SUPERSPACE AND CLASSICAL FIELD THEORY

by

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ABSTRACT

An axiomatic formulation for the theory of Classical Tensorial Fields is constructed based on three principles, the Relativity, the Chronos and the Variational Principle.

After a short historical introduction, we begin the construction of superspace starting from the product space of the space of metric functions and the space of any number of scalar functions and any number of vector functions. The scalar and vector fields are here defined on a three-dimensional manifold with a positive definite metric.

Using then the above mentioned principles we are able to deduce the most general Lagrangian which is compatible with them. This turns out to include the Einstein and Jordan theories as regards its gravitational content and the Yang-Mills and Chiral theories as far as its field theoretic content is concerned. The parameters of the gauge group are in one to one correspondence with the vector fields and the group acts on the space of scalar fields as a group of motions.

We then discuss the removal of massless scalar fields and the corresponding acquisition of mass of the vector fields. In this connection we distinguish the transitive from the intransitive group case (spontaneous symmetry breaking).

Finally we restrict our attention to the cases where the dimension of the space of scalar fields is one, two and three. For these cases we discuss all possible Lagrangians of the above mentioned form.

PREFACE

The work described in this thesis was performed under the supervision of Professor Abdus Salam, at Imperial College, between October 1973 and October 1975, and in collaboration with Demetrios Christodoulou. Unless otherwise stated the work is original and has not been presented for a degree of this or any other University.

The author wishes to thank his supervisor for his continuous encouragement while this work was in progress and is indebted to Demetrios Christodoulou for his collaboration and to Christos Ktorides, Sergio Hojman, Patricio Cordero, for many helpful discussions as well as to Maria Teresa Ruiz for helping him writing this thesis.

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CHAPTER I

INTRODUCTION

Superspace, that is, the space of all geometries of a 3-dimensional manifold, has originally attracted attention in connection with the canonical approach to the quantization of General Relativity. The work of Dirac[1], De Witt[2], Higgs[3] and many others towards this direction revealed that superspace is the domain manifold for the quantum mechanical state-functional. On the other hand, at the classical level and mainly due to the work of A.D.M.[4] it became clear that the dynamical variable of General Relativity is the 3-geometry of space. Both the above facts have been illuminated by Wheeler[5] who was the first to realize the importance of superspace and to clarify its role as the proper configuration space of General Relativity.

This having been done, it became highly desirable to obtain a better understanding of the structure of this space, the hope being that this would lead to deeper insights both at the classical and quantum level.

The first investigations of superspace were done by De Witt[2] who recognised the metric that General Relativity dictates to be introduced on it. He also investigated its geodetic structure and found that it was incomplete (geodetically). This was a rather discouraging result and was not to be clarified until Fischer's[6] work which came later. Meanwhile, Stern[7] studied the topological structure of superspace and found it to be Haussdorff. Almost at the same time Ebin[8] proved, with the help of some remarks of Palàis'[9], the so-called slice-theorem for superspace. This theorem was successfully used by Fischer[6] who showed that this space is a metrizable topological space and it inherits from the action of the group of diffeomorphisms Diff ($\mathcal{M}$) a "stratified" manifold structure. The same author proved that though superspace is not a proper manifold, it can be extended in such a way as to become a proper manifold.

Most of this work, however, was too complicated and mathematically involved to be of any direct use in physical applications. However, De Witt[10] was able, with a simpler analysis, to obtain space-time obeying Einstein's equations as a sheaf of geodesics in superspace.

In another development Christodoulou,[11] by introducing what he calls "The Chronos Principle", has shown how one can use superspace in order to obtain physical theories starting from very few principles.

This was our motivation in the first place for studying superspace. The idea was to use the methodology of Christodoulou's earlier work and find out where such an axiomatic basis would lead us. The principles on which we rely are the "Chronos Principle", the Variational Principle, and the Relativity Principle. By Variational Principle we mean that physical "histories" are obtained by stationarization of the action defined as a line integral in superspace. (By superspace we shall from now on mean the "generalized superspace" which includes, apart from the geometry, any number of scalar and any number of vector fields, defined on a three dimensional manifold).

For Relativity Principle we use its more physically appealling (and best suited in our case) formulation as given by Hojman, Kuchar and Teitelboim:[12] The laws of physics should be independent of the way that space-time is sliced into 3-dimensional space-like hypersurfaces". Finally, we can roughly define the Chronos Principle as: "time is a measure of the changing of the spatial configuration of the physical system".

In chapter one we investigate the structure of superspace and give the mathematical form of the above principles. (In this and the subsequent chapters we shall always assume that the three-dimensional manifold on which the spatial geometry and all fields are defined is compact).

Chapter two is devoted to the search of the form that the Lagrangian should have in accordance with the principles established in chapter one.

The gauge group is introduced and its structure investigated. It is shown that its action on the scalar fields is that of a group of motions of their space and so omission of the vector fields brings us to Isham's[13] theory.

Finally, in the last chapter we apply the results obtained earlier in order to get some general information about symmetry breaking and to obtain all gauge invariant Lagrangian in the case when we have one, two and three scalar fields.

In conclusion, we see that by using our set of axioms we are in position to deduce a physical theory in an economical way. We do not claim that this is the best approach possible. But we hope, however, that some better understanding of Classical Field Theory has been gained this way.

CHAPTER II

FUNDAMENTALS

1. Definitions

Let $\mathcal{M}$ be a C^∞ 3-dimensional manifold, which is compact and orientable, and let $T(\mathcal{M})$ be its tangent bundle. We construct over $\mathcal{M}$ the following three fibre bundles:

(1) The subspace $L_s^{2+}(T(\mathcal{M}))$ of positive definite forms of the tensor bundle of continuous symmetric bilinear forms (bundle of 2-covariant tensors $L_s^2(T(\mathcal{M}))$.

(2) The iterate cotangent bundle $T^{*(N)}(\mathcal{M})$, each fibre $T_x^*(N)$ of which over a point $x \in \mathcal{M}$ is the product of the cotangent space T_x^* to $\mathcal{M}$ at x with itself N times:

$$T_x^{*(N)} = \underbrace{T_x^* \times \ldots \times T_x^*}_{\text{N factors}} \tag{1.1}$$

(3) The bundle $\mathcal{K}(\mathcal{M})$, each fibre of which is an n-dimensional manifold $\mathcal{K}$.

We form then the product bundle $\mathcal{C}(\mathcal{M})$ of the above three fibrations:

$$\mathcal{C}(\mathcal{M}) = L_s^{2+}(T(\mathcal{M})) \times T^{*(N)}(\mathcal{M}) \times \mathcal{K}(\mathcal{M}) \tag{I.2}$$

Dach C^∞ cross-section of the bundle $L_s^{2+}(T(\mathcal{M}))$ is a C^∞ positive definite Riemannian metric on $\mathcal{M}$ and Riem $(\mathcal{M})$ is defined to be the space of such sections. Each C^∞ cross-section of $T^{*(N)}(\mathcal{M})$ is an is an N-tuple $W = (W^1, \ldots, W^N)$ of C^∞ 1-forms on $\mathcal{M}$, and we shall denote by Form $(\mathcal{M})$ the space of these sections.

A C^∞ cross-section of $\mathcal{K}(\mathcal{M})$ is a C^∞ map:

$$\Psi: \mathcal{M} \to \mathcal{K} \tag{I.3}$$

From the C^∞ nature of Ψ it follows that for every point $x \varepsilon \mathcal{M}$ and for every neighbourhood U of that point, there exists another neighborhood U_1, of x contained in U, such that Ψ U_1 (restricted to U_1) sends U_1 into a coordinate neighbourhood $\mathcal{U}$ of $\mathcal{K}(1)$. If then:

$$h : \mathcal{U} \to R^n,$$

is a local chart of $\mathcal{K}$,

$$\phi = h \circ \Psi \; U_1 : U_1 \to R^n \tag{I.4}$$

is an n-tuple of functions $(\phi_1, \ldots, \phi_n)$ on $U_1 \subset \mathcal{M}$. In particular, if y is a point contained in U_1, then $\phi(y) = (\phi_1(y), \ldots, \phi_n(y))$ are the coordinates of the point $p = \Psi(y) \varepsilon \mathcal{U}$. We shall denote by Map $(\mathcal{M} \to \mathcal{K})$ the space of C^∞ sections of the fibration $\mathcal{K}(\mathcal{M})$

We finally define the space Conf$(\mathcal{M})$: "space of configurations of $\mathcal{M}$" to be the space of C^∞ sections of the product bundle $\mathcal{C}(\mathcal{M})$

We introduce in the usual way tangent vectors associated with C^1 curves in Conf$(\mathcal{M})$: Let $c(\sigma)$ be a C^1 curve in Conf$(\mathcal{M})$. The vector X tangent to the curve $c(\sigma)$ at the point $c(\sigma_0)$ is the operator which maps every C^1 function F on Conf($\mathcal{M}$) into the number

$$XF = \frac{d}{d\sigma} F_o c(\sigma)\Big|_{\sigma=\sigma_o} \qquad (I.5)$$

Since points in Conf($\mathcal{M}$) are triplets:

$$(g,W,\Psi)$$

the vector X can be expressed as:

$$X = \int_{\mathcal{M}} n\{(\frac{dg}{d\sigma})\cdot \frac{\delta}{\delta g} + (\frac{dW^A}{d\sigma})\cdot \frac{\delta}{\delta W^A} + (\frac{d\phi_a}{d\sigma}) \frac{\delta}{\delta\phi_a}\}\Big|_{c(\sigma_o)}$$

$$A = 1\ldots N, \quad a = 1\ldots n, \qquad (I.6)$$

where ϕ_a are the functions defined by (1.4), and n in local coordinates is given by $dx^1\wedge dx^2\wedge dx^3$.

2. The metric structure of Conf($\mathcal{M}$)

A metric on Conf($\mathcal{M}$) is a smooth assignment of a bilinear symmetric form to its tangent bundle T(Conf($\mathcal{M}$)) which sends any two vectors $X^1, X^2 \epsilon T_c$(Conf($\mathcal{M}$)) to their inner product $G_c(X_1,X_2)$.

The most general form of this inner product is given by:

$$G_c(X^1,X^2) = \int_{\mathcal{M}} dV_g \int_{\mathcal{M}} dV'_g \{G^{ijmn}(x,x') \frac{dg^1_{ij}(x)}{d\sigma} \frac{dg^2_{mn}(x')}{d\sigma}$$

$$+ G^{ija}(x,x')(\frac{dg^1_{ij}(x)}{d\sigma} \frac{dW^2_a(x')}{d\sigma} + \frac{dW^1_a(x)}{d\sigma} \frac{dg^2_{ij}(x')}{d\sigma})$$

$$+G^{ijm}_A(x,x')(\frac{dg^1_{ij}(x)}{d\sigma} \frac{dW^{2A}_m(x')}{d\sigma} + \frac{dW^{1A}_m(x)}{d\sigma} \frac{dg^2_{ij}(x')}{d\sigma})$$

$$+G^{am}_A(x,x')(\frac{d\phi^1_a(x)}{d\sigma} \frac{dW^{2A}_m(x')}{d\sigma} + \frac{dW^{1A}_m(x)}{d\sigma} \frac{d\phi^2_a(x')}{d\sigma}$$

$$+G^{ab}(x,x') \frac{d\phi^1_a(x)}{d\sigma} \frac{d\phi^2_b(x')}{d\sigma} + G^{mn}_{AB}(x,x')\frac{dW^{1A}_n(x)}{d\sigma} \frac{dW^{2b}_n(x')}{d\sigma} \quad (I.7)$$

where dV_g denotes the volume element.
Here, each of the coefficients G is a C^∞ map which sends each element C of Conf($\mathcal{M}$) into a bitensor distribution in $\mathcal{M}$. These coefficients will be called "metric coefficients of Conf($\mathcal{M}$)".

Let X_u be a tangent vector at a point $c \in$ Conf($\mathcal{M}$), the components of which

$$(\frac{dg}{d\sigma}, \frac{dW}{d\sigma}, \frac{d\phi}{d\sigma})$$

have support only in a region $\mathcal{U} \subset \mathcal{M}$. We shall call such tangent vectors "local in $\mathcal{U}$".

Let then $Y_{\mathcal{V}}$ be another tangent vector at c, which is local in another region $\mathcal{V} \subset \mathcal{M}$. We introduce the following postulate:

$$G(X_{\mathcal{U}},Y_{\mathcal{V}}) = 0 \quad \text{if} \quad \mathcal{U} \cap \mathcal{V} = \emptyset \qquad \text{(Postulate I)}$$

From the above postulate, taking into account the fact that is Hausdorff, it follows that

$$G(x,x') = 0 \quad \text{if} \quad x \neq x'. \tag{I.8}$$

It is a well known result of the theory of distributions that a distribution which vanishes outside a certain point is a linear combination of the δ-function and its derivatives. Thus:

$$G(x,x') = G(x)\delta(x,x') + \sum_{n=1}^{\ell} G^{k_1 \ldots k_\nu}(x)\delta_{,k_1,\ldots,k_\nu}(x,x') \tag{I.9}$$

where each of the coefficients $G, G^{k_1,\ldots,k_\nu}$ appearing on the right is a C^∞ map which sends each element $c \in \mathrm{Conf}(\mathcal{M})$ into a C^∞ tensor field on $\mathcal{M}$

A fibre of the bundle $\mathcal{C}(\mathcal{M})$ over a point $x \in \mathcal{M}$ is the space:

$$\mathcal{C}_x = L_s^{2+}(T_x) \times T_x^{*(N)} \times \mathcal{K} \tag{I.10}$$

A vector X_x tangent to a C^1 curve $C_x(\sigma)$ in this fibre, at the point $c_x(\sigma_0)$ can be expressed in the form

$$X_x = \left\{ \left(\frac{dg(x)}{d\sigma}\right) \cdot \frac{\partial}{\partial g(x)} + \left(\frac{dW^A(x)}{d\sigma}\right) \cdot \frac{\partial}{\partial W^A(x)} + \left(\frac{d\phi_a(x)}{d\sigma}\right) \frac{\partial}{\partial \phi_a(x)} \right\} \Bigg|_{c_x(\sigma_0)} \tag{I.11}$$

The space $\mathcal{C}_x$, being an ordinary 6+3N+n dimensional manifold, admits a (pseudo) Riemannian metric which sends any two vectors

X^1, X_x^2 $T_{c_x}(\mathcal{C}_x)$ into their inner product:

$$G_{c_x}(X_x^1,X_x^2) = (G^{ijmn}\frac{dg^1_{ij}}{d}\frac{dg^2_{mn}}{d} + G^{ija}(\frac{dg^1_{ij}}{d}\frac{d\phi^2_a}{d} + \frac{d\phi^1_a}{d}\frac{dg^2_{ij}}{d})$$
$$+G^{ijm}_A(\frac{dg^1_{ij}}{d\sigma}\frac{dW^{2A}_m}{d\sigma} + \frac{dW^{1A}_m}{d\sigma}\frac{dg^2_{ij}}{d\sigma})+G^{ab}\frac{d\phi^1_a}{d\sigma}\frac{d\phi^2_a}{d\sigma} + G^{mn}_{AB}\frac{dW^{1A}_m}{d\sigma}\frac{dW^{2B}_m}{d\sigma} +$$
$$+G^{an}_A(\frac{d\phi^1_a}{d\sigma}\frac{dW^{2A}_m}{d\sigma} + \frac{dW^{1A}_m}{d\sigma}\frac{d\phi^2_a}{d\sigma}))\ (x)\Big|_{cx} \qquad \text{(I.12)}$$

In the above expression, each of the coefficients G(x) is a C^∞ map which sends each element c_x of $\mathcal{C}_x$ into a tensor at x. These coefficients will be called "metric coefficients of the fibre $\mathcal{C}_x$".

