HYPOTHESIS TESTING AND REGION ESTIMATION IN NONLINEAR REGRESSION

BY

A.H. POOI

THESIS SUBMITTED FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN THE UNIVERSITY OF LONDON

ACKNOWLEDGEMENTS

I would like to express my gratitude to my supervisor, Professor E.M.L. Beale, for his guidance and encouragement. I would also like to thank the University of Malaya for its financial assistance.

Abstract

The present work is concerned with the problems of hypothesis testing and region estimation concerning subsets of components of the parameter vector in nonlinear regression models.

A fundamental approach to these problems is to devise methods for indicating when the use of usual linear theory results as approximations is justified. Measures of nonlinearity are proposed in Beale (1960) for this purpose. In the present work, the problem of finding bounds for these measures of nonlinearity within which it is justifiable to use linear theory results is investigated. The use of nonlinear transformations of the parameter vector for making a model more nearly linear is also discussed.

The main approach considered here is based on general maximum likelihood (m.l.) ratios. The derivation of truncated series expansions of the significance probabilities and power functions of the general m.l. ratio tests is considered. The use of a computer to do the algebraic manipulation involved in this derivation is also illustrated.

The above approaches are then compared by means of numerical examples.

Contents

CHAPTER 1

INTRODUCTION

Section 1.1 Nonlinear regression model

The present work is concerned with hypothesis testing and region estimation concerning one or more components of the parameter vector in the nonlinear regression model which can be described as follows. Suppose that we have a set of observations y_u $(u = 1,2,\ldots,n)$ and a set of corresponding theoretical mean values which we may write as $\eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta})$. This notation indicates that the theoretical mean values depend on the conditions under which the u^{th} observation was taken, represented by a vector $\underset{\sim}{\xi}_u$ of independent variables, and also on the parameter vector $\underset{\sim}{\theta} = (\theta_1,\theta_2,\ldots,\theta_p)^T$ which is assumed to lie in a certain set Ω, the parameter space. Then the model can be expressed as

$$(1.1.1) \qquad y_u = \eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta}) + \varepsilon_u,$$

where ε_u are random errors with zero means and some statistical distribution.

It is convenient to consider a set of observations y_u as a point $\underset{\sim}{y} = (y_1,y_2,\ldots,y_n)^T$ in an n-dimensional Euclidean space. Within this space there is a subset consisting of points each of which fits the theoretical model exactly for some value of $\underset{\sim}{\theta}$. We call this subset the Solution Locus. In symbols a point in the solution locus has its u^{th} coordinate η_u defined by

$$(1.1.2) \qquad \eta_u = \eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta}) \text{ for all } u,$$

for some values of $\underset{\sim}{\theta}$. Let $P(\underset{\sim}{\theta})$ denote the point whose coordinates are defined by (1.1.2). The least squares estimate for $\underset{\sim}{\theta}$ is then the value of $\underset{\sim}{\theta}$ for which $P(\underset{\sim}{\theta})$ is nearest to the observed sample point $\underset{\sim}{y}$. In the present work we make the assumption that the ε_u are independently normally distributed with a common variance σ^2. This assumption implies that the least squares estimate of the parameter vector $\underset{\sim}{\theta}$ is also the maximum likelihood estimate $\hat{\underset{\sim}{\theta}}$.

Each of the function $\eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta})$ can be either linear or nonlinear in $\underset{\sim}{\theta}$. If

(i) all $\eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta})$ are linear in $\underset{\sim}{\theta}$,

(ii) all real $\underset{\sim}{\theta}$ are in Ω,

(iii) the (n×p) matrix $\{c_{uj}\} = \left\{\dfrac{\partial\eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta})}{\partial\theta_j}\right\}$ is of rank p,

then the corresponding solution locus is a p-dimensional hyperplane in sample space. Furthermore the components of $\underset{\sim}{\theta}$ define a Cartesian, i.e. uniform, system of coordinates in this hyperplane. If one or more $\eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta})$ are nonlinear in $\underset{\sim}{\theta}$, then the derivatives c_{uj} may become functions $c_{uj}(\underset{\sim}{\theta})$ of $\underset{\sim}{\theta}$. The solution locus may then be a distorted hyperplane. We refer to this solution locus as an "unconstrained" solution locus, and the corresponding model as an "unconstrained" model. This solution locus is to be distinguished from one which has boundary points, in which case the solution locus and the model are "constrained". It is important to realize that not all models are unconstrained. Typically parameters must lie between zero and infinity, and the solution locus may have a definite boundary where a function of $\underset{\sim}{\theta}$ tends to a limit. For example consider the models with $\eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta})$ given by

(A) $$\eta(\xi_u,\underset{\sim}{\theta}) = \frac{\theta_1}{\theta_1-\theta_2}\left(e^{-\theta_2\xi_u} - e^{-\theta_1\xi_u}\right),$$

where $0 < \theta_1 < \infty$, $0 < \theta_2 < \infty$, and

$\xi_u = 0.25, 0.5, 1, 1.5, 2, 4,$

(B) $$\eta(\xi_u,\theta) = 1 - \frac{1}{\theta_1-\theta_2}(\theta_1 e^{-\theta_2\xi_u} - \theta_2 e^{-\theta_1\xi_u}),$$

where $0 < \theta_1 < \infty$, $0 < \theta_2 < \infty$, and

$\xi_u = 1,2,3,4,5,6.$

(c.f. Guttman and Meeter (1965)).

We observe that as θ_1 or θ_2 tends to zero, the $\eta(\xi_u,\theta)$ in models (A) and (B) tend to finite limits. This implies that the solution loci of these models are constrained. In particular as θ_2 tends to zero, the $\eta(\xi_u,\theta)$ in model (A) tend to finite limits which depend on θ_1. Thus the solution locus of model (A) has a boundary where θ_2 tends to zero. An important feature of this boundary is that the $c_{u1}(\theta)$ tend to zeros as θ_2 tends to zero. We next note that as θ_1 tends to infinity, the $\eta(\xi_u,\theta)$ in models (A) and (B) tend to finite limits which depend on θ_2. This implies that each of the corresponding solution loci has a boundary where θ_1 tends to infinity. For each of these models, the $c_{u1}(\theta)$ tend to zeros as $P(\theta)$ approaches this boundary. We also observe that $P(\theta)$ in the solution locus of model (B) remains the same if we interchange θ_1 and θ_2. This suggests that we may impose the constraint that $\theta_1 \geq \theta_2$. Furthermore we note that as $(\theta_1 - \theta_2)$ tends to zero, the $\eta(\xi_u,\theta)$ tend to finite limits which depend on θ_1. Therefore the solution locus has a boundary where $(\theta_1 - \theta_2)$ tends to zero. Finally we note that the matrix $\{c_{uj}(\theta)\}$ becomes of rank one, which is less than p, as $(\theta_1 - \theta_2)$ tends to zero. We can examine the solution loci of these models more closely if we apply an orthogonal transformation of coordinates in sample space. The details of this transformation will be described in Chapter 2.

The effects of this transformation are that the point $P(\underset{\sim}{\theta}_T)$ associated with the true value $\underset{\sim}{\theta}_T$ of $\underset{\sim}{\theta}$, where for these models we choose $\underset{\sim}{\theta}_T = (1.4, 0.4)^T$, becomes the new origin corresponding to the transformed coordinates $\underset{\sim}{z} = 0$, and the plane tangent to the solution locus at $P(\underset{\sim}{\theta}_T)$ consists of points for which $z_i = 0$ for $i = 3,4,5,6$. In Fig. (1.1.1) and (1.1.2) we display the first two coordinates z_1 and z_2 of a number of points in the solution loci of these models. Each line in these figures corresponds to some constant value of θ_1 or θ_2. The values of R are for indicating the severity of an aspect of nonlinearity of the model.

The problems of hypothesis testing and region estimation for a model with a constrained solution locus are still open questions. Similar problems for a model with an unconstrained solution locus are less difficult. The present work is mainly concerned with the discussion of methods appropriate to the latter problem, and situations under which these methods may be applied if the model is constrained.

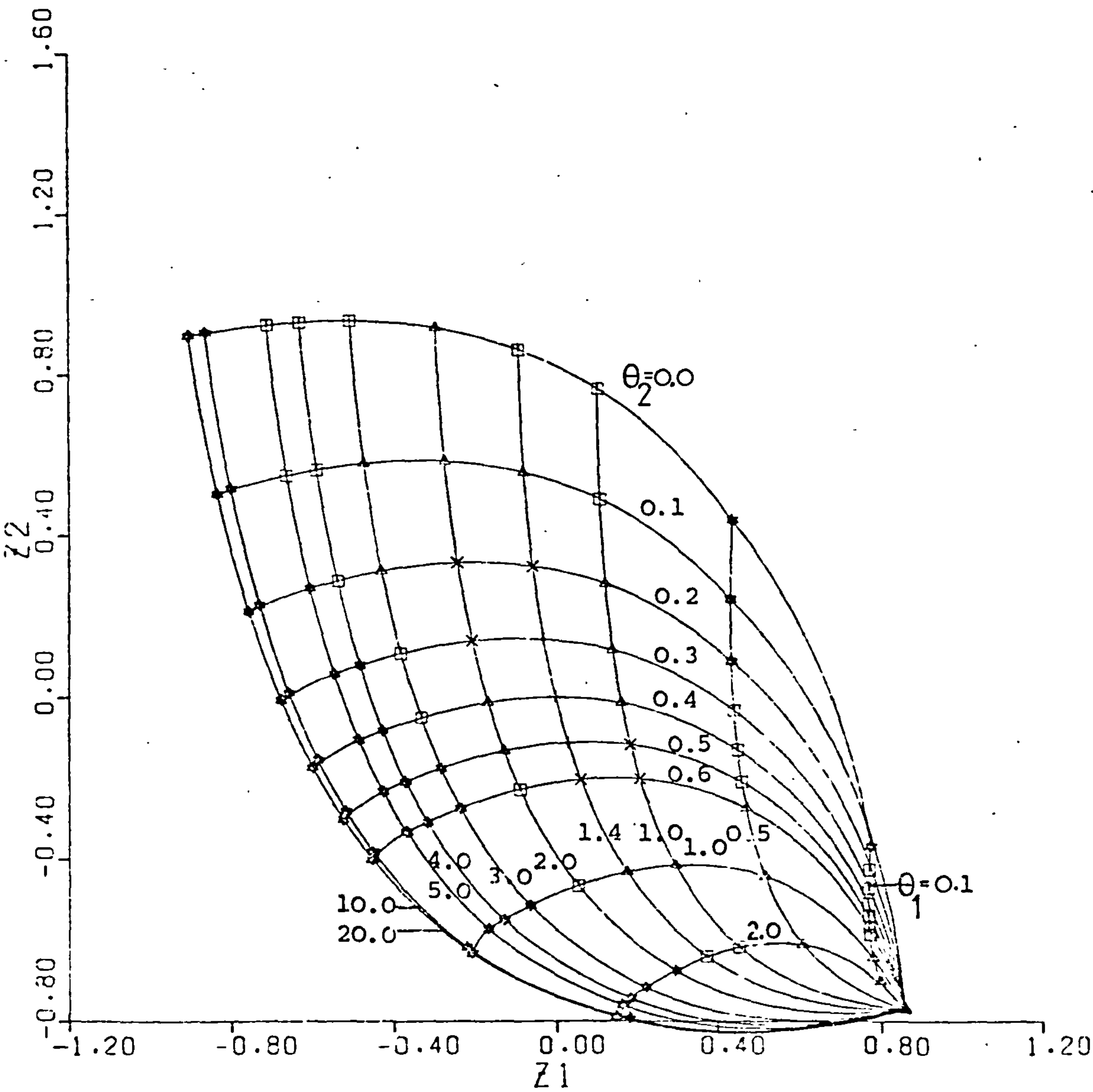
Section 1.2 The problems of hypothesis testing and region estimation

Many computer programs, using a variety of numerical methods, have been written to find point estimates of the parameter vector $\underset{\sim}{\theta}$ using the least squares criterion. Attention has also been paid to the problems of hypothesis testing and region estimation concerning k^* $(1 \leq k^* \leq p)$ components of interest in the parameter vector, treating the other components, if any, as nuisance parameters. Although large sample methods have been proposed, and justified asymptotically as n tends to infinity, these problems are known to be rather intractable when n is small and the functions $\eta(\underset{\sim}{\xi}_u, \underset{\sim}{\theta})$ have no special properties that simplify the analysis.

FIGURE (1.1.1)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI= 0.25.0.5.1.0.1.5.2.0.4.0

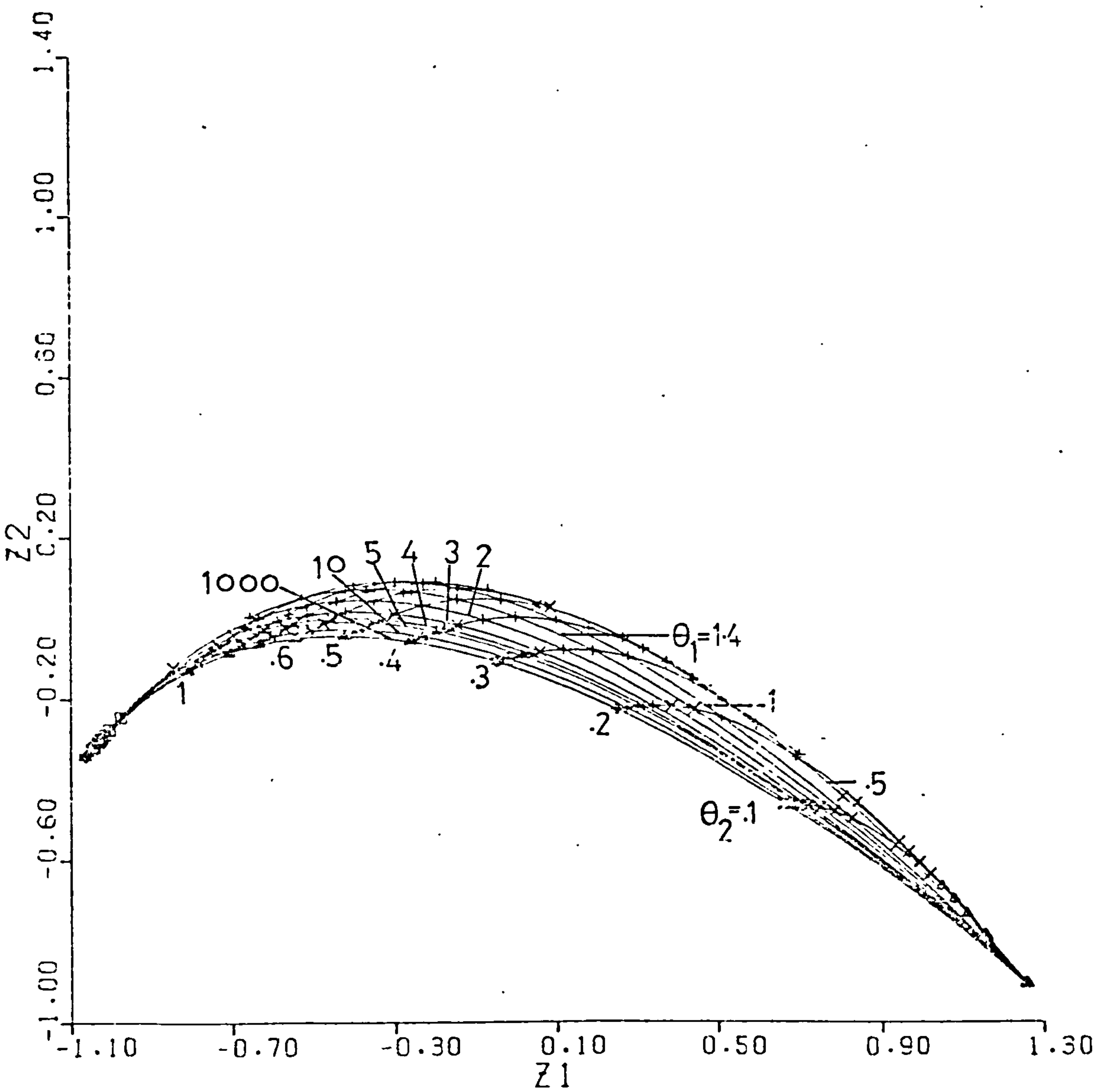
THETAI TRUE ARE 1.4000 0.4000



R=SQUARE ROOT ((SUM FROM P+1 TO N OF ZI SQUARE)/
(SUM FROM 1 TO P OF ZI SQUARE))
+:0≤R≤0.05:×:0.05<R≤0.1:△:0.1<R≤0.2:□:0.2<R≤0.3:✡:R>0.3

FIGURE (1.1.2)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1.2.3.4.5.6

THETAI TRUE ARE 1.4000 0.4000



R=SQUARE ROOT ((SUM FROM P+1 TO N OF ZI SQUARE)/
(SUM FROM 1 TO P OF ZI SQUARE))
+:0≤R≤0.05:X:0.05<R≤0.1:△:0.1<R≤0.2:□:0.2<R≤0.3:✡:R>0.3

We first explain the following phrases before outlining some approaches to these problems. We refer to a model as being "approximately linear in the parameter vector $\underset{\sim}{\theta}$" if and only if the solution locus can be approximated, for statistical purposes, by a p-dimensional linear manifold in which the components of $\underset{\sim}{\theta}$ define a uniform system of coordinates. We next refer to a model as being "approximately intrinsically linear" if and only if the solution locus can be approximated, for statistical purposes, by a p-dimensional linear manifold in sample space. We further refer to a model as being "approximately intrinsically linear in the parameters $\theta_{i_1}, \theta_{i_2}, \ldots, \theta_{i_{p-k*}}$", where $k* < p$, if and only if the set of points in solution locus such that $\theta_{i_j} = \theta_{Ti_j}$ for all $j > p-k*$ can be approximated, for statistical purposes, by a $(p-k*)$-dimensional linear manifold in sample space.

If the model is approximately linear in the parameter vector $\underset{\sim}{\theta}$, these problems are straightforward. If not there are two approaches that can be considered:

<u>either</u> make nonlinear transformations of the original parameters in such a way that there are at least k* of the transformed parameters which depend only on the original parameters of interest,

<u>or</u> carry out general maximum likelihood ratio tests and derive region estimates based on general maximum likelihood ratio criterion.

The former approach is based implicitly on the assumption that the model is approximately linear in the transformed parameter vector. The latter is based implicitly on the assumptions that

(a) if $k* = p$, then the model is approximately intrinsically linear,

(b) if $k^* < p$, then the model apart from being approximately intrinsically linear, is also approximately intrinsically linear in the nuisance parameters.

Section 1.3 Layout of the thesis

In Chapter 2, we investigate the former approach. A more efficient and more illuminating method of computing Beale's measures of nonlinearity using Householder transformations is first described. Then nonlinear transformations of the parameters, in particular power transformations, are used to reduce the nonlinearity for inference purposes.

In Chapter 3, we investigate the latter approach. General m.l. ratio tests are used to test a number of composite nonlinear hypotheses for making inference about subsets of components of the parameter vector. The derivation of the significance probabilities and power functions of the tests for these hypotheses is considered. The estimation of the coverage probabilities of the region estimates based on these tests is also discussed.

In Chapter 4, the derivation of series expansions of the power functions of the tests in Chapter 3 using a computer is discussed. Computer programs for deriving these series expansions truncated after some finite number of terms are presented.

Chapter 5 is devoted to the comparison of the methods of constructing interval and region estimates in Chapters 2 and 3 by means of numerical examples.

CHAPTER 2

MEASURES OF NONLINEARITY

Section 2.1 Definitions of measures of nonlinearity

Throughout this chapter, we make the assumptions that the particular observation $\underset{\sim}{Y}$ that we have obtained is such that $\hat{\underset{\sim}{\theta}}$ obtained by minimizing the residual sum of squares

$$S(\underset{\sim}{\theta}) = \sum_{u=1}^{n} \{y_u - \eta(\underset{\sim}{\xi}_u, \underset{\sim}{\theta})\}^2$$

is the unique unconstrained minimum, and for each $\underset{\sim}{\xi}_u$, $\eta(\underset{\sim}{\xi}_u, \underset{\sim}{\theta})$ is continuous in $\underset{\sim}{\theta}$ at $\underset{\sim}{\theta} = \hat{\underset{\sim}{\theta}}$.

A point $P(\underset{\sim}{\theta}^*)$ associated with a feasible $\underset{\sim}{\theta}^*$ will be referred to in the present work as a "nonsingular point" in the solution locus if and only if

(a) $P(\underset{\sim}{\theta}^*) = P(\underset{\sim}{\theta}_A)$ implies that $\underset{\sim}{\theta}^* = \underset{\sim}{\theta}_A$

(b) there exists an $\varepsilon > 0$ such that

$$|\underset{\sim}{\theta} - \underset{\sim}{\theta}^*| < \varepsilon$$

implies that $\underset{\sim}{\theta} \in \Omega$

(c) for each $\underset{\sim}{\xi}_u$, $\eta(\underset{\sim}{\xi}_u, \underset{\sim}{\theta})$ is a function of $\underset{\sim}{\theta}$ differentiable up to the second order at $\underset{\sim}{\theta} = \underset{\sim}{\theta}^*$

and (d) the $(n \times p)$ matrix $\{c_{uj}(\underset{\sim}{\theta}^*)\}$ is of rank p.

Thus the point $P(\hat{\underset{\sim}{\theta}})$ associated with the $\hat{\underset{\sim}{\theta}}$ is a nonsingular point in the solution locus if and only if (b), (c) and (d) with $\underset{\sim}{\theta}^*$ replaced by $\hat{\underset{\sim}{\theta}}$ hold.

In Beale (1960), numerical measures of nonlinearity in a sufficiently small neighbourhood of a nonsingular point $P(\hat{\underset{\sim}{\theta}})$ in the solution locus are introduced as functions of the experimental design, the parameterization of the model and the least-squares estimate $\hat{\underset{\sim}{\theta}}$.

The original and revised versions of the definitions of the measures shall now be introduced. First we note that the linear approximations of $\eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta})$ as a function of $\underset{\sim}{\theta}$, valid in the neighbourhood of $\hat{\underset{\sim}{\theta}}$, can be written as

$$(2.1.1) \qquad \eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta}) = \eta(\underset{\sim}{\xi}_u,\hat{\underset{\sim}{\theta}}) + \sum_{j=1}^{p} c_{uj}t_j + O(t^2),$$

where $\qquad c_{uj} = c_{uj}(\hat{\underset{\sim}{\theta}})$,

$$(2.1.2) \qquad t_j = \theta_j - \hat{\theta}_j$$

and $\qquad t^2 = \sum_{j=1}^{p} t_j^{\,2}.$

The plane tangent to the solution locus at $P(\hat{\underset{\sim}{\theta}})$ is then defined parametrically by

$$(2.1.3) \qquad \eta_u = \eta(\underset{\sim}{\xi}_u,\hat{\underset{\sim}{\theta}}) + \sum_{j=1}^{p} c_{uj}t_j \quad .$$

Let $T(\underset{\sim}{\theta})$ be the point whose coordinates are given by (2.1.3) when $t_j = \theta_j - \hat{\theta}_j$. Further, let $T^*(\underset{\sim}{\theta})$ be the point on the tangent plane such that the line joining $T^*(\underset{\sim}{\theta})$ and $P(\underset{\sim}{\theta})$ is perpendicular to the tangent plane. Next, consider a set of parameter values $\underset{\sim}{\theta}_1,\underset{\sim}{\theta}_2,\ldots,\underset{\sim}{\theta}_W$ near $\hat{\underset{\sim}{\theta}}$. Let t_{wj} $(j = 1,2,\ldots,p)$ be the j^{th} component of $\underset{\sim}{t}_w = \underset{\sim}{\theta}_w - \hat{\underset{\sim}{\theta}}$ and η_{uw}, $\hat{\eta}_u$ $(u = 1,2,\ldots,n)$ be the values $\eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta}_w)$, $\eta(\underset{\sim}{\xi}_u,\hat{\underset{\sim}{\theta}})$ respectively. We can regard the expression

$$(2.1.4)\qquad Q_\theta = \sum_{w=1}^{W} \sum_{u=1}^{n} (\eta_{uw} - \hat{\eta}_u - \sum_{j=1}^{p} c_{uj} t_{wj})^2 = \sum_{w=1}^{W} |P(\underset{\sim}{\theta}_w) - T(\underset{\sim}{\theta}_w)|^2$$

as a crude measure of the total nonlinearity of the model in terms of the parameter vector $\underset{\sim}{\theta}$ in the neighbourhood of $P(\hat{\underset{\sim}{\theta}})$. This Q_θ is essentially the sum of squares of distances from the points $P(\underset{\sim}{\theta}_w)$ in the solution locus to the associated point $T(\underset{\sim}{\theta}_w)$ on the tangent plane at $P(\hat{\underset{\sim}{\theta}})$.

As the measure Q_θ depends on the number of points $P(\underset{\sim}{\theta}_w)$ that one uses and on their distances from $P(\hat{\underset{\sim}{\theta}})$, it is necessary to normalize this measure. In the neighbourhood of $P(\hat{\underset{\sim}{\theta}})$, $\eta_{uw} - \hat{\eta}_u - \sum_{j=1}^{p} c_{uj} t_{wj}$ can be expected to be roughly proportional to the square of the distance of $P(\underset{\sim}{\theta}_w)$ from $P(\hat{\underset{\sim}{\theta}})$ i.e. proportional to $\sum_{u=1}^{n} (\eta_{uw} - \hat{\eta}_u)^2$. So it is natural to divide Q_θ by

$$(2.1.5)\qquad D = \sum_{w=1}^{W} \{ \sum_{u=1}^{n} (\eta_{uw} - \hat{\eta}_u)^2 \}^2 = \sum_{w=1}^{W} |P(\underset{\sim}{\theta}_w) - P(\hat{\underset{\sim}{\theta}})|^4.$$

As Q_θ has the dimensions of the square of an observation, and D has the dimensions of the fourth power of an observation, the quantity

$$(2.1.6)\qquad \hat{N}_\theta = ps^2 Q_\theta / D,$$

where s^2 is an estimate of σ^2, is a dimensionless quantity, and can be regarded as an estimated normalized measure of the total nonlinearity of the model in terms of the parameter vector $\underset{\sim}{\theta}$ in the neighbourhood of $P(\hat{\underset{\sim}{\theta}})$. The reason for the factor p in (2.1.6) is given in Beale (1960).

The empirical measure of nonlinearity $\hat{N}_\theta$ given by (2.1.6) has the theoretical measure of nonlinearity N_θ as its counterpart. In Beale (1960), N_θ is derived from $\hat{N}_\theta$ by altering s^2 to σ^2, and changing the finite set of values of $\underset{\sim}{\theta}_w$ to an infinite set of values of $\underset{\sim}{\theta}$ such that the points $T(\underset{\sim}{\theta})$ have a p-dimensional spherical normal distribution about $P(\hat{\underset{\sim}{\theta}})$ with an arbitrarily small variance. But it now seems preferable to replace the set of points $T(\underset{\sim}{\theta})$ by the points $T^*(\underset{\sim}{\theta})$. This ensures that no transformation of the parameter vector $\underset{\sim}{\theta}$ can change the chosen set of points $P(\underset{\sim}{\theta})$. Both N_θ and $\hat{N}_\theta$ are invariant under any linear transformation of $\underset{\sim}{\theta}$ (for $\hat{N}_\theta$, the values $\underset{\sim}{\theta}_1, \underset{\sim}{\theta}_2, \ldots, \underset{\sim}{\theta}_W$ are held constant in these transformations), and also invariant under any orthogonal transformation of coordinates in sample space.

Now suppose we fix the model and the experimental design and then make arbitrary transformations of $\underset{\sim}{\theta}$, say $\underset{\sim}{\psi} = \underset{\sim}{\psi}(\underset{\sim}{\theta})$. Suppose the minimum value of N_θ under these transformations is attained by using the transformation $\underset{\sim}{\phi} = \underset{\sim}{\phi}(\underset{\sim}{\theta})$, and is denoted by N_ϕ. In Beale (1960), N_ϕ is referred to as the intrinsic nonlinearity of the model in the neighbourhood of $P(\hat{\underset{\sim}{\theta}})$. The geometrical interpretation of N_ϕ is that it is the value of N_θ when the parameter vector $\underset{\sim}{\theta}$ is transformed in such a way that $T(\underset{\sim}{\theta})$ is always at the point $T^*(\underset{\sim}{\theta})$. The difference $N_\theta - N_\phi$ may be called the removable nonlinearity of the model in terms of parameter vector $\underset{\sim}{\theta}$ in the neighbourhood of $P(\hat{\underset{\sim}{\theta}})$.

The theoretical measures of nonlinearity N_θ and N_ϕ are derived in Beale (1960) from the first and second partial derivatives of $\eta(\underset{\sim}{\xi}_u, \underset{\sim}{\theta})$ with respect to the θ_j. These theoretical measures together with the empirical measures were investigated by Guttman and Meeter (1965). These authors concluded that the empirical measures may significantly underestimate the nonlinearity of the model, and are therefore unreliable, but that the corresponding theoretical measures give an indication of the severity of the nonlinearity - although the interpretation of the measures suggested in Beale (1960) is unduly conservative. In latter sections and in Chapter 5 we will continue the investigation of the theoretical measures of nonlinearity.

Section 2.2 Computation of theoretical measures of nonlinearity

A method based on Householder transformations for computing the theoretical measures of nonlinearity for a given model in the neighbourhood of a nonsingular point $P(\hat{\underset{\sim}{\theta}})$ will now be described.

As each $\eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta})$ as a function of $\underset{\sim}{\theta}$ is differentiable up to the second order at $\theta = \hat{\underset{\sim}{\theta}}$, we can obtain second order approximations to $\eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta})$ as a function of $\underset{\sim}{\theta}$, valid in the neighbourhood of $P(\hat{\underset{\sim}{\theta}})$, as

$$(2.2.1) \qquad \eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta}) = \eta(\underset{\sim}{\xi}_u,\hat{\underset{\sim}{\theta}}) + \sum_{j=1}^{p} c_{uj}t_j + \sum_{j=1}^{p}\sum_{k=1}^{p} c_{ujk}t_jt_k + o(t^2),$$

where $c_{ujk} = \frac{1}{2}\left[\frac{\partial^2\eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta})}{\partial\theta_j\partial\theta_k}\right]_{\underset{\sim}{\theta}=\hat{\underset{\sim}{\theta}}}$.

Now let $\underset{\sim}{\eta}$, $\hat{\underset{\sim}{\eta}}$, $\underset{\sim}{c}_j$ and $\underset{\sim}{c}_{jk}$ denote the $(n\times 1)$ vectors whose u^{th} components are $\eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta})$, $\eta(\underset{\sim}{\xi}_u,\hat{\underset{\sim}{\theta}})$, c_{uj} and c_{ujk} respectively. Further, let $\underset{\sim}{C}$ be the $(n\times p)$ matrix $\{c_{uj}\}$, and $\underset{\sim}{H}$ an $(n\times n)$ orthogonal matrix such that $\underset{\sim}{H}\underset{\sim}{C}$ is an upper triangular $(p\times p)$ nonsingular matrix $\underset{\sim}{D}$ with an $((n-p)\times p)$ zero matrix beneath it. $\underset{\sim}{H}$ can be written as a product of p orthogonal $(n\times n)$ matrices $\underset{\sim}{H}^{(p)},\underset{\sim}{H}^{(p-1)},\ldots,\underset{\sim}{H}^{(2)},\underset{\sim}{H}^{(1)}$ corresponding to p Householder transformations. Each $\underset{\sim}{H}^{(j)}$ can be written as

$$\underset{\sim}{H}^{(j)} = \underset{\sim}{I} - [\underset{\sim}{v}^{(j)}][\underset{\sim}{v}^{(j)}]^T ,$$

where the $(n\times 1)$ vectors $\underset{\sim}{v}^{(j)}$ are computed as shown in Appendix 1.

We then apply an orthogonal transformation

$$(2.2.2) \qquad \underset{\sim}{H}(\underset{\sim}{y} - \hat{\underset{\sim}{\eta}}) = \underset{\sim}{z}$$

of coordinates in sample space such that the point $P(\hat{\underset{\sim}{\theta}})$ in the solution locus becomes the new origin $\underset{\sim}{z} = \underset{\sim}{O}$ and the plane tangent to the solution locus at $P(\hat{\theta})$ consists of points for which $z_i = 0$ for $i = p+1, p+2, \ldots, n$. We refer to z as the rotated coordinates of the sample point y.

For a point $P(\theta)$ in the solution locus, the rotated coordinates are given by

$$(2.2.3) \quad z_i = \begin{cases} \sum_{j=i}^{p} d_{ij} t_j + \sum_{j=1}^{p} \sum_{k=1}^{p} d_{ijk} t_j t_k + o(t^2), & (i = 1,2,\ldots,p) \\ \sum_{j=1}^{p} \sum_{k=1}^{p} d_{ijk} t_j t_k + o(t^2), & (i = p+1,p+2,\ldots,n), \end{cases}$$

where d_{ij} and d_{ijk} are the i^{th} components of the $(n \times 1)$ vectors $H c_j$ and $H c_{jk}$ respectively.

If we apply the linear transformation

$$(2.2.4) \quad \tau = Dt$$

of parameter vector t, then (2.2.3) becomes

$$(2.2.5) \quad z_i = \begin{cases} \tau_i + \sum_{\ell=1}^{p} \sum_{m=1}^{p} f_{i\ell m} \tau_\ell \tau_m + o(\tau^2), & (i = 1,2,\ldots,p) \\ \sum_{\ell=1}^{p} \sum_{m=1}^{p} f_{i\ell m} \tau_\ell \tau_m + o(\tau^2), & (i = p+1,p+2,\ldots,n), \end{cases}$$

where $\tau^2 = \sum_{i=1}^{p} \tau_i^2$,

$$(2.2.6) \quad f_{i\ell m} = f_{im\ell} = \sum_{j=1}^{\ell} \sum_{k=1}^{m} d_{ijk} d^{j\ell} d^{km},$$

and $d^{j\ell}$ is the (j,ℓ) entry of the inverse of D.

For the purpose of deriving the measures, we consider that u_i given by

$$u_i = z_i = \tau_i + \sum_{\ell=1}^{p} \sum_{m=1}^{p} f_{i\ell m} \tau_\ell \tau_m + o(\tau^2)$$

are independently normally distributed with mean zero and variance V where V is arbitrarily small. As Q_θ is given by

$$Q_\theta = \sum_{i=1}^{n} \left(\sum_{\ell=1}^{p} \sum_{m=1}^{p} f_{i\ell m} \tau_\ell \tau_m \right)^2 + o(\tau^4)$$

$$= \sum_{i=1}^{n} \left(\sum_{\ell=1}^{p} \sum_{m=1}^{p} f_{i\ell m} u_\ell u_m \right)^2 + o(u^4),$$

where

$$u^4 = \left(\sum_{i=1}^{p} u_i^2 \right)^2,$$

the mean value of Q_θ is given by

$$\bar{Q}_\theta = \int_{u_1=-\infty}^{\infty} \int_{u_2=-\infty}^{\infty} \cdots \int_{u_p=-\infty}^{\infty} \left\{ \sum_{i=1}^{n} \left(\sum_{\ell=1}^{p} \sum_{m=1}^{p} f_{i\ell m} u_\ell u_m \right)^2 + o(u^4) \right\} \left\{ \prod_{j=1}^{p} \frac{1}{\sqrt{2\pi V}} e^{-u_j^2/(2V)} \right\} du_1 du_2 \ldots du_p$$

$$= E_{\underset{\sim}{u}} \left\{ \sum_{i=1}^{n} \left(\sum_{\ell=1}^{p} \sum_{m=1}^{p} f_{i\ell m} u_\ell u_m \right)^2 + o(u^4) \right\}$$

i.e.

$$\text{(2.2.7)} \qquad \bar{Q}_\theta = V^2 \sum_{i=1}^{n} \sum_{\ell=1}^{p} \sum_{m=1}^{p} (f_{i\ell\ell} f_{imm} + 2f_{i\ell m}^2) + o(V^2).$$

Next, as the fourth power of the distance from $P(\underset{\sim}{\theta})$ to $P(\hat{\underset{\sim}{\theta}})$ is given by

$$|P(\underset{\sim}{\theta}) - P(\hat{\underset{\sim}{\theta}})|^4 = \left(\sum_{i=1}^{p} \tau_i^2 \right)^2 + o(\tau^4)$$

$$= \left(\sum_{i=1}^{p} u_i^2 \right)^2 + o(u^4),$$

the mean value of this fourth power is given by

$$\overline{D} = E_{\underset{\sim}{u}}\{(\sum_{i=1}^{p} u_i^2)^2 + o(u^4)\}$$

$$= p(p+2)V^2 + o(V^2).$$

Then we see that

$$(2.2.8) \qquad N_\theta = \frac{\sigma^2}{p+2} \sum_{i=1}^{n} \sum_{\ell=1}^{p} \sum_{m=1}^{p} (f_{i\ell\ell} f_{imm} + 2f_{i\ell m}^2).$$

We next derive N_ϕ. As N_θ is based only on second order approximations of $\eta(\underset{\sim}{\xi}_u, \underset{\sim}{\theta})$ as functions of $\underset{\sim}{\theta}$, valid in the neighbourhood of $\hat{\underset{\sim}{\theta}}$, the terms of order higher than two in the expansions of the arbitrary transformations $\psi_i = \psi_i(\underset{\sim}{\theta})$, valid in the neighbourhood of $\hat{\underset{\sim}{\theta}}$, are not relevant as far as reduction of total nonlinearity by means of the transformation $\underset{\sim}{\psi}$ is concerned. Therefore, in general, we can write ψ_i as

$$\psi_i = \psi_{i0} + \sum_{j=1}^{p} \left(\frac{\partial \psi_i}{\partial \theta_j}\right)_{\underset{\sim}{\theta}=\hat{\underset{\sim}{\theta}}} t_j + \frac{1}{2} \sum_{j=1}^{p} \sum_{k=1}^{p} \left(\frac{\partial^2 \psi_i}{\partial \theta_j \partial \theta_k}\right)_{\underset{\sim}{\theta}=\hat{\underset{\sim}{\theta}}} t_j t_k + o(t^2).$$

As N_θ is not changed by any linear transformation of parameter vector, we can restrict our attention to $\underset{\sim}{\psi}$ such that

$$\psi_{io} = 0$$

and

$$\left(\frac{\partial \psi_i}{\partial \theta_j}\right)_{\underset{\sim}{\theta}=\hat{\underset{\sim}{\theta}}} = d_{ij} \ .$$

Then ψ_i can be written as

$$\psi_i = \sum_{j=i}^{p} d_{ij} t_j + \frac{1}{2} \sum_{j=1}^{p} \sum_{k=1}^{p} \left(\frac{\partial^2 \psi_i}{\partial \theta_j \partial \theta_k}\right)_{\underset{\sim}{\theta}=\hat{\underset{\sim}{\theta}}} t_j t_k + o(t^2) \quad .$$

After applying the transformation $\underset{\sim}{\tau} = \underset{\sim}{D}\underset{\sim}{t}$, we can write ψ_i as

$$(2.2.9)\qquad \psi_i = \tau_i - \sum_{\ell=1}^{p}\sum_{m=1}^{p} g_{i\ell m}\tau_\ell\tau_m + o(\tau^2),$$

where $g_{i\ell m} = g_{im\ell}$.

If we apply this transformation, then (2.2.5) becomes

$$(2.2.10)\qquad z_i = \begin{cases} \psi_i + \sum_{\ell=1}^{p}\sum_{m=1}^{p} (f_{i\ell m} + g_{i\ell m})\psi_\ell\psi_m + o(\psi^2), & (i = 1,2,\ldots,p) \\ \sum_{\ell=1}^{p}\sum_{m=1}^{p} f_{i\ell m}\psi_\ell\psi_m + o(\psi^2), & (i = p+1,p+2,\ldots,n) \end{cases}$$

where $\psi^2 = \sum_{i=1}^{p} \psi_i^{\,2}$.

The theoretical measure of total nonlinearity in terms of parameter vector $\underset{\sim}{\psi}$ in the neighbourhood of $P(\hat{\underset{\sim}{\theta}})$ is

$$(2.2.11)\qquad N_\psi = \frac{\sigma^2}{p+2}\left\{ \sum_{i=1}^{p}\sum_{\ell=1}^{p}\sum_{m=1}^{p} [(f_{i\ell\ell}+g_{i\ell\ell})(f_{imm}+g_{imm}) + 2(f_{i\ell m}+g_{i\ell m})^2] + \sum_{i=p+1}^{n}\sum_{\ell=1}^{p}\sum_{m=1}^{p} [f_{i\ell\ell}f_{imm}+2f_{i\ell m}^2]\right\}.$$

Minimizing N_ψ with respect to $g_{i\ell m}$ $(i,\ell,m = 1,2,\ldots,p)$, we obtain the theoretical measure of intrinsic nonlinearity in the neighbourhood of $P(\hat{\underset{\sim}{\theta}})$ as

$$(2.2.12)\qquad N_\phi = \frac{\sigma^2}{p+2} \sum_{i=p+1}^{n}\sum_{\ell=1}^{p}\sum_{m=1}^{p} (f_{i\ell\ell}f_{imm} + 2f_{i\ell m}^2).$$

The theoretical measure of removable nonlinearity in terms of parameter $\underset{\sim}{\theta}$ in the neighbourhood of $P(\hat{\underset{\sim}{\theta}})$ is now given by

$$(2.2.13)\quad N_\theta - N_\phi = \frac{\sigma^2}{p+2} \sum_{i=1}^{p} \sum_{\ell=1}^{p} \sum_{m=1}^{p} (f_{i\ell\ell} f_{imm} + 2f_{i\ell m}^2).$$

If we set $g_{p\ell m}$ ($\ell,m = 1,2,\ldots,p$) in (2.2.9) to be zeros so that the last component ψ_p of the transformed parameter vector $\underset{\sim}{\psi}$ is a linear function of θ_p, and choose $g_{i\ell m}$ ($i = 1,2,\ldots,p-1$; $\ell,m = 1,2,\ldots,p$) such that the total nonlinearity is minimized, then the resulting minimum value of the total nonlinearity may be called the measure of nonlinearity associated with θ_p in the neighbourhood of $P(\hat{\underset{\sim}{\theta}})$ and is given by

$$(2.2.14)\quad N_{\theta_p} = \frac{\sigma^2}{p+2} \sum_{i=p}^{n} \sum_{\ell=1}^{p} \sum_{m=1}^{p} (f_{i\ell\ell} f_{imm} + 2f_{i\ell m}^2).$$

By permuting the positions of the components of $\underset{\sim}{\theta}$, each component θ_i ($i = 1,2,\ldots,p-1$) can be in the last position and N_{θ_i} ($i = 1,2,\ldots,p-1$) can be obtained in a similar way as N_{θ_p}.

Section 2.3 Significance of measures of nonlinearity

We now investigate how the total nonlinearity in terms of the parameter vector $\underset{\sim}{\psi}$, where $\underset{\sim}{\psi}$ is either the original or the transformed parameter vector, can be used to conclude, or to indicate, that the model is approximately linear in the parameter vector $\underset{\sim}{\psi}$.

Suppose σ^2 is known and all $P(\underset{\sim}{\theta})$ are nonsingular points. Further, let $N_{\psi_{crit}}$ be the value such that if N_ψ at the true value $\underset{\sim}{\psi}_T$ of $\underset{\sim}{\psi}$ is less than $N_{\psi_{crit}}$, then the use of linear theory results as approximations is justified. Now if all N_ψ evaluated at feasible $\underset{\sim}{\psi}$ is less than $N_{\psi_{max}}$, then N_ψ at $\underset{\sim}{\psi}_T$ is less than $N_{\psi_{max}}$. Thus if $N_{\psi_{max}}$ is less than $N_{\psi_{crit}}$, then we can conclude that the model is approximately linear in the parameter vector $\underset{\sim}{\psi}$. The problem of finding this $N_{\psi_{crit}}$ will be considered in Chapter 5.

In practice, we may use N_ψ evaluated at the least squares estimate $\hat{\underset{\sim}{\psi}}$ of $\underset{\sim}{\psi}$, and $\sigma^2 = s^2$ to estimate N_ψ at $\underset{\sim}{\psi}_T$. Information concerning the reliability of this estimate may be derived from N_ψ evaluated at feasible $\underset{\sim}{\psi}$ in the neighbourhood of $\hat{\underset{\sim}{\psi}}$, and σ^2 in the neighbourhood of s^2. Suppose this estimate is small enough and is reliable. Then it is plausible to believe that linear theory results can be applied, with negligible errors, to make inference about $\underset{\sim}{\psi}$. The estimation of N_ψ at $\underset{\sim}{\psi}_T$ when $\underset{\sim}{\psi}$ is the original parameter vector $\underset{\sim}{\theta}$ will be investigated in Chapter 5.

Section 2.4 Reduction of nonlinearity for inference purposes

The total nonlinearity N_θ in the neighbourhood of $P(\hat{\underset{\sim}{\theta}})$ may be large but the corresponding N_ϕ of intrinsic nonlinearity may be fairly small. In these circumstances, we can apply nonlinear transformation of the parameter vector $\underset{\sim}{\theta}$ to reduce the total nonlinearity. The choice of transformation depends on the type of inference that we want to make about $\underset{\sim}{\theta}$. If we want to obtain region estimate for the last k* (k* = 1,2,...,p) components of the parameter vector $\underset{\sim}{\theta}$, then it is convenient to use a one to one transformation $\underset{\sim}{\gamma} = \underset{\sim}{\gamma}(\underset{\sim}{t})$ of the form

(2.4.1) $\gamma_i = \gamma_i(\underset{\sim}{t})$, (i = 1,2,...,p) in the case when k* = p,

and in the case when k* < p,

(2.4.2) $\gamma_i = \gamma_i(\underset{\sim}{t})$, (i = 1,2,...,p-k*),

(2.4.3) $\gamma_i = \gamma_i(t_{p-k^*+1}, t_{p-k^*+2}, \ldots, t_p)$, (i = p-k*+1,p-k*+2,...,p),

where γ_i are differentiable up to the second order.

We choose $\underset{\sim}{\gamma} = \underset{\sim}{\gamma}(\underset{\sim}{t})$ to be such that the nonlinearity N_γ in terms of the transformed parameter $\underset{\sim}{\gamma}$ is minimized. For reasons similar to those given in the derivation of N_ϕ in section 2.2, we can restrict our attention to γ_i of the form

$$(2.4.4) \qquad \gamma_i = t_i - \sum_{\ell=1}^{p} \sum_{m=1}^{p} g_{i\ell m} t_\ell t_m + o(t^2),$$

where $g_{i\ell m} = g_{im\ell}$. Suppose $g_{i\ell m} = g_{i\ell m}^{(k*)}$ are such that the corresponding $\underset{\sim}{\gamma}$ achieves maximum reduction of nonlinearity. Then we refer to $g_{i\ell m}^{(k*)}$ as the optimal $g_{i\ell m}$. To find $g_{i\ell m}^{(k*)}$, we first write (2.4.4) as

$$(2.4.5) \qquad t_i = \gamma_i + \sum_{\ell=1}^{p} \sum_{m=1}^{p} g_{i\ell m} \gamma_\ell \gamma_m + o(\gamma^2),$$

where $\qquad \gamma^2 = \sum_{i=1}^{p} \gamma_i^2.$

Substituting these expressions for t_i into (2.2.3), we obtain

$$(2.4.6) \qquad z_i = \begin{cases} \sum_{j=i}^{p} d_{ij}\gamma_j + \sum_{j=1}^{p} \sum_{k=1}^{p} d_{ijk}^{+}\gamma_j\gamma_k + o(\gamma^2), & (i = 1,2,\ldots,p) \\ \sum_{j=1}^{p} \sum_{k=1}^{p} d_{ijk}^{+}\gamma_j\gamma_k + o(\gamma^2), & (i = p+1,p+2,\ldots,n), \end{cases}$$

where

$$(2.4.7) \qquad d_{ijk}^{+} = \begin{cases} d_{ijk} + \sum_{\ell=i}^{p} d_{i\ell} g_{\ell jk}, & (i = 1,2,\ldots,p) \\ d_{ijk}, & (i = p+1,p+2,\ldots,n). \end{cases}$$

We now apply the transformation $\underset{\sim}{\tau}^{+} = \underset{\sim}{D}\underset{\sim}{\gamma}$ so that (2.4.6) becomes

$$(2.4.8)\quad z_i = \begin{cases} \tau_i^+ + \sum_{\ell=1}^{p}\sum_{m=1}^{p} f_{i\ell m}^+ \tau_\ell^+ \tau_m^+ + o((\tau^+)^2), & (i = 1,2,\dots,p) \\ \sum_{\ell=1}^{p}\sum_{m=1}^{p} f_{i\ell m}^+ \tau_\ell^+ \tau_m^+ + o((\tau^+)^2), & (i = p+1,p+2,\dots,n), \end{cases}$$

where

$$(2.4.9)\quad f_{i\ell m}^+ = \sum_{j=1}^{\ell}\sum_{k=1}^{m} d_{ijk}^+ d^{j\ell} d^{km},$$

and

$$(\tau^+)^2 = \sum_{i=1}^{p} (\tau_i^+)^2.$$

The total nonlinearity in terms of parameter vector $\underset{\sim}{\gamma}$ in the neighbourhood of $P(\hat{\underset{\sim}{\theta}})$ is then given by

$$(2.4.10)\quad N_\gamma = \frac{\sigma^2}{p+2} \sum_{i=1}^{n}\sum_{\ell=1}^{p}\sum_{m=1}^{p} [f_{i\ell\ell}^+ f_{imm}^+ + 2(f_{i\ell m}^+)^2].$$

N_γ is now minimized with respect to g_{ijk} $(i,j,k = 1,2,\dots,p)$. It can be shown that if $k^* = p$, $g_{ijk}^{(p)}$ are given by

$$(2.4.11)\quad d_{ijk} + \sum_{\ell=i}^{p} d_{i\ell} g_{\ell jk}^{(p)} = 0, \quad (i,j,k = 1,2,\dots,p),$$

and if $k^* < p$, $g_{ijk}^{(k^*)}$ are given by

$$(2.4.12)\quad g_{ijk}^{(k^*)} = \begin{cases} \sum_{\ell=p-k^*+1}^{j}\sum_{m=p-k^*+1}^{k} s_{i\ell m} d_{\ell j} d_{mk}, & (i,j,k=p-k^*+1,p-k^*+2,\dots,p) \\ 0, & (i=p-k^*+1,p-k^*+2,\dots,p; \\ & j,k=1,2,\dots,p-k^*), \end{cases}$$

$$(2.4.13)\quad d_{ijk} + \sum_{\ell=i}^{p} d_{i\ell} g_{\ell jk}^{(k^*)} = 0, \quad (i = 1,2,\ldots,p-k^*;\ j,k = 1,2,\ldots,p),$$

$$(2.4.14)\quad 2(f_{ijj} + \sum_{k=i}^{p} d_{ik} s_{kjj}) + \sum_{\ell=1}^{p} f_{i\ell\ell} + \sum_{m=p-k^*+1}^{p} \sum_{k=i}^{p} d_{ik} s_{kmm} = 0,$$

$$(i,j = p-k^*+1, p-k^*+2, \ldots, p),$$

and

$$(2.4.15)\quad f_{i\ell m} + \sum_{k=i}^{p} d_{ik} s_{k\ell m} = 0, \quad (i,\ell,m = p-k^*+1, p-k^*+2, \ldots, p \text{ and } \ell \neq m).$$

We note that if $k^* = 1$, then from (2.4.12) and (2.4.14), we obtain

$$(2.4.16)\quad g_{ppp}^{(1)} = -\frac{d^{pp}}{3}\left(\sum_{m=1}^{p-1} f_{pmm} + 3f_{ppp}\right),$$

and the nonlinearity associated with γ_p in the neighbourhood of $P(\hat{\underset{\sim}{\theta}})$ is given by

$$(2.4.17)\quad N_{\gamma_p} = \frac{\sigma^2}{p+2}\left[\frac{2}{3}\left(\sum_{\ell=1}^{p-1} f_{p\ell\ell}\right)^2 + 2\sum_{\ell=1}^{p} \sum_{m=1}^{p} f_{p\ell m}^2 - 2f_{ppp}^2\right.$$

$$\left. + \sum_{i=p+1}^{n} \sum_{\ell=1}^{p} \sum_{m=1}^{p} (f_{i\ell\ell} f_{imm} + 2f_{i\ell m}^2)\right].$$

Now the task of finding $\underset{\sim}{\gamma}$ will be complete if the terms of order higher than two in (2.4.4) can be found. For the case when $k^* = 1$ and θ_p is non-negative, it is convenient to use a power transformation of the form

$$(2.4.18)\qquad \psi_p = \begin{cases} \dfrac{\theta_p^{\lambda_p} - 1}{\lambda_p} & \text{if } \lambda_p \neq 0 \\ \ln \theta_p & \text{if } \lambda_p = 0, \end{cases}$$

where the parameter of transformation λ_p is defined in terms of $g^{(1)}_{ppp}$ by the equation

$$(2.4.19)\qquad \lambda_p = 1 - 2g^{(1)}_{ppp}\hat{\theta}_p .$$

We need a further linear transformation to derive γ_p from ψ_p. The parameters γ_i for $i < p$ are yet not found. But as far as finding an interval estimate for θ_p is concerned, we need not find these γ_i explicitly. In fact an interval estimate of θ_p is readily seen to be the set of feasible values of θ_p which lie in the following interval:

$$(2.4.20)\qquad \begin{cases} [\hat{\theta}_p^{\lambda_p} - k_\alpha |\lambda_p \hat{\theta}_p^{\lambda_p - 1}| \sigma_{\theta_p}]^{1/\lambda_p} \leq \theta_p \leq [\hat{\theta}_p^{\lambda_p} + k_\alpha |\lambda_p \hat{\theta}_p^{\lambda_p - 1}| \sigma_{\theta_p}]^{1/\lambda_p}, & (\lambda_p \neq 0;\ \sigma^2 \text{ is known}) \\ \exp[\ln \hat{\theta}_p - k_\alpha \sigma_{\theta_p}/\hat{\theta}_p] \leq \theta_p \leq \exp[\ln \hat{\theta}_p + k_\alpha \sigma_{\theta_p}/\hat{\theta}_p], & (\lambda_p = 0;\ \sigma^2 \text{ is known}) \end{cases}$$

or

$$(2.4.21)\qquad \begin{cases} [\hat{\theta}_p^{\lambda_p} - t_\alpha(n-p) |\lambda_p \hat{\theta}_p^{\lambda_p - 1}| \hat{\sigma}_{\theta_p}]^{1/\lambda_p} \leq \theta_p \leq [\hat{\theta}_p^{\lambda_p} + t_\alpha(n-p) |\lambda_p \hat{\theta}_p^{\lambda_p - 1}| \hat{\sigma}_{\theta_p}]^{1/\lambda_p}, & (\lambda_p \neq 0;\ \sigma^2 \text{ is unknown}) \\ \exp[\ln \hat{\theta}_p - t_\alpha(n-p)\hat{\sigma}_{\theta_p}/\hat{\theta}_p] \leq \theta_p \leq \exp[\ln \hat{\theta}_p + t_\alpha(n-p)\hat{\sigma}_{\theta_p}/\hat{\theta}_p], & (\lambda_p = 0;\ \sigma^2 \text{ is unknown}), \end{cases}$$

where

$$\sigma_{\theta_p} = |d^{pp}|\sigma,$$

$$\hat{\sigma}_{\theta_p} = |d^{pp}| \sqrt{S(\hat{\underset{\sim}{\theta}})/(n-p)} \quad ,$$

k_α is the $100(1 - \frac{1}{2}\alpha)$ percentage point of a standard normal distribution, and $t_\alpha(n-p)$ is the $100(1 - \frac{1}{2}\alpha)$ percentage point of a t-distribution with n-p degrees of freedom.

For the case when the θ_i are not non-negative, the terms of order higher than two in (2.4.4) are not yet found. However if we are satisfied with the second order approximations of $\underset{\sim}{\gamma}$, then a region estimate of $\theta_{p-k*+1}, \theta_{p-k*+2}, \ldots, \theta_p$ is seen to be the set of feasible values of these components which lie in the following region:

$$\sum_{i=p-k*+1}^{p} \{ \sum_{j=i}^{p} d_{ij} [(\theta_j - \hat{\theta}_j) - \sum_{\ell=p-k*+1}^{p} \sum_{m=p-k*+1}^{p} g_{j\ell m}^{(k*)} (\theta_\ell - \hat{\theta}_\ell)(\theta_m - \hat{\theta}_m)]\}^2$$

$$\text{(2.4.22)} \qquad \leq \sigma^2 \chi^2_{k*,\alpha} \qquad \text{if } \sigma^2 \text{ is known,}$$

or

$$\sum_{i=p-k*+1}^{p} \{ \sum_{j=i}^{p} d_{ij} [(\theta_j - \hat{\theta}_j) - \sum_{\ell=p-k*+1}^{p} \sum_{m=p-k*+1}^{p} g_{j\ell m}^{(k*)} (\theta_\ell - \hat{\theta}_\ell)(\theta_m - \hat{\theta}_m)]\}^2$$

$$\text{(2.4.23)} \qquad \leq \frac{k*}{n-p} S(\hat{\underset{\sim}{\theta}}) F_\alpha(k*, n-p) \qquad \text{if } \sigma^2 \text{ is unknown,}$$

where

$\chi^2_{k*,\alpha}$ is the $100(1-\alpha)$ percentage point of a χ^2-distribution with k* degrees of freedom

and

$F_\alpha(k^*,n-p)$ is the $100(1-\alpha)$ percentage point of an F-distribution with k^* and $n-p$ degrees of freedom.

Whether an interval or region estimate derived in this section will cover the true values of the corresponding components of $\underset{\sim}{\theta}$ with the nominal probability $(1-\alpha)$ depends on the adequacy of the approximations that the model is linear in the corresponding $\underset{\sim}{\gamma}$.

In Chapter 5 we shall investigate the estimates in this section by means of some numerical examples.

Section 2.5 Region estimate of a different subset of components of the parameter vector

Suppose now we are interested in the region estimate of a different subset of k^* $(1 \leq k^* < p)$ components $\theta_{i_1},\theta_{i_2},\ldots,\theta_{i_{k^*}}$. For any component for which i_s does not satisfy $p-k^*+1 \leq i_s \leq p$, we interchange its position in the vector $(\theta_1,\theta_2,\ldots,\theta_p)^T$ with another component θ_j (where $p-k^*+1 \leq j \leq p$) which are not of interest so that in the resulting vector, $\theta_{i_1},\theta_{i_2},\ldots,\theta_{i_{k^*}}$ form the last k^* components, and the methods in section 2.4 for obtaining region estimate can be applied. A method for doing the above interchanging of the position of θ_{i_s} with θ_j will now be described.

Define

$$(2.5.1) \qquad \underset{\sim}{H}^{[i_s]} = \underset{\sim}{I} - [\underset{\sim}{v}^{[i_s]}][\underset{\sim}{v}^{[i_s]}]^T$$

to be a Householder transformation such that

$$\underset{\sim}{H}^{[i_s]} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ d_{i_s\ i_s+1} \\ d_{i_s+1\ i_s+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is a column vector whose only nonzero entry is at i_s^{th} position.

Let $\underset{\sim}{z}^{(i_s)}$ be given by

$$(2.5.2)\quad z_i^{(i_s)} = \begin{cases} \sum_{j=i}^{p} d_{ij} t_j + \sum_{j=1}^{p} \sum_{k=1}^{p} d_{ijk} t_j t_k + o(t^2), & (i = 1,2,\ldots,p) \\ \sum_{j=1}^{p} \sum_{k=1}^{p} d_{ijk} t_j t_k + o(t^2), & (i = p+1,p+2,\ldots,n) \end{cases}$$

If we apply the transformation

$$\underset{\sim}{H}^{[i_s]} \underset{\sim}{z}^{(i_s)} = \underset{\sim}{z}^{(i_s+1)}$$

of coordinates in sample space, then (2.5.2) becomes

$$(2.5.3)\quad z_i^{(i_s+1)} = \begin{cases} \sum_{j=i}^{p} d_{ij}^{(i_s+1)} t_j^{(i_s+1)} + \sum_{j=1}^{p} \sum_{k=1}^{p} d_{ijk}^{(i_s+1)} t_j^{(i_s+1)} t_k^{(i_s+1)} + o(t^2), & (i = 1,2,\ldots,p), \\ \sum_{j=1}^{p} \sum_{k=1}^{p} d_{ijk}^{(i_s+1)} t_j^{(i_s+1)} t_k^{(i_s+1)} + o(t^2), & (i = p+1,p+2,\ldots,n), \end{cases}$$

where

$$(2.5.4) \qquad t_j^{(i_s+1)} = \begin{cases} t_{i_s+1}, & j = i_s \\ t_{i_s}, & j = i_s+1 \\ t_j, & \text{otherwise.} \end{cases}$$

We note that an effect of this transformation is to interchange the position of t_{i_s} with the next component t_{i_s+1} in $(t_1\ t_2 \ldots t_{i_s}\ t_{i_s+1} \ldots t_p)$.

We then repeat the above process until t_{i_s} is at the j^{th} position.

Section 2.6 Conditions for the existence of the power transformation which achieves maximum reduction of the nonlinearity associated with an individual parameter

In section 2.4 we have shown that the power transformation given by (2.4.18) achieves maximum reduction of the nonlinearity associated with the p^{th} parameter in the neighbourhood of a nonsingular point $P(\hat{\underset{\sim}{\theta}})$. And in section 2.5 we have discussed how the corresponding power transformation for reducing nonlinearity associated with the i^{th} parameter, where $i < p$, can be found. As for some models, e.g. models (A) and (B) in Chapter 1, the solution loci are bounded, it is of interest to investigate whether these power transformations for reducing the non-linearity in the neighbourhood of $P(\underset{\sim}{\theta})$ will exist as the parameter vector $\underset{\sim}{\theta}$ tends to a value which may correspond to a point on a boundary of a solution locus.

Let $\{\underset{\sim}{\theta}_m\}$ be a sequence of feasible value of the parameter vector converging to $\underset{\sim}{\theta}_B$. It can be shown that the following conditions are a set of sufficient conditions for the existence of $\lambda_i = \lambda_i(\underset{\sim}{\theta}_m)$ of the power transformation which achieves maximum reduction of the nonlinearity associated with the i^{th} parameter in the neighbourhood of $P(\underset{\sim}{\theta}_m)$, as m tends to infinity:

[1] the $c_{uk}(\underset{\sim}{\theta}_m)$ can be expressed as products of two functions of $\underset{\sim}{\theta}_m$ as follows:

$$c_{uk}(\underset{\sim}{\theta}_m) = c^*_{uk}(\underset{\sim}{\theta}_m)h_k(\underset{\sim}{\theta}_m) \;, \quad (u = 1,2,\ldots,n;\ k = 1,2,\ldots,p),$$

where $c^*_{uk}(\underset{\sim}{\theta}_m)$ are functions with

$$\lim_{m\to\infty} c^*_{uk}(\underset{\sim}{\theta}_m) = c_{uk},$$

and the c_{uk} are finite numbers,

[2] the (n×p) matrix $\underset{\sim}{C} = \{c_{uk}\}$ is of rank p,

[3] $\lim_{m\to\infty} \dfrac{c_{ujk}(\underset{\sim}{\theta}_m)}{h_j(\underset{\sim}{\theta}_m)h_k(\underset{\sim}{\theta}_m)} h_i(\underset{\sim}{\theta}_m)\theta_{mi} = q_{uijk}, \quad (u = 1,2,\ldots,n;\ j,k = 1,2,\ldots,p),$

where $c_{ujk}(\underset{\sim}{\theta}_m) = \frac{1}{2}\left[\dfrac{\partial^2\eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta})}{\partial\theta_j\partial\theta_k}\right]_{\underset{\sim}{\theta}=\underset{\sim}{\theta}_m}$,

θ_{mi} is the i^{th} component of $\underset{\sim}{\theta}_m$,

and the q_{uijk} are finite numbers.

To show that these are the sufficient conditions, we first let $\underset{\sim}{H}_m$ be the (n×n) matrix such that $\underset{\sim}{H}_m\underset{\sim}{C}_m$, where $\underset{\sim}{C}_m = \{c_{uk}(\underset{\sim}{\theta}_m)\}$, is a (p×p) upper triangular matrix $\underset{\sim}{D}_m = \{d_{ik}(\underset{\sim}{\theta}_m)\}$ with an [(n-p)×p] zero matrix beneath it (c.f. (2.2.2)). $\underset{\sim}{H}_m$ can be written as

$$\underset{\sim}{H}_m = \prod_{j=1}^{p} (\underset{\sim}{I} - [\underset{\sim}{v}^{(p+1-j)}(\underset{\sim}{\theta}_m)][\underset{\sim}{v}^{(p+1-j)}(\underset{\sim}{\theta}_m)]^T) \;,$$

where $\underset{\sim}{v}^{(j)}(\underset{\sim}{\theta}_m)$ are (n×1) vectors whose u^{th} components are $v_u^{(j)}(\underset{\sim}{\theta}_m)$. We note that (c.f. Appendix 1)

$$v_1^{(1)}(\underset{\sim}{\theta}_m) = \sqrt{1 + \frac{c_{11}(\underset{\sim}{\theta}_m)}{r_1(\underset{\sim}{\theta}_m)}}$$

and

$$v_u^{(1)}(\underset{\sim}{\theta}_m) = \frac{c_{u1}(\underset{\sim}{\theta}_m)}{v_1^{(1)}(\underset{\sim}{\theta}_m) r_1(\underset{\sim}{\theta}_m)} , \quad (u = 2,3,\ldots,n),$$

where

$$r_1(\underset{\sim}{\theta}_m) = \sqrt{\sum_{u=1}^{n} [c_{u1}(\underset{\sim}{\theta}_m)]^2}$$

is chosen to have the same sign as $c_{11}(\underset{\sim}{\theta}_m)$. Then because of condition [1], we have

$$\lim_{m\to\infty} v_1^{(1)}(\underset{\sim}{\theta}_m) = \sqrt{1 + \frac{c_{11}}{r_1}} \tag{2.6.1}$$

and

$$\lim_{m\to\infty} v_u^{(1)}(\underset{\sim}{\theta}_m) = \frac{c_{u1}}{\sqrt{1 + \frac{c_{11}}{r_1}}\, r_1} , \quad (u = 2,3,\ldots,n), \tag{2.6.2}$$

where

$$r_1 = \sqrt{\sum_{u=1}^{n} c_{u1}^2} .$$

$r_1 \neq 0$ because $\underset{\sim}{C}$ is of rank p. Therefore the limits in (2.6.1) and (2.6.2) are finite. Similarly because of conditions [1] and [2], $\lim_{m\to\infty} v_u^{(j)}(\underset{\sim}{\theta}_m)$ for $j = 2,3,\ldots,p$ and $u = 1,2,\ldots,n$ are finite.

Next we have

$$d_{11}(\underset{\sim}{\theta}_m) = c_{11}(\underset{\sim}{\theta}_m) - v_1^{(1)}(\underset{\sim}{\theta}_m) \sum_{u=1}^{n} c_{u1}(\underset{\sim}{\theta}_m) v_u^{(1)}(\underset{\sim}{\theta}_m).$$

This together with (2.6.1), (2.6.2) and condition [1] imply that $d_{11}(\underset{\sim}{\theta}_m)$ can be expressed in the form

$$(2.6.3) \qquad d_{11}(\underset{\sim}{\theta}_m) = d^*_{11}(\underset{\sim}{\theta}_m) h_1(\underset{\sim}{\theta}_m),$$

where $d_{11}(\underset{\sim}{\theta}_m)$ is a function with

$$(2.6.4) \qquad \lim_{m\to\infty} d^*_{11}(\underset{\sim}{\theta}_m) = d_{11}$$

and d_{11} is a finite number. Similarly it can be shown that $d_{ik}(\underset{\sim}{\theta}_m)$ can be expressed in the form

$$(2.6.5) \qquad d_{ik}(\underset{\sim}{\theta}_m) = d^*_{ik}(\underset{\sim}{\theta}_m) h_k(\underset{\sim}{\theta}_m), \quad (i = 1,2,\ldots,p,\ k = 2,3,\ldots,p, \text{ and } i \leq k),$$

where $d^*_{ik}(\underset{\sim}{\theta}_m)$ are functions with

$$(2.6.6) \qquad \lim_{m\to\infty} d^*_{ik}(\underset{\sim}{\theta}_m) = d_{ik}$$

and the d_{ik} are finite numbers. Further because of [2], $\underset{\sim}{D}$ is nonsingular and consequently $d_{ii} \neq 0$ for all $i = 1,2,\ldots,p$.

Now for any $\underset{\sim}{\theta}_m$ such that $\underset{\sim}{C}_m$ is of rank p, $\underset{\sim}{D}_m$ is non-singular and its inverse $\underset{\sim}{D}_m^{-1} = \{d^{ik}(\underset{\sim}{\theta}_m)\}$ is given by

$$d^{ii}(\underset{\sim}{\theta}_m) = \frac{1}{d_{ii}(\underset{\sim}{\theta}_m)}, \quad (i = 1,2,\ldots,p),$$

and

$$d^{ik}(\underset{\sim}{\theta}_m) = -\frac{1}{d_{ii}(\underset{\sim}{\theta}_m)} \sum_{j=i}^{k} d_{ij}(\underset{\sim}{\theta}_m) d^{jk}(\underset{\sim}{\theta}_m), \quad (i = 1,2,\ldots,p-1,\ k = 2,3,\ldots,p \text{ and } i < k).$$

Then because of (2.6.3) and (2.6.4), $d^{ik}(\underset{\sim}{\theta}_m)$ can be expressed in the form

$$(2.6.7) \qquad d^{ik}(\underset{\sim}{\theta}_m) = d^{+}_{ik}(\underset{\sim}{\theta}_m)[h_i(\underset{\sim}{\theta}_m)]^{-1}, \quad (i,k = 1,2,\ldots,p),$$

where $d^{+}_{ik}(\underset{\sim}{\theta}_m)$ are functions with

$$(2.6.8) \qquad \lim_{m\to\infty} d^{+}_{ik}(\underset{\sim}{\theta}_m) = s_{ik}$$

and the s_{ik} are finite numbers.

Define

$$\underset{\sim}{d}_{jk}(\underset{\sim}{\theta}_m) = \underset{\sim}{H}_m \underset{\sim}{c}_{jk}(\underset{\sim}{\theta}_m), \quad (j,k = 1,2,\ldots,p)$$

where $\underset{\sim}{c}_{jk}(\underset{\sim}{\theta}_m)$ is the (n×1) vector whose u^{th} component is $c_{ujk}(\underset{\sim}{\theta}_m)$. Then because $\lim_{m\to\infty} v_u^{(j)}(\underset{\sim}{\theta}_m)$ are finite, the i^{th} component of $\underset{\sim}{d}_{jk}(\underset{\sim}{\theta}_m)$ can be expressed in the form

$$(2.6.9) \qquad d_{ijk}(\underset{\sim}{\theta}_m) = \sum_{u=1}^{n} v_{ijku}(\underset{\sim}{\theta}_m) c_{ujk}(\underset{\sim}{\theta}_m),$$

where $v_{ijku}(\underset{\sim}{\theta}_m)$ are functions with

$$\lim_{m\to\infty} v_{ijku}(\underset{\sim}{\theta}_m) = v_{ijku} \tag{2.6.10}$$

and the v_{ijku} are finite numbers.

Applying the method in section 2.4 for finding λ_p at $\hat{\underset{\sim}{\theta}}$ to find the corresponding $\lambda_p(\underset{\sim}{\theta}_m)$ at $\underset{\sim}{\theta}_m$ for which $\underset{\sim}{C}_m$ is of rank p, we obtain

$$\begin{aligned}\lambda_p(\underset{\sim}{\theta}_m) = 1 + \frac{2}{3} d_{pp}(\underset{\sim}{\theta}_m) \Bigg[& \sum_{\ell=1}^{p-1} \sum_{j=1}^{\ell} \sum_{k=1}^{\ell} d_{pjk}(\underset{\sim}{\theta}_m) d^{j\ell}(\underset{\sim}{\theta}_m) d^{k\ell}(\underset{\sim}{\theta}_m) \\ & + 3 \sum_{j=1}^{p} \sum_{k=1}^{p} d_{pjk}(\underset{\sim}{\theta}_m) d^{jp}(\underset{\sim}{\theta}_m) d^{kp}(\underset{\sim}{\theta}_m) \Bigg] \theta_{mp}.\end{aligned} \tag{2.6.11}$$

Then by using (2.6.3)-(2.6.10), we obtain

$$\begin{aligned}\lim_{m\to\infty} \lambda_p(\underset{\sim}{\theta}_m) = 1 + \frac{2}{3} d_{pp} \lim_{m\to\infty} \Bigg\{ h_p(\underset{\sim}{\theta}_m) \Bigg[& \sum_{\ell=1}^{p-1} \sum_{j=1}^{\ell} \sum_{k=1}^{\ell} \sum_{u=1}^{n} v_{pjku} c_{ujk}(\underset{\sim}{\theta}_m) \\ \times \frac{s_{j\ell}}{h_j(\underset{\sim}{\theta}_m)} \frac{s_{k\ell}}{h_k(\underset{\sim}{\theta}_m)} \theta_{mp} & + 3 \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{u=1}^{n} v_{pjku} c_{ujk}(\underset{\sim}{\theta}_m) \frac{s_{jp}}{h_j(\underset{\sim}{\theta}_m)} \frac{s_{kp}}{h_k(\underset{\sim}{\theta}_m)} \theta_{mp} \Bigg] \Bigg\}.\end{aligned} \tag{2.6.12}$$

From (2.6.12) we see that $\lim_{m\to\infty} \lambda_p(\underset{\sim}{\theta}_m)$ is finite if

$$\lim_{m\to\infty} \frac{c_{ujk}(\underset{\sim}{\theta}_m)}{h_j(\underset{\sim}{\theta}_m) h_k(\underset{\sim}{\theta}_m)} hp(\underset{\sim}{\theta}_m) \theta_{mp}$$

is finite for u = 1,2,...,n and j,k = 1,2,...,p.

Similarly $\lim_{m\to\infty} \lambda_i(\underset{\sim}{\theta}_m)$, where $i < p$, is finite if

$$\lim_{m\to\infty} \frac{c_{ujk}(\underset{\sim}{\theta}_m)}{h_j(\underset{\sim}{\theta}_m) h_k(\underset{\sim}{\theta}_m)} h_i(\underset{\sim}{\theta}_m) \theta_{mi}$$

is finite for u = 1,2,...,n and j,k = 1,2,...,p.

We now show that in models (A) and (B), the sufficient conditions for the existence of the power transformation which achieves maximum reduction of the nonlinearity associated with the first parameter are satisfied when θ_1 tends to infinity. By differentiating the $\eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta})$ of these models with respect to θ_1 and θ_2, we see that [1] and [2] are satisfied. Next we can show that for model (A),

$$h_1(\underset{\sim}{\theta}_m) = \frac{1}{\theta_{m1}^2},$$

$$h_2(\underset{\sim}{\theta}_m) = 1,$$

$$c_{u11}(\underset{\sim}{\theta}_m) = \frac{v_{u11}(\underset{\sim}{\theta}_m)}{\theta_{m1}^3},$$

$$c_{u12}(\underset{\sim}{\theta}_m) = \frac{v_{u12}(\underset{\sim}{\theta}_m)}{\theta_{m1}^2},$$

$$c_{u22}(\underset{\sim}{\theta}_m) = v_{u22}(\underset{\sim}{\theta}_m),$$

where $v_{uij}(\underset{\sim}{\theta}_m)$ are functions which converge to finite limits as m tends to infinity, and for model (B),

$$h_1(\underset{\sim}{\theta}_m) = \frac{1}{\theta_{m1}^2},$$

$$h_2(\underset{\sim}{\theta}_m) = 1,$$

$$c_{u11}(\underset{\sim}{\theta}_m) = \frac{w_{u11}(\underset{\sim}{\theta}_m)}{\theta_{m1}^3},$$

$$c_{u12}(\underset{\sim}{\theta}_m) = \frac{w_{u12}(\underset{\sim}{\theta}_m)}{\theta_{m1}^2},$$

$$c_{u22}(\underset{\sim}{\theta}_m) = w_{u22}(\underset{\sim}{\theta}_m),$$

where $w_{uij}(\underset{\sim}{\theta}_m)$ are functions which converge to finite limits as m tends to infinity. Condition [3] can then be readily shown to be satisfied in each of these models.

For model (B), the matrix $\underset{\sim}{C}$ for the case when $\underset{\sim}{\theta}_B$ is such that $\theta_{B1} - \theta_{B2} = 0$ is of rank one, which is less than p. However, if we let

$$\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix},$$

then the corresponding sufficient conditions for the existence of the power transformation of β_2 are satisfied. This indicates that whenever [2] is not satisfied, there may exist a linear transformation $\underset{\sim}{\beta} = \underset{\sim}{L}\underset{\sim}{\theta}$, where $\underset{\sim}{L}$ is a (pxp) non-singular matrix, such that the corresponding sufficient conditions for the existence of the power transformation of β_i are satisfied.

Section 2.7 Alternative transformations to reduce nonlinearity

In many situations, it will be much easier to appreciate the physical significance of transformations of individual parameters rather than transformations such that the new parameters are functions of more than one original parameter. We therefore investigate transformation $\underset{\sim}{\beta}$ of the form

$$\beta_i = \beta_i(t_i), \quad (i = 1,2,\ldots,p).$$

We choose $\beta_i = \beta_i(t_i)$ to be one such that the total nonlinearity is reduced as much as possible. For reasons similar to those given in the derivation of N_ϕ in section 2.2, we can restrict our attention to β_i of the form

$$\beta_i = t_i - g_i t_i^2 + o(t_i^2), \quad (i = 1,2,\ldots,p). \tag{2.7.1}$$

Suppose g_i^* are such that the corresponding $\underset{\sim}{\beta}$ achieves the maximum reduction of the total nonlinearity. To find g_i^*, we first apply the orthogonal transformation $\underset{\sim}{H}$ (c.f. (2.2.2)) of coordinates in sample space and then use the linear transformation $\underset{\sim}{\tau}^{(\beta)} = \underset{\sim}{D}\underset{\sim}{\beta}$ to obtain

$$N_\beta = \frac{\sigma^2}{p+2} \sum_{i=1}^{n} \sum_{\ell=1}^{p} \sum_{m=1}^{p} [f^+_{i\ell\ell} f^+_{imm} + 2(f^+_{i\ell m})^2] \tag{2.7.2}$$

where

$$f^+_{i\ell m} = \begin{cases} f_{i\ell m} + \sum\limits_{i \le j \le \min[\ell,m]} d_{ij} g_j d^{j\ell} d^{jm}, & (i = 1,2,\ldots,p) \\ f_{i\ell m}, & (i = p+1,p+2,\ldots,n). \end{cases} \tag{2.7.3}$$

By differentiating (2.7.2) with respect to the g_j, it can be shown that the g_j^* are given by

$$\sum_{j=1}^{p} [\sum_{i=1}^{\min[j,s]} d_{is} [(\sum_{k=j}^{p} d_{ij} (d^{jk})^2) (\sum_{\ell=s}^{p} (d^{s\ell})^2) \tag{2.7.4}$$

$$+ 2\, d_{ij} (\sum_{\ell=\max[j,s]}^{p} d^{j\ell} d^{s\ell})^2]] g_j^*$$

$$+ \sum_{i=1}^{s} d_{is}[(\sum_{\ell=1}^{p} f_{i\ell\ell})(\sum_{k=s}^{p} (d^{sk})^2) + 2 \sum_{\ell=s}^{p} \sum_{m=s}^{p} f_{i\ell m} d^{s\ell} d^{sm}] = 0$$

$$(s = 1,2,\ldots,p) .$$

For the case when the θ_i are non-negative, the power transformations

$$(2.7.5) \qquad \psi_i = \begin{cases} \dfrac{\theta_i^{\lambda_i} - 1}{\lambda_i} , & \lambda_i \neq 0, \\ \ln \theta_i , & \lambda_i = 0 \end{cases} , \quad (i = 1,2,\ldots,p),$$

where $\lambda_i = 1 - 2g_i^*\hat{\theta}_i$, and the transformations β_i given by (2.7.1) with g_i changed to g_i^* are equivalent in the sense that they reduce the total nonlinearity by the same amount.

Interval and region estimates based on these power transformations are investigated numerically in Chapter 5.

Section 2.8 Effects of design of experiments on nonlinearity

This section is concerned with the effects of design of experiments on nonlinearity.

It is geometrically obvious that if the number, n, of observations is less than or equal to the number, p, of components of the parameter vector $\underset{\sim}{\theta}$, then the solution locus of the model is a subset of an n-dimensional linear manifold. If n is larger than p, but the number, s, of distinct experimental conditions adopted in the model is less than or equal to p, then the solution locus can be shown as follows to be a subset of an s-dimensional linear manifold.

Let $y_{u_1u_2}$ be the observed value of the u_2th of r_{u_1} responses in the u_1th of s ($s \leq p$) distinct experiments. We then have

$$(2.8.1)\qquad E(y_{u_1u_2}) = \eta(\underset{\sim}{\xi}_{u_1}, \underset{\sim}{\theta}),\ (u_1 = 1,2,\ldots,s;\ u_2 = 1,2,\ldots,r_{u_1};\ \sum_{u_1=1}^{s} r_{u_1} = n).$$

Let $\underset{\sim}{H}_k$ and $\underset{\sim}{H}^+$ be the (k×k) and (n×n) orthogonal matrices given by

$$(2.8.2)\qquad \underset{\sim}{H}_k = \begin{bmatrix} \frac{1}{\sqrt{k}} & \frac{1}{\sqrt{k}} & \frac{1}{\sqrt{k}} & \cdots & \frac{1}{\sqrt{k}} & \frac{1}{\sqrt{k}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \cdots & 0 & 0 \\ \frac{1}{\sqrt{2\times 3}} & \frac{1}{\sqrt{2\times 3}} & -\frac{2}{\sqrt{2\times 3}} & 0 \cdots & 0 & 0 \\ \vdots & & & & & \\ \frac{1}{\sqrt{(k-1)k}} & \frac{1}{\sqrt{(k-1)k}} & \frac{1}{\sqrt{(k-1)k}} & \cdots & \frac{1}{\sqrt{(k-1)k}} & \frac{-(k-1)}{\sqrt{(k-1)k}} \end{bmatrix}$$

and

$$(2.8.3)\qquad \underset{\sim}{H}^+ = \begin{bmatrix} \underset{\sim}{H}_{r_1} & & & \bigcirc \\ & \underset{\sim}{H}_{r_2} & & \\ & & \ddots & \\ \bigcirc & & & \underset{\sim}{H}_{r_s} \end{bmatrix} .$$

Further, let $\underset{\sim}{y}^*$ be the (n×1) vector given by

$$\underset{\sim}{y}^* = [y_{11}y_{12}\cdots y_{1r_1} \,\vdots\, y_{21}y_{22}\cdots y_{2r_2} \,\vdots\, \cdots \,\vdots\, y_{s1}y_{s2}\cdots y_{sr_s}]^T .$$

If we apply the orthogonal transformation

(2.8.4) $\underset{\sim}{H}^{+}\underset{\sim}{y}^{*} = \underset{\sim}{y}^{+}$

of coordinates in sample space, then (2.8.1) becomes

(2.8.5) $$E(y^{+}_{u_1u_2}) = \begin{cases} \sqrt{r_{u_1}}\,\eta(\underset{\sim}{\xi}_{u_1},\underset{\sim}{\theta}), & (u_1 = 1,2,\ldots,s;\ u_2 = 1) \\ 0, & (u_1 = 1,2,\ldots,s;\ u_2 = 2,3,\ldots,r_{u_1}), \end{cases}$$

where $y^{+}_{u_1u_2}$ is the $(\sum_{i=1}^{u_1} r_i - r_{u_1} + u_2)^{th}$ component of $\underset{\sim}{y}^{+}$.

From (2.8.5), it is clear that the solution locus of the model is a subset of an s-dimensional linear manifold.

We next investigate the effects of replication of experiments on the measures of nonlinearity in the neighbourhood of a non-singular point $P(\hat{\underset{\sim}{\theta}})$.

Let s, r_{u_1} and n be changed to n, r and nr respectively. We have

(2.8.6) $$E(y_{u_1u_2}) = \eta(\underset{\sim}{\xi}_{u_1},\hat{\underset{\sim}{\theta}}) + \sum_{j=1}^{p} c_{u_1j}t_j + \sum_{j=1}^{p}\sum_{k=1}^{p} c_{u_1jk}t_jt_k + o(t^2),$$

$$(u_1 = 1,2,\ldots,n;\ u_2 = 1,2,\ldots,r).$$

Let $\underset{\sim}{\eta}^{*}$ be the (nrx1) vector whose $((u_1 - 1)r + u_2)^{th}$ component is $\eta(\underset{\sim}{\xi}_{u_1},\hat{\underset{\sim}{\theta}})$.

After applying the orthogonal transformation

$$\underset{\sim}{H}^{+}(\underset{\sim}{y}^{*} - \underset{\sim}{\eta}^{*}) = \underset{\sim}{y}^{(r+)}$$

of coordinates in sample space, (2.8.6) becomes

$$(2.8.7)\qquad E(y^{(r+)}_{u_1u_2}) = \begin{cases} \sum_{j=1}^{p} \sqrt{r}\, c_{u_1j}t_j + \sum_{j=1}^{p}\sum_{k=1}^{p} \sqrt{r}\, c_{u_1jk}t_jt_k + o(t^2), & (u_1=1,2,\ldots,n;\ u_2 = 1) \\ 0, & (u_1=1,2,\ldots,n;\ u_2=2,3,\ldots,r), \end{cases}$$

where $y^{(r+)}_{u_1u_2}$ is the $((u_1-1)r + u_2)^{th}$ component of $\mathbf{y}^{(r+)}$.

Let $\mathbf{I}_{n(r-1)}$ be an $(n(r-1)\times n(r-1))$ identity matrix and $\mathbf{y}^{(r)}$ be the $(nr\times 1)$ vector given by

$$\mathbf{y}^{(r)} = [y_{11}^{(r+)}\ y_{21}^{(r+)}\ \ldots\ y_{n1}^{(r+)}\ 00\ldots 0]^T.$$

If we apply the orthogonal transformation

$$(2.8.8)\qquad \left[\begin{array}{c:c} \mathbf{H} & \bigcirc \\ \hdashline \bigcirc & \mathbf{I}_{n(r-1)} \end{array}\right] \mathbf{y}^{(r)} = \mathbf{z}^* \quad (\text{c.f. } (2.2.2))$$

of coordinates in sample space, then (2.8.7) becomes

$$(2.8.9)\qquad E(z_{i_1i_2}) = \begin{cases} \sum_{j=1}^{p} \sqrt{r}\, d_{i_1j}t_j + \sum_{j=1}^{p}\sum_{k=1}^{p} \sqrt{r}\, \ddot{d}_{i_1jk}t_jt_k + o(t^2), & (i_1 = 1,2,\ldots,p;\ i_2 = 1) \\ \sum_{j=1}^{p}\sum_{k=1}^{p} \sqrt{r}\, \ddot{d}_{i_1jk}t_jt_k + o(t^2), & (i_1=p+1,p+2,\ldots,n;\ i_2 = 1) \\ 0, & (i_1=1,2,\ldots,n;\ i_2=2,3,\ldots,r), \end{cases}$$

where $z_{i_1i_2}$ is the $(i_1 + (i_2-1)n)^{th}$ component of $\mathbf{z}^*$.

After applying the linear transformation

(2.8.10) $$\underset{\sim}{\tau}^{(r)} = \sqrt{r}\, \underset{\sim}{D}\, \underset{\sim}{t},$$

(2.8.9) becomes

(2.8.11)
$$E(z_{i_1 i_2}) = \begin{cases} \tau_{i_1}^{(r)} + \sum_{\ell=1}^{p} \sum_{m=1}^{p} \frac{1}{\sqrt{r}} f_{i_1 \ell m} \tau_\ell^{(r)} \tau_m^{(r)} + o((\tau^{(r)})^2), & (i_1 = 1,2,\ldots,p;\ i_2 = 1) \\ \sum_{\ell=1}^{p} \sum_{m=1}^{p} \frac{1}{\sqrt{r}} f_{i_1 \ell m} \tau_\ell^{(r)} \tau_m^{(r)} + o((\tau^{(r)})^2), & (i_1=p+1,p+2,\ldots,n;\ i_2=1) \\ 0 & ,\ (i_1=1,2,\ldots,n;\ i_2=2,3,\ldots,r). \end{cases}$$

We then have

(2.8.12) $$N_\theta = \frac{1}{r} \left[\frac{\sigma^2}{p+2} \sum_{i_1=1}^{n} \sum_{\ell=1}^{p} \sum_{m=1}^{p} (f_{i_1 \ell\ell} f_{i_1 mm} + 2f_{i_1 \ell m}^2) \right]$$

and

(2.8.13) $$N_\phi = \frac{1}{r} \left[\frac{\sigma^2}{p+2} \sum_{i_1=p+1}^{n} \sum_{\ell=1}^{p} \sum_{m=1}^{p} (f_{i_1 \ell\ell} f_{i_1 mm} + 2f_{i_1 \ell m}^2) \right].$$

Therefore replication of each of the experiments r times shrinks the measures of total nonlinearity and intrinsic nonlinearity in the neighbourhood of $P(\hat{\underset{\sim}{\theta}})$ by a factor of r.

We next see from (2.8.10) and (2.8.11) that the values of $g_{ijk}^{(k*)}$ in section 2.4 and $g_i{}^*$ in section 2.7 do not change due to replication of each of the experiments r times.

CHAPTER 3

HYPOTHESIS TESTING AND REGION ESTIMATION BASED ON GENERAL MAXIMUM LIKELIHOOD RATIOS

Section 3.1 Introduction

In this chapter, we consider the problem of testing a number of hypotheses for making inference about subsets of components of the parameter vector in nonlinear models. General maximum likelihood (m.l.) ratio tests are used for testing these hypotheses. The estimation of the coverage probabilities of the region estimates based on these tests is also discussed.

Section 3.2 Hypothesis testing in unconstrained nonlinear models

We shall restrict our attention to unconstrained nonlinear models and consider the problem of testing the null hypothesis

$$H_i \text{ that } (\underset{\sim}{\theta},\sigma) \in \Omega_{H_i}$$

against the alternative hypothesis

$$K_i \text{ that } (\underset{\sim}{\theta},\sigma) \in \Omega_{K_i}\ , \quad (i = 1,2,\ldots,5),$$

where

$$\Omega_{H_1} = \{(\underset{\sim}{\theta},\sigma)\colon \theta_j=\theta_{0j} \text{ for } j = p-k^*+1, p-k^*+2,\ldots,p, \text{ and } \sigma = \sigma_0\}\ ,$$

$$(k^* = 1,2,\ldots,p-1),$$

$$\Omega_{K_1} = \{(\underset{\sim}{\theta},\sigma): \theta_j \neq \theta_{0j} \text{ for } j = p-k^*+1, p-k^*+2, \ldots, p, \text{ and } \sigma = \sigma_0\},$$

$$\Omega_{H_2} = \{(\underset{\sim}{\theta},\sigma): \underset{\sim}{\theta} = \underset{\sim}{\theta}_0, \text{ and } \sigma = \sigma_0\},$$

$$\Omega_{K_2} = \{(\underset{\sim}{\theta},\sigma): \underset{\sim}{\theta} \neq \underset{\sim}{\theta}_0, \text{ and } \sigma = \sigma_0\},$$

$$\Omega_{H_3} = \{(\underset{\sim}{\theta},\sigma): \theta_j = \theta_{0j} \text{ for } j = p-k^*+1, p-k^*+2, \ldots, p\},$$

$$\Omega_{K_3} = \{(\underset{\sim}{\theta},\sigma): \theta_j \neq \theta_{0j} \text{ for } j = p-k^*+1, p-k^*+2, \ldots, p\},$$

$$\Omega_{H_4} = \{(\underset{\sim}{\theta},\sigma): \underset{\sim}{\theta} = \underset{\sim}{\theta}_0\},$$

$$\Omega_{K_4} = \{(\underset{\sim}{\theta},\sigma): \underset{\sim}{\theta} \neq \underset{\sim}{\theta}_0\},$$

$$\Omega_{H_5} = \{(\underset{\sim}{\theta},\sigma): \sigma = \sigma_0\},$$

$$\Omega_{K_5} = \{(\underset{\sim}{\theta},\sigma): \sigma \neq \sigma_0\},$$

$\underset{\sim}{\theta}_0$ and σ_0 are particular values of $\underset{\sim}{\theta}$ and σ respectively, and θ_{0j} is the j^{th} component of $\underset{\sim}{\theta}_0$.

The usual monotonic functions of the general m.l. ratios for testing these hypotheses are given respectively by

$$T_1(\underset{\sim}{y}) = [S^M(\theta_{0p-k^*+1}, \theta_{0p-k^*+2}, \ldots, \theta_{0p}) - S(\hat{\underset{\sim}{\theta}})],$$

$$T_2(\underset{\sim}{y}) = [S(\underset{\sim}{\theta}_0) - S(\hat{\underset{\sim}{\theta}})],$$

$$T_3(\underset{\sim}{y}) = [S^M(\theta_{0p-k^*+1}, \theta_{0p-k^*+2}, \ldots, \theta_{0p}) - S(\hat{\underset{\sim}{\theta}})]/S(\hat{\underset{\sim}{\theta}}),$$

$$T_4(\underset{\sim}{y}) = [S(\underset{\sim}{\theta}_0) - S(\hat{\underset{\sim}{\theta}})]/S(\hat{\underset{\sim}{\theta}})$$

and

$$T_5(\underset{\sim}{y}) = S(\hat{\underset{\sim}{\theta}}),$$

where $S(\underset{\sim}{\theta}) = \sum_{u=1}^{n} \{y_u - \eta(\underset{\sim}{\xi}_u, \underset{\sim}{\theta})\}^2$,

and $S^M(\theta_{0p-k^*+1}, \theta_{0p-k^*+2}, \ldots, \theta_{0p})$ is the minimum value of $S(\underset{\sim}{\theta})$ with respect to $\underset{\sim}{\theta}$ where $\underset{\sim}{\theta}$ are such that $(\underset{\sim}{\theta}, \sigma_0) \in \Omega_{H_1}$.

Acceptance regions of the general m.l. ratio tests are

$$\omega_i = \{\underset{\sim}{z} : S_i^D(\underset{\sim}{z}) \leq d_i^{*2}\}, \quad (i = 1,2,\ldots,5),$$

where

$$S_1^D(\underset{\sim}{z}) = [S^M(\theta_{0p-k^*+1}, \theta_{0p-k^*+2}, \ldots, \theta_{0p}) - S(\hat{\underset{\sim}{\theta}})]/\sigma_0^2,$$

$$S_2^D(\underset{\sim}{z}) = [S(\underset{\sim}{\theta}_0) - S(\hat{\underset{\sim}{\theta}})]/\sigma_0^2,$$

$$S_3^D(\underset{\sim}{z}) = [S^M(\theta_{0p-k^*+1}, \theta_{0p-k^*+2}, \ldots, \theta_{0p}) - (1 + \frac{k^*F_\alpha(k^*, n-p)}{n-p})S(\hat{\underset{\sim}{\theta}})]/\sigma_0^2 + \frac{k^*F_\alpha(k^*, n-p)}{n-p} \sum_{j=p+1}^{n} z_j^2/\sigma_0^2,$$

$$S_4^D(\underset{\sim}{z}) = [S(\underset{\sim}{\theta}_0) - (1 + \frac{pF_\alpha(p,n-p)}{n-p})S(\hat{\underset{\sim}{\theta}})]/\sigma_0^2 + \frac{pF_\alpha(p,n-p)}{n-p} \sum_{j=p+1}^{n} z_j^2/\sigma_0^2,$$

$$S_5^D(\underset{\sim}{z}) = S(\hat{\underset{\sim}{\theta}})/\sigma_0^2,$$

$(\underset{\sim}{\theta}_0, \sigma_0) \in \Omega_{H_i}$, $(i = 1,2,3,4,5)$,

$$d_i{*}^2 = (d_i{*})^2 = \begin{cases} \chi^2_{k^*,\alpha} & , (i = 1) \\ \chi^2_{p,\alpha} & , (i = 2) \\ \dfrac{k^*F_\alpha(k^*,n-p)}{n-p} \displaystyle\sum_{j=p+1}^{n} z_j^{\,2}/\sigma_0^{\,2} & , (i = 3) \\ \dfrac{pF_\alpha(p,n-p)}{n-p} \displaystyle\sum_{j=p+1}^{n} z_j^{\,2}/\sigma_0^{\,2} & , (i = 4) \\ \chi^2_{n-p,\alpha} & , (i = 5), \end{cases}$$

and the z_i are rotated coordinates of a point $\underset{\sim}{y}$ in sample space with $P(\underset{\sim}{\theta}_0)$ as origin (c.f. section 2.2).

A number which is not larger than the probability

$$I_i(\underset{\sim}{\theta}_0,\sigma_0) = \Pr\{\underset{\sim}{z} \in \omega_i \mid \underset{\sim}{\theta} = \underset{\sim}{\theta}_0 \text{ and } \sigma = \sigma_0\}$$

for all $(\underset{\sim}{\theta}_0,\sigma_0) \in \Omega_{H_i}$ is the significance probability of the test for the hypothesis H_i, and the probability

$$\beta_i(\underset{\sim}{\theta}_A,\sigma_A) = 1 - \Pr\{\underset{\sim}{z} \in \omega_i \mid \underset{\sim}{\theta} = \underset{\sim}{\theta}_A \text{ and } \sigma = \sigma_A\} ,$$

where $(\underset{\sim}{\theta}_A,\sigma_A) \in \Omega_{K_i}$, is the power function of this test.

Note that we have chosen $\chi^2_{k^*,\alpha}$, $\chi^2_{p,\alpha}$, $F_\alpha(k^*,n-p)$, $F_\alpha(p,n-p)$ and $\chi^2_{n-p,\alpha}$ to be the corresponding constants appearing in the expressions for $d_i{*}^2$. A reason for choosing them is that if the model is linear, then α is the level of significance of each of the tests.

Section 3.3 Significance probabilities of the general m.l. ratio tests

In this section we restrict our attention to the $\eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta})$ which are functions of $\underset{\sim}{\theta}$ differentiable up to the third order and consider the derivation of approximations of the probabilities $I_i(\underset{\sim}{\theta}_O,\sigma_O)$ (c.f. section 3.2), where $\underset{\sim}{\theta}_O$ is such that $P(\underset{\sim}{\theta}_O)$ is a non-singular point in the solution locus.

The functions $\eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta})$ can be written as

$$(3.3.1)\qquad \eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta}) = \eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta}_O) + \sum_{j=1}^{p} c_{uj}t_j + \underset{\sim}{t}^T\underset{\sim}{C}_u\underset{\sim}{t} + \sum_{j=1}^{p} [\underset{\sim}{t}^T\underset{\sim}{C}_{uj}\underset{\sim}{t}]t_j + o(t^3),\quad (u = 1,2,\ldots,n),$$

where

$\underset{\sim}{t}$ is a (p×1) vector whose j^{th} component is $t_j = \theta_j - \theta_{Oj}$,

$\underset{\sim}{C}_u$ is a (p×p) symmetric matrix whose (j,k) entry is c_{ujk}

$\underset{\sim}{C}_{uj}$ is a (p×p) symmetric matrix whose (k,ℓ) entry is $c_{ujk\ell}$

and

$t^3 = (\sum_{j=1}^{p} t_j^2)^{3/2}$. Though t_j, c_{uj}, c_{ujk} are also used to denote the corresponding terms that arise in expanding $\eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta})$ in the neighbourhood of $\hat{\underset{\sim}{\theta}}$ in Chapter 2, we hope that no confusion should arise.

Let $\underset{\sim}{\eta}_O$ be the (n×1) vector whose u^{th} component is $\eta(\underset{\sim}{\xi}_u,\underset{\sim}{\theta}_O)$. Further, let $\underset{\sim}{H}$ (c.f. section 2.2) be an (n×n) orthogonal matrix such that $\underset{\sim}{H}\underset{\sim}{C}$, where $\underset{\sim}{C} = \{c_{uj}\}$, is an upper triangular (p×p) matrix $\underset{\sim}{D}$ with an ((n-p)×p) zero matrix beneath it.

As in section 2.2, we can apply the orthogonal transformation

$$\underset{\sim}{H}(\underset{\sim}{y} - \underset{\sim}{\eta}_O) = \underset{\sim}{z}$$

in sample space and the linear transformation

$$\tau = D t$$

of parameter vector t. After these transformations, the rotated coordinates z_i^* of a point $P(\theta)$ in the solution locus can be written as

$$(3.3.2) \qquad z_i^* = \begin{cases} \tau_i + \tau^T F_i \tau + \sum_{j=1}^{p} [\tau^T F_{ij} \tau]\tau_j + o(\tau^3), & (i = 1,2,\ldots,p) \\ \tau^T F_i \tau + \sum_{j=1}^{p} [\tau^T F_{ij} \tau]\tau_j + o(\tau^3), & (i = p+1,p+2,\ldots,n) \end{cases}$$

where

$$\tau^3 = (\sum_{i=1}^{p} \tau_i^2)^{3/2} ,$$

and

$$F_i = \{f_{ijk}\},$$

$$F_{ij} = \{f_{ijk\ell}\}$$

are $(p \times p)$ symmetric matrices.

After applying the nonlinear transformations

$$(3.3.3) \qquad \phi_i = \tau_i + \tau^T F_i \tau + \sum_{j=1}^{p} [\tau^T F_{ij} \tau]\tau_j + o(\tau^3), \quad (i = 1,2,\ldots,p),$$

of the parameters τ_i, (3.3.2) becomes

$$(3.3.4) \qquad z_i^* = \begin{cases} \phi_i & , (i = 1,2,\ldots,p) \\ \phi^T A_i \phi + \sum_{j=1}^{p} [\phi^T A_{ij} \phi]\phi_j + o(\phi^3), & (i = p+1,p+2,\ldots,n), \end{cases}$$

where $\underset{\sim}{A}_i = \underset{\sim}{F}_i$,

$$\underset{\sim}{A}_{ij} = \underset{\sim}{F}_{ij} - 2 \sum_{k=1}^{p} f_{ijk} \underset{\sim}{F}_k$$

and $\phi^3 = (\sum_{i=1}^{p} \phi_i^2)^{3/2}$.

The sum of squares of the residuals of an observed sample point $\underset{\sim}{z}$ is given by

$$(3.3.5) \qquad S = \sum_{i=1}^{p} (z_i - \phi_i)^2 + \sum_{i=p+1}^{n} \{z_i - \underset{\sim}{\phi}^T \underset{\sim}{A}_i \underset{\sim}{\phi} - \sum_{j=1}^{p} [\underset{\sim}{\phi}^T \underset{\sim}{A}_{ij} \underset{\sim}{\phi}] \phi_j + o(\phi^3)\}^2.$$

Approximations of the least squares estimates $\hat{\phi}_m$ for the components ϕ_m of the parameter vector $\underset{\sim}{\phi}$ can be obtained by minimizing S in (3.3.5) with respect to the ϕ_i, and it is found that

$$(3.3.6) \qquad \hat{\phi}_m = z_m + 2 \sum_{i=p+1}^{n} \sum_{j=1}^{p} a_{ijm} z_i z_j$$

$$+ 4 \sum_{i=p+1}^{n} \sum_{h=p+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} a_{ijm} a_{hjk} z_i z_h z_k$$

$$- 2 \sum_{i=p+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{\ell=1}^{p} a_{ijk} a_{i\ell m} z_j z_k z_\ell$$

$$+ 3 \sum_{i=p+1}^{n} \sum_{j=1}^{p} \sum_{\ell=1}^{p} a_{ijm\ell} z_i z_j z_\ell + o(z^3), \quad (m = 1,2,\ldots,p),$$

where $z^3 = (\sum_{i=1}^{p} z_i^2)^{3/2}$. The corresponding sum of squares of residuals is

$$(3.3.7)\quad S(\hat{\underset{\sim}{\theta}}) = \sum_{j=p+1}^{n} z_j^2 - 2\sum_{i=p+1}^{n}\sum_{j=1}^{p}\sum_{k=1}^{p} a_{ijk} z_i z_j z_k$$

$$+ \sum_{i=p+1}^{n}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{\ell=1}^{p}\sum_{m=1}^{p} a_{ijk} a_{i\ell m} z_j z_k z_\ell z_m$$

$$-4 \sum_{h=p+1}^{n}\sum_{i=p+1}^{n}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{\ell=1}^{p} a_{hj\ell} a_{ik\ell} z_h z_i z_j z_k$$

$$-2 \sum_{i=p+1}^{n}\sum_{j=1}^{p}\sum_{k=1}^{p}\sum_{\ell=1}^{p} a_{ijk\ell} z_i z_j z_k z_\ell \qquad + o(z^4).$$

Under the hypothesis H_1 or H_3, the components τ_j $(j = p-k^*+1, p-k^*+2,\ldots,p)$ are zero, and consequently (3.3.2) becomes

$$(3.3.8)\quad z_i^* = \begin{cases} \tau_i + \underset{\sim}{\tau}_M^T \underset{\sim}{F}_{Mi} \underset{\sim}{\tau}_M + \sum_{j=1}^{p-k^*} [\underset{\sim}{\tau}_M^T \underset{\sim}{F}_{Mij} \underset{\sim}{\tau}_M]\tau_j + o(\tau_M^3), & (i = 1,2,\ldots,p-k^*) \\ \underset{\sim}{\tau}_M^T \underset{\sim}{F}_{Mi} \underset{\sim}{\tau}_M + \sum_{j=1}^{p-k^*} [\underset{\sim}{\tau}_M^T \underset{\sim}{F}_{Mij} \underset{\sim}{\tau}_M]\tau_j + o(\tau_M^3), & (i = p-k^*+1, p-k^*+2,\ldots,n) \end{cases}$$

where

$$\underset{\sim}{\tau}_M = [\tau_1 \tau_2 \ldots \tau_{p-k^*}]^T,$$

$$\tau_M^3 = \left(\sum_{i=1}^{p-k^*} \tau_i^2\right)^{3/2},$$

and

$$\underset{\sim}{F}_{Mi} = \{f_{ijk}\},$$

$$\underset{\sim}{F}_{Mij} = \{f_{ijk\ell}\}$$

are $((p-k^*)\times(p-k^*))$ symmetric matrices.

