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PHASE TRANSITIONS AND MAGNETIC PROPERTIES OF SUPERCONDUCTING-NORMAL SANDWICHES

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ABSTRACT

When a superconductor S and a normal metal N are put into contact, the Cooper pairs which make up superconductivity in S can leak into N and induce superconductivity into N by proximity effects.

We demonstrate that the order parameter at the outer surface of N, where the supercurrent is required to vanish, can take one of two possible values. This arbitrariness leads to two different modes for the superconducting in the sandwich, whose respective energies are differently affected by variation of the external conditions such as the temperature and the magnetic field.

In this thesis, within the domain of validity of the local Ginzburg-Landau GL theory, we investigate the occurrence of first order phase transitions in superconducting-normal S/N sandwiches. Experimentally, such a phase transition has been observed as a response to the external magnetic field, whereby induced superconductivity in which the pair potential is finite throughout the N, gives way at and above a given field (the breakdown field, H_b) to a second mode for which the field penetrates into N and destroys the induced superconductivity.

In the absence of a magnetic field, a first order phase transition (which has not yet been observed experimentally) is predicted due to competition between the two possible modes.

In addition to the calculation of upper critical field of type II superconductors we study the extreme type II S in a magnetic field. The effective penetration depth and the barrier field are also evaluated.

It is shown that in the presence of a magnetic field, the S/N binary layers in limiting case for which GL parameters of S and N both are much smaller than unity, yield a first order phase change (due to the aforementioned arbitrariness of boundary conditions) as a function of the field, the temperature or the normal sample thickness. Thus the breakdown field effect is accounted for as a direct extension of the previous phase transition.

This is compared with theoretical work and experiments carried out by the Orsay and the Imperial College groups. We conclude that all phase transitions in S/N binary layers can be simply and explicitly explained by the interplay of the two possible boundary conditions at the N-vacuum interface.

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CHAPTER 1

PHASE TRANSITIONS IN SUPERCONDUCTING-NORMAL SANDWICHES

1.I Introduction:

In (1950) Ginzburg and Landau (GL) established the theory of superconductivity, based on the pioneering work of Landau (1937) on second-order phase transitions. GL started their argument by introducing the quantity $\Delta(r)$ (called the "order parameter") to characterize the degree of superconductivity at various points in the material. The order parameter is defined so as to be zero for the normal phase for which the temperature T is greater than the critical temperature T_C of the bulk material in zero magnetic field (now bearing the name of T_C) and *non-zero* for the superconducting phase for which $T < T_C$. At $T = T_C$ a transition occurs from the normal phase to the superconductive phase.

The superconductivity arises from an attraction (Frölich, 1952) electrons interacting via the lattice which results in the pairing of electrons in opposite spin and momentum states (Cooper pairs). This was proposed by Bardeen, Cooper and Schrieffer (BCS) in 1957 and is the starting point of the BCS microscopic theory. According to this theory, the strength of such attractive interaction is constant and is characterized by the magnitude of the interaction parameter V which is positive for a superconductor in the convention we adopt. The quantity V can be related to T_C by $k_B T_C = 1.14\hbar\omega_D \exp\{-1/N(0)V\}$ for all weak-coupling superconductors for which $N(0)V << 1$. ω_D is the Debye cut-off frequency and N(0) is the density of states at the Fermi level for electrons of one spin.

This chapter is devoted to general discussion of the occurrence of phase transitions in the binary system which consists of two metals one with higher critical temperature than the other. The one with lower critical temperature which is called "normal" metal (N) is placed in good electrical contact with the one with higher critical temperature (called superconductor (S)). A good electrical contact between these two materials does not allow inter-diffusion, or the formation of an intermetallic compound. The S/N sandwich is studied in the temperature range of $T_{CN} \leq T \leq T_{CS}$, where T_{Ci} is the bulk critical temperature of the sample in zero magnetic field. The subscripts (i = S,N) are used to refer to the superconductor and the normal samples respectively, throughout this thesis.

By proximity effect, the Cooper pairs, from S, can leak into N (induced superconductivity) and the electrons, in N, penetrate into S to preserve the superconducting properties of S. This diffusion from both S and N side gives rise to the following consequences:
(i) The superconducting properties of S/N are reduced with respect to the isolated S. This means that, the critical currents are lowered (Meissner, 1960) and also the critical temperature of the system T_{CSN} will be smaller than T_{CS}. In fact, the depression of the critical temperature, due to the proximity effect, strongly depends on the thickness of S (Smith et al, 1961, Simmons and Douglass, 1962).
(ii) In the presence of a magnetic field, the screening currents which develop in S in the neighbourhood of the S/N interface are weaker than the currents which develop in the vicinity of the free surface of the bulk S. This results to an unscreened field in N which penetrates more deeply in the S side of the S/N system than in the isolated superconductor (De Gennes and Matricon, 1965).

The existence of the induced superconductivity can be detected by the experiment of tunnelling effects on the N part, by measuring the Meissner effect in N and by the measurement of the micro-waves absorption in N. Moreover, the aforementioned proximity effect measurements have often been used in an endeavour to determine whether or not N is superconductive, if it is, the value of its critical temperature.

The range of the proximity effect, that is, the distance into N that the Cooper pairs from S leak into N is of the order of a few thousand angstroms. Consequently, in order to observe the proximity effects, it is necessary to work with rather thin films. Theoretically, all calculations are done for the case of a sharp boundary between two uniform metals, whereas, in practice, the boundary is likely to differ considerably from this and therefore the comparison of the experimental observations with the theoretical results will be dealt with difficultly. This can be improved by putting the two metals in good electrical contact (this point has already been discussed in this section). Fig. 1.1 shows schematically the geometry of a S/N sandwich.

The state of N (e.g. its free energy density given by (A.1)) due to induced superconductivity, strongly depends on external conditions such as temperature, magnetic field and normal sample thickness. In particular, these external conditions affect the value of the order parameter of N at free surface $f(l_N)$ (Fig. 1.1), where f and l_N are the order parameter and the thickness of N in dimensionless units (see, Appendix A). Hook and Battilana (1976) have shown that $f(l_N)$ is monotonically decreasing function of l_N, when there is no external magnetic field.

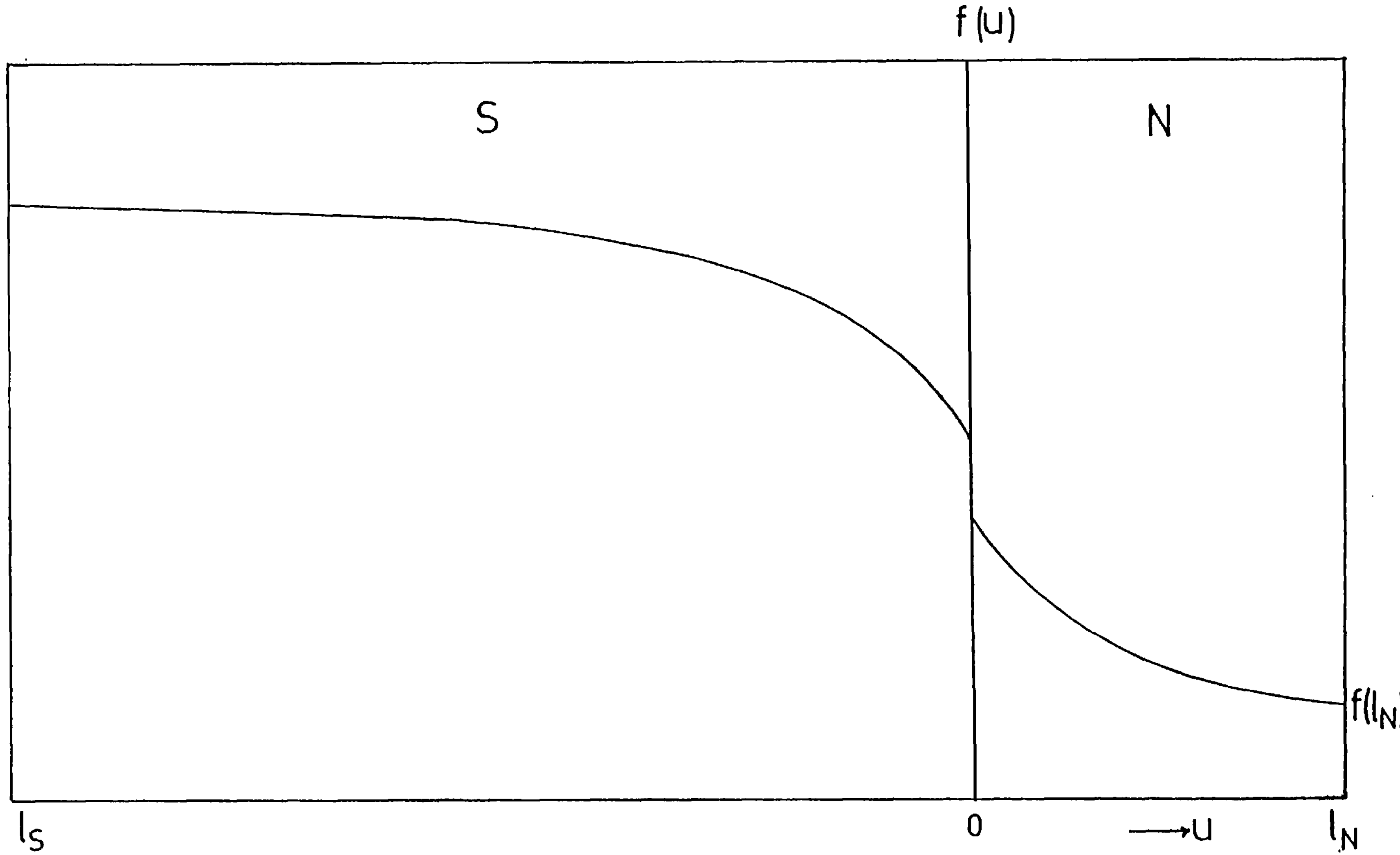

Figure 1.1: Qualitative behaviour of the order parameters (in reduced units) in S and N in the S/N binary layer. We note that the order parameter in S, at intermetallic interface, reduces due to the proximity effect.

In the following section, it will be shown how these conditions affect the boundary of normal-vacuum interface (free surface of N).

1.II The boundary condition at normal-vacuum interface:

The probability amplitude of finding Cooper pair which is proportional to the order parameter (in homogeneous medium, the proportionality coefficient $\frac{1}{V}$ is constant, i.e. position independent) is exponentially decreasing function of temperature (Hurault, 1966 and De Gennes, 1969) and is maximum in the vicinity of intermetallic interface. This amplitude will be reduced to its minimum value at free surface of N. One expects to observe no Cooper pair on that surface (at any temperature and for any thicknesses). In other words, the probability amplitude of finding Cooper pair, and hence the order parameter at free surface, should vanish. But this does not always happen because for a given thickness, $f(l_N)$ decreases smoothly as T increases and vanishes abruptly at a certain temperature $T = T^*$ (Hurault, 1968). For $T > T^*$, $f(l_N)$ will be always zero. Consequently, a first order phase change is predicted at $T = T^*$.

It is shown that the occurrence of this phase transition is due to competition between two boundary conditions imposed at normal-vacuum interface leading to two different configurations (modes) of the order parameter (Fig. 1.2), thus either $f'(l_N) = 0$, mode (I) or $f(l_N) = 0$, mode (II). The above point can be verified by refering to free energies F_I and F_{II} (see, Appendix A) of modes (I) and (II) respectively. It can be seen that, even if (I) is the mode of lowest energy at sufficiently low temperatures, it is much more vulnerable than mode (II) to a rise in T. In fact, the variation of F_I with respect to T will be greater than the variation of F_{II} with respect to T, partly because of the quadratic term in f which is the only explicit function of T in the GL free energy.

As was mentioned previously, the phase transition, in the absence of an external magnetic field, takes place for a certain value of thickness and temperature obtained by matching F_I and F_{II}.

Moreover, in the presence of the external magnetic field, Hurault (1968) has demonstrated that at $l_N = l_N^*$ where a first order phase transition could be observed, $f(l_N)$ practically sharply vanishes at a certain value of magnetic field called "breakdown field" h_b ($\frac{\partial f}{\partial h_a} = \infty$ at $l_N = l_N^*$ and $h_a = h_b$, where h_a is the applied external magnetic field). This effect is again due to competition between two modes. For $h_a < h_b$, $G_I < G_{II}$, where G_I and G_{II} are Gibbs' free energies of modes (I) and (II) respectively (see chapter 4). Therefore first mode would be more stable than the second one. On the other hand, for $h_a > h_b$ which implies $G_I > G_{II}$, mode (II) would be energetically more favourable than mode (I). It is, however, clear that h_b is defined by $G_I(h_b) = G_{II}(h_b)$. The breakdown field has been observed by Burger et al (1965) using tunnelling characteristic on the normal side of a InBi/Zn double layer (InBi = S and Zn = N) and by the resonance frequency measurements. They have experimentally shown that, the slope of the tunnelling characteristic essentially returns to its normal value (see, Fig. 1 of Burger et al (1965)) at and above the breakdown field.

1.III Conclusion:

We conclude that the occurrence of a first order phase transition in S/N sandwiches placed in an external magnetic field parallel to the surface of N is an extension of the phase change previously mentioned (in zero field) which was due to arbitrariness of boundary conditions at N-vacuum interface (the phase transitions in the absence and in the presence of an external magnetic field will be studied in chapters 2 and 4 respectively). Moreover, it has to be mentioned that the second penetration mode (II), appeared in the calculations done by Hurault (1968) and the Orsay Group (1966,1967), has been chosen for computational simplicity and is somewhat arbitrary although its general features can certainly be justified on physical grounds. But we believe that, the order parameter at free surface of N vanishes for certain values of the temperature, thickness of normal material and for a certain value of the magnetic field (the breakdown field, h_b). Our belief relies on the fact that the super-current (see Appendix A) does not pass through the free surface of the sample. In other words, the super-current vanishes on that surface. The two boundary conditions ($f' = 0$ and/or $f = 0$ at $u = l_N$), however, are necessary consequences of Ginzburg-Landau superconductivity, i.e. both conditions make the super-current (see, Appendix A) vanish on the free surface of N. Therefore the two possible configurations for the order parameter are actually expected (Fig. 1.2).

This section is followed by applying the same boundary conditions to an isolated superconductor. It will be illustrated that for an isolated S, the second mode will have no physical significance.

1.IV Outer surface condition for the isolated superconductor:

We consider an isolated superconductor (S) in zero magnetic field and we let the thickness of S be d_S (or l_S in reduced units, where all lengths are measured in units of λ) and its geometry be of the form shown in Fig. 1.3.

We discuss in a somewhat detailed manner the choice of the boundary condition of the order parameter at outer suface of the sample. The requirement of zero super-current at the free surface of the specimen leads to $\frac{d\Delta}{dx} = 0$ or $\Delta = 0$ at that surface (see, Appendix A). The quantum mechanical state of a particle in an infinite potential well follows the latter condition. It will, however, be seen that, the second condition, for an isolated superconductor yields a minimum thickness below which no superconductivity can exist, whereas thinner superconductive films have been practically made which have very nearly the same critical temperatures as the bulk materials. This is because, the first Ginzburg-Landau (GL) equation is a non-linear differential equation (see, Appendix A) and the order parameter must find its own amplitude, whereas the quantum mechanical wave function is normalized.

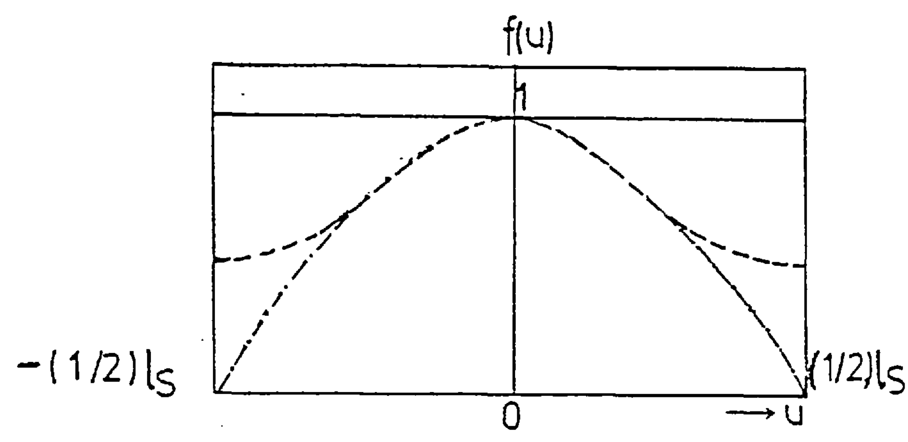

Figure 1.3: The schematical representation of an isolated superconductor with thickness d_S. This figure has been chosen for convenience to examine the boundary conditions.

This statement can be verified by solving the GL equation in the absence of the magnetic field, for a superconducting film with thickness l_S (Fig. 1.3 is chosen for convenience) and by examining both boundary conditions. Beside the above assertion, such a calculation suggests that, Δ can not vary more rapidly than the coherence length $\xi_S(T)$.

Since we will investigate the problem in one dimensional space where the order parameter Δ depends only on x, the GL equation is given by (Appendix A);

$$\kappa^{-2} f'' = f(-1+f^2) \qquad (1.1)$$

where $f = \frac{\Delta(x)}{\Delta_{SB}}$, $f' = \frac{df}{du}$, $u = \frac{x}{\lambda}$, and Δ_{SB} is the value of $\Delta(x)$ for bulk material at equilibrium. $\kappa = \frac{\lambda}{\xi}$ is called the GL parameter and is the ratio of the penetration depth λ over the coherence length ξ.

The first integration of (1.1) reads

$$\kappa^{-2} f'^2 = \tfrac{1}{2}(1-f^2)^2 + C \qquad (1.2)$$

With no loss of generality, f(x) can be chosen symmetrical about u = 0, i.e. $\frac{df}{du}\big|_{u=0} = 0$. This condition determines C;

$$C = -\tfrac{1}{2}(1-f_0^2)^2 \qquad \text{where } f_0 = f(u=0) \qquad (1.3)$$

The equation (1.2) in connection with equation (1.3) must be solved exactly. By introducing a new variable ϕ, such that $f = f_0 \sin\phi$ (1.4) and using (1.3), equation (1.2) can be written in the following form:

$$\pm\, \phi'/\kappa = \frac{f_0}{\sqrt{2}\, C_1}(1-C_1^2 \sin^2\phi)^{\frac{1}{2}} \quad , \quad C_1^2 = f_0^2/(2-f_0^2) \qquad (1.5)$$

or $$\pm \kappa \, du = \frac{\sqrt{2}C_1}{f_0} \frac{d\phi}{(1-C_1{}^2\sin^2\phi)^{\frac{1}{2}}} \tag{1.6}$$

We are now at a position to test two boundary conditions.

<u>(I) $f = 0$ at $u = \pm l_S/2 \Rightarrow \phi = 0$:</u>

The equation (1.6) can be integrated, the result for the lowest free energy (with a fewest nodes) would be (Abramowitz and Stegun (AS), 1972, P.569)

$$\kappa\left(\frac{l_S}{2} - |u|\right) = \frac{\sqrt{2C_1}}{f_0} F(\phi|C_1) \quad , \quad \phi = \sin^{-1}\left\{\frac{f(u)}{f_0}\right\} \tag{1.7}$$

where F is the incomplete elliptic integral of the first kind with modulus (parameter) C_1. The value of f_0 is determined by an implicit relation obtained from (1.7) by setting $u = 0$, i.e.

$$\frac{\kappa l_S}{2} = \frac{\sqrt{2C_1}}{f_0} F\left(\frac{\pi}{2}\middle|C_1\right) = \frac{\sqrt{2C_1}}{f_0} K(C_1) = \frac{\sqrt{2C_1}}{f_0} \frac{\pi}{2} M(\tfrac{1}{2},\tfrac{1}{2};1;C_1) \tag{1.8}$$

where $K(C_1)$ is a quarter period of the elliptic sine or cosine function (AS, P.569) and M is the hypergeometric function (AS, P.591). The right hand side of the equation (1.8) is a monotonically increasing function of f_0 and is always greater than (or equal to) $\pi/2$. It turns out that, $l_S\kappa/2 \geq \pi/2$ (the equality being reached for $f_0 = 0$ which implies $f(u) = 0$, and hence the material never becomes superconductor) which means that the thickness is greater than a certain value $d \geq d_C = \pi\xi$.,This result is incompatible with experiment which suggests that a superconductor with a thickness even thinner than the aforementioned value can be made. Therefore the boundary condition $f = 0$

at the surface is not valid for an isolated superconductor. The above derivation is due to Werthamer, 1969.

The boundary condition $f' = 0$ at the surface which must produce $f(u) = 1$, however, has to be examined.

<u>(II) $f' = 0$ at $u = \pm\frac{1}{2}l_S \Rightarrow \phi' = 0$:</u>

We apply this condition to the equation (1.2) to determine C;

$$C = -\tfrac{1}{2}(1-f_b^{\,2})^2 \tag{1.9}$$

where f_b denotes the value of reduced order parameter at boundary, i.e. $f_b = f(u=\pm\frac{1}{2}l_S)$. The comparison between equations (1.3) and (1.9) gives rise to

$$f_0 = f(u=0) = f_b \equiv f(u=\pm\tfrac{1}{2}l_S)$$

Thus, $f(u) = 1$ as we expected, the material can always be in superconducting state and there is no geometrical restriction to superconductivity in this case.

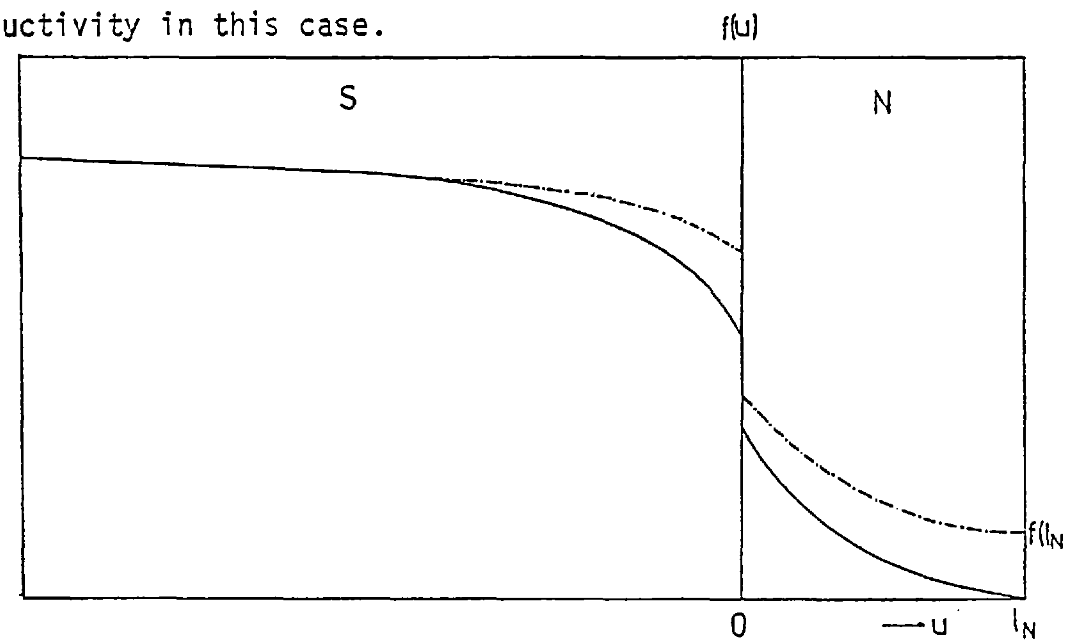

Figure 1.2: The two possible configurations for the order parameter f (in reduced units) of a S/N sandwich. Chain curve: the traditional mode (I); full curve: the new mode (II).

CHAPTER 2

SUPERCONDUCTING SANDWICHES IN ZERO MAGNETIC FIELD

2.I Introduction:

When a superconductor (S) is deposited on a normal metal (N), the superconductivity will be induced in N as a result of the leakage of Cooper pairs into it. Apart from giving information on the potential superconductivity of metals with hitherto inaccessible superconducting transition termperatures, the proximity effects could be used as a tool to study the properties of magnetic impurities which often display remarkable behaviour (Kondo effect, spin glass). The problem of making superconductivity and magnetic impurities coexist, that is finding a potential superconductor into which impurities are magnetic and which remains superconducting even at moderate impurity concentrations, is bypassed by inducing the superconductivity. In other words, if N is a magnetic metal induced superconductivity disappears in N very close to intermetallic interface and superconductivity may be completely quenched (Hauser et al 1966). Thus the coexistence of magnetism and superconductivity can be investigated by the proximity effects.

Pioneering theoretical and experimental work has been devised to reveal and clarify this proximity effect over the past years (see for review, Deutscher and De Gennes, 1969). Theoretically the concepts have centred (with the simple exception of the McMillan model (tunnelling model) introduced for the proximity effect which can be solved completely, McMillan 1968a, 1968b) on the generalized Ginzburg-Landau (GL) equations of the microscopic theory derived by Gorkov (1959), usually in the dirty limit. Most of the field of proximity effect can be said to be qualitatively well understood. One exception is the occurrence of a first order phase transition as a response to an external magnetic field (Orsay Group 1966, 1967), whereby conventional induced superconductivity, in which the order parameter is finite throughout N (Fig. 2.1a) gives way at and above a given magnetic field (the breakdown field, H_b) to a second mode for which the field penetrates part of the N side of the S/N sandwich where superconductivity is entirely destroyed. The particular configuration chosen by the original investigators (Orsay Group 1966,1967), (Fig. 2.1b), appears to have been chosen for "computational simplicity" and is somewhat arbitrary, although its general features are certainly justifiable on physical grounds. Moreover, our colleagues at Imperial College have experienced considerable difficulties in interpreting on those lines several recent experiments (Adatia 1976, Tai and Park 1978). These reasons have compelled us to look at the problem afresh. We feel that the main reason for the occurrence of the first order phase transition is the interplay of the two possible boundary conditions at the free surface of N discussed below.

This chapter is mainly concerned with the "complete" solutions of the generalized GL equations in zero external magnetic field leading to the occurrence of a "new first" order phase transition (as a function of temperature T and or normal slab thickness d_N). Specifically, we shall investigate the situation in which the superconducting part of the S/N sandwich is infinite, and neglect the non-linear term in the superconducting order parameter in the normal part of the sandwich. An analytic solution can be obtained in this case.

First, however, we discuss in a somewhat detailed manner the possibility of the two boundary conditions of the order parameter at outer surface of N in S/N binary layers.

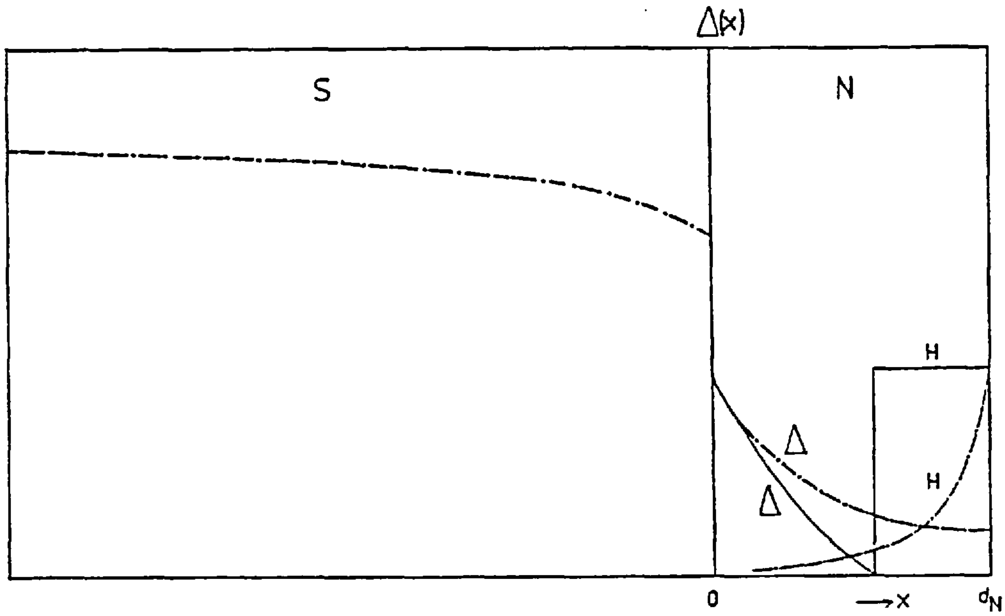

Figure 2.1: The two configurations suggested in Orsay (1967) for the order parameter Δ of a S/N sandwich in a magnetic field. (a) Chain curve : $H< H_b$. (b) full curve : $H >H_b$ (H_b is the breakdown field).

2.II The boundary conditions:

The GL equations, being local differential equation, must be solved by imposing certain boundary conditions on the order parameter Δ (r). Since we will be concerned with the solutions of the GL equations in one dimensional space where the order parameter Δ depends only on one coordinate ($\Delta(x)$), we use the boundary conditions at intermetallic interface derived by (De Gennes 1964)

$$\left.\begin{matrix} \dfrac{\Delta}{N(0)V} \\ \\ \dfrac{D}{V}\dfrac{d\Delta}{dx} \end{matrix}\right\} \text{ are continuous at intermetallic interface} \qquad (2.1)$$

where N(0) is the density of states at the Fermi level (per unit energy and per unit volume), V is electron-electron interaction and D is diffusion constant of a conduction electron. But the boundary condition at outer surface of N in S/N sandwich can be

$$\begin{matrix} \text{either} & \dfrac{d\Delta}{dx} = 0 \\ & \\ \text{or} & \Delta = 0 \end{matrix} \quad \text{at outer surface of N} \qquad (2.2)$$

for the following reasons:

(i) These two conditions are necessary consequences of GL superconductivity (they make the superconducting current vanish at free surface of N, (Appendix A). The above conditions lead to two different configurations (modes) for the order parameter. The one with zero slope was called "mode(I)" and the one with zero order parameter at free surface was called "mode(II)" in the preceding chapter.

(ii) Blackburn et al (1975) have shown that, the order parameter in N decreases by increasing the temperature and does actually vanish at the free surface of N at a certain temperature and for a given normal sample thickness.

(iii) De Gennes (1964) has argued that, if the electron-electron interaction V_N in the normal metal is negligible, then $\Delta=0$ at the outer surface of N. Therefore the second boundary condition can safely be used at least for the normal material with low BCS coupling constant N(0)V. In fact, De Gennes and Saint-James (1963), with the help of microscopic theory, have shown that even if the normal sample thickness is very small ($d_N \ll \xi_{0S}$, ξ_{0S} is the coherence length of S at zero temperature), no energy gap in the excitation spectrum will be detected in the tunnelling experiment, if V is negligibly small. In this situation, it turns out that the order parameter $\Delta=0$ at free surface of N. The boundary conditions at the S/N interface (2.1), however, are not in contention here and are those derived by De Gennes (1964). See, however, Silvert (1975) for a critical discussion of (2.1). Throughout this thesis, the GL equations will be used which means that we stay within the domain of validity of the local GL superconductivity (discussed in Appendix A).