To each point $c = (g,W,\Psi)\varepsilon$ Conf($\mathcal{M}$) there corresponds a point $c_x = (g(x),W(x),\Psi(x)\varepsilon\mathcal{C}_x$ (the point where the given section of the bundle $\mathcal{C}$ ($\mathcal{M}$) intersects the fibre over x). Also to each local vector $X_{U_x}\varepsilon T_c(\mathrm{Conf}(\mathcal{M}))$ which has components

$$(\frac{dg}{d\sigma}, \frac{dW}{d\sigma}, \frac{d\phi_a}{d\sigma})$$

with support in a neighbourhood U_x of x, there corresponds a vector $X_x\varepsilon T_{c_x}(\mathcal{C}_x)$ with components

$$(\frac{dg(x)}{d\sigma}, \frac{dW(x)}{d\sigma}, \frac{d\phi_a(x)}{d\sigma})$$

We now introduce the postulate:

$$\lim_{U_x\to x} \frac{G_c(X_{U_x},X_{U_x})}{\int_U dV} = G_{c_x}(X_x,X_x) \qquad \text{(Postulate II)}$$

where the limit is taken in the Moore-Smith sense with respect to the directed set of neighbourhoods of the point x.

It follows from the above postulate, in view of Eq. (I.12) that for any two tangent vectors $X, Y \epsilon T_c(\mathrm{Conf}(\mathcal{M}))$,

$$G_c(X,Y) = \int_{\mathcal{M}} G_{c_x}(X_x, Y_x)dV \tag{I.13}$$

where X_x, Y_x are the corresponding tangent vectors in $T_{c_x}(\mathcal{C}_x)$.

We may thus express the "element of arc length" dL in $\mathrm{Conf}(\mathcal{M})$, where

$$(\frac{dL}{d\sigma})^2 = G_c(X,X), \tag{I.14}$$

in terms of the "element of arc length $d\ell(x)$ in C_x:

$$(\frac{d\ell(x)}{d\sigma})^2 = G_{c_x}(X_x, X_x) \tag{I.15}$$

as the integral:

$$dL^2 = \int_{\mathcal{M}} d\ell(x^2)dV \tag{I.16}$$

The formalism with which we are working is invariant under the point transformations:

$$\phi_a \rightarrow \tilde{\phi}_a = \tilde{\phi}_a(\phi)$$

$$W^A \rightarrow \tilde{W}^A = W^A{}_B(\phi)W^B \tag{I.17}$$

the first of which is a coordinate transformation in the manifold $\mathcal{X}$ and the second is a transformation in the linear space $T_x^{*(N)}$.

3. The group of diffeomorphisms and the introduction of superspace

Consider now the group Diff($\mathcal{M}$) of C^∞ orientation-preserving diffeomorphisms

$$f : \mathcal{M} \to \mathcal{M}$$

of the base manifold $\mathcal{M}$. The group structure of Diff($\mathcal{M}$) as defined by composition, has C^∞ group operations. It is thus a Lie group, which, as a manifold, is modeled on the space V($\mathcal{M}$), namely the space of C^∞ vector fields on $\mathcal{M}$.

If $f \varepsilon \mathrm{Diff}(\mathcal{M})$, f acts on T($\mathcal{M}$) by its tangent map

$$T_x f : \quad T_x \to T_{f(x)}$$

defined as follows: For each $\upsilon_x \varepsilon T_x$ tangent to a curve $k(t)$ at $k(t_o)=x$, $T_x f(\upsilon_x)$ is the vector which is tangent to the curve $f \circ k(t)$ at $f \circ k(t_o)=f(x)$.

Diff($\mathcal{M}$) acts as a transformation group on Conf($\mathcal{M}$):

$$\mathrm{Diff}(\mathcal{M}) \times \mathrm{Conf}(\mathcal{M}) \to \mathrm{Conf}(\mathcal{M}) \qquad \text{(I.18)}$$

where the action sends $(f,c) \longrightarrow f^*c$. Here c is the point

$$(g, W, \ ) \in \mathrm{Conf}(\mathcal{M})$$

and f^*c is the point

$$(f^*g, \quad f^*W, \quad f^*\Psi) \in \mathrm{Conf}(\mathcal{M})$$

defined by:

$$(f^*g)_x(\mathcal{V}_x, \mathcal{U}_x) = g_{f(x)}(T_x f(\mathcal{V}_x), T_x f(\mathcal{U}_x)) \tag{I.19}$$

for every $\mathcal{V}_x, \mathcal{U}_x \in T_x$,

$$<f^*W^A, \mathcal{V}_x>x \ = <W^A, \ T_x f(\mathcal{V}_x)>f(x) \tag{I.20}$$

for every $\mathcal{V}_x \in T_x$, and

$$f^*\Psi(x) = \Psi(f(x)). \tag{I.21}$$

For a fixed point $c \in \mathrm{Conf}(\mathcal{M})$, the above action embeds $\mathrm{Diff}(\mathcal{M})$ as a differentiable submanifold in $\mathrm{Conf}(\mathcal{M})$ through the orbit map:

$$0_c^D : \mathrm{Diff}(\mathcal{M}) \to \mathrm{Conf}(\mathcal{M}), \tag{I.22}$$

where:

$$0_c^D(f) = f^*c \tag{I.23}$$

and the image of $\mathrm{Diff}(\mathcal{M})$ by 0_C^D is the "orbit of the group of diffeomorphisms through c":

$$O^D(c) = \{f^* c \mid f \in \mathrm{Diff}(\mathcal{M})\} \qquad (I.24)$$

Let f_t where $t \in [-1,1]$ and $f_o = id$ be smooth curve in $\mathrm{Diff}(\mathcal{M})$. As is well known, to every such curve corresponds a vector field $\xi \in V(\mathcal{M})$ defined by requiring that $\xi(x)$ be the vector tangent to the curve $f_t(x)$ in $\mathcal{M}$. If in particular f_t is a one-parameter subgroup of $\mathrm{Diff}(\mathcal{M})$ then for every differentiable function ϕ on

$$\phi(f_t(x)) = \exp(t\xi)\phi(x) \qquad (I.25)$$

It thus turns out that every vector field generates a diffeomorphism and therefore $V(\mathcal{M})$ is the Lie algebra of $\mathrm{Diff}(\mathcal{M})$.

The tangent at the identity to the orbit map (I.22) is the map $T_{id}O^D_C$ which sends any vector field ξ tangent at $f_o = id$ to a curve f_t in $\mathrm{Diff}(\mathcal{M})$, into the following vector in $\mathrm{Conf}(\mathcal{M})$:

$$P_{\xi|c} = \int \eta\{(\frac{d(f_t^* g}{dt}) \cdot \frac{\delta}{\delta g} + (\frac{df_t^* W^A}{dt}) \cdot \frac{\delta}{\delta W^A} + (\frac{df_t^* \phi_a}{dt}) \frac{\delta}{\delta \phi_a}\}\Big|_{t=0}$$

$$= \int_{\mathcal{M}} \eta\{(\pounds_\xi\, g) \cdot \frac{\delta}{\delta g} + (\pounds_\xi\, W^A) \cdot \frac{\delta}{\delta W^A} + (\pounds_\xi \phi_a) \frac{\delta}{\delta \phi_a}\}\Big|_c \qquad (I.26)$$

In the case that f_t is a one-parameter subgroup of $\mathrm{Diff}(\mathcal{M})$, then for every differentiable functional Φ on $\mathrm{Conf}(\mathcal{M})$,

$$\Phi(f_t^* c) = \exp(tp_\xi)\Phi(c) \qquad (I.27)$$

Thus, the operator P_ξ which (integrating by parts in (I.26) may also be expressed in the form:

$$P_\xi = \int_{\mathcal{M}} \eta(\xi\cdot p) \tag{I.28}$$

where p is an operator having the tensor character of an 1-form density whose components in a local coordinate system are expressed by:

$$p_k = -2g_{km}\nabla_n \frac{\delta}{\delta g_{mn}} - W^A_k\nabla_m \frac{\delta}{\delta W^A_m} - f^A_{km}\frac{\delta}{\delta W^A_m} + \phi_{a,k}\frac{\delta}{\delta\phi_a} \tag{I.29}$$

is the generator of an action of Diff($\mathcal{M}$) on Conf($\mathcal{M}$). In (I.29) f^A_{mn} are the components of the 2-forms which are the exterior derivatives of the 1-forms W^A:

$$f^A_{mn} = W^A_{m,n} - W^A_{n,m} \tag{I.30}$$

By its orbit map (I.22) the group of diffeomorphisms of $\mathcal{M}$ induces an equivalence relation on Conf($\mathcal{M}$): two configurations c_1, c_2 are equivalent if they lie on the same orbit of Diff($\mathcal{M}$). Considering the fact that this equivalence implies that the configurations are physically indistinguishable, we define "superspace" to be the identification space:

$$\mathcal{S}(\mathcal{M}) = \frac{\mathrm{Conf}(\mathcal{M})}{\mathrm{Diff}(\mathcal{M})} \tag{I.31}$$

where the quotient denotes that there exists a continuous, open projection Π which maps each orbit $O^D(c)$ in Conf($\mathcal{M}$) into a point $s\epsilon S$. A point of superspace will be called an "intrinsic configuration of $\mathcal{M}$".

Fischer has studied the structure of the space of geometries (Wheeler's original notion of superspace) which is defined to be quotient space:

$$\mathcal{G}(\mathcal{M}) = \frac{\mathrm{Riem}(\mathcal{M})}{\mathrm{Diff}(\mathcal{M})} \tag{I.32}$$

In his penetrating analysis he showed that $\mathcal{G}(\mathcal{M})$ is not a proper manifold. This is due to the existence in Riem($\mathcal{M}$) of orbits 0^D, which are such that the metrics g contained in them are left invariant under the action of some non-trivial subgroup of Diff($\mathcal{M}$).

$$I_g = \{f \in \mathrm{Diff}(\mathcal{M}) \mid f^* g = g\} \tag{I.33}$$

namely metrics which admit an isometry group I_g. These orbits are projected into points in $\mathcal{G}(\mathcal{M})$ which have neighbourhoods that are not homeomorphic to those of the points in $\mathcal{G}(\mathcal{M})$ which correspond to "generic" orbits in Riem($\mathcal{M}$) (that is to say orbits, the metrics of which do not admit any group of isometries). As a consequence, the space of geometries decomposes into strata, where each stratum contains geometries of the same symmetry type. Fischer has further shown that each of these strata is a proper manifold and the strata containing geometries of a higher symmetry (larger isometry group) form the boundary of the strata which contain geometries of lower symmetry.

Fischer's reasoning should carry over to (generalized) superspace since Conf($\mathcal{M}$) and Riem($\mathcal{M}$) have basically the same manifold structure and the quotient is taken with respect to the same group.

Thus, the orbits O^D in Conf($\mathcal{M}$), whose points c are left invariant under the action of a subgroup:

$$I_c = \{f \in \mathrm{Diff}(\mathcal{M}) \mid f^* c = c\} \qquad \text{(I.34)}$$

of Diff($\mathcal{M}$) and therefore represent symmetric configurations, shall have neighbourhoods in superspace which are not homeomorphic to those of generic orbits. The stratified structure should also exist in the present case.

4. The requirement that $\mathcal{D}$ is an isometry of the metric of Conf($\mathcal{M}$)

We shall now proceed to construct a definition of arc length in superspace from the already obtained definition of arc length in Conf($\mathcal{M}$).

First we not that consistency with the equivalence relation induced by Diff($\mathcal{M}$) on Conf($\mathcal{M}$) requires us to demand that the group of transformations (I.18) induced by Diff($\mathcal{M}$) on Conf($\mathcal{M}$) is an isometry of the metric of Conf($\mathcal{M}$). This demand is expressed by

$$\forall \; f \in \mathrm{Diff}(\mathcal{M}), \quad f^* G = G \qquad \text{(I.35)}$$

where

$$(f^* G)_c(X,Y) = G_{f^* c}(T_c f^*(X), T_c f^*(Y)) \qquad \text{(I.36)}$$

In the above relation, $T_c f^*$ is the map tangent to f^* at the point $c \in \mathrm{Conf}(\mathcal{M})$. (Thus, if X is the vector tangent to the curve $c(\sigma)$ at $c(\sigma_o)=c$, $T_c f^*(X)$ is the vector tangent to the curve $f^* c(\sigma)$ at $f^*(\sigma_o)=f^* c$).

The requirement expressed by Eq. (I.35) is readily satisfied if the tensor character of the metric coefficients of $\mathcal{C}_x$ is chosen so that the integral (I.13) is independent of the choice of coordinate systems.