After applying the nonlinear transformations

$$\phi_{Mi} = \tau_i + \tau_M^T F_{Mi} \tau_M + \sum_{j=1}^{p-k^*} [\tau_M^T F_{Mij} \tau_M]\tau_j + o(\tau_M^3), \quad (i = 1,2,\ldots,p-k^*)$$

of the parameters τ_i, (3.3.8) becomes

(3.3.9)
$$z_i^* = \begin{cases} \phi_{Mi} & , (i = 1,2,\ldots,p-k^*) \\ \phi_M^T A_{Mi} \phi_M + \sum_{j=1}^{p-k^*} [\phi_M^T A_{Mij} \phi_M]\phi_{Mj} + o(\phi_M^3), & (i = p-k^*+1, p-k^*+2,\ldots,n), \end{cases}$$

where

$$A_{Mi} = F_{Mi},$$

$$A_{Mij} = F_{Mij} - 2\sum_{k=1}^{p-k^*} f_{ijk} F_{Mk}$$

and

$$\phi_M^3 = \left(\sum_{i=1}^{p-k^*} \phi_{Mi}^2\right)^{3/2} .$$

For the same observed sample point z, we can find the values $\hat{\phi}_{Mm}$ of the ϕ_{Mm} such that the sum of squares

$$\sum_{i=1}^{p-k^*} (z_i-\phi_{Mi})^2 + \sum_{i=p-k^*+1}^{n} \{z_i - \phi_M^T A_{Mi} \phi_M - \sum_{j=1}^{p-k^*} [\phi_M^T A_{Mij} \phi_M]\phi_{Mj} + o(\phi_M^3)\}^2$$

is minimized, and it is found that

$$(3.3.10)\quad \hat{\phi}_{Mm} = z_m + 2 \sum_{i=p-k^*+1}^{n} \sum_{j=1}^{p-k^*} a_{ijm} z_i z_j$$

$$+ 4 \sum_{i=p-k^*+1}^{n} \sum_{h=p-k^*+1}^{n} \sum_{j=1}^{p-k^*} \sum_{k=1}^{p-k^*} a_{ijm} a_{hjk} z_i z_h z_k$$

$$- 2 \sum_{i=p-k^*+1}^{n} \sum_{j=1}^{p-k^*} \sum_{k=1}^{p-k^*} \sum_{\ell=1}^{p-k^*} a_{ijk} a_{i\ell m} z_j z_k z_\ell$$

$$+ 3 \sum_{i=p-k^*+1}^{n} \sum_{j=1}^{p-k^*} \sum_{\ell=1}^{p-k^*} a_{Mijm\ell} z_i z_j z_\ell + o(z^3),$$

$$(m = 1,2,\ldots,p-k^*),$$

and

$$(3.3.11)\quad S^M(\theta_{Op-k^*+1}, \theta_{Op-k^*+2}, \ldots, \theta_{Op})$$

$$= \sum_{j=p-k^*+1}^{n} z_j^2 - 2 \sum_{i=p-k^*+1}^{n} \sum_{j=1}^{p-k^*} \sum_{k=1}^{p-k^*} a_{ijk} z_i z_j z_k$$

$$+ \sum_{i=p-k^*+1}^{n} \sum_{j=1}^{p-k^*} \sum_{k=1}^{p-k^*} \sum_{\ell=1}^{p-k^*} \sum_{m=1}^{p-k^*} a_{ijk} a_{i\ell m} z_j z_k z_\ell z_m$$

$$- 4 \sum_{h=p-k^*+1}^{n} \sum_{i=p-k^*+1}^{n} \sum_{j=1}^{p-k^*} \sum_{k=1}^{p-k^*} \sum_{\ell=1}^{p-k^*} a_{hj\ell} a_{ik\ell} z_h z_i z_j z_k$$

$$- 2 \sum_{i=p-k^*+1}^{n} \sum_{j=1}^{p-k^*} \sum_{k=1}^{p-k^*} \sum_{\ell=1}^{p-k^*} a_{Mijk\ell} z_i z_j z_k z_\ell + o(z^4).$$

Provided that σ_O, the a_{hjk}, and the $a_{hjk\ell}$ or $a_{Mhjk\ell}$ are sufficiently small, the quartic approximations of $S(\hat{\underset{\sim}{\theta}})$ and $S^M(\theta_{Op-k^*+1}, \theta_{Op-k^*+2}, \ldots, \theta_{Op})$ are adequate for most of the $\underset{\sim}{z}$, and $S_i^D(\underset{\sim}{z})$ $(i = 1,2,\ldots,5)$ can thus be approximated by the following expressions for most of the $\underset{\sim}{z}$:

(3.3.12)
$$S_1^D(\underset{\sim}{z}) = \sum_{j=p-k^*+1}^{p} z_j^2/\sigma_O^2$$

$$- \frac{2}{\sigma_O^3} \sum_{i=p-k^*+1}^{p} \sum_{j=1}^{p-k^*} \sum_{k=1}^{p-k^*} a^*_{ijk} z_i z_j z_k$$

$$+ \frac{2}{\sigma_O^3} \sum_{i=p+1}^{n} \sum_{j,k=1,2,\ldots,p} a^*_{ijk} z_i z_j z_k$$

and at least one of j,k is equal to p-k*+1 or p-k*+2,... or p

$$+ \frac{1}{\sigma_O^4} \sum_{i=p-k^*+1}^{p} \left(\sum_{j=1}^{p-k^*} \sum_{k=1}^{p-k^*} a^*_{ijk} z_j z_k \right)^2$$

$$- \frac{1}{\sigma_O^4} \sum_{i=p+1}^{n} \sum_{j,k,\ell,m=1,2,\ldots,p} a^*_{ijk} a^*_{i\ell m} z_j z_k z_\ell z_m$$

and at least one of j,k,ℓ,m is equal to p-k*+1 or p-k*+2,..., or p

$$- \frac{4}{\sigma_O^4} \sum_{h,i=p-k^*+1,p-k^*+2,\ldots,n} \sum_{j=1}^{p-k^*} \sum_{k=1}^{p-k^*} \sum_{\ell=1}^{p-k^*} a^*_{hj\ell} a^*_{ik\ell} z_h z_i z_j z_k$$

and at least one of h,i is equal to p-k*+1 or p-k*+2,..., or p

$$+ \frac{4}{\sigma_O^4} \sum_{h=p+1}^{n} \sum_{i=p+1}^{n} \sum_{j,k,\ell=1,2,\ldots,p} a^*_{hj\ell} a^*_{ik\ell} z_h z_i z_j z_k$$

and at least one of j,k,ℓ is equal to p-k*+1 or p-k*+2,..., or p

$$- \frac{2}{\sigma_O^4} \sum_{i=p-k^*+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{\ell=1}^{p} a_{MCijk\ell} z_i z_j z_k z_\ell ,$$

where

$a^*_{ijk} = a_{ijk}\sigma_O$,

$a_{MCijk\ell} = a_{Mijk\ell}\sigma_O^2$, $(i = p-k^*+1,\ p-k^*+2,\ldots,p;\ j,k,\ell = 1,2,\ldots,p-k^*)$,

$a_{MCijk\ell} = 0$, $(i = p-k^*+1, p-k^*+2,\ldots,p;\ j,k,\ell = 1,2,\ldots,p$ and at least one of j,k,ℓ is equal to $p-k^*+1$ or $p-k^*+2,\ldots,$ or $p)$

$a_{MCijk\ell} = a_{Mijk\ell}\sigma_O^2 - a_{ijk\ell}\sigma_O^2$, $(i = p+1,p+2,\ldots,n;\ j,k,\ell = 1,2,\ldots,p-k^*)$

and

$a_{MCijk\ell} = -a_{ijk\ell}\sigma_O^2$, $(i = p+1,p+2,\ldots,n;\ j,k,\ell = 1,2,\ldots,p$ and at least one of j,k,ℓ is equal to $p-k^*+1$ or $p-k^*+2,\ldots,$ or $p)$;

(3.3.13)
$$S_2^D(\underset{\sim}{z}) = \sum_{j=1}^{p} z_j^2/\sigma_O^2$$

$$+ \frac{2}{\sigma_O^3} \sum_{i=p+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} a^*_{ijk} z_i z_j z_k$$

$$- \frac{1}{\sigma_O^4} \sum_{i=p+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{\ell=1}^{p} \sum_{m=1}^{p} a^*_{ijk} a^*_{i\ell m} z_j z_k z_\ell z_m$$

$$+ \frac{4}{\sigma_O^4} \sum_{h=p+1}^{n} \sum_{i=p+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{\ell=1}^{p} a^*_{hj\ell} a^*_{ik\ell} z_h z_i z_j z_k$$

$$+ \frac{2}{\sigma_O^4} \sum_{i=p+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{\ell=1}^{p} a^*_{ijk\ell} z_i z_j z_k z_\ell ,$$

where $a^*_{ijk\ell} = a_{ijk\ell}\sigma_0^2$;

(3.3.14) $$S_3^D(\underset{\sim}{z}) = \sum_{j=p-k^*+1}^{p} z_j^2/\sigma_0^2$$

$$- \frac{2}{\sigma_0^3} \sum_{i=p-k^*+1}^{p} \sum_{j=1}^{p-k^*} \sum_{k=1}^{p-k^*} a^*_{ijk} z_i z_j z_k$$

$$+ \frac{2}{\sigma_0^3} \sum_{i=p+1}^{n} \sum_{j=1}^{p-k^*} \sum_{k=1}^{p-k^*} \frac{k^*F}{n-p} a^*_{ijk} z_i z_j z_k$$

$$+ \frac{2}{\sigma_0^3} \sum_{i=p+1}^{n} \sum_{j,k=1,2,\ldots,p} \left(1 + \frac{k^*F}{n-p}\right) a^*_{ijk} z_i z_j z_k$$

and at least one of j,k is equal to p-k*+1 or p-k*+2,..., or p

$$+ \frac{1}{\sigma_0^4} \sum_{i=p-k^*+1}^{p} \left(\sum_{j=1}^{p-k^*} \sum_{k=1}^{p-k^*} a^*_{ijk} z_j z_k \right)^2$$

$$- \frac{1}{\sigma_0^4} \sum_{i=p+1}^{n} \left(\sum_{j=1}^{p-k^*} \sum_{k=1}^{p-k^*} \sqrt{\frac{k^*F}{n-p}}\, a^*_{ijk} z_j z_k \right)^2$$

$$- \frac{1}{\sigma_0^4} \sum_{i=p+1}^{n} \sum_{j,k,\ell,m=1,2,\ldots,p,} \left(1 + \frac{k^*F}{n-p}\right) a^*_{ijk} a^*_{i\ell m} z_j z_k z_\ell z_m$$

and at least one of j,k,ℓ,m is equal to p-k*+1 or p-k*+2,..., or p

$$- \frac{4}{\sigma_0^4} \sum_{h,i=p-k^*+1,p-k^*+2,\ldots,n} \sum_{j=1}^{p-k^*} \sum_{k=1}^{p-k^*} \sum_{\ell=1}^{p-k^*} a^*_{hj\ell} a^*_{ik\ell} z_h z_i z_j z_k$$

and at least one of h,i is equal to p-k*+1 or p-k*+2,..., or p

$$+ \frac{4}{\sigma_0^4} \sum_{h=p+1}^{n} \sum_{i=p+1}^{n} \sum_{j=1}^{p-k^*} \sum_{k=1}^{p-k^*} \sum_{\ell=1}^{p-k^*} \frac{k^*F}{n-p} a^*_{hj\ell} a^*_{ik\ell} z_h z_i z_j z_k$$

$$+ \frac{4}{\sigma_0^4} \sum_{h=p+1}^{n} \sum_{i=p+1}^{n} \sum_{j,k,\ell=1,2,\ldots,p} (1+ \frac{k^*F}{n-p}) a^*_{hj\ell} a^*_{ik\ell} z_h z_i z_j z_k$$

and at least one of j,k,ℓ is equal to $p-k^*+1$ or $p-k^*+2,\ldots,$ or p.

$$- \frac{2}{\sigma_0^4} \sum_{i=p-k^*+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{\ell=1}^{p} a_{MFijk\ell} z_i z_j z_k z_\ell ,$$

where

$F = F_\alpha(k^*, n-p)$,

$a_{MFijk\ell} = a_{Mijk\ell}\sigma_0^2$, $(i = p-k^*+1, p-k^*+2,\ldots,p;\ j,k,\ell = 1,2,\ldots,p-k^*)$,

$a_{MFijk\ell} = 0$, $(i = p-k^*+1, p-k^*+2,\ldots,p;\ j,k,\ell = 1,2,\ldots,p$ and at least one of j,k,ℓ is equal to $p-k^*+1$ or $p-k^*+2,\ldots,$ or $p)$,

$a_{MFijk\ell} = a_{Mijk\ell}\sigma_0^2 - (1+ \frac{k^*F}{n-p}) a_{ijk\ell}\sigma_0^2$, $(i = p+1, p+2,\ldots,n;\ j,k,\ell = 1,2,\ldots,p-k^*)$,

$a_{MFijk\ell} = -(1+ \frac{k^*F}{n-p}) a_{ijk\ell}\sigma_0^2$, $(i = p+1,p+2,\ldots,n;\ j,k,\ell = 1,2,\ldots,p$ and at least one of j,k,ℓ is equal to $p-k^*+1$ or $p-k^*+2,\ldots,$ or $p)$;

(3.3.15) $$S_4^D(\underset{\sim}{z}) = \sum_{j=1}^{p} z_j^2$$

$$+ \frac{2}{\sigma_0^3} \sum_{i=p+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} (1 + \frac{pF}{n-p}) a^*_{ijk} z_i z_j z_k$$

$$- \frac{1}{\sigma_0^4} \sum_{i=p+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{\ell=1}^{p} \sum_{m=1}^{p} (1+ \frac{pF}{n-p}) a^*_{ijk} a^*_{i\ell m} z_j z_k z_\ell z_m$$

$$+ \frac{4}{\sigma_0^4} \sum_{h=p+1}^{n} \sum_{i=p+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{\ell=1}^{p} \left(1 + \frac{pF}{n-p}\right) a^*_{hj\ell} a^*_{ik\ell} z_h z_i z_j z_k$$

$$+ \frac{2}{\sigma_0^4} \sum_{i=p+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{\ell=1}^{p} \left(1+ \frac{pF}{n-p}\right) a^*_{ijk\ell} z_i z_j z_k z_\ell ,$$

where

$$F = F_\alpha(p, n-p);$$

and

$$(3.3.16) \quad S_5^D(\underset{\sim}{z}) = \sum_{j=p+1}^{n} z_j^2$$

$$- \frac{2}{\sigma_0^3} \sum_{i=p+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} a^*_{ijk} z_i z_j z_k$$

$$+ \frac{1}{\sigma_0^4} \sum_{i=p+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{\ell=1}^{p} \sum_{m=1}^{p} a^*_{ijk} a^*_{i\ell m} z_j z_k z_\ell z_m$$

$$- \frac{4}{\sigma_0^4} \sum_{h=p+1}^{n} \sum_{i=p+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{\ell=1}^{p} a^*_{hj\ell} a^*_{ik\ell} z_h z_i z_j z_k$$

$$- \frac{2}{\sigma_0^4} \sum_{i=p+1}^{n} \sum_{j=1}^{p} \sum_{k=1}^{p} \sum_{\ell=1}^{p} a^*_{ijk\ell} z_i z_j z_k z_\ell .$$

Given that $\underset{\sim}{\theta} = \underset{\sim}{\theta}_0$ and $\sigma = \sigma_0$, z_j/σ_0 $(j = 1,2,\ldots,n)$ are independently distributed as $N(0,1)$. Therefore as far as deriving approximations of the probabilities $I_i(\underset{\sim}{\theta}_0, \sigma_0)$ is concerned, we can set σ_0 appearing in the expressions of $S_i^D(\underset{\sim}{z})$ given by (3.3.12)-(3.3.16) to be one while leaving the σ_0 appearing in the expressions of the $d_i^{*2}, a^*_{ijk}, a^*_{ijk\ell}, a_{MCijk\ell}$ and $a_{MFijk\ell}$ unchanged, and treat the z_j as being distributed as $N(0,1)$.

Let $z_j^{(s)}$ denote z_j^2. Each of $z_j^{(s)}$ is then distributed as χ_1^2, and conditional on z_j being negative (or non-negative), $z_j^{(s)}$ is still distributed as χ_1^2. Further, let $E_{z_{i_1},z_{i_2},\ldots,z_{i_k}}\{f(z_{i_1},z_{i_2},\ldots,z_{i_k})\}$ denote the expectation of $f(z_{i_1},z_{i_2},\ldots,z_{i_k})$. The probabilities $I_i(\underset{\sim}{\theta}_0,\sigma_0)$ can now be written as

(3.3.17) $$I_i(\underset{\sim}{\theta}_0,\sigma_0) = E_{z_{p+1},z_{p+2},\ldots,z_n,z_1,z_2,\ldots,z_{p-k^*}}$$

$$\Bigg\{ \sum_{s_{p-k^*+1}=-1,+1} \; \sum_{s_{p-k^*+2}=-1,+1} \cdots \sum_{s_p=-1,+1}$$

$$\Bigg[\frac{1}{2^{k^*}} \int_{z^{(s)}_{p-k^*+1}} \int_{z^{(s)}_{p-k^*+2}} \cdots \int_{z^{(s)}_p} \chi_1^{\,2}(z^{(s)}_{p-k^*+1})\chi_1^{\,2}(z^{(s)}_{p-k^*+2})\cdots\chi_1^{\,2}(z^{(s)}_p)$$

$$(z^{(s)}_{p-k^*+1},z^{(s)}_{p-k^*+2},\ldots,z^{(s)}_p)ez_i^*$$

$$\times\, dz^{(s)}_{p-k^*+1}dz^{(s)}_{p-k^*+2}\cdots dz^{(s)}_p \Bigg]\Bigg\} \quad , \ (i = 1 \text{ and } 3),$$

(3.3.18) $$I_i(\underset{\sim}{\theta}_0,\sigma_0) = E_{z_{p+1},z_{p+2},\ldots,z_n}$$

$$\Bigg\{ \sum_{s_1=-1,+1} \; \sum_{s_2=-1,+1} \cdots \sum_{s_p=-1,+1}$$

$$\Bigg[\frac{1}{2^p} \int_{z_1^{(s)}} \int_{z_2^{(s)}} \cdots \int_{z_p^{(s)}} \chi_1^2(z_1^{(s)})\chi_1^2(z_2^{(s)})\cdots\chi_1^2(z_p^{(s)})$$

$$(z_1^{(s)},z_2^{(s)},\ldots,z_p^{(s)})ez_i^*$$

$$\times\, dz_1^{(s)}dz_2^{(s)}\cdots dz_p^{(s)} \Bigg]\Bigg\} \quad , \ (i = 2 \text{ and } 4),$$

and

$$(3.3.19)\quad I_5(\underset{\sim}{\theta}_0,\sigma_0) = E_{z_1,z_2,\ldots,z_p}$$

$$\left\{ \sum_{s_{p+1}=-1,+1} \; \sum_{s_{p+2}=-1,+1} \cdots \sum_{s_n=-1,+1} \right.$$

$$\left[\frac{1}{2^{n-p}} \underset{(z_{p+1}^{(s)},z_{p+2}^{(s)},\ldots,z_n^{(s)}) \in Z_5^*}{\int_{z_{p+1}^{(s)}} \int_{z_{p+2}^{(s)}} \cdots \int_{z_n^{(s)}}} \chi_1^2(z_{p+1}^{(s)})\chi_1^2(z_{p+2}^{(s)})\cdots\chi_1^2(z_n^{(s)})dz_{p+1}^{(s)}dz_{p+2}^{(s)}\cdots dz_n^{(s)}\right]\Bigg\} ,$$

where

$$(3.3.20)\quad Z_i^* = \Big\{ (z_{p-k^*+1}^{(s)},z_{p-k^*+2}^{(s)},\ldots,z_p^{(s)}) :$$

$$S_i^D(z_1,z_2,\ldots,z_{p-k^*},s_{p-k^*+1}\sqrt{z_{p-k^*+1}^{(s)}},\; s_{p-k^*+2}\sqrt{z_{p-k^*+2}^{(s)}},\ldots,$$

$$s_p\sqrt{z_p^{(s)}},z_{p+1},z_{p+2},\ldots,z_n) \le d_i^{*2}\Big\},$$

(i = 1 and 3),

$$(3.3.21)\quad Z_i^* = \Big\{ (z_1^{(s)},z_2^{(s)},\ldots,z_p^{(s)}) :$$

$$S_i^D(s_1\sqrt{z_1^{(s)}},\; s_2\sqrt{z_2^{(s)}},\ldots,s_p\sqrt{z_p^{(s)}},z_{p+1},z_{p+2},\ldots,z_n) \le d_i^{*2}\Big\},$$

(i = 2 and 4),

(3.3.22) $$z_5^* = \left\{ (z_{p+1}^{(s)}, z_{p+2}^{(s)}, \ldots, z_n^{(s)}) : \right.$$

$$\left. S_5^D(z_1, z_2, \ldots, z_p, s_{p+1}\sqrt{z_{p+1}^{(s)}}, s_{p+2}\sqrt{z_{p+2}^{(s)}}, \ldots, s_n\sqrt{z_n^{(s)}}) \le d_5^{*2} \right\},$$

(3.3.23) $$s_i = \begin{cases} -1 & \text{if } z_i < 0 \\ +1 & \text{if } z_i \ge 0 \end{cases}, \qquad (i = 1,2,\ldots,n),$$

and for $k \ge 1$,

$$\chi_k^2(t) = \frac{e^{-t/2}\, t^{\frac{k}{2}-1}}{2^{k/2}\,\Gamma(k/2)}, \quad (t \ge 0) .$$

For $i = 1$ and 3, we apply the transformations

(3.3.24) $$r^{(s)} = \sum_{j=p-k^*+1}^{p} z_j^{(s)}$$

and

(3.3.25) $$\bar{z}_j^{(s)} = \frac{1}{r^{(s)}} z_j^{(s)} \qquad \text{for } j = p-k^*+1,\ p-k^*+2, \ldots, p-1.$$

For $i = 2$ and 4, we apply the transformations

(3.3.26) $$r^{(s)} = \sum_{j=1}^{p} z_j^{(s)}$$

and

(3.3.27) $$\bar{z}_j^{(s)} = \frac{1}{r^{(s)}} z_j^{(s)} \qquad \text{for } j = 1,2,\ldots,p-1.$$

For i = 5, we apply the transformations

(3.3.28) $$r^{(s)} = \sum_{j=p+1}^{n} z_j^{(s)}$$

and

(3.3.29) $$\bar{z}_j^{(s)} = \frac{1}{r^{(s)}} z_j^{(s)} \quad \text{for } j = p+1, p+2, \ldots, n-1.$$

The probabilities $I_i(\underset{\sim}{\theta}_0, \sigma_0)$ can then be written as

(3.3.30) $$I_i(\underset{\sim}{\theta}_0, \sigma_0) = E_{z_{p+1}, z_{p+2}, \ldots, z_n, z_1, z_2, \ldots, z_{p-k^*}}$$

$$\Bigg\{ \sum_{s_{p-k^*+1}=-1,+1} \; \sum_{s_{p-k^*+2}=1,+1} \cdots \sum_{s_p=-1,+1}$$

$$\Bigg[\frac{1}{2^{k^*}} \int_{\bar{z}^{(s)}_{p-k^*+1}=0}^{1} \; \int_{\bar{z}^{(s)}_{p-k^*+2}=\bar{z}^{(s)}_{p-k^*+1}}^{1} \cdots \int_{\bar{z}^{(s)}_{p-1}=\bar{z}^{(s)}_{p-2}}^{1}$$

$$\chi_1^2(\bar{z}^{(s)}_{p-k^*+1}) \chi_1^2(\bar{z}^{(s)}_{p-k^*+2}) \cdots \chi_1^2(\bar{z}^{(s)}_{p-1}) \chi_1^2(1 - \sum_{j=p-k^*+1}^{p-1} \bar{z}_j^{(s)}) [\chi_{k^*}^2(1)]^{-1}$$

$$\times \Big(\int_{R_i^*} \chi_{k^*}^2(r^{(s)}) d(r^{(s)}) \Big)$$

$$\times \, d\bar{z}^{(s)}_{p-k^*+1} d\bar{z}^{(s)}_{p-k^*+2} \cdots d\bar{z}^{(s)}_{p-1} \Bigg] \Bigg\} \quad , \quad (i = 1 \text{ and } 3),$$

(3.3.31)

$$I_i(\underset{\sim}{\theta}_0, \sigma_0) = E_{z_{p+1}, z_{p+2}, \ldots, z_n}$$

$$\left\{ \sum_{s_1=-1,+1} \sum_{s_2=-1,+1} \cdots \sum_{s_p=-1,+1} \right.$$

$$\left[\frac{1}{2^p} \int_{\bar{z}_1^{(s)}=0}^{1} \int_{\bar{z}_2^{(s)}=\bar{z}_1^{(s)}}^{1} \cdots \int_{\bar{z}_{p-1}^{(s)}=\bar{z}_{p-2}^{(s)}}^{1} \chi_1^2(\bar{z}_1^{(s)}) \chi_1^2(\bar{z}_2^{(s)}) \ldots \chi_1^2(\bar{z}_{p-1}^{(s)}) \right.$$

$$\times \chi_1^2(1 - \sum_{j=1}^{p-1} \bar{z}_j^{(s)}) [\chi_p^2(1)]^{-1}$$

$$\times (\int_{R_i^*} \chi_p^2(r^{(s)}) d(r^{(s)}))$$

$$\left. \left. \times d\bar{z}_1^{(s)} d\bar{z}_2^{(s)} \ldots d\bar{z}_{p-1}^{(s)} \right] \right\} , \quad (i = 2 \text{ and } 4),$$

and

(3.3.32)

$$I_5(\underset{\sim}{\theta}_0, \theta_0) = E_{z_1, z_2, \ldots, z_p}$$

$$\left\{ \sum_{s_{p+1}=-1,+1} \sum_{s_{p+2}=-1,+1} \cdots \sum_{s_n=-1,+1} \right.$$

$$\left[\frac{1}{2^{n-p}} \int_{\bar{z}_{p+1}^{(s)}=0}^{1} \int_{\bar{z}_{p+2}^{(s)}=\bar{z}_{p+1}^{(s)}}^{1} \int_{\bar{z}_{n-1}^{(s)}=\bar{z}_{n-2}^{(s)}}^{1} \chi_1^2(\bar{z}_{p+1}^{(s)}) \chi_1^2(\bar{z}_{p+2}^{(s)}) \ldots \chi_1^2(\bar{z}_{n-1}^{(s)}) \right.$$

$$\times \chi_1^2(1 - \sum_{j=p+1}^{n-1} \bar{z}_j^{(s)}) \; [\chi_{n-p}^2(1)]^{-1}$$

$$\times \left(\int_{R_5^*} \chi^2_{n-p}(r^{(s)})\,d(r^{(s)}) \right)$$

$$\times\, d\bar{z}^{(s)}_{p+1}\, d\bar{z}^{(s)}_{p+2} \cdots d\bar{z}^{(s)}_{n-1} \Big] \Big\} ,$$

where

(3.3.33)

$$R_i^* = \{r^{(s)} : S_i^D(z_1, z_2, \ldots, z_{p-k^*}, s_{p-k^*+1}\sqrt{r^{(s)}\bar{z}^{(s)}_{p-k^*+1}}, s_{p-k^*+2}\sqrt{r^{(s)}\bar{z}^{(s)}_{p-k^*+2}}, \ldots,$$

$$\ldots, s_{p-1}\sqrt{r^{(s)}\bar{z}^{(s)}_{p-1}}, s_p\sqrt{r^{(s)}(1-\sum_{j=p-k^*+1}^{p-1}\bar{z}^{(s)}_j)}, z_{p+1}, z_{p+2}, \ldots, z_n) \le d_i^{*2}\}$$

$$, \quad (i = 1 \text{ and } 3),$$

(3.3.34)

$$R_i^* = \{r^{(s)} : S_i^D(s_1\sqrt{r^{(s)}\bar{z}^{(s)}_1}, s_2\sqrt{r^{(s)}\bar{z}^{(s)}_2}, \ldots, s_{p-1}\sqrt{r^{(s)}\bar{z}^{(s)}_{p-1}},\ s_p\sqrt{r^{(s)}(1-\sum_{j=1}^{p-1}\bar{z}^{(s)}_j)},$$

$$z_{p+1}, z_{p+2}, \ldots, z_n) \le d_i^{*2}\}, \quad (i = 2 \text{ and } 4),$$

and

(3.3.35)

$$R_5^* = \{r^{(s)} : S_5^D(z_1, z_2, \ldots, z_p, s_{p+1}\sqrt{r^{(s)}\bar{z}^{(s)}_{p+1}},\ s_{p+2}\sqrt{r^{(s)}\bar{z}^{(s)}_{p+2}},$$

$$\ldots, s_{n-1}\sqrt{r^{(s)}\bar{z}^{(s)}_{n-1}},\ s_n\sqrt{r^{(s)}(1-\sum_{p+1}^{n-1}\bar{z}^{(s)}_j)}) \le d_5^{*2}\} .$$

To the extent that the approximations of $S_i^D(\underset{\sim}{z})$ given by (3.3.12)-(3.3.16) are adequate, we can obtain a truncated series expansion of the probability $I_i(\underset{\sim}{\theta}_0, \sigma_0)$ as follows:

(3.3.36)

$$I_i(\underset{\sim}{\theta}_O,\sigma_O) \simeq (1-\alpha) + \sum_{j=1}^{n^*} I_{ia_j^*}\, a_j^* + \frac{1}{2}\sum_{j=1}^{n^*}\sum_{k=1}^{n^*} I_{ia_j^* a_k^*}\, a_j^* a_k^*$$

where $n^* = \frac{1}{2}\, p(p+1)n$,

$\{a_1^*, a_2^*, \ldots, a_{n^*}^*\}$ is the set containing all the a_{hjk}^* for which $j \le k$,

$$I_{ia_j^*} = \left[\frac{\partial I_i(\underset{\sim}{\theta}_O,\sigma_O)}{\partial\, a_j^*}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} ,$$

$$I_{ia_j^* a_k^*} = \left[\frac{\partial^2 I_i(\underset{\sim}{\theta}_O,\sigma_O)}{\partial a_j^*\ \partial a_k^*}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}}$$

and $\underset{\sim}{a}^*$ is a column vector whose components are the a_{hjk}^*, and the $a_{hjk\ell}^*$, $a_{MChjk\ell}$ or $a_{MFhjk\ell}$ for which $j \le k \le \ell$.

We now consider how $I_{ia_j^*}$ and $I_{ia_j^* a_k^*}$ can be derived. We first note that provided that the magnitude $|\underset{\sim}{a}^*|$ of $\underset{\sim}{a}^*$ is sufficiently small, the sets R_i^* can be written as

$$R_i^* = \{r^{(s)} : 0 \le r^{(s)} \le r_i^{(*s)}\} ,$$

where $r_i^{(*s)}$ are such that

(3.3.37)

$$S_i^D\Big(z_1, z_2, \ldots, z_{p-k^*}, s_{p-k^*+1}\sqrt{r_i^{(*s)}\bar{z}_{p-k^*+1}^{(s)}},\ s_{p-k^*+2}\sqrt{r_i^{(*s)}\bar{z}_{p-k^*+2}^{(s)}}, \ldots$$
$$\ldots, s_{p-1}\sqrt{r_i^{(*s)}\bar{z}_{p-1}^{(s)}}, s_p\sqrt{r_i^{(*s)}\Big(1 - \sum_{j=p-k^*+1}^{p-1}\bar{z}_j^{(s)}\Big)}, z_{p+1}, z_{p+2}, \ldots, z_n\Big) = d_i^{*2}$$

$$, \quad (i = 1,3),$$

(3.3.38)

$$s_i^D(s_1\sqrt{r_i^{(*s)}\bar{z}_1^{(s)}}, s_2\sqrt{r_i^{(*s)}\bar{z}_2^{(s)}}, \ldots, s_{p-1}\sqrt{r_i^{(*s)}\bar{z}_{p-1}^{(s)}},$$

$$s_p\sqrt{r_i^{(*s)}(1 - \sum_{j=1}^{p-1}\bar{z}_j^{(s)})},\ z_{p+1}, z_{p+2}, \ldots, z_n) = d_i^{*2},$$

$$(i = 2,4),$$

and

(3.3.39)

$$s_5^D(z_1, z_2, \ldots, z_p, s_{p+1}\sqrt{r_5^{(*s)}\bar{z}_{p+1}^{(s)}}, s_{p+2}\sqrt{r_5^{(*s)}\bar{z}_{p+2}^{(s)}}, \ldots, s_{n-1}\sqrt{r_5^{(*s)}\bar{z}_{n-1}^{(s)}},$$

$$s_n\sqrt{r_5^{(*s)}(1 - \sum_{j=p+1}^{n-1}\bar{z}_j^{(s)})}) = d_5^{*2}, \quad (i = 5).$$

The equations (3.3.37), (3.3.38) and (3.3.39) are of the form

$$(3.3.40)\quad r_i^{(*s)} + \sum_{j=1}^{n*} h_j^*(r_i^{(*s)})a_j^* + \sum_{j=1}^{n*}\sum_{k=1}^{n*} h_{jk}^*(r_i^{(*s)})a_j^*a_k^* + b^* = d_i^{*2},$$

where $h_j^*(r_i^{(*s)})$ and $h_{jk}^*(r_i^{(*s)})$ are functions of $r_i^{(*s)}$, the z_ℓ and $\bar{z}_m^{(s)}$, and b^* is a function of the z_ℓ and $\bar{z}_m^{(s)}$, and the $a_{ijk\ell}^*$, $a_{MCijk\ell}$ or $a_{MFijk\ell}$. Note that as b^* does not depend on the a_j^*, we have

$$\left[\frac{\partial b^*}{\partial a_j^*}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} = 0, \quad (j = 1,2,\ldots,n^*),$$

and

$$\left[\frac{\partial^2 b^*}{\partial a_j^*\,\partial a_k^*}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} = 0, \quad (j,k = 1,2,\ldots,n^*).$$

By differentiating both sides of (3.3.40) with respect to a_j^*, we get

$$(3.3.41)\quad \frac{\partial r_i^{(*s)}}{\partial a_j^*} + h_j^*(r_i^{(*s)}) + \sum_{j_1=1}^{n^*} \frac{\partial h_{j_1}^*(r_i^{(*s)})}{\partial r_i^{(*s)}} \frac{\partial r_i^{(*s)}}{\partial a_j^*} a_{j_1}^*$$

$$+ 2\sum_{k=1}^{n^*} h_{jk}^*(r_i^{(*s)}) a_k^* + \sum_{j_1=1}^{n^*} \sum_{k=1}^{n^*} \frac{\partial h_{j_1 k}^*(r_i^{(*s)})}{\partial r_i^{(*s)}} \frac{\partial r_i^{(*s)}}{\partial a_j^*} a_{j_1}^* a_k^*$$

$$+ \frac{\partial b^*}{\partial a_j^*} = 0.$$

Hence $$\left[\frac{\partial r_i^{(*s)}}{\partial a_j^*}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} = -h_j^*(d_i^{*2}) .$$

By differentiating both sides of (3.3.41) with respect to a_k^* and subsequently setting $\underset{\sim}{a}^*$ to $\underset{\sim}{0}$, we get

$$(3.3.42)\quad \left[\frac{\partial^2 r_i^{(*s)}}{\partial a_j^* \partial a_k^*}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} = -\left[\frac{\partial h_j^*(r_i^{(*s)})}{\partial r_i^{(*s)}} \frac{\partial r_i^{(*s)}}{\partial a_k^*}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} - \left[\frac{\partial h_k^*(r_i^{(*s)})}{\partial r_i^{(*s)}} \frac{\partial r_i^{(*s)}}{\partial a_j^*}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} -2h_{jk}^*(d_i^{*2}).$$

From the integrability and continuity of the function $\chi_k^2(r^{(s)})$ and the differentiability of $r_i^{(*s)}$ with respect to the a_j^*, we have

$$\left[\frac{\partial}{\partial a_j^*} \int_{r^{(s)}=0}^{r_i^{(*s)}} \chi_k^2(r^{(s)})\,d(r^{(s)})\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} = \chi_k^2(d_i^{*2}) \left[\frac{\partial r_i^{(*s)}}{\partial a_j^*}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}}$$

and

$$\left[\frac{\partial^2}{\partial a_j^* \partial a_k^*} \int_{r^{(s)}=0}^{r_i^{(*s)}} \chi_k^2(r^{(s)})\, d(r^{(s)})\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}}$$

$$= \chi_k^2(d_i^{*2}) \left[\left[\left(\frac{k}{2}-1\right)\frac{1}{d_i^{*2}} - \frac{1}{2}\right]\left[\frac{\partial r_i^{(*s)}}{\partial a_j^*}\,\frac{\partial r_i^{(*s)}}{\partial a_k^*}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} + \left[\frac{\partial^2 r_i^{(*s)}}{\partial a_j^* \partial a_k^*}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}}\right],$$

where $k = k^*$, p or $n-p$.

Then by showing that conditions similar to those which justify the validity of

$$\left[\frac{\partial}{\partial a} \int_{-\infty}^{\infty} g(z,a)\,dz\right]_{a=a_0} = \int_{-\infty}^{\infty} \left[\frac{\partial g(z,a)}{\partial a}\right]_{a=a_0} dz ,$$

where $\int_{-\infty}^{\infty} g(z,a)\,dz$ exists, are satisfied, we get

(3.3.43)

$$I_{ia_j^*} = E_{z_{p+1},z_{p+2},\ldots,z_n,z_1,z_2,\ldots,z_{p-k^*}}$$

$$\left\{ \sum_{s_{p-k^*+1}=-1,+1} \; \sum_{s_{p-k^*+2}=-1,+1} \cdots \sum_{s_p=-1,+1} \right.$$

$$\left[\frac{1}{2^{k^*}} \int_{\bar{z}_{p-k^*+1}^{(s)}=0}^{1} \; \int_{\bar{z}_{p-k^*+2}^{(s)}=\bar{z}_{p-k^*+1}^{(s)}}^{1} \cdots \int_{\bar{z}_{p-1}^{(s)}=\bar{z}_{p-2}^{(s)}}^{1} \chi_1^2(\bar{z}_{p-k^*+1}^{(s)})\,\chi_1^2(\bar{z}_{p-k^*+2}^{(s)}) \right.$$

$$\cdots \chi_1^2(\bar{z}_{p-1}^{(s)})$$

$$\times\, \chi_1^2\Big(1 - \sum_{j_1=p-k^*+1}^{p-1} \bar{z}_{j_1}^{(s)}\Big)\,[\chi_{k^*}^2(1)]^{-1}\,\chi_{k^*}^2(d_i^{*2}) \left[\frac{\partial r_i^{(*s)}}{\partial a_j^*}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}}$$

$$\times \; d\bar{z}^{(s)}_{p-k^*+1} d\bar{z}^{(s)}_{p-k^*+2} \cdots d\bar{z}^{(s)}_{p-1} \Bigg] \Bigg\} \qquad , \; (i = 1 \text{ and } 3),$$

(3.3.44)

$$I_{ia^*_j} = E_{z_{p+1}, z_{p+2}, \ldots, z_n}$$

$$\Bigg\{ \sum_{s_1=-1,+1} \; \sum_{s_2=-1,+1} \cdots \sum_{s_p=-1,+1}$$

$$\Bigg[\frac{1}{2^p} \int_{\bar{z}^{(s)}_1=0}^{1} \int_{\bar{z}^{(s)}_2=\bar{z}^{(s)}_1}^{1} \cdots \int_{\bar{z}^{(s)}_{p-1}=\bar{z}^{(s)}_{p-2}}^{1} \chi^2_1(\bar{z}^{(s)}_1) \chi^2_1(\bar{z}^{(s)}_2) \ldots \chi^2_1(\bar{z}^{(s)}_{p-1})$$

$$\times \; \chi^2_1 (1 - \sum_{j_1=1}^{p-1} \bar{z}^{(s)}_{j_1}) \; [\chi^2_p(1)]^{-1} \; \chi^2_p \; (d^{*2}_i) \left[\frac{\partial r^{(*s)}_i}{\partial a^*_j} \right]_{\underset{\sim}{a}^*=\underset{\sim}{0}}$$

$$\times \; d\bar{z}^{(s)}_1 \; d\bar{z}^{(s)}_2 \; \ldots \; d\bar{z}^{(s)}_{p-1} \Bigg] \Bigg\}, \qquad (i = 2 \text{ and } 4),$$

(3.3.45)

$$I_{5a^*_j} = E_{z_1, z_2, \ldots, z_p}$$

$$\Bigg\{ \sum_{s_{p+1}=-1,+1} \; \sum_{s_{p+2}=-1,+1} \cdots \sum_{s_n=-1,+1}$$

$$\Bigg[\frac{1}{2^{n-p}} \int_{\bar{z}^{(s)}_{p+1}=0}^{1} \int_{\bar{z}^{(s)}_{p+2}=\bar{z}^{(s)}_{p+1}}^{1} \cdots \int_{\bar{z}^{(s)}_{n-1}=\bar{z}^{(s)}_{n-2}}^{1} \chi^2_1(\bar{z}^{(s)}_{p+1}) \chi^2_1(\bar{z}^{(s)}_{p+2}) \ldots \chi^2_1(\bar{z}^{(s)}_{n-1})$$

$$\times \; \chi^2_1 (1 - \sum_{j_1=p+1}^{n-1} \bar{z}^{(s)}_{j_1}) [\chi^2_{n-p}(1)]^{-1} \; \chi^2_{n-p} (d^{*2}_5) \left[\frac{\partial r^{(*s)}_5}{\partial a^*_j} \right]_{\underset{\sim}{a}^*=\underset{\sim}{0}}$$

$$\times d\bar{z}_{p+1}^{(s)}\, d\bar{z}_{p+2}^{(s)} \ldots d\bar{z}_{n-1}^{(s)} \Big] \Bigg\} \quad , \qquad (i = 5),$$

(3.3.46)

$$I_{ia_j^*a_k^*} = E_{z_{p+1}z_{p+2},\ldots,z_n,z_1,z_2,\ldots,z_{p-k^*}} \Bigg\{ \sum_{s_{p-k^*+1}=-1,+1} \sum_{s_{p-k^*+2}=-1,+1} \cdots \sum_{s_p=-1,+1}$$

$$\Bigg[\frac{1}{2^{k^*}} \int_{\bar{z}_{p-k^*+1}^{(s)}=0}^{1} \int_{\bar{z}_{p-k^*+2}^{(s)}=\bar{z}_{p-k^*+1}^{(s)}}^{1} \cdots \int_{\bar{z}_{p-1}^{(s)}=\bar{z}_{p-2}^{(s)}}^{1}$$

$$\chi_1^2(\bar{z}_{p-k^*+1}^{(s)})\,\chi_1^2(\bar{z}_{p-k^*+2}^{(s)}) \ldots \chi_1^2(\bar{z}_{p-1}^{(s)})$$

$$\times \chi_1^2\Big(1 - \sum_{j_1=p-k^*+1}^{p-1} \bar{z}_{j_1}^{(s)}\Big)[\chi_{k^*}^2(1)]^{-1}\chi_{k^*}^2(d_i^{*2})\Bigg[\Big[\Big(\frac{k^*}{2}-1\Big)\frac{1}{d_i^{*2}}-\frac{1}{2}\Big]\Big[\frac{\partial r_i^{(*s)}}{\partial a_j^*}\,\frac{\partial r_i^{(*s)}}{\partial a_k^*}\Big]_{\underset{\sim}{a}^*=\underset{\sim}{0}}$$

$$+ \Big[\frac{\partial^2 r_i^{(*s)}}{\partial a_j^*\partial a_k^*}\Big]_{\underset{\sim}{a}^*=\underset{\sim}{0}}\Bigg]$$

$$\times\, d\bar{z}_{p-k^*+1}^{(s)}\, d\bar{z}_{p-k^*+2}^{(s)} \ldots d\bar{z}_{p-1}^{(s)} \Bigg] \Bigg\} \quad , \quad (i = 1 \text{ and } 3),$$

(3.3.47)

$$I_{ia_j^*a_k^*} = E_{z_{p+1}z_{p+2},\ldots,z_n}$$

$$\Bigg\{ \sum_{s_1=-1,+1} \sum_{s_2=-1,+1} \cdots \sum_{s_p=-1,+1}$$

$$\Bigg[\frac{1}{2^p} \int_{\bar{z}_1^{(s)}=0}^{1} \int_{\bar{z}_2^{(s)}=\bar{z}_1^{(s)}}^{1} \cdots \int_{\bar{z}_{p-1}^{(s)}=\bar{z}_{p-2}^{(s)}}^{1} \chi_1^2(\bar{z}_1^{(s)})\,\chi_1^2(\bar{z}_2^{(s)}) \ldots \chi_1^2(\bar{z}_{p-1}^{(s)})$$

$$\chi_1^2(1 - \sum_{j_1=1}^{p-1} \bar{z}_{j_1}^{(s)})[\chi_p^2(1)]^{-1}\chi_p^2(d_i^{*2}) \Bigg[\Big[(\frac{p}{2}-1)\frac{1}{d_i^{*2}} - \frac{1}{2}\Big]\Big[\frac{\partial r_i^{(*s)}}{\partial a_j^*}\frac{\partial r_i^{(*s)}}{\partial a_k^*}\Big]_{\underset{\sim}{a}^*=\underset{\sim}{0}} + \Big[\frac{\partial^2 r_i^{(*s)}}{\partial a_j^* \partial a_k^*}\Big]_{\underset{\sim}{a}^*=\underset{\sim}{0}}\Bigg]$$

$$\times\, d\bar{z}_1^{(s)}\, d\bar{z}_2^{(s)} \ldots d\bar{z}_{p-1}^{(s)} \Bigg]\Bigg\} \qquad , \ (i = 2 \text{ and } 4),$$

and

(3.3.48)

$$I_{5a_j^*a_k^*} = E_{z_1,z_2,\ldots,z_p}$$

$$\Bigg\{ \sum_{s_{p+1}=-1,+1} \; \sum_{s_{p+2}=-1,+1} \cdots \sum_{s_n=-1,+1}$$

$$\Bigg[\frac{1}{2^{n-p}} \int_{\bar{z}_{p+1}^{(s)}=0}^{1} \int_{\bar{z}_{p+2}^{(s)}=\bar{z}_{p+1}^{(s)}}^{1} \cdots \int_{\bar{z}_{n-1}^{(s)}=\bar{z}_{n-2}^{(s)}}^{1} \chi_1^2(\bar{z}_{p+1}^{(s)})\chi_1^2(\bar{z}_{p+2}^{(s)}) \ldots \chi_1^2(\bar{z}_{n-1}^{(s)})$$

$$\times\, \chi_1^2(1 - \sum_{j_1=p+1}^{n-1} \bar{z}_{j_1}^{(s)})[\chi_{n-p}^2(1)]^{-1}\chi_{n-p}^2(d_5^{*2}) \Bigg[\Big[(\frac{n-p}{2}-1)\frac{1}{d_5^{*2}} - \frac{1}{2}\Big]$$

$$\times \Big[\frac{\partial r_5^{(*s)}}{\partial a_j^*}\frac{\partial r_5^{(*s)}}{\partial a_k^*}\Big]_{\underset{\sim}{a}^*=\underset{\sim}{0}} + \Big[\frac{\partial^2 r_5^{(*s)}}{\partial a_j^* \partial a_k^*}\Big]_{\underset{\sim}{a}^*=\underset{\sim}{0}}\Bigg]$$

$$\times\, d\bar{z}_{p+1}^{(s)}\, d\bar{z}_{p+2}^{(s)} \ldots d\bar{z}_{n-1}^{(s)} \Bigg]\Bigg\} \qquad , \ (i = 5).$$

In deriving $I_{ia_j^*}$ and $I_{ia_j^*a_k^*}$, we can make use of

(3.3.49)

$$\int_{\bar{z}_1^{(s)}=0}^{1}\int_{\bar{z}_2^{(s)}=\bar{z}_1^{(s)}}^{1}\cdots\int_{\bar{z}_{m-1}^{(s)}=\bar{z}_{m-2}^{(s)}}^{1}\left[\bar{z}_{i_1}^{(s)}\right]^{\frac{m_1}{2}}\left[\bar{z}_{i_2}^{(s)}\right]^{\frac{m_2}{2}}\cdots\left[\bar{z}_{i_q}^{(s)}\right]^{\frac{m_q}{2}}$$

$$\times\,\chi_1^2(\bar{z}_1^{(s)})\chi_1^2(\bar{z}_2^{(s)})\;\cdots\;\chi_1^2(\bar{z}_m^{(s)})\,d\bar{z}_1^{(s)}d\bar{z}_2^{(s)}\;\cdots\;d\bar{z}_{m-1}^{(s)}$$

$$=\left[\prod_{k_1=1}^{q}\frac{2^{\frac{m_{k_1}+1}{2}}\Gamma\left(\frac{m_{k_1}+1}{2}\right)}{\sqrt{2\pi}}\right]\chi^2_{m+\sum_{\ell=1}^{q}m_\ell}(1)\;,$$

where

$m = 2,3,\ldots,$

$q = 1,2,\ldots,m,$

$m_\ell = 0,1,2,\ldots,$ for $\ell = 1,2,\ldots,q,$

$i_\ell = 1,2,\ldots,m$ for $\ell = 1,2,\ldots,q$ and these i_ℓ are all distinct,

and

$$\bar{z}_m^{(s)} = 1 - \sum_{\ell=1}^{m-1}\bar{z}_\ell^{(s)}\;.$$

The derivation of $I_{1a_j^*}$ and $I_{1a_j^*a_k^*}$ is shown in Appendix 2. From the expressions of $I_{1a_j^*}$ and $I_{1a_j^*a_k^*}$, it is found that $I_1(\underset{\sim}{\theta}_0,\sigma_0)$ given by (3.3.36) is

(3.3.50)

$$I_1(\underset{\sim}{\theta}_0,\sigma_0) \simeq (1-\alpha) + \sigma_0^2\chi_3^2[\chi^2_{1,\alpha}]\left[\sum_{j=1}^{p-1}\sum_{k=1}^{p-1}(2a^2_{pjk}-a_{pjj}a_{pkk})\right.$$

$$\left.-\sum_{i=p+1}^{n}[a^2_{ipp}+2(\sum_{j=1}^{p-1}(2a^2_{ijp}-a_{ijj}a_{ipp}))]\right],\quad (p \geq 2;\ k^* = 1),$$

(3.3.51)

$$I_1(\underset{\sim}{\theta}_0,\sigma_0) \simeq (1-\alpha) + \sigma_0^2\chi^2_{k^*+2}[\chi^2_{k^*,\alpha}]\Bigg[\sum_{i=p-k^*+1}^{p}\sum_{j=1}^{p-k^*}\sum_{k=1}^{p-k^*}(2a^2_{ijk}-a_{ijj}a_{ikk})$$

$$-\sum_{i=p+1}^{n}[\sum_{j=p-k^*+1}^{p}\sum_{k=p-k^*+1}^{p}(2a^2_{ijk}-a_{ijj}a_{ikk})$$

$$+2\sum_{j=1}^{p-k^*}\sum_{k=p-k^*+1}^{p}(2a^2_{ijk}-a_{ijj}a_{ikk})]\Bigg]$$

$$(p \geq 3;\ k^* \geq 2).$$

The derivation of $I_i(\underset{\sim}{\theta}_0,\sigma_0)$ $(i = 2,3,4,5)$ given by (3.3.36) is similar to that of $I_1(\underset{\sim}{\theta}_0,\sigma_0)$ and it can be shown that

(3.3.52)

$$I_2(\underset{\sim}{\theta}_0,\sigma_0) \simeq (1-\alpha)-\sigma_0^2\chi^2_{p+2}[\chi^2_{p,}\alpha]\sum_{i=p+1}^{n}\sum_{j=1}^{p}\sum_{k=1}^{p}(2a^2_{ijk}-a_{ijj}a_{ikk}),$$

(3.3.53)

$$I_3(\underset{\sim}{\theta}_0,\sigma_0) \simeq (1-\alpha) + \sigma_0^2 U_{0,3,n-p}(F_1)\sum_{j=1}^{p-1}\sum_{k=1}^{p-1}(2a^2_{pjk}-a_{pjj}a_{pkk})$$

$$-\sigma_0^2\frac{1}{n-p}U_{0,3,n-p}(F_1)\sum_{i=p+1}^{n}\sum_{j=1}^{p-1}\sum_{k=1}^{p-1}(2a^2_{ijk}-a_{ijj}a_{ikk})$$

$$+\sigma_0^2\sum_{i=p+1}^{n}\Big[[(1+F_1)(3F_1-1)U_{2,3,n-p}(F_1)-3(1+F_1)^2U_{2,5,n-p}(F_1)$$

$$+3(1+F_1)U_{0,5,n-p}(F_1)]a^2_{ipp}$$

$$+[4(1+F_1)F_1U_{2,1,n-p}(F_1)-4(1+F_1)(2+F_1)U_{2,3,n-p}(F_1)$$

$$+4(1+F_1)U_{0,3,n-p}(F_1)]\sum_{j=1}^{p-1}a^2_{ijp}$$

$$+[2F_1(1+F_1)U_{2,1,n-p}(F_1)+2(1+F_1)U_{0,3,n-p}(F_1)$$

$$\left. -2F_1(1+F_1)U_{2,3,n-p}(F_1)] \sum_{j=1}^{p-1} a_{ijj}a_{ipp} \right] .$$

$$, \quad (p \geq 2; \quad k^* = 1),$$

(3.3.54)

$$I_3(\underset{\sim}{\theta}_0,\sigma_0) \simeq (1-\alpha)+\sigma_0^2 U_{0,k^*+2,n-p}(F_2) \sum_{i=p-k^*+1}^{p} \sum_{j=1}^{p-k^*} \sum_{k=1}^{p-k^*} (2a_{ijk}^2 - a_{ijj}a_{ikk})$$

$$-\sigma_0^2 \frac{k^*}{n-p} U_{0,k^*+2,n-p}(F_2) \sum_{i=p+1}^{n} \sum_{j=1}^{p-k^*} \sum_{k=1}^{p-k^*} (2a_{ijk}^2 - a_{ijj}a_{ikk})$$

$$+\sigma_0^2 \sum_{i=p+1}^{n} \Big[[(1+F_2)(3F_2-1)U_{2,k^*+2,n-p}(F_2)-3(1+F_2)^2 U_{2,k^*+4,n-p}(F_2)$$

$$+3(1+F_2)U_{0,k^*+4,n-p}(F_2)] \sum_{j=p-k^*+1}^{p} a_{ijj}^2$$

$$+[2(1+F_2)(F_2-1)U_{2,k^*+2,n-p}(F_2)-2(1+F_2)^2 U_{2,k^*+4,n-p}(F_2)$$

$$+2(1+F_2)U_{0,k^*+4,n-p}(F_2)] \sum_{\substack{j,k=p-k^*+1,p-k^*+2,\ldots,p \\ \text{and } j\neq k}} a_{ijk}^2$$

$$+[(1+F_2)^2 U_{2,k^*+2,n-p}(F_2)-(1+F_2)^2 U_{2,k^*+4,n-p}(F_2)$$

$$+(1+F_2)U_{0,k^*+4,n-p}(F_2)] \sum_{\substack{j,k=p-k^*+1,p-k^*+2,\ldots,p \\ \text{and } j\neq k}} a_{ijj}a_{ikk}$$

$$+[4(1+F_2)F_2 U_{2,k^*,n-p}(F_2)-4(1+F_2)(2+F_2)U_{2,k^*+2,n-p}(F_2)$$

$$+4(1+F_2)U_{0,k^*+2,n-p}(F_2)] \sum_{j=1}^{p-k^*} \sum_{k=p-k^*+1}^{p} a_{ijk}^2$$

$$+[2F_2(1+F_2)U_{2,k^*,n-p}(F_2)+2(1+F_2)U_{0,k^*+2,n-p}(F_2)$$

$$-2F_2(1+F_2)U_{2,k^*+2,n-p}(F_2)]\sum_{j=1}^{p-k^*}\sum_{k=p-k^*+1}^{p}a_{ijj}a_{ikk}\Bigg]$$

$$,\ (p\geq 3;\ k^*\geq 2),$$

(3.3.55)

$$I_4(\underset{\sim}{\theta}_0,\sigma_0)\simeq(1-\alpha)+\sigma_0^2\sum_{i=p+1}^{n}\Bigg[[(1+F_3)(3F_3-1)U_{2,p+2,n-p}(F_3)-3(1+F_3)^2U_{2,p+4,n-p}(F_3)$$

$$+3(1+F_3)U_{0,p+4,n-p}(F_3)]\sum_{j=1}^{p}a_{ijj}^2$$

$$+[2(1+F_3)(F_3-1)U_{2,p+2,n-p}(F_3)-2(1+F_3)^2U_{2,p+4,n-p}(F_3)$$

$$+2(1+F_3)U_{0,p+4,n-p}(F_3)]\sum_{\substack{j,k=1,2,\ldots,p\\ \text{and } j\neq k}}a_{ijk}^2$$

$$+[(1+F_3)^2U_{2,p+2,n-p}(F_3)-(1+F_3)^2U_{2,p+4,n-p}(F_3)$$

$$+(1+F_3)U_{0,p+4,n-p}(F_3)]\sum_{\substack{j,k=1,2,\ldots,p\\ \text{and } j\neq k}}a_{ijj}a_{ikk}\Bigg]$$

and

(3.3.56)

$$I_5(\underset{\sim}{\theta}_0,\sigma_0)\simeq(1-\alpha)+\sigma_0^2\chi^2_{n-p+2}[\chi^2_{n-p,\alpha}]\sum_{i=p+1}^{n}\sum_{j=1}^{p}\sum_{k=1}^{p}[2a_{ijk}^2-a_{ijj}a_{ikk}]\ ,$$

where

$$F_1=\frac{F_\alpha(1,n-p)}{n-p}\ ,$$

$$F_2 = \frac{k^* F_\alpha(k^*, n-p)}{n-p},$$

$$F_3 = \frac{p F_\alpha(p, n-p)}{n-p},$$

$$U_{k_1,k_2,k_3}(v) = E_{z_{i_1}, z_{i_2} \ldots z_{i_{k_3}}} [z_{i_j}^{k_1} \chi^2_{k_2}(v \sum_{j^*=1}^{k_3} z^2_{i_{j^*}})],$$

$$(k_1 = 0,1,2,\ldots;\ k_2,k_3 = 1,2,\ldots,;\ j = 1,2,\ldots,k_3),$$

$$(3.3.57)\quad U_{0,k_2,k_3}(v) = \frac{v^{\frac{k_2-2}{2}} \Gamma(\frac{k_2+k_3-2}{2})}{2(1+v)^{\frac{k_2+k_3-2}{2}} \Gamma(\frac{k_2}{2}) \Gamma(\frac{k_3}{2})},$$

$$(3.3.58)\quad U_{2,k_2,k_3}(v) = \frac{v^{\frac{k_2-2}{2}} \Gamma(\frac{k_2+k_3}{2})}{k_3(1+v)^{\frac{k_2+k_3}{2}} \Gamma(\frac{k_2}{2}) \Gamma(\frac{k_3}{2})},$$

and v is a constant.

We further find, after some reduction, that

(3.3.59)

$$I_3(\theta_0, \sigma_0) \simeq (1-\alpha) + \sigma_0^2 U_{0,3,n-p}(F_1) \Bigg[\sum_{j=1}^{p-1} \sum_{k=1}^{p-1} (2a^2_{pjk} - a_{pjj} a_{pkk})$$

$$- \frac{1}{n-p} \sum_{i=p+1}^{n} \sum_{j=1}^{p-1} \sum_{k=1}^{p-1} (2a^2_{ijk} - a_{ijj} a_{ikk})$$

$$- \frac{n-p+1}{n-p} \sum_{i=p+1}^{n} [a^2_{ipp} + 2(\sum_{j=1}^{p-1} 2a^2_{ijp} - a_{ijj} a_{ipp})] \Bigg],$$

$$(p \geq 2;\ k^*=1)$$

(3.3.60)

$$I_3(\underset{\sim}{\theta}_0,\sigma_0) \simeq (1-\alpha)+\sigma_0^2 U_{0,k^*+2,n-p}(F_2)\left[\sum_{i=p-k^*+1}^{p}\sum_{j=1}^{p-k^*}\sum_{k=1}^{p-k^*}(2a_{ijk}^2-a_{ijj}a_{ikk})\right.$$

$$-\frac{k^*}{n-p}\sum_{i=p+1}^{n}\sum_{j=1}^{p-k^*}\sum_{k=1}^{p-k^*}(2a_{ijk}^2-a_{ijj}a_{ikk})$$

$$-\frac{n-p+k^*}{n-p}\sum_{i=p+1}^{n}\left[\sum_{j=p-k^*+1}^{p}\sum_{k=p-k^*+1}^{p}(2a_{ijk}^2-a_{ijj}a_{ikk})\right.$$

$$\left.\left.+\,2\sum_{j=1}^{p-k^*}\sum_{k=p-k^*+1}^{p}(2a_{ijk}^2-a_{ijj}a_{ikk})\right]\right]$$

$$,\ (p \geq 3;\ k^* \geq 2),$$

and

(3.3.61)

$$I_4(\underset{\sim}{\theta}_0,\sigma_0) \simeq (1-\alpha) - \frac{n}{n-p}\,\sigma_0^2 U_{0,p+2,n-p}(F_3)\sum_{i=p+1}^{n}\sum_{j=1}^{p}\sum_{k=1}^{p}(2a_{ijk}^2-a_{ijj}a_{ikk}).$$

Note that the approximations of $I_2(\underset{\sim}{\theta}_0,\sigma_0)$, $I_4(\underset{\sim}{\theta}_0,\sigma_0)$ and $I_5(\underset{\sim}{\theta}_0,\sigma_0)$ given by (3.3.52), (3.3.61) and (3.3.56) respectively agree with the corresponding approximations derived in Beale (1960).

Section 3.4 Power functions of the general m.l. ratio tests

This section is concerned with the derivation of approximations of the power functions $\beta_i(\underset{\sim}{\theta}_A,\sigma_A)$ (c.f. section 3.2). Note that $\sigma_A = \sigma_O$ if i = 1 and 2, and $\sigma_A \neq \sigma_O$ if i = 5.

The acceptance regions

$$\omega_i = \{\underset{\sim}{z} : S_i^D(\underset{\sim}{z}) \leq d_i^{*2}\} , \qquad (i = 1,2,\ldots,5)$$

defined in section 3.2 can also be written as

$$\omega_i = \{\underset{\sim}{z} : S_i^{DA}(\underset{\sim}{z}) \leq d_i^{+2}\}, \qquad (i = 1,2,\ldots,5),$$

where

$$S_1^{DA}(\underset{\sim}{z}) = [S^M(\theta_{Op-k^*+1},\theta_{Op-k^*+2},\ldots,\theta_{Op}) - S(\hat{\underset{\sim}{\theta}})]/\sigma_A{}^2,$$

$$S_2^{DA}(\underset{\sim}{z}) = [S(\underset{\sim}{\theta}_O) - S(\hat{\underset{\sim}{\theta}})]/\sigma_A{}^2,$$

$$S_3^{DA}(\underset{\sim}{z}) = [S^M(\theta_{Op-k^*+1},\theta_{Op-k^*+2},\ldots,\theta_{Op}) - (1+\frac{k^*F_\alpha(k^*,n-p)}{n-p})S(\hat{\underset{\sim}{\theta}})]/\sigma_A^2$$

$$+ \frac{k^*F_\alpha(k^*,n-p)}{n-p} \sum_{j=p+1}^{n} z_j^2/\sigma_A{}^2,$$

$$S_4^{DA}(\underset{\sim}{z}) = [S(\underset{\sim}{\theta}_O)-(1+\frac{pF_\alpha(p,n-p)}{n-p} S(\hat{\underset{\sim}{\theta}})]/\sigma_A^2 + \frac{pF_\alpha(p,n-p)}{n-p} \sum_{j=p+1}^{n} z_j^2/\sigma_A^2 ,$$

$$S_5^{DA}(\underset{\sim}{z}) = S(\hat{\underset{\sim}{\theta}})/\sigma_A^2 ,$$

and

$$d_i^{+2} = (d_i^+)^2 = \begin{cases} \chi^2_{k^*,\alpha} & , (i = 1) \\ \chi^2_{p,\alpha} & , (i = 2) \\ \dfrac{k^* F_\alpha(k^*, n-p)}{n-p} \sum_{j=p+1}^{n} z_j^2/\sigma_A^2 & , (i = 3) \\ \dfrac{p F_\alpha(p, n-p)}{n-p} \sum_{j=p+1}^{n} z_j^2/\sigma_A^2 & , (i = 4) \\ \dfrac{\sigma_O^2}{\sigma_A^2} \chi^2_{n-p,\alpha} & , (i = 5). \end{cases}$$

The corresponding approximations of $S_i^{DA}(\underset{\sim}{z})$ $(i = 1,2,\ldots,5)$ can be expressed respectively in terms of the right hand side of (3.3.12)-(3.3.16) with σ_O, a^*_{ijk}, $a^*_{ijk\ell}$, $a_{MCijk\ell}$ and $a_{MFijk\ell}$ changed to σ_A, a^+_{ijk}, $a^+_{ijk\ell}$, $a_{MCAijk\ell}$ and $a_{MFAijk\ell}$ respectively.

Let $\underset{\sim}{\eta}_A$ be the $(n\times 1)$ vector whose u^{th} component is $\eta(\underset{\sim}{\xi}_u, \underset{\sim}{\theta}_A)$. Given that $\underset{\sim}{\theta} = \underset{\sim}{\theta}_A$ and $\sigma = \sigma_A$, z_j/σ_A $(j = 1,2,\ldots,n)$ are independently distributed as $N(z_{Aj}, 1)$, where z_{Aj} is the j^{th} component of $\frac{1}{\sigma_A} \underset{\sim}{H}(\underset{\sim}{\eta}_A - \underset{\sim}{\eta}_O)$. As far as obtaining approximations of the $\beta_i(\underset{\sim}{\theta}_A, \sigma_A)$ is concerned, we can set σ_A appearing in the expressions of the $S_i^{DA}(\underset{\sim}{z})$ to be one while leaving the σ_A appearing in the expressions of the d_i^{+2}, a^+_{ijk}, $a^+_{ijk\ell}$, $a_{MCAijk\ell}$ and $A_{MFAijk\ell}$ unchanged, and treat the z_j as being distributed as $N(z_{Aj}, 1)$. Each of the $z_j^{(s)}$ is then distributed as non-central χ^2 with one degree of freedom and parameter $\lambda_j = z_{Aj}^2$. Conditional on z_j being negative, i.e. $s_j = -1$ (or non-negative, i.e. $s_j = +1$), the probability density function (p.d.f.) of $z_j^{(s)}$ is

$$\frac{1}{2\sqrt{2\pi}} \frac{\exp[-\frac{1}{2}(s_j\sqrt{z_j^{(s)}}-z_{Aj})^2]}{\sqrt{z_j^{(s)}}} \Bigg/ \int_{z_j^{(s)}=0}^{\infty} \frac{1}{2\sqrt{2\pi}} \frac{\exp[-\frac{1}{2}(s_j\sqrt{z_j^{(s)}} - z_{Aj})^2]}{\sqrt{z_j^{(s)}}} dz_j^{(s)} .$$

The power functions $\beta_i(\underset{\sim}{\theta}_A,\sigma_A)$ can now be written as

(3.4.1)

$$\beta_i(\underset{\sim}{\theta}_A,\sigma_A) = 1 - E_{z_{p+1},z_{p+2},\ldots,z_n,z_1,z_2,\ldots,z_{p-k^*}}$$

$$\Bigg\{ \sum_{s_{p-k^*+1}=-1,+1} \quad \sum_{s_{p-k^*+2}=-1,+1} \cdots \sum_{s_p=-1,+1}$$

$$\Bigg[\underset{(z_{p-k^*+1}^{(s)},z_{p-k^*+2}^{(s)},\ldots,z_p^{(s)})\in Z_i^+}{\int_{z_{p-k^*+1}^{(s)}} \int_{z_{p-k^*+2}^{(s)}} \cdots \int_{z_p^{(s)}}} \Bigg[\prod_{j=p-k^*+1}^{p} \Bigg[\frac{1}{2\sqrt{2\pi}} \frac{\exp[-\frac{1}{2}(s_j\sqrt{z_j^{(s)}} - z_{Aj})^2]}{\sqrt{z_j^{(s)}}} \Bigg]\Bigg]$$

$$dz_{p-k^*+1}^{(s)} dz_{p-k^*+2}^{(s)} \cdots dz_p^{(s)} \Bigg]\Bigg\}$$

, (i = 1 and 3),

(3.4.2)

$$\beta_i(\underset{\sim}{\theta}_A,\sigma_A) = 1 - E_{z_{p+1},z_{p+2},\ldots,z_n}$$

$$\Bigg\{ \sum_{s_1=-1,+1} \quad \sum_{s_2=-1,+1} \cdots \sum_{s_p=-1,+1}$$

$$\left[\int_{z_1^{(s)}} \int_{z_2^{(s)}} \cdots \int_{z_p^{(s)}} \left[\prod_{j=1}^{p} \left[\frac{1}{2\sqrt{2\pi}} \frac{\exp[-\frac{1}{2}(s_j\sqrt{z_j^{(s)}} - z_{Aj})^2]}{\sqrt{z_j^{(s)}}}\right]\right]\right.$$
$$(z_1^{(s)}, z_2^{(s)}, \ldots, z_p^{(s)}) \in Z_i^+$$
$$\left. dz_1^{(s)} dz_2^{(s)} \ldots dz_p^{(s)}\right]\Bigg\}, \quad (i = 2 \text{ and } 4),$$

and

(3.4.3)

$$\beta_5(\underset{\sim}{\theta}_A, \sigma_A) = 1 - E_{z_1, z_2, \ldots, z_p}$$

$$\Bigg\{ \sum_{s_{p+1}=-1,+1} \sum_{s_{p+2}=-1,+1} \cdots \sum_{s_n=-1,+1}$$

$$\left[\int_{z_{p+1}^{(s)}} \int_{z_{p+2}^{(s)}} \cdots \int_{z_n^{(s)}} \left[\prod_{j=p+1}^{n} \left[\frac{1}{2\sqrt{2\pi}} \frac{\exp[-\frac{1}{2}(s_j\sqrt{z_j^{(s)}} - z_{Aj})^2]}{\sqrt{z_j^{(s)}}}\right]\right]\right.$$
$$(z_{p+1}^{(s)}, z_{p+2}^{(s)}, \ldots, z_n^{(s)}) \in Z_5^+$$
$$\left. dz_{p+1}^{(s)}\, dz_{p+2}^{(s)} \ldots dz_n^{(s)}\right]\Bigg\},$$

where Z_i^+ are given by the right hand side of (3.3.20)-(3.3.22) with S_i^D and d_i^* changed to S_i^{DA} and d_i^+ respectively.

After applying the transformations given by (3.3.24)-(3.3.29), the power functions $\beta_i\ (\underset{\sim}{\theta}_A, \sigma_A)$ can be written as

(3.4.4)

$$\beta_i(\underset{\sim}{\theta}_A,\sigma_A) = 1 - E_{z_{p+1},z_{p+2},\ldots,z_n,z_1,z_2,\ldots,z_{p-k^*}}$$

$$\left\{ \sum_{s_{p-k^*+1}=-1,+1} \; \sum_{s_{p-k^*+2}=-1,+1} \cdots \sum_{s_p=-1,+1} \right.$$

$$\left[\int_{\bar{z}^{(s)}_{p-k^*+1}=0}^{1} \int_{\bar{z}^{(s)}_{p-k^*+2}=\bar{z}^{(s)}_{p-k^*+1}}^{1} \cdots \int_{\bar{z}^{(s)}_{p-1}=\bar{z}^{(s)}_{p-2}}^{1} \right.$$

$$\chi_1^2(\bar{z}^{(s)}_{p-k^*+1})\chi_1^2(\bar{z}^{(s)}_{p-k^*+2})\cdots\chi_1^2(\bar{z}^{(s)}_{p-1})\chi_1^2(1 - \sum_{j=p-k^*+1}^{p-1} \bar{z}^{(s)}_j)$$

$$\times \left[\int_{R_i^+} e^{1/2} e^{-r^{(s)}/2} [r^{(s)}]^{\frac{k^*}{2}-1} \left[\prod_{k=p-k^*+1}^{p} \frac{1}{2} e^{-z_{Ak}^2/2} e^{s_k\sqrt{r^{(s)}\bar{z}_k^{(s)}}\, z_{Ak}} \right] dr^{(s)} \right]$$

$$\left. \left. \times \; d\bar{z}^{(s)}_{p-k^*+1}\, d\bar{z}^{(s)}_{p-k^*+2} \cdots d\bar{z}^{(s)}_{p-1} \right] \right\}$$

, (i = 1 and 3),

(3.4.5)

$$\beta_i(\underset{\sim}{\theta}_A,\sigma_A) = 1 - E_{z_{p+1},z_{p+2},\ldots,z_n}$$

$$\left\{ \sum_{s_1=-1,+1} \; \sum_{s_2=-1,+1} \cdots \sum_{s_p=-1,+1} \right.$$

$$\left[\int_{\bar{z}_1^{(s)}=0}^{1} \int_{\bar{z}_2^{(s)}=\bar{z}_1^{(s)}}^{1} \cdots \int_{\bar{z}^{(s)}_{p-1}=\bar{z}^{(s)}_{p-2}}^{1} \chi_1^2(\bar{z}_1^{(s)})\chi_1^2(\bar{z}_2^{(s)}) \cdots \chi_1^2(\bar{z}^{(s)}_{p-1})\chi_1^2(1-\sum_{j=1}^{p-1} \bar{z}_j^{(s)}) \right.$$

$$\times \left[\int_{R_i^+} e^{1/2} e^{-r^{(s)}/2} [r^{(s)}]^{\frac{p}{2}-1} \left[\prod_{k=1}^{p} \frac{1}{2} e^{-z_{Ak}^2/2} e^{s_k \sqrt{r^{(s)} \bar{z}_k^{(s)}}\, z_{Ak}} \right] dr^{(s)} \right]$$

$$\times\, d\bar{z}_1^{(s)}\, d\bar{z}_2^{(s)} \ldots d\bar{z}_{p-1}^{(s)} \Big] \Big\} \quad , \quad (i = 2 \text{ and } 4),$$

(3.4.6)

$$\beta_5(\underset{\sim}{\theta}_A, \sigma_A) = 1 - E_{z_1, z_2, \ldots, z_p}$$

$$\Big\{ \sum_{s_{p+1}=-1,+1} \; \sum_{s_{p+2}=-1,+1} \cdots \sum_{s_n=-1,+1}$$

$$\Big[\int_{\bar{z}_{p+1}^{(s)}=0}^{1} \; \int_{\bar{z}_{p+2}^{(s)}=\bar{z}_{p+1}^{(s)}}^{1} \cdots \int_{\bar{z}_{n-1}^{(s)}=\bar{z}_{n-2}^{(s)}}^{1}$$

$$\chi_1^2(\bar{z}_{p+1}^{(s)})\chi_1^2(\bar{z}_{p+2}^{(s)}) \ldots \chi_1^2(\bar{z}_{n-1}^{(s)})\chi_1^2(1 - \sum_{j=p+1}^{n-1} \bar{z}_j^{(s)})$$

$$\times \left[\int_{R_5^+} e^{1/2} e^{-r^{(s)}/2} [r^{(s)}]^{\frac{n-p}{2}-1} \left[\prod_{k=p+1}^{n} \frac{1}{2} e^{-z_{Ak}^2/2} e^{s_k \sqrt{r^{(s)} \bar{z}_k^{(s)}}\, z_{Ak}} \right] dr^{(s)} \right]$$

$$\times\, d\bar{z}_{p+1}^{(s)}\, d\bar{z}_{p+2}^{(s)} \ldots d\bar{z}_{n-1}^{(s)} \Big] \Big\}$$

where R_i^+ are given by the right hand side of (3.3.33)-(3.3.35) with S_i^D and d_i^* changed to S_i^{DA} and d_i^+ respectively.

We can obtain a truncated series expansion of the power function $\beta_i(\underset{\sim}{\theta}_A, \sigma_A)$ as follows:

$$(3.4.7) \qquad \beta_i(\underset{\sim}{\theta}_A,\sigma_A) \simeq \alpha_A + \sum_{j=1}^{n^*} \beta_{ia_j^+} a_j^+ + \frac{1}{2}\sum_{j=1}^{n^*}\sum_{k=1}^{n^*} \beta_{ia_j^+ a_k^+} a_j^+ a_k^+ ,$$

where $\{a_1^+, a_2^+, \ldots, a_{n^*}^+\}$ is the set containing all the a_{hjk}^+,

$$\alpha_A = [\beta_i(\underset{\sim}{\theta}_A,\sigma_A)]_{\underset{\sim}{a}^+=\underset{\sim}{0}} ,$$

$$\beta_{ia_j^+} = \left[\frac{\partial\beta_i(\underset{\sim}{\theta}_A,\sigma_A)}{\partial a_j^+}\right]_{\underset{\sim}{a}^+=\underset{\sim}{0}} ,$$

$$\beta_{ia_j^+a_k^+} = \left[\frac{\partial^2\beta_i(\underset{\sim}{\theta}_A,\sigma_A)}{\partial a_j^+\,\partial a_k^+}\right]_{\underset{\sim}{a}^+=\underset{\sim}{0}}$$

and $\underset{\sim}{a}^+$ is a column vector whose components are the a_{hjk}^+, and the $a_{hjk\ell}^+$, $a_{MCAhjk\ell}$ or $a_{MFAhjk\ell}$ for which $j \leq k \leq \ell$.

We now consider how $\beta_{ia_j^+}$ and $\beta_{ia_j^+a_k^+}$ can be derived. We first note that provided that the magnitude $|\underset{\sim}{a}^+|$ of $\underset{\sim}{a}^+$ is sufficiently small, the sets R_i^+ can be written as

$$R_i^+ = \{r^{(s)} = 0 \leq r^{(s)} \leq r_i^{(+s)}\} ,$$

where $r_i^{(+s)}$ are given by (3.3.37)-(3.3.39) with S_i^D, $r_i^{(*s)}$ and d_i^* changed to S_i^{DA}, $r_i^{(+s)}$ and d_i^+ respectively. We next note that the equations defining the $r_i^{(+s)}$ are of the form

$$(3.4.8) \qquad r_i^{(+s)} + \sum_{j=1}^{n^*} h_j^+(r_i^{(+s)})a_j^+ + \sum_{j=1}^{n^*}\sum_{k=1}^{n^*} h_{jk}^+(r_i^{(+s)})a_j^+a_k^+ + b^+ = d_i^{+2} ,$$

where $h_j^+(r_i^{(+s)})$ and $h_{jk}^+(r_i^{(+s)})$ are functions of $r_i^{(+s)}$, the z_ℓ and $\bar{z}_m^{(s)}$, and b^+ is a function of the z_ℓ and $\bar{z}_m^{(s)}$, and the $a_{ijk\ell}^+$, $a_{MCAijk\ell}$ or $a_{MFAijk\ell}$.

As for $r_i^{(*s)}$ in section 3.3, we have

$$(3.4.9) \qquad \left[\frac{\partial r_i^{(+s)}}{\partial a_j^+}\right]_{\underset{\sim}{a}^+=\underset{\sim}{0}} = -h_j^+(d_i^{+2})$$

and

$$(3.4.10) \qquad \left[\frac{\partial^2 r_i^{(+s)}}{\partial a_j^+ \partial a_k^+}\right]_{\underset{\sim}{a}^+=\underset{\sim}{0}} = -\left[\frac{\partial h_j^+(r_i^{(+s)})}{\partial r_i^{(+s)}}\,\frac{\partial r_i^{(+s)}}{\partial a_k^+}\right]_{\underset{\sim}{a}^+=\underset{\sim}{0}}$$

$$- \left[\frac{\partial h_k^+(r_i^{(+s)})}{\partial r_i^{(+s)}}\,\frac{\partial r_i^{(+s)}}{\partial a_j^+}\right]_{\underset{\sim}{a}^+=\underset{\sim}{0}} - 2h_{jk}^+(d_i^{+2}) .$$

Using the arguments similar to those in deriving $I_{ia_j^*}$ and $I_{ia_j^*a_k^*}$, we obtain

(3.4.11)

$$\beta_{ia_j^+} = -E_{z_{p+1},z_{p+2},\ldots,z_n,z_1,z_2,\ldots,z_{p-k^*}}$$

$$\Bigg\{ \sum_{s_{p-k^*+1}=-1,+1} \quad \sum_{s_{p-k^*+2}=-1,+1} \cdots \sum_{s_p=-1,+1}$$

$$\left[\int_{\bar z^{(s)}_{p-k^*+1}=0}^{1} \int_{\bar z^{(s)}_{p-k^*+2}=\bar z^{(s)}_{p-k^*+1}}^{1} \cdots \int_{\bar z^{(s)}_{p-1}=\bar z^{(s)}_{p-2}}^{1}\right.$$

$$\chi_1^2(\bar z^{(s)}_{p-k^*+1})\chi_1^2(\bar z^{(s)}_{p-k^*+2})\cdots\chi_1^2(\bar z^{(s)}_{p-1})\chi_1^2(1-\sum_{j=p-k^*+1}^{p-1}\bar z^{(s)}_j)$$

$$\times e^{1/2} e^{-d_i^{+2}/2}[d_i^{+2}]^{\frac{k^*}{2}-1}\left[\prod_{k=p-k^*+1}^{p}\frac{1}{2}e^{-z_{Ak}^2/2}e^{s_k d_i^+\sqrt{\bar z_k^{(s)}}z_{Ak}}\right]\left[\frac{\partial r_i^{(+s)}}{\partial a_j^+}\right]_{\underset{\sim}{a}^+=\underset{\sim}{0}}$$

$$\left.\left.\times\, d\bar z^{(s)}_{p-k^*+1}\, d\bar z^{(s)}_{p-k^*+2}\cdots d\bar z^{(s)}_{p-1}\right]\right\}, \quad (i = 1 \text{ and } 3),$$

(3.4.12)

$$\beta_{ia_j^+} = -E_{z_{p+1},z_{p+2},\ldots,z_n}$$

$$\left\{\sum_{s_1=-1,+1}\ \sum_{s_2=-1,+1}\cdots\sum_{s_p=-1,+1}\right.$$

$$\left[\int_{\bar z_1^{(s)}=0}^{1}\int_{\bar z_2^{(s)}=\bar z_1^{(s)}}^{1}\cdots\int_{\bar z^{(s)}_{p-1}=\bar z^{(s)}_{p-2}}^{1}\right.$$

$$\chi_1^2(\bar z_1^{(s)})\chi_1^2(\bar z_2^{(s)})\cdots\chi_1^2(\bar z_{p-1}^{(s)})\chi_1^2(1-\sum_{j=1}^{p-1}\bar z_j^{(s)})$$

$$\times e^{1/2} e^{-d_i^{+2}/2}[d_i^{+2}]^{\frac{p}{2}-1}\left[\prod_{k=1}^{p}\frac{1}{2}e^{-z_{Ak}^2/2}e^{s_k d_i^+\sqrt{\bar z_k^{(s)}}z_{Ak}}\right]\left[\frac{\partial r_i^{(+s)}}{\partial a_j^+}\right]_{\underset{\sim}{a}^+=\underset{\sim}{0}}$$

$$\left.\left.\times\, d\bar z_1^{(s)}\, d\bar z_2^{(s)}\cdots d\bar z_{p-1}^{(s)}\right]\right\}, \quad (i = 2 \text{ and } 4),$$

(3.4.13)

$$\beta_{5a_j^+} = - E_{z_1 \cdot z_2, \ldots, z_p}$$

$$\Bigg\{ \sum_{s_{p+1}=-1,+1} \sum_{s_{p+2}=-1,+1} \cdots \sum_{s_n=-1,+1}$$

$$\Bigg[\int_{\bar{z}^{(s)}_{p+1}=0}^{1} \int_{\bar{z}^{(s)}_{p+2}=\bar{z}^{(s)}_{p+1}}^{1} \cdots \int_{\bar{z}^{(s)}_{n-1}=\bar{z}^{(s)}_{n-2}}^{1}$$

$$\chi_1^2(\bar{z}^{(s)}_{p+1})\chi_1^2(\bar{z}^{(s)}_{p+2}) \ldots \chi_1^2(\bar{z}^{(s)}_{n-1})\chi_1^2(1 - \sum_{j=p+1}^{n-1} \bar{z}^{(s)}_j)$$

$$\times\ e^{1/2}\ e^{-d_5^{+2}/2}\ [d_5^{+2}]^{\frac{n-p}{2}-1} \left[\prod_{k=p+1}^{n} \frac{1}{2}\ e^{-z_{Ak}^2/2}\ e^{s_k d_5^+ \sqrt{\bar{z}^{(s)}_k} z_{Ak}} \right] \left[\frac{\partial r_5^{(+s)}}{\partial a_j^+} \right]_{\underset{\sim}{a}^+=\underset{\sim}{0}}$$

$$\times\ d\bar{z}^{(s)}_{p+1}\ d\bar{z}^{(s)}_{p+2} \ldots d\bar{z}^{(s)}_{n-1} \Bigg] \Bigg\}$$

(3.4.14)

$$\beta_{1a_j^+a_k^+} = - E_{z_{p+1}, z_{p+2}, \ldots, z_n, z_1, z_2, \ldots, z_{p-k^*}}$$

$$\Bigg\{ \sum_{s_{p-k^*+1}=-1,+1} \sum_{s_{p-k^*+2}=-1,+1} \cdots \sum_{s_p=-1,+1}$$

$$\left[\int_{\bar{z}^{(s)}_{p-k^*+1}=0}^{1} \int_{\bar{z}^{(s)}_{p-k^*+2}=\bar{z}^{(s)}_{p-k^*+1}}^{1} \cdots \int_{\bar{z}^{(s)}_{p-1}=\bar{z}^{(s)}_{p-2}}^{1} \right.$$

$$\chi_1^2(\bar{z}^{(s)}_{p-k^*+1})\chi_1^2(\bar{z}^{(s)}_{p-k^*+2}) \ldots \chi_1^2(\bar{z}^{(s)}_{p-1})\chi_1^2(1 - \sum_{j=p-k^*+1}^{p-1} \bar{z}^{(s)}_j)$$

$$\times e^{1/2} e^{-d_i^{+2}/2} [d_i^{+2}]^{\frac{k^*}{2}-1} \left[\prod_{k=p-k^*+1}^{p} \frac{1}{2} e^{-z_{Ak}^2/2} e^{s_k d_i^+ \sqrt{\bar{z}^{(s)}_k} z_{Ak}} \right]$$

$$\times \left[\left[\left(\frac{k^*}{2}-1\right)\frac{1}{d_i^{+2}} - \frac{1}{2} + \frac{1}{2d_i^+} \sum_{\ell=p-k^*+1}^{p} s_\ell \sqrt{\bar{z}^{(s)}_\ell}\, z_{A\ell} \right] \left[\frac{\partial r_i^{(+s)}}{\partial a_j^+} \frac{\partial r_i^{(+s)}}{\partial a_k^+} \right]_{\underset{\sim}{a}^+=\underset{\sim}{0}} \right.$$

$$\left. + \left[\frac{\partial^2 r_i^{(+s)}}{\partial a_j^+ \partial a_k^+} \right]_{\underset{\sim}{a}^+=\underset{\sim}{0}} \right]$$

$$\left. \times\, d\bar{z}^{(s)}_{p-k^*+1}\, d\bar{z}^{(s)}_{p-k^*+2} \ldots d\bar{z}^{(s)}_{p-1} \right] \Bigg\} \quad , \quad (i = 1 \text{ and } 3),$$

(3.4.15)

$$\beta_{ia_j^+a_k^+} = - E_{z_{p+1}, z_{p+2}, \ldots, z_n}$$

$$\left\{ \sum_{s_1=-1,+1} \sum_{s_2=-1,+1} \cdots \sum_{s_p=-1,+1} \right.$$

$$\left[\int_{\bar{z}^{(s)}_1=0}^{1} \int_{\bar{z}^{(s)}_2=\bar{z}^{(s)}_1}^{1} \cdots \int_{\bar{z}^{(s)}_{p-1}=\bar{z}^{(s)}_{p-2}}^{1} \chi_1^2(\bar{z}^{(s)}_1)\chi_1^2(\bar{z}^{(s)}_2) \ldots \chi_1^2(\bar{z}^{(s)}_{p-1})\chi_1^2(1-\sum_{j=1}^{p-1} \bar{z}^{(s)}_j) \right.$$

$$\times\ e^{1/2}\ e^{-d_i^{+2}/2}\,[d_i^{+2}]^{\frac{p}{2}-1}\left[\prod_{k=1}^{p} \frac{1}{2}\, e^{-z_{Ak}^2/2}\, e^{s_k d_i^+ \sqrt{\bar{z}_k^{(s)}}\, z_{Ak}}\right]$$

$$\times\left[\left[\left(\frac{p}{2}-1\right)\frac{1}{d_i^{+2}} - \frac{1}{2} + \frac{1}{2d_i^+}\sum_{\ell=1}^{p} s_\ell \sqrt{\bar{z}_\ell^{(s)}}\ z_{A\ell}\right]\left[\frac{\partial r_i^{(+s)}}{\partial a_j^*}\ \frac{\partial r_i^{(+s)}}{\partial a_k^*}\right]_{a^+=0}\right.$$

$$\left. + \left[\frac{\partial^2 r_i^{(+s)}}{\partial a_j^+ \partial a_k^+}\right]_{a^+=0}\right]$$

$$\left.\times\, d\bar{z}_1^{(s)}\, d\bar{z}_2^{(s)} \ldots\ d\bar{z}_{p-1}^{(s)}\ \right]\Bigg\} \qquad ,\quad (i = 2 \text{ and } 4)$$

and

(3.4.16)

$$\beta_{5a_j^+a_k^+} = -\ E_{z_1,z_2,\ldots,z_p}$$

$$\Bigg\{\sum_{s_{p+1}=-1,+1}\ \sum_{s_{p+2}=-1,+1} \cdots \sum_{s_n=-1,+1}$$

$$\left[\int_{\bar{z}_{p+1}^{(s)}=0}^{1}\ \int_{\bar{z}_{p+2}^{(s)}=\bar{z}_{p+1}^{(s)}}^{1} \cdots \int_{\bar{z}_{n-1}^{(s)}=\bar{z}_{n-2}^{(s)}}^{1} \chi_1^2(\bar{z}_{p+1}^{(s)})\,\chi_1^2(\bar{z}_{p+2}^{(s)}) \ldots \chi_1^2(\bar{z}_{n-1}^{(s)})\,\chi_1^2\Big(1-\sum_{j=p+1}^{n-1}\bar{z}_j^{(s)}\Big)\right.$$

$$\times\ e^{1/2}\ e^{-d_5^{+2}/2}\,[d_5^{+2}]^{\frac{n-p}{2}-1}\left[\prod_{k=p+1}^{n} \frac{1}{2}\, e^{-z_{Ak}^2/2}\, e^{s_k d_5^+ \sqrt{\bar{z}_k^{(s)}}\, z_{Ak}}\right]$$

$$\times\left[\left[\left(\frac{n-p}{2}-1\right)\frac{1}{d_5^{+2}} - \frac{1}{2} + \frac{1}{2d_5^+}\sum_{\ell=p+1}^{n} s_\ell \sqrt{\bar{z}_\ell^{(s)}}\ z_{A\ell}\right]\left[\frac{\partial r_5^{(+s)}}{\partial a_j^+}\ \frac{\partial r_5^{(+s)}}{\partial a_k^+}\right]_{a^+=0}\right.$$

$$\left. + \left[\frac{\partial^2 r_5^{(+s)}}{\partial a_j^+ \partial a_k^+}\right]_{a^+=0}\right]$$

$$\times\ d\bar{z}^{(s)}_{p+1}\ d\bar{z}^{(s)}_{p+2} \cdots d\bar{z}^{(s)}_{n-1} \Big] \Big\}.$$

In deriving $\beta_{ia_j^+}$ and $\beta_{ia_j^+a_k^+}$, we can make use of

$$(3.4.17) \qquad E_{z_j}(z_j) = z_{Aj},$$

$$(3.4.18) \qquad E_{z_j}(z_j^2) = 1 + z_{Aj}^2,$$

$$(3.4.19) \qquad E_{z_j}(z_j^3) = z_{Aj}(3 + z_{Aj}^2),$$

$$(3.4.20) \qquad E_{z_j}(z_j^4) = 3 + 6z_{Aj}^2 + z_{Aj}^4,$$

$$(3.4.21) \quad \sum_{s_1=-1,+1} \sum_{s_2=-1,+1} \cdots \sum_{s_k=-1,+1} \int_{\bar{z}_1^{(s)}=0}^{1} \int_{\bar{z}_2^{(s)}=\bar{z}_1^{(s)}}^{1} \cdots \int_{\bar{z}_{k-1}^{(s)}=\bar{z}_{k-2}^{(s)}}^{1}$$

$$\chi_1^2(\bar{z}_1^{(s)})\chi_1^2(\bar{z}_2^{(s)}) \cdots \chi_1^2(\bar{z}_k^{(s)})$$

$$\times\ e^{1/2}\ e^{-d^{+2}/2}\ [d^{+2}]^{\frac{k}{2}-1} \Big[\prod_{j=1}^{k} \frac{1}{2}\ e^{-z_{Aj}^2/2}\ e^{s_j d^+ \sqrt{\bar{z}_j^{(s)}}\, z_{Aj}} \Big]$$

$$\times \Big[\prod_{a=1}^{b} \big[\bar{z}_{i_a}^{(s)}\big]^{m_a/2} \Big] \Big[\prod_{c=1}^{d} \big[s_{j_c}\sqrt{\bar{z}_{j_c}^{(s)}}\big]^{n_c} \Big]$$

$$\times\ d\bar{z}_1^{(s)}\ d\bar{z}_2^{(s)} \cdots d\bar{z}_{k-1}^{(s)}$$

$$= \sum_{i_1^*=0}^{\infty} \sum_{i_2^*=0}^{\infty} \cdots \sum_{i_k^*=0}^{\infty} \left[\prod_{c=1}^{d} (z_{Aj_c})^{1-\delta_{0n_c}} \right] [d^+]^{-\sum_{a=1}^{b} m_a - \sum_{c=1}^{d} n_c}$$

$$\times \left[\prod_{e=1}^{k} 2\chi^2_{2+2i_e^*}(z^2_{Ae}) \right]$$

$$\times \left[\chi^2_{k+\sum_{a=1}^{b} m_a + \sum_{c=1}^{d} (n_c+1-\delta_{0n_c}) + 2\sum_{e=1}^{k} i_e^*} (d^{+2}) \right] \left[\prod_{a=1}^{b} p_a \right] \left[\prod_{c=1}^{d} q_c \right]$$

where

$$\bar{z}_k^{(s)} = 1 - \sum_{j=1}^{k-1} \bar{z}_j^{(s)},$$

$k = 2,3,\ldots,$

$b,d = 1,2,\ldots,k,$

$i_1, i_2, \ldots, i_b,\ j_1, j_2, \ldots, j_d$ are distinct elements of $\{1,2,\ldots,k\}$,

$m_a = 0$ or even positive integer,

$n_c = 0$ or odd positive integer,

$$p_a = \begin{cases} 1 & \text{if } m_a = 0 \\ \prod_{f=1}^{m_a/2} (2f-1+2i^*_{i_a}) & \text{if } m_a = 2,4,6,\ldots, \end{cases}$$

and

$$q_c = \begin{cases} 1 & \text{if } n_c = 0 \text{ and } 1 \\ \prod_{g=2}^{(n_c+1)/2} (2g-1+2i^*_{j_c}) & \text{if } n_c = 3,5,7,\ldots; \end{cases}$$

(3.4.22)

$$\sum_{i_1^*=0}^{\infty} \sum_{i_2^*=0}^{\infty} \cdots \sum_{i_k^*=0}^{\infty} \left[\prod_{e=1}^{k} 2\chi^2_{2+2i_e^*}(z^2_{Ae}) \right] \chi^2_{k'+2\sum_{e=1}^{k} i_e^*} (d^{+2}) = \chi'^2_{k',\, \sum_{e=1}^{k} z^2_{Ae}} (d^{+2}),$$

where $k = 1,2,\ldots$ and $k' \geq k$;

(3.4.23)

$$\sum_{i_1^*=0}^{\infty} \sum_{i_2^*=0}^{\infty} \cdots \sum_{i_k^*=0}^{\infty} \left[\prod_{s=1}^{\ell} i_{j_s}^* (i_{j_s}^* - 1) \cdots (i_{j_s}^* - r_{j_s}) \right] \left[\prod_{e=1}^{k} 2\chi^2_{2+2i_e^*}(z^2_{Ae}) \right] \chi^2_{k'+2\sum_{e=1}^{k} i_e^*}(d^{+2})$$

$$= \left[\prod_{s=1}^{\ell} \frac{z^2_{Aj_s}}{2}^{r_{j_s}+1} \right] \chi'^2_{k'+2\sum_{s=1}^{\ell}(r_{j_s}+1),\ \sum_{e=1}^{k} z^2_{Ae}}(d^{+2}) ,$$

where

$k = 2,3,\ldots,$

$k' \geq k,$

$\ell = 1,2,\ldots,k,$

$j_1, j_2, \ldots, j_\ell$ are distinct elements of $\{1,2,\ldots,k\}$

and $r_{j_s} = 0,1,2,\ldots$;

(3.4.24) $$\int_{r^{(s)}=0}^{\infty} [r^{(s)}]^{\frac{k_0}{2}} \chi'^2_{k_1',\lambda_1}(r^{(s)}) \chi'^2_{k_2',\lambda_2}(vr^{(s)})\, dr^{(s)}$$

$$= \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} 2^{\frac{k_0}{2}+1} \chi^2_{2+2i_1}(\lambda_1) \chi^2_{2+2i_2}(\lambda_2) \frac{v^{\frac{k_2+2i_2-2}{2}}}{(1+v)^{\frac{k_0+k_1+k_2+2i_1+2i_2-2}{2}}}$$

$$\times \frac{\Gamma\left(\frac{k_0+k_1+k_2+2i_1+2i_2-2}{2}\right)}{\Gamma\left(\frac{k_1+2i_1}{2}\right) \Gamma\left(\frac{k_2+2i_2}{2}\right)} ,$$

where k_0 is an integer,

$k_1, k_2 = 1,2,\ldots,$

λ_1, λ_2 are the parameters of the non-central χ^2 distributions,

and v is a constant.

The evaluation of the approximations of $\beta_i(\underset{\sim}{\theta}_A, \sigma_A)$ using a computer will be considered in Chapter 4.

Section 3.5 Region estimation in unconstrained nonlinear models

Consider the problem of finding region estimates for

(1) $\theta_{p-k^*+1}, \theta_{p-k^*+2}, \ldots, \theta_p$, where $1 \leq k^* < p$, when σ is known to be equal to σ_0,

(2) $\underset{\sim}{\theta}$ when σ is known to be equal to σ_0,

(3) $\theta_{p-k^*+1}, \theta_{p-k^*+2}, \ldots, \theta_p$, where $1 \leq k^* < p$, when σ is unknown,

(4) $\underset{\sim}{\theta}$ when σ is unknown,

and

(5) σ^2 when $\underset{\sim}{\theta}$ is unknown.

For each $i = 1,2,3,4$, let $R_i(\underset{\sim}{z})$ be the totality of values $\theta_{0p-k^*+1}, \theta_{0p-k^*+2}, \ldots, \theta_{0p}$ (where $1 \leq k^* < p$ if $i = 1,3$ and $k^* = p$ if $i = 2,4$) for which H_i is accepted when the general m.l. ratio test is carried out for the observed $\underset{\sim}{z}$,

i.e.

$$R_1(\underset{\sim}{z}) = \{\theta_{p-k^*+1}, \theta_{p-k^*+2}, \ldots, \theta_p : S^M(\theta_{p-k^*+1}, \theta_{p-k^*+2}, \ldots, \theta_p) - S(\hat{\underset{\sim}{\theta}}) \leq \sigma_0^2 \chi^2_{k^*,\alpha}\},$$

$$R_2(\underset{\sim}{z}) = \{\underset{\sim}{\theta}: S(\underset{\sim}{\theta}) - S(\hat{\underset{\sim}{\theta}}) \leq \sigma_0^2 \chi^2_{p,\alpha}\},$$

$$R_3(\underset{\sim}{z}) = \{\theta_{p-k^*+1},\theta_{p-k^*+2},\ldots,\theta_p : S^M(\theta_{p-k^*+1},\theta_{p-k^*+2},\ldots,\theta_p) - S(\hat{\underset{\sim}{\theta}}) \leq \frac{k^*}{n-p} S(\hat{\underset{\sim}{\theta}})F_\alpha(k^*,n-p)\},$$

and

$$R_4(\underset{\sim}{z}) = \{\underset{\sim}{\theta} : S(\underset{\sim}{\theta}) - S(\hat{\underset{\sim}{\theta}}) \leq \frac{p}{n-p} S(\hat{\underset{\sim}{\theta}})F_\alpha(p,n-p)\}.$$

We next denote $R_5(\underset{\sim}{z})$ to be the totality of values σ_0^2 for which H_5 is accepted when the general m.l. ratio test is carried out for the observed $\underset{\sim}{z}$, i.e.

$$R_5(\underset{\sim}{z}) = \{\sigma^2 : S(\hat{\underset{\sim}{\theta}}) \leq \sigma^2\chi^2_{n-p,\alpha}\}.$$

Then $R_i(\underset{\sim}{z})$ can be regarded as a region estimate for the parameters in (i). If the true value of σ is σ_T, then the region estimate $R_i(\underset{\sim}{z})$ will cover the true values of the corresponding parameters with probability $I_i(\underset{\sim}{\theta}_T,\sigma_T)$. This probability is an important quantity associated with the region estimate $R_i(\underset{\sim}{z})$. Usually we do not know this probability as we do not know $\underset{\sim}{\theta}_T$ and σ_T. In the case when σ_T is known to be equal to σ_0, we could use $I_i(\hat{\underset{\sim}{\theta}},\sigma_0)$ to estimate this probability. Information concerning the reliability of this estimate could be derived from $I_i(\underset{\sim}{\theta}_f,\sigma_0)$ where $\underset{\sim}{\theta}_f$ are feasible $\underset{\sim}{\theta}$ in the neighbourhood of $\hat{\underset{\sim}{\theta}}$. The estimation of the probability $I_i(\underset{\sim}{\theta}_T,\sigma_0)$ will be investigated in Chapter 5.

Section 3.6 Inference of functions of the parameter vector based on general maximum likelihood ratios

In previous sections, we have considered the problem of making inference about subsets of components of the parameter vector θ in an unconstrained nonlinear model. In practice we may also be interested in making inference about nonlinear functions of this parameter vector θ. For example, we may be interested in hypothesis testing and interval estimation concerning $\eta(\xi^*,\theta)$ where $\xi^* \neq \xi_u$ for all u = 1,2,...,n. In general we can denote these functions of θ by $\alpha_{p-k^*+1}(\theta), \alpha_{p-k^*+2}(\theta), \ldots, \alpha_p(\theta)$, where $1 \leq k^* \leq p$. We shall restrict our attention to the $\alpha_i(\theta)$ which have the following properties:

(a) $\alpha_i(\theta)$ are differentiable up to the third order,

(b) if $k^* = p$, then $\alpha = \alpha(\theta) = [\alpha_1(\theta), \alpha_2(\theta), \ldots, \alpha_p(\theta)]^T$ is a one to one function of θ, and if $k^* < p$, then there exist differentiable functions $\alpha_i(\theta)$, where $1 \leq i \leq p-k^*$, such that the resulting α is a one to one function of θ.

We now consider the problem of testing the hypothesis

$$H_i^{(\alpha)} \text{ that } (\alpha, \sigma) \in \Omega_{H_i^{(\alpha)}}$$

against the alternative

$$K_i^{(\alpha)} \text{ that } (\alpha, \sigma) \in \Omega_{K_i^{(\alpha)}} , \quad (i = 1,2,3,4),$$

where $\Omega_{H_i^{(\alpha)}}$ and $\Omega_{K_i^{(\alpha)}}$ are respectively the same as Ω_{H_i} and Ω_{K_i} (cf. section 3.2) if we change θ to α and θ_0 to α_0. If we can find the expressions $\theta_i(\alpha)$ for the θ_i in terms of α, then the model is one in which the theoretical means of y_u are $\eta(\underset{\sim}{\xi}_u, \underset{\sim}{\theta}(\underset{\sim}{\alpha}))$, where $\underset{\sim}{\theta}(\underset{\sim}{\alpha}) = [\theta_1(\underset{\sim}{\alpha}), \theta_2(\underset{\sim}{\alpha}), \ldots, \theta_p(\underset{\sim}{\alpha})]^T$, and the problem of testing the hypotheses $H_i^{(\alpha)}$ is similar to that of testing the hypotheses H_i. In practice, it is not necessary to find the expressions $\theta_i(\underset{\sim}{\alpha})$ for the purpose of testing the hypotheses $H_i^{(\alpha)}$. We note that the appropriate quantities which need to be computed are $\underset{\sim}{\theta}(\underset{\sim}{\alpha}_O)$, $\hat{\underset{\sim}{\theta}}$, $S(\underset{\sim}{\theta}(\underset{\sim}{\alpha}_O))$, $S(\hat{\underset{\sim}{\theta}})$, and $S_\alpha^M(\alpha_{Op-k^*+1}, \alpha_{Op-k^*+2}, \ldots, \alpha_{Op})$ which is the minimum value of $S(\underset{\sim}{\theta}(\underset{\sim}{\alpha}))$ with respect to $\underset{\sim}{\alpha}$ where $\underset{\sim}{\alpha}$ are such that $(\underset{\sim}{\alpha}, \sigma_O) \in \Omega_{H_i^{(\alpha)}}$. This could be achieved by using some technique of constrained and unconstrained minimization. The remaining quantities which need to be computed are the significance probabilities and the power of the general m.l. ratio tests. To derive approximations of these probabilities, it suffices to express $\eta(\underset{\sim}{\xi}_u, \underset{\sim}{\theta}(\underset{\sim}{\alpha}))$ as cubic functions of the $\alpha_j^* = \alpha_j - \alpha_{Oj}$ similar to (3.3.1), as then the problem of deriving these approximations under the parameterization $\underset{\sim}{\alpha}$ is similar to that under the parameterization $\underset{\sim}{\theta}$.

As $\alpha_i(\underset{\sim}{\theta})$, where $p-k^*+1 \leq i \leq p$, are differentiable up to the third order, we can obtain cubic approximations of the corresponding α_i^* as follows:

$$(3.6.1) \qquad \alpha_i^* = \sum_{j=1}^{p} b_{ij} t_j + \underset{\sim}{t}^T \underset{\sim}{B}_i \underset{\sim}{t} + \sum_{j=1}^{p} [\underset{\sim}{t}^T \underset{\sim}{B}_{ij} \underset{\sim}{t}] t_j + o(t^3),$$

$$(i = p-k^*+1, p-k^*+2, \ldots, p),$$

where $B_i = \{b_{ijk}\}$ and $B_{ij} = \{b_{ijk\ell}\}$ are symmetric $(p \times p)$ matrices.

As there exist differentiable functions $\alpha_i(\underset{\sim}{\theta})$, where $1 \leq i \leq p-k^*$, such that the resulting $\underset{\sim}{\alpha}$ is a one to one function of $\underset{\sim}{\theta}$, we can find the numbers b_{ij}, where $1 \leq i \leq p-k^*$ and $1 \leq j \leq p$, such that the $(p \times p)$ matrix $B = \{b_{ij}\}$ is non-singular. After finding these numbers, we define

(3.6.2) $$\alpha_i^* = \sum_{j=1}^{p} b_{ij} t_j \quad , \quad (i = 1,2,\ldots,p-k^*).$$

Let $\underset{\sim}{H}_B = \{h_{Bjk}\}$ be a $(p\times p)$ orthogonal matrix such that $\underset{\sim}{H}_B \underset{\sim}{B}$ is an upper triangular $(p\times p)$ matrix $\underset{\sim}{E} = \{e_{ij}\}$. Next, let γ_i be the i^{th} component of

(3.6.3) $$\underset{\sim}{\gamma} = \underset{\sim}{H}_B \underset{\sim}{\alpha}^*.$$

Further, let $\underset{\sim}{b}_{jk}$ and $\underset{\sim}{b}_{jk\ell}$ be the $(p\times 1)$ column vectors whose i^{th} components are b_{ijk} and $b_{ijk\ell}$ respectively if $p-k^*+1 \leq i \leq p$, and zeros if $1 \leq i \leq p-k^*$. Then from (3.6.1)-(3.6.3), we have

(3.6.4) $$\gamma_i = \sum_{j=i}^{p} e_{ij} t_j + \underset{\sim}{t}^T \underset{\sim}{E}_i \underset{\sim}{t} + \sum_{j=1}^{p} [\underset{\sim}{t}^T \underset{\sim}{E}_{ij} \underset{\sim}{t}] t_j + o(t^3),$$

$$(i = 1,2,\ldots,p),$$

where

$$\underset{\sim}{E}_i = \{e_{ijk}\},$$

$$\underset{\sim}{E}_{ij} = \{e_{ijk\ell}\}.$$

These e_{ijk} and $e_{ijk\ell}$ are respectively the i^{th} components of $\underset{\sim}{H}_B \underset{\sim}{b}_{jk}$ and $\underset{\sim}{H}_B \underset{\sim}{b}_{jk\ell}$.