We solve the linearized GL equations for N of finite thickness and the non-linear GL equation for S of infinite thickness. Having done that, we predict that a first-order phase transition should occur in a S/N sandwich, even in the absence of an external magnetic field. This is a new prediction which has not yet been observed experimentally.

Moreover, the non-linear GL equations will be solved completely for both isolated S and N of finite thicknesses.

2.III Complete solutions of the GL equations:

Our sandwich has infinite transverse dimensions and we have a one dimensional problem that Δ depends on the longitudinal coordinate x only. Our starting point is the GL free energy in one dimensional space. For real Δ, it is given by (Appendix A);

$$F = \int dx \, \{C \left(\frac{d\Delta}{dx}\right)^2 + A\Delta^2 + \tfrac{1}{2}B\Delta^4\} \qquad (2.3)$$

This is formally valid for both S and N. The GL equation is obtained by minimizing F with respect to Δ;

$$C\Delta'' - A\Delta - B\Delta^3 = 0 \qquad (2.4)$$

where primed means the derivative of Δ with respect to x. It is more convenient to introduce the following quantities

$$\Delta_0^2 = \frac{|A|}{B}, \quad f(x) = \frac{\Delta(x)}{\Delta_0} \quad \text{and} \quad \xi^2 = \frac{C}{|A|} \qquad (2.5)$$

where ξ is the GL coherence length. In a clean sample where the microscopic theory is valid, ξ and Δ_0 are given by †

$$\xi = \xi_0 |T/T_C - 1|^{-\frac{1}{2}} \quad , \quad \Delta_0 = 3.1 k_B T_C |1 - T/T_C|^{\frac{1}{2}} \qquad (*)$$

where $\xi_0 = 0.13 \hbar \, v_F / k_B T_C$ (2.6)

and v_F, T_C are Fermi velocity and critical temperature of the specimen respectively.

† An asterisk labels the equations valid exclusively within the microscopic GL theory. The other equations can all accommodate phenomenological parameters.

The superconductivity will be induced from S into N, if $T_{CN} \leq T \leq T_{CS}$ (the subscripts S and N are refered to superconductive and normal materials respectively), so that A_S is naturally negative and A_N is positive.

Using (2.5), (2.4) for S and N becomes

$$\xi_S^2 f_S'' + f_S - f_S^3 = 0 \tag{2.7}$$

$$\xi_N^2 f_N'' - f_N - f_N^3 = 0 \tag{2.8}$$

These non-linear GL equations for S and N can be solved and the solutions will be expressed in terms of elliptic functions. The problem will remain to find a matching combination of elliptic functions consistent with the boundary conditions.

First integrations of the above equations will be

$$2\xi_S^2 f_S'^2 = f_S^4 - 2f_S^2 + k_S \tag{2.9}$$

$$2\xi_N^2 f_N'^2 = f_N^4 + 2f_N^2 + k_N \tag{2.10}$$

where k's are integration constants which are determined by appropriate boundary conditions.

Assuming that the superconducting film extends from $-d_S \leq x \leq 0$ and the normal film from $0 \leq x \leq d_N$, where d_S and d_N are the thicknesses of S and N respectively, the above equations are solved separately.

If we let a_S be the value of f_S at free surface of S, then by requiring $\Delta'_S\ (x=-d_S) = 0$ we can evaluate k_S,

$$k_S = a_S^2(2-a_S^2) \equiv a_S^2 b_S^2$$

where $a_S = f(x=-d_S)$ and $b_S^2 = 2-a_S^2$ (2.11)

The final solution for (2.9) is, (Abramowitz and Stegun (AS), 1972, P.596)

$$f_S(x) = a_S \text{ sn } \{ \frac{b_S}{\sqrt{2}\xi_S} (x_0-x)|m\} \qquad (2.12)$$

where m is the parameter (modulus) of elliptic functions which here is

$$m = a_S^2/b_S^2 \qquad (2.13)$$

The "extrapolation length" x_0 (the distance from the interface x=0 at which the order parameter vanishes) is expressed in terms of d_S and K(m) (the complete elliptic integral, AS, P.569) by applying the condition $\frac{df_S}{dx} = 0$ at $x = -d_S$ on (2.12). Thus x_0 will be

$$x_0 = -d_S + \frac{K(m)}{\gamma_S}, \qquad \gamma_S = \frac{b_S}{\sqrt{2}\ \xi_S} \qquad (2.14)$$

It is advantageous to return to the original order parameter Δ

$$\Delta_S(x) = \Delta_S \text{ sn}(\gamma_S(x_0-x)|m) \qquad (2.15)$$

where $\Delta_S = \Delta_S(x=-d_S)$

and $\gamma_S = \xi_S^{-1} \{1 - \frac{1}{2} (\frac{\Delta_S}{\Delta_{0S}})^2\}^{\frac{1}{2}}$, $\Delta_{0S}^2 = \frac{|A_S|}{B_S}$

$$m = (\Delta_S/\Delta_{0S})^2/(2-(\Delta_S/\Delta_{0S})^2) \qquad (2.16)$$

x_0 is given by (2.14) and the expression (2.11) has been used. (Δ_{0S} is the order parameter for the same but isolated and uniform superconductor). The quantity Δ_S will be determined by using the boundary conditions at metallic interface (2.1). The equation (2.15) is formally valid for both modes (I) and (II) (obtained from different boundary conditions (2.2)), but the matching parameter Δ_S takes different values for (I) and (II), Δ_{SI} and Δ_{SII}.

We now solve equation (2.10) for the two aforementioned modes.

(I) First mode:

The condition $\frac{df_N}{dx} = 0$ at $x = d_N$ implies that k_N is negative (from expression (2.10)). Assume that

$a_{NI} = f(x=d_N)$ is the value of f_N at free surface of N,

$$k_{NI} = -a_{NI}^2(a_{NI}^2+2) \tag{2.17}$$

and for the sake of simplicity $b_{NI}^2 = a_{NI}^2+2$

Having imposed the condition $\frac{df_N}{dx} = 0$ at $x=d_N$ on (2.10), the solution will be (AS, P.596)

$$f_{NI}(x) = a_{NI}\ \mathrm{nc}\ \{\xi_N^{-1} \left(\frac{a_{NI}^2+b_{NI}^2}{2}\right)^{\frac{1}{2}} (d_N-x)\ |\ m_{NI}\} \tag{2.18}$$

where $\mathrm{nc}(\nu|m)= \frac{1}{\mathrm{cn}(\nu|m)}$ and parameter m_{NI} here is

$$m_{NI} = b_{NI}^2\ /\ (a_{NI}^2 + b_{NI}^2) \tag{2.19}$$

Equation (2.18) in the conventional form would be

$$\Delta_{NI}(x) = \Delta_{NI}\ \mathrm{nc}\{\gamma_{NI}(d_N-x)|m_{NI}\} \tag{2.20}$$

where

Δ_{NI} is the value of Δ at $x=d_N$ and

$$\gamma_{NI} = \xi_N^{-1} \{1 + (\frac{\Delta_{NI}}{\Delta_{ON}})^2\}^{\frac{1}{2}} , \qquad \Delta_{ON}^2 = |\frac{A_N}{B_N}| \tag{2.21}$$

$$m_{NI} = \frac{2 + (\Delta_{NI}/\Delta_{ON})^2}{2\{1+(\Delta_{NI}/\Delta_{ON})^2\}} \tag{2.22}$$

Here Δ_{NI} is the parameter to be matched.

In order to visualize our two solutions (2.15) and (2.20), we recall that an elliptic function is the intermediate between a circular (m=0) and hyperbolic (m=1) function (AS P.571). Thus, for (2.15)

$$sn\ (\nu|m=0) = \sin\nu \quad , \quad sn\ (\nu|m=1) = \tanh\nu$$

whereas, for (2.20)

$$nc\ (\nu|m=0) = \sec\nu \quad , \quad nc\ (\nu|m=1) = \cosh\nu$$

Since some of the elliptic functions go through zero after a quarter period K(m), m can be directly related to the thickness of the sample (see, e.g. (2.14)).

Making use of boundary conditions (2.1) produces two implicit expressions which determine Δ_S and Δ_{NI}.

$$\Delta_{SI} cd(\gamma_{SI} d_S | m_{SI}) = \alpha\Delta_{NI}\ nc(\gamma_{NI} d_N | m_{NI}) \tag{2.23}$$ *

where α is the ratio of coupling constants i.e.

$$\alpha = \frac{N_S V_S}{N_N V_N} \quad , \quad N_i = N(0) \quad , \quad i = S,N \tag{2.24}$$

* Note; $dn(\nu|m=0) = 1$ and $dn(\nu|m=1) = \text{sech}\nu$

and

$$\sigma_S \Delta_{SI} \gamma_{SI} m'_{SI} \; sd(\gamma_{SI} d_S | m_{SI}) \; nd(\gamma_{SI} d_S | m_{SI})$$

$$= \sigma_N \alpha \Delta_{NI} \gamma_{NI} \; sc(\gamma_{NI} d_N | m_{NI}) \; dc(\gamma_{NI} d_N | m_{NI}) \tag{2.25}$$

where σ is the normal state conductivity defined by $\sigma = 2e^2 ND$
$m'_{SI} = 1 - m_{SI}$.

Clearly, equations (2.23) and (2.25) are implicit functions of Δ_{SI} and Δ_{NI} and can not be solved analytically for unknown Δ_{SI} and Δ_{NI}.

(II) Second mode:

We now solve equation (2.10) under the condition $f_N=0$ at $x=d_N$. This implies that k_N is positive in this case,

$$k_{NII} = 2\xi_N^2 \, f'^2_{NII}(x=d_N)$$

Under the above boundary condition, the solution of (2.10) will be

$$f_{NII}(x) = a_{NII} \; sc \left\{ \frac{b_{NII}}{\sqrt{2}\,\xi_N} (d_N - x) \,|\, m_{NII} \right\} \tag{2.26}$$

where a_{NII} is a quantity related to the gradient of order parameter at $x=d_N$ by

$$2\xi_N^2 \, f'^2_{NII}(x=d_N) = a^2_{NII}(2-a^2_{NII})$$

and $$b^2_{NII} = 2 - a^2_{NII} \tag{2.27}$$

here $$m_{NII} = 1 - a^2_{NII}/b^2_{NII} \tag{2.28}$$

In terms of the order parameter, (2.26) reads,

$$\Delta_{NII}(x) = \Delta_{NII} \; sc(\gamma_{NII}(d_N - x) \,|\, m_{NII}) \tag{2.29}$$

where Δ_{NII} is the parameter to be matched and

$$\gamma_{NII} = \xi_N^{-1} \quad \{1-\tfrac{1}{2}\,(\frac{\Delta_{NII}}{\Delta_{ON}})^2\}^{\frac{1}{2}} \tag{2.30}$$

$$m_{NII} = 2\,\{\frac{1-(\Delta_{NII}/\Delta_{ON})^2}{2-(\Delta_{NII}/\Delta_{ON})^2}\} \tag{2.31}$$

(Note again that;

$$sc(\nu|m=0) = \tan\nu \qquad \text{and} \qquad sc(\nu|m=1) = \sinh\nu$$

(AS, P.571)).

Having applied the conditions (2.1) (to determine Δ_{SII} and Δ_{NII} for the present case) on (2.15) and (2.29), we obtain the following implicit relations between Δ_{SII} and Δ_{NII};

$$\Delta_{SII}\; cd(\gamma_{SII}d_S|m_{SII}) = \alpha\Delta_{NII}\; sc(\gamma_{NII}d_N|m_{NII}) \tag{2.32}$$

$$\sigma_S\Delta_{SII}\gamma_{SII}m'_{SII}\; sd(\gamma_{SII}d_S|m_{SII})\; nd(\gamma_{SII}d_S|m_{SII}) = \sigma_N\alpha\Delta_{NII}\gamma_{NII}\; dc(\gamma_{NII}d_N|m_{NII})\; nc(\gamma_{NII}d_N|m_{NII}) \tag{2.33}$$

where $m'_{SII} = 1-m_{SII}$, m_{SII} is given by (2.16).

We have used (2.14) and changed the arguments of elliptic functions (AS, P.572) to find the above relationships. Unfortunately (2.32) and (2.33) are the implicit relations between matching parameters Δ_{SII} and Δ_{NII}. In other words, they are not analytically soluble; were they such, we would be able to calculate the total free energies of the S/ N system defined by $F_T = F_S + F_N$ for both modes (although the required integral may be a formidable mathematical task too), where F_S and F_N are given by (2.3). Then equating F_{TI} and F_{TII} could give rise to the prediction of a new first order phase transition in the binary system of finite thickness in the absence of an external magnetic field, without any simplification of the GL equations. By restricting ourselves to a special case, in which (2.32) and (2.33) are analytically soluble, we can demonstrate this assertion.

In the forthcoming section the GL equation will be solved in its general form for S of infinite thickness and $\Delta_N(x)$ will be found from linearized GL equation for N with finite thickness.

2.IV First-order phase transition in S/N sandwiches:

In section (2.III), we derived the variation of order parameters $\Delta_S(x)$ and $\Delta_N(x)$ in S/N binary systems using the generalized one dimensional GL equations. Moreover, we concluded that a first order phase transition can be predicted by matching the total energies of the system F_{TI} and F_{TII} corresponding to the modes (I) and (II) respectively (section 2.III). To demonstrate this assertion we restrict ourselves to a special case in this section. First, the GL equation will be solved for infinite S and for N with finite thickness d_N, then we evaluate F_T's to predict a first order phase transition.

2.IVa The solution of the GL equation for infinite S:

Let the superconducting side of the sandwich have infinite thickness and occupy the whole region x<0 (Fig. 2.2). Then on the S side the solution of (2.4) would be

$$\Delta_S(x) = \Delta_{SB}\tanh\{(x_0-x)/\sqrt{2}\xi_S(T)\} \quad , \quad x<0 \qquad (2.34)$$

where the conventional boundary condition (I) has been taken at $x=-\infty$, x_0 is a length parameter to be matched (the equation (2.34) is the limit of (2.15) for m=1, AS, P.571). $\Delta_{SB}=\Delta_{0S}$ is the order parameter for the bulk S matter and $\xi_S(T)$ is the coherence length of S. These are given by (2.6);

$$\Delta_{SB}(T) = 3.1k_BT_{CS}(1-T/T_{CS})^{\frac{1}{2}} \qquad (*)$$

$$\xi_S(T) = \xi_{0S}(1-T/T_{CS})^{-\frac{1}{2}} \quad , \quad \xi_{0S} = 0.13\hbar v_{FS}/k_BT_{CS} \qquad (2.35)$$

The parameter x_0 will be determined later on by using the conditions (2.1).

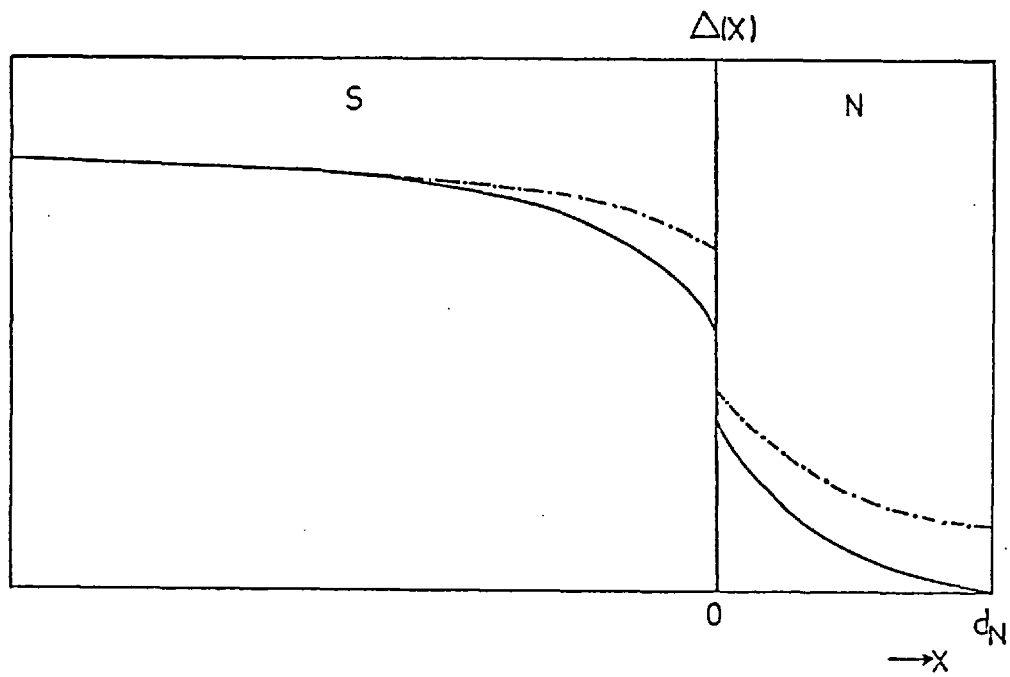

Figure 2.2: The two possible configurations for the order parameter Δ of a S/N sandwich. Chain curve: the traditional mode (I); full curve: the new mode (II).

2.IVb The solutions of the linearized GL equation for N:

Let N side of the sandwich extend over a distance d_N (Fig. 2.2) and neglect on the normal side the nonlinear term in the GL equation (2.4). The theory is therefore of dubious validity when $T \approx T_{CN}$, but should remain qualitatively valid in a reasonably wide range of temperature below T_{CS}. On the N side, for either of the alternative modes (boundary conditions given by (2.2))

$$\Delta_{NI}(x) = \{\Delta_{NI}(0)/\cosh(d_N/\xi_N(T))\} \cosh\{(d_N-x)/\xi_N(T)\} \qquad (2.36)$$

$$x>0$$

$$\Delta_{NII}(x) = \{\Delta_{NII}(0)/\sinh(d_N/\xi_N(T))\}\sinh\{(d_N-x)/\xi_N(T)\} \qquad (2.37)$$

Here $\Delta_N(0)$ is the parameter to be matched and $\xi_N(T)$ is the coherence length of N given by

$$\xi_N(T) = \xi_{ON}(T/T_{CN}-1)^{-\frac{1}{2}} \quad , \quad \xi_{ON} = 0.13\,\hbar v_{FN}/k_B T_{CN} \;^{(*)} \qquad (2.38)$$

We are now at a position to calculate x_0 and $\Delta_N(0)$'s by using boundary conditions at metallic interface (De Gennes 1964). These conditions are given by (2.1). The continuity of $\frac{\Delta}{NV}$ at metallic interface for either of the alternative modes leads to

$$\begin{aligned} \Delta_{SI}(0) &= \alpha\Delta_{NI}(0) \\ \Delta_{SII}(0) &= \alpha\Delta_{NII}(0) \end{aligned} \qquad (2.39)$$

where α is

$$\alpha = \frac{N_S V_S}{N_N V_N} \qquad (2.40)$$

and $$\Delta_S(0) = \Delta_{SB} \tanh(x_0/\sqrt{2}\xi_S) \quad (2.41)$$

It is more convenient to express the relations in terms of the quantity

$$\tau = \frac{\Delta_S(0)}{\Delta_{SB}} = \tanh(x_0/\sqrt{2}\xi_S) \leqslant 1 \quad (2.42)$$

The second condition (2.1) obtains

$$1 - \tau_I^2 = 2\mu\tau_I \tanh u \quad (2.43)$$

$$1 - \tau_{II}^2 = 2\mu\tau_{II} \coth u \quad (2.44)$$

where $$\mu = \frac{\sigma_N}{\sqrt{2}\sigma_S} \cdot \frac{\xi_S(T)}{\xi_N(T)}, \quad u = d_N/\xi_N(T) \quad (2.45)$$

σ is normal state conductivity given by $\sigma = 2e^2ND$, μ is a monotonically increasing function of temperature between T_{CN} and T_{CS} and u is the normalized thickness.

From (2.43) and (2.44) one obtains;

$$\tau_I = -\mu\tanh u + (1+\mu^2\tanh^2 u)^{\frac{1}{2}} \quad (2.46)$$

$$\tau_{II} = -\mu\coth u + (1+\mu^2\coth^2 u)^{\frac{1}{2}} \quad (2.47)$$

Thus $\tau_{II} \leqslant \tau_I$, the equality being reached only when $d_N/\xi_N(T) = \infty$. In the Orsay papers (Orsay Group on Superconductivity 1966, 1967) $\tau \propto \Delta_N(0)$ was taken to be the same for both modes and therefore so was the total free energy F in the absence of an external magnetic field so that the possible existence of this phase transition was overlooked.

The parameter x_0 in terms of T and d_N is given by (equation 2.42)

$$x_0 = \frac{\xi_S(T)}{\sqrt{2}} \ln\left(\frac{1+\tau}{1-\tau}\right) \quad (2.48)$$

and from (2.39) and (2.42) $\Delta_N(0)$ for both modes is

$$\Delta_N(0) = \frac{\Delta_{SB}}{\alpha}\tau \tag{2.49}$$

where τ's are given by (2.46) and (2.47). Figure (2.3) shows the variation of $\Delta_N(0)$ with the normalized thickness u, where $\Delta_N^{as}(0)$ is

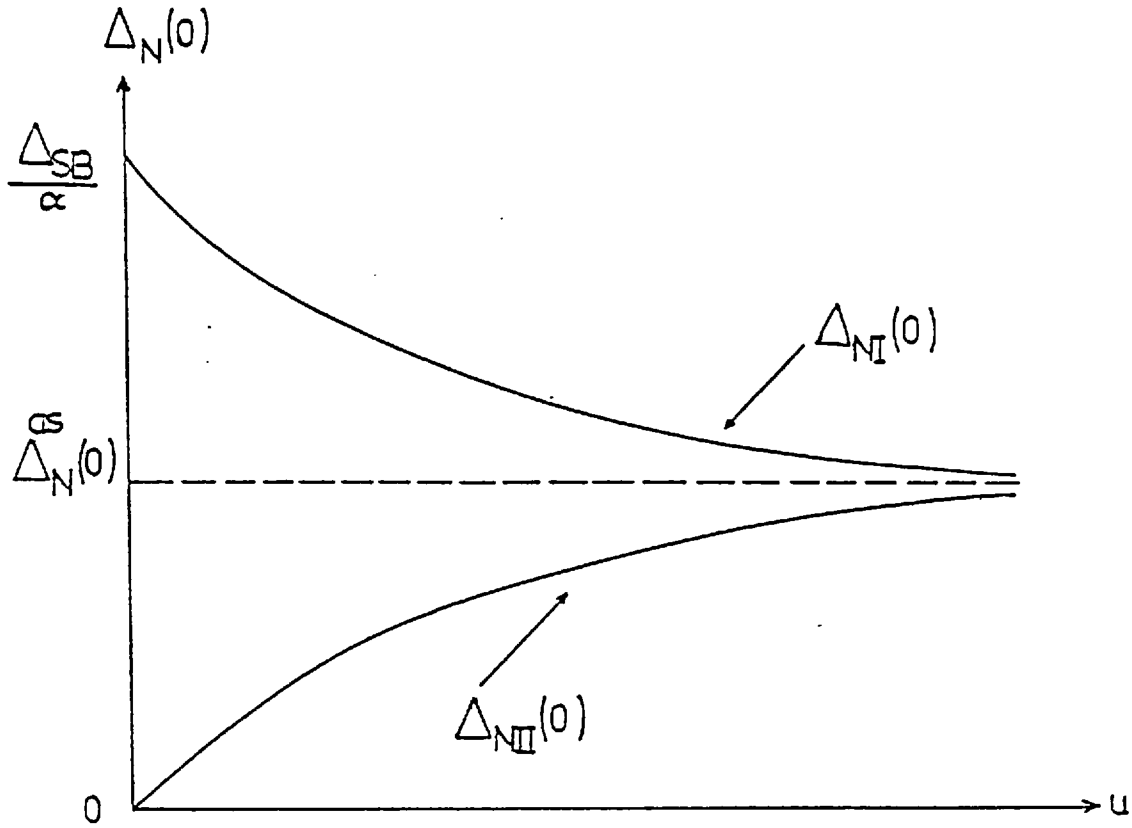

Figure 2.3: The variation of the order parameter for N at metallic interface $\Delta_N(0)$ versus the normalized normal slab thickness u.

the value of $\Delta_N(0)$ at u=∞ and is given by (from (2.46) and (2.49)),

$$\Delta_N^{as}(0) = \frac{\Delta_{SB}}{\alpha}\{-\mu + (1+\mu^2)^{\frac{1}{2}}\} \tag{2.50}$$

2.IVc The calculation of total free energies:

The total GL free energy given by (2.3) can be written as

$$F = C\Delta(x)\Delta'(x)|_0 - \tfrac{1}{2}\int dx\, B\Delta^4(x) \tag{2.51}$$

where equation (2.4) has been used. The first term in (2.51) is evaluated at the S/N interface (its value is zero on the outer boundaries of either mode in N and for S). C depends on the material (Appendix A) but is independent of temperature, thickness and boundary conditions. On the N side, the second term is neglected to be consistent with the linear approximation in equation (2.4)). The total free energy of S/N systems $F_T = F_N + F_S$ is then given by

$$F_T = -C_N\Delta_N(0)\Delta_N'(0) + C_S\Delta_S(0)\Delta_S'(0) - \tfrac{1}{2}\int_{\infty}^{0} dx\; B_S\Delta_S^{4}(x) \tag{2.52}$$

note x<0 for S side and x>0 for N side.

In the above equation the first term is the total energy of N side F_N and the last two terms are the total energy of S side. These two energies will be calculated separately:

$$F_S = C_S\Delta_S(0)\Delta_S'(0) - \tfrac{1}{2}\int_{-\infty}^{0} dx\; B_S\Delta_S^{4}(X) \tag{2.53}$$

From equation (2.34) one deduces

$$F_S = C_S\Delta_S(0)\Delta_S'(0) + F_{S0} - \frac{3}{4}P\left(-\frac{4}{3} + \tau + \frac{1}{3}\tau^3\right) \tag{2.54}$$

where (2.42) has been used, $F_{S0} = \int -\tfrac{1}{2}B_S\Delta_{SB}^{4}dx$ is the free energy of the isolated superconducting material, and the parameter P is defined by

$$P = (4/3)C_S\Delta_{SB}^{2}/\sqrt{2}\,\xi_S(T) \tag{2.55}$$

and $\Delta_S'(0) = -\frac{\Delta_{SB}}{\sqrt{2}\xi_S}(1-\tau^2)$, $\Delta_S(0) = \Delta_{SB}\tau$

Substituting these expressions into (2.54) gives

$$F_S = F_{SO} + P\{1 - \tfrac{3}{2}(1-\tau^2/3)\tau\} \tag{2.56}$$

In order to calculate F_N we might use either (2.36) or (2.37). From (2.36) one has

$$\Delta_N'(0) = -\frac{1}{\xi_N}\Delta_N(0) \tanh u$$

where $\Delta_N(0)$, from (2.49) is $\Delta_N(0) = \frac{\Delta_{SB}}{\alpha}\tau$. The definition of μ and equation (2.43) enable us to write

$$\Delta_N'(0) = -\frac{\Delta_{SB}}{\alpha}\frac{\sigma_S}{\sigma_N}\frac{1}{\sqrt{2}\,\xi_S}(1-\tau^2)$$

Now F_N becomes;

$$F_N = \frac{3}{4}\frac{P}{\alpha^2}\frac{C_N}{C_S}\frac{\sigma_S}{\sigma_N}(1-\tau^2)\tau \tag{2.57}$$

where (2.55) has been used. Since $C \propto \frac{n}{T_C^2}$ (*) (Appendix A) where n is the number of conduction electrons per unit volume, it is more convenient to introduce a parameter a such that

$$a = 1-(4/3)/\left(\frac{\varepsilon_S}{\varepsilon_N} - 2/3\right) \tag{2.58}$$

which is expressed in terms of a mismatch ratio $\varepsilon_S/\varepsilon_N$ where $\varepsilon = \sigma T_C^2/nN^2V^2$. Thus (2.57) becomes

$$F_N = P\left(\tfrac{1}{2} + \frac{1}{1-a}\right)(1-\tau^2)\tau \tag{2.59}$$

Summing up F_S and F_N given by (2.56) and (2.59) respectively gives the total energy of the S/N sandwich F_T, formally valid for either mode and is

$$F_T = F_{SO} + P\,\{1 + \tau(a-\tau^2)/(1-a)\} \tag{2.60}$$

It is of interest to note that the limiting case a=1 ($\frac{\varepsilon_S}{\varepsilon_N} \to \infty$) corresponds to the fully linearized GL equation, on both N and S sides. It will be shown later on that for normal materials with extremely low critical temperature such as Copper the mismatch ratio $\frac{\varepsilon_S}{\varepsilon_N}$ will be high, and hence the parameter a is nearly unity.

2.IVd The first-order phase transition:

So far we have evaluated the total free energy (2.60) of S/N binary layers. The total free energies for modes (I) and (II) respectively are (from 2.60);

$$\begin{aligned} F_{TI} &= F_{SO} + P\{1 + \tau_I(a-\tau_I^2)/(1-a)\} \\ F_{TII} &= F_{SO} + P\{1 + \tau_{II}(a-\tau_{II}^2)/(1-a)\} \end{aligned} \tag{2.61}$$

A first order phase transition occurs when $F_{TI} = F_{TII}$; i.e. when

$$(\tau_I - \tau_{II})(\tau_I^2 + \tau_I\tau_{II} + \tau_{II}^2 - a) = 0$$

We have already pointed out that, if $\tau_I = \tau_{II}$, the solution can be achieved only if the normal sample thickness is infinity i.e. $d_N/\xi_N(T) = \infty$ (see, equations (2.46) and (2.47)). Therefore for finite d_N, the phase diagram obeys;

$$\tau_I^2 + \tau_I\tau_{II} + \tau_{II}^2 = a \tag{2.62}$$

where τ_I and τ_{II} are given by (2.46) and (2.47) respectively. The phase diagram in the (u,μ) or (d_N,T) plane is given by eliminating τ_I and τ_{II} from equations (2.46), (2.47) and (2.62). Since $\tau_I \geqslant \tau_{II}$, we introduce the following quantities

$$\tau_I = r\cos\omega \qquad \tau_{II} = r\sin\omega \qquad \text{where } \omega \leqslant \pi/4 \tag{2.63}$$

and $\qquad \phi = \sin\omega\cos\omega$.