Let us consider a curve $s(\sigma)$ in superspace, namely a map of the closed interval $[\sigma_1,\sigma_2]$ of the real lime into $\mathcal{S}(\mathcal{M})$. Since $\mathcal{S}(\mathcal{M})$ is not a manifold, we cannot specify directly the continuity and differentiability properties of the curve. We can, however, specify these properties in the following indirect fashion:

The inverse π^{-1} of the projection map sends the curve $s(\sigma)$ into a one parameter family of orbits $0^D(\sigma)$ in Conf($\mathcal{M}$) which is such that there exists a smooth curve $c(\sigma)$ (C^∞ map of $[\sigma_1,\sigma_2]$ into Conf($\mathcal{M}$), which crosses each orbit of the family once and only once. Any such curve $s(\sigma)$ will be called "smooth" curve in superspace.

Every other curve in Conf($\mathcal{M}$) with the same properties may be obtained from the particular curve $c(\sigma)$ in the following manner: Let f_σ be a smooth 1-parameter family of diffeomorphisms, namely a C^∞ curve in Diff($\mathcal{M}$). Let us then move each point $c(\sigma_o)$ of the curve $c(\sigma)$, along the orbit $0^D(\sigma_o)$ on which it lies, through the action induced by the diffeomorphism f_{σ_o}. The curve $f^*_\sigma c(\sigma)$ which results in this way is another C^∞ curve in Conf($\mathcal{M}$) which crosses each orbit of the family $0^D(\sigma)$ once and only once.

Thus we see that to a given curve $s(\sigma)$ in $\mathcal{S}(\mathcal{M})$ corresponds an infinity of smooth curves $f^*_\sigma c(\sigma)$ in Conf($\mathcal{M}$), one for every smooth curve f_σ in Diff($\mathcal{M}$). The need of assigning in a unique way an arc length to the curve $s(\sigma)$ motivates us to introduce the following definition: The arc length of a curve $s(\sigma)$ in superspace is equal to the stationary value of the arc length of the curves $f^{*}c(\sigma)$ in Conf($\mathcal{M}$) as f_σ ranges over the space of smooth curves in Diff($\mathcal{M}$):

$$\tilde{L}(s(\sigma)) = \text{stat. } L(f^{*}c(\sigma)) \qquad \text{(I.37)}$$

We must, however, demonstrate that this stationary value exists for any curve $s(\sigma)$ in $\mathcal{S}(\mathcal{M})$ of the kind defined above, and also that it is unique and that it tends to zero if the curve $s(\sigma)$ is allowed to contract to a singel point in $\mathcal{S}(\mathcal{M})$.

5. Point correspondence equations

Let $\xi(\sigma)$ be a smooth one parameter family of vector fields on $\mathcal{M}$ (C^∞ curve in V(M)), defined as follows: For any particular value $\sigma_o \varepsilon [\sigma_1, \sigma_2]$ of the parameter, $\xi(\sigma_o)$ is the vector field which is tangent at the identity to the curve $e_\sigma = f_\sigma \circ f_{\sigma_0}^{-1}$ in Diff($\mathcal{M}$).

Through any point $c(\sigma_o) = c$ of the curve $c(\sigma)$, let us define a curve $e_\sigma^{*}c(\sigma)$ from which the curve $f_\sigma^{*}c(\sigma)$ results through a motion induced by the diffeomorphism f_{σ_o} (since $e_\sigma^{*} = f_{\sigma_o}^{*-1} \text{of}_\sigma^{*}$). If X_{σ_o} is the vector tangent to the curve $c(\sigma)$ at c, then the vector Y_{σ_o} tangent to the curve $e_\sigma^{*}c(\sigma)$ at the same point is given by

$$Y_{\sigma_o} = X_{\sigma_o} + T_\xi(\sigma_o) \qquad \text{(I.38)}$$

where

$$T_\xi(\sigma_o) = T_{id}O_C^D(\xi(\sigma_o)) \qquad (I.39)$$

Thus, if

$$\left(\frac{dg}{d\sigma}, \frac{dW}{d\sigma}, \frac{d\phi}{d\sigma}\right)\bigg|_{\sigma_o}$$

are the components of X_{σ_o}, then

$$\left(\frac{dg}{d\sigma} + \mathcal{L}_\xi g, \frac{dW}{d\sigma} + \mathcal{L}_\xi W, \frac{d}{d\sigma} + \mathcal{L}_\xi \phi\right)$$

are the components of Y_{σ_o}. Finally, the vector tangent to the curve $f^*_\sigma c(\sigma)$ at $f^*_{\sigma_o} c(\sigma_o) = f^*_{\sigma_o} c$ is evidently $T_c f^*_{\sigma_o}(Y_{\sigma_o})$.

In terms of the vector Y_σ defined as above at any point along the curve $c(\sigma)$, the arc length of the curve $f^*_\sigma c(\sigma)$ is expressed as:

$$L(f^*_\sigma c(\sigma)) = \int_{\sigma_1}^{\sigma_2} d\sigma \left[G_{f^*c}(T_c f^*_\sigma(Y_\sigma), T_C f^*_\sigma(Y_\sigma))\right]^{1/2} \qquad (I.40)$$

Taking, however, into account Eqs. (I.35) and (I.36) the above expression reduces to:

$$L(f^*_\sigma c(\sigma)) = \int_{\sigma_1}^{\sigma_2} d\sigma \left[G_c(Y_\sigma, Y_\sigma)\right]^{1/2} \equiv L(c(\sigma), \xi(\sigma)) \qquad (I.41)$$

where the integration is now carried over along the original fixed curve $c(\sigma)$.

A given pair $(c(\sigma), \xi(\sigma))$ represents not a single curve $f^*_\sigma c(\sigma)$ in Conf($\mathcal{M}$) but a class of curves $h^* \circ f_\sigma c(\sigma)$ one for every $h \varepsilon$ Diff($\mathcal{M}$), all of which possess the same arc length. For any point on any given orbit of the family $O^D(\sigma)$, there is one and only one curve of this flass which passes through that point. As a consequence a pair $(c(\sigma)$,

$\xi(\sigma)$) establishes a one to one correspondence between the points of any two orbits $O^D(\sigma')$ and $O^D(\sigma'')$ of the family.

From Eq. (I.41) we conclude that the variation implied by Eq. (I.37) may equivalently be carried over the space of smooth curves on $V(\mathcal{M})$. The stationarization conditions

$$\frac{\delta L(c(\sigma);\xi(\sigma))}{\delta\xi(\sigma)} \tag{I.42}$$

assume the form of a linear inhomogeneous equation for the vector field ξ at any given value of the parameter σ

$$A\cdot\xi = p \tag{I.43}$$

where A is a (2-covariant) self-adjoint tensor operator which in a local coordinate chart is expressed by:

$$\begin{aligned} A_{ij} &= 2g_{nj}(-2g_{ik}\nabla_\ell G^{k\ell mn}+\phi_{a,i}G^{mna} + G_A^{mnk}f^A{}_{ki} - W_i^A\nabla_k G_A^{mnk}\nabla_n) \\ &+(G_{AB}^{mn}f^A_{mi}-W^A_{mi}\nabla_m G^{mn}_{AB} - 2g_{ik}\nabla_\ell G_B^{k\ell n} + \phi_{a,i}G_B^{an})(W^B_{n;j}+W^B_j\nabla_n) \\ &+(G^{ab}\phi_{b,i}-2g_{ik}\nabla_\ell G^{k\ell a}+G_A^{am}f^A_{mi}-W_i^A\nabla_m G_A^{am})\phi_{a,j} \end{aligned} \tag{I.44}$$

while P is a 1-form given locally by:

$$P_i = -2g_{ik}\left(G^{k\ell mn}\frac{dg_{mn}}{d\sigma}\right)_{;\ell} + \phi_{a,i}\, G^{ab}\frac{d\phi_b}{d\sigma}$$

$$+ G^{mn}_{AB}\, f^A_{mi}\frac{dW^B_n}{d\sigma} - W^A_i\left(G^{mn}_{AB}\frac{dW^B_n}{d\sigma}\right)_{;m} - 2g_{im}\left(G^{mna}\frac{d\phi_a}{d\sigma}\right)_{;n} + \phi_{a,i}G^{mna}\frac{dg_{mn}}{d\sigma}$$

$$-2g_{im}\left(G^{mnj}_A\frac{dW^A_j}{d\sigma}\right)_{;n} + G^{mnj}_A f^A_{ji}\frac{dg_{mn}}{d\sigma} - W^A_i\left(G^{mnj}_A\frac{dg_{mn}}{d\sigma}\right)_{;j}$$

$$+G^{ma}_A f^A_{mi}\frac{d\phi_a}{d\sigma} - W^A_i\left(G^{ma}_A\frac{d\phi_a}{d\sigma}\right)_{;m} + \phi_{a,i}G^{ma}_A\frac{dW^A_n}{d\sigma} \qquad \text{(I.45)}$$

The question of the existence and uniqueness of the stationary value in Eq. (I.37) evidently reduces to the question of the existence and uniqueness of the solutions of Eq. (I.43). We discuss this problem in Appendix I.

Equation (I.43) shall be called "global point correspondence equation" for a reason which will become apparent in the following.

6. The action as a path integral in superspace

We shall now formulate the Variational Principle in $\mathcal{S}(\mathcal{M})$. Let us be given two fixed points s_1 and s_2 in $\mathcal{S}(\mathcal{M})$. Let us consider the set of "smooth" (in the sence of preceding paragraph) curves which join these points. The subset of "physically acceptable" curves in $\mathcal{S}(\mathcal{M})$ which have s_1 and s_2 as end points, are those for which the line integral:

$$S = \int_{s(\sigma)} \tilde{A} d\tilde{L} \left[= \int_{\sigma_2}^{\sigma_1} (\tilde{A}(s(\sigma)) \frac{d\tilde{L}}{d\sigma}) d\sigma \right] \qquad \text{(I.46)}$$

is stationary. Here $\tilde{A}$ is a functional on $\mathcal{S}(\mathcal{M})$ which is such that $\tilde{A} \circ \pi = A$ is a C^{∞} functional on Conf($\mathcal{M}$) which is constant along the orbits of the group of diffeomorphisms. Such functionals will be called "smooth" functionals on superspace.

We can also formulate the Variational Principle in the original space Conf($\mathcal{M}$) as follows: Consider the orbits O_1^D and O_2^D in Conf($\mathcal{M}$) into which the points s_1 and s_2 are sent by the inverse π^{-1} of the projection map. The subset of physically acceptable curves, of the set of smooth curves in Conf($\mathcal{M}$) which have end points in the orbits O_1^D and O_2^D, are those for which the line integral:

$$S = \int_{c(\sigma)} A dL \left[= \int_{\sigma_1}^{\sigma_2} (A(c(\sigma) \frac{dL}{d\sigma}) d\sigma \right] \qquad \text{(I.47)}$$

(Variational Principle)

is stationary. Evidently, in this formulation the end points c_1 and c_2 of the curves $c(\sigma)$ are not fixed but are allowed to vary along their orbits (O_1^D and O_2^D respectively).

It follows from the definition of arc length in superspace and the fact that A is constant along the orbits of Diff($\mathcal{M}$), that the two formulations of the variational principle are equivalent. This is because stationarization of the above form of S with respect to the class of curves $c(\sigma)$ which cross the one-parameter family of orbits $O^D(\sigma)$ which π sends into a given curve $s(\sigma)$ in $\mathcal{S}(\mathcal{M})$, will bring us back to the form of S expressed by Eq. (I.46).

The quantity S defined above plays evidently the role of an action. Its form as given by Eqs. (I.46) and (I.47) will be called the "global form of the action" to distinguish it from the "local form" of which we shall speak later.

7. The Chronos Principle

We are now ready to introduce the Chronos Principle. Consider a smooth curve $c(\sigma)$ in Conf($\mathcal{M}$) and let $c(\sigma_1)$, $c(\sigma_2)$ be the end points of this curve. We define the "global time interval" T between these points to be the integral

$$T = \int_{c(\sigma)} \frac{dL}{B^{1/2}} \left[= \int_{\sigma_1}^{\sigma_2} \frac{1}{B^{1/2}} \frac{dL}{d\sigma}\, d\sigma \right] \tag{I.48}$$

where B is a smooth functional on Conf($\mathcal{M}$).

The above definition may be expressed formally in infinitesimal form by:

$$dT^2 = \frac{dL^2}{B} \tag{I.49}$$

We now postulate that the functional B has the form of a simple integral over the base manifold $\mathcal{M}$:

$$B = \int_{\mathcal{M}} b\, dV \qquad \text{(Postulate III)}$$

Remembering that dL^2 also has the form of a simple integral over $\mathcal{M}$ (cf. Eq. (I.16)), we may, by restricting the region of integration in dL^2 and B, to a neighbourhood $U \subset \mathcal{M}$, define the element of local time in U

$$dT_U^2 = \frac{dL_U^2}{B_U} \tag{I.50}$$

where

$$dL_U^2 = \int_U d\ell^2 dV, \quad B_U = \int_U b dV$$

In the Moore-Smith limit of U converging to a point $x \epsilon U$ definition (I.50) gives the element of "local time at x":

$$d\tau(x)^2 = \frac{d\ell(x)^2}{b(x)} \quad \text{(Chronos Principle)} \tag{I.51}$$

8. Construction of space-time

We shall now construct the 4-dimensional spacetime manifold and its Lorentzian metric. Consider a smooth curve $c^*(\sigma)$ in Conf($\mathcal{M}$), such that $(d\tau^*/d\tau)^2$ is positive at every point on the curve and for any $x \epsilon \mathcal{M}$. We define spacetime to be the 4-dimensional manifold ${}^{(4)}\mathcal{M} = \mathcal{M} \times [\sigma_1, \sigma_2]$ associated with such a curve. Its metric is constructed as follows: Let us define the function τ^* on ${}^{(4)}\mathcal{M}$ by:

$$\tau^*(x,\sigma) = \int_{\sigma_1}^{\sigma} N^*(x) d\sigma, \tag{I.52}$$

where:

$$N^* = \frac{d\tau^*}{d\sigma} \tag{I.53}$$

Evidently, $\tau^*(x,\sigma)$ is the local time at x. The metric of ${}^{(4)}\mathcal{M}$ is then define by:

$$ {}^{(4)}g^* = -d\tau^* \otimes d\tau^* + g^* \qquad (I.54)$$

Here $g^*(\sigma)$ is the projection on Riem($\mathcal{M}$) of the line $c^*(\sigma)$ in Conf($\mathcal{M}$), and d represents the exterior derivative operator.

