614

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STATIONARY AND REGENERATIVE MULTIVARIATE POINT PROCESSES

by

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To my wife, my mother, and in memory
of my late father

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ABSTRACT

A multivariate point process is a stochastic process of point events of several types. Various aspects of the theory of stationary multivariate point processes are developed. These include some general theory, a few models for stationary multivariate point processes, and the statistical analysis of some of these models.

First, some general relationships are derived for the counting and interval processes of stationary multivariate point processes under various initial conditions. Regenerative multivariate point processes are then defined. Some important properties of such processes are derived and these are used to analyse, in detail, three bivariate examples, that is examples where there are only two types of events.

Tests for discriminating between several different regenerative bivariate point process models are derived under two conditions: first, when both types of events are observable, and secondly, when the events of one type only are observable. The effect of assuming models which are either too general or too specific is then investigated.

Finally, a problem involving a special type of regenerative multivariate point process is considered. This is the Markov renewal process. Specifically, consideration is given to some single server queues whose departure processes are Markov renewal processes. Necessary and sufficient conditions are found for the departure processes of these queues to be renewal processes.

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TABLE OF CONTENTS

Page

LIST OF TABLES

CHAPTER 1: INTRODUCTION

A multivariate point process is a stochastic process of point events of several types occurring in a one-dimensional continuum, usually time. The classification of the events may be by a qualitative variable attached to each event, or by their arising in a common time scale but in different physical locations. Multivariate point processes should not be confused with what are often termed multi-dimensional point processes, that is events of one type occurring in an n dimensional continuum. It is, of course, possible to combine the two notions and consider multivariate multidimensional point processes, but this generalization will not be considered.

Multivariate point processes arise in a variety of fields. In queueing theory, the joint properties of the input and output processes are sometimes of interest (Daley, 1968). On other occasions, it is useful to consider complicated input or output processes of queueing systems as multivariate point processes (Neuts, 1971). In neuro-physiology, there is interest in the relationships between several neuronal spike trains, and these are sometimes modelled as multivariate point processes (Ten Hoopen and Reuver, 1965; Lawrance, 1970, 1971). In reliability, there is sometimes interest in relating the times of failure of various machine components; multivariate point process models have been proposed for analysing such events (Lewis, 1964, 1970). The occurrence of earthquakes also has been modelled as a multivariate point process (Vere-Jones, 1970).

All the multivariate point processes mentioned above are stationary, that is, all joint and marginal distributions of the multivariate process are translation invariant. A more formal definition of stationarity is given in Chapter 2. This thesis is solely concerned with stationary multivariate point processes.

Various aspects of stationary multivariate point processes are considered. These include (a) general theoretical results, (b) the building of fairly general models which incorporate a variety of more specific models, and which are amenable to probabilistic and statistical analysis, and (c) the statistical analysis of some stationary multivariate point processes.

Prior to the 1970's, there appears to have been little investigation into the general theoretical properties of stationary multivariate point processes; most of the earlier work is concerned with the probabilistic, and sometimes statistical, analysis of very specific models. A review of work done prior to that time and of most of the general results then known is given by Cox and Lewis (1972). In recent years, useful contributions to the general theory of stationary multivariate point processes have been made by Milne (1971), Wisniewski (1972) and Daley and Milne (1975). In Chapter 2, the results of Daley and Milne are extended. Some relationships are derived for the counting and interval processes of stationary multivariate point processes under various initial conditions. A few examples are given to illustrate some of these relationships.

Not much consideration appears to have been given to deriving the properties of fairly general multivariate point process models which themselves incorporate a large variety of more specific models. The only general multivariate point process which appears to have been studied in great detail is the Markov renewal process; see Çinlar (1969) for details. Many of the stationary multivariate point process models which have been proposed, including Markov renewal processes, have used the idea of regeneration points. Any event which occurs prior to a regeneration point is statistically independent of any event which occurs after that point. The assumption of regeneration points usually leads to simplified probabilistic and statistical analysis of the process under study. In Chapter 3, many such stationary multivariate point process models are generalized by the consideration of what are called regenerative multivariate point processes. The central idea is that events of one type, say type 0 events, occur in a renewal process, i.e. that the intervals between successive type 0 events are independent and identically distributed random variables. Further, the numbers and positions of the events of other types occurring in any such "renewal interval" depend only on that interval and are independent of occurrences in other intervals. That is, type 0 events form regeneration points for the entire multivariate point process. Some important properties of such processes are derived in Chapter 3 including the joint probability generating function for numbers of events in an interval of fixed length and the asymptotic behaviour of the counts of such processes. The results of Chapter 2 are used to obtain some of the results under various initial conditions.

In Chapter 4, three simple examples of regenerative bivariate point processes are considered, two of them apparently new, the third an example from neurophysiology, previously considered by Ten Hoopen and Reuver (1965) and Lawrance (1970, 1971). All three examples are considered in some detail and the results of Chapter 3 are applied. The first example, which is discussed in Section 4.2, is obtained by imbedding between successive events of a renewal process (type 0 events), a simple non-stationary point process - the inhomogeneous Poisson process (type 1 events). The second example, discussed in Section 4.3, is obtained by imbedding, between successive events of a renewal process (type 0 events), a simple stationary point process - another renewal process (type 1 events). The neurophysiological example, discussed in Section 4.4, is presented to illustrate how easily the general theory of regenerative multivariate point processes leads to previously known and to new results.

Chapter 5 is concerned with discriminating between the regenerative bivariate point processes of Sections 4.2 and 4.3. The problem is considered under two different conditions: first, when both the type 0 and type 1 events are observed (and are identified as such), and secondly, when only the type 1 events are observed. When the events of both types are observed, we can condition on the occurrence times of the type 0 events. By the definition of a regenerative bivariate point process, the type 1 events can then be treated as a series of independent realizations of the same underlying process. The problem of distinguishing between the two bivariate point processes is then a simple generalization of the problem of distinguishing between an inhomogeneous Poisson process and a renewal process. In Section 5.2, this problem is considered briefly under various assumptions about the imbedded process. In Section 5.3, the more difficult problem of discriminating between the two processes when the type 0 events are unobserved is considered. It is shown that, in this case, some of the processes considered in Section 4.2 are almost indistinguishable from other processes, considered in Section 4.3, when the sample size is moderately large (say, about 100 type 1 events). The remainder of the section is concerned with finding a suitable restriction on the processes of Sections 4.2 and 4.3. so that they are distinguishable with moderately large sample sizes, and then deriving a test for discriminating between the two processes. The test is applied to several artificial sets of data.

In Section 5.2.4, very specific models, which involve a nuisance parameter, are assumed for both the inhomogeneous Poisson process and the renewal process. The theory of similar tests leads easily to a particular statistic which is used to condition out the effect of the nuisance parameter. However, a simple generalization of the models leads to a different conditioning statistic. The question then arises: what is the effect of assuming models which are either too general or too specific, and hence conditioning on the incorrect statistic? This problem, which has broad implications, is considered in detail in Chapter 6, where the problem is generalized to bivariate exponential families. A few particular examples of the general theory are given, including the example first mentioned in Section 5.2.4.

Chapter 7 is concerned with a very special regenerative multivariate point process sometimes found in queueing theory and other branches of stochastic processes. This is the Markov renewal process. A number of single server queueing systems which have been discussed in the queueing literature have arrival processes which are Poisson processes and departure processes which are Markov renewal processes. This chapter is concerned with determining when the departure processes of such queues are also renewal processes. The general question of when the departure processes of queues are renewal processes is important in the theory of tandem queues, that is where there are several queueing systems and where the departure process of one queue provides the arrival process of another queue. If all the departure processes (and hence all the arrival processes) are renewal processes, then the whole system is usually easier to model. In Chapter 7, some results are obtained for various types of Markov renewal processes and queueing systems, and these are combined to obtain necessary and sufficient conditions for the departure processes of some single server queues to be renewal processes.

After Chapter 7, there are four appendices. Appendices 1 and 2 prove some results used in Chapter 4. Appendix 3 proves a result used in Chapter 5. Appendix 4 gives details of four artificial sets of data to which some tests are applied in Chapter 5. The times of all the events in the four data sets are given, as well as some of the theoretical properties of the underlying processes.

CHAPTER 2: SOME MULTIVARIATE GENERALIZATIONS OF RESULTS IN STATIONARY UNIVARIATE POINT PROCESSES

2.1 Introduction

The distributions associated with the counting and interval processes of stationary multivariate point processes are usually dependent on initial conditions, that is the method of choosing the time origin. One method is to choose the time origin independently of the multivariate process. This is often called asynchronous sampling. Another method is to choose the time origin to coincide with a random event of specified type. This is called here partially synchronous sampling. The terms synchronous and semisynchronous are commonly used instead of partially synchronous when discussing univariate and bivariate point processes respectively. For a detailed discussion of synchronous, semisynchronous and asynchronous sampling, see Cox and Lewis (1972), Sections 2.3 and 2.4.

Daley and Milne (1975) have derived an equation for stationary multivariate point processes, which relates the distribution of numbers of events in intervals in the asynchronous case to those in the partially synchronous case. In this chapter, their equation is used to derive a number of generalizations of some well-known univariate results.

2.2 Assumptions and definitions

The definitions and assumptions which follow will be used throughout the thesis. It is assumed that there are m + 1 point processes on the real line, where m is some non-negative integer; the reason for using the number m + 1 rather than the number m will become clear in subsequent chapters. Let $N_r(A)$ $(r = 0,1,\ldots,m)$ be the number of type r events in the set A, where A is any set in the family $\mathcal{B}$ of Borel sets on the real line. Then the vector

$$\underset{\sim}{N}(\cdot) \equiv \{N_0(\cdot),\ldots,N_m(\cdot)\} \qquad (2.2.1)$$

defines a multivariate point process, for which $N_r(\cdot)$ is the rth <u>marginal</u> point process.

We shall be concerned only with multivariate point processes which are <u>(completely) stationary</u>, which means that the joint distributions of the random variables

$$N_{i_0}(A_0 + t),\ldots,N_{i_s}(A_s + t) \qquad (2.2.2)$$

for $i_r \in \{0,\ldots,m\}$, $A_r \in \mathcal{B}$ $(r = 0,\ldots,s)$, do not depend on t, where $A + t = \{(x + t) : x \in A\}$. Often, the interest will be in the distribution of the numbers of events in an interval $(t, x + t]$. By stationarity, this depends only on x. Hence, for the numbers of events in such an interval we will usually write $\underset{\sim}{N}(x) \equiv \{N_0(x),\ldots,N_m(x)\}$.

It is assumed that the process is <u>non-degenerate</u> in the sense that

$$P\{N_i(-\infty,0] = N_i(0,\infty) = \infty,\ i = 0,\ldots,m\} = 1. \qquad (2.2.3)$$

The ith point process is said to be <u>orderly</u> if

$$\lim_{h\downarrow 0} \frac{P\{N_i(h) > 1\}}{h} = 0. \qquad (2.2.4)$$

It will be assumed throughout the thesis that each of the m + 1 point processes, taken separately, is orderly. Orderliness implies that two or more events cannot occur simultaneously. The univariate point process

$$N(\cdot) \quad = \quad N_0(\cdot) + \ldots + N_m(\cdot) \tag{2.2.5}$$

is called the <u>superposition</u> point process of all the marginals. If the superposition of the m + 1 point processes is orderly, then the multivariate point process is said to be <u>strongly orderly</u>. For the sake of economy and notational simplicity, it will be assumed in Sections 2.2 to 2.5 that the process is strongly orderly. In Section 2.6, some of the results are generalized to the case where the process is not strongly orderly.

Let the finite intensity of the ith marginal process be

$$\lambda_i \quad = \quad \lim_{h \downarrow 0} \frac{P\{N_i(h) > 0\}}{h} \quad (i = 0,1,\ldots,m). \tag{2.2.6}$$

Let

$$Q^i_{\underset{\sim}{j}}(\underset{\sim}{x}) \quad = \quad \lim_{h \downarrow 0} P\{N_r(x_r) \leq j_r,\ r = 0,.,\ldots,m \mid N_i(-h,0]>0\}, \tag{2.2.7}$$

where $\underset{\sim}{j} = (j_0,\ldots,j_m)$, and

$$Q_{\underset{\sim}{j}}(\underset{\sim}{x}) \quad = \quad \frac{\sum_{i=0}^{m} \lambda_i Q^i_{\underset{\sim}{j}}(\underset{\sim}{x})}{\sum_{i=0}^{m} \lambda_i}\,. \tag{2.2.8}$$

The stationarity assumption implies the existence of both the limits (2.2.6) and (2.2.7). We can think of $Q_{\underset{\sim}{j}}^{i}(\underset{\sim}{x})$ as $P\{N_r(x_r) \leq j_r, r = 0,\dots,m \mid \text{type } i \text{ event at the origin}\}$, and $Q_{\underset{\sim}{j}}(\underset{\sim}{x})$ as $P\{N_r(x_r) \leq j_r, r = 0,\dots,m \mid \text{an event (of any type) at the origin}\}$. Write $Q_{\underset{\sim}{j}}^{e}(\underset{\sim}{x})$ for the asynchronous (equilibrium) case analogous to the partially synchronous definition of (2.2.7), i.e.

$$Q_{\underset{\sim}{j}}^{e}(\underset{\sim}{x}) = P\{N_r(x_r) \leq j_r,\ r = 0,\dots,m\}. \tag{2.2.9}$$

By considering all possible events in the small interval $(-h,0]$, dividing the subsequent equations by h, and letting $h \to 0$, Daley and Milne have shown for strongly orderly processes that

$$Q_{\underset{\sim}{j}}^{e}(\underset{\sim}{x}) = \int_0^\infty \sum_{i=0}^{m} \lambda_i \{Q_{\underset{\sim}{j}}(\underset{\sim}{x} + u\underset{\sim}{1}) - Q^{i}_{\underset{\sim}{j}-\underset{\sim}{\varepsilon}_i}(\underset{\sim}{x} + u\underset{\sim}{1})\}du, \tag{2.2.10a}$$

where $\underset{\sim}{\varepsilon}_i$ is the $(m + 1)$-vector with units in the ith position and zeros elsewhere; $\underset{\sim}{x} + u\underset{\sim}{1} = (x_0 + u,\dots,x_m + u)$; and $Q^{i}_{\underset{\sim}{j}-\underset{\sim}{\varepsilon}_i}(\underset{\sim}{x} + u\underset{\sim}{1})$ is set equal to zero in the case where any coordinate of $\underset{\sim}{j} - \underset{\sim}{\varepsilon}_i$ is negative.

Incidentally, in (2.10) and (2.11) of Daley and Milne, the sign on the right-hand side of both equations should be reversed. Using (2.2.8), one can easily rewrite (2.2.10a) as

$$Q_{\underset{\sim}{j}}^{e}(\underset{\sim}{x}) = \int_0^\infty \sum_{i=0}^{m} \lambda_i \{Q^{i}_{\underset{\sim}{j}}(\underset{\sim}{x} + u\underset{\sim}{1}) - Q^{i}_{\underset{\sim}{j}-\underset{\sim}{\varepsilon}_i}(\underset{\sim}{x} + u\underset{\sim}{1})\}du. \tag{2.2.10b}$$

Let

$$R_{\underset{\sim}{j}}^{i}(\underset{\sim}{x}) = \lim_{h \downarrow 0} P\{N_r(x_r) = j_r,\ r = 0,\dots,m \mid N_i(-h,0]>0\}, \tag{2.2.11}$$

and

$$G_{\underset{\sim}{j}}(\underset{\sim}{x}) = \lim_{h \downarrow 0} P\{N_r(x_r) \geq j_r,\ r = 0,\ldots,m \mid N_i(-h,0] > 0\}. \qquad (2.2.12)$$

Then, using analogous derivations to that for $Q^i_{\underset{\sim}{j}}(\underset{\sim}{x})$, it can be shown that

$$R^e_{\underset{\sim}{j}}(\underset{\sim}{x}) = \int_0^\infty \sum_{i=0}^m \lambda_i \{R^i_{\underset{\sim}{j}}(\underset{\sim}{x} + u\underset{\sim}{1}) - R^i_{\underset{\sim}{j}-\underset{\sim}{\varepsilon}_i}(\underset{\sim}{x} + u\underset{\sim}{1})\}du, \qquad (2.2.13)$$

$$G^e_{\underset{\sim}{j}}(\underset{\sim}{x}) = \int_0^\infty \sum_{i=0}^m \lambda_i \{G^i_{\underset{\sim}{j}}(\underset{\sim}{x} + u\underset{\sim}{1}) - G^i_{\underset{\sim}{j}-\underset{\sim}{\varepsilon}_i}(\underset{\sim}{x} + u\underset{\sim}{1})\}du. \qquad (2.2.14)$$

Equations (2.2.13) and (2.2.14) will be used later to obtain other general results for stationary multivariate point processes.

2.3 Some results for the counting process

Let

$$\phi_i(\underset{\sim}{\zeta};\underset{\sim}{x}) = \sum_{j_0} \cdots \sum_{j_m} R^i_{\underset{\sim}{j}}(\underset{\sim}{x})\zeta_0^{j_0} \cdots \zeta_m^{j_m}$$

be the joint probability generating function (p.g.f.) of $\underset{\sim}{N}(\underset{\sim}{x})$. From (2.2.11) the following relation for p.g.f.'s is obtained:

$$\phi_e(\underset{\sim}{\zeta};\underset{\sim}{x}) = \sum_{i=0}^m \lambda_i(1 - \zeta_i) \int_0^\infty \phi_i(\underset{\sim}{\zeta};\underset{\sim}{x} + u\underset{\sim}{1})du. \qquad (2.3.2)$$

Equation (2.3.2) is a generalization of a well-known result in stationary univariate point processes (Cox and Lewis, 1966, Section 4.3, equation (8)). It is not too difficult to show that, under fairly mild assumptions, $\int_0^\infty R^i_{\underset{\sim}{j}}(\underset{\sim}{x} + u\underset{\sim}{1})du$ is uniformly bounded (independently of j).

It then follows that, provided $0 \leq \zeta_r < 1$ $(r = 0,\ldots,m)$, $\int_0^\infty \phi_i(\underset{\sim}{\zeta};\underset{\sim}{x} + u\underset{\sim}{1})du$ is finite, and hence that the right-hand side of (2.3.2) exists. However, from (2.2.11) and (2.3.1), it is seen that $\phi_i(\underset{\sim}{1};\underset{\sim}{x}) = \underset{\sim}{1}$, the $(m + 1)$-vector of ones, and hence $\int_0^\infty \phi_i(\underset{\sim}{1};\underset{\sim}{x} + u\underset{\sim}{1})du$ is infinite. Consequently, we cannot put $\eta_r = \zeta_r - 1$ $(r = 0,1,\ldots,m)$ in (2.3.2) in order to obtain a relation between factorial moment generating functions (m.g.f.'s) and therefore (expanding about $\underset{\sim}{\eta} = \underset{\sim}{0}$, the $(m + 1)$-vector of zeros) relations between factorial moments.

We can, however, proceed further easily in one special case. <u>If all the x_r's are equal (with common value x)</u>, then (2.3.2) becomes

$$\phi_e(\underset{\sim}{\zeta};x) = \sum_{i=0}^{m} \lambda_i(1 - \zeta_i) \int_x^\infty \phi_i(\underset{\sim}{\zeta};y)dy, \tag{2.3.3}$$

where the "x" in $\phi_e(\underset{\sim}{\zeta};x)$ is shorthand for the vector $x\underset{\sim}{1}$. Now, from (2.2.11), note that

$$\begin{aligned} R^i_{\underset{\sim}{j}}(\underset{\sim}{0}) &= 1, \quad \text{if } \underset{\sim}{j} = \underset{\sim}{0}\,, \\ &= 0, \quad \text{otherwise,} \end{aligned} \tag{2.3.4}$$

whence, from (2.3.1) and (2.3.4), it follows that

$$\begin{aligned} 1 &= \phi_e(\underset{\sim}{\zeta};0) \\ &= \sum_{i=0}^{m} \lambda_i(1 - \zeta_i) \int_0^\infty \phi_i(\underset{\sim}{\zeta};y)dy \end{aligned} \tag{2.3.5}$$

by (2.3.3). Then (2.3.3) and (2.3.5) give

$$\phi_e(\underset{\sim}{\zeta};x) = 1 + \sum_{i=0}^{m} \lambda_i(\zeta_i - 1) \int_0^x \phi_i(\underset{\sim}{\zeta};y)dy. \tag{2.3.6}$$

Equation (2.3.6) is a very simple generalization of the univariate result. Because of the finite upper limit in the integral, the right-hand side of (2.3.6) will always exist for $0 \le \zeta_r \le 1$ $(r = 0,1,\ldots,m)$. Then putting $\eta_r = \zeta_r - 1$ $(r = 0,1,\ldots,m)$ in (2.3.6), we obtain a relation between the factorial m.g.f.'s which leads to the following relations between multivariate factorial moments. Let

$$\mu_{\underset{\sim}{j}}^{i}(x) \quad = \quad E\{\prod_{r=0}^{m} N_r(x)^{[j_r]} \,|\text{type } i \text{ event at the origin}\}, \tag{2.3.7}$$

where $a^{[n]} = a(a-1)\ldots.(a-n+1)$; $a^{[0]} \equiv 1$.

Then we obtain from (2.3.6)

$$\mu_{\underset{\sim}{j}}^{e}(x) \quad = \quad \sum_{i=0}^{m} \lambda_i j_i \int_0^x \mu_{\underset{\sim}{j}-\underset{\sim}{\varepsilon}_i}^{i}(y)dy, \tag{2.3.8}$$

where $\mu_{\underset{\sim}{0}}^{i}(x) \equiv 1$ $(i = 0,1,\ldots,m;e)$, and $\mu_{\underset{\sim}{j}-\underset{\sim}{\varepsilon}_i}^{i}(x) \equiv 0$, if any of the co-ordinates of $\underset{\sim}{j} - \underset{\sim}{\varepsilon}_i$ is negative.

It will sometimes be more helpful to use the Laplace transform of (2.3.6), which can be expressed neatly in matrix notation. Denote the Laplace transform of any function $L(\cdot)$ defined on the non-negative real line by

$$L^*(s) \quad = \quad \int_0^\infty e^{-sx}L(x)dx. \tag{2.3.9}$$

Let $\underset{\sim}{\lambda}^T = (\lambda_0,\ldots,\lambda_m)$, $\underset{\sim}{\xi} = \text{diag}(\zeta_0,\ldots,\zeta_m)$, $\underset{\sim}{I}$ the $(m+1) \times (m+1)$ identity matrix, and $\underset{\sim}{\phi}^*(\underset{\sim}{\zeta};s)^T = (\phi_0^*(\underset{\sim}{\zeta};s),\ldots,\phi_m^*(\underset{\sim}{\zeta};s))$. Then (2.3.6) becomes

$$s\phi_e^*(\underset{\sim}{\zeta};s) \; = \; 1 + \underset{\sim}{\lambda}^T(\underset{\sim}{\xi} - \underset{\sim}{I})\underset{\sim}{\phi}^*(\underset{\sim}{\zeta};s). \tag{2.3.10}$$

Equations (2.3.6), (2.3.8) and (2.3.10) are the most useful results derived in this chapter and will be used in the examples of Section 2.5.

2.4 The interval process

Let $S^{(r)}_{j_r}$ be the sum of the lengths of the first j_r intervals of the rth point process ($r = 0,1,\ldots,m$). Noting that $N_r(x_r) \geq j_r$ is equivalent to $S^{(r)}_{j_r} \leq x_r$ and using the definition (2.2.12), it follows that

$$G^i_{\underset{\sim}{j}}(\underset{\sim}{x}) = \lim_{h\downarrow 0} P\{S^{(r)}_{j_r} \leq x_r\ r = 1,\ldots,m \mid N_i(-h,0] > 0\}. \tag{2.4.1}$$

Hence (2.2.14) immediately gives the fundamental relationship between interval distributions. If it is assumed that each distribution function $G^i_{\underset{\sim}{j}}(\underset{\sim}{x})$ has a corresponding probability density function (p.d.f.) $g^i_{\underset{\sim}{j}}(\underset{\sim}{x})$, the relations between joint p.d.f.'s follow easily:

$$g^e_{\underset{\sim}{j}}(\underset{\sim}{x}) = \sum_{i=0}^{m} \lambda_i \int_0^\infty \{g^i_{\underset{\sim}{j}}(\underset{\sim}{x} + u\underset{\sim}{1}) - g^i_{\underset{\sim}{j}-\underset{\sim}{\varepsilon}_i}(\underset{\sim}{x} + u\underset{\sim}{1})\}du, \tag{2.4.2}$$

which is a generalization of equation (10), Section 4.2, Cox and Lewis (1966).

Again if all the x_r's are equal (with common value x), then, using arguments similar to those used in the previous section, it can be shown that

$$G^e_{\underset{\sim}{j}}(x) = \sum_{i=0}^{m} \lambda_i \int_0^x \{G^i_{\underset{\sim}{j}-\underset{\sim}{\varepsilon}_i}(y) - G^i_{\underset{\sim}{j}}(y)\}dy \quad (\underset{\sim}{j} \neq \underset{\sim}{0}), \tag{2.4.3}$$

Also note from (2.2.12) that

$$G^i_{\underset{\sim}{0}}(x) \equiv 1 \quad (i = 0,\ldots,m;e).$$

Hence, from (2.4.3) or the (corrected version of) relation (2.10) of Daley and Milne (1975), it follows that

$$D_x^+ G_{\underset{\sim}{j}}^e(x) = \sum_{i=0}^{m} \lambda_i \{G^i_{\underset{\sim}{j}-\underset{\sim}{\varepsilon}_i}(x) - G^i_{\underset{\sim}{j}}(x)\} \quad (\underset{\sim}{j} \neq 0), \tag{2.4.4}$$

where D_x^+ denotes a right-hand derivative with respect to x. This is another generalization of equation (10) Section 4.2, Cox and Lewis (1966). Notice however that $D_x^+ G_{\underset{\sim}{j}}^e(x)$ is not the same as $g_{\underset{\sim}{j}}^e(x)$ except in the univariate case.

Equation (2.4.2) does not appear to give rise to simple moment or m.g.f. relations.

2.5 Three examples

2.5.1 Markov renewal processes

Çinlar (1969) gives the equation for the partially synchronous joint p.g.f. of the counting process of a Markov renewal process (i.e. the number of visits to the different states in the interval (0,t] given that a transition to state j occurs at the origin). Assume that the process is strongly orderly with a state space of finite dimension m + 1.

Let $T_0 \equiv 0$, $T_1, T_2, \ldots$ be the epochs of jumps and define X_n to be the state into which the particle enters at epoch T_n. For j, k = 0,1,...,m, let

$$F_{jk}(t) = P(X_1 = k,\ T_1 \leq t | X_0 = j), \quad f_{jk}(t) = d|dt\{F_{jk}(t)\}.$$

For j = 0,1,...,m, let

$$B_j(t) = \sum_{k=0}^{m} F_{jk}(t), \quad C_j(t) = B_j(\infty) - B_j(t).$$

Let $\underset{\sim}{F}(t)$, $\underset{\sim}{f}(t)$, $\underset{\sim}{B}(t)$ and $\underset{\sim}{C}(t)$ be the respective corresponding matrices and vectors. The assumption of strong orderliness implies that

$F_{jk}(0+) = 0$ $(j,k = 0,\ldots,m)$. Let $\underset{\sim}{f}^*(s)$ and $\underset{\sim}{C}^*(s)$ be the matrices whose elements are the Laplace transforms of the corresponding elements of $\underset{\sim}{f}(t)$ and $\underset{\sim}{C}(t)$ respectively. Let $\underset{\sim}{\phi}(\underset{\sim}{\zeta};t)^T = \{\phi_0(\underset{\sim}{\zeta};t),\ldots,\phi_m(\underset{\sim}{\zeta};t)\}$. Then, in our notation, Çinlar's equation (8.11), p.164, is

$$\underset{\sim}{\phi}(\underset{\sim}{\zeta};t) = \underset{\sim}{C}(t) + \int_{0}^{t} \underset{\sim}{f}(u)\underset{\sim}{\xi}\underset{\sim}{\phi}(\underset{\sim}{\zeta};t-u)du, \tag{2.5.1a}$$

whence, taking Laplace transforms and rearranging, one obtains

$$\underset{\sim}{\phi}^*(\underset{\sim}{\zeta};s) = \{\underset{\sim}{I} - \underset{\sim}{f}^*(s)\underset{\sim}{\xi}\}^{-1}\underset{\sim}{C}^*(s). \tag{2.5.1b}$$

In particular, if $B_j(\infty) = 1$ $(j = 0,1,\ldots,m)$, it is easily found that

$$\underset{\sim}{C}^*(s) = \frac{1}{s}\{\underset{\sim}{I} - \underset{\sim}{f}^*(s)\}\underset{\sim}{1}.$$

Hence,

$$\underset{\sim}{\phi}^*(\underset{\sim}{\zeta};s) = \frac{1}{s}\{\underset{\sim}{I} - \underset{\sim}{f}^*(s)\underset{\sim}{\xi}\}^{-1}\{\underset{\sim}{I} - \underset{\sim}{f}^*(s)\}\underset{\sim}{1}, \tag{2.5.1c}$$

which is the multivariate generalization of a well-known result in renewal theory (Cox, 1962, Section 3.2, equation (4)). Then from (2.3.10) it is found that

$$\begin{aligned}\phi_e^*(\underset{\sim}{\zeta};s) &= \frac{1}{s}[1 + \underset{\sim}{\lambda}^T(\underset{\sim}{\xi} - \underset{\sim}{I})\{\underset{\sim}{I} - \underset{\sim}{f}^*(s)\underset{\sim}{\xi}\}^{-1}\underset{\sim}{C}^*(s)] \\ &= \frac{1}{s} + \frac{1}{s^2}\underset{\sim}{\lambda}^T(\underset{\sim}{\xi} - \underset{\sim}{I})\{\underset{\sim}{I} - \underset{\sim}{f}^*(s)\underset{\sim}{\xi}\}^{-1}\{\underset{\sim}{I} - \underset{\sim}{f}^*(s)\}\underset{\sim}{1},\end{aligned} \tag{2.5.2}$$

if $B_j(\infty) = 1$ $(j = 0,1,\ldots,m)$, where $\underset{\sim}{\lambda}$ is the vector of inverses of the mean recurrence times for the m + 1 states. This is a generalization of the univariate result (Cox, 1962, Section 3.2, equation (6)).

2.5.2 Exchangeable processes

By an exchangeable multivariate point process, we mean that any property of the process is invariant under arbitrary permutations of the indices defining the event types. For such processes, the ϕ_i's can be found from ϕ_e which is sometimes more easily determined. In this case (assuming strong orderliness), (2.3.6) becomes

$$\phi_e(\underset{\sim}{\zeta};x) = 1 + \lambda_0\{(\sum_{r=0}^{m} \zeta_r) - (m + 1)\} \int_0^x \phi_0(\underset{\sim}{\zeta};y)dy \tag{2.5.3}$$

or

$$\phi_0(\underset{\sim}{\zeta};x) = \frac{d|dx\{\phi_e(\underset{\sim}{\zeta};x)\}}{\lambda_0\{(\sum_{r=0}^{m} \zeta_r) - (m + 1)\}} .$$

As an example of (2.5.3), consider a special case of a bivariate point process discussed by Cox and Lewis (1972, Section 4). Suppose there is an unobservable main or generating Poisson process of rate μ. Events from the main process are delayed (independently) by random amounts Y_0 with common distribution $F_0(t)$ and superposed on a "noise" process which is Poisson with rate ν_0. The resulting process is the observed marginal process of type 0 events. Similarly, the events in the main process are delayed (independently) by random amounts with common distribution $F_1(t)$ and superposed with another independent noise process which is Poisson with rate ν_1. The resulting process is the observed marginal process of type 1 events. Cox and Lewis give the logarithm of the joint p.g.f. of the asynchronous bivariate counting process (see their equation (4.25)). If we take $F_0(t) = F_1(t) = F(t)$ (assuming $F(0+) = 0$ to ensure strong orderliness), and $\nu_0 = \nu_1 = \nu$, then $\lambda_0 = \mu + \nu$ and their equation becomes

$$\log \phi_e(\underset{\sim}{\zeta};x) = \lambda_0 x(\zeta_0 + \zeta_1 - 2)$$
$$+ \mu(\zeta_0 - 1)(\zeta_1 - 1)[\int_0^x \{F(y)\}^2 dy + \int_0^\infty \{F(y+x) - F(y)\}^2 dy]. \tag{2.5.4}$$

This equation can be used with (2.5.3) to obtain ϕ_0 (and ϕ_1).