Therefore (2.62) gives

$$r = \left(\frac{a}{1+\phi}\right)^{\frac{1}{2}} \tag{2.64}$$

ω can be written as (2.63);

$$\sin\omega = \left\{\frac{1-\sqrt{(1-4\phi^2)}}{2}\right\}^{\frac{1}{2}} \tag{2.65}$$

substituting these two expressions into (2.63) to obtain τ_{II},

$$\tau_{II} = \{\tfrac{1}{2}(a/(1+\phi))\cdot(1-\sqrt{(1-4\phi^2)})\}^{\frac{1}{2}} \tag{2.66}$$

We have to evaluate ϕ in terms of a and $4\mu^2$. From (2.43) and (2.44) one can write

$$(1-\tau_{II}^2)/2\mu\tau_{II} = 2\mu\tau_I/(1-\tau_I^2)$$

This expression produces a quadratic equation in ϕ by using (2.63) and (2.64). Its solutions are

$$\phi = \{a-2+4\mu^2a\pm a\left[(4\mu^2-1)^2-4(1-a)\right]^{\frac{1}{2}}\} \, / \left[2(a^2+1-4\mu^2a)\right] \tag{2.67}$$

The phase diagram in the (u,μ) or (d_N,T) plane, therefore is given by (equation 2.44)

$$u = \coth^{-1} \{(1-\tau_{II}^{\,2})/2\mu\tau_{II}\} \tag{2.68}$$

By introducing the quantity;

$$H = (1-\tau_{II}^{\,2})/2\mu\tau_{II} \tag{2.69}$$

we can show that u is a logarithmic function of μ via,

$$u = \tfrac{1}{2} \ln(1 + \frac{2}{H-1}) \tag{2.70}$$

where H, τ_{II} and ϕ are respectively given by (2.69), (2.66) and (2.67).

The phase diagram depends solely on the parameter a. Since $\varepsilon_S/\varepsilon_N \geqslant 0$ and $\mu \geqslant 0$, physically, one is restricted to $a \geqslant 3$ or $0 \leqslant a \leqslant 1$. However we have found a solution (2.70) to equations (2.46), (2.47) and (2.62) and thus a phase transition is only in the region $0 \leqslant a \leqslant 1$, namely if

$$4\mu^2 \geqslant a/3 - 2 + 3/a \qquad \text{for } 0 \leqslant a \leqslant a_C$$

and

$$4\mu^2 \geqslant 1 + 2\sqrt{1-a} \qquad \text{for } a_C \leqslant a \leqslant 1$$

with $a_C = \frac{3}{2}(\sqrt{13} - 3) \simeq 0.908$

The quantity ϕ (equation 2.67) is double-valued in the region $a_C \leqslant a \leqslant 1$ and

$$1 + 2\sqrt{1-a} \leqslant 4\mu^2 \leqslant a/3 - 2 + 3/a$$

wherein u is a double-valued function of the temperature. Outside, only the minus sign solution for ϕ is physical and the phase diagram is single-valued.

The phase diagram in the (u,μ) plane is plotted in figure (2.4). We recall that to have a phase transition, the parameter a must lie between 0 and 1. As can be seen in table (2.1), several of the sandwiches commonly investigated have a outside this range. In this region the conventional mode (I) has then the lowest free energy and the standard theory is applicable in this case, in zero magnetic field.

Since $u = d_N/\xi_N(T)$ we use (2.38) to write the interphase as

$$2d_N/\xi_{ON} = (T/T_{CN}-1)^{-\frac{1}{2}}\ln\{1+2/(H-1)\} \qquad (2.71)$$

where (2.70) has been used. We note that from (2.6) and (2.45), μ as a function of temperature would be

$$\mu = \frac{\sigma_N}{\sqrt{2}\sigma_S}\,\frac{v_{FN}}{v_{FS}}\,\frac{T_{CN}}{T_{CS}}\left\{\frac{T/T_{CN}-1}{1-T/T_{CS}}\right\}^{\frac{1}{2}} \qquad (*) \qquad (2.72)$$

The equation (2.71) has been used to plot the phase diagram of d_N and T, with the parameters relevant to a PbZn sandwich (*) in figure (2.5). It turns out that for a given thickness and at a temperaure much below T_{CS}, the mode (I) is energetically more favourable than mode (II) i.e. $F_{TI} < F_{TII}$. By increasing T, the mode (I) still remains stable till the temperature reaches a certain value (on phase diagram) at which the phase transition occurs, namely when $F_{TI} = F_{TII}$ and the order parameter at outer surface of N falls down to zero. We now again increase T but still below T_{CS}. It is seen that in this range of temperature the mode (II) would be more stable than mode (I), i.e. $F_{TII} < F_{TI}$ until T attains to T_{CS} at and above which the whole system of S/N becomes normal. It is of interest to note that the region (II), where mode (II) is energetically more favourable than mode (I), is broadened by increasing the thickness of normal specimen.

The phase diagram has the following salient features:

(i) As $4\mu^2 \to \infty$, i.e. as $T \to T_{CS}$

$$u = a^{-\frac{1}{2}} (1-a)(2\mu)^{-1} + O\left(\frac{1}{4\mu^2}\right) \tag{2.73}$$

or, within the microscopic Ginzburg-Landau-Gor'kov Theory

$$\frac{d_N}{\xi_{0S}} = \frac{1-a}{\sqrt{(2a)}} \frac{D_S}{D_N} \frac{N_N}{N_S} \frac{T_{CS}}{T_{CN}} (1-T_{CN}/T_{CS})^{-1}(1-T/T_{CS})^{\frac{1}{2}} \quad (*) \tag{2.74}$$

The scale of the d_N axis is therefore the coherence length of the superconducting metal ξ_{0S} at T=0, enhanced if T_{CN}/T_{CS} is either very small or very close to one. The former is the case of all the Cu sandwiches (even though $\sigma_N/\sigma_S >> 1$) and the latter that of InTl. The phase transition may then be too close to the critical temperature T_{CS} of the whole sandwich to be observable for all except the thickest sandwiches.

(ii) As $4\mu^2 \to 4\mu_\infty^2 = \frac{3}{a}(1-a/3)^2$, u diverges logarithmically (expression (2.70)):

$$u = -\tfrac{1}{4} \ln |4\mu^2 - 4\mu_\infty^2| + O(1) \tag{2.75}$$

$4\mu_\infty^2$ and thus the corresponding temperature T_∞ decreases with increasing a.

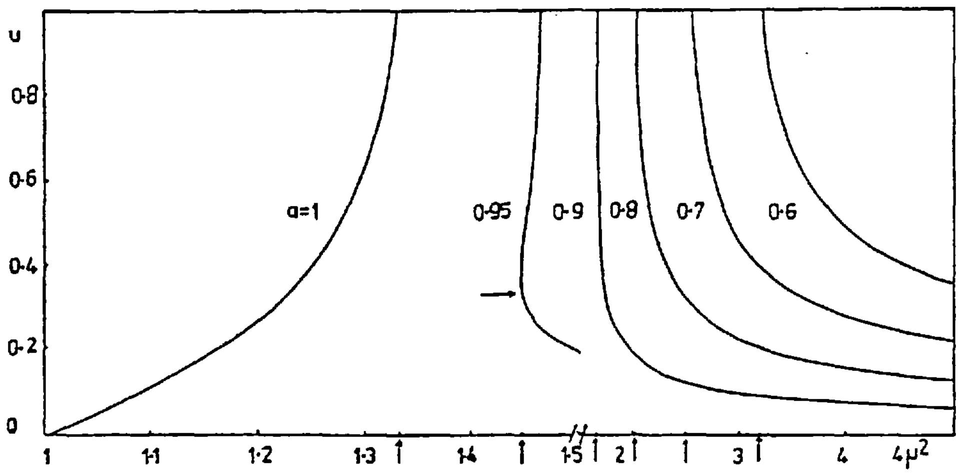

Figure 2.4: The phase diagram of a S/N sandwich for several values of the parameter a. $u = d_N/\xi_N(T)$ is proportional to the thickness d_N of the normal material, and $4\mu^2 \propto (T-T_{CN})/T_{CS}-T)$ (*) to the temperature. Note the change of scale in the abscissa.

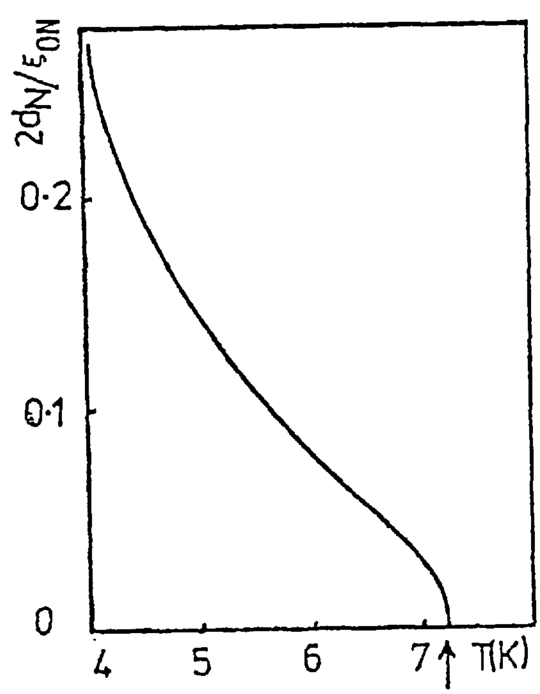

Figure 2.5: The same in terms of d_N and T, with the parameters relevant to a PbZn sandwich (*).

Table 2.1 Predicted occurrence of first-order phase transition

Data from equations (2.75) and (2.76) and from Kittel (1971).

N S (T_{CN}) (T_{CS})	Pb (7.2)	Sn (3.7)	In (3.4)	Tl (2.4)	Ga (1.1)	Zn 0.85)	Cd (0.56)	Cu (0.015?)
Nb(9.2)	No	No	No	0.75 8.2	0.90 8.2	0.81 6.2	0.91 7.2 7.2	1.0 7.8 7.5
Pb(7.2)		No	No	0.51 6.2	0.83 5.7	0.63 4.0	0.83 4.5	1.0 4.9 4.5
Sn(3.7)			No	0.51 3.6	0.83 3.3	0.63 2.7	0.83 2.9	1.0 3.0 2.8
In(3.4)				0.52 3.3	0.83 3.2	0.63 2.7	0.83 2.8	1.0 2.9 2.8
Tl(2.4)					0.26 2.3	No	0.29 2.0	1.0 1.3 1.2
Ga(1.1)		a T_∞(°K) T_{min}(°K)				No	No	0.99 0.44 0.41
Zn(0.85)							0.03 0.85	0.99 0.65 0.62
Cd(0.56)								0.99 0.33 0.32

(iii) As $4\mu^2 \to 4\mu^2_{min} = 1 + 2(1-a)^{\frac{1}{2}}$, when $a \geqslant a_C$, u approaches a turning point;

$$(4\mu^2 - 4\mu^2_{min}) = Q(u-u_{min})^2 \qquad (2.76)$$

Both u_{min} and Q are complicated but monotonic functions of a: u_{min}, $4\mu^2_{min}$ and the corresponding T_{min} decrease and Q increases with increasing a. $u_{min}(a_C) = \infty$, $Q(a_C) = 0$, whereas $u_{min}(1) = 0$, $Q(1) = \infty$; the interphase is linear.

To fix the respective scales, these are the conversion formulae between (u,μ) and (d_N,T) in the microscopic theory:

$$1-T/T_{CS} = (1-T_{CN}/T_{CS})/\{1+2(D_S/D_N)^2(T_{CS}/T_{CN})\mu^2\} \qquad (*)$$

$$d_N/\xi_{OS} = (T_{CS}/T_{CN})(N_N/N_S)(T/T_{CN}-1)^{-\frac{1}{2}}u \qquad (*) \qquad (2.77)$$

where equations (2.6) and (2.45) have been used.

Since the interphase never reaches T_{CN} (see table 2.1) $T/T_{CN} >> 1$, the second expression of (2.77), in a good approximation can be written as

$$d_N/\xi_{OS} \simeq (T_{CS}/T_{CN})^{\frac{1}{2}}u$$

It turns out that for a normal specimen with thickness greater than or of order of the coherence length of N, (ξ_N) i.e. $u \gtrsim 1$, the temperature of the phase transition is roughly independent of the thickness. This means that the difference between free energies is a very weak function of temperature and the two modes become indistinguishable. Consequently, the relevant thickness to investigate the phase transition are those within

$$0 \leq d_N \leq (T_{CS}/T_{CN})^{\frac{1}{2}} \xi_{OS}$$

2.V Conclusion:

We predict, within the local GL theory, the occurrence of a first order phase transition in several superconducting sandwiches (in the absence of an external magnetic field), due to the competition between two configuration modes for the order parameter compatible with local Ginzburg-Landau superconductivity. This phase transition occurs only if the mismatch parameter $\varepsilon_S/\varepsilon_N$ is large enough:

$$\varepsilon_S/\varepsilon_N \geq 2 \qquad \text{where} \qquad \varepsilon = \sigma T_C^{\,2}/nN^2V^2$$

It can be seen in table (2.1) that several, but not all, the sandwiches hitherto studied experimentally do not exhibit this phase transition and have the conventional configuration. This may be the reason why the phase transition has not so far been observed experimentally. Moreover, it may be difficult to observe if the normal material is too thick since the latent heat involved in the transition goes to zero as the normalized thickness u tends to infinity.

The same energetic arguments can be extended to include the effect of an external magnetic field. The conventional mode (I) is indeed more vulnerable to the application of a magnetic field on the normal side of the sandwich than mode (II) and should give way to the latter through a first-order phase transition at some finite breakdown field below the critical field of the superconductor.

The effect of an external magnetic field will be studied in the forthcoming chapter.

The phase transition predicted here should also affect the behaviour of Josephson junctions S/N/S which can be viewed as our S/N sandwich with its mirror image juxtaposed. At sufficiently high temperatures and for thick enough ($\simeq \xi_{0S}$) normal material, the order parameter is in the mode (II) which is antisymmetrical across the junction. There is thus a phase shift of π.

Finally, by reducing the electronic mean free path, i.e. σ_N, in the normal side of the sandwich, the parameters $^{\varepsilon}S/^{\varepsilon}N$ and a are increased and a phase transition from mode (I) to mode (II) can be induced in the sandwich at a fixed temperature and thickness, as indicated in figure (2.4).

CHAPTER 3

SUPERCONDUCTOR IN AN EXTERNAL MAGNETIC FIELD

3.I Introduction:

In earlier chapters, we restricted ourselves to the study of the systems such as S/N sandwiches in the absence of an external magnetic field. This restriction clearly excludes the consideration of most of the outstanding properties of an isolated superconductor, i.e. those occuring in an external magnetic field. In this chapter we treat these properties.

3.II Upper critical field of the superconductors:

When a superconducting sample is placed in a uniform and sufficiently high magnetic field, the superconductivity will be destroyed and the microscopic field will be homogeneous and equal to h_a (h_a is applied magnetic field in reduced units) throughout the superconductor. We now decrease h_a until the superconductivity begins to appear. This value of h_a is called the upper critical field and is denoted by h_{C2}. It will be shown that h_{C2} is quite different from the thermodynamic critical field h_C and it can be either greater or smaller than h_C depending on the type of the material.

In the vicinity of h_{C2}, the order parameter is so small that the non-linear terms in f (in GL equations) can be ignored and the linearized GL equations could be used to determine h_{C2}. Since all screening effects due to supercurrents are proportional to $h' = \frac{dh}{du}$ (see Appendix A) and therefore to f^2, the resulting modifications to the field could safely be ignored in linear approximation. This enables us to assume that the microscopic field is equal to h_a. Then the coupled GL equations (Appendix A) become,

$$\begin{aligned} \kappa^{-2} f'' &= f(-1+a^2) \\ a'' &= 0 \end{aligned} \tag{3.1}$$

where $f' = \frac{df}{du}$ and $h = a' = \frac{da}{du}$. Now if we use the following boundary condition,

$$a = 0 \quad , \quad \text{at some position } u = u_0 \tag{3.2}$$

then the second equation of (3.1) will be

$$a = h_a(u-u_0) \tag{3.3}$$

and the first one becomes,

$$\{-\kappa^{-2} \frac{d^2}{du^2} + h_a{}^2(u-u_0)^2\} f = f \tag{3.4}$$

This is formally identical to a Schrödinger equation for the motion of a particle in a uniform magnetic field, or bound in a harmonic oscillator potential. This problem is analogous to the problem of finding the eigen-states of a charged particle in h_a, leading to the so-called Landau levels. The frequency ω of the harmonic oscillator (3.4) and its eigen-energy ε are, here

$$\hbar\omega = \frac{2h_a}{\kappa} \quad , \quad \varepsilon = 1 \tag{3.5}$$

In general the harmonic oscillator eigen-energies are given by,

$$\varepsilon_n = (n+\tfrac{1}{2})\hbar\omega \ ; \qquad (3.6)$$

Using (3.5), we obtain $$h_a = \frac{\kappa}{2n+1} \qquad (3.7)$$

The highest field for which the superconductivity begins to appear (i.e. $f \neq 0$) will be determined by setting n=0 in equation (3.7). Thus

$$h_{c2} = \kappa \quad \text{in dimensionless units}$$

and is, $$H_{c2} = \sqrt{2}\,\kappa H_c = \frac{\phi_0}{2\pi\xi^2} \quad \text{in conventional units} \cdot \qquad (3.8)$$

(The thermodynamical critical field H_c has been defined and determined in Appendix A).

The derivation of the upper critical field, H_{c2}, is valid for all values of κ. Evidently, for type I superconductor for which $\kappa < 1/\sqrt{2}$, the upper critical field is smaller than H_c, whereas for type II material ($\kappa > 1/\sqrt{2}$), H_{c2} exceeds H_c; $H_{c2} = H_c$ for the specimen with $\kappa = 1/\sqrt{2}$. De Gennes (1966) has commented that, in bulk type II superconductor, the condensed phase ($f \neq 0$) will appear for any $H_a < H_{c2}$. Since a complete Meissner effect is not energetically favoured at any applied field higher than H_c, this phase can not coincide with a perfect exclusion of the magnetic flux. On the other hand, in bulk type I material a perfect Meissner effect occurs at H_c, greater than H_{c2}.

Saint-James et al (1969) have determined the field H_n for nucleation of superconductivity. Since expression (3.8) gives H_{C2} for an infinite sample and the edge effect makes no contribution to it, they have shown that nucleation, for finite specimen, occurs at a nucleation field $H_{C3} > H_{C2}$ and on the surfaces of the sample parallel to H_a (H_a is the applied external magnetic field in conventional units). In other words, very close to these surfaces, a superconducting sheath exists and the rest of the material remains in normal state.

It has to be mentioned that the determination of H_{C2} in three dimensional space which leads to the same result, yields a three dimensional solution for the order parameter corresponding to an eigen-value which is highly degenerate (Saint-James et al (1969)). Many authors, Abrikosov (1957), De Gennes (1966), Saint-James (1969) and Tinkham (1975) have used this property to obtain the Abrikosov vortex structure of the type II superconductor close to H_{C2}.

We have, however, used a very simple quantum mechanical method to evaluate H_{C2} for superconductors of any type.

3.III Magnetic field effects on superconductors:

One of the interesting features of the bulk superconductor with $\kappa \gtrsim 1$ (κ is the Ginzburg-Landau (GL) parameter) when placed in an external magnetic field, is the extension of the magnetic perturbation well over the size of a Cooper pair. On the other hand, the extension of the magnetic perturbation, in the materials with $\kappa < 1$, is less than the size of a Cooper pair. As usual, the GL parameter κ is defined by: $\kappa = \frac{\lambda_T}{\xi_T}$, where λ_T and ξ_T are the penetration depth and the coherence length. It is of great importance to notice that the penetration depth depends on the applied magnetic field H_a. In fact, the calculations carried out by GL (1950) show that the effective penetration depth λ_{eff} is seven percent higher than the penetration depth in zero magnetic field, λ_T, for the superconductor with $\kappa = 1/\sqrt{2}$ and for $H_a = H_C$ (H_C is the thermodynamic critical field of the bulk material). In forthcoming sections λ_{eff} will be calculated for the extreme type II superconductor ($\kappa \to \infty$), placed in an external magnetic field parallel to the surface of the superconductor. It will be shown that $\lambda_{eff} = \sqrt{2}\lambda_T$ for the extreme type II superconductor of infinite thickness, by using the GL equations.

The type II superconductor, in a low external magnetic field H_a, displays a complete Meissner effect. The specimen will remain in the Meissner state, even if H_a is higher than the bulk lower critical field H_{C1} (H_{C1} was first calculated by Abrikosor, 1957, and is given by $H_{C1} = \frac{H_C}{\sqrt{2\kappa}}$ ($\ln\kappa + 0.08$) . This means that the vortex line can not enter into the specimen, even if $H_a > H_{C1}$. The reason is that, a vortex line near the surface of the sample interacts with the external field and the associated screening currents. This leads to a surface barrier effect impeding the entry of the flux lines into an ideal specimen. It turns out that the Meissner state can subsist up to a certain field H_S (the barrier field) above H_{C1}. The surface barrier effect first was predicted independently by Bean and Livingston and by De Gennes and Matricon (1964). The resulting delayed flux entry has been observed by Joseph and Tomasch (1964), De Blois and De Sorbo (1964), Boato et al (1965) and Bôbel and Ratto (1965). Moreover, De Gennes (1966) has commented that, although a vortex line gains the energy, and hence reaches the deep inside of the sample, when $H_S > H_a > H_{C1}$, nonetheless there is a barrier near the surface of the sample, and the line will not enter the surface of the sample if the surface is clean.

The calculations done by Bean and Livingston (1964), for the bulk superconductors with $\kappa >> 1$, show that $H_S = H_C/\sqrt{2}$ which is not in agreement with the experiments carried out by De Blois and De Sorbo (1964) on NbTa alloys which give $H_S \approx H_C$. Furthermore, the Orsay Group On Superconductivity (1966), by using the London equation, has calculated H_S for the bulk superconductor with $\kappa >> 1$ and the result has been the same as the result due to Bean and Livingston (1964). The discrepancy between the aforementioned theories and the experiments of De Blois and De Sorbo (1964) is due to the fact that the London equation is valid only in low fields, i.e. $H_a << H_C$ (Saint-James et al 1969). In particular, the London penetration depth is taken to be field independent.

Fink (1966) has reported that the surface barrier effect can be predicted for the superconductors of arbitrary κ, if the thickness of the sample is much larger than the coherence length ξ_T. We restrict ourselves to the limit $\kappa \to \infty$ (the GL parameter of an extreme type II superconductor.) By using the GL equations, it will be shown that the barrier field H_S is exactly equal to the bulk thermodynamic critical field H_C (for an extreme type II superconductor with infinite thickness), i.e. $H_S = H_C$ which is in excellent agreement with the experimental result of De Blois and De Sorbo (1964).

In the GL theory, H_S plays the role of the boundary for superheating of the superconducting phase. In fact, when H_a is increased above H_C, the metastable state occurs in the superconductor. This is accounted for as superheating (Burger and Saint-James, 1969). Therefore, according to our previous discussion about H_S, in this section, the superheating is pictured as the permanence of the Meissner effect above the corresponding critical field. This means that the type II material remains in the Meissner state instead of going to the mixed state at $H_a = H_S$ (what we called the barrier field). Consequently H_S is the maximum field up to which the superconductor remains in the metastable state.

Within the domain of validity of the GL theory, we calculate the effective penetration depth λ_{eff} and the barrier field (the superheating field) H_S for the extreme type II superconductors ($\kappa \rightarrow \infty$). It will be shown that the order parameter is reduced considerably near the surface of the sample, when H_a is of the order of H_C.

3.IV The GL equations for an extreme type II superconductor:

The advantage of the London equation used by many authors (Orsay Group, 1966 and Bean and Livingston,1964) in explaining the surface barriers, is that it gives us a tool to study the nucleation process. When H_a is of the order of H_C, the super-current (or the order parameter) is considerably lowered even when a vortex has not yet appeared (Orsay Group, 1966). Therefore the London equation, with a constant penetration depth λ_T, can be used everywhere, except for the hard cores of the vortices (the hard core radius is ξ_T); this disregards the sensitivity of H_S to the coherence length ξ_T (De Gennes

1965). Thus the London equation is not applicable whenever H_a is high. Since the GL equations are applicable even when H_a is of the order H_C, they will be used to determine H_S. It will be shown, by using the GL equations, that for an infinite type II superconductor with $\kappa \to \infty$, $H_S = H_C$ which is in good agreement with the experimental results of De Blois and De Sorbo (1964).

This section, using the GL equations, is devoted only to the derivation of a differential equation for the magnetic vector potential (for an extreme type II material) which will be used everywhere throughout this chapter. It is mentioned that this chapter deals only with the high κ limit.

Since no vortex has yet appeared, we can use the GL equations in one-dimensional space in the limiting case $\kappa >> 1$. The GL equations in dimensionless forms are (Appendix A),

$$\kappa^{-2} f'' = f(-1+f^2+a^2) \tag{3.9}$$

$$a'' = f^2 a \tag{3.10}$$

In the limit $\kappa >> 1$ equation (3.9) takes the form

$$f(-1+f^2+a^2) = 0. \tag{3.11}$$

This implies that either $f = 0$, which describes the normal state of the system, or

$$f^2+a^2 = 1 \tag{3.12}$$

Since in our present work, the order parameter f extends over a distance approximately equal to the penetration depth i.e. $f'' \sim f$, then the neglected terms are of order κ^{-2}.

The order parameter f as given by equation (3.12) does not generally satisfy the boundary condition $f' = 0$ at the surface of the sample but this does not have a great significance in the case of the specimen with $\kappa >> 1$. However, it has been proven by the Orsay Group On Superconductivity (1966) that the correction to f in equation (3.12) needed in order to satisfy the above boundary condition, is of order of κ^{-2} which can be neglected in the limit $\kappa >> 1$.

Combining equations (3.12) and (3.10), we obtain an equation of motion given by:

$$a'' - a + a^3 = 0 \tag{3.13}$$

which is the equation of motion of the standard ϕ^4 field theory. This is equivalent to the equation of motion of a particle of mass unity in a potential

$$V(a) = \tfrac{1}{4}a^4 - \tfrac{1}{2}a^2 \tag{3.14}$$

where the "time" and "position" have been replaced by u and a respectively, (Fig. 3.1). The corresponding energy of the particle (E) is obtained by the first integral of (3.13) which is given by

$$\tfrac{1}{2}a'^2 + V(a) = E = \text{constant} \tag{3.15}$$

All the solutions of this equation are elliptic functions which are all oscillatory, except for one (a solitary wave), which corresponds to E = 0. In this case, it takes an infinite time for the particle to reach to the apex point 0 (Fig. 3.1). There are two other possible solutions of

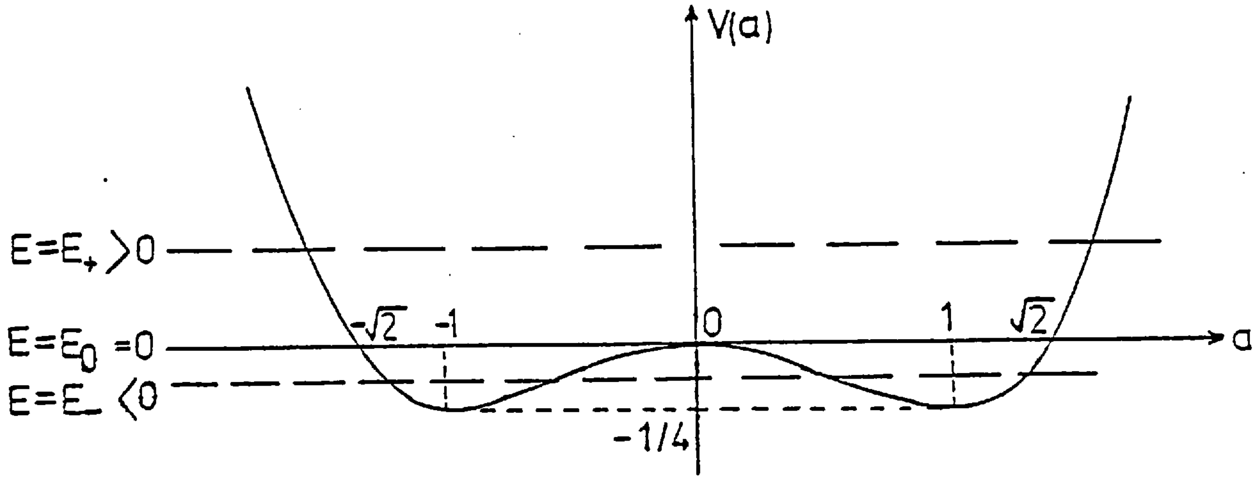

Fig. 3.1: Potential well of a particle with mass unity (we note that the abscissa is a). This is the solution of (3.14). The particle is bound to move in the physical regions where $|a| \leq 1$, $1 \geq f \geq 0$ (equation (3.12)). The corresponding energy E of the particle can be either positive E_+ or zero E_0 or negative E_-.

(3.15) which correspond to $E < 0$ and $E > 0$. All the possible solutions (corresponding to $E = 0$, $E < 0$ and $E > 0$) will be investigated separately. Moreover, we will use mechanical arguments in analogy with the motion of a particle in the potential well (Fig. 3.1) to obtain the physically acceptable regions at which the solutions of (3.15) are such that $|a| \leq 1$ and $f \geq 0$; we recall that, since the reduced order parameter $0 \leq f \leq 1$, from equation (3.12), the absolute value of the magnetic vector potential a, for the physical regions, must be less than one, i.e. $|a| \leq 1$. The regions at which a exceeds unity are not accepted on physical grounds (Fig. 3.1).