If now $c^*(\sigma) = f_\sigma^* c(\sigma)$, and we perform the 4-dimensional diffeomorphism

$$ h : \mathcal{M} \times [\sigma_1,\sigma_2] \to \mathcal{M} \times [\sigma_1,\sigma_2] $$

with

$$ h(x,\sigma) = (f_\sigma^{-1}(x),\sigma), \qquad (I.55)$$

then, we can show that $h^{*(4)}g^* = {}^{(4)}g$ is expressed by:

$$ {}^{(4)}g = -(N^2 - g(\xi,\xi))d \otimes d\sigma - \Xi \otimes d\sigma - d\sigma \otimes \Xi + g \quad , \qquad (I.56)$$

where Ξ is the 1-form related (through the metric g) to the vector ξ and N is defined by: $N^* = f_\sigma^* N$. This is shown as follows:

Consider a curve $(x(t),\sigma(t))$ in ${}^{(4)}\mathcal{M}$. Let a be its tangent vector at (x_o,σ_o). Then if a is the vector tangent at x_o to the projection x(t) of this curve on $\mathcal{M}$,

$$ \tilde{a} = a + \frac{d\sigma(t)}{dt}\frac{\partial}{\partial\sigma} . \qquad (I.57)$$

The 4-dim. diffeomorphism h send the curve $(x(t),\sigma(t))$ into the curve $(f^{-1}_{\sigma(t)}(x(t)),\sigma(t))$. Let e_σ be the aforementioned curve $e_\sigma = f_\sigma o f^{-1}_{\sigma_o}$ in Diff($\mathcal{M}$) which passes through the identity at $\sigma=\sigma_o$. Let us then construct a curve $e^{-1}_{\sigma(t)}(x(t))$ in $\mathcal{M}$ through the point x_o, from which the curve $f^{-1}_{\sigma(t)}(x(t))$ is obtained by acting with the diffeomorphism $f^{-1}_{\sigma_o}$. The vector b tangent to the curve $e^{-1}_{\sigma(t)}(x(t))$ at x_o is obtained from the vector a by:

$$b = a + \zeta \tag{I.58}$$

where ζ is the vector tangent at x_o to the curve $e^{-1}_{\sigma(t)}(x_o)$. It can be easily seen that

$$\zeta = -\xi \frac{d\sigma(t)}{dt}, \tag{I.59}$$

where ξ is the vector $\xi(x_o,\sigma_o)$ defined previously. The vector $T_{(x_o,\sigma_o)}h(a)$ tangent to the curve $(f^{-1}_{\sigma(t)}(x(t)), \sigma(t)$ in ${}^{(4)}\mathcal{M}$ at $(f^{-1}_\sigma(x_o),\sigma_o)$ is then simply expressed as

$$T_{(x_o,\sigma_o)}h(\tilde{a}) = T_{x_o} f^{-1}_{\sigma_o}(b) + \frac{d\sigma(t)}{dt}\frac{\partial}{\partial\sigma} \tag{I.60}$$

Now, the 4-metric γ is related to the 4-metric γ^* by:

$$\begin{aligned} {}^{(4)}g_{(x_o,\sigma o)}(\tilde{a},\tilde{a}) &= {}^{(4)}g_{(f^{-1}_{\sigma_o}(x_o),\sigma_o)}(T_{(x_o,\sigma o)}h(\tilde{a}),T_{(x_o,\sigma o)}h(\tilde{a})) \\ &= -(N\ )^2_{f^{-1}_{\sigma_o}(x_o)}\left(\frac{d\sigma(t)}{dt}\right)^2 + g^*_{f^{-1}_{\sigma_o}(x_o)}(T_{x_o} f^{-1}_{\sigma_o}(b),\ T_{x_o} f^{-1}_{\sigma_o}{}_o(b)) \end{aligned} \tag{I.61}$$

Considering then the definitions of N and g, the above reduces to:

$$^{(4)}g_{(x_o,\sigma_o)}(\tilde{a},\tilde{a}) = -N^2_{x_o}\left(\frac{d\sigma(t)}{dt}\right)^2 + g_{x_o}(b,b)$$

$$= -N^2_{x_o}\left(\frac{d\sigma(t)}{dt}\right)^2 + g_{x_o}\left(a-\xi\frac{d\sigma(t)}{dt}\ ,\ a-\xi\frac{d\sigma(t)}{dt}\right) \qquad (I.62)$$

from which Eq. (I.56) follows immediately (Q.E.D.).

It is evident from Eq. (I.56) that the (time-like) vector field of norm N which is normal to the hypersurfaces $\mathcal{M}_\sigma = \mathcal{M}\times\{\sigma\}$ in $^{(4)}\mathcal{M}$ is given by:

$$n = \xi + \frac{\partial}{\partial\sigma} \qquad (I.23)$$

We thus see that the one parameter family $\xi(\sigma)$ of vector fields on $\mathcal{M}$ gives the vector field n on $^{(4)}\mathcal{M}$ which establishes the normal point correspondence of any two hypersurfaces $\mathcal{M}_{\sigma'}$ and $\mathcal{M}_{\sigma''}$ $(\sigma',\sigma''\varepsilon[\sigma_1,\sigma_2])$ defined by requiring that the integral curves of n pass through the corresponded points on the two hypersurfaces. Therefore, where the intrinsic geometry of the hypersurfaces $\mathcal{M}_\sigma$ is given by the line $\mathcal{G}(\sigma)$ in $\mathcal{G}(\mathcal{M})$ to which both curves $g(\sigma)$ and $g^*(\sigma)$ in Riem($\mathcal{M}$) are projected by the orbit projection map, it is the line $\xi(\sigma)$ in V($\mathcal{M}$), specifying the particular orbit correspondence in Conf($\mathcal{M}$), which, together with the local time τ^*, prescribes the way in which the space-like hypersurfaces $\mathcal{M}_\sigma$ are to be embedded in the spacetime manifold $^{(4)}\mathcal{M}$.

The second fundamental form (or extrinsic curvature) of any hypersurface $\mathcal{M}_{\sigma_o}$ is given by:

$$K^*(\sigma_o) = \frac{1}{2}\left(\frac{dg^*}{d\tau^*}\right)_{\sigma=\sigma_o} \qquad \text{(I.64)}$$

from which by acting with the diffeomorphism $f_{\sigma_o}^{-1}$ we obtain:

$$K(\sigma_o) = \frac{1}{2}\left[\frac{1}{N}\left(\frac{dg}{d\sigma} + \pounds_{\xi} g\right)\right]_{\sigma=\sigma_o} \qquad \text{(I.65)}$$

where $K^*(\sigma_o) = f^*_{\sigma_o} K(\sigma_o)$.

We have thus seen that to an equivalence class of diffeomorphically related curves $\{f^* c(\sigma) \mid f \in \mathrm{Diff}(\mathcal{M})\}$ in Conf($\mathcal{M}$), or, equivalently, to a curve $s(\sigma)$ in $\mathcal{S}(\mathcal{M})$ plus a "connection" along that curve (namely, a correspondence of the orbits $O^D(\sigma)$), is associated a spacetime of determined geometry, plus a slicing of that spacetime into space-like hypersurfaces.

Let us suppose now that from a given curve $c(\sigma)$ in Conf($\mathcal{M}$), in addition to the metric ${}^{(4)}g$, an N-tuple of 1-forms ${}^{(4)}W$ can be constructed on ${}^{(4)}\mathcal{M}$ as well as a map ${}^{(4)}$ from ${}^{(4)}\mathcal{M}$ to $\mathcal{K}$, which are such that the N-tuple of 1-forms induced on each space-like hypersurface $\mathcal{M}_{\sigma_o}$ is $W(\sigma_o)$ and the map restricted to $\mathcal{M}_{\sigma_o}$ is $\Psi(\sigma_o)$. The Relativity principle is the statement that the actions which correspond to any two curves in Conf($\mathcal{M}$) which give the same intrinsic configuration of ${}^{(4)}\mathcal{M}$ sliced in two different ways into spacelike hypersurfaces, are equal (Principle of Path-Independence).

A change of the slicing of spacetime is a 4-dim. diffeomorphism

$$(x,\sigma) \to (x,\rho(x,\sigma)) \qquad \text{(I.66)}$$

which is orthochronous" (namely $\partial\rho/\partial\sigma$ is positive everywhere) and sends each spacelike section σ=const. into a spacelike section ρ=const.

We note that the action S is, with the introduction of ξ, manifestly invariant under any 4-dim. diffeomorphism

$$(x,\sigma) \to (f(x,\sigma),\sigma), \tag{I.67}$$

which reduces to a 3-dim. orientation preserving diffeomorphism on each $\mathcal{M}_{\sigma_o}$. On the other hand a diffeomorphism (I.66) can be composed on the right with a diffeomorphism of the above type to give a generic 4-dim. diffeomorphism

$$h: \mathcal{M} \times [\sigma_1,\sigma_2] \to \mathcal{M} \times [\sigma_1,\sigma_2]$$

which preserves the orientations of $\mathcal{M}$ and $[\sigma_1,\sigma_2]$ and sends the space-like sections into space-like sections. Consequently, the Relativity Principle is equivalent to the demand that the action be invariant under the group of 4-dim. diffeomorphisms of this kind.

9. The space-time form of the action

Using the established correspondence between a line in Conf($\mathcal{M}$) and a spacetime manifold, we shall now transform the action from a line integral in Conf($\mathcal{M}$) into a space-time integral.

The form of the action given by Eq.(I.47) shall be valid in the case that the slicing of spacetime is such that

$$d(x) = dT, \quad \text{for every} \quad x \in \mathcal{M}, \qquad (I.68)$$

namely any two space-like sections have constant normal separation ("Global inertial frame").

In transforming this global form of the action into a space-time integral we must take into account the following two consistency requirements:

1) That the variation of S with respect to the restricted class of smooth curves in Conf($\mathcal{M}$) which cross each orbit in a fixed one-parameter family of orbits $0^D(\sigma)$, should recover the global point correspondence equation (Eq. I.43).

2) That the variation of S with respect to $dT/d\sigma$ should recover the definition of global time (Eq. I.49).

It follows from the first requirement that the tangent vector to the curve $c(\sigma)$ over which S is defined, enters S only in terms of its norm $dL/d\sigma$. We shall now make use of Eq. (I.49) in replacing the measure dL of the integration in Eq. (I.47) by the measure dT. The most general expansion of S which satisfies requirement 1) is the following:

$$S = \int_{c(\sigma)} A f(\lambda) B^{1/2} dT,$$

where

$$\lambda = \frac{1}{B} \left(\frac{dL}{dT}\right)^2 \qquad (I.70)$$

Here f is a C^∞ function of λ satisfying

$$f(1) = 1 \quad . \tag{I.71}$$

Evidently, if Eq. (I.49) holds, $\lambda=1$ and the expression for S given by Eq. (I.69) reduces to that of Eq. (I.47). It is easy to show that requirement 2 demands that

$$f(\lambda) = 2\lambda \frac{df(\lambda)}{d\lambda} = 0 \quad \text{implies } \lambda=1. \tag{I.72}$$

We now take into consideration the Relativity Principle in requiring that, since in Eq. (I.69) only a single integration over $[\sigma_1,\sigma_2]$ occurs, S must have the form

$$S = \int_{^{(4)}\mathcal{M}} \mathcal{L}\, dV^{(4)}, \tag{I.73}$$

of a simple integral over the spacetime manifold $^{(4)}\mathcal{M} = \mathcal{M}\times[\sigma_1,\sigma_2]$ Here

$$dV^{(4)} = d\tau dV \tag{I.74}$$

is the measure of $^{(4)}\mathcal{M}$ constructed from the 4-metric $^{(4)}g$ which corresponds to the line $c(\sigma)$ in Conf($\mathcal{M}$).

Comparing Eqs. (I.69) and (I.73) and taking into account the fact that both dL^2 and B are simple integrals over $\mathcal{M}$ we conclude that the two equations are compatible if and only if:

$$\left.\begin{array}{ll} f = c_1 + c_2\lambda, & \text{(a)} \\ A = B^{1/2} \text{ (up to an inessential multiplicative constant)} & \text{(b)} \end{array}\right\} \tag{I.75}$$

From (I.71) and (I.72) it then follows that

$$c_1 = c_2 = 1/2.$$

Hence, the action, in the case that spacetime is sliced in accordance with (I.68), is expressed as

$$S = \frac{1}{2} \int_{c(\sigma)} \{(\frac{dL}{dT})^2 + B\} dT \qquad (I.76)$$

The action in the case of a general slicing of spacetime into spacelike hypersurfaces, is then evidently given by:

$$S = \frac{1}{2} \int_{\mathcal{M}} \int_{c(\sigma)} \{(\frac{d\ell}{d\tau})^2 + b\} \, d\tau dV \qquad (I.77)$$

This is the "local (generic) form of the action", and if we vary it with respect to $d\tau/d\sigma$ we recover the definition of local time (Eq.(I.51)). We see that it has the form of a spacetime integral, and the Lagrangian ($\mathcal{L}$) is equal to:

$$\mathcal{L} = (\frac{d\ell}{d\tau})^2 + b \qquad (I.78)$$

Transforming back the local form of the action into a line integral in Conf($\mathcal{M}$) by making use of Eq. (I.51), we obtain:

$$S = \frac{1}{2} \int_{\mathcal{M}} \int_{c(\sigma)} a \, d\ell dV, \qquad (I.79)$$

where:

$$a = b^{1/2} \qquad (I.80)$$

Finally, the "local point correspondence equation" is obtained in the same way as the global one, with the difference that the local instead of the global form of the action is being varied.

CHAPTER III

DERIVATION OF THE LAGRANGIAN

We shall now derive the most general form of the Lagrangian compatible with the principles established in the previous chapter.