Let $\underset{\sim}{\tau} = \underset{\sim}{E}\underset{\sim}{t}$. Then we have

(3.6.5) $$\underset{\sim}{t} = \underset{\sim}{E}^{-1} \underset{\sim}{\tau},$$

where $\underset{\sim}{E}^{-1} = \{e^{ij}\}$ is the inverse of $\underset{\sim}{E}$. From (3.6.4), we have

(3.6.6) $$\gamma_i = \tau_i + \underset{\sim}{\tau}^T \underset{\sim}{R}_i \underset{\sim}{\tau} + \sum_{j=1}^{p} [\underset{\sim}{\tau}^T \underset{\sim}{R}_{ij} \underset{\sim}{\tau}][\sum_{k=j}^{p} e^{jk} \tau_k] + o(\tau^3) \quad (i = 1,2,\ldots,p),$$

where

$$(3.6.7) \qquad \underset{\sim}{R}_i = (\underset{\sim}{E}^{-1})^T \underset{\sim}{E}_i (\underset{\sim}{E}^{-1}) = \{r_{ijk}\},$$

$$(3.6.8) \qquad \underset{\sim}{R}_{ij} = (\underset{\sim}{E}^{-1})^T \underset{\sim}{E}_{ij} (\underset{\sim}{E}^{-1}) = \{r_{ijk\ell}\},$$

and $\tau^3 = (\sum_{j=1}^{p} \tau_j^2)^{3/2}$.

From (3.6.6) we have

$$(3.6.9) \qquad \tau_i = \gamma_i - \underset{\sim}{\gamma}^T \underset{\sim}{R}_i \underset{\sim}{\gamma} - \sum_{k=1}^{p} [\underset{\sim}{\gamma}^T (\sum_{j=1}^{k} e^{jk} \underset{\sim}{R}_{ij} - 2 \sum_{j=1}^{p} r_{ijk} \underset{\sim}{R}_j) \underset{\sim}{\gamma}] \gamma_k + o(\gamma^3)$$

$$, \ (i = 1,2,\ldots,p),$$

where $\gamma^3 = (\sum_{j=1}^{p} \gamma_j^2)^{3/2}$.

Finally, by using (3.3.1), (3.6.5), (3.6.9) and (3.6.3), we can express the $\eta(\underset{\sim}{\xi}_u, \underset{\sim}{\theta}(\underset{\sim}{\alpha}))$ as cubic functions of the α_i^*. In what follows, we present these functions truncated after the quadratic terms:

$$(3.6.10) \qquad \eta(\underset{\sim}{\xi}_u, \underset{\sim}{\theta}(\underset{\sim}{\alpha})) = \eta(\underset{\sim}{\xi}_u, \underset{\sim}{\theta}(\underset{\sim}{\alpha}_0)) + \sum_{j=1}^{p} [\sum_{j_1=1}^{p} \sum_{j_2=j_1}^{p} c_{uj_1} e^{j_1 j_2} h_{Bj_2 j}] \alpha_j^*$$

$$+ (\underset{\sim}{\alpha}^*)^T [\underset{\sim}{H}_B^T (\underset{\sim}{E}^{-1})^T \underset{\sim}{C}_u \underset{\sim}{E}^{-1} \underset{\sim}{H}_B - \sum_{j_1=1}^{p} \sum_{j_2=j_1}^{p} c_{uj_1} e^{j_1 j_2} \underset{\sim}{H}_B^T \underset{\sim}{R}_{j_2} \underset{\sim}{H}_B] \underset{\sim}{\alpha}^* + o((\alpha^*)^3),$$

where $(\alpha^*)^3 = (\sum_{i=1}^{p} (\alpha_i^*)^2)^{3/2}$.

Section 3.7 Hypothesis testing and region estimation in constrained nonlinear models

In previous sections we have considered the problems of testing nonlinear hypotheses and obtaining region estimates in the case when the solution locus is unconstrained. In some models, for example models (A) and (B) described in Chapter 1, the solution loci are constrained. However, if the point $P(\underset{\sim}{\theta}_T)$ is sufficiently far away from any boundary of a solution locus of these models, and the standard error σ of the observations is sufficiently small, then we can still treat this solution locus as being unconstrained, and obtain inference based on general maximum likelihood ratios. Thus if the number p of parameters is at most two, then we can first attempt to display the first p rotated coordinates of the points in the solution locus of a constrained model (c.f. Fig. (1.1.1) and (1.1.2)) in order to find out in what way the solution locus is constrained. We can then attempt to judge, from the point $P(\hat{\underset{\sim}{\theta}})$ and the value $S(\hat{\underset{\sim}{\theta}})/(n-p)$, whether the model could be treated as being unconstrained for statistical purposes.

In the case when we cannot treat the constrained nonlinear models as being unconstrained for statistical purposes, the problems of hypothesis testing and region estimation are still open questions for most of these models.

CHAPTER 4

DERIVATION OF APPROXIMATIONS OF THE POWER OF THE GENERAL MAXIMUM LIKELIHOOD RATIO TESTS USING A COMPUTER

Section 4.1 Introduction

In Chapter 3, the power functions $\beta_i(\underset{\sim}{\theta}_A, \sigma_A)$ of the general maximum likelihood ratio tests for the various nonlinear hypotheses (c.f. section 3.2) are approximated by series expansions truncated after some finite number of terms (c.f. (3.4.7)). These truncated series expansions can also be written as

$$(4.1.1) \qquad \beta_i(\underset{\sim}{\theta}_A, \sigma_A) \simeq \alpha_A + \beta_i^{(1)} \sigma_A + \beta_i^{(2)} \sigma_A^2 \quad ,$$

where

$$\beta_i^{(1)} = \sum_{h=1}^{n} \sum_{j=1}^{p} \sum_{k=j}^{p} \beta_{i\, a^+_{hjk}} a_{hjk}$$

and

$$\beta_i^{(2)} = \frac{1}{2} \sum_{h_1=1}^{n} \sum_{h_2=1}^{n} \sum_{j_1=1}^{p} \sum_{k_1=j_1}^{p} \sum_{j_2=1}^{p} \sum_{k_2=j_2}^{p} \beta_{i a^+_{h_1 j_1 k_1} a^+_{h_2 j_2 k_2}} a_{h_1 j_1 k_1} a_{h_2 j_2 k_2} .$$

In this chapter, we consider the case when i = 1 and 2, i.e. the case in which the corresponding hypotheses are concerned with one or more components of the parameter vector $\underset{\sim}{\theta}$ when σ is known to be equal to σ_0. An outline of how the approximate values of the corresponding $\beta_i(\underset{\sim}{\theta}_A, \sigma_A)$ can be calculated is as follows. Initially we have the model and the values k^*, $(\underset{\sim}{\theta}_O, \sigma_O)$ and $(\underset{\sim}{\theta}_A, \sigma_A)$, where in this case $\sigma_A = \sigma_O$. We can next calculate the values $\eta(\underset{\sim}{\xi}_u, \underset{\sim}{\theta}_A) - \eta(\underset{\sim}{\xi}_u, \underset{\sim}{\theta}_O)$ and the first and second derivatives $c_{uj}(\underset{\sim}{\theta}_O)$ and $c_{ujk}(\underset{\sim}{\theta}_O)$. From the values $\eta(\underset{\sim}{\xi}_u, \underset{\sim}{\theta}_A) - \eta(\underset{\sim}{\xi}_u, \underset{\sim}{\theta}_O)$ we can calculate $\underset{\sim}{z}_A$ (c.f. section 3.4), and from the values $c_{uj}(\underset{\sim}{\theta}_O)$ and $c_{ujk}(\underset{\sim}{\theta}_O)$, we can calculate a_{hjk} (c.f. section 3.3), where $h = 1,2,\ldots,n$ and $j,k = 1,2,\ldots,p$. Then from the values σ_O, $d_i^{+2} = \chi^2_{k^*,\alpha}$ or $\chi^2_{p,\alpha}$, $\underset{\sim}{z}_A$ and the a_{hjk}, we can calculate $\beta_i^{(1)}$ and $\beta_i^{(2)}$ (c.f. (3.4.11)-(3.4.16)). With these values of $\beta_i^{(1)}$ and $\beta_i^{(2)}$, the approximate value of $\beta_i(\underset{\sim}{\theta}_A, \sigma_A)$ can be found if we further calculate α_A and the right hand side of (4.1.1).

For the purpose of illustrating how we can use a computer to do the algebraic manipulation and calculation involved in finding $\beta_i(\underset{\sim}{\theta}_A, \sigma_A)$, we shall not consider all the calculation outlined above. Instead we assume that we already know the values of

(i) n, p, k^*,

(ii) σ_O ,

(iii) d_i^{+2}

(iv) $\underset{\sim}{z}_A$

and (v) the a_{ijk},

and we wish to calculate $\beta_i^{(1)}$ and $\beta_i^{(2)}$. There are two stages in the calculation of $\beta_i^{(1)}$ and $\beta_i^{(2)}$. First we derive algebraic expressions for the first two derivatives of $r_i^{(+s)}$ (c.f. (3.4.9) and (3.4.10)). The details of this derivation are described in sections 4.2-4.11, and the programs for actually deriving the derivatives of $r_i^{(+s)}$ are given in Appendix 3. We then make use of the derivatives of $r_i^{(+s)}$ and the equations (3.4.11)-(3.4.23) to calculate $\beta_i^{(1)}$ and $\beta_i^{(2)}$. The programs for deriving $\beta_i(\underset{\sim}{\theta}_A, \sigma_A)$ in terms of $\alpha_A, \beta_i^{(1)}$, $\beta_i^{(2)}$ and σ_A are given in Appendix 4.

Section 4.2 Representation of algebraic expressions by one dimensional arrays in a computer

We note that the algebraic expressions that we are dealing with in deriving the coefficients $\beta_{ia_j^+}$ and $\beta_{ia_j^+a_k^+}$ are mainly sums of a number of terms each of which is of the form

$$\text{constant} \times \prod_{i=1}^{m} \beta_i^{\ell_i}$$

where m is a positive integer, ℓ_i are integers and β_i are symbols which are common for all the terms. Each term can be represented in a computer by storing the corresponding constant, $\ell_1,\ell_2,\ldots,\ell_m$ in one dimensional arrays.

Section 4.3 Algebraic manipulation done on a computer

The addition of two terms with the same values of $\ell_1,\ell_2,\ldots,\ell_m$ can be achieved by adding together the constants of these terms and storing the sum of these constants, as well as $\ell_1,\ell_2,\ldots,\ell_m$ in one dimensional arrays.

The multiplication of two terms represented by

$$\text{constant A}, \quad \ell_{A1},\ell_{A2},\ldots,\ell_{Am}$$

and

$$\text{constant B}, \quad \ell_{B1},\ell_{B2},\ldots,\ell_{Bm}$$

can be achieved by calculating

$$\text{constant} = \text{constant A} \times \text{constant B}$$

$$\ell_i = \ell_{Ai} + \ell_{Bi}, \quad (i = 1,2,\ldots,m),$$

and storing constant, $\ell_1, \ell_2, \ldots, \ell_m$ in one dimensional arrays.

Section 4.4 Representation of the equation $S_1^{DA}(\underset{\sim}{z}) = d_1^{+2}$ in a computer

The case when i = 1 shall be chosen to illustrate the ideas behind the derivation of $\beta_{ia_j^+}$ and $\beta_{ia_j^+a_k^+}$, where i = 1,2. Suppose we are interested in finding $\beta_{1a^+_{i_1j_1k_1}}$, $\beta_{1a^+_{i_2j_2k_2}}$ and $\beta_{1a^+_{i_1j_1k_1}a^+_{i_2j_2k_2}}$. To find these coefficients, the equation

$$[S_1^{DA}(\underset{\sim}{z})]_{\text{all components of } \underset{\sim}{a}^+ \text{ other than } a^+_{i_1j_1k_1} \text{ and } a^+_{i_2j_2k_2} \text{ are zero}} = d_1^{+2}$$

is first represented in a computer as follows

(i) set II to be 1,

(ii) find out the IIth term (in $S_1^{DA}(\underset{\sim}{z})$) containing $a^+_{i_1j_1k_1}$, $a^+_{i_2j_2k_2}$ or $a^+_{i_1j_1k_1}a^+_{i_2j_2k_2}$,

(iii) a term found out in (ii) is of the form

$$\text{constant} \quad z_1^{\ell_1} \, z_2^{\ell_2} \ldots z_n^{\ell_n} \times a^{+m_1}_{i_1j_1k_1} a^{+m_2}_{i_2j_2k_2}$$

and can be represented in a computer by storing

constant in AC(II),

$\ell_1, \ell_2, \ldots, \ell_n$ in AZ(II,1),AZ(II,2),...,AZ(II,n)

and m_1, m_2 in AA(II,1),AA(II,2),

(iv) Increase the current value of II by 1 and continue the process same as for II = 1 until all the terms (in $S_1^{DA}(\underset{\sim}{z})$) containing $a^+_{i_1j_1k_1}$, $a^+_{i_2j_2k_2}$ or $a^+_{i_1j_1k_1}a^+_{i_2j_2k_2}$ have been found out, and represented by storing the appropriate values in AC(II), AZ(II,1), AZ(II,2),...,AZ(II,n),AA(II,1),AA(II,2).

Section 4.5 Representation of the equation

$$\underline{s_1^{DA}(z_1,z_2,\ldots,z_{p-k^*},s_{p-k^*+1}\sqrt{r_1^{(+s)-(s)}z_{p-k^*+1}},s_{p-k^*+2}\sqrt{r_1^{(+s)-(s)}z_{p-k^*+2}},\ldots,}$$

$$\underline{s_p\sqrt{r_1^{(+s)-(s)}z_p},z_{p+1},z_{p+2},\ldots,z_n) = d_1^{+2}}$$

in a computer

For each II, store 1.0 in each of SIZN(II,1), SIZN(II,2),...,SIZN (II,p-k*), SIZN(II,p+1), SIZN(II,p+2),...,SIZN(II,n) and ss_e in SIZN(II,e) for e = p-k*+1,p-k*+2,..., z_p, where

$$ss_e = \begin{cases} 1.0 & \text{if AZ(II,e) is even} \\ -1.0 & \text{if AZ(II,e) is odd.} \end{cases}$$

Further, store [AZ(II,p-k*+1) + AZ(II,p-k*+2)+...+AZ(II,p)] in R(II). Then the terms in

$$\left[s_1^{DA}(z_1,z_2,\ldots,z_{p-k^*},s_{p-k^*+1}\sqrt{r_1^{(+s)-(s)}z_{p-k^*+1}},s_{p-k^*+2}\sqrt{r_1^{(+s)-(s)}z_{p-k^*+2}},\ldots,s_p\sqrt{r_1^{(+s)-(s)}z_p}, z_{p+1},z_{p+2},\ldots,z_n)\right]_{\substack{\text{all components of } \underset{\sim}{a}^+ \\ \text{other than } a^+_{i_1j_1k_1} \text{ and } \\ a^+_{i_2j_2k_2} \\ \text{are zero}}}$$

involving $a^{+}_{i_1 j_1 k_1}$ or $a^{+}_{i_2 j_2 k_2}$ are represented by

AC(II),R(II),SIZN(II,1),AZ(II,1),SIZN(II,2),AZ(II,2),...,SIZN(II,n),
AZ(II,n),AA(II,1),AA(II,2)

where II = 1,2,...,IIMAX and IIMAX is the total number of such terms.

Section 4.6 Representation of $\left[\frac{\partial r_1^{(+s)}}{\partial a^{+}_{i_1 j_1 k_1}}\right]_{\underset{\sim}{a}^{+}=\underset{\sim}{0}}$, $\left[\frac{\partial r_1^{(+s)}}{\partial a^{+}_{i_2 j_2 k_2}}\right]_{\underset{\sim}{a}^{+}=\underset{\sim}{0}}$ and $\left[\frac{\partial^2 r_1^{(+s)}}{\partial a^{+}_{i_1 j_1 k_1} \partial a^{+}_{i_2 j_2 k_2}}\right]_{\underset{\sim}{a}^{+}=\underset{\sim}{0}}$ in a computer

By using (3.4.9) and (3.4.10), we can obtain this representation straightforwardly from the representation of the equation in section 4.5.

Section 4.7 Computation of $\beta_{1a^{+}_{i_1 j_1 k_1}}$ and $\beta_{1a^{+}_{i_1 j_1 k_1} a^{+}_{i_2 j_2 k_2}}$ in a computer

By using (3.4.11)-(3.4.23), we can find the values of $\beta_{1a^{+}_{i_1 j_1 k_1}}$ and $\beta_{1a^{+}_{i_1 j_1 k_1} a^{+}_{i_2 j_2 k_2}}$ from the representation of the derivatives in section 4.6.

Section 4.8 Partition of the set of all possible a^{+}_{ijk} into subsets such that in each subset, different a^{+}_{ijk} have similar expressions of $\beta_{1a^{+}_{ijk}}$

It is not necessary to derive all the $\beta_{1a^{+}_{ijk}}$ because different a^{+}_{ijk} may have similar expressions of $\beta_{1a^{+}_{ijk}}$. For example for the case when $k^* \geq 2$, $p-k^* \geq 2$, i_1, i_2 belong to $ST2 = \{p-k^*+1, p-k^*+2, \ldots, p\}$, j_1, j_2 belong to $ST1 = \{1, 2, \ldots, p-k^*\}$ (note that $ST3 = \{p+1, p+2, \ldots, n\}$), $i_1 \neq i_2$ and $j_1 \neq j_2$, we have

$$(4.8.1) \quad \beta_{1a^{+}_{i_1 j_1 j_1}} = - z_{Ai_1}(1 + z^2_{Aj_1})\chi'^{2}_{k^*+2,\lambda}(d^{+2}_1), \text{ where } \lambda = \sum_{j=p-k^*+1}^{p} z^2_{Aj},$$

and

$$(4.8.2) \quad \beta_{1a^{+}_{i_2 j_2 j_2}} = - z_{Ai_2}(1 + z^2_{Aj_2})\chi'^{2}_{k^*+2,\lambda}(d^{+2}_1),$$

and these expressions can be summarised by

$$(4.8.3) \quad \beta_{1a^{+}_{ijk}} = f_1(i,j,k), \quad (i = i_1, i_2;\ j = j_1, j_2;\ k = j_1, j_2;\ j = k),$$

where $f_1(i,j,k) = -z_{Ai}(1 + z^2_{Aj})\chi'^{2}_{k^*+2,\lambda}(d^{+2}_1)$.

We refer to $a^{+}_{i_1 j_1 k_1}$ and $a^{+}_{i_2 j_2 k_2}$ as being in the same subset provided that i_1, i_2 belong to the same STi, $(i = 1,2,3)$

j_1, j_2 belong to the same STi,

k_1, k_2 belong to the same STi,

and if the expressions of $\beta_{1a^{+}_{i_1 j_1 k_1}}$ and $\beta_{1a^{+}_{i_2 j_2 k_2}}$ are represented by $f_{11}(i_1, j_1, k_1)$ and $f_{12}(i_2, j_2, k_2)$ respectively, then $f_{11}(i,j,k) = f_{12}(i,j,k)$.

It is straightforward to see from the expressions of $S_i^{DA}(\underset{\sim}{z})$ that the sufficient conditions for $a^+_{i_1 j_1 k_1}$ and $a^+_{i_2 j_2 k_2}$ to be in the same subset are that

$$JP(1,i) = JP(2,i), \quad (i = 1,2,\ldots,6),$$

where

$$JP(m,1) = \begin{cases} 1 & \text{if } i_m \in ST1 \\ 2 & \text{if } i_m \in ST2 \\ 3 & \text{if } i_m \in ST3 \end{cases}, \qquad (m = 1,2),$$

$$JP(m,2) = \begin{cases} 1 & \text{if } j_m \in ST1 \\ 2 & \text{if } j_m \in ST2 \end{cases},$$

$$JP(m,3) = \begin{cases} 1 & \text{if } k_m \in ST1 \\ 2 & \text{if } k_m \in ST2 \end{cases},$$

$$JP(m,4) = \begin{cases} 1 & \text{if } i_m = j_m \\ 0 & \text{if } i_m \neq j_m \end{cases},$$

$$JP(m,5) = \begin{cases} 1 & \text{if } i_m = k_m \\ 0 & \text{if } i_m \neq k_m \end{cases},$$

and

$$JP(m,6) = \begin{cases} 1 & \text{if } j_m = k_m \\ 0 & \text{if } j_m \neq k_m \end{cases}.$$

Section 4.9 Partition of the set of all possible $(a^+_{i_1j_1k_1}, a^+_{i_2j_2k_2})$ into subsets such that in each subset, different $(a^+_{i_1j_1k_1}, a^+_{i_2j_2k_2})$ have similar expressions of $\beta_{1a^+_{i_1j_1k_1}a^+_{i_2j_2k_2}}$

As for $\beta_{1a^+_{ijk}}$, we refer to $(a^+_{i_{11}j_{11}k_{11}}, a^+_{i_{12}j_{12}k_{12}})$ and $(a^+_{i_{21}j_{21}k_{21}}, a^+_{i_{22}j_{22}k_{22}})$ as being in the same subset provided that if the expressions of $\beta_{1a^+_{i_{11}j_{11}k_{11}}a^+_{i_{12}j_{12}k_{12}}}$ and $\beta_{1a^+_{i_{21}j_{21}k_{21}}a^+_{i_{22}j_{22}k_{22}}}$ are represented by $f_{21}(i_{11},j_{11},k_{11},i_{12},j_{12},k_{12})$ and $f_{22}(i_{21},j_{21},k_{21},i_{22},j_{22},k_{22})$ respectively, then $f_{21}(i_1,j_1,k_1,i_2,j_2,k_2) = f_{22}(i_1,j_1,k_1,i_2,j_2,k_2)$. It is seen from the expressions of $S^{DA}_i(\underset{\sim}{z})$ that sufficient conditions for $(a^+_{i_{11}j_{11}k_{11}}, a^+_{i_{12}j_{12}k_{12}})$ and $(a^+_{i_{21}j_{21}k_{21}}, a^+_{i_{22}j_{22}k_{22}})$ to be in the same subset are that

$$KP(1,i) = KP(2,i), \qquad (i = 1,2,\ldots,21),$$

where

$$KP(m,1) = \begin{cases} 1 & \text{if } i_{m1} \in ST1 \\ 2 & \text{if } i_{m1} \in ST2 \\ 3 & \text{if } i_{m1} \in ST3 \end{cases}, \qquad (m = 1,2),$$

$$KP(m,2) = \begin{cases} 1 & \text{if } j_{m1} \in ST1 \\ 2 & \text{if } j_{m1} \in ST2 \end{cases},$$

$$KP(m,3) = \begin{cases} 1 & \text{if } k_{m1} \in ST1 \\ 2 & \text{if } k_{m1} \in ST2 \end{cases},$$

$$KP(m,4) = \begin{cases} 1 & \text{if } i_{m2} \in ST1 \\ 2 & \text{if } i_{m2} \in ST2 \\ 3 & \text{if } i_{m2} \in ST3 \end{cases} ,$$

$$KP(m,5) = \begin{cases} 1 & \text{if } j_{m2} \in ST1 \\ 2 & \text{if } j_{m2} \in ST2 \end{cases}$$

$$KP(m,6) = \begin{cases} 1 & \text{if } k_{m2} \in ST1 \\ 2 & \text{if } k_{m2} \in ST2 \end{cases} ,$$

$$KP(m,7) = \begin{cases} 1 & \text{if } i_{m1} = j_{m1} \\ 0 & \text{if } i_{m1} \neq j_{m1} \end{cases} ,$$

$$KP(m,8) = \begin{cases} 1 & \text{if } i_{m1} = k_{m1} \\ 0 & \text{if } i_{m1} \neq k_{m1} \end{cases} ,$$

$$KP(m,9) = \begin{cases} 1 & \text{if } i_{m1} = i_{m2} \\ 0 & \text{if } i_{m1} \neq i_{m2} \end{cases} ,$$

$$KP(m,10) = \begin{cases} 1 & \text{if } i_{m1} = j_{m2} \\ 0 & \text{if } i_{m1} \neq j_{m2} \end{cases} ,$$

$$KP(m,11) = \begin{cases} 1 & \text{if } i_{m1} = k_{m2} \\ 0 & \text{if } i_{m1} \neq k_{m2} \end{cases} ,$$

$$KP(m,12) = \begin{cases} 1 & \text{if } j_{m1} = k_{m1} \\ 0 & \text{if } j_{m1} \neq k_{m1} \end{cases} ,$$

$$KP(m,13) = \begin{cases} 1 & \text{if } j_{m1} = i_{m2} \\ 0 & \text{if } j_{m1} \neq i_{m2} \end{cases},$$

$$KP(m,14) = \begin{cases} 1 & \text{if } j_{m1} = j_{m2} \\ 0 & \text{if } j_{m1} \neq j_{m2} \end{cases},$$

$$KP(m,15) = \begin{cases} 1 & \text{if } j_{m1} = k_{m2} \\ 0 & \text{if } j_{m1} \neq k_{m2} \end{cases},$$

$$KP(m,16) = \begin{cases} 1 & \text{if } k_{m1} = i_{m2} \\ 0 & \text{if } k_{m1} \neq i_{m2} \end{cases},$$

$$KP(m,17) = \begin{cases} 1 & \text{if } k_{m1} = j_{m2} \\ 0 & \text{if } k_{m1} \neq j_{m2} \end{cases},$$

$$KP(m,18) = \begin{cases} 1 & \text{if } k_{m1} = k_{m2} \\ 0 & \text{if } k_{m1} \neq k_{m2} \end{cases},$$

$$KP(m,19) = \begin{cases} 1 & \text{if } i_{m2} = j_{m2} \\ 0 & \text{if } i_{m2} \neq j_{m2} \end{cases}$$

$$KP(m,20) = \begin{cases} 1 & \text{if } i_{m2} = k_{m2} \\ 0 & \text{if } i_{m2} \neq k_{m2} \end{cases},$$

and

$$KP(m,21) = \begin{cases} 1 & \text{if } j_{m2} = k_{m2} \\ 0 & \text{if } j_{m2} \neq k_{m2} \end{cases}.$$

Section 4.10 Programs for deriving $\left[\frac{\partial r_1^{(+s)}}{\partial a^+_{i_1j_1k_1}} \times \frac{\partial r_1^{(+s)}}{\partial a^+_{i_2j_2k_2}}\right]_{\underset{\sim}{a}^+=\underset{\sim}{0}}$

and $\left[\frac{\partial^2 r_1^{(+s)}}{\partial a^+_{i_1j_1k_1}\,\partial a^+_{i_2j_2k_2}}\right]_{\underset{\sim}{a}^+=\underset{\sim}{0}}$.

The programs for deriving these expressions are programs PARTIT and POWPRO, subroutines POWSUO, POWSUA, POWSUB and POWSUC as shown in Appendix 3. Note that these programs can also be used to derive the expressions for the case when $k^* = p$. Further, if POWSUC is replaced by SIGSUC (c.f. Appendix 3), the resulting set of programs can also be used to derive the approximations of the probabilities $I_1(\underset{\sim}{\theta}_0,\sigma_0)$ and $I_2(\underset{\sim}{\theta}_0,\sigma_0)$. These probabilities are equal to $\beta_1(\underset{\sim}{\theta}_0,\sigma_0)$ and $\beta_2(\underset{\sim}{\theta}_0,\sigma_0)$ respectively.

Section 4.11 Expressions of $\left[\frac{\partial r_1^{(+s)}}{\partial a^+_{i_1j_1k_1}} \times \frac{\partial r_1^{(+s)}}{\partial a^+_{i_2j_2k_2}}\right]_{\underset{\sim}{a}^+=\underset{\sim}{0}}$

and $\left[\frac{\partial^2 r_1^{(+s)}}{\partial a^+_{i_1j_1k_1}\,\partial a^+_{i_2j_2k_2}}\right]_{\underset{\sim}{a}^+=\underset{\sim}{0}}$

Each of these expressions has one or more terms. Each term is of the form

$$\text{constant}\quad [d^+]^{\ell_0} z_1^{\ell_1} z_2^{\ell_2} \cdots z_{p-k^*}^{\ell_{p-k^*}} \left[s_{p-k^*+1}\sqrt{z^{(s)}_{p-k^*+1}}\right]^{\ell_{p-k^*+1}}$$

$$\left[s_{p-k^*+2}\sqrt{z^{(s)}_{p-k^*+2}}\right]^{\ell_{p-k^*+2}} \cdots \left[s_p\sqrt{z^{(s)}_p}\right]^{\ell_p} z_{p+1}^{\ell_{p+1}} z_{p+2}^{\ell_{p+2}} \cdots z_n^{\ell_n} .$$

These terms obtained by using the programs in section 4.10 are presented in Table (4.11.1)- Table (4.11.20). The numbers in columns [11] to [31] of a table are the codes XP(·,·) for the subsets containing elements whose corresponding expressions are nonzero. The subscripts $i_1, j_1, k_1, i_2, j_2, k_2$ of a typical element $(a^+_{i_1 j_1 k_1}, a^+_{i_2 j_2 k_2})$ of a subset are in columns [5] to [10]. Each number in column [2] is the total number of terms of an expression. The numbers in columns [3] and [4] are respectively ℓ_0 and constant. Each number in column $[i_1^+]$ (where $i_1^+ = 32, 34, 36$ and 38) is such that if it is j^+, then the number in the next column (i.e. column $[i_1^+ + 1]$) of the same row is ℓ_{k^+} of $z_{k^+}^{\ell_{k^+}}$ where

$$(4.11.1) \qquad k^+ = \begin{cases} i_1 & \text{if} \quad j^+ = 1 \\ j_1 & \text{if} \quad j^+ = 2 \\ k_1 & \text{if} \quad j^+ = 3 \\ i_2 & \text{if} \quad j^+ = 4 \\ j_2 & \text{if} \quad j^+ = 5 \\ k_2 & \text{if} \quad j^+ = 6 \end{cases} ,$$

and $i_1, j_1, k_1, i_2, j_2, k_2$ all belong to ST1. All values of $\ell_1, \ell_2, \ldots, \ell_{p-k^*}$ other than those in columns [33], [35], [37] and [39] are zeros. These values are not presented in the tables. Each number in column $[i_2^+]$ (where $i_2^+ = 40, 42, 44$ and 46) is such that if it is j^+, then the number in the next column of the same row is ℓ_{k^+} of $\left[s_{k^+}\sqrt{z_{k^+}^{(s)}}\right]^{\ell_{k^+}}$ where k^+ and j^+ are related as shown in (4.11.1), and $i_1, j_1, k_1, i_2, j_2, k_2$ in (4.11.1) all belong to ST2. All values of $\ell_{p-k^*+1}, \ell_{p-k^*+2}, \ldots, \ell_p$ other than those in columns [41], [43], [45] and [47] are zeros. These values are not presented in the tables. Each number in column $[i_3^+]$ (where $i_3^+ = 48, 50, 52$ and 54) is such that if it is j^+, then the number in the next column of the same row is ℓ_{k^+} of $z_{k^+}^{\ell_{k^+}}$ where k^+ and j^+ are related as shown in (4.11.1) and $i_1, j_1, k_1, i_2, j_2, k_2$ in (4.11.1) all belong to ST3. All values of $\ell_{p+1}, \ell_{p+2}, \ldots, \ell_n$ other than those in columns [49], [51], [53] and [55] are zeros. These values are not presented in the tables. For example from Table (4.11.1), we get

$$\left[\frac{\partial r_2^{(+s)}}{\partial a_{211}^{+}}\right]_{\underset{\sim}{a}^+=\underset{\sim}{0}} = 4d_2^{*4} \left[s_1\sqrt{\bar{z}_1^{(s)}}\right]^4 z_2^2$$

and

$$\left[\frac{\partial r_2^{(+s)}}{\partial a_{211}^{+}} \frac{\partial r_2^{(+s)}}{\partial a_{311}^{+}}\right]_{\underset{\sim}{a}^+=\underset{\sim}{0}} = 4d_2^{*4} \left[s_1\sqrt{\bar{z}_1^{(s)}}\right]^4 z_2 z_3 \ .$$

While from Table (4.11.2), we get

$$\left[\frac{\partial^2 r_2^{(+s)}}{\partial a_{211}^{+2}}\right]_{\underset{\sim}{a}^+=\underset{\sim}{0}} = 2d_2^{*4} \left[s_1\sqrt{\bar{z}_1^{(s)}}\right]^4 - 8d_2^{*2}\left[s_1\sqrt{\bar{z}_1^{(s)}}\right]^2 z_2^2 + 8d_2^{*2}\left[s_1\sqrt{\bar{z}_1^{(s)}}\right]^4 z_2^2$$

and

$$\left[\frac{\partial^2 r_2^{(+s)}}{\partial a_{211}^{+}\ a_{311}^{+}}\right]_{\underset{\sim}{a}^+=\underset{\sim}{0}} = -8d_2^{*2}\left[s_1\sqrt{\bar{z}_1^{(s)}}\right]^2 z_2 z_3 + 8d_2^{*2}\left[s_1\sqrt{\bar{z}_1^{(s)}}\right]^4 z_2 z_3 \ .$$

Table (4.11.1) Representation of the terms in the expressions of the product of the first partial derivatives of $r_2^{(+s)}$ for the case when p=1 and k*=1

1	1	4.0	4.0	211211	322322001001011011001	00000000	24000000	12000000
2	1	4.0	4.0	211311	322322000001011011001	00000000	24000000	11410000

Table (4.11.2) Representation of the terms in the expressions of the second partial derivatives of $r_2^{(+s)}$ for the case when p=1 and k*=1

1	3	4.0	2.0	211211	322322001001011011001	00000000	24000000	00000000
2	3	2.0	-8.0	211211	322322001001011011001	00000000	22000000	12000000
3	3	2.0	8.0	211211	322322001001011011001	00000000	24000000	12000000
4	2	2.0	-8.0	211311	322322000001011011001	00000000	22000000	11410000
5	2	2.0	8.0	211311	322322000001011011001	00000000	24000000	11410000

Table (4.11.3) Representation of the terms in the expressions of the product of the first partial derivatives of $r_1^{(+s)}$ for the case when p=2 and k*=1

1	1	2.0	4.0	211211	211211001001011011001	24000000	12000000	00000000
2	1	2.0	-8.0	211312	211312000011010010000	23000000	12000000	41000000
3	1	3.0	-4.0	211322	211322000111000000001	22000000	13000000	41000000
4	1	2.0	16.0	312312	312312001000010001000	22000000	32000000	12000000
5	1	3.0	8.0	312322	312322001000000011001	21000000	33000000	12000000
6	1	3.0	8.0	322312	322312001001001001000	51000000	23000000	12000000
7	1	4.0	4.0	322322	322322001001011011001	00000000	24000000	12000000
8	1	2.0	16.0	312412	312312000000010001000	22000000	32000000	11410000
9	1	3.0	8.0	312422	312322000000000011001	21000000	33000000	11410000
10	1	3.0	8.0	322412	322312000001001001000	51000000	23000000	11410000
11	1	4.0	4.0	322422	322322000001011011001	00000000	24000000	11410000

Table (4.11.4) Representation of the terms in the expressions of the second partial derivatives of $r_1^{(+s)}$ for the case when p=2 and k*=1

1	3	0	-2.0	211211	211211001001011011001	24000000	00000000	00000000
2	3	2.0	8.0	211211	211211001001011011001	22000000	12000000	00000000
3	3	0	4.0	211211	211211001001011011001	24000000	12000000	00000000
4	1	1.0	8.0	211311	211311000001011011001	22000000	11000000	41000000
5	1	0	-8.0	211312	211312000011010010000	23000000	12000000	41000000
6	1	1.0	-6.0	211322	211322000111000000001	22000000	13000000	41000000
7	2	1.0	4.0	311312	311312001001010010000	23000000	61000000	00000000
8	2	1.0	-8.0	311312	311312001001010010000	21000000	61000000	12000000
9	1	2.0	2.0	311322	311322001001000000001	22000000	52000000	00000000
10	2	1.0	4.0	312311	312311001000011000001	23000000	31000000	00000000
11	2	1.0	-8.0	312311	312311001000011000001	21000000	31000000	12000000
12	4	2.0	8.0	312312	312312001000010001000	22000000	32000000	00000000
13	4	0	-8.0	312312	312312001000010001000	22000000	00000000	12000000
14	4	2.0	-8.0	312312	312312001000010001000	00000000	32000000	12000000
15	4	0	16.0	312312	312312001000010001000	22000000	32000000	12000000
16	3	3.0	4.0	312322	312322001000000011001	21000000	33000000	00000000
17	3	1.0	-8.0	312322	312322001000000011001	21000000	31000000	12000000
18	3	1.0	12.0	312322	312322001000000011001	21000000	33000000	12000000
19	1	2.0	2.0	322311	322311001001000000001	52000000	22000000	00000000
20	3	3.0	4.0	322312	322312001001001001000	51000000	23000000	00000000
21	3	1.0	-8.0	322312	322312001001001001000	51000000	21000000	12000000
22	3	1.0	12.0	322312	322312001001001001000	51000000	23000000	12000000
23	3	4.0	2.0	322322	322322001001011011001	00000000	24000000	00000000
24	3	2.0	-8.0	322322	322322001001011011001	00000000	22000000	12000000
25	3	2.0	8.0	322322	322322001001011011001	00000000	24000000	12000000
26	1	1.0	-8.0	311412	311312000001010010000	21000000	61000000	11410000
27	1	1.0	-8.0	312411	312311000000011000001	21000000	31000010	11410000
28	3	0	-8.0	312412	312312000000010001000	22000000	00000000	11410000
29	3	2.0	-8.0	312412	312312000000010001000	00000000	32000000	11410000
30	3	0	16.0	312412	312312000000010001000	22000000	32000000	11410000
31	2	1.0	-8.0	312422	312322000000000011001	21000000	31000000	11410000
32	2	1.0	12.0	312422	312322000000000011001	21000000	33000000	11410000
33	2	1.0	-8.0	322412	322312000001001001000	51000000	21000000	11410000
34	2	1.0	12.0	322412	322312000001001001000	51000000	23000000	11410000
35	2	2.0	-8.0	322422	322322000001011011001	00000000	22000000	11410000
36	2	2.0	8.0	322422	322322000001011011001	00000000	24000000	11410000

Table (4.11.5) Representation of the terms in the expressions of the product of the first partial derivatives of $r_2^{(+s)}$ for the case when p=2 and k*=2

1	1	4.0	4.0	311311	322322001001011011001	00000000	24000000	12000000
2	1	4.0	8.0	311312	322322001001010010000	00000000	23610000	12000000
3	1	4.0	4.0	311322	322322001001000000001	00000000	22520000	12000000
4	1	4.0	8.0	312311	322322001000011000001	00000000	23310000	12000000
5	1	4.0	16.0	312312	322322001000010001000	00000000	22320000	12000000
6	1	4.0	8.0	312322	322322001000000011001	00000000	21330000	12000000
7	1	4.0	8.0	322312	322322001001001001000	00000000	51230000	12000000
8	1	4.0	4.0	311411	322322000001011011001	00000000	24000000	11410000
9	1	4.0	8.0	311412	322322000001010010000	00000000	23610000	11410000
10	1	4.0	4.0	311422	322322000001000000001	00000000	22520000	11410000
11	1	4.0	8.0	312411	322322000000011000001	00000000	23310000	11410000
12	1	4.0	16.0	312412	322322000000010001000	00000000	22320000	11410000
13	1	4.0	8.0	312422	322322000000000011001	00000000	21330000	11410000
14	1	4.0	8.0	322412	322322000001001001000	00000000	51230000	11410000

Table (4.11.6) Representation of the terms in the expressions of the second partial derivatives of $r_2^{(+s)}$ for the case when p=2 and k*=2

1	3	4.0	2.0	311311	322322001001011011001	00000000	24000000	00000000
2	3	2.0	-8.0	311311	322322001001011011001	00000000	22000000	12000000
3	3	2.0	8.0	311311	322322001001011011001	00000000	24000000	12000000
4	3	4.0	4.0	311312	322322001001010010000	00000000	23610000	00000000
5	3	2.0	-8.0	311312	322322001001010010000	00000000	21610000	12000000
6	3	2.0	16.0	311312	322322001001010010000	00000000	23610000	12000000
7	2	4.0	2.0	311322	322322001001000000001	00000000	22520000	00000000
8	2	2.0	8.0	311322	322322001001000000001	00000000	22520000	12000000
9	3	4.0	4.0	312311	322322001000011000001	00000000	23310000	00000000
10	3	2.0	-8.0	312311	322322001000011000001	00000000	21310000	12000000
11	3	2.0	16.0	312311	322322001000011000001	00000000	23310000	12000000
12	4	4.0	8.0	312312	322322001000010001000	00000000	22320000	00000000
13	4	2.0	-8.0	312312	322322001000010001000	00000000	22000000	12000000
14	4	2.0	-8.0	312312	322322001000010001000	00000000	32000000	12000000
15	4	2.0	32.0	312312	322322001000010001000	00000000	22320000	12000000
16	3	4.0	4.0	312322	322322001000000011001	00000000	21330000	00000000
17	3	2.0	-8.0	312322	322322001000000011001	00000000	21310000	12000000
18	3	2.0	16.0	312322	322322001000000011001	00000000	21330000	12000000
19	3	4.0	4.0	322312	322322001001001001000	00000000	51230000	00000000
20	3	2.0	-8.0	322312	322322001001001001000	00000000	51210000	12000000
21	3	2.0	16.0	322312	322322001001001001000	00000000	51230000	12000000
22	2	2.0	-8.0	311411	322322000001011011001	00000000	22000000	11410000
23	2	2.0	8.0	311411	322322000001011011001	00000000	24000000	11410000
24	2	2.0	-8.0	311412	322322000001010010000	00000000	21610000	11410000
25	2	2.0	16.0	311412	322322000001010010000	00000000	23610000	11410000
26	1	2.0	8.0	311422	322322000001000000001	00000000	22520000	11410000
27	2	2.0	-8.0	312411	322322000000011000001	00000000	21310000	11410000
28	2	2.0	16.0	312411	322322000000011000001	00000000	23310000	11410000
29	3	2.0	-8.0	312412	322322000000010001000	00000000	22000000	11410000
30	3	2.0	-8.0	312412	322322000000010001000	00000000	32000000	11410000
31	3	2.0	32.0	312412	322322000000010001000	00000000	22320000	11410000
32	2	2.0	-8.0	312422	322322000000000011001	00000000	21310000	11410000
33	2	2.0	16.0	312422	322322000000000011001	00000000	21330000	11410000
34	2	2.0	-8.0	322412	322322000001001001000	00000000	51210000	11410000
35	2	2.0	16.0	322412	322322000001001001000	00000000	51230000	11410000

Table (4.11.7) Representation of the terms in the expressions of the product of the first partial derivatives of $r_1^{(+s)}$ for the case when p=3 and k*=1

1	1	2.0	4.0	311311	211211001001011011001	24000000	12000000	00000000
2	1	2.0	8.0	311312	211211001001010010000	23510000	12000000	00000000
3	1	2.0	4.0	311322	211211001001000000001	22520000	12000000	00000000
4	1	2.0	8.0	312311	211211001000011000001	23510000	12000000	00000000
5	1	2.0	16.0	312312	211211001000010001000	22320000	12000000	00000000
6	1	2.0	8.0	312322	211211001000000001101	21330000	12000000	00000000
7	1	2.0	8.0	322312	211211001001001001000	51230000	12000000	00000000
8	1	2.0	-8.0	311413	211312000011010010000	23000000	12000000	41000000
9	1	2.0	-8.0	311423	211312000011000000000	22510000	12000000	41000000
10	1	3.0	-4.0	311433	211322000111000000001	22000000	13000000	41000000
11	1	2.0	-16.0	312413	211312000010010000000	22310000	12000000	41000000
12	1	2.0	-16.0	312423	211312000010000010000	21320000	12000000	41000000
13	1	3.0	-8.0	312433	211322000110000000001	21310000	13000000	41000000
14	1	2.0	16.0	413413	312312001000010001000	22000000	32000000	12000000
15	1	2.0	16.0	413423	312312001000000001000	21510000	32000000	12000000
16	1	3.0	8.0	413433	312322001000000001101	21000000	33000000	12000000
17	1	3.0	8.0	433413	322312001001001001000	51000000	23000000	12000000
18	1	4.0	4.0	433433	322322001001011011001	00000000	24000000	12000000
19	1	2.0	16.0	413513	312312000000010001000	22000000	32000000	11410000
20	1	2.0	16.0	413523	312312000000000001000	21510000	32000000	11410000
21	1	3.0	8.0	413533	312322000000000001101	21000000	33000000	11410000
22	1	3.0	8.0	433513	322312000001001001000	51000000	23000000	11410000
23	1	4.0	4.0	433533	322322000001011011001	00000000	24000000	11410000

Table (4.11.8) Representation of the terms in the expressions of the second partial derivatives of $r_1^{(+s)}$ for the case when p=3 and k*=1

1	3	0	-2.0	311311	211211001001011011001	24000000	00000000	00000000
2	3	2.0	8.0	311311	211211001001011011001	22000000	12000000	00000000
3	3	0	4.0	311311	211211001001011011001	24000000	12000000	00000000
4	3	0	-4.0	311312	211211001001010010000	23610000	00000000	00000000
5	3	2.0	8.0	311312	211211001001010010000	21610000	12000000	00000000
6	3	0	8.0	311312	211211001001010010000	23610000	12000000	00000000
7	2	0	-2.0	311322	211211001001000000001	22520000	00000000	00000000
8	2	0	4.0	311322	211211001001000000001	22520000	12000000	00000000
9	3	0	-4.0	312311	211211001000011000001	23310000	00000000	00000000
10	3	2.0	8.0	312311	211211001000011000001	21310000	12000000	00000000
11	3	0	8.0	312311	211211001000011000001	23310000	12000000	00000000
12	4	0	-8.0	312312	211211001000010001000	22320000	00000000	00000000
13	4	2.0	8.0	312312	211211001000010001000	22000000	12000000	00000000
14	4	2.0	8.0	312312	211211001000010001000	32000000	12000000	00000000
15	4	0	16.0	312312	211211001000010001000	22320000	12000000	00000000
16	3	0	-4.0	312322	211211001000000001101	21330000	00000000	00000000
17	3	2.0	8.0	312322	211211001000000001101	21310000	12000000	00000000
18	3	0	8.0	312322	211211001000000001101	21330000	12000000	00000000
19	3	0	-4.0	322312	211211001001001001000	51230000	00000000	00000000
20	3	2.0	8.0	322312	211211001001001001000	51210000	12000000	00000000
21	3	0	8.0	322312	211211001001001001000	51230000	12000000	00000000
22	1	1.0	8.0	311411	211311000001011011001	22000000	11000000	41000000
23	1	1.0	8.0	311412	211311000001010010000	21610000	11000000	41000000
24	1	0	-8.0	311413	211312000011010010000	23000000	12000000	41000000
25	1	0	-8.0	311423	211312000011000000000	22510000	12000000	41000000
26	1	1.0	-6.0	311433	211322000111000000001	22000000	13000000	41000000
27	1	1.0	0.0	312411	211311000000011000001	21310000	11000000	41000000
28	2	1.0	0.0	312412	211311000000010001000	22000000	11000000	41000000
29	2	1.0	8.0	312412	211311000000010001000	32000000	11000000	41000000
30	1	0	-16.0	312413	211312000010010000000	22310000	12000000	41000000

Table (4.11.8) contd.

31	1	1.0	8.0	312422	211311000000000011001	21310000	11000000	41000000
32	1	0	-16.0	312423	211312000010000010000	21320000	12000000	41000000
33	1	1.0	-12.0	312433	211322000110000000001	21310000	13000000	41000000
34	1	1.0	8.0	322412	211311000001001001000	51210000	11000000	41000000
35	2	1.0	4.0	411413	311312001001010010000	23000000	61000000	00000000
36	2	1.0	-8.0	411413	311312001001010010000	21000000	61000000	12000000
37	1	1.0	4.0	411423	311312001001000000000	22510000	61000000	00000000
38	1	2.0	2.0	411433	311322001001000000001	22000000	52000000	00000000
39	2	1.0	8.0	412413	311312001000010000000	22310000	61000000	00000000
40	2	1.0	-8.0	412413	311312001000010000000	31000000	61000000	12000000
41	2	1.0	8.0	412423	311312001000000010000	21320000	61000000	00000000
42	2	1.0	-8.0	412423	311312001000000010000	21000000	61000000	12000000
43	1	2.0	4.0	412433	311322001000000000001	21310000	52000000	00000000
44	2	1.0	4.0	413411	312311001000011000001	23000000	31000000	00000000
45	2	1.0	-8.0	413411	312311001000011000001	21000000	31000000	12000000
46	2	1.0	8.0	413412	312311001000010000000	22610000	31000000	00000000
47	2	1.0	-8.0	413412	312311001000010000000	61000000	31000000	12000000
48	4	2.0	8.0	413413	312312001000010001000	22000000	32000000	00000000
49	4	0	-8.0	413413	312312001000010001000	22000000	00000000	12000000
50	4	2.0	-8.0	413413	312312001000010001000	00000000	32000000	12000000
51	4	0	16.0	413413	312312001000010001000	22000000	32000000	12000000
52	1	1.0	4.0	413422	312311001000000000001	21520000	31000000	00000000
53	3	2.0	8.0	413423	312312001000000001000	21510000	32000000	00000000
54	3	0	-8.0	413423	312312001000000001000	21510000	00000000	12000000
55	3	0	16.0	413423	312312001000000001000	21510000	32000000	12000000
56	3	3.0	4.0	413433	312322001000000011001	21000000	33000000	00000000
57	3	1.0	-8.0	413433	312322001000000011001	21000000	31000000	12000000
58	3	1.0	12.0	413433	312322001000000011001	21000000	33000000	12000000
59	2	1.0	8.0	423412	312311001000001000000	51220000	31000000	00000000
60	2	1.0	-8.0	423412	312311001000001000000	51000000	31000000	12000000
61	1	2.0	2.0	433411	322311001001000000001	52000000	22000000	00000000
62	1	2.0	4.0	433412	322311001001000000000	51610000	22000000	00000000
63	3	3.0	4.0	433413	322312001001001001000	51000000	23000000	00000000
64	3	1.0	-8.0	433413	322312001001001001000	51000000	21000000	12000000
65	3	1.0	12.0	433413	322312001001001001000	51000000	23000000	12000000
66	3	4.0	2.0	433433	322322001001011011001	00000000	24000000	00000000
67	3	2.0	-8.0	433433	322322001001011011001	00000000	22000000	12000000
68	3	2.0	8.0	433433	322322001001011011001	00000000	24000000	12000000
69	1	1.0	-8.0	411513	311312000001010010000	21000000	61000000	11410000
70	1	1.0	-8.0	412513	311312000000010000000	31000000	61000000	11410000
71	1	1.0	-8.0	412523	311312000000000010000	21000000	61000000	11410000
72	1	1.0	-8.0	413511	312311000000011000001	21000000	31000000	11410000
73	1	1.0	-8.0	413512	312311000000010000000	61000000	31000000	11410000
74	3	0	-8.0	413513	312312000000010001000	22000000	00000000	11410000
75	3	2.0	-9.0	413513	312312000000010001000	00000000	32000000	11410000
76	3	0	16.0	413513	312312000000010001000	22000000	32000000	11410000
77	2	0	-8.0	413523	312312000000000001000	21510000	00000000	11410000
78	2	0	16.0	413523	312312000000000001000	21510000	32000000	11410000
79	2	1.0	-8.0	413533	312322000000000011001	21000000	31000000	11410000
80	2	1.0	12.0	413533	312322000000000011001	21000000	33000000	11410000
81	1	1.0	-8.0	423512	312311000000001000000	51000000	31000000	11410000
82	2	1.0	-8.0	433513	322312000001001001000	51000000	21000000	11410000
83	2	1.0	12.0	433513	322312000001001001000	51000000	23000000	11410000
84	2	2.0	-8.0	433533	322322000001011011001	00000000	22000000	11410000
85	2	2.0	8.0	433533	322322000001011011001	00000000	24000000	11410000

Table (4.11.9) Representation of the terms in the expressions of the product of the first partial derivatives of $r_1^{(+s)}$ for the case when p=3 and k*=2

1	1	2.0	4.0	211211	211211001001011011001	24000000	12000000	00000000
2	1	2.0	4.0	211311	211211000001011011001	24000000	11410000	00000000
3	1	2.0	-8.0	211412	211312000011010010000	23000000	12000000	41000000
4	1	2.0	-8.0	211413	211312000001010110000	23000000	11610000	41000000
5	1	3.0	-4.0	211422	211322001110000010001	22000000	13000000	41000000
6	1	3.0	-8.0	211423	211322001010100000000	22000000	12610000	41000000
7	1	3.0	-4.0	211433	211322000001000000001	22000000	11520000	41000000
8	1	3.0	-8.0	311423	211322000011000000000	22000000	51120000	41000000
9	1	2.0	16.0	412412	312312001000010001000	22000000	32000000	12000000
10	1	2.0	16.0	412413	312312001000010000000	22000000	31610000	12000000
11	1	3.0	8.0	412422	312322001000000011001	21000000	33000000	12000000
12	1	3.0	16.0	412423	312322001000000010000	21000000	32610000	12000000
13	1	3.0	8.0	412433	312322001000000000001	21000000	31520000	12000000
14	1	3.0	16.0	413423	312322001000000001000	21000000	51320000	12000000
15	1	3.0	8.0	422412	322312001001001001000	51000000	23000000	12000000
16	1	3.0	8.0	422413	322312001001000000000	51000000	22610000	12000000
17	1	4.0	4.0	422422	322322001001011011001	00000000	24000000	12000000
18	1	4.0	8.0	422423	322322001001010010000	00000000	23610000	12000000
19	1	4.0	4.0	422433	322322001001000000001	00000000	22520000	12000000
20	1	3.0	16.0	423412	322312001000001000000	51000000	22310000	12000000
21	1	3.0	16.0	423413	322312001000000001000	51000000	21320000	12000000
22	1	4.0	8.0	423422	322322001000011000001	00000000	23310000	12000000
23	1	4.0	16.0	423423	322322001000010001000	00000000	22320000	12000000
24	1	4.0	8.0	423433	322322001000000011001	00000000	21330000	12000000
25	1	4.0	8.0	433423	322322001001001001000	00000000	51230000	12000000
26	1	2.0	16.0	412512	312312000000001001000	22000000	32000000	11410000
27	1	2.0	16.0	412513	312312000000001000000	22000000	31610000	11410000
28	1	3.0	8.0	412522	312322000000000011001	21000000	33000000	11410000
29	1	3.0	16.0	412523	312322000000000010000	21000000	32610000	11410000
30	1	3.0	8.0	412533	312322000000000000001	21000000	31520000	11410000
31	1	3.0	16.0	413523	312322000000000001000	21000000	51320000	11410000
32	1	3.0	8.0	422512	322312000001001001000	51000000	23000000	11410000
33	1	3.0	8.0	422513	322312000001000000000	51000000	22610000	11410000
34	1	4.0	4.0	422522	322322000001011011001	00000000	24000000	11410000
35	1	4.0	8.0	422523	322322000001010010000	00000000	23610000	11410000
36	1	4.0	4.0	422533	322322000001000000001	00000000	22520000	11410000
37	1	3.0	16.0	423512	322312000000001000000	51000000	22310000	11410000
38	1	3.0	16.0	423513	322312000000000001000	51000000	21320000	11410000
39	1	4.0	8.0	423522	322322000000011000001	00000000	23310000	11410000
40	1	4.0	16.0	423523	322322000000010001000	00000000	22320000	11410000
41	1	4.0	8.0	423533	322322000000000011001	00000000	21330000	11410000
42	1	4.0	8.0	433523	322322000001001001000	00000000	51230000	11410000

Table (4.11.10) Representation of the terms in the expressions of the second partial derivatives of $r_1^{(+s)}$ for the case when p=3 and k*=2

1	3	0	-2.0	211211	211211001001011011001	24000000	00000000	00000000
2	3	2.0	8.0	211211	211211001001011011001	22000000	12000000	00000000
3	3	0	4.0	211211	211211001001011011001	24000000	12000000	00000000
4	2	2.0	8.0	211311	211211000001011011001	22000000	11410000	00000000
5	2	0	4.0	211311	211211000001011011001	24000000	11410000	00000000
6	1	1.0	8.0	211411	211311000001011011001	22000000	11000000	41000000
7	1	0	-8.0	211412	211312000011010010000	23000000	12000000	41000000
8	1	0	-8.0	211413	211312000001010010000	23000000	11610000	41000000
9	1	1.0	-6.0	211422	211322001110000000011	22000000	13000000	41000000
10	1	1.0	-12.0	211423	211322001010000000000	22000000	12610000	41000000
11	1	1.0	-6.0	211433	211322000001000000001	22000000	11520000	41000000
12	1	1.0	-12.0	311423	211322000011000000000	22000000	51120000	41000000
13	2	1.0	4.0	411412	311312001001010010000	23000000	61000000	00000000
14	2	1.0	-8.0	411412	311312001001010010000	21000000	61000000	12000000
15	1	2.0	2.0	411422	311322001001000000001	22000000	52000000	00000000
16	1	2.0	4.0	411423	311322001001000000000	22000000	51610000	00000000
17	2	1.0	4.0	412411	312311001000011000001	23000000	31000000	00000000
18	2	1.0	-8.0	412411	312311001000011000001	21000000	31000000	12000000
19	4	2.0	8.0	412412	312312001000010001000	22000000	32000000	00000000
20	4	0	-8.0	412412	312312001000010001000	22000000	00000000	12000000
21	4	2.0	-8.0	412412	312312001000010001000	00000000	32000000	12000000
22	4	0	16.0	412412	312312001000010001000	22000000	32000000	12000000
23	3	2.0	8.0	412413	312312001000010000000	22000000	31610000	00000000
24	3	2.0	-8.0	412413	312312001000010000000	00000000	31610000	12000000
25	3	0	16.0	412413	312312001000010000000	22000000	31610000	12000000
26	3	3.0	4.0	412422	312322001000000011001	21000000	33000000	00000000
27	3	1.0	-8.0	412422	312322001000000011001	21000000	31000000	12000000
28	3	1.0	12.0	412422	312322001000000011001	21000000	33000000	12000000
29	3	3.0	8.0	412423	312322001000000010000	21000000	32610000	00000000
30	3	1.0	-8.0	412423	312322001000000010000	21000000	61000000	12000000

Table (4.11.10) contd.

31	3	1.0	24.0	412423	312322001000000001000	21000000	32610000	12000000
32	2	3.0	4.0	412433	312322001000000000001	21000000	31520000	00000000
33	2	1.0	12.0	412433	312322001000000000001	21000000	31520000	12000000
34	3	3.0	8.0	413423	312322001000000001000	21000000	51320000	00000000
35	3	1.0	-8.0	413423	312322001000000001000	21000000	51000000	12000000
36	3	1.0	24.0	413423	312322001000000001000	21000000	51320000	12000000
37	1	2.0	2.0	422411	322311001001000000001	52000000	22000000	00000000
38	3	3.0	4.0	422412	322312001001001001000	51000000	23010000	00000000
39	3	1.0	-8.0	422412	322312001001001001000	51000000	21000000	12000000
40	3	1.0	12.0	422412	322312001001001001000	51000000	23000000	12000000
41	2	3.0	4.0	422413	322312001001000000000	51000000	22610000	00000000
42	2	1.0	12.0	422413	322312001001000000000	51000000	22610000	12000000
43	3	4.0	2.0	422422	322322001001011011001	00000000	24000000	00000000
44	3	2.0	-8.0	422422	322322001001011011001	00000000	22000000	12000000
45	3	2.0	8.0	422422	322322001001011011001	00000000	24000000	12000000
46	3	4.0	4.0	422423	322322001001010010000	00000000	23610000	00000000
47	3	2.0	-8.0	422423	322322001001010010000	00000000	21610000	12000000
48	3	2.0	16.0	422423	322322001001010010000	00000000	23610000	12000000
49	2	4.0	2.0	422433	322322001001000000001	00000000	22520000	00000000
50	2	2.0	8.0	422433	322322001001000000001	00000000	22520000	12000000
51	1	2.0	4.0	423411	322311001000000000001	52000000	21310000	00000000
52	3	3.0	8.0	423412	322312001000001000000	51000000	22310000	00000000
53	3	1.0	-8.0	423412	322312001000001000000	51000000	31000000	12000000
54	3	1.0	24.0	423412	322312001000001000000	51000000	22310000	12000000
55	3	3.0	8.0	423413	322312001000000001000	51000000	21320000	00000000
56	3	1.0	-8.0	423413	322312001000000001000	51000000	21000000	12000000
57	3	1.0	24.0	423413	322312001000000001000	51000000	21320000	12000000
58	3	4.0	4.0	423422	322322001000011000001	00000000	23310000	00000000
59	3	2.0	-8.0	423422	322322001000011000001	00000000	21310000	12000000
60	3	2.0	16.0	423422	322322001000011000001	00000000	23310000	12000000
61	4	4.0	8.0	423423	322322001000010001000	00000000	22320000	00000000
62	4	2.0	-8.0	423423	322322001000010001000	00000000	22000000	12000000
63	4	2.0	-8.0	423423	322322001000010001000	00000000	32000000	12000000
64	4	2.0	32.0	423423	322322001000010001000	00000000	22320000	12000000
65	3	4.0	4.0	423433	322322001000000011001	00000000	21330000	00000000
66	3	2.0	-8.0	423433	322322001000000011001	00000000	21310000	12000000
67	3	2.0	16.0	423433	322322001000000011001	00000000	21330000	12000000
68	3	4.0	4.0	433423	322322001001001001000	00000000	51230000	00000000
69	3	2.0	-8.0	433423	322322001001001001000	00000000	51210000	12000000
70	3	2.0	16.0	433423	322322001001001001000	00000000	51230000	12000000
71	1	1.0	-8.0	411512	311312000001010010000	21000000	61000000	11410000
72	1	1.0	-8.0	412511	312311000000011000001	21000000	31000000	11410000
73	3	0	-8.0	412512	312312000000010001000	22000000	00000000	11410000
74	3	2.0	-8.0	412512	312312000000010001000	00000000	32000000	11410000
75	3	0	16.0	412512	312312000000010001000	22000000	32000000	11410000
76	2	2.0	-8.0	412513	312312000000010000000	00000000	31610000	11410000
77	2	0	16.0	412513	312312000000010000000	22000000	31610000	11410000
78	2	1.0	-8.0	412522	312322000000000011001	21000000	31000000	11410000
79	2	1.0	12.0	412522	312322000000000011001	21000000	33010000	11410000
80	2	1.0	-8.0	412523	312322000000000001000	21000000	61000000	11410000
81	2	1.0	24.0	412523	312322000000000001000	21000000	32610000	11410000
82	1	1.0	12.0	412533	312322000000000000001	21000000	31520000	11410000
83	2	1.0	-8.0	413523	312322000000000001000	21000000	51000000	11410000
84	2	1.0	24.0	413523	312322000000000001000	21000000	51320000	11410000
85	2	1.0	-8.0	422512	322312000001001001000	51000000	21000000	11410000
86	2	1.0	12.0	422512	322312000001001001000	51000000	23000000	11410000
87	1	1.0	12.0	422513	322312000001000000000	51000000	22610000	11410000
88	2	2.0	-8.0	422522	322322000001011011001	00000000	22000000	11410000
89	2	2.0	8.0	422522	322322000001011011001	00000000	24000000	11410000
90	2	2.0	-8.0	422523	322322000001010010000	00000000	21610000	11410000
91	2	2.0	16.0	422523	322322000001010010000	00000000	23610000	11410000
92	1	2.0	8.0	422533	322322000001000000001	00000000	22520000	11410000
93	2	1.0	-8.0	423512	322312000000001000000	51000000	31000000	11410000
94	2	1.0	24.0	423512	322312000000001000000	51000000	22310000	11410000
95	2	1.0	-8.0	423513	322312000000000001000	51000000	21000000	11410000
96	2	1.0	24.0	423513	322312000000000001000	51000000	21320000	11410000
97	2	2.0	-8.0	423522	322322000000011000001	00000000	21310000	11410000
98	2	2.0	16.0	423522	322322000000011000001	00000000	23310000	11410000
99	3	2.0	-8.0	423523	322322000000010001000	00000000	22000000	11410000
100	3	2.0	-8.0	423523	322322000000010001000	00000000	32000000	11410000
101	3	2.0	32.0	423523	322322000000010001000	00000000	22320000	11410000
102	2	2.0	-8.0	423533	322322000000000011001	00000000	21310000	11410000
103	2	2.0	16.0	423533	322322000000000011001	00000000	21330000	11410000
104	2	2.0	-8.0	433523	322322000001001001000	00000000	51210000	11410000
105	2	2.0	16.0	433523	322322000001001001000	00000000	51230000	11410000

Table (4.11.11) Representation of the terms in the expressions of the product of the first partial derivatives of $r_2^{(+s)}$ for the case when p=3 and k*=3

1	1	4.0	4.0	411411	322322001001011011001	00000000	24000000	12000000
2	1	4.0	8.0	411412	322322001001010010000	00000000	23610000	12000000
3	1	4.0	4.0	411422	322322001001000000001	00000000	22520000	12000000
4	1	4.0	8.0	411423	322322001001000000000	00000000	22516100	12000000
5	1	4.0	8.0	412411	322322001000011000001	00000000	23310000	12000000
6	1	4.0	16.0	412412	322322001000010001000	00000000	22320000	12000000
7	1	4.0	16.0	412413	322322001000010000000	00000000	22316100	12000000
8	1	4.0	8.0	412422	322322001000000011001	00000000	21330000	12000000
9	1	4.0	16.0	412423	322322001000000010000	00000000	21326100	12000000
10	1	4.0	8.0	412433	322322001000000000001	00000000	21315200	12000000
11	1	4.0	16.0	413423	322322001000000001000	00000000	21513200	12000000
12	1	4.0	8.0	422412	322322001001001001000	00000000	51230000	12000000
13	1	4.0	16.0	423412	322322001000001000000	00000000	51223100	12000000
14	1	4.0	4.0	411511	322322000001011011001	00000000	24000000	11410000
15	1	4.0	8.0	411512	322322000001010010000	00000000	23610000	11410000
16	1	4.0	4.0	411522	322322000001000000001	00000000	22520000	11410000
17	1	4.0	8.0	411523	322322000001000000000	00000000	22516100	11410000
18	1	4.0	8.0	412511	322322000000011000001	00000000	23310000	11410000
19	1	4.0	16.0	412512	322322000000010001000	00000000	22320000	11410000
20	1	4.0	16.0	412513	322322000000010000000	00000000	22316100	11410000
21	1	4.0	8.0	412522	322322000000000011001	00000000	21330000	11410000
22	1	4.0	16.0	412523	322322000000000010000	00000000	21326100	11410000
23	1	4.0	8.0	412533	322322000000000000001	00000000	21315200	11410000
24	1	4.0	16.0	413523	322322000000000001000	00000000	21513200	11410000
25	1	4.0	8.0	422512	322322000001001001000	00000000	51230000	11410000
26	1	4.0	16.0	423512	322322000000001000000	00000000	51223100	11410000

Table (4.11.12) Representation of the terms in the expressions of the second partial derivatives of $r_2^{(+s)}$ for the case when p=3 and k*=3

1	3	4.0	2.0	411411	322322001001011011001	00000000	24000000	00000000
2	3	2.0	-8.0	411411	322322001001011011001	00000000	22000000	12000000
3	3	2.0	8.0	411411	322322001001011011001	00000000	24000000	12000000
4	3	4.0	4.0	411412	322322001001010010000	00000000	23610000	00000000
5	3	2.0	-8.0	411412	322322001001010010000	00000000	21610000	12000000
6	3	2.0	16.0	411412	322322001001010010000	00000000	23610000	12000000
7	2	4.0	2.0	411422	322322001001000000001	00000000	22520000	00000000
8	2	2.0	8.0	411422	322322001001000000001	00000000	22520000	12000000
9	2	4.0	4.0	411423	322322001001000000000	00000000	22516100	00000000
10	2	2.0	16.0	411423	322322001001000000000	00000000	22516100	12000000
11	3	4.0	4.0	412411	322322001000011000001	00000000	23310000	00000000
12	3	2.0	-8.0	412411	322322001000011000001	00000000	21310000	12000000
13	3	2.0	16.0	412411	322322001000011000001	00000000	23310000	12000000
14	4	4.0	8.0	412412	322322001000010001000	00000000	22320000	00000000
15	4	2.0	-8.0	412412	322322001000010001000	00000000	22000000	12000000
16	4	2.0	-8.0	412412	322322001000010001000	00000000	32000000	12000000
17	4	2.0	32.0	412412	322322001000010001000	00000000	22320000	12000000
18	3	4.0	8.0	412413	322322001000010000000	00000000	22316100	00000000
19	3	2.0	-8.0	412413	322322001000010000000	00000000	31610000	12000000
20	3	2.0	32.0	412413	322322001000010000000	00000000	22316100	12000000
21	3	4.0	4.0	412422	322322001000000011001	00000000	21330000	00000000
22	3	2.0	-8.0	412422	322322001000000011001	00000000	21310000	12000000
23	3	2.0	16.0	412422	322322001000000011001	00000000	21330000	12000000
24	3	4.0	8.0	412423	322322001000000010000	00000000	21326100	00000000
25	3	2.0	-8.0	412423	322322001000000010000	00000000	21610000	12000000
26	3	2.0	32.0	412423	322322001000000010000	00000000	21326100	12000000
27	2	4.0	4.0	412433	322322001000000000001	00000000	21315200	00000000
28	2	2.0	16.0	412433	322322001000000000001	00000000	21315200	12000000
29	3	4.0	8.0	413423	322322001000000001000	00000000	21513200	00000000
30	3	2.0	-8.0	413423	322322001000000001000	00000000	21510000	12000000

Table (4.11.12) contd.

31	3	2.0	32.0	413423	322322001000000001000	00000000	21513200	12000000
32	3	4.0	4.0	422412	322322110010010010000	00000000	51230000	00000000
33	3	2.0	-8.0	422412	322322001001001001000	00000000	51210000	12000000
34	3	2.0	16.0	422412	322322001001001001000	00000000	51230000	12000000
35	3	4.0	8.0	423412	322322001000001000000	00000000	51223100	00000000
36	3	2.0	-8.0	423412	322322001000001000000	00000000	51310000	12000000
37	3	2.0	32.0	423412	322322001000001000000	00000000	51223100	12000000
38	2	2.0	-8.0	411511	322322000001011011001	00000000	22000000	11410000
39	2	2.0	8.0	411511	322322000001011011001	00000000	24000000	11410000
40	2	2.0	-8.0	411512	322322000001010010000	00000000	21610000	11410000
41	2	2.0	16.0	411512	322322000001010010000	00000000	23610000	11410000
42	1	2.0	8.0	411522	322322000001000000001	00000000	22520000	11410000
43	1	2.0	16.0	411523	322322000001000000000	00000000	22516100	11410000
44	2	2.0	-8.0	412511	322322000000011000001	00000000	21310000	11410000
45	2	2.0	16.0	412511	322322000000011000001	00000000	23310000	11410000
46	3	2.0	-8.0	412512	322322000000010001000	00000000	22000000	11410000
47	3	2.0	-8.0	412512	322322000000010001000	00000000	32000000	11410000
48	3	2.0	32.0	412512	322322000000010001000	00000000	22320000	11410000
49	2	2.0	-8.0	412513	322322000000010000000	00000000	31610000	11410000
50	2	2.0	32.0	412513	322322000000010000000	00000000	22316100	11410000
51	2	2.0	-8.0	412522	322322000000000011001	00000000	21310000	11410000
52	2	2.0	16.0	412522	322322000001000011001	00000000	21330000	11410000
53	2	2.0	-8.0	412523	322322000000000010000	00000000	21610000	11410000
54	2	2.0	32.0	412523	322322000000000010000	00000000	21326100	11410000
55	1	2.0	16.0	412533	322322000000000000001	00000000	21315200	11410000
56	2	2.0	-8.0	413523	322322000000000001000	00000000	21510000	11410000
57	2	2.0	32.0	413523	322322000000000001000	00000000	21513200	11410000
58	2	2.0	-8.0	422512	322322000001001001000	00000000	51210000	11410000
59	2	2.0	16.0	422512	322322000001001001000	00000000	51230000	11410000
60	2	2.0	-8.0	423512	322322000000001000000	00000000	51310000	11410000
61	2	2.0	32.0	423512	322322000000001000000	00000000	51223100	11410000

Table (4.11.13) Representation of the terms in the expressions of the product of the first partial derivatives of $r_1^{(+s)}$ for the case when p=4 and k*=1

1	1	2.0	4.0	411411	211211001001011011001	24000000	12000000	00000000
2	1	2.0	8.0	411412	211211001001010010000	23610000	12000000	00000000
3	1	2.0	4.0	411422	211211001001000000001	22520000	12000000	00000000
4	1	2.0	8.0	411423	211211001001000000000	22516100	12000000	00000000
5	1	2.0	8.0	412411	211211001000011000001	23310000	12000000	00000000
6	1	2.0	16.0	412412	211211001000010001000	22320000	12000000	00000000
7	1	2.0	16.0	412413	211211001000010000000	22316100	12000000	00000000
8	1	2.0	8.0	412422	211211001000000011001	21330000	12000000	00000000
9	1	2.0	16.0	412423	211211001000000010000	21326100	12000000	00000000
10	1	2.0	8.0	412433	211211001000000000001	21315200	12000000	00000000
11	1	2.0	16.0	413423	211211001000000001000	21513200	12000000	00000000
12	1	2.0	8.0	422412	211211001001001001000	51230000	12000000	00000000
13	1	2.0	16.0	423412	211211001000001000000	51223100	12000000	00000000
14	1	2.0	-8.0	411514	211312000011010010000	23000000	12000000	41000000
15*	1	2.0	-8.0	411524	211312000011000000000	22510000	12000000	41000000
16	1	3.0	-4.0	411544	211322000111000000001	22000000	13000000	41000000
17	1	2.0	-16.0	412514	211312000010010000000	22310000	12000000	41000000
18	1	2.0	-16.0	412524	211312000010000010000	21320000	12000000	41000000
19	1	2.0	-16.0	412534	211312000010000000000	21315100	12000000	41000000
20	1	3.0	-8.0	412544	211322000110000000001	21310000	13000000	41000000
21	1	2.0	16.0	514514	312312001000010001000	22000000	32000000	12000000
22	1	2.0	16.0	514524	312312001000000001000	21510000	32000000	12000000
23	1	3.0	8.0	514544	312322001000000011001	21000000	33000000	12000000
24	1	3.0	8.0	544514	322312001001001001000	51000000	23000000	12000000
25	1	4.0	4.0	544544	322322001001011011001	00000000	24000000	12000000
26	1	2.0	16.0	514614	312312000000010001000	22000000	32000000	11410000
27	1	2.0	16.0	514624	312312000000000001000	21510000	32000000	11410000
28	1	3.0	8.0	514644	312322000000000011001	21000000	33000000	11410000
29	1	3.0	8.0	544614	322312000001001001000	51000000	23000000	11410000
30	1	4.0	4.0	544644	322322000001011011001	00000000	24000000	11410000

Table (4.11.14) Representation of the terms in the expressions of the second partial derivatives of $r_1^{(+s)}$ for the case when p=4 and k*=1

1	3	0	-2.0	411411	211211001001011011001	24000000	00000000	00000000
2	3	2.0	8.0	411411	211211001001011011001	22000000	12000000	00000000
3	3	0	4.0	411411	211211001001011011001	24000000	12000000	00000000
4	3	0	-4.0	411412	211211001001010010000	23610000	00000000	00000000
5	3	2.0	8.0	411412	211211001001010010000	21610000	12000000	00000000
6	3	0	8.0	411412	211211001001010010000	23610000	12000000	00000000
7	2	0	-2.0	411422	211211001001000000001	22520000	00000000	00000000
8	2	0	4.0	411422	211211001001000000001	22520000	12000000	00000000
9	2	0	-4.0	411423	211211001001000000000	22516100	00000000	00000000
10	2	0	8.0	411423	211211001001000000000	22516100	12000000	00000000
11	3	0	-4.0	412411	211211001000011000001	23310000	00000000	00000000
12	3	2.0	8.0	412411	211211001000011000001	21310000	12000000	00000000
13	3	0	8.0	412411	211211001000011000001	23310000	12000000	00000000
14	4	0	-8.0	412412	211211001000010001000	22320000	00000000	00000000
15	4	2.0	8.0	412412	211211001000010001000	22000000	12000000	00000000
16	4	2.0	8.0	412412	211211001000010001000	32000000	12000000	00000000
17	4	0	16.0	412412	211211001000010001000	22320000	12000000	00000000
18	3	0	-8.0	412413	211211001000010000000	22316100	00000000	00000000
19	3	2.0	8.0	412413	211211001000010000000	31610000	12000000	00000000
20	3	0	16.0	412413	211211001000010000000	22316100	12000000	00000000
21	3	0	-4.0	412422	211211001000000011001	21330000	00000000	00000000
22	3	2.0	8.0	412422	211211001000000011001	21310000	12000000	00000000
23	3	0	8.0	412422	211211001000000011001	21330000	12000000	00000000
24	3	0	-8.0	412423	211211001000000010000	21326100	00000000	00000000
25	3	2.0	8.0	412423	211211001000000010000	21610000	12000000	00000000
26	3	0	16.0	412423	211211001000000010000	21326100	12000000	00000000
27	2	0	-4.0	412433	211211001000000000001	21315200	00000000	00000000
28	2	0	8.0	412433	211211001000000000001	21315200	12000000	00000000
29	3	0	-8.0	413423	211211001000000001000	21513200	00000000	00000000
30	3	2.0	8.0	413423	211211001000000001000	21510000	12000000	00000000
31	3	0	16.0	413423	211211001000000001000	21513200	12000000	00000000
32	3	0	-4.0	422412	211211001001001001000	51230000	00000000	00000000
33	3	2.0	8.0	422412	211211001001001001000	51210000	12000000	00000000
34	3	0	8.0	422412	211211001001001001000	51230000	12000000	00000000
35	3	0	-8.0	423412	211211001000001000000	51223100	00000000	00000000
36	3	2.0	8.0	423412	211211001000001000000	51310000	12000000	00000000
37	3	0	16.0	423412	211211001000001000000	51223100	12000000	00000000
38	1	1.0	8.0	411511	211311000001011011001	22000000	11000000	41000000
39	1	1.0	8.0	411512	211311000001010010000	21610000	11000000	41000000
40	1	0	-8.0	411514	211312000011010010000	23000000	12000000	41000000
41	1	0	-8.0	411524	211312000011000000000	22510000	12000000	41000000
42	1	1.0	-6.0	411544	211322000111000000001	22000000	13000000	41000000
43	1	1.0	8.0	412511	211311000000011000001	21310000	11000000	41000000
44	2	1.0	8.0	412512	211311000000010001000	22000000	11000000	41000000
45	2	1.0	8.0	412512	211311000000010001000	32000000	11000000	41000000
46	1	1.0	8.0	412513	211311000000010000000	31610000	11000000	41000000
47	1	0	-16.0	412514	211312000010010000000	22310000	12000000	41000000
48	1	1.0	8.0	412522	211311000000000011001	21310000	11000000	41000000
49	1	1.0	8.0	412523	211311000000000010000	21610000	11000000	41000000
50	1	0	-16.0	412524	211312000010000010000	21320000	12000000	41000000
51	1	0	-16.0	412534	211312000010000000000	21315100	12000000	41000000
52	1	1.0	-12.0	412544	211322000110000000001	21310000	13000000	41000000
53	1	1.0	8.0	413523	211311000000000001000	21510000	11000000	41000000
54	1	1.0	8.0	422512	211311000001001001000	51210000	11000000	41000000
55	1	1.0	8.0	423512	211311000000001000000	51310000	11000000	41000000
56	2	1.0	4.0	511514	311312001001010010000	23000000	61000000	00000000
57	2	1.0	-8.0	511514	311312001001010010000	21000000	61000000	12000000
58	1	1.0	4.0	511524	311312001001000000000	22510000	61000000	00000000
59	1	2.0	2.0	511544	311322001001000000001	22000000	52000000	00000000
60	2	1.0	8.0	512514	311312001000010000000	22310000	61000000	00000000

Table (4.11.14) contd.

61	2	1.0	-8.0	512514	311312001000010000000	31000000	61000000	12000000
62	2	1.0	8.0	512524	311312001000000010000	21320000	61000000	00000000
63	2	1.0	-8.0	512524	311312001000000010000	21000000	61000000	12000000
64	1	1.0	8.0	512534	311312001000000000000	21315100	61000000	00000000
65	1	2.0	4.0	512544	311322001000000000001	21310000	52000000	00000000
66	2	1.0	4.0	514511	312311001000011000001	23000000	31000000	00000000
67	2	1.0	-8.0	514511	312311001000011000001	21000000	31000000	12000000
68	2	1.0	8.0	514512	312311001000010000000	22610000	31000000	00000000
69	2	1.0	-8.0	514512	312311001000010000000	61000000	31000000	12000000
70	4	2.0	8.0	514514	312312001000010001000	22000000	32000000	00000000
71	4	0	-8.0	514514	312312001000010001000	22000000	00000000	12000000
72	4	2.0	-8.0	514514	312312001000010001000	00000000	32000000	12000000
73	4	0	16.0	514514	312312001000010001000	22000000	32000000	12000000
74	1	1.0	4.0	514522	312311001000000000001	21520000	31000000	00000000
75	1	1.0	8.0	514523	312311001000000000000	21516100	31000000	00000000
76	3	2.0	8.0	514524	312312001000000001000	21510000	32000000	00000000
77	3	0	-8.0	514524	312312001000000001000	21510000	00000000	12000000
78	3	0	16.0	514524	312312001000000001000	21510000	32000000	12000000
79	3	3.0	4.0	514544	312322001000000011001	21000000	33000000	00000000
80	3	1.0	-8.0	514544	312322001000000011001	21000000	31000000	12000000
81	3	1.0	12.0	514544	312322001000000011001	21000000	33000000	12000000
82	2	1.0	8.0	524512	312311001000001000000	51220000	31000000	00000000
83	2	1.0	-8.0	524512	312311001000001000000	51000000	31000000	12000000
84	1	2.0	2.0	544511	322311001001000000001	52000000	22000000	00000000
85	1	2.0	4.0	544512	322311001001000000000	51610000	22000000	00000000
86	3	3.0	4.0	544514	322312001001001001000	51000000	23000000	00000000
87	3	1.0	-8.0	544514	322312001001001001000	51000000	21000000	12000000
88	3	1.0	12.0	544514	322312001001001001000	51000000	23000000	12000000
89	3	4.0	2.0	544544	322322001001011011001	00000000	24000000	00000000
90	3	2.0	-8.0	544544	322322001001011011001	00000000	22000000	12000000
91	3	2.0	8.0	544544	322322001001011011001	00000000	24000000	12000000
92	1	1.0	-8.0	511614	311312000001010010000	21000000	61000000	11410000
93	1	1.0	-8.0	512614	311312000000010000000	31000000	61000000	11410000
94	1	1.0	-8.0	512624	311312000000000010000	21000000	61000000	11410000
95	1	1.0	-8.0	514611	312311000000011000001	21000000	31000000	11410000
96	1	1.0	-8.0	514612	312311000000010000000	61000000	31000000	11410000
97	3	0	-8.0	514614	312312000000010001010	22000000	00000000	11410000
98	3	2.0	-8.0	514614	312312000000010001010	00000000	32000000	11410000
99	3	0	16.0	514614	312312000000010001000	22000000	32000000	11410000
100	2	0	-8.0	514624	312312000000000001000	21510000	00000000	11410000
101	2	0	16.0	514624	312312000000000001000	21510000	32000000	11410000
102	2	1.0	-8.0	514644	312322000000000011001	21000000	31000000	11410000
103	2	1.0	12.0	514644	312322000000000011001	21000000	33000000	11410000
104	1	1.0	-8.0	524612	312311000000001000000	51000000	31000000	11410000
105	2	1.0	-8.0	544614	322312000001001001010	51000000	21000000	11410000
106	2	1.0	12.0	544614	322312000001001001010	51000000	23000000	11410000
107	2	2.0	-8.0	544644	322322000001011011001	00000000	22000000	11410000
108	2	2.0	8.0	544644	322322000001011011001	00000000	24000000	11410000

Table (4.11.15) Representation of the terms in the expressions of the product of the first partial derivatives of $r_1^{(+s)}$ for the case when p=4 and k*=2

1	1	2.0	4.0	311311	211211001001011011001	24000000	12000000	00000000
2	1	2.0	8.0	311312	211211001001010010000	23610000	12000000	00000000
3	1	2.0	4.0	311322	211211001001000000001	22520000	12000000	00000000
4	1	2.0	8.0	312311	211211001000011000001	23310000	12000000	00000000
5	1	2.0	16.0	312312	211211001000010001000	22320000	12000000	00000000
6	1	2.0	8.0	312322	211211001000000011001	21330000	12000000	00000000
7	1	2.0	8.0	322312	211211001001001001000	51230000	12000000	00000000
8	1	2.0	4.0	311411	211211000001011011001	24000000	11410000	00000000
9	1	2.0	8.0	311412	211211000001010010000	23610000	11410000	00000000
10	1	2.0	4.0	311422	211211000001000000001	22520000	11410000	00000000
11	1	2.0	8.0	312411	211211000000011000001	23310000	11410000	00000000
12	1	2.0	16.0	312412	211211000000010001000	22320000	11410000	00000000
13	1	2.0	8.0	312422	211211000000000011001	21330000	11410000	00000000
14	1	2.0	8.0	322412	211211000001001001000	51230000	11410000	00000000
15	1	2.0	-8.0	311513	211312000011010010000	23000000	12000000	41000000
16	1	2.0	-8.0	311514	211312000001010010000	23000000	11610000	41000000
17	1	2.0	-8.0	311523	211312000011000000000	22510000	12000000	41000000
18	1	2.0	-8.0	311524	211312000001000000000	22510000	11610000	41000000
19	1	3.0	-4.0	311533	211322000111000000001	22000000	13000000	41000000
20	1	3.0	-8.0	311534	211322000101000000000	22000000	12610000	41000000
21	1	3.0	-4.0	311544	211322000001000000001	22000000	11520000	41000000
22	1	2.0	-16.0	312513	211312000010010000000	22310000	12000000	41000000
23	1	2.0	-16.0	312514	211312000000010000000	22310000	11610000	41000000
24	1	2.0	-16.0	312523	211312000010000010000	21320000	12000000	41000000
25	1	2.0	-16.0	312524	211312000000000010000	21320000	11610000	41000000
26	1	3.0	-8.0	312533	211322000110000000001	21310000	13000000	41000000
27	1	3.0	-16.0	312534	211322000100000000000	21310000	12610000	41000000
28	1	3.0	-8.0	312544	211322000000000000001	21310000	11520000	41000000
29	1	3.0	-8.0	411534	211322000011000000000	22000000	51120000	41000000
30	1	3.0	-16.0	412534	211322000010000000000	21310000	51120000	41000000
31	1	2.0	16.0	513513	312312001000010001000	22000000	32000000	12000000
32	1	2.0	16.0	513514	312312001000010000000	22000000	31610000	12000000
33	1	2.0	16.0	513523	312312001000000001000	21510000	32000000	12000000
34	1	2.0	16.0	513524	312312001000000000000	21510000	31610000	12000000
35	1	3.0	8.0	513533	312322001000000011001	21000000	33000000	12000000
36	1	3.0	16.0	513534	312322001000000010000	21000000	32610000	12000000
37	1	3.0	8.0	513544	312322001000000000001	21000000	31520000	12000000
38	1	3.0	16.0	514534	312322001000000001000	21000000	51320000	12000000
39	1	3.0	8.0	533513	322312001001001001000	51000000	23000000	12000000
40	1	3.0	8.0	533514	322312001001000000000	51000000	22610000	12000000
41	1	4.0	4.0	533533	322322001001011011001	00000000	24000000	12000000
42	1	4.0	8.0	533534	322322001001010010000	00000000	23610000	12000000
43	1	4.0	4.0	533544	322322001001000000001	00000000	22520000	12000000
44	1	3.0	16.0	534513	322312001000001000000	51000000	22310000	12000000
45	1	3.0	16.0	534514	322312001000000001000	51000000	21320000	12000000
46	1	4.0	8.0	534533	322322001000011000001	00000000	23310000	12000000
47	1	4.0	16.0	534534	322322001000010001000	00000000	22320000	12000000
48	1	4.0	8.0	534544	322322001000000011001	00000000	21330000	12000000
49	1	4.0	8.0	544534	322322001001001001000	00000000	51230000	12000000
50	1	2.0	16.0	513613	312312000000010001000	22000000	32000000	11410000
51	1	2.0	16.0	513614	312312000000010000000	22000000	31610000	11410000
52	1	2.0	16.0	513623	312312000000000001000	21510000	32000000	11410000
53	1	2.0	16.0	513624	312312000000000000000	21510000	31610000	11410000
54	1	3.0	8.0	513633	312322000000000011001	21000000	33000000	11410000
55	1	3.0	16.0	513634	312322000000000010000	21000000	32610000	11410000
56	1	3.0	8.0	513644	312322000000000000001	21000000	31520000	11410000
57	1	3.0	16.0	514634	312322000000000001000	21000000	51320000	11410000
58	1	3.0	8.0	533613	322312000001001001000	51000000	23000000	11410000
59	1	3.0	8.0	533614	322312000001000000000	51000000	22610000	11410000
60	1	4.0	4.0	533633	322322000001011011001	00000000	24000000	11410000
61	1	4.0	8.0	533634	322322000001010010000	00000000	23610000	11410000
62	1	4.0	4.0	533644	322322000001000000001	00000000	22520000	11410000
63	1	3.0	16.0	534613	322312000000001000000	51000000	22510000	11410000
64	1	3.0	16.0	534614	322312000000000001000	51000000	21320000	11410000
65	1	4.0	8.0	534633	322322000000011000001	00000000	23310000	11410000
66	1	4.0	16.0	534634	322322000000010001000	00000000	22320000	11410000
67	1	4.0	8.0	534644	322322000000000011001	00000000	21330000	11410000
68	1	4.0	8.0	544634	322322000001001001000	00000000	51230000	11410000

Table (4.11.16) Representation of the terms in the expressions of the second partial derivatives of $r_1^{(+s)}$ for the case when p=4 and k*=2

1	3	0	-2.0	311311	211211001001011011001	24000000	00000000	00000000
2	3	2.0	8.0	311311	211211001001011011001	22000000	12000000	00000000
3	3	0	4.0	311311	211211001001011011001	24000000	12000000	00000000
4	3	0	-4.0	311312	211211001001010010000	23610000	00000000	00000000
5	3	2.0	8.0	311312	211211001001010010000	21610000	12000000	00000000
6	3	0	8.0	311312	211211001001010010000	23610000	12000000	00000000
7	2	0	-2.0	311322	211211001001000000001	22520000	00000000	00000000
8	2	0	4.0	311322	211211001001000000001	22520000	12000000	00000000
9	3	0	-4.0	312311	211211001000011000001	23310000	00000000	00000000
10	3	2.0	8.0	312311	211211001000011000001	21310000	12000000	00000000
11	3	0	8.0	312311	211211001000011000001	23310000	12000000	00000000
12	4	0	-8.0	312312	211211001000010001000	22320000	00000000	00000000
13	4	2.0	8.0	312312	211211001000010001000	22000000	12000000	00000000
14	4	2.0	8.0	312312	211211001000010001000	32000000	12000000	00000000
15	4	0	16.0	312312	211211001000010001000	22320000	12000000	00000000
16	3	0	-4.0	312322	211211001000000011001	21330000	00000000	00000000
17	3	2.0	8.0	312322	211211001000000011001	21310000	12000000	00000000
18	3	0	8.0	312322	211211001000000011001	21330000	12000000	00000000
19	3	0	-4.0	322312	211211001001001001000	51230000	00000000	00000000
20	3	2.0	8.0	322312	211211001001001001000	51210000	12000000	00000000
21	3	0	8.0	322312	211211001001001001000	51230000	12000000	00000000
22	2	2.0	8.0	311411	211211000001011011001	22000000	11410000	00000000
23	2	0	4.0	311411	211211000001011011001	24000000	11410000	00000000
24	2	2.0	8.0	311412	211211000001010010000	21610000	11410000	00000000
25	2	0	8.0	311412	211211000001010010000	23610000	11410000	00000000
26	1	0	4.0	311422	211211000001000000001	22520000	11410000	00000000
27	2	2.0	8.0	312411	211211000000011000001	21310000	11410000	00000000
28	2	0	8.0	312411	211211000000011000001	23310000	11410000	00000000
29	3	2.0	8.0	312412	211211000000010001000	22000000	11410000	00000000
30	3	2.0	8.0	312412	211211000000010001000	32000000	11410000	00000000
31	3	0	16.0	312412	211211000000010001000	22320000	11410000	00000000
32	2	2.0	8.0	312422	211211000000000011001	21310000	11410000	00000000
33	2	0	8.0	312422	211211000000000011001	21330000	11410000	00000000
34	2	2.0	8.0	322412	211211000001001001000	51210000	11410000	00000000
35	2	0	8.0	322412	211211000001001001000	51230000	11410000	00000000
36	1	1.0	8.0	311511	211311000001011011001	22000000	11000000	41000000
37	1	1.0	8.0	311512	211311000001010010000	21610000	11000000	41000000
38	1	0	-8.0	311513	211312000011010010000	23000000	12000000	41000000
39	1	0	-8.0	311514	211312000001010010000	23000000	11610000	41000000
40	1	0	-8.0	311523	211312000011000000000	22510000	12000000	41000000
41	1	0	-8.0	311524	211312000001000000000	22510000	11610000	41000000
42	1	1.0	-6.0	311533	211322000111000000001	22000000	13000000	41000000
43	1	1.0	-12.0	311534	211322000101000000000	22000000	12610000	41000000
44	1	1.0	-6.0	311544	211322000001000000001	22000000	11520000	41000000
45	1	1.0	8.0	312511	211311000000011000001	21310000	11000000	41000000
46	2	1.0	8.0	312512	211311000000010001000	22000000	11000000	41000000
47	2	1.0	8.0	312512	211311000000010001000	32000000	11000000	41000000
48	1	0	-16.0	312513	211312000001001000000	22310000	12000000	41000000
49	1	0	-16.0	312514	211312000000001000000	22310000	11610000	41000000
50	1	1.0	8.0	312522	211311000000000011001	21310000	11000000	41000000
51	1	0	-16.0	312523	211312000010000010000	21320000	12000000	41000000
52	1	0	-16.0	312524	211312000000000010000	21320000	11610000	41000000
53	1	1.0	-12.0	312533	211322000110000000001	21310000	13000000	41000000
54	1	1.0	-24.0	312534	211322000100000000000	21310000	12610000	41000000
55	1	1.0	-12.0	312544	211322000000000000001	21310000	11520000	41000000
56	1	1.0	8.0	322512	211311000001001001000	51210000	11000000	41000000
57	1	1.0	-12.0	411534	211322000011000000000	22000000	51120000	41000000
58	1	1.0	-24.0	412534	211322000010000000000	21310000	51120000	41000000
59	2	1.0	4.0	511513	311312001001010010000	23000000	61000000	00000000
60	2	1.0	-8.0	511513	311312001001010010000	21000000	61000000	12000000

Table (4.11.16) contd.

61	1	1.0	4.0	511523	311312001001000000000	22510000	61000000	00000000
62	1	2.0	2.0	511533	311322001001000000001	22000000	52000000	00000000
63	1	2.0	4.0	511534	311322001001000000000	22000000	51610000	00000000
64	2	1.0	8.0	512513	311312001000010000000	22310000	61000000	00000000
65	2	1.0	-8.0	512513	311312001000010000000	31000000	61000000	12000000
66	2	1.0	8.0	512523	311312001000001010000	21320000	61000000	00000000
67	2	1.0	-8.0	512523	311312001000000010000	21000000	61000000	12000000
68	1	2.0	4.0	512533	311322001000000000001	21310000	52000000	00000000
69	1	2.0	8.0	512534	311322001000000000000	21310000	51610000	00000000
70	2	1.0	4.0	513511	312311001000011000001	23000000	31000000	00000000
71	2	1.0	-8.0	513511	312311001000011000001	21000000	31000000	12000000
72	2	1.0	8.0	513512	312311001000010000000	22510000	31000000	00000000
73	2	1.0	-8.0	513512	312311001000010000000	61000000	31000000	12000000
74	4	2.0	8.0	513513	312312001000010001000	22000000	32000000	00000000
75	4	0	-8.0	513513	312312001000010001000	22000000	00000000	12000000
76	4	2.0	-8.0	513513	312312001000010001000	00000000	32000000	12000000
77	4	0	16.0	513513	312312001000010001000	22000000	32000000	12000000
78	3	2.0	8.0	513514	312312001000010000000	22000000	31610000	00000000
79	3	2.0	-8.0	513514	312312001000010000000	00000000	31610000	12000000
80	3	0	16.0	513514	312312001000010000000	22000000	31610000	12000000
81	1	1.0	4.0	513522	312311001000000000001	21520000	31000000	00000000
82	3	2.0	8.0	513523	312312001000000001000	21510000	32000000	00000000
83	3	0	-8.0	513523	312312001000000001000	21510000	00000000	12000000
84	3	0	16.0	513523	312312001000000001000	21510000	32000000	12000000
85	2	2.0	8.0	513524	312312001000000000000	21510000	31610000	00000000
86	2	0	16.0	513524	312312001000000000000	21510000	31610000	12000000
87	3	3.0	4.0	513533	312322001000000011001	21000000	33000000	00000000
88	3	1.0	-8.0	513533	312322001000000011001	21000000	31000000	12000000
89	3	1.0	12.0	513533	312322001000000011001	21000000	33000000	12000000
90	3	3.0	8.0	513534	312322001000000010000	21000000	32610000	00000000
91	3	1.0	-8.0	513534	312322001000000010000	21000000	61000000	12000000
92	3	1.0	24.0	513534	312322001000000010000	21000000	32610000	12000000
93	2	3.0	4.0	513544	312322001000000000001	21000000	31520000	00000000
94	2	1.0	12.0	513544	312322001000000000001	21000000	31520000	12000000
95	3	3.0	8.0	514534	312322001000000001000	21000000	51320000	00000000
96	3	1.0	-8.0	514534	312322001000000001000	21000000	51000000	12000000
97	3	1.0	24.0	514534	312322001000000001000	21000000	51320000	12000000
98	2	1.0	8.0	523512	312311001000001000000	51220000	31000000	00000000
99	2	1.0	-8.0	523512	312311001000001000000	51000000	31000000	12000000
100	1	2.0	2.0	533511	322311001001000000001	52000000	22000000	00000000
101	1	2.0	4.0	533512	322311001001000000000	51610000	22000000	00000000
102	3	3.0	4.0	533513	322312001001001001000	51000000	23000000	00000000
103	3	1.0	-8.0	533513	322312001001001001000	51000000	21000000	12000000
104	3	1.0	12.0	533513	322312001001001001000	51000000	23000000	12000000
105	2	3.0	4.0	533514	322312001001000000000	51000000	22610000	00000000
106	2	1.0	12.0	533514	322312001001000000000	51000000	22610000	12000000
107	3	4.0	2.0	533533	322322001001011011001	00000000	24000000	00000000
108	3	2.0	-8.0	533533	322322001001011011001	00000000	22000000	12000000
109	3	2.0	8.0	533533	322322001001011011001	00000000	24000000	12000000
110	3	4.0	4.0	533534	322322001001010010000	00000000	23610000	00000000
111	3	2.0	-8.0	533534	322322001001010010000	00000000	21610000	12000000
112	3	2.0	16.0	533534	322322001001010010000	00000000	23610000	12000000
113	2	4.0	2.0	533544	322322001001000000001	00000000	22520000	00000000
114	2	2.0	8.0	533544	322322001001000000001	00000000	22520000	12000000
115	1	2.0	4.0	534511	322311001000000000001	52000000	21310000	00000000
116	1	2.0	8.0	534512	322311001000000000000	51610000	21310000	00000000
117	3	3.0	8.0	534513	322312001000001000000	51000000	22310000	00000000
118	3	1.0	-8.0	534513	322312001000001000000	51000000	31000000	12000000
119	3	1.0	24.0	534513	322312001000001000000	51000000	22310000	12000000
120	3	3.0	8.0	534514	322312001000000001000	51000000	21320000	00000000

Table (4.11.16) contd.