2.5.3 The asymptotic covariance matrix of an asynchronous bivariate point process

Under fairly general conditions, (2.3.8) can be used to obtain the asymptotic form of the asynchronous covariance matrix of a strongly orderly bivariate point process. Let $N_r^{(q)}(x)$ be the number of type r events in (t, x+t] given an event of type q at time t (q,r = 0,1). Assume that (i) the bivariate point process is strongly orderly, (ii) a strong law of large numbers applies to both marginal processes, and (iii) the Laplace transform of $E\{N_r^{(q)}(x)\}$ (denoted by $E^*_{r,q}(s)$) can be represented as

$$E^*_{r,q}(s) = \lambda_r s^{-2} + \omega_{rq} s^{-1} + \nu_{rq} + o(1) \qquad \text{as } s \to 0+. \tag{2.5.5}$$

Assumption (ii) is necessary for $E\{N_r^{(q)}(x)\} = \lambda_r x + o(x)$ as $x \to \infty$, independently of q (Cox and Lewis, 1966, Section 4.3), whence the leading term in (2.5.5). Putting $\underset{\sim}{j} = (1,0)$ or $(0,1)$ in (2.3.8), for example, it is easily found that

$$E\{N_r^{(e)}(x)\} = \lambda_r x, \tag{2.5.6}$$

as one would expect. Then, we have, for instance

$$\text{var}\{\underset{\sim}{N}_r^{(e)}(x)\} = 2\lambda_r \int_0^x E\{N_r^{(r)}(y)\}dy + \lambda_r x - \lambda_r^2 x^2 \tag{2.5.7}$$

by (2.3.8) and (2.5.6). Taking Laplace transforms of (2.5.7):

$$
\begin{aligned}
V_r^*(s) &= 2\lambda_r E^*_{r,r}(s)s^{-1} + \lambda_r s^{-1} - 2\lambda_r^2 s^{-3} \\
&= \lambda_r(2\omega_{rr} + 1)s^{-2} + 2\lambda_r \nu_{rr} s^{-1} + o(s^{-1}) \text{ as } s \to 0+. \qquad (2.5.8)
\end{aligned}
$$

by (2.5.5). Under fairly general conditions (Widder, 1946, Section 5.4), one can invert (2.5.8) to obtain

$$\mathrm{var}\{N_r^{(e)}(x)\} = \lambda_r(2\omega_{rr} + 1)x + 2\lambda_r\nu_{rr} + o(1) \quad \text{as } x \to \infty.$$

The asymptotic covariances can be obtained similarly. The asymptotic asynchronous covariance matrix is then

$$\mathrm{var}\{\underset{\sim}{N}^{(e)}(x)\} = x\underset{\sim}{\Gamma} + \underset{\sim}{\Delta} + \underset{\sim}{o}(1) \quad \text{as } x \to \infty, \qquad (2.5.9)$$

where

$$\underset{\sim}{\Gamma} = \begin{bmatrix} \lambda_0(2\omega_{00} + 1) & \lambda_0\omega_{10} + \lambda_1\omega_{01} \\ \lambda_0\omega_{10} + \lambda_1\omega_{01} & \lambda_1(2\omega_{11} + 1) \end{bmatrix}$$

$$\underset{\sim}{\Delta} = \begin{bmatrix} 2\lambda_0\nu_{00} & \lambda_0\nu_{10} + \lambda_1\nu_{01} \\ \lambda_0\nu_{10} + \lambda_1\nu_{01} & 2\lambda_1\nu_{11} \end{bmatrix} .$$

This establishes a very general asymptotic relation between mean values in the various semisynchronous processes and second moments in the asynchronous process.

2.6 Multivariate point processes which are marginally, but not strongly, orderly

Some of the results given in previous sections of this chapter can, in principle, be extended easily to multivariate point processes which are marginally, but not strongly, orderly. One means of doing this is by using the analogue of (2.2.10a) for processes which are not strongly orderly (this equation is derived by Daley and Milne (1975)) and proceeding in an analogous manner. However, we will use here another method, which illustrates the relationship between point processes which are strongly orderly and those which are not. The results require more complex notation; consequently general results will not be given here. Instead, the approach will be demonstrated by considering the simplest case: bivariate point processes which are not strongly orderly. We can think of such a process as a trivariate strongly orderly point process, the three types of events being (i) type 0 events occurring alone (henceforth called type a events), (ii) type 1 events occurring alone (type b events), and (iii) type 0 and type 1 events occurring simultaneously (type c events). Then

$$\begin{aligned} N_0(x) &= N_a(x) + N_c(x) , \\ N_1(x) &= N_b(x) + N_c(x) . \end{aligned} \tag{2.6.1}$$

Before proceeding further, it must be shown that the stationarity of $\{N_0(\cdot), N_1(\cdot)\}$ implies the stationarity of $\{N_a(\cdot), N_b(\cdot), N_c(\cdot)\}$. The stationarity of $\{N_0(\cdot), N_1(\cdot)\}$, the finiteness of λ_0 and λ_1, and the strong orderliness of $\{N_a(\cdot), N_b(\cdot), N_c(\cdot)\}$ imply, for instance, that

$$\lim_{h\downarrow 0} \frac{P\{N_c(y,y+h] > 0\}}{h} = \lim_{h\downarrow 0} \frac{P\{N_0(y,y+h] > 0,\ N_1(y,y+h] > 0\}}{h}$$

exists, is finite, and is independent of y. Similar results follow for all other joint and marginal probabilities involving N_a, N_b and N_c. We now use arguments similar to those used by, amongst others, Leadbetter (1971) and Oakes (1972). If $A + y$ is an interval $(y, y + x]$, we can divide $A + y$ into n equal subintervals $A_{ni} + y$, $i = 1,\ldots,n$, and write $\chi_{ni} = 1$ if $N_c(A_{ni} + y) \geq 1$, $\chi_{ni} = 0$ otherwise. Let $E_n(y) = \{\sum_{i=1}^{n} \chi_{ni} \geq m\}$. Then $E_n(y) \subset E_{n+1}(y)$ and so, by orderliness, $\Pr\{N_c(A + y) \geq m\} = \Pr\{\lim E_n(y)\} = \lim \Pr\{E_n(y)\}$. However, since all the joint and marginal probabilities involving N_a, N_b and N_c in "small" intervals are asymptotically independent of y, it follows that $\lim \Pr\{E_n(y)\}$, and hence $\Pr\{N_c(A + y) \geq m\}$, is independent of y. Analogous arguments prove the stationarity of $\{N_a(\cdot), N_b(\cdot), N_c(\cdot)\}$.

Because the stationarity of $\{N_0(\cdot), N_1(\cdot)\}$ implies the stationarity of the strongly orderly process $\{N_a(\cdot), N_b(\cdot), N_c(\cdot)\}$, one can use earlier results and (2.6.1) to generalize to point processes which are not strongly orderly. For instance, using notation analogous to that used previously, a simple heuristic argument can be used to obtain the analogue of (2.3.6). For $i = a,b,c,e$, we have (by (2.2.11) and (2.3.1)):

$$\begin{aligned}\phi_i(\zeta_0,\zeta_1;x) &= E\{\zeta_0^{N_0(x)}\,\zeta_1^{N_1(x)}\,|\text{type } i \text{ event at the origin}\}\\ &= E\{\zeta_0^{N_a(x)}\,\zeta_1^{N_b(x)}\,(\zeta_1\zeta_2)^{N_c(x)}\,|\text{type } i \text{ event at the origin}\}\\ &= \phi_i(\zeta_0,\zeta_1,\zeta_0\zeta_1;x)\ ,\end{aligned} \tag{2.6.2}$$

where the ϕ_i on the right-hand side of (2.6.2) is the joint p.g.f. of a trivariate strongly orderly point process. Substituting (2.6.2) into (2.3.6), the following relation for the bivariate non-strongly orderly point process is obtained:

$$\phi_0(\zeta_0,\zeta_1;x) = 1 + \lambda_a(\zeta_0-1)\int_0^x \phi_a(\zeta_0,\zeta_1;y)dy + \lambda_b(\zeta_1-1)\int_0^x \phi_b(\zeta_0,\zeta_1;y)dy$$
$$+ \lambda_c(\zeta_0\zeta_1 -1)\int_0^x \phi_c(\zeta_0,\zeta_1;y)dy \; . \tag{2.6.3}$$

The extension of (2.3.8) is in general more complex. However, the results of Section 2.5.3 are easily extended. Firstly, note from (2.6.1) that

$$\text{var}\{N_0(x)\} = \text{var}\{N_a(x)\} + \text{var}\{N_c(x)\} + 2\,\text{cov}\{N_a(x),\ N_c(x)\} \tag{2.6.4}$$

and

$$\text{cov}\{N_0(x),\ N_1(x)\} = \text{cov}\{N_a(x),\ N_b(x)\} + \text{cov}\{N_a(x),\ N_c(x)\}$$
$$+ \text{cov}\{N_b(x),\ N_c(x)\} + \text{var}\{N_c(x)\}. \tag{2.6.5}$$

The results of Section 2.5.3 can be applied to N_a, N_b and N_c and combined (using (2.6.4) and (2.6.5)) to obtain the asymptotic asynchronous second moments of a non-strongly orderly bivariate point process. Under the assumptions of (and using notation analogous to) that subsection, these are

$$\text{var}\{N_0^e(x)\} = 2\{\lambda_a(\omega_{aa} + \omega_{ca} + \tfrac{1}{2}) + \lambda_c(\omega_{cc} + \omega_{ac} + \tfrac{1}{2})\}x$$
$$+ 2\{\lambda_a(\nu_{aa} + \nu_{ca}) + \lambda_c(\nu_{cc} + \nu_{ac})\} + o(1) \quad \text{as } x \to \infty, \tag{2.6.6}$$

$$\text{cov}\{N_0^e(x), N_1^e(x)\} = \{\lambda_a(\omega_{ba} + \omega_{ca}) + \lambda_b(\omega_{ab} + \omega_{cb}) + \lambda_c(2\omega_{cc}+\omega_{ac}+\omega_{bc} + 1)\}x$$
$$+ \{\lambda_a(\nu_{ba} + \nu_{ca}) + \lambda_b(\nu_{ab} + \nu_{cb}) + \lambda_c(2\nu_{cc} + \nu_{ac} + \nu_{bc})\} + o(1)$$
$$\text{as } x \to \infty. \tag{2.6.7}$$

Similarly, $\text{var}\{N_1^e(x)\}$ can be obtained.

CHAPTER 3: REGENERATIVE MULTIVARIATE POINT PROCESSES

3.1 Introduction

This chapter is concerned with a special class of multivariate point processes. The central idea is that events of one type, the type 0 events say, occur in a renewal process, i.e. the intervals between successive type 0 events are independent and identically distributed random variables. Further the numbers and positions of the events of other types occurring in any such "renewal interval" depend only on that interval and are independent of occurrences in other renewal intervals. That is, the type 0 events form regeneration points for the whole process (Smith, 1955) and hence such a multivariate point process will be called a regenerative multivariate point (r.m.p.) process.

The interest of such processes partly arises from the possibility of obtaining some general properties and partly because quite a number of special processes studied in connection with queueing theory, reliability theory, neurophysiology and other fields are special cases (Gaver, 1963; Ten Hoopen and Reuver, 1965; Lawrance, 1970,1971; Neuts, 1971; Rudemo, 1972; Rohde and Grandell, 1972; Grandell, 1976, Section 2.3). In particular, Markov renewal processes are a very special case of r.m.p. processes; for the general theory of Markov renewal processes, see Çinlar (1969).

In this chapter, some of the general properties of r.m.p. processes are examined. In Chapter 4, three bivariate examples of such processes are studied, two apparently new and one an application of the general results to the work of Ten Hoopen and Reuver (1965) and Lawrance (1970, 1971).

3.2 Assumptions and definitions

Again assume that there are m + 1 point processes (where m is a positive integer, which we label $\{0,1,\ldots,m\}$. Let $\mathcal{H}_t$ be the complete history of the multivariate process at time t, i.e. the full specification of occurrences in $(-\infty,t]$. Then the formal definition of a regenerative multivariate point process is:

Given a type 0 event at t, then for all u > 0, the joint distribution of $\{N_0(t,t+u],\ldots,N_m(t,t+u]\}$ is independent of $\mathcal{H}_t$ and depends only on u, and not on t.

The immediate consequences of the definition of the r.m.p. process are that, firstly, the type 0 events marginally, i.e. on their own, form a renewal process; and secondly, conditional on two successive type 0 events occurring at s and t (s < t), the joint distribution of $\{N_1(s,t],\ldots,N_m(s,t]\}$ is independent of $\mathcal{H}_s$ and is a function of t - s only. It further follows from the definition that r.m.p. processes are completely stationary.

For simplicity, assume that the intervals between successive type 0 events have an absolutely continuous distribution with p.d.f. $f(\cdot)$ and survivor function $\mathcal{F}(\cdot)$, with $\mathcal{F}(0+) = 1$. Because of the stationarity assumption, one can, as in Chapter 2, write $\underset{\sim}{N}(x) \equiv \{N_0(x),\ldots,N_m(x)\}$ for the numbers of events in an interval of length x.

In the following sections of this chapter we will often, for simplicity, write formulae when m = 1 or 2, so that there are two or three types of events in all, but all the results can be given for general m.

In studying the properties of, for example, the numbers of events in given intervals, the specification of initial conditions is important. These will be indicated by an appropriate subscript or superscript. The letter "$\mathcal{A}$" will be used to denote any specified initial conditions. Of particular interest will be the asynchronous and partially synchronous cases; as in Chapter 2, the letter "e" will be used for the asynchronous (equilibrium) case, and the number "i" when a type i event occurs at the origin.

3.3 The distribution of numbers of events in a fixed interval

Some general relations can be obtained for the Laplace transform of the joint p.g.f. of $\underset{\sim}{N}(x)$ for any specified initial conditions. It is convenient to obtain these for m = 1, writing $M = M(x) = N_0(x)$, $N = N(x) = N_1(x)$, with joint p.g.f.

$$\phi_{\mathcal{A}}(\zeta,\eta;x) = E(\zeta^M\eta^N|\mathcal{A}) . \tag{3.3.1}$$

Let $\mathcal{R}(x,dx)$ denote the event that the first type 0 event after the origin is in (x,x+dx]. Then let

$$\psi_{\mathcal{A}}(\eta;x) = \lim_{\delta x\to 0} \frac{E\{\eta^N|\mathcal{R}(x,\delta x),\mathcal{A}\}.\Pr\{\mathcal{R}(x,\delta x)|\mathcal{A}\}}{\delta x} \tag{3.3.2}$$

determine the distribution of the number of type 1 events in (0,x] given that the interval ends with the first type 0 event after the origin. Then, a familiar renewal argument gives that

$$\phi_{\mathcal{A}}(\zeta,\eta;x) = \phi_{\mathcal{A}}(0,\eta;x) + \zeta \int_0^x \psi_{\mathcal{A}}(\eta;y)\phi_0(\zeta,\eta;x-y)dy, \tag{3.3.3}$$

where ϕ_0 denotes a p.g.f. for an interval starting with a type 0 event. Taking Laplace transforms of (3.3.3) one obtains

$$\phi^*_{\mathcal{A}}(\zeta;\eta;s) = \phi^*_{\mathcal{A}}(0,\eta;s) + \zeta\psi^*_{\mathcal{A}}(\eta;s)\phi^*_0(\zeta,\eta;s). \tag{3.3.4}$$

In particular, for an interval starting with a type O event, one obtains from (3.3.4)

$$\phi_O^*(\zeta,\eta;s) = \frac{\phi_O^*(0,\eta;s)}{1-\zeta\psi_O^*(\eta;s)} . \tag{3.3.5}$$

Finally, substituting (3.3.5) into (3.3.4):

$$\phi_{\mathcal{A}}^*(\zeta,\eta;s) = \phi_{\mathcal{A}}^*(0,\eta;s) + \frac{\zeta\psi_{\mathcal{A}}^*(\eta;s)\phi_O^*(0,\eta;s)}{1-\zeta\psi_O^*(\eta;s)} . \tag{3.3.6}$$

The Laplace transform of the marginal p.g.f. of N(x) is then obtained by putting $\zeta = 1$ in (3.3.6). Putting $\eta = 1$ in (3.3.6), one obtains the Laplace transform of the p.g.f. of type O events, i.e. the p.g.f. of a (modified) renewal process. This will agree with Cox (1962), Section 3.2, equation (5), if we note that $\psi_{\mathcal{A}}(1;x)$ is the p.d.f. of the time from the origin to the first type O event (by (3.3.2)), and that $s\phi_{\mathcal{A}}^*(0,1;s) = 1 - \psi_{\mathcal{A}}^*(0,1;s)$ (by (3.3.1) and (3.3.2)).

An important special case, covering the three examples of Chapter 4, is when the type O events form what will be called a <u>strongly Poisson</u> process, of rate ρ, say. Strongly Poisson processes are characterized by the property that the probability of a type O event occurring at any time x is a constant independent of $\mathfrak{H}_x$, the complete history of the bivariate point process prior to x. Then for any specified initial conditions

$$\psi_{\mathcal{A}}(\eta;x) = \rho\phi_{\mathcal{A}}(0,\eta;x),$$

whence (3.3.6) becomes

$$\phi_{\mathcal{A}}^*(\zeta,\eta;s) = \frac{\phi_{\mathcal{A}}^*(0,\eta;s)}{1-\rho\zeta\phi_O^*(0,\eta;s)} . \tag{3.3.7}$$

If $m > 1$, we have in all the above formulae only to replace η by a vector $\underset{\sim}{\eta} = (\eta_1, \ldots, \eta_m)$.

Equation (3.3.6) shows that $\phi_{\mathcal{A}}(\underset{\sim}{\zeta}, \underset{\sim}{\eta}; x)$ is dependent on $\phi_{\mathcal{A}}(0, \underset{\sim}{\eta}; x)$, $\psi_{\mathcal{A}}(\underset{\sim}{\eta}; x)$, $\phi_0(0, \underset{\sim}{\eta}; x)$ and $\phi_0(\underset{\sim}{\eta}; x)$ only. These generating functions must be determined for the specified initial conditions. Usually, interest will be in the $m + 2$ cases $\mathcal{A} = 0, 1, \ldots, m; e$. As we have seen in Chapter 2, knowledge of $\phi_{\mathcal{A}}(0, \underset{\sim}{\eta}; x)$ for $\mathcal{A} = 0, 1, \ldots, m$ enables us to determine $\phi_e(0, \underset{\sim}{\eta}; x)$; if it is assumed that the r.m.p. process is strongly orderly (a property possessed by all three examples of Chapter 4) this is done by putting $\underset{\sim}{\zeta} = (0, \underset{\sim}{\eta})$ in (2.3.6) or (2.3.10). Using this result in (3.3.6), one can obtain a similar relationship between the asynchronous and partially synchronous ψ's:

$$s\psi_e^*(\underset{\sim}{\eta}; s) = \lambda_0\{1 - \psi_0^*(\underset{\sim}{\eta}; s)\} + \sum_{i=1}^{m} \lambda_i(\eta_i - 1)\psi_i^*(\underset{\sim}{\eta}; s), \qquad (3.3.8)$$

where λ_i^{-1} is the mean interval length between successive type i events; exact formulae for λ_i are given in Section 3.4. Equations (2.3.6) and (3.3.8) will be used later for the bivariate examples of Chapter 4 to obtain $\phi_{\mathcal{A}}(\zeta, \eta; x)$ and $\psi_{\mathcal{A}}(\eta; x)$ under certain initial conditions.

Analogous formulae to (2.3.6) and (3.3.8) exist when the r.m.p. process is not strongly orderly; see Section 2.6.

3.4 Moments of the counting process

The Laplace transforms of the factorial moments (both marginal and joint) of $\underset{\sim}{N}(x)$ can be obtained, in principle, by differentiating (3.3.6) the appropriate number of times and putting $\zeta = \eta = 1$. These transforms will not be given here because they are generally rather complicated expressions, apparently providing little constructive information. However, some useful results concerning moments can be obtained. Again, it is convenient to obtain these for $m = 1$ as this does not restrict the generality of the results. First, some additional notation is required. For given initial conditions $\mathcal{A}$, let

$$H_\ell^{(\mathcal{A})}(x) = E[\{N(x)\}^\ell \mid \text{initial conditions } \mathcal{A} \text{ and first type 0 event after origin is at } x], \tag{3.4.1}$$

$$V^{(\mathcal{A})}(x) = \operatorname{var}\{N(x) \mid \text{initial conditions } \mathcal{A} \text{ and first type 0 event after origin is at } x\}. \tag{3.4.2}$$

Note that $H_\ell^{(\mathcal{A})}(x)$ is a non-negative, non-decreasing function of x for all positive integers ℓ. Assume that $H_\ell^{(\mathcal{A})}(0+) = 0$. When $\ell = 1$, we shall drop the subscript; when $\mathcal{A} = 0$, i.e. a type 0 event occurs at the origin, we shall drop the superscript in both $H_\ell^{(\mathcal{A})}(x)$ and $V^{(\mathcal{A})}(x)$. Let

$$\mu_\ell = \int_0^\infty x^\ell f(x)dx, \tag{3.4.3}$$

$$\kappa_\ell = \int_0^\infty H_\ell(x)f(x)dx. \tag{3.4.4}$$

The quantity μ_ℓ is the ℓth moment of the interval length between successive type 0 events, while κ_ℓ is the ℓth moment of the number of type 1 events between successive type 0 events.

Using simple probabilistic arguments (or the asymptotics of Section 3.5), it is found that in the asynchronous case, the means are $E\{M^{(e)}(x)\} = \lambda_0 x = \mu_1^{-1}x$, $E\{N^{(e)}(x)\} = \lambda_1 x = \kappa_1 x/\mu_1$, where the superscript "e" denotes equilibrium (asynchronous) initial conditions. When $m > 1$, there are analogous formulae which give the λ_i's required in (2.3.6) or (3.3.7).

A useful inequality is now given. It shows that the variance to mean ratio of $N(x)$ in a renewal interval is enlarged in an r.m.p. process.

Theorem 3.4.1: Suppose that there exists an $\alpha > 0$ such that, for given initial conditions $\mathcal{A}$,

$$V^{(\mathcal{A})}(x) \geq \alpha H^{(\mathcal{A})}(x), \tag{3.4.5}$$

$$V(x) \geq \alpha H(x). \tag{3.4.6}$$

almost everywhere (a.e.) in $\{x: \mathcal{J}(x) > 0\}$. Then, for all $x \geq 0$

$$\mathrm{var}\{N^{(\mathcal{A})}(x)\} \geq \alpha E\{N^{(\mathcal{A})}(x)\}. \tag{3.4.7}$$

Proof: Suppose that in the interval $(0,x]$ there are k type 0 events situated at $\underset{\sim}{t}_k = (t_1,\ldots,t_k)$ $(t_i \leq t_{i+1},\ i = 1,2,\ldots,k-1;\ t_1 \geq 0,\ t_k \leq x)$. Then by the definition of an r.m.p. process given in Section 3.2, and (3.4.5) and (3.4.6), it follows that $\mathrm{var}\{N^{(\mathcal{A})}(x)|k,\underset{\sim}{t}_k\} \geq \alpha E\{N^{(\mathcal{A})}(x)|k,\underset{\sim}{t}_k\}$, for all k and $\underset{\sim}{t}_k$. Hence

$$\begin{aligned}
\mathrm{var}\{N^{(\mathcal{A})}(x)\} &= E_{k,\underset{\sim}{t}_k}[\mathrm{var}\{N^{(\mathcal{A})}(x)|k,\underset{\sim}{t}_k\}] + \mathrm{var}_{k,\underset{\sim}{t}_k}[E\{N^{(\mathcal{A})}(x)|k,\underset{\sim}{t}_k\}] \\
&\geq E_{k,\underset{\sim}{t}_k}[\mathrm{var}\{N^{(\mathcal{A})}(x)|k,\underset{\sim}{t}_k\}] \\
&\geq \alpha E_{k,\underset{\sim}{t}_k}[E\{N^{(\mathcal{A})}(x)|k,\underset{\sim}{t}_k\}] \\
&= \alpha E\{N^{(\mathcal{A})}(x)\}.
\end{aligned}$$

3.5 Asymptotics

The processes $N_i(t)$, $i = 0,1,\ldots,m$ are all examples of cumulative processes (Smith, 1955, Section 5). His asymptotic results can be particularized to r.m.p. processes to obtain Theorem 3.5.1, which, in turn, can be used to obtain some useful asymptotic moment inequalities for regenerative multivariate point processes.

First, we note that all the asymptotic results for cumulative processes are independent of the initial conditions. Hence, initial conditions are not specified for any results given in this section. Again it is convenient to obtain results for $m = 1$ for the time being, although later it will be assumed that $m = 2$ when considering the asymptotic covariance of two subsidiary counting processes of a r.m.p. process. We do, however, require additional notation. Let

$$\omega = \int_0^\infty xH(x)f(x)dx, \tag{3.5.1}$$

where $H(x)$ is the mean number of type 1 events between successive type 0 events which occur at times t and $t + x$ respectively. The quantity $\omega - \mu_1\kappa_1$ is then the covariance of the interval length between successive type 0 events and the number of type 1 events in that interval. Using Theorems 7 to 10 of Smith, we find

<u>Theorem 3.5.1</u>: If $\mu_1 < \infty$ and $\kappa_1 < \infty$, then as $x \to \infty$

(i) $E\{N_i(x)\} = \lambda_i x + o(x)$,

(ii) $\lim_{x\to\infty} N_i(x)/x = \lambda_i$ with probability one,

where $\lambda_0 = \mu_1^{-1}$, $\lambda_1 = \kappa_1\mu_1^{-1}$.

If, in addition, $\mu_2 < \infty$ and $\kappa_2 < \infty$, then as $x \to \infty$

(iii) $\mathrm{cov}\{N_i(x), N_j(x)\} = \sigma_{ij}x + o(x)$,

where $\sigma_{00} = \mu_1^{-3}(\mu_2 - \mu_1^2)$, $\sigma_{11} = \mu_1^{-3}(\mu_1^2\kappa_2 + \mu_2\kappa_1^2 - 2\mu_1\kappa_1\omega)$, $\sigma_{01} = \mu_1^{-3}(\mu_2\kappa_1 - \mu_1\omega)$.

(iv) $\{\underset{\sim}{N}(x) - x\underset{\sim}{\lambda}\}/x^{1/2}$ converges in distribution to a normal distribution with zero mean and covariance matrix $\underset{\sim}{\Sigma} = (\sigma_{ij})$, where $\underset{\sim}{N}(x) = \{M(x), N(x)\}$ and $\underset{\sim}{\lambda} = (\lambda_0, \lambda_1)$.

The asymptotic results of Theorem 3.5.1 extend immediately to the case $m > 1$, even when the asymptotic covariance of two subsidiary counting processes of an r.m.p. process is required. This last quantity is obtained by using the results of Theorem 3.5.1 on the superposition of two subsidiary point processes and then "breaking up" the result into its component parts. The result is given in Theorem 3.5.6.

The results of Theorem 3.5.1 give rise to a few other results which now follow.

<u>Theorem 3.5.2</u>: Assume $\mu_2 < \infty$ and $\kappa_2 < \infty$. Let

$$\beta = \frac{\int_0^\infty V(x)f(x)dx}{\int_0^\infty H(x)f(x)dx}. \tag{3.5.2}$$

Then

$$\lim_{x\to\infty} \frac{\operatorname{var}\{N(x)\}}{E\{N(x)\}} \geq \beta, \tag{3.5.3}$$

the equality holding if and only if $H(x)$ is proportional to x a.e. on the support of f.

<u>Proof</u>: By Theorem 3.5.1(i) and (iii)

$$\lim_{x\to\infty} x^{-1}[\operatorname{var}\{N(x)\} - E\{N(x)\}] = \mu_1^{-3}\{\mu_1^2(\kappa_2 - \beta\kappa_1) + \mu_2\kappa_1^2 - 2\mu_1\kappa_1\omega\}$$

$$= \mu_1^{-3}[\{\mu_1(\kappa_2 - \beta\kappa_1)^{1/2} - \mu_2\kappa_1\}^2 + 2\mu_1\kappa_1\{\mu_2^{1/2}(\kappa_2 - \beta\kappa_1)^{1/2} - \omega\}] \tag{3.5.4}$$

after rearranging. Now using the Cauchy-Schwarz inequality on (3.5.1), it follows by (3.4.3), (3.4.4) and (3.5.2) that

$$\omega \leq \mu_2^{1/2}(\kappa_2 - \beta\kappa_1)^{1/2}. \tag{3.5.5}$$

Therefore, substituting (3.5.5) into (3.5.4), we conclude that $\lim_{x\to\infty} x^{-1}[\text{var}\{N(x)\} - E\{N(x)\}] \geq 0$, which is equivalent to (3.5.3).

For equality to hold, it is necessary that the Cauchy-Schwarz inequality (3.5.5) is an equality. This is true if and only if $H(x)$ is proportional to x a.e. on the support of f. For the sufficiency, it must be shown that the first expression on the right-hand side of (3.5.4) is zero when $H(x) = \nu x$ a.e. on the suppose of f, for some $\nu > 0$. If this is true, it follows, by (3.4.3), (3.4.4) and (3.5.2) that $\kappa_1 = \nu\mu_1$ and $\kappa_2 = \nu^2\mu_2 + \nu\beta\mu_1$, whence $\mu_1(\kappa_2 - \beta\kappa_1)^{1/2} - \mu_2^{1/2}\kappa_1 = 0$; and so the sufficiency of the equality is proved.

Theorem 3.5.2 says approximately that the asymptotic variance to mean ratio of $N(x)$ is greater than or equal to the ratio of the "average" variance to the "average" mean of the number of type 1 events between successive type 0 events. The equality holds if and only if the dependence between the type 0 and type 1 events is very weak in that $H(x)$ is proportional to x a.e. on the support of f.

Theorem 3.5.2 leads to a corollary which is the asymptotic equivalent of Theorem 3.4.1.

<u>Corollary 3.5.3</u>: Assume $\mu_2 < \infty$ and $\kappa_2 < \infty$. If $V(x) \geq \alpha H(x)$ for some constant α a.e. on the support of f, then

$$\lim_{x\to\infty} \frac{\text{var}\{N(x)\}}{E\{N(x)\}} \geq \alpha.$$

<u>Proof</u>: Now $V(x) \geq \alpha H(x)$ implies $\int_0^\infty V(x)f(x)dx \geq \alpha\int_0^\infty H(x)f(x)dx$. The result then follows by a proof similar to that for Theorem 3.5.2.

If $H(x)$ is differentiable with derivative $h(x)$, then $h(x)dx$ can be interpreted as the probability of a type 1 event in the interval $[x,x+dx)$ given that the last type 0 event occurred at the origin. Therefore, if

$h(x)$ is an increasing function of x, then the occurrence of type 0 events acts as an inhibitor of type 1 events, and so we would expect $M(x)$ and $N(x)$ to be negatively correlated; conversely, if $h(x)$ is a decreasing function of x, one would expect $M(x)$ and $N(x)$ to be positively correlated. This can be shown at least for the asymptotic covariance $(\sigma_{01}x)$ given in Theorem 3.5.1. We first, however, require the following lemma, which is itself quite interesting. For simplicity of the proof, assume an absolutely continuous distribution. The result however holds for all distributions defined on the positive real line.

Lemma 3.5.4: Let $f(x)$ be the p.d.f. of an absolutely continuous distribution defined on $[0,\infty)$. Let $\mu_i = \int_0^\infty x^i f(x)dx$. Suppose $\mu_j < \infty$. Then

$$W_{i,j}(x) \equiv \int_x^\infty \int_y^\infty (\mu_i z^{j-1} - \mu_j z^{i-1}) f(z)dzdy \tag{3.5.6a}$$

$$= \int_x^\infty (\mu_i z^{j-1} - \mu_j z^{i-1})(z - x)f(z)dz \tag{3.5.6b}$$

is non-negative for all $x \geq 0$ and $i = 1,2,\ldots,j-1$.

Proof: Because μ_j (and hence μ_i) is finite, we can interchange the order of integration in (3.5.6a) to obtain (3.5.6b). Let $r = (\mu_j/\mu_i)^{1/j-1}$ (where the positive root is taken). If $x > r$, it follows that $z > x$ implies that $\mu_i z^{j-1} - \mu_j z^{i-1} \geq 0$, and hence, by representation (3.5.6b), $W_{i,j}(x) \geq 0$. If $0 \leq x \leq r$, then by (3.5.6a), $W''_{i,j}(x) = \mu_i x^{j-1} - \mu_j x^{i-1} \leq 0$, and so $W_{i,j}(x)$ is concave in $[0,r]$. Hence, for all x in $[0,r]$, $W_{i,j}(x) \geq \min\{W_{i,j}(0), W_{i,j}(r)\}$. It has already been shown that $W_{i,j}(r) \geq 0$, and it is easily shown that $W_{i,j}(0) = 0$. Hence the lemma is proved.