I. Introduction of the gauge group

We first draw attention to the fact that although relativistic invariance requires W^A_m to be the three-dimensional components of four-vectors W^A defined on the four-dimensional space-time manifold, no components W^A_0 enter the action. The only way in which this contradiction can be removed is if the space $\mathrm{Conf}(\mathcal{M})|_F$* is replaced by the space of "invariant fields"

$$\mathcal{S}_F = \frac{\mathrm{Conf}(\mathcal{M})|_F}{\mathcal{T}} \qquad (2.I)$$

where $\mathcal{T}$ is a N-parameter local group of continuous transformations. By applying Fischer's reasoning to the operation of taking this quotient we see that this operation destroys the manifold properties of $\mathcal{S}_F$ at the neighborhoods of those points $\mathcal{F}_0$ which correspond to orbits in $\mathrm{Conf}(\mathcal{M})|_F$ through points F_0 which are left invariant by the action of some subgroup $\mathcal{T}_0$ of $\mathcal{T}$. Thus such points $\mathcal{F}_0$ will be boundary points on $\mathcal{S}_F$.

The parameters of the group are "gauge" functions χ^A (A=I...N) which are as many as the number of vector fields so that they are in one-to-one correspodence with the W_0's needed.

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*$\mathrm{Conf}(\mathcal{M})|_F$ is the space of C^∞ sections of the product bundle $\mathcal{C}(\mathcal{M})|_F = T^{*(N)}(\mathcal{M}) \times \mathcal{K}(\mathcal{M})$

So on $\mathrm{Conf}(\mathcal{M})|_F$ two groups act: the group of diffeomorphisms $\mathrm{Diff}(\mathcal{M})$ and the gauge group $\mathcal{T}$. Then $d\tilde{L}$ in S represents not the element of distance in the original superspace $\mathcal{S}$ but rather that in the "invariant superspace"

$$\bar{\mathcal{S}} = \frac{\mathcal{S}}{\mathcal{T}} \tag{2.2}$$

and $\tilde{A}$ is a functional on $\bar{\mathcal{S}}$. It will be shown in the following that the correlation of "gauges" removes the afore-mentioned contradiction by introducing the W_o^A's as gauge correlation funtions, just as the g_{oi}'s are introduced by the correspodence of points as the vectors ξ_i establishing this correspondence.

In order that relativistic invariance of the Lagrangian is ensured it is also necessary that the following consistency requirement is satisfied: that to the afore-mentioned group of spatial gauge transformations corresponds a group of space-time gauge transfomations which acts on the four-dimensional field forms and is such that its action on the (spatial) field forms generated on each space-like hypersurface reduces to the action of the corresponding three-dimensional group.

It can be readily seen that the above requirement restricts the infinitesimal form of the four-dimensional gauge transfomation to the following:

$$\delta\varphi_a = J_{Aa}\chi^A \qquad \text{(a)}$$
$$\delta W^A_\mu = g^A{}_{BR} W^B_\mu \chi^R + e^{Aa}{}_B \varphi_{a,\mu}\chi^B + h^A{}_B \chi^B{}_{,\mu} \qquad \text{(b)} \qquad (2.3)$$

where J_{Aa}, $g^A{}_{BR}$, $e^{Aa}{}_B$ and $h^A{}_B$ are functions of $\varphi_1, \ldots, \varphi_n$ only.

We now assume that $h^A{}_B$ is a non-degenerate matrix. We can then redefine the vector fields

$$\tilde{W}^A_\mu = (h^{-1})^A{}_B W^B_\mu \tag{2.4}$$

and in terms of the new vector fields, the gauge transformation takes the form

$$\delta\tilde{\varphi}_a = \mathfrak{J}_{Aa}\chi^A \qquad \text{(a)}$$
$$\delta\tilde{W}^A_\mu = \tilde{g}^A{}_{BR}\tilde{W}^B_\mu\chi^R + \tilde{e}^{Aa}{}_B\,\varphi_{a,\mu}\chi^B + \chi^A{}_{,\mu} \qquad \text{(b)} \tag{2.5}$$

where

$$\tilde{g}^A{}_{BR} = \left[(h^{-1})^A{}_P\, g^P{}_{QR} + \partial(h^{-1})^A{}_Q/\partial\varphi_a \mathfrak{J}_{Ra}\right]h^Q{}_B$$
$$\tilde{e}^{Aa}{}_B = (h^{-1})^A{}_P\, e^{Pa}{}_B \tag{2.6}$$

In the following we will use the form (2.5) of the gauge transformations, dropping the $\sim$ for simplicity.

The possible reparametrizations (I.17) are now restricted to

$$\varphi_a \rightarrow \tilde{\varphi}_a = \tilde{\varphi}_a(\varphi_1, \ldots \varphi_n) \qquad \text{(a)}$$
$$W^A_\mu \rightarrow \tilde{W}^A_\mu = \lambda^A{}_B W^B_\mu \qquad \text{(b)} \tag{2.7}$$

where the $\lambda^A{}_B$ are constants. These will be the reparametrizations that will be allowed from now on. Under this transformation the metric coefficients transform as

$$\tilde{G}^{ijmn} = G^{ijmn} \qquad \tilde{G}^{ab} = \frac{\partial\varphi_c}{\partial\tilde{\varphi}_a}\frac{\partial\varphi_d}{\partial\tilde{\varphi}_b}G^{cd}$$

$$\tilde{G}^{ija} = \frac{\partial\varphi_b}{\partial\tilde{\varphi}_a}G^{ijb} \qquad \tilde{G}^{mn}_{AB} = (\lambda^{-1})^P{}_A(\lambda^{-1})^Q{}_B\, G^{mn}_{PQ} \tag{2.8}$$

$$\tilde{G}^{ijm}_A = (\lambda^{-1})^B{}_A G^{ijm}_B \qquad \tilde{G}^{am}_A = (\lambda^{-1})^B{}_A \frac{\partial\varphi_b}{\partial\tilde{\varphi}_a}G^{bm}_B$$

If then the gauge functions χ^A are taken to transform as

$$\chi^A \to \tilde{\chi}^A = \lambda^A{}_B \, \chi^B \tag{2.9}$$

in order that the form(2.5) of the gauge transformation remains invariant, J_{Aa}, $g^A{}_{BR}$, and $e^{Aa}{}_B$ should transform as

$$\begin{aligned} \tilde{J}_{Aa} &= (\lambda^{-1})^B{}_A \, \partial\tilde{\varphi}_a / \partial\varphi_b \, J_{Bb} \\ \tilde{g}^A{}_{BR} &= \lambda^A{}_P (\lambda^{-1})^Q{}_B (\lambda^{-1})^S{}_R \, g^P{}_{QS} \\ \tilde{e}^{Aa}{}_B &= \lambda^A{}_P (\lambda^{-1})^Q{}_B \, \partial\varphi_b / \partial\tilde{\varphi}_a \, e^{Pb}{}_Q \end{aligned} \tag{2.10}$$

From eqs.(2.10) it is evident that the J_{Aa}'s transform as components of a contravariant vector in $\mathcal{K}$

22. The requirement that $\mathcal{J}$ is an isometry of $Conf(\mathcal{M})$

We will now begin the construction of a definition of a distance in $\bar{\mathcal{S}}$ from the already obtained definition of distance in $\mathcal{S}$.

Let us consider two nearby elements $\bar{\mathcal{S}}_0$ and $\bar{\mathcal{S}}_0'$ of $\bar{\mathcal{S}}$. We are looking for a definition of their distance. Now, to these two points correspond two orbits $O(\mathcal{J})$ and $O'(\mathcal{J})$ of the group $\mathcal{J}$ in $\mathcal{S}$ respectively. Consider a particular point $\mathcal{S}_0$ on the orbit $O(\mathcal{J})$. The distance of this point from a point on the orbit $O'(\mathcal{J})$

as given by $\tilde{L}$ of eq.(I.37) obviously varies in general as the other point traces the orbit $O'(\mathfrak{J})$. However, the distance between the points $\bar{\xi}_0$ and $\bar{\xi}_0'$ should be unique. Hence it must be defined equal to the distance $d\tilde{L}(\xi_0,\xi_0')$ between ξ_0 and some particular point ξ_0' of the orbit $O'(\mathfrak{J})$. Thus a particular correspondence $\xi_0 \longleftrightarrow \xi_0'$ should be defined. We shall return to this problem later. For the moment we note that if such a correspondence has been established then a one-to-one correspondence between all other points of the orbits $O(\mathfrak{J})$ and $O'(\mathfrak{J})$ is fixed by the rule

$$\mathcal{P}\xi_0 \longleftrightarrow \mathcal{P}\xi_0' \tag{2.11}$$

where $\mathcal{P}$ is any element of the group $\mathfrak{J}$. Since the original point ξ_0 on $O(\mathfrak{J})$ was picked arbitrarily, we must demand that the distance between corresponding points on $O(\mathfrak{J})$ and $O'(\mathfrak{J})$ is invariant along the orbit

$$\tilde{L}(\mathcal{P}\xi_0, \mathcal{P}\xi_0') = \tilde{L}(\xi_0, \xi_0') \tag{2.12}$$

Thus we must demand that if we vary the field functions which enter the metric coefficients G in accordance with the spatial transformation corresponding to that given by eq.(2.5) and the differentials of the field functions in accordance with*

$$\delta d\varphi_\alpha = d\delta\varphi_\alpha = \partial\mathfrak{J}_{A\alpha}/\partial\varphi_b \, d\varphi_b \chi^A \qquad (a)$$

$$(2.13)$$

$$\delta dW^A_m = d\delta W^A_m = (g^A{}_{BR} dW^B_m + \partial g^A{}_{BR}/\partial\varphi_a W^B_m \qquad (b)$$

$$+ e^{Aa}{}_R d\varphi_{a,m} + \partial e^{Aa}{}_R/\partial\varphi_b \, \varphi_{a,m} d\varphi_b)\chi^R \quad (d\chi^A = 0)$$

*From now on we shall be working in coordinates $\xi_i = 0$.

then the corresponding variation of $\tilde{L}$ vanishes

$$\delta \tilde{L} = 0 \tag{2.I4}$$

This determines the way that the metric coefficients G transform under the action of the gauge group. Even before writing down these transformation laws, it is evident that $e^{Aa}{}_B$ must be set equal to zero

$$e^{Aa}{}_B = 0 \tag{2.I5}$$

since this term would introduce derivatives of $d\varphi_a$ and φ_a in L and no such derivatives are present in the original expression. With eq.(2.I5) taken into account the equations for the six metric coefficients are

$$\delta G^{ijmn} = 0 \tag{a}$$

$$\delta G^{ija} + \left(G^{ijb} \frac{\partial J_{Rb}}{\partial \varphi_a} + G^{ijm}_{A} \frac{\partial g^{A}{}_{QR}}{\partial \varphi_a} W^{Q}_{m} \right) \chi^{R} = 0 \tag{b}$$

$$\delta G^{ijm}_{A} + G^{ijm}_{B}\, g^{B}{}_{AR} \chi^{R} = 0 \tag{c}$$

$$\begin{aligned} \delta G^{ab} + \Big(G^{ac} \frac{\partial J_{Rc}}{\partial \varphi_b} + G^{bc} \frac{\partial J_{Rc}}{\partial \varphi_a} + G^{am}_{A} \frac{\partial g^{A}{}_{BR}}{\partial \varphi_b} W^{B}_{m} \qquad & \qquad \text{(d)} \\ + G^{bm}_{A} \frac{\partial g^{A}{}_{BR}}{\partial \varphi_a} W^{B}_{m} \Big) \chi^{R} = 0 & \qquad \text{(2.I6)} \end{aligned}$$

$$\delta G^{mn}_{AB} + \left(G^{mn}_{AQ}\, g^{Q}{}_{BR} + G^{mn}_{BQ}\, g^{Q}{}_{AR} \right) \chi^{R} = 0 \tag{e}$$

$$\delta G^{am}_{A} + \left(G^{bm}_{A} \frac{\partial J_{Rb}}{\partial \varphi_a} + G^{am}_{B}\, g^{B}{}_{AR} + G^{mn}_{AB} \frac{\partial g^{B}{}_{QR}}{\partial \varphi_a} \right) \chi^{R} = 0 \tag{f}$$

Now since the variations of the metric coefficients G are given by (dropping the indices of the G's)

$$\delta G = \frac{\partial G}{\partial \varphi_a} \cdot \delta\varphi_a + \frac{\partial G}{\partial W^A_m} \cdot \delta W^A_m$$

$$= \left(\frac{\partial G}{\partial \varphi_a} \mathfrak{J}_{Ra} + \frac{\partial G}{\partial W^A_m} g^A_{BR} W^B_m \right) \chi^R + \frac{\partial G}{\partial W^R_m} \chi^R{}_{,m} \qquad (2.17)$$

we obtain two equations from each of eqs. 2.16 one being the condition that the coefficient of $\chi^R{}_{,m}$ vanishes and the other that the coefficient of χ^R vanishes.

The equations which arise from the vanishing of the coefficient of $\chi^R{}_{,m}$ are trivial

$$\frac{\partial G}{\partial W^A_m} = 0 \qquad (2.18)$$

If we now take into account the most general possible form of the G's as restricted by postulate I we deduce

$$G^{ijmn} = h_1 g^{ij} g^{mn} + h_2 g^{i(m} g^{jn)} \qquad (a)$$

$$G^{ija} = f^a g^{ij} \qquad (b)$$

$$G^{ijm}_A = 0 \qquad (c)$$

$$G^{ab} = G^{ab} \qquad (d)$$

$$G^{mn}_{AB} = f_{AB} g^{mn} \qquad (e) \qquad (2.19)$$

$$G^{am}_A = 0 \qquad (f)$$

where $h_{1,2}$, f^a, G^{ab} and f_{AB} are now functions of $\varphi_1,\ldots\varphi_n$ only. Then the vanishing of the coefficient of χ^R in eqs. 2.16(a),(b) and (d),(f) gives the following equations for these functions

$$\mathcal{J}_{Ra}\frac{\partial h_{1,2}}{\partial\varphi_a}=0 \tag{a}$$

$$\mathcal{J}_{Rb}\frac{\partial f^a}{\partial\varphi_b}+f^b\frac{\partial\mathcal{J}_{Rb}}{\partial\varphi_a}=0 \tag{b}$$

$$\mathcal{J}_{Rc}\frac{\partial G^{ab}}{\partial\varphi_c}+G^{ac}\frac{\partial\mathcal{J}_{Rc}}{\partial\varphi_b}+G^{bc}\frac{\partial\mathcal{J}_{Rc}}{\partial\varphi_a}=0 \qquad (c) \qquad (2.20)$$

$$\mathcal{J}_{Ra}\frac{\partial f_{AB}}{\partial\varphi_a}+f_{AQ}g^Q{}_{BR}+f_{BQ}g^Q{}_{AR}=0 \tag{d}$$

Finally, the vanishing of the coefficient of χ^R in eq 2.16(f) gives

$$\frac{\partial g^A{}_{BR}}{\partial\varphi_a}=0 \tag{2.21}$$

provided that the matrix f_{AB} is non-degenerate.