121	3	1.0	-8.0	534514	322312001000000001000	51000000	21000000	12000000
122	3	1.0	24.0	534514	322312001000000001000	51000000	21320000	12000000
123	3	4.0	4.0	534533	322322001000011000001	00000000	23310000	00000000
124	3	2.0	-8.0	534533	322322001000011000001	00000000	21310000	12000000
125	3	2.0	16.0	534533	322322001000011000001	00000000	23310000	12000000
126	4	4.0	8.0	534534	322322001000010001000	00000000	22320000	00000000
127	4	2.0	-8.0	534534	322322001000010001000	00000000	22000000	12000000
128	4	2.0	-8.0	534534	322322001000010001000	00000000	32000000	12000000
129	4	2.0	32.0	534534	322322001000010001000	00000000	22320000	12000000
130	3	4.0	4.0	534544	322322001000000011001	00000000	21330000	00000000
131	3	2.0	-8.0	534544	322322001000000011001	00000000	21310000	12000000
132	3	2.0	16.0	534544	322322001000000011001	00000000	21330000	12000000
133	3	4.0	4.0	544534	322322001001001001000	00000000	51230000	00000000
134	3	2.0	-8.0	544534	322322001001001001000	00000000	51210000	12000000
135	3	2.0	16.0	544534	322322001001001001000	00000000	51230000	12000000
136	1	1.0	-8.0	511613	311312000001010010000	21000000	61000000	11410000
137	1	1.0	-8.0	512613	311312000000010000000	31000000	61000000	11410000
138	1	1.0	-8.0	512623	311312000000000010000	21000000	61000000	11410000
139	1	1.0	-8.0	513611	312311000000011000001	21000000	31000000	11410000
140	1	1.0	-8.0	513612	312311000000010000000	61000000	31000000	11410000
141	3	0	-8.0	513613	312312000000010001000	22000000	00000000	11410000
142	3	2.0	-8.0	513613	312312000000010001000	00000000	32000000	11410000
143	3	0	16.0	513613	312312000000010001000	22000000	32000000	11410000
144	2	2.0	-8.0	513614	312312000000010000000	00000000	31610000	11410000
145	2	0	16.0	513614	312312000000010000000	22000000	31610000	11410000
146	2	0	-8.0	513623	312312000000000001000	21510000	00000000	11410000
147	2	0	16.0	513623	312312000000000001000	21510000	32000000	11410000
148	1	0	16.0	513624	312312000000000000000	21510000	31610000	11410000
149	2	1.0	-8.0	513633	312322000000000011001	21000000	31000000	11410000
150	2	1.0	12.0	513633	312322000000000011001	21000000	33000000	11410000
151	2	1.0	-8.0	513634	312322000000000010000	21000000	61000000	11410000
152	2	1.0	24.0	513634	312322000000000010000	21000000	32610000	11410000
153	1	1.0	12.0	513644	312322000000000000001	21000000	31520000	11410000
154	2	1.0	-8.0	514634	312322000000000001000	21000000	51000000	11410000
155	2	1.0	24.0	514634	312322000000000001000	21000000	51320000	11410000
156	1	1.0	-8.0	523612	312311000000001000000	51000000	31000000	11410000
157	2	1.0	-8.0	533613	322312000001001001000	51000000	21000000	11410000
158	2	1.0	12.0	533613	322312000001001001000	51000000	23000000	11410000
159	1	1.0	12.0	533614	322312000001000000000	51000000	22610000	11410000
160	2	2.0	-8.0	533633	322322000001011011001	00000000	22000000	11410000
161	2	2.0	8.0	533633	322322000001011011001	00000000	24000000	11410000
162	2	2.0	-8.0	533634	322322000001010010000	00000000	21610000	11410000
163	2	2.0	16.0	533634	322322000001010010000	00000000	23610000	11410000
164	1	2.0	8.0	533644	322322000001000000001	00000000	22520000	11410000
165	2	1.0	-8.0	534613	322312000000001000000	51000000	31000000	11410000
166	2	1.0	24.0	534613	322312000000001000000	51000000	22310000	11410000
167	2	1.0	-8.0	534614	322312000000000001000	51000000	21000000	11410000
168	2	1.0	24.0	534614	322312000000000001000	51000000	21320000	11410000
169	2	2.0	-8.0	534633	322322000000011000001	00000000	21310000	11410000
170	2	2.0	16.0	534633	322322000000011000001	00000000	23310000	11410000
171	3	2.0	-8.0	534634	322322000000010001000	00000000	22000000	11410000
172	3	2.0	-8.0	534634	322322000000010001000	00000000	32000000	11410000
173	3	2.0	32.0	534634	322322000000010001000	00000000	22320000	11410000
174	2	2.0	-8.0	534644	322322000000000011001	00000000	21310000	11410000
175	2	2.0	16.0	534644	322322000000000011001	00000000	21330000	11410000
176	2	2.0	-8.0	544634	322322000001001001000	00000000	51210000	11410000
177	2	2.0	16.0	544634	322322000001001001000	00000000	51230000	11410000

Table (4.11.17) Representation of the terms in the expressions of the product of the first partial derivatives of $r_1^{(+s)}$ for the case when p=4 and k*=3

1	1	2.0	4.0	211211	211211001001011011001	24000000	12000000	00000000
2	1	2.0	4.0	211311	211211000001011011001	24000000	11410000	00000000
3	1	2.0	-8.0	211512	211312000011010010000	23000000	12000000	41000000
4	1	2.0	-8.0	211513	211312000001010010000	23000000	11610000	41000000
5	1	3.0	-4.0	211522	211322000111000000001	22000000	13000000	41000000
6	1	3.0	-4.0	211523	211322000101010000000	22000000	12610000	41000000
7	1	3.0	-4.0	211533	211322000000100000001	22000000	11520000	41000000
8	1	3.0	-8.0	211534	211322000001000000000	22000000	11516100	41000000
9	1	3.0	-8.0	311523	211322000011000000000	22000000	51120000	41000000
10	1	2.0	16.0	512512	312312001000010001000	22000000	32000000	12000000
11	1	2.0	16.0	512513	312312001000010000000	22000000	31610000	12000000
12	1	3.0	8.0	512522	312322001000000011001	21000000	33000000	12000000
13	1	3.0	16.0	512523	312322001000000010000	21000000	32610000	12000000
14	1	3.0	8.0	512533	312322001000000000001	21000000	31520000	12000000
15	1	3.0	16.0	512534	312322001000000000000	21000000	31516100	12000000
16	1	3.0	16.0	513523	312322001000000001000	21000000	51320000	12000000
17	1	3.0	8.0	522512	322312001001001001000	51000000	23000000	12000000
18	1	3.0	8.0	522513	322312001001000000000	51000000	22610000	12000000
19	1	4.0	4.0	522522	322322001001011011001	00000000	24000000	12000000
20	1	4.0	8.0	522523	322322001001010010000	00000000	23610000	12000000
21	1	4.0	4.0	522533	322322001001000000001	00000000	22520000	12000000
22	1	4.0	8.0	522534	322322001001000000000	00000000	22516100	12000000
23	1	3.0	16.0	523512	322312001001001000000	51000000	22310000	12000000
24	1	3.0	16.0	523513	322312001000000001000	51000000	21320000	12000000
25	1	3.0	16.0	523514	322312001000000000000	51000000	21316100	12000000
26	1	4.0	8.0	523522	322322001000011000001	00000000	23310000	12000000
27	1	4.0	16.0	523523	322322001000010001000	00000000	22320000	12000000
28	1	4.0	16.0	523524	322322001000010000000	00000000	22316100	12000000
29	1	4.0	8.0	523533	322322001000000011001	00000000	21330000	12000000
30	1	4.0	16.0	523534	322322001000000010000	00000000	21326100	12000000
31	1	4.0	8.0	523544	322322001000000000001	00000000	21315200	12000000
32	1	4.0	16.0	524534	322322001000000001000	00000000	21513200	12000000
33	1	4.0	8.0	533523	322322001001001001000	00000000	51230000	12000000
34	1	4.0	16.0	534523	322322001000001000000	00000000	51223100	12000000
35	1	2.0	16.0	512612	312312000000010001000	22000000	32000000	11410000
36	1	2.0	16.0	512613	312312000000010000000	22000000	31610000	11410000
37	1	3.0	8.0	512622	312322000000000011001	21000000	33000000	11410000
38	1	3.0	16.0	512623	312322000000000010000	21000000	32610000	11410000
39	1	3.0	8.0	512633	312322000000000000001	21000000	31520000	11410000
40	1	3.0	16.0	512634	312322000000000000000	21000000	31516100	11410000
41	1	3.0	16.0	513623	312322000000000001000	21000000	51320000	11410000
42	1	3.0	8.0	522612	322312000001001001000	51000000	23000000	11410000
43	1	3.0	8.0	522613	322312000001000000000	51000000	22610000	11410000
44	1	4.0	4.0	522622	322322000001011011001	00000000	24000000	11410000
45	1	4.0	8.0	522623	322322000001010010000	00000000	23610000	11410000
46	1	4.0	4.0	522633	322322000001000000001	00000000	22520000	11410000
47	1	4.0	8.0	522634	322322000001000000000	00000000	22516100	11410000
48	1	3.0	16.0	523612	322312000000001000000	51000000	22310000	11410000
49	1	3.0	16.0	523613	322312000000000001000	51000000	21320000	11410000
50	1	3.0	16.0	523614	322312000000000000000	51000000	21316100	11410000
51	1	4.0	8.0	523622	322322000000011000001	00000000	23310000	11410000
52	1	4.0	16.0	523623	322322000000010001000	00000000	22320000	11410000
53	1	4.0	16.0	523624	322322000000010000000	00000000	22316100	11410000
54	1	4.0	8.0	523633	322322000000000011001	00000000	21330000	11410000
55	1	4.0	16.0	523634	322322000000000010000	00000000	21326100	11410000
56	1	4.0	8.0	523644	322322000000000000001	00000000	21315200	11410000
57	1	4.0	16.0	524634	322322000000000001000	00000000	21513200	11410000
58	1	4.0	8.0	533623	322322000001001001000	00000000	51230000	11410000
59	1	4.0	16.0	534623	322322000000001000000	00000000	51223100	11410000

Table (4.11.18) Representation of the terms in the expressions of the second partial derivatives of $r_1^{(+s)}$ for the case when p=4 and k*=3

1	3	0	-2.0	211211	211211001001011011001	24000000	00000000	00000000
2	3	2.0	8.0	211211	211211001001011011001	22000000	12000000	00000000
3	3	0	4.0	211211	211211001001011011001	24000000	12000000	00000000
4	2	2.0	8.0	211311	211211000001011011001	22000000	11410000	00000000
5	2	0	4.0	211311	211211000001011011001	24000000	11410000	00000000
6	1	1.0	8.0	211511	211311000001011011001	22000000	11000000	41000000
7	1	0	-8.0	211512	211312000001101001000	23000000	12000000	41000000
8	1	0	-8.0	211513	211312000001010010000	23000000	11610000	41000000
9	1	1.0	-6.0	211522	211322000111000000001	22000000	13000000	41000000
10	1	1.0	-12.0	211523	211322000101000000000	22000000	12610000	41000000
11	1	1.0	-6.0	211533	211322000001000000001	22000000	11520000	41000000
12	1	1.0	-12.0	211534	211322000001000000000	22000000	11516100	41000000
13	1	1.0	-12.0	311523	211322000011000000000	22000000	51120000	41000000
14	2	1.0	4.0	511512	311312001001010010000	23000000	61000000	00000000
15	2	1.0	-8.0	511512	311312001001010010000	21000000	61000000	12000000
16	1	2.0	2.0	511522	311322001001000000001	22000000	52000000	00000000
17	1	2.0	4.0	511523	311322001001000000000	22000000	51610000	00000000
18	2	1.0	4.0	512511	312311001000011000001	23000000	31000000	00000000
19	2	1.0	-8.0	512511	312311001000011000001	21000000	31000000	12000000
20	4	2.0	8.0	512512	312312001000010001000	22000000	32000000	00000000
21	4	0	-8.0	512512	312312001000010001000	22000000	00000000	12000000
22	4	2.0	-8.0	512512	312312001000010001000	00000000	32000000	12000000
23	4	0	16.0	512512	312312001000010001000	22000000	32000000	12000000
24	3	2.0	8.0	512513	312312001000010000000	22000000	31610000	00000000
25	3	2.0	-8.0	512513	312312001000010000000	00000000	31610000	12000000
26	3	0	16.0	512513	312312001000010000000	22000000	31610000	12000000
27	3	3.0	4.0	512522	312322001000000011001	21000000	33000000	00000000
28	3	1.0	-8.0	512522	312322001000000011001	21000000	31000000	12000000
29	3	1.0	12.0	512522	312322001000000011001	21000000	33000000	12000000
30	3	3.0	8.0	512523	312322001000000010000	21000000	32610000	00000000
31	3	1.0	-8.0	512523	312322001000000010000	21000000	61000000	12000000
32	3	1.0	24.0	512523	312322001000000010000	21000000	32610000	12000000
33	2	3.0	4.0	512533	312322001000000000001	21000000	31520000	00000000
34	2	1.0	12.0	512533	312322001000000000001	21000000	31520000	12000000
35	2	3.0	8.0	512534	312322001000000000000	21000000	31516100	00000000
36	2	1.0	24.0	512534	312322001000000000000	21000000	31516100	12000000
37	3	3.0	8.0	513523	312322001000000001000	21000000	51320000	00000000
38	3	1.0	-8.0	513523	312322001000000001000	21000000	51000000	12000000
39	3	1.0	24.0	513523	312322001000000001000	21000000	51320000	12000000
40	1	2.0	2.0	522511	322311001001000000001	52000000	22000000	00000000
41	3	3.0	4.0	522512	322312001001001001000	51000000	23000000	00000000
42	3	1.0	-8.0	522512	322312001001001001000	51000000	21000000	12000000
43	3	1.0	12.0	522512	322312001001001001000	51000000	23000000	12000000
44	2	3.0	4.0	522513	322312001001000000000	51000000	22610000	00000000
45	2	1.0	12.0	522513	322312001001000000000	51000000	22610000	12000000
46	3	4.0	2.0	522522	322322001001011011001	00000000	24000000	00000000
47	3	2.0	-8.0	522522	322322001001011011001	00000000	22000000	12000000
48	3	2.0	8.0	522522	322322001001011011001	00000000	24000000	12000000
49	3	4.0	4.0	522523	322322001001010010000	00000000	23610000	00000000
50	3	2.0	-8.0	522523	322322001001010010000	00000000	21610000	12000000
51	3	2.0	16.0	522523	322322001001010010000	00000000	23610000	12000000
52	2	4.0	2.0	522533	322322001001000000001	00000000	22520000	00000000
53	2	2.0	8.0	522533	322322001001000000001	00000000	22520000	12000000
54	2	4.0	4.0	522534	322322001001000000000	00000000	22516100	00000000
55	2	2.0	16.0	522534	322322001001000000000	00000000	22516100	12000000
56	1	2.0	4.0	523511	322311001000000000001	52000000	21310000	00000000
57	3	3.0	8.0	523512	322312001000001000000	51000000	22310000	00000000
58	3	1.0	-8.0	523512	322312001000001000000	51000000	31000000	12000000
59	3	1.0	24.0	523512	322312001000001000000	51000000	22310000	12000000
60	3	3.0	8.0	523513	322312001000000001000	51000000	21320000	00000000
61	3	1.0	-8.0	523513	322312001000000001000	51000000	21000000	12000000
62	3	1.0	24.0	523513	322312001000000001000	51000000	21320000	12000000
63	2	3.0	8.0	523514	322312001000000000000	51000000	21316100	00000000
64	2	1.0	24.0	523514	322312001000000000000	51000000	21316100	12000000
65	3	4.0	4.0	523522	322322001000011000001	00000000	23310000	00000000

Table (4.11.18) contd.

66	3	2.0	-8.0	523522	322322001000011000001	00000000	21310000	12000000
67	3	2.0	16.0	523522	322322001000011000001	00000000	23310000	12000000
68	4	4.0	8.0	523523	322322001000010001000	00000000	22320000	00000000
69	4	2.0	-8.0	523523	322322001000010001000	00000000	22000000	12000000
70	4	2.0	-8.0	523523	322322001000010001000	00000000	32000000	12000000
71	4	2.0	32.0	523523	322322001000010001000	00000000	22320000	12000000
72	3	4.0	8.0	523524	322322001000010000000	00000000	22316100	00000000
73	3	2.0	-8.0	523524	322322001000010000000	00000000	31610000	12000000
74	3	2.0	32.0	523524	322322001000010000000	00000000	22316100	12000000
75	3	4.0	4.0	523533	322322001000000011001	00000000	21330000	00000000
76	3	2.0	-8.0	523533	322322001000000011001	00000000	21310000	12000000
77	3	2.0	16.0	523533	322322001000000011001	00000000	21330000	12000000
78	3	4.0	8.0	523534	322322001000000010000	00000000	21326100	00000000
79	3	2.0	-8.0	523534	322322001000000010000	00000000	21610000	12000000
80	3	2.0	32.0	523534	322322001000000010000	00000000	21326100	12000000
81	2	4.0	4.0	523544	322322001000000000001	00000000	21315200	00000000
82	2	2.0	16.0	523544	322322001000000000001	00000000	21315200	12000000
83	3	4.0	8.0	524534	322322001000000001000	00000000	21513200	00000000
84	3	2.0	-8.0	524534	322322001000000001000	00000000	21510000	12000000
85	3	2.0	32.0	524534	322322001000000001000	00000000	21513200	12000000
86	3	4.0	4.0	533523	322322001001001001000	00000000	51230000	00000000
87	3	2.0	-8.0	533523	322322001001001001000	00000000	51210000	12000000
88	3	2.0	16.0	533523	322322001001001001000	00000000	51230000	12000000
89	3	4.0	8.0	534523	322322001000001000000	00000000	51223100	00000000
90	3	2.0	-8.0	534523	322322001000001000000	00000000	51310000	12000000
91	3	2.0	32.0	534523	322322001000001000000	00000000	51223100	12000000
92	1	1.0	-8.0	511612	311312000010100100000	21000000	61000000	11410000
93	1	1.0	-8.0	512611	312311000000011000001	21000000	31000000	11410000
94	3	0	-8.0	512612	312312000000010001000	22000000	00000000	11410000
95	3	2.0	-8.0	512612	312312000000010001000	00000000	32000000	11410000
96	3	0	16.0	512612	312312000000010001000	22000000	32000000	11410000
97	2	2.0	-8.0	512613	312312000000010000000	00000000	31610000	11410000
98	2	0	16.0	512613	312312000000010000000	22000000	31610000	11410000
99	2	1.0	-8.0	512622	312322000000000011001	21000000	31000000	11410000
100	2	1.0	12.0	512622	312322000000000011001	21000000	33000000	11410000
101	2	1.0	-8.0	512623	312322000000000010000	21000000	61000000	11410000
102	2	1.0	24.0	512623	312322000000000010000	21000000	32610000	11410000
103	1	1.0	12.0	512633	312322000000000000001	21000000	31520000	11410000
104	1	1.0	24.0	512634	312322000000000000000	21000000	31516100	11410000
105	2	1.0	-8.0	513623	312322000000000001000	21000000	51000000	11410000
106	2	1.0	24.0	513623	312322000000000001000	21000000	51320000	11410000
107	2	1.0	-8.0	522612	322312000001001001000	51000000	21000000	11410000
108	2	1.0	12.0	522612	322312000001001001000	51000000	23000000	11410000
109	1	1.0	12.0	522613	322312000001000000000	51000000	22610000	11410000
110	2	2.0	-8.0	522622	322322000001011011001	00000000	22000000	11410000
111	2	2.0	8.0	522622	322322000001011011001	00000000	24000000	11410000
112	2	2.0	-8.0	522623	322322000001010010000	00000000	21610000	11410000
113	2	2.0	16.0	522623	322322000001010010000	00000000	23610000	11410000
114	1	2.0	8.0	522633	322322000001000000001	00000000	22520000	11410000
115	1	2.0	16.0	522634	322322000001000000000	00000000	22516100	11410000
116	2	1.0	-8.0	523612	322312000000001000000	51000000	31000000	11410000
117	2	1.0	24.0	523612	322312000000001000000	51000000	22310000	11410000
118	2	1.0	-8.0	523613	322312000000000001000	51000000	21000000	11410000
119	2	1.0	24.0	523613	322312000000000001000	51000000	21320000	11410000
120	1	1.0	24.0	523614	322312000000000000000	51000000	21316100	11410000
121	2	2.0	-8.0	523622	322322000000011000001	00000000	21310000	11410000
122	2	2.0	16.0	523622	322322000000011000001	00000000	23310000	11410000
123	3	2.0	-8.0	523623	322322000000010001000	00000000	22000000	11410000
124	3	2.0	-8.0	523623	322322000000010001000	00000000	32000000	11410000
125	3	2.0	32.0	523623	322322000000010001000	00000000	22320000	11410000
126	2	2.0	-8.0	523624	322322000000010000000	00000000	31610000	11410000
127	2	2.0	32.0	523624	322322000000010000000	00000000	22316100	11410000
128	2	2.0	-8.0	523633	322322000000000011001	00000000	21310000	11410000
129	2	2.0	16.0	523633	322322000000000011001	00000000	21330000	11410000
130	2	2.0	-8.0	523634	322322000000000010000	00000000	21610000	11410000
131	2	2.0	32.0	523634	322322000000000010000	00000000	21326100	11410000
132	1	2.0	16.0	523644	322322000000000000001	00000000	21315200	11410000
133	2	2.0	-8.0	524634	322322000000000001000	00000000	21510000	11410000
134	2	2.0	32.0	524634	322322000000000001000	00000000	21513200	11410000
135	2	2.0	-8.0	533623	322322000001001001000	00000000	51210000	11410000
136	2	2.0	16.0	533623	322322000001001001000	00000000	51230000	11410000
137	2	2.0	-8.0	534623	322322000000001000000	00000000	51310000	11410000
138	2	2.0	32.0	534623	322322000000001000000	00000000	51223100	11410000

Table (4.11.19) Representation of the terms in the expressions of the product of the first partial derivatives of $r_2^{(+s)}$ for the case when p=4 and k*=4

1	1	4.0	4.0	511511	322322001001011011001	00000000	24000000	12000000
2	1	4.0	8.0	511512	322322001001010010000	00000000	23610000	12000000
3	1	4.0	4.0	511522	322322001001000000001	00000000	22520000	12000000
4	1	4.0	8.0	511523	322322001001000000010	00000000	22516100	12000000
5	1	4.0	8.0	512511	322322001000011000001	00000000	23310000	12000000
6	1	4.0	16.0	512512	322322001000010001000	00000000	22320000	12000000
7	1	4.0	16.0	512513	322322001000010000000	00000000	22316100	12000000
8	1	4.0	8.0	512522	322322001000000011001	00000000	21330000	12000000
9	1	4.0	16.0	512523	322322001000000010010	00000000	21326100	12000000
10	1	4.0	8.0	512533	322322001000000000001	00000000	21315200	12000000
11	1	4.0	16.0	512534	322322001000000000000	00000000	21315161	12000000
12	1	4.0	16.0	513523	322322001000000001000	00000000	21513200	12000000
13	1	4.0	8.0	522512	322322001001001001000	00000000	51230000	12000000
14	1	4.0	16.0	523512	322322001000001000000	00000000	51223100	12000000
15	1	4.0	4.0	511611	322322000001011011001	00000000	24000000	11410000
16	1	4.0	8.0	511612	322322000001010010000	00000000	23610000	11410000
17	1	4.0	4.0	511622	322322000001000000001	00000000	22520000	11410000
18	1	4.0	8.0	511623	322322000001000000010	00000000	22516100	11410000
19	1	4.0	8.0	512611	322322000000011000001	00000000	23310000	11410000
20	1	4.0	16.0	512612	322322000000010001000	00000000	22320000	11410000
21	1	4.0	16.0	512613	322322000000010000000	00000000	22316100	11410000
22	1	4.0	8.0	512622	322322000000000011001	00000000	21330000	11410000
23	1	4.0	16.0	512623	322322000000000010000	00000000	21326100	11410000
24	1	4.0	8.0	512633	322322000000000000001	00000000	21315200	11410000
25	1	4.0	16.0	512634	322322000000000000000	00000000	21315161	11410000
26	1	4.0	16.0	513623	322322000000000001000	00000000	21513200	11410000
27	1	4.0	8.0	522612	322322000001001001000	00000000	51230000	11410000
28	1	4.0	16.0	523612	322322000000001000000	00000000	51223100	11410000

Table (4.11.20) Representation of the terms in the expressions of the second partial derivatives of $r_2^{(+s)}$ for the case when p=4 and k*=4

1	3	4.0	2.0	511511	322322001001011011001	00000000	24000000	00000000
2	3	2.0	-8.0	511511	322322001001011011001	00000000	22000000	12000000
3	3	2.0	8.0	511511	322322001001011011001	00000000	24000000	12000000
4	3	4.0	4.0	511512	322322001001010010000	00000000	23610000	00000000
5	3	2.0	-8.0	511512	322322001001010010000	00000000	21610000	12000000
6	3	2.0	16.0	511512	322322001001010010000	00000000	23610000	12000000
7	2	4.0	2.0	511522	322322001001000000001	00000000	22520000	00000000
8	2	2.0	8.0	511522	322322001001000000001	00000000	22520000	12000000
9	2	4.0	4.0	511523	322322001001000000000	00000000	22516100	00000000
10	2	2.0	16.0	511523	322322001001000000000	00000000	22516100	12000000
11	3	4.0	4.0	512511	322322001000011000001	00000000	23310000	00000000
12	3	2.0	-8.0	512511	322322001000011000001	00000000	21310000	12000000
13	3	2.0	16.0	512511	322322001000011000001	00000000	23310000	12000000
14	4	4.0	8.0	512512	322322001000010001000	00000000	22320000	00000000
15	4	2.0	-8.0	512512	322322001000010001000	00000000	22000000	12000000
16	4	2.0	-8.0	512512	322322001000010001000	00000000	32000000	12000000
17	4	2.0	32.0	512512	322322001000010001000	00000000	22320000	12000000
18	3	4.0	8.0	512513	322322001000010000000	00000000	22316100	00000000
19	3	2.0	-8.0	512513	322322001000010000000	00000000	31610000	12000000
20	3	2.0	32.0	512513	322322001000010000000	00000000	22316100	12000000
21	3	4.0	4.0	512522	322322001000000011001	00000000	21330000	00000000
22	3	2.0	-8.0	512522	322322001000000011001	00000000	21310000	12000000
23	3	2.0	16.0	512522	322322001000000011001	00000000	21330000	12000000
24	3	4.0	8.0	512523	322322001000000010000	00000000	21326100	00000000
25	3	2.0	-8.0	512523	322322001000000010000	00000000	21610000	12000000
26	3	2.0	32.0	512523	322322001000000010000	00000000	21326100	12000000
27	2	4.0	4.0	512533	322322001000000000001	00000000	21315200	00000000
28	2	2.0	16.0	512533	322322001000000000001	00000000	21315200	12000000
29	2	4.0	8.0	512534	322322001000000000000	00000000	21315161	00000000
30	2	2.0	32.0	512534	322322001000000000000	00000000	21315161	12000000

Table (4.11.20) contd.

31	3	4.0	8.0	513523	322322001000000001000	00000000	21513200	00000000
32	3	2.0	-8.0	513523	322322001000000001000	00000000	21510000	12000000
33	3	2.0	32.0	513523	322322001000000001000	00000000	21513200	12000000
34	3	4.0	4.0	522512	322322001001001001000	00000000	51230000	00000000
35	3	2.0	-8.0	522512	322322001001001001000	00000000	51210000	12000000
36	3	2.0	16.0	522512	322322001001001001000	00000000	51230000	12000000
37	3	4.0	8.0	523512	322322001000001000000	00000000	51223100	00000000
38	3	2.0	-8.0	523512	322322001000001000000	00000000	51310000	12000000
39	3	2.0	32.0	523512	322322001000001000000	00000000	51223100	12000000
40	2	2.0	-8.0	511611	322322000010110110001	00000000	22000000	11410000
41	2	2.0	8.0	511611	322322000010110110001	00000000	24000000	11410000
42	2	2.0	-8.0	511612	322322000001010010000	00000000	21610000	11410000
43	2	2.0	16.0	511612	322322000001010010000	00000000	23610000	11410000
44	1	2.0	8.0	511622	322322000010000000001	00000000	22520000	11410000
45	1	2.0	16.0	511623	322322000010000000000	00000000	22516100	11410000
46	2	2.0	-8.0	512611	322322000000011000001	00000000	21310000	11410000
47	2	2.0	16.0	512611	322322000000011000001	00000000	23310000	11410000
48	3	2.0	-8.0	512612	322322000000010001000	00000000	22000000	11410000
49	3	2.0	-8.0	512612	322322000000010001000	00000000	32000000	11410000
50	3	2.0	32.0	512612	322322000000010001000	00000000	22320000	11410000
51	2	2.0	-8.0	512613	322322000000010000000	00000000	31610000	11410000
52	2	2.0	32.0	512613	322322000000010000000	00000000	22316100	11410000
53	2	2.0	-8.0	512622	322322000000000011001	00000000	21310000	11410000
54	2	2.0	16.0	512622	322322000000000011001	00000000	21330000	11410000
55	2	2.0	-8.0	512623	322322000000000010000	00000000	21610000	11410000
56	2	2.0	32.0	512623	322322000000000010000	00000000	21326100	11410000
57	1	2.0	16.0	512633	322322000000000000001	00000000	21315200	11410000
58	1	2.0	32.0	512634	322322000000000000000	00000000	21315161	11410000
59	2	2.0	-8.0	513623	322322000000000001000	00000000	21510000	11410000
60	2	2.0	32.0	513623	322322000000000001000	00000000	21513200	11410000
61	2	2.0	-8.0	522612	322322000001001001000	00000000	51210000	11410000
62	2	2.0	16.0	522612	322322000001001001000	00000000	51230000	11410000
63	2	2.0	-8.0	523612	322322000000001000000	00000000	51310000	11410000
64	2	2.0	32.0	523612	322322000000001000000	00000000	51223100	11410000

Section 4.12 Programs for deriving $\beta_1(\underset{\sim}{\theta}_A, \sigma_A)$

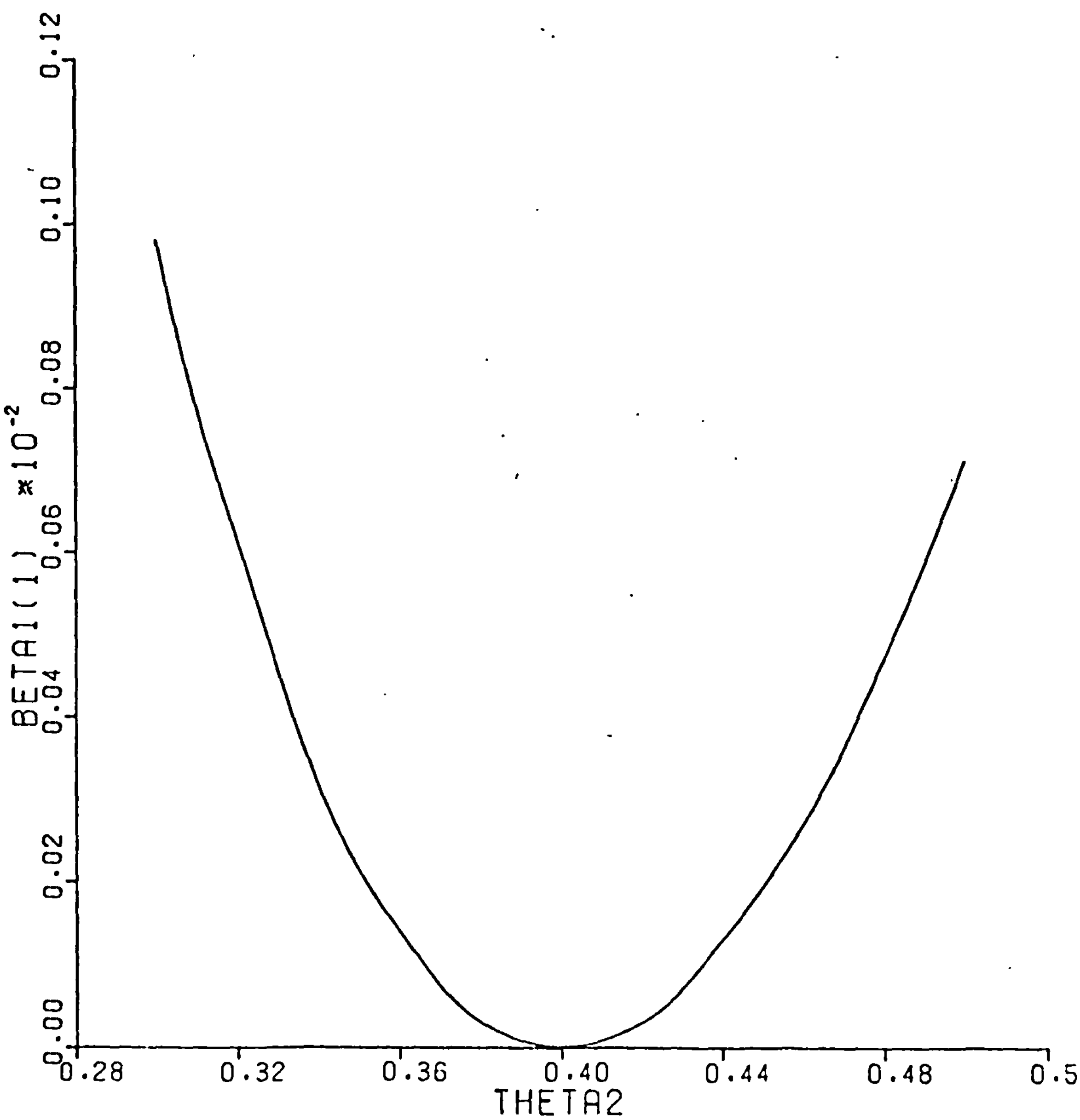
The programs for deriving $\beta_1(\underset{\sim}{\theta}_A, \sigma_A)$ are program POWCAL and subroutines COEF11, COEF1, COEF2, E1000, E2000, etc. as shown in Appendix 4. Note that these programs can also be used to derive $\beta_2(\underset{\sim}{\theta}_A, \sigma_A)$.

Section 4.13 Numerical examples

In this section, we make use of the computer programs in section 4.12 to evaluate $\beta_i^{(1)}$ and $\beta_i^{(2)}$ $(i = 1,2)$ in models (A) and (B) described in Chapter 1. For all the corresponding hypotheses in these models, we choose $\underset{\sim}{\theta}_0$ to be $(1.4, 0.4)^T$. For the case when the hypothesis is concerned with $\theta_1 = 1.4$, we consider $\underset{\sim}{\theta}_A$ of the form $(1.4, \theta_{A2})^T$, and for the case when the hypothesis is concerned with $\theta_2 = 0.4$, we consider $\underset{\sim}{\theta}_A$ of the form $(\theta_{A1}, 0.4)^T$. If the hypothesis is concerned with $\underset{\sim}{\theta} = (1.4, 0.4)^T$, we consider $\underset{\sim}{\theta}_A$ of the forms $(1.4, \theta_{A2})^T$ and $(\theta_{A1}, 0.4)^T$. In Fig. (4.13.1)-(4.13.16), the variation of $\beta_i^{(1)}$ and $\beta_i^{(2)}$ $(i = 1,2)$ with respect to $\underset{\sim}{\theta}_A$ is shown.

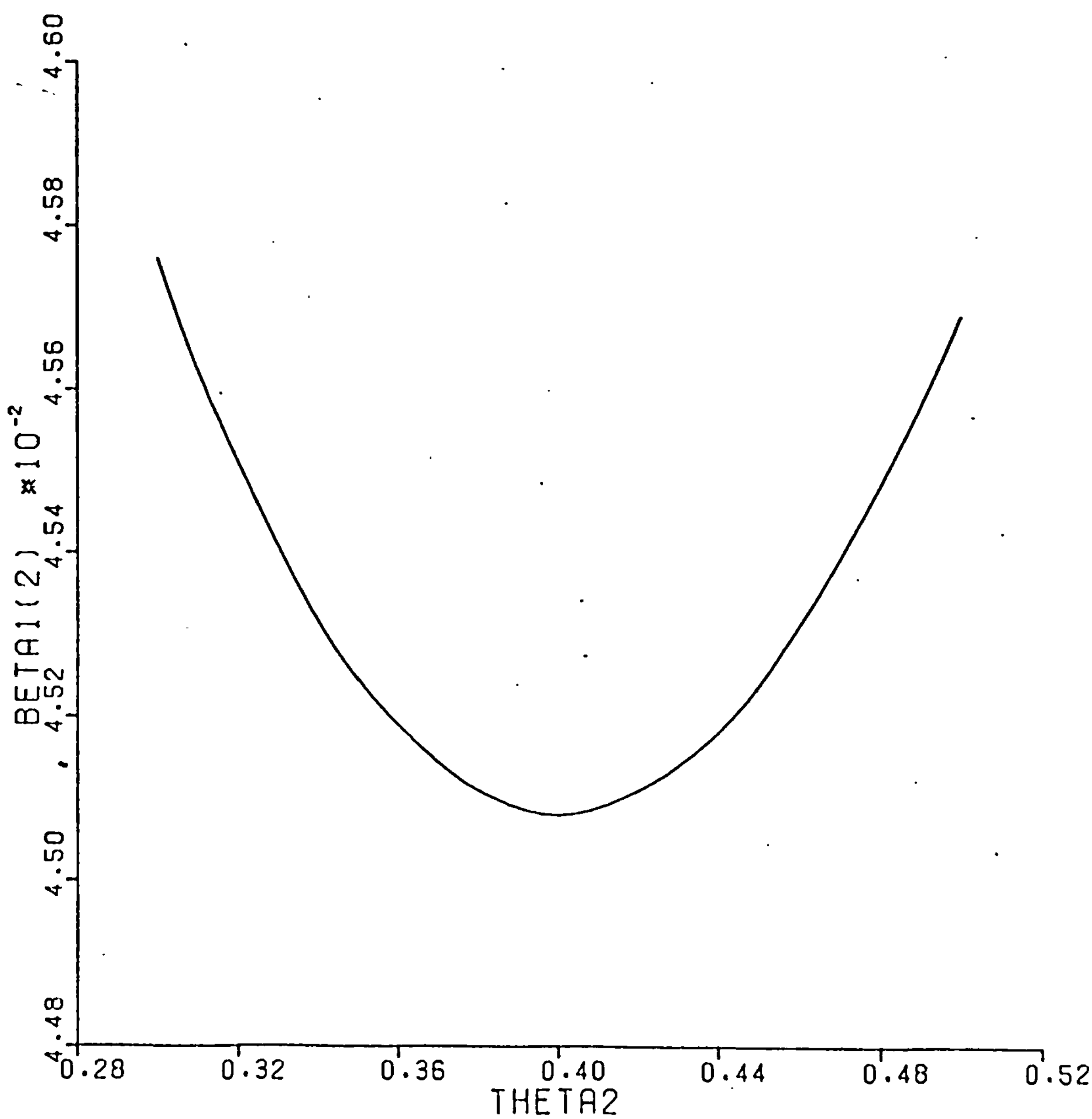
FIGURE(4.13.1)
BETA1(J) IN THE SERIES EXPANSION OF THE
POWER FUNCTION OF THE GENERAL MAXIMUM
LIKELIHOOD RATIO TEST CONCERNING THETA 1
WHEN SIGMA IS KNOWN
MODEL IS .
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETA1 ZERO ARE 1.4000 0.4000



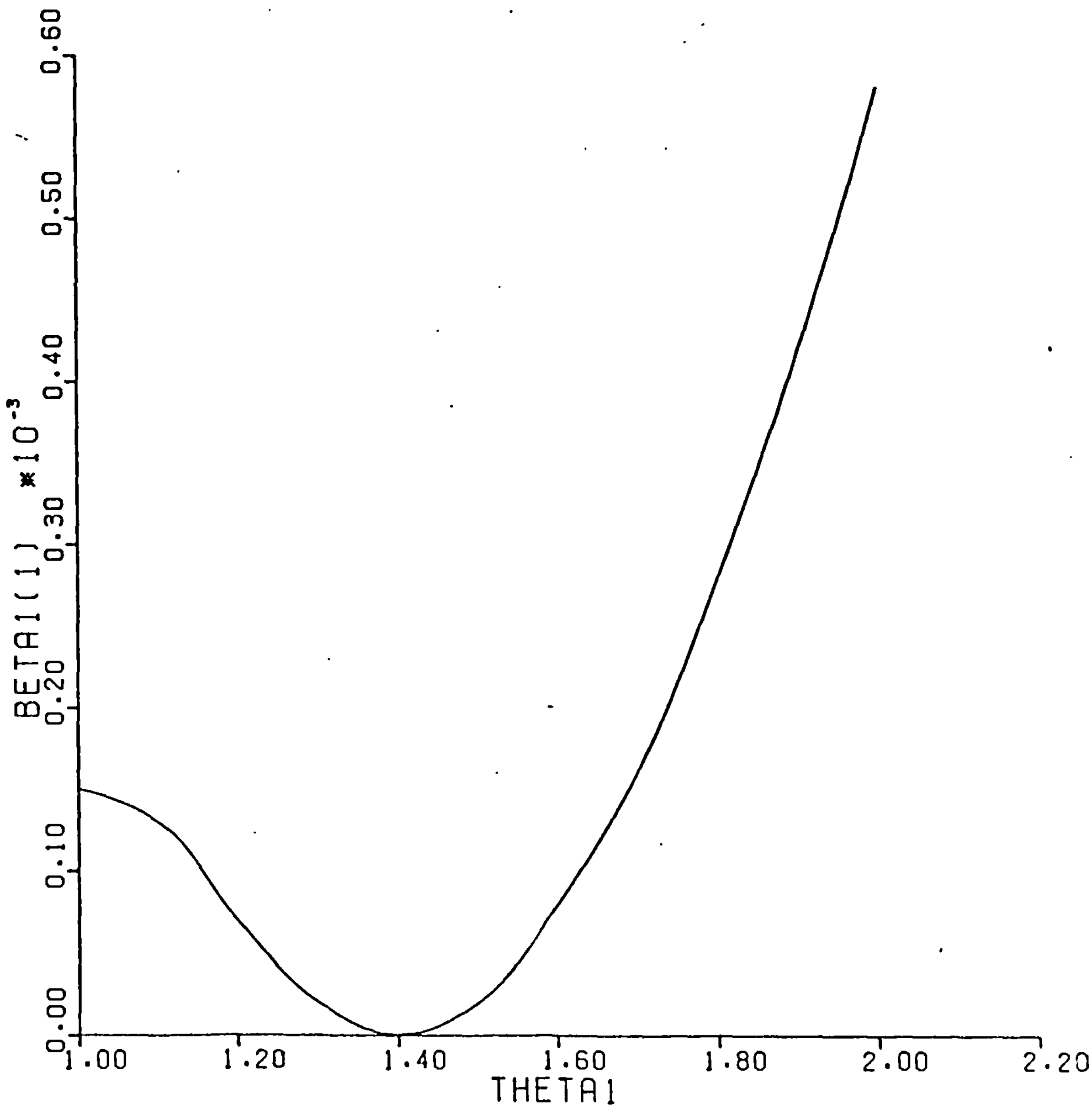
FIGURE(4.13.2)
BETAI(J) IN THE SERIES EXPANSION OF THE
POWER FUNCTION OF THE GENERAL MAXIMUM
LIKELIHOOD RATIO TEST CONCERNING THETA 1
WHEN SIGMA IS KNOWN
MODEL IS
E(Y)=(THETA1/(THETA1-THETA2))
     *(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI=   0.25,0.5,1.0,1.5,2.0,4.0
THETAI ZERO ARE 1.4000 0.4000



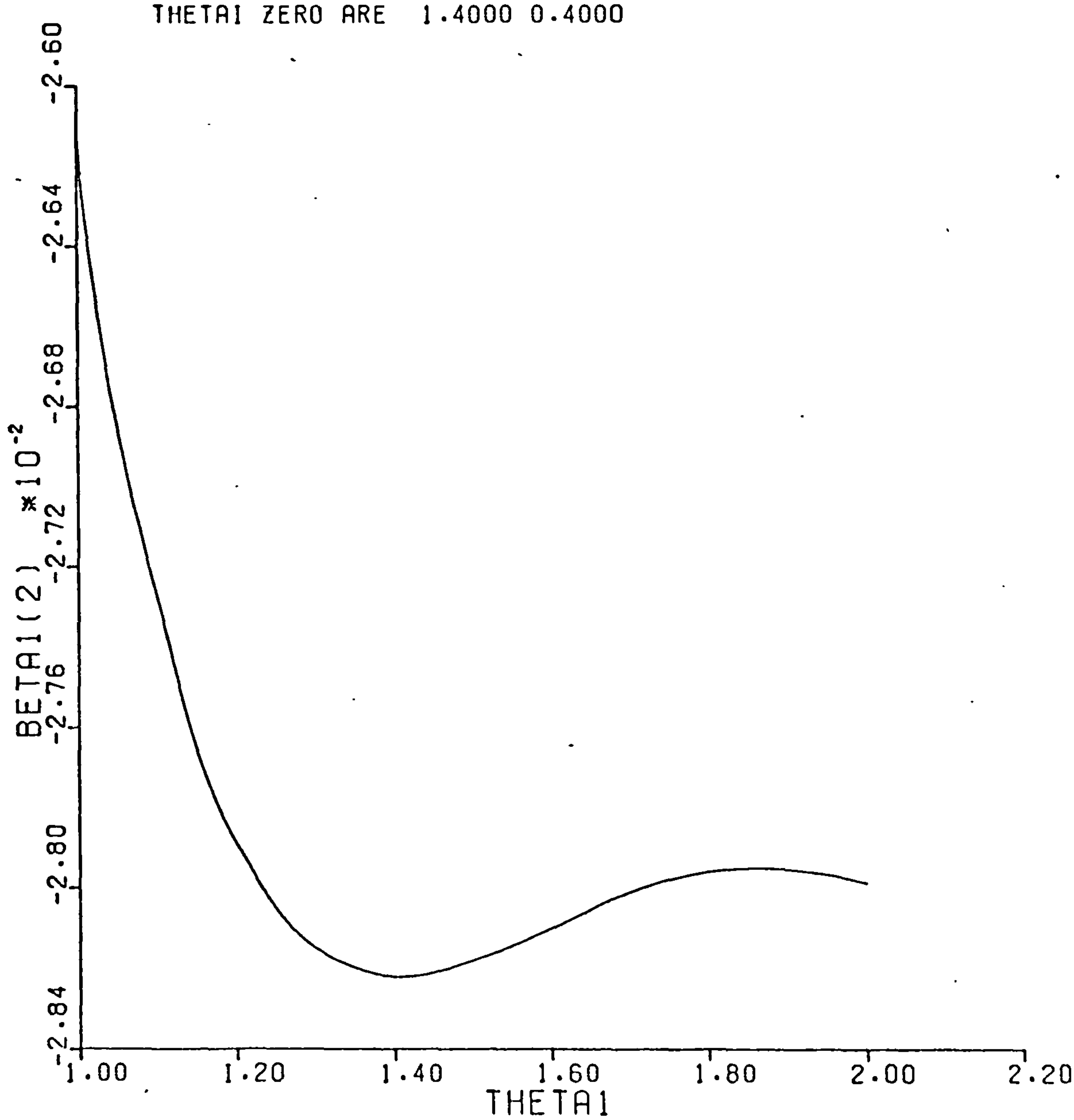
FIGURE(4.13.3)
BETAI(J) IN THE SERIES EXPANSION OF THE
POWER FUNCTION OF THE GENERAL MAXIMUM
LIKELIHOOD RATIO TEST CONCERNING THETA 2
WHEN SIGMA IS KNOWN
MODEL IS
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETAI ZERO ARE 1.4000 0.4000



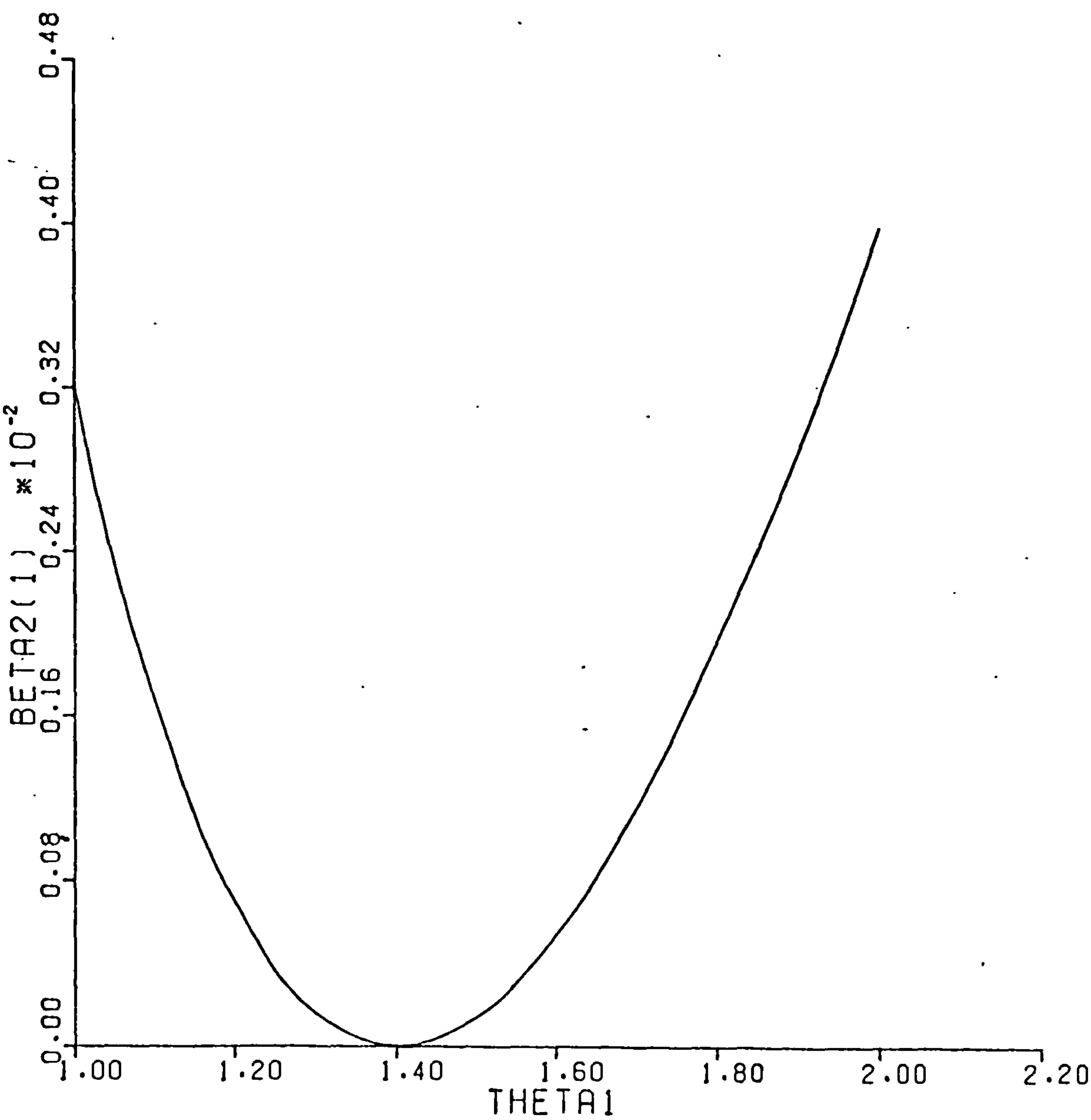
FIGURE(4.13.4)
BETAI(J) IN THE SERIES EXPANSION OF THE
POWER FUNCTION OF THE GENERAL MAXIMUM
LIKELIHOOD RATIO TEST CONCERNING THETA 2
WHEN SIGMA IS KNOWN
MODEL IS
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI= 0.25.0.5.1.0.1.5.2.0.4.0
THETAI ZERO ARE 1.4000 0.4000



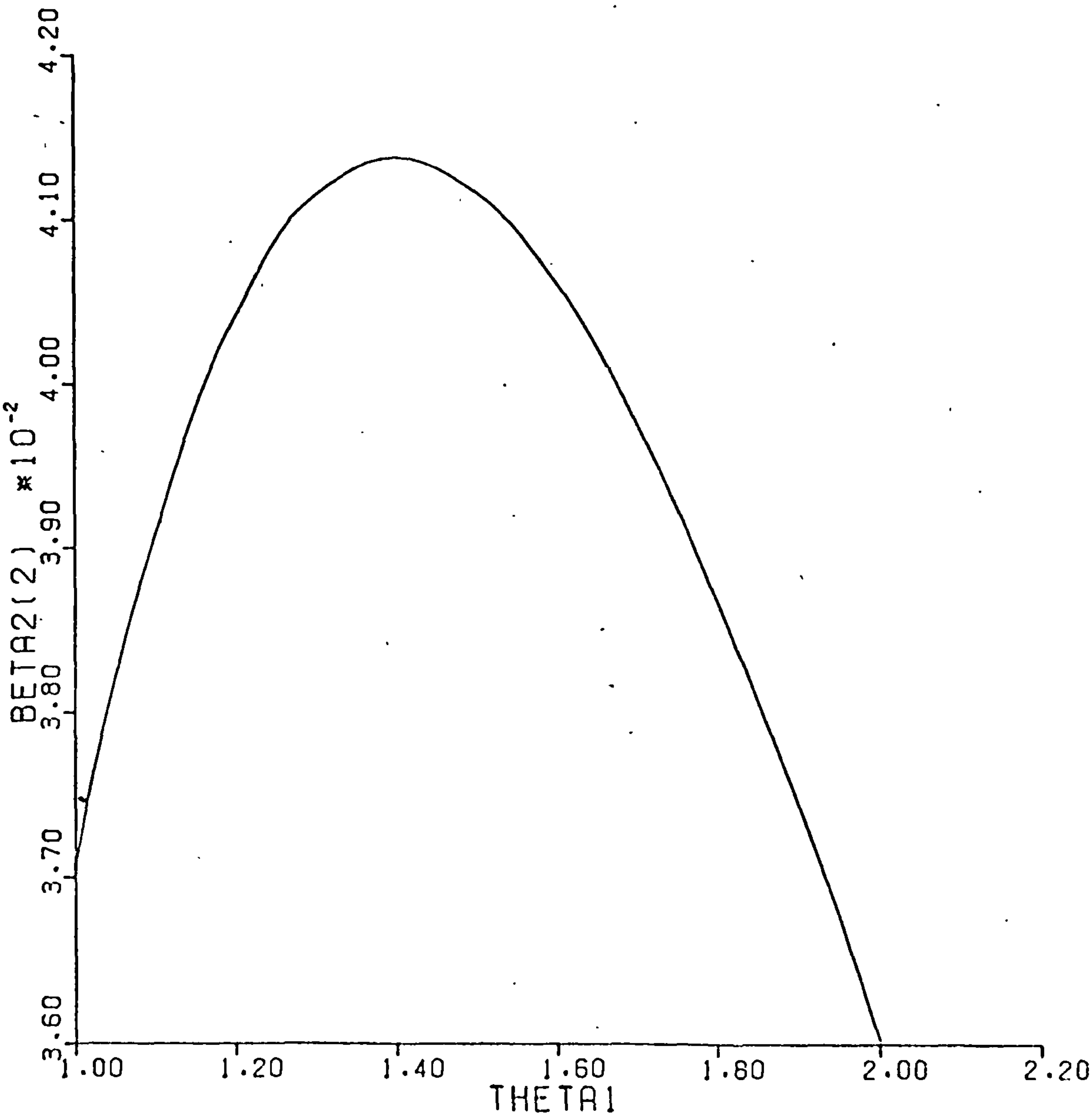
FIGURE(4.13.5)
BETAI(J) IN THE SERIES EXPANSION OF THE
POWER FUNCTION OF THE GENERAL MAXIMUM
LIKELIHOOD RATIO TEST CONCERNING THETA
WHEN SIGMA IS KNOWN
MODEL IS
E(Y)=(THETA1/(THETA1-THETA2))
. *(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETAI ZERO ARE 1.4000 0.4000



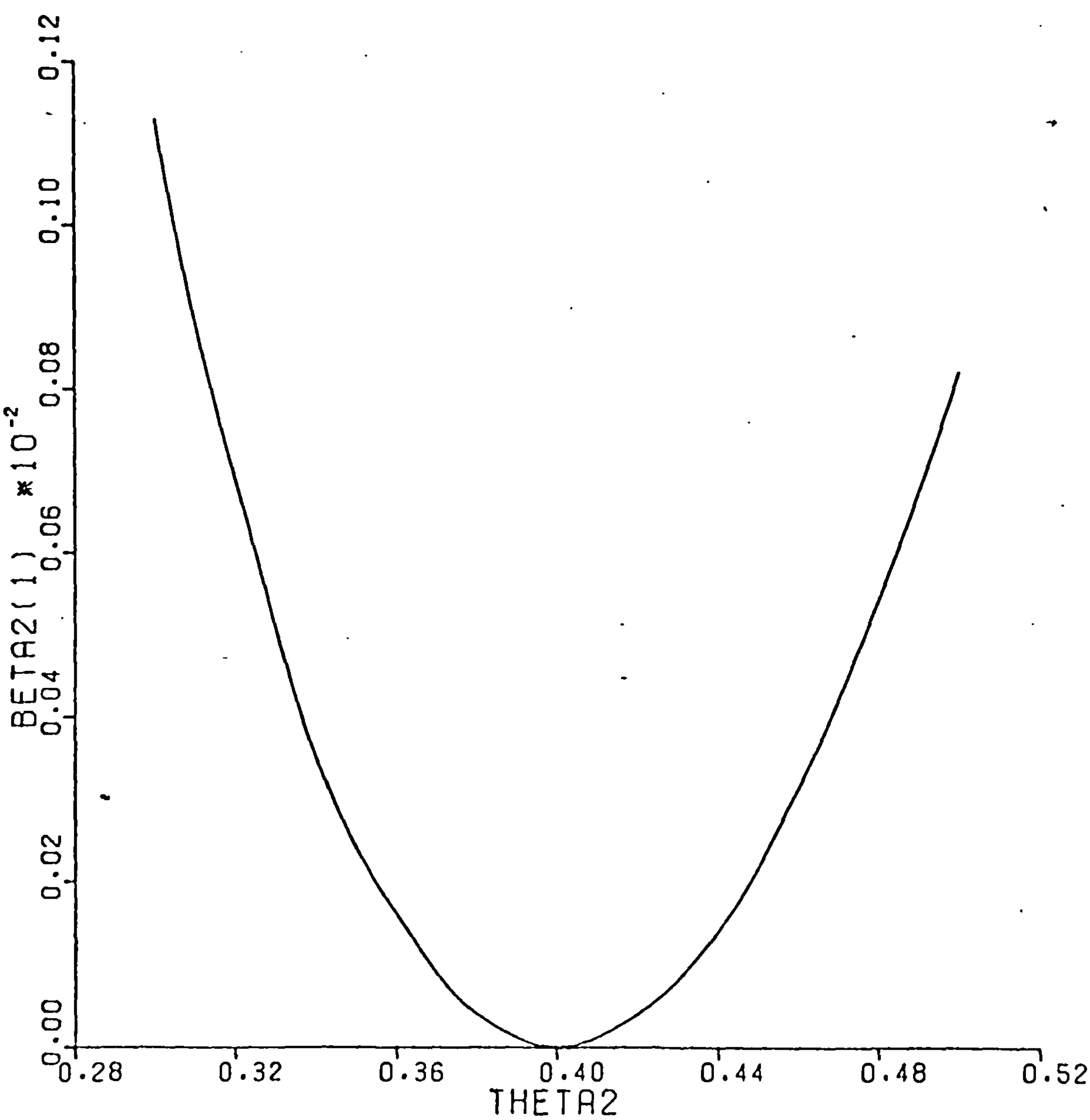
FIGURE(4.13.6)
BETA1(J) IN THE SERIES EXPANSION OF THE
POWER FUNCTION OF THE GENERAL MAXIMUM
LIKELIHOOD RATIO TEST CONCERNING THETA
WHEN SIGMA IS KNOWN
MODEL IS
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETA1 ZERO ARE 1.4000 0.4000



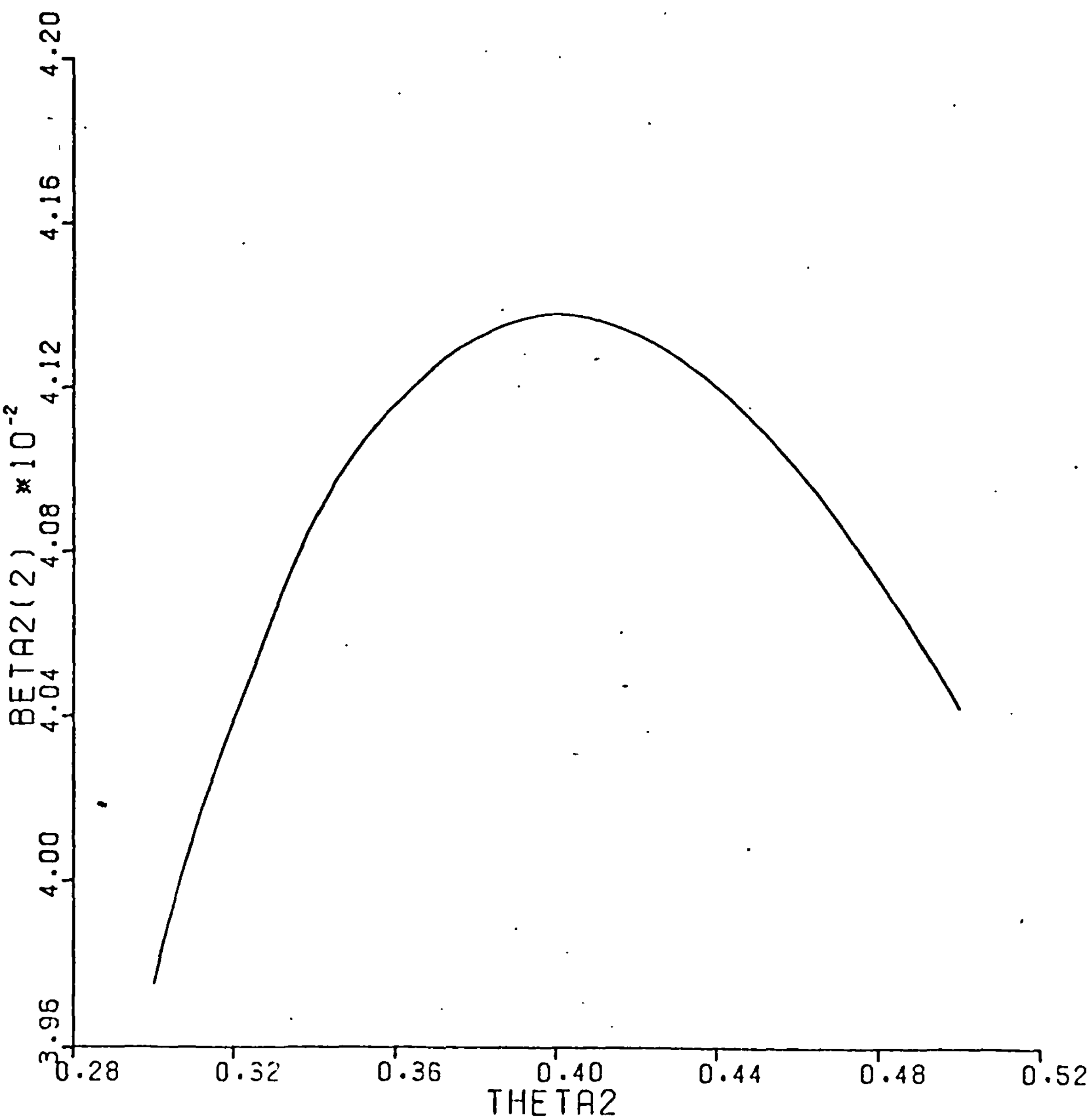
FIGURE(4.13.7)
BETA1(J) IN THE SERIES EXPANSION OF THE
POWER FUNCTION OF THE GENERAL MAXIMUM
LIKELIHOOD RATIO TEST CONCERNING THETA
WHEN SIGMA IS KNOWN
MODEL IS
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETA1 ZERO ARE 1.4000 0.4000

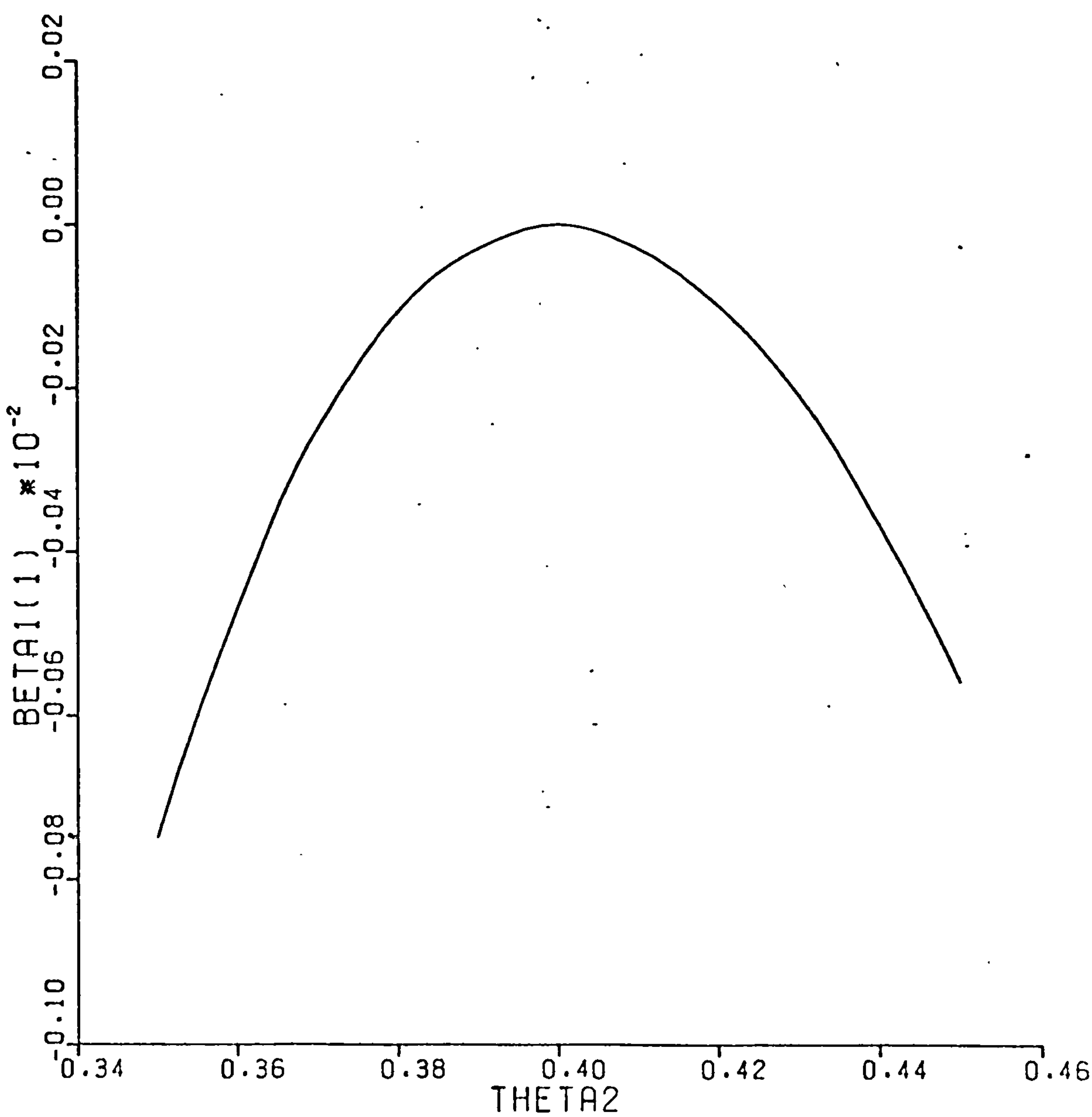


FIGURE(4.13.8)
BETAI(J) IN THE SERIES EXPANSION OF THE
POWER FUNCTION OF THE GENERAL MAXIMUM
LIKELIHOOD RATIO TEST CONCERNING THETA
WHEN SIGMA IS KNOWN
MODEL IS
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

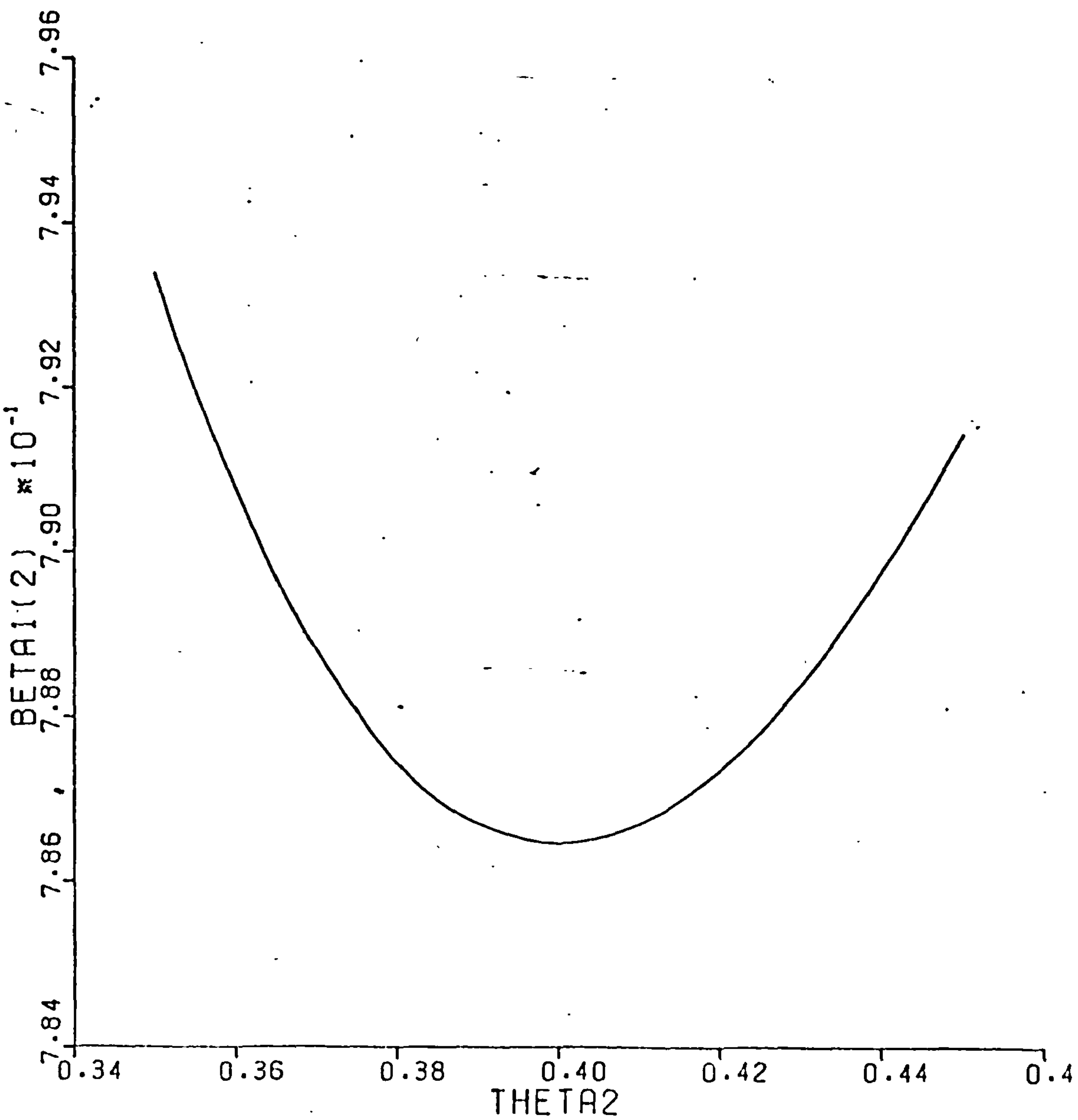
XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETAI ZERO ARE 1.4000 0.4000



FIGURE(4.13.9)
BETAI(J) IN THE SERIES EXPANSION OF THE
POWER FUNCTION OF THE GENERAL MAXIMUM
LIKELIHOOD RATIO TEST CONCERNING THETA 1
WHEN SIGMA IS KNOWN
MODEL IS
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1,2,3,4,5,6
THETAI ZERO ARE 1.4000 0.4000



FIGURE(4.13.10)
BETA1(J) IN THE SERIES EXPANSION OF THE
POWER FUNCTION OF THE GENERAL MAXIMUM
LIKELIHOOD RATIO TEST CONCERNING THETA 1
WHEN SIGMA IS KNOWN
MODEL IS
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1,2,3,4,5,6
THETA1 ZERO ARE 1.4000 0.4000



FIGURE(4.13.11)
BETAI(J) IN THE SERIES EXPANSION OF THE
POWER FUNCTION OF THE GENERAL MAXIMUM
LIKELIHOOD RATIO TEST CONCERNING THETA 2
WHEN SIGMA IS KNOWN
MODEL IS
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1,2,3,4,5,6
THETAI ZERO ARE 1.4000 0.4000

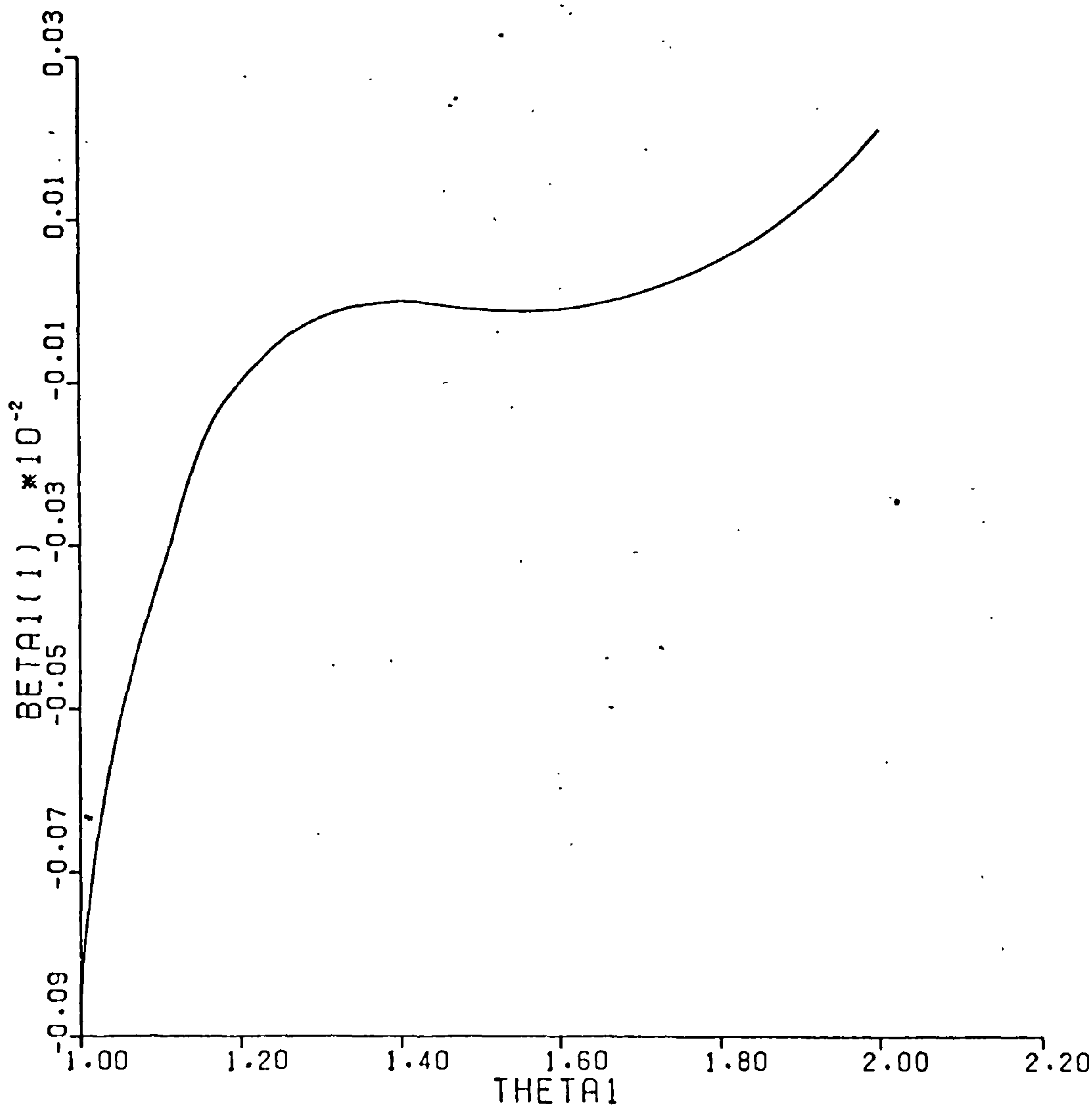
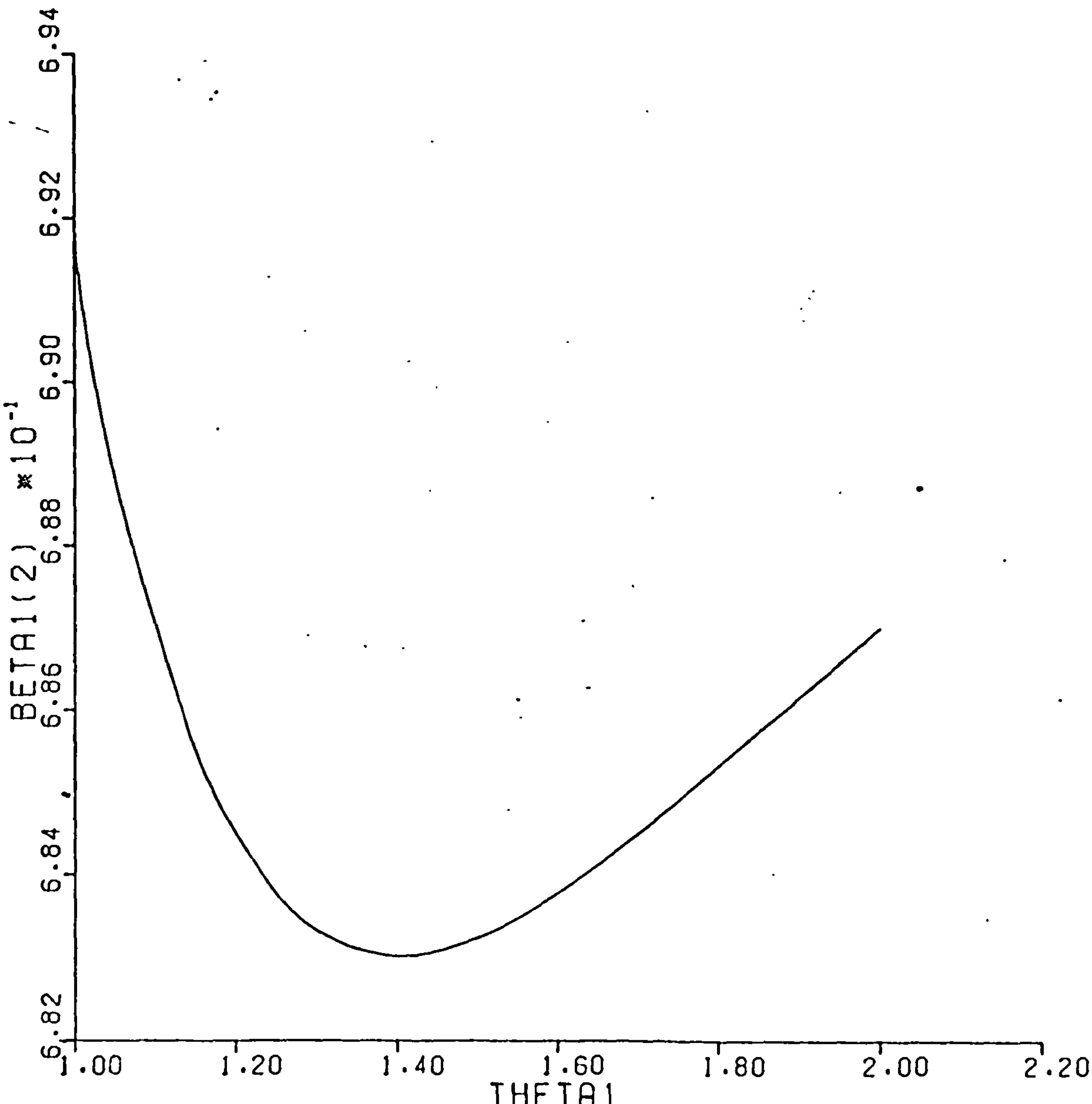
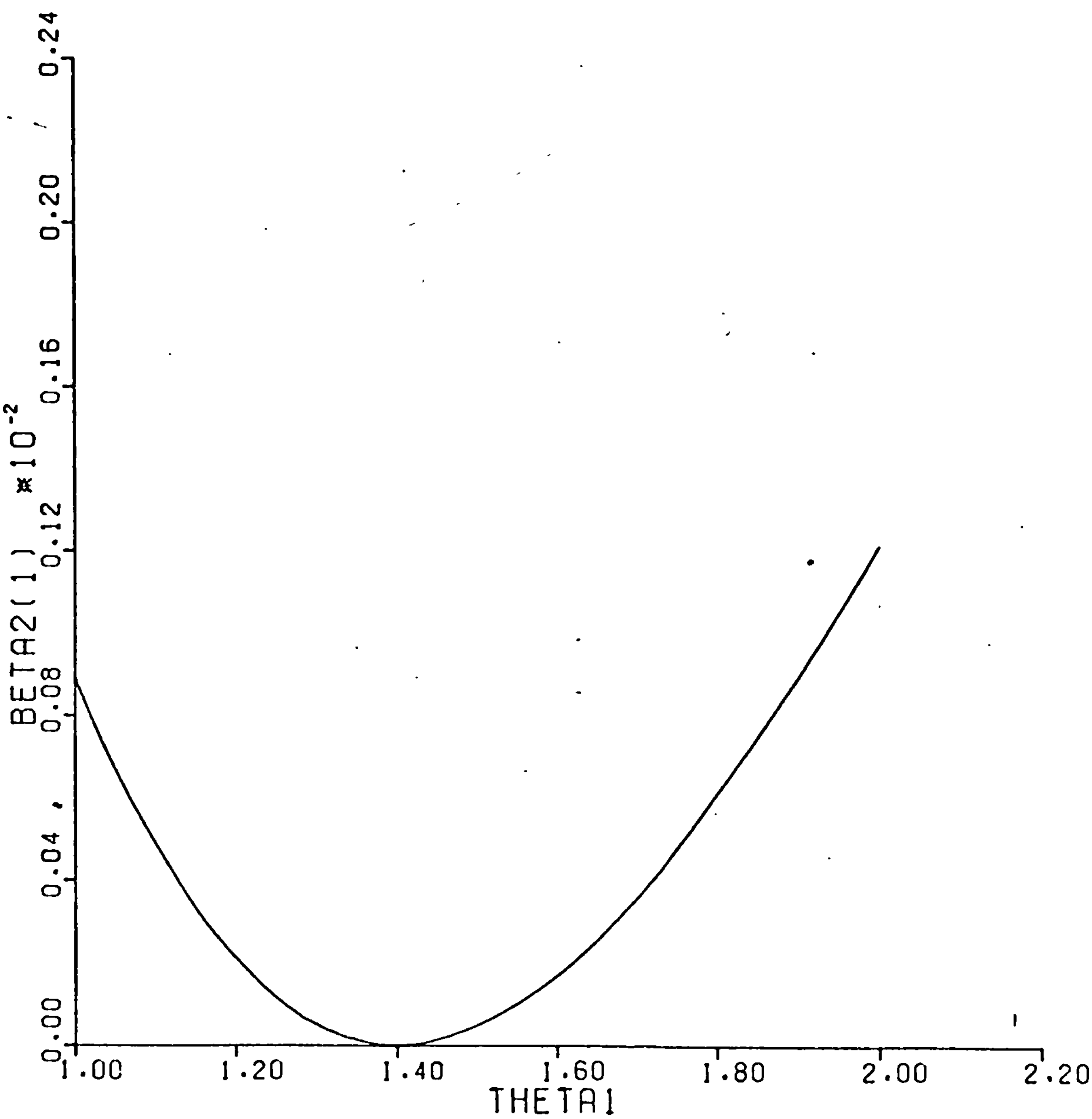


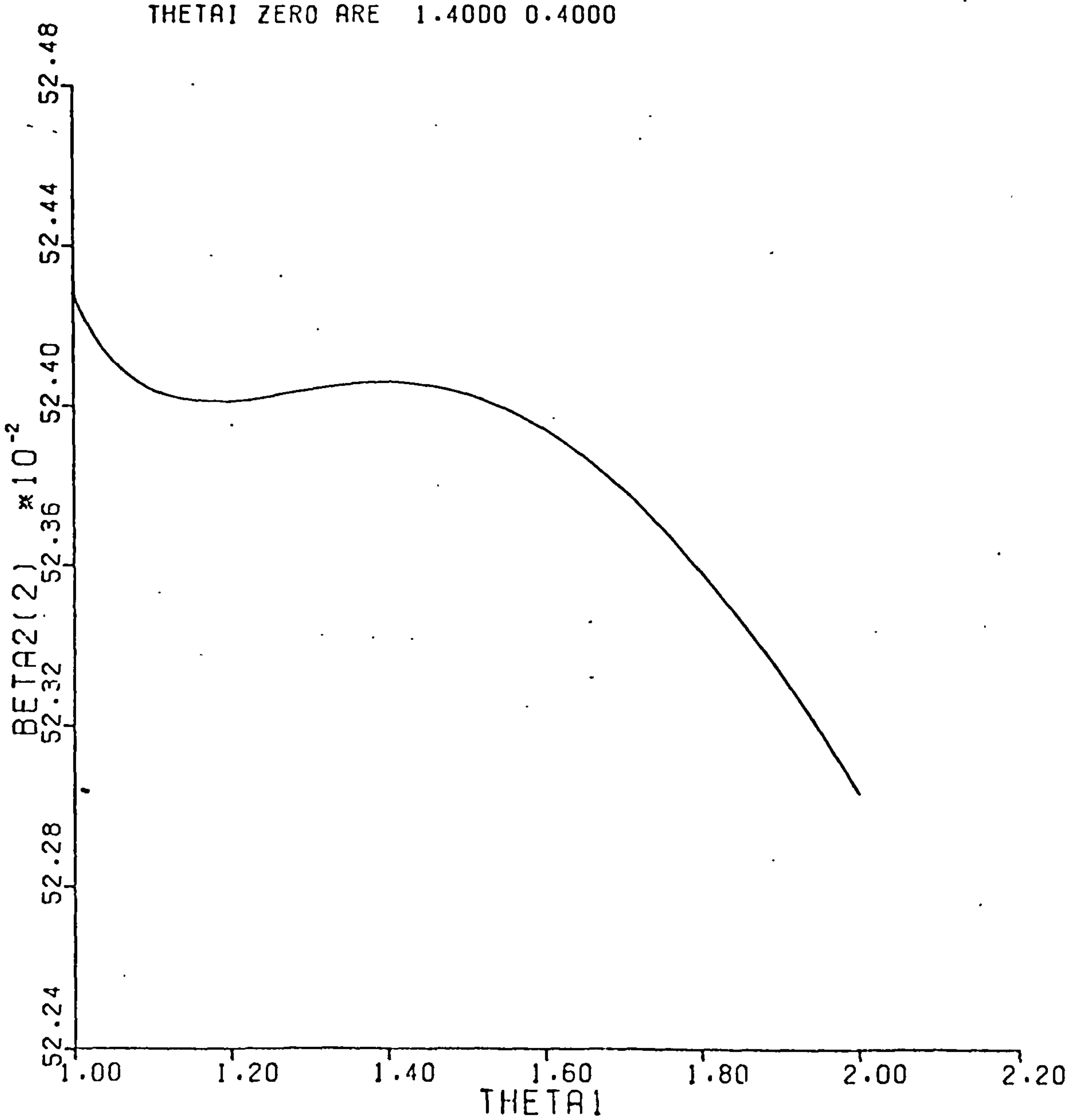
FIGURE (4.13.12)
BETAI(J) IN THE SERIES EXPANSION OF THE
POWER FUNCTION OF THE GENERAL MAXIMUM
LIKELIHOOD RATIO TEST CONCERNING THETA 2
WHEN SIGMA IS KNOWN
MODEL IS
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1,2,3,4,5,6
THETAI ZERO ARE 1.4000 0.4000



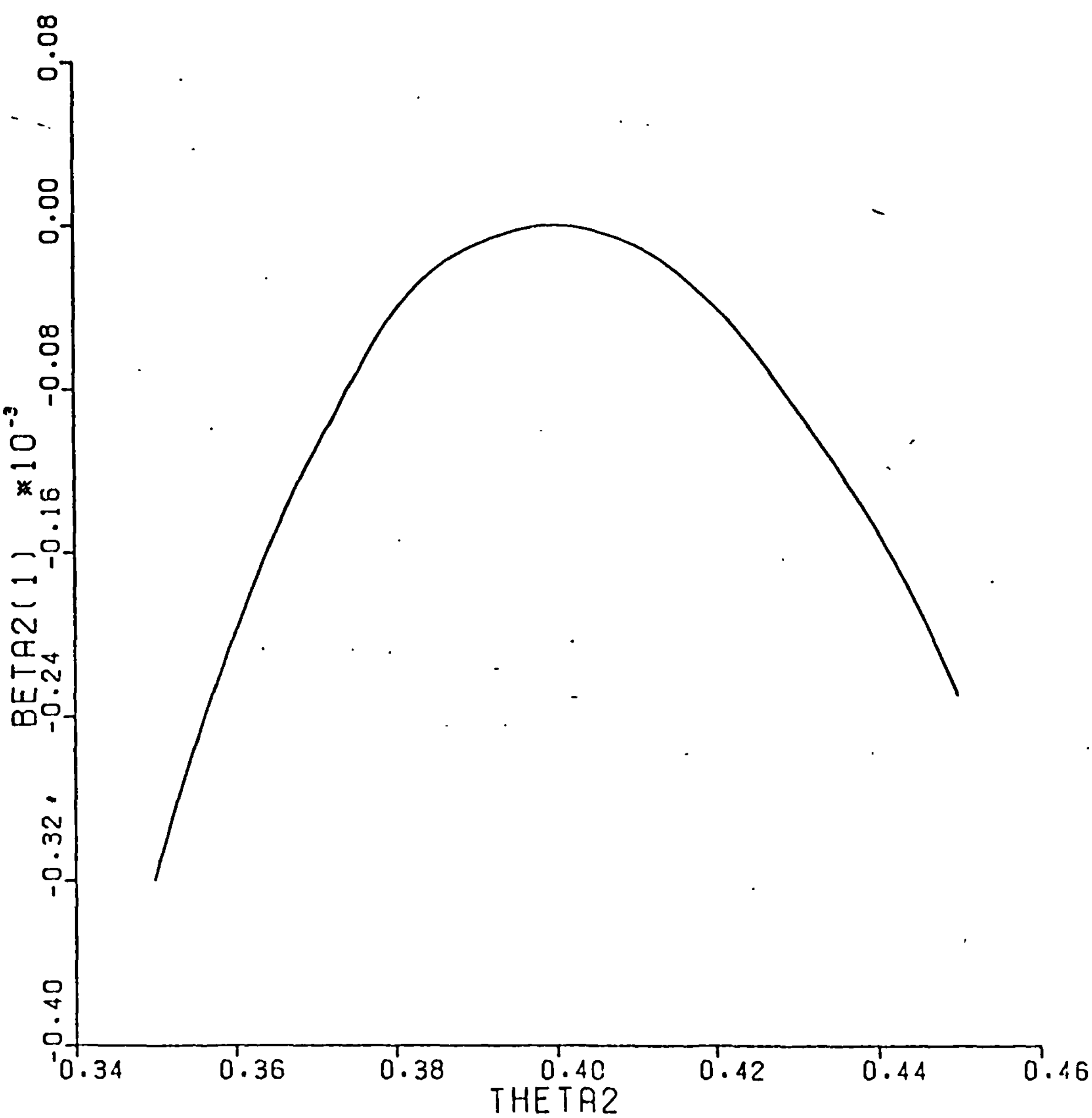
FIGURE(4.13.13)
BETA1(J).IN THE SERIES EXPANSION OF THE
POWER FUNCTION OF THE GENERAL MAXIMUM
LIKELIHOOD RATIO TEST CONCERNING THETA
WHEN SIGMA IS KNOWN
MODEL IS
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1,2,3,4,5,6
THETA1 ZERO ARE 1.4000 0.4000



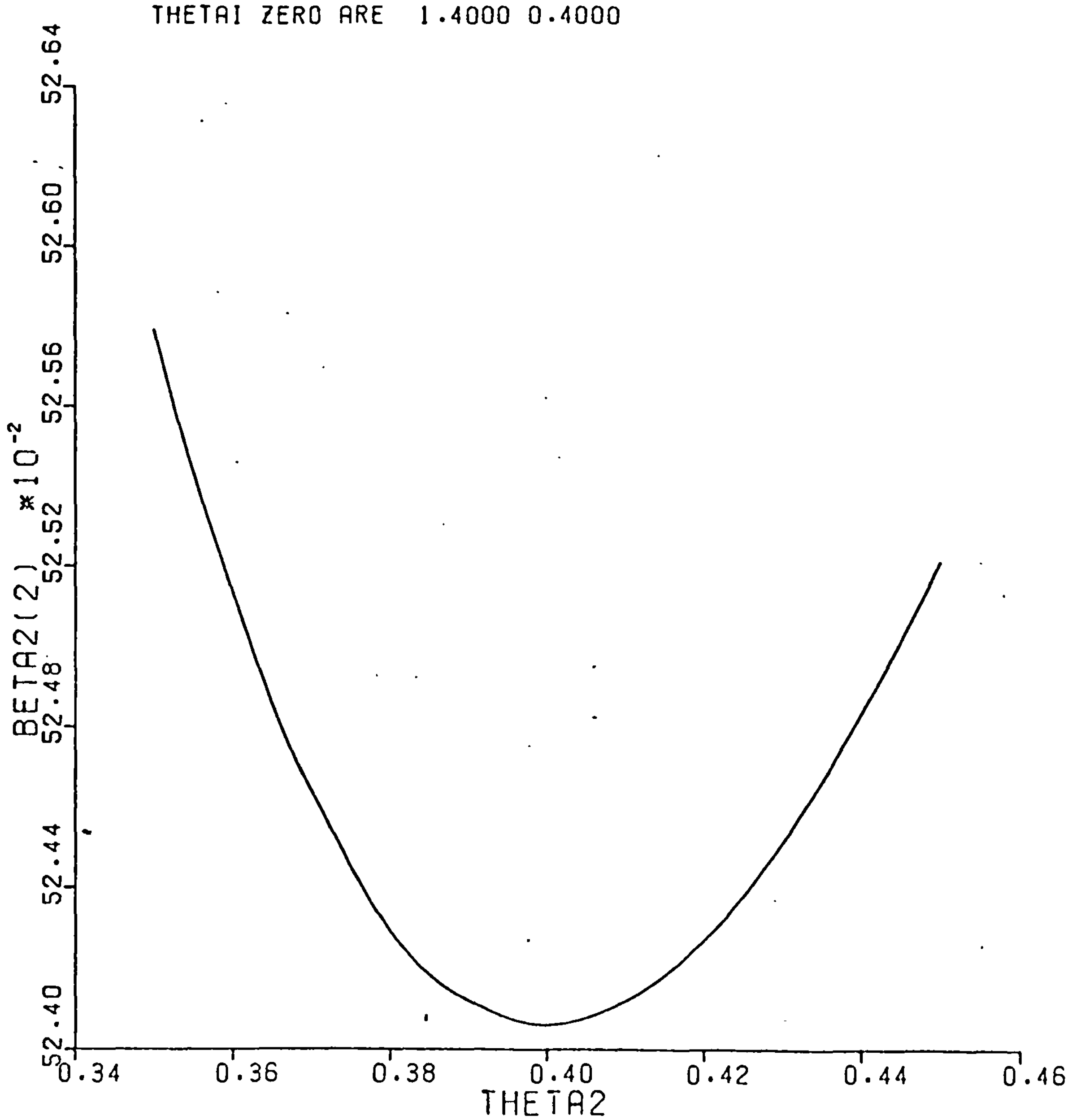
FIGURE(4.13.14)
BETAI(J) IN THE SERIES EXPANSION OF THE
POWER FUNCTION OF THE GENERAL MAXIMUM
LIKELIHOOD RATIO TEST CONCERNING THETA
WHEN SIGMA IS KNOWN
MODEL IS
E(Y)=1-(THETA1*EXP(-THETA2*XI)
      -THETA2*EXP(-THETA1*XI))
      /(THETA1-THETA2)
XI=   1,2,3,4,5,6
THETAI ZERO ARE 1.4000 0.4000



FIGURE(4.13.15)
BETAI(J) IN THE SERIES EXPANSION OF THE
POWER FUNCTION OF THE GENERAL MAXIMUM
LIKELIHOOD RATIO TEST CONCERNING THETA
WHEN SIGMA IS KNOWN
MODEL IS
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1,2,3,4,5,6
THETA1 ZERO ARE 1.4000 0.4000



FIGURE(4.13.16)
BETAI(J) IN THE SERIES EXPANSION OF THE
POWER FUNCTION OF THE GENERAL MAXIMUM
LIKELIHOOD RATIO TEST CONCERNING THETA
WHEN SIGMA IS KNOWN
MODEL IS
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1,2,3,4,5,6
THETAI ZERO ARE 1.4000 0.4000



CHAPTER 5

COMPARISON OF VARIOUS METHODS OF OBTAINING REGION ESTIMATES BY MEANS OF NUMERICAL EXAMPLES

Section 5.1 Introduction

In Chapters 2 and 3, various methods of obtaining region estimates for a subset of k* ($1 \leq k^* \leq p$) components of the parameter vector $\underset{\sim}{\theta}$ have been described. We refer to these methods as methods 1-4 as follows:

Method 1 is based on the approximations that the model is linear in the original parameter vector $\underset{\sim}{\theta}$.

Method 2 is based on power transformations of all the individual parameters (c.f. section 2.7).

Method 3 is based on general transformations in which the transformed parameters of interest depend only on the original parameters of interest, but the remaining transformed parameters may depend on all the original parameters (c.f. section 2.4).

Method 4 is based on general maximum likelihood ratios (c.f. Chapter 3).

In this chapter we restrict our attention to the case when σ is known, and apply the four methods to obtain region estimates in models (A) and (B) described in Chapter 1. These regions are compared in two aspects:

(i) the estimation of the coverage probability and the values of nonlinearity associated with the regions,

(ii) the boundaries of these regions.

In comparing (ii) we see how we can suggest bounds for the values of nonlinearity within which the use of linear theory to obtain the corresponding region estimates is justifiable.

Section 5.2 Estimation of coverage probability and nonlinearity

The coverage probability of the region estimate given by method 4 for k^* ($1 \leq k^* \leq p$) components of the parameter vector $\underset{\sim}{\theta}$ is an important quantity associated with this region. As $\underset{\sim}{\theta}_T$ is not known, we usually do not know the value of the coverage probability. In this section, we shall investigate the feasibility of estimating the actual coverage probability by using the coverage probability evaluated at $\underset{\sim}{\theta} = \hat{\underset{\sim}{\theta}}$.

We choose $\underset{\sim}{\theta}_T = (1.4, 0.4)^T$. We then set $\alpha = 0.05$ and use (3.3.50) and (3.3.52) to calculate the coverage probabilities $I_1(\underset{\sim}{\theta},\sigma)$ and $I_2(\underset{\sim}{\theta},\sigma)$ for various values of $\underset{\sim}{\theta}$. In Fig. (5.2.1)-(5.2.6) we display the absolute values of

$$[I_i(\underset{\sim}{\theta},\sigma) - I_i(\underset{\sim}{\theta}_T,\sigma)]/[I_i(\underset{\sim}{\theta}_T,\sigma) - 0.95]$$

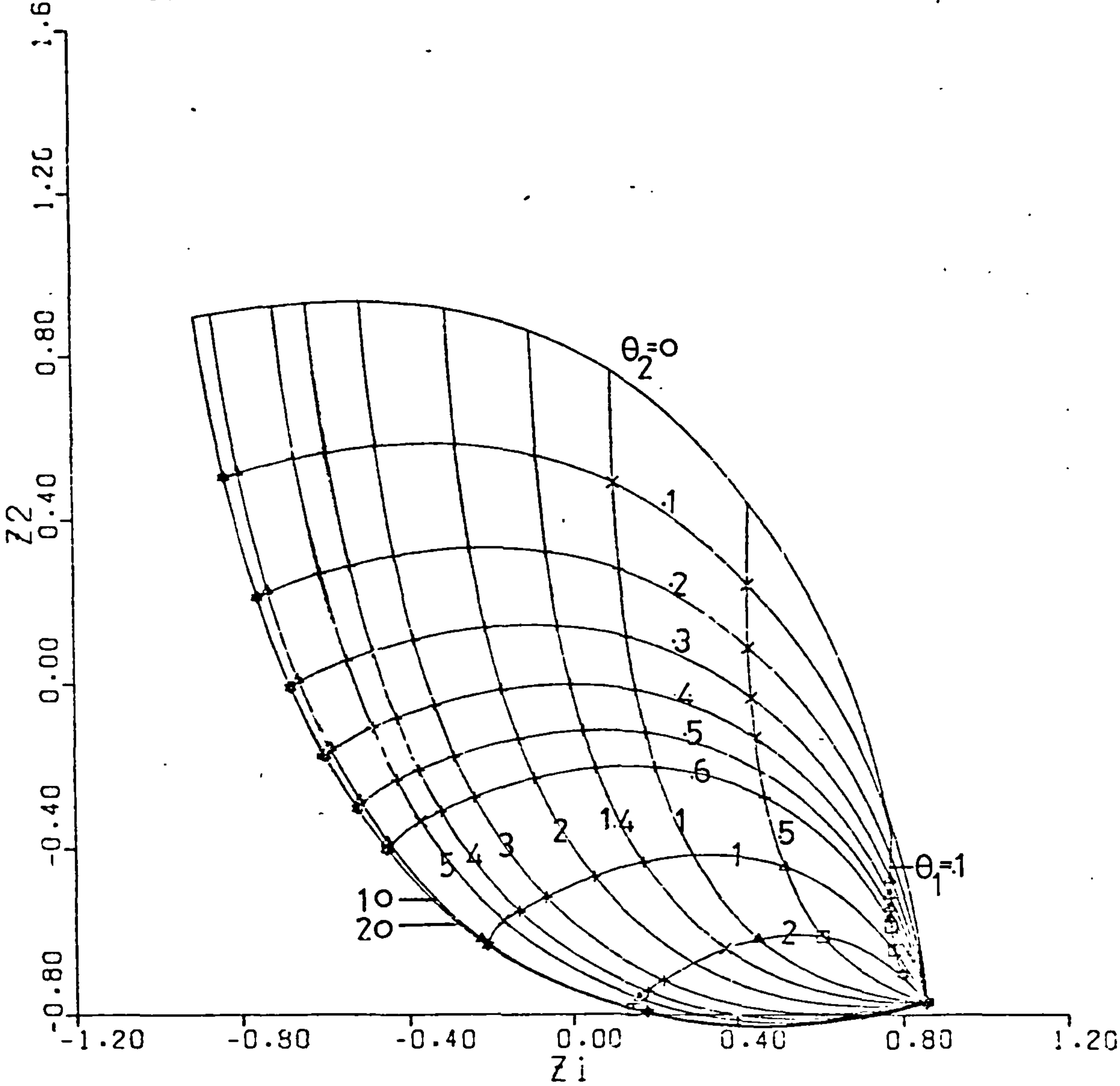
for $i = 1,2$. These absolute values are classified into five categories each of which is represented by a symbol (c.f. footnotes of the figures).