In view of the analogy of the magnetic energy $h^2/2$ (we note that the gradient of the vector potential at any point is the magnetic field) of an extreme type II superconductor with the kinetic energy $\frac{a'^2}{2}$ of a particle with mass unity moving in the potential well (Fig. 3.1), it seems useful to discuss 'mechanically' the three possible solutions of

(3.15). The magnetic energy of such a superconductor, from (3.15), is $\frac{h^2}{2} = E - V$ (which is equivalently the kinetic energy of the above particle $\frac{a'^2}{2} = E - V$). The three possible solutions of (3.15) which correspond to $E = E_+ > 0$, $E = E_0 = 0$ and $E = E_- < 0$ (see Fig. 3.1) are discussed as follows:

(1) When the energy of the particle is E_+ and at some time which corresponds to the point $a = 1$ (where the potential $V(a)$ is minimum), the kinetic energy of the particle (or accordingly the magnetic energy of the specimen) is maximum and is given by $\frac{a'^2}{2} = \frac{h^2}{2} = E_+ + \frac{1}{4}$. In our present investigation of an extreme type II material, this gives the maximum allowed field, $h_S = (2E_+ + \frac{1}{2})^{\frac{1}{2}}$, the barrier field in reduced units. (From Appendix A, the bulk thermodynamic critical field $h_C = 1/\sqrt{2}$. Therefore $h_S > h_C$ when $E = E_+ > 0$, which means that the superconductor remains in the Meissner state even when the field is greater than the bulk thermodynamic critical field. In other words, the interaction between the vortex line with the external field or with the surface of the sample reduces considerably the energy of the vortex such that it can not enter in the superconductor). At later times ($a < 1$), when the particle moves towards the maximum value of $V(a)$, its kinetic energy is decreased (the magnetic field is lowered) and it takes some definite time for the particle to reach to the point $V(a) = 0$ at $a = 0$. At this point the particle has its minimum kinetic energy $\frac{a'^2}{2} = \frac{h^2}{2} = E_+$. This means that the magnetic field h does not vanish in the case when $E = E_+ > 0$ and in the physical region ($a \leqslant 1$, $f \geqslant 0$ from (3.12)). Therefore, when $E > 0$, the minimum field at which (3.15) has an acceptable solution in the physical region of interest ($a \leqslant 1$) is given by $h = \sqrt{2E_+}$. Clearly E is determined by the appropriate boundary condition. Since $V(a)$ is symmetric with respect to a, the above arguments hold for negative a. We notice that the solution of (3.15),

for $E > 0$, is oscillatory and goes beyond the point where $|a| = 1$ (unphysical region) at some applied field.

(2) When the particle has zero energy, i.e. $E = E_0 = 0$ (Fig. 3.1) and is at the point $a = 1$ (say zero time), its kinetic energy is maximum and is given by $a'^2 = \frac{1}{4}$. At later times, its energy is reduced such that it takes an infinite time for the particle to have zero kinetic energy (when it reaches the apex point 0 at which the potential V(a) vanishes, Fig. 3.1). Consequently, in this case, i.e. $E = 0$, the maximum admissible field, for which $|a| \leq 1$ and $0 \lesssim f \leq 1$ are satisfied, is $h_S = 1/\sqrt{2} = h_C$. This means that the barrier field, in this case ($E = 0$), is exactly equal to the bulk thermodynamic field. It is clear that when $h_S = h_C$,($a = 1$), the order parameter vanishes at the point where the magnetic field is applied (at the surface of the sample). The same arguments can be applied for negative a, since V(a) is invariant under the sign transformation of a. We note that the solution of (3.15) is not oscillatory ($E = 0$) and, at some applied field, it goes beyond the point where $|a| = 1$ (unacceptable region).

(3) The third possible solution of (3.15) is the one in which $E = E_- < 0$. Assume that the particle is, at some specific time (say zero), at the point where $a = 1$ (for $a = 1$, the potential is minimum, i.e. $V(1) = -\frac{1}{4}$). The kinetic energy of the particle, at this point, is maximum and is given by $a'^2/2 = E_- + \frac{1}{4}$. (Correspondingly the maximum permissible field is $h_S = (2E_- + \frac{1}{2})^{\frac{1}{2}}$ which is obviously less than $h_C = 1/\sqrt{2}$, since $E_- < 0$. This means that the superconductor remains in the Meissner state for the field less than the bulk thermodynamic critical field, contrary to the case where E is positive). At later times (as a decreases towards zero) the kinetic energy reduces and vanishes at some point. The value

of a at which the particle has no kinetic energy is calculated from $\frac{1}{4}a_m^4 - \frac{1}{2}a_m^2 = -E_-$. For a less than the value determined from the above equation (later time) the kinetic energy will be negative. Therefore the particle has to oscillate between a = 1 and the positive value of a_m obtained from the above equation. This means that when $E = E_- < 0$, the solution of (3.15) is not physically acceptable for $|a| < |a_m|$, since the magnetic energy will be negative in this region. However, the solution of (3.15) is oscillatory and it goes beyond the point $|a| = 1$ (unphysical region), at some applied field.

In this section we discussed the three possible solutions of (3.15) corresponding to the cases where $E > 0$, $E = 0$ and $E < 0$, by using the mechanical arguments of the motion of a particle in a potential well. The equation (3.15) will be solved for all the above cases in the forthcoming sections which will enable us to determine h_S in each case. It will be shown that h_S depends strongly on the thickness of the superconductor of finite thickness. We note again that all the sections will deal with high κ.

3.IV.a Semi-infinite superconductor with high κ:

We consider a semi-infinite sample which occupies the half-space $u \geq 0$. The external magnetic field h_a (in reduced units) is applied at $u = 0$ and the other boundary conditions are,

$$a' = 0 \quad \text{and} \quad a = \text{constant} = 0 \text{ say, at } u = \infty \tag{3.16}$$

Applying these conditions to equation (3.15) gives $E = 0$ and then equation (3.15) becomes

$$a'^2 = a^2 - a^4/2 \tag{3.17}$$

The solution of this equation is simply given by ,

$$a(u) = \sqrt{2}\ \mathrm{sech}(u+u_0), \tag{3.18}$$

where u_0 is the parameter to be determined by using the following condition:

$$h = -\frac{da}{du} = h_a \quad \text{at} \quad u = 0. \tag{3.19}$$

The variation of the magnetic field, however, is given by

$$h(u) = \sqrt{2}\ \mathrm{sech}(u+u_0)\tanh(u+u_0) \tag{3.20}$$

Therefore u_0 is related to h_a by

$$h_a = \sqrt{2}\ \mathrm{sech}\ u_0 \tanh u_0 \tag{3.21}$$

Since in reduced units (Appendix A) any magnetic field is measured in the units of $\sqrt{2}H_C$ (H_C being the critical magnetic field for the bulk sample in conventional units). So, equation (3.21) in terms of conventional units is

$$\frac{H_a}{H_C} = 2\ \mathrm{sech}\ u_0 \tanh u_0, \tag{3.22}$$

or

$$\sinh u_0 = \left(\frac{H_C}{H_a}\right) + \left\{\left(\frac{H_C}{H_a}\right)^2 - 1\right\}^{\frac{1}{2}} \tag{3.23}$$

Hence the possible existence of a solution only when $H_a/H_C \leqslant 1$ is evident. Nevertheless, we mention that for all $u \geqslant 0$, the solution must remain "physical" (from expression (3.12), $|a| \leqslant 1$); obviously this is not possible for all the oscillatory solutions of the equation (3.13). (We note that the solution of (3.13) is not oscillatory for $E = 0$, section

3.III). This means that, they all go beyond the point where $|a| = 1$ at some applied field. For semi-infinite specimen, the maximum permissible value of H_a is H_C (corresponding to $\sinh u_o = 1$) and the only acceptable solution is the one in which the condition $u_o \geq \sinh^{-1}(1)$ is satisfied (Fig. 3.2), so that $a(u \geq \theta) \leq 1$.

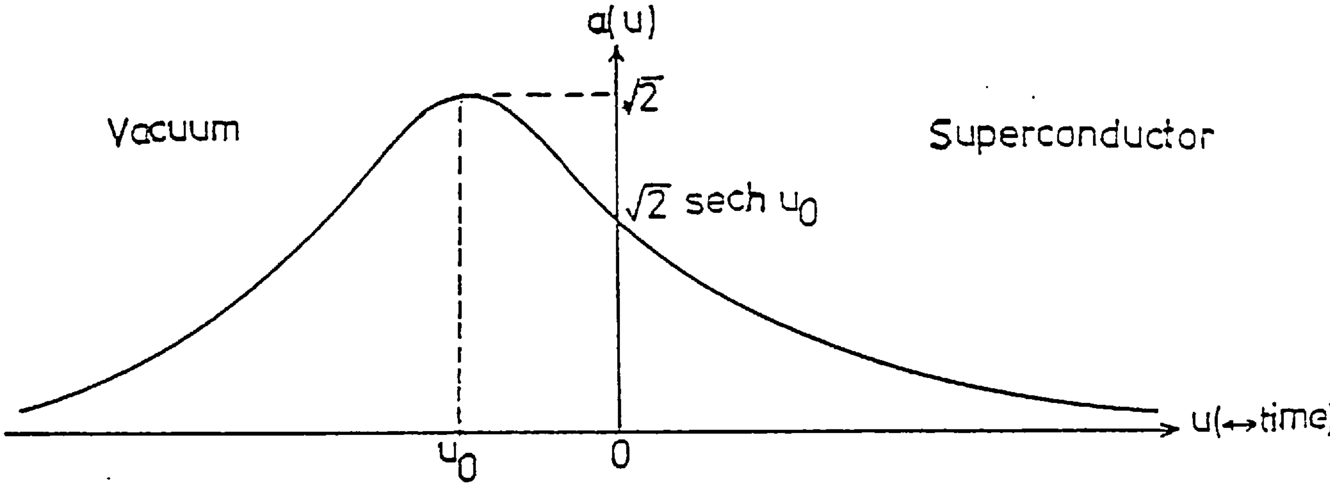

Figure 3.2: Vector potential distribution in the extreme type II superconductor ($\kappa >> 1$). (Solution of the equation 3.18). Note that the slope $a' = -h$ is the field to be matched at the boundaries of the sample.

Since for the field $H_a > H_C$, the surface barrier disappears and hence the vortices enter into the specimen, the GL equations must be solved in two dimensional space. We must conclude that the solutions which are functions of y as well must be the only solutions of the GL equations for $H_a > H_C$. The system which is studied in two dimensional space buckles, which leads to the creation of the vortices.

The one dimensional solution of the GL equation for the material with high κ however, carries further interesting results which can be obtained by resorting to the field distribution and the variation of order parameter in the sample.

Combining equations (3.18) and (3.12), we get

$$f^2(u) = 2\tanh^2(u + u_o) - 1 \tag{3.24}$$

Using expression (3.22), the value of f at u = 0, in terms of field, becomes

$$f^2(0) = \{1 - (\frac{H_a}{H_C})^2\}^{\frac{1}{2}} \tag{3.25}$$

Therefore, for $H_a = H_C$, the order parameter at the surface i.e. f(0) falls down to zero and hence we are in fact dealing with a situation of strong field effects near the surface (strongly metastable states, section 3.III). De Blois and De Sorbo (1964) and Joseph and Tomasch (1964) have experimentally shown that these metastable states can be preserved for a considerable time. This means that the one dimensional solution (in the present work) always corresponds to a local minimum of the thermodynamic potential for $H_a < H_C$. Figure (3.3) shows the field distribution in a sample with high κ for various values of H_a in metastable states before penetration of vortices ($H_a > H_{C1}$).

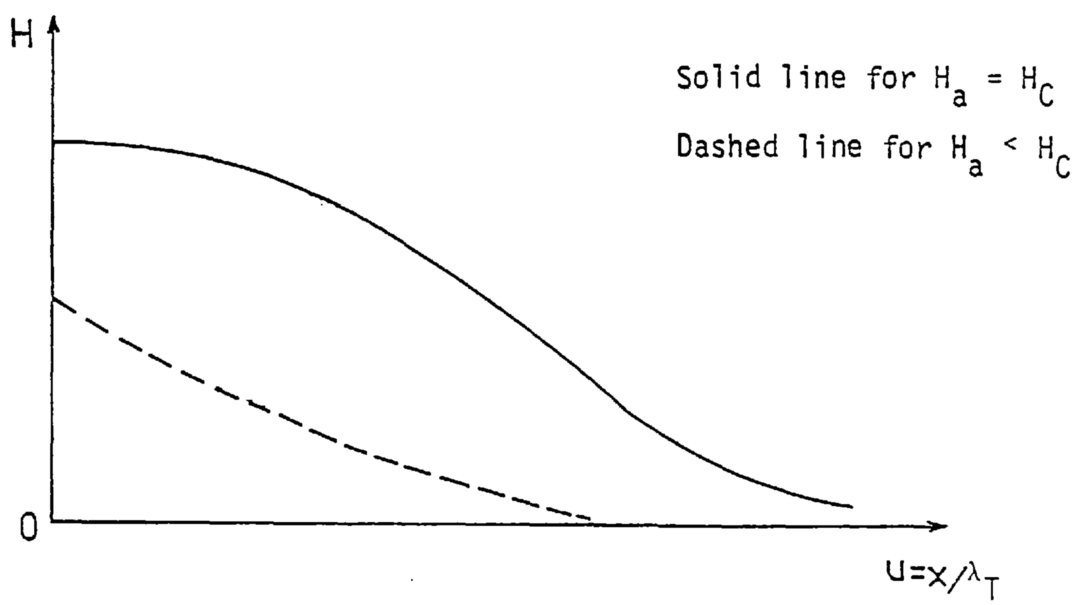

Figure 3.3: Distribution of the field in an extreme type II superconductor for different applied fields ($H_a > H_{C1}$).

The distribution of the current J in the sample, for $H_a = H_C$, is of a particular interest. Since the current is proportional to $\frac{dh}{du} = -a'' = a^3 - a$. it vanishes at the surfaces of the sample (for a = 1, f = 0 and for a = 0, f = 1, see (3.12)) and has a maximum, J_m, at a distance $u_m = \ln(\frac{\sqrt{6} + \sqrt{5}}{\sqrt{2} + 1}) \simeq 0.663$ from the origin. At this point $H_a = \frac{\sqrt{5}}{3} H_C$, $a = a_m = 1/\sqrt{3}$ and $f = f_m = \sqrt{(2/3)}$. J_m is the same as the conventional critical current for a thin film (see for a review, De Gennes, 1966 and Grassie, 1975). As is shown in Fig. (3.4), the current close to the surface is small. First it increases up to J_m, $u = u_m$ and then decreases to zero by increasing u from $u = u_m$ to $u = \infty$.

The variation of order parameter is plotted in Fig. (3.5) for the same conditions ($H_a = H_C$; $\kappa \to \infty$). Since the vector potential at the surface $a(0) = \sqrt{2}$ sech u_o is unity for $H_a = H_C$, the slope of f at that point is infinity. This slope is said to remain finite for finite κ (Orsay Group On Superconductivity, 1966).

3.IV.b The barrier field:

Another remarkable result obtained from the solution of the GL equations is the highest field H_S for which the Meissner state can subsist. In type II superconductor, this means that the specimen remains in the Meissner state instead of going to the mixed state at $H_a = H_{C1}$. H_S is indeed "the barrier field" which opposes the entry of the vortex line or more precisely, it is the maximum field less than which the sample remains in the metastable state.

The problem now is to calculate H_S for semi-infinite materials with high κ.

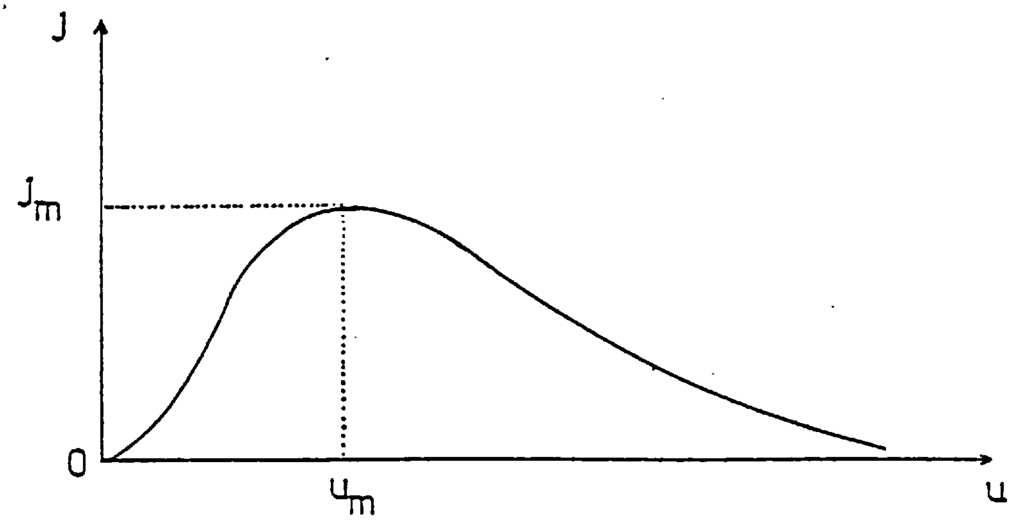

Figure 3.4: Current distribution for a metastable state of an extreme type II superconductor ($\kappa >> 1$) for $H_a = H_C$.

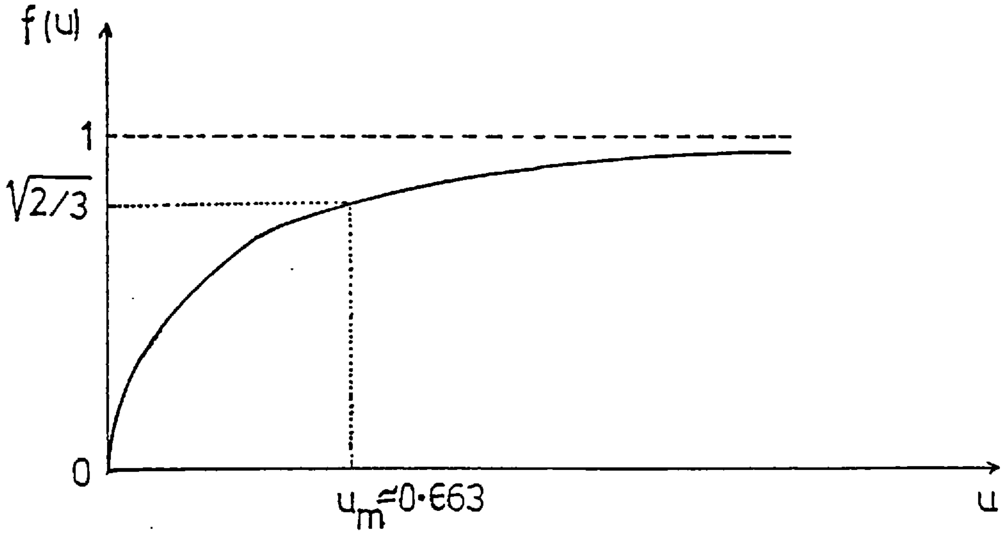

Figure 3.5: Variation of the order parameter f for the same conditions for $\kappa \to \infty$.

Combination of first integrations of equations (3.9) and (3.10) gives us (for arbitrary κ):

$$a'^2 = h^2 = -\kappa^{-2}f'^2 + f^2(-1 + f^2/2 + a^2) + c \qquad (3.26)$$

The integration constant c is determined by imposing the following boundary conditions, $h = -a' = 0$, a = constant = 0, say, $f = 1$ and $f' = 0$ at $u = \infty$ which will be $c = \frac{1}{2}$. Substituting the value of c into expression (3.26) and using equation (3.9) we obtain

$$h^2 = \tfrac{1}{2}(1 - f^4) + (f/\kappa)^2 \frac{d^2}{du^2}\ln f \qquad (3.27)$$

This is valid for all values of κ (for semi-infinite materials). For an extreme type II superconductor ($\kappa >> 1$) one gets

$$h^2 \cong \tfrac{1}{2}(1 - f^4) \qquad (3.28)$$

We note that expressions (3.20) and (3.24) satisfy this equation. Since $h = h_a$, $f = f(0) \equiv f_o$ at the surface i.e. at $u = 0$; then h can be replaced by h_a if

$$h_a^2 \cong \tfrac{1}{2}(1 - f_o^4) \qquad (3.29)$$

The maximum applied field $h_a)_{max} \equiv h_S$ is the one which makes f_o zero. Therefore the barrier field is

$h_S = 1/\sqrt{2}$ in dimensionless units, or

$H_S = H_C$ in conventional units.

This is comparable to the barrier field obtained by Joseph and Thomasch (1964). They used "London equation" and showed that $H_S = H_C/\sqrt{2}$. But the experiment carried out by De Blois and De Sorbo (1964) shows that H_S = 320 Oe for the NbTa alloys with H_C = 310 Oe. Moreover, Matricon and Saint-James (1967) have numerically calculated H_S from equations (3.9) and (3.10), for all values of the GL parameter . It has been pointed out that

$$H_S = (\kappa\sqrt{2})^{-\frac{1}{2}} H_C \qquad \text{for low } \kappa$$

$$H_S = H_C \qquad \text{for } \kappa \to \infty$$

The result of their calculation is plotted in Fig. (3.6).

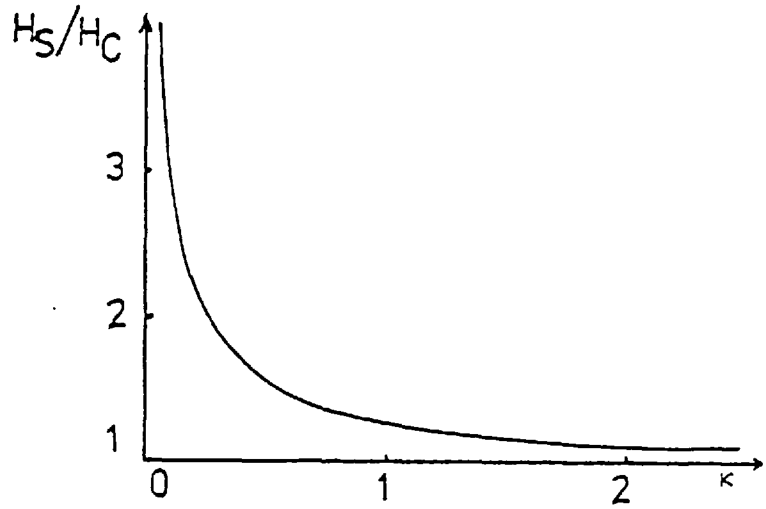

Figure 3.6: Variation of H_S versus κ for semi-infinite superconductor (Matricon and Saint-James, 1967).

3.IV.c The effective penetration depth in semi-infinite superconductors:

We have already mentioned in section (3.III) that the penetration depth of the magnetic field in superconducting material depends strikingly on the applied field, what we call "effective penetration depth" denoted by λ_{eff}. We now proceed to calculate λ_{eff} for semi-infinite materials for the aforementioned conditions. In the present case it is defined by

$$\lambda_{eff} = \frac{1}{h_a}\int_0^{\infty} h \, du \qquad \text{in reduced units.} \qquad (3.30)$$

Since we defined the field $h = -\frac{da}{du}$, where a is the vector potential in reduced units; equation (3.30) gives

$$\lambda eff = \frac{a(0)}{h_a} \quad , \qquad a(0) = a(u = 0).$$

(The condition a = 0 at u = ∞ has been used). We make use of expressions (3.18) and (3.20) to write λ_{eff} as

$$\lambda_{eff} = \coth u_o \tag{3.31}$$

and using equation (3.21), this quantity (in dimensionless units) can be related to the applied field by

$$\lambda_{eff} = \{2/(1 + \sqrt{1 - 2h_a^2}\}^{\frac{1}{2}} \tag{3.32}$$

Equation (3.32), in conventional units, is given by

$$\lambda_{eff}/\lambda_T = \{2/(1 + \sqrt{1 - (H_a/H_C)^2})\}^{\frac{1}{2}} \tag{3.33}$$

This leads to:

(i) $\lambda_{eff}/\lambda_T \rightarrow 1$ as $H_a/H_C \rightarrow 0$ as expected (De Gennes and Matricon (1965) and Matricon (1966)).

(ii) $\lambda_{eff}/\lambda_T = \sqrt{2}$ for $H_a/H_C = 1$, (3.34)

Figure (3.7) presents the variation of the penetration depth for different applied magnetic fields.

We conclude that the penetration depth increases from λ_T to $\sqrt{2}\lambda_T$ as the applied field increases from zero to H_C which is equal to the barrier field. (Note that at $H_a = H_S = H_C$, the order parameter falls down to zero at the surface of the sample).

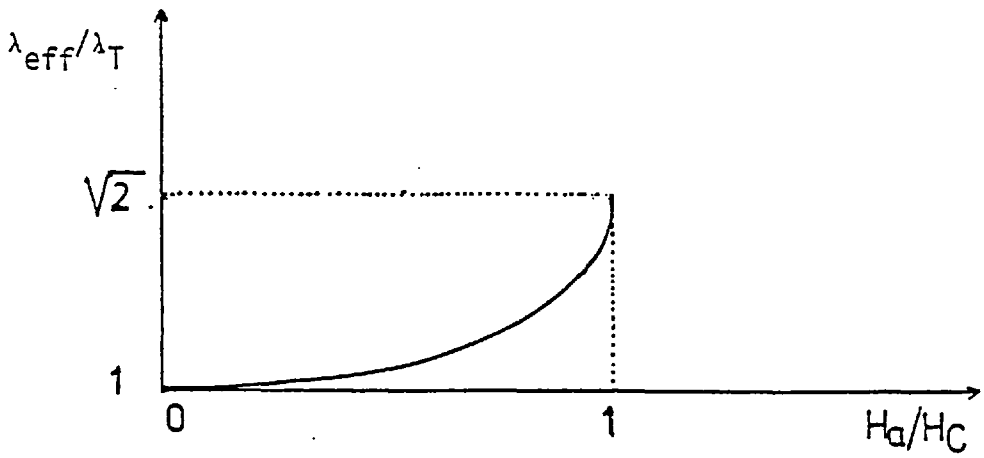

Figure 3.7: The effective penetration depth as a function of applied magnetic field for a semi-infinite superconductor with high κ.

3.V Superconductor with high κ and finite thickness:

In the previous section, we showed that H_S is exactly equal to H_C for semi-infinite superconducting materials (for $\kappa \to \infty$). For the sample with finite thickness in parallel field , the results are quite different. We now proceed to study such thin films. The thickness of the specimen is assumed to be d_S in physical units or $l_S = \frac{d_S}{\lambda_T}$ in reduced units. We will deduce different results depending on whether the magnetic field is applied only on one side or on both sides of the sample.

3.V.a Thin film with zero field on one side and H_a on the other side:

The problem is to solve the equation (3.13) for the following boundary conditions,

$$h = \frac{da}{du} = 0 \qquad \text{at } u = 0$$
$$h = \frac{da}{du} = h_a \qquad \text{at } u = l_S \tag{3.35}$$

Since the only allowable solution is the one which satisfies $|a| \leq 1$, then from equations (3.14) and (3.15) in conjunction with the above boundary conditions, it is inferred that $E < 0$ (equation (3.15)). The equation (3.15) becomes

$$a'^2 = \tfrac{1}{2}(a^2 - a_o^2)(b_o^2 - a^2) \qquad (3.36)$$

where $a_o = a(u = 0)$ is the value of vector potential at origin and

$$b_o^2 = 2 - a_o^2, \qquad E = \tfrac{1}{4}a_o^2(a_o^2 - 2) \qquad (3.37)$$

We note that $0 \leq a_o \leq 1$ and $1 \leq b_o \leq \sqrt{2}$. A solution of equation (3.36) is (Abramowitz and Stegun, (AS), 1972, p. 596):

$$a(u) = a_o \mathrm{nd}(\tfrac{b_o u}{\sqrt{2}}|m) \qquad *(3.38)$$

where nd is the reciprocal of dn (elliptic function) and m is the "parameter" of the elliptic function, defined here as

$$m = 1 - a_o^2/b_o^2 \qquad (3.39)$$

Using $h = \frac{da}{du}$ and equation (3.12) to deduce the magnetic field h and the order parameter f, we get

$$h(u) = \frac{a_o b_o m}{\sqrt{2}} \mathrm{sd}(\tfrac{b_o u}{\sqrt{2}}|m)\mathrm{cd}(\tfrac{b_o u}{\sqrt{2}}|m), \qquad *(3.40)$$

$$f(u) = \{1 - a_o^2 \mathrm{nd}^2(\tfrac{b_o u}{\sqrt{2}}|m)\}^{\frac{1}{2}} \qquad *(3.41)$$

The value of the order parameter $f(0) = (1 - a_o^2)^{\frac{1}{2}}$ is generally less than unity, except for the semi-infinite sample for which $m = 1$ (which suggests $a_o = 0$ and the problem must be solved by the same method as used

* We note that (AS, p. 571):

$\mathrm{nd}(\nu|m) = 1$ and $\mathrm{sd}(\nu|m) = \tfrac{1}{2}\sin 2\nu$ for $m = 0$

$= \cosh\nu$ $\qquad = \sinh\nu$ for $m = 1$

$\mathrm{cd}(\nu|m) = \cos\nu$ for $m = 0$

$= 1$ for $m = 1$

in the section 3.IV.a) giving f(0) = 1. The variation of the vector potential a with respect to u (equation 3.38) is presented in the figure (3.8). The physical regions within which the whole specimen must be combined, are those at which $|a| \lesssim 1$ and also $f \gtrsim 0$ (see equation (3.12)).

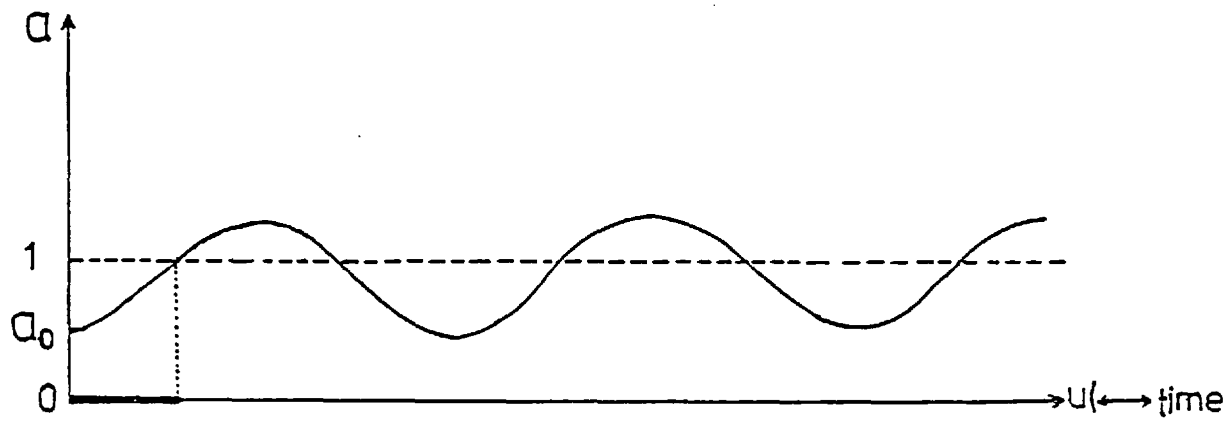

Figure 3.8: The distribution of the vector potential in the superconductor with high κ (solution of 3.38). The slope $a' = h$ is the field to be matched at the boundary of the sample. The heavy line on the u axis shows the physical region within which the whole specimen must be combined.