3. The finite form of the gauge transformations

Taking into account eqs.(2.15) and(2.21) ,the form of the infinitesimal gauge transformation is now reduced to

$$\delta\varphi_a=\mathcal{J}_{Aa}\chi^A \qquad (a)$$

$$\delta W^A_m=g^A{}_{BR}W^B_m\chi^R+\chi^A{}_{,m} \qquad (b) \qquad (2.22)$$

with $\delta\, dW^A_m = g^A{}_{BR}\, dW^B_m \chi^R$ (c)

where the $g^A{}_{BR}$'s are now constants. From the above form of the infinitesimal gauge transformations and their group property we may deduce their finite form

$$\varphi'_a = e^{\int d^3x\, \chi^R Z^{\varphi}_R} \varphi_a \qquad \text{(a)}$$

(2.23)

$$W'^A_m = e^{\int d^3x\, \chi^R Z^{W}_R} W^A_m \qquad \text{(b)}$$

$$dW'^A_m = e^{\int d^3x\, \chi^R Z^{dW}_R} dW^A_m \qquad (2.24)$$

$$Z^{\varphi}_A = \mathcal{T}_{Aa} \frac{\delta}{\delta \varphi_a} \qquad \text{(a)}$$

$$Z^{W}_A = -\nabla_m \frac{\delta}{\delta W^A_m} + g^R{}_{BA} W^B_m \frac{\delta}{\delta W^R_m} \qquad \text{(b)} \qquad (2.25)$$

$$Z^{dW}_A = g^R{}_{BA}\, dW^B_m \frac{\delta}{\delta\, dW^R_m} \qquad (2.26)$$

Here,

$$Z_A = Z^{\varphi}_A + Z^{W}_A \qquad (2.27)$$

are the generators of the gauge group $\mathcal{T}$.

Since eq.(2.23) says that the gauge transformation of the φ's does not involve the W's and the gauge transformation of the W's does not involve the φ's we may think of the set of gauge transformations of the φ's as the group $\mathcal{T}(\varphi)$ induced by $\mathcal{T}$ on $\mathcal{K}$ and the set of gauge transformations of the W's as the group $\mathcal{T}(W)$ induced by $\mathcal{T}$ on $\mathcal{N}$.

Let then $\mathcal{T}(dW)$ be the group induced on the cotangent bundle of $\mathcal{N}$ (transformations (2.24))

Since the scalar fields $\varphi_a({}^3x')$ and $\varphi_a({}^3x'')$ and the differentials of the vector fields $dW^A({}^3x')$ and $dW^A({}^3x'')$ at two different space points ${}^3x'$ and ${}^3x''$ transform independently, the groups $\mathcal{T}(\varphi)$ and $\mathcal{T}(dW)$ are direct products of groups $\mathcal{T}(\varphi({}^3x))$ and $\mathcal{T}(dW({}^3x))$ which act on $\mathcal{K}$ and $\mathcal{N}$, respectively

$$\mathcal{T}(\varphi) = \prod_{{}^3x \in \mathcal{M}}^{\text{direct}} \mathcal{T}(\varphi({}^3x))$$

$$\mathcal{T}(dW) = \prod_{{}^3x \in \mathcal{M}}^{\text{direct}} \mathcal{T}(dW({}^3x)) \tag{2.28}$$

Thus, while the groups $\mathcal{T}(\varphi)$ and $\mathcal{T}(dW)$ are like $\mathcal{T}$ itself infinite parameter groups the groups $\mathcal{T}(\varphi({}^3x))$ and $\mathcal{T}(dW({}^3x))$ are finite (M and N) parameter groups.

In terms of the generators X_R and Y_R of these groups which are given by

$$X_R = \mathcal{T}_{Ra} \frac{\partial}{\partial \varphi_a} \qquad \text{(a)}$$

$$(Y_R)^A{}_B = g^A{}_{BR} \qquad \text{(b)} \tag{2.29}$$

the equations of finite gauge transformations of the φ's and the dW's given by eqs2.23(a)and(2.24)are reexpressed as

$$\varphi'_a = e^{\chi^R X_R}\varphi_a \qquad (a)$$

$$dW'^A_m = \left(e^{\chi^R Y_R}\right)^A{}_B\, dW^B_m \qquad (b) \qquad (2.30)$$

From the first of eqs. 2.I0 it follows that under a reparametrization the generators X_A transform as

$$X_A \rightarrow \tilde{X}_A = (\lambda^{-1})^B{}_A X_B \qquad (2.3I)$$

namely as covariant vectors in $\mathcal{N}_x$

Actually only the non vanishing X_A's should be considered as generators of $\mathcal{T}(\varphi(^3x))$. Thus if N-M of the X_A's vanish

$$X_{\dot{A}} = 0 \qquad \dot{A} = M+I,\ldots\ldots,N \qquad (2.32)$$

only the remaining M $X_{\bar{A}}$ ($\bar{A}$ =I,....,M) should be properly considered as generators.

Eq.(2.32) limits the allowed re parametrizations by the condition

$$(\lambda^{-1})^{\bar{B}}{}_{\dot{A}} = 0 \qquad (\text{or} \quad \lambda^{\bar{B}}{}_{\dot{A}}=0) \quad (\text{so that } \tilde{X}_{\dot{A}}=0 \text{ also}) \qquad (2.33)$$

The generators $X_{\bar{R}}$ and Y_R should obey

$$[X_{\bar{A}}, X_{\bar{B}}] = s^{\bar{R}}{}_{\bar{A}\bar{B}} X_{\bar{R}} \quad (a) \qquad [Y_P, Y_Q] = t^R{}_{PQ} \quad (b) \qquad (2.34)$$

$s^{\bar{R}}{}_{\bar{A}\bar{B}}$ and $t^R{}_{PQ}$ being the structure constants of $\mathcal{T}(\varphi(^3x))$ and $\mathcal{T}(d\,W(^3x))$.

From eqs.(2.30) and (2.34) it follows that

$$g^A{}_{RB}\, t^B{}_{PQB} = g^A{}_{BP}\, g^B{}_{RQ} - g^A{}_{BQ}\, g^B{}_{RP} \qquad (2.35)$$

4. Gauge correspodence equations

We now return to the problem of defining the correspodence $\zeta_0 \leftrightarrow \zeta_0'$, which will give finally the required definition of distance in $\bar{\zeta}$. Let us first pick the point ζ_0' to be any point on the orbit $O(\mathcal{J})$ which is in an infinitesimal neighborhood of ζ_0.

Let us then change the arbitrary correspondence just obtained

$$\zeta_0 \leftrightarrow \zeta_0' \qquad (2.36)$$

by acting on ζ_0 and ζ_0' by the two different elements $\mathcal{P}$ and $\mathcal{P}'$ of $\mathcal{J}$ respectively, thus establishing a new correspondence

$$\mathcal{P}\zeta_0 \leftrightarrow \mathcal{P}'\zeta_0' \qquad (2.37)$$

We may think of $\bar{\zeta}_0$ and $\bar{\zeta}_0'$ to be the points σ and $\sigma+d\sigma$ on the path

$$\bar{\zeta} = \bar{\zeta}(\sigma) \qquad (2.38)$$

in the invariant superspace and $\mathcal{P}$ and $\mathcal{P}'$ to correspond respectively to the points σ and $\sigma+d\sigma$ on the path

$$\chi^A = \chi^A(\sigma) \qquad (2.39)$$

in the space of gauge functions χ^A

Defining

$$\eta^A = \frac{\partial \chi^A}{\partial \sigma} d\sigma \qquad (2.40)$$

wesee that the change in correspondence is manifested in the presence of these η^A's. It follows from eqs. (2.30) that the new metric and field differences are then given in terms of the original ones by

$$dg'_{ij} = dg_{ij} \qquad \text{(a)} \qquad (2.41)$$

$$d\varphi'_a = d\varphi_a + d\delta\varphi_a$$

$$= \frac{\partial \mathfrak{J}_{Aa}}{\partial \varphi_b} d\varphi_b \chi^A + \bar{d}\varphi_a \qquad \text{(b)}$$

$$dW'^A_i = dW^A_i + d\delta W^A_i \qquad \text{(c)} \qquad (2.41)$$

$$= g^A{}_{BR}\, dW^B_i \chi^R + \bar{d}W^A_i$$

where

$$\bar{d}\varphi_a = d\varphi_a + \mathfrak{J}_{Aa} n^A \qquad \text{(a)}$$

$$\bar{d}W^A_i = dW^A_i + g^A{}_{BR} W^B_i n^R + n^A{}_{,i} \qquad \text{(b)} \qquad (2.42)$$

The distance between $\pi^{-1}(\mathcal{P}S_0)$ and $\pi^{-1}(\mathcal{P}'S'_0)$ is then given by

$$dL'^2 = \int (h_1 g^{ij} g^{mn} + h_2 g^{i(m} g^{jn)})\, dg_{ij}\, dg_{mn}$$

$$+ 2 g^{ij} f^a \bar{d}g_{ij} \bar{d}\varphi_a + G^{ab} \bar{d}\varphi_a \bar{d}\varphi_b + f_{AB} g^{mn} \bar{d}W^A_m \bar{d}W^B_n \qquad (2.43)$$

We finally define the distance dL^2 between S_0 and S'_0 to be the extremal distance between $\mathcal{P}S_0$ and $\mathcal{P}'S'_0$ as the correspondence is varied

$$dL = \text{extr.}_n (dL) \qquad (2.44)$$

The extremization conditions are

$$\frac{1}{2} \frac{\delta dL'^2}{\delta n^A} = g^{ij} f^a dg_{ij} \mathfrak{J}_{Aa} + G^{ab} \bar{d}\varphi_b \mathfrak{J}_{Aa}$$

$$+ f_{BQ} g^Q{}_{RA} \bar{d}W^B_m W^{Rm} - (f_{AB} \bar{d}W^B_m)^{;m} = 0 \qquad (2.45)$$

or

$$H_{AB} n^B = S_A \qquad (2.46)$$

Here H_{AB} is the self-adjoint operator

$$H_{AB} = D^{+a}{}_{Am} D_{aB}{}^{m} + G^{ab} J_{Aa} J_{Bb} \qquad (2.47)$$

where

$$D_{ABm} = f_{Aa} D^{a}{}_{Bm}$$
$$D^{a}{}_{Bm} = \delta^{a}{}_{B} \nabla_m + g^{a}{}_{RB} W^{R}_{m} \qquad (2.48)$$
$$D^{+a}{}_{Bm} = -\delta^{a}{}_{B} \nabla_m + g^{a}{}_{RB} W^{R}_{m}$$

The source term S_A in eq.(2.46) is given by

$$S_A = J_{Aa} f^{a} g^{ij} dg_{ij} + G^{ab} J_{Aa} d\varphi_b$$
$$+ f_{Ba} g^{a}{}_{RA} dW^{B}_{m} W^{Rm} - (f_{AB}\, dW^{B}_{m})^{;m} \qquad (2.49)$$

Three conditions should be satisfied by the given definition of distance in $\bar{\mathcal{S}}$ in order that the definition is meaningful.

I) That it exists for any pair of points in $\bar{\mathcal{S}}$ which correspond to nearby orbits in $\mathcal{S}$

2) That it is unique

3) That if the orbits in $\mathcal{S}$ to which $\bar{\mathcal{S}}_0$ and $\bar{\mathcal{S}}_0'$ correspond coincide, the distance between $\bar{\mathcal{S}}_0$ and $\bar{\mathcal{S}}_0'$ should be zero.

In this case however, $\mathcal{S}_0$ and $\mathcal{S}_0'$ lie on the same orbit and therefore $P\mathcal{S}_0$ and $P'\mathcal{S}_0'$ may be chosen to coincide, in which case we have $dg_{ij} = d\varphi_a = dW^{A}_{m} = 0$ which satisfies eqs.(2.45) and $dL^2 = 0$. Hence, if condition 2) is satisfied, condition 3) is always satisfied. Thus we need only demonstrate the existence and uniqueness of the solution η^A of eq2.46for any source term S_A of the form given by eq.(2.49). This will be done in appendixII.

5. Construction of the four-dimensional field forms

We are now finally in position to determine the four-dimensional one-forms $^{(4)}\rho^A$ .Consider a path $S(\sigma)$ in $\bar{S}$.
Let the spatial one-forms $^{(3)}\rho^A$ be the intrinsic one-forms of the three-dimensional hypersurfaces σ = constant of the space-time manifold defined earlier. Let then the gauges of the spatial one-forms belonging to adjacent hypersurfaces (σ and $\sigma+d\sigma$) be correlated by the gauge correlation defined by eq.(2.46). We then define the space-time one-form to be

$$^{(4)}\rho^A = W^{*\prime A}_{\ i}\, dx^{*i} \tag{2.50}$$

remembering that since page 42 we have been working in the correct* correspondence without actually putting $*$ for the shake of simplicity.

