from theorem 3, we obtain that the vertex group of $\Gamma(G,H)$ at the origin is an HNN group with base-part some tree product of groups.

That is we have characterised the group H.

5.3 The theorem

From the remarks just made we have the following result.

Theorem 7

Let $\mathcal{G} = \{\theta_\alpha : U_\alpha \to V_\alpha, \alpha \in A\text{-}o\}$ be any tree of groups G_α ($\alpha \in A$), and let G be the tree product of $\mathcal{G}$, and H be any subgroup of G. Choose any regular representative system $\{Q_\alpha : \alpha \in A\}$ for G mod H, and for each element α of $A\text{-}o$ choose any complement for U_α.

Consider any element α of $A\text{-}o$ and any α-representative q which is either a double coset representative for G mod (H, G_α) or a o-representative, and consider the isomorphism given by $quq^{-1} \longrightarrow qu\theta_\alpha q^{-1}$ as u ranges through $q^{-1}Hq \cap U_\alpha$. Let $\mathcal{\xi}$ be the set of all these isomorphisms.

Then $\mathcal{\xi}$ is a tree of groups $H \cap qG_\alpha q^{-1}$ where q ranges through the double coset representatives for G mod (H, G_α) and α ranges through A.

Also H is the HNN group with base-part the tree product of $\mathcal{\xi}$ and free-part generated by the elements $q(q^\beta)^{-1}$ where q ranges through the complement for U_α and β is the predecessor of α and α ranges through $A\text{-}o$.

Finally consider any element α of $A\text{-}o$, with predecessor β say, and any element q of the complement for U_α. Then the isomorphism associated with the generator $q(q^\beta)^{-1}$ is given by $q^\beta u\theta_\alpha (q^\beta)^{-1} \longrightarrow quq^{-1}$ as u ranges through $q^{-1}Hq \cap U_\alpha$.

Finally we can characterise the subgroups of any HNN group. To do this we use corollary 2 of proposition 8 and theorem 6, and we obtain results similar to those of Karrass & Solitar [6] and Cohen [2]. The method is straightforward, and we omit the details.

REFERENCES

1. I.M. Chiswell, On groups acting on trees, Ph.D. dissertation, Michigan 1973.

2. D. E. Cohen, Subgroups of HNN groups, Unpublished.

3. P.J. Higgins, Presentations of groupoids, with applications to groups, Proc.Camb.Phil.Soc. 60 (1964) 7-20

4. " , Notes on categories & groupoids, Van Nostrand, New York 1971.

5. A. Karrass & D. Solitar, The subgroups of a free product of two groups with an amalgamated subgroup, Trans.Amer.Math.Soc. 150 (1970) 227-255.

6. " , Subgroups of HNN groups & groups with one defining relator, Canadian J.Math. 23 (1971) 627-643.

7. W. Magnus, A. Karrass & D. Solitar, Combinatorial Group Theory, Interscience, New York 1966.

8. H. Neumann, Generalised free products with amalgamated subgroups II, Amer. J.Math. 71 (1949) 491-540.

9. E.T. Ordman, On subgroups of amalgamated free products, Proc.Camb.Phil.Soc. 69 (1971) 13-23.

APPENDIX

AN EXAMPLE OF A CONNECTED TREE PRODUCT OF GROUPOIDS

Here we give an example of a connected tree product of groupoids, G say, and we show that the presentation we obtain for the vertex group of G, G_{ii} say, using simply a representative system, does not enable us to describe G_{ii} precisely as an HNN group with base-part some tree product of groups.

To begin with put $I = \{1,2\}$ and $A=\{\alpha,\beta,\gamma\}$.

Let G_α, G_β and G_γ be I-groupoids with G_β and G_γ connected and G_α discrete.

Suppose the group $(G_\gamma)_{11}$, $(G_\beta)_{11}$ has a presentation $\langle\{c\},\emptyset\rangle$, $\langle\{b_1,b_2\},\{b_1^2b_2^2b_1^2b_2^2\}\rangle$ respectively (here $\emptyset$ denotes the empty set), and that $(G_\alpha)_{11}$ is the trivial group, and $(G_\alpha)_{22}$ has a presentation $\langle\{a\},\emptyset\rangle$.

Choose any edge y, z of $(G_\beta)_{12}$, $(G_\gamma)_{12}$ respectively, and let $\theta_\gamma,\theta_\beta$ be the groupoid I-isomorphisms generated by $z^{-1}c^2z \longrightarrow y^{-1}b_1^2y$, $y^{-1}b_2^2y \longrightarrow a^2$ respectively. Note then that the domain, range of θ_γ is a discrete subgroupoid of G_γ, G_β respectively, and that the domain, range of θ_β is a discrete subgroupoid of G_β, G_α respectively.