We make use of the boundary conditions (3.35) to find an implicit equation which determines a_o. It is, from equation (3.40)

$$h_a = \frac{a_o b_o m}{\sqrt{2}} \, sd(\frac{b_o l_S}{\sqrt{2}}|m) cd(\frac{b_o l_S}{\sqrt{2}}|m) \qquad (3.42)$$

and

$$b_o = (2 - a_o^2)^{\frac{1}{2}} \quad , \quad m = 2(1 - a_o^2)/(2 - a_o^2),$$

where equations (3.37) and (3.39) have been used.

The mechanical arguments of the motion of a particle in the potential well (Fig 3.1) discussed in the section (3.IV) for $E < 0$ (see, equation (3.37) implies that the maximum permissible field h_S is less than h_C, for finite thickness of an extreme type II material. This means that, the order parameter does not fall down to zero at the surface of the sample, i.e. at $u = l_S$, when $h_a = h_S$.

The barrier field h_S depends on the thickness l_S (equation (3.42)) such that as l_S tends to infinity, h_S approaches h_C (h_C is the bulk critical field of the superconductor in reduced units) which is compatible with the result of the last section. For finite l_S, $h_S(l_S)$ can not be found explicitly. This field, however, can be evaluated for a very thin film $l_S << 1$.

3.V.a.I Calculation of the barrier field for thin film:

It is shown that for very thin sample, i.e. $l_S << 1$, the barrier field varies linearly with thickness. We expand the elliptic functions which appear in the equation (3.40) up to u^3, giving

$$h(u) \simeq (a_o - a_o^3)\{1 + (1 - 3a_o^2)u^2/6\}u, \qquad u < 1 \tag{3.43}$$

Since $h(u) = h_a$ at $u = l_S$, then

$$h_a \simeq (a_o - a_o^3)\{1 + (1 - 3a_o^2)l_S^2/6\}l_S \tag{3.44}$$

for $l_S << 1$. This yields

$$h_a \simeq (a_o - a_o^3)\, l_S \qquad , \qquad l_S << 1.$$

For a given thickness (l_S), the maximum value of the above field, i.e. h_S is such that the condition $(d/da_o)h_a = 0$ is satisfied, this implies $a_o = 1/\sqrt{3}$. Hence $h_S = h_a)_{max} = \frac{2}{3\sqrt{3}}\, l_S$ in reduced units, or in conventional units

$$\frac{H_S}{H_C} = \left(\frac{2}{3}\right)^{3/2} \frac{d_S}{\lambda_T} \quad , \qquad d_S << \lambda_T. \tag{3.45}$$

It is seen that the current density is uniform within $d_S << \lambda_T$ and this agrees with the conventional critical current (Ginzburg, 1958 and Ginzburg and Landau, 1950).

3.V.a.II The effective penetration depth for thin film:

We are now in a position to calculate the effective penetration depth λ_{eff} for a thin film placed in the applied field. λ_{eff} is defined, generally, by (3.30). By using $h = \frac{da}{du}$, (3.42) and (3.38), we obtain

$$\lambda_{eff} = \frac{\sqrt{2}}{b_o m} ds\left(\frac{b_o l_S}{\sqrt{2}}\middle|m\right) dc\left(\frac{b_o l_S}{\sqrt{2}}\middle|m\right)\left\{nd\left(\frac{b_o l_S}{\sqrt{2}}\middle|m\right) - 1\right\} \tag{*(3.46)}$$

in units of λ_T. The above equation indicates that λ_{eff} depends strongly on the thickness of the sample l_S. For a very thin film ($l_S << 1$) the elliptic functions in (3.46) can be expanded and by truncating the expansion series after the first two terms, λ_{eff}, in the physical units, will be

$$\frac{\lambda_{eff}}{\lambda_T} \simeq \frac{1}{2} \frac{d_S}{\lambda_T} \quad , \qquad \text{for } d_S << \lambda_T. \tag{3.47}$$

We conclude that the effective penetration depth for this particular case is less than the zero field penetration depth, i.e. $\lambda_{eff} < \lambda_T$.

*We recall that (AS, p. 571):

$$ds(\nu|m)dc(\nu|m) = \frac{1}{\sin\nu\cos\nu} \quad \text{and} \quad nd(\nu|m) = 1 \quad \text{for } m = 0$$

$$= \frac{1}{\sinh\nu} \qquad\qquad = \cosh\nu \quad \text{for } m = 1$$

3.V.b Films in the symmetrical fields:

We now consider the case in which the sample with finite thickness l_S is placed in a symmetrical magnetic field. The field h_a is applied symmetrically at both surfaces of the sample and the origin is at the mid-point of the specimen. In other words

$$h(\tfrac{1}{2}l_S) = h_a = h(-\tfrac{1}{2}l_S). \tag{3.48}$$

This field symmetry implies that the vector potential a(u) is an odd function of u, i.e.

$$a(0) = 0. \tag{3.49}$$

This boundary condition in conjunction with the equation (3.15) gives $h_o^2 = h^2(0) = 2E$ (where h_o is the value of h at origin) which means $E > 0$ and hence the particle with mass unity moving in the potential $V(a) = \frac{1}{4}a^2(a^2 - 2)$, will have enough energy to reach $|a| = 1$. At $|a|=1$, the magnetic field will be (from equation (3.15))

$$h = \frac{1}{\sqrt{2}}(1 + 2h_o^2)^{\frac{1}{2}} = \frac{1}{\sqrt{2}}(1 + 4E)^{\frac{1}{2}}. \tag{3.50}$$

Since, from (3.12), $a \lesseqgtr 1$, the field at which $|a| = 1$ is in fact the maximum admissible field, that is h_S, so here where $E > 0$, $H_S/H_C > 1$ and therefore the order parameter f is destroyed at the sample surfaces.

We now solve equation (3.15) for the boundary conditions given by expressions (3.48) and (3.49). Equation (3.15) can be written in the following form,

$$\left(\frac{da}{du}\right)^2 = \tfrac{1}{2}(a_o^2 - a^2)(a^2 + b_o^2) \tag{3.51}$$

where

$$a_o^2 - b_o^2 = 2 \tag{3.52a}$$

$$a_o^2 b_o^2 = 4E = 2h_o^2 \tag{3.52b}$$

The solution of equation (3.51) is (AS, p. 596):

$$a(u) = a_o(1 - m)^{\frac{1}{2}} \mathrm{sd}\{\sqrt{(a_o^2 - 1)}\ u|m\} \qquad *(3.53)$$

the parameter m is

$$m = a_o^2/2(a_o^2 - 1). \tag{3.54}$$

The figure (3.9) shows the variation of the vector potential a with respect to u. The heavy line on the u axis in the figure (3.9) represents the physical region within which the whole sample must be combined.

We note that the magnetic field $h = \frac{da}{du}$ vanishes when $a = a_o$ and in this specific case $a_o \geqslant \sqrt{2}$. The field at origin h_o depends on the sample thickness. It is immediately seen from expression (3.52b) that a_o is thickness dependent. It turns out that a(u) can go beyond its maximum allowed value i.e. $|a(u)| = 1$; for instance, at the point where $h = 0$, a would be $a(u) = a_o \geqslant \sqrt{2}$ and is therefore unphysical ($f^2 < 0$ from equation (3.12)). In the region where $|a|$ exceeds one, the two dimensional solutions of the GL equation are required. The solutions which are function of x and y buckle which lead to the creation of the vortices.

The magnetic field $h = \frac{da}{du}$ and the order parameter $f = (1 - a^2)^{\frac{1}{2}}$ are derived from equation (3.53),

*We note that: $\mathrm{sd}(\nu|m) = \sin \nu$ for $m = 0$
$= \sinh \nu$ for $m = 1$

$$h(u) = a_o\left(\frac{a_o^2}{2} - 1\right)^{\frac{1}{2}} \,\mathrm{cd}\{(a_o^2 - 1)^{\frac{1}{2}}u|m\}\mathrm{nd}\{(a_o^2 - 1)^{\frac{1}{2}}u|m\} \tag{3.55}$$

$$f(u) = \{1 - a_o^2(1 - m)^{\frac{1}{2}} \,\mathrm{sd}^2((a_o^2 - 1)^{\frac{1}{2}}|m)\}^{\frac{1}{2}} \tag{3.56}$$

The quantity a_o is obtained by imposing the boundary condition given by the equation (3.48) on expression (3.55). It is then given by the implicit relation,

$$h_a = a_o(a_o^2/2 - 1)^{\frac{1}{2}} \,\mathrm{cd}\{(a_o^2 - 1)^{\frac{1}{2}}\frac{l_S}{2}|m\}\mathrm{nd}\{(a_o^2 - 1)^{\frac{1}{2}}\frac{l_S}{2}|m\}. \tag{3.57}$$

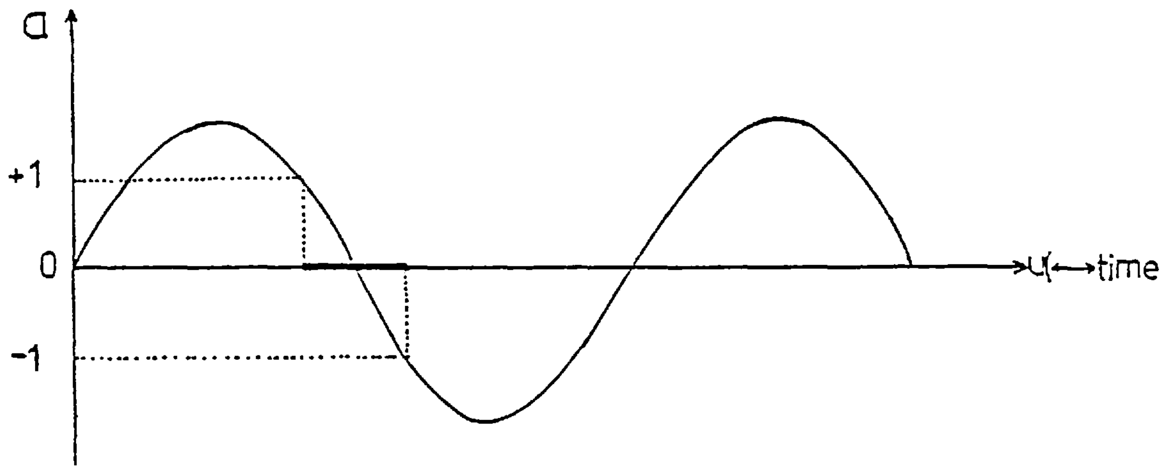

Figure 3.9: The variation of the vector potential in an extreme type II superconductor placed in a symmetrical field (the solution of (3.53)). The slope a' = h is the field to be matched at the boundary of the specimen.

3.V.b.I The barrier field for the thin specimen in the symmetrical fields:

As in the case (3.IV.b), the barrier field is produced when the order parameter vanishes, i.e. $f = 0$ on the surface of the specimen, which means $f = 0$ at $u = \pm\frac{1}{2}\, l_S$. This corresponds to $a = 1$ at $u = \pm\frac{1}{2}l_S$. In other words

$$a_o(1 - m)^{\frac{1}{2}} \,\mathrm{sd}\{(a_o^2 - 1)^{\frac{1}{2}}\frac{l_S}{2}|m\} = 1, \tag{3.58}$$

or equivalently,

$$\mathrm{cn}\{(a_o^2 - 1)^{\frac{1}{2}}\frac{l_S}{2}|m\} = (1 - \frac{2}{a_o^2})^{\frac{1}{2}} \quad \text{and} \quad \mathrm{dn}\{(a_o^2 - 1)^{\frac{1}{2}}\frac{l_S}{2}|m\} = (\frac{a_o^2 - 2}{a_o^2 - 1})^{\frac{1}{2}} \qquad *(3.59)$$

Inserting these quantities into equation (3.57), we get

$$h_S = \frac{a_o^2 - 1}{\sqrt{2}} \qquad \text{in dimensionless units} \qquad (3.60a)$$

$$H_S/H_C = a_o^2 - 1 \qquad \text{in conventional units.} \qquad (3.60b)$$

Since $a_o \geqslant \sqrt{2}$, then $H_S \geqslant H_C$ and the equality is for the case when the sample has infinite thicknesss, which agrees with the case (3.IV.b).. From equations (3.59) and (3.60b) we have

$$\frac{d_S}{\lambda_T} = 2(\frac{H_S}{H_C})^{-\frac{1}{2}} \mathrm{cn}^{-1}(\frac{H_S/H_C - 1}{H_S/H_C + 1}|m), \qquad (3.61a)$$

where m as a function of H_S/H_C is

$$m = \tfrac{1}{2}\{1 + (\frac{H_S}{H_C})^{-1}\}. \qquad (3.61b)$$

For an infinite sample for which $a_o = \sqrt{2}$, (or $m = 1$), H_S turns out to be equal to the bulk critical field H_C as expected, which is exactly what was obtained in section (3.III.b). Plotting the variations of H and H_S versus u and thickness, requires the numerical evaluation of quantities appearing in the equations (3.55) and (3.61a). Fig. (3.10) indicates the distribution of field in film placed in the symmetrical applied fields expressed above. The curve (C) represents a specific solution for $a_o = (1 + \sqrt{5})^{\frac{1}{2}}$. At point M, a = 0 and f = 1, then a increases from zero to $|a| = 1$ at point N (the outer surface of the specimen) as u increases from zero to $u = \frac{1}{2}l_S$, whereas the order

*The elliptic functions are intermediate between circular and hyperbolic functions (AS, p. 571), i.e.

$$\mathrm{cn}(\nu|m) = \cos\nu \quad \text{and} \quad \mathrm{dn}(\nu|m) = 1 \qquad \text{for } m = 0$$
$$\phantom{\mathrm{cn}(\nu|m)} \doteq \mathrm{sech}\,\nu \qquad\qquad\quad = \mathrm{sech}\,\nu \qquad \text{for } m = 1$$

parameter f decreases from unity at point M to zero at N. In other words f falls down to zero at the sample surfaces. The dotted curve corresponds to the unphysical region for which the vector potential a goes beyond $|a| = 1, (f^2 < 0)$.

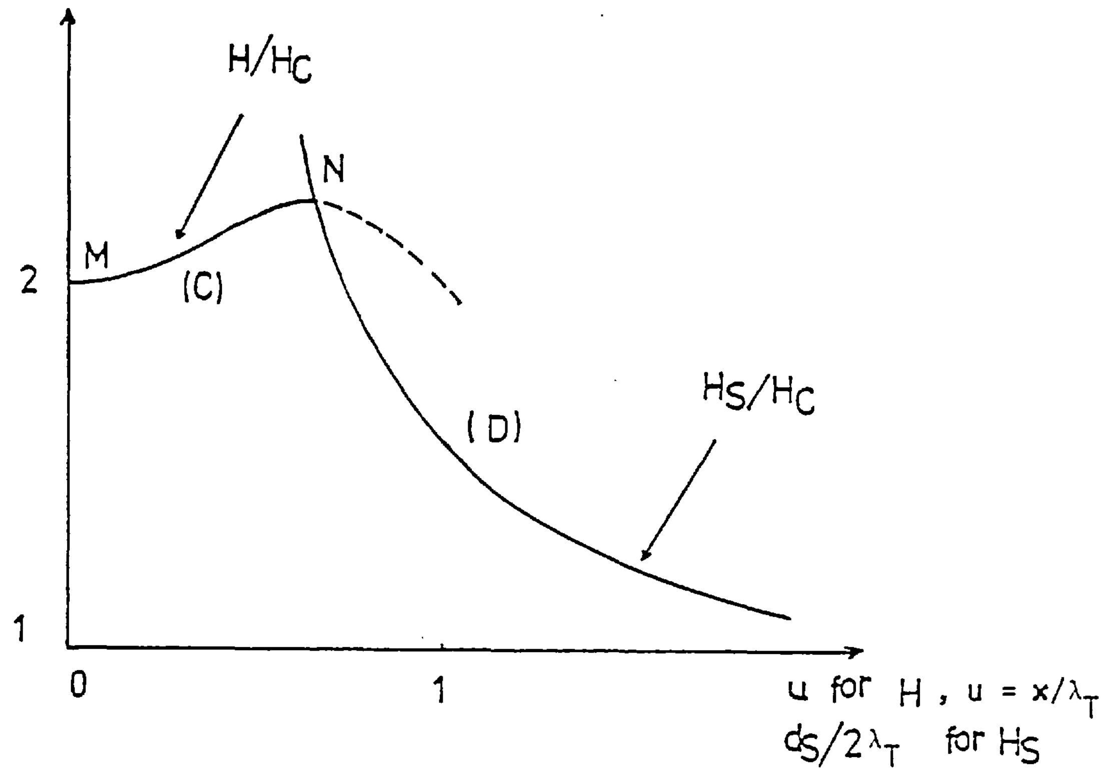

Figure 3.10: The distribution of the field in an extreme type II superconductor placed in a symmetrical field. The curve (C) is a particular solution of (3.55) for $a_0 = (1 + \sqrt{5})^{\frac{1}{2}}$ and is physically meaningful up to the point N where the vector potential a = 1 and hence the order parameter f = 0 (we note that at the point N, the field is maximum). The barrier field H_S as a function of the film thickness d_S (curve (D)) is the locus of the maxima of different solutions of H obtained by varying d_S.

Since the value of the field at the origin $h_o = a_o(\frac{a_o^2}{2} - 1)^{\frac{1}{2}}$ depends strongly on the thickness of the material as well as a_o, different solutions for H can be deduced as thickness (or a_o) is varied. It follows that the maximum allowed value of H i.e. H_S decreases with increasing l_S. Therefore H_S is the "locus" of the maxima of different solutions for H. Moreover, the barrier field H_S, for very thin film, is much larger than the bulk critical field H_C. In particular, such a thin film remains in the metastable state as the applied field is increased to a value above H_S (it is maintained in the superconducting state (in the mixed state)).

The thickness dependence of a_o will become clearer by expanding the expression (3.59) up to second order. Indeed, we find

$$(1 - \frac{2}{a_o^2})^{\frac{1}{2}} = 1 - (a_o^2 - 1)\frac{l_S^2}{8} + O(l_S^4).$$

For $l_S < 1$ we keep only the first two terms and obtain,

$$a_o^2 - 1 \simeq \tfrac{1}{2}\{-1 + (1 + \frac{32}{l_S^2})^{\frac{1}{2}}\}, \quad l_S < 1. \tag{3.62}$$

Clearly a_o is a decreasing function of thickness l_S (in our reduced units). The left hand side of the above expression is just H_S/H_C, in physical units (equation (3.60b)),

$$\frac{H_S}{H_C} = \tfrac{1}{2}\{-1 + (1 + \frac{32}{l_S^2})^{\frac{1}{2}}\} \quad , \quad l_S < 1.$$

This can be written as

$$\frac{H_S}{H_C} \simeq \sqrt{8}\frac{\lambda_T}{d_S} + (\frac{\sqrt{2}}{32}\ \frac{d_S}{\lambda_T} - \tfrac{1}{2}), \quad l_S < 1. \tag{3.63}$$

This is in good agreement with the calculations done by De Gennes (1965). The above bracket could be a correction to his result. Nevertheless for an extremely thin sample this bracket can safely be neglected, so one has

$$\frac{H_S}{H_C} \simeq \sqrt{8}\frac{\lambda_T}{d_S} \qquad \text{when } d_S \ll \lambda_T.$$

This is to be dimensionally compared with the upper critical field $H_{||}$

of a thin type I superconductor with $d_S/\xi_T < 1$ as well as $d_S/\lambda_T < 1$. Abrikosov (1952) and Rickayzen (1965) have shown that

$$H_{11}/H_C = \sqrt{24}\,\frac{\lambda_T}{d_S}$$

But H_S and H_{11} given by the above expression refer to quite differnt situations. In our present case, the thin film is maintained in the superconducting state (it remains in the metastable state) as the applied field is increased to a value above H_S (De Gennes, 1965), whereas the thin film of the type I material remains in the superconducting state as the applied field is lowered to a value below H_{11}.

3.V.b.II The effective penetration depth of symmetrical fields:

In the last section, we showed that H_S is larger than H_C, when an extreme type II material is placed in the symmetrical fields. Since $H_S > H_C$, the effective penetration depth λ_{eff} is not expected to be large. This can be experimentally verified by noticing that the sample remains superconducting as H_a is increased to a value greater than H_S. The effective penetration depth however, can reach its maximum value $\sqrt{2}\lambda_T$ for the superconductor with infinite thickness, as expected. λ_{eff} is defined generally by (3.30). In the present case, we use $h = \frac{da}{du}$ and (3.49) to obtain

$$\lambda_{eff} = \frac{1}{h_a} \quad , \qquad \text{in reduced units} \tag{3.64}$$

where the condition $a(u = \frac{1}{2}l_S) = 1$ has been implied. The equation (3.57) changes the above expression formally to

$$\lambda_{eff} = (\tfrac{1}{2}a_o^4 - a_o^2)^{-\frac{1}{2}}\, \mathrm{dc}\{(a_o^2 - 1)^{\frac{1}{2}}\tfrac{l_S}{2}|m\}\ \mathrm{dn}\{(a_o^2 - 1)^{\frac{1}{2}}\tfrac{l_S}{2}|m\} \qquad *$$

*We note again that (AS, p. 571):

$$\mathrm{dc}(\nu|m)\,\mathrm{dn}(\nu|m) = \sec\nu \quad \text{for } m = 0$$
$$= \mathrm{sech}\,\nu \quad \text{for } m = 1$$

A straightforward simplification of this expression with the aid of equations (3.59) will give

$$\lambda_{eff} = \sqrt{2}/(a_o^2 - 1) \qquad \text{in unit of } \lambda_T,$$

and conventionally

$$\lambda_{eff}/\lambda_T = \sqrt{2}/(a_o^2 - 1); \tag{3.65}$$

Since $a_o^2 - 1 = H_S/H_C$, from equation (3.60b), then this becomes

$$\lambda_{eff}/\lambda_T = \sqrt{2}\ \left(\frac{H_S}{H_C}\right)^{-1}. \tag{3.66}$$

For very thin films for which $H_S/H_C >> 1$, the effective penetration depth would be much smaller than λ_T i.e. $\lambda_{eff}/\lambda_T << 1$. Two limiting cases can immediately be seen:

(i) $\lambda_{eff}/\lambda_T \to 0$, as $l_S \to 0$ which implies $H_S/H_C \to \infty$,

(ii) $\lambda_{eff}/\lambda_T \to \sqrt{2}$, as $l_S \to \infty$ corresponding to $H_S/H_C \to 1$.

By substitution of the right hand side of equation (3.66) into the expression (3.61a), we arrive at the general functional dependence of λ_{eff} on thickness d_S,

$$\frac{d_S}{2\lambda_T} = \left(\frac{\lambda_{eff}}{\sqrt{2}\lambda_T}\right)^{\frac{1}{2}}\ cn^{-1}\left[\frac{1 - (\lambda_{eff}/\sqrt{2}\lambda_T)}{1 + (\lambda_{eff}/\sqrt{2}\lambda_T)}\middle| m\right] \quad , \tag{3.67}$$

here $m = \frac{1}{2}(1 + \frac{\lambda_{eff}}{\sqrt{2}\lambda_T})$. In the limit $d_S << \lambda_T$, equation (3.67) can be reduced to the simple form

$$\frac{\lambda_{eff}}{\lambda_T} \simeq \frac{d_S}{2\lambda_T} \qquad \text{for } d_S << \lambda_T. \tag{3.68}$$

This implies that for very thin films, λ_{eff} is much smaller than λ_T. Accordingly H_S is higher than H_C and hence the vortices can not enter into the sample, if the applied field is less than H_S. The variation of λ_{eff} with respect to d_S is shown in Fig. (3.11).

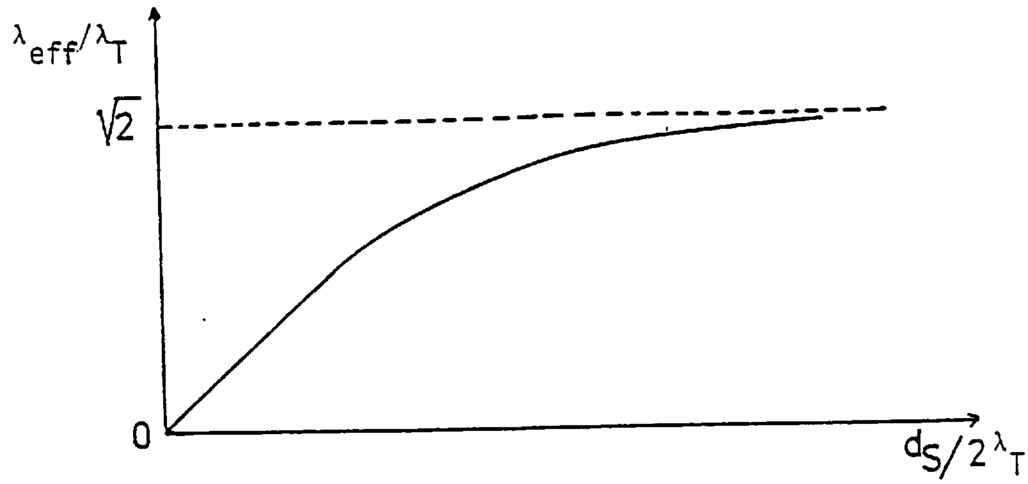

Fig. 3.11: The variation of the effective penetration depth of a superconductor with high κ placed in the symmetrical fields with respect to the thickness of the sample (solution of (3.67)).

3.VI Conclusion:

Apart from the complete solutions of the one dimensional dependence of the GL equations for the extreme type II superconductors, we studied the surface barrier effects in such materials. It was shown that the value of the barrier field, in comparison with the bulk critical field, depends strongly on the applied field geometry and the thickness of the specimens. This argument was used to determine the effective penetration depth for different cases. We now mention the most interesting results:

(i) For semi-infinite or infinite type II superconductors with $\kappa >> 1$, the barrier field H_S is exactly equal to H_C. The same results are produced for different field geometries. In other words. if one applies the magnetic field only on one side of the sample one will obtain the same result as if one applies H_a on both sides of an infinite sample. Moreover, the effective penetration depth, at $H_a = H_C$, is $\lambda_{eff} = \sqrt{2}\lambda_T$, which is larger than the penetration depth in the absence of the magnetic field, λ_T.

(ii) When we apply H_a only on one side of a sample of finite thickness, then H_S will be smaller than H_C and the order parameter f at the sample surface does not drop to zero, contrary to the preceding case. Under the above circumstances, λ_{eff} is smaller than λ_T.

(iii) If we apply equal fields on both sides of the superconductor with finite thickness, then $H_S > H_C$ and $\lambda_{eff} < \sqrt{2}\lambda_T$. But the order parameter f at the surfaces vanishes and when H_a reaches a value above H_S, the film remains superconducting (see, the section 3.V.b.I).

It is of great importance to notice that our argument about surface barriers does not involve flux trapping. In fact, experimentally, in order to observe H_S, one must avoid any source of trapped flux in the initial states when the specimen has finite thickness (De Gennes, 1965).

CHAPTER 4

THE S/N SYSTEM IN AN EXTERNAL MAGNETIC FIELD

4.I Introduction:

In chapter two we studied S/N binary layers in zero magnetic field. The overall conclusion was the prediction of a first-order phase transition due to two possible configurations of the order parameter Δ of a S/N system. As an extension of previous work (in the absence of a magnetic field), in this chapter, the magnetic properties of S/N sandwiches are studied in the framework of the local GL approximation in the limit $\kappa_N << 1$ for the normal material and $\kappa_S << 1$ for the superconductor. We find, as a function of the magnetic field, temperature or thickness of the normal sample, a first order phase transition between two possible configurations of the matching order parameter.

This is shown to be due to the arbitrariness of the boundary conditions of the outer surface of the normal specimen. As was pointed out in chapter one, both an order parameter with zero slope and a vanishing order parameter will make the super-current vanish at that point. Either one or the other boundary condition leads to a matching configuration of lesser energy in definite regions of magnetic field, temperature and normal slab thickness.

The study of a S/N double layer in an applied magnetic field parallel to the S/N interface leads us to the following remarkable effects.(Deutscher and De Gennes, 1969):

(i) In low field and under specific conditions, N exhibits a Meissner effect. On the other hand, in rather high fields, the superconductor properties induced in N by the proximity of S are destroyed. Hook (1974) has shown that the proximity effect in N disappears with increasing field, even at temperatures below T_{CN}. Practically, this disappearance takes place in fields significantly smaller than H_{CS} (H_C is the bulk critical field of the sample). As a corollary this destruction, therefore, could give valuable information about the superconductive properties of N to experimentalists.

(ii) H_{CS} and also, in the case of type II material, the upper critical field of S, i.e. H_{C2S} could be modified by the proximity effect.

As was mentioned in the first chapter, the electrical contact between S and N must be very good, otherwise N will be partially oxidized at the S/N interface. For instance, Van Gurp (1963) observed no proximity effects in Sn/Al and In/Al (the critical temperature of S/N system was not changed in his experiments). This might be due to partial oxidation of Al.

The second point to be emphasized here is that the atoms of S and N must not migrate to each other. Rose-Innes and Serin (1961) have experimentally shown that a small concentration of N in S depresses T_{CS}. Although this will not be a very strong effect if N is a non-magnetic material, it will become really catastrophic if N is magnetic. The atomic migrations in superimposed films can influence the superconductive properties of the system. This seems to be probably the most serious effect for the S/N systems with non-magnetic normal metals. Therefore the atomic migrations must be minimized. Among all

S and N metals, the metals which are not miscible and do not form intermetallic compounds such as In/Zn and Pb/Al, are those which minimize the migration effects. Moreover, the insensitivity of "dirty" S and N to a small diffusion of one to another leads to use dirty S and N specimens. Nevertheless we will be concerned with "pure" unmiscible samples, assuming that these materials are not too sensitive to the interatomic diffusion.

Under the above circumstances, the GL parameter κ is constant (temperature and position-independent). On the other hand in the dirty limit, since the penetration depth λ depends on temperature and on position (Orsay Group On Superconductivity, 1966), κ varies with the aforementioned parameters such that κ_N tends to zero as $T \to T_{CN}$ and approaches infinity as $T \to T_{CS}$. For the intermediate T, the temperature-dependence of $\kappa_N$depends on the boundary conditions at the S/N interface. Orsay Group (1966) have argued that, if T_{CN} is not very low, then at $T > T_{CN}$, κ_N will be smaller than unity. This assertion has been confirmed for the InBi/Zn system where Zn has been taken as the normal metal with $T_{CN} = 0.9$ (Burger et al, 1965).