Using now the four-dimensional gauge transformations (noting that(2.50) denotes $W^{*\prime A}_{\ 0} = 0$) we can write the four-dimensional form(2.50) in the gauge of the original gauge correlation obtaining

$$\begin{aligned} ^{(4)}\rho^A &= -\eta^{*A} + W^{*A}_{\ i}\, dx^{*i} \\ &= -\eta^{*A} + W^A_{\ m}(dx^m - N^m dx^0) = W^A_{\ \mu}\, dx^\mu \end{aligned} \tag{2.51}$$

*By correct we mean the one defined on page 40.

where in the last expressions we have also returned to the coordinate system of the original arbitrary point correspondence. From eq. (2.51) we deduce

$$\overset{*}{n}{}^{A} = N^{\mu} \frac{\partial x^{A}}{\partial x^{\mu}} dx^{0} = -N^{\mu} W^{A}_{\mu} dx^{0} \qquad (2.52)$$

6. Derivation of the Lagrangian

We now express the Lagrangian in a general coordinate system and gauge obtaining

$$\mathcal{L} = \left(\frac{d\hat{\ell}(^3x)}{d\tau(^3x)} \right)^2 + b(^3x) \tag{2.53}$$

where the local element of distance $d\ell(^3x)$ in $\bar{S}$ is given by

$$\begin{aligned} d\hat{\ell}(^3x)^2 &= (h_1 g^{ij} g^{mn} + h_2 g^{i(m} g^{jn)})\, \bar{d}g_{ij}\, \bar{d}g_{mn} \\ &+ 2 g^{ij} f^a \bar{d}g_{ij} \bar{d}\varphi_a + G^{ab} \bar{d}\varphi_a \bar{d}\varphi_b + f_{AB} g^{mn} \bar{d}W^A_m \bar{d}W^B_n \end{aligned} \tag{2.54}$$

where $\bar{d}$ is the "convective" derivative

$$\bar{d} = d + L^{\mathrm{Diff}}_{\xi} + L^{\mathcal{T}}_{n'} \tag{2.55}$$

Here L^{Diff}_{ξ} is the Lie derivative with respect to the group of diffeomorfisms along ξ and $L^{\mathcal{T}}_{n^*}$ is the Lie derivative with respect to the gauge group along n^i where

$$\begin{aligned} \xi_i &= N_i\, dx^0 \\ n^{*A} &= -N^\mu W^A_\mu\, dx^0 . \end{aligned} \tag{2.56}$$

Thus

$$\begin{aligned} \bar{d}g_{ij} &= dg_{ij} + \xi_{i;j} + \xi_{j;i} \\ \bar{d}\varphi_a &= d\varphi_a + \varphi_{a,i}\xi^i + J_{Aa} n^A \\ \bar{d}W^A_m &= dW^A_m + W^A_{m;i}\xi^i + W^A_i \xi^i_{;m} + g^A{}_{PR} W^P_m n^{*R} + n^{*A}{}_{;m} \end{aligned} \tag{2.57}$$

From eqs. 2.53,54,55,56 and(2.57) we conclude that the problem of the introduction of the W^A_o's in the Lagrangian has finally been solved.

We now seek a $b(^3x)$ which is such that when added to the already obtained expression for $(d\ell(^3x)/d\tau(^3x))^2$ makes the resulting Lagrangian relativistic invariant. Let us suppose that a particular $b_o(^3x)$ is found which satisfies this requirement. Then if the most general $b(^3x)$ is written in the form

$$b(^3x) = b_o(^3x) - e(^3x) \tag{2.58}$$

it is obvious that $e(^3x)$ must also be by itself relativistic invariant. But the only spacial invariants we can form which are also space-time invariants are functions of $\varphi_1, \cdots, \varphi_n$ only:

$$e(^3x) = e(\varphi_1, \cdots, \varphi_n) \tag{2.59}$$

From eqs.(2.58) and (2.59) we conclude that $b(^3x)$ is up to a function of the scalars, determined by the relativistic invariance of the Lagrangian.

Let us first look at the third term in eq.(2.54). Defining

$$D_{a\mu} = \varphi_{a,\mu} - \mathcal{J}_{Aa} W^A_\mu \tag{2.60}$$

we can bring this term to the form

$$G^{ab} \frac{\bar{d}\varphi_a}{d\tau(^3x)} \frac{\bar{d}\varphi_b}{d\tau(^3x)} = \frac{G^{ab}}{N^2} (D_{ao} + N^i D_{ai})(D_{bo} + N^j D_{bj}) \tag{2.6I}$$

Taking then into account the fact that the components of the contravariant four-dimensional metric tensor are given by

$$g^{00} = \frac{-1}{N^2} \ , \ g^{0i} = \frac{-N^i}{N^2} \quad {}^{(4)}g^{ij} = g^{ij} - \frac{N^i N^j}{N^2} \tag{2.62}$$

it becomes clear that if the purely spatial expression (part of $b(^3x)$)

$$- G^{ab} g^{ij} D_{ai} D_{bj} \tag{2.63}$$

is added to the expression 2.6I the sum which result is the relativistic invariant expression

$$- g^{\mu\nu} G^{ab} D_{a\mu} D_{b\nu} \tag{2.64}$$

Let us then turn to the fourth term in eq. 2.54 . Defining now

$$F^A_{\mu\nu} = f^A_{\mu\nu} - g^A{}_{PQ} W^P_\mu W^Q_\nu \tag{2.65}$$

where $f^A_{\mu\nu} = W^A_{\mu,\nu} - W^A_{\nu,\mu}$ (2.66)

we express the term as

$$f_{AB} g^{mn} \frac{\bar{d} W^A_m}{d\tau(^3x)} \frac{\bar{d} W^B_n}{d\tau(^3x)} = f_{AB} \frac{g^{mn}}{N^2} (F^A_{m0} + N^i F^A_{mi})(F^B_{n0} + N^j F^B_{nj}) \tag{2.67}$$

The above expression differs from a relativistic invariant one by a purely spacial term if and only if $F^A_{\mu\nu}$ is antisymmetric in μ and ν .Considering eq.2.65 this implies that $g^A{}_{PQ}$ is antisymmetric in P and Q

$$g^A{}_{PQ} + g^A{}_{QP} = 0 \tag{2.68}$$

Then the addition of the purely spatial expression

$$-\frac{1}{2} f_{AB} g^{mn} g^{ij} F^A_{mi} F^B_{nj} \tag{2.69}$$

to the term (2.67) gives the rel.invariant expression

$$-\frac{1}{2} f_{AB} g^{\mu\nu} g^{\rho\sigma} F^A_{\mu\rho} F^B_{\nu\sigma} \tag{2.70}$$

We finally consider the first and second terms in eq.(2.54) .

It can be shown that the only space-time invariant which differs by a purely spatial quantity from the sum of these two terms,and this only after an integration by parts has taken place (see below) is up to a function of the scalars

$$^{(4)}R h \tag{2.7I}$$

where h is a function of the scalars.

The part of the action which corresponds to expression(2.7I) is then

$$\frac{1}{2}\int {}^{(4)}R h \sqrt{-{}^{(4)}g}\, d^4x$$

$$= \frac{1}{2}\int d^3x \int d\tau({}^3x) \Big\{ -2g_{ij}\, h \frac{\bar{d}}{d\tau({}^3x)}\Big(\sqrt{g}\, \hat{G}^{ijmn} \frac{\bar{d} g_{mn}}{d\tau({}^3x)}\Big) - h\sqrt{g}\, \hat{G}^{ijmn} \frac{\bar{d} g_{ij}}{d\tau({}^3x)} \frac{\bar{d} g_{mn}}{d\tau({}^3x)} + {}^{(3)}R h \sqrt{g} \Big\} \tag{2.72}$$

where

$$\hat{G}^{ijmn} = \frac{1}{4}(g^{i(m} g^{n)j} - g^{ij} g^{mn} \tag{2.73}$$

Performing an integration by parts with respect to x^0 on the first term in the integrant of eq.(2.72) we obtain

$$\frac{1}{2}\int {}^{(4)}R h \sqrt{-{}^{(4)}g}\, d^4x =$$

$$\frac{1}{2}\int \sqrt{g}\, d^3x \int d\tau({}^3x) \Big\{ h\, \hat{G}^{ijmn} \frac{\bar{d} g_{ij}}{d\tau({}^3x)} \frac{\bar{d} g_{mn}}{d\tau({}^3x)} - \frac{1}{2} g^{mn} \frac{\bar{d} g_{mn}}{d\tau({}^3x)} \frac{\bar{d}\varphi_a}{d\tau({}^3x)} \frac{\partial h}{\partial \varphi_a} + {}^{(3)}R h \Big\} \tag{2.74}$$

Comparing the above expression with the first and second terms of eq.(2.54) we conclude that

$$h_2 = -h_1 = \frac{h}{4} \tag{2.75}$$

$$f^a = -\frac{1}{2}\frac{\partial h}{\partial \varphi_a}$$

and

$$J_{Aa}\, \partial h / \partial \varphi_a = 0 \tag{2.76}$$

The above equations contain eqs. 2.20(a) and 2.20(b) for h_1, h_2 and f^a. Hence in the following we need only take into account eqs.(2.75) and (2.76).

Considering eqs.(2.74),(2.70) and (2.64) we conclude that we have found a particular $b_0(^3x)$ which when added to $\left(d\ell(^3x)/d\tau(^3x)\right)^2$ makes the resulting Lagrangian relativistic invariant. The most general $b(^3x)$ is then given by (2.58) and the most general form of the Lagrangian is

$$\mathcal{L} = {}^{(4)}R\,h - e - G^{ab} D_{a\mu} D_b{}^{\mu} - \frac{1}{2} f_{AB} F^A_{\mu\nu} F^{B\mu\nu} \tag{2.77}$$

the spatial function $b(^3x)$ being just the three-dimensional analogue of the above four-dimensional Lagrangian

$$b(^3x) = {}^{(3)}R\,h - e - G^{ab} D_{am} D_b{}^{m} - \frac{1}{2} f_{AB} F^A_{mn} F^{Bmn} \tag{2.78}$$

(where all contractions are with respect to the spatial contravariant metric tensor g^{mn})

To the Lagrangian(2.75) we must now impose the final consistency requirement, that of invariance under the four-dimensional gauge transformations (2.5) .(We could equivalently, impose on $b({}^3x)$ the condition of invariance under the group of spatial gauge transformatios. Because of the identity in form of(2.77) and(2.78) as well as the identity in form of the transformations (2.3) and (2.5) this would lead to identical results).

Taking into account eqs(2.22) and(2.68) the quantities introduced by eqs.(2.6o) and (2.65) transform as follows under the action of an infinitesimal gauge transformation

$$\delta D_{a\mu} = \Xi_{Ra\mu} \chi^R$$
$$\delta F^A_{\mu\nu} = \Pi^A{}_{R\mu\nu} \chi^R \tag{2.79}$$

where

$$\Xi_{Ra\mu} = \frac{\partial J_{Ra}}{\partial \varphi_b} \varphi_{b,\mu} - \frac{\partial J_{Ba}}{\partial \varphi_b} J_{Rb} W^B_\mu - J_{Aa} g^A{}_{BR} W^B_\mu$$
$$\Pi^A{}_{R\mu\nu} = g^A{}_{BR} f^B{}_{\mu\nu} - (g^A{}_{QB} g^B{}_{RP} + g^A{}_{PB} g^B{}_{QR}) W^P_\mu W^Q_\nu \tag{2.80}$$

The requirement of gauge invariance of the Lagrangian takes then the form

$$\delta\mathcal{L} = \Big\{ R\frac{\partial h}{\partial\varphi_a}\mathcal{J}_{Ra} - \frac{\partial G^{ab}}{\partial\varphi_c}D_{a\mu}D_b^{\mu}\mathcal{J}_{Rc} - G^{ab}\Xi_{Ra\mu}D_b^{\mu}$$

$$- G^{ab}D_{a\mu}\Xi_{Rb}{}^{\mu} - \frac{1}{2}\frac{\partial f_{AB}}{\partial\varphi_a}F^{A}_{\mu\nu}F^{B\mu\nu}\mathcal{J}_{Ra} - \frac{1}{2}f_{AB}\Pi^{A}_{R\mu\nu}F^{B\mu\nu} \tag{2.81}$$

$$- \frac{1}{2}f_{AB}F^{A}_{\mu\nu}\Pi^{B}{}_{R}{}^{\mu\nu} - \frac{\partial e}{\partial\varphi_a}\mathcal{J}_{Ra}\Big\}\chi^R = 0$$

In the above equation the coefficients of

R	a)	$\Phi_{a,\mu}W^{A\mu}$	d)
$\Phi_{a,\mu}\Phi_b{}^{,\mu}$	b)	$f^{A}_{\mu\nu}W^{B\mu}W^{R\nu}$	e)
$f^{A}_{\mu\nu}f^{B\mu\nu}$	c)	$W^{A}_{\mu}W^{B\mu}$	f)

$W^{A}_{\mu}W^{B\mu}W^{P}_{\nu}W^{Q\nu}$ g)

must clearly vanish separately. The vanishing of the coefficients of a), b), and c) reproduces eqs. 2.76,2.20(c)and 2.20(d) respectively, while the vanishing of the coefficients of d) and e) gives (taking into account eqs.2.20(c) and2.20(d))

$$\mathcal{J}_{Bb}\,\partial\mathcal{J}_{Ra}/\partial\varphi_b - \mathcal{J}_{Rb}\,\partial\mathcal{J}_{Ba}/\partial\varphi_b = \mathcal{J}_{Aa}g^{A}{}_{BR}$$

$$g^{A}{}_{QB}g^{B}{}_{RP} + g^{A}{}_{PB}g^{B}{}_{QR} + g^{A}{}_{RB}g^{B}{}_{PQ} = 0 \tag{2.82}$$

From the above equations it now follows that

$$\Xi_{Ra\mu} = \partial\mathcal{J}_{Ra}/\partial\varphi_b\,D_{b\mu}$$

$$\Pi^{A}{}_{R\mu\nu} = g^{A}{}_{BR}F^{B}_{\mu\nu} \tag{2.83}$$

From eqs.(2.83) (taking again into account eqs2.20(c)and (d)) it then follows that the coefficients of f) and g) vanish identically.The only remaining condition so that eq.(2.81)is satisfied is the vanishing of the last term

$$\mathcal{J}_{Ra}\,\partial e/\partial\varphi_a = 0 \tag{2.84}$$

7. Discussion

We now look at eqs.(2.82). In terms of the generators $X_{\bar{A}}$ of the group $\mathcal{J}(\varphi(\beta x))$ the first of eqs.(2.82) assumes the form

$$
\begin{aligned}
&[X_{\bar{B}}, X_{\bar{R}}] = g^{\bar{A}}{}_{\bar{B}\bar{R}} X_{\bar{A}} && \text{(a)} \\
&g^{\bar{A}}{}_{B\dot{R}} = 0 && \text{(b)}
\end{aligned}
\qquad (2.85)
$$

Comparing eqs. 2.34(a) and 2.85(a) we conclude that the $g^{\bar{A}}{}_{\bar{B}\bar{R}}$'s are the structure constants of the group $\mathcal{J}(\varphi(\beta x))$

$$S^{\bar{A}}{}_{\bar{B}\bar{R}} = g^{\bar{A}}{}_{\bar{B}\bar{R}} \qquad (2.86)$$

Comparing then eq.(2.35) with the second of eqs.(2.82) we obtain

$$t^{R}{}_{PQ} = -g^{R}{}_{PQ} \qquad (2.87)$$

namely that the $g^{R}{}_{PQ}$'s are minus the structure constants of the group $\mathcal{J}(dW(\beta x))$. It follows that the second of eqs(2.82) is simply the Jacobi identity for the structure constants of the group .