We now proceed to the theorem concerning the asymptotic covariance of $M(x)$ and $N(x)$.

Theorem 3.5.5: Assume that $H(x)$ is twice differentiable on the support of f, and that $\mu_2 < \infty$, $\kappa_2 < \infty$. If $H''(x) \geq 0$ a.e. on the support of f, then $\sigma_{01} \leq 0$, the inequality being reversed if $H''(x) \leq 0$.

Proof: By (3.4.3), (3.4.4), (3.5.1) and Theorem 3.5.1(iii), one obtains

$$
\begin{aligned}
\mu_1{}^3\sigma_{01} &= \int_0^\infty (\mu_2 - \mu_1 x) f(x) H(x) dx \\
&= -\int_0^\infty W_{1,2}(x) H''(x) dx
\end{aligned}
\tag{3.5.7}
$$

after a double integration by parts, where $W_{1,2}(x)$ is given by (3.5.6). By Lemma 3.5.4, $W_{1,2}(x) \geq 0$, all $x \geq 0$, and hence, by (3.5.7), σ_{01} has the opposite sign to $H''(x)$.

At this point it is convenient to assume $m = 2$ because we are interested in the behaviour of the asymptotic covariance of two subsidiary counting processes of an r.m.p. process. Extensions of earlier definitions are required. Let $H_{\ell;i}(x)$ be the ℓth moment of the number of type i events between successive type 0 events at times 0 and x respectively. Let

$$
\begin{aligned}
\kappa_{\ell;i} &= \int_0^\infty H_{\ell;i}(x) f(x) dx, \\
& \qquad\qquad\qquad\qquad (i = 1,2) \\
\omega_i &= \int_0^\infty x H_{\ell;i}(x) f(x) dx.
\end{aligned}
$$

$$
S(x) = E\{N_1(x) N_2(x) \mid \text{successive type 0 events at 0 and } x\},
$$

$$
\tau = \int_0^\infty S(x) f(x) dx .
$$

Then, using Theorem 10 of Smith (1955) or using Theorem 3.5.1 on the superposition of N_1 and N_2, we obtain

<u>Theorem 3.5.6</u>: If $\mu_2 < \infty$, $\kappa_{2;1} < \infty$ and $\kappa_{2;2} < \infty$, then as $x \to \infty$

$$\operatorname{cov}\{N_1(x),\ N_2(x)\} \;=\; \sigma_{12}(x) + o(x)\ ,$$

where

$$\sigma_{12} = \mu_1^{-3}\{\mu_1^{\,2}\tau + \mu_2\kappa_{1;1}\kappa_{1;2} - \mu_1(\omega_1\kappa_{1;2} + \omega_2\kappa_{1;1})\}.$$

The random variables $M(x)$, $N_1(x)$ and $N_2(x)$ are also asymptotically jointly normally distributed. In fact Theorem 3.5.1 (iv) extends to general m in an obvious fashion.

Theorem 3.5.6 enables us to obtain conditions for the asymptotic covariance of two subsidiary counting processes of an r.m.p. process to be non-negative or non-positive.

<u>Theorem 3.5.7</u>: Assume that $H_{1;1}(x)$ and $H_{1;2}(x)$ are both twice differentiable in $\{x: \mathfrak{F}(x) > 0\}$ and that $\mu_2 < \infty$, $\kappa_{2;1} < \infty$ and $\kappa_{2;2} < \infty$. Let

$$d \;=\; \int_0^\infty \{S(x) - H_{1;1}(x)H_{1;2}(x)\}f(x)dx. \tag{3.5.8}$$

(i) If $d \geq 0$, and $H''_{1;1}(x)$ and $H''_{1;2}(x)$ are either both non-negative or both non-positive a.e. in $\{x:\ (x) > 0\}$, then $\sigma_{12} \geq 0$;

(ii) If $d \leq 0$, and $H''_{1;1}(x) \leq 0 \leq H''_{1;2}(x)$ a.e. in $\{x:\ (x) > 0\}$, then $\sigma_{12} \leq 0$.

<u>Proof</u>: Using (3.5.8) and Theorem 3.5.6, one finds after a double integration by parts that

$$\mu_1^{\;3}\sigma_{12} \;=\; d + \int_0^\infty H''_{1;2}(x)B(x)dx\ , \tag{3.5.9}$$

where

$$B(x) \;=\; \mu_1 \int_x^\infty \int_y^\infty \{\mu_1^{\;2}\sigma_{01} + \mu_1 H_{1;1}(z) - \kappa_{1;1}z\}f(z)dzdy.$$

It can be shown by arguments similar to, but more involved than, those of Lemma 3.5.4 that if $H''_{1;1}(x) \geq 0$ a.e. in $\{x: \mathfrak{F}(x) \geq 0\}$, then $B(x) \geq 0$, all $x \geq 0$; the inequality is reversed if $H''_{1;1}(x) \leq 0$. The various results of the theorem then follow easily from (3.5.9).

Theorems 3.5.5 and 3.5.7 together state, for instance, that under conditions which ensure that $N_1(x)$ and $N_2(x)$ are both positively correlated with $M(x)$ when x is large, $N_1(x)$ will also be positively correlated with $N_2(x)$ if their "average" conditional covariance taken between two successive type 0 events, d, is also positive.

3.6 The intervals between successive events of subsidiary processes

Knowledge of the Laplace transform of the joint p.g.f. of $\underset{\sim}{N}(x)$ (and hence of the marginal p.g.f. of $N_i(x)$, $i = 0,1,\ldots,x$) enables us, in principle, to determine the properties of the interval process of type i events. The interval properties of the type 0 events are just those of a renewal process and will not be given here. Again, for convenience, we assume that m = 1 and obtain results for the process of type 1 events; results for other processes when m > 1 are analogous.

General methods for determining the properties of the interval process of a stationary univariate point process from its counting process are given in Cox and Lewis (1966), Chapter 4. However, because of the structure of (3.3.6), many results for the interval process give rise to very complicated expressions which will not be reproduced here. For given initial conditions $\mathcal{A}$, let $X_{\mathcal{A}}$ be the time from the origin to the first type 1 event after the origin. Some of the properties of $X_{\mathcal{A}}$, in particular $E(X_{\mathcal{A}})$, $E(X_{\mathcal{A}}^2)$ and $\chi_{\mathcal{A}}(s) = E\{\exp(-sX_{\mathcal{A}})\}$, the m.g.f. of $X_{\mathcal{A}}$, are not too complicated to derive and are given below. From (3.3.5) and (3.3.6) one easily finds

$$\chi_{\mathcal{A}}(s) \quad = \quad 1 - s\phi^*_{\mathcal{A}}(0,0;s) - \psi^*_{\mathcal{A}}(0;s)\{1 - \chi_0(s)\}, \tag{3.6.1}$$

where

$$\chi_0(s) \quad = \quad 1 - \frac{s\psi^*_0(0,0;s)}{1 - \psi^*_0(0;s)} \tag{3.6.2}$$

is the m.g.f. of X_0, the interval between a type 0 event and the first type 1 event after it, and where $\phi_{\mathcal{A}}$ and $\psi_{\mathcal{A}}$ are given by (3.3.1) and (3.3.2) respectively. Then it follows that

$$E(X_{\mathcal{A}}) \quad = \quad \phi^*_{\mathcal{A}}(0,0;0) + \psi^*_{\mathcal{A}}(0;0)E(X_0) , \tag{3.6.3}$$

where

$$E(X_0) \quad = \quad \frac{\phi^*_0(0,0;0)}{1 - \psi^*_0(0;0)} . \tag{3.6.4}$$

In particular, note that $E(X_1) = \lambda_1^{-1} = \mu_1/\kappa_1$, where X_1 is the interval between successive type 1 events, and μ_1 and κ_1 are given by (3.4.3) and (3.4.4) respectively. Second moments are given by

$$E(X_{\mathcal{A}}^2) \quad = \quad 2\{\Omega_{\mathcal{A}}(0) + \Pi_{\mathcal{A}}(0)E(X_0)\} + \psi^*_{\mathcal{A}}(0;0)E(X_0^2) , \tag{3.6.5}$$

where

$$E(X_0^{\,2}) = \frac{2\{\Omega_0(0) + \Pi_0(0)E(X_0)\}}{1 - \psi_0^*(0;0)}\,, \tag{3.6.6}$$

and $\Omega_{\mathcal{A}}(s) = -d|ds\ \phi^*_{\mathcal{A}}(0,0;s)$, $\Pi_{\mathcal{A}}(s) = -d|ds\ \psi^*_{\mathcal{A}}(0;s)$.

In particular (Cox and Lewis, 1966, Section 4.2, equation (5))

$$\begin{aligned} E(X_1^{\,2}) &= 2E(X_1)E(X_e) \\ &= 2\mu_1\{\phi_e^*(0,0;0) + \psi_e^*(0,0;0)E(X_0)\}/\kappa_1\,, \end{aligned} \tag{3.6.7}$$

where X_e is the time to the first type 1 event when the process is in statistical equilibrium, i.e. the asynchronous case.

CHAPTER 4: THREE SIMPLE EXAMPLES OF REGENERATIVE BIVARIATE POINT PROCESSES

4.1 Introduction

In this chapter, three simple examples of regenerative bivariate point processes are considered, two of them apparently new, the third having been previously studied by Ten Hoopen and Reuver (1965), and Lawrance (1970, 1971). Since all the examples are bivariate point processes, we shall use the notation of Chapter 3 with $m = 1$. The events of the renewal process (which are regeneration points for the bivariate point process) will again be called type 0 events.

All the examples discussed in this chapter are particularly amenable to treatment using the theory of r.m.p. processes, because in each of the three cases, the process of type 0 events forms what will be called a strongly renewal process. Such processes are characterized by the property that the probability of a type 0 event occurring at any time t is dependent on the past history of the whole bivariate point process only through the time to the previous type 0 event. This property makes the calculation of certain quantities, which are useful in r.m.p. processes, relatively easy in many cases.

With each example discussed, we shall give the generating functions $\phi_{\mathcal{A}}(0,\eta;x)$ and $\psi_{\mathcal{A}}(\eta;x)$ for initial conditions $\mathcal{A} = 0$, 1 and e (see (3.3.1) and (3.3.2)), as well as $H(x)$ and $V(x)$, the mean and variance respectively of the number of type 1 events between two successive type 0 events which are a distance x apart (see (3.4.1), (3.4.2) and the following discussion). With these quantities one can obtain the Laplace transform of the joint p.g.f. of $M(x)$ and $N(x)$ (see Section 3.3) as well as the interval properties of the process of type 1 events (see Section 3.6); they also enable us to use results in Sections 3.4 and 3.5 to obtain properties specific to each of the three processes. For each of the three examples, we consider conditions under which either the bivariate point process or just the marginal point process of type 1 events is a member of another class of point processes (e.g. Poisson process, bivariate Markov renewal process).

4.2 Imbedding an inhomogeneous Poisson process

In this example, we imbed, between successive events of a renewal process (type 0 events), an inhomogeneous Poisson process (type 1 events) whose rate function is a function of the time to the previous type 0 event. More formally, the bivariate point process can be defined by

$$
\begin{aligned}
&P\{\text{type 0 event in } [x,x+dx) \,|\, \mathfrak{H}_x \text{ and previous type 0 event at } y \leq x\} \\
&\quad = f(x-y)dx + o(dx), \\
&P\{\text{type 1 event in } [x,x+dx) \,|\, \mathfrak{H}_x \text{ and previous type 0 event at } y \leq x\} \\
&\quad = \lambda(x-y)dx + o(dx).
\end{aligned}
\tag{4.2.1}
$$

Here $f(\cdot)$ (and $\mathfrak{F}(\cdot)$) are assumed to satisfy the conditions given in Section 3.2, while $\lambda(\cdot)$ is assumed to be a finite, non-negative, deterministic function with at most a countable number of discontinuities.

Note the following:

(i) If $\lambda(x) = \lambda$, all $x \in \{x: \mathfrak{F}(x) > 0\}$, the process of type 1 events is a homogeneous Poisson (λ) process and is independent of the process of type 0 events;

(ii) the process of type 1 events forms a doubly stochastic Poisson (d.s.P.) process; see Cox and Lewis (1966, Section 7.2) or Grandell (1976) for results concerning d.s.P. processes. If $\nu(x)$ is the stochastic rate function at time x of the process of type 1 events (considered as a d.s.P. process), then given the last type 0 event prior to x was at y, we have the relation (from (4.2.1)):

$$\nu(x) = \lambda(x - y); \tag{4.2.2}$$

(iii) $p_j(x) \equiv P\{N(t,x+t] = j \,|\, \text{successive type 0 events at } t \text{ and } x + t\}$

$$= e^{-\Lambda(x)} \frac{\{\Lambda(x)\}^j}{j!} \equiv P_j\{\Lambda(x)\}, \quad j = 0,1,2,\ldots, \tag{4.2.3}$$

where $\Lambda(x) = \int_0^x \lambda(y)dy$.

It is easily found that the conditional mean and variance of (4.2.3) are

$$H(x) = V(x) = \Lambda(x), \tag{4.2.4}$$

whence by (3.4.4) and (3.5.1)

$$\kappa_1 = \int_0^\infty \Lambda(x)f(x)dx\,,$$

$$\kappa_2 = \int_0^\infty \Lambda(x)\{1 + \Lambda(x)\}f(x)dx\,, \tag{4.2.5}$$

$$\omega = \int_0^\infty x\Lambda(x)f(x)dx.$$

We now derive $\phi_{\mathcal{A}}(0,\eta,x)$ for $\mathcal{A} = 0$, 1 and e. By definition of the process, the probability that there are no type 0 events and j type 1 events in (0,x], given that a type 0 event occurs at the origin, is just $\mathfrak{F}(x)P_j\{\Lambda(x)\}$, by (4.2.3). Hence, using (3.3.1), one obtains

$$\phi_0(0,\eta;x) = \mathfrak{F}(x)\, \exp\{\Lambda(x)(\eta - 1)\}. \tag{4.2.6}$$

For the equilibrium process, we argue in the following way. The joint probability that the last type 0 event prior to the origin was in $[-y, -y+dy)$ and that there are no type 0 events in $(0,x]$ is easily seen to be $\mu_1^{-1}\mathfrak{F}(x+y)dy$. Conditional on these events, the probability that there are j type 1 events in $(0,x]$ is $P_j\{\Lambda(x+y) - \Lambda(y)\}$. Multiplying these two probabilities, integrating over all possible y's and using (3.3.1), we have

$$\phi_e(0,\eta;x) = \mu_1^{-1} \int_0^{\infty} \mathfrak{F}(x+y)\exp[\{\Lambda(x+y) - \Lambda(y)\}(\eta-1)]dy. \qquad (4.2.7)$$

One can derive $\phi_1(0,\eta;x)$ from first principles or using (2.3.6). It is

$$\phi_1(0,\eta;x) = \kappa_1^{-1} \int_0^{\infty} \mathfrak{F}(x+y)\lambda(y)\exp[\{\Lambda(x+y) - \Lambda(y)\}(\eta-1)]dy. \qquad (4.2.8)$$

In each of the above cases, $\psi_{\mathcal{A}}(\eta;x)$ (which is derived in an analogous fashion), is the same as $\phi_{\mathcal{A}}(0,\eta;x)$ except that whenever $\mathfrak{F}(\cdot)$ appears in $\phi_{\mathcal{A}}(0,\eta;x)$, it is replaced by $f(\cdot)$ in $\psi_{\mathcal{A}}(\eta;x)$.

Using (4.2.6) to (4.2.8), one can obtain a number of the counting and interval properties of the process; using (4.2.5) and Theorem (3.5.1), one can obtain the asymptotic counting properties. Then, assuming $\mu_2 < \infty$ and $\kappa_2 < \infty$, and using the results of Sections 3.4 and 3.5, it is easy to prove the following results for the process:

Theorem 4.2.1: (i) For all initial conditions $\mathcal{A}$ and all $x \geq 0$, $\mathrm{var}\{N^{(\mathcal{A})}(x)\} \geq E\{N^{(\mathcal{A})}(x)\}$.

(ii) $$\lim_{x\to\infty} \frac{\mathrm{var}\{N(x)\}}{E\{N(x)\}} \geq 1,$$

the equality holding if and only if $\Lambda(x)$ is proportional to x a.e. on the support of f.

(iii) Assume that $\lambda(x)$ is differentiable on the support of f. If $\lambda(x)$ is non-decreasing a.e. on the support of f, then $\sigma_{01} \leq 0$ (where σ_{01} is given in Theorem 3.5.1 (iii)). The inequality is reversed if $\lambda(x)$ is non-increasing.

Proof: Part (i) can be proved for various initial conditions $\mathcal{A}$ by appealing to Theorem 3.4.1. However, it can be more directly and more generally proved by noting the fact that d.s.P. processes (of which this process is one) are never underdispersed. From (4.2.4), it is seen that β of (3.5.2) is equal to 1, and hence part (ii) follows from Theorem 3.5.2. Part (iii) is merely a restatement of Theorem 3.5.5.

Part (ii) of Theorem 4.2.1 implies that a necessary condition for the process of type 1 events to be a Poisson process is that $\Lambda(x) = \lambda x$, some $\lambda > 0$, a.e. on the support of f. In fact we can obtain the following stronger result.

Theorem 4.2.2: The process of type 1 events is a Poisson process if and only if there exists a $\lambda > 0$ such that $\lambda(x) = \lambda$ a.e. in $\{x: \mathcal{F}(x) > 0\}$.

Proof: This follows from the easily proved fact (Kingman, 1964) that a stationary d.s.P. process is a Poisson process if and only if $\nu(x) = \lambda$ for all x, and the fact that the process discussed here is a d.s.P. process with $\nu(x)$ given by (4.2.2).

As an example of the above results, consider the case when $f(x) = \rho e^{-\rho x}$, $\rho > 0$, $x \geq 0$ (i.e. the process of type 0 events is strongly Poisson), and $\lambda(x) = \gamma e^{-\rho x}$, $\gamma > 0$, $x \geq 0$. We give below some of the most important marginal properties of the process of type 1 events. Using (3.3.7), Theorem 3.5.1, (3.6.7), (4.2.5) to (4.2.8), and equations (5) and (7) of Cox and Lewis (1966), Section 4.6, we obtain

$$E\{N^{(1)}(x)\} = \frac{1}{2}\gamma\{x + \frac{1}{6\rho}(1 - e^{-2\rho x})\}; \tag{4.2.9}$$

$$E(X_1) = 2/\gamma; \ \mathrm{var}(X_1) = \frac{4\{\pi(\alpha)(\alpha+e^{-\alpha}) - \alpha + e^{-\alpha} + 1\}}{\gamma^2(\alpha+e^{-\alpha} - 1)}; \tag{4.2.10}$$

$$\rho_i \equiv \mathrm{corr}(X_{1;j}, X_{1;j+i}) = \frac{E(X_1)\{a_i - E(X_1)\}}{\mathrm{var}(X_1)}, \ i = 1,2, \tag{4.2.11}$$

where $X_{1;j}$ is the jth interval in the process of type 1 events,

$$a_1 = \frac{\alpha\{\alpha[\pi(\alpha) - 1] + 1 - e^{-\alpha}\}\{1 - e^{-\alpha}\}}{\gamma\{\alpha + e^{-\alpha} - 1\}^2}, \tag{4.2.12}$$

$$a_2 = \frac{\alpha}{\gamma}\left\{\frac{[\pi(\alpha) - \frac{3}{2} + \frac{1}{2}e^{-\alpha}(3+\alpha)]}{[\alpha + e^{-\alpha} - 1]} + \frac{[\pi(\alpha)(2-e^{-\alpha}(2+2\alpha+\frac{1}{2}\alpha^2))+1-e^{-\alpha}(1+\alpha)]}{[\alpha + e^{-\alpha} - 1]^2}\right. \tag{4.2.13}$$

$$\left. + \frac{\pi(\alpha)[1 - e^{-\alpha}(1+\alpha)]^2}{[\alpha + e^{-\alpha} - 1]^3}\right\},$$

$$\pi(\alpha) = C + \log\alpha - \mathrm{Ei}(-\alpha) \equiv \int_0^1 \frac{(1 - e^{-\alpha y})}{y}\,dy, \tag{4.2.14}$$

where $\alpha = \gamma/\rho$, and where $C = .5772$ is Euler's constant.

A numerical example of the above results is given in Section 5.3.2 in order to show the similarity of the process to another one whose properties are given at the end of Section 4.3. Note, from (2.5.7), that there is a one-to-one correspondence between $E\{N^{(1)}(x)\}$ and $\mathrm{var}\{N^{(e)}(x)\}$, so that $E\{N^{(1)}(x)\}$ is a second order property of the counting process.

4.3 Imbedding an ordinary renewal process

In this example, we imbed, between successive events of a renewal process (type 0 events), an ordinary renewal process (type 1 events); see Cox (1962), Section 2.1. Formally, we define the process in the following way:

$$P\{\text{type 0 event in } [x, x+dx) \mid \mathcal{H}_x \text{ and previous type 0event at } y \leq x\}$$
$$= f(x-y)dx + o(dx), \qquad (4.3.1)$$

$$P\{\text{type 1 event in } [x, x+dx) \mid \mathcal{H}_x, \text{ previous type 0 event at } y_1 \leq x, \text{ and}$$
$$\text{previous type 1 event at } y_2 \leq x\} = g(x - \max(y_1,y_2))dx + o(dx).$$

Equations (4.3.1) imply that each type 0 event generates a series of subsidiary events, the intervals between successive subsidiary events (including the interval between a type 0 event and the first subsidiary event after it) being independent and identically distributed random variables with p.d.f. $g(\cdot)$. Each type 0 event generates these subsidiary events until the next type 0 event occurs, at which point the previous type 0 event ceases generating its "cluster" of subsidiary events, and the latest type 0 event begins generating its cluster. The superposition of these subsidiary events forms the process of type 1 events, and it is easily seen that the resulting bivariate point process is a regenerative one.

The model here is similar to one discussed by Bartlett (1963) and Lewis (1964) (the so-called "Bartlett-Lewis" process) and generalized by Lewis (1970), where each type 0 event generates s following subsidiary events as an ordinary renewal process. The Bartlett-Lewis process, however, is not a regenerative process with respect to its cluster centres, because overlapping of the clusters is allowed.

Note that, for the process defined by (4.3.1), if $g(x) = \lambda e^{-\lambda x}$, the process of type 1 events is a Poisson (λ) process and is independent of the process of type 0 events.

Before determining the properties of this regenerative bivariate point process, we shall require some additional assumptions and notation. Assume that $g(\cdot)$ is the p.d.f. of an absolutely continuous distribution with survivor function $\mathcal{G}(\cdot)$, and that $\mathcal{G}(0+) = 1$. Then it follows from general renewal theory that $H(x)$ (the mean number of type 1 events between successive type 0 events which are a distance x apart) is the renewal function associated with the density $g(\cdot)$ and is differentiable a.e. (with derivative $h(x)$). Let $\beta = \int_0^\infty xg(x)dx$. Assume that $0 < \beta < \infty$. Also assume that there exists a $c > 1$ such that $\int_0^\infty \{g(x)\}^c dx < \infty$. Then it is known that $\lim_{x\to\infty} h(x) = \beta^{-1}$ (Smith, 1954, Theorem 12).

The properties of $H(x)$, $H_2(x)$ and $V(x)$ (see (3.4.1), (3.4.2) and the following discussion) are readily available from Cox (1962), Sections 4.1 and 4.5. In particular

$$H^*(s) = \frac{g^*(s)}{s\{1 - g^*(s)\}}, \tag{4.3.2}$$

$$H_2(x) = H(x) + 2\int_0^x H(x - y)h(y)dy, \tag{4.3.3}$$

$$V(x) = H(x)\{1 - H(x)\} + 2\int_0^x H(x - y)h(y)dy. \tag{4.3.4}$$

The quantities κ_1, κ_2 and ω (see (3.4.4) and (3.5.1)) required for asymptotic first and second moments of the counting process can then be found using (4.3.2) and (4.3.3).

We now briefly derive $\phi_{\mathcal{A}}(0,\eta;x)$ for $\mathcal{A} = 0$, 1 and e. We however first require the following. Let $\theta(\eta;x)$ be the p.g.f. of the number of events in the interval $(0,x]$ of the ordinary renewal process whose interval lengths have common p.g.f. $g(\cdot)$. Then we have (Cox, 1962, Section 3.1, equation (4))

$$\theta^*(\eta;s) = \frac{1 - g^*(s)}{s\{1 - \eta g^*(s)\}}. \tag{4.3.5}$$

The probability that there are no type 0 events and j type 1 events in $(0,x]$, given that a type 0 event occurs at the origin is just $\mathfrak{F}(x)p_j(x)$, where $p_j(x)$ is the probability that j events of the above-mentioned ordinary renewal process occur in $(0,x]$. Hence, using (3.3.1), one obtains

$$\phi_0(0,\eta;x) = \mathfrak{F}(x)\theta(\eta;x). \tag{4.3.6}$$

If there is a type 1 event at the origin, we need to condition on the time of occurrence of the last type 0 event before the origin. Using arguments similar to those used in deriving (4.2.7), one obtains

$$\phi_1(0,\eta;x) = \kappa_1^{-1}\theta(\eta;x)\int_0^\infty \mathfrak{F}(x + y)h(y)dy. \tag{4.3.7}$$

For $\mathscr{A} = 0$ and 1, $\psi_{\mathscr{A}}(\eta;x)$ is the same as $\phi_{\mathscr{A}}(0,\eta;x)$ except that whenever $\mathfrak{F}(\cdot)$ appears in $\phi_{\mathscr{A}}(0,\eta;x)$, it is replaced by $f(\cdot)$ in $\psi_{\mathscr{A}}(\eta;x)$. For $\mathscr{A} = e$, $\phi_e(0,\eta;x)$ and $\psi_e(\eta;x)$ can be obtained via the bivariate forms of (2.3.6) and (3.3.8) respectively; the resulting formulae do not appear to simplify considerably.

Using (4.3.5) to (4.3.7), one can obtain a number of the counting and interval properties of the process; using (4.3.2), (4.3.3) and Theorem 3.5.1, one can obtain the asymptotic counting properties. Then, assuming that $\mu_2 < \infty$ and $\kappa_2 < \infty$, we can use the results of Sections 3.4

and 3.5 to obtain some interesting results. Some of those which follow require results given in Appendices 1 and 2. We also require the following definition given by Barlow and Proschan (1975): $G(=1-\mathcal{G})$ is said to be New Worse than Used (NWU) if for non-negative x and y

$$\mathcal{G}(x+y) \geq \mathcal{G}(x)\mathcal{G}(y) \ . \tag{4.3.8}$$

If the inequality is reversed, G is said to be New Better than Used (NBU). For some discussion of NWU and NBU distributions and their relations with other types of distributions, see Appendix 2. We have

<u>Theorem 4.3.1</u>: (a) If $h(x)$ is non-increasing or (b) G is NWU, then for all $x \geq 0$, $\mathrm{var}\{N^{(\mathcal{A})}(x)\} \geq E\{N^{(\mathcal{A})}(x)\}$, for $\mathcal{A} = 0$, 1 and e.

<u>Proof</u>: Using definitions (3.4.1) and (3.4.2), observe that if either (a) or (b) is satisfied, then (by Theorems A2.3 and A2.2 respectively) $V(x) \geq H(x)$ and $V^{(1)}(x) \geq H^{(1)}(x)$, $x \geq 0$; and that if either (a) or (b) is satisfied, then (by Theorems A1.1 and A2.1 respectively) $V^{(e)}(x) \geq H^{(e)}(x)$, $x \geq 0$. The theorem then follows from Theorem 3.4.1.

As a simple example of the last theorem consider the case where $g(x) = \theta\alpha_1 e^{-\alpha_1 x} + (1-\theta)\alpha_2 e^{-\alpha_2 x}$, $x \geq 0$; $0 < \theta < 1$; $\alpha_1, \alpha_2 > 0$ (i.e. the mixture of two exponentials). Using (4.3.2) (or its equivalent) one finds that $h'(x) = -\theta(1-\theta)(\alpha_1-\alpha_2)^2\exp\{-[(1-\theta)\alpha_1 + \theta\alpha_2]x\}$, and so $h(x)$ is decreasing; it can also be shown that G is NWU, although the details will not be given here. Hence, by Theorem 4.3.1, $\mathrm{var}\{N^{(\mathcal{A})}(x)\} \geq E\{N^{(\mathcal{A})}(x)\}$, all $x \geq 0$, for $\mathcal{A} = 0$, 1 and e. Further, by Theorem 3.5.5, $\mathrm{cov}\{M(x), N(x)\}$ is asymptotically positive. Both these results hold independently of $f(\cdot)$, the p.d.f. of the interval lengths of the process of type 0 events.

4.3.1 When is the process a bivariate Markov renewal process?

As mentioned in Section 3.1, Markov renewal (or semi-Markov) processes are r.m.p. processes. The essential property of bivariate Markov renewal processes is that the distribution of the time from either a type 0 event or a type 1 event to the next event is independent of the past history of the process. In the process of Section 4.3, this property (of independence of the past) is possessed by type 0 events; it is also possessed by type 1 events conditional on the next event also being a type 1 event. Therefore there are similarities between the process of Section 4.3 and bivariate Markov renewal processes. Hence, it is interesting to ask the question: which processes belong to both classes of bivariate point processes? This question is answered in the following theorem.

Theorem 4.3.2: The bivariate point process of Section 4.3 is a bivariate Markov renewal process if and only if the intervals of the process of type 0 events have a survivor function of the form

$$\begin{aligned} \mathfrak{F}(x) &= \mathfrak{F}_0(x), \quad x < \delta, \\ &= \mathfrak{F}_0(\delta)e^{-\rho(x-\delta)}, \quad x \geq \delta \quad (\text{some } \rho > 0), \end{aligned} \tag{4.3.9}$$

where

$$\delta = \inf_{x \geq 0}\{x: \mathfrak{G}(x) < 1\}, \tag{4.3.10}$$

and $\mathfrak{F}_0(x)$ is any survivor function with $\mathfrak{F}_0(\delta) > 0$.

Proof: Necessity: For the process discussed in Section 4.3, we have that

$$P(\text{first event after origin is of type 1 and occurs in } [x, x+dx) \mid \text{type 0 event at origin}) = \mathcal{F}(x)g(x)dx + o(dx), \tag{4.3.11}$$

$$\begin{aligned} &P(\text{first event after origin is of type 1 and occurs in } [x, x+dx), \\ &\quad \text{second event after origin is of type 0 and occurs in} \\ &\quad [x+y, x+y+dy) \mid \text{type 0 event at origin}) \\ &\qquad = f(x+y)g(x)\,\mathcal{G}(y)dxdy + o(dx \times dy), \end{aligned} \tag{4.3.12}$$

$$\begin{aligned} &P(\text{first event after } x \text{ is of type 0 and occurs in } [x+y, x+y+dy) \\ &\quad \mid \text{type 1 event at } x) = \psi_1(0;y)dy + o(dy) \quad (\text{by } (3.3.2)) \\ &\qquad = \kappa_1^{-1}\mathcal{G}(y)\int_0^\infty f(z+y)h(z)dzdy + o(dy) \end{aligned} \tag{4.3.13}$$

by (4.3.5), (4.3.7) and the following discussion.

However, if the process is a bivariate Markov renewal process, we must have the left-hand side of (4.3.12) equalling the product of the left-hand side of (4.3.11) and the left-hand side of (4.3.13), that is

$$f(x+y)g(x)\,\mathcal{G}(y) = \kappa_1^{-1}\mathcal{F}(x)g(x)\,\mathcal{G}(y)\int_0^\infty f(y+z)h(z)dz$$

or

$$\frac{f(x+y)}{\mathcal{F}(x)} = \kappa_1^{-1}\int_0^\infty f(z+y)h(z)dz \tag{4.3.14}$$

provided $g(x) > 0$ and $\mathcal{G}(y) > 0$.