It has been discussed by the Orsay Group On Superconductivity (1966) that, for $\kappa_N > 1$, even a weak magnetic field applied (on the N side) parallel to the S/N interface, penetrates freely in N up to the S/N interface. It turns out that the field will not be essentially screened in N and hence N does not exhibit the Meissner effect if $\kappa_N > 1$. On the other hand, if $\kappa_N << 1$ (the penetration depth is much smaller than the coherence length) the field will be screened by N. The applied magnetic field H_a does not perturb the order parameter (associated with the "first" penetration mode (I)) until it reaches a certain value $H_a = H_b$ (the so called "breakdown field") when the mode (II) becomes energetically more favourable than the mode (I) (see chapter 1). At $H_a = H_b$ a first-order phase transition occurs. The study of this phase transition is the main purpose of this chapter. This will be qualitatively compared with the calculations done by Orsay Group (1967). Moreover the perfect Meissner effect is studied for the S/N sandwiches with $\kappa_S << 1$ and $\kappa_N << 1$.

4.II The solutions of GL equations in the presence of the field:

The pair of the GL equations for both S and N sides in one dimension are given by (A.11),

$$\kappa^{-2} f'' = f(\pm 1 + f^2 + a^2) \tag{4.1a}$$

$$a'' = f^2 a \tag{4.1b}$$

where $\begin{cases} T > T_C, & \text{positive sign} \\ T < T_C, & \text{negative sign} \end{cases}$ in (4.1a)

and the magnetic field is applied parallel to the surface of N.

The Gibbs' free energy of N in reduced units would be (Fetter & Walecka, 1971):

$$G = \int_{\text{specimen}} du(F - 2\,h\,h_a) \tag{4.2}$$

where h_a is the applied magnetic field in reduced units and

$$F = \kappa_N^{-2} f'^2 + f^2(+1 + \tfrac{1}{2} f^2 + a^2) + h^2 \quad , \quad h = \frac{da}{du}$$

is the Helmholtz' free energy density. Replacing the second term of F by $\kappa^{-2} f f''$ (from 4.1a), gives

$$G = \kappa_N^{-2} ff' \Big|_{\text{specimen}} + \int_{\text{specimen}} (-\tfrac{1}{2}f^4 + h^2 - 2h\, h_a)du \qquad (4.3)$$

The S is assumed to have infinite thickness and occupies the entire region $x < 0$, whereas N has finite thickness d_N and extends over the distance $0 \leq x \leq d_N$. Emphasis should be made here that the thickness of S, i.e. d_S plays a very important role in S/N binary layers. This means that the transition temperature of S/N system T_{CSN} strongly depends on d_S (Chapter 1). For instance, if S is infinitely thick, T_{CSN} will be very close to T_{CS} for any value of d_N. Werthamer (1963) has demonstrated that for a given d_N, T_{CSN} decreases with decreasing the thickness of the superconductive sample. Of course T_{CSN} depends on d_N as well. It is depressed by increasing d_N (Jacob and Ginsberg, 1968). We shall discuss the importance of thickness dependent of T_{CSN} in somewhat more detailed fashion at the end of this chapter. It will be argued that the appropriate interpretation of the phase transition of S/N sandwiches can be done by resorting to thickness dependent of the transition temperature of the sandwiches.

Now in reduced units the thickness of N would be $1 = \dfrac{d_N}{\lambda_N(T)}$ and (4.3) could be written in the following convenient form:

$$G_N + 1h_a^2 = \kappa_N^{-2} ff' \Big|_0 - \tfrac{1}{2}\int_0^1 f^4 du + h_a^2 \int_0^1 (1 - h/h_a)^2 du \qquad (4.4)$$

This equation formally holds for either penetration mode. The first term on the right hand side of (4.4) is computed at the S/N interface (its value is zero on the outer boundaries of either mode in N).

We used a Taylor expansion of $f_S(u)$ and $f_N(u)$ about the outer boundaries to get, from (4.1a):

$$f_S(u) = 1 \qquad (4.5a)$$

$$f_{NI}(u) = f_o = \text{constant} \tag{4.5b}$$

$$f_{NII}(u) = f_o(1 - u/l) \tag{4.5c}$$

where the terms of the order κ^2 in the expansions have been neglected, since $\kappa \ll 1$, the boundary conditions $f'(u) = 0$ at outer surfaces has been used for S and mode (I), whereas $f_{NII}(u)$ has been obtained for the mode (II) using the condition $f_N(u) = 0$ at $u = 1$ and $f_o = \frac{1}{\alpha}\frac{\Delta_{SB}}{\Delta_{ON}}$ ($\Delta_{ON} = \left|\frac{A_N}{B_N}\right|$ see, Appendix A) has been calculated by using the continuity of $\frac{\Delta}{NV}$ at the interface of S/N. The other boundary condition, i.e. the continuity of $\frac{D}{V}\frac{d\Delta}{dx}$ at the interface of S/N, need not be imposed here. It is an identity for mode (I), and can not be satisfied for mode (II). Moreover $\alpha = \frac{N_S V_S}{N_N V_N}$ is the ratio of the coupling constants and $\Delta_{SB} = \frac{|A_S|}{B_S}$ is the bulk value of Δ for superconductive matter. By using the expression Δ_{SB} and Δ_{ON} (Appendix A), f_o can be written as

$$f_o = \frac{1}{\alpha}\frac{T_{CS}}{T_{CN}}\left|\frac{1 - T/T_{CS}}{T/T_{CN} - 1}\right|^{\frac{1}{2}} \qquad (*). \tag{4.6}$$

The figure (4.1) indicates the two possible configurations for the order parameter Δ of a S/N sandwich (with κ_S and κ_N much less than unity).

Since at $H_a = H_b$ no effect is observed on S side (Krätzig and Schreiber, 1973) the following boundary conditions will be used to solve (4.1b):

$$h = h_a \quad \text{at} \quad u = 1 \tag{4.7}$$

$$h = 0 \quad \text{at} \quad u = 0 \tag{4.8}$$

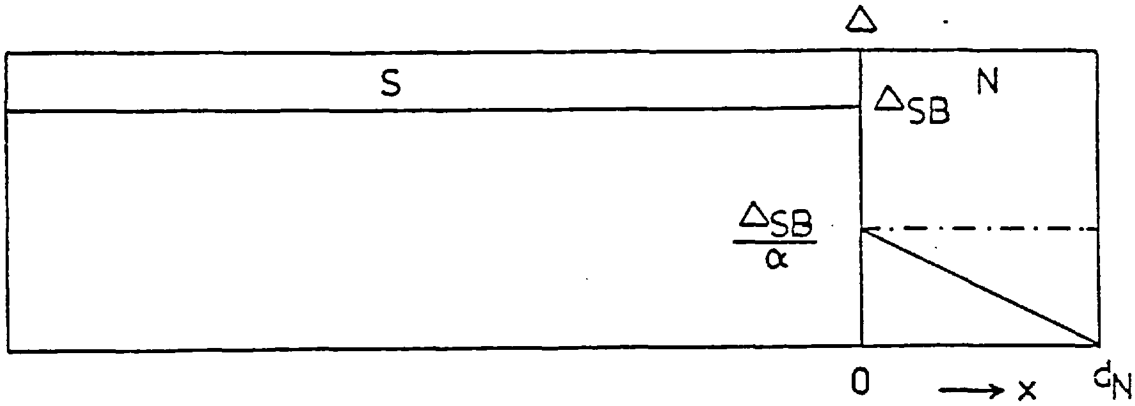

Figure 4.1: The two possible configurations for the order parameter Δ of a S/N sandwich. Chain curve: the traditional mode (I); full curve: the new mode (II).

4.III The magnetic field distribution in N:

By combining (4.1b) and (4.5b), one can derive the variation of the vector potential a and the magnetic field h in N for the mode (I);

$$a_I(u) = h_a \cosh f_o u / f_o \sinh f_o l \quad (4.9)$$

$$h_I(u) = h_a \sinh f_o u / \sinh f_o l \quad (4.10)$$

where the conditions (4.7) and (4.8) have been used. In order to solve (4.1b) for the mode (II) we make the following substitution

$$Z = Z_o(1 - u/l) \quad , \quad Z_o = \sqrt{2f_o l} \quad (4.11)$$

into (4.1b). With the aid of (4.5c), we obtain

$$\frac{d^2a}{dZ^2} - \frac{Z^2}{4}a = 0 \quad (4.12)$$

The solution of this differential equation is the Parabolic Cylinder function which can be written in terms of Confluent Hypergeometric (Kummer's) function F (Abramowitz and Stegun, (AS), 1972, p. 686);

$$a_{II}(Z) = e^{-\frac{1}{4}Z^2}\{C_1F(\tfrac{1}{4};\tfrac{1}{2};Z^2/2) + C_2ZF(\tfrac{3}{4};3/2;Z^2/2)\} \quad (4.13)$$

where C_1 and C_2 are constants to be determined by the boundary conditions given by (4.7) and (4.8). Since the field is $h = \frac{da}{du}$, then from (4.13), we obtain

$$h_{II}(Z) = h_a\, e^{-\frac{1}{4}Z^2} \left\{ \frac{C_1 Z}{2C_2} \left[F(5/4;3/2;Z^2/2) - F(\tfrac{1}{4};\tfrac{1}{2};Z^2/2) \right] + \left[\frac{Z^2}{2} F(7/4;5/2;Z^2/2) - \frac{Z^2}{2} F(\tfrac{3}{4};3/2;Z^2/2) + F(\tfrac{3}{4};\tfrac{3}{2};Z^2/2) \right] \right\} \quad (4.14)$$

where (4.7) has been imposed on h after differentiation and

$$C_2 = -1h_a/Z_o \quad , \quad Z_o = (2f_o 1)^{\frac{1}{2}} \quad (4.15)$$

Using (4.8), the constant C_1 is given by

$$C_1 = -\frac{1h_a\{ F(7/4;5/2;Z_o^2/2) - F(\tfrac{3}{4};3/2;Z_o^2/2) + \frac{2}{Z_o^2} F(\tfrac{3}{4};3/2;Z_o^2/2)\}}{F(5/4;3/2;Z_o^2/2) - F(\tfrac{1}{4};\tfrac{1}{2};Z_o^2/2)} \quad (4.16)$$

The expressions (4.13) and (4.14) in conjunction with (4.15) and (4.16) are general forms of the distribution of the vector potential and the field, for the second penetration mode (II) respectively.

The total Gibbs' free energy of the S/N sandwich is the sum of total Gibbs' free energies of S and N parts, i.e. $G_T = G_S + G_N$. As was pointed out in chapter one, for any applied field H_a less than a certain field H_b (what we called the breakdown field) $G_{TI} < G_{TII}$ and consequently, the mode (I) would be more stable than the mode (II). On the other hand, when H_a exceeds H_b, then $G_{TII} < G_{TI}$ and hence the second penetration mode (II) would be energetically more favourable than the first one (I). Clearly H_b is the field at which $G_{TI} = G_{TII}$, where a first order phase transition takes place. Therefore the problem remains to calculate G's. It is evident that the total Gibbs' free energy G_T must be minimum. The analytical calculations done by Orsay

Group (1967) and also the numerical computations carried out by Hook (1970) neglect G_S in G_T. The disagreement of the Orsay Group's results with the tunnelling experiments done by Martinet, 1966 is believed to be partly due to neglecting the total Gibbs' free energy of S side. Although Hook (1970) has mentioned that the effect of S can be included in some effective boundary conditions at the S/N interface, nevertheless there is no evidence how this modification can be done.

In the present work, since $f_s(u)$ is constant (equation 4.5a) and is the same for both modes, then $G_{SI} = G_{SII}$. Hence only G_N can safely be minimized. G_N given by (4.4) is easily evaluated for the mode (I) but G_{NII} can not be analytically exactly calculated due to complexity of $h_{II}(u)$. Had it not been so, we would have been able to find a first order phase transition in general for the S/N sandwiches, in the limiting case for which κ_S and κ_N are much smaller than unity. However, G_{NII} can be well approximated for the limiting cases.

4.III.a The distribution of $h_{II}(Z)$ in N for $1f_o << 1$:

The equation (4.14) can be expanded for the limiting case when $\frac{1}{2}Z^2 << 1$. Since $1f_o \propto \frac{d_N}{\lambda_{ON}}\sqrt{1 - T/T_{CS}}$ (λ_{ON} is the penetration depth at T = 0) and from equation (4.11), it can be seen that $\frac{1}{2}Z^2$ would be much smaller than unity throughout the normal sample for the following cases:

(a) $d_N > \lambda_{ON}$ and T close to T_{CS}.

(b) $d_N << \lambda_{ON}$ and T takes any value, $T_{CN} \leqslant T \leqslant T_{CS}$.

(c) If $d_N > \lambda_{ON}$ and for any T, the hypergeometric function F can be expanded in the vicinity of the free surface of N, i.e. $u \sim 1$. For any other values of d_N and T, the argument of F i.e. $\frac{1}{2}Z^2$ is greater than one and it will be more convenient to work with the asymptotic solution of (4.12). It will be shown that, if $\frac{1}{2}Z^2 \to \infty$ which requires taking semi-infinite N and working at the temperatures not too close to T_{CS}, h_{II} will vary linearly with u in the neighbourhood of the S/N interface with vanishingly small slope.

We use series expansion of F's (AS, p.504)

$$F(\alpha,\beta,x) = 1 + \frac{\alpha}{\beta}x + \frac{\alpha(\alpha + 1)}{\beta(\beta + 1)}\frac{x^2}{2!} + \ldots\ldots + \frac{(\alpha)_n}{(\beta)_n}\frac{x^n}{n!}$$

where $(\alpha)_n = \alpha(\alpha + 1)(\alpha + 2) \ldots\ldots (\alpha + n - 1)$, $(\alpha)_0 = 1$
in (4.14) up to Z^3 to get

$$h_{II}(Z) = h_a\left[1 - \left(\frac{Z}{Z_0}\right)^3\right] \tag{4.17}$$

where the boundary conditions (4.7) and (4.8) have been used, or from (4.11):

$$h_{II}(u) = h_a\left[1 - (1 - u/l)^3\right] \tag{4.18}$$

This is the distribution of the magnetic field for the second mode of penetration (II) in the limiting case when $\frac{1}{2}Z^2 = f_0 l(1 - u/l)^2 << 1$ and is in a good agreement with the numerical integrations of the GL equations done by Hook (1970). The slope of $h_{II}(u)$, from (4.18), at the S/N interface is $\frac{3h_a}{l}$. It turns out that, if the conditions (a) hold, this slope will be small and approaches zero as $l \to \infty$. The same argument can be made for $h_I(u)$. The equation (4.10) gives the slope of $h_I(u)$ at $u = 0$ which is $f_0 h_a/\sinh f_0 l$. Thus when $l \to \infty$, $h_I(u)$ tends to zero at the S/N interface much faster than $h_{II}(u)$. This means that the field h_I is screened by N further from the S/N interface. The complete screening of h_I by N will be investigated later on (perfect Meissner effect in S/N binary layers).

The variations of two magnetic fields corresponding to the two possible configurations of the order parameter Δ of a S/N sandwich with $\kappa_i \ll 1$ (i = S,N) is shown in figure (4.2).

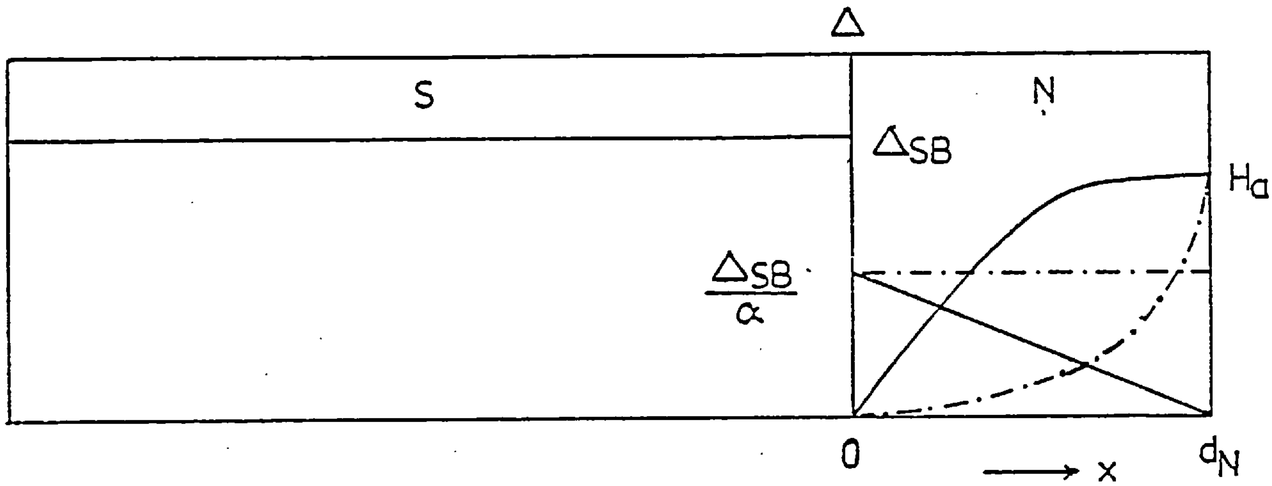

Figure 4.2: The two possible configurations for the order parameter Δ of a S/N sandwich with $\kappa_i \ll 1$ in a magnetic field. Chain curve: $H_a < H_b$, mode I; full curve: $H_a > H_b$; mode II, for ($lf_o \ll 1$).

4.III.b The distribution of h_{II} in N for $lf_o \gg 1$:

For the case when the expansion parameter $\frac{1}{2}Z^2 \gg 1$, the appropriate solution of (4.12) can be written in terms of Whittaker's function (AS, p. 687)

$$U(0,Z) = a_{II}(Z) = C\sqrt{\pi}2^{-\frac{1}{4}}e^{-\frac{1}{4}Z^2}\left\{\frac{F(\frac{1}{4};\frac{1}{2};Z^2)}{\Gamma(3/4)} - \frac{\sqrt{2}ZF(\frac{3}{4};3/2;\frac{1}{2}Z^2)}{\Gamma(\frac{1}{4})}\right\} \qquad (4.19)$$

where C is a constant to be determined by the boundary condition (4.7) and Z is given by (4.11).

The asymptotic expansion of (4.19) is (AS, p.689)

$$a_{II}(Z) \sim Z^{-\frac{1}{2}}e^{-\frac{1}{4}Z^2}\{1 + O(Z^{-2})\} \quad , \quad Z \to \infty \quad , \qquad (4.20)$$

which does not diverge to this order. This shows that at the S/N interface ($u = 0$, $Z = Z_0$) when $Z_0 = \surd(2lf_0) \to \infty$, the magnetic field vanishes and therefore the boundary condition (4.8) is satisfied.

Since, by definition $h = \frac{da}{du} = -\frac{Z_0}{l}\frac{da}{dZ}$, the magnetic field h can be evaluated from (4.19) and with the aid of the following recurrence relation (AS, p. 688);

$$\frac{d}{dZ}U(0,Z) = -\tfrac{1}{2}\{ZU(0,Z) + U(1,Z)\}$$

which with the help of (4.7), h_{II} becomes:

$$h_{II}(Z) = h_a e^{-\frac{1}{4}Z^2}\{F(\tfrac{3}{4};\tfrac{1}{2};\tfrac{1}{2}Z^2) - \surd 2\,\frac{\Gamma(5/4)}{\Gamma(\frac{3}{4})}\,Z\Big[F(5/4;3/2;\tfrac{1}{2}Z^2) - F(\tfrac{1}{4};\tfrac{1}{2};\tfrac{1}{2}Z^2)\Big] - \frac{Z^2}{2}F(\tfrac{3}{4};3/2;\tfrac{1}{2}Z^2)\} \qquad (4.21)$$

This is the field distribution for the case when $\frac{1}{2}Z^2 >> 1$. It can be seen that, at a large distance from the S/N interface, h has a linear behaviour in u and when $lf_0 \to \infty$, its slope will be vanishingly small. This implies that h, for the computational simplicity can be taken zero at the point $u = u_0$ (which will be determined later on). The assumption that $h_{II}(u)$ is taken zero, can be made safely, if N is not too thick. It might have the rather important consequences in the stability for semi-infinite N. It will be shown, however, that h_b is too small when N is infinitely thick (Fig. 4.5).At the other side of the normal sample i.e. close to $u = 1$, the condition $\frac{1}{2}Z^2 >> 1$ is no longer valid, but $\frac{1}{2}Z^2 << 1$. Thus the variation of h_{II} for the case when $lf_0 >> 1$ is reminiscent of (4.17) in the region $Z' \leqslant Z \leqslant 0$ where $Z' = Z_0(1 - u_0/l)$, and hence

$$h_{II}(Z) = h_a\{1 - (\frac{Z}{Z'})^3\} \quad , \qquad Z' \leqslant Z \leqslant 0 \qquad (4.22)$$

or

$$h_{II}(u) = h_a\{1 - (\frac{1 - u/1}{1 - u_o/1})^3\} , \quad u_o \leq u \leq 1 \qquad (4.23)$$

and from (4.20),

$$h_{II}(u) \sim e^{-Z_o^2/4} Z_o^{5/2} u/1 \quad , \quad 0 \leq u \leq u_o \qquad (4.24)$$

The distribution of the field in the case for which $1f_o >> 1$ is indicated in figure (4.3).

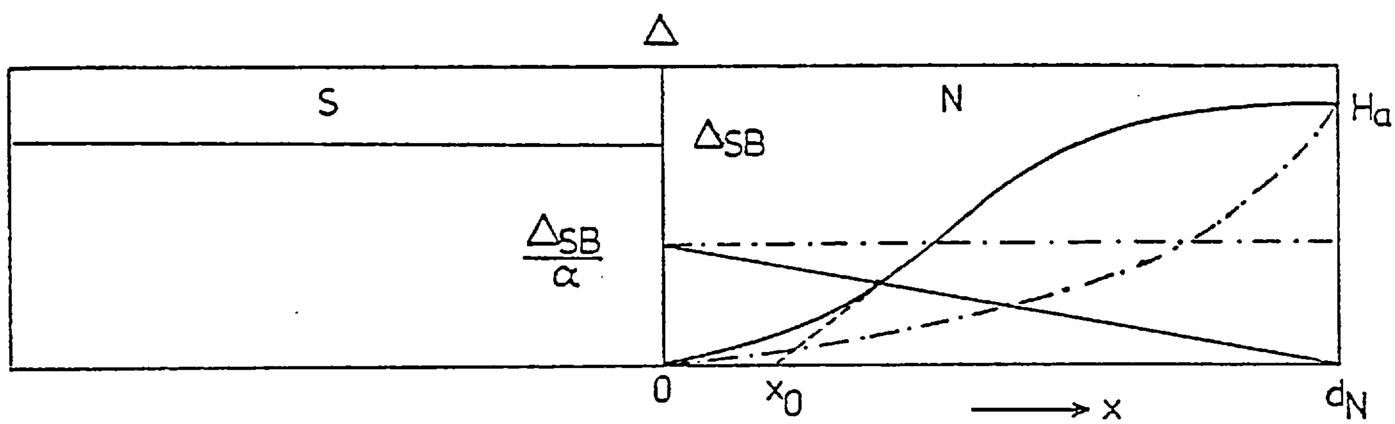

Figure 4.3: The same as figure (4.2) but for $1f_o >> 1$. Note $u_o = x_o/\lambda_N(T)$ by a priori definition.

In order to compute $Z' = Z_o(1 - u_o/1)$ we adjust the curvature of the approximate solution (4.23) and the exact solution (4.14) of h(u) at u = 1. Equation (4.23) yields,

$$h_{II}'''(u) = 6h_a/(1 - u_o)^3 = \text{constant} \quad , \quad \forall, u \qquad (4.25)$$

and from (4.12) one has

$$h_{II}'''(u) = \tfrac{1}{2}(\frac{Z_o}{T})^4 a_{II}(u)$$

At u = 1, (Z = 0), this reads

$$h_{II}'''(u)|_{u=1} = \sqrt{2}\,\frac{\Gamma(5/4)}{\Gamma(\frac{3}{4})}\,(\frac{Z_o}{T})^3 h_a \qquad (4.26)$$

where equations (4.19) and (4.21) determined the constant C appearing in $a_{II}(u)$. Equating (4.25) and (4.26) gives Z'

$$Z' = Z_0(1 - u_0/l) = \left[3\sqrt{2}\ \Gamma(\tfrac{3}{4})/\Gamma(5/4)\right]^{1/3} \simeq 1.80 \tag{4.27}$$

4.IV Calculation of Gibbs' free energies for $lf_0 \ll 1$:

The Gibbs' free energy associated with the first mode of penetration (I) is obtained by (4.4), using the expression (4.10) for the field, which is:

$$G_{NI} + lh_a^2 = -\tfrac{1}{2} l f_0^4 + lh_a^2\left[\frac{3}{2} - \frac{3}{2}\,\frac{\coth lf_0}{lf_0} - \frac{\coth^2 lf_0}{2} + \frac{2}{lf_0 \sinh lf_0}\right] \tag{4.28}$$

$$\forall, lf_0$$

This is valid for all normal specimen thicknesses and temperatures. For $lf_0 < 1$, (4.28) is converted to

$$G_{NI} + lh_a^2 = -\tfrac{1}{2} l f_0^4 + lh_a^2\{\tfrac{1}{3} + \tfrac{7}{180}(lf_0)^2\}\ ,\ lf_0 < 1 \tag{4.29}$$

The Gibbs' free energy associated with the second penetration mode (II) is calculated by substituting (4.5c) and (4.18) into (4.4);

$$GN_{II} + lh_a^2 = \kappa_N^{-2} l^{-1} f_0^2 - \tfrac{1}{10} l f_0^4 + \tfrac{1}{7} l h_a^2\ ,\quad lf_0 < 1 \tag{4.30}$$

In order to compare G_{NI} and G_{NII}, it is more convenient to introduce the quantities G_I' and G_{II}'.

$$G_I' = -(\kappa_N^{-2} l^{-1} f_0^2 + \tfrac{2}{5} l f_0^4) \tag{4.31}$$

$$, lf_0 < 1$$

$$G_{II}' = -lh_a^2\{\tfrac{4}{21} + \frac{7(lf_0)^2}{180}\} \tag{4.32}$$

such that $G_{NI} - G_{NII} = G_I' - G_{II}'$.

So G'_I and G'_{II} are both negative. The former is independent of the applied field, whereas the latter is a quadratic function of h_a. The field dependence of the G' 's is shown in figure (4.4). The breakdown field h_b is the field at which $G'_I = G'_{II}$. It is seen, from figure (4.4), that for $h_a < h_b$ the first mode (I) with lower energy than the mode (II) is more stable than the second mode. For $h_a > h_b$, on the other hand, $G_{NII} < G_{NI}$ and hence the mode (II) with lower energy will be more favourable than (I). Moreover, it is inferred from figure (4.4) that the breakdown field h_b decreases as the thickness of the normal matter l increases.

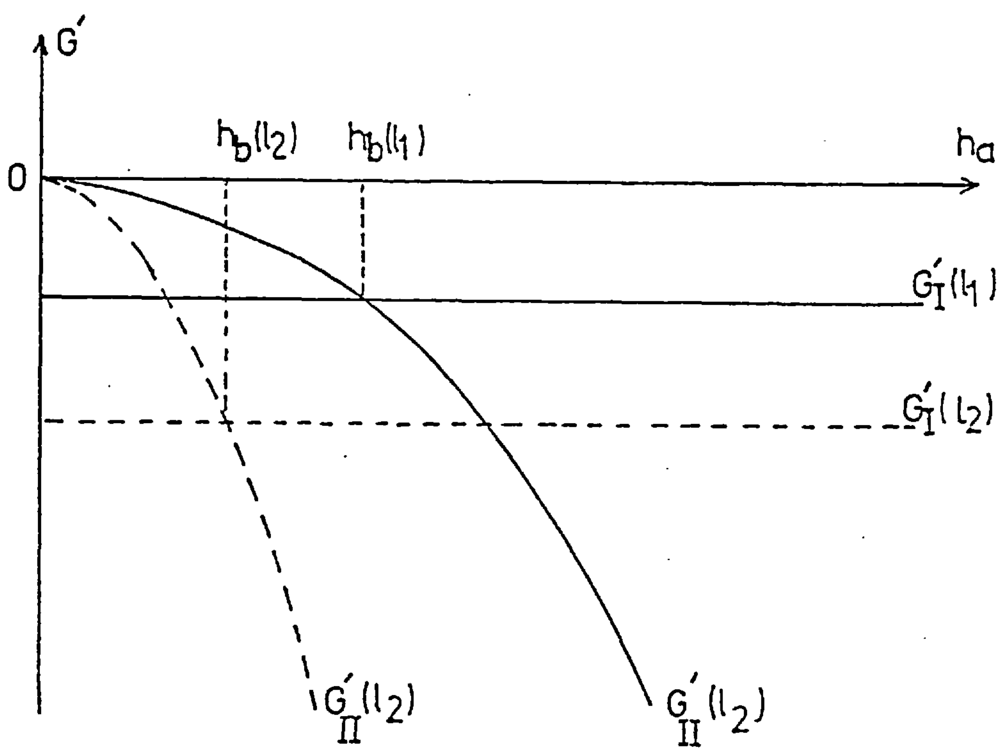

Figure 4.4: The variation of the G' 's with respect to h_a for two different thicknesses, $l_2 > l_1$.

4.V The calculation of the breakdown field for $lf_o << 1$:

The breakdown field h_b is the field at which $G'_I = G'_{II}$. From

(4.31) and (4.32) and for $lf_o \ll 1$, we obtain, in dimensionless units:

$$h_b = \sqrt{5.25}\,\frac{f_o}{l\kappa_N}\{1 - \frac{49}{480}(lf_o)^2\} \quad , \quad lf_o \ll 1 \qquad (4.33)$$

This implies that for a very thin N, the breakdown field, at low temperature is high and varies linearly with the inverse of the thickness of the normal specimen. At temperatures close to T_{CS}, on the other hand, h_b is very low and vanishes at $T = T_{CS}$, where $f_o = 0$ (equation 4.6). At $T = T_{CS}$, the superconductivity in S is completely destroyed, and hence the S/N sandwich turns suddenly normal (a transition from superconducting state to normal state occurs). Consequently, in order to observe the breakdown field effect, the applied field must not be higher than H_{CS} (the thermodynamic critical field of the bulk S). (See Fig. 4.6). We note that at $T = T_{CS}$, H_{CS} vanishes (see Appendix A).