Then $\Theta = \{\theta_\gamma,\theta_\beta\}$ is a tree of I-groupoids G_α, G_β and G_γ.

Let G be the tree product of Θ. Obviously G is a connected I-groupoid.

Put $\alpha=0$, and call 1 the 'origin' of I, and let Q_0, Q_β, Q_γ be the graphs $\{z\}$, $\{y\}$ and $\{z\}$ respectively. Clearly $\{Q_0, Q_\beta, Q_\gamma\}$ is a representative system.

Using the general procedure outlined in the introduction to chapter 2, with the representative system $\{Q_o, Q_\beta, Q_\gamma\}$, we obtain a presentation for the vertex group of G at the origin, as follows.

First, from proposition 7, we have that G_o, G_β, and G_γ has an I-presentation $\langle\{a\}, \emptyset\rangle$, $\langle\{b_1, b_2, y\}, \{b_1^2 b_2^2 b_1^2 b_2^2\}\rangle$ and $\langle\{c, z\}, \emptyset\rangle$ respectively (here $\emptyset$ is an empty graph).

And so, from lemma 1 and the first part of lemma 2, we obtain an I-presentation for G.

$$\langle \{a, b_1, b_2, c\} \cup \{y, z\},\ \{b_1^2 b_2^2 b_1^2 b_2^2\} \cup \{c^2 z y^{-1} b_1^{-2} y z^{-1}, b_2^2 y a^{-2} y^{-1}\} \rangle$$

Now, let $F(\{a, b_1, b_2, c\} \cup \{y, z\})$ be the free groupoid on $\{a, b_1, b_2, c\} \cup \{y, z\}$. Then, using theorem 1 and its corollary, with the representative system $\{Q_o, Q_\beta, Q_\gamma\}$, we see that the vertex group of $F(\{a, b_1, b_2, c\} \cup \{y, z\})$ at the origin is freely generated by the elements $(zaz^{-1}), b_1, b_2, (yz^{-1}), c$.

Then, from the second part of lemma 2, rewriting the relators in this I-presentation for G in terms of this set of free generators, we obtain a presentation for the vertex group of G at the origin, G_{11},

$$\langle \{(zaz^{-1}), b_1, b_2, c\} \cup \{yz^{-1}\},\ \{b_1^2 b_2^2 b_1^2 b_2^2\} \cup \{c^2 (yz^{-1})^{-1} b_1^{-2} (yz^{-1}), b_2^2 (yz^{-1}) (zaz^{-1})^{-2} (yz^{-1})^{-1}\} \rangle$$

From this presentation we see that G_{11} is the HNN group with free-part generated by (yz^{-1}) and base-part presented by,

$$\langle \{(zaz^{-1}), b_1, b_2, c\} \; , \; \{b_1^2 b_2^2 b_1^2 b_2^2\} \cup \{c^2 (zaz^{-1})^2 \; c^2 (zaz^{-1})^2\} \rangle.$$

However we cannot describe this base-part as a tree product of the groups $(G_\beta)_{11}$, $(G_\gamma)_{11}$ and $z(G_o)_{22}z^{-1}$. All we can do is write K for the subgroup of G_{11} generated by $(G_\gamma)_{11} \cup (z(G_o)_{22}z^{-1})$, and then say that the base-part is a tree product of the groups $(G_\beta)_{11}$ and K (of course the base-part is in fact the free product of $(G_\beta)_{11}$ and K).

Now let us follow through this procedure again, this time using a regular representative system.

So put $Q_o' = \{y\}$. Then, clearly, $\{Q_o', Q_\beta, Q_\gamma\}$ is a regular representative system. And so, using theorem 1 and its corollary, with the regular representative system $\{Q_o', Q_\beta, Q_\gamma\}$, we see that the vertex group of $F(\{a, b_1, b_2, c\} \cup \{y, z\})$ at the origin is freely generated by the elements $(yay^{-1}), b_1, b_2, c, (zy^{-1})$. Then, rewriting the relators in the given I-presentation for G in terms of this new set of free generators, we obtain another presentation for G_{11},

$$\langle \{(yay^{-1}), b_1, b_2, c\} \cup \{zy^{-1}\} \; , \; \{b_1^2 b_2^2 b_1^2 b_2^2\} \cup \{c^2 (zy^{-1}) b_1^{-2} (zy^{-1})^{-1}, b_2^2 (yay^{-1})^{-2}\} \rangle.$$

And from this second presentation we see that G_{11} is the HNN group with free-part generated by (zy^{-1}) and base-part some tree product of the groups $(G_\beta)_{11}$, $(G_\gamma)_{11}$ and $(y(G_o)_{22}y^{-1})$.

This example, then, indicates the necessity of choosing a regular representative system to help us to describe the vertex group of any connected tree product of groupoids.