The right-hand side of (4.3.14) is independent of x, and so $f(x+y)/\mathfrak{F}(x)$ must also be independent of x, provided $g(x) > 0$ and $\mathcal{G}(y) > 0$. These conditions are found to be equivalent to

$$\mathfrak{F}(x) = \mathfrak{F}_0(\delta)e^{-\rho(x-\delta)}, \; x \in [\delta, 2\varepsilon] \quad (\text{some } \rho > 0), \tag{4.3.15}$$

where

$$\varepsilon = \sup_{x \geq 0} \{x: \mathcal{G}(x) > 0\},$$

and δ is given by (4.3.10).

It is easily seen that for all bivariate Markov renewal processes, there is a positive probability of n type 1 events lying between any two successive type 0 events, for all positive integers n. By considering n-fold generalizations of the above probability relations (for each positive integer n), we find that the upper limit in (4.3.15), 2ε, must be extended to infinity, in order to satisfy the n-fold analogue of (4.3.14) and its accompanying proviso. This gives (4.3.9).

Sufficiency: By the definition of r.m.p. processes, the time from a type 0 event to the next event is independent of everything before that event. For type 1 events: given that a type 1 event has occurred (at time x, say), we know by (4.3.10) that no type 0 events have occurred in $(x-\delta, x]$. Hence, by (4.3.9), the time from x to the next type 0 event is exponentially distributed, and so "lacks a memory". Therefore, the time from a type 1 event to the next event is independent of everything before that event and so the bivariate point process is a Markov renewal process.

If a multivariate point process is a Markov renewal process then it is easily seen that each marginal point process is a renewal process. Hence, in the bivariate point process of Section 4.3, if the process of type 0 events has a survivor function of the form (4.3.9), then the process of type 1 events is a renewal process irrespective of $g(\cdot)$. Of particular interest is the case when $\underline{\delta} = 0$, in which case the distribution in (4.3.9) is exponential. The renewal process of type 1 events then has some interesting properties. These are derived in the following section.

4.3.2 Properties of the type 1 events when the type 0 events form a Poisson process

Assume that the type 0 events have an interval p.d.f. $f(x) = \rho e^{-\rho x}$, $x \geq 0$, $\rho > 0$. We will be particularly interested in relating the properties of the marginal renewal process of type 1 events to those of the imbedded renewal process. To distinguish between the imbedded renewal process (with p.d.f. $g(\cdot)$ and renewal density $h(\cdot)$) and the resulting marginal process of type 1 events, we shall use the same notation for the latter process as for the former one, except that we will use the subscript "m" (e.g. $g_m(\cdot)$ and $h_m(\cdot)$). Then using the lack of memory property of the Poisson process, and standard renewal arguments, one obtains

$$h_m(x) = e^{-\rho x}h(x) + \rho \int_0^x e^{-\rho y}h(y)dy\,, \tag{4.3.16}$$

or, taking Laplace transforms,

$$sh_m^*(s) = (\rho + s)h^*(\rho + s)\,. \tag{4.3.17}$$

If h(x) is differentiable, then, from (4.3.16)

$$h_m'(x) = e^{-\rho x} h'(x), \tag{4.3.18}$$

so that if h(x) is non-decreasing or non-increasing, $h_m(x)$ exhibits the same behaviour. This interesting fact will be used later. Using (4.3.17) and (4.3.2) (which holds for $H_m^*(s)$ and $g_m^*(s)$ also since the formula is a general one for renewal processes), we obtain the relations between p.d.f.'s and survivor functions:

$$g_m^*(s) = \frac{(\rho + s)g^*(\rho + s)}{\rho g^*(\rho + s) + s}, \tag{4.3.19}$$

$$\mathcal{G}_m^*(s) = \frac{\mathcal{G}^*(\rho + s)}{1 - \rho\mathcal{G}^*(\rho + s)}. \tag{4.3.20}$$

Then the mean and second moment of the interval length distribution of the process of type 1 events are given by

$$\begin{aligned} E(X_1) &= \frac{\mathcal{G}^*(\rho)}{1 - \rho\mathcal{G}^*(\rho)} = \frac{1 - g^*(\rho)}{\rho g^*(\rho)}, \\ E(X_1^2) &= \frac{-2d/d\rho\, \mathcal{G}^*(\rho)}{\{1 - \rho\mathcal{G}^*(\rho)\}^2}. \end{aligned} \tag{4.3.21}$$

In Theorem 4.3.1, we obtained conditions under which the process of type 1 events is overdispersed. With the Poisson assumption for the process of type 0 events, we can obtain analogous conditions under which the process of type 1 events is underdispersed. The results are given in the next two theorems.

<u>Theorem 4.3.3</u>: If h(x) is differentiable and non-decreasing, then for all $x \geq 0$, $\mathrm{var}\{N^{(\mathcal{A})}(x)\} \leq E\{N^{(\mathcal{A})}(x)\}$ for $\mathcal{A} = 1$ and e.

Proof: By (4.3.18), if $h(x)$ is non-decreasing, so is $h_m(x)$. But $h_m(x)$ is the renewal density of a renewal process. The result then follows immediately from Theorems A1.1 and A2.3.

Theorem 4.3.4: If G is NBU, then

$$\lim_{x\to\infty} \frac{\operatorname{var}\{N(x)\}}{E\{N(x)\}} \leq 1 .$$

Proof: As the process of type 1 events is a renewal process, it is well known that $\lim_{x\to\infty}\{\operatorname{var} N(x)\}/E\{N(x)\} = \operatorname{var}(X_1)/\{E(X_1)\}^2$. Hence the result to be proved is equivalent to $E(X_1{}^2) \leq 2\{E(X_1)\}^2$. Consider now two independent non-negative random variables, U and V, with respective survivor functions $e^{-\rho u}$ and $\mathcal{G}(v)$. Let $W = \min(U,V)$. Then $P(W > w) = e^{-\rho w}\mathcal{G}(w)$. Further

$$\begin{aligned} E(W) &= \int_0^\infty e^{-\rho w}\mathcal{G}(w)dw = E(X_1)\{1 - \rho\mathcal{G}^*(\rho)\}, \\ E(W^2) &= 2\int_0^\infty we^{-\rho w}\mathcal{G}(w)dw = E(X_1{}^2)\{1 - \rho\mathcal{G}^*(\rho)\}^2 \end{aligned} \tag{4.3.22}$$

by (4.3.21). Now, by (4.3.8), G is NBU implies $e^{-\rho(x+y)}\mathcal{G}(x+y) \leq e^{-\rho x}\mathcal{G}(x).e^{-\rho y}\mathcal{G}(y)$, i.e. $P(W > x+y) \leq P(W > x).P(W > y)$ and therefore W also has an NBU distribution. Hence, using Theorem A2.2 on the renewal process whose interval distribution is the same as that of W, and letting $x \to \infty$, we obtain $\operatorname{var}(W)/\{E(W)\}^2 \leq 1$, or $E(W^2) \leq 2\{E(W)\}^2$. The result of the theorem then follows by applying (4.3.22).

We briefly consider two examples. First, let $g(x) = \alpha^2 xe^{-\alpha x}$, $x \geq 0$, $\alpha > 0$. This is the p.d.f. of a Gamma $(2,\alpha)$ variable. Using (4.3.2), it can be shown that $h(x) = \frac{1}{2}\alpha(1 - e^{-2\alpha x})$ which is an increasing function of x; Barlow and Proschan (1975, Section 3.5) show that the Gamma $(2,\alpha)$ distribution is also NBU. Hence, by Theorem 4.3.3, $\mathrm{var}\{N^{(\mathcal{A})}(x)\} \leq E\{N^{(\mathcal{A})}(x)\}$ for $\mathcal{A} = 1$ and e. Using (4.3.17) and (4.3.19) it can also be shown that

$$g_m(x) = \frac{\alpha^2\{e^{-Bx} - e^{-Ax}\}}{A - B},$$

where

$$A, B = \alpha + \frac{1}{2}\rho \pm \frac{1}{2}(\rho^2 + 4\alpha\rho)^{1/2},$$

and

$$h_m(x) = \frac{\alpha^2}{(\rho + 2\alpha)}\{1 - e^{-(\rho+2\alpha)x}\}.$$

As a second example, we return to the example considered just before Section 4.3.1, i.e. with $g(x) = \theta\alpha_1 e^{-\alpha_1 x} + (1 - \theta)\alpha_2 e^{-\alpha_2 x}$, $x \geq 0$; $0 < \theta < 1$; $\alpha_1, \alpha_2 > 0$. It was shown that, for this process, $\mathrm{var}\{N^{(\mathcal{A})}(x)\} \geq E\{N^{(\mathcal{A})}(x)\}$, all $x \geq 0$, for $\mathcal{A} = 0$, 1 and e. With the added assumption of $f(x) = \rho e^{-\rho x}$, we find by (4.3.19) that

$$g_m(x) = \omega\xi_1 e^{-\xi_1 x} + (1 - \omega)\xi_2 e^{-\xi_2 x}, \tag{4.3.23}$$

where

$$\xi_1, \xi_2 = \frac{1}{2}\{\alpha_1 + \alpha_2 + \rho \pm \Delta\} \qquad (\xi_1 > \xi_2),$$

$$\omega = \frac{\frac{1}{2}\{(2\theta - 1)(\alpha_1 - \alpha_2) - \rho + \Delta\}}{\Delta}, \tag{4.3.24}$$

$$\Delta = [(\alpha_1 - \alpha_2)^2 + \rho^2 + 2\rho(1 - 2\theta)(\alpha_1 - \alpha_2)]^{1/2}.$$

It can be shown after some manipulation that ξ_1, $\xi_2 > 0$ and $0 < \omega < 1$. Hence, by imbedding a renewal process whose interval length distribution is a mixture of exponentials between the type 0 events, the resulting marginal renewal process of type 1 events has an interval length distribution which is still a mixture of exponentials. Using (4.3.23), it is easily found that

$$E(X_1) \equiv \mu_X = \frac{\omega}{\xi_1} + \frac{(1-\omega)}{\xi_2}, \quad E(X_1^{\,2}) = 2\left\{\frac{\omega}{\xi_1^{\,2}} + \frac{(1-\omega)}{\xi_2^{\,2}}\right\}. \tag{4.3.25}$$

Using (4.3.23) in (4.3.2), one obtains

$$H_m(x) \equiv E\{N^{(1)}(x)\} = \frac{x}{\mu_X} + \frac{\omega(1-\omega)(\xi_1-\xi_2)^2\{1-\exp(-\xi_1\xi_2 x)\}.}{(\xi_1\xi_2\mu_X)^2} \tag{4.3.26}$$

Note the structural similarity of (4.3.26) to (4.2.9). The interval lengths of the process are of course uncorrelated. A numerical example of the above results is given in Section 5.3.2 in order to show the similarity of this process to the process whose main properties are given at the end of Section 4.2. In that process, the type 0 events form a Poisson process of rate ρ; between the type 0 events is imbedded an inhomogeneous Poisson process with rate function $\lambda(x) = \gamma e^{-\rho x}$.

4.4 Selective interaction of a Poisson process and a renewal process

Ten Hoopen and Reuver (1965) have studied the following process: the type 0 events form a renewal process with interval length p.d.f. $f(\cdot)$. The type 1 events form a Poisson (λ) process except that the first event of the Poisson process after each type 0 event is deleted; if no Poisson events occur between successive type 0 events, then the process is unaltered. It is easily seen that the type 1 events satisfy the conditions necessary for the bivariate process to be an r.m.p. process. Ten Hoopen and Reuver derived only the Laplace transform of the p.d.f. of the interval length between successive type 1 events; Lawrance (1970, 1971) derived many more results for both the interval and counting processes. Some of the results of these authors can be obtained more simply using the general theory of r.m.p. processes, notably the Laplace transforms of both the p.g.f. of $N(x)$ (when the process is in statistical equilibrium) and the interval length p.d.f., as well as the asymptotic mean and variance of $N(x)$. The general theory also easily leads to several results not given by the above authors.

By the definition of the process it is easily seen that

$$\begin{aligned} p_j(x) &\equiv P\{N(t, x+t] = j \mid \text{successive type 0 events at } t \text{ and } x+t\} \\ &= e^{-\lambda x}(1+\lambda x), \quad j = 0, \qquad (4.4.1) \\ &= e^{-\lambda x}\frac{(\lambda x)^{j+1}}{(j+1)!}, \quad j = 1,2,\ldots, \end{aligned}$$

whence, using the notation of (3.4.1), (3.4.2) and the following discussion

$$H(x) = e^{-\lambda x} + \lambda x - 1, \qquad (4.4.2)$$

$$H_2(x) = (\lambda x)^2 - H(x), \qquad (4.4.3)$$

$$V(x) - H(x) = 1 - 2\lambda x e^{-\lambda x} - e^{-2\lambda x} > 0, \; x \geq 0. \qquad (4.4.4)$$

Hence, by (3.4.3), (3.4.4), (3.5.1), (4.4.2) and (4.4.3)

$$\begin{aligned} \kappa_1 &= \mu_1\lambda - 1 + f^*(\lambda) \;, \\ \kappa_2 &= \mu_2\lambda^2 - \kappa_1 , \\ \omega &= \mu_2\lambda - \mu_1 - d/d\lambda\{f^*(\lambda)\}. \end{aligned} \tag{4.4.5}$$

We now derive $\phi_{\mathcal{A}}(0,\eta;x)$ for $\mathcal{A} = 0$, 1 and e. By definition of the process, the probability that there are no type 0 events and j type 1 events in $(0,x]$, given that a type 0 event occurs at the origin is just $\mathfrak{F}(x)p_j(x)$, where $p_j(x)$ is given by (4.4.1). Hence, using (3.3.1), one obtains

$$\phi_0(0,\eta;x) = \mathfrak{F}(x)\{e^{-\lambda x}(1 - \eta^{-1}) + \eta^{-1}e^{\lambda x(\eta-1)}\}. \tag{4.4.6}$$

If there is a type 1 event at the origin, we need to condition on the time of occurrence of the last type 0 event before the origin. The joint probability that the last type 0 event prior to the origin was in $[-y, -y+dy)$ and that there are no type 0 events in $(0,x]$ is easily seen to be $\mu_1^{-1}\mathfrak{F}(x+y)dy$. Conditional on these events, the joint probability of a type 1 event in $[0,dx)$ and j type 1 events in $[dx, x+dx)$ is $\{\lambda(1 - e^{-\lambda y})e^{-\lambda x}(\lambda x)^j/j!\}dx$, while the marginal probability of a type 1 event in $[0,dx)$ is just $(\kappa_1/\mu_1)dx$ (see Section 3.4). Combining these probabilities appropriately, integrating over all possible y's and using (3.3.1), one obtains

$$\phi_1(0,\eta;x) = \lambda\kappa_1^{-1}e^{\lambda x(\eta-1)} \int_0^\infty \mathfrak{F}(x+y)(1 - e^{-\lambda y})dy. \tag{4.4.7}$$

We can then derive $\phi_e(0,\eta;x)$ from (2.3.6) or first principles. It is

$$\phi_e(0,\eta;x) = \mu_1^{-1} \int_x^\infty \mathfrak{F}(y)\{e^{\lambda x(\eta-1)} + (\eta^{-1} - 1)(e^{\lambda x\eta} - 1)e^{-\lambda y}\}dy. \tag{4.4.8}$$

In each of the above cases $\psi_{\mathcal{A}}(\eta;x)$ is the same as $\phi_{\mathcal{A}}(0,\eta;x)$ except that whenever $\mathcal{F}(\cdot)$ appears in $\phi_{\mathcal{A}}(0,\eta;x)$, it is replaced by $f(\cdot)$ in $\psi_{\mathcal{A}}(\eta;x)$.

With $\phi_{\mathcal{A}}(0,\eta;x)$ and $\psi_{\mathcal{A}}(\eta;x)$, we have, in principle, the Laplace transform of the joint p.g.f. of $M(x)$ and $N(x)$, through (3.3.6), for $\mathcal{A} = 0$, 1 and e. Lawrance (1970) gives the Laplace transform of the p.g.f. of $N(x)$ when the process is in statistical equilibrium (i.e. $\mathcal{A} = e$) in his equation (2.6); we note its "structural" similarity to our equation (3.3.6), when $\zeta = 1$; in fact, Lawrance's functions α_1, α_2, β_1 and β_2 are equivalent to our ψ_0, ϕ_0, ψ_e and ϕ_e, respectively. Similarly, using (3.6.1), (4.4.6) and (4.4.7), we can find the Laplace transform of the p.d.f. of the interval length between successive type 1 events, as given by Ten Hoopen and Reuver (1965, p.290).

By substituting (4.4.5) into the appropriate parts of Theorem 3.5.1, one can obtain the asymptotic mean and variance of $N(x)$ ($\lambda_1 x$ and $\sigma_{11} x$ respectively); in particular, note that the asymptotic variance of $N(x)$ is the same as the first term in equation (3.3) of Lawrance (1970), except that his symbols differ from ours (his ν, ν_2, μ and $\psi^*(\mu)$ are replaced in our formulae by μ_1, μ_2, λ and $f^*(\lambda)$ respectively). Using the results of Chapter 3 and this section, we can obtain some of Lawrance's other results for the counting and interval processes. Assuming that $\mu_2 < \infty$ and $\kappa_2 < \infty$, we can also obtain the following additional results for the process.

<u>Theorem 4.4.1</u>: $\mathrm{var}\{N^{(\mathcal{A})}(x)\} > E\{N^{(\mathcal{A})}(x)\}$, all $x \geq 0$, for $\mathcal{A} = 0$ and 1.

<u>Proof</u>: By (4.4.4), $V(x) > H(x)$, all $x \geq 0$, and so, by a modification of Theorem 3.4.1, $\mathrm{var}\{N^{(0)}(x)\} > E\{N^{(0)}(x)\}$, all $x \geq 0$. If there is a type 1 event at the origin (i.e. $\mathcal{A} = 1$), and if there are no type 0 events in $(0,x]$, then $N(x)$ has a Poisson (λx) distribution; hence

$V^{(1)}(x) = H^{(1)}(x)$. Therefore, by (4.4.4) and a modification of Theorem 3.4.1, we have $\mathrm{var}\{N^{(1)}(x)\} > E\{N^{(1)}(x)\}$, all $x \geq 0$.

Theorem 4.4.2: (i) $\lim_{x\to\infty} \frac{\mathrm{var}\{N(x)\}}{E\{N(x)\}} > 1$.

(ii) $\sigma_{01} < 0$.

Proof: Part (i) follows by a modification of Corollary 3.5.3, as $V(x) > H(x)$, all $x \geq 0$ (by (4.4.4)). If we differentiate (4.4.2), we find that $H''(x) = \lambda^2 e^{-\lambda x}$ which is positive for all $x \geq 0$. Part (ii) then follows from a modification of Theorem 3.5.5.

An immediate corollary of the above theorems is that the type 1 events can never form a Poisson process. The results of the above theorems are intuitively reasonable as the occurrence of type 0 events causes the deletion of some events of a Poisson process; this would increase the variance of $N(x)$ relative to its mean; also, the greater the number of type 0 events in $(0,x]$, the greater the number of deletions from the Poisson process, which suggests that $M(x)$ and $N(x)$ ought to be negatively correlated.

CHAPTER 5: TESTS FOR DISCRIMINATION BETWEEN TWO CLASSES OF REGENERATIVE BIVARIATE POINT PROCESSES

5.1 Introduction

Discrimination between alternative point process models by the analysis of empirical data is notoriously difficult. Although no thorough study will be attempted, this chapter illustrates some of these difficulties by considering two alternative classes of regenerative bivariate point processes. In the first class, the process of type 1 events is formed by imbedding a Poisson process, usually inhomogeneous, between the type 0 events. In the second, the process of type 1 events is formed by imbedding an ordinary renewal process between the type 0 events. Full details concerning the two classes are given in Sections 4.2 and 4.3 respectively. Note that if the imbedded Poisson process were homogeneous, or the ordinary renewal process were a Poisson process, the two models would be identical and the whole process of type 1 events would be a Poisson process.

There are two distinct forms in which the data may be available. In the first, both types of events are observed. Assume that the type 0 and type 1 events can be distinguished from each other. Then, conditionally on the time of occurrence of the type 0 events, we have so far as the type 1 events are concerned, a series of independent realizations of variable length to be examined for consistency with the above models. In the second and clearly more difficult situation, the type 0 events are not observed. These two situations are discussed separately, in Sections 5.2 and 5.3 respectively.

In both situations, it appears very difficult to obtain reasonably powerful general tests for discriminating between the two classes of point processes. Therefore, in both Sections 5.2 and 5.3, emphasis is placed on discrimination in the presence of suitable restrictions on one or other of the two classes. In Section 5.2, it is assumed that the imbedded Poisson process has a monotone trend; in Section 5.3, it is assumed that the imbedded renewal process has local behaviour of a particular form.

5.2 Type 0 events observed

5.2.1 Introduction

This section is concerned with discriminating between the two classes of regenerative bivariate point processes mentioned in Section 5.1 when both the type 0 and type 1 events are observed. It is assumed that the type 0 events can be distinguished from the type 1 events. Then, by the theory of r.m.p. processes, the process of type 1 events can be broken up into a series of independent realizations of the same "mechanism". Hence, simple generalizations of well-known tests for Poisson and/or renewal processes can be used to discriminate between the two classes. Such tests are thoroughly discussed by Cox and Lewis (1966, Chapters 3 and 6), and there seems little point in most cases in giving the details of the relevant generalizations here, because they are mostly very simple extensions of the original argument. Therefore, in this section, there is only a brief outline of some methods of discrimination under several different restrictions on the classes of processes involved. In Section 5.2.2, the only assumption is that the imbedded process is either an inhomogeneous Poisson process or that it is a renewal process. In Section 5.2.3, it is assumed that the inhomogeneous Poisson process has a monotone rate function. In Section 5.2.4, it is assumed that the inhomogeneous Poisson process has a monotone rate function of a specific parametric form, and that the renewal process is a homogeneous Poisson process.

The following notation and assumptions will be required. Assume that the type 0 and type 1 events are observed in the interval $[0,T)$, that, for simplicity, there is a type 0 event at the origin and that there are $M - 1$ type 0 events in $(0,T)$ $(M \geq 1)$. Denote the time of the ith type 0 event by x_i $(i = 1,2,\ldots,M-1)$. For notational convenience, define $x_0 \equiv 0$ and $x_M \equiv T$. Let n_i be the number of type 1 events in $[x_{i-1}, x_i)$, and let y_{ij} be the occurrence times of these events.

5.2.2 Some general aspects of model discrimination

The problem of discriminating between a renewal process and an inhomogeneous Poisson process is rather difficult if no extra assumptions are made about the latter process. It would appear natural to use the intervals between successive events because, under the renewal hypothesis, they are independent and identically distributed. However, the class of inhomogeneous Poisson processes contains processes whose intervals behave in a large variety of ways. It is difficult to incorporate this variety into one model for the intervals. In such a case it appears preferable to consider counts in preassigned intervals, especially when the expected number of type 1 events between successive type 0 events is reasonably large. One method of utilizing counts is via a generalization of the well-known dispersion statistic.

Before this statistic is given, some further notation is required. For notational ease, assume that the interval lengths between successive type 0 events, $z_i = x_i - x_{i-1}$ $(i = 1,\ldots,M)$, decrease in size as i increases. There is no loss of generality in this assumption, because the type 0 events are observed. Let $(W_j,\ j = 0,\ldots,L)$ be an arbitrary set of increasing constants with $W_0 \equiv 0$ and $L = \sup(j: W_j \leq z_1)$. Let $r_{ij} = N[x_{i-1} + W_{j-1}, x_{i-1} + W_j)$. Then a possible statistic for testing the general hypothesis that the imbedded process is an inhomogeneous Poisson process is

$$D = \sum_{j=1}^{L} \sum_{i=1}^{\ell_j} \frac{(r_{ij} - \bar{r}_j)^2}{\bar{r}_j}, \tag{5.2.1}$$

where ℓ_j is the number of intervals for which $W_j \leq z_i$, and $\bar{r}_j = \Sigma_i r_{ij}/\ell_j$.

If the imbedded process is an inhomogeneous Poisson process with rate function $\lambda(\cdot)$, and if $\Lambda(x) = \int_0^x \lambda(y)dy$, then

$$r_{ij} \sim \text{Poisson}\{\Lambda(W_j - W_{j-1})\}. \tag{5.2.2}$$

In particular, the parameter of the Poisson distribution in (5.2.2) is independent of i. Further, if $i \neq i'$, r_{ij} and $r_{i'j}$ are independent of each other. Hence, provided the $\bar{r}_j$'s are not too small, it follows that D is well approximated by a chi-squared distribution with $q = \Sigma(\ell_j - 1)$ degrees of freedom. Rao and Chakravarti (1956) mention that the chi-squared approximation will generally be satisfactory if $\bar{r}_j$ is greater than 3. Consequently, the W_j's should be chosen to satisfy this criterion. However, this will inevitably lead to the discarding of the end bits of some intervals since the z_i's are usually variable. In many cases, it ought to be possible to choose the W_j's astutely, so that the above criterion is satisfied and also so that only an insignificant proportion of the data is discarded.

Suppose now that the imbedded process is a renewal process with finite interval mean and variance denoted by ν and τ^2, respectively. Let $C = \tau/\nu$. Some calculations concerning the asymptotic power of the dispersion test have been made; the details will not be given. The general conclusion is that the asymptotic power of the test is an increasing function both of $|C - 1|$ and q, the degrees of freedom of the test. However, the power of the test is low in comparison to the tests of the following two sections, where more specific assumptions are made.

5.2.3 Testing for trend

It is now assumed that the imbedded inhomogeneous Poisson process has a trend, that is an increasing or decreasing rate function. For the moment, no assumptions are made about the form of $G(\cdot)$, the distribution function of the intervals under the alternative hypothesis of an imbedded renewal process. In this case, perhaps the most obvious approach is to regress the lengths of the intervals between successive type 1 events (which lie in the same type 0 interval) on some other independent variable. Let $V_{ij} = y_{ij} - y_{i,j-1}$ $(j = 1,\ldots,n_i;$ $i = 1,\ldots,M)$ be these interval lengths (where $y_{i0} \equiv x_{i-1}$) and let ξ_{ij} be the independent variable. Usually $\xi_{ij} = j$, $y_{i,j-1}$ or $\frac{1}{2}(y_{i,j-1} + y_{i,j})$. Typically, a model of the following form is assumed:

$$\Psi(V_{ij}) = \beta_0 + \beta_1\xi_{ij} + \varepsilon_{ij}, \tag{5.2.3}$$

where $\Psi(V)$ is some monotonic function of V, and ε_{ij} is the error term. Often $\Psi(V) = \log V$ since this ensures that V is always positive. Because under the renewal hypothesis, the interval lengths are independent and identically distributed, the ε_{ij}'s are usually assumed to be independent with common variance under both hypotheses. These assumptions are certainly not true under the hypothesis of a Poisson process with a trend. If normality approximations are made, testing for trend (i.e. $\beta_1 = 0$) leads to standard t tests. Elaborations and variations of regression analysis are given in Cox and Lewis (1966, Sections 3.2(i) and 3.3(ii)); they often make Poisson assumptions, but many of their arguments still hold approximately without this assumption.

Note that in the regression model (5.2.3), the "censored" intervals at the end of each of the M independent series (i.e. $x_i - y_{i,n_i}$) have been discarded. If the expected number of type 1 events between successive type 0 events is reasonably large, very little is lost by doing this. However, if this is not the case, the censored intervals should be incorporated into the analysis via methods commonly used for censored data. The details will not be given here.

If some assumptions about the form of $G(\cdot)$ are made, more specific techniques are sometimes available (Cox, 1972). In Section 5.2.4, this is illustrated with the assumption that $G(x) = 1 - e^{-\lambda x}$, $x \geq 0$ (i.e. the imbedded process is a homogeneous Poisson process).

5.2.4 Trend analysis for Poisson processes

It is assumed that between each successive pair of type 0 events, the same, possibly inhomogeneous, Poisson process is imbedded with common rate function $\lambda(u) = \exp(\alpha + \beta u)$. We wish to test whether the imbedded process has a trend (i.e. $\beta = 0$). Cox and Lewis (1966, Section 3.3(i)) discuss this problem in detail for the case $M = 1$ (i.e. a type 0 event at the origin and no type 0 events in $(0,T)$), while Cox (1972, Section 4) discusses the case $M > 1$. A test statistic is easily obtained for these very specific hypotheses. Power approximations are also straightforward, and are the main concern of the present section. For this reason, the derivation of the test statistic will be outlined briefly and some of its properties obtained under the null and alternative hypotheses. Let $u_{ij} = y_{ij} - x_{i-1}$ be the times of the type 1 events as measured from the previous type 0 event. Then the likelihood of the data is easily found to be

$$L(\underset{\sim}{u}) = \exp\{\alpha \sum_{i=1}^{M} n_i + \beta \sum_{i=1}^{M} \sum_{j=1}^{n_i} u_{ij} - e^{\alpha} \sum_{i=1}^{M} (e^{\beta z_i} - 1)/\beta\}, \qquad (5.2.4)$$

where $z_i = x_i - x_{i-1}$ is the length of the interval between the $(i - 1)$st and ith type 0 events in $(0,T)$ with $x_0 \equiv 0$. Let $N = \Sigma n_i$ and $S = \Sigma_{i,j} u_{ij}$. Suppose that a test is required of the null hypothesis H_0: $\beta = 0$ against the one-sided alternative H_1: $\beta < 0$, say. Since α is a nuisance parameter, the theory of similar tests states that the uniformly most powerful (UMP) similar test of the above hypothesis is obtained by considering the null distribution of S conditional on N, the minimal sufficient statistic for α when β is known.

Some comments follow regarding the conditional distribution of S under both the null and alternate hypotheses. Consider first the distribution of $s_i = \Sigma_j u_{ij}$ conditional on n_i, where the u_{ij}'s are the times of events from an inhomogeneous Poisson process with, for the moment, a general rate function $\lambda(u)$. Then it is easily shown that the u_{ij}'s, conditional on n_i, are the n_i order statistics of a random sample from the distribution with density $g(u) = \lambda(u) \Big/ \int_0^{z_i} \lambda(v)dv$ $(0 \leq u < z_i)$. Therefore s_i is the sum of n_i independent and identically distributed random variables (i.i.d.r.v.'s) with p.d.f. $g(u)$. Now, because of the Poisson nature of the process, it is easily seen that, conditional on N, the vector $\underset{\sim}{n} = (n_1,\ldots,n_M)$ has a multinomial distribution with parameters N and $\underset{\sim}{p} = (p_1,\ldots,p_M)$, where $p_i = \theta_0(z_i)/\Sigma_j\theta_0(z_j)$ $(i = 1,\ldots,M)$, and

$$\theta_j(z) = \int_0^{z} u^j \lambda(u)du. \qquad (5.2.5)$$

Using these two conditional distributions, it is easily shown that S conditional on N is asymptotically normally distributed with mean and variance

$$E(S|N) = N \frac{\sum_{i=1}^{M} \theta_1(z_i)}{\sum_{i=1}^{M} \theta_0(z_i)}, \quad \mathrm{var}(S|N) = N \frac{\sum_{i=1}^{M} \theta_2(z_i)}{\sum_{i=1}^{M} \theta_0(z_i)} - \frac{\{E(S|N)\}^2}{N}. \tag{5.2.6}$$

In particular, under the null hypothesis (i.e. $\lambda(u) = e^{\alpha}$),

$$E_0 \equiv E(S|N, \beta = 0) = N \frac{\sum_{i=1}^{m} z_i^{\,2}}{2T},$$

$$V_0 \equiv \mathrm{var}(S|N, \beta = 0) = N \left\{ \frac{\sum_{i=1}^{M} z_i^{\,3}}{3T} - \left(\frac{\sum_{i=1}^{M} z_i^{\,2}}{2T} \right)^2 \right\}. \tag{5.2.7}$$

Therefore, the test statistic is

$$Z = \frac{S - E_0}{V_0^{\,1/2}}. \tag{5.2.8}$$

Under the null hypothesis, the distribution of Z should tend fairly rapidly to the standard normal form, because, conditional on n_i at least, s_i is the sum of n_i i.i.d.r.v.'s from a uniform distribution on $[0,z_i)$.