FIGURE (5.2.1)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=(THETA1/(THETA1-THETA2))
  *(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI= 0.25.0.5.1.0.1.5.2.0.4.0

THETAI TRUE ARE 1.4000 0.4000

II(THETA TRUE.SIGMA)-0.95= -451 *SIGMA SQUARE/100



R=ABSOLUTE VALUE OF((II(THETA.SIGMA)-II(THETA TRUE.SIGMA))/(II(THETA TRUE.SIGMA)-0.95))
PARAMETER OF INTEREST IS THETA1

+:0≤R≤0.5 ;X:0.5 <R≤1 ;△:1 <R≤10 ;□:10 <R≤100;✿:R>100

FIGURE (5.2.2)

ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS

MODEL IS

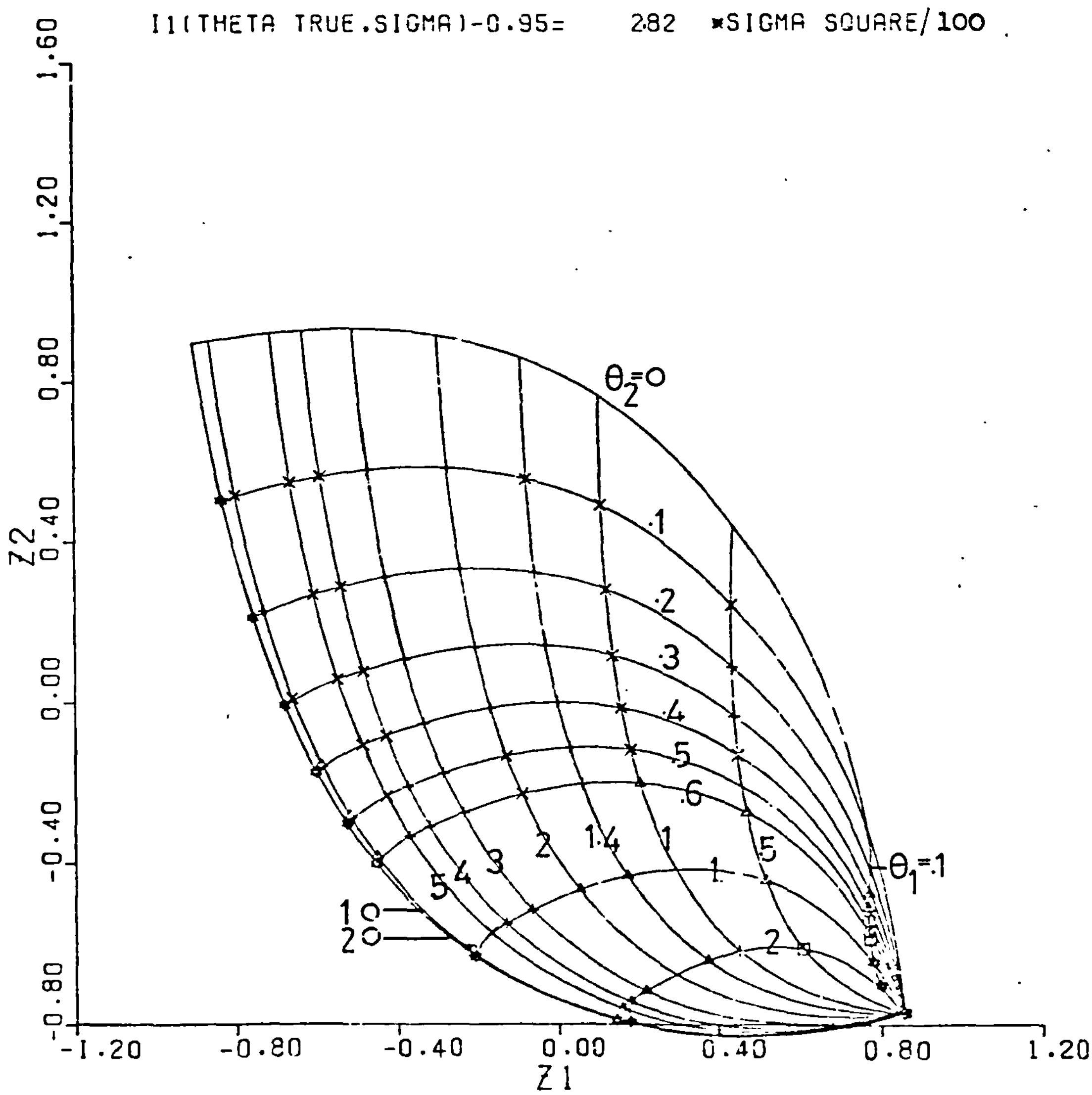
E(Y)=(THETA1/(THETA1-THETA2))

 *(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI= 0.25,0.5,1.0,1.5,2.0,4.0

THETAI TRUE ARE 1.4000 0.4000

I1(THETA TRUE,SIGMA)-0.95= 282 *SIGMA SQUARE/100



R=ABSOLUTE VALUE OF((I1(THETA,SIGMA)-I1(THETA TRUE,SIGMA))/(I1(THETA TRUE,SIGMA)-0.95))
PARAMETER OF INTEREST IS THETA2

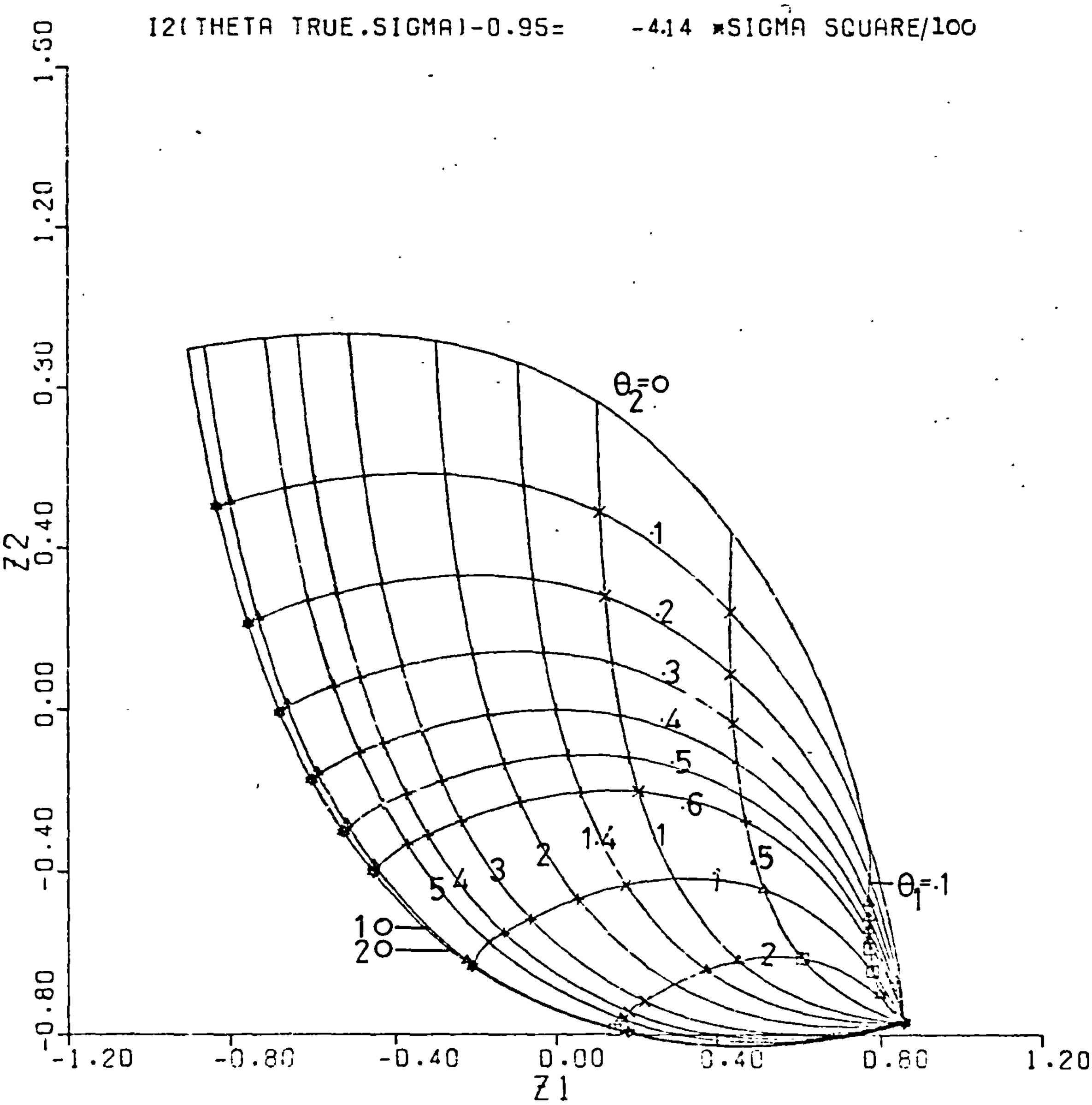
+:0≤R≤0.5 :X:0.5 <R≤1 :△:1 <R≤10 :□:10 <R≤100:✡:R>100

FIGURE (5.2.3)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=(THETA1/(THETA1-THETA2))
    *(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI= 0.25.0.5.1.0.1.5.2.0.4.0

THETAI TRUE ARE 1.4000 0.4000

I2(THETA TRUE.SIGMA)-0.95= -4.14 *SIGMA SQUARE/100



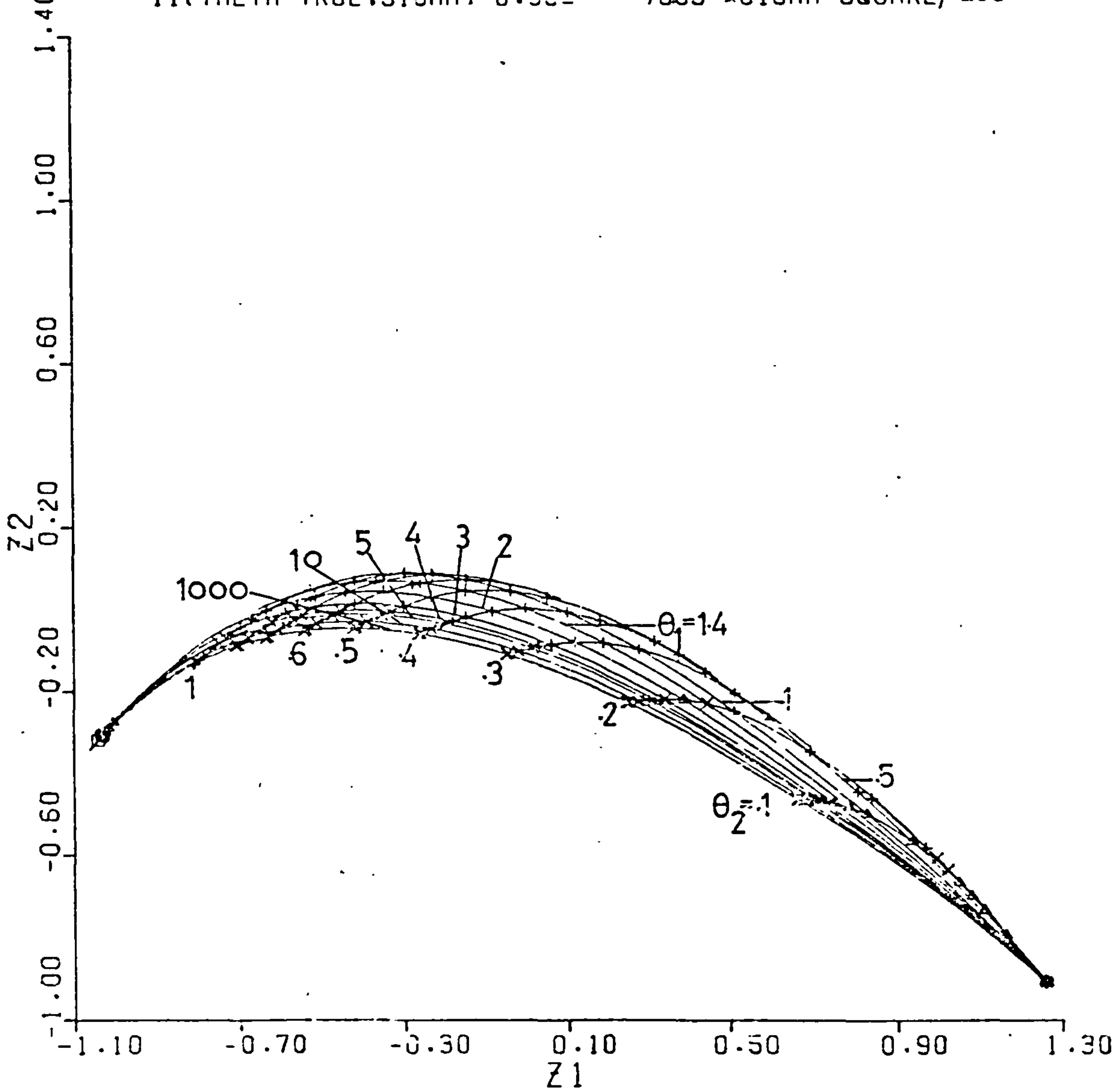
R=ABSOLUTE VALUE OF((I2(THETA.SIGMA)-I2(THETA TRUE.SIGMA))/(I2(THETA TRUE.SIGMA)-0.95))

+:0≤R≤0.5 :X:0.5 <R≤1 :△:1 <R≤10 :□:10 <R≤100:✡:R>100

FIGURE (5.2.4)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1.2.3.4.5.6

THETAI TRUE ARE 1.4000 0.4000

I1(THETA TRUE.SIGMA)-0.95= -7865 *SIGMA SQUARE/ 100



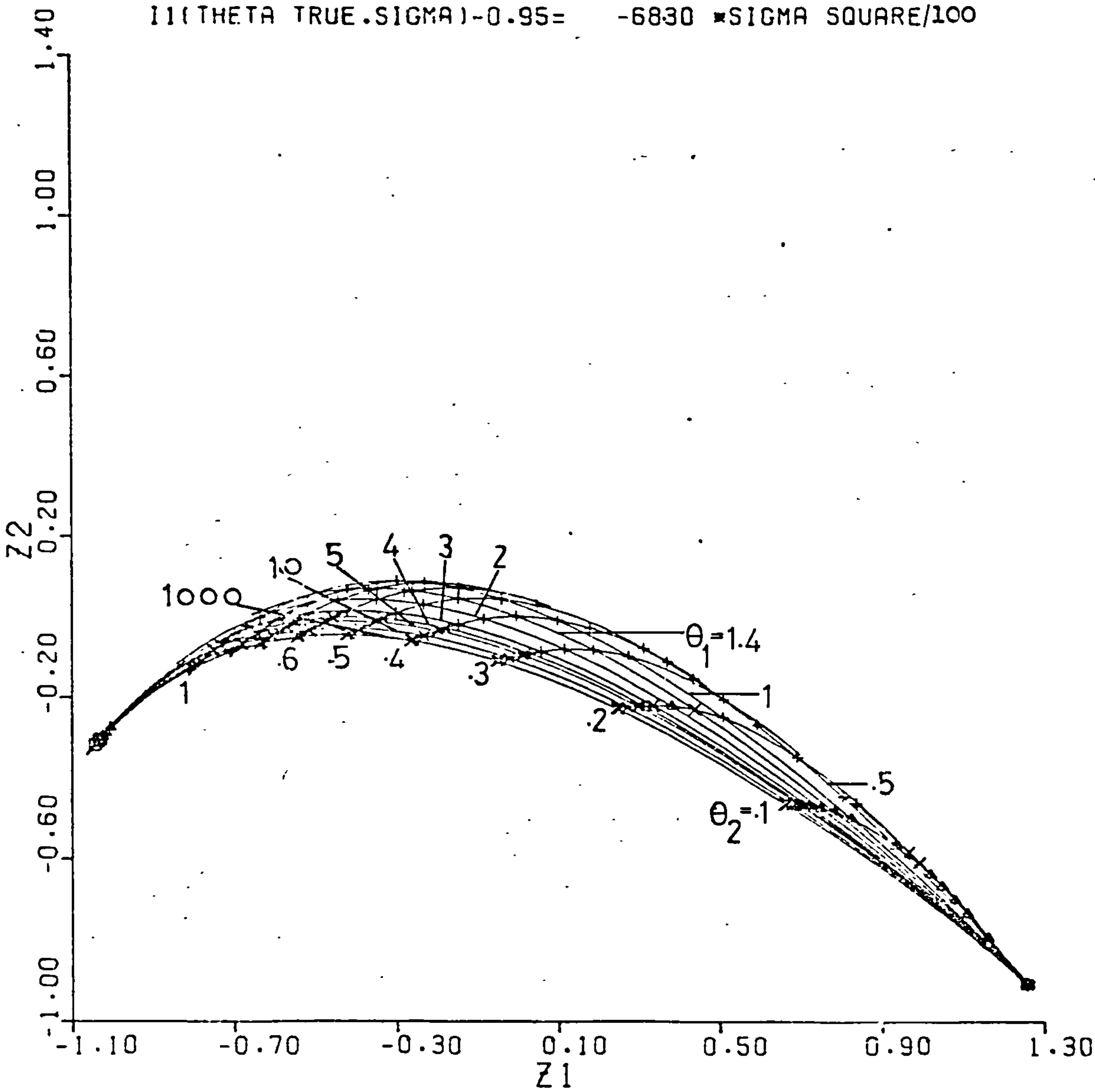
R=ABSOLUTE VALUE OF((I1(THETA.SIGMA)-I1(THETA TRUE.SIGMA))/(I1(THETA TRUE.SIGMA)-0.95))
PARAMETER OF INTEREST IS THETA1

+:0≤R≤0.5 ;X:0.5 <R≤1 :△:1 <R≤10 :□:10 <R≤100:✡:R>100

FIGURE (5.2.5)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=1-(THETA1*EXP(-THETA2*XI)
      -THETA2*EXP(-THETA1*XI))
      /(THETA1-THETA2)
XI=   1.2.3.4.5.6

THETAI TRUE ARE 1.4000 0.4000

II(THETA TRUE.SIGMA)-0.95=   -6830 *SIGMA SQUARE/100



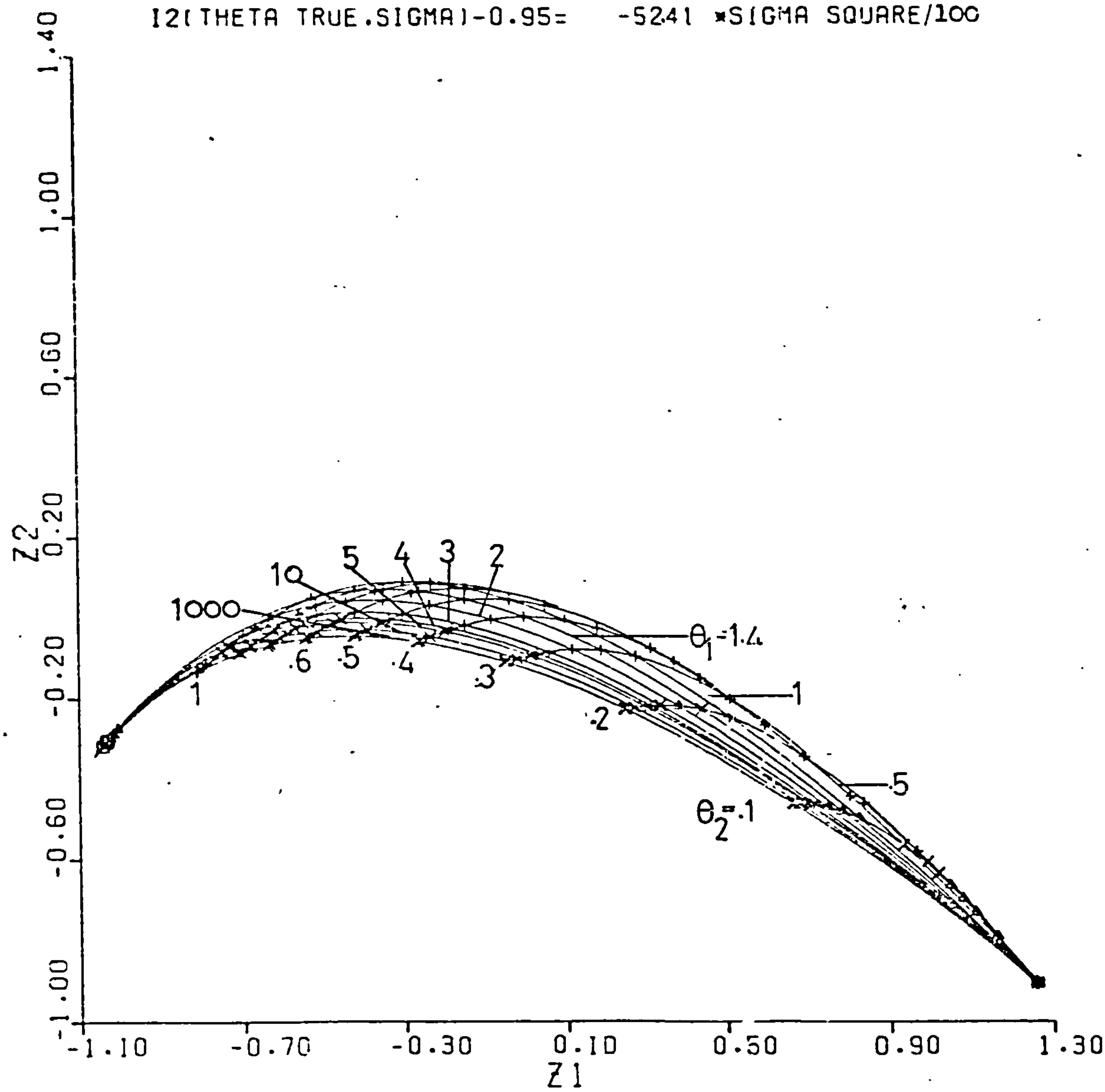
R=ABSOLUTE VALUE OF((I(THETA.SIGMA)-I(THETA TRUE.SIGMA))/(I(THETA TRUE.SIGMA)-0.95))
PARAMETER OF INTEREST IS THETA2

+:0≤R≤0.5 ;X:0.5 <R≤1 :Δ:1 <R≤10 :□:10 <R≤100:✡:R>100

FIGURE (5.2.6)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1,2,3,4,5,6

THETAI TRUE ARE 1.4000 0.4000

I2(THETA TRUE.SIGMA)-0.95= -52.41 *SIGMA SQUARE/100



R=ABSOLUTE VALUE OF((I2(THETA.SIGMA)-I2(THETA TRUE.SIGMA))/(I2(THETA TRUE.SIGMA)-0.95))

+:0≤R≤0.5 :X:0.5 <R≤1 ;△:1 <R≤10 :□:10 <R≤100:✡:R>100

These figures indicate that provided that σ is sufficiently small, the difference between $I_i(\underset{\sim}{\theta}_T, \sigma)$ and $I_i(\hat{\underset{\sim}{\theta}}, \sigma)$ is small for most of the $\hat{\underset{\sim}{\theta}}$, and the estimation of $I_i(\underset{\sim}{\theta}_T, \sigma)$ using $I_i(\hat{\underset{\sim}{\theta}}, \sigma)$ can thus be regarded as feasible. For example consider model (A) with $\sigma = 0.1$ and model (B) with $\sigma = 0.02$. These figures indicate that for most of the $\hat{\theta}$,

$$|I_i(\hat{\underset{\sim}{\theta}}, \sigma) - I_i(\underset{\sim}{\theta}_T, \sigma)| \leq 0.00005$$

for model (A), and

$$|I_i(\hat{\underset{\sim}{\theta}}, \sigma) - I_i(\underset{\sim}{\theta}_T, \sigma)| \leq 0.00032$$

for model (B). Thus if we use $I_i(\hat{\underset{\sim}{\theta}}, \sigma)$ to estimate $I_i(\underset{\sim}{\theta}_T, \sigma)$, the errors involved are small for most of the $\hat{\underset{\sim}{\theta}}$.

In practice, after calculating $\hat{\underset{\sim}{\theta}}$ and $I_i(\hat{\underset{\sim}{\theta}}, \sigma)$, we may wish to get some indication of whether σ is small enough for the estimation of the unknown value of $I_i(\underset{\sim}{\theta}_T, \sigma)$ to be feasible. One suggestion is to examine the values of $I_i(\underset{\sim}{\theta}, \sigma)$ evaluated at $\underset{\sim}{\theta}$ which are such that the distance between the point $P(\underset{\sim}{\theta})$ and $P(\hat{\underset{\sim}{\theta}})$ is less than δ, where $\delta > 0$. A plausible value of δ is 2σ.

Suppose we have obtained a particular $\hat{\underset{\sim}{\theta}}$ and there is indication that σ is small enough for the estimation of $I_i(\underset{\sim}{\theta}_T, \sigma)$ to be feasible. Then we may refer to the region estimate based on this particular $\hat{\underset{\sim}{\theta}}$ as an "approximately 100 $I_i(\hat{\underset{\sim}{\theta}}, \sigma)$ - c%" region estimate, where c is any number which can be regarded as negligible, in particular, $c = 0$.

We next consider the estimation of the following values of nonlinearity which are multiples of the measures of nonlinearity:

(5.2.1) $M_{\theta} = 100(p+2)\chi^2_{p+2}(\chi^2_{p,\alpha})N_{\theta}$,

(5.2.2) $M_{\psi} = 100(p+2)\chi^2_{p+2}(\chi^2_{p,\alpha})N_{\psi}$,

(5.2.3) $M_{\phi} = 100(p+2)\chi^2_{p+2}(\chi^2_{p,\alpha})N_{\phi}$,

(5.2.4) $M_{\theta_i} = 100(p+2)\chi^2_3(\chi^2_{1,\alpha})N_{\theta_i}$,

and

(5.2.5) $M_{\psi_i} = 100(p+2)\chi^2_3(\chi^2_{1,\alpha})N_{\psi_i}$,

where $\underset{\sim}{\psi}$ is the transformation based on method 2 and the ψ_i are transformations based on methods 2 and 3. Note that the values of M_{θ}, M_{ψ} and M_{ϕ} evaluated at $\underset{\sim}{\theta} = \underset{\sim}{\theta}_A$ are the upper bounds of $|J_2(\underset{\sim}{\theta}_A,\sigma)|$, and the values of M_{θ_i} evaluated at $\underset{\sim}{\theta} = \underset{\sim}{\theta}_A$ are the upper bounds of the corresponding $|J_1(\underset{\sim}{\theta}_A,\sigma)|$, where

(5.2.6) $J_i(\underset{\sim}{\theta}_A,\sigma) = 100(I_i(\underset{\sim}{\theta}_A,\sigma) - (1-\alpha))$, $(i = 1,2)$,

and the $I_i(\underset{\sim}{\theta}_A,\sigma)$ are evaluated by using (3.3.50) and (3.3.52). In fact

(5.2.7) $M_{\theta} \geq M_{\psi} \geq M_{\phi} \geq |J_2(\underset{\sim}{\theta}_A,\sigma)|$.

Furthermore we have

(5.2.8) $M_{\theta_i} \geq M_{\psi_i} \geq |J_1(\underset{\sim}{\theta}_A,\sigma)|$

if the transformation ψ_i is based on method 3.

We choose $\underset{\sim}{\theta}_T = (1.4, 0.4)^T$. We then set $\alpha = 0.05$ and calculate these values of nonlinearity at various values of $\underset{\sim}{\theta}$. Let these values of nonlinearity be denoted by $M_\theta(\underset{\sim}{\theta})$, $M_\psi(\underset{\sim}{\theta})$, $M_\phi(\underset{\sim}{\theta})$, $M_{\theta_i}(\underset{\sim}{\theta})$ and $M_{\psi_i}(\underset{\sim}{\theta})$. In Fig. (5.2.7) to (5.2.24), we display the absolute values of

$$\frac{M_\beta(\underset{\sim}{\theta}) - M_\beta(\underset{\sim}{\theta}_T)}{M_\beta(\underset{\sim}{\theta}_T)} ,$$

where $\beta = \theta, \psi, \phi, \theta_i$ and ψ_i.

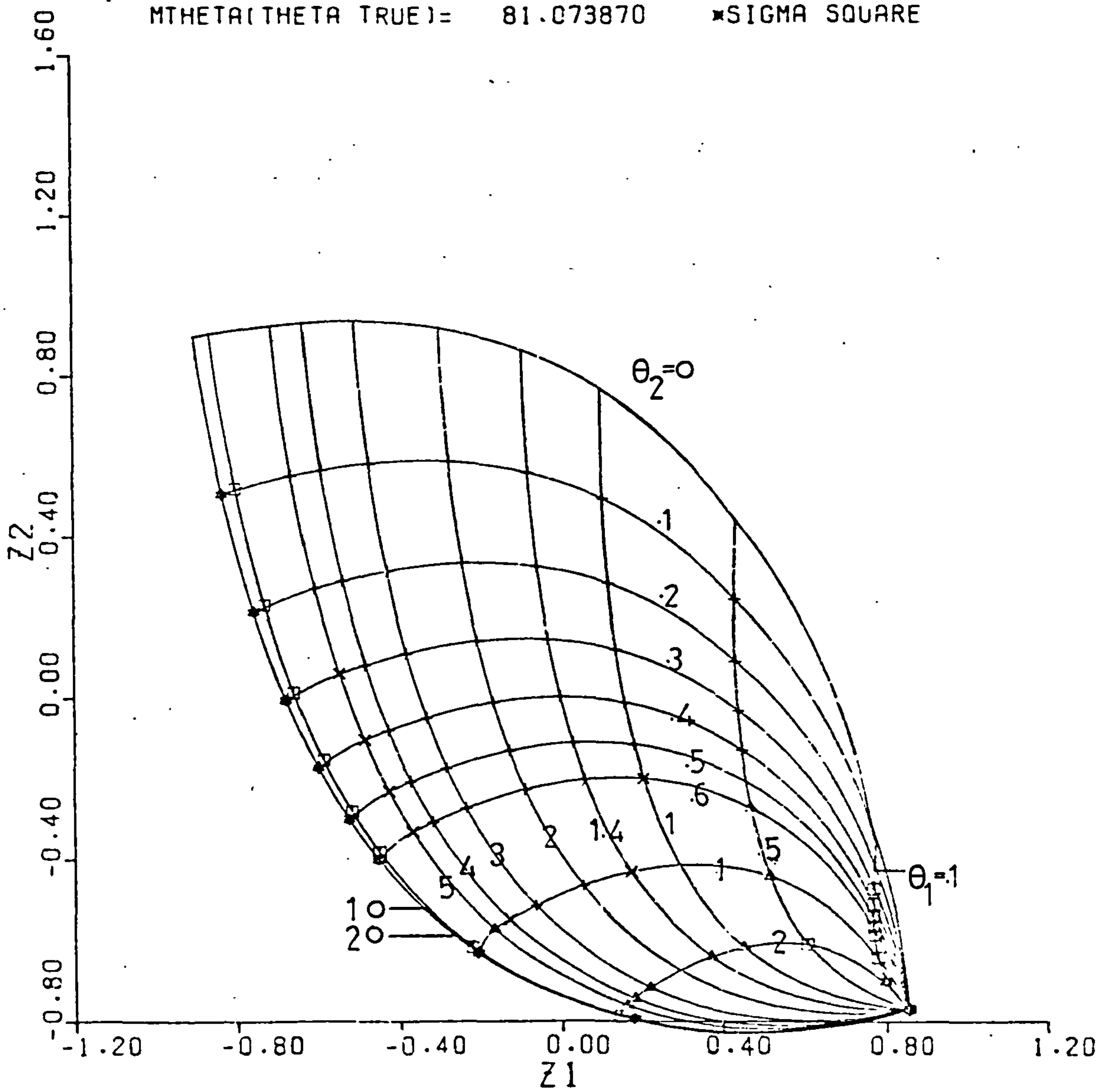
These figures indicate that, for a given β, provided that σ is sufficiently small, the difference between $M_\beta(\hat{\underset{\sim}{\theta}})$ and $M_\beta(\underset{\sim}{\theta}_T)$ is small for most of the $\hat{\underset{\sim}{\theta}}$, and the estimation of $M_\beta(\underset{\sim}{\theta}_T)$ using $M_\beta(\hat{\underset{\sim}{\theta}})$ can thus be regarded as feasible. Note that the value of σ needs to be much smaller in the present case than in the case when we estimate $I_i(\underset{\sim}{\theta}_T, \sigma)$. Further, the larger the value of $M_\beta(\underset{\sim}{\theta}_T)/\sigma^2$, the smaller the value of σ should be for the estimation of $M_\beta(\underset{\sim}{\theta}_T)$ to be feasible.

FIGURE (5.2.7)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI= 0.25,0.5,1.0,1.5,2.0,4.0

THETAI TRUE ARE 1.4000 0.4000

MTHETA(THETA TRUE)= 81.073870 *SIGMA SQUARE



R=ABSOLUTE VALUE OF((MTHETA(THETA)-MTHETA(THETA TRUE))/MTHETA(THETA TRUE))

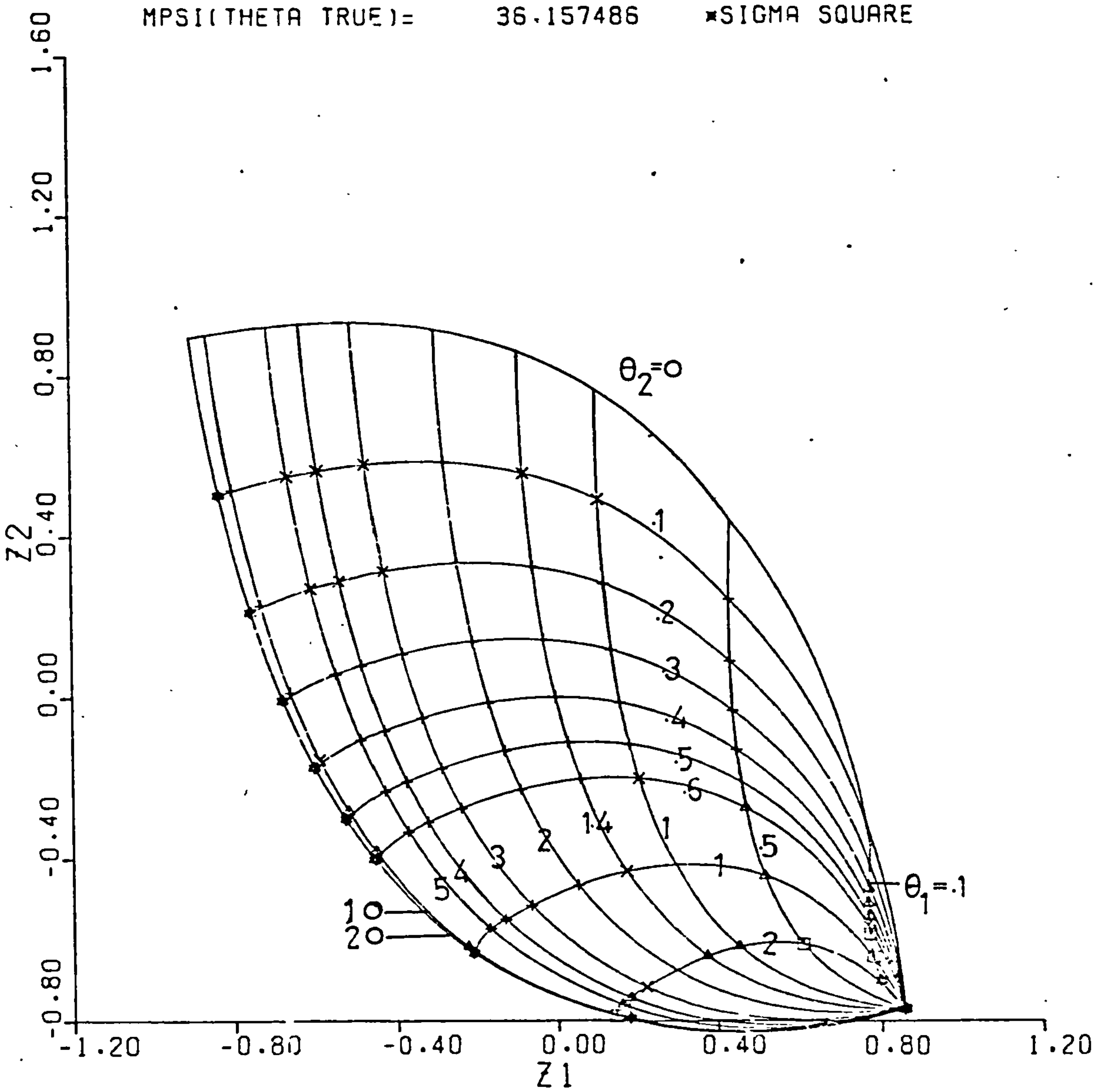
+:0≤R≤0.5 ;X:0.5 <R≤1 ;△:1 <R≤10 ;□:10 <R≤100;✡:R>100

FIGURE (5.2.8)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI= 0.25.0.5.1.0.1.5.2.0.4.0

THETAI TRUE ARE 1.4000 0.4000

MPSI(THETA TRUE)= 36.157486 *SIGMA SQUARE



R=ABSOLUTE VALUE OF((MPSI(THETA)-MPSI(THETA TRUE))/MPSI(THETA TRUE))
PSI IS TRANSFORMATION BASED ON METHOD 2

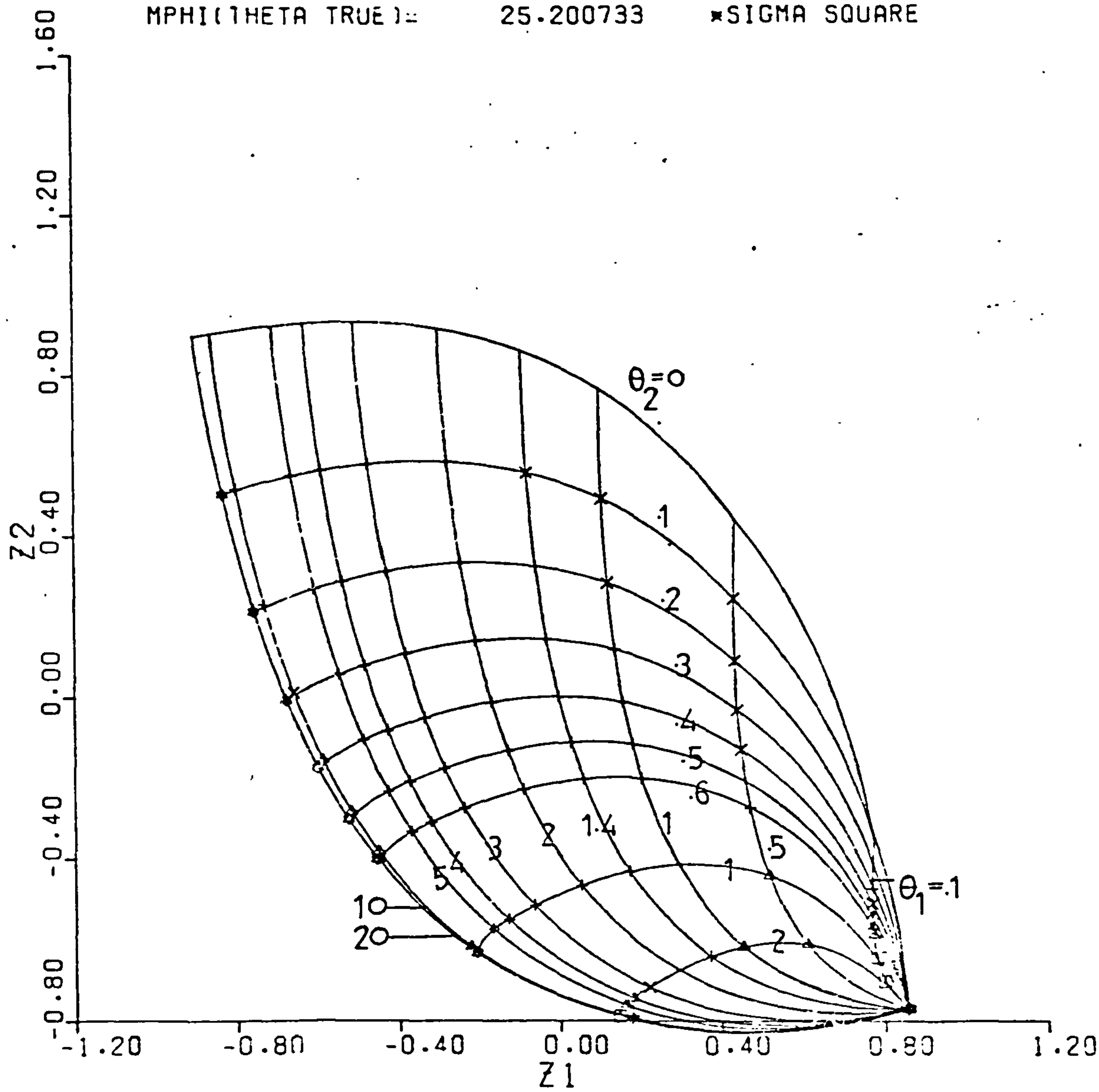
+:0≤R≤0.5 ;X:0.5 <R≤1 :△:1 <R≤10 :□:10 <R≤100;☆:R>100

FIGURE (5.2.9)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(I)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI= 0.25,0.5,1.0,1.5,2.0,4.0

THETAI TRUE ARE 1.4000 0.4000

MPHI(THETA TRUE)= 25.200733 *SIGMA SQUARE



R=ABSOLUTE VALUE OF((MPHI(THETA)-MPHI(THETA TRUE))/MPHI(THETA TRUE))

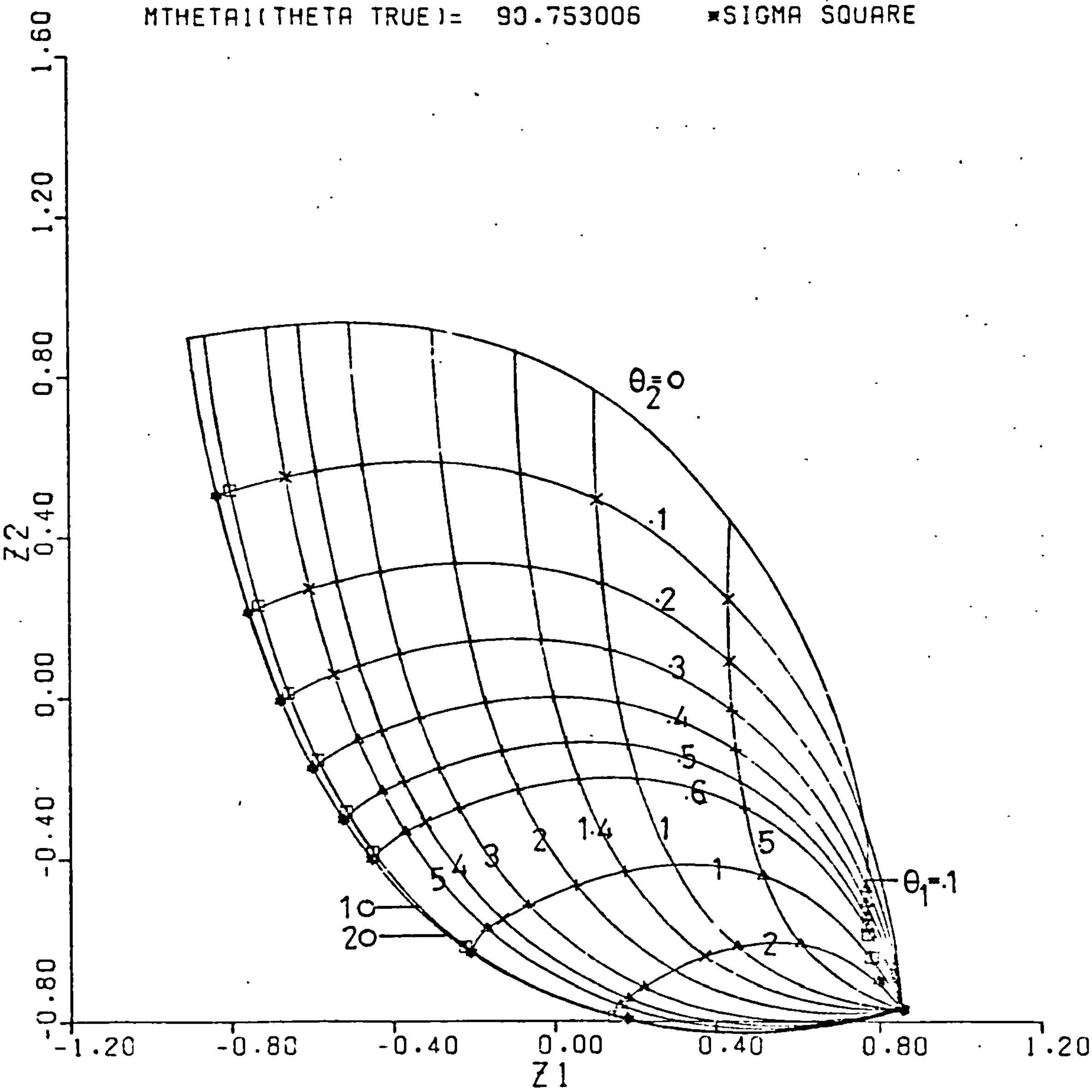
+:0≤R≤0.5 :X:0.5 <R≤1 :△:1 <R≤10 :□:10 <R≤100;✿:R>100

FIGURE (5.2.10)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI= 0.25,0.5,1.0,1.5,2.0,4.0

THETA1 TRUE ARE 1.4000 0.4000

MTHETA1(THETA TRUE)= 90.753006 *SIGMA SQUARE



R=ABSOLUTE VALUE OF((MTHETA1(THETA)-MTHETA1(THETA TRUE))/MTHETA1(THETA TRUE))

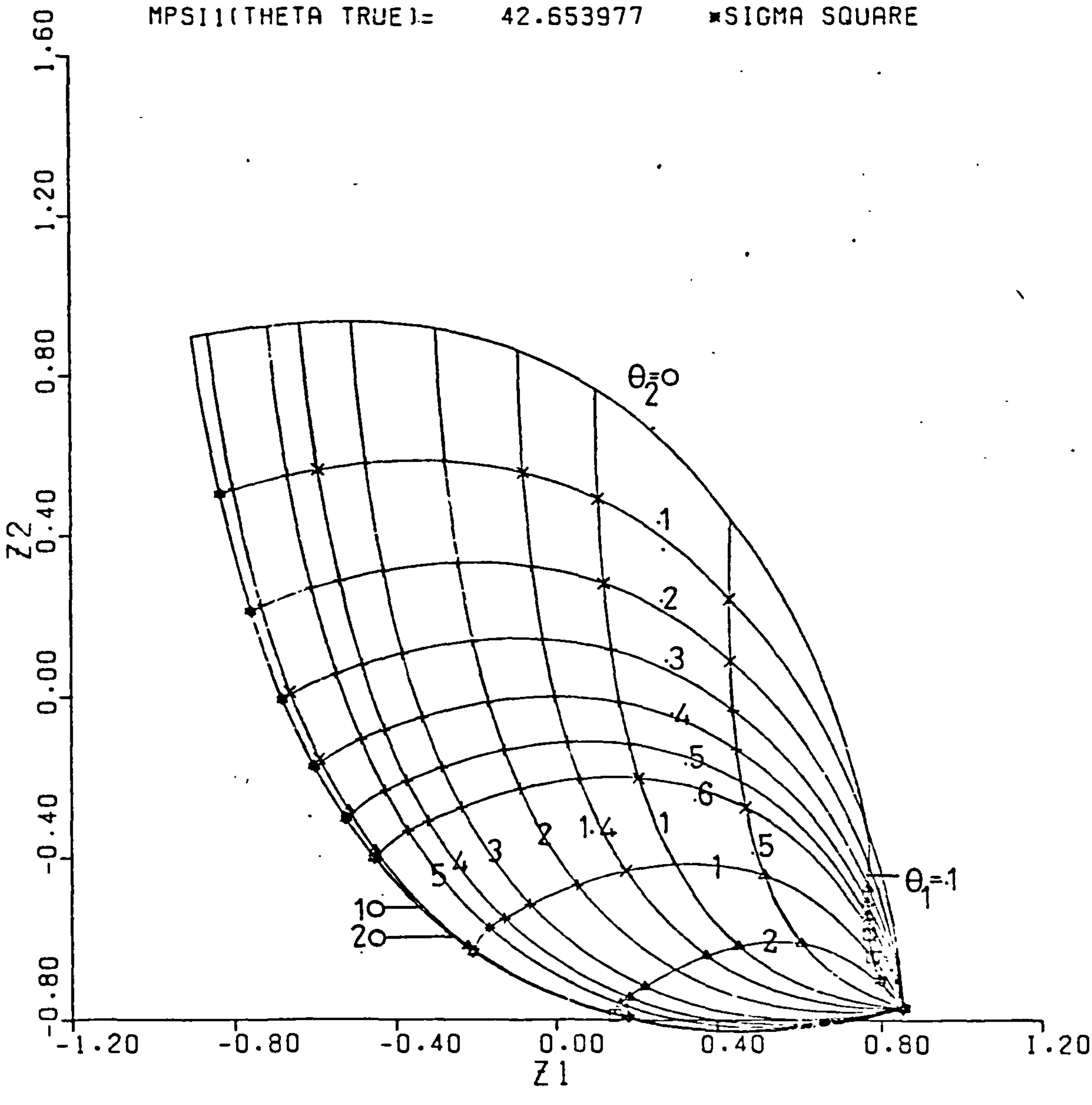
+:0≤R≤0.5 :X:0.5 <R≤1 ;△:1 <R≤10 ;□:10 <R≤100;✡:R>100

FIGURE (5.2.11)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI= 0.25,0.5,1.0,1.5,2.0,4.0

THETAI TRUE ARE 1.4000 0.4000

MPSI1(THETA TRUE)= 42.653977 *SIGMA SQUARE



R=ABSOLUTE VALUE OF((MPSI1(THETA)-MPSI1(THETA TRUE))/MPSI1(THETA TRUE))
PSI1 IS POWER TRANSFORMATION BASED ON METHOD 2

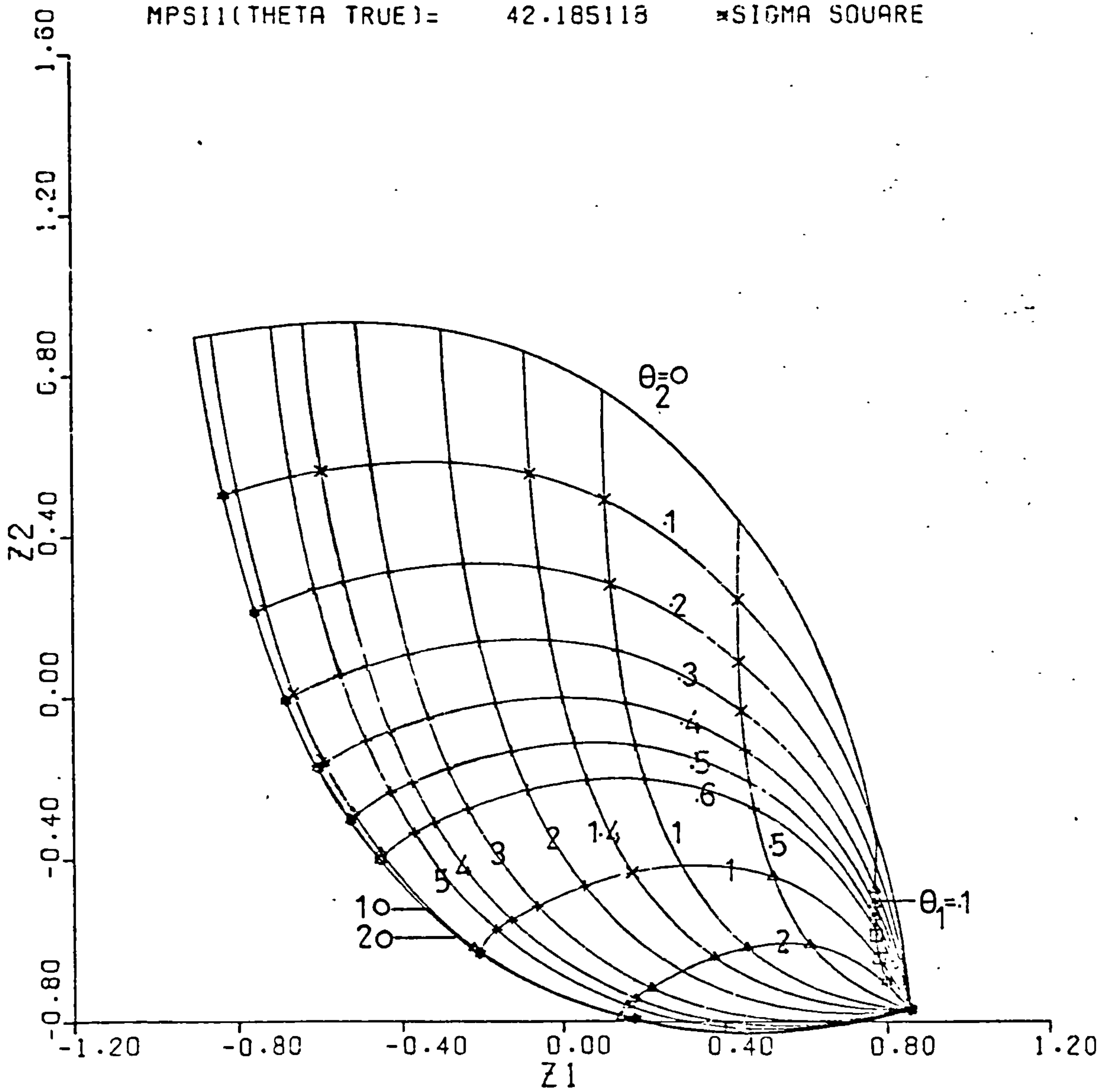
+: 0≤R≤0.5 ; ✕: 0.5 <R≤1 ; △: 1 <R≤10 ; ◻: 10 <R≤100 ; ✡: R>100

FIGURE (5.2.12)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI= 0.25,0.5,1.0,1.5,2.0,4.0

THETAI TRUE ARE 1.4000 0.4000

MPSI1(THETA TRUE)= 42.185118 *SIGMA SQUARE



R=ABSOLUTE VALUE OF((MPSI1(THETA)-MPSI1(THETA TRUE))/MPSI1(THETA TRUE))
PSI1 IS POWER TRANSFORMATION BASED ON METHOD 3

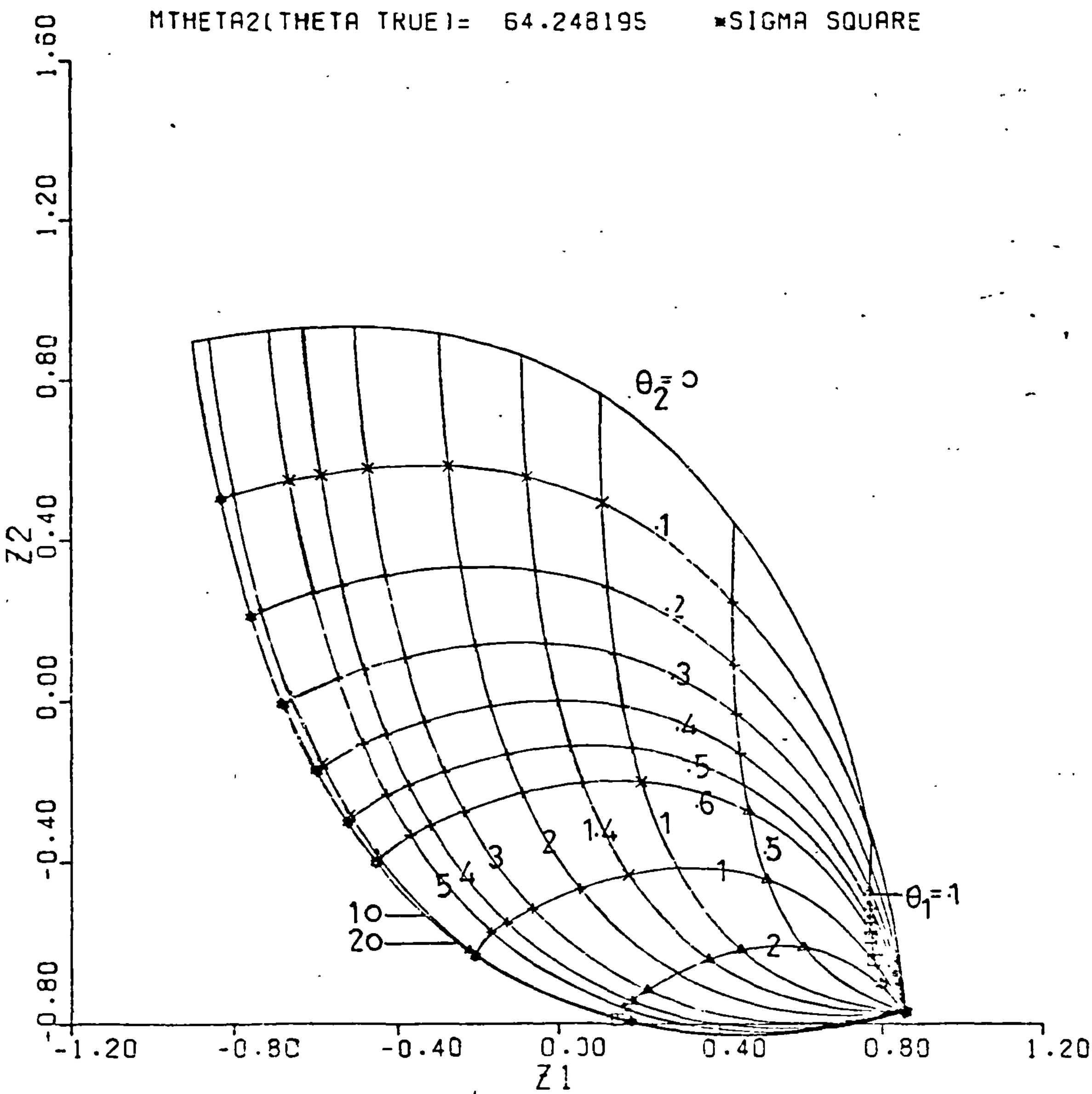
+:0≤R≤0.5 ;X:0.5 <R≤1 ;△:1 <R≤10 ;□:10 <R≤100;☆:R>100

FIGURE (5.2.13)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI= 0.25.0.5,1.0,1.5,2.0,4.0

THETAI TRUE ARE 1.4000 0.4000

MTHETA2(THETA TRUE)= 64.248195 *SIGMA SQUARE



R=ABSOLUTE VALUE OF ((MTHETA2(THETA)-MTHETA2(THETA TRUE))/MTHETA2(THETA TRUE))

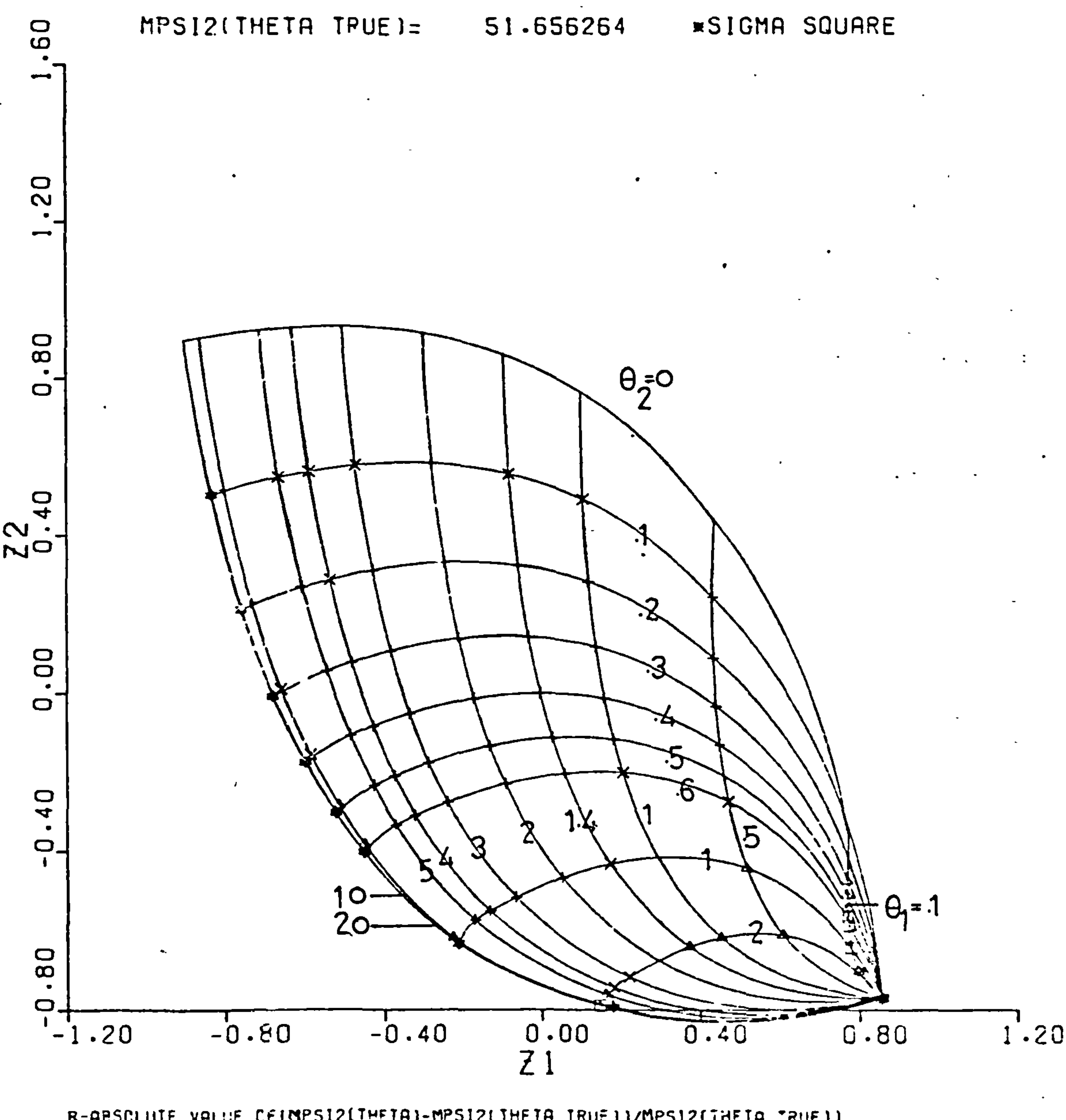
+:0≤R≤0.5 ;X:0.5 <R≤1 ;△:1 <R≤10 ;□:10 <R≤100;☆:R>100

FIGURE (5.2.14)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI= 0.25,0.5,1.0,1.5,2.0,4.0

THETAI TRUE ARE 1.4000 0.4000

MPSI2(THETA TRUE)= 51.656264 *SIGMA SQUARE



R=ABSOLUTE VALUE OF(MPSI2(THETA)-MPSI2(THETA TRUE))/MPSI2(THETA TRUE))
PSI2 IS POWER TRANSFORMATION BASED ON METHOD 2

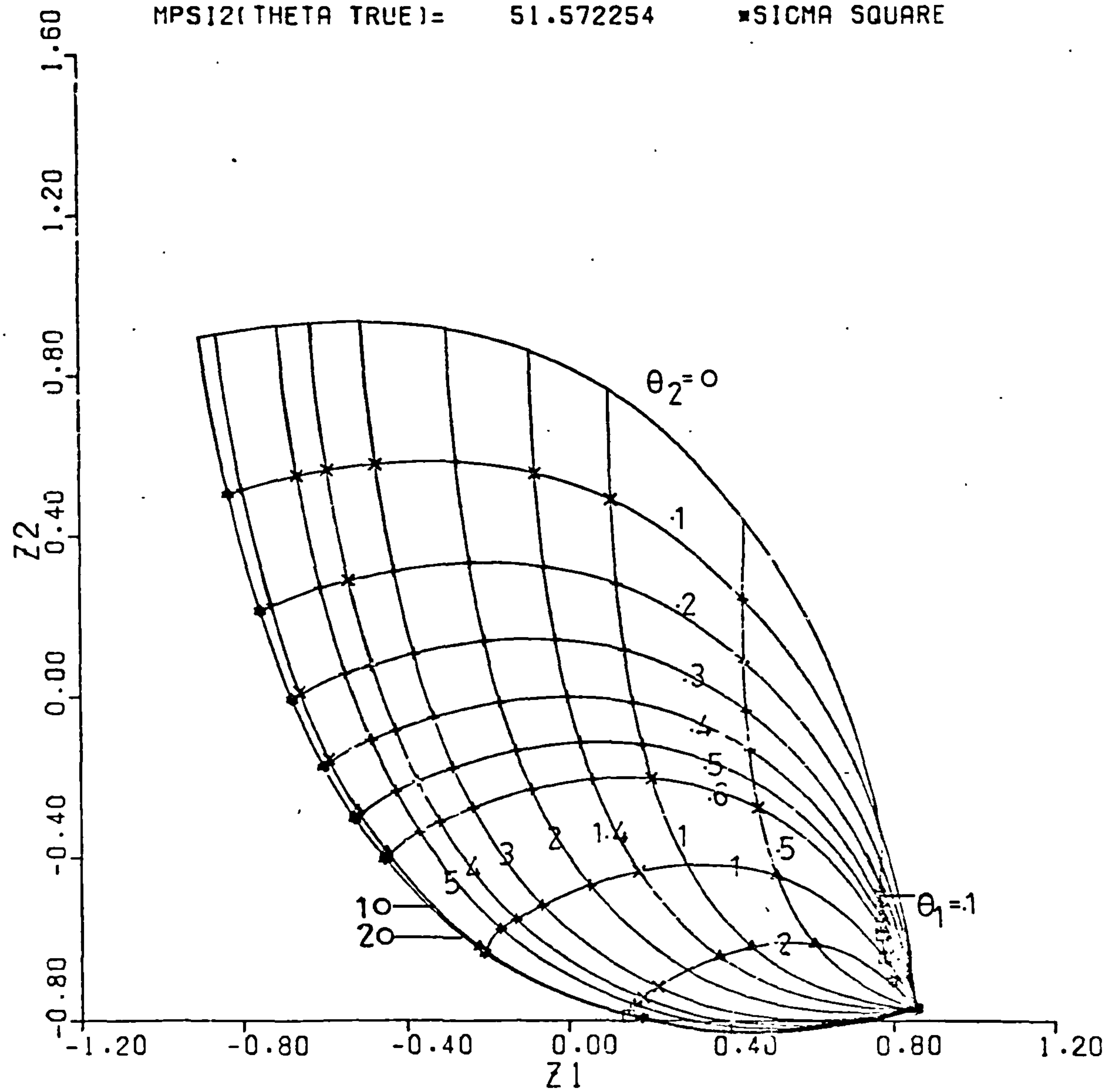
+:0≤R≤0.5 :×:0.5 <R≤1 :△:1 <R≤10 :□:10 <R≤100:✩:R>100

FIGURE (5.2.15)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI= 0.25,0.5,1.0,1.5,2.0,4.0

THETAI TRUE ARE 1.4000 0.4000

MPSI2(THETA TRUE)= 51.572254 *SIGMA SQUARE



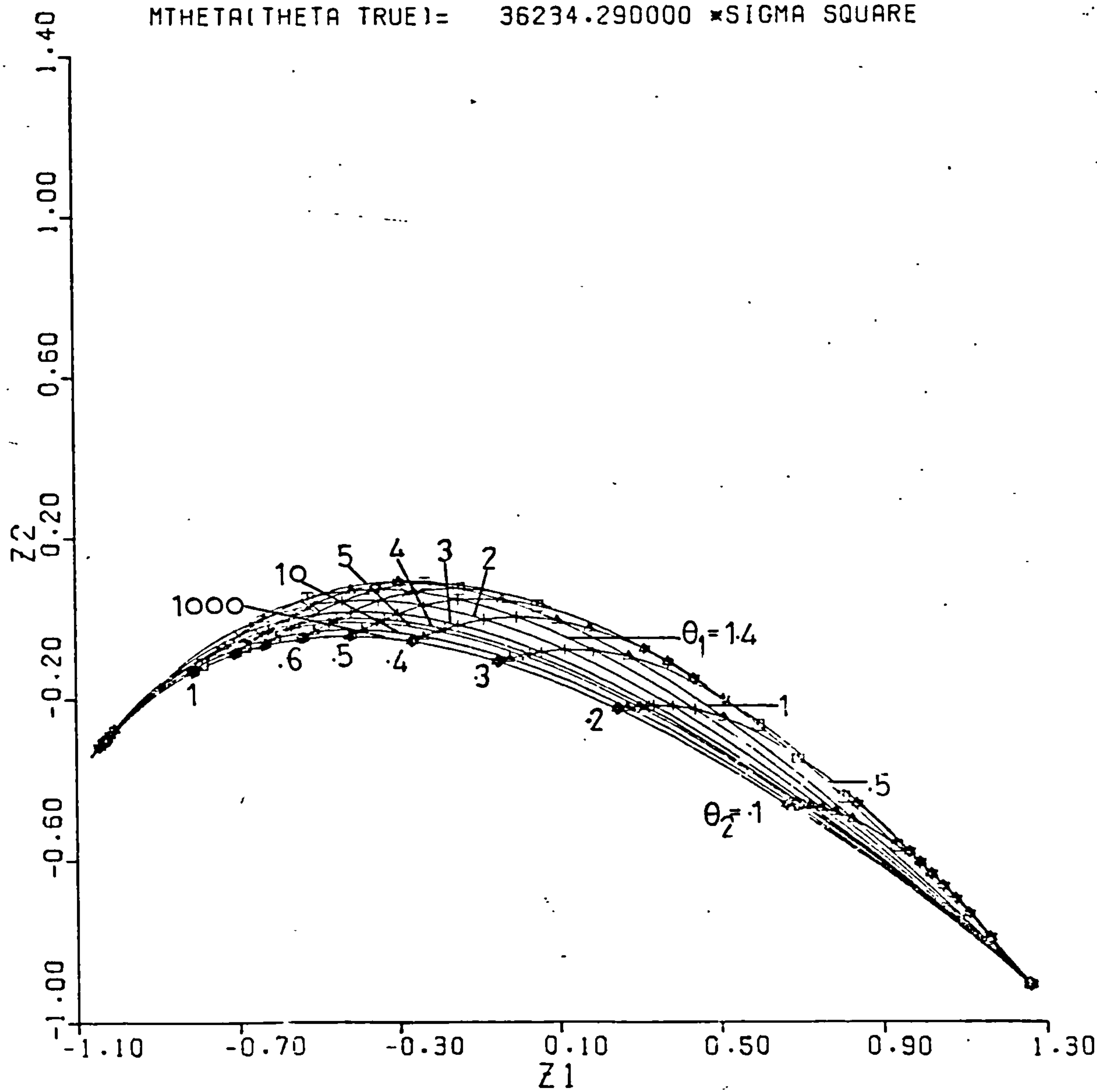
R=ABSOLUTE VALUE OF((MPSI2(THETA)-MPSI2(THETA TRUE))/MPSI2(THETA TRUE))
PSI2 IS POWER TRANSFORMATION BASED ON METHOD 3

+:0≤R≤0.5 ;X:0.5 <R≤1 :△:1 <R≤10 :□:10 <R≤100:✿:R>100

FIGURE (5.2.16)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1.2.3.4.5.6

THETAI TRUE ARE 1.4000 0.4000

MTHETA(THETA TRUE)= 36234.290000 *SIGMA SQUARE



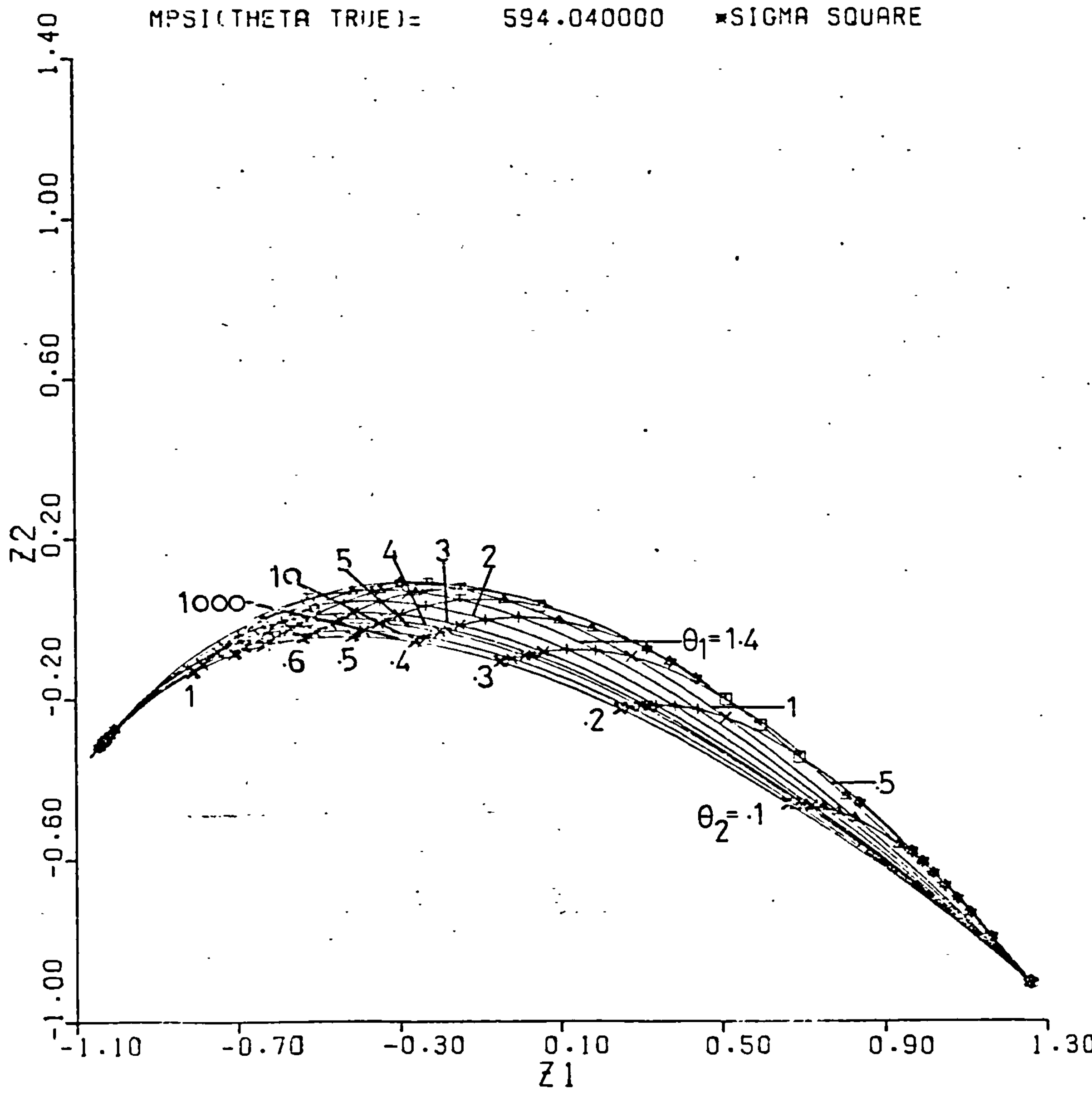
R=ABSOLUTE VALUE OF((MTHETA(THETA)-MTHETA(THETA TRUE))/MTHETA(THETA TRUE))

+:0≤R≤0.5 :X:0.5 <R≤1 :△:1 <R≤10 :□:10 <R≤100:✡:R>100

FIGURE (5.2.17)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=1-(THETA1*EXP(-THETP2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1.2.3.4.5.6

THETAI TRUE ARE 1.4000 0.4000

MPSI(THETA TRUE)= 594.040000 *SIGMA SQUARE



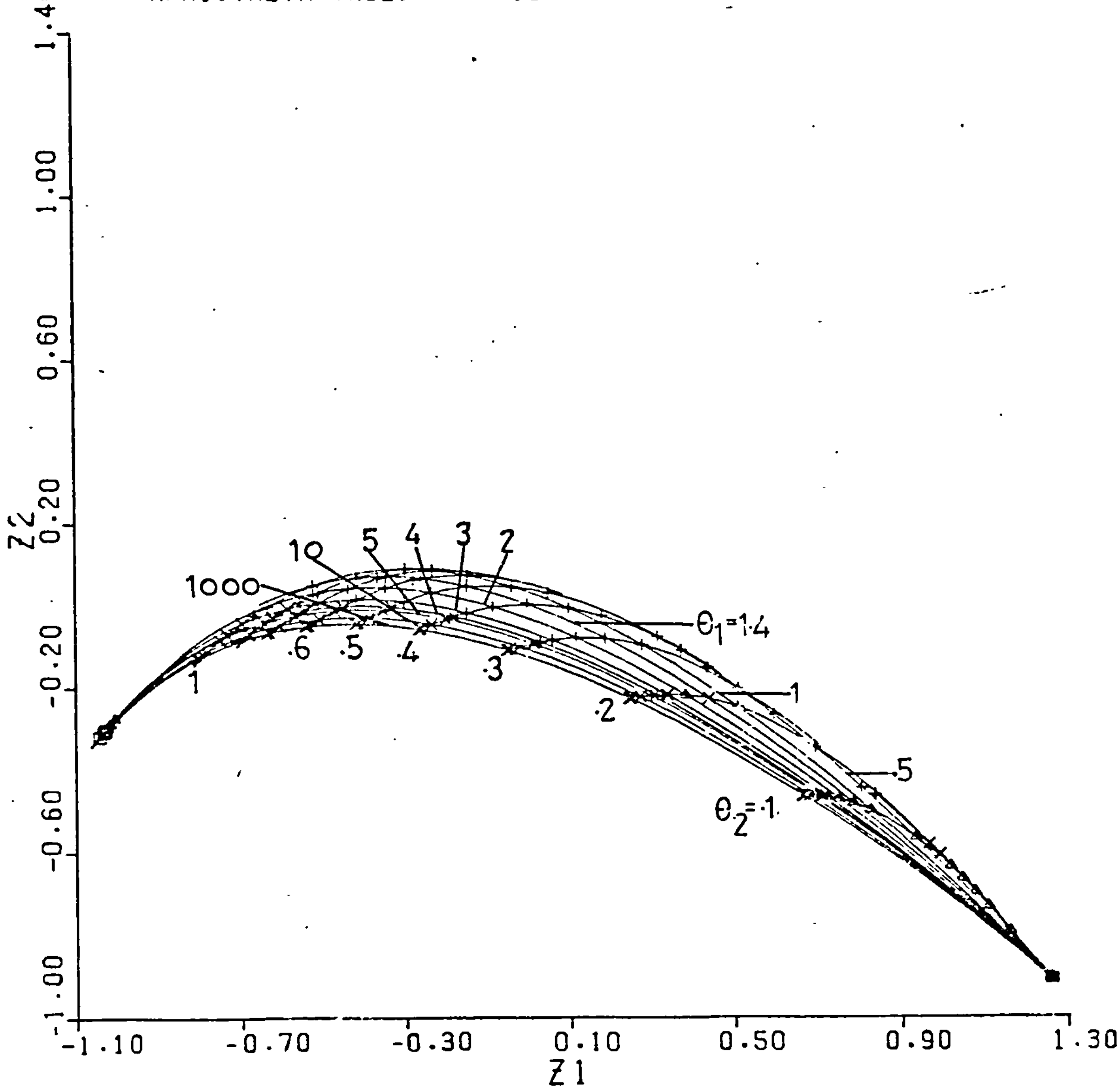
R=ABSOLUTE VALUE OF((MPSI(THETA)-MPSI(THETA TRUE))/MPSI(THETA TRUE))
PSI IS TRANSFORMATION BASED ON METHOD 2

+:0≤R≤0.5 ;×:0.5 <R≤1 :△:1 <R≤10 :□:10 <R≤100:✡:R>100

FIGURE (5.2.18)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1.2.3.4.5.6

THETAI TRUE ARE 1.4000 0.4000

MPHI(THETA TRUE)= 137.040000 *SIGMA SQUARE



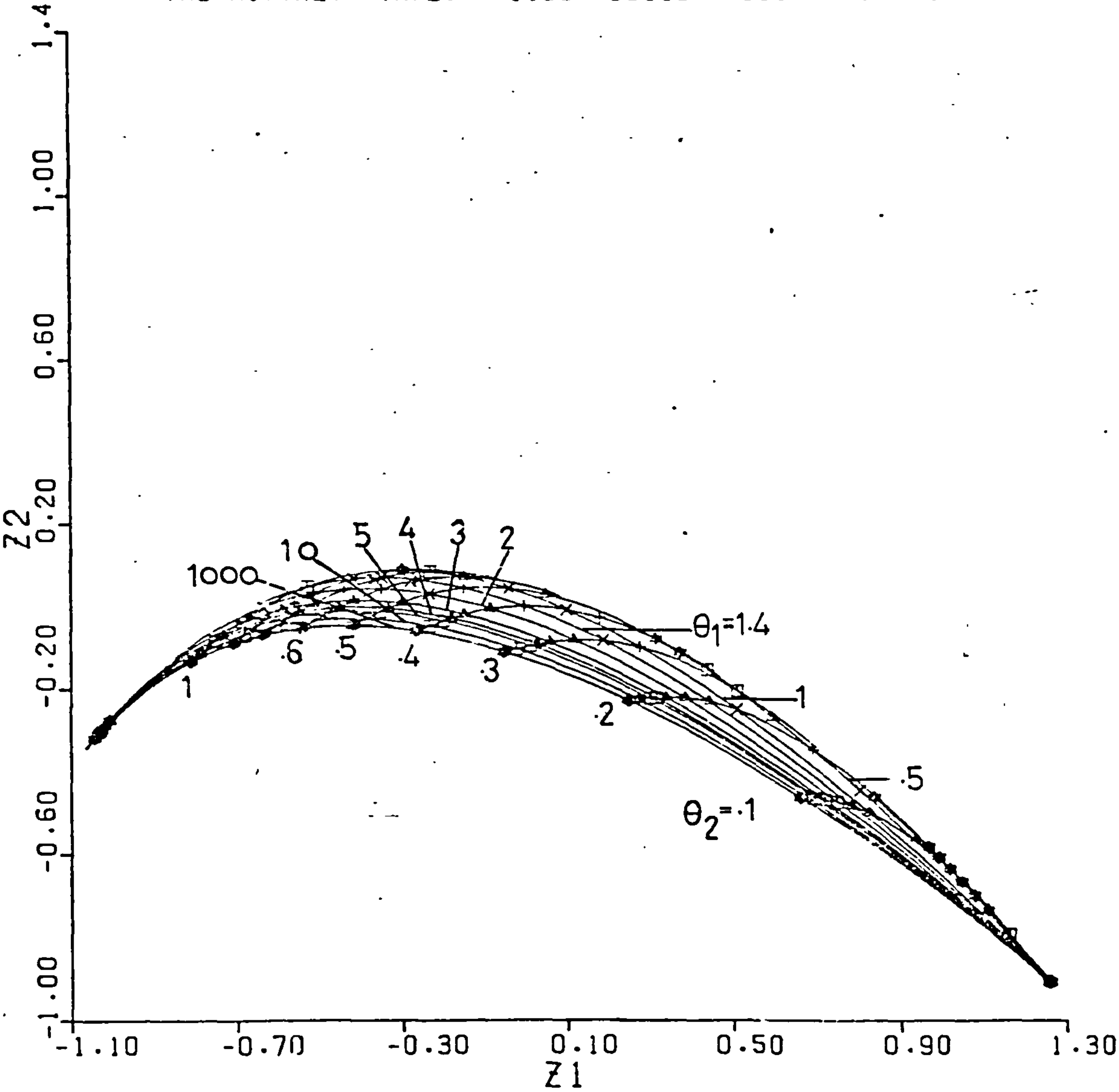
R=ABSOLUTE VALUE OF((MPHI(THETA)-MPHI(THETA TRUE))/MPHI(THETA TRUE))

+:0≤R≤0.5 ;X:0.5 <R≤1 :△:1 <R≤10 :□:10 <R≤100:✪:R>100

FIGURE (5.2.19)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1.2.3.4.5.6

THETAI TRUE ARE 1.4000 0.4000

MTHETA1(THETA TRUE)= 1195.460000 *SIGMA SQUARE



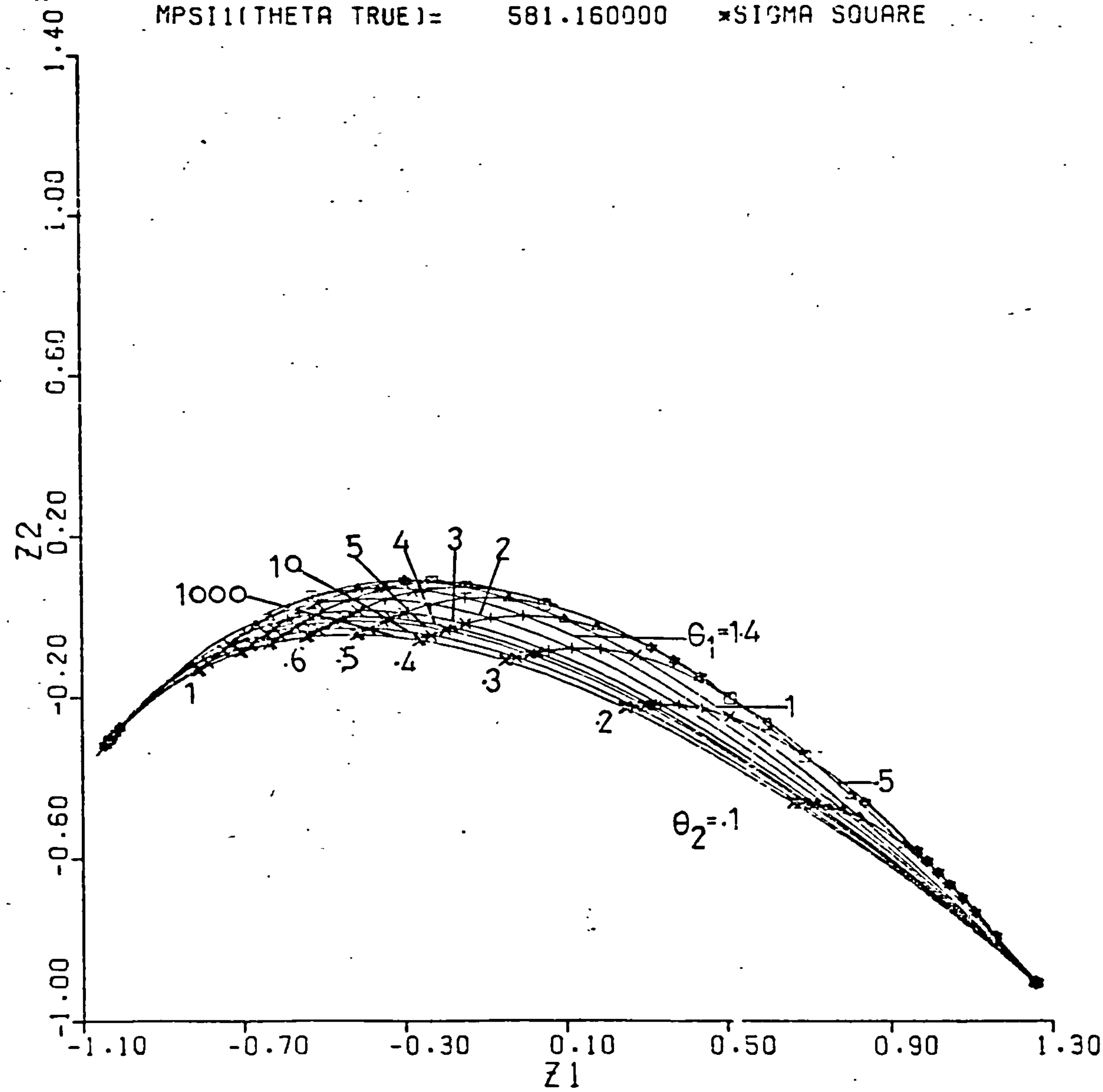
R=ABSOLUTE VALUE OF((MTHETA1(THETA)-MTHETA1(THETA TRUE))/MTHETA1(THETA TRUE))

+:0≤R≤0.5 ;X:0.5 <R≤1 :△:1 <R≤10 :□:10 <R≤100:✡:R>100

FIGURE (5.2.20)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1.2.3.4.5.6

THETAI TRUE ARE 1.4000 0.4000

MPSI1(THETA TRUE)= 581.160000 *SIGMA SQUARE



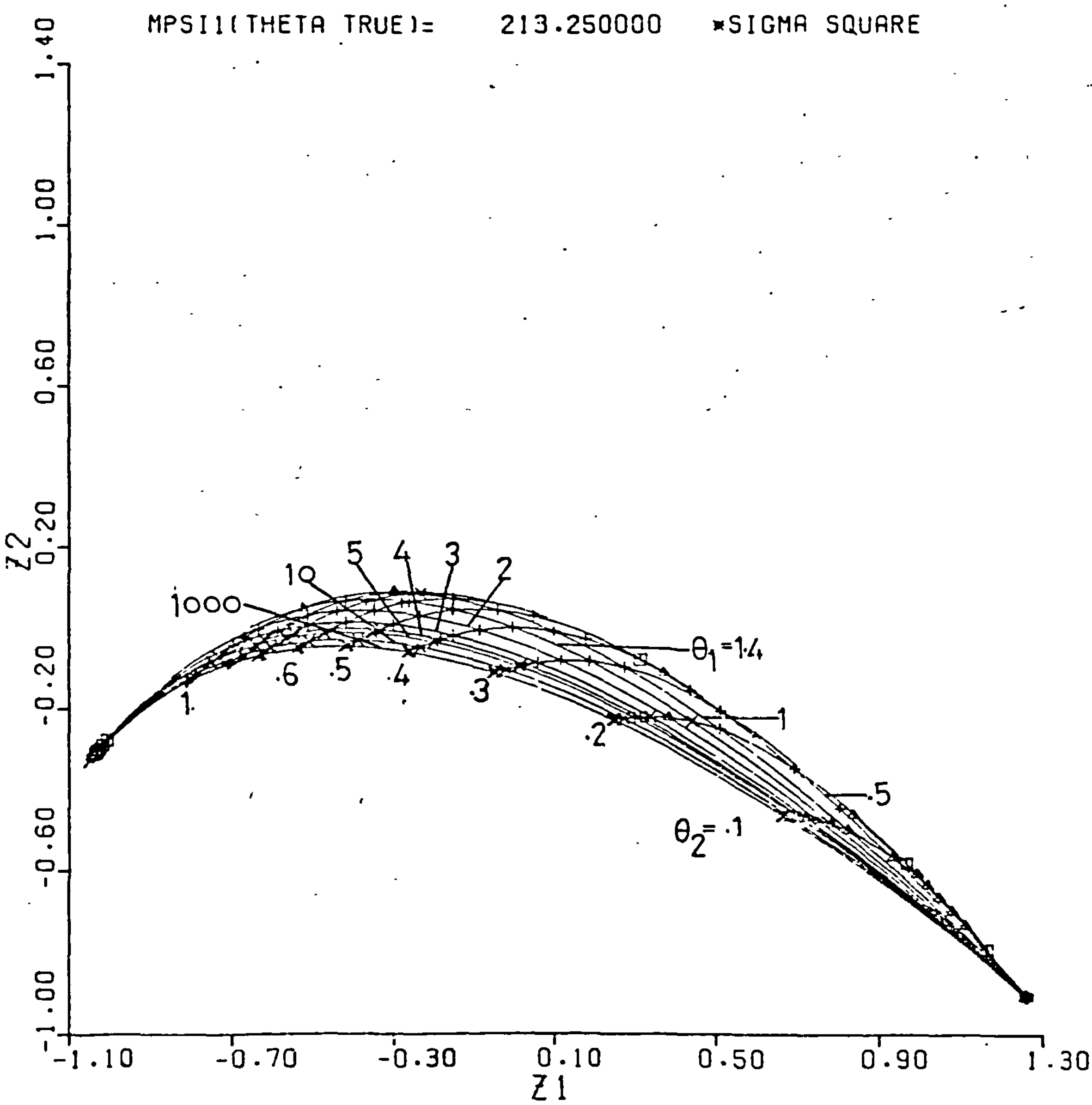
R=ABSOLUTE VALUE OF(MPSI1(THETA)-MPSI1(THETA TRUE))/MPSI1(THETA TRUE))
PSI1 IS POWER TRANSFORMATION BASED ON METHOD 2

+:0≤R≤0.5 ;X:0.5 <R≤1 :△:1 <R≤10 :□:10 <R≤100:✡:R>100

FIGURE (5.2.21)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1.2.3.4.5.6

THETAI TRUE ARE 1.4000 0.4000

MPSI1(THETA TRUE)= 213.250000 *SIGMA SQUARE



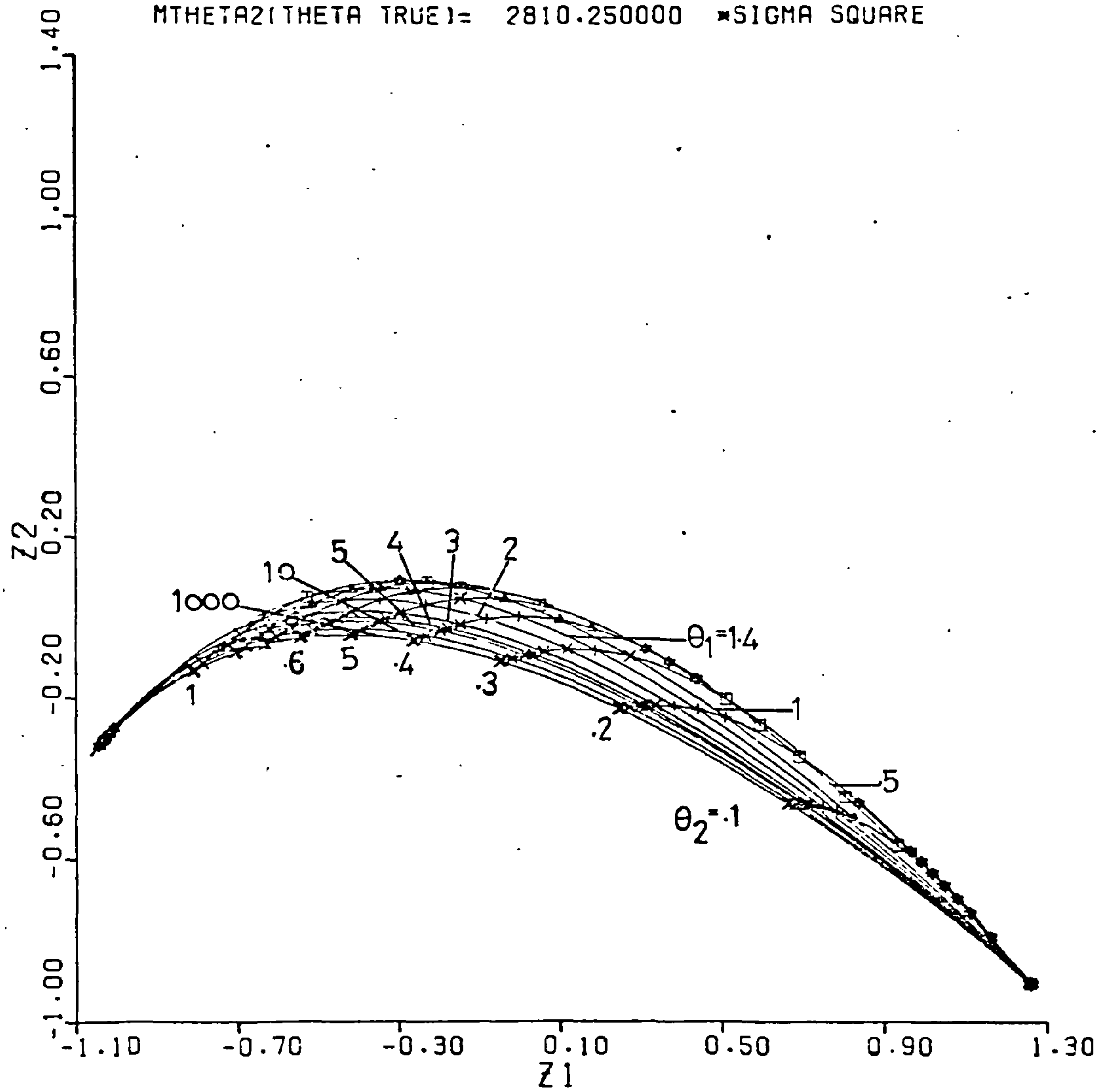
R=ABSOLUTE VALUE OF((MPSI1(THETA)-MPSI1(THETA TRUE))/MPSI1(THETA TRUE))
PSI1 IS POWER TRANSFORMATION BASED ON METHOD 3

+:0≤R≤0.5 ;X:0.5 <R≤1 :△:1 <R≤10 :□:10 <R≤100:✿:R>100

FIGURE (5.2.22)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1.2.3.4.5.6

THETAI TRUE ARE 1.4000 0.4000

MTHETA2(THETA TRUE)= 2810.250000 *SIGMA SQUARE



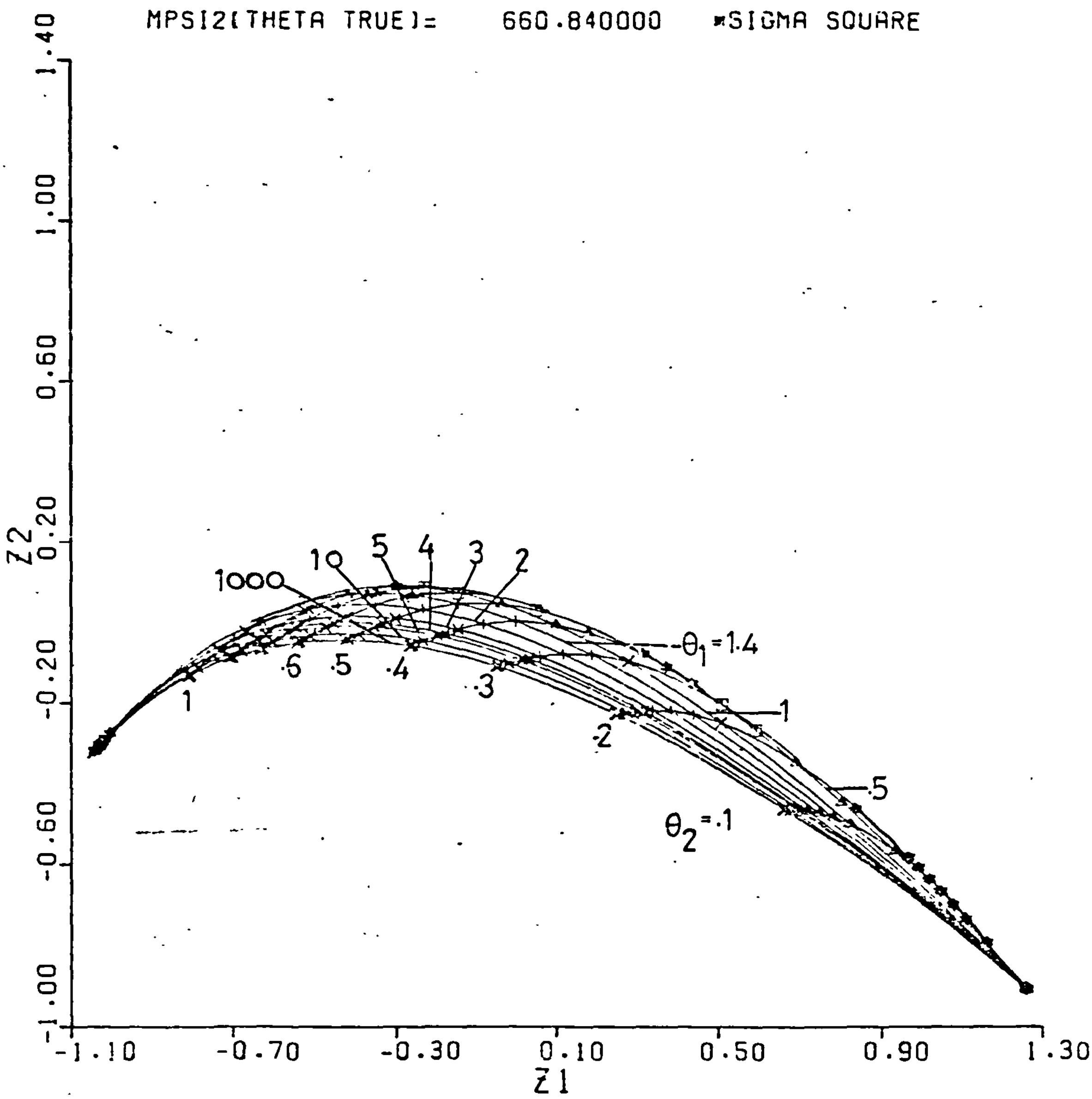
R=ABSOLUTE VALUE OF((MTHETA2(THETA)-MTHETA2(THETA TRUE))/MTHETA2(THETA TRUE))

+:0≤R≤0.5 ;X:0.5 <R≤1 :△:1 <R≤10 :□:10 <R≤100:✡:R>100

FIGURE (5.2.23)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1.2.3.4.5.6

THETA1 TRUE ARE 1.4000 0.4000

MPSI2(THETA TRUE)= 660.840000 *SIGMA SQUARE



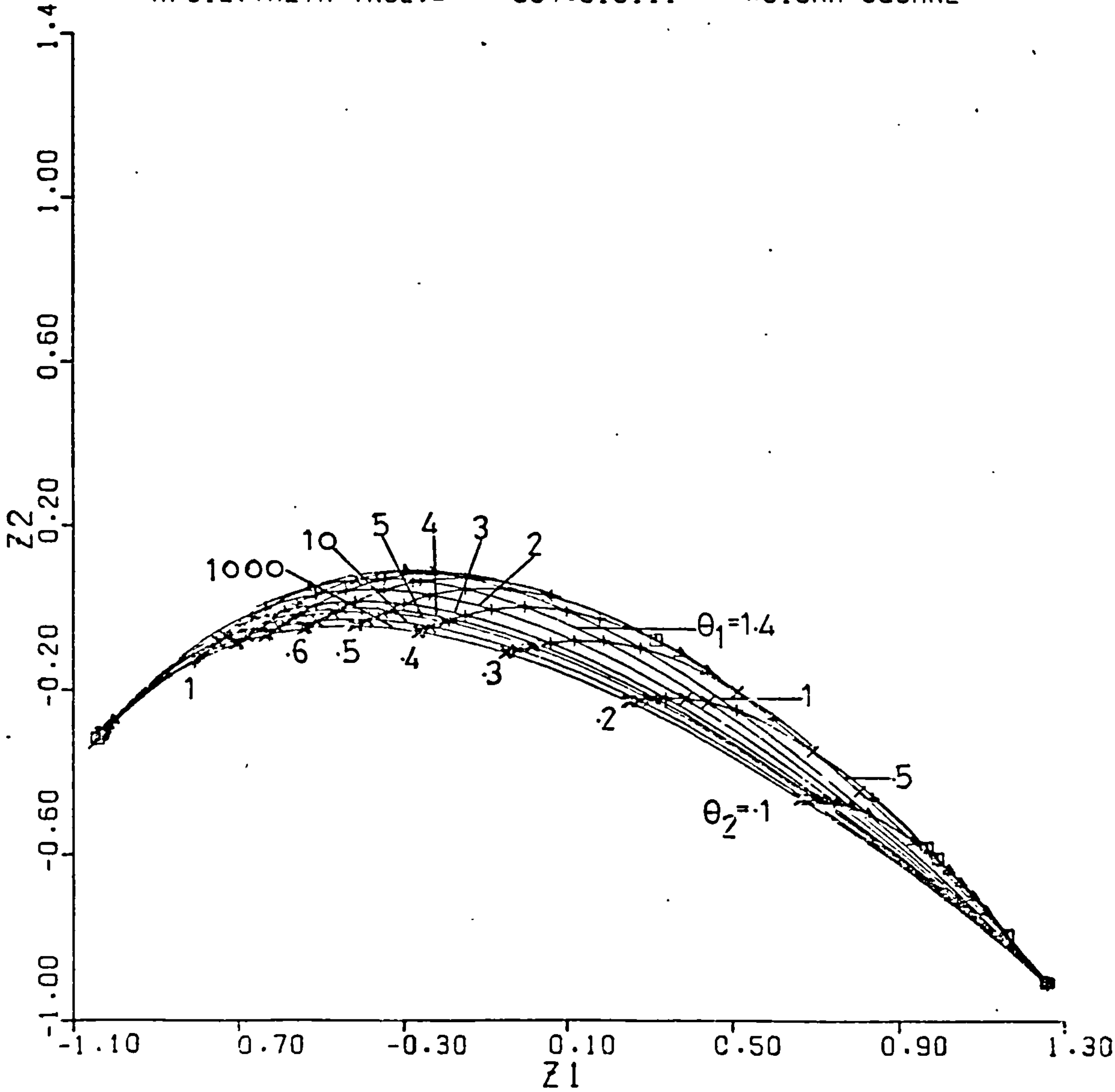
R=ABSOLUTE VALUE OF(MPSI2(THETA)-MPSI2(THETA TRUE))/MPSI2(THETA TRUE))
PSI2 IS POWER TRANSFORMATION BASED ON METHOD 2

+:0≤R≤0.5 ;X:0.5 <R≤1 :△:1 <R≤10 :□:10 <R≤100:✡:R>100

FIGURE (5.2.24)
ROTATED COORDINATES OF POINTS IN SOLUTION LOCUS
MODEL IS
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1.2.3.4.5.6

THETAI TRUE ARE 1.4000 0.4000

MPSI2(THETA TRUE)= 384.619111 *SIGMA SQUARE



R=ABSOLUTE VALUE OF((MPSI2(THETA)-MPSI2(THETA TRUE))/MPSI2(THETA TRUE))
PSI2 IS POWER TRANSFORMATION BASED ON METHOD 3

+:0≤R≤0.5 ;X:0.5 <R≤1 ;△:1 <R≤10 ;□:10 <R≤100;✿:R>100

Section 5.3 Region estimates of $\underset{\sim}{\theta}$

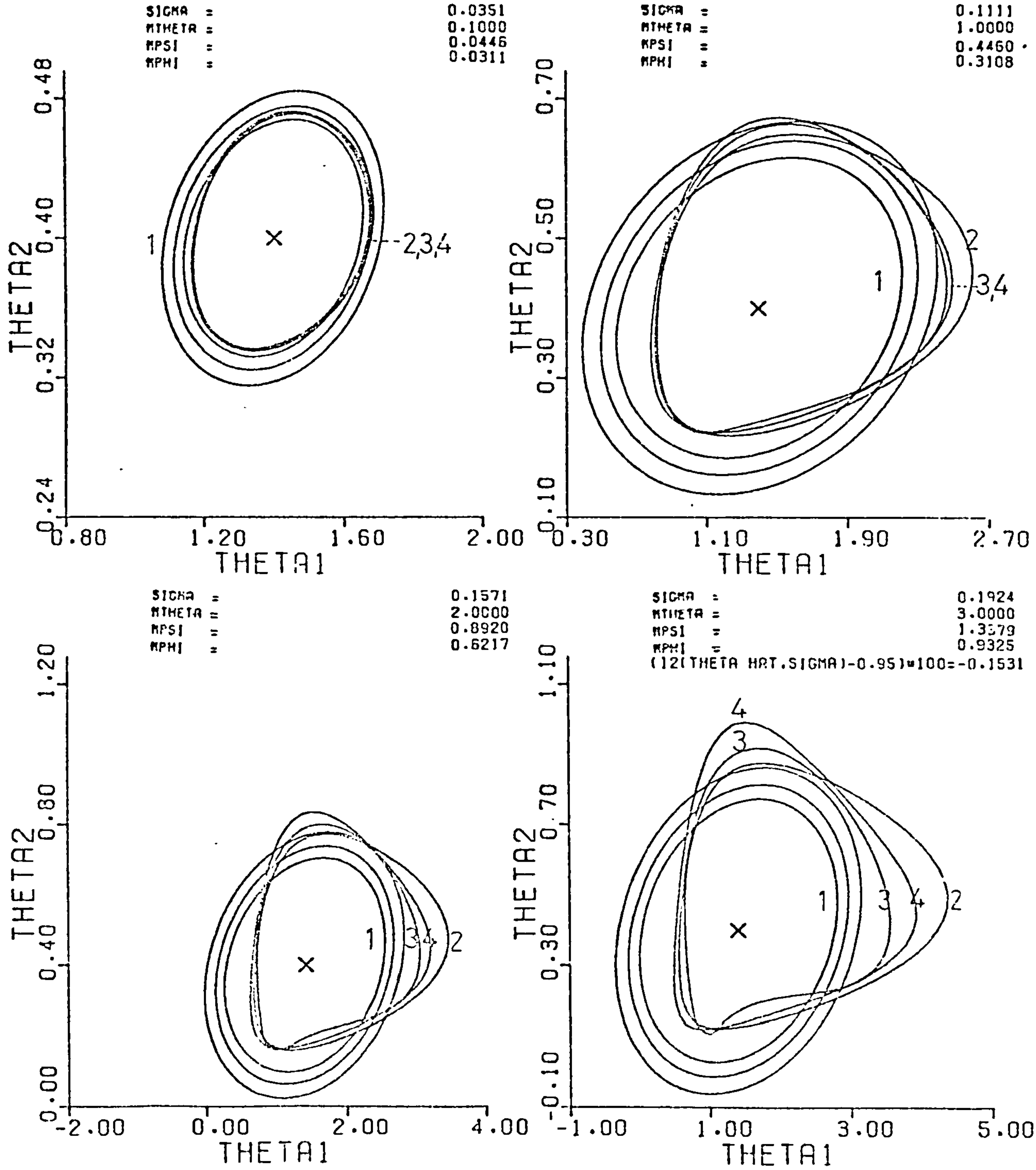
In this section we compare the boundaries of the region estimates given by the four methods. The levels of σ are set to be sufficiently small to ensure that models (A) and (B) with $\underset{\sim}{\theta}_T = (1.4,0.4)^T$ can be treated as unconstrained models for statistical purposes. For model (A), we consider three values of $\hat{\underset{\sim}{\theta}}$, namely $(1.4, 0.4)^T$, $(2.0,0.2)^T$ and $(1.0,0.8)^T$, while for model (B), we consider $(1.4,0.4)^T$, $(2.0,0.4)^T$ and $(1.0,0.35)^T$. For a value of $\hat{\underset{\sim}{\theta}}$, we find two observations $\underset{\sim}{y}_i$ $(i = 1,2)$ such that the rotated coordinates are

$$\underset{\sim}{z}_i = (0,0,\frac{s_i}{2},\frac{s_i}{2},\frac{s_i}{2},\frac{s_i}{2})^T,$$

where $s_1 = \sqrt{0.01}$, $s_2 = \sqrt{0.02}$ in model (A), and $s_1 = \sqrt{0.0001}$, $s_2 = \sqrt{0.0002}$ in model (B). With the $\underset{\sim}{y}_i$, this value of $\hat{\underset{\sim}{\theta}}$ is a least squares estimate of $\underset{\sim}{\theta}$ and the residual sums of squares are s_i^2. We then set $\alpha = 0.05$ and apply the various methods to obtain region estimates of $\underset{\sim}{\theta}$. The boundaries of these regions are shown in Fig. (5.3.1)-(5.3.12).