Since $h = \frac{2\pi\lambda\xi}{\phi_o} H$ (H is the magnetic field in conventional units), the equation (4.33) takes the following form:

$$H_b = \frac{C_1}{d_N}(1 - T/T_{CS})^{\frac{1}{2}}\{1 - C_2 d_N^2(1 - T/T_{CS})\} \qquad (*) \qquad (4.34)$$

where the expression (4.6) and the following quantities have been used:

$$\kappa_N = \frac{\lambda_N(T)}{\xi_N(T)},\ l = \frac{d_N}{\lambda_N(T)},\ \xi_N(T) = \xi_{ON}(\frac{T}{T_{CN}} - 1)^{-\frac{1}{2}},\ \lambda_N(T) = \lambda_{ON}(\frac{T}{T_{CN}} - 1)^{-\frac{1}{2}}$$

$$\text{and } \lambda_{ON} = \frac{\lambda_L(0)}{\sqrt{2}} \qquad (*) \qquad (4.35)$$

(where $\lambda_L(0) = (mc^2/4\pi ne^2)^{\frac{1}{2}}$ is the London penetration depth in the pure metal at zero temperature (De Gennes, 1966, p.223).and n is the total number of conduction electrons per cubic centimeter) and

$$C_1 = \sqrt{10.5}\,\frac{1}{\alpha}\,\frac{T_{CS}}{T_{CN}}\,H_{CN}(0)\xi_{ON}\ ,\ \xi_{ON} = \frac{0.13\hbar V_{FN}}{k_B T_{CN}} \qquad (*) \qquad (4.36)$$

$H_{CN}(0)$ is the thermadynamic critical field of N at T = 0 (see Appendix A)

$$C_2 = \frac{49}{240}\left[\frac{1}{\alpha}\frac{T_{CS}}{T_{CN}}\frac{1}{\lambda_L(0)}\right]^2 \tag{4.37}$$

By neglecting the term $O(1f_o)^2$ in (4.33), the equation (4.34) becomes

$$H_b = \frac{C_1}{d_N}(1 - T/T_{CS})^{\frac{1}{2}} \tag{4.38}$$

This expression shows clearly the variation of the breakdown field with respect to the normal sample thickness d_N and the temperature T. Figure (4.7) indicates the variation of H_b versus T. In order to interpret the diagram (4.7) which has been plotted for a given thickness, we take a temperature below T_{CS} and we increase the field. For any field less than H_b, no phase transition takes place and the mode (I), with lower Gibbs' free energy, is more stable than the mode (II). When the field reaches H_b (on diagram 4.7), the order parameter at the free surface of N vanishes and the transition from mode (I) to mode (II) occurs, i.e. $G_{NI} = G_{NII}$. Finally, in the region for which the field is greater than H_b, the mode (II) is energetically favoured. It will be shown that the domain of existence of H_b is restricted to the certain values of d_N and T. This means that not for all d_N and T can the breakdown field be observed.

H_b must not exceed the thermodynamic critical field of the superconductor H_{CS}. Fischer et al (1965) have experimentally reported that at $H_a = H_{CS}$, the bulk of the superconductor suddenly becomes normal, and therefore the whole S/N system turns suddenly normal. This is indeed a second-order phase transition which is not related to the breakdown field effect. It can be inferred from figure (4.6) that, for a given d_N and at rather high temperature $G_{NI} < G_{NII}$, for $H_a < H_{CS}$ and

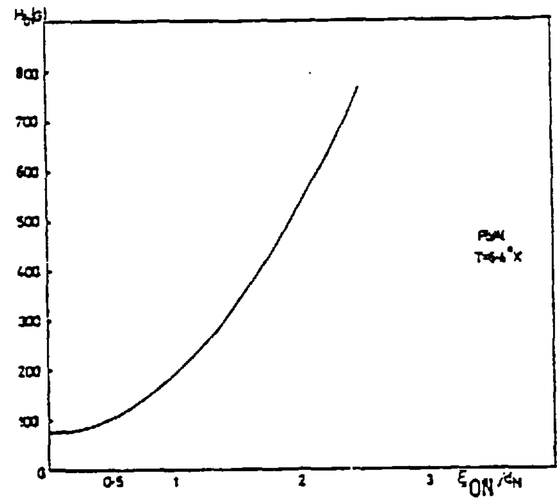

Fig. 4.5: Variation of the breakdown field with the inverse of the normal sample thickness (solution of the equation 4.45) in Pb/Al sandwich. (Note Pb has infinite thickness.

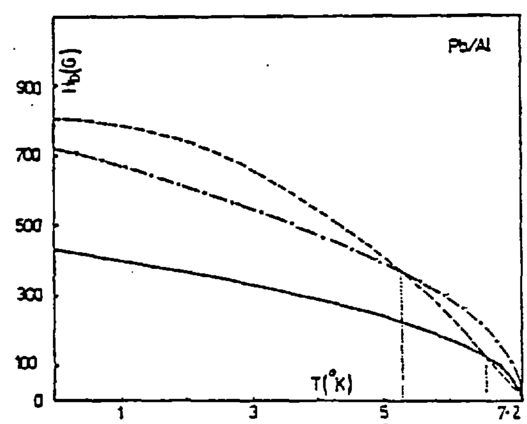

Fig. 4.6: The breakdown field versus temperature T for two different thickness of N·—·—·— for $d_N = 1.2\xi_{ON}$, ——— for $d_N = 2\xi_{ON}$ and ------ the bulk critical field of Pb.

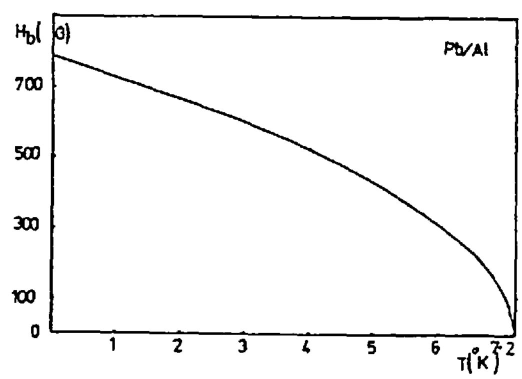

Fig. 4.7: The breakdown field H_b versus temperature T in Pb/Al binary layer for $d_N = 1.1\xi_{ON}$ (note Pb is infinitely thick.

the magnetic field penetrates gradually into N as if it was an ordinary type I superconductor until the magnetic field reaches H_{CS} at which a second-order phase transition occurs (Krätzig and Schreiber, 1973). This process remains the same till the temperature is lowered to a certain value $T = T^*$ at which $H_b = H_{CS}$. At lower temperatures the magnetic field penetrates abruptly into N at the breakdown field $H_b < H_{CS}$, i.e. a first-order phase transition takes place (Krätzig and Schreiber, 1973).

4.VI The calculation of Gibbs' free energies for $lf_o >> 1$:

The Gibbs' free energy associated with the mode (I), for all lf_o is given by (4.28). In the limiting case where $lf_o >> 1$, it becomes

$$G_{NI} + lh_a^2 = -\tfrac{1}{2}lf_o^4 + lh_a^2(1 - 3/2lf_o) \quad , \quad lf_o >> 1 \tag{4.39}$$

where $\exp(-lf_o)$ has been neglected. For this limit, G_{NII} (the Gibbs' free energy of the mode (II)) is calculated by (4.4). Using (4.5c), (4.11), (4.15), (4.23) and (4.27), G_{NII} becomes

$$G_{NII} + lh_a^2 = \kappa_N^{-2}l^{-1}f_o^2 - \frac{1}{10}lf_o^4 + \frac{1.8}{7}/\surd(2lf_o) \quad , \quad lf_o >> 1 \tag{4.40}$$

As in section IV, we introduce the following quantities:

$$G_I' = -(\kappa_N^{-2}l^{-1}f_o^2 + \frac{2}{5}lf_o^4) \tag{4.41}$$

$$G_{II}' = -lh_a^2(1 - \frac{3}{2lf_o} - \frac{1.8}{7\sqrt{2lf_o}}) \tag{4.42}$$

$$, \quad lf_o >> 1$$

such that $G_{NI} - G_{NII} = G_I' - G_{II}'$. Both G' and G_{II}' are negative. The former is independent of h_a, whereas the latter varies quadratically with the field h_a. G_I' and G_{II}' have been schematically plotted against h_a for two different thicknesses, i.e. $l_2 > l_1$ (Fig 4.4) and $lf_o \to \infty$. It is seen that the breakdown field h_b (the field at which $G_I' = G_{II}'$ and a first-order phase transition occurs) decreases as the thickness of the nornal sample l increases.

4.VII The breakdown field for $lf_o >> 1$:

By equating the equations (4.41) and (4.42), we obtain the following dimensionless expression for the breakdown field h_b, for the limit $lf_o >> 1$:

$$h_b = \sqrt{2/5}f_o^2(1 + \tfrac{5}{4}\kappa_N^{-2}l^{-2}f_o^{-2})(1 + \frac{3}{4f_o l} + .9/7\sqrt{2}lf_o) + O(l^{-3}f_o^{-3}) \quad (4.43)$$

Since $\kappa_N << 1$, the above equation can be approximately written as

$$h_b = \sqrt{(2/5)}f_o^2(1 + \tfrac{5}{4}\kappa_N^{-2}l^{-2}f_o^{-2}) \quad , \quad lf_o >> 1 \quad (4.44)$$

The expression $h = \frac{2\pi\lambda\xi}{\phi_o}H$ (Appendix A) and the equations (4.6) and (4.35) transform the above equation to its physical form:

$$H_b/H_{CN}(0) = \frac{2}{\sqrt{5}}\left[(\frac{1}{\alpha}\frac{T_{CS}}{T_{CN}})^2(1 - \frac{T}{T_{CS}}) + \frac{5}{4}(\xi_{ON}/d_N)^2\right] \quad (4.45)$$

$H_{CN}(0)$ is the thermodynamic critical field of N at absolute zero which is given by $H_{CN}(0) = \frac{\phi_o}{2\pi\sqrt{2}\lambda_{ON}\xi_{ON}}$ (Appendix A), where λ_{ON} and ξ_{ON} are the penetration depth and the coherence length of N (at T = 0) respectively which, in the microscopic GL theory, are given by (4.35) and (4.36). The equation (4.45) explicitly shows that the breakdown field H_b decreases as the thickness of the normal specimen increases and tends to a constant value independent of d_N (the first term in the expression 4.45) with zero slope as d_N/ξ_{ON} approaches infinity (Fig. 4.5).

The full phase diagram, in three dimensional space of H, T and d_N has been plotted in figure (4.8) for the simplest case which we have been studying so far (S has infinite thickness and $\kappa_i << 1$). In this figure, the broken curve represents the thermodynamic critical field of the bulk superconductor given by the well known expression $H_{CS}(T) = H_{CS}(0)\left[1 - (T/T_{CS})^2\right]$ and the full curve is H_b. Figure (4.8) enables us to discuss the ingredients in a somewhat more detailed manner:

(1) At a constant temperature T where H_b is a function of d_N only:
As the thickness of the normal sample d_N is increased, the breakdown field H_b decreases (Fig. 4.5). This is qualitatively in a good agreement with the experiments done by Tai and Park,(1978) measuring the change of the ultrasonic attenuation in N part of the S/N sandwiches (Pb/Cu and Pb/Ag). We note that, if N is very thin, no breakdown field effect will be observed, since $H_b > H_{CS}$ in this case.

(2) For a fixed d_N where H_b varies only with T:
By increasing T, H_b decreases and vanishes at $T = T_{CS}$ (Fig. 4.7).

(3) H = 0:
It can be seen from Chapter 2 that, in the limiting case, where $\kappa_i << 1$, the total free energy associated with the mode (I) is always less than the one associated with mode (II). Therefore there is no phase transition in this case and (I) is always more stable than (II) for any T and d_N.

4.VIII The maximum T below which a first-order phase transition occurs:

(a) As was mentioned in the section (4.VII), the breakdown field effect can be observed for certain values of d_N and T. Therefore, for a given d_N there is a maximum temperature T* below which the transition from the mode (I) to the mode (II) takes place. Clearly T* is a function of d_N. In order to visualize this characteristic temperature $T^*(d_N)$ we take a constant d_N (in Fig. 4.8). For a given temperature, the breakdown field is less than H_{CS} and hence the transition from (I) to (II) is expected. By increasing T, this process is maintained up to a certain temperature for which $H_b = H_{CS}$. We denote this temperature by T*. At T > T*, the field $H_b > H_{CS}$ and no phase transition occurs. This means that, in this region, the mode (II) with lower energy is favoured. Finally at $T = T_{CS}$, the superconducting properties of the S/N system are completely destroyed leading to the occurrence of a second-order transition in the whole S/N binary layer, from the superconductive phase to the normal phase. The intersection of H_b and H_{CS}, in figure (4.8), is marked by a cross (x) and the projection of this point on the (T,d_N) plane is marked by empty circle (O). It is of great importance to notice that, for a given d_N, as T is lowered below T*, the transition from (I) to (II) becomes sharper, as the applied field parallel to the surface of N is increased. The measurements of the initial amplitude of the tunnelling characteristic done by Martinet (1966) on the InBi/Zn system (Fig. 4.9) reveals clearly the sharpness of the transition at T and below T* (for a given d_N). Moreover, from figure (4.9), it follows that there is no transition from (I) to (II) for T > T* in the presence of the magnetic field.

(b) For a given d_N and at constant T, the low applied magnetic field penetrates gradually into N up to the S/N interface. This means that for the low fields, the mode (I) is more stable than the mode (II). We now increase the field. At $H_a = H_b$, the field destroys the superconductivity at the free surface of N and hence (II) will be energetically favoured at and above H_b (Fig. 4.8).

(c) Now let us fix T and increase d_N. For small d_N, there is no a phase transition, since H_b is much higher than H_{CS} (Fig. 4.8). By increasing d_N, H_b decreases and for a certain value of d_N for which the corresponding T* is greater than or equal to the fixed T, a first-order phase transition from (I) to (II) occurs. This transition takes place by further increase in d_N. As is indicated in Fig. (4.8), as d_N is increased, H_b decreases and T*, which is the projection of the point at which $H_b = H_{CS}$ on the (T, d_N) plane, approaches T_{CS} such that $T^* = T_{CS}$ as the thickness of N tends to infinity. Joining the empty circles in Fig. (4.8) which have been obtained by projecting the intersections of H_{CS} and H_b for different d_N's, on the (T, d_N) plane, produces Fig. (4.10). This figure determines the region at which the mode (I), by increasing the field, changes to the mode (II) (under the curve in Fig. (4.10)). The solid curve indicates the variation of T* (the maximum temperature above which no first-order phase transition in the presence of the field is observed) with d_N. In the region above the curve, the mode (II) will be more stable than (I) for any T less than T_{CS}. At and above T_{CS} the superconducting properties of the S/N system are completely quenched.

The most important consequence of the breakdown field effect is attributed to Fig. (4.10) as it shows the domain of the existence of a first-order phase change due to the arbitrariness of the boundary conditions at the N-vacuum interface. This figure which has been plotted for the parameters relevant to Pb/Al double layer obeys the following explicit relation between d_N and T*

$$d_N = (1 - t^*)^{-\frac{1}{2}}\left[-W + (W^2 + 1/C_2)^{\frac{1}{2}}\right] \qquad (4.45^*)$$

where $t^* = T^*/T_{CS}$, $W = \frac{H_{CS}(0)}{2C_1C_2}(1 + t^*)$, C_1 and C_2 are given by (4.36) and (4.37) respectively. $H_{CS}(0)$ is the bulk critical field of S at absolute zero. It has to be mentioned that the above phase diagram is

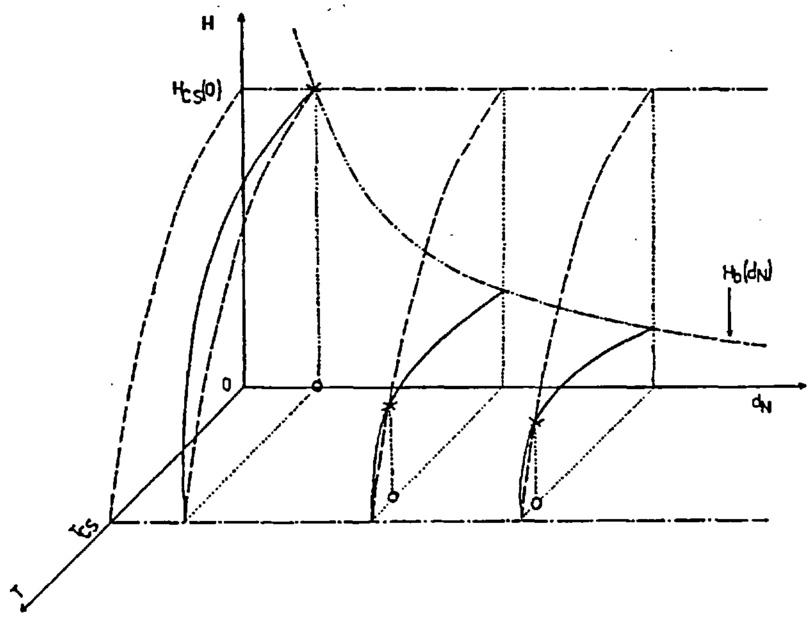

Fig. 4.8: The typical three dimensional phase diagram (H,T,d_N) of the breakdown field effect in a S/N system where S has infinite thickness. ------ the bulk critical field of S, ——— the breakdown field and O shows the maximum T below which a first-order phase transition occurs.

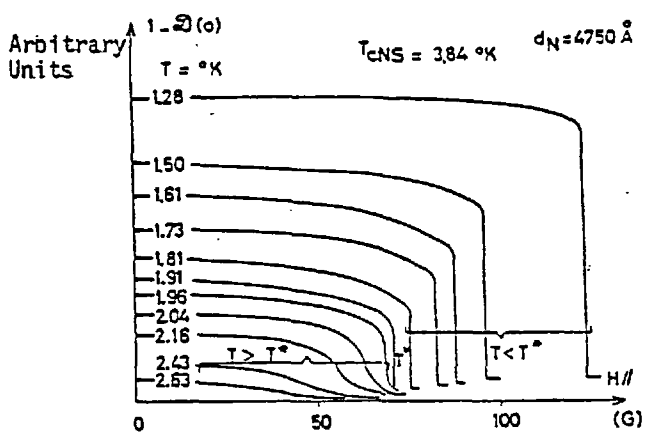

Fig. 4.9: The initial slope of the tunnelling characteristic measurements on normal side of InBi/Zn as a function of the applied field parallel to the metallic interface (Martinet, 1966). Note that for T > T* no breakdown field effect is observed.

for the case when S has infinite thickness and is derived by equation (4.34) and H_{CS} given by $H_{CS}(T) = H_{CS}(0)\left[1 - (T/T_{CS})^2\right]$.

(d) The values of H_b calculated from (4.38) for Pb/Al, for different T and d_N, are higher than the experimental results of Krätzig and Schreiber (1973). As an example, at $T = 7.1^0$K and for d_N = 3000 Å, we obtain $H_b \simeq 250$G for Pb/Al, whereas, for the same d_N and H_b, they have measured $T = 1.9^0$K. Therefore the present work predicts T* higher than the corresponding experimental observation. We recall that our calculation of H_b is valid only when S has infinite thickness and $\kappa_i << 1$ (in the experiment of Krätzig and Schreiber, 1973, d_S = 2500 Å and κ_N varies with temperature.) Therefore the comparison between our result with their measurements can hardly be made, since in their experiment S is finite and κ_N varies with T. Taking infinite S, however, causes two problems:
(1) T* is too high; (2) T* does not go down for large d_N. Martinet (1966) has measured $T^*(d_N)$ for InBi/Zn superimposed films, by using the tunnelling effect (Fig. 4.11). According to his result T* increases first by increasing d_N and then decreases as d_N is sufficiently increased. In the next section, it will be shown how T*, in our present work, can be modified by taking the thickness dependence of the critical temperature of the S/N double layer T_{CSN} into account.

4.IX Generalization to finite thickness of S:

In the earlier section we studied the breakdown field effect in the S/N superimposed films where S had infinite thickness. The most important result was Fig. (4.10) which could clearly determine the domain of the existence of a first-order phase transition. In this section we generalize the previous investigations to the case when S has finite thickness d_S.

T_{CSN}, for a given d_N, is depressed by decreasing d_S (Hauser et al, 1964) and since the energy gap is proportional to the critical temperature, it is significantly smaller than the BCS gap (Guyon et al, 1966). Thus the thickness dependence of the critical temperature must be taken into account. Werthamer (1963) has used the Gor'kov self-consistent equation to derive T_{CSN} (in zero field). He has arrived at the following formulae which we shall use:

$$T_{CSN} = T_{CS}\left[1 + \frac{\pi^2}{4}\frac{d_N}{d_S}\right]^{-1} \tag{4.46}$$

Therefore the critical temperature of the S/N sandwich T_{CSN} decreases as $\frac{d_N}{d_S}$ is increased. This means that the S/N system becomes normal at T_{CSN} lower than T_{CS} for finite d_S and d_N. Since the critical temperature appears in the critical field of S which is given by (see, for example Lynton, 1971)

$$H_{CS}(T) = H_{CS}(0)\left[1 - (T/T_{CS})^2\right] \text{ , for the bulk superconductor} \tag{4.47}$$

where $H_{CS}(0)$ is related to the energy gap of S, i.e. $2\Delta_{0S}$ at T = 0 by (Fetter and Walecka, 1971):

$$H_{CS}(0) = \left[4\pi N_S(0)\ \Delta_{0S}^2\right]^{\frac{1}{2}} \tag{4.48}$$

and $N_S(0)$ is the density of states of S at Fermi level, the thickness dependence of T_{CSN} affect the critical field. In other words, H_{CS}, for finite d_S, is lower than the critical field of the bulk S. Therefore in order to generalize the problem which we studied in the preceding section, we must scale H_{CS} and H_b due to finite thickness of S. There are two ways of scaling:

(i) Scaling the whole phase diagram (4.8) by the law of corresponding states (BCS, 1957). This law states that all superconductors have identical properties when these properties are expressed in reduced units, for example, the critical curve $H_C(T)/H_C(0)$ is a universal function of the reduced temperature T/T_C (see, (4.49)) such that it is maximum at $T = 0$ and vanishes at $T = T_C$. Moreover, according to that law $2\Delta_{0S}/k_BT_{CS} = 3.5$. We substitute this expression into (4.48) (we note that (4.48) is derived by the law of the corresponding states, BCS, 1957) to obtain $H_{CS}(0) = 3.5\left[\pi N_S(0)\right]^{\frac{1}{2}}k_BT_{CS}$. Setting $T_{CS} = T_{CNS}$ given by (4.46) only for the case when $d_N/d_S >> 1$, we arrive at the following equation for the critical field of the S/N system:

$$H_{CSN}(T)/H_{CSN}(0) = \left[1 - (T/T_{CSN})^2\right] \quad \text{for } d_N/d_S >> 1 \qquad (4.49)$$

where $H_{CSN}(0) = 3.5\left[\pi N_S(0)\right]^{\frac{1}{2}}k_BT_{CSN}$.

This enables us to scale the thermodynamic critical field by using (4.49). It is evident that $H_{CSN}(0)$ decreases as d_N/d_S is increased.

(ii) Scaling the T-axis only. This is applied for scaling H_b, since H_b is a matter of the boundary conditions at normal-vacuum interface and it should not be much affected by d_S.

With the above scaling for the thermodynamic critical field and H_b, the phase diagram (4.8) will be generalized for a given d_S (Fig. 4.12). It is seen, from this diagram that, for a fixed T and by increasing the d_N, T* which is again the projection of the cross over point of H_b and H_{CSN} on the (T,d_N) plane, first increases up to a certain value of d_N and then decreases as d_N is increased further. The turning point gives the maximum value of T* for a given d_S. If we increase d_N further, T* will decrease (Fig. 4.12). As was pointed out in the last section, the breakdown field effect can be observed for the certain values of T and d_N. The domain of existence of this effect is determined by equating H_b and the thermodynamic critical field of S which gives a relation between T ≡ T* and d_N. But the breakdown field, for finite d_S, must be calculated exactly and is no longer given simply by (4.34) and (4.45) which were derived for infinite S. The calculation of H_b for finite d_S requires the complete solutions of the GL equations for both S and N. This is a formidable mathematical task (as well as the evaluations of the Gibbs' free energies). So the quantitative comparison with the experiments can not be made. Nonetheless, the qualitative comparison of the present work with the experiments can easily be made by resorting to Fig. (4.12). In other words, from this figure, it is seen that, T* is not too high (for finite d_S) and has a maximum for a certain d_N which is qualitatively in agreement with the experiments of Martinet (1966) (tunnelling and permeability measurements, see Fig. 4.11).

In order to understand the difference between calculated/measured T*, it seems useful to point out the following remarks which are concerned with both experiments and theory:

(1) Our calculations are based on the assumption that the transmission coefficient is unity. The existence of even a very thin oxide layer at the S/N interface does actually reduce the transmission coefficient leading to lowering T* experimentally.

(2) T* can be very sensitive to any trapped flux. In particular, T* is lowered due to the presence of trapped flux (Deutscher et al, 1969).

(3) The occurrence of some diffusion at the S/N interface is expected (Orsay Group, 1967). This leads to a local D_N (diffusion coefficient of N) somewhat higher than the one which is directly measured.

(4) The expansion of Δ in the power series (equation 4.5) of Δ_{SB} is valid only at high temperatures. In fact, we are dealing with rather low temperatures. So the saturation effect can take place and can reduce T*.

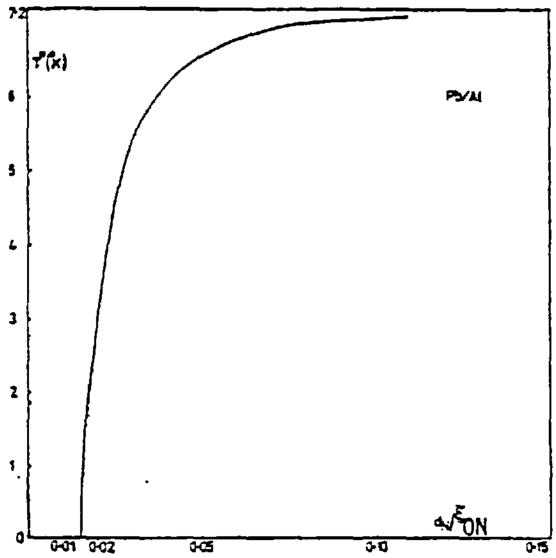

Fig. 4.10: The maximum temperature T* below which a first-order phase transition takes place in Pb/Al binary layer (Pb has infinite thickness) due to breakdown field effect versus the reduced thickness of Al i.e. d_N/ξ_{ON} (from equation 4.45*).

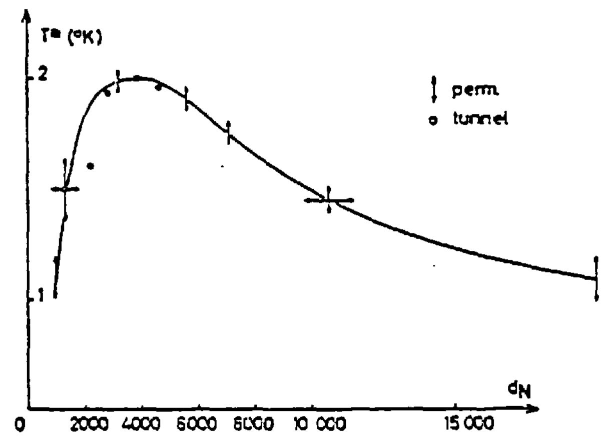

Fig. 4.11: The same as Fig. 4.10, but for InBi/Zn superimposed films where InBi is 2μ thick (the tunnelling and permeability measurements of Martinet, 1966).

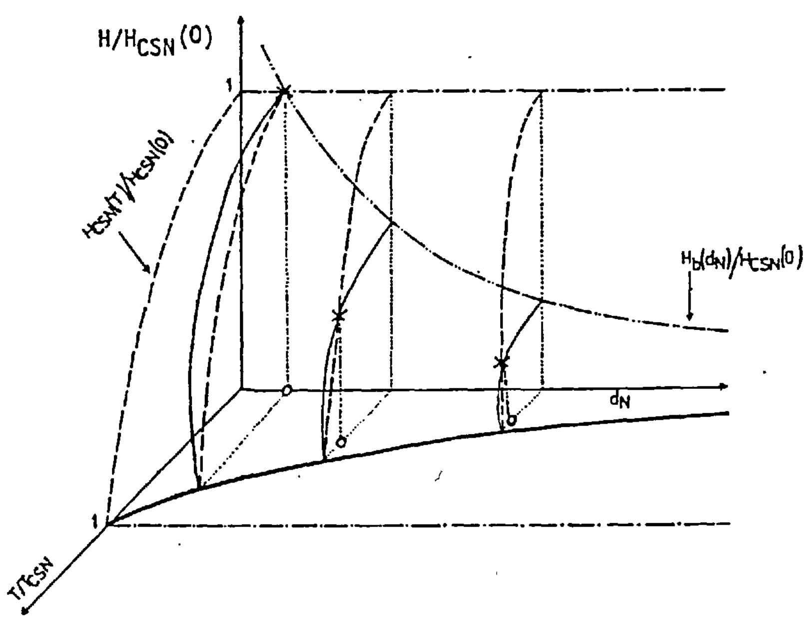

Fig. 4.12: The typical three dimensional phase diagram (H,T,d_N) of the breakdown field effect in a S/N sandwich where S has finite given thickness. ----- the scaled critical field of S, ——— the scaled breakdown field, 0 shows the maximum temperature T* below which a first-order phase transition can be observed and the heavy curve represents the variation of the critical temperature of the S/N system T_{CSN} (for a given d_S) with respect to d_N (from equation 4.46).

(5) Finally, in the limiting case $\kappa_i << 1$, we matched G_{NI} and G_{NII}, as G_{SI} and G_{SII} (the Gibbs' free energies of S side of the system for (I) and (II) respectively) were equal. For larger κ_S, $G_{SI} \neq G_{SII}$ and hence the total Gibbs' free energy $G_T = G_N + G_S$ must be minimized. The contribution of G_S to G_T certainly modifies T*.

4.X Perfect Meissner effect in N:

The case $\kappa_N << 1$ is of particular interest as it is associated with the study of the Meissner effect in the S/N binary layers like Pb/Ag, and with the determination of T_{CN}, (if N is intrinsically superconductor), of Ag or other noble metals.

We assume that, for the low external magnetic field for which the first penetration mode (I) is energetically favoured, N does not give way to the field to enter. This means that the microscopic field is zero everywhere inside N. For rather high field for which (II) is more stable than (I), N does not screen the field at all, that is, the applied field H_a is assumed to penetrate uniformly into N up to the S/N interface. This is the simplest case of the investigation of the breakdown field effect which confirms the preceding studies. In the above circumstances, it will be shown that, the breakdown field in this case H_{bM} is less than H_b calculated in section (4.V).