Under the alternative hypothesis (i.e. $\lambda(u) = \exp(\alpha + \beta u)$), it follows from (5.2.6) that

$$E_\beta \equiv E(S|N,\beta) = \frac{N \sum_{i=1}^{M} \{1 + e^{\beta z_i}(\beta z_i - 1)\}}{\beta \sum_{i=1}^{M} (e^{\beta z_i} - 1)}, \tag{5.2.9}$$

$$V_\beta \equiv \mathrm{var}(S|N,\beta) = \frac{2N\sum_{i=1}^{M}\{e^{\beta z_i}(\frac{1}{2}\beta^2 z_i^2 - \beta z_i + 1) - 1\}}{\beta^2 \sum_{i=1}^{M}(e^{\beta z_i} - 1)} - \frac{E_\beta^2}{N} .$$

Hence, if we have a one-sided alternative H_1: $\beta < 0$, and we use the asymptotic test: reject H_0: $\beta = 0$ if $Z < -Z_\gamma$, where Z_γ is the upper γ point of the standard normal distribution, then the asymptotic power of this test is

$$P_\beta \overset{\cdot}{\sim} \Phi\left(\frac{E_0 - E_\beta - Z_\gamma V_0^{1/2}}{V_\beta^{1/2}}\right) , \tag{5.2.10}$$

as $N \to \infty$, where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution.

The results of this section are now applied to two artificial sets of data. These data sets are numbered (i) and (ii) in Appendix 4, where full details concerning the data sets can be found. The imbedded process of data set (i) is an inhomogeneous Poisson process with rate function $\lambda(u) = 10e^{-u}$ (i.e. $\alpha = \log 10$, $\beta = -1$); the imbedded process of data set (ii) is a homogeneous Poisson process. For both sets the period of observation, T, is 20, while M (the number of type 0 events in [0,T)) is 14. For data set (i), N = 94, while for data set (ii), N = 100. For data set (i), Z = -4.85, which indicates strong rejection of the null hypothesis. This is not surprising, since the power of a .05 lower-tail test is, for $\beta = -1$, approximately $\Phi(4.8) \doteqdot 1.00$, by (5.2.10). For data set (ii), a value of Z = 1.21 was obtained, which indicates acceptance of the null hypothesis.

Finally, note that the model which has been used in this section is an unusual one. Because of the regenerative nature of the process, there are M independent Poisson processes each with the same rate function $\lambda(u) = \exp(\alpha + \beta u)$. A more common situation might be M independent

Poisson processes, the ith process having a rate function $\lambda(u) = \exp(\alpha_i + \beta u)$, where β is common to all M processes, and the α_i's are unknown and possibly different. In this case, $\underset{\sim}{\alpha} = (\alpha_1, \ldots, \alpha_M)$ are the nuisance parameters when β is the parameter of interest. The theory of similar tests then leads one to the conclusion that the appropriate statistic on which to condition is $\underset{\sim}{n}$, and not N. It is therefore interesting to consider the effect of conditioning on $\underset{\sim}{n}$ when all the α_i's are equal, and conversely the effect of conditioning on N when the α_i's are not all equal. This question is considered in greater generality in Chapter 6.

5.3 Type 0 events unobserved

5.3.1 Introduction

This section is concerned with discriminating between the two classes of regenerative bivariate point processes mentioned in Section 5.1 when the type 0 events are unobserved. This is a very difficult problem, even when the sample sizes are moderately large (say about 100 type 1 events). Some of the problems of discrimination are illustrated in Section 5.3.2 where a specific example of each of the two classes of processes is considered and the properties of their type 1 processes are compared. Discrimination may be made easier by restricting the behaviour of one or other of the two classes in a suitable manner. One possible form of restriction is on the local behaviour of the process of type 1 events of one of the two classes. In the present section, the examination of the local behaviour of the two classes of processes leads to a test for discriminating between the two classes under suitable restrictions. The test is applied to four artificial sets of data.

5.3.2 A comparison of two stationary univariate point processes

In Table 5.1 below, some of the marginal properties of the process of type 1 events are given for two quite different regenerative bivariate point processes.

Process (a) is a numerical case of the example at the end of Section 4.2: the type 0 events form a Poisson process of rate ρ; the type 1 events are generated by imbedding, between successive type 0 events, an inhomogeneous Poisson process with rate function $\lambda(x) = \lambda e^{-\rho x}$. The particular parameter values used are $\rho = 1$, $\lambda = 10$. The results given in Table 5.1 are based on equations (4.2.9) to (4.2.14).

Process (b) is a numerical case of the example at the end of Section 4.3: the type 0 events form a Poisson process of rate ρ, the type 1 events are generated by imbedding, between successive type 0 events, an ordinary renewal process with p.d.f. $g(x) = \theta\alpha_1 e^{-\alpha_1 x} + (1 - \theta)\alpha_2 e^{-\alpha_2 x}$. The particular parameter values used are $\rho = 1$; $\theta = \frac{1}{2}(1 + 17/\sqrt{409})$; $\alpha_1, \alpha_2 = \frac{1}{6}(23 \pm \sqrt{409})$. The results given in Table 5.1 are based on equations (4.3.23) to (4.3.26).

In Table 5.1, the following quantities are given for the two processes: $U(x) \equiv E\{N^{(1)}(x)\}$, the expected number of type 1 events in $(0,x]$ given that a type 1 event occurs at the origin; $E(X_1)$ and $\mathrm{var}(X_1)$, the mean and variance, respectively, of the interval lengths between successive type 1 events; ρ_1 and ρ_2, the first two serial correlation coefficients of the intervals of the process of type 1 events. For the interval properties, results are given correct to 4 decimal places.

Many of the properties of the interval lengths of process (a) appear extremely difficult to determine. Hence, only the mean, variance and first two serial correlation coefficients have been given in Table 5.1.

Table 5.1 Some properties of the type 1 process for two regenerative multivariate point processes

	Process (a)	Process (b)
$U(x)$	$5\{x + \frac{1}{6}(1 - e^{-2x})\}$	$5\{x + \frac{1}{6}(1 - e^{-2x})\}$
$E(X_1)$	.2000	.2000
$\text{var}(X_1)$	.0880	.1067
ρ_1	.1009	.0000
ρ_2	.0363	.0000

The function $U(x) \equiv E\{N^{(1)}(x)\}$ will be called the synchronous mean function. For renewal processes, $U(x)$ is called the renewal function. Note, from Table 5.1, that processes (a) and (b) have identical synchronous mean functions. This implies, by (2.5.7), that the two processes also have identical asynchronous variance functions. Hence, the first and second order counting properties of the two processes are identical.

It has been shown in Section 4.3.1 that process (b) is a renewal process and hence all its serial correlation coefficients are zero. The first two serial correlation coefficients of process (a) are, to two decimal places, .10 and .04 respectively. Serial correlation coefficients of lag greater than 2 have not been given, because they are difficult to determine theoretically; however, they are undoubtedly even smaller. For a sample size of 100 type 1 events, it appears difficult to distinguish process (a) from a renewal process. For instance, if $\tilde{\rho}_j$ is the sample serial correlation coefficient of lag j and if the process is a renewal process, it is known that $\tilde{\rho}_j\sqrt{N - j}$ has asymptotically a standard normal distribution as N, the sample size, approaches infinity. A test based on $\tilde{\rho}_1$ will not be very powerful when $\rho_1 = .10$ and when the sample size is about 100. Other more general tests for renewal processes, such as those based on the spectrum of intervals (see Cox and Lewis, 1966, Section 6.4), will only be appreciably more powerful if the serial correlation coefficients die away very slowly; this appears unlikely in the case of process (a), since the type 0 events form a Poisson process.

The ratio of the two interval variances is .1067/.0880 = 1.21. Since this is so close to one, it would be difficult to distinguish between the variances of the two processes with sample sizes of about 100.

Thus, it is difficult to distinguish between processes (a) and (b) when the sample size is about 100. It is conceivable that there might be other processes belonging to the classes discussed in Sections 4.2 and 4.3 which are difficult to distinguish with moderately large sample sizes. The implication of this is that one or other of the two classes should be restricted in a manner which makes their discrimination somewhat easier. One possible restriction is on the local behaviour of one of the two classes.

5.3.3 The local behaviour of some stationary univariate point processes

This section is concerned with the local behaviour of two functions useful in stationary point processes: the distribution function of the interval lengths which will be denoted by $B(x)$, and the synchronous mean function, defined in Section 5.3.2 and denoted by $U(x)$. It is easily shown that

$$U(x) = B(x) + \sum_{n=2}^{\infty} \Pr\{N^{(1)}(x) \geq n\}. \tag{5.3.1}$$

For many point processes with little long term dependence, it can be shown that the series on the right-hand side of (5.3.1) is negligible compared with $B(x)$ when x is small. Hence, for such processes, $U(x) \doteqdot B(x)$. In particular, if $B(x) = \beta x^{\gamma} + o(x^{\gamma})$ as $x \to 0$ ($\beta > 0$, $\gamma > 0$), then, for many processes, $U(x) = \beta x^{\gamma} + o(x^{\gamma})$ as $x \to 0$, and vice versa.

It can be shown that for each of the two classes of processes being considered in this chapter, $U(x)$ exhibits the same local behaviour as $B(x)$. The details, however, are somewhat involved and will not be given here. For both classes, the local behaviour of only one of $U(x)$ and $B(x)$ will be derived.

(a) Doubly stochastic Poisson processes

As was pointed out in Section 4.2, the process of type 1 events discussed in that section (i.e. an inhomogeneous Poisson processes imbedded between the type 0 events) is a doubly stochastic Poisson (d.s.P.) process. In the present section, the local behaviour of stationary d.s.P. processes is investigated. Apart from the general assumptions of Chapter 2, the following definitions and assumptions are made. Let $\nu(\cdot)$ be the stationary stochastic rate function of the d.s.P. process. Let ν, σ^2 and $\rho(x)$ be, respectively, the mean, variance and autocorrelation function of $\nu(\cdot)$; both ν and σ^2 are assumed finite. Suppose that $\rho(x)$ is continuous at and in the neighbourhood of the origin, so that $\rho(x) = 1 - o(1)$ as $x \to 0$. Cox and Lewis (1966, Section 7.2, equation 3) give, for stationary d.s.P. processes, the following equation for $V'(x) \equiv d|dx[\mathrm{var}\{N^{(e)}(x)\}]$:

$$V'(x) = \nu + 2\sigma^2 \int_0^x \rho(u)du. \tag{5.3.2}$$

Differentiating (2.5.7) and equating it to (5.3.2), one obtains

$$U(x) = \nu x + \frac{\sigma^2}{\nu} \int_0^x \rho(u)du. \tag{5.3.3}$$

Equation (5.3.3) has two immediate consequences. First, because of the assumed behaviour of $\rho(u)$ about zero, we have

$$U(x) = (\nu + \frac{\sigma^2}{\nu})x + o(x) = \frac{E\{\nu^2(u)\}}{E\{\nu(u)\}} x + o(x) \tag{5.3.4}$$

as $x \to 0$, that is $U(x) = \beta x^\gamma + o(x^\gamma)$ as $x \to 0$, where $\gamma = 1$ and $\beta \geq \nu$, the equality holding if and only if $\sigma^2 = 0$, i.e. the process is a homogeneous Poisson process. The second consequence, which will not be used but which is interesting nevertheless, is obtained by differentiating (5.3.3). Since $U(x)$ is a non-decreasing function of x, its derivative is non-negative. This leads to the inequality

$$\rho(x) \geq -\nu^2/\sigma^2 \tag{5.3.5}$$

for all x. Equation (5.3.5) is vacuous unless $\sigma > \nu$.

For completeness, we use (5.3.4) to derive the local behaviour of the process of type 1 events in Section 4.2. Using the notation of that section, we have

$$\begin{aligned} \Pr\{\nu(x) = \lambda(y)\} &= \Pr\{\text{last type 0 event prior to } x \text{ was in } [x-y, x-y+dy)\} \\ &= \mu_1^{-1} \mathcal{F}(y)dy + o(dy). \end{aligned}$$

Therefore

$$E(\nu^i(x)) \;=\; \mu_1^{-1} \int_0^\infty \mathfrak{F}(y)\lambda^i(y)dy,$$

and hence, by (5.3.4)

$$U(x) \;=\; \frac{\int_0^\infty \mathfrak{F}(y)\lambda^2(y)dy}{\int_0^\infty \mathfrak{F}(y)\lambda(y)dy}\; x + o(x) \tag{5.3.6}$$

as $x \to 0$. It can be shown, more tediously, that, for the class of processes considered in Section 4.2, $B(x)$ exhibits the same local behaviour as $U(x)$.

Equation (5.3.4) allows us, in principle, to distinguish stationary d.s.P. processes from those point processes whose local behaviour is $U(x) = \beta x^\gamma + o(x^\gamma)$ as $x \to 0$ with either (i) $\gamma \neq 1$ or (ii) $\gamma = 1$ and $\beta < \nu$, where ν is the mean occurrence rate of the process (i.e. the reciprocal of the mean interval length). Process (a) in Section 5.3.2 is an example of the class of processes considered in Section 4.2, and hence satisfies (5.3.4) and (5.3.6). This can be verified from the first line of Table 5.1, where $U(x) \equiv E\{N^{(1)}(x)\}$ is given. However, Table 5.1 also shows that processes (a) and (b) in Section 5.3.2 have identical synchronous mean functions, $U(x)$, and hence the local behaviour of $U(x)$ (and also $B(x)$) is identical for the two processes. Therefore, the two processes cannot be distinguished by their local behaviour. However, as we are about to see, there are many processes in the class considered in Section 4.3 which can be distinguished from stationary d.s.P. processes by applying criteria (i) and (ii) above.

(b) The regenerative bivariate point process with an imbedded renewal process

In this section, the local behaviour of $B(x)$ and $U(x)$ is derived for the process of type 1 events considered in Section 4.3. In this process, the type 0 events form a renewal process with distribution function $F(\cdot)$. The process of type 1 events is formed by imbedding between successive type 0 events an ordinary renewal process with distribution function $G(\cdot)$ and p.d.f. $g(\cdot)$. There are a number of ways of determining the local behaviour of $B(x)$ and $U(x)$. For instance, one can differentiate (3.3.6) to obtain the Laplace transform of $U(x)$, expand as $s \to \infty$, and invert the transform to find the behaviour of $U(x)$ as $x \to 0$. Such an approach however may be very complicated. A simpler heuristic approach is given here.

Suppose that $G(x) = \beta x^{\gamma} + o(x^{\gamma})$ as $x \to 0$ $(\beta, \gamma > 0)$. Now, it is easily seen that the process considered in Section 4.3 is strongly orderly, and hence a type 0 event and a type 1 event cannot occur simultaneously. Therefore, as $x \to 0$,

$$
\begin{aligned}
B(x) &= P(\text{at least one type 1 event in } (0,x] \mid \text{a type 1 event at } 0) \\
&\sim P(\text{at least one type 1 event and no type 0 events in } (0,x] \mid \text{a type 1 event at } 0) \\
&= P(\text{at least one type 1 event in } (0,x] \mid \text{no type 0 events in } (0,x], \text{ a type 1 event at } 0) \\
&\quad \times P(\text{no type 0 events in } (0,x] \mid \text{a type 1 event at } 0).
\end{aligned}
$$

The first probability in the above product is just $G(x) = \beta x^{\gamma} + o(x^{\gamma})$; the second probability is easily shown to be $1 - o(1)$. Hence multiplying the two probabilities, one obtains

$$B(x) = \beta x^{\gamma} + o(x^{\gamma}) \tag{5.3.7}$$

as $x \to 0$. Note that this is the same as the local behaviour of $G(x)$ and is independent of $F(\cdot)$, the distribution function of the intervals between successive type 1 events. It can be shown also that $U(x) = \beta x^\gamma + o(x^\gamma)$ as $x \to 0$.

Equations (5.3.6) and (5.3.7) together indicate that, if the type 0 events are unobserved, one can, in principle, distinguish between the process considered in Section 4.2 and the process considered in Section 4.3 by examining the local behaviour of the two processes, provided that the process considered in Section 4.3 does not have a $G(\cdot)$ of the form $G(x) = \beta x + o(x)$ as $x \to 0$ with $\beta \geq$ (mean interval length)$^{-1}$. As we shall see in the next section, a test based on the local behaviour of $B(x)$ gives a general method for distinguishing between the two processes when the sample size is moderately large provided the constraint just mentioned is satisfied.

5.3.4 A test for discriminating between the local behaviour of some stationary univariate point processes

A test based on the local behaviour of $B(x)$ is now derived for discriminating between stationary d.s.P. processes (including the processes of Section 4.2) and those stationary point processes which satisfy the constraint mentioned at the end of Section 5.3.3. With modifications which will become obvious, the test might be used for distinguishing between other classes of processes. Assume that the stationary point process being investigated has an interval distribution function of the form $B(x) = \beta x^\gamma + o(x^\gamma)$ as $x \to 0$ and that $U(x) \sim \nu x$ as $x \to \infty$. There are two things we wish to test: first, whether $\gamma = 1$, and if so, whether $\beta \geq \nu$.

Suppose that there is some number Δ such that, for $0 \leq x \leq \Delta$, the approximation $B(x) = \beta x^\gamma$ is a reasonable one; later, we shall discuss a method of choosing Δ from the data. Let X be an interval of the stationary point process. Then, approximately, $\Pr(X < x | X < \Delta) = (x/\Delta)^{\gamma}$, independently of β, or writing $Y = \log(\Delta/X)$ given $X < \Delta$, we find that Y has approximately an exponential distribution with mean γ^{-1}. Suppose we observe a stationary point process over the interval (O,T] and observe N events in this interval (and hence N - 1 intervals between successive events). If the process is a renewal process, and if there are L intervals whose length X_j $(j = 1,\ldots,L)$ is less than Δ, then the (approximate) maximum likelihood estimator of γ is $\hat{\gamma} = L/\Sigma Y_j = \bar{Y}^{-1}$, where $Y_j = \log(\Delta/X_j)$. This is also the obvious estimator even if the process is not renewal. In fact for processes exhibiting the behaviour $B(x) = \beta x^{\gamma} + o(x^{\gamma})$ as $x \to 0$, it is easily shown that $\lim_{\Delta\to 0} E\{\log(\Delta/X) | X < \Delta\} = \gamma^{-1}$. <u>If the process is a renewal process</u>, then $2\gamma\Sigma Y_j$ has approximately a chi-squared distribution with 2L degrees of freedom. Hence, reject the hypothesis $H_{\gamma}: \gamma = 1$ at the α level if

$$\sum_{j=1}^{L} Y_j < \frac{1}{2}\chi^2_{2L,1-\alpha/2} \quad \text{or} \quad \sum_{j=1}^{L} Y_j > \frac{1}{2}\chi^2_{2L,\alpha/2}, \tag{5.3.8}$$

where $\chi^2_{2L,\alpha}$ is the upper α point of the chi-squared distribution with 2L degrees of freedom. If the process is not a renewal process, the first few sample serial correlation coefficients for the <u>transformed</u> data (i.e. the Y_j's) should be calculated. If these are not appreciable, the test (5.3.8) should still be used. If, however, there are some appreciable estimated serial correlation coefficients, these can be used to alter the variance of ΣY_j and adjust the chi-squared test accordingly; details will not be given here for the sake of brevity and because all four data sets considered later are either renewal processes or indistinguishable from them.

If one accepts the hypothesis $H : \gamma = 1$, the hypothesis H_β: $\beta \geq \nu$ must then be tested. This involves estimating both β and ν and testing their difference. The obvious estimator of ν is $\hat{\nu} = N/T$. Provided that T is reasonably large, the variance of $\hat{\nu}$ will, in general, be small compared with the variance of the estimator of β, and so $\hat{\nu}$ can be treated as being a fixed constant. This will be assumed in what follows. <u>If the process is a renewal process</u>, then, conditional on N, L is binomial with parameters N - 1 and (if the hypothesis $\gamma = 1$ is accepted) approximately $\beta\Delta$. The approximate maximum likelihood estimator of β is $\hat{\beta} = L/\{(N - 1)\Delta\}$ and one obtains an immediate test of the null hypothesis. If N is large enough, a normal approximation can be used; H_β is rejected at <u>less than</u> the α level (as under the null hypothesis the exact value of β is unspecified) if

$$\frac{L - (N-1)\hat{\nu}\Delta}{\{(N-1)\hat{\nu}\Delta(1-\hat{\nu}\Delta)\}^{1/2}} < -Z_\alpha, \tag{5.3.9}$$

where Z_α is the upper α point of the standard normal distribution.

If the process is not a renewal process, then the first few sample serial correlation coefficients for the <u>untransformed</u> X_i's should be calculated. If the first k of them are appreciable, and if we define the indicator variable W_i by $W_i = 0$ if $X_i \leq \Delta$, and $W_i = 1$ if $X_i > \Delta$, then the W_i's form approximately a two state kth order Markov chain with stationary probabilities $(\beta\Delta, 1 - \beta\Delta)$. Testing the hypothesis $H : \beta \geq \hat{\nu}$ is then equivalent to testing the stationary probabilities of a two state kth order Markov chain with unspecified transition probabilities. This problem is discussed by Billingsley (1961); the special case of the two state 1st order Markov chain is discussed extensively by Klotz (1973).

Suppose that H_γ: $\gamma = 1$ is rejected at the α_γ level, and that, conditional on accepting H_γ, H_β: $\beta \geq \nu$ is rejected at less than the α_β level. Then it is easily seen that the significance level obtained by combining the two tests into one is less than $\alpha_\gamma + (1 - \alpha_\gamma)\alpha_\beta$. However, it would be misleading to use only this significance level as a method of testing the combined hypothesis. This is because the significance level is conditional on a nested hypothesis, i.e. there is some preassigned region where H_γ: $\gamma = 1$ is rejected regardless of the estimated value of β. It seems best to carry out the two tests separately, and to make an intelligent qualitative conclusion based on their separate results.

There now remains the question of a suitable choice for Δ, the number such that, for $0 \leq x \leq \Delta$, the approximation $B(x) = \beta x^\gamma$ is a reasonable one. The equation $B(x) = \beta x^\gamma$ will usually only be an approximation to the true distribution except in the case of the power function distribution where the equation is exact. In general, the approximation becomes poorer as x increases. As the sample size increases, $B(x)$ can be estimated more precisely, and hence, in general, Δ ought to be a decreasing function of the sample size. Trying to estimate Δ by eye from a plot of the empirical distribution function, $\tilde{B}_N(x)$, gives an estimate which is much too large, as do various tests of fit that have been tried; such estimates use observations which swamp the local behaviour of the distribution. A more conservative approach which appears to work quite well is based on the following idea. Assume that $U(x)$ exhibits the same local behaviour as $B(x)$. If we have reasonable estimates of $B(x)$ and $U(x)$, respectively, for a set of data, these estimates should be equal, or almost equal, in the same region that $B(x) = U(x) = \beta x^\gamma$ is a reasonable approximation. The obvious estimator of $B(x)$ is the empirical distribution function $\tilde{B}_N(x)$. An obvious estimator of $U(x)$ also exists. Let $y_1,\ldots,y_N$ be the times of the N events observed in $(0,T]$. Let $K(x)$ ($\le N$) be the number of events such that $y_i + x \le T$ ($i = 1,\ldots,K(x)$). Then the obvious estimator of $U(x)$ is

$$\tilde{U}(x) = \frac{\sum_{i=1}^{K(x)} N(y_i, y_i + x]}{K(x)} . \tag{5.3.10}$$

Because $K(x)$ is a random variable, $\tilde{U}(x)$ is a biased estimator of $U(x)$. However, for fixed x, $\tilde{U}(x)$ will be asymptotically unbiased as $T \to \infty$ for all stationary point processes. If x is sufficiently small, then $N(y_i, y_i + x]$ will be either 0 or 1 for all i. It then follows that, in this region, $\tilde{U}(x)$ equals $\tilde{B}_{K(x)}(x)$, the empirical distribution function based on the first $K(x)$ intervals. It also follows from the definitions of $\tilde{B}_{K(x)}(x)$ and $\tilde{U}(x)$ that these two functions are equal if and only if $x < \min(X_i + X_{i+1},\ i = 1,\ldots,N-2)$. Hence, a suggested choice of Δ is $\Delta = \min(X_i + X_{i+1},\ i = 1,\ldots,N-2)$. This suggestion is undoubtedly conservative, but it is probably prudent to err on the side of conservatism. For this choice of Δ, it can be shown that, for a renewal process, $E(L) \sim \{N\pi/(4a)\}^{1/2}$, $\mathrm{var}(L) \sim N(1 - \pi/4)/a$ as $N \to \infty$, where L is the number of intervals whose length is less than Δ, and $a = \{\Gamma(\gamma + 1)\}^2/\Gamma(2\gamma + 1)$. Hence $\lim_{N\to\infty} \mathrm{var}(L)/\{E(L)\}^2 = (4/\pi) - 1$ and so the asymptotic distribution is non-normal. A heuristic derivation of the asymptotic mean and variance of L (when the process is renewal) is given in Appendix 3. As an example, if $N = 100$ and $\gamma = 1$, then the mean and standard deviation of L are approximately 12.5 and 6.6, respectively.

Even though the suggested choice of Δ is precisely defined, there should be some flexibility in its choice. For instance, there is a small probability that the jth and $(j+1)$st intervals are the two shortest intervals, in which case $\Delta = X_j + X_{j+1}$. There is also a small probability that $X_i > \Delta$, all $i \neq j$ or $j + 1$. Then $L = 2$. In such a case it might be appropriate to let $\Delta = \min(X_i + X_{i+1})_{i=1,i\neq j}^{N-2}$.

5.3.5 Results for some artificial data

The test suggested in Section 5.3.4 is now applied to four artificial sets of data. Full details concerning these data sets can be found in Appendix 4. Data sets (i) and (ii) are examples of the regenerative bivariate point process considered in Section 4.2 and so both have $\gamma = 1$ and $\beta \geq \nu =$ (mean interval length)$^{-1}$. In fact data set (ii) is a homogeneous Poisson process and hence $\beta = \nu$. Data sets (iii) and (iv) are examples of the regenerative bivariate point process considered in Section 4.3. As can be seen from Table A1 in Appendix 4, data set (iii) is generated by a process with $\gamma = \frac{1}{2}$, while data set (iv) is generated by a process with $\gamma = 2$. The sample sizes for all four data sets are between 86 and 100. Data sets (ii), (iii) and (iv) are all renewal processes; data set (i) is indistinguishable from a renewal process with its sample size of 94.

For each data set, the results of the suggested test are summarized in Table 5.2. In each case, the table gives: N, the sample size; $\tilde{\rho}_1\sqrt{N - 1}$, the standardized sample serial correlation coefficient of lag 1 (under a renewal hypothesis this has asymptotically a standard normal distribution); Δ, the number below which a power function approximation is used for the distribution function; L, the number of intervals whose length is less than Δ; the true value of γ; and the suggested estimates of γ, ν and β. For each data set, two tests were carried out: a test of H_γ: $\gamma = 1$, and (assuming $\gamma = 1$) a test of H_β: $\beta \geq \nu$. A double asterisk above the value of $\hat{\beta}$ or $\hat{\gamma}$ indicates rejection of the null hypothesis at a level less than .01, and a single asterisk indicates rejection at a level between .05 and .01.

Table 5.2 shows that the test of H_γ: $\gamma = 1$ is more powerful against alternatives H_A: $\gamma = 1 - \delta$ than alternatives H_A: $\gamma = 1 + \delta$. For instance, for $L = 20$, the acceptance region of a .05 equal-tailed test is $\hat{\gamma} \in (.68, 1.67)$. For the values $\gamma = .5, 1.5, 2.0$, the power of this test takes the values .91, .35, .82 respectively. It is fortuitous, however, that for many distributions with $\gamma > 1$, $B(x) < \nu x$ for small x; this behaviour is shown, for instance, by all Gamma and Weibull distributions with $\gamma > 1$. In such cases, $E(\hat{\beta}) = E(L/N\Delta) = B(\Delta)/\Delta < \nu$. This behaviour is opposite to that of stationary d.s.P. processes, so that even if the hypothesis H_γ: $\gamma = 1$ is accepted (when in fact $\gamma > 1$), there is a reasonable chance that the hypothesis H_β: $\beta \geq \nu$ will be rejected. This is illustrated by data set (iv); the hypothesis H_γ: $\gamma = 1$ is just accepted at the .05 level, but not at the .01 level; however, the hypothesis H_β: $\beta \geq \nu$ is decisively rejected.

On the basis of the two tests, the correct conclusion is reached for data sets (i), (iii) and (iv). For data set (ii), the homogeneous Poisson process, the hypothesis H_γ: $\gamma = 1$ is strongly accepted. However, there is some doubt about the hypothesis H_β: $\beta \geq \nu$. This is possibly due to an atypical realization of the Poisson process.

Table 5.2 Results of a test applied to four sets of data

	Data Set No.			
	(i)	(ii)	(iii)	(iv)
N	94	100	91	86
$\tilde{\rho}_1\sqrt{N-1}$	.41	-.26	-.62	.43
Δ	.05	.05	.01	.09
L	21	17	20	17
γ (true value)	1	1	.5	2
$\hat{\gamma} = \overline{Y}^{-1}$	1.06	.68	.54**	1.71
$\hat{\nu} = N/T$	4.70	5.00	4.55	4.30
$\hat{\beta} = L/(N-1)\Delta$	4.52	3.43*	22.22	2.24**

Footnote to Table 5.2: Data sets (i) and (ii) are d.s.P. processes; data sets (iii) and (iv) are not d.s.P. processes.

CHAPTER 6: THE EFFECT OF CONDITIONING ON STATISTICS WHICH ARE NOT MINIMAL SUFFICIENT

6.1 Introduction

In this chapter, a generalization of the problem mentioned at the end of Section 5.2.4 is considered, that is the problem of conditioning on statistics which are not minimal sufficient.

Suppose that M independent but not necessarily identically distributed bivariate random variables (s_i, t_i) $(i = 1, \ldots, M)$ are observed with the following exponential family distribution:

$$f(s_i, t_i) = \exp\{\alpha_i t_i + \beta s_i + a_i(s_i, t_i) + b_i(\alpha_i, \beta)\}. \tag{6.1.1}$$

Note that β is common to the M distributions, while each distribution has a possibly different α_i. Typically s_i and t_i will themselves be the sums of other, perhaps identically distributed, random variables. The ranges of s_i and t_i are unspecified but we assume them to be independent of the parameters α_i and β.

It is desired to test H_0: $\beta = \beta_0$ against H_1: $\beta < \beta_0$ (or H_1': $\beta > \beta_0$). The vector of parameters $\underset{\sim}{\alpha} = (\alpha_1, \ldots, \alpha_M)$ is then a nuisance parameter. From (6.1.1), the joint likelihood of the 2M random variables (s_i, t_i) $(i = 1, \ldots, M)$ is

$$L(\underset{\sim}{s}, \underset{\sim}{t}) = \exp\{\sum_{i=1}^{M} \alpha_i t_i + \beta \sum_{i=1}^{M} s_i + \sum_{i=1}^{M} a_i(s_i, t_i) + \sum_{i=1}^{M} b_i(\alpha_i, \beta)\}. \tag{6.1.2}$$

By the theory of similar tests, the uniformly most powerful (UMP) similar test of H_0: $\beta = \beta_0$ against H_1: $\beta < \beta_0$ is obtained by considering the null distribution of $S = \Sigma s_i$ conditional on the minimal sufficient statistic for $\underset{\sim}{\alpha}$ when β is known. If all the α_i's are taken to be unequal, then this statistic is $\underset{\sim}{t} = (t_1,\ldots,t_M)$, by (6.1.2). If, however, all the α_i's are treated as being equal then, from (6.1.2), the minimal sufficient statistic is $T = \Sigma t_i$.

In this chapter, the effect of assuming that the α_i's are unequal (and hence conditioning on $\underset{\sim}{t}$), when in fact they are equal, is investigated, as well as the effect of assuming that the α_i's are equal (and hence conditioning on T), when in fact they are unequal. It is assumed that s_i and t_i are each the sum of n_i random variables. Let $N = \Sigma n_i$. It is further assumed that, under both forms of conditioning, the conditional distribution of s is asymptotically normal as $N \to \infty$ (whether the α_i's are equal or not, and under both the null and alternative hypotheses). This assumption will allow us to obtain simple quantities for measuring the asymptotic effect of making an incorrect assumption about the α_i's.

In Section 6.2 some useful general theory is given. In Section 6.3, three examples illustrate various aspects of the general theory.