FIGURE (5.3.1)
REGION ESTIMATES IN THE MODEL
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

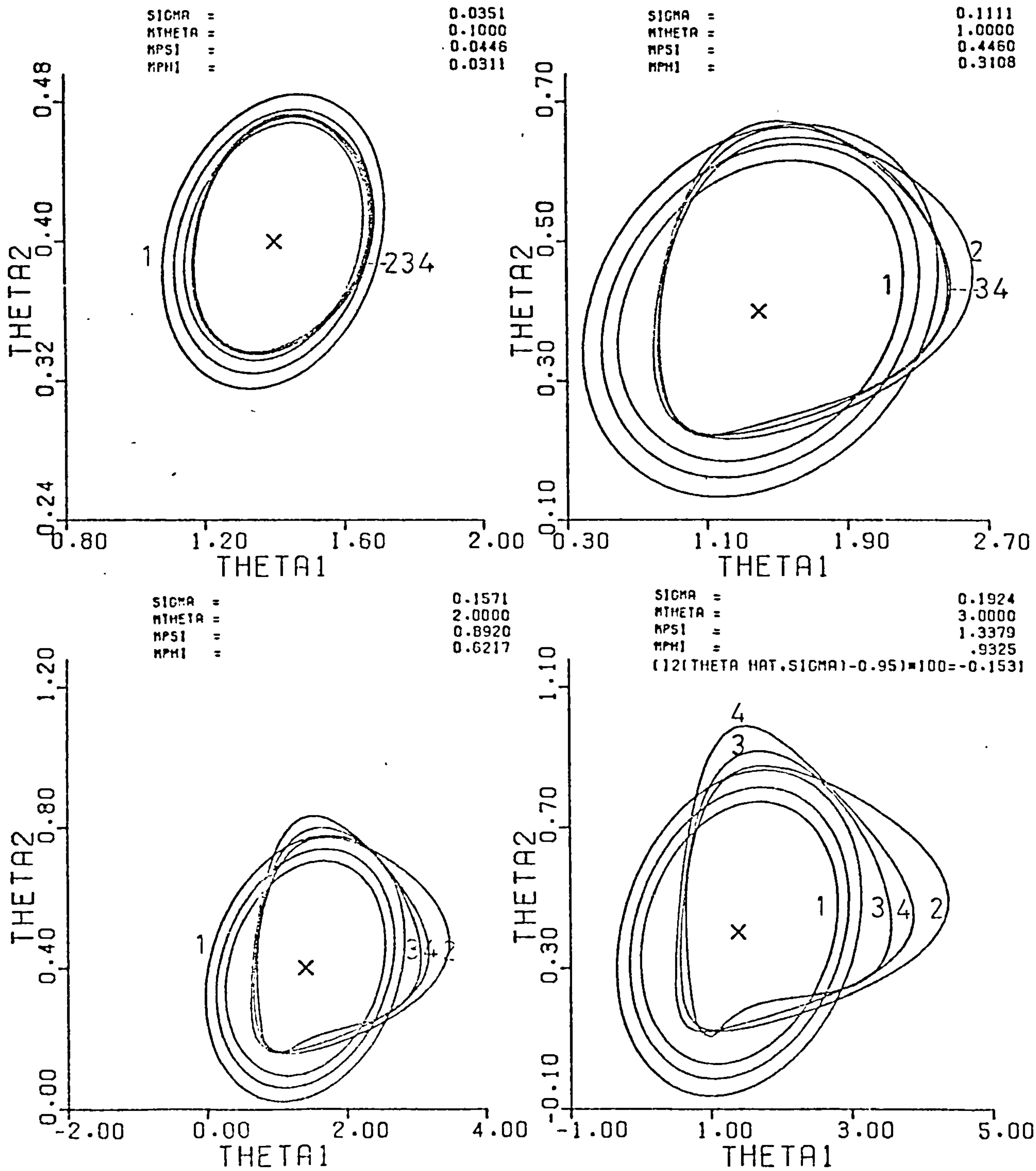
XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETA1 HAT ARE 1.4000 0.4000
RESIDUAL SUM OF SQUARES = 0.0100



(1) : REGION ESTIMATES (NOMINAL 95,97.5,99 PERCENT) GIVEN BY METHOD 1
(I) : REGION ESTIMATES (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=2,3,4)

FIGURE (5.3.2)
REGION ESTIMATES IN THE MODEL
E(Y)=(THETA1/(THETA1-THETA2))
    *(EXP(-THETA2*XI)-EXP(-THETA1*XI))

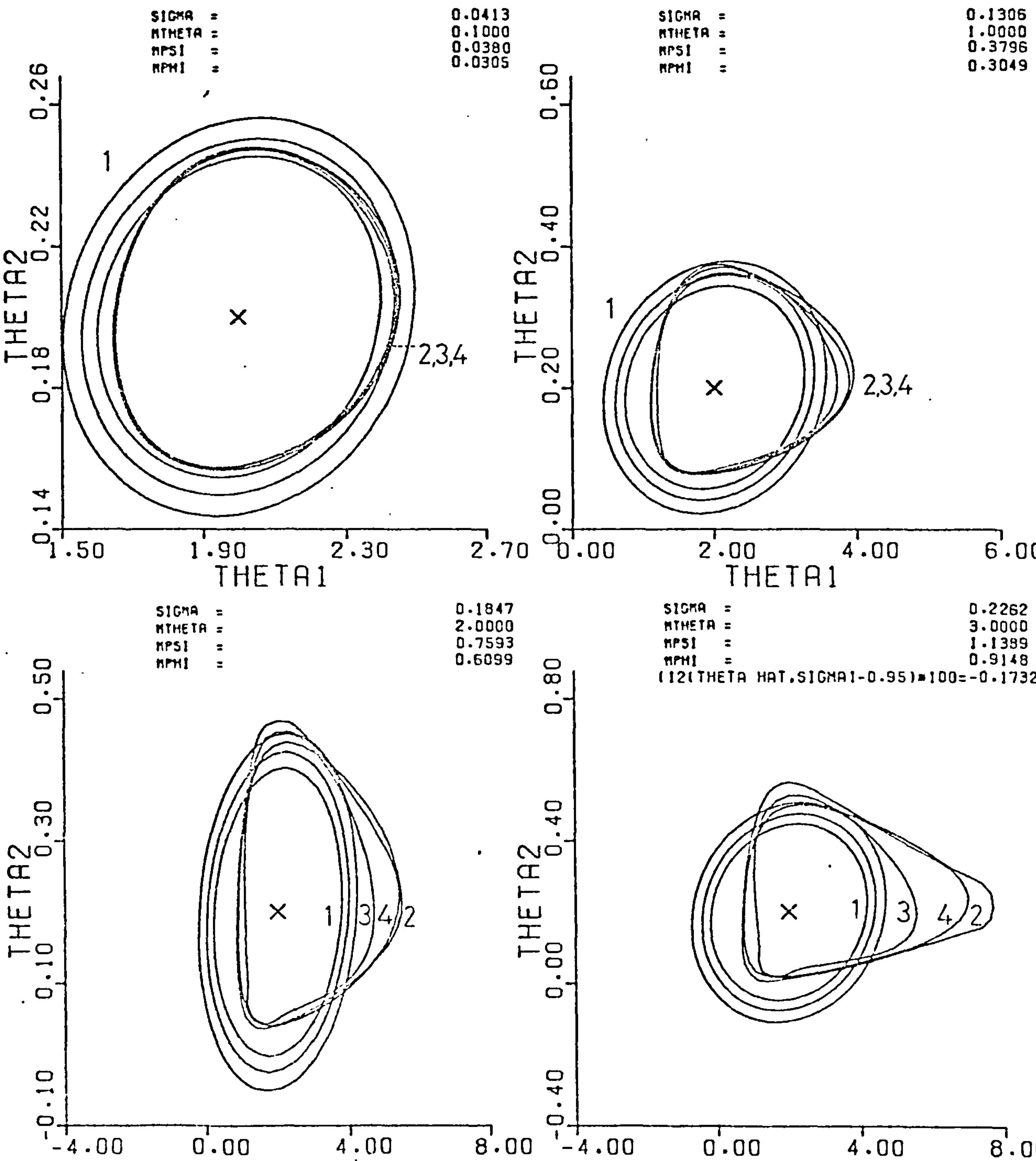
XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETA HAT ARE 1.4000   0.4000
RESIDUAL SUM OF SQUARES = 0.0200



(1) : REGION ESTIMATES (NOMINAL 95,97.5,99 PERCENT) GIVEN BY METHOD 1
(I) : REGION ESTIMATES (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=2,3,4)

FIGURE (5.3.3)
REGION ESTIMATES IN THE MODEL
E(Y)=(THETA1/(THETA1-THETA2))
    *(EXP(-THETA2*XI)-EXP(-THETA1*XI))

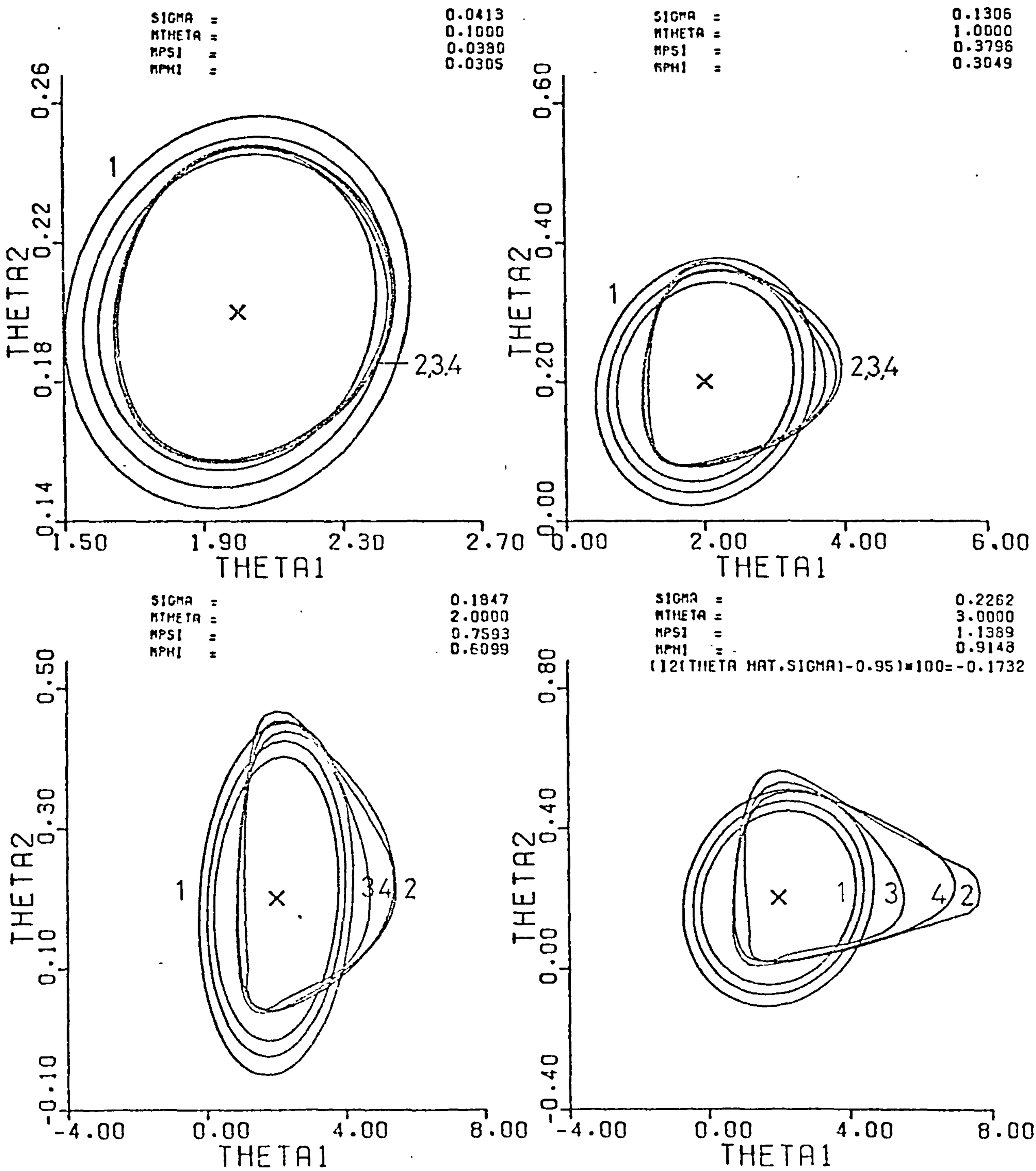
XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETAI HAT ARE 2.0000   0.2000
RESIDUAL SUM OF SQUARES = 0.0100



(1) : REGION ESTIMATES (NOMINAL 95,97.5,99 PERCENT) GIVEN BY METHOD 1
(I) : REGION ESTIMATES (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=2,3,4)

FIGURE (5.3.4)
REGION ESTIMATES IN THE MODEL
E(Y)=(THETA1/(THETA1-THETA2))
    *(EXP(-THETA2*XI)-EXP(-THETA1*XI))

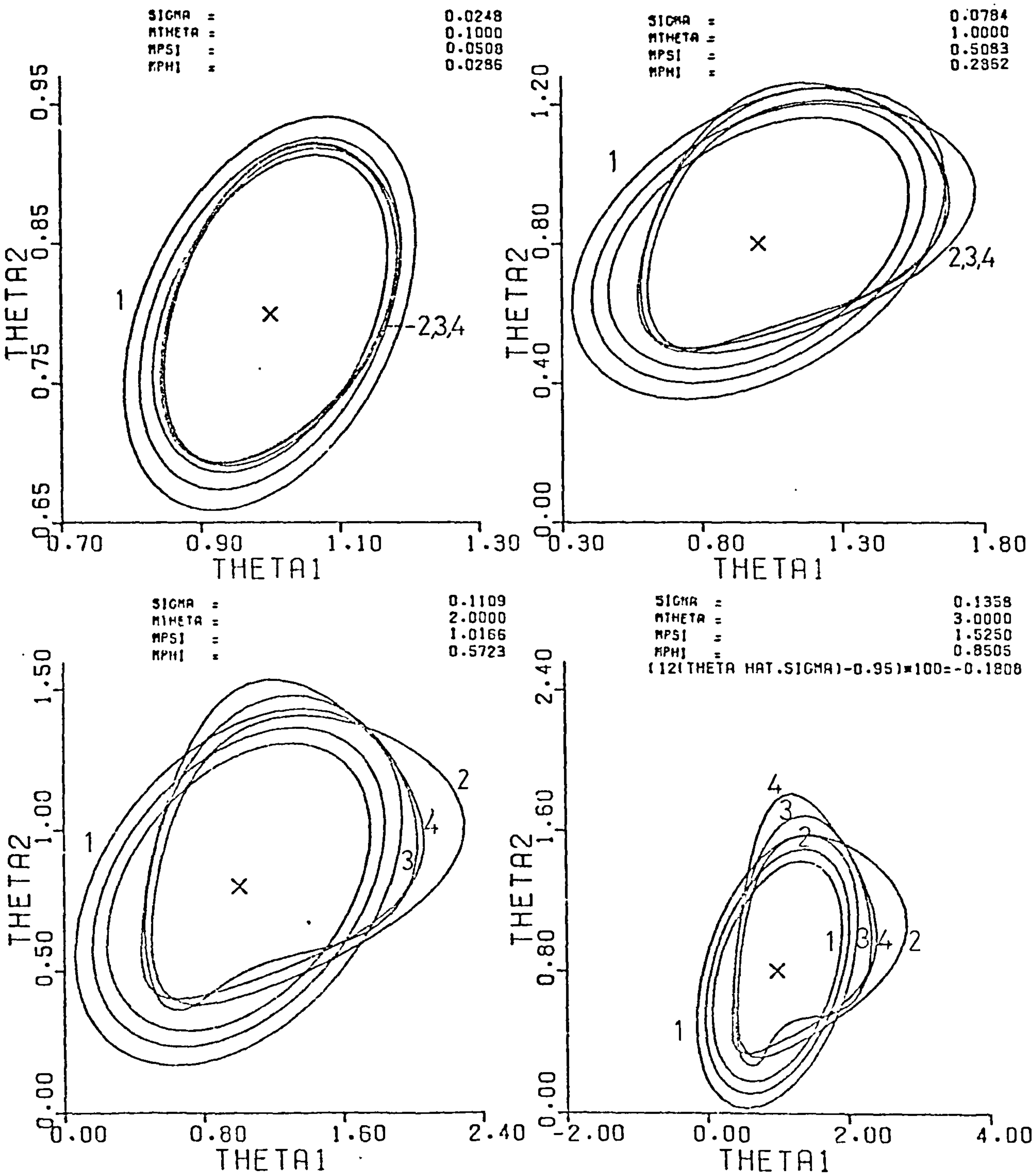
XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETA1 HAT ARE 2.0000   0.2000
RESIDUAL SUM OF SQUARES = 0.0200



(1) : REGION ESTIMATES (NOMINAL 95,97.5,99 PERCENT) GIVEN BY METHOD 1
(I) : REGION ESTIMATES (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=2,3,4)

FIGURE (5.3.5)
REGION ESTIMATES IN THE MODEL
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

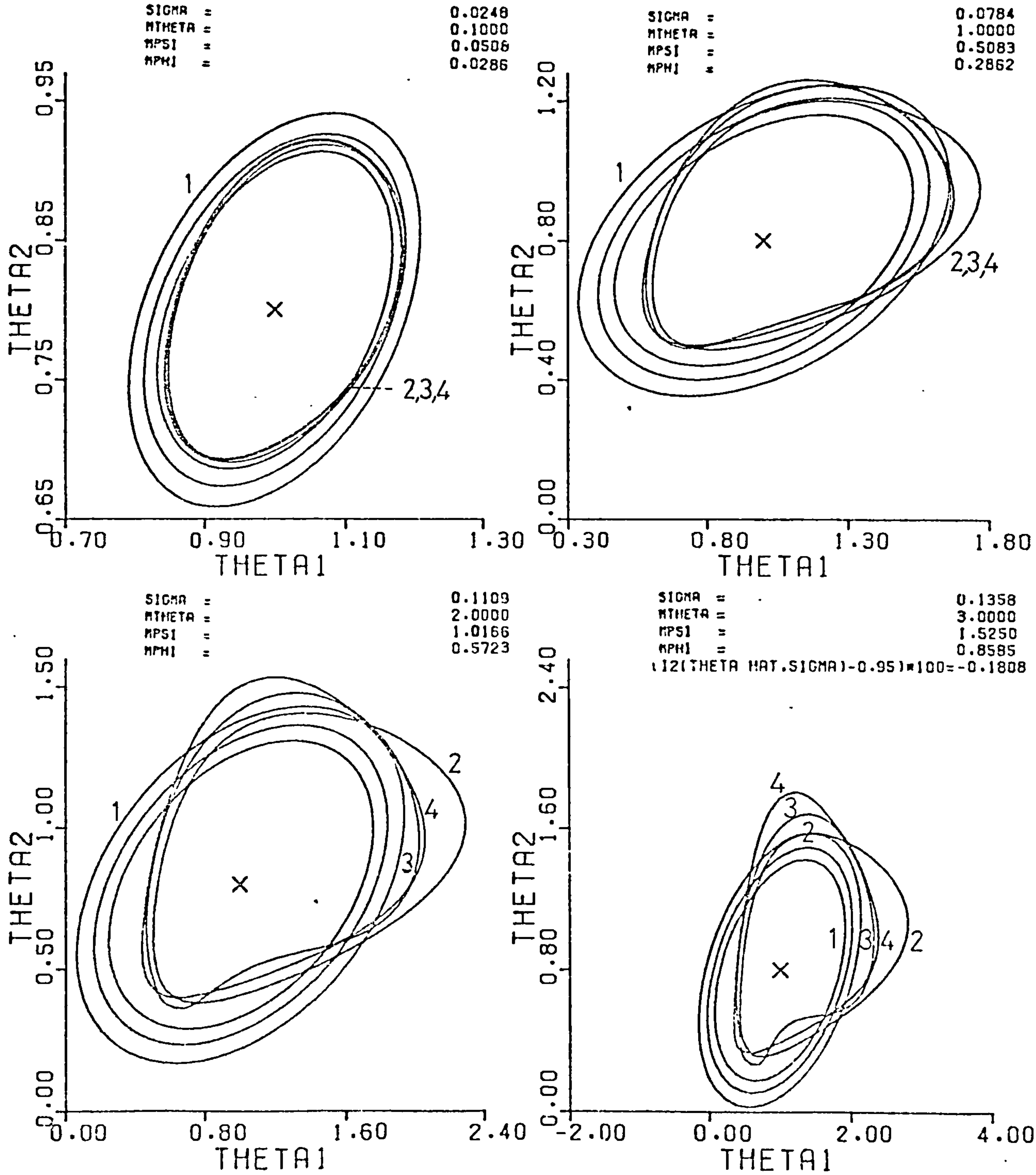
XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETAI HAT ARE 1.0000 0.8000
RESIDUAL SUM OF SQUARES = 0.0100



(1) : REGION ESTIMATES (NOMINAL 95.97.5.99 PERCENT) GIVEN BY METHOD 1
(I) : REGION ESTIMATES (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=2.3.4)

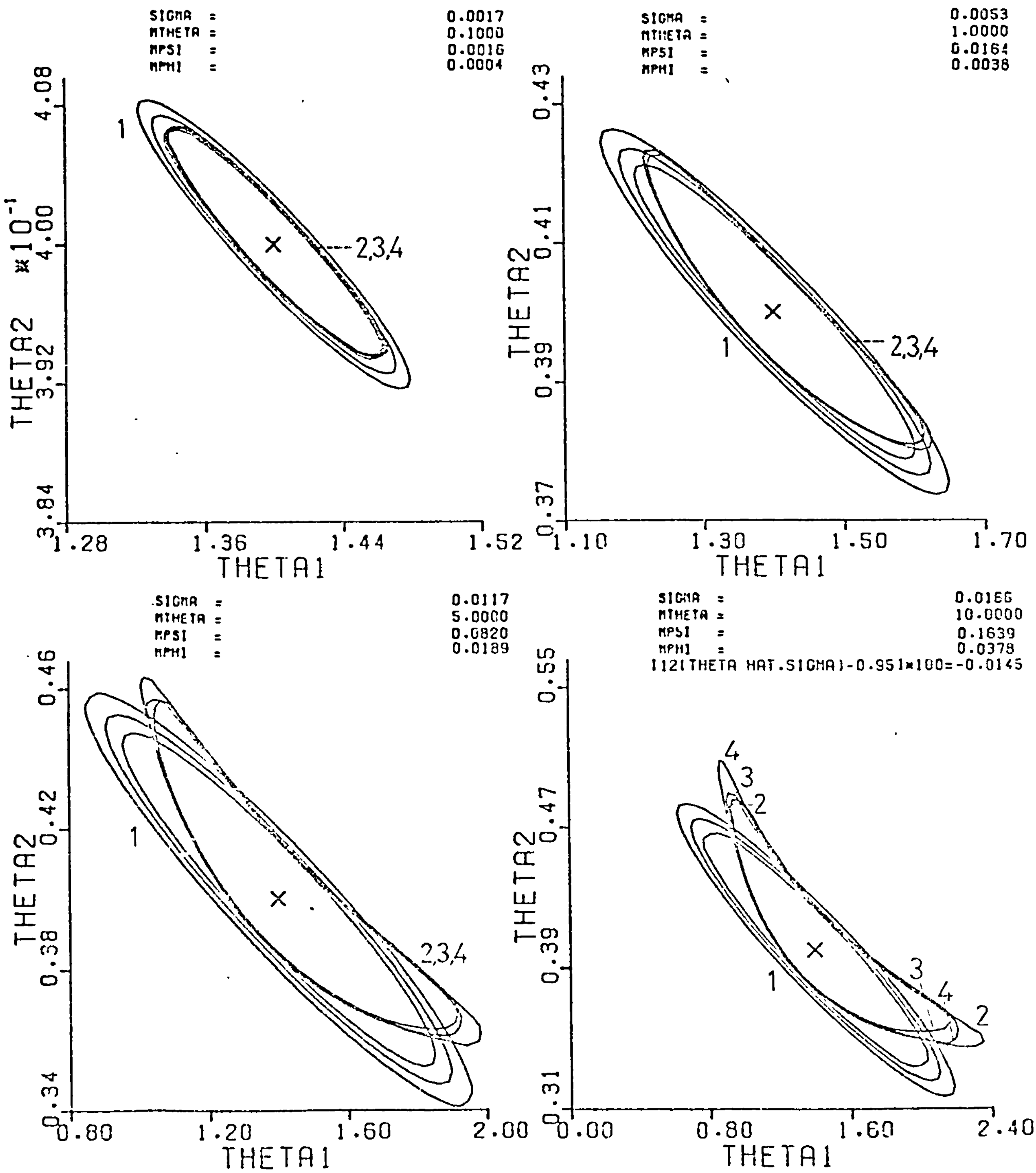
FIGURE (5.3.6)
REGION ESTIMATES IN THE MODEL
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI= 0.25.0.5.1.0.1.5.2.0.4.0
THETAI HAT ARE 1.0000 0.8000
RESIDUAL SUM OF SQUARES = 0.0200



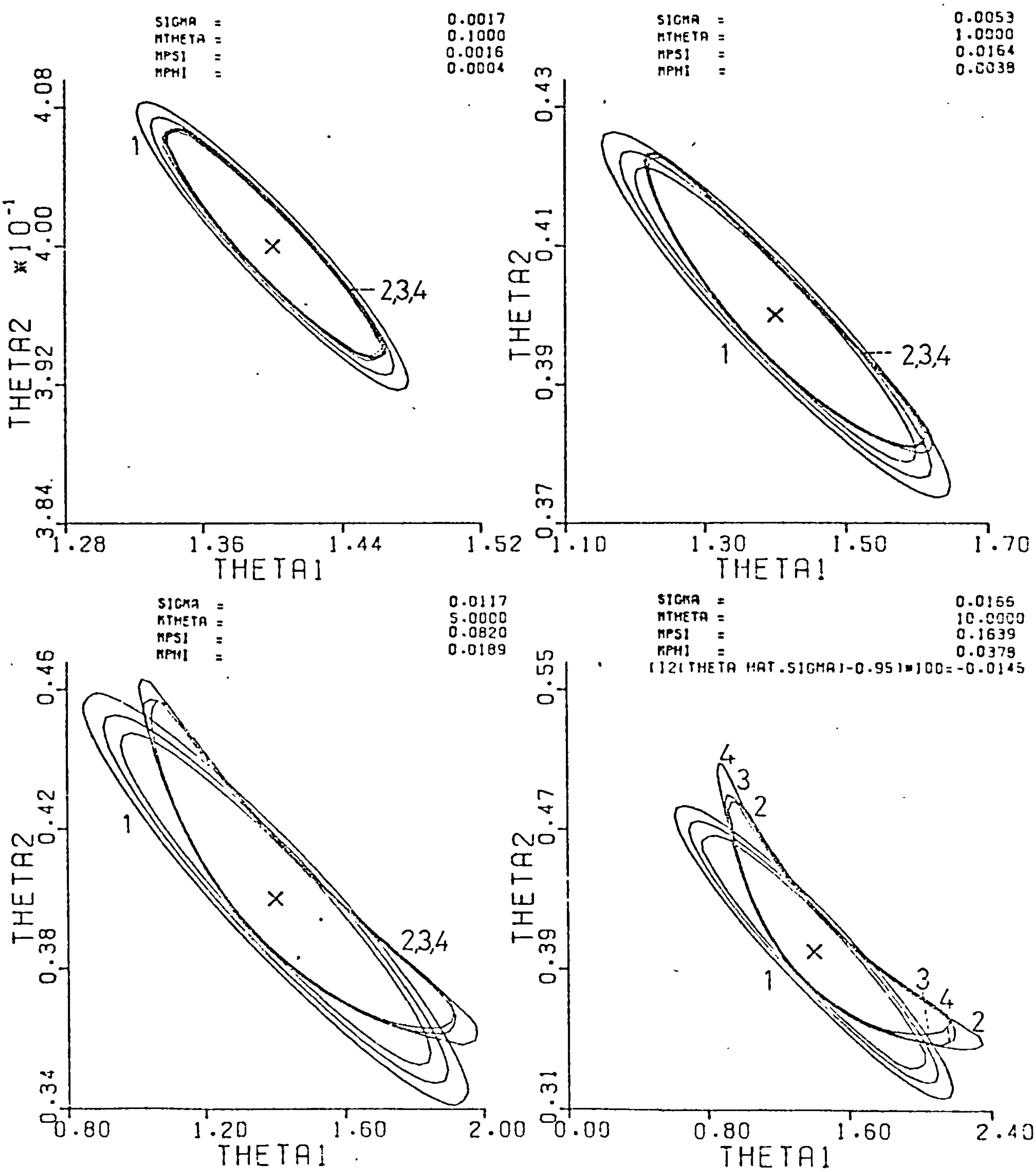
(1) : REGION ESTIMATES (NOMINAL 95.97.5.99 PERCENT) GIVEN BY METHOD 1
(I) : REGION ESTIMATES (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=2.3.4)

FIGURE (5.3.7)
REGION ESTIMATES IN THE MODEL
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1.2,3,4.5.6
THETAI HAT ARE 1.4000 0.4000
RESIDUAL SUM OF SQUARES = 0.0001



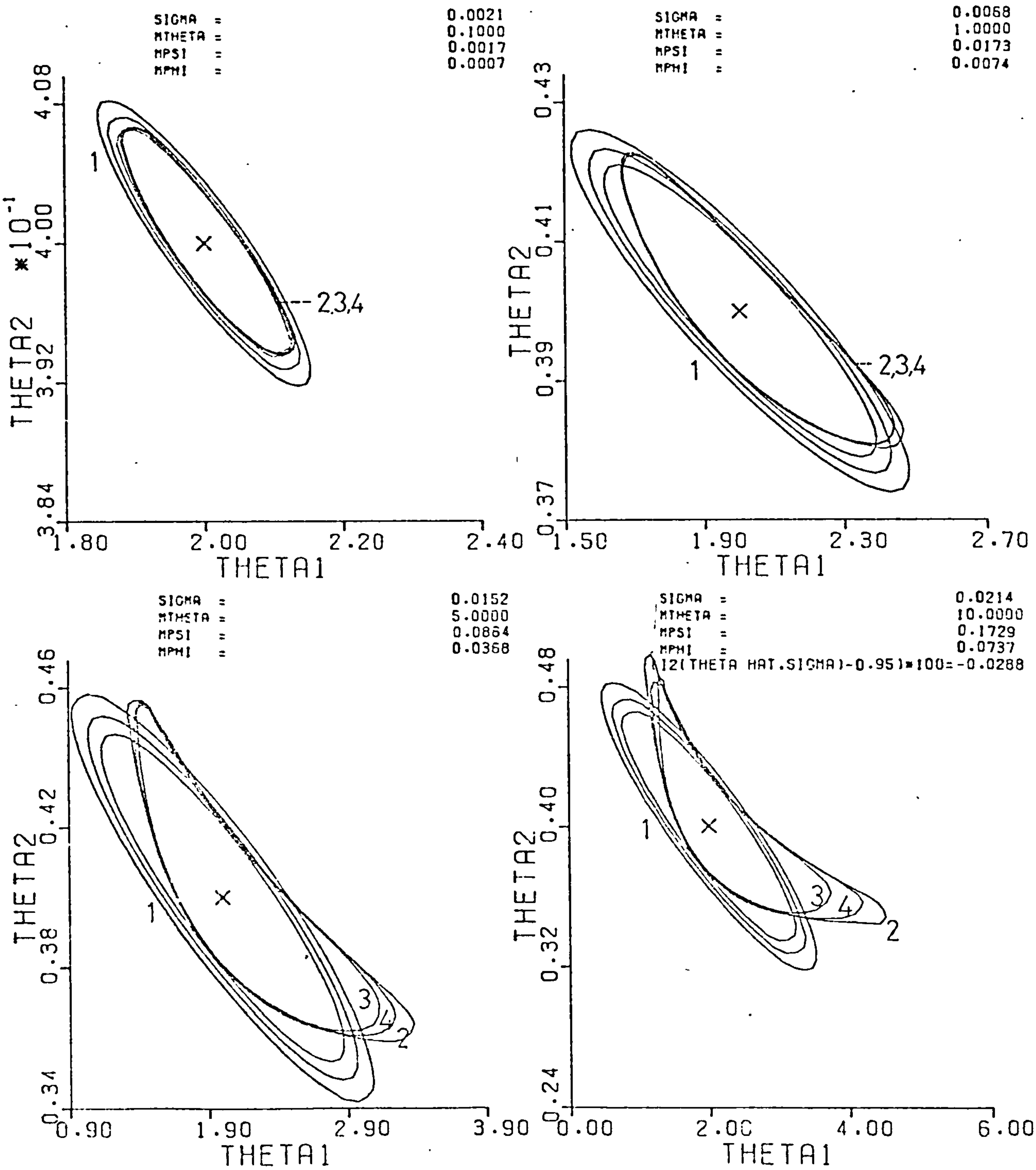
(1) : REGION ESTIMATES (NOMINAL 95.97.5.99 PERCENT) GIVEN BY METHOD 1
(I) : REGION ESTIMATES (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=2.3.4)

FIGURE (5.3.8)
REGION ESTIMATES IN THE MODEL
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1.2.3.4.5.6
THETA1 HAT ARE 1.4000 0.4000
RESIDUAL SUM OF SQUARES = 0.0002



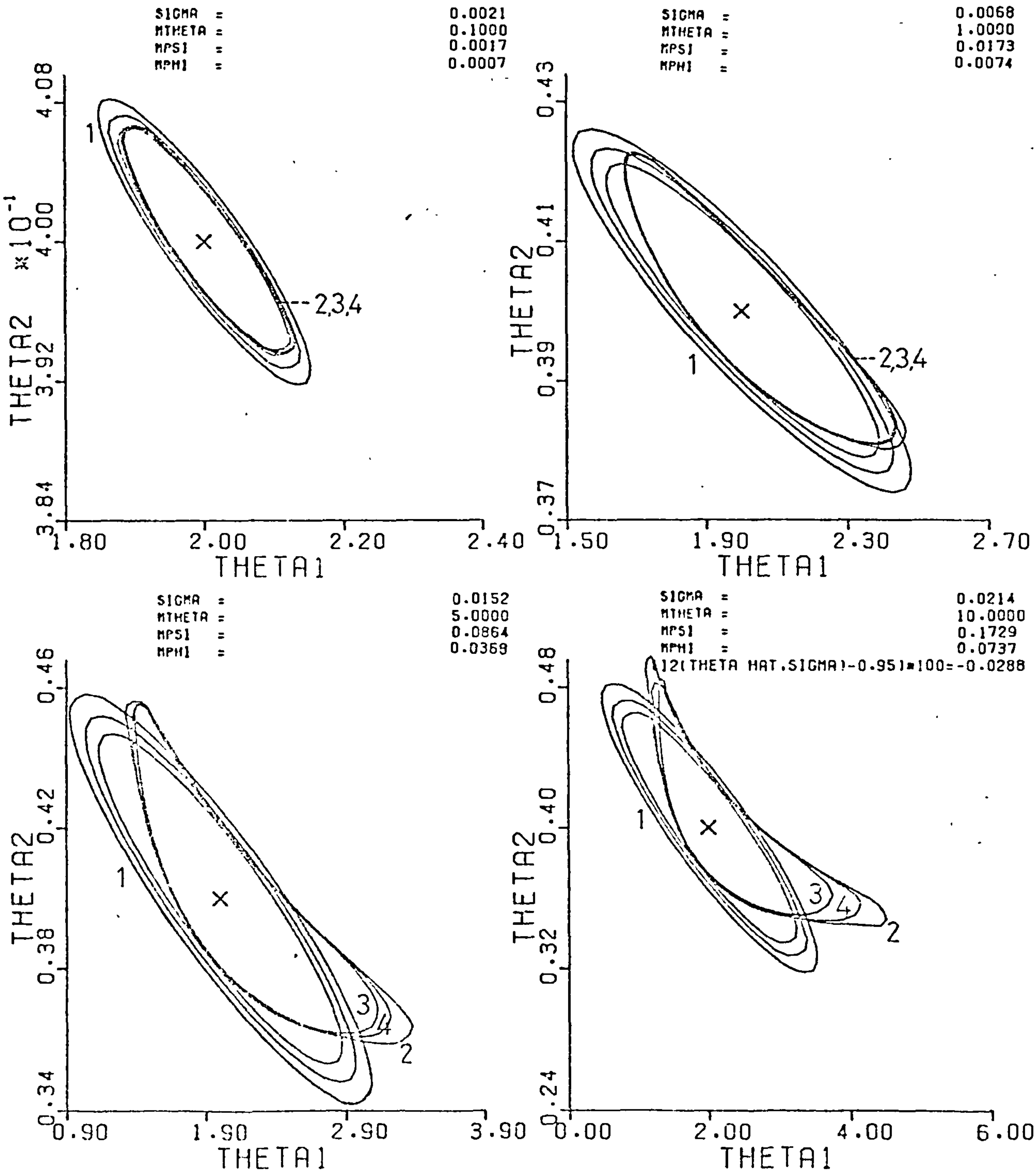
(1) : REGION ESTIMATES (NOMINAL 95.97.5.99 PERCENT) GIVEN BY METHOD 1
(I) : REGION ESTIMATES (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=2.3.4)

FIGURE (5.3.9)
REGION ESTIMATES IN THE MODEL
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1.2.3.4.5.6
THETA1 HAT ARE 2.0000 0.4000
RESIDUAL SUM OF SQUARES = 0.0001



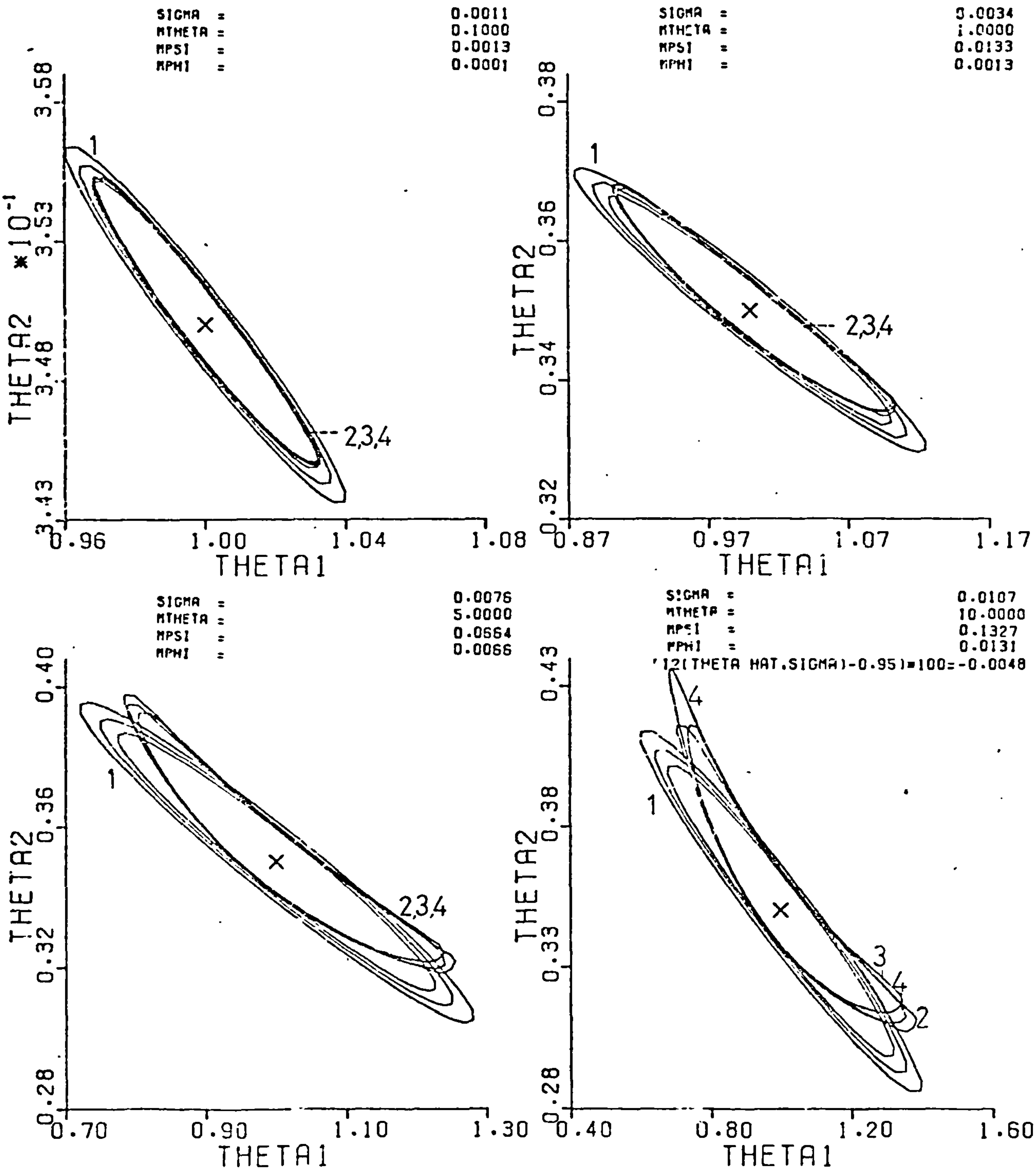
(1) : REGION ESTIMATES (NOMINAL 95.97.5.99 PERCENT) GIVEN BY METHOD 1
(I) : REGION ESTIMATES (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=2.3.4)

FIGURE (5.3.10)
REGION ESTIMATES IN THE MODEL
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1,2,3,4,5,6
THETA1 HAT ARE 2.0000 0.4000
RESIDUAL SUM OF SQUARES = 0.0002



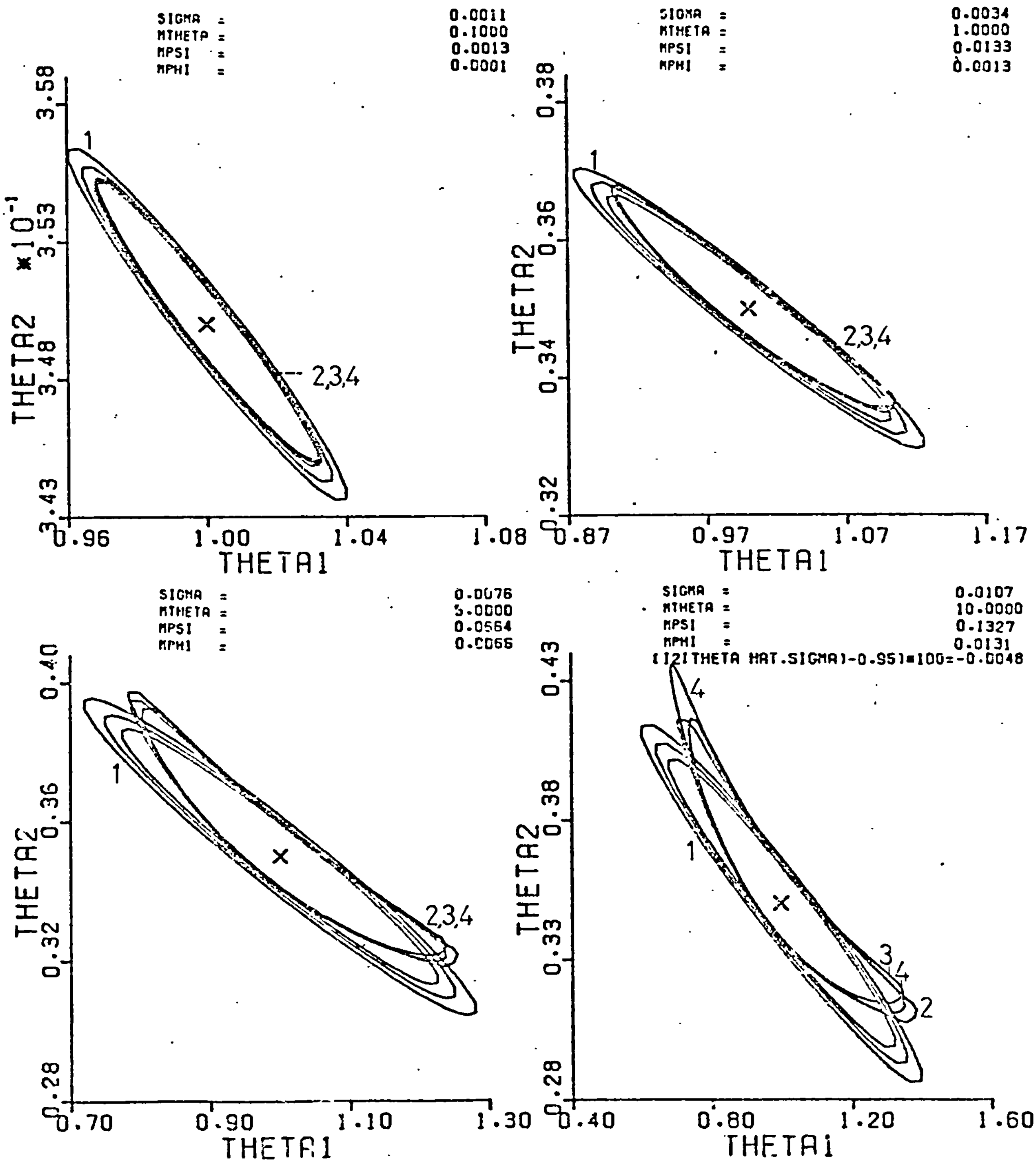
(1) : REGION ESTIMATES (NOMINAL 95.97.5.99 PERCENT) GIVEN BY METHOD 1
(I) : REGION ESTIMATES (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=2.3.4)

FIGURE (5.3.11)
REGION ESTIMATES IN THE MODEL
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1,2,3,4,5,6
THETAI HAT ARE 1.0000 0.3500
RESIDUAL SUM OF SQUARES = 0.0001



(1) : REGION ESTIMATES (NOMINAL 95,97.5,99 PERCENT) GIVEN BY METHOD I
(I) : REGION ESTIMATES (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=2,3,4)

FIGURE (5.3.12)
REGION ESTIMATES IN THE MODEL
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1.2.3.4.5.6
THETAI HAT ARE 1.0000 0.3500
RESIDUAL SUM OF SQUARES = 0.0002



(1) : REGION ESTIMATES (NOMINAL 95.97.5.99 PERCENT) GIVEN BY METHOD 1
(I) : REGION ESTIMATES (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=2.3.4)

We note that for each of the $\hat{\underset{\sim}{\theta}}$, M_θ is very much larger than $|J_2(\hat{\underset{\sim}{\theta}},\sigma)|$ while M_ψ and M_ϕ are fairly close to this number. Furthermore as shown in the figures, the order given by (5.2.7) with $\underset{\sim}{\theta}_A = \hat{\underset{\sim}{\theta}}$ is fairly well preserved in the closeness of the region estimates given by methods 1, 2 and 3 to that given by method 4. We also note that the change in the residual sum of squares of an observation from s_1^2 to s_2^2 has very slight effect on the region estimate given by method 4.

We next observe that at these levels of σ, the absolute values of the differences between $J_2(\hat{\underset{\sim}{\theta}},\sigma)$ and $J_2(\underset{\sim}{\theta}_T,\sigma)$ are small fractions of a percent. This implies that our choice of the values of σ, $\hat{\underset{\sim}{\theta}}$ and $\underset{\sim}{\theta}_T$ has resulted in situations in which we can refer to the method 4 region estimates based on these values of $\hat{\underset{\sim}{\theta}}$ as approximately 100 $I_2(\hat{\underset{\sim}{\theta}},\sigma)$% region estimates.

We further observe that the four methods give almost identical region estimates when $M_\theta = 0.1$. As the region estimates given by method 4 are approximately 95% region estimates, the region estimates given by methods 1, 2 and 3 are also approximately 95% region estimates.

When $M_\theta = 1$, methods 1 and 4 give slightly different region estimates, and whenever the corresponding M_ψ and M_ϕ are less than 0.1, methods 2, 3 and 4 give almost identical region estimates. This observation is consistent with the observation in the case when $M_\theta = 0.1$.

When $M_\theta > 1$, the region estimates given by methods 1 and 4 are fairly significantly different. The region estimates given by method 2 or 3 differ slightly from those given by method 4 whenever M_ψ or M_ϕ is less than 1. This observation is consistent with the observation in the case when $M_\theta = 1$.

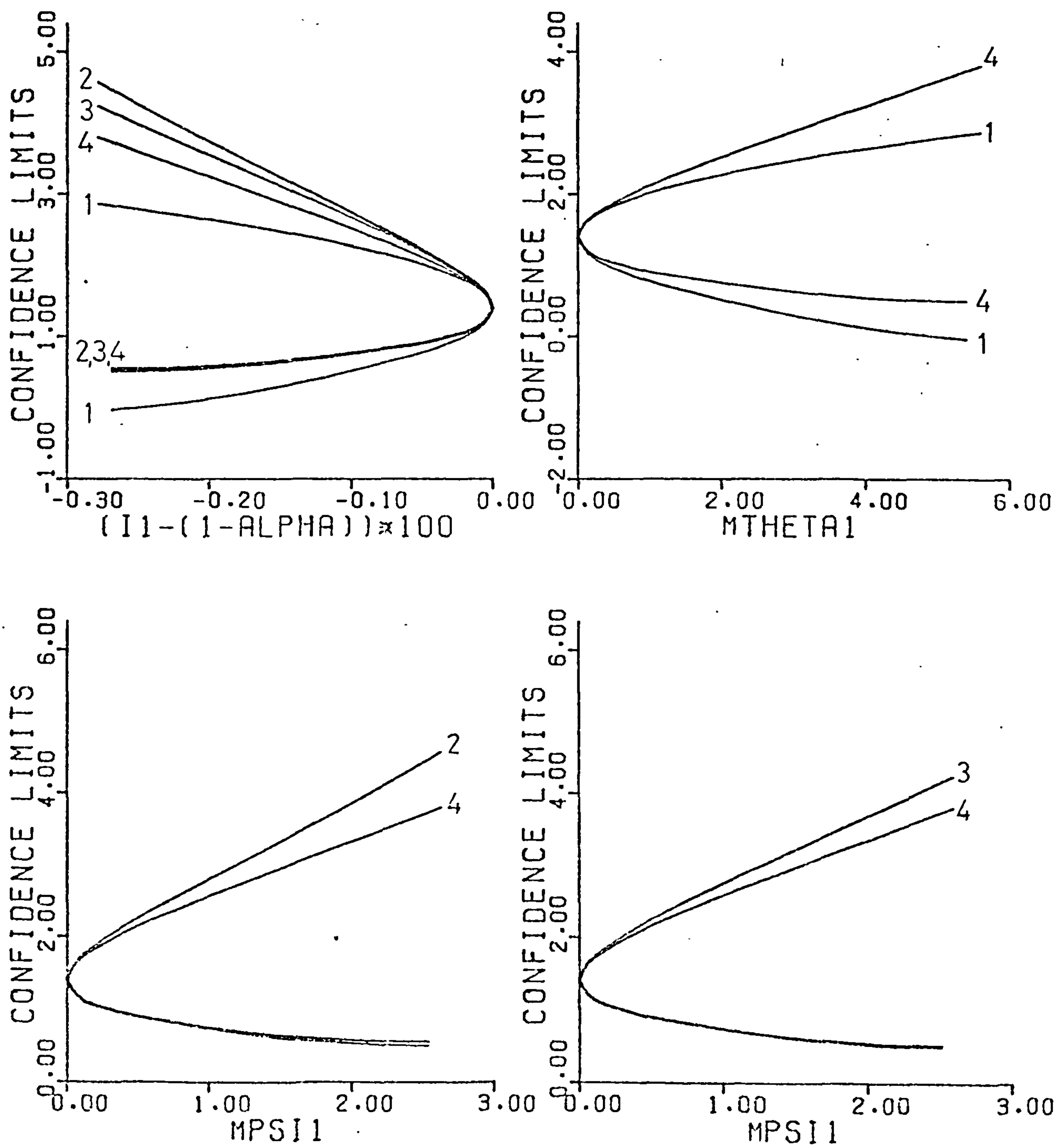
The above observations indicate that for these models with $\underset{\sim}{\theta}_T$ near $(1.4,0.4)^T$ and the levels of σ similar to those considered before, a region estimate of the parameter vector $\underset{\sim}{\theta}$ based on method 1, 2, or 3 is an approximately 95% region estimate provided that the value of the corresponding nonlinearity M_β, where $\beta = \theta$, ψ or ϕ, is less than or equal to 0.1.

Section 5.4 Interval estimates of θ_i

In this section we compare the limits of the interval estimates given by the four methods. The values of $\underset{\sim}{\theta}_T$, $\hat{\underset{\sim}{\theta}}$, $\underset{\sim}{y}_i$, s_i and α are chosen to be the same as in section 5.3. The various methods are then applied to obtain interval estimates of $\underset{\sim}{\theta}$. The limits of these intervals are shown in Fig. (5.4.1)-(5.4.24).

FIGURE (5.4.1)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

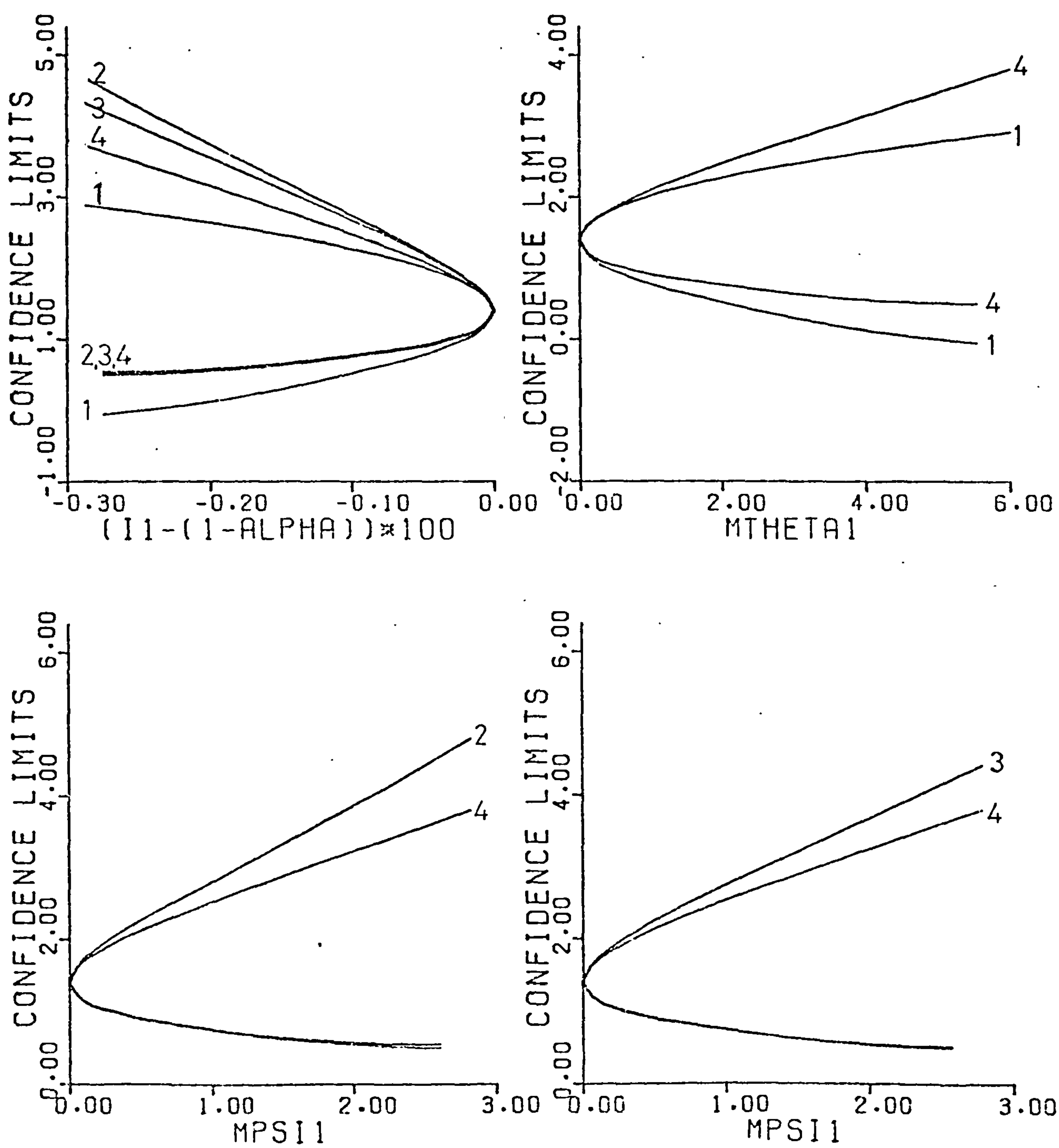
XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETAI HAT ARE 1.4000 0.4000
RESIDUAL SUM OF SQUARES = 0.1000



(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

FIGURE (5.4.2)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

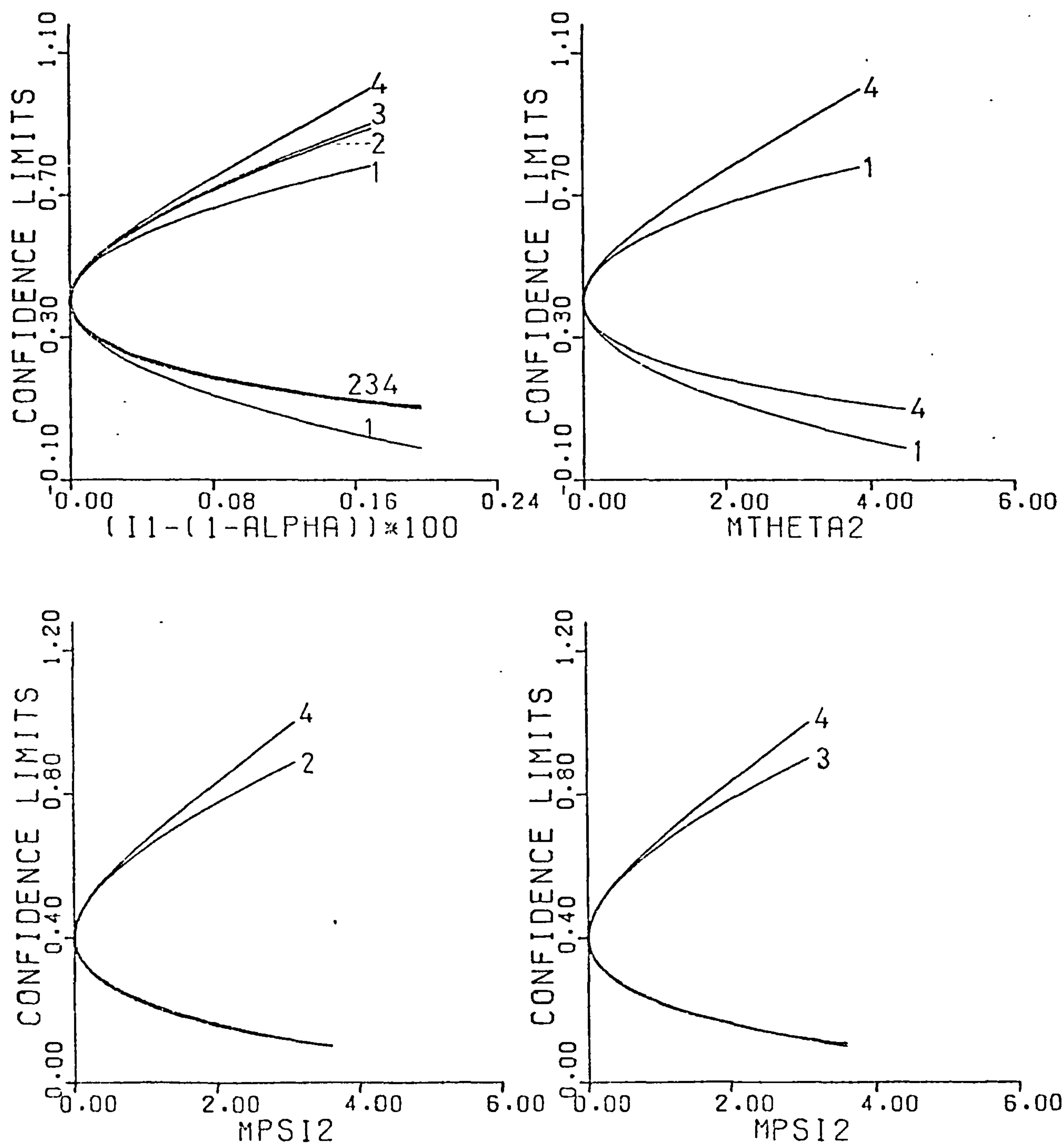
XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETAI HAT ARE 1.4000 0.4000
RESIDUAL SUM OF SQUARES = 0.2000



(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

FIGURE (5.4.3)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=(THETA1/(THETA1-THETA2))
    *(EXP(-THETA2*XI)-EXP(-THETA1*XI))

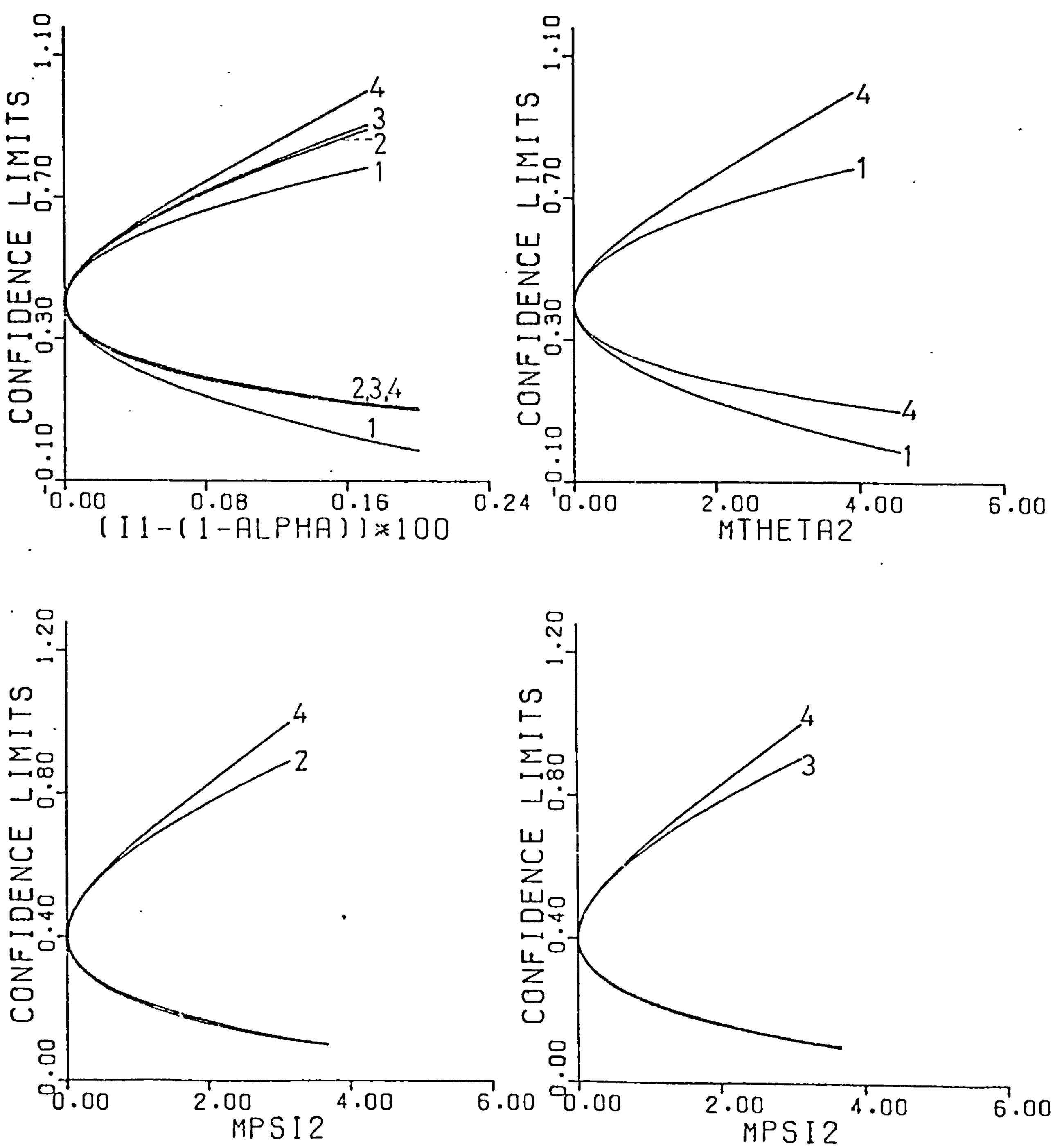
XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETAI HAT ARE 1.4000   0.4000
RESIDUAL SUM OF SQUARES = 0.1000



(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

FIGURE (5.4.4)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

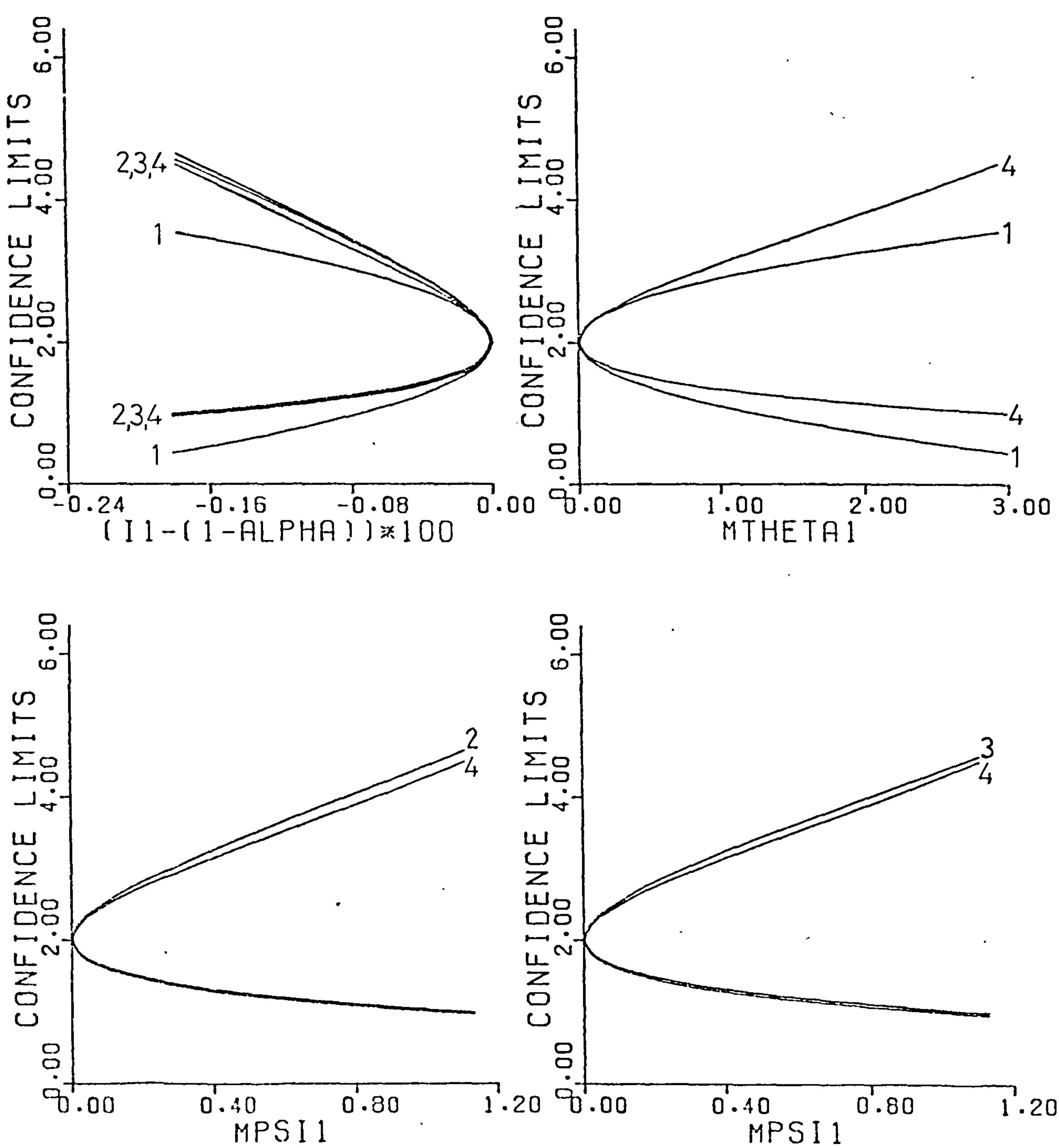
XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETA1 HAT ARE 1.4000 0.4000
RESIDUAL SUM OF SQUARES = 0.2000



(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

FIGURE (5.4.5)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

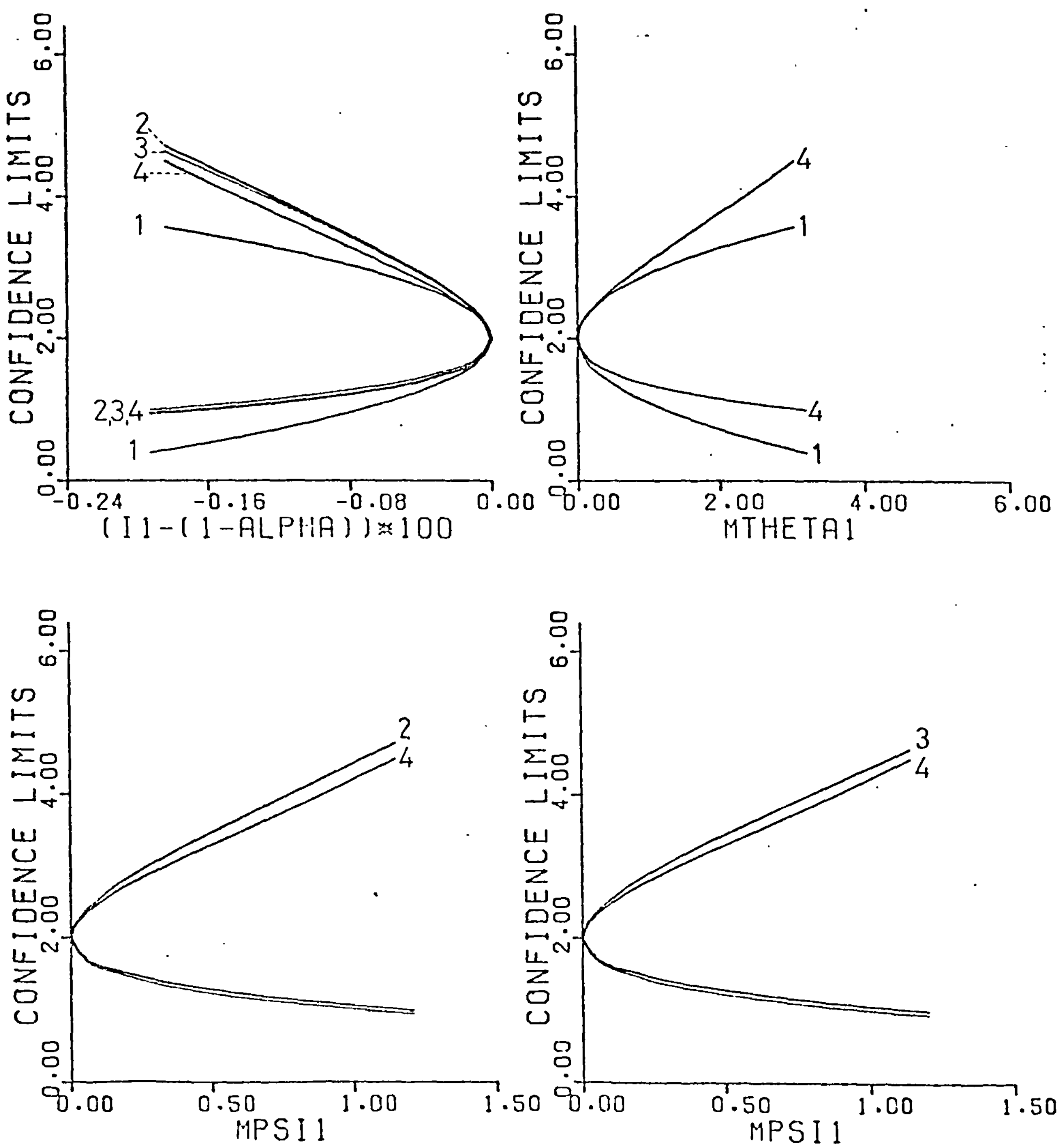
XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETAI HAT ARE 2.0000 0.2000
RESIDUAL SUM OF SQUARES = 0.1000



(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

FIGURE (5.4.6)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

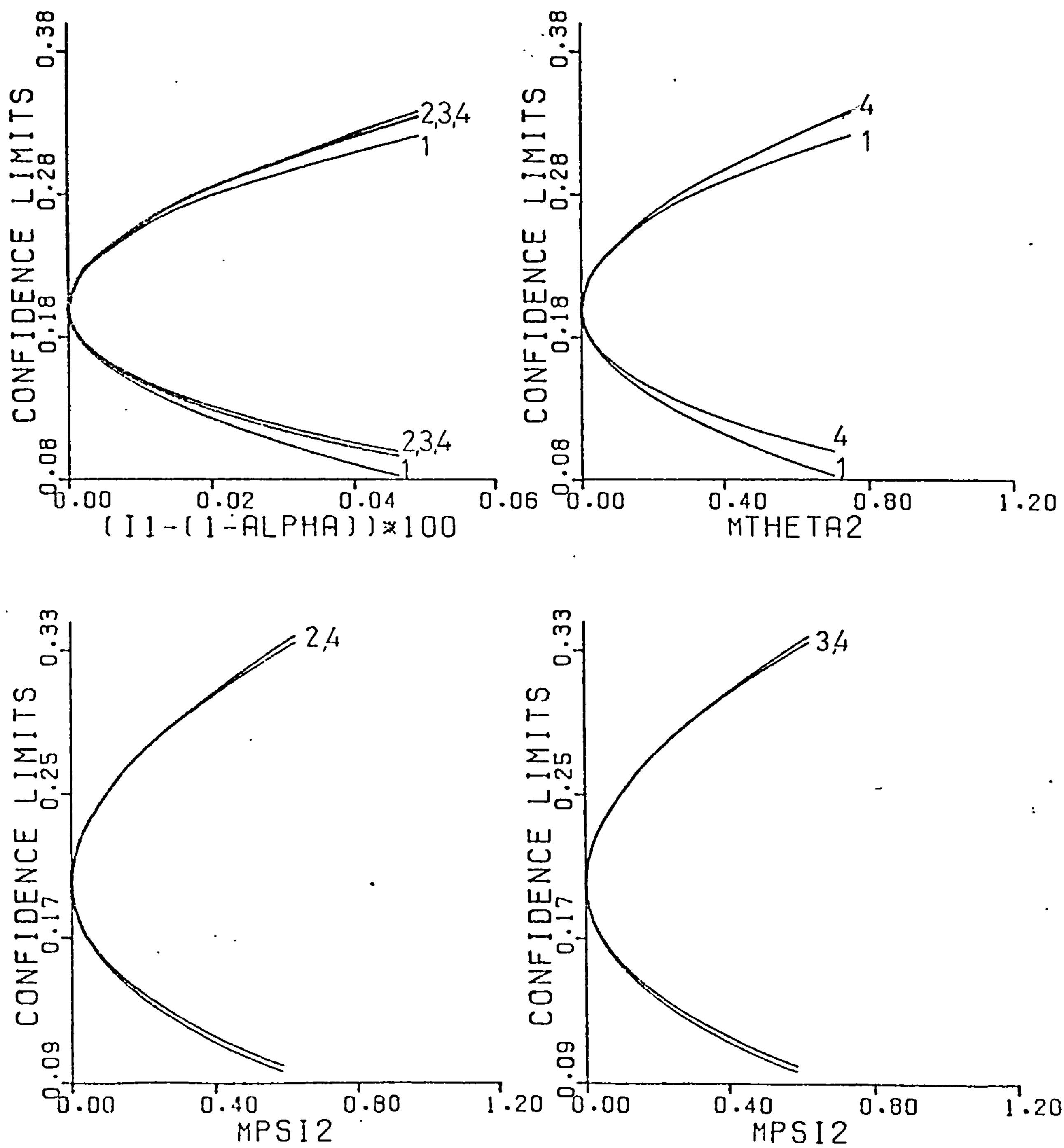
XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETAI HAT ARE 2.0000 0.2000
RESIDUAL SUM OF SQUARES = 0.2000



(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

FIGURE (5.4.7)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=(THETA1/(THETA1-THETA2))
    *(EXP(-THETA2*XI)-EXP(-THETA1*XI))

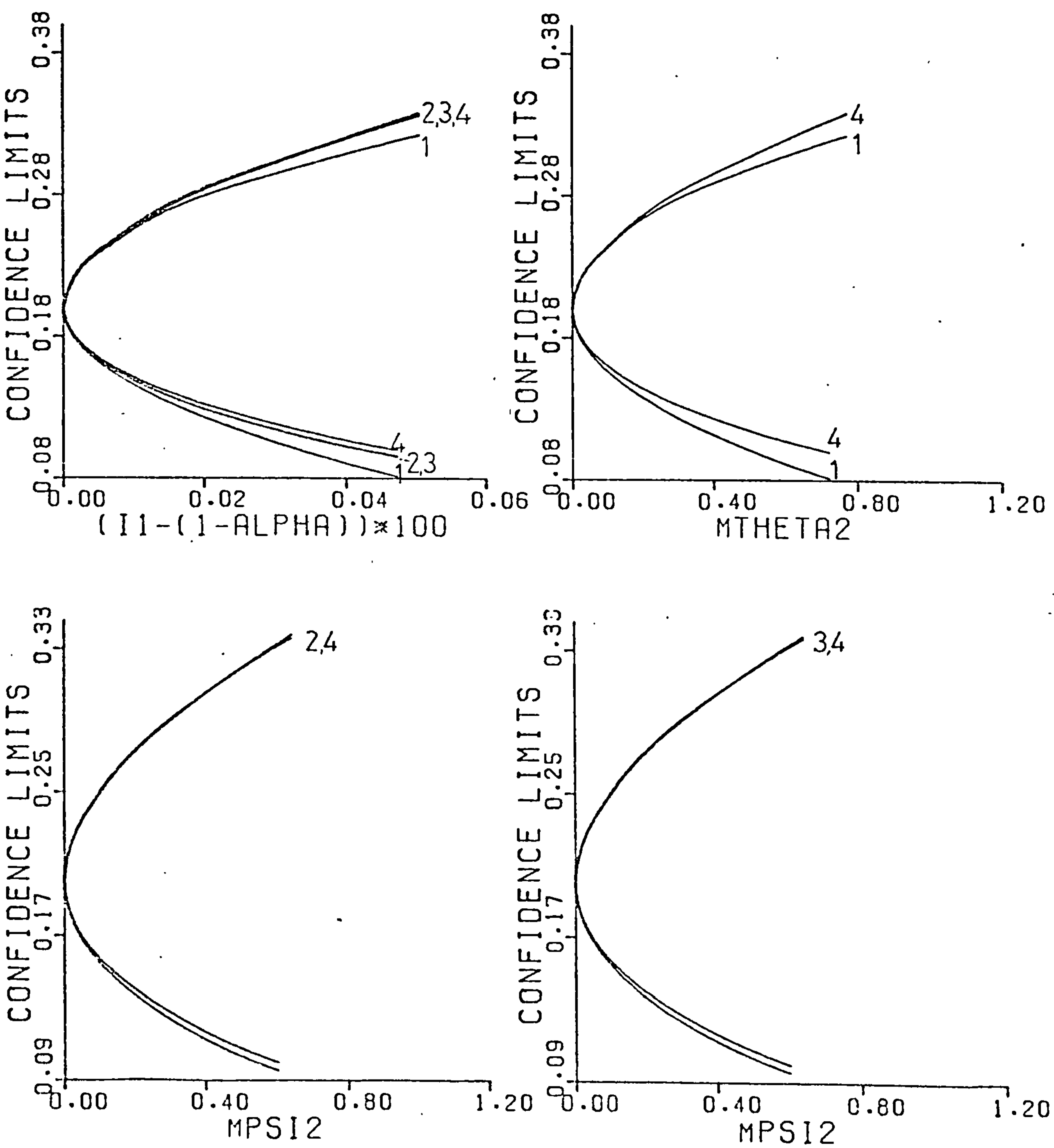
XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETAI HAT ARE 2.0000   0.2000
RESIDUAL SUM OF SQUARES = 0.1000



(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

FIGURE (5.4.8)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

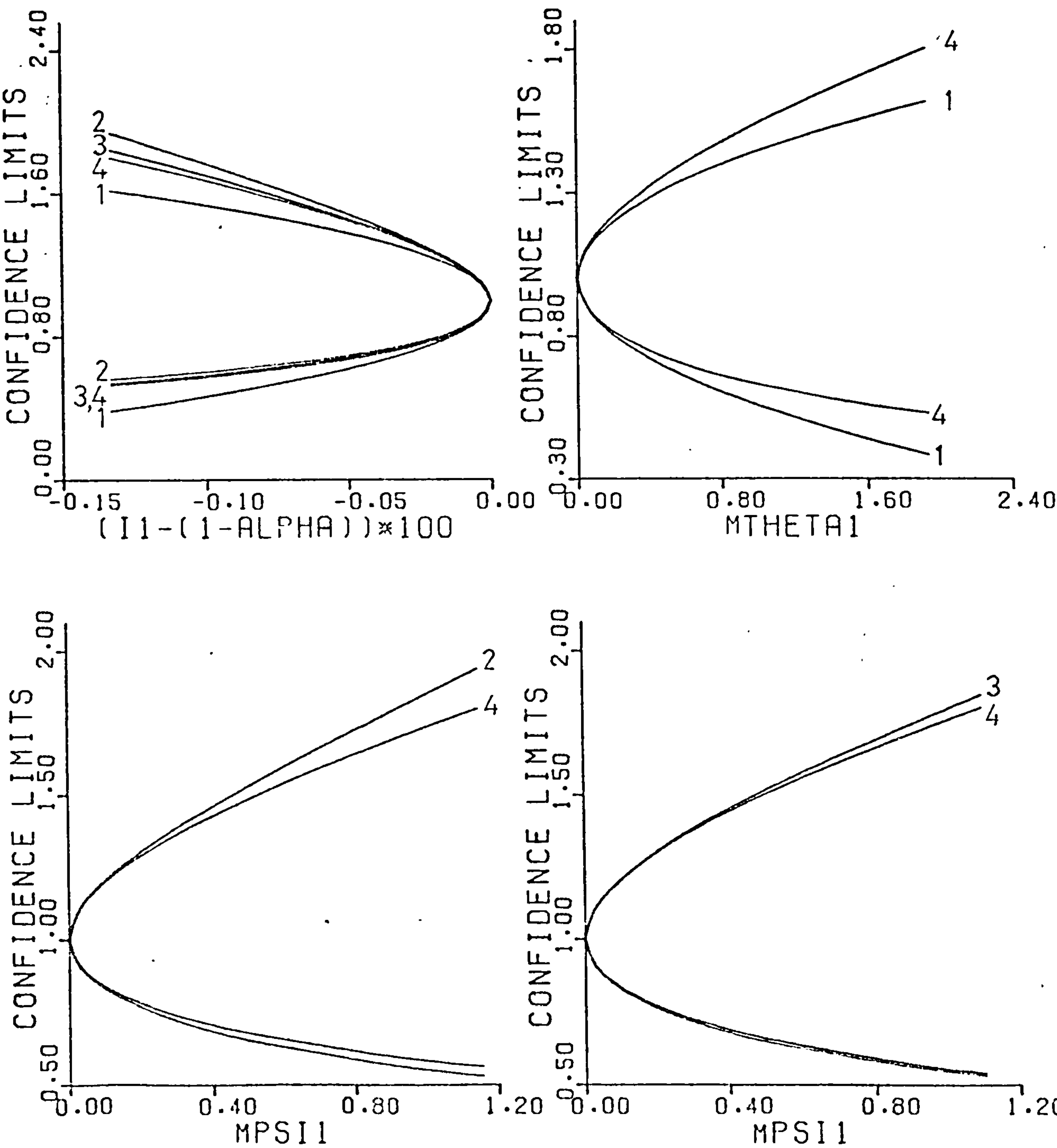
XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETAI HAT ARE 2.0000 0.2000
RESIDUAL SUM OF SQUARES = 0.2000



(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

FIGURE (5.4.9)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=(THETA1/(THETA1-THETA2))
    *(EXP(-THETA2*XI)-EXP(-THETA1*XI))

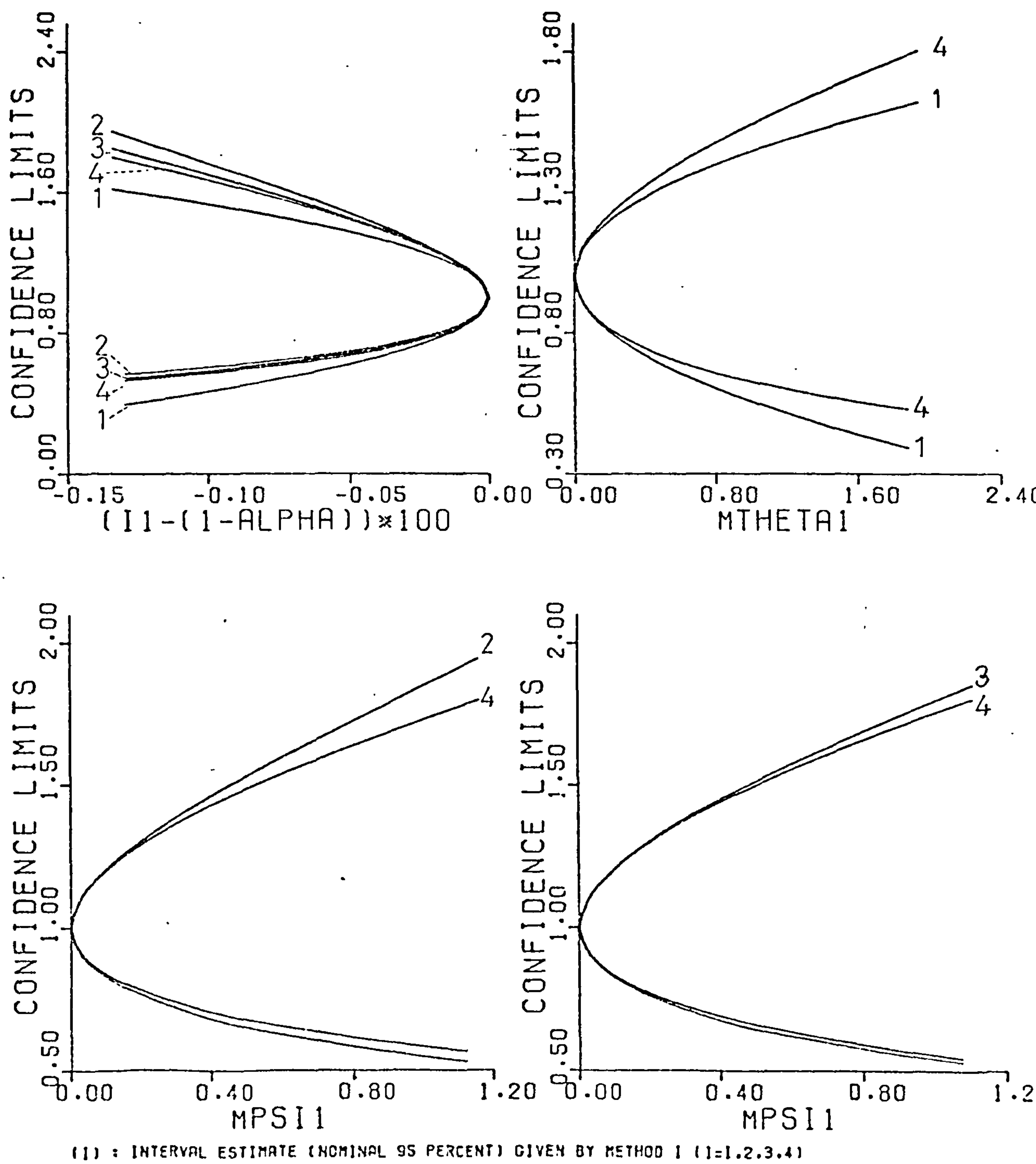
XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETAI HAT ARE 1.0000   0.8000
RESIDUAL SUM OF SQUARES = 0.1000



(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

FIGURE (5.4.10)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

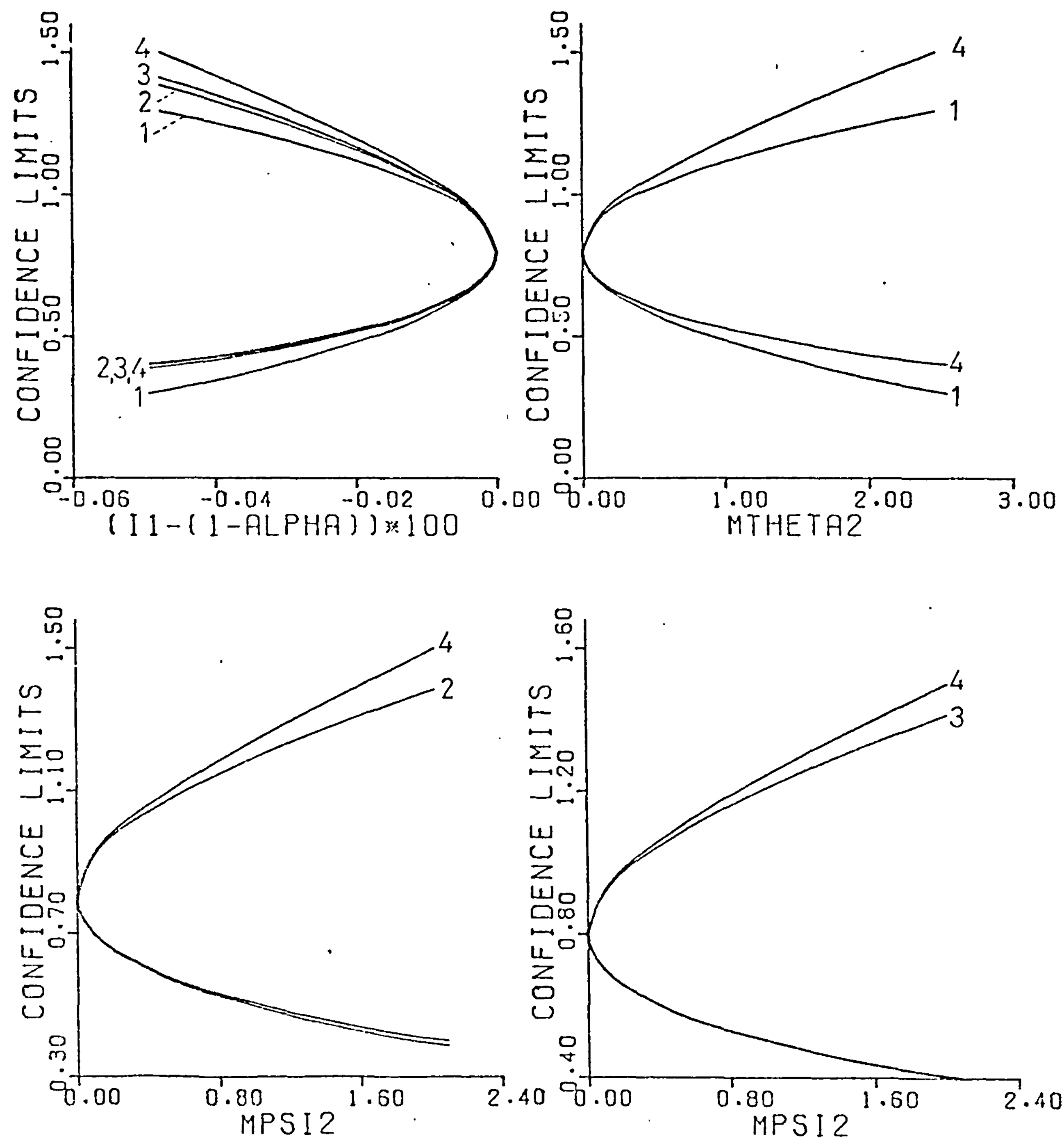
XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETAI HAT ARE 1.0000 0.8000
RESIDUAL SUM OF SQUARES = 0.2000



(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

FIGURE (5.4.11)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=(THETA1/(THETA1-THETA2))
    *(EXP(-THETA2*XI)-EXP(-THETA1*XI))

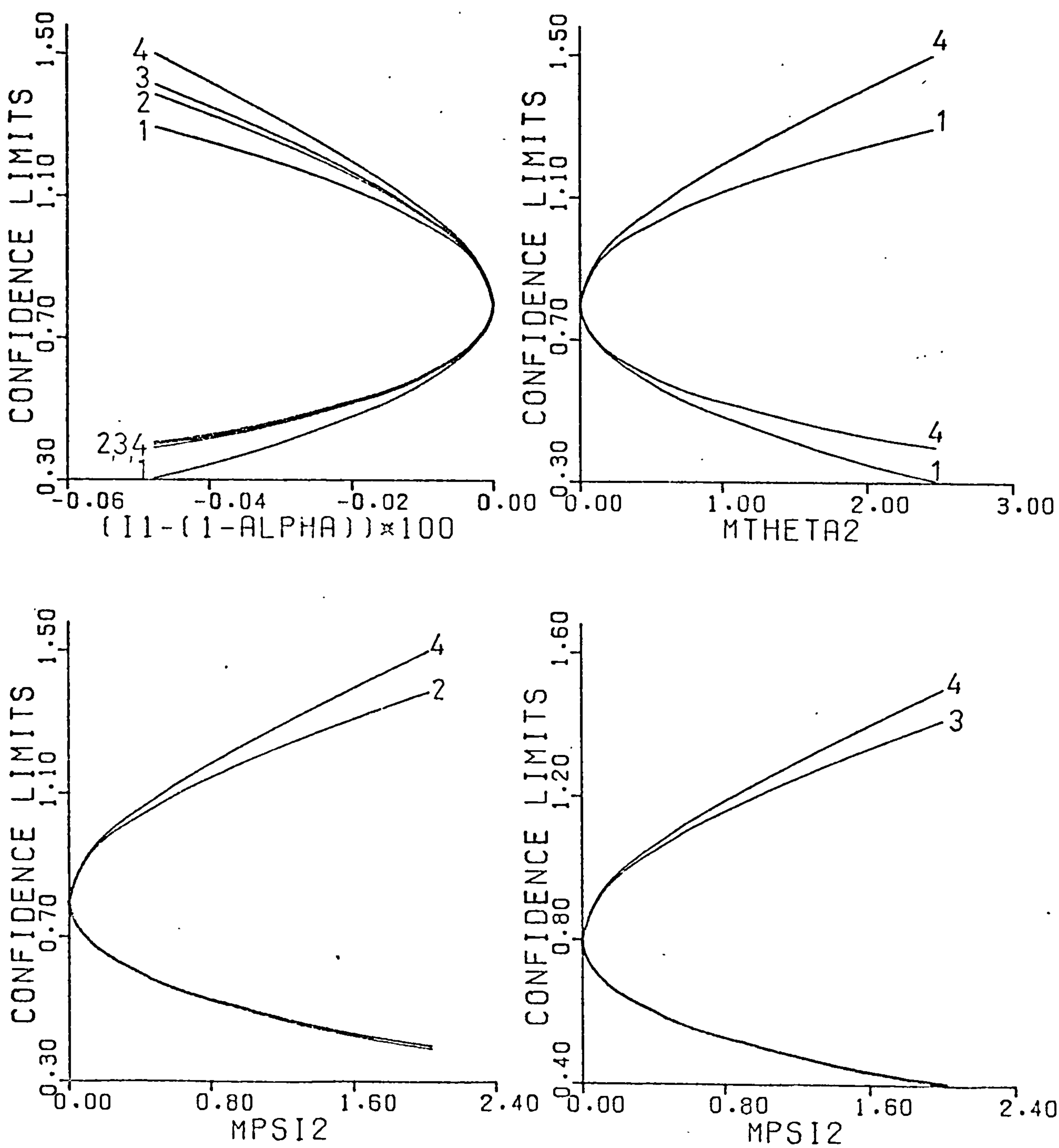
XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETAI HAT ARE 1.0000   0.8000
RESIDUAL SUM OF SQUARES = 0.1000



(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

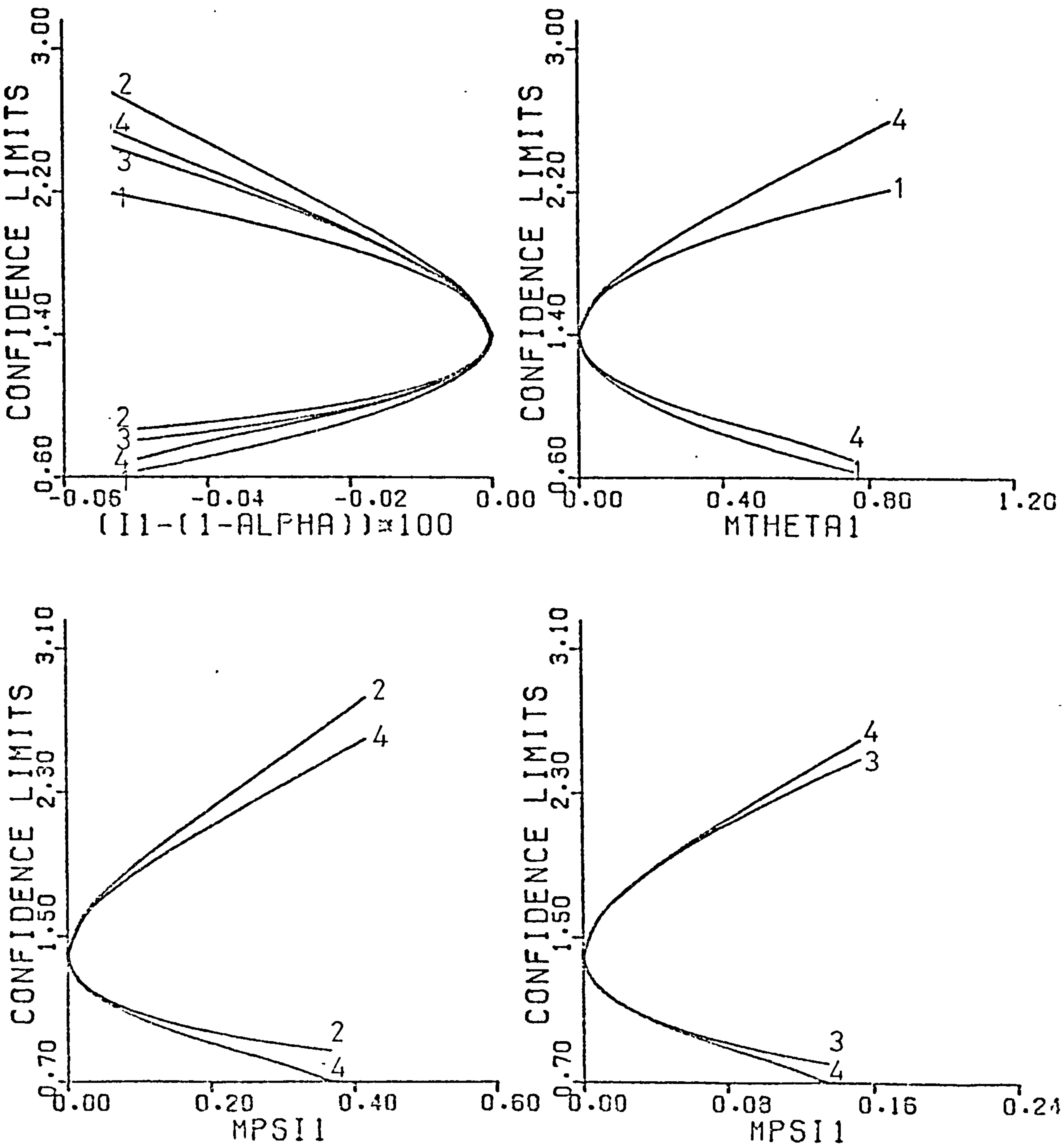
FIGURE (5.4.12)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETAI HAT ARE 1.0000 0.8000
RESIDUAL SUM OF SQUARES = 0.2000



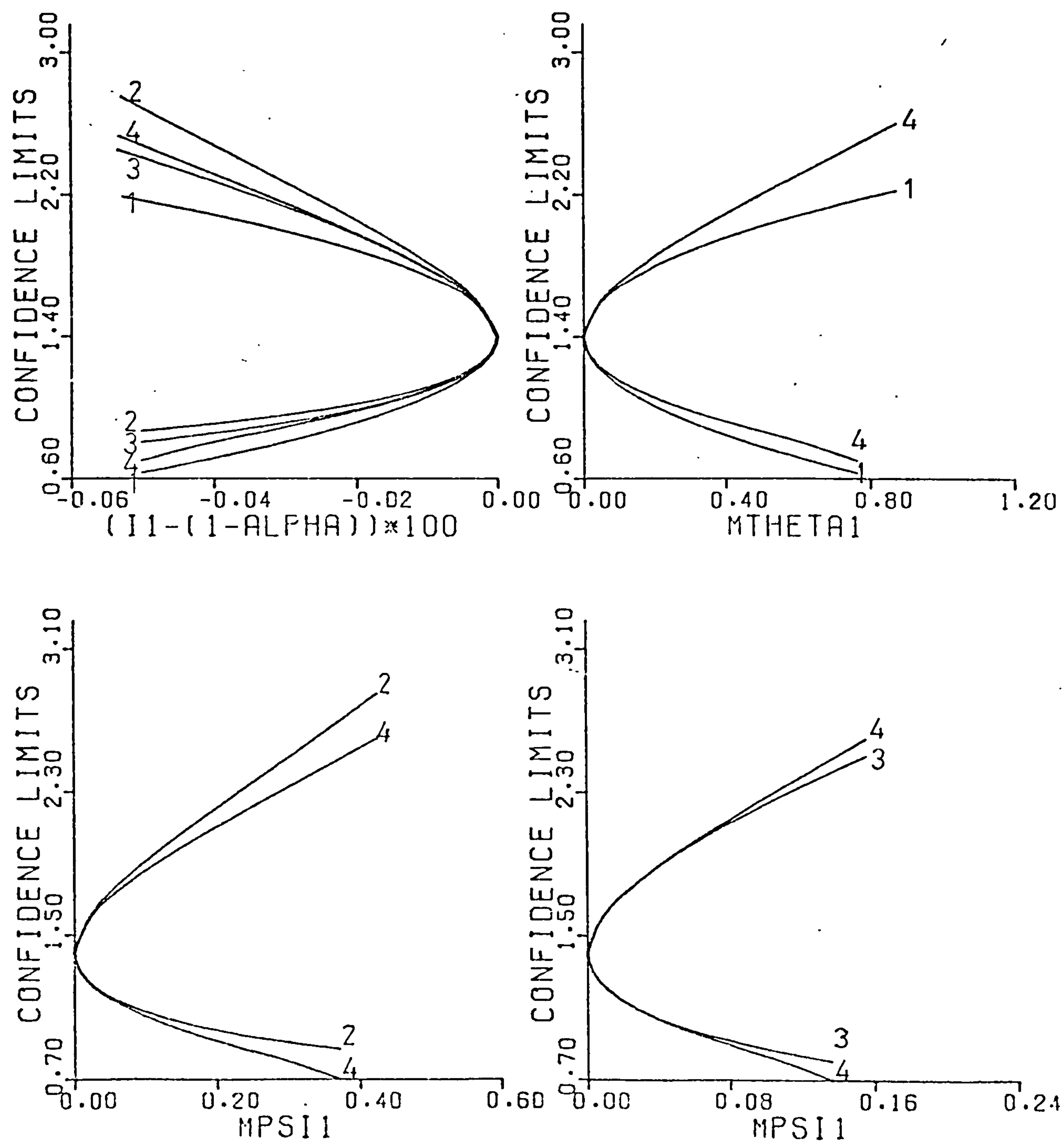
(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

FIGURE (5.4.13)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1.2.3.4.5.6
THETAI HAT ARE 1.4000 0.4000
RESIDUAL SUM OF SQUARES = 0.0001



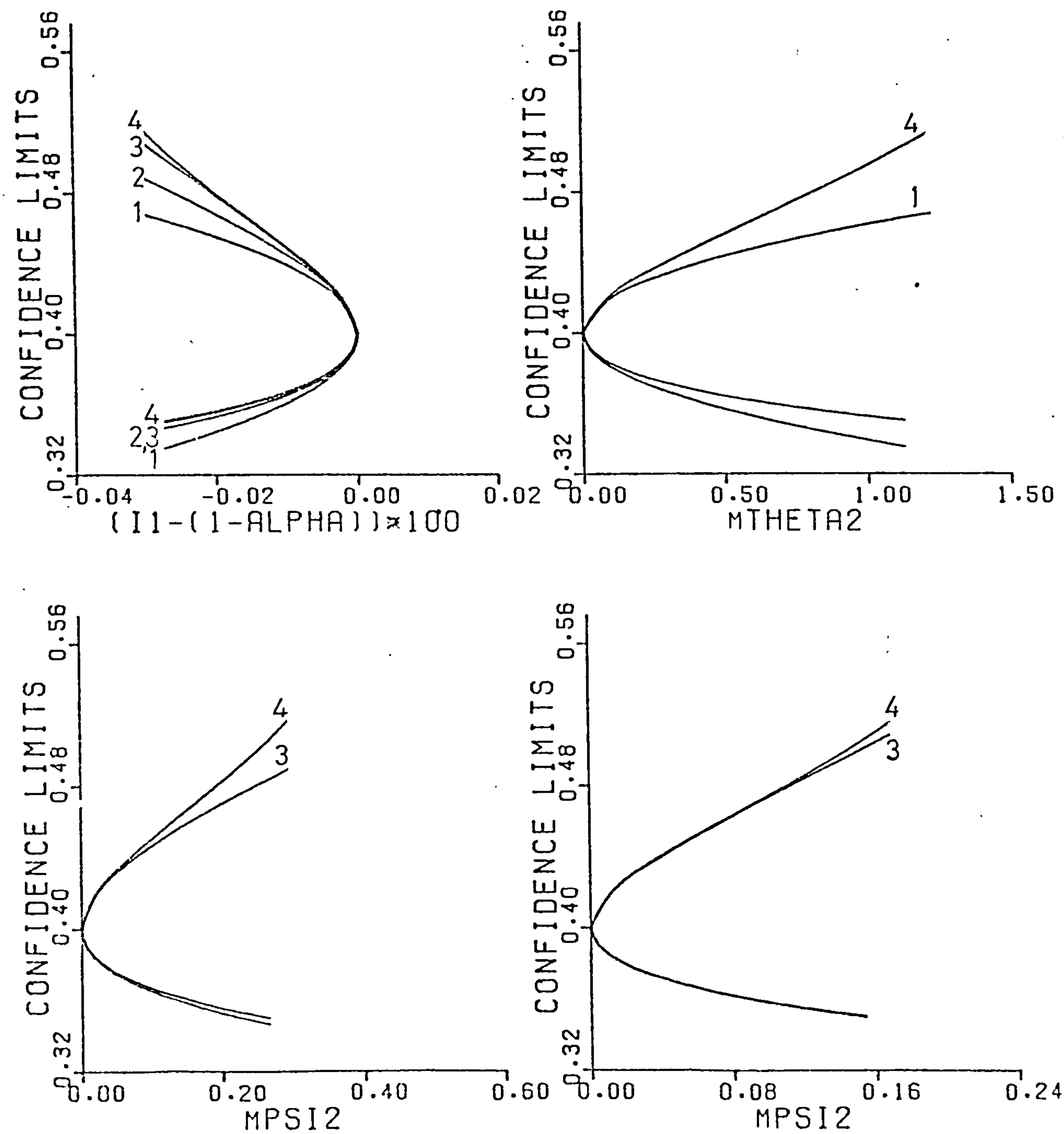
(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1.2.3.4)

FIGURE (5.4.14)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1,2,3,4,5,6
THETAI HAT ARE 1.4000 0.4000
RESIDUAL SUM OF SQUARES = 0.0002



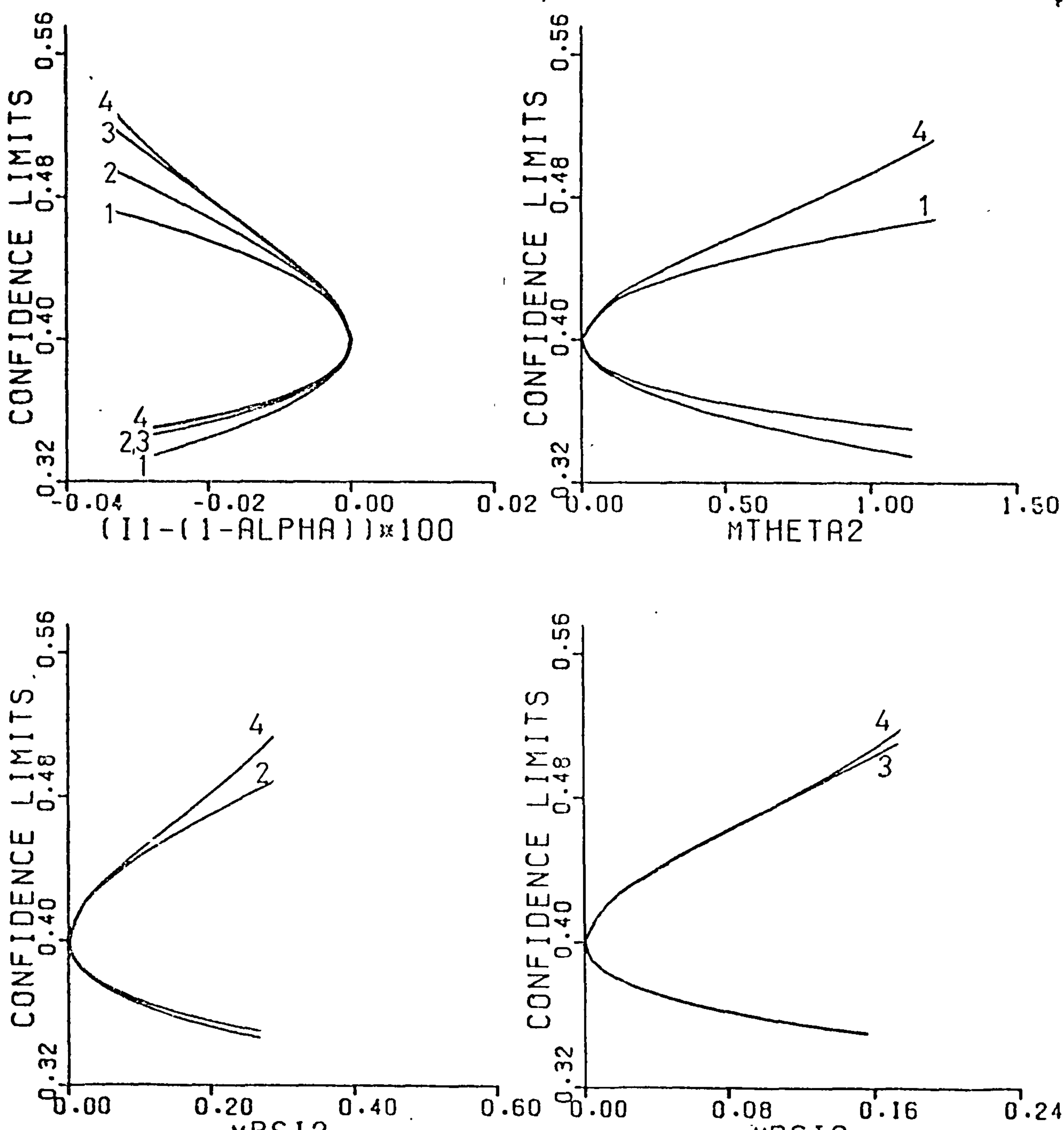
(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

FIGURE (5.4.15)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1,2,3,4,5,6
THETAI HAT ARE 1.4000 0.4000
RESIDUAL SUM OF SQUARES = 0.0001



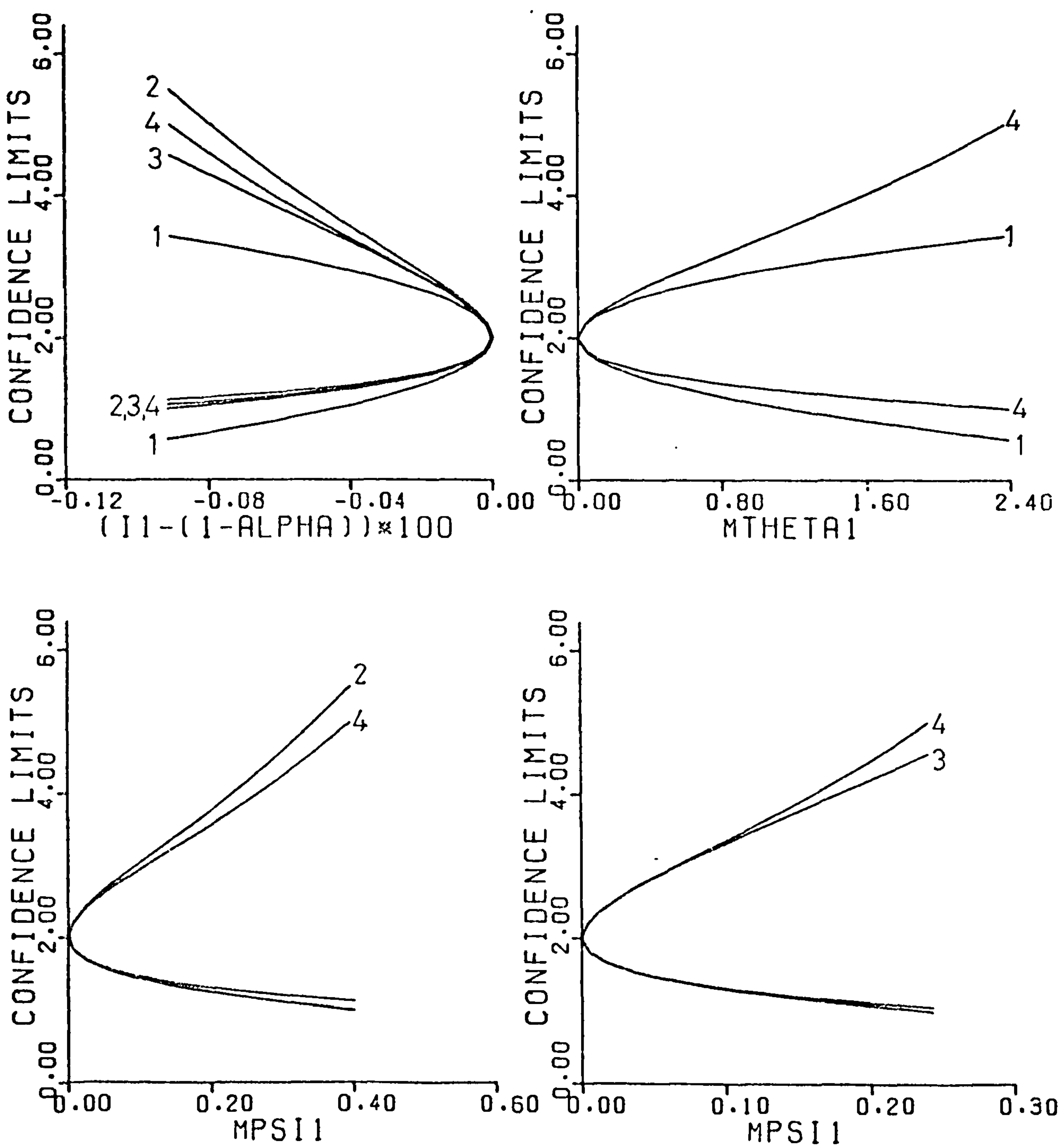
(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