Our starting point is the calculation of the Gibbs' free energy (4.4) for both configuration (I) and (II), using the following conditions:

$$h_I = 0 \qquad \text{everywhere inside N for (I) and}$$
$$h_{II} = h_a \qquad \text{everywhere inside N for (II).} \qquad (4.50)$$

We recall that all quantities are given in dimensionless units (see Appendix A) and h_a is the external field applied on the N part parallel to the S/N interface. Using (4.5) and (4.50), G_{NI} and G_{NII}, from (4.4), read

$$G_{NI} = -\tfrac{1}{2} l f_o^4 \qquad (4.51)$$

$$G_{NII} = \kappa_N^{-2} l^{-1} f_o^2 - \frac{1}{10} l f_o^4 - l h_a^2 \qquad (4.52)$$

where κ_N is the GL parameter of N, l is the thickness of N in reduced units and f_o is given by (4.6). It is seen from (4.51) and (4.52) that G_{NI} is independent of h_a, whereas G_{NII} is quadratically a decreasing function of h_a. Furthermore, the breakdown field h_{bM}, which is determined by equating G_{NI} and G_{NII}, decreases as the thickness of N is increased. The breakdown field h_{bM} is the field at which $G_{NI} = G_{NII}$. Thus

$$h_{bM} = f_o^2 \left[\frac{2}{5} + \frac{1}{(l f_o \kappa_N)^2}\right]^{\frac{1}{2}} \qquad (4.53)$$

which is valid for all l. It is more convenient to introduce the following quantities:

$$X = \sqrt{2}\xi_N / f_o \qquad (4.54)$$

$$H_o = H_{CN} f_o^2$$

Then (4.53) in terms of physical units becomes (we recall that $h = \frac{2\pi}{\phi_o}\lambda_N \xi_N H$ in the conventional units and $H_{CN} = \frac{\phi_o}{2\pi\sqrt{2}\xi_N\lambda_N}$, Appendix A):

$$H_{bM} = H_o \left[\frac{4}{5} + \left(\frac{X}{d_N}\right)^2\right]^{\frac{1}{2}} \qquad (4.56)$$

This equation yields two asymptotic expressions for H_{bM} where $d_N \ll X$ and $d_N \gg X$:

$$H_{bM} \simeq H_o X/d_N \quad , \quad d_N << X \qquad (4.57)$$

$$H_{bM} \simeq \frac{2}{\sqrt{5}} H_o \left[1 + \frac{5}{8} \left(\frac{X}{d_N} \right)^2 \right] \quad , \quad d_N >> X \qquad (4.58)$$

The expression (4.57) is exactly the same result obtained by the Orsay Group (1967) which has been confirmed by experiment and a good agreement within 10% has been found. Since the above derivations are for the case that S has infinite thickness, the expression (4.58) differs from the results of the Orsay Group obtained for finite S.

H_{bM}, using (4.56), can be plotted against $1/d_N$ which has the similar behaviour as the figure (4.5). This means that H_{bM} goes to infinity with the slope unity as $d_N/\xi_N \to 0$, for a given T, and approaches $2H_o/\sqrt{5}$ with zero slope as $d_N/\xi_N \to \infty$. Moreover, making use of (4.36) and substituting (4.54) and (4.55) into (4.38) converts the breakdown field H_b (4.38) to $H_b = \sqrt{5.25}\, H_o X/d_N$, therefore $H_b = \sqrt{5.25}\, H_{bM}$ (H_{bM} is given by 4.57). This implies that excluding the field out of the N in mode (I) has been the main cost in energy.

4.XI Conclusion:

In this chapter, within the domain of validity of the GL equations, we showed that all phase transitions in S/N binary layers (placed in an external magnetic field parallel to the intermetallic interface) with the GL parameter much smaller than unity could be simply and qualitatively explained by the interplay of the two different boundary conditions on the order parameter Δ at the normal-vacuum interface. In particular, we demonstrated that a first order phase transition in a S/N sandwich could be observed, at a certain field (the breakdown field), even if the normal sample was very thick.

The two different boundary conditions mentioned above are locally associated with the exchange stability of the field, i.e. they have different values in the field. This provides the simplest reason for the phase transition (of course, since we have non-linear equations, there is no reason why the simplest answer should provide the truth). This is the attraction of the problem presented in this thesis.

The domain of existence of the above transition $T^*(d_N)$ which is the projection of the cross over points of the thermodynamic critical field and the breakdown field on the (T,d_N) plane makes the most comprehensible link with the experiment.

We have a problem, qualitatively and quantitatively, with the case in which the superconductor has infinite thickness. Thus we had to wave somewhat our hands (using the result of Werthamer, 1963, i.e. thickness dependence of the critical temperature of S) to generalize our phase diagram to the case of finite d_S. The results are meant to show a trend (how the phase diagram is modified by a finite d_S) and have not been put forward in order to be fitted to experiment. This being said, we believe that the general qualitative and quantitative agreement is fairly satisfactory.

GENERAL CONCLUSION:

This thesis was mainly devoted to the investigation of the phase transitions in the superconductive-normal binary layers in the framework of the local GL theory of superconductivity. We presented the simplest reason for the occurrence of the phase changes in the superimposed materials; that is, the only two possible boundary conditions of the order parameter at the normal-vacuum interface provide the simplest reason for a phase transition.

In chapter two, we predicted a first-order phase transition in the S/N sandwiches which has not so far been observed experimentally, in the absence of an external magnetic field.

As a direct extension of the aforementioned phase change, in chapter four, we studied the S/N double layer in the presence of an external field for the limiting case where the GL parameters of both S and N sides were much smaller than unity. It was concluded that a first-order phase transition takes place at a certain field (the breakdown field) again due to the competition between the two possible configurations of the order parameter and we could achieve to a reasonable qualitative agreement with the experiment.

In general, in the GL equations, two things matter. One is the boundary conditions and the other is the nonlinearity of the equations. It is quite true that nonlinearity can cause the phase transition or other collective behaviour even if the boundary conditions remain unaltered. Nonetheless, we think that it is interesting that the simple reason can, in principle, accommodate observed and hitherto unobserved phase transitions.

We realize that, as one progresses through the thesis, although the situation becomes more general (addition of a field, etc.), the assumptions and the approximations needed to solve the mathematical problems become more and more drastic, so that it can be argued that our case for interplay of the two boundary conditions becomes weaker rather than stronger as one reads along. We feel, however, that it is unlikely that the qualitative results have been forced by the approximations needed to obtain an analytical solution. Moreover, we were compelled to proceed to a sufficient trend of generality and complexity ($H_a \neq 0$) by the varied experimental situations.

APPENDIX A

THE GINZBURG-LANDAU (GL) EQUATIONS AND THE GAUGE TRANSFORMATION

A.I Introduction:

The (GL) equations are derived by minimizing the free energy of the superconducting state with respect to the pair potential and the magnetic vector potential.

The free energy density, F_{SH}, of the superconducting state is given by (De Gennes, 1966)

$$F_{SH} = F_{NO} + A|\Delta|^2 + \frac{B}{2}|\Delta|^4 + C|(-i\nabla - \frac{e^*\underset{\sim}{A}}{\hbar c})\Delta|^2 + \frac{H^2}{8\pi}, \qquad (A.1)$$

$$e^* = 2e$$

where F_{NO} is the free energy density of the normal state, $\underset{\sim}{A}$ is vector potential related to magnetic field H, by $\vec{H} = \nabla X \vec{\underset{\sim}{A}}$, A, B and C are given (in BCS approximation, De Gennes 1966) by †

$$A = N(0)|1-T/T_C| \quad , \quad B = 0.098N(0)/(k_BT_C)^2$$

$$C = N(0)\xi_0^2 \quad , \quad \xi_0 = 0.13\frac{\hbar v_F}{k_BT_C} \quad (*) \qquad (A.2)$$

ξ_0 is the coherence length at absolute zero and N(0) is the density of states at the Fermi level for electrons of one spin.

† An asterisk labels the expressions valid exclusively within the microscopic GL theory. The other expressions can all accommodate phenomenological parameters.

In view of the analogy of the superconducting order parameter with the wave function of ordinary quantum mechanics (one of the GL equations is a non-linear Schrödinger equation), we set

$$\Delta(r) = \left\{\frac{\hbar^2}{2m^*C}\right\}^{\frac{1}{2}} \psi(r)$$

$$\alpha = \frac{\hbar^2}{2m^*}\frac{A}{C} \quad , \quad \beta = \left(\frac{\hbar^2}{2m^*}\right)^2 \frac{B}{C^2} \quad \text{and} \qquad (A.3)$$

$$H_C{}^2 = 4\pi\,\alpha^2/\beta \text{ is the bulk critical field} \quad , \; m^*=2m$$

Hence F_{SH} could be written in the following form:

$$F_{SH} = F_{NO} + \alpha|\psi|^2 + \tfrac{1}{2}\beta|\psi|^4 + \frac{1}{2m^*}\left|\left(-i\hbar\nabla - \frac{e^*A}{c}\right)\psi\right|^2 + \frac{H^2}{8\pi} \qquad (A.4)$$

It is more convenient to work with dimensionless units forms of the GL equations. First we derive them in one dimensional space with the geometry shown in Fig. (A.1). ψ is assumed to be real (choice of the gauge, see below) and to depend only on x. The z-axis is taken along the uniform applied magnetic field H_a, so $\vec{H} = (0,0,H)$.

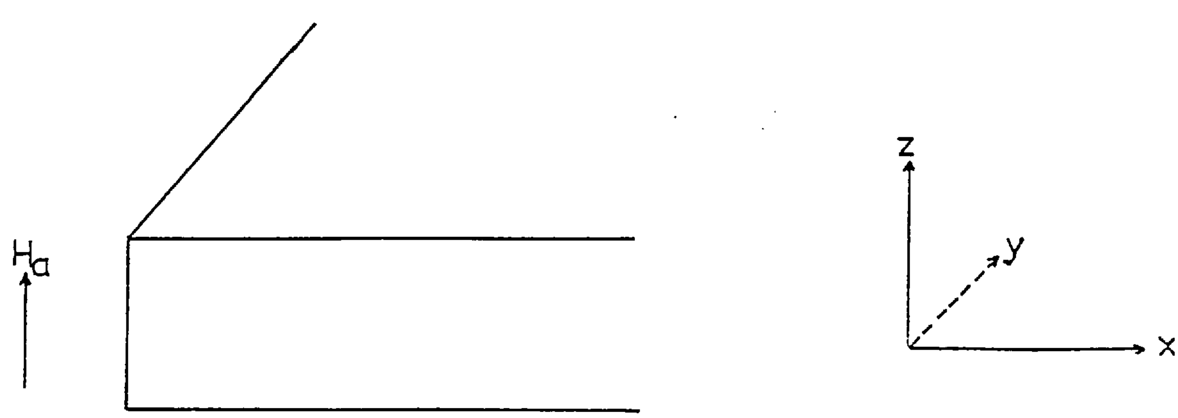

Figure A.1: Schematical representation of a superconductor placed in an external magnetic field H_a. The induction field H is equal to H_a at x = 0.

Now there is considerable freedom in the choice of gauge for $\underset{\sim}{A}$. It is assumed to be directed along y-axis (asymmetric gauge);

$$\vec{\underset{\sim}{A}} = (0,Hx,0) \qquad (A.5)$$

which means the vector potential depends only on x (we have made two assumptions so far: (1) the gauge is irrelevant, (2) ψ and $\vec{\underset{\sim}{A}}$ are functions of x only).

An alternative would be (-Hy,0,0) , and indeed any linear combination of this equation with (A.5) would serve too. However all these choices satisfy the "London gauge" i.e.

$$\nabla.\vec{\underset{\sim}{A}} = 0 \quad \text{and} \quad \vec{H} = \vec{\nabla}x\vec{\underset{\sim}{A}}$$

The physical quantities such as H_{C2} (the upper critical field, see chapter 3) are independent of the choice of the gauge).

As has been pointed out in chapter 3, the assumption that ψ and $\vec{A}$ are functions of x only is inadequate for type II superconductors with high κ and untenable for $H>H_C$ (H_C is the thermodynamic critical field). Even for $H_{C1}<H<H_C$ (H_{C1} is the lower critical field), individual vortices are nucleated into the superconductor whose state is no longer described by this assumption.

Under the above circumstances, by introducing the following dimensionless quantities

$$u = \frac{x}{\lambda} \;, \quad \vec{a} = \frac{2\pi\xi}{\phi_0}\vec{\underset{\sim}{A}} \;, \quad f = \frac{\psi}{\psi_0} \quad \text{and} \quad \vec{h} = \frac{2\pi\lambda\xi}{\phi_0}\vec{H} \qquad (A.6)$$

where $\phi_0 = \frac{ch}{2e}$ is a quantum of flux and $\psi_0{}^2 = \frac{|\alpha|}{\beta}$

for position, vector potential, order parameter and magnetic field respectively, (A.4) becomes:

$$F_{SH} = F_{NO} + \frac{H_C{}^2}{4\pi}\,(\pm f^2+\tfrac{1}{2}f^4+\kappa^{-2}f'^2+f^2a^2+a'^2) \qquad (A.7)$$

where $$\frac{H_C{}^2}{4\pi} = \frac{\alpha^2}{\beta} \quad \text{and} \quad \begin{cases} T>T_C, & \text{positive sign} \\ T<T_C, & \text{negative sign} \end{cases} \qquad (A.8)$$

we note that $h=\frac{da}{du}=a'$ and the GL parameter κ is defined, as usual, by $\kappa = \lambda/\xi$

where

$$\xi = \left\{\frac{\hbar^2}{2m^*}\,\frac{1}{|\alpha|}\right\}^{\frac{1}{2}}$$ is the coherence length (*) and

$$\lambda = \left\{\frac{m^*c^2}{4\pi e^{*2}}\,\frac{\beta}{|\alpha|}\right\}^{\frac{1}{2}}$$ is the penetration depth, so the critical field for the bulk material would be $H_C = \frac{\phi_0}{2\pi\sqrt{2}\lambda\xi}$.

It is more convenient to measure the energy density in units of $H_C{}^2/4\pi$, then equation (A.7) reads

$$\tilde{F}_{SH} = \tilde{F}_{NO} + (\pm f^2+\tfrac{1}{2}f^4+\kappa^{-2}f'^2+f^2a^2+a'^2) \qquad (A.9)$$

where $\tilde{F} = F/H_C{}^2/4\pi$. The expression (A.9) is the free energy density of superconducting state.

A.II Equilibrium (GL) equations in one dimensional space:

We must now minimize the free energy with respect to the order parameter and the magnetic field distribution, or $\vec{a}$. Since the free energy density (A.9) is an explicit function of two variables, i.e. f and $\vec{a}$, the equilibrium (GL) equations are the Euler-Lagrange equations for f and $\vec{a}$, namely

$$\frac{\partial \tilde{F}_{SH}}{\partial f} - \frac{\partial}{\partial u}\left(\frac{\partial \tilde{F}_{SH}}{\partial f'}\right) = 0$$
$$\frac{\partial \tilde{F}_{SH}}{\partial a} - \frac{\partial}{\partial u}\left(\frac{\partial \tilde{F}_{SH}}{\partial a'}\right) = 0 \qquad \text{(A.10)}$$

Hence

$$\kappa^{-2} f'' = f(\pm 1 + f^2 + a^2)$$
$$a'' = f^2 a \qquad \text{(A.11)}$$

These are the GL equations in dimensionless units. Dimensionless (reduced) units correspond to measuring lengths, magnetic fields, the vector potential, the order parameter and the energy density in the units of λ, $\sqrt{2}H_C$, $\sqrt{2}H_C\lambda$, $|\psi_0|$ and $H_C^2/4\pi$ respectively.

The GL equations are a pair of non linear, second order differential equations, linking the order parameter and the magnetic vector potential. These equations must be supplemented with the appropriate boundary conditions at the free surface of the sample. The Euler-Lagrange equations, i.e. (A.10) are derived in such a way that the following conditions are satisfied;

$$\frac{\partial \tilde{F}_{SH}}{\partial a'}\,\delta a = 0 \qquad \text{(A.12)}$$

$$\frac{\partial \tilde{F}_{SH}}{\partial f'}\,\delta f = 0 \qquad \text{(A.13)}$$

Since the vector potential and the magnetic field are always fixed at the surface of the specimen, the condition (A.12) is always satisfied. Using (A.9) the condition (A.13) becomes;

$$f'\delta f = 0 \quad , \quad \text{at the surface}$$

This implies that either $f = 0$, and hence $\delta f = 0$ or $f' = 0$ at the surface. The former condition corresponds to vanishing the wave function (in ordinary quantum mechanics) at the boundary. As was shown in chapter one the condition $f = 0$ (at the free surface of the superconductor) is not valid for an isolated superconductor, but $f' = 0$.

This argument does not hold for a S/N system (see, chapter 2), i.e. the order parameter at N-vacuum interface vanishes for a given thickness of the normal sample and at high temperatures, whereas at low temperatures the gradient of the order parameter vanishes at normal-vacuum interface.

It has to be mentioned that, both conditions $f = 0$ and $f' = 0$ are necessary consequences of GL superconductivity (they make the super-current vanish at the free surface of the sample). This point will be discussed in section A.III .

A.III Equilibrium (GL) equations in three dimensional space:

So far we have derived GL equations in one dimensional space. The second equation in (A.11) is not adequate to talk generally about vanishing supercurrent at the surface of the sample, even though it is proportional to $a' = \frac{dh}{du}$. Therefore it is essential to calculate GL equations in general forms. The prescription of deriving these equations is the same as before (Euler-Lagrange equations will be used).

Since ψ is generally complex, we can minimize F_{SH} (equation (A.4)) with respect to ψ^* (or ψ) and $\vec{\underset{\sim}{A}}$. So the Euler- Lagrange equations for ψ^* and $\vec{\underset{\sim}{A}}$ are

$$\frac{\partial F_{SH}}{\partial \psi^*} - \sum_k \frac{\partial}{\partial x_k} \frac{\partial F_{SH}}{\partial(\nabla_k \psi^*)} = 0 \qquad (A.14)$$

$$\sum_i \left\{\frac{\partial F_{SH}}{\partial \underset{\sim}{A}_i} - \sum_k \frac{\partial}{\partial x_k} \frac{\partial F_{SH}}{\partial(\partial \underset{\sim}{A}_i / \partial x_k)}\right\} = 0 \qquad (A.15)$$

where $\nabla_k \psi^*$ is the component of the gradient in the direction k and $\underset{\sim}{A}_i$ is the component of $\vec{\underset{\sim}{A}}$ in the ith direction.

By using (A.4), the equation (A.14) gives

$$\alpha\psi + \beta|\psi|^2\psi + \frac{1}{2m^*} (-i\hbar\nabla - \frac{e^*}{c} A)^2\psi = 0 \qquad (A.16)$$

This is called the first GL equation in general form and is in fact the non linear Schrödinger equation of a particle with mass $m^*=2m$ and charge $e^*=2e$. It is clear that the minimization of F_{SH} with respect to ψ rather than ψ^* yields the complex-conjugate equation of (A.16).

We must now minimize F_{SH} with respect to variation of the magnetic vector potential $\vec{\underset{\sim}{A}}$ (equation (A.15)). For the sake of simplicity we split (A.15) to its three components. The x-component of the first and the second terms of (A.15) are

$$\frac{\partial F_{SH}}{\partial A_x} = \frac{1}{2m^*} \{ i\hbar\frac{e^*}{c}(\psi^*\nabla_x\psi - \psi\nabla_x\psi^*) + \frac{2e^{*2}}{c^2} \underset{\sim}{A}_x|\psi|^2 \} \tag{A.17}$$

and
$$\sum_k \frac{\partial}{\partial x_k} \frac{\partial F_{SH}}{\partial(\partial \underset{\sim}{A}_x/\partial x_k)} = \frac{1}{4\pi}(\nabla^2\underset{\sim}{A}_x - \nabla_x\nabla\cdot\underset{\sim}{A}) \tag{A.18}$$

respectively, where (A.4) has been used and $\nabla_x\psi$ (for example) denotes the partial differentiation of ψ with respect to x. The similar equations can easily be obtained for the other components.

Since $\vec{H} = \text{curl } \vec{\underset{\sim}{A}}$, expression (A.18) is just the x-component of $-\vec{J}/c$, where $\vec{J}$ is the super-current related to the magnetic field via Maxwell's equation, i.e. curl $\vec{H} = \frac{4\pi}{c}\vec{J}$.

The full expressions for the first and second terms of (A.15) are

$$\sum_i \frac{\partial F_{SH}}{\partial \underset{\sim}{A}_i} = \frac{1}{2m^*} \{ i\hbar\frac{e^*}{c}(\psi^*\nabla\psi - \psi\nabla\psi^*) + \frac{2e^{*2}}{c^2} \underset{\sim}{A}|\psi|^2 \}$$

and
$$\sum_i \sum_k \frac{\partial}{\partial x_k} \frac{\partial F_{SH}}{\partial(\partial \underset{\sim}{A}_i/\partial x_k)} = -\frac{\vec{J}}{c}$$

respectively. Substituting these two expressions into (A.15) leads to the celebrated second GL equation

$$\vec{J} = -\frac{i\hbar e^*}{2m^*}(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*) - \frac{e^{*2}}{m^*c}\psi^*\psi\vec{\underset{\sim}{A}} \tag{A.19}$$

which holds for a general gauge.

The first term of (A.19) has exactly the form of the conventional quantum-mechanical current for a particle (of mass $m^*=2m$ and of charge e*=2e) with a wave function $\psi(r)$ and the second term represents a response of a superconductor. In fact these two terms are related by a gauge transformation.

If we set $\psi = \psi_0 e^{i\theta/\hbar}$, where ψ_0 and θ are real, then (A.19) becomes;

$$\vec{J} = \frac{e^*}{m^*} \psi_0^2 (\nabla\vec{\theta} - \frac{e^*}{c} \vec{\underset{\sim}{A}})$$

By introducing the following gauge transformation

$$\vec{\underset{\sim}{A}} = \frac{c}{e^*} (\nabla\vec{\theta} - \vec{\underset{\sim}{A}}') \tag{A.20}$$

the above expression would be

$$\vec{J} = \frac{e^*}{m^*} \psi_0^2 \vec{\underset{\sim}{A}}' \tag{A.21}$$

So the first and second terms of (A.19)are related by (A.20).

The coupling between matter and electromagnetic field is only effected through the gradient in the free energy, F_{SH}, (minimal gauge coupling) and this yields the standard expression for the current. (It also implies the gauge invariance, Josephson effect, collective modes with a gap, etc.).

There is of course one basic difference for which ψ (in the present investigation) is not normalized, on contrast to the wave function in quantum mechanics. This is clearly due to using different mass and charge, i.e. m* and e*. Moreover, (A.19) is a "local" expression for the London type, i.e. $\vec{J}(r)$ is given by the values of $\nabla\vec{\psi}$ and $\vec{\underset{\sim}{A}}$ at the point r.

For a "pure" superconductor (the electronic mean free path is greater than the coherence length ξ) the range of validity of the theory is that of the expansion in powers of ψ, and the temperature has to be close to the critical temperature of the bulk material T_C. For a "dirty" specimen for which the electronic mean free path is much less than ξ, on the other hand, a local theory is adequate even if ψ is large, and the temperaure need not be restricted to the neighbourhood of T_C.

As was stated previously, the equation (A.16), apart from the nonlinear term, has the form of a Schrödinger equation for Cooper pairs with eigen-energy - α (note that for superconductive state, α is negative). The fact that, ψ turns out to be as uniformly as possible in space (in addition to the kinetic energy term $(\nabla\psi)^2$) is due to the contribution of the nonlinear term in (A.16). Tinkham (1975) has pointed out that the uniform distribution of ψ in space is due to action of the nonlinear term as a repulsive potential of ψ on itself.

So far we have derived GL equations. These two equations must be supplemented with the appropriate boundary conditions at the free surface of the specimen. As usual, the Euler-Lagrange equations, i.e. (A.14) and (A.15) are obtained by requiring the following expressions:

$$\frac{\partial F_{SH}}{\partial(\nabla\underset{\sim}{A})} \; \delta\underset{\sim}{A} = 0 \qquad (A.22)$$

$$\frac{\partial F_{SH}}{\partial(\nabla\psi^*)} \quad \delta\psi^* = 0 \qquad \text{(A.23)}$$

These are indeed the integrands of the surface integrals appearing in minimization of total free energy $\int_{\text{all space}} F_{SH} d^3\underline{r}$ when $\underset{\sim}{A}$ and ψ^* are varied by $\delta\underset{\sim}{A}$ and $\delta\psi^*$ respectively. With help of (A.4) the expression (A.22) takes the following form;

$$\vec{n}.\ \frac{\partial F_{SH}}{\partial(\nabla\underset{\sim}{A})} \quad \delta\vec{\underset{\sim}{A}} = (\text{curl } \vec{\underset{\sim}{A}} \times \delta\vec{\underset{\sim}{A}}).\vec{n} = 0 \qquad \text{(A.24)}$$

where $\vec{n}$ is the unit vector perpendicular to the sample surface. The surface integral of this quantity at infinity where the magnetic field has dropped to zero, will be zero.(Grassie, 1975). $\vec{\underset{\sim}{A}}$ and $\vec{H}$ = curl $\vec{\underset{\sim}{A}}$ are always fixed at the surface so that (A.24) is always satisfied. (A.4) also transforms (A.23) into

$$\vec{n}.\ (-i\hbar\vec{\nabla} - \frac{e^*}{c}\vec{\underset{\sim}{A}})\psi \ \delta\psi^* = 0 \text{ , at the free surface} \qquad \text{(A.25)}$$

The calculus of variations formalism usually is carried out with the subsidiary condition ψ^*=0, and hence $\delta\psi^*$=0 at the surface. It was proved in chapter one, that the order parameter can not be zero at free surface of an isolated superconductor, even though the condition ψ^*=0 is sufficient to make the supercurrent (A.19) vanish at the free surface. So the appropriate boundary condition at an insulating surface which assures that no supercurrent passes through the surface would be

$$\vec{n}.\ (-i\hbar\vec{\nabla} - \frac{e^*}{c}\vec{\underset{\sim}{A}})\psi = 0 \qquad \text{(A.26)}$$

Now if we set $\vec{J}.\vec{n} = 0$ (from (A.19)), we would not obtain the above condition, but only

$$\vec{n}.\ (-i\hbar\vec{\nabla} - \frac{e^*}{c}\vec{\underset{\sim}{A}})\psi = ib\psi \qquad (A.27)$$

where b is a real constant and depends on the nature of the material to which contact is made. De Gennes (1966) has discussed that (A.27) is applicable for a superconductor-normal metal junction. We have, however, demonstrated (see chapter 2) that above conditions have had important implications in the onset of superconductivity.

A.IV The gauge transformation of GL equations:

It is more convenient to work with dimensionless units forms of GL equations. For this purpose we use (A.6) in three dimensional space (for example $\psi=\psi(r)$) to convert (A.16) and (A.19) to:

$$-f + f|f|^2 + (-i\kappa^{-1}\vec{\nabla}-\vec{a})^2 f = 0 \qquad (A.28)$$

$$\text{curl curl } \vec{a} + |f|^2\vec{a} + \frac{i\kappa^{-1}}{2}(f^*\vec{\nabla}f - f\vec{\nabla}f^*) = 0 \qquad (A.29)$$

where the Maxwell's equation $\vec{J} = \frac{c}{4\pi}$ curl $\vec{H}$ has been used and here all differentiations are taken with respect to the quantity $\vec{u} = \frac{\vec{r}}{\lambda}$.

We now introduce $f = f_0 e^{i\theta}$ where f_0 is the modulus and θ is the phase of f to simplify (A.29) to

$$\text{curl curl } \vec{a} + f_0^{\,2}(\vec{a} - \kappa^{-1}\nabla\vec{\theta}) = 0 \qquad (A.30)$$

Performing a gauge transformation

$$\vec{a}_0 = \vec{a} - \kappa^{-1}\nabla\vec{\theta} \qquad (A.31)$$

converts (A.28) and (A.30) to

$$-f_0 + f_0^3 + (-i\kappa^{-1}\vec{\nabla}-\vec{a}_0)^2 f_0 = 0 \qquad \text{(A.32)}$$

$$\text{curl curl } \vec{a}_0 + f_0^2\vec{a}_0 = 0 \qquad \text{(A.33)}$$

where the magnetic field is given by $\vec{h}$ = curl $\vec{a}_0$.
With the help of London gauge $\vec{\nabla}.\vec{a}_0 = 0$ and (A.33) we can show that (A.32) is simply given by

$$-\kappa^{-2}\nabla^2 f_0 + f_0(-1+f_0^2+a_0^2) = 0 \qquad \text{(A.34)}$$

and (A.33) becomes

$$-\nabla^2 a_0 + f_0^2 a_0 = 0 \qquad \text{(A.35)}$$

where the identity curl curl a_0 = grad div $a_0 - \nabla^2 a_0$ has been used. The above equations are a couple of nonlinear differential equations (GL equations in reduced units) in three dimensional space and are a direct generalization of (A.11), where both f and a were assumed to be functions of x only (section A.I).

It is of interest to apply the above expressions for the extreme type II superconductor ($\kappa >> 1$). For this limiting case (A.34) and (A.35) become;

$$f_0^2 + a_0^2 - 1 = 0 \qquad \text{(A.36)}$$

$$\nabla^2 a_0 - f_o^2 a_0 = 0 \qquad \text{(A.37)}$$

which are general forms of (3.10) and (3.12). Here again $|a_0|$ has to be less than unity in the physical regions of superconductivity. Solving the above equations for a_0, we deduce;

$$\nabla^2 a_0 - (1-a_0^2)a_0 = 0 \qquad \text{(A.38)}$$

This is a vector differential equation in three dimensional space and is the general form of (3.13). A complete solution of (A.38) for an arbitrary geometry has not been obtained up to now, in contrary to the solution of (3.13). If we could solve it completely and without making any assumption we would have been able to study the structure of Abrikosov vortices for an extreme type II material.

The equation (A.38) is a generalized London equation. Indeed,

$$\text{curl curl } \vec{a}_0 + (1-a_0^{\,2})\vec{a}_0 = 0 \qquad \text{(A.39)}$$

Since the first term is proportional to current in reduced units, then (A.39) would be general form of London equation for the materials with high κ.

However, equation (A.38) can be expected to be solved for a particular geometry of specimen and for certain direction of the magnetic field or vector potential $\vec{a}_0$ (chapter 3).

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