6.2 General theory

From (6.1.1), it follows that the marginal distribution of t_i and the conditional distribution of s_i given t_i are, respectively,

$$f(t_i) = \exp\{\alpha_i t_i + b_i(\alpha_i,\beta) + c_i(t_i,\beta)\}, \qquad (6.2.1)$$

$$f(s_i,t_i) = \exp\{\beta s_i + a_i(s_i,t_i) - c_i(t_i,\beta)\}, \qquad (6.2.2)$$

where

$$c_i(t_i,\beta) = \log \int \exp\{\beta s_i + a_i(s_i,t_i)\}ds_i. \qquad (6.2.3)$$

Here and elsewhere, a sum is used rather than an integral if the random variables are discrete. Then the cumulant generating function of s_i conditional on t_i has the usual exponential family form

$$\log E\{\exp(zs_i)|t_i\} = c_i(t_i,\beta+z) - c_i(t_i,\beta), \tag{6.2.4}$$

whence, as in the usual theory for the exponential family, the conditional mean and variance of s_i for any particular β are

$$E(s_i|t_i,\beta) = \frac{\partial}{\partial\beta} c_i(t_i,\beta),\ \mathrm{var}(s_i|t_i,\beta) = \frac{\partial^2}{\partial\beta^2} c_i(t_i,\beta) = \frac{\partial}{\partial\beta} E(s_i|t_i,\beta). \tag{6.2.5}$$

Hence, because of independence, it follows that

$$E(S|\underset{\sim}{t},\beta) = \frac{\partial}{\partial\beta} \sum_{i=1}^{M} c_i(t_i,\beta), \quad \mathrm{var}(S|\underset{\sim}{t},\beta) = \frac{\partial}{\partial\beta} E(S|\underset{\sim}{t},\beta). \tag{6.2.6}$$

From (6.1.2), it is not too difficult to show that the joint distribution of S and T is

$$f(S,T) = \exp\{\beta S + \alpha_k T + A(S,T;\underset{\sim}{\delta}) + \sum_{i=1}^{M} b_i(\alpha_i,\beta)\}, \tag{6.2.7}$$

where $\underset{\sim}{\delta} = (\alpha_2 - \alpha_1,\ldots,\alpha_M - \alpha_{M-1})$, and $A(S,T;\underset{\sim}{\delta})$ is defined by

$$\exp\{-\sum_{i=1}^{M} b_i(\alpha_i,\beta)\} = \iint \exp\{\beta S + \alpha_k T + A(S,T;\underset{\sim}{\delta})\}dSdT. \tag{6.2.8}$$

Note that the vector $\underset{\sim}{\delta}$ is the zero vector if and only if all the α_i's are equal. Equation (6.2.7) has a similar form to (6.1.1) and hence similar arguments to those above can be used to show that

$$E(S|T,\beta) = \frac{\partial}{\partial\beta} C(T,\beta;\underset{\sim}{\delta}), \quad \mathrm{var}(S|T,\beta) = \frac{\partial}{\partial\beta} E(S|T,\beta), \tag{6.2.9}$$

where

$$C(T,\beta;\underset{\sim}{\delta}) = \log \int \exp\{\beta S + A(S,T;\underset{\sim}{\delta})\}dS. \tag{6.2.10}$$

Note from (6.2.1) that $c_i(t_i,\beta)$ appears in $f(t_i)$. Hence, by (6.2.5), knowledge of the marginal p.d.f. of t_i is enough to determine the mean and variance of s_i conditional on t_i. Similarly, knowledge of the marginal p.d.f. of T is enough to determine the mean and variance of S conditional on T.

6.2.1 Conditioning on $\underset{\sim}{t}$ when the α_i's are equal

When the α_i's are equal (with a common value α, say), the vector $\underset{\sim}{t}$ is, for given β, still sufficient for α, even though it is not minimal sufficient. Hence the conditional distribution of S given $\underset{\sim}{t}$ is independent of α and so provides a similar test of β with a mean and variance which are known under the null hypothesis. The test will however be either identical to or less powerful than the test based on the distribution of S conditional on T which will be UMP similar. Hence, the asymptotic local power of the two tests (as $N \to \infty$) can be compared by considering the asymptotic relative efficiency (ARE) of the two tests.

Let U_1 and U_2 both be the random variable S but conditional on $\underset{\sim}{t}$ and T respectively. Now in the definition of the ARE, the alternative hypothesis approaches the null hypothesis as $N \to \infty$. Hence, $\underset{\sim}{t}$ and T (suitably standardized) will, with probability one, approach their <u>null</u> expected values. Using this fact together with (6.2.6) and (6.2.9), it follows that the ARE of U_1 to U_2 is

$$\xi = \lim_{N\to\infty} E\left\{\frac{\mathrm{var}(S|\underset{\sim}{t},\ \beta_0)}{\mathrm{var}(S|T,\ \beta_0)}\right\}, \tag{6.2.11}$$

where the expectation in (6.2.11) is for $\underset{\sim}{t}$ and T under the null hypothesis. Note that if two statistics U_1 and U_2 are both unbiased estimates of β, then their ARE would be the inverse of the ARE in (6.2.11).

Since $\mathrm{var}(S|T,\beta) = E\{\mathrm{var}(S|\underset{\sim}{t},\beta)|T\} + \mathrm{var}\{E(S|\underset{\sim}{t},\beta)|T\}$, it follows from (6.2.11) that $\xi \leq 1$, which reflects the fact that U_2 is the statistic giving rise to the UMP similar test. Finally, note that if $E(s_i|t_i,\ \beta_0) = \pi t_i$ $(i = 1,\ldots,M)$ where π is independent of i, then it follows easily that $\xi = 1$, which says that asymptotically no local power is lost by conditioning on $\underset{\sim}{t}$ instead of T. The loss in small samples could in principle be investigated.

6.2.2 Conditioning on T when the α_i's are unequal

If the α_i's are unequal, then T is not, for given β, sufficient for $\underset{\sim}{\alpha}$. Hence the conditional distribution of S given T will depend on $\underset{\sim}{\alpha}$; in fact it depends on $\underset{\sim}{\delta}$, the vector of differences between the α_i's. Let $\mu(\underset{\sim}{\delta}) = \frac{\partial}{\partial\beta} C(T,\beta;\underset{\sim}{\delta})$ and $\sigma^2(\underset{\sim}{\delta}) = \frac{\partial^2}{\partial\beta^2} C(T,\beta;\underset{\sim}{\delta})$, both expressions being evaluated at $\beta = \beta_0$. If the false assumption is made that the α_i's are equal, then the asymptotic null distribution of the statistic

$$Z = \frac{S - \mu(\underset{\sim}{0})}{\sigma(\underset{\sim}{0})} \tag{6.2.12}$$

is normal, not with zero mean and unit variance, but with mean $\{\mu(\underset{\sim}{\delta}) - \mu(\underset{\sim}{0})\}/\sigma(\underset{\sim}{0})$ and variance $\sigma^2(\underset{\sim}{\delta})/\sigma^2(\underset{\sim}{0})$. Hence, one meaningful measure of making an incorrect assumption about the α_i's in this case would be to calculate the actual asymptotic size of a test based on the mistaken assumption that Z is asymptotically normal with zero mean and unit variance. Consider the one-sided test:

$$\text{reject } H_0: \beta = \beta_0 \qquad \text{if } Z < -Z_\gamma, \tag{6.2.13}$$

where Z_γ is the upper γ point of the standard normal distribution. In general, $C(T,\beta;\underset{\sim}{\delta}) = O(N)$ for all $\underset{\sim}{\delta}$ and hence, if $\mu(\underset{\sim}{\delta}) - \mu(\underset{\sim}{0})$ is not asymptotically zero (i.e. $E(S|T,\beta_0)$ is not asymptotically independent of $\underset{\sim}{\alpha}$), $E(Z|T,\beta_0) = O(\sqrt{N})$ and $\mathrm{var}(Z|T,\beta_0) = O(1)$ as $N \to \infty$. In such a case, the asymptotic size of the test (6.2.13) would be either zero or unity depending on the sign of $\mu(\underset{\sim}{\delta}) - \mu(\underset{\sim}{0})$. However, in the case that $\mu(\underset{\sim}{\delta}) - \mu(\underset{\sim}{0}) \to 0$ as $N \to \infty$, the asymptotic size of the test (6.2.13) is actually $\Phi(-Z_\gamma\Sigma)$, where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution and $\Sigma = \lim_{N\to\infty} \sigma(\underset{\sim}{0})/\sigma(\underset{\sim}{\delta})$. Note that if $E(s_i|t_i,\beta_0) = \pi t_i$ $(i = 1,\ldots,M)$, where π is a constant independent of i, then $E(S|T,\beta_0)$ is independent of $\underset{\sim}{\alpha}$. Therefore $\mu(\underset{\sim}{\delta}) = \mu(\underset{\sim}{0})$, whence the asymptotic size of the test is, in general, neither zero nor unity.

6.3 Three examples

6.3.1 Testing the common variance of independent normal samples with possibly different means

Suppose that M independent data sets are observed. The ith data set contains n_i observations $(X_{ij},\ j = 1,\ldots,n_i)$ which are each independently normally distributed with mean μ_i and variance σ^2 (which is independent of i). The parameters σ^2 and $\underset{\sim}{\mu} = (\mu_1,\ldots,\mu_M)$ are all unknown. It is desired to test $H_0: \sigma^2 = \sigma_0^2$ against the alternative $H_1: \sigma^2 > \sigma_0^2$. The likelihood of the data is

$$L(\underset{\sim}{X}) = (2\pi\sigma^2)^{-N/2}\exp\{-\tfrac{1}{2}\sigma^{-2}(\sum_{i=1}^{M}\sum_{j=1}^{n_i} X_{ij}^2 - 2\sum_{i=1}^{M}\mu_i\sum_{j=1}^{n_i} X_{ij} + \sum_{i=1}^{M} n_i\mu_i^2)\}, \tag{6.3.1}$$

where $N = \Sigma n_i$. In the notation of (6.1.2), $\alpha_i = \mu_i/\sigma^2$, $\beta = -\frac{1}{2}\sigma^{-2}$, $\mathcal{S} = \Sigma\Sigma X_{ij}^{\ 2}$, $t_i = \Sigma_j X_{ij} = n_i\overline{X}_i$, and $T = \Sigma\Sigma X_{ij} = N\overline{X}..$.

For the moment, assume that the μ_i's are all equal. Then using well-known normal theory it can show that, in the notation of Section 6.2, $\text{var}(S|\underset{\sim}{t}, \beta_0)/\text{var}(S|T,\beta_0) = (N-M)/(N-1)$, which is independent of $\underset{\sim}{t}$ and T. For simplicity, assume now that the n_i's are all equal with common value n. Then $N = nM$. If we fix M, and let n increase to infinity, then we obtain $\mathcal{E} = 1$. Therefore, for large sample sizes, little power is lost by assuming that the μ_i's are unequal when in fact they are equal. However, if we fix n and let M increase to infinity, then we obtain $\mathcal{E} = 1 - \frac{1}{n}$. Hence, if there are many samples with small sample sizes, appreciable local power can be lost by assuming that the μ_i's are unequal when in fact they are equal. Both these conclusions are easily interpreted in terms of the degrees of freedom of the relevant estimates of the variance.

Now assume that the μ_i's are unequal. Again using normal theory, it is easily shown that $E(S|\overline{X}.., \sigma^2) = N\overline{X}..^2 + \sigma^2\{N-1 + \Sigma n_i(\mu_i - \overline{\mu})^2\}$, where $\overline{\mu} = \Sigma n_i\mu_i/N$. Excluding the degenerate case $\sigma^2 = 0$, $E(S|\overline{X}..,\sigma^2)$ is independent of the μ_i's if and only if all μ_i's are equal. Hence, by the remarks of Section 6.2.2, the test (6.2.13) will <u>always</u> have an asymptotic size of zero or unity.

6.3.2 Trend analysis for Poisson processes

Suppose that M, possibly inhomogeneous, Poisson processes are observed. The ith process is observed over a time z_i during which n_i events are observed $(i = 1,\ldots,M)$. Let the occurrence times of the events of the ith process, as measured from the beginning of the period of observation of the process, be $(u_{ij}, j = 1,\ldots,n_i)$. Specifically, assume that the ith process has a rate function $\lambda(u) = \exp(\alpha_i + \beta u)$.

This situation was studied in some detail in Section 5.2.4, except that there it was known a priori that all the α_i's were equal.

Testing for the existence of a general trend in the M data sets is equivalent to testing the null hypothesis $\beta = 0$; the α_i's are then nuisance parameters. The likelihood of the data is

$$L(\underset{\sim}{u}) = \exp\{\sum_{i=1}^{M} \alpha_i n_i + \beta \sum_{i=1}^{M} \sum_{j=1}^{n} u_{ij} - \sum_{i=1}^{M} e^{\alpha_i}(e^{\beta z_i} - 1)/\beta\}. \tag{6.3.2}$$

In the notation of (6.1.2), $\alpha_i = \alpha_i$, $\beta = \beta$, $s_i = \Sigma_j u_{ij}$, $t_i = n_i$. Under the null hypothesis (i.e. $\beta = 0$), the times u_{ij} $(j = 1,\ldots,n_i)$, conditional on n_i, are the order statistics from a uniform distribution on $[0,z_i]$. Therefore, s_i is the sum of n_i i.i.d.r.v.'s from a uniform distribution on $[0,z_i]$. Hence, if $\underset{\sim}{n} = (n_1,\ldots,n_M)$, then

$$E(S|\underset{\sim}{n}, \beta = 0) = \frac{1}{2} \sum_{i=1}^{M} n_i z_i, \quad \mathrm{var}(S|\underset{\sim}{n}, \beta = 0) = \frac{1}{12} \sum_{i=1}^{M} n_i z_i^{\,2}. \tag{6.3.3}$$

Let $N = \Sigma n_i$ and $Y = \Sigma z_i$. Now because of the Poisson nature of the process, it is easily seen that under the null hypothesis and conditional on N, the vector $\underset{\sim}{n}$ has a multinomial distribution, with parameters N and $\underset{\sim}{p} = (p_1,\ldots,p_M)$, where $p_i = z_i e^{\alpha_i}/\Sigma_j z_j e^{\alpha_j}$; note that if all the α_j's are equal, then $p_i = z_i/\Sigma z_j$. Then

$$\begin{aligned} E(S|N, \beta = 0) &= \frac{1}{2} N \sum_{i=1}^{M} p_i z_i, \\ \mathrm{var}(S|N, \beta = 0) &= N\{\frac{1}{3} \sum_{i=1}^{M} p_i z_i^{\,2} - (\frac{1}{2} \sum_{i=1}^{M} p_i z_i)^2\}. \end{aligned} \tag{6.3.4}$$

Now assume that all the α_i's are equal. Then, using (6.3.3) and (6.3.4), it is found that

$$\frac{\mathrm{var}(S|\underset{\sim}{n}, \beta = 0)}{\mathrm{var}(S|N, \beta = 0)} = \frac{\Sigma q_i r_i^2}{4\Sigma r_i^3 - 3(\Sigma r_i^2)^2}, \tag{6.3.5}$$

where $q_i = n_i/N$ and $r_i = z_i/\Sigma z_j$. Note that $\Sigma q_i = \Sigma r_i = 1$. The expression (6.3.5) will be unchanged as $N \to \infty$ if we fix M and let the $n_i \to \infty$ with the proportions q_i fixed, and will be independent of N. Under the null hypothesis, $E(q_i) = r_i$, so that, taking expectations of (6.3.5), one obtains the following ARE:

$$\xi = \frac{\Sigma r_i^3}{4\Sigma r_i^3 - 3(\Sigma r_i^2)^2}. \tag{6.3.6}$$

The quantity ξ is never greater than unity. Note that $E(s_i|n_i) = \frac{1}{2} n_i z_i$, so that, by a remark in Section 6.2.1, if all the z_i's (and hence all the r_i's) are equal, then $\xi = 1$ (as can be verified from (6.3.6)). As an example of (6.3.6), consider data set (i), full details of which are given in Appendix 4. When the type 0 events are observed, the type 1 events can be broken up into 14 independent Poisson processes, each with the same rate function $\lambda(u) = \exp(\alpha + \beta u)$. In Section 5.2.4, a test of $\beta = 0$ was carried out using the correct model for $\lambda(u)$. If the incorrect model, $\lambda(u) = \exp(\alpha_i + \beta u)$ for the ith process, is assumed, then, using (6.3.6), it is found that the resulting test is only 72% asymptotically efficient.

It is interesting to find a lower bound for ξ for each M. The method of solution is quite complicated but involves the use of Lagrange multipliers and the temporary assumption that M is a continuous parameter; the details will not be given here. However, the solution is that the minimum value of ξ is achieved when one of the proportions, say r_1, assumes a particular value R, and the other M - 1 proportions all assume the value $(1 - R)/(M - 1)$, where R has one of the two values $R_1 = [M + 2\{M(M - 1)\}^{1/2} \cos(\frac{1}{3}\theta)]^{-1}$, or

$R_2 = [M - 2\{M(M - 1)\}^{1/2}\cos(\frac{1}{3}\pi + \frac{1}{3}\theta)]^{-1}$, and $\theta = \cos^{-1}\{(1 - \frac{1}{M})^{1/2}\}$.
Of the two values, R is chosen to be the one which minimizes (6.3.6).
It then follows that for each fixed M

$$\min \xi = \frac{\{(1 - R)^2 + (k - 1)R^2\}^2}{(1 - R)^3 + (k - 1)^2R^3} . \tag{6.3.7}$$

For M = 2, 3 and 4, (R, min ξ) are correct to two decimal places (.79, .75), (.68, .65) and (.61, .59) respectively. If we choose, say, $r_1 = (M - 1)^{-1/2}$ and $r_i = (1 - r_1)/(k - 1)$ $(i = 2,\ldots,k)$, then from (6.3.6) an expression is obtained which approaches zero as $M \to \infty$. Hence, the universal infimum of ξ must be zero.

The conclusion is that, if the α_i's are assumed to be unequal when in fact they are equal, appreciable local power can be lost, especially if M is large. However, when all the z_i's are equal (with a common value z), s_i is under the null hypothesis and conditional on n_i, the sum of n_i i.i.d.r.v.'s from a uniform distribution on [0,z], and hence $S = \Sigma s_i$ is the sum of N i.i.d.r.v.'s from the same distribution. The distribution of S, therefore, depends on $\underset{\sim}{n}$ through N only, and so conditioning on $\underset{\sim}{n}$ and N will produce identical tests. Consequently, no power at all is lost in this case by assuming that the α_i's are unequal when in fact they are all equal.

Now assume that the α_i's are unequal. As can be seen from (6.3.4), $E(S|N, \beta = 0)$ will usually depend on the α_i's. Hence, a test such as (6.2.13) based on the incorrect assumption that the α_i's are equal will usually have an asymptotic size of zero or one. However, if all the z_i's are equal, the argument of the preceding paragraph can be repeated (because it is independent of the values of the α_i's), so that, in this special case, conditioning on $\underset{\sim}{n}$ and N still produces identical tests, with identical power functions, which are independent of the values of the α_i's.

6.3.3 Binary regression

Suppose that M sets of independent binary random variables $(Y_{ij},\ j = 1,\ldots,n_i)$ are observed, where n_i is the number of observations in the ith data set. Without loss of generality, assume that the Y_{ij}'s are all either zero or one. Corresponding to each Y_{ij} there is also an explanatory variable, x_{ij}, which also only takes the values zero or one. For the ith data set, suppose that ℓ_i of the x_{ij}'s equal one, and the remaining $(n_i-\ell_i)$ x_{ij}'s equal zero. Assume the following logistic relationship between the Y_{ij}'s and x_{ij}'s:

$$\log \frac{P(Y_{ij} = 1)}{P(Y_{ij} = 0)} = \alpha_i + \beta x_{ij} , \tag{6.3.8}$$

where β (which is common to all the data sets) and the α_i's are all unknown. It is required to test whether the distribution of the Y_{ij}'s depends on the x_{ij}'s, that is whether $\beta = 0$. The α_i's are then nuisance parameters. The likelihood for the data is

$$L(\underset{\sim}{Y}) = \frac{\exp\left(\sum_{i=1}^{M} \alpha_i \sum_{j=1}^{n_i} Y_{ij} + \beta \sum_{i=1}^{M} \sum_{j=1}^{n_i} x_{ij} Y_{ij}\right)}{\prod_{i=1}^{M} \prod_{j=1}^{n_i} \{1 + \exp(\alpha_i + \beta x_{ij})\}} . \tag{6.3.9}$$

In the notation of (6.1.2), $\alpha_i = \alpha_i$, $\beta = \beta$, $s_i = \Sigma_j x_{ij} Y_{ij}$, $t_i = \Sigma_j Y_{ij}$. The quantity t_i is just the number of observations in the ith data set with $Y_{ij} = 1$, while s_i is the number of observations with both x_{ij} and Y_{ij} equal to 1. Hence, under the null hypothesis (i.e. $\beta = 0$), s_i conditional on t_i has a hypergeometric distribution. Therefore

$$E(s_i|t_i,\ \beta = 0) = \frac{t_i \ell_i}{n_i},\quad \mathrm{var}(s_i|t_i,\ \beta = 0) = \frac{t_i(n_i-t_i)\ell_i(n_i-\ell_i)}{n_i^2(n_i - 1)} . \tag{6.3.10}$$

Note that if the ratio $\ell_i/n_i = \pi$, a constant independent of i, then $E(s_i|t_i, \beta = 0) = \pi t_i$ $(i = 1,\ldots,M)$. Therefore, by the remarks of Section 6.2.1, if all the α_i's are equal, no asymptotic local power will be lost if we assume them to be unequal. Further, if the α_i's are unequal, and we mistakenly assume them to be equal, the actual asymptotic size of the test (6.2.13) will not, in general be zero, unity or γ; the actual size will be investigated later when $M = 2$.

Now assume that all the α_i's are equal. Then, by similar arguments to those previously, $S = \Sigma s_i$ conditional on $T = \Sigma t_i$ has a null distribution which is hypergeometric with

$$E(S|T, \beta = 0) = \frac{TL}{N}, \quad \mathrm{var}(S|T, \beta = 0) = \frac{T(N-T)L(N-L)}{N^2(N-1)}, \tag{6.3.11}$$

where $L = \Sigma\ell_i$, $N = \Sigma n_i$. Also, as both t_i and T have, under the null hypothesis, binomial distributions, it is easily seen that the null distribution of t_i conditional on T is also hypergeometric with

$$E(t_i|T, \beta = 0) = \frac{Tn_i}{N}, \quad \mathrm{var}(t_i|T, \beta = 0) = \frac{T(N-T)n_i(N-n_i)}{N^2(N-1)}. \tag{6.3.12}$$

Therefore, if we let $\pi_i = \ell_i/n_i$ and $q_i = n_i/N$, and if the ℓ_i's, n_i's and N are allowed to go to infinity with the proportions π_i, q_i and M fixed, then the ARE is

$$\xi = \frac{\Sigma q_i\pi_i(1 - \pi_i)}{\Sigma q_i\pi_i(1 - \Sigma q_i\pi_i)}. \tag{6.3.13}$$

The quantity ξ is never greater than unity, but by an earlier remark will be equal to unity if the π_i's are all equal (but not all equal to either zero or one). For each M, the minimum attainable value of ξ is zero. This occurs when some of the π_i's, but not all M of them, equal zero (i.e. $\ell_i = 0$) and the rest of the π_i's equal one (i.e. $\ell_i = n_i$). This is intuitively reasonable because it says that, in each data set, all the explanatory variables have the same value and so no data set by itself has any information about β. However, the fact that the π_i's are not all zero or one implies that information about β can be obtained by pooling the M data sets and treating them as one data set.

Now assume that the α_i's are unequal. By the remarks of Section 6.2.2, the asymptotic size of the test (6.2.13) will usually be either zero or unity. However, by a previous remark, this will not be so if the π_i's are all equal. The mathematics which follows is useful in finding the asymptotic size of the test (6.2.13) in this special case. Because the argument is quite complicated, we restrict ourselves to the case $M = 2$. As we are only concerned with the size of the test, we shall henceforth assume that $\beta = 0$.

Let $\pi = \pi_1 = \pi_2$, and $p_i = e^{\alpha_i}/(1 + e^{\alpha_i})$ $(i = 1, 2)$. Now it follows directly from (6.3.10) that $E(S|\underset{\sim}{t}, \beta = 0) = E(S|T, \beta = 0) = T$, where $\underset{\sim}{t} = (t_1,t_2)$ and $T = t_1 + t_2$. However, $\text{var}(S|\underset{\sim}{t}, \beta = 0)$ is much more difficult to obtain. Since t_1 and t_2 have independent binomial distributions (T is only binomially distributed if $p_1 = p_2$), the conditional distribution of t_1 or t_2 given T is found easily. Using this fact and (6.3.10), it follows that

$$\text{var}(S|T, \beta=0) = \frac{\pi(1-\pi)}{D(n_1,n_2,T,\delta)}\left\{\begin{array}{l} n_1 e^{\delta}\{D(n_1-1,n_2,T-1,\delta)-e^{\delta}D(n_1-2,n_2,T-2,\delta)\} \\ +n_2\{D(n_1,n_2-1,T-1,\delta)-D(n_1,n_2-2,T-2,\delta)\} \end{array}\right\}, \tag{6.3.14}$$

where

$$D(n_1,n_2,T,\delta) = \sum_{j=0}^{T} \binom{n_1}{j} \binom{n_2}{T-j} e^{\delta j} \tag{6.3.15}$$

and $\delta = \alpha_1 - \alpha_2$. When $\delta = 0$, $D(n_1,n_2,T,0) = \binom{n_1+n_2}{T}$, and the variance in (6.3.11) is obtained. The equations (6.3.14) and (6.3.15) do not give any insight into the asymptotic behaviour of $\mathrm{var}(S|T, \beta = 0)$ when $\delta \neq 0$. We need therefore an asymptotic form for $D(n_1,n_2,T,\delta)$ as n_1, n_2 and T approach infinity. First, note the following:

$$P(T = t) = \exp\{\alpha_2 t + B(\alpha_2,\delta) + C(t;\delta)\}, \tag{6.3.16}$$

where

$$B(\alpha_2,\delta) = -n_1 \log(1+e^{\delta+\alpha_2})-n_2 \log(1+e^{\alpha_2});\ \exp\{C(t;\delta)\} = D(n_1,n_2,T,\delta). \tag{6.3.17}$$

The quantity $C(t;\delta)$ is the same as that in (6.2.10) in the special case $\beta = 0$. Assume that n_1, n_2 and T approach infinity together in fixed proportions. Then, modifying the approach of Daniels (1954), we can obtain asymptotic expansions of functions of the form (6.3.15), where $T = \Sigma V_i + \Sigma W_j$, the V_i's being i.i.d.r.v.'s, the W_j's being i.i.d.r.v.'s and all the V_i's and W_j's being independent of each other. The following asymptotic expansion can then be obtained for $\exp\{C(t;\delta)\}$:

$$\exp\{C(t;\delta)\} \sim \frac{\exp\{-\hat{\alpha}t - B(\hat{\alpha},\delta)\}}{\{-2\pi \frac{\partial^2}{\partial\alpha^2} B(\alpha,\delta)\Big|_{\alpha=\hat{\alpha}}\}^{1/2}}, \tag{6.3.18}$$

where $\hat{\alpha}$ is the solution of

$$-\frac{\partial}{\partial\alpha} B(\alpha,\delta) = t. \tag{6.3.19}$$

Substituting (6.3.17) into (6.3.18) and (6.3.19), it is found that

$$D(n_1,n_2,T,\delta) \sim \frac{\nu_2^{-T}(1+\nu_1)^{n_1}(1+\nu_2)^{n_2}}{\left[2\pi\left\{\frac{n_1\nu_1}{(1+\nu_1)^2} + \frac{n_2\nu_2}{(1+\nu_2)^2}\right\}\right]^{1/2}} , \tag{6.3.20}$$

where

$$\nu_1 = \frac{[\{(n_1-T)+e^{\delta}(n_2-T)\}^2+4e^{\delta}T(n_1+n_2-T)]^{1/2} - (n_1-T)-e^{\delta}(n_2-T)}{2e^{\delta}(n_1+n_2-T)} \tag{6.3.21}$$

and $\nu_2 = e^{\delta}\nu_1$. If $\delta = 0$, then the asymptotic expansion obtained is

$$D(n_1,n_2,T,0) \sim \frac{N^{N+\frac{1}{2}}}{(2\pi)^{\frac{1}{2}} T^{T+\frac{1}{2}} (N-T)^{N-T+\frac{1}{2}}} , \tag{6.3.22}$$

which is also obtained using Stirling's formula on $N = n_1 + n_2$, T and $N - T$. Note that if n_1, n_2 and T approach infinity in fixed proportions, then ν_1 is asymptotically the same whether we use (n_1, n_2, T) or, say $(n_1 - 1, n_2, T - 1)$. Then substituting (6.3.20) and (6.3.21) into (6.3.14), it follows that

$$\operatorname{var}(S|T, \beta = 0) \sim \frac{\pi(1-\pi)}{n_i}\{T(n_i-T) + 2n_jT\left(\frac{\nu_i}{1+\nu_i}\right) - n_jN\left(\frac{\nu_i}{1+\nu_i}\right)^2\} , \tag{6.3.23}$$

where $(i,j) = (1,2)$ or $(2,1)$. If we incorrectly assume that the α_i's are all equal, then, from (6.3.11) and noting that $\pi = L/N$, the test (6.2.13) becomes: reject H_0: $\beta = 0$ if

$$Z = \frac{(N-1)^{1/2}(S-\pi T)}{\{(1-\pi)T(N-T)\}^{1/2}} < -Z_\gamma. \tag{6.3.24}$$

The actual asymptotic size of this test can then be determined from (6.3.23) (as, by a previous remark, $E(S|T) = \pi T$, even if the α_i's are unequal). However, one final adjustment needs to be made. As n_1, n_2 and T approach infinity, T/N converges with probability one to $E(T/N) = (n_1p_1 + n_2p_2)/N$, and this should be substituted in (6.3.21), (6.3.23) and (6.3.24) for T/N when calculating the asymptotic size of the test. When this substitution is carried out, it turns out that $\nu_1/(1 + \nu_1) = p_2$, so that after some manipulation, we obtain

$$\mathrm{var}(Z|T,\ \beta = 0) \to \Sigma^{-2} = \frac{q_1p_1(1-p_1) + q_2p_2(1-p_2)}{\{q_1p_1+q_2p_2\}\{q_1(1-p_1)+q_2(1-p_2)\}} \tag{6.3.25}$$

as $n_1, n_2 \to \infty$ with $q_i = n_i/N$ fixed. The actual asymptotic size of the test (6.3.24) is then $\Phi(-Z_\gamma\Sigma)$. Note that if $p_1 = p_2$, then the right-hand side of (6.3.25) is 1, so that the asymptotic size of the test (6.3.24) is γ. This is in fact the maximum possible asymptotic value for $\mathrm{var}(Z|T,\ \beta = 0)$. The minimum possible value is zero which is attained when $p_1 = 0$, $p_2 = 1$, or vice versa; this is equivalent to the degenerate case $S = \ell_2$ with probability one. Therefore, since $\mathrm{var}(Z|T,\ \beta = 0)$ always lies in $[0,1]$, it follows that for all possible values of p_1 and p_2, the actual size of the test (6.3.24) lies in the interval $[0,\gamma]$.

6.4 Conclusions

The general theory and examples show that, for exponential families, much power can be lost by assuming that the nuisance parameters are unequal when in fact they are all equal. Conversely, if the nuisance parameters are unequal, the assumption of equality will, in general, lead to inconsistent tests. However, the theory and examples also show that if the samples are "balanced" in the sense that $E(s_i|t_i, \beta_0)$ is proportional to t_i, it matters very little (and sometimes not at all - see Section 6.3.2) whether the nuisance parameters are assumed equal or not. This conclusion is similar to that of Welch (1937) in relation to the Behrens-Fisher problem. In this problem, it is desired to test the difference between the means of two normal samples which have unknown and possibly unequal variances. Welch investigated the effect of assuming that the two population variances are equal (and so using a pooled estimate of the variance), when in fact the two variances are unequal (implying that each variance should be estimated separately). His broad conclusion was that, provided the two sample sizes are equal, no great harm is done by ignoring the inequality of the variance. Although this problem does not fit into the theoretical framework of Section 6.2, it is interesting to note the similarity of the conclusions in relation to "balanced" samples.

CHAPTER 7: ON SINGLE SERVER QUEUES WITH MARKOV RENEWAL DEPARTURE PROCESSES

7.1 Introduction

In this chapter, we study a number of single server queues, all of which have a Poisson arrival process, and a departure process which is a Markov renewal process; as was mentioned in the introduction to Chapter 3, the Markov renewal process is a very special r.m.p. process. The relatively simple structure of the departure process is, in all the queues we shall study, partly a consequence of the even simpler structure of the arrival process. The aim of this chapter is to find necessary and sufficient conditions for the departure process of such queues to be a renewal process. The solution of this problem has general importance in the modelling of tandem queueing systems, that is, where several queues occur in series, the departure process of any queueing system providing the arrival process for the following queueing system. If the departure processes are renewal, then the modelling of tandem queueing systems will, in general, be greatly simplified. In this chapter, we develop a general approach which applies to all the examples, and leads to results, some of which are well known and others which are apparently new.