FIGURE (5.4.16)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1,2,3,4,5,6
THETAI HAT ARE 1.4000 0.4000
RESIDUAL SUM OF SQUARES = 0.0002



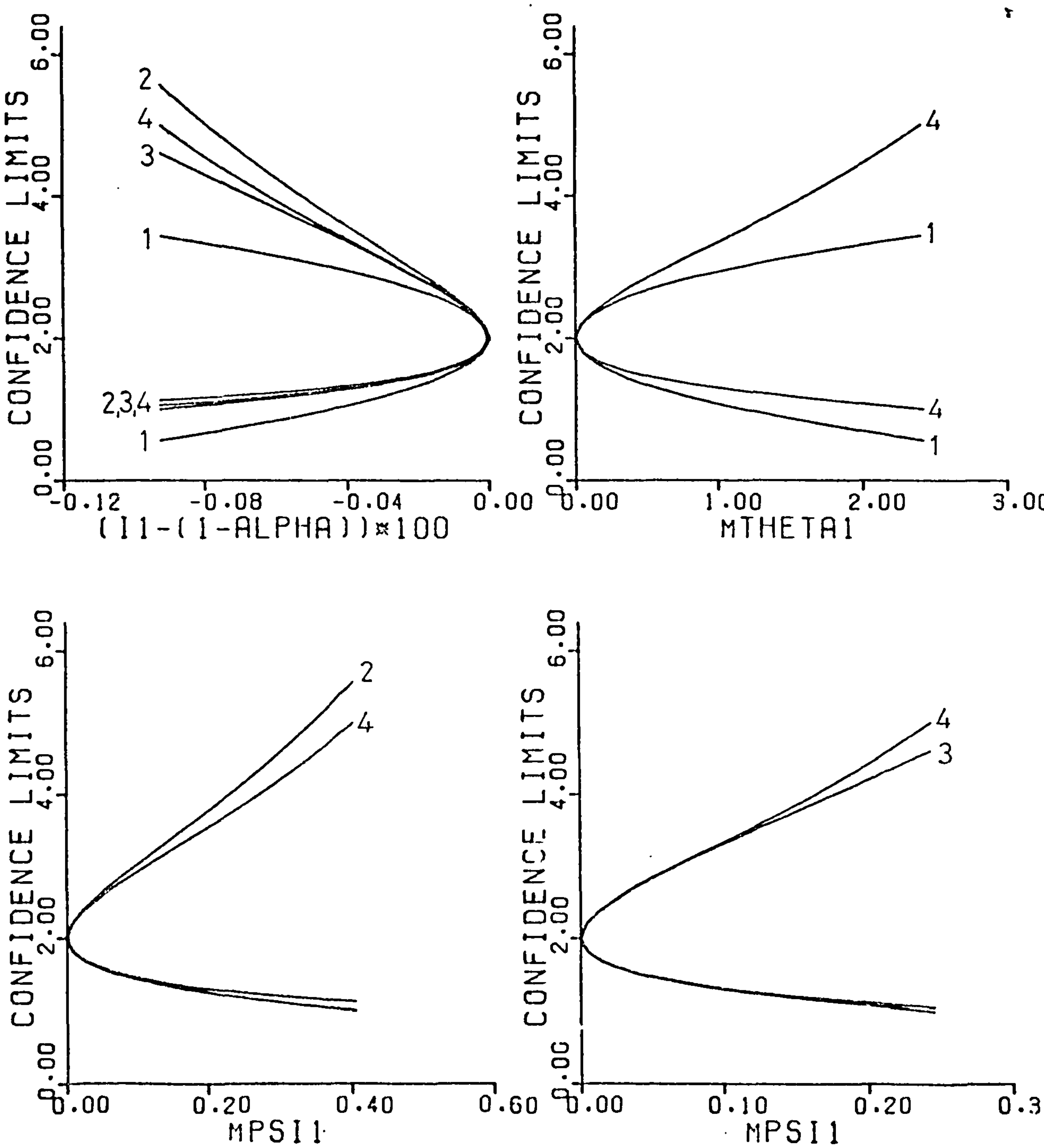
(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

FIGURE (5.4.17)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1,2,3,4,5,6
THETAI HAT ARE 2.0000 0.4000
RESIDUAL SUM OF SQUARES = 0.0001



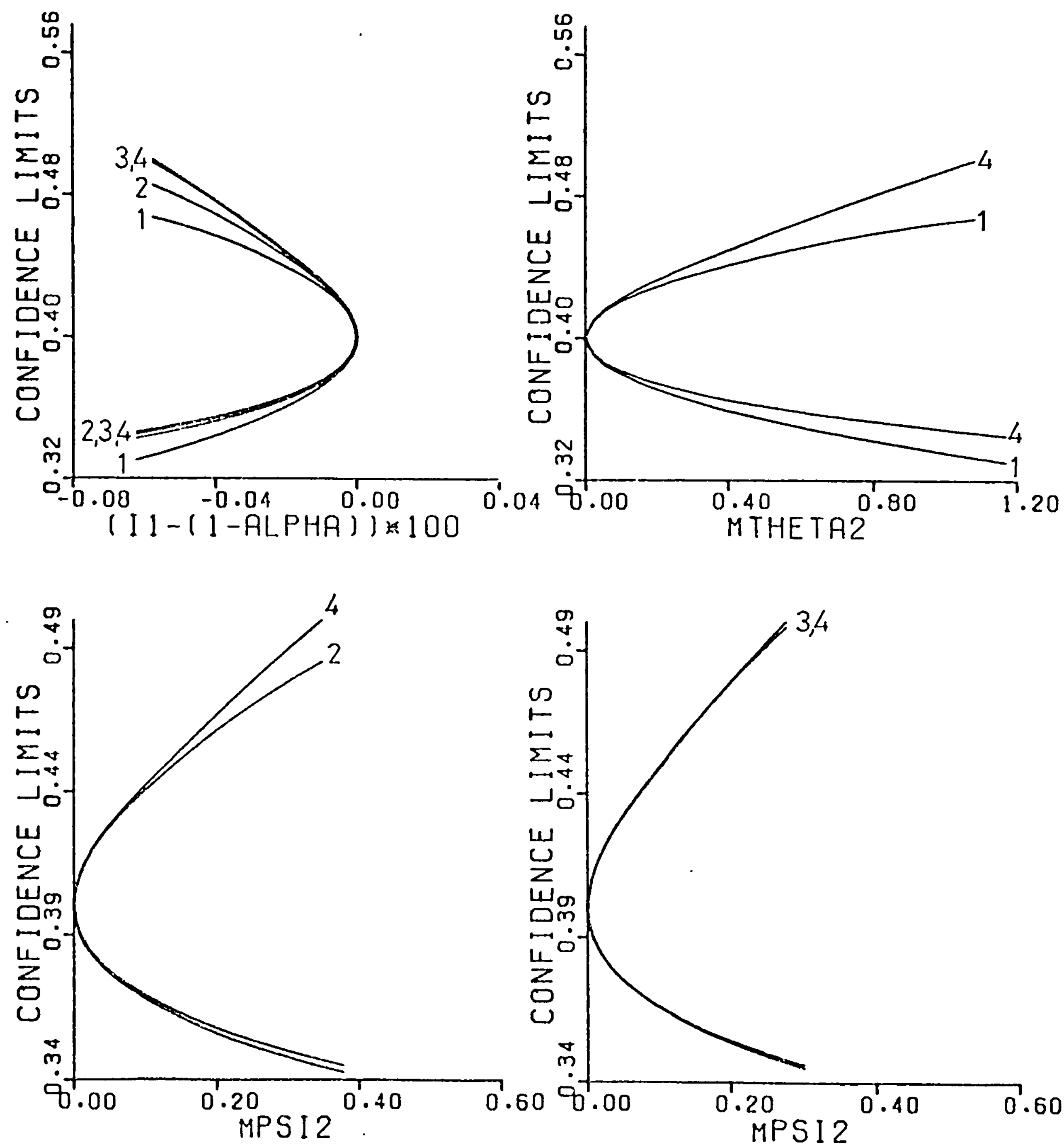
(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

FIGURE (5.4.18)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1,2,3,4,5,6
THETAI HAT ARE 2.0000 0.4000
RESIDUAL SUM OF SQUARES = 0.0002



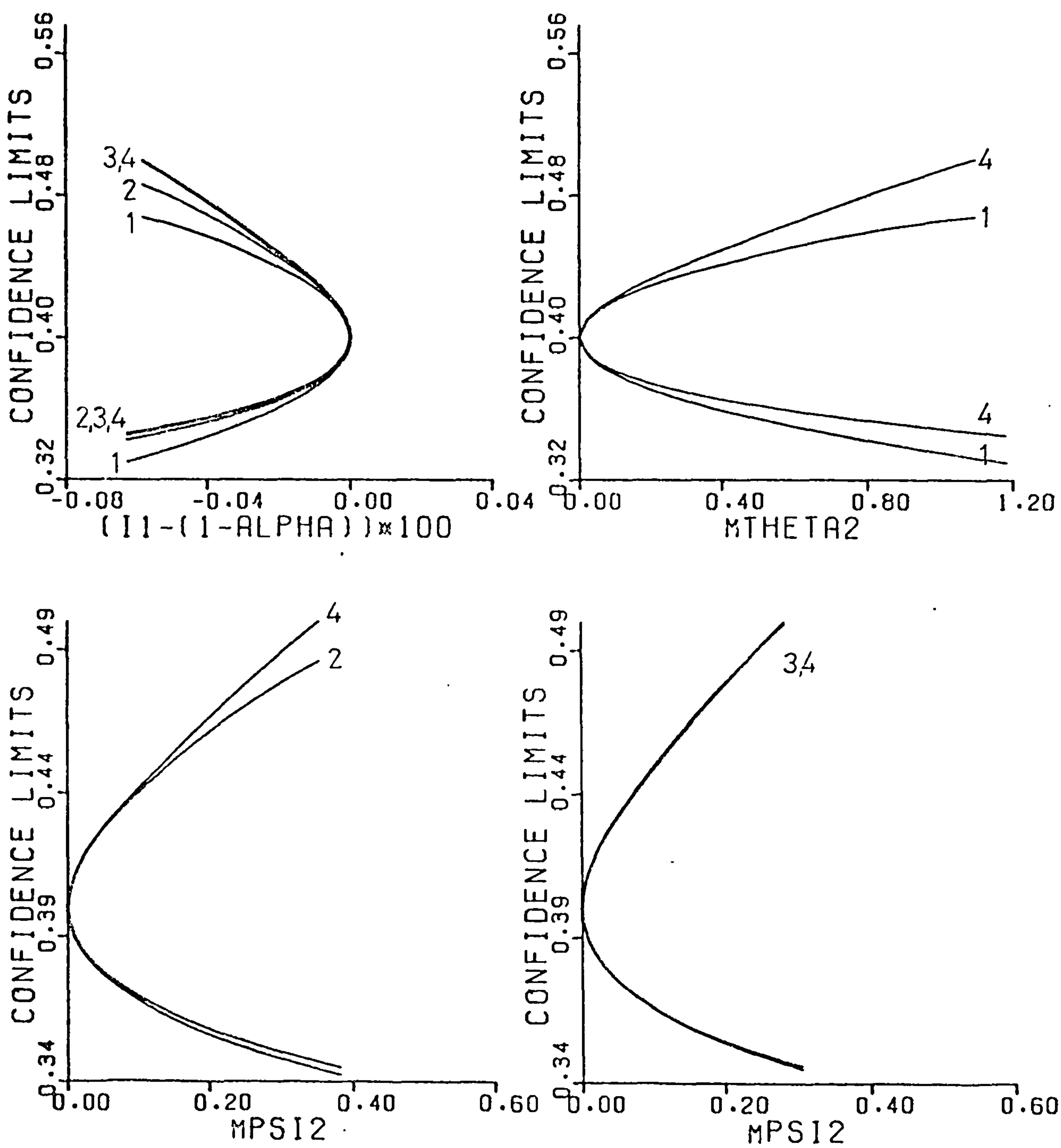
(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

FIGURE (5.4.19)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1,2,3,4,5,6
THETAI HAT ARE 2.0000 0.4000
RESIDUAL SUM OF SQUARES = 0.0001



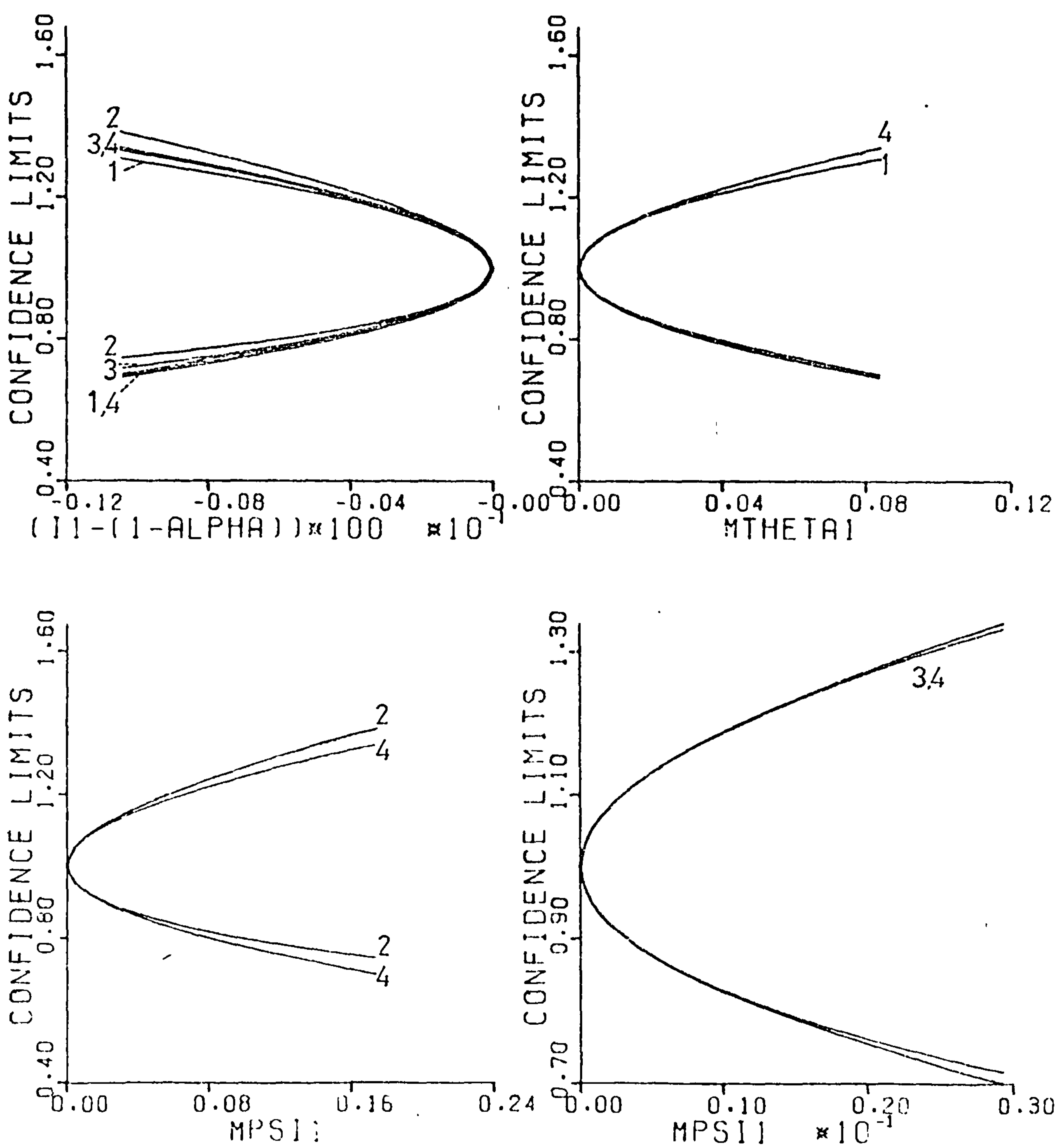
(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

FIGURE (5.4.20)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1,2,3,4,5,6
THETAI HAT ARE 2.0000 0.4000
RESIDUAL SUM OF SQUARES = 0.0002



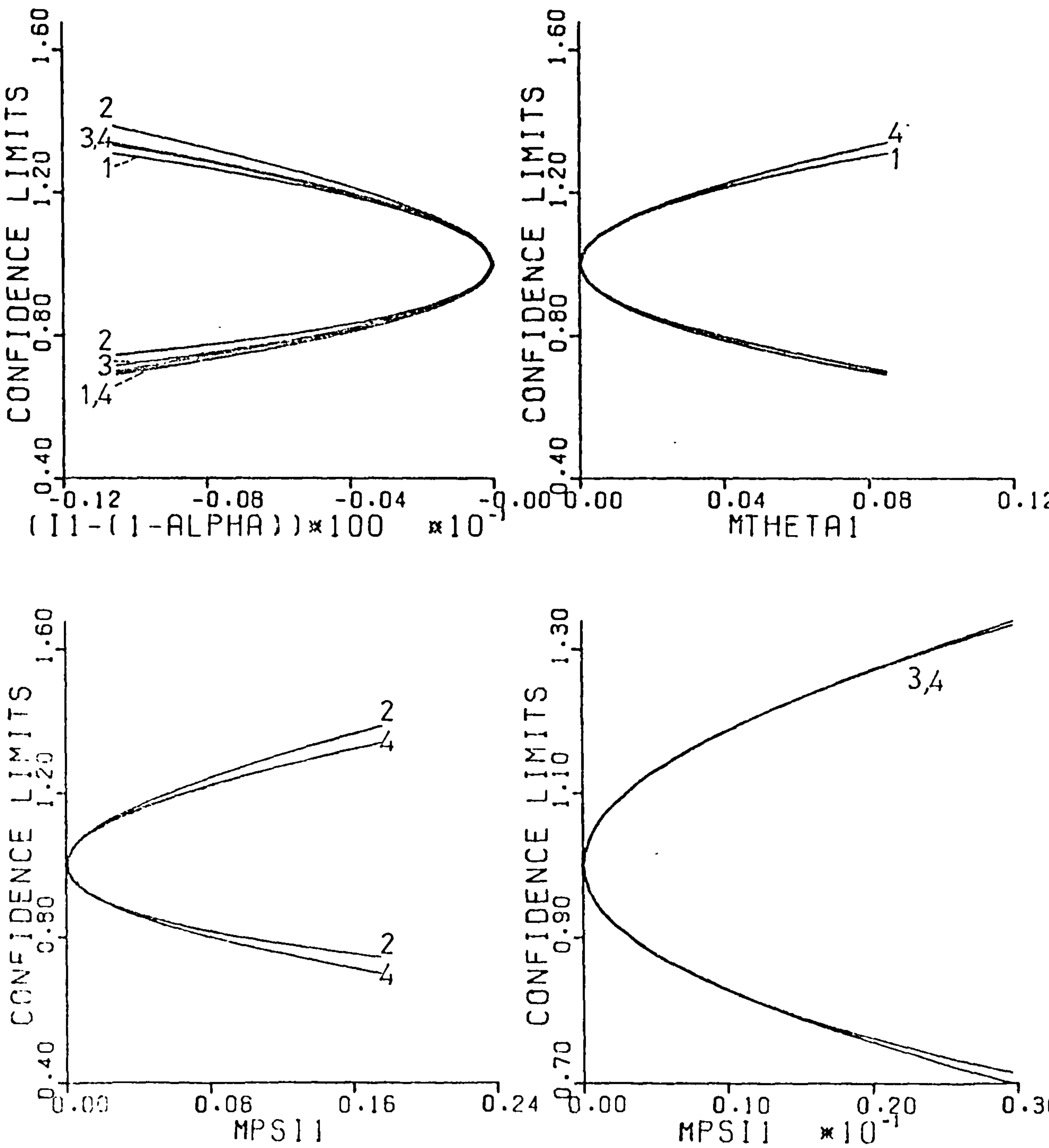
(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

FIGURE (5.4.21)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1,2,3,4,5,6
THETAI HAT ARE 1.0000 0.3500
RESIDUAL SUM OF SQUARES = 0.0001



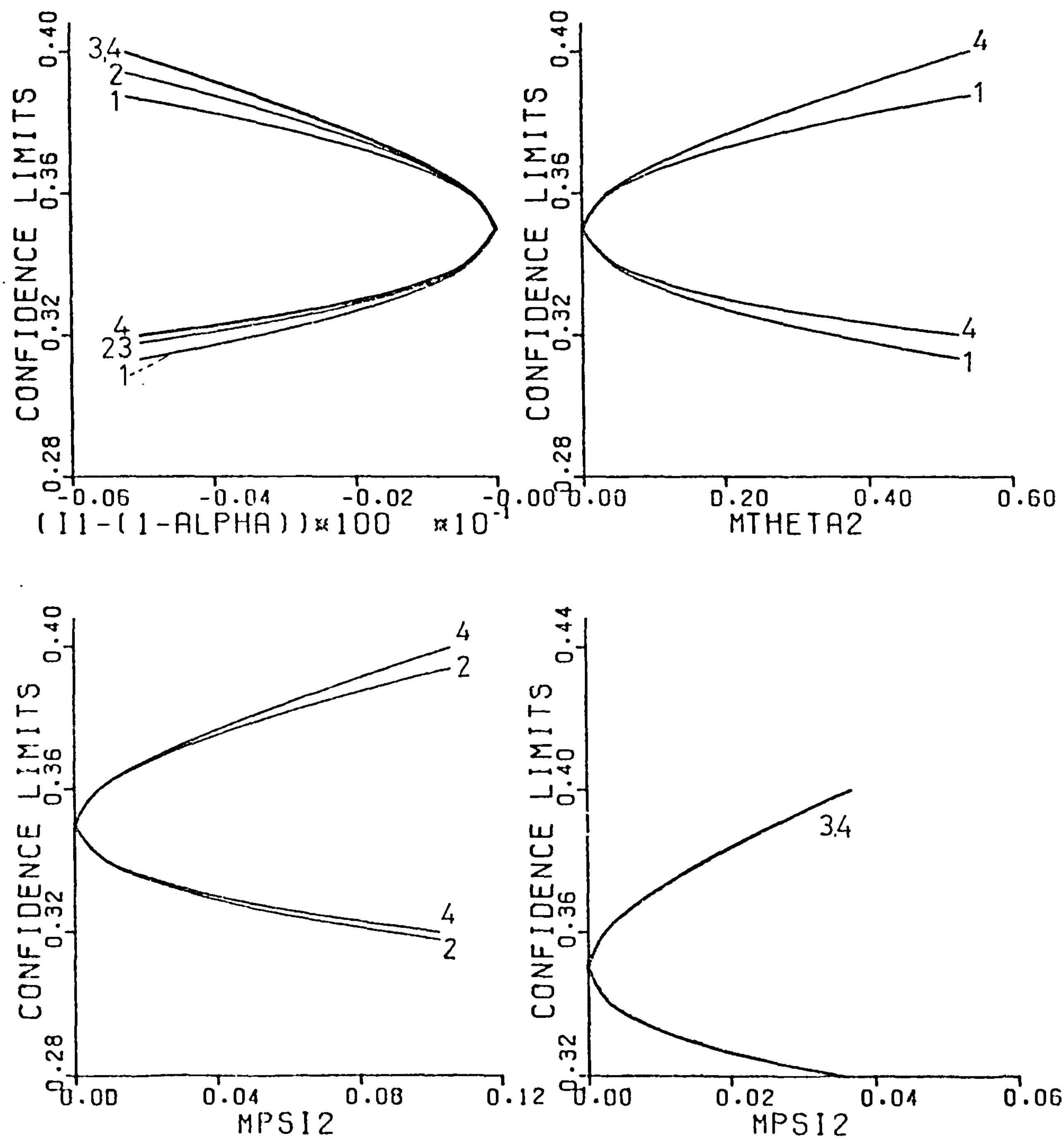
(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

FIGURE (5.4.22)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1,2,3,4,5,6
THETAI HAT ARE 1.0000 0.3500
RESIDUAL SUM OF SQUARES = 0.0002



(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

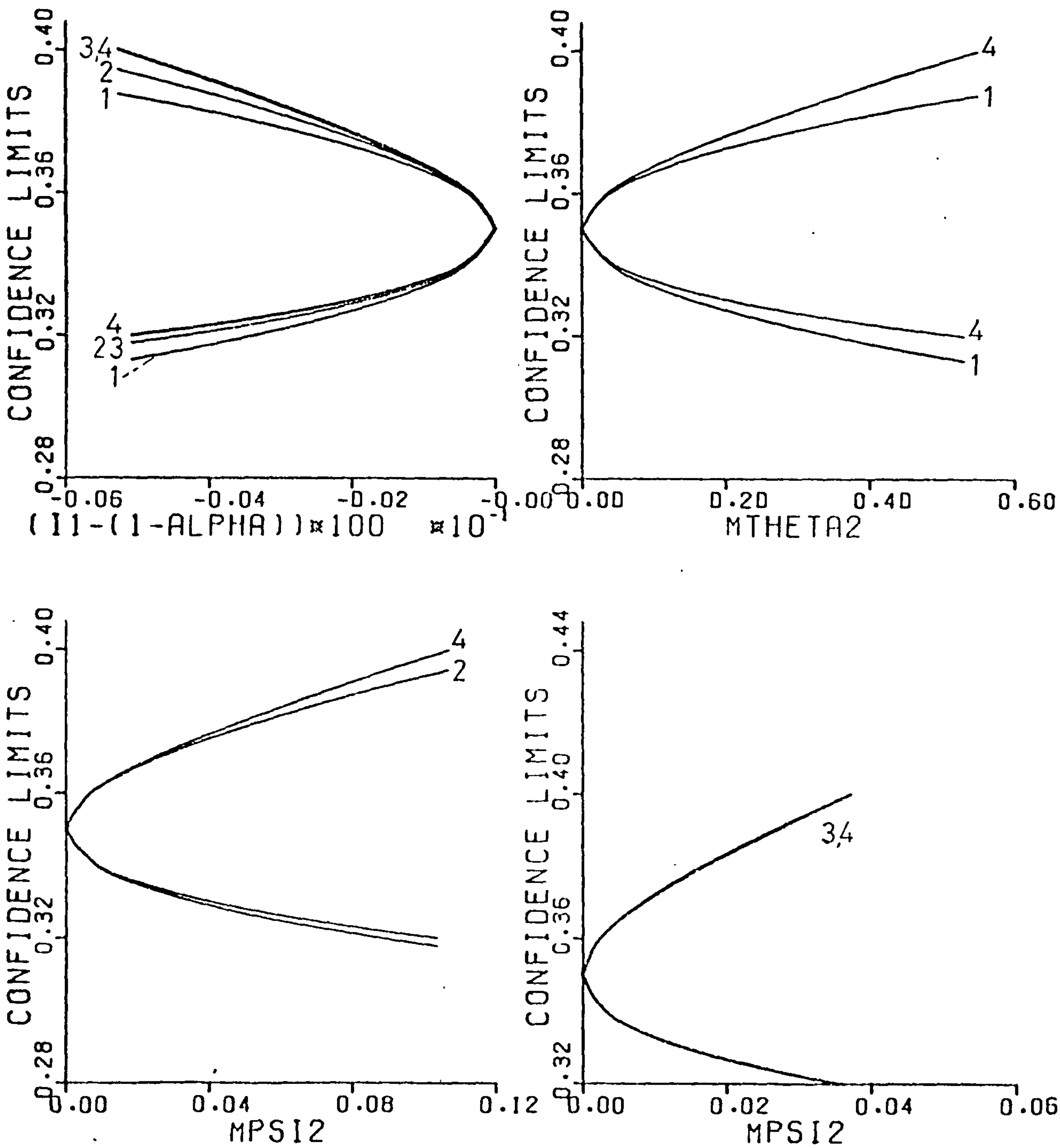
FIGURE (5.4.23)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=1-(THETA1*EXP(-THETA2*XI)
-THETA2*EXP(-THETA1*XI))
/(THETA1-THETA2)
XI= 1,2,3,4,5,6
THETA1 HAT ARE 1.0000 0.3500
RESIDUAL SUM OF SQUARES = 0.0001



(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

FIGURE (5.4.24)
INTERVAL ESTIMATES IN THE MODEL
E(Y)=(THETA1/(THETA1-THETA2))
*(EXP(-THETA2*XI)-EXP(-THETA1*XI))

XI= 0.25,0.5,1.0,1.5,2.0,4.0
THETAI HAT ARE 1.0000 0.3500
RESIDUAL SUM OF SQUARES = 0.0002



(I) : INTERVAL ESTIMATE (NOMINAL 95 PERCENT) GIVEN BY METHOD I (I=1,2,3,4)

We note that for each of the $\hat{\theta}$, the order given by (5.2.8) with $\theta_A = \hat{\theta}$ is fairly well preserved in the closeness of the interval estimates given by methods 1 and 3 to those given by method 4. We also note that the change in the residual sum of squares of an observation from s_1^2 to s_2^2 has very slight effect on the interval estimate given by method 4.

We next observe that the values of $J_1(\theta_T,\sigma)$ and $J_1(\hat{\theta},\sigma)$ as shown in the figures are fractions of a percent. This implies that our choice of the values of σ, $\hat{\theta}$ and θ_T has resulted in situations in which we can refer to the method 4 interval estimates based on these values of $\hat{\theta}$ as approximately 100 $I_1(\hat{\theta},\sigma)$% interval estimates.

We further observe that the four methods give almost identical interval estimates when $M_{\theta_i} \leq 0.1$. As the interval estimates given by method 4 are approximately 95% interval estimates, the interval estimates given by methods 1, 2 and 3 are also approximately 95% interval estimates.

When $M_{\psi_i} \leq 0.1$, an interval estimate based on the transformation ψ_i is almost identical to that given by method 4. This interval estimate based on ψ_i is therefore an approximately 95% interval estimate.

The above observations indicate that for these models with θ_T near $(1.4,0.4)^T$ and the levels of σ similar to those considered before, an interval estimate of the parameter θ_i based on method 1, 2, or 3 is an approximately 95% interval estimate provided that the value of the corresponding nonlinearity M_β, where $\beta = \theta_i$ or ψ_i, is less than or equal to 0.1. Note that this value of 0.1 for the bounds of M_β is the same as that for the case when $p = k^* = 2$ (c.f. section 5.3).

For other models which are unconstrained or which can be treated as unconstrained for statistical purposes, we can likewise obtain bounds for the values of nonlinearity within which it is justifiable to use linear theory to obtain the corresponding interval and region estimates. It is expected that the values of these bounds are fractions of one.

Appendix 1 Householder Transformations

Let $\underset{\sim}{c}_i$ be the i^{th} column of the $(n\times p)$ matrix $\underset{\sim}{C}$ of rank p. We wish to find the Householder transformations

$$\underset{\sim}{H}^{(j)} = \underset{\sim}{I} - [\underset{\sim}{v}^{(j)}][\underset{\sim}{v}^{(j)}]^T , \qquad (j = 1,2,\ldots,p),$$

such that $\underset{\sim}{H}^{(p)}\underset{\sim}{H}^{(p-1)}\ldots\underset{\sim}{H}^{(1)}\underset{\sim}{C}$ is a $(p\times p)$ upper triangular matrix with an $((n-p)\times p)$ matrix beneath it.

We first find $\underset{\sim}{v}^{(1)}$ such that $\underset{\sim}{H}^{(1)}\underset{\sim}{c}_1$ is a column vector with only one nonzero component, and this component is at the first position. The computational procedure involved in finding the components $v_j^{(1)}$ of $\underset{\sim}{v}^{(1)}$ is as follows

(i) compute $r_1 = \sqrt{\sum_{j=1}^{n} c_{j1}^2}$ which is chosen to have the same sign as c_{11},

(ii) compute $v_1^{(1)} = \sqrt{1 + c_{11}/r_1}$,

(iii) compute $v_j^{(1)} = c_{j1}/(r_1 v_1^{(1)})$ for $j \neq 1$.

Now let

$$\underset{\sim}{c}_T^{(j)} = \underset{\sim}{H}^{(j-1)}\underset{\sim}{H}^{(j-2)}\ldots\underset{\sim}{H}^{(1)}\underset{\sim}{c}_j,$$

$$c_{Mu}^{(j)} = 0 \qquad \text{for } u < j,$$

and

$$c_{Mu}^{(j)} = c_{Tu}^{(j)} \quad \text{for } u \geq j,$$

where $j = 2,3,\ldots,p$, and $c_{Tu}^{(j)}$ and $c_{Mu}^{(j)}$ are the u^{th} components of $\underset{\sim}{c}_T^{(j)}$ and $\underset{\sim}{c}_M^{(j)}$ respectively.

We next find $\underset{\sim}{v}^{(j)}$, where $j = 2,3,\ldots,p$, such that $\underset{\sim}{H}^{(j)}\underset{\sim}{c}_M^{(j)}$ is a column vector with only one nonzero component, and this component is at the j^{th} position. The computational procedure involved in finding $\underset{\sim}{v}^{(j)}$ is as follows

(i) compute $r_j = \sqrt{\sum_{u=j}^{n} (c_{Mu}^{(j)})^2}$ chosen to have the same sign as $c_{Mj}^{(j)}$,

(ii) compute $v_j^{(j)} = \sqrt{1 + c_{Mj}^{(j)}/r_j}$,

(iii) compute $v_k^{(j)} = c_{Mk}^{(j)}/(r_j v_j^{(j)})$ for $k \neq j$.

With these $\underset{\sim}{v}^{(j)}$, the corresponding $\underset{\sim}{H}^{(j)}$ are the required Householder transformations.

Appendix 2 Derivation of $I_{1a_j^*}$ and $I_{1a_j^*a_k^*}$ in (3.3.43)-(3.3.48)

Let

$$ST1 = \{1,2,\ldots,p-k^*\}$$

$$ST2 = \{p-k^*+1,p-k^*+2,\ldots,p\}$$

and

$$ST3 = \{p+1,p+2,\ldots,n\}.$$

Let $i \in ST2$ and $j \in ST1$. We have

$$\left[\frac{\partial r_1^{(*s)}}{\partial a_{ijj}^*}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} = 2d_1^* \, s_i \sqrt{\bar{z}_i^{(s)}} \, z_j^2 \,,$$

$$\left[\frac{\partial^2 r_1^{(*s)}}{\partial a_{ijj}^{*2}}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} = 2[2\bar{z}_i^{(s)} z_j^4 - z_j^4 + 4d_1^{*2}\bar{z}_i^{(s)} z_j^2],$$

$$I_{1a_{ijj}^*} = 0,$$

and

$$I_{1a_{ijj}^*a_{ijj}^*} = 2\chi^2_{k^*+2}(d_1^{*2}) \,.$$

Let $i \in ST2$ and $j,k \in ST1$ where $j \neq k$. We have

$$\left[\frac{\partial r_1^{(*s)}}{\partial a^*_{ijk}}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} = 4\, s_i \sqrt{\bar{z}_i^{(s)}}\, d_1^* z_j z_k, \quad \left[\frac{\partial r_1^{(*s)}}{\partial a^*_{ijj}}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} = 2\, s_i \sqrt{\bar{z}_i^{(s)}}\, d_1^* z_j^2,$$

$$\left[\frac{\partial^2 r_1^{(*s)}}{\partial a^{*2}_{ijk}}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} = 16\bar{z}_i^{(s)} z_j^2 z_k^2 - 8 z_j^2 z_k^2 + 8\bar{z}_i^{(s)} d_1^{*2} (z_j^2 + z_k^2),$$

$$\left[\frac{\partial^2 r_1^{(*s)}}{\partial a^*_{ijj}\,\partial a^*_{ikk}}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} = 4\bar{z}_i^{(s)} z_j^2 z_k^2 - 2 z_j^2 z_k^2\,,$$

$$I_{1}a^*{}_{ijk} = 0$$

$$I_{1a^*_{ijk}a^*_{ijk}} = 8\chi^2_{k^*+2}(d_1^{*2})$$

and

$$I_{1a^*_{ijj}a^*_{ikk}} = -2\chi^2_{k^*+2}(d_1^{*2}).$$

Let $i \in ST3$ and $j,k \in ST2$ where $j \neq k$. We have

$$\left[\frac{\partial r_1^{(*s)}}{\partial a^*_{ijj}}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} = -2 d_1^{*2} z_i \bar{z}_j^{(s)}\,,$$

$$\left[\frac{\partial r_1^{(*s)}}{\partial a^*_{ijk}}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} = -4 d_1^{*2} z_i s_j s_k \sqrt{z_j^{(s)}}\sqrt{z_k^{(s)}}$$

$$\left[\frac{\partial^2 r_1^{(*s)}}{\partial a^{*2}_{ijj}}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} = 8 z_i^2 (\bar{z}_j^{(s)})^2 d_1^{*2} - 2\bar{z}_j^{(s)} d_1^{*2} (4 z_i^2 - \bar{z}_j^{(s)} d_1^{*2}),$$

$$\left[\frac{\partial^2 r_l^{(*s)}}{\partial a^*_{ijk}\partial a^*_{ijk}}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} = 32z_i^2\bar{z}_j^{(s)}\bar{z}_k^{(s)}d_l^{*2} - 8z_i^2(\bar{z}_j^{(s)}+\bar{z}_k^{(s)})d_l^{*2} + 8\bar{z}_j^{(s)}\bar{z}_k^{(s)}d_l^{*4} ,$$

$$\left[\frac{\partial^2 r_l^{(*s)}}{\partial a^*_{ijj}\partial a^*_{ikk}}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} = 8z_i^2\bar{z}_j^{(s)}\bar{z}_k^{(s)}d_l^{*2} + 2\bar{z}_j^{(s)}\bar{z}_k^{(s)}d_l^{*4} ,$$

$$I_{la^*_{ijj}} = 0,$$

$$I_{la^*_{ijk}} = 0,$$

$$I_{la^*_{ijj}a^*_{ijj}} = -2\chi^2_{k^*+2}(d_l^{*2})$$

$$I_{la^*_{ijk}a^*_{ijk}} = -8\chi^2_{k^*+2}(d_l^{*2})$$

and

$$I_{la^*_{ijj}a^*_{ikk}} = 2\chi^2_{k^*+2}(d_l^{*2}) .$$

Let $i \in ST3$, $j \in ST1$ and $k \in ST2$. We have

$$\left[\frac{\partial r_l^{(*s)}}{\partial a^*_{ijj}}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} = 0,$$

$$\left[\frac{\partial r_l^{(*s)}}{\partial a^*_{ijk}}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} = -4d_l^*z_iz_j\ s_k\sqrt{\bar{z}_k^{(s)}}$$

$$\left[\frac{\partial^2 r_l^{(*s)}}{\partial a^{*2}_{ijj}}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} = 0,$$

$$\left[\frac{\partial^2 r_1^{(*s)}}{\partial a_{ijk}^{*2}}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} = 16 z_i^2 z_j^2 \bar{z}_k^{2(s)} - 8(z_i^2 z_j^2 + z_i^2 \bar{z}_k^{2(s)} d_1^{*2} - z_j^2 \bar{z}_k^{2(s)} d_1^{*2}),$$

$$\left[\frac{\partial^2 r_1^{(*s)}}{\partial a_{ijj}^* \partial a_{ikk}^*}\right]_{\underset{\sim}{a}^*=\underset{\sim}{0}} = 2 z_j^2 \bar{z}_k^{2(s)} d_1^{*2} \quad ,$$

$$I_{1a_{ijj}^* a_{ijj}^*} = 0,$$

$$I_{1a_{ijk}^* a_{ijk}^*} = -8\chi^2_{k^*+2}(d_1^{*2}),$$

and

$$I_{1a_{ijj}^* a_{ikk}^*} = 2\chi^2_{k^*+2}(d_1^{*2}) \ .$$

Next, it can be shown that all the $I_{1a_{ijk}^*}$ and $I_{1a_{i_1j_1k_1}^* a_{i_2j_2k_2}^*}$ other than those already derived are zero.

APPENDIX 3 Programs PARTIT, POWPRO, and subroutines POWSUO, POWSUA,

POWSUB, POWSUC and SIGSUC

```
76/09/17   IMPERIAL  COLLEGE  FORTRAN  COMPILER       KRONOS  2.1.X  PSR2+    77/05/03. 21.22.28.

        MNF(B=PARTIT)
                   C
                   C     PROGRAM PARTIT IS FOR PARTITIONING THE SET OF ALL
                   C     (A+I1J1K1,A+I2J2K2) INTO SUBSETS(C.F. SECTION (4.9))
                   C
 000000B                  PROGRAM PARTIT(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT,TAPE7)
 006060B                  COMMON/MSKP/MS(6),KP(500,21)
 006060B                  IREAD=5
 006060B                  IPRINT=6
 006061B                  ITAPE7=7
 006062B                  READ(IREAD,2) KCH,IN,NPAR,NOBS,KSTAR
                   C
                   C     NPAR IS TOTAL NUMBER OF COMPONENTS IN THE PARAMETER VECTOR
                   C     NOBS IS TOTAL NUMBER OF OBSERVATIONS
                   C     KSTAR IS NUMBER OF COMPONENTS OF INTEREST IN THE PARAMETER VECTOR
                   C
 006074B                2 FORMAT(A6,I4,14I5)
 006074B                  CALL CHECIN(KCH,IN,6HNPAROB,0)
                   C
                   C     ROUTINE CHECIN CHECKS THAT DATA CARD IS CORRECT
                   C
 006076B                  WRITE(IPRINT,3) NPAR,NOBS,KSTAR
 006105B                3 FORMAT(/1X,5HNPAR=,I2,1H/,5HNOBS=,I2,1H/,6HKSTAR=,I2/)
 006105B                  NPMK=NPAR-KSTAR
 006106B                  NPMKP1=NPMK+1
 006107B                  NPARP=NPAR+1
 006110B                  NPMKP2=NPMK+2
 006111B                  NPARP2=NPAR+2
 006112B                  IP=0
                   C
                   C     THESE DO LOOPS ARE FOR PARTITIONING THE SET OF ALL
                   C     (A+I1J1K1,A+I2J2K2) INTO SUBSETS. THE SUBSETS ARE INDEXED BY IP
                   C
 006113B                  DO 160 I1=NPMKP1,NOBS
 006116B                  DO 150 I2=NPMKP1,NOBS
 006120B                  DO 140 J1=1,NPAR
 006123B                  DO 130 K1=J1,NPAR
 006124B                  DO 120 J2=1,NPAR
 006126B                  DO 110 K2=J2,NPAR
 006127B                  IF(.NOT.(I1.LE.I2)) GO TO 110
 006132B                  IP=IP+1
 006134B                  MS(1)=I1
 006134B                  MS(2)=J1
 006135B                  MS(3)=K1
 006137B                  MS(4)=I2
 006140B                  MS(5)=J2
 006141B                  MS(6)=K2
 006142B                  DO 10 IM=1,6
 006144B                  I=MS(IM)
 006144B                  IF(I.GE.1.AND.I.LE.NPMK) GO TO 7
 006152B                  IF(I.GE.NPMKP1.AND.I.LE.NPAR) GO TO 8
 006156B                  IF(I.GE.NPARP.AND.I.LE.NOBS) GO TO 9
 006162B                7 KP(IP,IM)=1
 006165B                  GO TO 10
 006166B                8 KP(IP,IM)=2
 006171B                  GO TO 10
 006172B                9 KP(IP,IM)=3
 006175B               10 CONTINUE
 006200B                  IT=6
 006201B                  DO 16 IM=1,5
 006203B                  IMP1=IM+1
 006203B                  DO 15 JM=IMP1,6
 006207B                  IT=IT+1
 006210B                  IF(MS(IM).EQ.MS(JM)) GO TO 12
```

```
006213B           IF(MS(IM).NE.MS(JM)) GO TO 13
006214B        12 KP(IP,IT)=1
006220B           GO TO 15
006221B        13 KP(IP,IT)=0
006224B        15 CONTINUE
006226B        16 CONTINUE
             C
             C      (KP(IP,IT),IT=1,21) COMPLETELY SPECIFY THE IP-TH SUBSET
             C
006231B           IF(IP.EQ.1) GO TO 40
006233B           KSAME=0
006233B           IPM1=IP-1
006234B           DO 20 IPM=1,IPM1
006237B           DO 18 IT=1,21
006241B           IF(KP(IPM,IT).NE.KP(IP,IT)) GO TO 20
006241B        18 CONTINUE
006250B           KSAME=1
006254B           GO TO 30
006254B        20 CONTINUE
006255B        30 IF(KSAME.EQ.0) GO TO 40
006261B           IP=IP-1
006262B           GO TO 110
006262B        40 LSKIP=1
006263B           IF(IP.NE.1) GO TO 54
006265B           WRITE(IPRINT,50) IP,(MS(IB),IB=:,6),(KP(IP,IB),IB=1,21)
006311B        50 FORMAT(1X,3HIP=,I5,1H/,21HI1,J1,K1,I2,J2,K2 ARE,6(I2,1X),1H/,12HKP
                 1(IP,*) ARE,21(I1,1X))
006311B           GO TO 58
006311B        54 LSKIP=1
006312B           WRITE(IPRINT,56) IP,(MS(IB),IB=1,6),(KP(IP,IB),IB=1,21)
006336B        56 FORMAT(1X,3HIP=,I5,1H/,21X,6(I2,1X),1H/,12X,21(I1,1X))
006336B        58 LSKIP=1
006336B           WRITE(ITAPE7,60) IP,(MS(IB),IB=1,6),(KP(IP,IB),IB=1,21)
006362B        60 FORMAT(1X,28I4)
006362B       110 CONTINUE
006364B       120 CONTINUE
006367B       130 CONTINUE
006372B       140 CONTINUE
006375B       150 CONTINUE
006400B       160 CONTINUE
006403B           STOP
006406B           END
             C
             C      PROGRAM POWPRO,SUBROUTINES POWSUO,POWSUA,POWSUB AND POWSUC ARE FOR
             C      REPRESENTING (1) PRODUCT OF FIRST PARTIAL DERIVATIVES OF (R+)**2
             C      W.R.T. A+I1J1K1 AND A+I2J2K2 EVALUATED AT A+=0 AND (2) SECOND
             C      PARTIAL DERIVATIVES OF (R+)**2 W.R.T. TO A+I1J1K1,A+I2J2K2
             C      EVALUATED AT A+=0(C.F.SECTIONS (4.2) TO (4.6))
             C
000000B           PROGRAM POWPRO(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT,TAPE9,TAPE10,
                 1TAPE7,TAPE11)
014140B           COMMON/MSKP/MS(6),KP(30,21)
014140B           COMMON/ACAAAZ/AC(30),R(30),D(30),AAC(30),AA(30,2),SIZN(30,10),AZ(3
                 10,10)
014140B           COMMON/NKPP/N(2,2,3),KPP(8,6)
014140B           COMMON/AC1AA1/AC1(4,10),D1(4,10),AA1(4,10,2),SIGN1(4,10,10),AZ1(4,
                 110,10)
014140B           COMMON/IGIGMX/IG(4),IGMAX(4)
014140B           COMMON/ACGAAG/ACG(4,10),DG(4,10),AAG(4,10,2),SIGNG(4,10,10),AZG(4,
                 110,10)
014140B           COMMON/AKSTAR/AKSTAR(50),DEGREE(50),GAMMA(10)
014140B           COMMON/IREAD/IREAD,IPRINT,ITAPE
014140B           COMMON/NOBS/NOBS,NPAR,KSTAR,NPMK,NPMKP1,NPARP
014140B           COMMON/KS/KS,IP,I1MAX,II
014140B           COMMON/I1J1K1/I1,J1,K1,I2,J2,K2
014140B           COMMON/LPRINT/LPN(50)
```

```
0141408          COMMON/IOAB/IOA,IOB
0141408          COMMON/KPOWER/KPOWER
0141408          IREAD=5
0141408          IPRINT=6
0141418          ITAPE7=7
0141428          READ(IREAD,4) KCH,IN,NPAR,NOBS,KSTAR
         C
         C     NPAR IS TOTAL NUMBER OF COMPONENTS IN THE PARAMETER VECTOR
         C     NOBS IS TOTAL NUMBER OF OBSERVATIONS
         C     KSTAR IS NUMBER OF COMPONENTS OF INTEREST IN THE
         C     PARAMETER VECTOR
         C
0141548        4 FORMAT(A6,I4,14I5)
0141548          CALL CHECIN(KCH,IN,6HNPAROB,0)
         C
         C     ROUTINE CHECIN CHECKS THAT THE DATA CARD IS CORRECT
         C
0141568          WRITE(IPRINT,5) NPAR,NOBS,KSTAR
0141658        5 FORMAT(/1X,@NPAR,NOBS,KSTAR ARE@,3(I3,1X))
0141658          NPMK=NPAR-KSTAR
0141668          NPMKP1=NPMK+1
0141678          NPARP=NPAR+1
         C
0141708          READ(IREAD,4) KCH,IN,(LPN(I),I=1,7)
0142038          CALL CHECIN(KCH,IN,6HLPN(1),0)
0142058          READ(IREAD,4) KCH,IN,(LPN(I),I=8,14)
0142208          CALL CHECIN(KCH,IN,6HLPN(8),0)
         C
         C     LPN(*) DECIDE  WHETHER THE INTERMEDIATE RESULTS WILL BE PRINTED
         C     OUT
         C
0142228          KAAC=6
0142228          READ(IREAD,6) KCH,IN,(AAC(I),I=1,KAAC)
0142378          WRITE(IPRINT,8) (AAC(I),I=1,KAAC)
         C
         C     AAC(*) ARE SOME NUMBERS ASSOCIATED WITH THE EQUATION
         C     SDA1(Z)=(D+)**2 (C.F. SECTION (4.4)). THEY ARE RESPECTIVELY
         C     2.0,-2.0,-1.0,1.0,4.0 AND -4.0
         C
0142518        6 FORMAT(A6,I4,7F10.6)
0142518        8 FORMAT(/1X,@AAC(*) ARE@,6(F10.4,1X)/)
0142518          CALL CHECIN(KCH,IN,6HAAC(I),0)
0142538          READ(IREAD,4) KCH,IN,KPOWER
0142628          CALL CHECIN(KCH,IN,6HKPOWER,0)
         C
         C     NORMALLY KPOWER=1 EXCEPT IN THE CASE WHEN WE WANT TO USE POWPRO,
         C     POWSUO,POWSUA,POWSUB AND SIGSUC TO DERIVE I1(THETA,SIGMA)
         C
0142648          READ(IREAD,4) KCH,IN,KSELEC,IS1,JS1,KS1,IS2,JS2,KS2
0143018          CALL CHECIN(KCH,IN,6HKSELEC,0)
0143038          READ(IREAD,4) KCH,IN,KSTART
0143128          CALL CHECIN(KCH,IN,6HKSTART,0)
         C
         C     NORMALLY KSELEC=0,KSTART=0 AND IS1,JS1,KS1,IS2,JS2,KS2 ARE ANY
         C     POSITIVE INTEGERS EXCEPT IN THE CASE WHEN WE WANT TO CHECK SOME
         C     PARTICULAR PART OF THE PROGRAM
         C
0143148          READ(IREAD,4) KCH,IN,IDMAX
         C
         C     IDMAX IS TOTAL NUMBER OF SUBSETS GENERATED BY PARTITIONING THE SET
         C     OF ALL (A+I1J1K1,A+I2J2K2) (C.F. PROGRAM PARTIT)
         C
0143238          CALL CHECIN(KCH,IN,5HIDMAX,0)
0143258          IP=0
0143258          IOA=0
0143258          IOB=0
0143268          DO 1910 ID=1,IDMAX
```

```
0143318          IP=IP+1
0143328          IF(KPOWER.EQ.1) IP=1
0143368          READ(ITAPE7,10) IDD,(MS(IB),IB=1,6),(KP(IP,IB),IB=1,21)
          C
          C      (KP(IP,IB),IB=1,21) COMPLETELY SPECIFY THE IP-TH SUBSET
          C
0143618       10 FORMAT(1X,28I4)
0143618          IF(.NOT.(KSELEC.EQ.0.OR.(I1.EQ.IS1.AND.J1.EQ.JS
                1.I2.EQ.IS2.AND.J2.EQ.JS2.AND.K2.EQ.KS2)))GO TO 1910
0144038          IF(LPN(1).EQ.0) GO TO 35
0144048          WRITE(IPRINT,33) (MS(I),I=1,6)
0144158       33 FORMAT(/1X,21HI1,J1,K1,I2,J2,K2 ARE,6(I2,1X))
0144158          WRITE(IPRINT,34) IP,(KP(IP,I),I=1,21)
0144328       34 FORMAT(1X,5HIP = ,I3,1H/,12HKP(IP,I) ARE,21(I1,1X))
0144328       35 LSKIP=1
0144328          I1=MS(1)
0144338          J1=MS(2)
0144358          K1=MS(3)
0144368          I2=MS(4)
0144408          J2=MS(5)
0144418          K2=MS(6)
0144438          KS=0
0144448          IF(I1.EQ.I2.AND.J1.EQ.J2.AND.K1.EQ.K2.AND.J1.EQ.K1) KS=1
0144558          IF(I1.EQ.I2.AND.J1.EQ.J2.AND.K1,EQ.K2.AND.J1.NE.K1) KS=2
0144668          KII=30
0144668          DO 38 II=1,KII
0144708          DO 36 IZ=1,NOBS
0144738          AZ(II,IZ)=0.0
0144738       36 CONTINUE
0145018       38 CONTINUE
          C
          C      TO REPRESENT THE EQUATION SDA1(Z)=(D+)**2 IN A COMPUTER
          C      (C.F. SECTION (4,4))
          C      NONLINEAR TERMS IN SDA1(Z) ARE INDEXED BY II
          C
0145048          II=0
0145048          IF(.NOT.(KP(IP,1).EQ.2.AND.KP(IP,2).EQ.1.AND.KP(IP,3).EQ.1)) GO TO
                 144
          C
          C      TO REPRESENT 2.0(SUM FROM I EQUAL TO P-KSTAR+1 TO P) (SUM FROM
          C      J EQUAL TO 1 TO P-KSTAR) (SUM FROM K EQUAL TO 1 TO P-KSTAR)
          C      OF A+IJKZIZJZK
          C
0145158          II=II+1
0145168          IF(J1.EQ.K1) AC(II)=AAC(1)
0145228          IF(J1.NE.K1) AC(II)=2.0*AAC(1)
0145268          AA(II,1)=1.0
0145268          AA(II,2)=0.0
0145308          DO 40 IZ=1,NOBS
0145338          IF(IZ.EQ.I1) AZ(II,IZ)=AZ(II,IZ)+1.0
0145428          IF(IZ.EQ.J1) AZ(II,IZ)=AZ(II,IZ)+1.0
0145508          IF(IZ.EQ.K1) AZ(II,IZ)=AZ(II,IZ)+1.0
0145568       40 CONTINUE
0145608          IF(LPN(5).EQ.0) GO TO 43
0145618          WRITE(IPRINT,41) (MS(I),I=1,6)
0145728       41 FORMAT(/1X,21HI1,J1,K1,I2,J2,K2 ARE,6(I2,1X))
0145728          WRITE(IPRINT,42)II,AC(II),AA(II,1),AA(II,2),(AZ(II,IB),IB=1,NOBS)
0146168       42 FORMAT(/1X,5HII = ,I2,1X,9HAC(II) = ,F4.1,2X,11HAA(II,1) = ,F4.1,2
                1X,11HAA(II,2) = ,F4.1,2X,13HAZ(II,IZ) ARE,10(F4.1,1X))
0146168       43 LSKIP=1
0146168       44 IF(.NOT.(KP(IP,4).EQ.2.AND.KP(IP,5).EQ.1.AND.KP(IP,6).EQ.1)) GO TO
                1 110
0146278          IF(.NOT.KS.EQ.0) GO TO 110
0146318          II=II+1
0146338          IF(J2.EQ.K2) AC(II)=AAC(1)
0146378          IF(J2.NE.K2) AC(II)=2.0*AAC(1)
```

```
014643B        AA(II,1)=0.0
014643B        AA(II,2)=1.0
014645B        DO 50 IZ=1,NOBS
014650B        IF(IZ.EQ.I2) AZ(II,IZ)=AZ(II,IZ)+1.0
014657B        IF(IZ.EQ.J2) AZ(II,IZ)=AZ(II,IZ)+1.0
014665B        IF(IZ.EQ.K2) AZ(II,IZ)=AZ(II,IZ)+1.0
014673B     50 CONTINUE
014675B        IF(LPN(5).EQ.0) GO TO 60
014676B        WRITE(IPRINT,41) (MS(I),I=1,6)
014707B        WRITE(IPRINT,42)II,AC(II),AA(II,1),AA(II,2),(AZ(II,IB),IB=1,NOBS)
014733B     60 LSKIP=1
014733B    110 LOUT=1
014734B        IF(.NOT.((KP(IP,1).EQ.3).AND.(KP(IP,2).EQ.2.OR.KP(IP,3).EQ.2))) GO
              1 TO 124
        C
        C      TO REPRESENT -2.0(SUM FROM I EQUAL TO P+1 TO N) (SUM FROM J EQUAL
        C      TO 1 TO P,SUM FROM K EQUAL TO 1 TO P, AND AT LEAST ONE OF J,K IS
        C      EQUAL TO  P-KSTAR+1 OR P-KSTAR+2,...,OR P) OF A+IJKZIZJZK
        C
014746B        II=II+1
014747B        IF(J1.EQ.K1) AC(II)=AAC(2)
014754B        IF(J1.NE.K1) AC(II)=2.0*AAC(2)
014760B        AA(II,1)=1.0
014760B        AA(II,2)=0.0
014762B        DO 120 IZ=1,NOBS
014765B        IF(IZ.EQ.I1) AZ(II,IZ)=AZ(II,IZ)+1.0
014774B        IF(IZ.EQ.J1) AZ(II,IZ)=AZ(II,IZ)+1.0
015002B        IF(IZ.EQ.K1) AZ(II,IZ)=AZ(II,IZ)+1.0
015010B    120 CONTINUE
015012B        IF(LPN(5).EQ.0) GO TO 122
015013B        WRITE(IPRINT,41) (MS(I),I=1,6)
015024B        WRITE(IPRINT,42)II,AC(II),AA(II,1),AA(II,2),(AZ(II,IB),IB=1,NOBS)
015050B    122 LSKIP=1
015050B    124 IF(.NOT.(KS.EQ.0)) GO TO 180
015053B        IF(.NOT.((KP(IP,4).EQ.3).AND.(KP(IP,5).EQ.2.OR.KP(IP,6).EQ.2))) GO
              1 TO 180
015065B        II=II+1
015066B        IF(J2.EQ.K2) AC(II)=AAC(2)
015072B        IF(J2.NE.K2) AC(II)=2.0*AAC(2)
015076B        AA(II,1)=0.0
015076B        AA(II,2)=1.0
015100B        DO 130 IZ=1,NOBS
015103B        IF(IZ.EQ.I2) AZ(II,IZ)=AZ(II,IZ)+1.0
015112B        IF(IZ.EQ.J2) AZ(II,IZ)=AZ(II,IZ)+1.0
015120B        IF(IZ.EQ.K2) AZ(II,IZ)=AZ(II,IZ)+1.0
015126B    130 CONTINUE
015130B        IF(LPN(5).EQ.0) GO TO 132
015131B        WRITE(IPRINT,41) (MS(I),I=1,6)
015142B        WRITE(IPRINT,42)II,AC(II),AA(II,1),AA(II,2),(AZ(II,IB),IB=1,NOBS)
015166B    132 LSKIP=1
015166B    180 LOUT=1
015167B        CALL POWSUU
015172B        CALL POWSUA
015174B   1910 CONTINUE
015176B        STOP
015201B        END
```

```
0000008           SUBROUTINE CHECIN(KA,I,KB,J)
           C
           C     ROUTINE CHECIN CHECKS THAT DATA CARD IS CORRECT
           C
000000B           IF(KA.NE.KB)GO TO 100
000001B           IF(I.EQ.J) GO TO 9000
000003B           IF(I.EQ.0.AND.J.EQ.1) GO TO 9000
000006B       100 WRITE(6,110) KA,I,KB,J
000016B       110 FORMAT(23H ERROR IN CARD LABELLED,2X,A6,I4,2X,15HPROGRAM EXPECTS,2
                 2X,A6,I4)
000016B      9000 RETURN
000020B           END

NUMBER AND NAME CROSS REFERENCE MAP (/N= N  REFERENCES IN STATEMENT)

VARIABLE AND ARRAY NAMES SORTED BY ADDRESS       (B=RELATIVE ADDRESS)
                                                 (C= RELATIVE TO  //)

000000B           SUBROUTINE POWSUO
           C
           C     ROUTINE POWSUO IS CALLED IN POWPRO
           C
000000B           COMMON/MSKP/MS(6),KP(30,21)
000000B           COMMON/ACAAAZ/AC(30),R(30),D(30),AAC(30),AA(30,2),SIZN(30,10),AZ(3
                 10,10)
000000B           COMMON/NKPP/N(2,2,3),KPP(8,6)
000000B           COMMON/AC1AA1/AC1(4,10),D1(4,10),AA1(4,10,2),SIGN1(4,10,10),AZ1(4,
                 110,10)
000000B           COMMON/IGIGMX/IG(4),IGMAX(4)
000000B           COMMON/ACGAAG/ACG(4,10),DG(4,10),AAG(4,10,2),SIGNG(4,10,10),AZG(4,
                 110,10)
000000B           COMMON/AKSTAR/AKSTAR(50),DEGREE(50),GAMMA(10)
000000B           COMMON/IREAD/IREAD,IPRINT,ITAPE
000000B           COMMON/NOBS/NOBS,NPAR,KSTAR,NPMK,NPMKP1,NPARP
000000B           COMMON/KS/KS,IP,IIMAX,II
000000B           COMMON/I1J1K1/I1,J1,K1,I2,J2,K2
000000B           COMMON/LPRINT/LPN(50)
000000B           COMMON/IQAB/IQA,IQB
000000B        41 FORMAT(/1X,21HI1,J1,K1,I2,J2,K2 ARE,6(I2,1X))
000000B        42 FORMAT(/1X,5HII = ,I2,1X,9HAC(II) = ,F4.1,2X,11HAA(II,1) = ,F4.1,2
                 1X,11HAA(II,2) = ,F4.1,2X,13HAZ(II,IZ) ARE,10(F4.1,1X))
000000B           IF(.NOT.(KP(IP,1).EQ.2.AND.KP(IP,2).EQ.1.AND.KP(IP,3).EQ.1.AND.KP(
                 1IP,4).EQ.2.AND.KP(IP,5).EQ.1.AND.KP(IP,6).EQ.1))GO TO 304
           C
           C     TO REPRESENT -1.0(SUM FROM I EQUAL TO P-KSTAR+1 TO P) OF (SQUARE OF
           C     (SUM FROM J EQUAL TO 1 TO P-KSTAR,SUM FROM K EQUAL TO 1 TO P-KSTAR)
           C     OF A+IJKZJZK)
           C
000017B           IF(.NOT.(KS.EQ.0)) GO TO 244
000021B           IF(I1.NE.I2) GO TO 304
000023B           II=II+1
000025B           IF(J1.EQ.K1.AND.J2.EQ.K2) AC(II)=2.0*AAC(3)
000033B           IF(J1.NE.K1.AND.J2.EQ.K2) AC(II)=4.0*AAC(3)
000042B           IF(J1.EQ.K1.AND.J2.NE.K2) AC(II)=4.0*AAC(3)
000051B           IF(J1.NE.K1.AND.J2.NE.K2) AC(II)=8.0*AAC(3)
000060B           AA(II,1)=1.0
000060B           AA(II,2)=1.0
000062B           DO 220 IZ=1,NOBS
000064B           IF(IZ.EQ.J1) AZ(II,IZ)=AZ(II,IZ)+1.0
000073B           IF(IZ.EQ.K1) AZ(II,IZ)=AZ(II,IZ)+1.0
000101B           IF(IZ.EQ.J2) AZ(II,IZ)=AZ(II,IZ)+1.0
000107B           IF(IZ.EQ.K2) AZ(II,IZ)=AZ(II,IZ)+1.0
000115B       220 CONTINUE
000117B           IF(LPN(5).EQ.0) GO TO 230
000120B           WRITE(IPRINT,41) (MS(I),I=1,6)
```

```
000131B       WRITE(IPRINT,42)II,AC(II),AA(II,1),AA(II,2),(AZ(II,IB),IB=1,NOBS)
000155B   230 LSKIP=1
000155B       GO TO 304
000156B   244 IF(.NOT.(KS.EQ.1)) GO TO 252
000162B       II=II+1
000163B       AC(II)=AAC(3)
000164B       AA(II,1)=2.0
000165B       AA(II,2)=0.0
000166B       DO 248 IZ=1,NOBS
000171B       IF(IZ.EQ.J1) AZ(II,IZ)=AZ(II,IZ)+1.0
000200B       IF(IZ.EQ.K1) AZ(II,IZ)=AZ(II,IZ)+1.0
000206B       IF(IZ.EQ.J2) AZ(II,IZ)=AZ(II,IZ)+1.0
000214B       IF(IZ.EQ.K2) AZ(II,IZ)=AZ(II,IZ)+1.0
000222B   248 CONTINUE
000224B       IF(LPN(5).EQ.0) GO TO 250
000225B       WRITE(IPRINT,41) (MS(I),I=1,6)
000236B       WRITE(IPRINT,42)II,AC(II),AA(II,1),AA(II,2),(AZ(II,IB),IB=1,NOBS)
000262B   250 LSKIP=1
000262B       GO TO 304
000263B   252 II=II+1
000265B       AC(II)=4.0*AAC(3)
000266B       AA(II,1)=2.0
000270B       AA(II,2)=0.0
000270B       DO 256 IZ=1,NOBS
000273B       IF(IZ.EQ.J1) AZ(II,IZ)=AZ(II,IZ)+1.0
000302B       IF(IZ.EQ.K1) AZ(II,IZ)=AZ(II,IZ)+1.0
000310B       IF(IZ.EQ.J2) AZ(II,IZ)=AZ(II,IZ)+1.0
000316B       IF(IZ.EQ.K2) AZ(II,IZ)=AZ(II,IZ)+1.0
000324B   256 CONTINUE
000326B       IF(LPN(5).EQ.0) GO TO 260
000327B       WRITE(IPRINT,41) (MS(I),I=1,6)
000340B       WRITE(IPRINT,42)II,AC(II),AA(II,1),AA(II,2),(AZ(II,IB),IB=1,NOBS)
000364B   260 LSKIP=1
000364B   304 LEND=1
000365B       IF(.NOT.(KP(IP,1).EQ.3.AND.KP(IP,4).EQ.3.AND.(KP(IP,2).EQ.2.OR.KP(
             1IP,3).EQ.2.OR.KP(IP,5).EQ.2.OR.K P(IP,6).EQ.2))) GO TO 410
          C
          C   TO REPRESENT 1.0(SUM FROM I EQUAL TO P+1 TO N) (SUM FROM J EQUAL TO
          C   1 TO P,SUM FROM K EQUAL TO 1 TO P,SUM FROM L EQUAL TO 1 TO P,SUM
          C   FROM M EQUAL TO 1 TO P, AND AT LEAST ONE OF J,K,L,M IS EQUAL TO
          C   P-KSTAR+1 OR P-KSTAR+2,...,OR P) OF A+IJKA+ILMZJZKZLZM
          C
000406B       IF(.NOT.(KS.EQ.0)) GO TO 344
000410B       IF(I1.NE.I2) GO TO 410
000412B       II=II+1
000413B       IF(J1.EQ.K1.AND.J2.EQ.K2) AC(II)=2.0*AAC(4)
000422B       IF(J1.NE.K1.AND.J2.EQ.K2) AC(II)=4.0*AAC(4)
000431B       IF(J1.EQ.K1.AND.J2.NE.K2) AC(II)=4.0*AAC(4)
000440B       IF(J1.NE.K1.AND.J2.NE.K2) AC(II)=8.0*AAC(4)
000447B       AA(II,1)=1.0
000447B       AA(II,2)=1.0
000451B       DO 320 IZ=1,NOBS
000453B       IF(IZ.EQ.J1) AZ(II,IZ)=AZ(II,IZ)+1.0
000462B       IF(IZ.EQ.K1) AZ(II,IZ)=AZ(II,IZ)+1.0
000470B       IF(IZ.EQ.J2) AZ(II,IZ)=AZ(II,IZ)+1.0
000476B       IF(IZ.EQ.K2) AZ(II,IZ)=AZ(II,IZ)+1.0
000504B   320 CONTINUE
000506B       IF(LPN(5).EQ.0) GO TO 330
000507B       WRITE(IPRINT,41) (MS(I),I=1,6)
000520B       WRITE(IPRINT,42)II,AC(II),AA(II,1),AA(II,2),(AZ(II,IB),IB=1,NOBS)
000544B   330 LSKIP=1
000544B       GO TO 410
000545B   344 IF(.NOT.(KS.EQ.1)) GO TO 352
000551B       II=II+1
000552B       AC(II)=AAC(4)
000553B       AA(II,1)=2.0
```

```
000554B        AA(II,2)=0.0
000555B        DO 348 IZ=1,NOBS
000560B        IF(IZ.EQ.J1) AZ(II,IZ)=AZ(II,IZ)+1.0
000567B        IF(IZ.EQ.K1) AZ(II,IZ)=AZ(II,IZ)+1.0
000575B        IF(IZ.EQ.J2) AZ(II,IZ)=AZ(II,IZ)+1.0
000603B        IF(IZ.EQ.K2) AZ(II,IZ)=AZ(II,IZ)+1.0
000611B    348 CONTINUE
000613B        IF(LPN(5).EQ.0) GO TO 350
000614B        WRITE(IPRINT,41) (MS(I),I=1,6)
000625B        WRITE(IPRINT,42)II,AC(II),AA(II,1),AA(II,2),(AZ(II,IB),IB=1,NOBS)
000651B    350 LSKIP=1
000651B        GO TO 410
000652B    352 II=II+1
000654B        AC(II)=4.0*AAC(4)
000655B        AA(II,1)=2.0
000657B        AA(II,2)=0.0
000657B        DO 356 IZ=1,NOBS
000662B        IF(IZ.EQ.J1) AZ(II,IZ)=AZ(II,IZ)+1.0
000671B        IF(IZ.EQ.K1) AZ(II,IZ)=AZ(II,IZ)+1.0
000677B        IF(IZ.EQ.J2) AZ(II,IZ)=AZ(II,IZ)+1.0
000705B        IF(IZ.EQ.K2) AZ(II,IZ)=AZ(II,IZ)+1.0
000713B    356 CONTINUE
000715B        IF(LPN(5).EQ.0) GO TO 360
000716B        WRITE(IPRINT,41) (MS(I),I=1,6)
000727B        WRITE(IPRINT,42)II,AC(II),AA(II,1),AA(II,2),(AZ(II,IB),IB=1,NOBS)
000753B    360 LSKIP=1
000753B    410 LOUT=1
000754B        RETURN
000757B        END
          C
          C     ROUTINE POWSUA IS CALLED IN POWPRO
          C
000000B        SUBROUTINE POWSUA
000000B        COMMON/MSKP/MS(6),KP(30,21)
000000B        COMMON/ACAAAZ/AC(30),R(30),D(30),AAC(30),AA(30,2),SIZN(30,10),AZ(3
              10,10)
000000B        COMMON/NKPP/N(2,2,3),KPP(8,6)
000000B        COMMON/AC1AA1/AC1(4,10),D1(4,10),AA1(4,10,2),SIGN1(4,10,10),AZ1(4,
              110,10)
000000B        COMMON/IGIGMX/IG(4),IGMAX(4)
000000B        COMMON/ACGAAG/ACG(4,10),DG(4,10),AAG(4,10,2),SIGNG(4,10,10),AZG(4,
              110,10)
000000B        COMMON/AKSTAR/AKSTAR(50),DEGREE(50),GAMMA(10)
000000B        COMMON/IREAD/IREAD,IPRINT,ITAPE
000000B        COMMON/NOBS/NOBS,NPAR,KSTAR,NPMK,NPMKP1,NPARP
000000B        COMMON/KS/KS,IP,IIMAX,II
000000B        COMMON/I1J1K1/I1,J1,K1,I2,J2,K2
000000B        COMMON/LPRINT/LPN(50)
000000B        COMMON/IQAB/IQA,IQB
000000B        COMMON/KPOWER/KPOWER
000000B        N(1,1,1)=I1
000000B        N(1,1,2)=J1
000002B        N(1,1,3)=K1
000003B        N(1,2,1)=I1
000004B        N(1,2,2)=K1
000005B        N(1,2,3)=J1
000006B        N(2,1,1)=I2
000010B        N(2,1,2)=J2
000011B        N(2,1,3)=K2
000013B        N(2,2,1)=I2
000014B        N(2,2,2)=K2
000014B        N(2,2,3)=J2
000016B        IF(.NOT.((KP(IP,1).NE.1.AND.KP(IP,4).NE.1).AND.(KP(IP,1).EQ.2.OR.K
              1P(IP,4).EQ.2).AND.KP(IP,2).EQ.1.AND.KP(IP,3).EQ.1.AND.KP(IP,5).EQ.
              11.AND.KP(IP,6).EQ.1)) GO TO 510
          C
          C     TO REPRESENT 4.0 (SUM FROM H EQUAL TO P-KSTAR+1 TO N,SUM FROM
          C     I EQUAL TO P-KSTAR+1 TO N,AND AT LEAST ONE OF H,I IS EQUAL TO
          C     P-KSTAR+1 OR OR P-KSTAR+2,...,OR P) (SUM FROM J EQUAL TO 1 TO
          C     P-KSTAR,SUM FROM K EQUAL TO 1 TO P-KSTAR,SUM FROM L EQUAL TO 1 TO
          C      P-KSTAR) OF A+HJLA+IKLZHZIZJZK
          C
```

```
000042B          IF(.NOT.(KS.EQ.0)) GO TO 460
000045B          IPP=0
000045B          DO 454 IPERM=1,2
000050B          IF(IPERM.EQ.1) GO TO 420
000052B          IF(IPERM.EQ.2) GO TO 424
000053B      420 IPER1=1
000054B          IPER2=2
000055B          GO TO 426
000056B      424 IPER1=2
000057B          IPER2=1
000060B      426 LSKIP=1
000061B          DO 450 J=1,2
000063B          DO 446 K=1,2
000065B          IF(N(IPER1,J,3).NE.N(IPER2,K,3)) GO TO 446
000074B          IPP=IPP+1
000076B          DO 428 IM=1,3
000100B          KPP(IPP,IM)=N(IPER1,J,IM)
000105B          IN=IM+3
000106B          KPP(IPP,IN)=N(IPER2,K,IM)
000113B      428 CONTINUE
000114B          IF(IPP.EQ.1) GO TO 436
000115B          KSAME=0
000116B          IPPM=IPP-1
000117B          DO 432 I=1,IPPM
000121B          DO 430 L=1,6
000123B          IF(KPP(IPP,L).NE.KPP(I,L)) GO TO 432
000123B      430 CONTINUE
000132B          KSAME=1
000136B          GO TO 434
000136B      432 CONTINUE
000137B      434 IF(KSAME.EQ.0) GO TO 436
000143B          IPP=IPP-1
000144B          GO TO 446
000144B      436 LSKIP=1
000145B          II=II+1
000147B          AC(II)=AAC(5)
000150B          AA(II,1)=1.0
000151B          AA(II,2)=1.0
000152B          DO 438 IZ=1,NOBS
000155B          IF(IZ.EQ.KPP(IPP,1)) AZ(II,IZ)=AZ(II,IZ)+1.0
000164B          IF(IZ.EQ.KPP(IPP,2)) AZ(II,IZ)=AZ(II,IZ)+1.0
000172B          IF(IZ.EQ.KPP(IPP,4)) AZ(II,IZ)=AZ(II,IZ)+1.0
000200B          IF(IZ.EQ.KPP(IPP,5)) AZ(II,IZ)=AZ(II,IZ)+1.0
000206B      438 CONTINUE
000210B          IF(LPN(5).EQ.0) GO TO 444
000211B          WRITE(IPRINT,440) (MS(I),I=1,6)
000222B      440 FORMAT(/1X,21HI1,J1,K1,I2,J2,K2 ARE,6(I2,1X))
000222B          WRITE(IPRINT,442)II,AC(II),AA(II,1),AA(II,2),(AZ(II,IB),IB=1,NOBS)
000246B      442 FORMAT(/1X,5HII = ,I2,1X,9HAC(II) = ,F4.1,2X,11HAA(II,1) = ,F4.1,2
                1X,11HAA(II,2) = ,F4.1,2X,13HAZ(II,IZ) ARE,10(F4.1,1X))
000246B      444 LSKIP=1
000246B      446 CONTINUE
000251B      450 CONTINUE
000254B      454 CONTINUE
000257B          GO TO 510
000260B      460 IF(.NOT.(KS.EQ.1)) GO TO 480
000263B          II=II+1
000264B          AC(II)=AAC(5)
000265B          AA(II,1)=2.0
000266B          AA(II,2)=0.0
000267B          DO 470 IZ=1,NOBS
000272B          IF(IZ.EQ.I1) AZ(II,IZ)=AZ(II,IZ)+1.0
000301B          IF(IZ.EQ.I2) AZ(II,IZ)=AZ(II,IZ)+1.0
000307B          IF(IZ.EQ.J1) AZ(II,IZ)=AZ(II,IZ)+1.0
000315B          IF(IZ.EQ.J2) AZ(II,IZ)=AZ(II,IZ)+1.0
000323B      470 CONTINUE
000325B          IF(LPN(5).EQ.0) GO TO 474
```

```
000326B          WRITE(IPRINT,440) (MS(I),I=1,6)
000337B          WRITE(IPRINT,442)II,AC(II),AA(II,1),AA(II,2),(AZ(II,IB),IB=1,NOBS)
000363B      474 LSKIP=1
000363B          GO TO 510
000364B      480 II=II+1
000366B          AC(II)=AAC(5)
000367B          AA(II,1)=2.0
000370B          AA(II,2)=0.0
000371B          DO 490 IZ=1,NOBS
000374B          IF(IZ.EQ.I1) AZ(II,IZ)=AZ(II,IZ)+1.0
000403B          IF(IZ.EQ.I2) AZ(II,IZ)=AZ(II,IZ)+1.0
000411B          IF(IZ.EQ.J1) AZ(II,IZ)=AZ(II,IZ)+1.0
000417B          IF(IZ.EQ.J2) AZ(II,IZ)=AZ(II,IZ)+1.0
000425B      490 CONTINUE
000427B          IF(LPN(5).EQ.0) GO TO 491
000430B          WRITE(IPRINT,440) (MS(I),I=1,6)
000441B          WRITE(IPRINT,442)II,AC(II),AA(II,1),AA(II,2),(AZ(II,IB),IB=1,NOPS)
000465B      491 LSKIP=1
000465B          II=II+1
000467B          AC(II)=AAC(5)
000470B          AA(II,1)=2.0
000471B          AA(II,2)=0.0
000472B          DO 492 IZ=1,NOBS
000475B          IF(IZ.EQ.I1) AZ(II,IZ)=AZ(II,IZ)+1.0
000504B          IF(IZ.EQ.I2) AZ(II,IZ)=AZ(II,IZ/+1.0
000512B          IF(IZ.EQ.K1) AZ(II,IZ)=AZ(II,IZ)+1.0
000520B          IF(IZ.EQ.K2) AZ(II,IZ)=AZ(II,IZ)+1.0
000526B      492 CONTINUE
000530B          IF(LPN(5).EQ.0) GO TO 510
000531B          WRITE(IPRINT,440) (MS(I),I=1,6)
000542B          WRITE(IPRINT,442)II,AC(II),AA(.I,1),AA(II,2),(AZ(II,IB),IB=1,NOPS)
000566B      510 LEND=1
000566B          IF(.NOT.((KP(IP,1).EQ.3.AND.KP(IP,4).EQ.3).AND.(KP(IP,2).EQ.2.OR.K
                1P(IP,3).EQ.2.OR.KP(IP,5).EQ.2.OR.KP(IP,6).EQ.2))) GO TO 610
           C
           C     TO REPRESENT -4.0(SUM FROM H EQUAL TO P+1 TO N) (SUM FROM I
           C     EQUAL TO P+1 TO N) (SUM FROM J EQUAL 1 TO P,SUM FROM K EQUAL TO 1
           C     TO P,SUM FROM L EQUAL TO 1 TO P,AND AT LEAST ONE OF J,K,L IS
           C     EQUAL TO P-KSTAR+1 OR P-KSTAR+2,...,OR P) OF A+HJLA+IKLZHZIZJZK
           C
000607B          IF(.NOT.(KS.EQ.0)) GO TO 540

000613B          IPP=0
000614B          DO 536 IPERM=1,2
000616B          IF(IPERM.EQ.1) GO TO 512
000620B          IF(IPERM.EQ.2) GO TO 514
000621B      512 IPER1=1
000622B          IPER2=2
000623B          GO TO 516
000624B      514 IPER1=2
000625B          IPER2=1
000626B      516 LSKIP=1
000627B          DO 534 J=1,2
000631B          DO 532 K=1,2
000633B          IF(N(IPER1,J,3).NE.N(IPER2,K,3)) GO TO 532
000642B          IPP=IPP+1
000644B          DO 518 IM=1,3
000646B          KPP(IPP,IM)=N(IPER1,J,IM)
000653B          IN=IM+3
000654B          KPP(IPP,IN)=N(IPER2,K,IM)
000661B      518 CONTINUE
000662B          IF(IPP.EQ.1) GO TO 528
000663B          KSAME=0
000664B          IPPM=IPP-1
000665B          DO 524 I=1,IPPM
000667B          DO 520 L=1,6
000671B          IF(KPP(IPP,L).NE.KPP(I,L)) GO TO 524
```

```
0006718   520 CONTINUE
000700B       KSAME=1
000704B       GO TO 526
000704B   524 CONTINUE
000705B   526 IF(KSAME.EQ.0) GO TO 528
000711B       IPP=IPP-1
000712B       GO TO 532
000712B   528 LSKIP=1
000713B       II=II+1
000715B       AC(II)=AAC(6)
000716B       AA(II,1)=1.0
000717B       AA(II,2)=1.0
000720B       DO 530 IZ=1,NOBS
000723B       IF(IZ.EQ.KPP(IPP,1)) AZ(II,IZ)=AZ(II,IZ)+1.0
000732B       IF(IZ.EQ.KPP(IPP,2)) AZ(II,IZ)=AZ(II,IZ)+1.0
000740B       IF(IZ.EQ.KPP(IPP,4)) AZ(II,IZ)=AZ(II,IZ)+1.0
000746B       IF(IZ.EQ.KPP(IPP,5)) AZ(II,IZ)=AZ(II,IZ)+1.0
000754B   530 CONTINUE
000756B       IF(LPN(5).EQ.0) GO TO 531
000757B       WRITE(IPRINT,440) (MS(I),I=1,6)
000770B       WRITE(IPRINT,442)II,AC(II),AA(II,1),AA(II,2),(AZ(II,IB),IB=1,NOBS)
001014B   531 LSKIP=1
001014B   532 CONTINUE
001017B   534 CONTINUE
001022B   536 CONTINUE
001025B       GO TO 610
001026B   540 IF(.NOT.(KS.EQ.1)) GO TO 560
001031B       II=II+1
001032B       AC(II)=AAC(6)
001033B       AA(II,1)=2.0
001034B       AA(II,2)=0.0
001035B       DO 550 IZ=1,NOBS
001040B       IF(IZ.EQ.I1) AZ(II,IZ)=AZ(II,IZ)+1.0
001047B       IF(IZ.EQ.I2) AZ(II,IZ)=AZ(II,IZ)+1.0
001055B       IF(IZ.EQ.J1) AZ(II,IZ)=AZ(II,IZ)+1.0
001063B       IF(IZ.EQ.J2) AZ(II,IZ)=AZ(II,IZ)+1.0
001071B
001071B   550 CONTINUE
001073B       IF(LPN(5).EQ.0) GO TO 554
001074B       WRITE(IPRINT,440) (MS(I),I=1,6)
001105B       WRITE(IPRINT,442)II,AC(II),AA(II,1),AA(II,2),(AZ(II,IB),IB=1,NOBS)
001131B   554 LSKIP=1
001131B       GO TO 610
001132B   560 II=II+1
001134B       AC(II)=AAC(6)
001135B       AA(II,1)=2.0
001136B       AA(II,2)=0.0
001137B       DO 570 IZ=1,NOBS
001142B       IF(IZ.EQ.I1) AZ(II,IZ)=AZ(II,IZ)+1.0
001151B       IF(IZ.EQ.I2) AZ(II,IZ)=AZ(II,IZ)+1.0
001157B       IF(IZ.EQ.J1) AZ(II,IZ)=AZ(II,IZ)+1.0
001165B       IF(IZ.EQ.J2) AZ(II,IZ)=AZ(II,IZ)+1.0
001173B   570 CONTINUE
001175B       IF(LPN(5).EQ.0) GO TO 574
001176B       WRITE(IPRINT,440) (MS(I),I=1,6)
001207B       WRITE(IPRINT,442)II,AC(II),AA(II,1),AA(II,2),(AZ(II,IB),IB=1,NOBS)
001233B   574 LSKIP=1
001233B       II=II+1
001235B       AC(II)=AAC(6)
001236B       AA(II,1)=2.0
001237B       AA(II,2)=0.0
001240B       DO 580 IZ=1,NOBS
001243B       IF(IZ.EQ.I1) AZ(II,IZ)=AZ(II,IZ)+1.0
001252B       IF(IZ.EQ.I2) AZ(II,IZ)=AZ(II,IZ)+1.0
001260B       IF(IZ.EQ.K1) AZ(II,IZ)=AZ(II,IZ)+1.0
001266B       IF(IZ.EQ.K2) AZ(II,IZ)=AZ(II,IZ)+1.0
```

```
001274B      580 CONTINUE
001276B          IF(LPN(5).EQ.0) GO TO 584
001277B          WRITE(IPRINT,440) (MS(I),I=1,6)
001310B          WRITE(IPRINT,442)II,AC(II),AA(II,1),AA(II,2),(AZ(II,IB),IB=1,NOBS)
001334B      584 LSKIP=1
001334B      610 LOUT=1
001335B          IIMAX=II
         C
         C     NOW WE HAVE AC(II),AA(II,1),AA(II,2) AND AZ(II,IZ) WHERE II=1,IIMAX
         C     AND IZ=1,NOBS
         C
001336B          CALL POWSUB
001341B          RETURN
001343B          END

         C
         C     ROUTINE POWSUB IS CALLED IN POWSUA
         C
000000B          SUBROUTINE POWSUB
000000B          COMMON/MSKP/MS(6),KP(30,21)
000000B          COMMON/ACAAAZ/AC(30),R(30),D(30),AAC(30),AA(30,2),SIZN(30,10),AZ(3
                1 0,10)
000000B          COMMON/NKPP/N(2,2,3),KPP(8,6)
000000B          COMMON/AC1AA1/AC1(4,10),D1(4,10),AA1(4,10,2),SIGN1(4,10,10),AZ1(4,
                1 10,10)
000000B          COMMON/IGIGMX/IG(4),IGMAX(4)
000000B          COMMON/ACGAAG/ACG(4,10),DG(4,10),AAG(4,10,2),SIGNG(4,10,10),AZG(4,
                1 10,10)
000000B          COMMON/AKSTAR/AKSTAR(50),DEGREE(50),GAMMA(10)
000000B          COMMON/IREAD/IREAD,IPRINT,ITAPE
000000B          COMMON/NOBS/NOBS,NPAR,KSTAR,NPMK,NPMKP1,NPARP
000000B          COMMON/KS/KS,IP,IIMAX,II
000000B          COMMON/I1J1K1/I1,J1,K1,I2,J2,K2
000000B          COMMON/LPRINT/LPN(50)
000000B          COMMON/IQAB/IQA,IQB
000000B          COMMON/KPOWER/KPOWER
000000B          IF(.NOT.IIMAX.EQ.0) GO TO 628
000002B          IP=IP-1
000003B          GO TO 812
000005B      628 LSKIP=1
         C
         C     TO REPRESENT THE EQUATION IN SECTION (4.5)
         C
000006B          DO 660 IV=1,IIMAX
000016B          II=IV
000016B          IF(NPMK.EQ.0) GO TO 633
000020B          DO 632 IZ=1,NPMK
000027B          SIZN(II,IZ)=1.0
000027B      632 CONTINUE
000050B      633 LSKIP=1
000053B          DO 634 IZ=NPARP,NOBS
000066B          SIZN(II,IZ)=1.0
000066B      634 CONTINUE
```

```
000107B           TS=0.0
000110B           DO 654 IZ=NPMKP1,NPAR
000123B           TS=TS+AZ(II,IZ)
000132B           IF(INT(AZ(II,IZ)).EQ.0) GO TO 640
000143B           NTEST=INT(AZ(II,IZ)/2.0)
000152B           VNTEST=NTEST
000154B           VTEST=(AZ(II,IZ)/2.0)-VNTEST
000163B           IF(VTEST.GT.0.0) SIZN(II,IZ)=-1.0
000175B           IF(.NOT.(VTEST.GT.0.0)) SIZN(II,IZ)=1.0
000206B           GO TO 654
000207B       640 SIZN(II,IZ)=1.0
000220B       654 CONTINUE
000225B           R(II)=TS
000231B           IF(LPN(4).EQ.0) GO TO 659
000233B           WRITE(IPRINT,803) (MS(I),I=1,6)
000253B           IF(.NOT.II.EQ.1) GO TO 658
000256B           WRITE(IPRINT,657)
000265B       657 FORMAT(/1X,60HII,II,R(II),AC(II),AA(II,1),AA(II,2),SIZN(II,*),AZ(I
                 1I,*) ARE)
000265B       658 LSKIP=1
000266B           WRITE(IPRINT,806) II,II,R(II),AC(II),AA(II,1),AA(II,2),((SIZN(II,I
                 1Z),AZ(II,IZ)),IZ=1,NOBS)
000356B       659 LSKIP=1
000357B       660 CONTINUE
            C
            C     TO REPRESENT FIRST PARTIAL DERIVATIVE OF (R+)**2 W.R.T. A+I1J1K1
            C     EVALUATED AT A+=0,FIRST PARTIAL DERIVATIVE OF (R+)**2 W.R.T.
            C     A+I2J2K2 EVALUATED AT A+=0 AND SECOND PARTIAL DERIVATIVE OF (R+)**2
            C     W.R.T. A+I1J1K1,A+I2J2K2 EVALUATED AT A+=0 (C.F. SECTION (4.6))
            C
000365B           II1=0
000365B           DO 700 IV=1,IIMAX
000374B           II=IV
000374B           IF(.NOT.((INT(AA(II,1)).EQ.1.AND.INT(AA(II,2)).EQ.0).OR.(INT(AA(II
                 1,1)).EQ.0.AND.INT(AA(II,2)).EQ.1))) GO TO 700
000420B           II1=II1+1
000421B           DO 670 IRD=1,4
000424B           VIRD=IRD
000424B           D1(IRD,II1)=R(II)+(VIRD-2.0)
000440B           AC1(IRD,II1)=0.5*VIRD*AC(II)
000453B           AA1(IRD,II1,1)=AA(II,1)
000465B           AA1(IRD,II1,2)=AA(II,2)
000476B           DO 666 IZ=1,NOBS
000506B           SIGN1(IRD,II1,IZ)=SIZN(II,IZ)
000524B           AZ1(IRD,II1,IZ)=AZ(II,IZ)
000541B       666 CONTINUE
000544B       670 CONTINUE
000547B       700 CONTINUE
000554B           II1MAX=II1
000554B           IF(LPN(6).EQ.0) GO TO 709
000556B           WRITE(IPRINT,803) (MS(I),I=1,6)
000576B           WRITE(IPRINT,702)
000605B       702 FORMAT(/1X,98HIRD,IRD,D1(IRD,II1),AC1(IRD,II1),AA1(IRD,II1,1),AA1(
                 1IRD,II1,2),SIGN1(IRD,II1,*),AZ1(IRD,II1,*) ARE)
000605B           DO 708 II1=1,II1MAX
000614B           DO 706 IRD=1,4
000617B           WRITE(IPRINT,806) IRD,IRD,D1(IRD,II1),AC1(IRD,II1),AA1(IRD,II1,1),
                 1AA1(IRD,II1,2),((SIGN1(IRD,II1,IB),AZ1(IRD,II1,IB)),IB=1,NOBS)
000705B       706 CONTINUE
000710B       708 CONTINUE
000715B       709 LSKIP=1
000716B           III=IIMAX
000720B           DO 730 IV=1,IIMAX
000727B           II=IV
000727B           IF(.NOT.((INT(AA(II,1)).EQ.1.AND.INT(AA(II,2)).EQ.0).OR.(INT(AA(II
                 1,1)).EQ.0.AND.INT(AA(II,2)).EQ.1))) GO TO 720
000753B           D(II)=R(II)
```

```
000757B             IF(INT(R(II)).EQ.0) GO TO 730
              C
              C    AC(II),AA(II,*),SIZN(II,IZ) AND AZ(II,IZ) ARE NOT CHANGED
              C
000765B             DO 714 II1=1,II1MAX
000774B             III=III+1
000775B             IRD=R(II)
001003B             D(III)=D1(IRD,II1)
001014B             AC(III)=AC(II)*AC1(IRD,II1)
001031B             AA(III,1)=AA(II,1)+AA1(IRD,II1,1)
001047B             AA(III,2)=AA(II,2)+AA1(IRD,II1,2)
001066B             DO 710 IZ=1,NOBS
001077B             SIZN(III,IZ)=SIZN(II,IZ)*SIGN1(IRD,II1,IZ)
001123B             AZ(III,IZ)=AZ(II,IZ)+AZ1(IRD,II1,IZ)
001147B         710 CONTINUE
001151B         714 CONTINUE
001156B             GO TO 730
001157B         720 D(II)=R(II)
              C
              C    AC(II),AA(II,*),SIGN(II,IZ) AND AZ(II,IZ) ARE NOT CHANGED
              C
001165B         730 CONTINUE
001173B             IIIMAX=III
              C
              C    PARTIAL DERIVATIVES OF (R+)**2 ARE INDEXED BY KKG
              C    FIND IG(I)-TH TERM OF THE I-TH DERIVATIVES
              C
001173B             DO 740 J=1,4
001177B             IG(J)=0
001177B         740 CONTINUE
001213B             DO 800 I=1,IIIMAX
001223B             IF(INT(AA(I,1)).EQ.1.AND.INT(AA(I,2)).EQ.0) GO TO 750
001240B             IF(INT(AA(I,1)).EQ.0.AND.INT(AA(I,2)).EQ.1) GO TO 760
001253B             IF(INT(AA(I,1)).EQ.2.AND.INT(AA(I,2)).EQ.0) GO TO 770
001266B             IF(INT(AA(I,1)).EQ.1.AND.INT(AA(I,2)).EQ.1) GO TO 776
001302B             GO TO 800
001303B         750 IG(1)=IG(1)+1
001305B             IIG=IG(1)
001305B             KKG=1
001306B             GO TO 780
001310B         760 IG(2)=IG(2)+1
001312B             IIG=IG(2)
001312B             KKG=2
001313B             GO TO 780
001315B         770 IG(3)=IG(3)+1
001317B             IIG=IG(3)
001317B             KKG=3
001320B             GO TO 780
001322B         776 IG(4)=IG(4)+1
001324B             IIG=IG(4)
001324B             KKG=4
001325B         780 DG(KKG,IIG)=D(I)
001342B             IF(KKG.EQ.3) ACG(KKG,IIG)=2.0*AC(I)
001356B             IF(KKG.NE.3) ACG(KKG,IIG)=AC(I)
001371B             AAG(KKG,IIG,1)=AA(I,1)
001402B             AAG(KKG,IIG,2)=AA(I,2)
001414B             DO 790 IZ=1,NOBS
001423B             SIGNG(KKG,IIG,IZ)=SIZN(I,IZ)
001441B             AZG(KKG,IIG,IZ)=AZ(I,IZ)
001451B         790 CONTINUE
001454B         800 CONTINUE
001460B             DO 802 I=1,4
001462B             IGMAX(I)=IG(I)
001462B         802 CONTINUE
001476B             IF(LPN(3).EQ.0) GO TO 812
001501B             KPRINT=0
001501B             DO 810 KKG=1,4
```

```
0015058          IF(IGMAX(KKG).EQ.0) GO TO 810
0015128          DO 808 IIG=1,IGMAX(KKG)
0015258          KPRINT=KPRINT+1
0015268          IF(.NOT.(KPRINT.EQ.1)) GO TO 805
0015318          WRITE(IPRINT,803) (MS(I),I=1,6)
0015528      803 FORMAT(/1X,21HI1,J1,K1,I2,J2,K2 ARE,6(I2,1X))
0015528          WRITE(IPRINT,804)
0015618      804 FORMAT(/1X,83HKKG,IIG,DG(KKG,IIG),ACG(KKG,IIG),AAG(KKG,IIG,*),SIGN
                1G(KKG,IIG,*),AZG(KKG,IIG,*) ARE)
0015618      805 LSKIP=1
0015628          WRITE(IPRINT,806) KKG,IIG,DG(KKG,IIG),ACG(KKG,IIG),AAG(KKG,IIG,1),
                1 AAG(KKG,IIG,2),((SIGNG(KKG,IIG,IZ),AZG(KKG,IIG,IZ)),IZ=1,NOBS)
0016618      806 FORMAT(/1X,2(1H/,I3),2(1H/,F4.1),1H/,2(F4.1,1X),10(1H/,F4.1,1X,F3.
                11))
0016618      808 CONTINUE
0016658      810 CONTINUE
0016708      812 LSKIP=1
0016728          IF(KPOWER.EQ.1) CALL POWSUC
0017018          IF(KPOWER.NE.1) CALL SIGSUC
0017068          RETURN
0017108          END

             C
             C     ROUTINE POWSUC IS CALLED IN POWSUB
             C
0000008          SUBROUTINE POWSUC
0000008          COMMON/MSKP/MS(6),KP(30,21)
0000008          COMMON/ACAAAZ/AC(30),R(30),D(30),AAC(30),AA(30,2),SIZN(30,10),AZ(3
                10,10)
0000008          COMMON/NKPP/N(2,2,3),KPP(8,6)
0000008          COMMON/AC1AA1/AC1(4,10),D1(4,10),AA1(4,10,2),SIGN1(4,10,10),AZ1(4,
                110,10)
0000008          COMMON/IGIGMX/IG(4),IGMAX(4)
0000008          COMMON/ACGAAG/ACG(4,10),DG(4,10),AAG(4,10,2),SIGNG(4,10,10),AZG(4,
                110,10)
0000008          COMMON/AKSTAR/AKSTAR(50),DEGREE(50),GAMMA(10)
0000008          COMMON/IREAD/IREAD,IPRINT,ITAPE
0000008          COMMON/NOBS/NOBS,NPAR,KSTAR,NPMK,NPMKP1,NPARP
0000008          COMMON/KS/KS,IP,IIMAX,II
0000008          COMMON/I1J1K1/I1,J1,K1,I2,J2,K2
0000008          COMMON/LPRINT/LPN(50)
0000008          COMMON/IQAB/IQA,IQB
0000008          COMMON/MS123/MS1(4),MS2(4),MS3(4),MMS1(4),MMS2(4),MMS3(4)
0000008          IF(IIMAX.EQ.0) GO TO 1020
             C
             C     WE HAVE FIRST AND SECOND ORDER PARTIAL DERIVATIVES OF (R+)**2
             C     TO FIND PRODUCT OF THE TWO FIRST ORDER DERIVATIVES
             C
0000018          IC=0
0000018          IA=1
0000028          IF(KS.EQ.0) JB=2
0000058          IF(KS.EQ.1) JB=1
0000108          IF(KS.EQ.2) JB=1
0000138          IF(.NOT.(KS.EQ.0)) GO TO 814
0000148          IGMAX2=IGMAX(2)
0000158          IF(IGMAX(1).EQ.0.OR.IGMAX(2).EQ.0) GO TO 864
0000208          GO TO 818
0000218      814 LSKIP=1
0000228          IGMAX2=IGMAX(1)
0000248          IF(IGMAX2.EQ.0) GO TO 864
0000258      818 LSKIP=1
0000278          DO 860 I=1,IGMAX(1)
0000408          DO 850 J=1,IGMAX2
0000508          IC=IC+1
0000518          D(IC)=DG(IA,I)+DG(JB,J)
0000708          AC(IC)=ACG(IA,I)*ACG(JB,J)
0001108          DO 822 IZ=1,NOBS
```

```
000121B            SIZN(IC,IZ)=SIGNG(IA,I,IZ)*SIGNG(JB,J,IZ)
000146B            AZ(IC,IZ)=AZG(IA,I,IZ)+AZG(JB,J,IZ)
000174B        822 CONTINUE
000176B            IF(IC.EQ.1) GO TO 830
000200B            ICM1=IC-1
000201B            ISAME=0
000201B            DO 826 IADD=1,ICM1
000211B            IF(INT(D(IADD)).NE.INT(D(IC))) GO TO 826
000224B            DO 824 IZ=1,NOBS
000233B            IF(INT(SIZN(IADD,IZ)).NE.INT(SIZN(IC,IZ))) GO TO 826
000256B            IF(INT(AZ(IADD,IZ)).NE.INT(AZ(IC,IZ))) GO TO 826
000256B        824 CONTINUE
000271B            ISAME=1
000276B            IADDS=IADD
000277B            GO TO 828
000277B        826 CONTINUE
000301B        828 IF(ISAME.EQ.0) GO TO 830
000310B            AC(IADDS)=AC(IADDS)+AC(IC)
000321B            IC=IC-1
000322B        830 LSKIP=1
000324B        850 CONTINUE
000332B        860 CONTINUE
000336B        864 LSKIP=1
000337B            ICMAXM=IC
          C
          C   NOW THE TERMS IN THE EXPRESSION OF THE PRODUCT OF THE FIRST
          C   PARTIAL DERIVATIVES OF (R+)**2 W.R.T. A+I1J1K1 AND A+I2J2K2
          C   EVALUATED AT A+=0 ARE REPRESENTED BY
          C   ((D(JC),AC(JC),((SIZN(JC,IB),AZ(JC,IB)),IB=1,NOBS)),JC=1,ICMAXM)
          C
000341B            IF(ICMAXM.EQ.0) GO TO 869
000342B            IF(LPN(11).EQ.0) GO TO 869
000343B            WRITE(IPRINT,865) (MS(IB),IB=1,6)
000363B            DO 868 JC=1,ICMAXM
000372B            WRITE(IPRINT,866) JC,D(JC),AC(JC),((SIZN(JC,IB),AZ(JC,IB)),IB=1,NO
                  1BS)
000441B        868 CONTINUE
000445B        865 FORMAT(/1X,21HI1,J1,K1,I2,J2,K2 ARE,6(I2,1X))
000445B        866 FORMAT( 1X,1H/,3HIC=,I2,1H/,6HD(IC)=,F4.1,1H/,7HAC(IC)=,F4.1/1X ,2
                  13HSIZN(IC,*),AZ(IC,*) ARE,10(F4.1,1H*,F3.1,1H/))
000445B        869 LSKIP=1
000446B            IF(ICMAXM.EQ.0) GO TO 930
000450B            DO 924 IC=1,ICMAXM
000457B            DO 882 I=1,4
000462B            MS1(I)=0
000466B            MS2(I)=0
000472B            MS3(I)=0
000476B            MMS1(I)=0
000502B            MMS2(I)=0
000506B            MMS3(I)=0
000512B        882 CONTINUE
000515B            IH=0
000515B            IF(NPMK.EQ.0) GO TO 887
000517B            DO 886 IZ=1,NPMK
000526B            IF(INT(AZ(IC,IZ)).EQ.0) GO TO 886
000537B            DO 884 I=1,6
000542B            IF(MS(I).NE.IZ) GO TO 884
000550B            IH=IH+1
000551B            MS1(IH)=I
000556B            MMS1(IH)=INT(AZ(IC,IZ))
000563B            GO TO 886
000563B        884 CONTINUE
000565B        886 CONTINUE
000575B        887 LSKIP=1
000576B            IH=0
000577B            DO 890 IZ=NPMKP1,NPAR
000612B            IF(INT(AZ(IC,IZ)).EQ.0) GO TO 890
```

```
000623B           DO 888 I=1,6
000626B           IF(MS(I).NE.IZ) GO TO 888
000634B           IH=IH+1
000635B           MS2(IH)=I
000642B           MMS2(IH)=INT(AZ(IC,IZ))
000647B           GO TO 890
000647B       888 CONTINUE
000651B       890 CONTINUE
000661B           IH=0
000661B           DO 894 IZ=NPARP,NOBS
000674B           IF(INT(AZ(IC,IZ)).EQ.0) GO TO 894
000705B           DO 892 I=1,6
000710B           IF(MS(I).NE.IZ) GO TO 892
000716B           IH=IH+1
000717B           MS3(IH)=I
000724B           MMS3(IH)=INT(AZ(IC,IZ))
000731B           GO TO 894
000731B       892 CONTINUE
000733B       894 CONTINUE
            C
            C     NOW A TERM IN THE EXPRESSION OF THE PRODUCT OF THE FIRST PARTIAL
            C     DERIVATIVES OF (R+)**2 W.R.T. A+I1J1K1 AND A+I2J2K2 EVALUATED AT
            C     A+=0 IS REPRESENTED BY
            C     D(IC),AC(IC),((MS1(I),MMS1(I)),I=1,4),((MS2(I),MMS2(I)),I=1,4)
            C     AND ((MS3(I),MMS3(I)),I=1,4)
            C
000743B           IQA=IQA+1
000744B           IF(LPN(12).EQ.0) GO TO 901
000745B           WRITE(IPRINT,900) IQA,((MS1(IB),MMS1(IB)),IB=1,4),((MS2(IB),MMS2(I
                 1B)),IB=1,4),((MS3(IB),MMS3(IB)),IB=1,4)
001027B       900 FORMAT(1X,4HIQA=,I5,22HMS1(*)-MMS1(*) ETC ARE,1H/,4(I1,1H*,I1,1H/)
                 1,2H//,4(I1,1H*,I1,1H/),2H//,4(I1,1H*,I1,1H/),2H//)
001027B       901 LSKIP=1
001030B           ITAPE9=9
001032B           WRITE(ITAPE9,910) IQA,ICMAXM,(MS(IB),IB=1,6),(KP(IP,IB),IB=1,21),(
                 1(MS1(IB),MMS1(IB)),IB=1,4),((MS2(IB),MMS2(IB)),IB=1,4),((MS3(IB),M
                 1MS3(IB)),IB=1,4),D(IC),AC(IC)
001160B       910 FORMAT(1X,I5,I5,51I2,2F5.1)
001160B       924 CONTINUE
001164B       930 LSKIP=1
            C
            C     TO REPRESENT THE EXPRESSION OF THE SECOND PARTIAL DERIVATIVE OF
            C     (R+)**2 W.R.T. A+I1J1K1 AND A+I2J2K2 EVALUATED AT A+=0
            C
001165B           IF(KS.EQ.0) IGMAX4=IGMAX(4)
001171B           IF(KS.EQ.1.OR.KS.EQ.2) IGMAX4=IGMAX(3)
001176B           IF(IGMAX4.EQ.0) GO TO 986
001177B           IC=0
001177B           DO 984 I=1,IGMAX4
001206B           IC=IC+1
001207B           IF(KS.EQ.0) D(IC)=DG(4,I)
001224B           IF(KS.EQ.1.OR.KS.EQ.2) D(IC)=DG(3,I)
001242B           IF(KS.EQ.0) AC(IC)=ACG(4,I)
001256B           IF(KS.EQ.1.OR.KS.EQ.2) AC(IC)=ACG(3,I)
001274B           DO 970 IZ=1,NOBS
001303B           IF(KS.EQ.0) SIZN(IC,IZ)=SIGNG(4,I,IZ)
001323B           IF(KS.EQ.1.OR.KS.EQ.2) SIZN(IC,IZ)=SIGNG(3,I,IZ)
001344B           IF(KS.EQ.0) AZ(IC,IZ)=AZG(4,I,IZ)
001363B           IF(KS.EQ.1.OR.KS.EQ.2) AZ(IC,IZ)=AZG(3,I,IZ)
001405B       970 CONTINUE
001411B           IF(IC.EQ.1) GO TO 984
001412B           ICM1=IC-1
001413B           ISAME=0
001414B           DO 976 IADD=1,ICM1
001423B           IF(INT(D(IADD)).NE.INT(D(IC))) GO TO 976
001436B           DO 974 IZ=1,NOBS
001445B           IF(INT(SIZN(IADD,IZ)).NE.INT(SIZN(IC,IZ))) GO TO 976
```

```
001470B          IF(INT(AZ(IADD,IZ)).NE.INT(AZ(IC,IZ))) GO TO 976
001470B      974 CONTINUE
001503B          ISAME=1
001510B          IADDS=IADD
001511B          GO TO 978
001511B      976 CONTINUE
001513B      978 IF(ISAME.EQ.0) GO TO 984
001522B          AC(IADDS)=AC(IADD)+AC(IC)
001536B          IC=IC-1
001540B      984 CONTINUE
001545B      986 LSKIP=1
001546B          ICMAX=IC
         C
         C       NOW THE TERMS IN THE EXPRESSION OF THE SECOND PARTIAL DERIVATIVE
         C       OF (R+)**2 W.R.T. A+I1J1K1 AND A+I2J2K2 EVALUATED AT A+=0
         C       ARE REPRESENTED BY
         C       (D(IC),AC(IC),((SIZN(IC,IZ),AZ(IC,IZ)),IZ=1,NOBS)
         C
001550B          IF(ICMAX.NE.0) GO TO 987
001551B          GO TO 1020
001552B      987 LSKIP=1
001554B          IF(LPN(11).EQ.0) GO TO 996
001556B          WRITE(IPRINT,990)
001565B      990 FORMAT(/1X,22HSECOND DERIVATIVES ARE)
001565B          WRITE(IPRINT,865) (MS(IB),IB=1,6)
001605B          DO 994 IC=1,ICMAX
001614B          WRITE(IPRINT,866) IC,D(IC),AC(IC),((SIZN(IC,IB),AZ(IC,IB)),IB=1,NO
                1BS)
001663B      994 CONTINUE
001667B      996 LSKIP=1
001670B          DO 1014 IC=1,ICMAX
001700B          DO 997 I=1,4
001703B          MS1(I)=0
001707B          MS2(I)=0
001713B          MS3(I)=0
001717B          MMS1(I)=0
001723B          MMS2(I)=0
001727B          MMS3(I)=0
001733B      997 CONTINUE
001736B          IH=0
001736B          IF(NPMK.EQ.0) GO TO 1001
001740B          DO 1000 IZ=1,NPMK
001747B          IF(INT(AZ(IC,IZ)).EQ.0) GO TO 1000
001760B          DO 998 I=1,6
001763B          IF(MS(I).NE.IZ) GO TO 998
001771B          IH=IH+1
001772B          MS1(IH)=I
001777B          MMS1(IH)=INT(AZ(IC,IZ))
002004B          GO TO 1000
002004B      998 CONTINUE
002006B     1000 CONTINUE
002016B     1001 LSKIP=1
002017B          IH=0
002020B          DO 1004 IZ=NPMKP1,NPAR
002033B          IF(INT(AZ(IC,IZ)).EQ.0) GO TO 1004
002044B          DO 1002 I=1,6
002047B          IF(MS(I).NE.IZ) GO TO 1002
002055B          IH=IH+1
002056B          MS2(IH)=I
002063B          MMS2(IH)=INT(AZ(IC,IZ))
002070B          GO TO 1004
002070B     1002 CONTINUE
002072B     1004 CONTINUE
002102B          IH=0
002102B          DO 1008 IZ=NPARP,NOBS
002115B          IF(INT(AZ(IC,IZ)).EQ.0) GO TO 1008
002126B          DO 1006 I=1,6
```

```
002131B              IF(MS(I).NE.IZ) GO TO 1006
002137B              IH=IH+1
002140B              MS3(IH)=I
002145B              MMS3(IH)=INT(AZ(IC,IZ))
002152B              GO TO 1008
002152B         1006 CONTINUE
002154B         1008 CONTINUE
             C
             C       NOW A TERM IN THE EXPRESSION OF THE SECOND PARTIAL DERIVATIVE OF
             C       (R+)**2 W.R.T. A+I1J1K1 AND A+I2J2K2 EVALUATED AT A+=0 IS
             C       REPRESENTED BY
             C       D(IC),AC(IC),((MS1(I),MMS1(I)),I=1,4),((MS2(I),MMS2(I)),I=1,4)
             C       AND ((MS3(I),MMS3(I)),I=1,4)
             C
002164B              IOB=IOB+1
002165B              IF(LPN(12).EQ.0) GO TO 1010
002166B              WRITE(IPRINT,1009)IOB,((MS1(IB),MMS1(IB)),IB=1,4),((MS2(IB),MMS2(I
                    1B)),IB=1,4),((MS3(IB),MMS3(IB)),IB=1,4)
002250B         1009 FORMAT(1X,4HIOB=,I5,22HMS1(*),MMS1(*) ETC ARE,1H/,4(I1,1H*,I1,1H/)
                    1,2H//,4(I1,1H*,I1,1H/),2H//,4(I1,1H*,I1,1H/),2H//)
002250B         1010 LSKIP=1
002251B              ITAPE10=10
002253B              WRITE(ITAPE10,910)IOB,ICMAX ,(MS(IB),IB=1,6),(KP(IP,IB),IB=1,21),(
                    1(MS1(IB),MMS1(IB)),IB=1,4),((MS2(IB),MMS2(IB)),IB=1,4),((MS3(IB),M
                    1MS3(IB)),IB=1,4),D(IC),AC(IC)
002401B         1014 CONTINUE
002405B         1020 LSKIP=1
002406B              RETURN
002411B              END
             C
             C       ROUTINE SIGSUC IS CALLED IN POWSUB AND IT IS FOR DERIVING
             C       I1(THETA,SIGMA)
             C
000000B              SUBROUTINE SIGSUC
000000B              COMMON/MSKP/MS(6),KP(30,21)
000000B              COMMON/ACAAAZ/AC(30),R(30),D(30),AAC(30),AA(30,2),SIZN(30,10),AZ(3
                    10,10)
000000B              COMMON/NKPP/N(2,2,3),KPP(8,6)
000000B              COMMON/AC1AA1/AC1(4,10),D1(4,10),AA1(4,10,2),SIGN1(4,10,10),AZ1(4,
                    110,10)
000000B              COMMON/IGIGMX/IG(4),IGMAX(4)
000000B              COMMON/ACGAAG/ACG(4,10),DG(4,10),AAG(4,10,2),SIGNG(4,10,10),AZG(4,
                    110,10)
000000B              COMMON/AKSTAR/AKSTAR(50),DEGREE(50),GAMMA(10)
000000B              COMMON/IREAD/IREAD,IPRINT,ITAPE
000000B              COMMON/NOBS/NOBS,NPAR,KSTAR,NPMK,NPMKP1,NPARP
000000B              COMMON/KS/KS,IP,IIMAX,II
000000B              COMMON/I1J1K1/I1,J1,K1,I2,J2,K2
000000B              COMMON/LPRINT/LPN(50)
000000B              COMMON/IOAB/IOA,IOB
000000B              IF(IIMAX.EQ.0) GO TO 1010
             C
             C       TO REPRESENT ((((KSTAR/2)-1)/((DE)**2))-0.5)*FIRST DERIVATIVE OF
             C       (R*)**2 W.R.T. A*I1J1K1 EVALUATED AT A*=0*FIRST DERIVATIVE OF
             C       (R*)**2 W.R.T. A*I2J2K2 EVALUATED AT A*=0
             C
000001B              IC=0
000001B              IA=1
000002B              IF(KS.EQ.0) JB=2
000005B              IF(KS.EQ.1) JB=1
000010B              IF(KS.EQ.2) JB=1
000013B              IF(.NOT.(KS.EQ.0)) GO TO 814
000014B              IGMAX2=IGMAX(2)
000015B              IF(IGMAX(1).EQ.0.OR.IGMAX(2).EQ.0) GO TO 864
000020B              GO TO 818
000021B          814 LSKIP=1
```

```
000022B         IGMAX2=IGMAX(1)
000024B         IF(IGMAX2.EQ.0) GO TO 864
000025B     818 LSKIP=1
000027B         DO 860 I=1,IGMAX(1)
000040B         DO 850 J=1,IGMAX2
000050B         IC=IC+1
000051B         AKSTAR(IC)=0.0
000055B         D(IC)=DG(IA,I)+DG(JB,J)
000075B         AC(IC)=-0.5*ACG(IA,I)*ACG(JB,J)
000116B         DO 822 IZ=1,NOBS
000127B         SIZN(IC,IZ)=SIGNG(IA,I,IZ)*SIGNG(JB,J,IZ)
000154B         AZ(IC,IZ)=AZG(IA,I,IZ)+AZG(JB,J,IZ)
000202B     822 CONTINUE
000204B         IF(IC.EQ.1) GO TO 830
000206B         ICM1=IC-1
000207B         ISAME=0
000207B         DO 826 IADD=1,ICM1
000217B         IF(INT(AKSTAR(IADD)).NE.INT(AKSTAR(IC))) GO TO 826
000232B         IF(INT(D(IADD)).NE.INT(D(IC))) GO TO 826
000244B         DO 824 IZ=1,NOBS
000253B         IF(INT(SIZN(IADD,IZ)).NE.INT(SIZN(IC,IZ))) GO TO 826
000276B         IF(INT(AZ(IADD,IZ)).NE.INT(AZ(IC,IZ))) GO TO 826
000276B     824 CONTINUE
000311B         ISAME=1
000316B         IADDS=IADD
000317B         GO TO 828
000317B     826 CONTINUE
000321B     828 IF(ISAME.EQ.0) GO TO 830
000330B         AC(IADDS)=AC(IADDS)+AC(IC)
000341B         IC=IC-1
000342B     830 IC=IC+1
000345B         AKSTAR(IC)=0.0
000351B         D(IC)=DG(IA,I)+DG(JB,J)-2.0
000373B         AC(IC)=-ACG(IA,I)*ACG(JB,J)
000414B         DO 832 IZ=1,NOBS
000425B         SIZN(IC,IZ)=SIGNG(IA,I,IZ)*SIGNG(JB,J,IZ)
000452B         AZ(IC,IZ)=AZG(IA,I,IZ)+AZG(JB,J,IZ)
000500B     832 CONTINUE
000502B         IF(IC.EQ.2) GO TO 840
000504B         ICM1=IC-1
000505B         ISAME=0
000505B         DO 836 IADD=1,ICM1
000515B         IF(INT(AKSTAR(IADD)).NE.INT(AKSTAR(IC))) GO TO 836
000530B         IF(INT(D(IADD)).NE.INT(D(IC))) GO TO 836
000542B         DO 834 IZ=1,NOBS
000551B         IF(INT(SIZN(IADD,IZ)).NE.INT(SIZN(IC,IZ))) GO TO 836
000574B         IF(INT(AZ(IADD,IZ)).NE.INT(AZ(IC,IZ))) GO TO 836
000574B     834 CONTINUE
000607B         ISAME=1
000614B         IADDS=IADD
000615B         GO TO 838
000615B     836 CONTINUE
000617B     838 IF(ISAME.EQ.0) GO TO 840
000626B         AC(IADDS)=AC(IADDS)+AC(IC)
000637B         IC=IC-1
000640B     840 IC=IC+1
000643B         AKSTAR(IC)=1.0
000647B         D(IC)=DG(IA,I)+DG(JB,J)-2.0
000671B         AC(IC)=0.5*ACG(IA,I)*ACG(JB,J)
000713B         DO 842 IZ=1,NOBS
000723B         SIZN(IC,IZ)=SIGNG(IA,I,IZ)*SIGNG(JB,J,IZ)
000750B         AZ(IC,IZ)=AZG(IA,I,IZ)+AZG(JB,J,IZ)
000776B     842 CONTINUE
001000B         IF(IC.EQ.3) GO TO 850
001002B         ICM1=IC-1
001003B         ISAME=0
001003B         DO 846 IADD=1,ICM1
```

```
001013B        IF(INT(AKSTAR(IADD)).NE.INT(AKSTAR(IC))) GO TO 846
001026B        IF(INT(D(IADD)).NE.INT(D(IC))) GO TO 846
001040B        DO 844 IZ=1,NOBS
001047B        IF(INT(SIZN(IADD,IZ)).NE.INT(SIZN(IC,IZ))) GO TO 846
001072B        IF(INT(AZ(IADD,IZ)).NE.INT(AZ(IC,IZ)))GO TO 846
001072B    844 CONTINUE
001105B        ISAME=1
001112B        IADDS=IADD
001113B        GO TO 848
001113B    846 CONTINUE
001115B    848 IF(ISAME.EQ.0) GO TO 850
001124B        AC(IADDS)=AC(IADDS)+AC(IC)
001135B        IC=IC-1
001136B    850 CONTINUE
001143B    860 CONTINUE
001147B    864 LSKIP=1
001150B        ICMAXM=IC
001152B        IF(ICMAXM.EQ.0) GO TO 868
001153B        IF(LPN(7).EQ.0) GO TO 868
001154B        WRITE(IPRINT,865) (MS(IB),IB=1,6)
001174B    865 FORMAT(/1X,21HI1,J1,K1,I2,J2,K2 ARE,6(I2,1X))
001174B        DO 867 JC=1,ICMAXM
001203B        WRITE(IPRINT,866) JC,AKSTAR(JC),D(JC),AC(JC),((SIZN(JC,IB),AZ(JC,I
               1B)),IB=1,NOBS)
001257B    866 FORMAT(/1X,1H/,3HIC=,I2,1H/,11HAKSTAR(IC)=,F4.1,1H/,6HD(IC)=,F4.1,
               11H/,7HAC(IC)=,F4.1/ 1X,23HSIZN(IC,*),AZ(IC,*) ARE,10(F4.1,F3.1,1H/
               1))
001257B    867 CONTINUE
001263B    868 LSKIP=1
          C
          C     TO REPRESENT SECOND PARTIAL DERIVATIVE OF (R*)**2 W.R.T.
          C     A*I1J1K1,A*I2J2K2 EVALUATED AT A*=0
          C
001264B        IF(KS.EQ.0) IGMAX4=IGMAX(4)
001270B        IF(KS.EQ.1.OR.KS.EQ.2) IGMAX4=IGMAX(3)
001275B        IF(IGMAX4.EQ.0) GO TO 886
001276B        DO 884 I=1,IGMAX4
001304B        IC=IC+1
001305B        AKSTAR(IC)=0.0
001311B        IF(KS.EQ.0) D(IC)=DG(4,I)
001326B        IF(KS.EQ.1.OR.KS.EQ.2) D(IC)=DG(3,I)
001344B        IF(KS.EQ.0) AC(IC)=ACG(4,I)
001360B        IF(KS.EQ.1.OR.KS.EQ.2) AC(IC)=ACG(3,I)
001376B        DO 870 IZ=1,NOBS
001405B        IF(KS.EQ.0) SIZN(IC,IZ)=SIGNG(4,I,IZ)
001425B        IF(KS.EQ.1.OR.KS.EQ.2) SIZN(IC,IZ)=SIGNG(3,I,IZ)
001446B        IF(KS.EQ.0) AZ(IC,IZ)=AZG(4,I,IZ)
001465B        IF(KS.EQ.1.OR.KS.EQ.2) AZ(IC,IZ)=AZG(3,I,IZ)
001507B    870 CONTINUE
001513B        IF(IC.EQ.1) GO TO 884
001514B        ICM1=IC-1
001515B        ISAME=0
001516B        DO 876 IADD=1,ICM1
001525B        IF(INT(D(IADD)).NE.INT(D(IC))) GO TO 876
001540B        IF(INT(AKSTAR(IADD)).NE.INT(AKSTAR(IC))) GO TO 876
001552B        DO 874 IZ=1,NOBS
001561B        IF(INT(SIZN(IADD,IZ)).NE.INT(SIZN(IC,IZ))) GO TO 876
001604B        IF(INT(AZ(IADD,IZ)).NE.INT(AZ(IC,IZ))) GO TO 876
001604B    874 CONTINUE
001617B        ISAME=1
001624B        IADDS=IADD
001625B        GO TO 878
001625B    876 CONTINUE
001627B    878 IF(ISAME.EQ.0) GO TO 884
001636B        AC(IADDS)=AC(IADD)+AC(IC)
001652B        IC=IC-1
001654B    884 CONTINUE
```

```
0016618       886 LSKIP=1
0016628           ICMAX=IC
0016648           IF(ICMAX.NE.0) GO TO 887
0016658           IP=IP-1
0016668           GO TO 1010
0016708       887 LSKIP=1
0016718           IF(ICMAX.LE.ICMAXM) GO TO 889
0016748           IF(LPN(7).EQ.0) GO TO 889
0016768           WRITE(IPRINT,865) (MS(IB),IB=1,6)
0017168           ICMXMP=ICMAXM+1
0017178           DO 888 IC=1,ICMAX
0017268           WRITE(IPRINT,866) IC,AKSTAR(IC),D(IC),AC(IC),((SIZN(IC,IB),AZ(IC,I
                 1B)),IB=1,NOBS)
0020028       888 CONTINUE
0020068       889 LSKIP=1
              C
              C     NOW ((((KSTAR/2)-1)/((D*)**2))-0.5)*FIRST PARTIAL DERIVATIVE OF
              C     (R*)**2 W.R.T. A*I1J1K1 EVALUATED AT A*=0*FIRST PARTIAL DERIVATIVE
              C     OF (R*)**2 W.R.T. A*I2J2K2 EVALUATED AT A*=0+SECOND PARTIAL
              C     DERIVATIVE OF (R*)**2 W.R.T. A*I1J1K1,A*I2J2K2 EVALUATED AT A*=0
              C     ARE REPRESENIED BT
              C     (AKSTAR(IC),D(IC),AC(IC),SIZN(IC,IZ),AZ(IC,IZ),WHERE IC=1,ICMAX)
              C
              C
              C     TO REPRESENT I1A*I1J1K1A*I2J2K2
              C
0020078           DO 960 IC=1,ICMAX
0020178           TP=1.0
0020178           IF(NPMK.EQ.0) GO TO 894
0020228           DO 890 IZ=1,NPMK
0020308           IF(INT(AZ(IC,IZ)).EQ.0) GO TO 890
0020418           IF(INT(AZ(IC,IZ)).EQ.1) TP=0.0
0020458           IF(INT(AZ(IC,IZ)).EQ.2) GO TO 890
0020468           IF(INT(AZ(IC,IZ)).EQ.3) TP=0.0
0020528           IF(INT(AZ(IC,IZ)).EQ.4) TP=3.0*TP
0020568       890 CONTINUE
0020628       894 LSKIP=1
0020638           DO 900 IZ=NPARP,NOBS
0020768           IF(INT(AZ(IC,IZ)).EQ.0) GO TO 900
0021078           IF(INT(AZ(IC,IZ)).EQ.1) TP=0.0
0021138           IF(INT(AZ(IC,IZ)).EQ.2) GO TO 900
0021148           IF(INT(AZ(IC,IZ)).EQ.3) TP=0.0
0021208           IF(INT(AZ(IC,IZ)).EQ.4) TP=3.0*TP
0021248       900 CONTINUE
0021308           AC(IC)=TP*AC(IC)
0021358           KSIGN=0
0021358           DO 910 IZ=NPMKP1,NPAR
0021508           IF(INT(SIZN(IC,IZ)).NE.-1) GO TO 910
0021618           KSIGN=1
0021628           GO TO 920
0021628       910 CONTINUE
0021648       920 IF(KSIGN.EQ.0) GO TO 930
0021728           AC(IC)=0.0
0021778           GO TO 960
0022018       930 TP=1.0
0022038           IT=0.0
0022038           RTPI=SQRT(3.1415926536)
0022108           GAMMA(1)=RTPI
0022108           GAMMA(2)=1.0
0022118           GAMMA(3)=RTPI/2.0
0022138           GAMMA(4)=2.0
0022158           GAMMA(5)=3.0*RTPI/4.0
0022178           GAMMA(6)=2.0
0022208           GAMMA(7)=15.0*RTPI/8.0
0022238           T1=0.3989422804
0022248           DO 950 IZ=NPMKP1,NPAR
0022378           T2=0.5*(AZ(IC,IZ)+1.0)
```

```
002247B          IGAM=AZ(IC,IZ)+1.0
002250B          TP=TP*(2.0**T2)*GAMMA(IGAM)*T1
002262B          TS=TS+AZ(IC,IZ)
002264B      950 CONTINUE
002266B          AC(IC)=TP*AC(IC)
002274B          DEGREE(IC)=TS
002301B          IF(.NOT.(INT(DEGREE(IC)).EQ.2.AND.INT(AKSTAR(IC)).EQ.1))  GO TO 96
                1O
002314B          AKSTAR(IC)=0.0
002320B          DEGREE(IC)=0.0
002324B      960 CONTINUE
002331B          IF(LPN(8).EQ.0) GO TO 967
002332B          WRITE(IPRINT,865) (MS(IB),IB=1,6)
002352B          DO 964 IC=1,ICMAX
002361B          WRITE(IPRINT,962) IC,AKSTAR(IC),D(IC),AC(IC),DEGREE(IC)
002416B      962 FORMAT(/1X,3HIC=,I2,1H/,11HAKSTAR(IC)=,F4.1,1H/,6HD(IC)=,F4.1,1H/,
                17HAC(IC)=,F4.1,1H/,11HDEGREE(IC)=,F4.1)
002416B      964 CONTINUE
002422B      967 LSKIP=1
           C
           C     THE IC-TH TERM IN I1A*I1J1K1A*I2J2K2 IS OF THE FORM
           C     (KSTAR**AKSTAR(IC))*((D*)**D(IC))*(AC(IC)*DENSITY OF CHI  SQUARE
           C     DISTRIBUTION ON KSTAR DEGREES OF FREEDOM EVALUATED AT (D*)**2)*
           C     (DENSITY OF CHI SQUARE DISTRIBUTION ON KSTAR + DEGREE(IC) DEGREES
           C     OF FREEDOM EVALUATED AT 1)/(DENSITY OF CHI SQUARE DISTRIBUTION ON
           C     KSTAR DEGREES OF FREEDOM EVALUATED AT 1)
           C
           C
002423B          IIC=1
002424B          IF(ICMAX.EQ.1) GO TO 978
002426B          DO 976 IC=2,ICMAX
002436B          DO 968 IADD=1,IIC
002446B          IF(INT(AKSTAR(IADD)).NE.INT(AKSTAR(IC)).OR.INT(D(IADD)).NE.INT(D(I
                1C)).OR.INT(DEGREE(IADD)).NE.INT(DEGREE(IC))) GO TO 968
002501B          AC(IADD)=AC(IADD)+AC(IC)
002512B          GO TO 976
002512B      968 CONTINUE
002514B          IIC=IIC+1
002520B          AC(IIC)=AC(IC)
002530B          AKSTAR(IIC)=AKSTAR(IC)
002541B          D(IIC)=D(IC)
002552B          DEGREE(IIC)=DEGREE(IC)
002564B      976 CONTINUE
002572B      978 IICMAX=IIC
002574B          KZERO=0
002574B      979 FORMAT(/1X,68HIP,I1,J1,K1,I2,J2,K2,KP(IP,*),AKSTAR(IC),D(IC),AC(IC
                1),DEGREE(IC) ARE)
002574B          DO 1000 IC=1,IICMAX
002604B          IF(INT(AC(IC)).EQ.0) GO TO 1000
002612B          WRITE(IPRINT,979)
002621B          WRITE(IPRINT,980) IP,(MS(I),I=1,6),(KP(IP,I),I=1,21),AKSTAR(IC),D(
                1IC),AC(IC),DEGREE(IC)
002705B      980 FORMAT(/1X,I3,1H/,6(I2,1X),1H/,21(I1,1X),4(1H/,F10.5))
002705B          KZERO=1
002705B     1000 CONTINUE
002712B          IF(KZERO.EQ.1) GO TO 1010
002713B          IP=IP-1
002715B     1010 LSKIP=1
002717B          RETURN
002722B          END
```