In Section 7.2, we introduce the various queueing systems considered in this chapter and briefly discuss previous work on the structure of their departure processes. In Sections 7.3 and 7.4, we prove some results concerning a special type of Markov renewal process and a special type of queue. These are used to analyse several queueing departure processes. Finally, in Section 7.5, we mention some possible developments of the work in this chapter.

7.2 Some single server queues with Markov renewal departure processes

Throughout this chapter, it will be assumed that all the queueing systems being studied are stationary. In general, this corresponds to the mean effective service time being less than the mean inter-arrival time.

Probably the best known single server queue is the M/G/1 queue. Its arrival process is a homogeneous Poisson process (with mean inter-arrival time λ^{-1}), while its service times are independent (of each other and of the arrival process) and identically distributed (with arbitrary distribution function $G(\cdot)$). It is known that the departure process of the queue is Markov renewal in both the finite and infinite capacity cases (Neuts, 1965). The capacity of the queue is just the maximum number of customers allowed to wait for service at any one time (excluding the customer being served); if the queueing capacity is K, we speak of the M/G/1/K queue. The M/G/1/K queue in which the service times are identically zero and the M/G/1/O queue (i.e. no customers allowed to wait for service) are trivially seen to have renewal departure processes, the latter because it is a Markov renewal process with one state (which is a renewal process by definition); we assume that neither of these conditions holds.

Burke (1956) and Reich (1957) showed that the departure process of the M/M/1/∞ queue (i.e. negative exponential service times and infinite capacity) is a Poisson process with the same rate as the arrival process. Finch (1959) studied the M/G/1/K queue ($1 \le K \le \infty$) with service time distributions possessing a continuous second derivative. He proved that the departure process is a renewal process if and only if $G = M$ and $K = \infty$. King (1971) showed that the M/D/1/1 queue (i.e. the service times are constant and the queue has capacity 1) has a renewal departure process; of course, the service time distribution is not then twice differentiable. Using Markov renewal methods, Disney, Farrell and de Morais (1973) showed that the two cases $M/M/1/\infty$ and $M/D/1/1$ exhaust the possibilities for renewal departure processes from non-trivial $M/G/1/K$ systems.

The other queueing system to be considered is a special single server queue with state-dependent feedback; its properties have been studied extensively by Davignon and Disney (1976). It is like an $M/G/1/\infty$ queue, except that customers after being served either immediately join the queue again with some probability or depart permanently with the complementary probability. Such a probability is conditioned upon whether or not the previous unit fed back, upon the increments in the queue length between two consecutive service completions, and upon the length of service received. Davignon and Disney (see their Theorem 6.2.1) show that the departure process for this queue is a Markov renewal process, although the stationary transition probabilities of the departure process are in general rather difficult to write down explicitly. The general problem of finding when the Davignon-Disney queue has a renewal departure process appears difficult, mainly because the probability of feedback is conditioned upon a variety of things. By limiting the nature of this dependence, necessary and sufficient conditions are found for the departure process to be a renewal process.

7.3 A special Markov renewal process

In this section, we prove a useful result concerning a special Markov renewal process. First, however, a few definitions and assumptions are required. Suppose that the Markov renewal process has state space (0,1,...,m). We can think of this process as a multivariate point process with m + 1 event types; it is a special type of r.m.p. process because each of the m + 1 event types act as regeneration points for the entire multivariate process. All the results that follow in this section apply whether m is finite or infinite. Let $T_0 \equiv 0$, $T_1, T_2, \ldots$ be the transition times of the process and define X_ℓ to be the state into which the particle enters at time T_ℓ. Then the Markov renewal process is defined by the transition probabilities

$$F_{jk}(t) = P(X_1 = k,\ T_1 \leq t \mid X_0 = j) \quad (j,k = 0,1,\ldots,m). \qquad (7.3.1)$$

Let

$$f^*_{jk}(s) = \int_0^\infty e^{-st} F_{jk}(dt) \qquad (7.3.2)$$

and let $\underset{\sim}{F}(s)$ be the $(m+1) \times (m+1)$ matrix whose (j,k)th element is $f^*_{jk}(s)$. Assume that $\Pr(T_1 < \infty) = 1$ and that the Markov renewal process is stationary; conditions under which the process is stationary will not be given here as they are not used directly, but such conditions can be found, for instance, in Pyke (1961) and Pyke and Schaufele (1966).

Associated with the Markov renewal process is a Markov chain with transition matrix $\underset{\sim}{P} \equiv \underset{\sim}{F}(0)$. Assume that the Markov chain is both ergodic (i.e. aperiodic and positive-recurrent) and irreducible. Then it is known (Feller, 1968, Section XV.7) that there exists a unique equilibrium probability vector $\underset{\sim}{\pi}^T = (\pi_0,\ldots,\pi_m)$ with $\underset{\sim}{\pi}^T\underset{\sim}{P} = \underset{\sim}{\pi}^T$, $\underset{\sim}{\pi}^T\underset{\sim}{1} = 1$, and $\pi_i > 0 (i = 0,1,\ldots,m)$. Note also that $\underset{\sim}{P}\underset{\sim}{1} = \underset{\sim}{1}$.

Let

$$p^*(s) \quad = \quad \underset{\sim}{\pi}^T\underset{\sim}{F}(s)\underset{\sim}{1}. \tag{7.3.3}$$

This is easily seen to be the Laplace transform of the p.d.f. of the interval between two successive transitions of the Markov renewal process (without regard to the types of transition). By considering the joint distribution of n successive intervals of the Markov renewal process, it is easily seen that a necessary and sufficient condition for the Markov renewal process to be a renewal process is that

$$\underset{\sim}{\pi}^T \prod_{i=1}^{n} \underset{\sim}{F}(s_i)\underset{\sim}{1} \quad = \quad \prod_{i=1}^{n} p^*(s_i) \tag{7.3.4}$$

for all $s_i > 0$ $(i = 1,\ldots,n)$, and all $n \geq 1$. The left-hand side of (7.3.4) represents the joint distribution of n successive intervals of a Markov renewal process, while the right-hand side represents the joint distribution of n successive intervals of a renewal process.

In general, (7.3.4) is not easy to solve for $\underset{\sim}{F}(s)$; the number of solutions appears to increase with the size of the state space. However, by imposing certain restrictions (which are satisfied by all our queueing examples), it is possible to solve (7.3.4). With these restrictions there are only two possible solutions: the distribution of T_1 is independent either of X_0 or of X_1. This is proved in Theorem 7.3.1 which follows. Let $A_j(t) = \Sigma_k F_{jk}(t) \equiv P(T_1 \leq t|X_0 = j)$ $(j = 0,1,\ldots,m)$.

Theorem 7.3.1: Suppose that (i) $P(X_1 = k|X_0 = j) = 0$ ($k \leq j-2$, $j = 2,\ldots,m$); (ii) $P(X_1 = j-1|X_0 = j) > 0$ ($j = 1,\ldots,m$); and (iii) $A_1(t) = A_2(t) = \ldots. = A_m(t)$. Then the Markov renewal process is a renewal process if and only if either (a) $A_0(t) = A_1(t)$, or (b) $P(X_1 = k, T_1 \leq t) = P(X_1 = k).\ P(T_1 \leq t)$ ($k = 0,1,\ldots,m$).

In terms of Laplace transforms the above conditions are respectively equivalent to (i) $f^*_{jk}(s) = 0$(($k \leq j-2$, $j = 2,\ldots,m$); (ii) $f^*_{j,j-1}(s) \neq 0$ ($j = 1,\ldots,m$); (iii) $a_1(s) = a_2(s) = \ldots. = a_m(s)$, where $\underset{\sim}{F}(s)\underset{\sim}{1} = \{a_0(s),\ldots,a_m(s)\}$; (a) $\underset{\sim}{F}(s)\underset{\sim}{1} = p^*(s)\underset{\sim}{1}$; and (b) $\underset{\sim}{\pi}^T\underset{\sim}{F}(s) = p^*(s)\underset{\sim}{\pi}^T$.

Proof: Although the theorem can be proved probabilistically, it is simpler to do so analytically. The proof is straightforward, although the details are somewhat involved. Consequently, only an outline will be given here. The sufficiency is trivial and is proved by substituting the alternative forms of (a) and (b) in (7.3.4). Let $\underset{\sim}{\pi}^T\underset{\sim}{F}(s) = \{b_0(s),\ldots,b_m(s)\}$. For the necessity, note first that (7.3.4) is true when $n = 1$ by definition (i.e. equation (7.3.3)). When $n = 2$, it is found after some manipulation that (7.3.4) is true if and only if either $a_0(s) = a_1(s)$ (which is easily seen to be equivalent to condition (a)) or if $b_0(s) = \pi_0 p^*(s)$. For the remainder of the proof, we assume that condition (a) does not hold and we proceed by induction. Assume that (7.3.4) is true for $n = 1,2,\ldots,L$, where $L < m$ (if m is finite) and that $b_i(s) = \pi_i p^*(s)$ ($i = 0,\ldots,L-2$). Then further manipulation shows that (7.3.4) is satisfied for $n = L+1$ if and only if $b_{L-1}(s) = \pi_{L-1}p^*(s)$. The necessity of condition (b) is then obvious.

The first part of the proof of Theorem 7.3.1 is also enough to prove

<u>Corollary 7.3.2</u>: A two state Markov renewal process is a renewal process if and only if either (a) $P(X_1 = 0, T_1 \leq t) = P(X_1 = 0).P(T_1 \leq t)$, or (b) $P(X_0 = 0, T_1 \leq t) = P(X_0 = 0).P(T_1 \leq t)$.

For two state Markov renewal processes, $m = 1$ (i.e. the two states are 0 and 1). Hence, conditions (i) and (iii) of Theorem 7.3.1 are redundant in this case. Condition (ii) is only used in the induction part of theorem 7.3.1. This part is not needed to prove corollary 7.3.2, as the conditions (7.3.3) and $b_0(s) = \pi_0 p^*(s)$ immediately imply $b_1(s) = (1 - \pi_0)p^*(s)$ for a two state Markov renewal process, that is, condition (b).

In the single server queues which will be studied, the states of the Markov renewal departure process are the number of customers waiting for service just prior to or just after a departure. At each transition (i.e. a departure), the queue size can be reduced by at most one customer. In Theorem 7.3.1, this fact is reflected in condition (i). The probability in condition (ii) is the probability that no customers entered the queue between two successive departures given that the queue size at the beginning of this interval was j (≥ 1); this will always be positive and independent of j in all the examples. Condition (iii) reflects the fact that, in all the examples, the service time distribution is independent of the queue size at the beginning of the service. This last fact implies a more specific transition matrix than that considered in Theorem 7.3.1. This more specific matrix will be used in Section 7.4 where some queues with infinite capacity are considered. However, Theorem 7.3.1 will be sufficient for the following finite capacity example, although it should be stressed that the conclusions of that theorem also apply to queues with infinite capacity.

7.3.1 The M/G/1/K queue

We assume that $1 \leq K < \infty$, and we use the notation of Section 7.2. It is easily seen that the departure process is a Markov renewal process with state space $(0,1,\ldots,K)$ (Disney, Farrell and de Morais, 1973, Theorem 2.4). Let $g^*(s) = \int_0^\infty e^{-st} G(dt)$, where $G(\cdot)$ is the distribution function of the service times. Then the entries of $\underset{\sim}{F}(s)$ are

$$f^*_{jk}(s) = \begin{cases} \dfrac{\lambda}{\lambda+s} \dfrac{(-\lambda)^k}{k!} \dfrac{d^k}{d\lambda^k} g^*(\lambda+s); \quad j = 0;\ k = 0,1,\ldots,K-1; \\[2ex] \dfrac{(-\lambda)^{k+1-j}}{(k+1-j)!} \dfrac{d^{k+1-j}}{d\lambda^{k+1-j}} g^*(\lambda+s);\ j = 1,2,\ldots,K;\ k = j-1,\ldots,K-1; \\[2ex] 0;\ j = 2,3,\ldots,K;\ k = 0,1,\ldots,j-2; \\[2ex] \dfrac{\lambda}{\lambda+s} g^*(s) - \displaystyle\sum_{k=0}^{K-1} f^*_{0k}(s);\ j = 0;\ k = K; \\[2ex] g^*(s) - \displaystyle\sum_{k=0}^{K-1} f^*_{jk}(s);\ j = 1,2,\ldots,K;\ k = K. \end{cases} \tag{7.3.5}$$

Therefore from (7.3.3) and (7.3.5), one obtains

$$\underset{\sim}{F}(s)\underset{\sim}{1} = g^*(s)\{\lambda/(\lambda+s),\ 1,1,\ldots,1\}^T;\ p^*(s) = g^*(s)\{1 - \pi_0 s/(\lambda+s)\}. \tag{7.3.6}$$

Assumptions (i), (ii) and (iii) of Theorem 7.3.1 are satisfied. However, for condition (a) to be satisfied, it is necessary that $\lambda/(\lambda+s) = 1$ for all $s > 0$ (by (7.3.6)), which is clearly not possible. Therefore the process is a renewal process if and only if condition (b) is satisfied. At this point, we consider the cases $K = 1$ and $K > 1$

separately.

When K = 1, one finds from (7.3.5) that $\underset{\sim}{\pi}^T\underset{\sim}{F}(s) = p^*(s)\{g^*(\lambda+s)/g^*(s), 1 - g^*(\lambda+s)/g^*(s)\}$ so that condition (b) is true if and only if $g^*(\lambda+s) = \pi_0 g^*(s)$, i.e. $g^*(s) = e^{-Cs}$, where $C = -\lambda^{-1} \log \pi_0$, which says that the service time is a constant, C. In fact any positive constant, C, will satisfy condition (b) of Theorem 7.3.1 (i.e. $\underset{\sim}{\pi}^T\underset{\sim}{F}(s) = p^*(s)\underset{\sim}{\pi}^T$) as can be verified by putting s = 0 in this equation, and solving for π_0 to obtain $\pi_0 = e^{-C\lambda}$.

When K > 1, it is necessary to write down only the first two equations of the K + 1 equations $\underset{\sim}{\pi}^T\underset{\sim}{F}(s) = p^*(s)\underset{\sim}{\pi}^T$. Using (7.3.5) and (7.3.6), these are

$$g^*(\lambda+s)\{\pi_1 + \pi_0\lambda/(\lambda+s)\} = \pi_0 p^*(s), \tag{7.3.7}$$

$$\pi_2 g^*(\lambda+s) - \lambda \frac{d}{d\lambda} g^*(\lambda+s)\{\pi_1 + \pi_0\lambda/(\lambda+s)\} = \pi_1 p^*(s). \tag{7.3.8}$$

Equations (7.3.7) and (7.3.8) are identical to equations (3.1) and (3.2) respectively of Disney, Farrell and de Morais (1973), except for the notation. These authors manipulate the two equations to show that G must be exponential and that $\pi_1 = \pi_0(1 - \pi_0)$. It is not too difficult to solve the Markov chain equilibrium equation, $\underset{\sim}{\pi}^T\underset{\sim}{P} = \underset{\sim}{\pi}^T$, when G is exponential; see, for instance, Finch (1958). The solution is that, if G has mean ρ^{-1}, then

$$\pi_i = \frac{(1-\mu)\mu^i}{1-\mu^{K+1}}, \quad \text{if } \rho \neq \lambda \;; \qquad i = 0,1,\ldots,N+1, \tag{7.3.9}$$

$$= 1/(K+1), \quad \text{if } \rho = \lambda;$$

where $\mu = \lambda/\rho$. It is easily seen that $\pi_1 = \pi_0(1-\pi_0)$ is never satisfied by (7.3.9) when K is finite. Hence, we have shown that, for $1 \leq K < \infty$, the M/G/1/K queue has a renewal departure process if and only if K = 1 and the service time is a constant. The case $K = \infty$ will be considered in the next section.

7.4 Some queues with infinite capacity

In this section, two main results are proved, one a special example of Theorem 7.3.1 for Markov renewal processes with infinitely many states, the other a very general result about the departure processes of certain types of queues. We combine these two results to find when the departure processes of some queues with infinite capacity are renewal processes. First, we have the theorem concerning Markov renewal processes.

Theorem 7.4.1: Suppose that we have a stationary Markov renewal process with infinitely many states and that the elements of $\underset{\sim}{F}(s)$ are of the form

$$f^*_{jk}(s) = \begin{cases} \dfrac{\lambda\Psi_k(s)}{\lambda+s} ; & j = 0;\ k = 0,1,2,\ldots ; \\ \Psi_{k+1-j}(s); & j = 1,2,\ldots;\ k = j-1,j,\ldots; \\ 0; & j = 2,3,\ldots ;\ k = 0,1,\ldots,j-2; \end{cases} \qquad (7.4.1)$$

and that $\Psi_0(s) \neq 0$. Let $\Phi(s;z) = \sum_{i=0}^{\infty} \Psi_i(s)z^i$. Then the process is a renewal process if and only if

$$\Phi(s;z) = \frac{(\lambda+s)p^*(s)\Phi(0;z)(z-1)}{s\{\Phi(0;z) - 1\} + \lambda(z-1)} , \qquad (7.4.2)$$

where

$$p^*(s) = \Phi(s;1)\{1 - \pi_0 s/(\lambda+s)\} \tag{7.4.3}$$

and $$\pi_0 = 1 - \sum_{i=0}^{\infty} i\Psi_i(0) \equiv \{\partial\Phi(0;z)/\partial_z\}_{z=1} .$$

Proof: From (7.4.1), we see that $\underset{\sim}{F}(s)\underset{\sim}{1} = \Phi(s;1)\{\lambda/(\lambda+s), 1,1,\ldots\}^T$. Assumptions (i), (ii) and (iii) of Theorem 7.3.1 are satisfied. It follows immediately that condition (a) can never be satisfied. Therefore, by Theorem 7.3.1, the process is a renewal process if and only if $\underset{\sim}{\pi}^T\underset{\sim}{F}(s) = p^*(s)\underset{\sim}{\pi}^T$. It is easily seen from (7.4.1) that $p^*(s)$ is given by (7.4.3).

Because of the special form of $\underset{\sim}{F}(s)$, the equation $\underset{\sim}{\pi}^T\underset{\sim}{F}(s) = p^*(s)\underset{\sim}{\pi}^T$ is easy to solve. The simplest method uses generating functions; Cox and Miller (1965, example 3.19) solve this equation when $s = 0$ for $\underset{\sim}{\pi}$ in terms of $\underset{\sim}{P}$ ($\equiv \underset{\sim}{F}(0)$). Let $\Pi(z) = \sum_{i=0}^{\infty} \pi_i z^i$. After some manipulation, one obtains the condition

$$\Pi(z) = \frac{\pi_0\Phi(s;z)[1-\lambda z/(\lambda+s)]}{\Phi(s;z) - p^*(s)z} . \tag{7.4.4}$$

The above equation is a defining equation for $\Pi(z)$ when $s = 0$. Substituting the right-hand side of (7.4.4) with $s = 0$ into the left-hand side of (7.4.4) leads to (7.4.2). The value of π_0 given above is obtained by Cox and Miller, Section 3.8, equation (122); they also show that the associated Markov chain is ergodic if and only if $\sum_{i=0}^{\infty} i\Psi_i(0) < 1$.

In each of the examples we shall consider, the function $\Psi_i(s)$ used in (7.4.1) is the Laplace transform of the joint probability that the effective service time (i.e. the time between successive departures) is in [t, t+dt) and that during this service time there are i arrivals. The quantity $\Psi_i(0)$ then is just the probability that there are i arrivals during a service.

We next give a useful result concerning the departure processes of certain types of stationary queues, but first we must define the input process of a queue. The input process of a queue (as distinct from the arrival process) is the set of arrival times of the customers who actually receive service. In some queues, where all customers receive service, the arrival and input processes are identical; in other queues, such as queues with finite capacity, some customers are turned away and so the arrival and input processes are different.

The result, given as Theorem 7.4.2, applies to those queues whose input process is (i) a renewal process and (ii) whose interval length distribution is uniquely determined by its moments, all of which are assumed to be finite. Conditions under which a distribution is uniquely determined by its moments are given, for instance, in Feller (1971, Section VII.6). For the sake of brevity, only an outline of the proof of the theorem will be given; the arguments used can be justified by the conditions of the theorem.

Let $N_I(x)$ and $N_D(x)$ be the number of customers who respectively enter and depart (after service) from a queueing system in the interval $(0,x]$. Let $W(x) = N_I(x) - N_D(x)$.

Theorem 7.4.2: Suppose that (i) the input process of a stationary queueing system is a renewal process whose interval length distribution, $Q(\cdot)$, is uniquely determined by its moments, all of which are assumed to be finite. Further, suppose that (ii) for some initial conditions, $W(x) \xrightarrow{d} W$ as $x \to \infty$, where W is a random variable, finite with probability one. Then a necessary condition for the departure process to be a renewal process is that its interval length distribution is also $Q(\cdot)$.

Proof: Smith (1959) has shown that if the first n interval moments of a renewal process are finite, then the first n cumulants of the counting process, $N(x)$, are all asymptotically proportional to x as $x \to \infty$. He also showed implicitly that there is a one-to-one correspondence between the set of interval moments and the set of asymptotic cumulants of the counting process. For a renewal process satisfying the conditions of the theorem, this result can be summarized using generating functions. Using Wald's lemma, Cox (1962, Section 4.6) has shown that

$$\lim_{x\to\infty} \frac{\log E[\exp\{zN(x)\}]}{x} = \sigma(z), \tag{7.4.5}$$

where $\sigma(z)$ is the unique solution for θ of

$$-\log q^*(\theta) = z \tag{7.4.6}$$

and $q^*(\theta) = \int_0^\infty e^{-\theta x} Q(dx)$. In particular, if $Q(\cdot)$ is uniquely determined by its moments, $Q(x)$ and $\sigma(z)$ uniquely determine each other. This applies by assumption to $N_I(x)$. Now, for all positive integers k

$$\begin{aligned} E[\exp\{zN_D(x)\}] &= E[\exp\{z(N_I(x) - W(x))\}] \\ &\geq \Pr\{|W(x)| \leq k\}.\ E[\exp\{z(N_I(x) - W(x))\}\,|\,|W(x)| \leq k] \\ &\geq \Pr\{|W(x)| \leq k\}.E[\exp\{zN_I(x)\}\,|\,|W(x)| \leq k]e^{-zk}. \end{aligned} \tag{7.4.7}$$

Taking logs of both sides, dividing by x and letting $x \to \infty$, one obtains

$$\liminf_{x\to\infty} \frac{\log E[\exp\{zN_D(x)\}]}{x} \geq \liminf_{x\to\infty} \frac{\log E[\exp\{zN_I(x)\}\,|\,|W(x)| \leq k]}{x} \tag{7.4.8}$$

for all positive integers k. Because $W(x) \xrightarrow{d} W$, a finite random variable, as $x \to \infty$, it follows, by (7.4.5), that the limit of the right-hand side of (7.4.8) as $k \to \infty$ is just $\sigma(z)$. Hence

$$\liminf_{x\to\infty} \frac{\log E[\exp\{zN_D(x)\}]}{x} \geq \sigma(z). \tag{7.4.9}$$

A similar argument holds if we interchange $N_I(x)$ and $N_D(x)$. Then we obtain

$$\limsup_{x\to\infty} \frac{\log E[\exp\{zN_D(x)\}]}{x} \leq \sigma(z). \tag{7.4.10}$$

Equations (7.4.9) and (7.4.10) together give

$$\lim_{x\to\infty} \frac{\log E[\exp\{zN_D(x)\}]}{x} = \sigma(z). \tag{7.4.11}$$

Therefore, if the departure process is a renewal process, then, by previous remarks and assumptions, there is only one interval length distribution which can give rise to an asymptotic cumulant generating function of the form (7.4.11). This distribution is $Q(\cdot)$.

Condition (ii) is satisfied by many queueing systems. For systems which must at some time become idle with probability one, we can choose the initial condition that the system is idle. Then W(x) is the number of customers being served or waiting for service at time x. Hence, for single-server queues, the existence of a stationary distribution for queue size will often be enough to satisfy condition (ii).

Theorems 7.4.1 and 7.4.2 lead to the following useful corollary.

<u>Corollary 7.4.3</u>: Suppose we have an infinite capacity queue whose input process is a Poisson process with mean interval length λ^{-1}. Further, suppose that the departure process is a Markov renewal process satisfying the conditions of Theorem 7.4.1 and that condition (ii) of Theorem 7.4.2 is satisfied. Then the departure process is a renewal process if and only if

$$\Phi(s;z) = \frac{\lambda\Phi(0;z)(z-1)}{s\{\Phi(0;z) - 1\} + \lambda(z-1)} . \tag{7.4.12}$$

The departure process is a Poisson process with mean inter-departure times λ^{-1} (i.e. $p^*(s) = \lambda/(\lambda+s)$) and $\Phi(s;1) = \rho/(\rho+s)$, where $\rho = \lambda/(1-\pi_0)$.

The proof of the above corollary involves a simple combination of Theorems 7.4.1 and 7.4.2, after noting that condition (i) of Theorem 7.4.2 is satisfied by the Poisson process.

Two examples of Corollary 7.4.3 are now given.

7.4.1 <u>The M/G/1/∞ queue</u>

We use the notation of Section 7.2. The M/G/1/∞ queue is stationary if and only if the mean service time is less than λ^{-1}, the mean inter-arrival time. It is easily seen that the departure process of this stationary queue is a Markov renewal process with infinite state space (Disney, Farrell and de Morais, 1973, Theorem 2.4). In particular the process satisfies the conditions of Theorem 7.4.1 with

$$\Psi_k(s) = \frac{(-\lambda)^k}{k!} \frac{d^k}{d\lambda^k} g^*(\lambda+s).$$

Hence $\Phi(s;z) = g^*(s+\lambda(1-z))$ and $p^*(s) = g^*(s)\{1 - \pi_0 s/(\lambda+s)\}$ by (7.4.3). Since the arrival process (which in this case is identical to the input process) is a Poisson (λ) process, it follows from Corollary 7.4.3 that a necessary condition for the process to be renewal is that $\Phi(s;1) = g^*(s) = \rho/(\rho+s)$; in this case $\Phi(s;z) = \rho/(\rho+s+\lambda(1-z))$. It is easily shown that this form does indeed satisfy (7.4.12). The stationarity assumption is equivalent to $\rho > \lambda$. We have therefore shown that the <u>M/G/1/∞ queue has a renewal departure process if and only if the service time is exponentially distributed with parameter $\rho > \lambda$; the departure process is then Poisson with mean inter-departure time λ^{-1}</u>.

7.4.2 The Davignon-Disney feedback queue

We again use the notation of Section 7.2, where a brief outline of the process is given. More formally, let S_k be the length of the kth service time. Each service time is assumed to be independent of everything else that happens in the system. Let Z_k be the number queueing for service just after the end of the kth service (excluding the customer currently being served). Without loss of generality, assume that a customer who is fed back after a service goes to the front of the queue and immediately receives service again; this assumption makes no difference to the departure process but it simplifies the algebra. Let

$$Y_k = \begin{cases} 1 \text{ if the customer feeds back after the kth service,} \\ 0 \text{ if the customer departs after the kth service.} \end{cases}$$

Y_k is a random variable defined by the following switching rules:

$$\Pr\{Y_k = v \mid Y_{k-1} = u,\ Z_{k-1} = i,\ Z_k = j,\ S_k = y\} = \begin{cases} r_{0v}(j;y) \text{ if } u+i = 0, \\ r_{uv}(\ell;y) \text{ if } u+i > 0; \end{cases} \qquad (7.4.13)$$

where $\ell = j+1-u-i$; for $u,v = 0,1$; $i,j = 0,1,2,\ldots$. Note that $r_{u0}(j;y) + r_{u1}(j;y) = 1$.

We exclude trivial and degenerate cases, such as zero probability of either departing or feeding back; the latter case is just the $M/G/1/\infty$ queue again. Davignon and Disney give conditions for the system to be stationary (Theorem 3.2.4). When this is so, they show that both the output process (i.e. the times when a service ends) and the departure process are both Markov renewal processes (Theorems 5.2.1 and 6.2.1 respectively). While the transition probabilities of the output process are simple to write down, those of the departure process are not, because of the complicated dependence on (i) the number of feedbacks a customer undergoes before departing the system and (ii) the number of customers who arrive during each of his services. However, after some reflection, it can be seen that the departure process has a transition matrix $\underset{\sim}{F}(s)$ of the form given in Theorem 7.4.1. The Davignon-Disney queue also has a Poisson input process. Hence, Corollary 7.4.3 can be applied to determine when the departure process is a renewal process. This is done now for two special forms of $r_{uv}(j;y)$.

(a) <u>Feedback dependent only on length of last service</u>

Here the probability of feeding back is independent of whether or not the previous unit fed back and the number of customers who arrived during the service just completed. Hence we can write $r_{u1}(j;y) = r(y)$, $r_{u0}(j;y) = 1 - r(y)$, $(u = 0,1;\ j = 0,1,2,\ldots)$. In this case, the <u>effective</u> service time of each customer is independent of everything else that happens in the system. Hence it is just a special case of the $M/G/1/\infty$ queue in the previous example. Let $g(y)$ be the service time density, let $R(y) = r(y)g(y)$ and let $g^*(s)$ and $R^*(s)$ be their respective Laplace transforms. Then the Laplace transform of the effective service time is given by

$$\{g^*(s) - R^*(s)\} \sum_{k=0}^{\infty} \{R^*(s)\}^k = \frac{g^*(s) - R^*(s)}{1 - R^*(s)} . \tag{7.4.14}$$

By the example in Section 7.4.1, the departure process is a renewal process if and only if the expression in (7.4.14) equals $\rho/(\rho+s)$, where $\rho > \lambda$. This equation is easily solved to give the result

$$1 - r(y) = \frac{\rho \int_y^{\infty} g(t)dt}{g(y)} \equiv \frac{\rho}{\text{failure rate}} . \tag{7.4.15}$$

The right-hand side of (7.4.15) is a probability if and only if the failure rate is not less than ρ for all y for which the failure rate is defined. Two simple examples are (i) $g(y) = \nu e^{-\nu y}$, $y \geq 0$, $\nu > \rho$, in which case $r(y) = 1 - \rho/\nu$, a constant; and (ii) $g(y) = 1$, $0 \leq y \leq 1$, $\rho < 1$, in which case $r(y) = 1 - \rho(1-y)$, $0 \leq y \leq 1$. Example (i) was noticed by Davignon and Disney, although they did not find the general form (7.4.15).

(b) Feedback dependent only on number of arrivals during last service

Here the probability of feeding back is independent of whether or not the previous unit fed-back and the length of the service just completed. Hence we can write $r_{u1}(j;y) = r_j$, $r_{u0}(j;y) = 1 - r_j$ ($u = 0,1$; $j = 0,1,2,\ldots$). This case is not a special example of the $M/G/1/\infty$ queue; we use the result of Corollary 7.4.3 to show that the departure process is a renewal process if and only if the service time distribution is exponential (with mean ν say) and all the r_j's are equal (to r, say); we require $\nu(1-r) > \lambda$ for stationarity. This case is equivalent to example (i) of case (a).

The proof is by induction. Let $\Psi_k \equiv \Psi_k(0)$. From the definition of $\Phi(s;z)$ and (7.4.12) we have

$$\Psi_0(s) \equiv \Phi(s;0) = \frac{\lambda\Psi_0}{\lambda + s(1-\Psi_0)} . \tag{7.4.16}$$

Now $\Psi_0(s)$ is the Laplace transform of the joint probability that the time between successive departures is in $[t, t+dt)$ and that there are no arrivals during this time. Hence

$$\Psi_0(s) = (1-r_0)g^*(s+\lambda) \sum_{k=0}^{\infty} \{r_0 g^*(s+\lambda)\}^k = \frac{(1-r_0)g^*(s+\lambda)}{1-r_0 g^*(s+\lambda)} . \tag{7.4.17}$$

Equating (7.4.16) and (7.4.17) and replacing $s + \lambda$ by s, one obtains easily the necessary condition that $g^*(s) = \nu/(\nu+s)$, where $\nu = \lambda\Psi_0/(1-r_0)(1-\Psi_0)$. Differentiating (7.4.12) k times and putting $s = 0$, one obtains the form

$$\Psi_k(s) = \frac{\sum_{i=0}^{k} \omega_{k,i}\, s^i}{\{\lambda + s(1-\Psi_0)\}^{k+1}} , \tag{7.4.18}$$

where the $\omega_{k,i}$ are unspecified constants. Now assume that $r_j = r$ $(j = 0,1,2,\ldots,k-1)$. Then knowing that $g(s) = \nu/(\nu+s)$ and by considering all possible ways in which k customers can arrive during the <u>effective</u> service time, it can be shown that

$$\Psi_k(s) = \nu\lambda^k \quad \frac{(1-r)}{\{\lambda+\nu(1-r)+s\}^{k+1}} + \frac{(r-r_k)(\lambda+s)}{(\lambda+\nu+s)^k\{\lambda+\nu(1-r)+s\}^2} . \tag{7.4.19}$$

It is easily seen that this can only be of the form (7.4.18) if $r_k = r$. It follows therefore that a necessary condition for the departure process to be a renewal process is that all the r_j's are equal. That this is also a sufficient condition follows from example (i) of case (a).