APPENDIX 4 Programs POWCAL, and subroutines COEF11, COEF2, COEF1, E1000, E2000, etc.

```
                C
                C    PROGRAM POWCAL,SUBROUTINES COEF11,COEF2,COEF1,E1000,E2000,ETC ARE
                C    FOR CALCULATING BETA1(THETA A,SIGMA A)
                C
000000B               PROGRAM POWCAL(INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT,TAPE9,TAPE10)
010100B               COMMON/MSKPAB/MSA(23,6),MSB(73,6),KPA(23,21),KPB(73,21)
010100B               COMMON/MMSA/MMSA1(23,4),MMSA2(23,4),MMSA3(23,4)
010100B               COMMON/MSAB12/MSA1(23,4),MSA2(23,4),MSA3(23,4)
010100B               COMMON/MSB/MSB1(73,4),MSB2(73,4),MSB3(73,4)
010100B               COMMON/MMSB/MMSB1(73,4),MMSB2(73,4),MMSB3(73,4)
010100B               COMMON/DAC/DA(23),ACA(23),DB(73),ACB(73)
010100B               COMMON/ICMAXM/ICMAXM( 50),ICMAX(100)
010100B               COMMON/MSKP/MS(6),KP(21)
010100B               COMMON/NPAROB/NPAR,NOBS,KSTAR
010100B               COMMON/DPLUS/DPLUS
010100B               COMMON/KSTRP/KSTRP(20)
010100B               COMMON/CHINON/CHINON(12)
010100B               COMMON/FACGAM/FAC(150),GAMMA(150)
010100B               COMMON/ZA/ZA(10)
010100B               COMMON/A/A(5,3,3)
010100B               COMMON/LPN/LPN(14)
010100B               COMMON/IPRINT/IREAD,IPRINT,ITAPE7,ITAPE8,ITAPE9,ITAP10
010100B               COMMON/SIGMA/SIGMA
010100B               DIMENSION ZAS(20)
010100B               IREAD=5
010124B               IPRINT=6
010125B               ITAPE9=9
010126B               ITAP10=10
010127B               READ(IREAD,4) KCH,IN,DPLUS
                C
                C    DPLUS**2 IS (((SIGMA ZERO)**2)/(SIGMA A)**2))*((1-ALPHA)100 UPPER
                C    PERCENTAGE POINT OF A CHI SQUARE DISTRIBUTION ON KSTAR DEGREES OF
                C    FREEDOM)
                C
010137B             4 FORMAT(A6,I4,7F10.4)
010137B             6 FORMAT(A6,I4,10F6.3)
010137B               CALL CHECIN(KCH,IN,5HDPLUS,0)
010141B               READ(IREAD,4) KCH,IN,SIGMA
010150B               CALL CHECIN(KCH,IN,5HSIGMA,0)
010152B               READ(IREAD,10) KCH,IN,NPAR,NOBS,KSTAR
010163B            10 FORMAT(A6,I4,14I5)
010163B               CALL CHECIN(KCH,IN,6HNPAROB,0)
010165B               WRITE(IPRINT,11) NPAR,NOBS,KSTAR
                C
                C    NPAR IS TOTAL NUMBER OF COMPONENTS IN THE PARAMETER VECTOR
                C    NOBS IS TOTAL NUMBER OF OBSERVATIONS
                C    KSTAR IS NUMBER OF COMPONENTS OF INTEREST IN THE PARAMETER VECTOR
                C
010174B            11 FORMAT(/1X,'NPAR=',I3,1X,'NOBS=',I3,1X,'KSTAR=',I3)
010174B               NPMK=NPAR-KSTAR
010175B               NPMKP1=NPMK+1
010176B               NPMKP2=NPMK+2
010177B               NPARP=NPAR+1
010200B               NPARP2=NPAR+2
010201B               READ(IREAD,4) KCH,IN,(ZAS(IB),IB=1,5)
010215B               CALL CHECIN(KCH,IN,3HZAS,0)
010217B               READ(IREAD,4) KCH,IN,(ZAS(IB),IB=6,10)
010232B               CALL CHECIN(KCH,IN,4HZAS5,0)
010234B               DO 12 IOBS=1,NOBS
010236B               ZA(IOBS)=ZAS(IOBS)/SIGMA
                C
                C    ZA(I) ARE ZAI
                C
010236B            12 CONTINUE
010241B               READ(IREAD,10) KCH,IN,IQAMAX,IQBMAX
```

```
                  C
                  C     IQAMAX IS TOTAL NUMBER OF TERMS IN THE EXPRESSIONS OF THE
                  C     PRODUCTS OF THE FIRST PARTIAL DERIVATIVES OF (R+)**2 W.R.T.
                  C     A+I1J1K1 AND A+I2J2K2 EVALUATED AT A+=0
                  C
                  C     IQBMAX IS TOTAL NUMBER OF TERMS IN THE EXPRESSIONS OF THE SECOND
                  C     PARTIAL DERIVATIVES OF (R+)**2 W.R.T. A+I1J1K1,A+I2J2K2 EVALUATED
                  C     AT A+=0
                  C
010252B                 CALL CHECIN(KCH,IN,6HIQAMAX,0)
010254B                 READ(IREAD,10) KCH,IN,KCHIMX
                  C
                  C     KCHIMX IS THE NUMBER OF TERMS USED IN REPRESENTING NON-CENTRAL
                  C     CHI SQUARE DISTRIBUTION BY A LINEAR COMBINATION OF CENTRAL CHI
                  C     SQUARE DISTRIBUTION
                  C
010263B                 CALL CHECIN(KCH,IN,6HKCHIMX,0)
010265B                 READ(IREAD,4) KCH,IN,VSMALL
                  C
                  C     VSMALL IS SOME SMALL POSITIVE NUMBER
                  C
010274B                 CALL CHECIN(KCH,IN,6HVSMALL,0)
010276B                 READ(IREAD,10) KCH,IN,(LPN(I),I=1,7)
010311B                 CALL CHECIN(KCH,IN,6HLPN(1),0)
010313B                 READ(IREAD,10) KCH,IN,(LPN(I),I=8,14)
                  C
                  C     LPN(*) DECIDES WHETHER THE INTERMEDIATE RESULTS WILL BE PRINTED OUT
                  C
010326B                 CALL CHECIN(KCH,IN,6HLPN(8),0)
010330B                 DO 14 IOBS=1,NOBS
010332B                 IF(.NOT.NPAR.GE.4) GO TO 13
010335B                 READ(IREAD,6) KCH,IN,((A(IOBS,JB,KB),KB=JB,NPAR),JB=1,NPAR)
010360B                 CALL CHECIN(KCH,IN,6HA(***),IOBS)
010362B                 GO TO 14
010362B              13 LSKIP=1
010363B                 READ(IREAD,4) KCH,IN,((A(IOBS,JB,KB),KB=JB,NPAR),JB=1,NPAR)
                  C
                  C     A(I,J,K) ARE A+IJK
                  C
010407B                 CALL CHECIN(KCH,IN,6HA(***),IOBS)
010411B              14 CONTINUE
010413B                 DO 24 IQQA=1,IQAMAX
010415B                 IQA=IQQA
010415B                 READ(ITAPE9,20) IQA,ICMAXM(IQA),(MSA(IQA,I),I=1,6),(KPA(IQA,I),I=1
                       1,21),((MSA1(IQA,I),MMSA1(IQA,I)),I=1,4),((MSA2(IQA,I),MMSA2(IQA,I)
                       1),I=1,4),((MSA3(IQA,I),MMSA3(IQA,I)),I=1,4),DA(IQA),ACA(IQA)
010517B              20 FORMAT(1X,I5,I5,51I2,2F5.1)
                  C
                  C     (KPA(IQA,I),I=1,21) COMPLETELY SPECIFY A SUBSET GENERATED
                  C     BY PARTITIONING THE SET OF ALL (A+I1J1K1,A+I2J2K2)
                  C     (C.F. SECTION (4.9))
                  C     (MSA(IQA,I),I=1,6) REPRESENT A TYPICAL ELEMENT OF THIS SUBSET
                  C     ICMAXM(IQA) IS THE TOTAL NUMBER OF TERMS IN THE EXPRESSION OF
                  C     THE PRODUCT OF THE FIRST PARTIAL DERIVATIVES OF (R+)**2 W.R.T.
                  C     A+I1J1K1 AND A+I2J2K2 EVALUATED AT A+=0
                  C
                  C
                  C     A TERM IN AN EXPRESSION IS REPRESENTED BY ACA(IQA)(C.F.COLUMN(4) IN
                  C     SECTION (4.11)) DA(IQA)(C.F.COLUMN(3))((MSA1(IQA,I),MMSA1(IQA,I)),
                  C     I=1,4)(C.F.COLUMNS (32) TO (39))(MSA2(IQA,I),MMSA2(IQA,I)),I=1,4)
                  C     (C.F.COLUMNS (40) TO (47)) AND ((MSA3(IQA,I),MMSA3(IQA,I)),I=1,4)
                  C     (C.F. COLUMNS (48) TO (55))
                  C
010517B              24 CONTINUE
010521B                 DO 28 IQQB=1,IQBMAX
010523B                 IQB=IQQB
```

```
010523B     75.      READ(ITAP10,20) IQB,ICMAX(IQB), (MSB(IQB,I),I=1,6),(KPB(IQB,I),I=1
                    1,21),((MSB1(IQB,I),MMSB1(IQB,I)),I=1,4),((MSB2(IQB,I),MMSB2(IQB,I)
                    1),I=1,4),((MSB3(IQB,I),MMSB3(IQB,I)),I=1,4),DB(IQB),ACB(IQB)
                C
                C   (KPB(IQB,I),I=1,21) COMPLETELY SPECIFY A SUBSET
                C   GENERATED BY PARTITIONING THE SET OF ALL (A+I1J1K1,A+I2J2K2)
                C   (C.F.SECTION(4,9))
                C   (MSB(IQB,I),I=1,6) REPRESENTS A TYPICAL ELEMENT OF THIS SUBSET
                C   ICMAX(IQB) IS THE TOTAL NUMBER OF TERMS IN THE EXPRESSION OF THE
                C   SECOND DERIVATIVE OF (R+)**2 W.R.T. A+I1J1K1,A+I2J2K2 EVALUATED AT
                C   A+=0
                C
                C   A TERM IN AN EXPRESSION IS REPRESENTED BY ACB(IQB) (C.F. COLUMN (4)
                C   IN SECTION (4,11)),DB(IQB) (C.F. COLUMN (3)),
                C   ((MSB1(IQB,I),MMSB1(IQB,I)),I=1,4)(C.F.COLUMNS (32) TO (39)),
                C   ((MSB2(IQB,I),MMSB2(IQB,I)),I=1,4)(C.F.COLUMNS (40) TO (47)) AND
                C   ((MSB3(IQB,I),MMSB3(IQB,I)),I=1,4)(C.F.COLUMNS (48) TO (55))
                C
010630B     76.    28 CONTINUE
010632B     77.       IF(LPN(13).EQ.0) GO TO 50
010633B     78.       WRITE(IPRINT,30)
010637B     79.    30 FORMAT(/1X,@IQA,ICMAXM(IQA),DA(IQA),ACA(IQA),MSA(IQA,*),KPA(IQA,*)
                    1,MSA1(IQA,*),MMSA1(IQA,*),ETC ARE@/)
010637B     80.       WRITE(IPRINT,31)
010643B     81.    31 FORMAT(/1X,@IQB,ICMAX(IQB),DB(IQB),ACB(IQB),MSB(IQB,*),KPB(IQB,*'M
                    1SB1(IQB,*),MMSB1(IQB,*),ETC ARE@//////)
010643B     82.       DO 34 IQA=1,IQAMAX
010645B     83.       WRITE(IPRINT,32) IQA,ICMAXM(IQA),DA(IQA),ACA(IQA),(MSA(IQA,I),I=1,
                    16),(KPA(IQA,I),I=1,21),((MSA1(IQA,I),MMSA1(IQA,I)),I=1,4),((MSA2(I
                    1QA,I),MMSA2(IQA,I)),I=1,4),((MSA3(IQA,I),MMSA3(IQA,I)),I=1,4)
010741B     84.    32 FORMAT(1X,I3,1H ,I2,1H ,2(F5.1,1H ),6I1,1H ,21I1,1H ,3(8I1,1X))
010741B     85.    34 CONTINUE
010743B     86.       WRITE(IPRINT,36)
010747B     87.    36 FORMAT(//////,1X,1H )
010747B     88.       DO 42 IQB=1,IQBMAX
010751B     89.       WRITE(IPRINT,32) IQB,ICMAX(IQB), DB(IQB),ACB(IQB),(MSB(IQB,I),I=1,
                    16),(KPB(IQB,I),I=1,21),((MSB1(IQB,I),MMSB1(IQB,I)),I=1,4),((MSB2(I
                    1QB,I),MMSB2(IQB,I)),I=1,4),((MSB3(IQB,I),MMSB3(IQB,I)),I=1,4)
011045B     90.    42 CONTINUE
011047B     91.       WRITE(IPRINT,36)
011053B     92.    43 FORMAT(1X,@FAC(*) ARE@,10(F10.5,1X))
011053B     93.    44 FORMAT(1X,@GAMMA(*) ARE@,10(F10.5,1X))
011053B     94.    45 FORMAT(/1X,@VNAMDA=@,F10.5,@EXP(-0.5*VNAMDA)*EXP(-0.5*DSTAR*DSTAR)
                    1=@,F10.5/1X,@ZA(*) ARE@,10(F10.5,1X)/)
011053B     95.    50 LSKIP=1
011053B     96.       KCHIMP=KCHIMX+13
011055B     97.       FAC(1)=1.0
011056B     98.       DO 52 I=2,KCHIMP
011061B     99.       IM1=I-1
011061B    100.       VIM1=IM1
011063B    101.       FAC(I)=VIM1*FAC(IM1)
011064B    102.    52 CONTINUE
                C
                C   FAC(I) IS FACTORIAL OF (I-1)
                C
011066B    103.       IF(LPN(1).EQ.0) GO TO 53
011067B    104.       WRITE(IPRINT,43) (FAC(I),I=1,10)
011100B    105.    53 LSKIP=1
011100B    106.       GAMMA(1)=SQRT(3.1415926536)
011103B    107.       DO 54 I=3,KCHIMP,2
011105B    108.       IM2=I-2
011106B    109.       VI=I
011107B    110.       GAMMA(I)=0.5*(VI-2.0)*GAMMA(IM2)
011113B    111.    54 CONTINUE
011115B    112.       DO 56 I=2,KCHIMP,2
011116B    113.       J=I/2
```

```
011116B        GAMMA(I)=FAC(J)
011120B     56 CONTINUE
011122B        IF(LPN(1).EQ.0) GO TO 57
011123B        WRITE(IPRINT,44) (GAMMA(I),I=1,10)
011134B     57 LSKIP=1
011134B        TV=0.0
011135B        DO 58 IOBS=1,NOBS
011140B        TV=TV+(ZA(IOBS)*ZA(IOBS))
011140B     58 CONTINUE
011144B        VNAMDA=TV
011144B        T1=EXP(-0.5*VNAMDA)
011151B        T2=EXP(-0.5*DPLUS*DPLUS)
011156B        TT12=T1*T2
011157B        IF(LPN(1).EQ.0) GO TO 59
011161B        WRITE(IPRINT,45) VNAMDA,TT12,(ZA(IB),IB=1,NOBS)
011175B     59 LSKIP=1
011175B        DO 70 ICHI=1,12
011177B        VICHI=ICHI
011177B        TC=0.0
011200B        I=-1
011201B     60 I=I+1
011205B        VI=I
011205B        IP1=I+1
011206B        T3=0.5*(VICHI+2.0*VI)
011211B        TCPRV=TC
011213B        TC=TC+(T1*((.5*VNAMDA)**I)/FAC(IP1))*((DPLUS*DPLUS)**(T3-1.))*T2/(
              1(2.0**T3)*GAMMA(ICHI+2*I))
011245B        IF(LPN(1).EQ.0) GO TO 61
011246B        WRITE(IPRINT,67) TCPRV,TC
011254B     61 LSKIP=1
011254B        IF(I.LE.20) GO TO 60
011256B        IF(I.GE.KCHIMX) GO TO 64
011260B        IF(ABS(TC-TCPRV).GT.VSMALL) GO TO 60
011264B        CHINON(ICHI)=TC
011265B        WRITE(IPRINT,62) ICHI,CHINON(ICHI)
011275B     62 FORMAT(1X,@ICHI=@,I2,@CHINON(ICHI)=@,F12.6)
011275B        GO TO 68
011275B     64 CHINON(ICHI)=TC
011276B        WRITE(IPRINT,66) ICHI,KCHIMX,TCPRV,TC
011310B     66 FORMAT(1X,@ICHI=@,I2,1H/,@KCHIMX=@,I4,1H/,@CHINON(ICHI)=@,2(F12.6,
              11X))
011310B     67 FORMAT(1X,@TCPRV=@,F10.5,@TC=@,F10.5)
011310B     68 LSKIP=1
011310B     70 CONTINUE
         C
         C     NOW DENSITIES OF NON-CENTRAL CHI SQUARE DISTRIBUTIONS OF 1,2,...,12
         C     DEGREES OF FREEDOM AND PARAMETER VNAMDA EVALUATED AT (D+)**2
         C     ARE FOUND
         C
         C
         C     TO CALCULATE (SUM FROM I1 EQUAL TO 1 TO N)(SUM FROM I2 EQUAL TO 1
         C     TO N)(SUM FROM J1 EQUAL TO 1 TO P)(SIM FROM K1 EQUAL TO J1 TO P)
         C     (SUM FROM J2 EQUAL TO 1 TO P)(SUM FROM K2 EGUAL J2 TO P) OF
         C     0.5*BETA1A+I1J1K1A+I2J2K2*A+I1J1K1*A+I2J2K2
         C
011313B        TS=0.0
011313B        DO 290 I1=NPMKP1,NOBS
011316B        DO 280 I2=NPMKP1,NOBS
011320B        DO 270 J1=1,NPAR
011323B        DO 260 K1=J1,NPAR
011324B        DO 250 J2=1,NPAR
011326B        DO 240 K2=J2,NPAR
011327B        MS(1)=I1
011327B        MS(2)=J1
011331B        MS(3)=K1
011332B        MS(4)=I2
```

```
011334B          MS(5)=J2
011335B          MS(6)=K2
011336B          IF(I1.LE.I2) GO TO 72
011341B          MS(1)=I2
011342B          MS(2)=J2
011343B          MS(3)=K2
011344B          MS(4)=I1
011345B          MS(5)=J1
011346B          MS(6)=K1
011347B       72 LSKIP=1
011350B          DO 74 IM=1,6
011352B          I=MS(IM)
011352B          IF(I.GE.1.AND.I.LE.NPMK) KP(IM)=1
011361B          IF(I.GE.NPMKP1.AND.I.LE.NPAR) KP(IM)=2
011367B          IF(I.GE.NPARP.AND.I.LE.NOBS) KP(IM)=3
011374B       74 CONTINUE
011376B          IT=6
011377B          DO 80 IM=1,5
011401B          IMP1=IM+1
011401B          DO 78 JM=IMP1,6
011405B          IT=IT+1
011406B          IF(MS(IM).EQ.MS(JM)) KP(IT)=1
011413B          IF(MS(IM).NE.MS(JM)) KP(IT)=0
011415B       78 CONTINUE
011417B       80 CONTINUE
011422B          IQAYES=0
011422B          DO 110 IQA=1,IQAMAX
011425B          DO 90 I=1,21
011427B          IF(KP(I).NE.KPA(IQA,I)) GO TO 110
011427B       90 CONTINUE
011434B          IQAYES=IQA
011440B          GO TO 120
011440B      110 CONTINUE
011441B      120 IF(IQAYES.EQ.0) GO TO 130
011446B          I=IQAYES
011446B          DO 124 IC=1,ICMAXM(I)
011452B          I=IQAYES-1+IC
011453B          CALL COEF11(I,UNDEF)
           C
           C     ROUTINE COEF11 CALCULATES BETA1A+I1J1K1A+I2J2K2 ASSUMING THAT
           C     SECOND PARTIAL DERIVATIVE OF (R+)**2 IS ZERO
           C
011457B          TS=TS+UNDEF*A(I1,J1,K1)*A(I2,J2,K2)
011472B      124 CONTINUE
011475B      130 LSKIP=1
011475B          IQBYES=0
011476B          DO 180 IQB=1,IQBMAX
011501B          DO 170 I=1,21
011503B          IF(KP(I).NE.KPB(IQB,I)) GO TO 180
011503B      170 CONTINUE
011510B          IQBYES=IQB
011514B          GO TO 190
011514B      180 CONTINUE
011515B      190 IF(IQBYES.EQ.0) GO TO 200
011522B          I=IQBYES
011522B          DO 194 IC=1,ICMAX(I)
011526B          I=IQBYES-1+IC
011527B          CALL COEF2(I,UNDEF)
           C
           C     ROUTINE COEF2 CALCULATES BETA1A+I1J1K1A+I2J2K2 ASSUMING THAT
           C     PRODUCT OF FIRST PARTIAL DERIVATIVE OF (R+)**2 IS ZERO
           C
011533B          TS=TS+UNDEF*A(I1,J1,K1)*A(I2,J2,K2)
011546B      194 CONTINUE
011551B      200 LSKIP=1
011551B      240 CONTINUE
```

```
011554B    250 CONTINUE
011557B    260 CONTINUE
011562B    270 CONTINUE
011565B    280 CONTINUE
011570B    290 CONTINUE
011573B        TS=0.5*TS*SIGMA*SIGMA
011576B        TS2NEG=-TS
          C
          C     TO CALCULATE (SUM FROM I EQUAL TO 1 TO N)(SUM FROM J EQUAL TO 1
          C     TO P)(SUM FROM K EQUAL TO J TO P) OF BETA1A+IJK*A+IJK
          C
011576B        DO 350 I=NPMKP1,NOBS
011601B        DO 340 J=1,NPAR
011604B        DO 330 K=J,NPAR
011605B        CALL COEF1(I,J,K,UNDEF)
          C
          C     ROUTINE COEF1 CALCULATES BETA1A+IJK
          C
011610B        TS=TS+UNDEF*A(I,J,K)*SIGMA
011617B    330 CONTINUE
011622B    340 CONTINUE
011624B    350 CONTINUE
011627B        TS1NEG=-(TS+TS2NEG)
011631B        TS12NG=-TS
011632B        WRITE(IPRINT,360) SIGMA,TS1NEG,TS2NEG,TS12NG
011643B    360 FORMAT(/1X,@SIGMA=@,F20.6,@POWER=ALPHA  A +(@,F10.6,@)+(@,F10.6,@)=
              1ALPHA  A @,@+(@,F10.6,@)@)
011643B        STOP
011645B        END

          C
          C     ROUTINE COEF11 CALCULATES BETA1A+I1J1K1A+I2J2K2 ASSUMING THAT
          C     SECOND PARTIAL DERIVATIVES OF (R+)**2 IS ZERO
          C
000000B        SUBROUTINE COEF11(II,UNDEF)
000000B        COMMON/MSKPAB/MSA(23,6),MSB(73,6),KPA(23,21),KPB(73,21)
000000B        COMMON/MMSA/MMSA1(23,4),MMSA2(23,4),MMSA3(23,4)
000000B        COMMON/MSAB12/MSA1(23,4),MSA2(23,4),MSA3(23,4)
000000B        COMMON/MSB/MSB1(73,4),MSB2(73,4),MSB3(73,4)
000000B        COMMON/MMSB/MMSB1(73,4),MMSB2(73,4),MMSB3(73,4)
000000B        COMMON/DAC/DA(23),ACA(23),DB(73),ACB(73)
000000B        COMMON/ICMAXM/ICMAXM( 50),ICMAX(100)
000000B        COMMON/MSKP/MS(6),KP(21)
000000B        COMMON/NPAROB/NPAR,NOBS,KSTAR
000000B        COMMON/DPLUS/DPLUS
000000B        COMMON/KSTRP/KSTRP(20)
000000B        COMMON/CHINON/CHINON(12)
000000B        COMMON/FACGAM/FAC(150),GAMMA(150)
000000B        COMMON/ZA/ZA(10)
000000B        COMMON/A/A(5,3,3)
000000B        COMMON/LPN/LPN(14)
000000B        COMMON/NS/NS(4)
000000B        COMMON/IPRINT/IREAD,IPRINT,ITAPE7,ITAPE8,ITAPE9,ITAP10
000000B        COMMON/TP/TP
000000B        I=II
000000B        IF(LPN(8).EQ.0) GO TO 8
000002B        WRITE(IPRINT,2) (MS(IB),IB=1,6),(KP(IB),IB=1,21)
000034B      2 FORMAT(1X,@I1,J1,K1,I2,J2,K2 ARE@,6(I1,1X),1H/,@KP(*) ARE@,21(I1,1
              1X))
000034B        WRITE(IPRINT,4)
000043B      4 FORMAT(1X,@I,ICMAXM(I),DA(I),ACA(I),MSA(*),KPA(*),MSA1(I,*),MMSA1(
              1I,*) ETC ARE@)
000043B        WRITE(IPRINT,6) I,ICMAXM(I),DA(I),ACA(I),(MSA(I,IB),IB=1,6),(KPA(I
              1,IB),IB=1,21),((MSA1(I,IB),MMSA1(I,IB)),IB=1,4),((MSA2(I,IB),MMSA2
              1(I,IB)),IB=1,4),((MSA3(I,IB),MMSA3(I,IB)),IB=1,4)
000220B      6 FORMAT(1X,I5,1H/,I2,1H/,2(F4.1,1H/),6I1,1H/,21I1,1H/,3(4(I1,1H*,I1
```

```
                 1,1H/)),2H//)
000220B         8 LSKIP=1
000221B           TP=(DPLUS**DA(I))*ACA(I)
000233B           IF(LPN(9).EQ.0) GO TO 12
000235B           WRITE(IPRINT,10) DPLUS,DA(I),ACA(I),TP
000260B        10 FORMAT(1X,a(DPLUS**DA(I))*ACA(I)=(a,F10.5,2H**,F4.1,2H)*,F4.1,1H=,
                 1F10.5)
000260B        12 LSKIP=1
        C
        C         TO CALCULATE EXPECTATION OF (Z(1)**L(1))*(Z(2)**L(2))*...*
        C         (Z(P-KSTAR)**L(P-KSTAR))
        C
000261B           DO 14 IH=1,4
000265B           IF(MMSA1(I,IH).EQ.0) GO TO 14
000276B           IH1=MSA1(I,IH)
000303B           IH2=MS(IH1)
000310B           TZ=ZA(IH2)
000315B           MMS=MMSA1(I,IH)
000324B           CALL EXPZ(TZ,MMS,TP)
        C
        C         ROUTINE EXPZ CALCULATES EXPECTATION OF Z,Z**2,Z**3 AND Z**4
        C
000331B        14 CONTINUE
000334B           IF(LPN(9).EQ.0) GO TO 18
000335B           WRITE(IPRINT,16) TP
000345B        16 FORMAT(1X,a(DPLUS**DA(I))*ACA(I)*EXPECTATION OF PRODUCT OF FIRST P
                 1-KSTAR R.V. =a,F10.5)
000345B        18 LSKIP=1
        C
        C         TO CALCULATE EXPECTATION OF (Z(P+1)**L(P+1))*(Z(P+2)*L(P+2))*...*
        C         (Z(N)*L(N))
        C
000346B           DO 20 IH=1,4
000352B           IF(MMSA3(I,IH).EQ.0) GO TO 20
000363B           IH1=MSA3(I,IH)
000370B           IH2=MS(IH1)
000375B           TZ=ZA(IH2)
000402B           MMS=MMSA3(I,IH)
000411B           CALL EXPZ(TZ,MMS,TP)
000416B        20 CONTINUE
000421B           IF(LPN(9).EQ.0) GO TO 23
000422B           WRITE(IPRINT,22) TP
000432B        22 FORMAT(1X,a(DPLUS**DA(I))*ACA(I)*EXPECTATION OF PRODUCT OF FIRST P
                 1-KSTAR R.V. AND LAST N-P R.V. =a,F10.5)
000432B        23 LSKIP=1
        C
        C         TO CALCULATE EXPECTATION OF ((S(P-KSTAR+1)SQRT(ZBARS(P-KSTAR+1)))
        C         **L(P-KSTAR+1))*((S(P-KSTAR+2)SQRT(ZBARS(P-KSTAR+2)))**L(P-KSTAR+2)
        C         )*...*((S(P)SQRT(ZBARS(P)))**L(P))
        C
000433B           CALL COEG11(II,UNDEF)
000442B           RETURN
000444B           END
```

```
000000B          SUBROUTINE COEG11(II,UNDEF)
               C
               C  ROUTINE COEG11 CALCULATES EXPECTATION OF
               C  ((S(P-KSTAR+1)SQRT(ZBARS(P-KSTAR+1)))**L(P-KSTAR+1))*((S(P-KSTAR+2)
               C  SQRT(ZBARS(P-KSTAR+2)))**L(P-KSTAR+2))*...*((S(P)SQRT(ZBARS(P)))**
               C  L(P))
               C
000000B          COMMON/MSKPAB/MSA(23,6),MSB(73,6),KPA(23,21),KPB(73,21)
000000B          COMMON/MMSA/MMSA1(23,4),MMSA2(23,4),MMSA3(23,4)
000000B          COMMON/MSAB12/MSA1(23,4),MSA2(23,4),MSA3(23,4)
000000B          COMMON/MSB/MSB1(73,4),MSB2(73,4),MSB3(73,4)
000000B          COMMON/MMSB/MMSB1(73,4),MMSB2(73,4),MMSB3(73,4)
000000B          COMMON/DAC/DA(23),ACA(23),DB(73),ACB(73)
000000B          COMMON/ICMAXM/ICMAXM( 50),ICMAX(100)
000000B          COMMON/MSKP/MS(6),KP(21)
000000B          COMMON/NPAROB/NPAR,NOBS,KSTAR
000000B          COMMON/DPLUS/DPLUS
000000B          COMMON/KSTRP/KSTRP(20)
000000B          COMMON/CHINON/CHINON(12)
000000B          COMMON/FACGAM/FAC(150),GAMMA(150)
000000B          COMMON/ZA/ZA(10)
000000B          COMMON/A/A(5,3,3)
000000B          COMMON/LPN/LPN(14)
000000B          COMMON/NS/NS(4)
000000B          COMMON/IPRINT/IREAD,IPRINT,ITAPE7,ITAPE8,ITAPE9,ITAP10
000000B          COMMON/TP/TP
000000B          I=II
000000B          NPMKP1=NPAR-KSTAR+1
000003B          DO 24 II=1,20
000006B          KSTRP(II)=KSTAR+II
000006B       24 CONTINUE
000023B          VKSTAR=KSTAR
000025B          DPLSA=((0.5*VKSTAR-1.0)/(DPLUS**2))-0.5
000033B          DPLSB=0.5/DPLUS
000033B          DO 30 II=1,4
000037B          NS(II)=0
000037B       30 CONTINUE
000053B          DO 50 IH=1,4
000057B          DO 40 IIH=1,4
000062B          IF(MMSA2(I,IH).EQ.IIH) NS(IIH)=NS(IIH)+1
000101B       40 CONTINUE
000104B       50 CONTINUE
000110B          IF(NS(1).EQ.1.AND.NS(2).EQ.0.AND.NS(3).EQ.0.AND.NS(4).EQ.0)
                1GO TO 100
000116B          IF(NS(1).EQ.0.AND.NS(2).EQ.1.AND.NS(3).EQ.0.AND.NS(4).EQ.0)
                1GO TO 120
000123B          IF(NS(1).EQ.2.AND.NS(2).EQ.0.AND.NS(3).EQ.0.AND.NS(4).EQ.0)
                1GO TO 140
000130B          IF(NS(1).EQ.0.AND.NS(2).EQ.0.AND.NS(3).EQ.1.AND.NS(4).EQ.0)
                1GO TO 160
000135B          IF(NS(1).EQ.1.AND.NS(2).EQ.1.AND.NS(3).EQ.0.AND.NS(4).EQ.0)
                1GO TO 180
000143B          IF(NS(1).EQ.3.AND.NS(2).EQ.0.AND.NS(3).EQ.0.AND.NS(4).EQ.0)
                1GO TO 200
000150B          IF(NS(1).EQ.0.AND.NS(2).EQ.0.AND.NS(3).EQ.0.AND.NS(4).EQ.1)
                1GO TO 220
000155B          IF(NS(1).EQ.1.AND.NS(2).EQ.0.AND.NS(3).EQ.1.AND.NS(4).EQ.0)
                1GO TO 240
000163B          IF(NS(1).EQ.0.AND.NS(2).EQ.2.AND.NS(3).EQ.0.AND.NS(4).EQ.0)
                1GO TO 260
000170B          IF(NS(1).EQ.2.AND.NS(2).EQ.1.AND.NS(3).EQ.0.AND.NS(4).EQ.0)
                1GO TO 280
000176B          IF(NS(1).EQ.4.AND.NS(2).EQ.0.AND.NS(3).EQ.0.AND.NS(4).EQ.0)
                1GO TO 300
               C
               C   1000
               C
```

```
000202B    100 IH1=MS(MSA2(I,1))
000214B        TZ=ZA(IH1)
000221B        CALL E1000(TZ,TS1)
000226B        TS3=0.0
000226B        DO 110 IPAR=NPMKP1,NPAR
000241B        IF(IPAR.NE.IH1) CALL E1100(ZA(IH1),ZA(IPAR),TS2)
000257B        IF(IPAR.EQ.IH1) CALL E2000(ZA(IH1),TS2)
000271B        TS3=TS3+ZA(IPAR)*TS2
000277B    110 CONTINUE
000304B        TP=TP*(DPLSA*TS1+DPLSB*TS3)
000310B        GO TO 420
           C
           C    2000
           C
000311B    120 IH1=MS(MSA2(I,1))
000324B        TZ=ZA(IH1)
000331B        CALL E2000(TZ,TS1)
000336B        TS3=0.0
000336B        DO 130 IPAR=NPMKP1,NPAR
000351B        IF(IPAR.EQ.IH1) CALL E3000(ZA(IH1),TS2)
000363B        IF(IPAR.NE.IH1) CALL E2100(ZA(IH1),ZA(IPAR),TS2)
000401B        TS3=TS3+ZA(IPAR)*TS2
000407B    130 CONTINUE
000414B        TP=TP*(DPLSA*TS1+DPLSB*TS3)
000420B        GO TO 420
           C
           C    1100
           C
000421B    140 IH1=MS(MSA2(I,1))
000434B        TZ1=ZA(IH1)
000441B        IH2=MS(MSA2(I,2))
000452B        TZ2=ZA(IH2)
000457B        CALL E1100(TZ1,TZ2,TS1)
000464B        TS3=0.0
000464B        DO 150 IPAR=NPMKP1,NPAR
000477B        IF(IPAR.EQ.IH1) CALL E2100(ZA(IH1),ZA(IH2),TS2)
000516B        IF(IPAR.EQ.IH2) CALL E2100(ZA(IH2),ZA(IH1),TS2)
000534B        IF(IPAR.NE.IH1.AND.IPAR.NE.IH2) CALL E1110(ZA(IH1),ZA(IH2),ZA(IPAR
              1),TS2)
000557B        TS3=TS3+ZA(IPAR)*TS2
000565B    150 CONTINUE
000572B        TP=TP*(DPLSA*TS1+DPLSB*TS3)
000576B        GO TO 420
           C
           C    3000
           C
000577B    160 IH1=MS(MSA2(I,1))
000612B        TZ=ZA(IH1)
000617B        CALL E3000(TZ,TS1)
000624B        TS3=0.0
000624B        DO 170 IPAR=NPMKP1,NPAR
000637B        IF(IPAR.EQ.IH1) CALL E4000(ZA(IH1),TS2)
000651B        IF(IPAR.NE.IH1) CALL E3100(ZA(IH1),ZA(IPAR),TS2)
000667B        TS3=TS3+ZA(IPAR)*TS2
000675B    170 CONTINUE
000702B        TP=TP*(DPLSA*TS1+DPLSB*TS3)
000706B        GO TO 420
           C
           C    2100
           C
000707B    180 IH1=MS(MSA2(I,1))
000722B        TZ1=ZA(IH1)
000727B        IH2=MS(MSA2(I,2))
000740B        TZ2=ZA(IH2)
000745B        IF(MMSA2(I,1).EQ.2) GO TO 184
000754B        IIH1=IH1
000755B        IIH2=IH2
```

```
000757B        IH1=IIH2
000757B        IH2=IIH1
000761B        TTZ1=TZ1
000762B        TTZ2=TZ2
000764B        TZ1=TTZ2
000764B        TZ2=TTZ1
000766B    184 LSKIP=1
000770B        CALL E2100(TZ1,TZ2,TS1)
000775B        TS3=0.0
000775B        DO 190 IPAR=NPMKP1,NPAR
001010B        IF(IPAR.EQ.IH1) CALL E3100(ZA(IH1),ZA(IH2),TS2)
001027B        IF(IPAR.EQ.IH2) CALL E2200(ZA(IH1),ZA(IH2),TS2)
001045B        IF(IPAR.NE.IH1.AND.IPAR.NE.IH2) CALL E2110(ZA(IH1),ZA(IH2),ZA(IPAR
              1),TS2)
001070B        TS3=TS3+ZA(IPAR)*TS2
001076B    190 CONTINUE
001103B        TP=TP*(DPLSA*TS1+DPLSB*TS3)
001107B        GO TO 420
          C
          C     1110
          C
001110B    200 IH1=MS(MSA2(I,1))
001123B        TZ1=ZA(IH1)
001130B        IH2=MS(MSA2(I,2))
001141B        TZ2=ZA(IH2)
001146B        IH3=MS(MSA2(I,3))
001160B        TZ3=ZA(IH3)
001165B        CALL E1110(TZ1,TZ2,TZ3,TS1)
001172B        TS3=0.0
001172B        DO 210 IPAR=NPMKP1,NPAR
001205B        IF(IPAR.EQ.IH1) CALL E2110(ZA(IH1),ZA(IH2),ZA(IH3),TS2)
001230B        IF(IPAR.EQ.IH2) CALL E2110(ZA(IH2),ZA(IH1),ZA(IH3),TS2)
001252B        IF(IPAR.EQ.IH3) CALL E2110(ZA(IH3),ZA(IH1),ZA(IH2),TS2)
001274B        IF(IPAR.NE.IH1.AND.IPAR.NE.IH2.AND.IPAR.NE.IH3) CALL E1111(ZA(IH1)
              1,ZA(IH2),ZA(IH3),ZA(IPAR),TS2)
001325B        TS3=TS3+ZA(IPAR)*TS2
001333B    210 CONTINUE
001340B        TP=TP*(DPLSA*TS1+DPLSB*TS3)
001344B        GO TO 420
          C
          C     4000
          C
001345B    220 IH1=MS(MSA2(I,1))
001360B        TZ=ZA(IH1)
001365B        CALL E4000(TZ,TS1)
001372B        TS3=0.0
001372B        DO 230 IPAR=NPMKP1,NPAR
001405B        IF(IPAR.EQ.IH1) CALL E5000(ZA(IH1),TS2)
001417B        IF(IPAR.NE.IH1) CALL E4100(ZA(IH1),ZA(IPAR),TS2)
001435B        TS3=TS3+ZA(IPAR)*TS2
001443B    230 CONTINUE
001450B        TP=TP*(DPLSA*TS1+DPLSB*TS3)
001454B        GO TO 420
          C
          C     3100
          C
001455B    240 IH1=MS(MSA2(I,1))
001470B        TZ1=ZA(IH1)
001475B        IH2=MS(MSA2(I,2))
001506B        TZ2=ZA(IH2)
001513B        IF(MMSA2(I,1).EQ.3) GO TO 244
001522B        IIH1=IH1
001523B        IIH2=IH2
001525B        IH1=IIH2
001525B        IH2=IIH1
001527B        TTZ1=TZ1
001530B        TTZ2=TZ2
```

```
001532B          TZ1=TTZ2
001532B          TZ2=TTZ1
001534B      244 LSKIP=1
001536B          CALL E3100(TZ1,TZ2,TS1)
001543B          TS3=0.0
001543B          DO 250 IPAR=NPMKP1,NPAR
001556B          IF(IPAR.EQ.IH1) CALL E4100(ZA(IH1),ZA(IH2),TS2)
001575B          IF(IPAR.EQ.IH2) CALL E3200(ZA(IH1),ZA(IH2),TS2)
001613B          IF(IPAR.NE.IH1.AND.IPAR.NE.IH2) CALL E3110(ZA(IH1),ZA(IH2),ZA(IPAR
               1),TS2)
001636B          TS3=TS3+ZA(IPAR)*TS2
001644B      250 CONTINUE
001651B          TP=TP*(DPLSA*TS1+DPLSB*TS3)
001655B          GO TO 420
           C
           C     2200
           C
001656B      260 IH1=MS(MSA2(I,1))
001671B          TZ1=ZA(IH1)
001676B          IH2=MS(MSA2(I,2))
001707B          TZ2=ZA(IH2)
001714B          CALL E2200(TZ1,TZ2,TS1)
001721B          TS3=0.0
001721B          DO 270 IPAR=NPMKP1,NPAR
001734B          IF(IPAR.EQ.IH1) CALL E3200(ZA(IH1),ZA(IH2),TS2)
001753B          IF(IPAR.EQ.IH2) CALL E3200(ZA(IH2),ZA(IH1),TS2)
001771B          IF(IPAR.NE.IH1.AND.IPAR.NE.IH2) CALL E2210(ZA(IH1),ZA(IH2),ZA(IPAR
               1),TS2)
002014B          TS3=TS3+ZA(IPAR)*TS2
002022B      270 CONTINUE
002027B          TP=TP*(DPLSA*TS1+DPLSB*TS3)
002033B          GO TO 420
           C
           C     2110
           C
002034B      280 IH1=MS(MSA2(I,1))
002047B          TZ1=ZA(IH1)
002054B          IH2=MS(MSA2(I,2))
002065B          TZ2=ZA(IH2)
002072B          IH3=MS(MSA2(I,3))
002104B          TZ3=ZA(IH3)
002111B          IF(MMSA2(I,1).EQ.2) GO TO 288
002120B          IF(MMSA2(I,2).EQ.2) GO TO 282
002126B          IF(MMSA2(I,3).EQ.2) GO TO 284
002133B      282 TTZ1=TZ1
002136B          TTZ2=TZ2
002137B          TZ1=TTZ2
002137B          TZ2=TTZ1
002141B          IIH1=IH1
002142B          IIH2=IH2
002143B          IH1=IIH2
002143B          IH2=IIH1
002145B          GO TO 288
002146B      284 TTZ1=TZ1
002151B          TTZ3=TZ3
002152B          TZ1=TTZ3
002152B          TZ3=TTZ1
002154B          IIH1=IH1
002155B          IIH3=IH3
002156B          IH1=IIH3
002156B          IH3=IIH1
002160B      288 LSKIP=1
002162B          CALL E2110(TZ1,TZ2,TZ3,TS1)
002167B          TS3=0.0
002167B          DO 290 IPAR=NPMKP1,NPAR
002202B          IF(IPAR.EQ.IH1) CALL E3110(ZA(IH1),ZA(IH2),ZA(IH3),TS2)
002225B          IF(IPAR.EQ.IH2) CALL E2210(ZA(IH1),ZA(IH2),ZA(IH3),TS2)
```

```
002247B          IF(IPAR.EQ.IH3) CALL E2210(ZA(IH1),ZA(IH3),ZA(IH2),TS2)
002271B          IF(IPAR.NE.IH1.AND.IPAR.NE.IH2.AND.IPAR.NE.IH3) CALL E2111(ZA(IH1)
                1,ZA(IH2),ZA(IH3),ZA(IPAR),TS2)
002322B          TS3=TS3+ZA(IPAR)*TS2
002330B      290 CONTINUE
002335B          TP=TP*(DPLSA*TS1+DPLSB*TS3)
002341B          GO TO 420
           C
           C     1111
           C
002342B      300 IH1=MS(MSA2(I,1))
002355B          TZ1=ZA(IH1)
002362B          IH2=MS(MSA2(I,2))
002373B          TZ2=ZA(IH2)
002400B          IH3=MS(MSA2(I,3))
002412B          TZ3=ZA(IH3)
002417B          IH4=MS(MSA2(I,4))
002431B          TZ4=ZA(IH4)
002436B          CALL E1111(TZ1,TZ2,TZ3,TZ4,TS1)
002443B          TS3=0.0
002443B          DO 310 IPAR=NPMKP1,NPAR
002456B          IF(IPAR.EQ.IH1) CALL E2111(ZA(IH1),ZA(IH2),ZA(IH3),ZA(IH4),TS2)
002505B          IF(IPAR.EQ.IH2) CALL E2111(ZA(IH2),ZA(IH1),ZA(IH3),ZA(IH4),TS2)
002533B          IF(IPAR.EQ.IH3) CALL E2111(ZA(IH3),ZA(IH1),ZA(IH2),ZA(IH4),TS2)
002561B          IF(IPAR.EQ.IH4) CALL E2111(ZA(IH4),ZA(IH1),ZA(IH2),ZA(IH3),TS2)
002607B          IF(IPAR.NE.IH1.AND.IPAR.NE.IH2.AND.IPAR.NE.IH3.AND.IPAR.NE.IH4)CAL
                1L E11111(ZA(IH1),ZA(IH2),ZA(IH3),ZA(IH4),ZA(IPAR),TS2)
002646B          TS3=TS3+ZA(IPAR)*TS2
002654B      310 CONTINUE
002661B          TP=TP*(DPLSA*TS1+DPLSB*TS3)
002665B      420 LSKIP=1
002667B          UNDEF=TP
002671B          IF(LPN(9).EQ.0) GO TO 440
002673B          WRITE(IPRINT,430) TP
002703B      430 FORMAT(1X,a(DSTAR**DA(I))*ACA(I)*(EXPECTATION OF PRODUCT OF FIRST
                1P-KSTAR R.V. AND LAST N-P R.V.)a/1X,a*EXPECTATION OF PRODUCT OF MI
                1DDLE KSTAR R.V.)=a,F10.5)
002703B      440 LSKIP=1
002704B          RETURN
002707B          END
           C
           C     ROUTINE COEF2 CALCULATES BETA1A+I1J1K1A+I2J2K2 ASSUMING THAT
           C     PRODUCT OF FIRST PARTIAL DERIVATIVES OF (R+)**2 IS ZERO
           C
000000B          SUBROUTINE COEF2(II,UNDEF)
000000B          COMMON/MSKPAB/MSA(23,6),MSB(73,6),KPA(23,21),KPB(73,21)
000000B          COMMON/MSAB12/MSA1(23,4),MSA2(23,4),MSA3(23,4)
000000B          COMMON/MMSA/MMSA1(23,4),MMSA2(23,4),MMSA3(23,4)
000000B          COMMON/MSB/MSB1(73,4),MSB2(73,4),MSB3(73,4)
000000B          COMMON/MMSB/MMSB1(73,4),MMSB2(73,4),MMSB3(73,4)
000000B          COMMON/DAC/DA(23),ACA(23),DB(73),ACB(73)
000000B          COMMON/ICMAXM/ICMAXM( 50),ICMAX(100)
000000B          COMMON/MSKP/MS(6),KP(21)
000000B          COMMON/NPAROB/NPAR,NOBS,KSTAR
000000B          COMMON/DPLUS/DPLUS
000000B          COMMON/KSTRP/KSTRP(20)
000000B          COMMON/CHINON/CHINON(12)
000000B          COMMON/FACGAM/FAC(150),GAMMA(150)
000000B          COMMON/ZA/ZA(10)
000000B          COMMON/A/A(5,3,3)
000000B          COMMON/LPN/LPN(14)
000000B          COMMON/NS/NS(4)
000000B          COMMON/IPRINT/IREAD,IPRINT,ITAPE7,ITAPE8,ITAPE9,ITAP10
000000B          I=II
000000B          IF(LPN(10).EQ.0) GO TO 8
000002B          WRITE(IPRINT,2) (MS(IB),IB=1,6),(KP(IB),IB=1,21)
000021B        2 FORMAT(1X,aI1,J1,K1,I2,J2,K2 AREa,6(I1,1X),1H/,aKP(*) AREa,21(I1,1
                1X))
```

```
000021B        WRITE(IPRINT,4)
000025B      4 FORMAT(1X,@I,ICAMX(I),DB(I),ACB(I),MSB(*),KPB(*),MSB1(I,*),MMSB1(I
              1,*) ETC ARE@)
000025B        WRITE(IPRINT,6) I,ICMAX(I),DB(I),ACB(I),(MSB(I,IB),IB=1,6),(KPB(I,
              1IB),IB=1,21),((MSB1(I,IB),MMSB1(I,IB)),IB=1,4),((MSB2(I,IB),MMSB2(
              1I,IB)),IB=1,4),((MSB3(I,IB),MMSB3(I,IB)),IB=1,4)
000131B      6 FORMAT(1X,I5,1H/,I2,1H/,2(F4.1,1H/),6I1,1H/,21I1,1H/,3(4(I1,1H*,I1
              1,1H/)),2H//)
000131B      8 LSKIP=1
000131B        TP=(DPLUS**DB(I))*ACB(I)
000136B        IF(LPN(11).EQ.0) GO TO 12
000140B        WRITE(IPRINT,10) DPLUS,DB(I),ACB(I),TP
000152B     10 FORMAT(1X,@(DPLUS**DB(I))*ACB(I)=(@,F10.5,2H**,F4.1,2H)*,F4.1,1H=,
              1F10.5)
000152B     12 LSKIP=1
          C
          C    TO CALCULATE EXPECTATION OF (Z(1)**L(1))*(Z(2)**L(2))*...*
          C    (Z(P-KSTAR)**L(P-KSTAR))
          C
000152B        DO 14 IH=1,4
000154B        IF(MMSB1(I,IH).EQ.0) GO TO 14
000161B        IH1=MSB1(I,IH)
000163B        IH2=MS(IH1)
000165B        TZ=ZA(IH2)
000166B        MMS=MMSB1(I,IH)
000171B        CALL EXPZ(TZ,MMS,TP)
          C
          C    ROUTINE EXPZ CALCULATES EXPECTATION OF Z,Z**2,Z**3 AND Z**4
          C
000174B     14 CONTINUE
000176B        IF(LPN(11).EQ.0 )GO TO 18
000177B        WRITE(IPRINT,16) TP
000204B     16 FORMAT(1X,@(DSTAR**DB(I))*ACB(I)*EXPECTATION OF PRODUCT OF FIRST P
              1-KSTAR R.V. =@,F10.5)
000204B     18 LSKIP=1
          C
          C    TO CALCULATE EXPECTATION OF (Z(P+1)**L(P+1))*(Z(P+2)*L(P+2))*...*
          C    (Z(N)*L(N))
          C
000204B        DO 20 IH=1,4
000206B        IF(MMSB3(I,IH).EQ.0) GO TO 20
000213B        IH1=MSB3(I,IH)
000215B        IH2=MS(IH1)
000217B        TZ=ZA(IH2)
000220B        MMS=MMSB3(I,IH)
000223B        CALL EXPZ(TZ,MMS,TP)
000226B     20 CONTINUE
000230B        IF(LPN(11).EQ.0) GO TO 23
000231B        WRITE(IPRINT,22) TP
000236B     22 FORMAT(1X,@(DSTAR**DB(I))*ACB(I)*EXPECTATION OF PRODUCT OF FIRST P
              1-KSTAR R.V. AND LAST N-P R.V. =@,F10.5)
000236B     23 LSKIP=1
          C
          C    TO CALCULATE EXPECTATION OF ((S(P-KSTAR+1)SQRT(ZBARS(P-KSTAR+1)))
          C    **L(P-KSTAR+1))*((S(P-KSTAR+2)SQRT(ZBARS(P-KSTAR+2)))**L(P-KSTAR+2)
          C    )*...*((S(P)SQRT(ZBARS(P)))**L(P))
          C
000236B        DO 30 II=1,4
000240B        NS(II)=0
000240B     30 CONTINUE
000242B        DO 50 IH=1,4
000244B        DO 40 IIH=1,4
000246B        IF(MMSB2(I,IH).EQ.IIH) NS(IIH)=NS(IIH)+1
000256B     40 CONTINUE
000260B     50 CONTINUE
```

```
000263B    66.       IF(NS(1).EQ.0.AND.NS(2).EQ.0.AND.NS(3).EQ.0.AND.NS(4).EQ.0)
                    1GO TO 80
000270B    67.       IF(NS(1).EQ.1.AND.NS(2).EQ.0.AND.NS(3).EQ.0.AND.NS(4).EQ.0)
                    1GO TO 100
000275B    68.       IF(NS(1).EQ.0.AND.NS(2).EQ.1.AND.NS(3).EQ.0.AND.NS(4).EQ.0)
                    1GO TO 120
000302B    69.       IF(NS(1).EQ.2.AND.NS(2).EQ.0.AND.NS(3).EQ.0.AND.NS(4).EQ.0)
                    1GO TO 140
000307B    70.       IF(NS(1).EQ.0.AND.NS(2).EQ.0.AND.NS(3).EQ.1.AND.NS(4).EQ.0)
                    1GO TO 160
000314B    71.       IF(NS(1).EQ.1.AND.NS(2).EQ.1.AND.NS(3).EQ.0.AND.NS(4).EQ.0)
                    1GO TO 180
000322B    72.       IF(NS(1).EQ.3.AND.NS(2).EQ.0.AND.NS(3).EQ.0.AND.NS(4).EQ.0)
                    1GO TO 200
000327B    73.       IF(NS(1).EQ.0.AND.NS(2).EQ.0.AND.NS(3).EQ.0.AND.NS(4).EQ.1)
                    1GO TO 220
000334B    74.       IF(NS(1).EQ.1.AND.NS(2).EQ.0.AND.NS(3).EQ.1.AND.NS(4).EQ.0)
                    1GO TO 240
000342B    75.       IF(NS(1).EQ.0.AND.NS(2).EQ.2.AND.NS(3).EQ.0.AND.NS(4).EQ.0)
                    1GO TO 260
000347B    76.       IF(NS(1).EQ.2.AND.NS(2).EQ.1.AND.NS(3).EQ.0.AND.NS(4).EQ.0)
                    1GO TO 280
000355B    77.       IF(NS(1).EQ.4.AND.NS(2).EQ.0.AND.NS(3).EQ.0.AND.NS(4).EQ.0)
                    1GO TO 300
                C
                C    0000
                C
000362B    78.    80 TP=TP*CHINON(KSTAR)
000363B    79.       GO TO 420
                C
                C    1000
                C
000364B    80.   100 IH1=MS(MSB2(I,1))
000367B    81.       TZ=ZA(IH1)
000370B    82.       CALL E1000(TZ,TS)
000373B    83.       TP=TP*TS
000374B    84.       GO TO 420
                C
                C    2000
                C
000375B    85.   120 IH1=MS(MSB2(I,1))
000377B    86.       TZ=ZA(IH1)
000400B    87.       CALL E2000(TZ,TS)
000403B    88.       TP=TP*TS
000404B    89.       GO TO 420
                C
                C    1100
                C
000405B    90.   140 IH1=MS(MSB2(I,1))
000407B    91.       TZ1=ZA(IH1)
000410B    92.       IH2=MS(MSB2(I,2))
000412B    93.       TZ2=ZA(IH2)
000414B    94.       CALL E1100(TZ1,TZ2,TS)
000417B    95.       TP=TP*TS
000420B    96.       GO TO 420
                C
                C    3000
                C
000421B    97.   160 IH1=MS(MSB2(I,1))
000423B    98.       TZ=ZA(IH1)
000424B    99.       CALL E3000(TZ,TS)
000427B   100.       TP=TP*TS
000430B   101.       GO TO 420
                C
                C    2100
                C
```

```
0004318      180 IH1=MS(MSB2(I,1))
0004338          TZ1=ZA(IH1)
0004348          IH2=MS(MSB2(I,2))
0004368          TZ2=ZA(IH2)
0004408          IF(MMSB2(I,1).EQ.2) GO TO 184
0004428          TTZ1=TZ1
0004438          TTZ2=TZ2
0004448          TZ1=TTZ2
0004458          TZ2=TTZ1
0004468      184 LSKIP=1
0004508          CALL E2100(TZ1,TZ2,TS)
0004538          TP=TP*TS
0004548          GO TO 420
            C
            C    1110
            C
0004558      200 IH1=MS(MSB2(I,1))
0004578          TZ1=ZA(IH1)
0004608          IH2=MS(MSB2(I,2))
0004628          TZ2=ZA(IH2)
0004648          IH3=MS(MSB2(I,3))
0004668          TZ3=ZA(IH3)
0004708          CALL E1110(TZ1,TZ2,TZ3,TS)
0004738          TP=TP*TS
0004748          GO TO 420
            C
            C    4000
            C
0004758      220 IH1=MS(MSB2(I,1))
0004778          TZ=ZA(IH1)
0005008          CALL E4000(TZ,TS)
0005038          TP=TP*TS
0005048          GO TO 420
            C
            C    3100
            C
0005058      240 IH1=MS(MSB2(I,1))
0005078          TZ1=ZA(IH1)
0005108          IH2=MS(MSB2(I,2))
0005128          TZ2=ZA(IH2)
0005148          IF(MMSB2(I,1).EQ.3) GO TO 244
0005168          TTZ1=TZ1
0005178          TTZ2=TZ2
0005208          TZ1=TTZ2
0005218          TZ2=TTZ1
0005228      244 LSKIP=1
0005248          CALL E3100(TZ1,TZ2,TS)
0005278          TP=TP*TS
0005308          GO TO 420
            C
            C    2200
            C
0005318      260 IH1=MS(MSB2(I,1))
0005338          TZ1=ZA(IH1)
0005348          IH2=MS(MSB2(I,2))
0005368          TZ2=ZA(IH2)
0005408          CALL E2200(TZ1,TZ2,TS)
0005438          TP=TP*TS
0005448          GO TO 420
            C
            C    2110
            C
0005458      280 IH1=MS(MSB2(I,1))
0005478          TZ1=ZA(IH1)
0005508          IH2=MS(MSB2(I,2))
0005528          TZ2=ZA(IH2)
```

```
000554B         IH3=MS(MSB2(I,3))
000556B         TZ3=ZA(IH3)
000560B         IF(MMSB2(I,1).EQ.2) GO TO 288
000562B         IF(MMSB2(I,2).EQ.2) GO TO 282
000564B         IF(MMSB2(I,3).EQ.2) GO TO 284
000565B     282 TTZ1=TZ1
000566B         TTZ2=TZ2
000570B         TZ1=TTZ2
000570B         TZ2=TTZ1
000572B         GO TO 288
000572B     284 TTZ1=TZ1
000573B         TTZ3=TZ3
000575B         TZ1=TTZ3
000575B         TZ3=TTZ1
000577B     288 LSKIP=1
000600B         CALL E2110(TZ1,TZ2,TZ3,TS)
000603B         TP=TP*TS
000604B         GO TO 420
          C
          C     1111
          C
000605B     300 IH1=MS(MSB2(I,1))
000607B         TZ1=ZA(IH1)
000610B         IH2=MS(MSB2(I,2))
000612B         TZ2=ZA(IH2)
000614B         IH3=MS(MSB2(I,3))
000616B         TZ3=ZA(IH3)
000620B         IH4=MS(MSB2(I,4))
000622B         TZ4=ZA(IH4)
000624B         CALL E1111(TZ1,TZ2,TZ3,TZ4,TS)
000627B         TP=TP*TS
000630B     420 LSKIP=1
000631B         UNDEF=TP
000632B         IF(LPN(11).EQ.0) GO TO 440
000634B         WRITE(IPRINT,430) TP
000641B     430 FORMAT(1X,@(DSTAR**DB(I))*ACB(I)*(EXPECTATION OF PRODUCT OF FIRST
               1P-KSTAR R.V. AND LAST N-P R.V.)@/1X,@*EXPECTATION OF PRODUCT OF MI
               1DDLE KSTAR R.V.)=@,F10.5)
000641B     440 LSKIP=1
000641B         RETURN
000644B         END
000000B         SUBROUTINE COEF1(II,JJ,KK,UNDEF)
          C
          C     ROUTINE COEF1 CALCULATES BETA1A+IJK
          C
000000B         COMMON/MSKPAB/MSA(23,6),MSB(73,6),KPA(23,21),KPB(73,21)
000000B         COMMON/MMSA/MMSA1(23,4),MMSA2(23,4),MMSA3(23,4)
000000B         COMMON/MSAB12/MSA1(23,4),MSA2(23,4),MSA3(23,4)
000000B         COMMON/MSB/MSB1(73,4),MSB2(73,4),MSB3(73,4)
000000B         COMMON/MMSB/MMSB1(73,4),MMSB2(73,4),MMSB3(73,4)
000000B         COMMON/DAC/DA(23),ACA(23),DB(73),ACB(73)
000000B         COMMON/ICMAXM/ICMAXM( 50),ICMAX(100)
000000B         COMMON/MSKP/MS(6),KP(21)
000000B         COMMON/NPAROB/NPAR,NOBS,KSTAR
000000B         COMMON/DPLUS/DPLUS
000000B         COMMON/KSTRP/KSTRP(20)
000000B         COMMON/CHINON/CHINON(12)
000000B         COMMON/FACGAM/FAC(150),GAMMA(150)
000000B         COMMON/ZA/ZA(10)
000000B         COMMON/A/A(5,3,3)
000000B         COMMON/LPN/LPN(14)
000000B         COMMON/IPRINT/IREAD,IPRINT,ITAPE7,ITAPE8,ITAPE9,ITAP10
000000B         I=II
000000B         J=JJ
000002B         K=KK
000003B         NPMK=NPAR-KSTAR
```

```
000005B          NPMKP1=NPMK+1
000006B          NPARP=NPAR+1
000007B          TS=0.0
000010B          IF(.NOT.(I.GE.NPMKP1.AND.I.LE.NPAR.AND.J.GE.1.AND.J.LE.NPMK.AND.K.
                1GE.1.AND.K.LE.NPMK)) GO TO 30
000022B          CALL E1000(ZA(I),TS1)
000026B          IF(.NOT.(J.NE.K)) GO TO 10
000031B          TS=4.0*ZA(J)*ZA(K)*DPLUS*TS1
000035B          GO TO 20
000037B       10 TS=2.0*(1.0+ZA(J)*ZA(J))*DPLUS*TS1
000043B       20 LSKIP=1
000044B          GO TO 100
000045B       30 IF(.NOT.(I.GE.NPARP.AND.I.LE.NOBS))GO TO 100
000052B          IF(.NOT.((J.GE.1.AND.J.LE.NPMK).AND.(K.GE.NPMKP1.AND.K.LE.NPAR)))G
                1O TO 40
000062B          CALL E1000(ZA(K),TS1)
000066B          TS=-4.0*ZA(I)*ZA(J)*DPLUS*TS1
000073B          GO TO 100
000074B       40 IF(.NOT.((J.GE.NPMKP1.AND.J.LE.NPAR).AND.(K.GE.1.AND.K.LE.NPMK)))
                1GO TO 50
000105B          CALL E1000(ZA(J),TS1)
000110B          TS=-4.0*ZA(I)*ZA(K)*DPLUS*TS1
000115B          GO TO 100
000116B       50 IF(.NOT.((J.GE.NPMKP1.AND.J.LE.NPAR).AND.(K.GE.NPMKP1.AND.K.LE.NPA
                1R))) GO TO 100
000126B          IF(.NOT.(J.NE.K)) GO TO 60
000131B          CALL E1100(ZA(J),ZA(K),TS1)
000136B          TS=-4.0*ZA(I)*DPLUS*DPLUS*TS1
000142B          GO TO 70
000143B       60 IF(.NOT.(J.EQ.K)) GO TO 70
000147B          CALL E2000(ZA(J),TS1)
000152B          TS=-2.0*ZA(I)*DPLUS*DPLUS*TS1
000155B       70 LSKIP=1
000156B      100 LSKIP=1
000157B          UNDEF=TS
000160B          RETURN
000163B          END
                 SUBROUTINE E1000(TZ,TS)
000000B          COMMON/KSTRP/KSTRP(20)
000000B          COMMON/DPLUS/DPLUS
000000B          COMMON/CHINON/CHINON(12)
000000B          COMMON/NPAROB/NPAR,NOBS,KSTAR
000000B          TS=(TZ/DPLUS)*CHINON(KSTRP(2))
000003B          RETURN
000005B          END
```

NUMBER AND NAME CROSS REFERENCE MAP (/N= N REFERENCES IN STATEMENT)

VARIABLE AND ARRAY NAMES SORTED BY ADDRESS   (B=RELATIVE ADDRESS)
                                                (C= RELATIVE TO //)

```
000000C CHINON     000000C DPLUS     000000C KSTRP     000000C NPAR     000001C NOBS
```

```
000000B          SUBROUTINE E2000(TZ,TS)
000000B          COMMON/KSTRP/KSTRP(20)
000000B          COMMON/DPLUS/DPLUS
000000B          COMMON/CHINON/CHINON(12)
000000B          COMMON/NPAROB/NPAR,NOBS,KSTAR
000000B          TS=(1.0/(DPLUS**2))*(CHINON(KSTRP(2))+TZ*TZ*CHINON(KSTRP(4)))
000007B          RETURN
000012B          END
```

```
000000B      SUBROUTINE E1100(TZ1,TZ2,TS)
000000B      COMMON/KSTRP/KSTRP(20)
000000B      COMMON/DPLUS/DPLUS
000000B      COMMON/CHINON/CHINON(12)
000000B      COMMON/NPAROB/NPAR,NOBS,KSTAR
000000B      TS=(TZ1*TZ2/(DPLUS**2))*CHINON(KSTRP(4))
000004B      RETURN
000007B      END

NUMBER AND NAME CROSS REFERENCE MAP (/N= N  REFERENCES IN STATEMENT)

VARIABLE AND ARRAY NAMES SORTED BY ADDRESS      (B=RELATIVE ADDRESS)
                                                (C= RELATIVE TO  //)
  000000C CHINON      000000C DPLUS      000000C KSTRP      000000C NPAR      000001C NOBS

000000B      SUBROUTINE E3000(TZ,TS)
000000B      COMMON/KSTRP/KSTRP(20)
000000B      COMMON/DPLUS/DPLUS
000000B      COMMON/CHINON/CHINON(12)
000000B      COMMON/NPAROB/NPAR,NOBS,KSTAR
000000B      TS=(TZ/(DPLUS**3))*(3.0*CHINON(KSTRP(4))+TZ*TZ*CHINON(KSTRP(6)))
000010B      RETURN
000013B      END

NUMBER AND NAME CROSS REFERENCE MAP (/N= N  REFERENCES IN STATEMENT)

VARIABLE AND ARRAY NAMES SORTED BY ADDRESS      (B=RELATIVE ADDRESS)
                                                (C= RELATIVE TO  //)
  000000C CHINON      000000C DPLUS      000000C KSTRP      000000C NPAR      000001C NOBS

000000B      SUBROUTINE E2100(TZ1,TZ2,TS)
000000B      COMMON/KSTRP/KSTRP(20)
000000B      COMMON/DPLUS/DPLUS
000000B      COMMON/CHINON/CHINON(12)
000000B      COMMON/NPAROB/NPAR,NOBS,KSTAR
000000B      TS=(TZ2/(DPLUS**3))*(CHINON(KSTRP(4))+TZ1*TZ1*CHINON(KSTRP(6)))
000007B      RETURN
000012B      END

NUMBER AND NAME CROSS REFERENCE MAP (/N= N  REFERENCES IN STATEMENT)

VARIABLE AND ARRAY NAMES SORTED BY ADDRESS      (B=RELATIVE ADDRESS)
                                                (C= RELATIVE TO  //)
  000000C CHINON      000000C DPLUS      000000C KSTRP      000000C NPAR      000001C NOBS

000000B      SUBROUTINE E1110(TZ1,TZ2,TZ3,TS)
000000B      COMMON/KSTRP/KSTRP(20)
000000B      COMMON/DPLUS/DPLUS
000000B      COMMON/CHINON/CHINON(12)
000000B      COMMON/NPAROB/NPAR,NOBS,KSTAR
000000B      TS=(TZ1*TZ2*TZ3/(DPLUS**3))*CHINON(KSTRP(6))
000005B      RETURN
000010B      END

NUMBER AND NAME CROSS REFERENCE MAP (/N= N  REFERENCES IN STATEMENT)

VARIABLE AND ARRAY NAMES SORTED BY ADDRESS      (B=RELATIVE ADDRESS)
                                                (C= RELATIVE TO  //)
  000000C CHINON      000000C DPLUS      000000C KSTRP      000000C NPAR      000001C NOBS

000000B      SUBROUTINE E4000(TZ,TS)
000000B      COMMON/KSTRP/KSTRP(20)
000000B      COMMON/DPLUS/DPLUS
000000B      COMMON/CHINON/CHINON(12)
000000B      COMMON/NPAROB/NPAR,NOBS,KSTAR
000000B      TS=(1.0/(DPLUS**4))*(3.0*CHINON(KSTRP(4))+6.0*TZ*TZ*CHINON(KSTRP(6
            1))+TZ*TZ*TZ*TZ*CHINON(KSTRP(8)))
000014B      RETURN
000017B      END
```

```
000000B      57.      SUBROUTINE E3100(TZ1,TZ2,TS)
000000B      58.      COMMON/KSTRP/KSTRP(20)
000000B      59.      COMMON/DPLUS/DPLUS
000000B      60.      COMMON/CHINON/CHINON(12)
000000B      61.      COMMON/NPAROB/NPAR,NOBS,KSTAR
000000B      62.      TS=(TZ1*TZ2/(DPLUS**4))*(3.0*CHINON(KSTRP(6))+TZ1*TZ1*CHINON(KSTRP
                     1(8)))
000011B      63.      RETURN
000014B      64.      END
```

NUMBER AND NAME CROSS REFERENCE MAP (/N= N REFERENCES IN STATEMENT)

```
VARIABLE AND ARRAY NAMES SORTED BY ADDRESS      (B=RELATIVE ADDRESS)
                                                (C= RELATIVE TO  //)
  000000C CHINON      000000C DPLUS      000000C KSTRP      000000C NPAR      000001C NOBS

000000B      65.      SUBROUTINE E2200(TZ1,TZ2,TS)
000000B      66.      COMMON/KSTRP/KSTRP(20)
000000B      67.      COMMON/DPLUS/DPLUS
000000B      68.      COMMON/CHINON/CHINON(12)
000000B      69.      COMMON/NPAROB/NPAR,NOBS,KSTAR
000000B      70.      TS=(1.0 /(DPLUS**4))*(CHINON(KSTRP(4))+(TZ1*TZ1+TZ2*TZ2)*CHINON(KST
                     1RP(6))+TZ1*TZ1*TZ2*TZ2*CHINON(KSTRP(8)))
000015B      71.      RETURN
000020B      72.      END
```

NUMBER AND NAME CROSS REFERENCE MAP (/N= N REFERENCES IN STATEMENT)

```
VARIABLE AND ARRAY NAMES SORTED BY ADDRESS      (B=RELATIVE ADDRESS)
                                                (C= RELATIVE TO  //)
  000000C CHINON      000000C DPLUS      000000C KSTRP      000000C NPAR      000001C NOBS

000000B      73.      SUBROUTINE E2110(TZ1,TZ2,TZ3,TS)
000000B      74.      COMMON/KSTRP/KSTRP(20)
000000B      75.      COMMON/DPLUS/DPLUS
000000B      76.      COMMON/CHINON/CHINON(12)
000000B      77.      COMMON/NPAROB/NPAR,NOBS,KSTAR
000000B      78.      TS=(TZ2*TZ3/(DPLUS**4))*(CHINON(KSTRP(6))+TZ1*TZ1*CHINON(KSTRP(8))
                     1)
000011B      79.      RETURN
000014B      80.      END
```

NUMBER AND NAME CROSS REFERENCE MAP (/N= N REFERENCES IN STATEMENT)

```
VARIABLE AND ARRAY NAMES SORTED BY ADDRESS      (B=RELATIVE ADDRESS)
                                                (C= RELATIVE TO  //)
  000000C CHINON      000000C DPLUS      000000C KSTRP      000000C NPAR      000001C NOBS

000000B      81.      SUBROUTINE E1111(TZ1,TZ2,TZ3,TZ4,TS)
000000B      82.      COMMON/KSTRP/KSTRP(20)
000000B      83.      COMMON/DPLUS/DPLUS
000000B      84.      COMMON/CHINON/CHINON(12)
000000B      85.      COMMON/NPAROB/NPAR,NOBS,KSTAR
000000B      86.      TS=(TZ1*TZ2*TZ3*TZ4/(DPLUS**4))*CHINON(KSTRP(8))
000006B      87.      RETURN
000011B      88.      END
```

NUMBER AND NAME CROSS REFERENCE MAP (/N= N REFERENCES IN STATEMENT)

```
VARIABLE AND ARRAY NAMES SORTED BY ADDRESS      (B=RELATIVE ADDRESS)
                                                (C= RELATIVE TO  //)
  000000C CHINON      000000C DPLUS      000000C KSTRP      000000C NPAR      000001C NOBS

000000B      89.      SUBROUTINE    E5000(TZ,TS)
000000B      90.      COMMON/KSTRP/KSTRP(20)
000000B      91.      COMMON/DPLUS/DPLUS
000000B      92.      COMMON/CHINON/CHINON(12)
000000B      93.      COMMON/NPAROB/NPAR,NOBS,KSTAR
000000B      94.      TS=(TZ/(DPLUS**5))*(15.0*CHINON(KSTRP(6))+10.0*TZ*TZ*CHINON(KSTRP(
                     18))+TZ*TZ*TZ*TZ*CHINON(KSTRP(10)))
000014B      95.      RETURN
000017B      96.      END
```

```
000000B      97.        SUBROUTINE E4100(TZ1,TZ2,TS)
000000B      98.        COMMON/KSTRP/KSTRP(20)
000000B      99.        COMMON/DPLUS/DPLUS
000000B     100.        COMMON/CHINON/CHINON(12)
000000B     101.        COMMON/NPAROB/NPAR,NOBS,KSTAR
000000B     102.        TS=(TZ2/(DPLUS**5))*(TZ1*TZ1*TZ1*TZ1*CHINON(KSTRP(10))+6.0*TZ1*TZ1
                       1*CHINON(KSTRP(8))+3.0*CHINON(KSTRP(6)))
000015B     103.        RETURN
000020B     104.        END
```

NUMBER AND NAME CROSS REFERENCE MAP (/N= N REFERENCES IN STATEMENT)

```
VARIABLE AND ARRAY NAMES SORTED BY ADDRESS       (B=RELATIVE ADDRESS)
                                                 (C= RELATIVE TO  //)
  000000C CHINON     000000C DPLUS      000000C KSTRP      000000C NPAR       000001C NOBS
```

```
000000B     105.        SUBROUTINE E3200(TZ1,TZ2,TS)
000000B     106.        COMMON/KSTRP/KSTRP(20)
000000B     107.        COMMON/DPLUS/DPLUS
000000B     108.        COMMON/CHINON/CHINON(12)
000000B     109.        COMMON/NPAROB/NPAR,NOBS,KSTAR
000000B     110.        TS=(TZ1/(DPLUS**5))*(3.0*CHINON(KSTRP(6))+TZ1*TZ1*CHINON(KSTRP(8))
                       1+3.0*TZ2*TZ2*CHINON(KSTRP(8))+TZ1*TZ1*TZ2*TZ2*CHINON(KSTRP(10)))
000016B     111.        RETURN
000021B     112.        END
```

NUMBER AND NAME CROSS REFERENCE MAP (/N= N REFERENCES IN STATEMENT)

```
VARIABLE AND ARRAY NAMES SORTED BY ADDRESS       (B=RELATIVE ADDRESS)
                                                 (C= RELATIVE TO  //)
  000000C CHINON     000000C DPLUS      000000C KSTRP      000000C NPAR       000001C NOBS
```

```
000000B     113.        SUBROUTINE E3110(TZ1,TZ2,TZ3,TS)
000000B     114.        COMMON/KSTRP/KSTRP(20)
000000B     115.        COMMON/DPLUS/DPLUS
000000B     116.        COMMON/CHINON/CHINON(12)
000000B     117.        COMMON/NPAROB/NPAR,NOBS,KSTAR
000000B     118.        TS=(TZ1*TZ2*TZ3/(DPLUS**5))*(3.0*CHINON(KSTRP(8))+TZ1*TZ1*CHINON(K
                       1STRP(10)))
000013B     119.        RETURN
000015B     120.        END
```

NUMBER AND NAME CROSS REFERENCE MAP (/N= N REFERENCES IN STATEMENT)

```
VARIABLE AND ARRAY NAMES SORTED BY ADDRESS       (B=RELATIVE ADDRESS)
                                                 (C= RELATIVE TO  //)
  000000C CHINON     000000C DPLUS      000000C KSTRP      000000C NPAR       000001C NOBS
```

```
000000B     121.        SUBROUTINE E2210(TZ1,TZ2,TZ3,TS)
000000B     122.        COMMON/KSTRP/KSTRP(20)
000000B     123.        COMMON/DPLUS/DPLUS
000000B     124.        COMMON/CHINON/CHINON(12)
000000B     125.        COMMON/NPAROB/NPAR,NOBS,KSTAR
000000B     126.        TS=(TZ3/(DPLUS**5))*(CHINON(KSTRP(6))+(TZ1*TZ1+TZ2*TZ2)*CHINON(KST
                       1RP(8))+TZ1*TZ1*TZ2*TZ2*CHINON(KSTRP(10)))
000015B     127.        RETURN
000020B     128.        END
```

```
000000B      129.        SUBROUTINE E2111(TZ1,TZ2,TZ3,TZ4,TS)
000000B      130.        COMMON/KSTRP/KSTRP(20)
000000B      131.        COMMON/DPLUS/DPLUS
000000B      132.        COMMON/CHINON/CHINON(12)
000000B      133.        COMMON/NPAROB/NPAR,NOBS,KSTAR
000000B      134.        TS=(TZ2*TZ3*TZ4/(DPLUS**5))*(CHINON(KSTRP(8))+TZ1*TZ1*CHINON(KSTRP
                        1(10)))
000012B      135.        RETURN
000015B      136.        END
```

NUMBER AND NAME CROSS REFERENCE MAP (/N= N REFERENCES IN STATEMENT)

```
VARIABLE AND ARRAY NAMES SORTED BY ADDRESS       (B=RELATIVE ADDRESS)
                                                 (C= RELATIVE TO  //)
  000000C CHINON      000000C DPLUS       000000C KSTRP       000000C NPAR        000001C NOBS
```

```
000000B      137.        SUBROUTINE E11111(TZ1,TZ2,TZ3,TZ4,TZ5,TS)
000000B      138.        COMMON/KSTRP/KSTRP(20)
000000B      139.        COMMON/DPLUS/DPLUS
000000B      140.        COMMON/CHINON/CHINON(12)
000000B      141.        COMMON/NPAROB/NPAR,NOBS,KSTAR
000000B      142.        TS=(TZ1*TZ2*TZ3*TZ4*TZ5/(DPLUS**5))*CHINON(KSTRP(10))
000007B      143.        RETURN
000012B      144.        END
```

NUMBER AND NAME CROSS REFERENCE MAP (/N= N REFERENCES IN STATEMENT)

```
VARIABLE AND ARRAY NAMES SORTED BY ADDRESS       (B=RELATIVE ADDRESS)
                                                 (C= RELATIVE TO  //)
  000000C CHINON      000000C DPLUS       000000C KSTRP       000000C NPAR        000001C NOBS
```

```
000000B      145.        SUBROUTINE EXPZ(TZZ,MMS,TPP)
000000B      146.        TZ=TZZ
000000B      147.        TP=TPP
000002B      148.        IF(MMS.EQ.1) TP=TP*TZ
000006B      149.        IF(MMS.EQ.2) TP=TP*(1.0+TZ*TZ)
000013B      150.        IF(MMS.EQ.3) TP=TP*(3.0*TZ+TZ*TZ*TZ)
000020B      151.        IF(MMS.EQ.4) TP=TP*(3.0+6.0*TZ*TZ+TZ*TZ*TZ*TZ)
000027B      152.        TPP=TP
000027B      153.        RETURN
000032B      154.        END
```

References

Bard, Y. (1974). Nonlinear parameter estimation. Academic Press. New York and London.

Barnard, G.A. (1960). Discussion on Mr. Beale's paper, J. Roy. Statist. Soc. Ser. B, 22, 78-79.

Bartholomew, D.J. (1970). A comparison of frequentist and Bayesian approaches to inference with prior knowledge. Foundations of statistical inference - Proceedings of a symposium held at University of Waterloo.

Bartholomew, D.J. (1961). A test of homogeneity of means under restricted alternatives. J. Roy. Statist. Soc. Ser. B. 23, 239-281.

Beale, E.M.L. (1960). Confidence regions in nonlinear estimation. J. Roy. Statist. Soc. Ser. B. 22, 41-76.

Box, G.E.P. and Cox, D.R. (1964). An analysis of transformations. J. Roy. Statist. Soc., Series B, 26, 211-252.

Box, M.J. (1971). Bias in nonlinear estimation. J. Roy. Statist. Soc. Ser. B. 171-190.

Graybill, F.A. (1961). An introduction to linear statistical models, Vol.I, McGraw-Hill: New York.

Guttman, I. and Meeter, D.A. (1965). On Beale's measures of nonlinearity. Technometric, vol.7, No.4, November, 1965.

Guttman, I., Pereyra, V. and Scolnik, H.D. (1973). Least squares estimation for a class of nonlinear models. Technometric. Vol.15, No.2, May, 1973.

Halperin, M. (1963). Confidence interval estimation in nonlinear regression. J.R. Statist. Soc. B, 25, 330-333.

Halperin, M. (1964). Note on interval estimation in nonlinear regression when responses are correlated. J. Roy. Statist. Soc. Series B, 26, 267-269.

Hartley, H.O. (1964). Exact confidence regions for the parameters in nonlinear regression laws. Biometrika. 51, 347-353.

Lehmann, E.L. (1959). Testing statistical hypotheses. Wiley: New York.

Linssen, H.N. (1975). Nonlinearity measures: a case study. Statistica Neerlandica Vol.29, no.3, Rotterdam.

Ross, G.J.S. (1970). The efficient use of function minimization in nonlinear maximum likelihood estimation. J.R. Statist. Soc., C. 19, 205-221.

Ross, G.J.S. (1975). Simple nonlinear modelling for the general user. Warszawa 1-9 IX. ISI/BS. Invited paper 81.

Seber, G.A.F. (1963). The non-central chi-squared and beta distributions. Biometrika, 50, 542-544.

Seber, G.A.F. (1964). The linear hypothesis and large sample theory. Annals Math. Statist., 35, 773-779.

Wald, A. (1943). Tests of statistical hypotheses concerning several parameters when the number of observations is large. Trans. Amer. Math. Soc., 54, 426-482.

Wilks, S.S. and Daly, J.F. (1939). An optimum property of confidence region associated with the likelihood function. Ann. Math. Statist., 10, 225-235.

Williams, E.J. (1962). Exact fiducial limits in non-linear estimation, J.R. Statist. Soc. B, 24, 125-139.