7.5 Developments

The work of this chapter raises the possibility of some interesting developments. In Theorem 7.3.1, we have found, under restrictive assumptions, necessary and sufficient conditions for a Markov renewal process to be a renewal process. In Corollary 7.3.2, necessary and sufficient conditions were given for a two state Markov renewal process to be a renewal process, without any restrictive assumptions. That these conditions are sufficient for Markov renewal processes with any number of states is easily shown using (7.3.4). However, they are not necessary conditions when there are more than two states; this is not too difficult to show for a Markov renewal process with three states. It would be of interest to find necessary and sufficient conditions for any Markov renewal process to be a renewal process; this might possibly be done using (7.3.4).

An interesting unsolved problem concerns when the GI/G/1/∞ queue has a renewal departure process. The GI/G/1/∞ queue has a renewal arrival process with arbitrary interval distribution; i.i.d. service times with arbitrary distribution, possibly different from the inter-arrival time distribution, and independent of the arrival process; and infinite capacity. Until recently, the only two non-trivial such queues which were known to have renewal departure processes were the M/M/1/∞ (mentioned previously) and the queue with constant service time, C, and an inter-arrival time which is with probability 1 greater than or equal to C. However, U. Zähle has shown, in an as yet unpublished paper, that there are at least two more GI/G/1/∞ queues with renewal departure processes. In the first queue, the inter-arrival time and the service time both have negative exponential distributions, but each is displaced to the right by the same constant amount. Zähle's second example is the discrete analogue of the first: the inter-arrival time and service time both have displaced geometric distributions.

Daley (1971, Section 5) has speculated that a useful first step in finding necessary and sufficient conditions for the $GI/G/1/\infty$ queue to have a renewal output might be to show that if the departure process is a renewal process, then its interval distribution must be the same as that of the arrival process. This has been shown to be true in Theorem 7.4.2 under the assumption that the inter-arrival time distribution is uniquely determined by its moments (assumption (i)). However, Theorem 7.4.2 has not so far enabled me to obtain the necessary and sufficient conditions even under the restriction of assumption (i) of that theorem. Finally, it would be useful to obtain the result of Theorem 7.4.2 without the constraint of assumption (i); it appears that the asymptotic argument used in the proof of that theorem will be insufficient to do this.

Appendix 1: A Property of Stationary Univariate Point Processes

In this appendix and Appendix 2, we prove a number of results which are used to prove Theorems 4.3.1, 4.3.3 and 4.3.4. In this appendix, we make use of equation (2.5.7) to obtain a simple, but useful, result for a wide class of stationary univariate point processes. Let $N^{(0)}(x)$ and $N^{(e)}(x)$ be the number of events in the interval $(0,x]$ in the synchronous and asynchronous cases respectively. Let $H(x) = E\{N^{(0)}(x)\}$ and let $E\{N^{(e)}(x)\} = \beta^{-1}x$, $0 < \beta < \infty$. Assume that $H(x)$ is differentiable a.e. with derivative $h(x)$, and that $\lim_{x\to\infty} h(x) = \beta^{-1}$. These assumptions are satisfied by a wide class of stationary univariate point processes. Then we have

Theorem A1.1: If $h(x)$ is non-decreasing, then for all $x \geq 0$

$$\mathrm{var}\{N^{(e)}(x)\} \leq E\{N^{(e)}(x)\}. \tag{A1.1}$$

The inequality is reversed if $h(x)$ is non-increasing.

Proof: Assume that $h(x)$ is non-decreasing; the proof for non-increasing $h(x)$ is analogous. Then, for all $x \geq 0$, $h(x) \leq \beta^{-1}$ and hence $\int_0^x H(y)dy \leq x^2/2\beta$. Then, using (2.5.7), we obtain (A1.1).

Appendix 2: Some Properties of Renewal Processes

In this appendix, we obtain some useful results for renewal processes. We use the same notation and make the same assumptions as in Appendix 1. We also require the following notation and assumptions. Let $g(\cdot)$ and $\mathcal{G}(\cdot)$ be the p.d.f. and survivor function respectively of the inter-event times of the renewal process. Then from the theory of renewal processes, it follows that $\beta = \int_0^\infty xg(x)dx$. Assume that there exists a $c > 1$ such that $\int_0^\infty [g(x)]^c dx < \infty$. Then it is known that $\lim_{x\to\infty} h(x) = \beta^{-1}$ (Smith, 1954, Theorem 12).

The function $a(x) = g(x)/\mathcal{G}(x)$ is usually called the *failure rate function*. If $a(x)$ is an increasing function of x, we say that $G(=1 - \mathcal{G})$ is IFR (increasing failure rate), while if $a(x)$ is decreasing, we say that G is DFR (decreasing failure rate).

G is said to be New Better than Used (NBU) if for all non-negative x and y

$$\mathcal{G}(x+y) \leq \mathcal{G}(x)\,\mathcal{G}(y). \tag{A2.1}$$

If the inequality is reversed, G is said to be New Worse than Used (NWU). G is said to be New Better than Used in Expectation (NBUE) if for all non-negative x

$$\int_x^\infty \mathcal{G}(y)dy \leq \beta\mathcal{G}(x). \tag{A2.2}$$

If the inequality is reversed, G is said to be New Worse than Used in Expectation (NWUE). The above terminology is in common use in reliability theory. Barlow and Proschan (1975, Section 6.2) give the implications

$$\begin{array}{lllll} \text{IFR} & \Rightarrow & \text{NBU} & \Rightarrow & \text{NBUE}\ , \\ \text{DFR} & \Rightarrow & \text{NWU} & \Rightarrow & \text{NWUE}\ . \end{array} \tag{A2.3}$$

Barlow and Proschan also give a number of results for renewal processes when G has some of the above properties. These lead to the following useful results.

Theorem A2.1: If G is NBUE, then for all $x \geq 0$

$$\mathrm{var}\{N^{(e)}(x)\} \leq E\{N^{(e)}(x)\}. \tag{A2.4}$$

The inequality is reversed if G is NWUE.

Proof: Assume that G is NBUE; the proof is analogous if G is NWUE. By Theorem 3.15(b), p.171, of Barlow and Proschan (1975), if G is NBUE, then $H(x) \leq \beta^{-1}x$, and so $\int_0^x H(y)dy \leq x^2/2\beta$, whence we obtain (A2.4) using (2.5.7).

Theorem A2.2: If G is NBU, then for all $x \geq 0$

$$\mathrm{var}\{N^{(0)}(x)\} \leq H(x) \leq \beta^{-1}x. \tag{A2.5}$$

The inequalities are reversed if G is NWU.

Proof: This is just a combination of Theorems 3.14(b), p.171, and 3.19, p.174, of Barlow and Proschan (1975), and (A2.3).

Theorem A2.3: If h(x) is non-decreasing, then for all $x \geq 0$

$$\mathrm{var}\{N^{(0)}(x)\} \leq H(x) \leq \beta^{-1}x. \tag{A2.6}$$

The inequalities are reversed if h(x) is non-increasing.

Proof: Assume that h(x) is non-decreasing; the proof for non-increasing h(x) is analogous. We have from (4.3.4)

$$\mathrm{var}\{N^{(0)}(x)\} - H(x) = 2\int_0^x \{H(x-y) - H(y)\}h(y)dy$$

$$= 2\int_0^{x/2} \{H(x-y) - H(y)\}\{h(y) - h(x-y)\}dy, \qquad (A2.7)$$

after breaking up the integral into two and a change of variable. Now $0 \leq y \leq x/2$ implies $y \leq x-y$ which implies (a) $H(x-y) - H(y) \geq 0$ (as H(x) is always non-decreasing), and (b) $h(x-y) - h(y) \geq 0$ (as we have assumed that h(x) is non-decreasing). Using these two facts and (A2.7), we obtain the left-hand inequality of (A2.6). The right-hand inequality follows easily since $h(x) \leq \beta^{-1}$, for all $x \geq 0$.

Since renewal processes are stationary point processes, it follows that Theorem A1.1 applies to renewal processes, and so it is interesting to note that h(x) non-decreasing or non-increasing give rise to the same inequalities concerning var{N(x)} and E{N(x)} as do G is NBU or NWU, respectively, in both the synchronous and asynchronous cases; compare Theorems A1.1 and A2.3 with Theorems A2.1 and A2.2, respectively.

Note, however, that G is NBU does not imply h(x) is non-decreasing. For instance, let $g(x) = \frac{1}{2}\alpha^3 x^2 e^{-\alpha x}$, $x \geq 0$, $\alpha > 0$. This is the p.d.f. of a Gamma $(3,\alpha)$ variable. Barlow and Proschan (1975, Section 3.5) show that this distribution is IFR, and hence (by (A2.3)) is NBU. However, Cox (1962, Section 4.3, equation (9)) gives H(x) for this distribution. Differentiating this twice we obtain

$$H''(x) = h'(x) = \frac{2}{\sqrt{3}}\alpha^2 \exp(-\frac{3\alpha x}{2}) \sin(\frac{\sqrt{3}\alpha x}{2}),$$

which can be either positive or negative (depending on x). Hence $h(x)$ is neither non-decreasing nor non-increasing. Whether G is NWU implies $h(x)$ is non-increasing is apparently unknown. However, recently Brown (1978) has proved that G is DFR implies $h(x)$ is non-increasing.

Appendix 3: The Asymptotic Mean and Variance of a Random Variable Used in Section 5.3

Suppose we observe n consecutive intervals of a renewal process whose interval p.d.f. and survivor function are given by $b(\cdot)$ and $\mathcal{B}(\cdot)$ respectively. Assume that $\mathcal{B}(x) = 1 - \beta x^{\gamma} + o(x^{\gamma})$ as $x \to 0$ ($\beta > 0$, $\gamma > 0$). Denote the ith interval by X_i ($i = 1,\ldots,n$), and let $\Delta = \min\{X_i + X_{i+1}, i = 1,\ldots,n-1\}$. Let L be the number of X_i's less than Δ. Both L and Δ are used in a test described in Section 5.3.4. In this appendix, a heuristic derivation of the asymptotic mean and variance of L is given. Now

$$\begin{aligned} P(\Delta > x) &= P\{\bigcap_{i=1}^{n-1} (X_i + X_{i+1} > x)\} \\ &= P(X_1 + X_2 > x) . \prod_{i=2}^{n-1} P(X_i + X_{i+1} > x | X_{i-1} + X_i > x) \end{aligned} \tag{A3.1}$$

by the renewal assumption. Then it follows easily that

$$P(X_1 + X_2 > x) \equiv \mathcal{B}_2(x) = 1 - \beta^2 a x^{2\alpha} + o(x^{2\gamma}), \tag{A3.2}$$

$$P(X_i+X_{i+1} > x|X_{i-1}+X_i > x) = \frac{\mathcal{B}(x) + \int_0^x b(y)\{\mathcal{B}(x-y)\}^2 dx}{\mathcal{B}_2(x)}$$

$$= 1 - \beta^2 a x^{2\gamma} + o(x^{2\gamma}) \tag{A3.3}$$

as $x \to 0$, where

$$a = \frac{\{\Gamma(\gamma + 1)\}^2}{\Gamma(2\gamma + 1)} . \tag{A3.4}$$

Then, combining (A3.1) to (A3.3), and using a standard technique for determining asymptotic extreme value distributions, one obtains $P(\Delta > x/\{\beta^2 a(n-1)\}^{1/2\gamma}) \to \exp\{-x^{2\gamma}\}$ as $n \to \infty$, or

$$P(\Delta > x) \sim \exp\{-\beta^2 a(n-1)x^{2\gamma}\} \tag{A3.5}$$

as $n \to \infty$.

Now suppose that $\Delta = x = X_j + X_{j+1}$. Then $X_j \leq x$, $X_{j+1} \leq x$, $X_i + X_{i+1} \geq x$, all $i \neq j$, and $L \geq 2$. The distribution of L conditional on $\Delta = x$ will be asymptotically independent of j. Hence, assume without loss of generality that $j = n - 1$. If we define the indicator variable W_i by $W_i = 0$ if $X_i \leq \Delta$, and $W_i = 1$ if $X_i > \Delta$ $(i = 1,\ldots,n-2)$, then, conditional on $\Delta = x$, it is easily seen that the W_i's form a two state first order Markov chain and that $L - 2$ is the number of times that $W_i = 0$. The transition probabilities of this Markov chain are given by

$$1 - p_0(x) \equiv P(W_i = 1 | W_{i-1} = 0,\ \Delta = x)$$

$$= P(X_i > x | X_{i-1} \leq x,\ X_{i-1} + X_i > x,\ X_i + X_{i+1} > x)$$

$$= \frac{\mathcal{B}_2(x)\,\mathcal{B}(x)\{1 - \mathcal{B}(x)\}}{\{\mathcal{B}(x) + \int_0^x b(y)\,\mathcal{B}(x-y)dy\}\{\mathcal{B}_2(x) - \mathcal{B}(x)\}}$$

$$= 1 - \beta x^{\gamma} + o(x^{\gamma}) \tag{A3.6}$$

as $x > 0$;

$$p_1(x) \equiv P(W_i = 1 | W_{i-1} = 1, \quad \Delta = x)$$

$$= P(X_i > x | X_{i-1} > x, X_{i-1} + X_i > x, X_i + X_{i+1} > x)$$

$$= \frac{\mathcal{B}(x)}{\mathcal{B}_2(x)}$$

$$= 1 - \beta x^{\gamma} + o(x^{\gamma}) \tag{A3.7}$$

as $x \to 0$. Note from (A3.6) and (A3.7) that as $p_0(x)$ and $1 - p_1(x)$ are both $\beta x^{\gamma} + o(x^{\gamma})$ as $x \to 0$, the W_i's are approximately Bernouilli random variables for small x. Hence, conditional on $\Delta = x$, $L - 2$ behaves approximately like a binomial random variable with parameters $n - 2$ and βx^{γ} as $x \to 0$. Now, by (A3.5), the unconditional mean of L as $n \to \infty$ is

$$E(L) \sim 2\gamma\beta^2 a(n-1) \int_0^{\infty} E(L|\Delta = x).x^{2\gamma-1} \exp\{-\beta^2 a(n-1)x^{2\gamma}\}dx. \tag{A3.8}$$

Making the substitution $y = x^{2\gamma}$, we see that the above integral is a Laplace transform with parameter $\beta^2 a(n-1)$. By a well-known Abelian theorem (Widder, 1946, Section 5, Corollary 1a) the asymptotic behaviour of $E(L|\Delta = x)$ for small x determines the asymptotic behaviour of $E(L)$ for large n. Hence, putting the asymptotic binomial approximation for $E(L|\Delta = x)$ in (A3.8), one obtains $E(L) \sim \{n\pi/(4a)\}^{1/2}$ as $n \to \infty$. By a similar argument, it can be shown that $\mathrm{var}(L) \sim n(1-\pi/4)/a$ as $n \to \infty$.

Appendix 4: Four Sets of Data

In this appendix details are given concerning the four artificial sets of data to which tests discussed in Chapter 5 are applied. In Table A1 some of the theoretical marginal properties of the process of type 1 events are given for the four processes. In Table A2, the actual occurrence times of the type 0 and type 1 events generated by simulating the four processes are listed.

Each of the four processes is an example of a regenerative bivariate point process. Processes (i) and (ii) are both examples of the process considered in Section 4.2; processes (ii), (iii) and (iv) are all examples of the process considered in Section 4.3. For all four processes, the process of type 0 events was generated by a Poisson process of rate 1 (i.e. $f(x) = e^{-x}$, $x \geq 0$). Process (i) was obtained by imbedding between successive type 0 events an inhomogeneous Poisson process with rate function $\lambda(x) = 10e^{-x}$, $x \geq 0$. Process (ii) was obtained by imbedding between successive type 0 events a homogeneous Poisson process of rate 5 (i.e. $g(x) = 5e^{-5x}$, $x \geq 0$). Process (iii) was obtained by imbedding between successive type 0 events an ordinary renewal process with p.d.f. $g(x) = \exp(-x/.44)/(.44\pi x)^{1/2}$, $x \geq 0$ (i.e. a Gamma $(.5, (.44)^{-1})$ distribution). Process (iv) was obtained by imbedding between successive type 0 events an ordinary renewal process with p.d.f. $g(x) = 100x^{-10x}$, $x \geq 0$ (i.e. a Gamma (2,10) distribution).

Beside each quantity in Table A1, a section reference is given. This gives the section where the notation is initially defined. For the interval properties, results are given correct to 4 decimal places. The synchronous mean function, $E\{N^{(1)}(x)\}$ is not given for process (iii) as it is too long and complicated to put in the table. The asymptotic form of var $N(x)$ is readily obtained from $E\{N^{(1)}(x)\}$ using equations (2.5.5) and (2.5.9). For processes (ii), (iii) and (iv), these are more easily obtained another way. Since all three processes have imbedded processes which are renewal processes and since in each case the process of type 0 events is a Poisson process, we know by Theorem 4.3.2 that the process of type 1 events forms a renewal process. Hence $\mathrm{var}\{N(x)\} \sim \mathrm{var}(X_1)x/\{E(X_1)\}^3$ as $x \to \infty$. Further the serial correlation coefficients for these three processes are all zero. Note that process (i) is the same as process (a) in Section 5.3.2. As in Table 5.1 of that section, only the first two serial correlation coefficients have been given as, for process (i), the theoretical correlation coefficients of lag greater than 2 are very complicated.

In Table A2, we list the actual occurrence times of the type 0 and type 1 events generated by simulating the four processes. For all four processes, data were generated over the interval [0,20]. Each of the four processes simulated used the same realization for the process of type 0 events. Each process had a type 0 event at the origin and 13 type 0 events in (0,20] (i.e. $M = 14$). Let x_i be the time of the ith type 0 event in (0,20] and define $x_0 \equiv 0$ and $x_M \equiv T$. In Table A2(j) (j = i,ii,iii,iv), on the ith line is given: n_i, the number of type 1 events in $[x_{i-1},x_i)$; x_{i-1}; and the times of the type 1 events in $[x_{i-1},x_i)$. At the bottom of the n_i column is $N = \Sigma_i n_i$, the number of type 1 events in [0,20]. All times are given correct to 3 decimal places, except where two consecutive type 1 events are less than 10^{-3} apart; in this case, times are given to as many decimal places as are necessary to distinguish the times of the two events.

Table A1 Some properties of the type 1 process for four regenerative multivariate point processes

	Section Ref.	Process (i)	Process (ii)	Process (iii)	Process (iv)
$E\{N^{(1)}(x)\}$	3.1, 3.2	$5\{x + \frac{1}{6}(1-e^{-2x})\}$	$5x$		$\frac{100}{21^2}\{21x-1+e^{-21x}\}$
" (for small x)		$6.6667x + o(x)$	$5x$	$1.7011x^{1/2} + o(x^{1/2})$	$50x^2 + o(x^2)$
$\lim_{x\to\infty} \mathrm{var}\{N(x)\}/x$	3.1	13.3333	5.0000	9.1667	2.6023
$E(X_1)$	3.5	.2000	.2000	.2000	.2100
$\mathrm{var}(X_1)$	3.5	.0880	.0400	.0733	.0241
ρ_1	4.1	.1009	.0000	.0000	.0000
ρ_2	4.1	.0363	.0000	.0000	.0000

Table A2(i) The times of events generated by process (i)

i	n_i	Type 0 Events	Type 1 Events
1	10	0.000	.030, .140, .178, .271, .281, .394, .512, .527, 1.463, 1.468;
2	3	2.588	2.687, 3.177, 3.209;
3	12	3.380	3.419, 3.854, 3.989, 4.067, 4.072, 4.199, 4.286, 4.322, 4.889, 5.074, 5.211, 5.286;
4	8	5.346	5.373, 5.404, 5.584, 5.739, 5.873, 5.882, 6.001, 6.193;
5	1	6.777	6.836;
6	6	7.028	7.153, 7.233, 7.907, 8.340, 9.101, 9.170;
7	6	9.302	9.612, 9.680, 9.737, 10.383, 10.417, 11.005;
8	6	11.599	11.605, 11.672, 11.704, 12.125, 12.175, 12.435;
9	7	12.704	12.768, 12.948, 13.149, 13.181, 13.505, 13.611, 14.021;
10	8	15.643	15.853, 16,006, 16.227, 16.345, 16.386, 16.487, 16.514, 16.764;
11	9	16.830	16.878, 16.883, 16.966, 17.017, 17.134, 17.312, 17.699, 17.830, 17.927;
12	7	17.928	17.960, 18.009, 18.074, 18.096, 18.133, 18.278, 18.395;
13	1	18.500	18.611;
14	10	18.718	18.727, 18.742, 18.830, 18.967, 19.076, 19.254, 19.439, 19.871, 19.878, 19.978.
	94		

Table A2(ii) The times of events generated by process (ii)

i	n_i	Type 0 Events	Type 1 Events
1	11	0.000	.019, .040, .097, .119, .530, .820, .937, 1.299, 1.361, 1.647, 2.262;
2	3	2.588	2.672, 2.773, 2.844;
3	9	3.380	3.550, 3.677, 3.823, 4.265, 4.437, 4.562, 4.716, 4.916, 5.346;
4	10	5.346	5.798, 5.892, 5.977, 6.106, 6.112, 6.300, 6.372, 6.511, 6.728, 6.734;
5	0	6.777	;
6	15	7.028	7.149, 7.153, 7.801, 7.868, 7.942, 8.211, 8.361, 8.364, 8.750, 8.761, 9.007, 9.178, 9.233, 9.240, 9.286;
7	13	9.302	9.484, 9.485, 9.614, 9.696, 9.835, 10.047, 10.327, 10.449, 10.606, 10.845, 11.218, 11.486, 11.526;
8	5	11.599	11.635, 11.708, 11.821, 11.892, 12.363;
9	18	12.704	12.847, 12.922, 13.678, 13.943, 14.087, 14.132, 14.155, 14.238, 14.242, 14.445, 14.455, 14.689, 14.762, 15.003, 15.244, 15.369, 15.451, 15.520;
10	3	15.643	15.814, 16.598, 16.747;
11	4	16.830	17.253, 17.472, 17.507, 17.669;
12	0	17.928	;
13	2	18.500	18.532, 18.688;
14	7	18.718	18.831, 19.115, 19.372, 19.469, 19.491, 19.571, 19.950.
	100		

Table A2(iii) The times of events generated by process (iii)

i	n_i	Type 0 Events	Type 1 Events
1	7	0.000	.003, .663, .948, 2.430, 2.491, 2.540, 2.550;
2	2	2.588	2.623, 2.788;
3	18	3.380	3.481, 3.565, 3.570, 3.5832, 3.5834, 3.593, 3.786, 3.852, 3.96488, 3.96490, 4.056, 4.059, 4.098, 4.279, 5.082, 5.144, 5.145, 5.285;
4	4	5.346	5.848, 6.283, 6.311, 6.686;
5	3	6.777	6.781, 6.785, 6.924;
6	16	7.028	7.054, 7.078, 7.163, 7.414, 7.474, 7.492, 7.8006, 7.8012, 7.885, 8.640, 8.802, 8.807, 8.815, 9.232, 9.252, 9.261;
7	9	9.302	9.761, 10.659, 10.692, 10.894, 10.897, 11.110, 11.214, 11.563, 11.595;
8	5	11.599	11.599, 11.679, 11.7320, 11.7324, 12.544;
9	11	12.704	12.934, 13.020, 13.027, 13.041, 13.847, 14.301, 14.358, 15.280, 15.281, 15.562, 15.620;
10	4	15.643	16.300, 16.302, 16.322, 16.403;
11	4	16.830	16.917, 16.921, 16.994, 17.132;
12	3	17.928	17.935, 17.985, 18.234;
13	2	18.500	18.58783, 18.58787;
14	3	18.718	18.834, 19.839, 19.863;
	91		

Table A2(iv) The times of events generated by process (iv)

i	n_i	Type 0 Events	Type 1 Events
1	12	0.000	.339, .842. 1.026, 1.192, 1.211, 1.279, 1.404, 1.619, 1.913, 2.080, 2.408, 2.586;
2	2	2.588	2.652, 3.035;
3	7	3.380	3.806, 3.981, 4.087, 4.395, 4.667, 4.880, 5.208;
4	5	5.346	5.654, 5.901, 6.136, 6.435, 6.703;
5	0	6.777	;
6	17	7.028	7.089, 7.141, 7.480, 7.512, 7.688, 7.768, 7.793, 7.982, 8.152, 8.242, 8.276, 8.587, 8.887, 8.932, 9.036, 9.101, 9.212;
7	10	9.302	9.555, 9.863, 9.904, 10.053, 10.197, 10.532, 10.782, 10.876, 11.225, 11.312;
8	6	11.509	11.875, 11.904, 12.160, 12.477, 12.539, 12.635;
9	8	12.704	13.059, 13.576, 13.993, 14.134, 14.352, 14.690, 14.812, 15.462;
10	5	15.643	15.867, 16.103, 16.355, 16.651, 16.819;
11	4	16.830	17.326, 17.495, 17.722, 17.830;
12	3	17.928	18.055, 18.199, 18.232;
13	1	18.500	18.634;
14	6	18.718	19.095, 19.249, 19.615, 19.677, 19.787, 19.968.
	86		

BIBLIOGRAPHY

BARLOW, R.E. and PROSCHAN, F. (1975). Statistical Theory of Reliability and Life Testing: Probability Models. Holt, Rinehart and Winston, New York.

BARTLETT, M.S. (1963). The spectral analysis of point processes. J.R. Statist. Soc. B 25, 264-296.

BILLINGSLEY, P. (1961). Statistical Inference for Markov Processes. University of Chicago Press.

BROWN, M. (1978). A study of renewal processes with IMRL and DFR interarrival times. Ann. Prob. 6.

BURKE, P.J. (1956). The output of a queueing system. Operat. Res. 4, 699-704.

ÇINLAR, E. (1969). Markov renewal theory, Adv. Appl. Prob. 1, 123-187.

COX, D.R. (1962). Renewal Theory. Methuen, London.

COX, D.R. (1972). The statistical analysis of dependencies in point processes. In Stochastic Point Processes (P.A.W. Lewis (Ed.)). Wiley, New York, 55-66.

COX, D.R. and LEWIS, P.A.W. (1966). The Statistical Analysis of Series of Events. Methuen, London.

COX, D.R. and LEWIS, P.A.W. (1972). Multivariate point processes. Proc. Sixth Berkeley Symp. Math. Statist. 3, 401-448.

COX, D.R. and MILLER, H.D. (1965). The Theory of Stochastic Processes. Chapman and Hall, London.

DALEY, D.J. (1968). The correlation structure of the output process of some single server queueing systems. Ann. Math. Statist. 39, 1007-1019.

DALEY, D.J. (1971). Some problems in the theory of point processes. University of North Carolina at Chapel Hill, Institute of Statistics, Mimeo Series no. 772.

DALEY, D.J. and MILNE, R.K. (1975). Orderliness, intensities and Palm-Khinchin equations for multivariate point processes. J. Appl. Prob. 12, 383-389.

DANIELS, H.E. (1954). Saddlepoint approximations in statistics. Ann. Math. Statist. 25, 631-650.

DAVIGNON, G.R. and DISNEY, R.L. (1976). Single server queues with state-dependent feedback. Can. J. Operat. Res. 14, 71-85.

DISNEY, R.L., FARRELL, R.L. and DE MORAIS, P.R. (1973). A characterization of M/G/1 queues with renewal departure processes. Management Sci. 19, 1222-1228.

FELLER, W. (1968). An Introduction to Probability Theory and its Applications, Vol. 1, 3rd ed. Wiley, New York.

FELLER, W. (1971). An Introduction to Probability Theory and its Applications, Vol. 2, 2nd ed. Wiley, New York.

FINCH, P.D. (1958). The effect of the size of the waiting room on a simple queue. J.R. Statist. Soc. B 20, 182-186.

FINCH, P.D. (1959). The output of the queueing system M/G/1. J.R. Statist. Soc. B 21, 375-380.

GAVER, D.P. (1963). Random hazard in reliability problems. Technometrics 5, 211-226.

GRANDELL, J. (1976). Doubly Stochastic Poisson Processes. Springer-Verlag, Berlin.

KING, R.A. (1971). The covariance structure of the departure process from M/G/1 queues with finite waiting lines. J.R. Statist. Soc. B 33, 401-406.

KINGMAN, J.F.C. (1964). On doubly stochastic Poisson processes. Proc. Camb. Phil. Soc. 60, 923-930.

KLOTZ, J. (1973). Statistical inference in Bernouilli trials with dependence. Ann. Stat. 1, 373-379.

LAWRANCE, A.J. (1970). Selective interaction of a Poisson process and a renewal process:- first-order stationary point results. J. Appl. Prob. 7, 359-372.

LAWRANCE, A.J. (1971). Selective interaction of a Poisson process and a renewal process: the dependency structure of the intervals between responses. J. Appl. Prob. 8, 170-183.

LEADBETTER, M.R. (1972). On basic results of point process theory. Proc. Sixth Berkeley Symp. Math. Statist. 3, 449-462.

LEWIS, P.A.W. (1964). A branching Poisson process model for the analysis of computer failure patterns. J.R. Statist. Soc. B 26, 398-456.

LEWIS, P.A.W. (1970). Asymptotic properties of branching renewal processes. IBM Research Report, RC2878. Yorktown Heights, N.Y.

MILNE, R.K. (1971). Stochastic analysis of multivariate point processes. Ph.D. Thesis, The Australian National University, Canberra.

NEUTS, M.F. (1965). Discussion following REICH, E. Departure processes. Proc. Symp. Congestion Theory. University of North Carolina Press, Chapel Hill.

NEUTS, M.F. (1971). A queue subject to extraneous phase changes. Adv. Appl. Prob. 3, 78-119.

OAKES, D. (1972). Semi-Markov representations of some stochastic point processes. Ph.D. Thesis, Imperial College, University of London.

PYKE, R. (1961a). Markov renewal processes! definitions and preliminary properties. Ann. Math. Statist. 32, 1231-1242.

PYKE, R. (1961b). Markov renewal processes with finitely many states. Ann. Math. Statist. 32, 1243-1259.

PYKE, R. and SCHAUFELE, R.A. (1966). The existence and uniqueness of stationary measures for Markov renewal processes. Ann. Math. Statist. 37, 1439-1462.

RAO, C.R. and CHAKRAVARTI, I.M. (1956). Some small sample tests of significance for a Poisson distribution. Biometrics 12, 264-282.

REICH, E. (1957). Waiting times when queues are in tandem. Ann. Math. Statist. 28, 768-773.

ROHDE, H. and GRANDELL, J. (1972). On the removal of aerosol particles from the atmosphere by precipitation scavenging. Tellus 24, 443-454.

RUDEMO, M. (1972). Doubly stochastic Poisson processes and process control. Adv. Appl. Prob. 4, 318-338.

SMITH, W.L. (1954). Asymptotic renewal theorems. Proc. R. Soc. Edinburgh A 64, 9-48.

SMITH, W.L. (1955). Regenerative stochastic processes. Proc. R. Soc. A 232, 6-31.

SMITH, W.L. (1959). On the cumulants of renewal processes. Biometrika 46, 1-29.

TEN HOOPEN, M. and REUVER, H.A. (1965). Selective interaction of two independent recurrent processes. J. Appl. Prob. 2, 286-292.

VERE-JONES, D. (1970). Stochastic models for earthquake occurrence. J.R. Statist. Soc. B 32, 1-62.

WELCH, B.L. (1937). The significance of the difference between two means when the population variances are unequal. Biometrika 29, 350-362.

WIDDER, D.V. (1946). The Laplace Transform. Princeton University Press.

WISNIEWSKI, T.K.M. (1972). Bivariate stationary point processes, fundamental relations, and first recurrence times. Adv. Appl. Prob. 4, 296-317.