From eqs. 2.85(b) and (2.87) it follows that the generators $Y_{\dot{A}}$ close an algebra

$$[Y_{\dot{P}}, Y_{\dot{Q}}] = -g^{\dot{R}}{}_{\dot{P}\dot{Q}} Y_{\dot{R}} \qquad (2.88)$$

thus they represent the generators of an N-M parameter subgroup of $\mathcal{J}(dW(\beta x))$. We shall call this subgroup $\mathcal{J}(d\dot{W}(\beta x))$

With eq.2.85(b) taken into account the second of eqs.(2.82) imposes the following conditions on $g^{\dot{R}}{}_{\dot{P}\bar{Q}}$ and $g^{\dot{R}}{}_{\bar{P}\bar{Q}}$

$$
\begin{aligned}
&g^{\dot{A}}{}_{\dot{P}\dot{B}}\, g^{\dot{B}}{}_{\dot{Q}\bar{R}} + g^{\dot{A}}{}_{\dot{Q}\dot{B}}\, g^{\dot{B}}{}_{\bar{R}\dot{P}} + g^{\dot{A}}{}_{\bar{R}\dot{B}}\, g^{\dot{B}}{}_{\dot{P}\dot{Q}} = 0 && \text{(a)} \\
&g^{\dot{A}}{}_{\dot{P}\dot{B}}\, g^{\dot{B}}{}_{\bar{Q}\bar{R}} + g^{\dot{A}}{}_{\dot{P}\bar{B}}\, g^{\bar{B}}{}_{\bar{Q}\bar{R}} + g^{\dot{A}}{}_{\bar{Q}\dot{B}}\, g^{\dot{B}}{}_{\bar{R}\dot{P}} + g^{\dot{A}}{}_{\bar{R}\dot{B}}\, g^{\dot{B}}{}_{\dot{P}\bar{Q}} = 0 && \text{(b)} \\
&g^{\dot{A}}{}_{\bar{P}\bar{B}}\, g^{\bar{B}}{}_{\bar{Q}\bar{R}} + g^{\dot{A}}{}_{\bar{Q}\bar{B}}\, g^{\bar{B}}{}_{\bar{R}\bar{P}} + g^{\dot{A}}{}_{\bar{R}\bar{B}}\, g^{\bar{B}}{}_{\bar{P}\bar{Q}} && \\
&\quad + g^{\dot{A}}{}_{\bar{P}\dot{B}}\, g^{\dot{B}}{}_{\bar{Q}\bar{R}} + g^{\dot{A}}{}_{\bar{Q}\dot{B}}\, g^{\dot{B}}{}_{\bar{R}\bar{P}} + g^{\dot{A}}{}_{\bar{R}\dot{B}}\, g^{\dot{B}}{}_{\bar{P}\bar{Q}} = 0 && \text{(c)}
\end{aligned}
\qquad (2.89)
$$

Comparing eq.2.89(a)with the Jacobi identity for the structure constants of the group   we conclude that

$$g^{\dot{R}}{}_{\dot{P}\bar{Q}} = g^{\dot{R}}{}_{\dot{P}\dot{Q}} K^{\dot{Q}}{}_{\bar{Q}} \quad (K\text{'s: arbitrary constants}) \tag{2.90}$$

Making then a reparametrization (2.9) satisfying eq.(2.33) and such that

$$(\lambda^{-1})^{\dot{Q}}{}_{\bar{R}} = -K^{\dot{Q}}{}_{\bar{Q}}(\lambda^{-1})^{\bar{Q}}{}_{\bar{R}} \quad (\text{or } \lambda^{\dot{Q}}{}_{\bar{R}} = \lambda^{\dot{Q}}{}_{\dot{P}} K^{\dot{P}}{}_{\bar{R}}) \tag{2.9I}$$

we can set

$$\tilde{g}^{\dot{R}}{}_{\dot{P}\bar{Q}} = 0 \tag{2.92}$$

Further reparametrizations are now limited by the condition

$$\lambda^{\bar{A}}{}_{\dot{B}} = \lambda^{\dot{A}}{}_{\bar{B}} = 0 \tag{2.93}$$

and therefore they do not mix the $\bar{W}$'s with the $\dot{W}$'s .

With eq.(2.92) taken into account eq.2.89(b)reduces to

$$g^{\dot{R}}{}_{\bar{P}\bar{Q}} = 0 \tag{2.94}$$

and with eqs.(2.92) and(2.94) satisfied eq2.89(c)is an identity.

From eqs.(2.94) and(2.92) it follows that the generators $Y_{\bar{R}}$ also close an algebra (that of the group $\mathcal{J}(d\bar{W}(3\lambda))$)

$$[Y_{\bar{P}}, Y_{\bar{Q}}] = -g^{\bar{R}}{}_{\bar{P}\bar{Q}} Y_{\bar{R}} \tag{2.95}$$

and

$$[Y_{\bar{P}}, Y_{\dot{Q}}] = 0 \tag{2.96}$$

We realize from eqs.(2.96), (2.95) and(2.88) that the group $\mathcal{J}(dW(3\lambda))$ is the direct product of the groups $\mathcal{J}(d\bar{W}(3\lambda))$ and $\mathcal{J}(d\dot{W}(3\lambda))$.

From eqs.2.29(b),2.85(b)2.92and(2.94)as well as eqs.2.34(a) and (2.86)we conclude that the group $\mathcal{J}(d\bar{W}(3\lambda))$ is simply the adjoint of the group $\mathcal{J}(\varphi(3\lambda))$.

We now go back to eqs2.20(c) and2.20(d). Since the space $\mathcal{K}$ is a n-dimensional Riemannian space, we can define its affine connection in the usual manner (torsion free)

$$C_a^{bc} = (G^{-1})_{ad}\, C^{bc,d}$$
$$C^{bc,d} = \frac{1}{2}\left(\frac{\partial G^{bd}}{\partial \varphi_c} + \frac{\partial G^{cd}}{\partial \varphi_b} - \frac{\partial G^{bc}}{\partial \varphi_d} \right) \qquad (2.97)$$

We can then write eq2.20(c)in a reparametrization covariant manner

$$J_{\bar{A}}^{\;a|b} + J_{\bar{A}}^{\;b|a} = 0 \qquad (2.98)$$

where $J^a_{\bar{A}}$ is the covariant vector corresponding to $J_{\bar{A}a}$

$$J^a_{\bar{A}} = G^{ab} J_{\bar{A}b} \qquad (2.99)$$

and "$|$" denotes the covariant derivative with respect to the connection

$$J_{\bar{A}}^{\;a|b} = \partial J^a_{\bar{A}} / \partial \varphi_b - C_c^{ab} J^c_{\bar{A}} \qquad (2.100)$$

Equation(2.98) is Killing's equation for the Riemannian space $\mathcal{K}$ .We conclude that $J_{\bar{A}a}$ are the components of a Killing vector in $\mathcal{K}$ and $X_{\bar{A}}$ generates an isometry of the space $\mathcal{K}$. Since the $X_{\bar{A}}$'s are the generators of the group $\mathcal{J}(\varphi(\beta\lambda))$ it follows that $\mathcal{J}(\varphi(\beta\lambda))$ is either the complete group of motions of the space $\mathcal{K}$ or one of its subgroups.

Conversly, the space $\mathcal{K}$ must be a n-dimensional Riemannian space admitting a M-parameter group of motions.

We now turn to eq. 2.20(d)

$$X_R f_{AB} + f_{AQ} g^Q{}_{BR} + f_{BQ} g^Q{}_{AR} = 0 \qquad (2.101)$$

This equation represents a system of differential equations generated by the X_A's for the metric coefficients f_{AB}. The integrability of this system is examined in appendix II .From that analysis it follows that eq(2.101)possesses acceptable solutions f_{AB} for any group of motions of a Riemannian space be it compact or not.

The remaining conditions on the Lagrangian given by eq.(2.77) which have yet to be discussed are those imposed on the functions h and e by eqs.(2.76) and(2.84) respectively

$$X_{\bar{A}}\, h = 0 \qquad X_{\bar{A}}\, e = 0 \qquad (2.102)$$

These equations denote the invariance of the functions h and e under the action of any element of the group $\mathcal{J}(\varphi(3x))$. Hence these functions must be constants along the orbits of the group $\mathcal{J}(\varphi(3x))$ on $\mathcal{K}$.

CHAPTER IV

APPLICATIONS

In this chapter we shall use the results obtained previously in order to get more specific information about the Lagrangians of our theory.

We shall only deal with the case that the unphysical W^A's are absent. For this case we study the removal of massless scalar fields and the corresponding acquisition of mass by the vector fields. This will be done in the first two sections. In the last sections we give all gauge invariant Lagrangians for the cases when the space of scalar fields is of dimension one, two and three. This is done for each dimension by first giving the group of motions of the φ-space then the corresponding metric of this space and finally the solutions of eqs. (2.101) for the f_{AB}'s.

A main distinction that is done in this chapter is between transitive and intransitive groups of motions. One can think of the transitive groups of motions as more physically acceptable for reasons like the exclusion of Jordan,[14] Brans-Dicke-type[15] of theories and the fact that Higgs bosons which are present in the case of intransitive groups have so far failed experimental verification.

1. Removal of massless scalar fields (transitive group)

In this case it follows from Eq. (2.84) that e is a constant, absorbed in the function h of $\mathcal{L}_G$ which is also a constant because of Eq. (2.76). It also turns out that the solutions for G^{ab} and f_{AB} of Eqs. (2.98) and (2.101) respectively contain only arbitrary constants and not functions of integration.

It is evident then that all the scalar fields are massless. Since for a transitive group,

$$\mathrm{rank}\,(\mathcal{J}_{Aa}) = n \tag{3.1}$$

everywhere on $\mathcal{K}$

we can use n of the generators of $\mathcal{J}(\varphi({}^3x))$ to move from any point on $\mathcal{K}$ to some fixed point ${}^n\varphi_1$ – the same for all space points ${}^3x$. After this fixation, the matter Lagrangian assumes the form :

$$\mathcal{L}_M = -(\mathcal{J}_{Aa}\mathcal{J}_B{}^a)_1 W^{A\mu} W^B_\mu - \tfrac{1}{2}(f_{AB})_1 F^A_{\mu\nu} F^{B\mu\nu} \tag{3.2}$$

Thus, all the scalar fields are removed.

The constant matrix $(f_{AB})_1$, which now appears in front of the kinetic part can be transformed into the unit matrix by a reparametrization (2.7(b)). Then it is evident that the matrix

$$\mu^2_{AB} = (\mathcal{J}_{Aa}\mathcal{J}_B{}^a)_1 \tag{3.3}$$

will be the mass-matrix of the vector fields. It can be diagnonalized by a further reparametrization using orthogonal matrices $\lambda^A{}_B$ which will leave unaltered the kinetic part.

If the group $\mathcal{J}(\varphi(3_x))$ is simply transitive ($n = N$) there is no remaining gauge symmetry after the fixation. If, on the other hand, it is multiply transitive ($N \geq n$) the reduced gauge symmetry is that of the $N - n$ parameter stability subgroup of the point of fixation ${}^n\varphi_1$.

2. Removal of massless scalar fields (intransitive group)

In this case the action of the group is transitive on the minimal invariant varieties which it defines on $\mathcal{K}$. Thus, if we redefine our coordinates in $\mathcal{K}$

$$
\begin{aligned}
\psi_{\bar{a}} &= \psi_{\bar{a}}(\varphi_1, \ldots, \varphi_n) \quad : \bar{a} = 1, \ldots, m \\
\theta_{\dot{a}} &= \theta_{\dot{a}}(\varphi_1, \ldots, \varphi_n) \quad : \dot{a} = 1, \ldots, n-m
\end{aligned}
\tag{3.4}
$$

such that the minimal invariant varieties are the m-dimensional subspace of $\mathcal{K}$ defined by the eqns :

$$d\theta_{\dot{a}} = 0 \tag{3.5}$$

then in this system of coordinates,

$$\mathcal{J}_{A\dot{a}} = 0 \tag{3.6}$$

$$X_A = \mathcal{J}_{A\bar{a}} \frac{\partial}{\partial \psi_{\bar{a}}} \tag{3.7}$$

The $\psi_{\bar a}$'s are the coordinates on the minimal invariant varieties and the vectors $\partial/\partial\theta_{\dot a}$ take us from one minimal invariant variety to another.

It follows then from Eq.(2.84) (2.76) that e, h are independent of the $\psi_{\bar a}$'s:

$$e = e(\theta_1, .., \theta_{n-m}) \qquad h = h(\theta_1, .., \theta_{n-m}) \tag{3.8}$$

and also the arbitrary functions of integration contained in f_{AB} and G^{ab} can depend on the θ's only.

Hence, the θ fields may have masses, but the ψ fields are necessarily massless. These fields can however be removed by a fixation of the point on the minimal invariant varieties, the argument of Case I applying here too, since on these varieties the group $\mathcal{J}(\varphi({}^3x))$ has transitive action. Let the point of fixation be ${}^m\psi_1$. The matter Lagrangian assumes then the form :

$$\begin{aligned} \mathcal{L}_M = -e - (G^{\dot a \dot b})_1\, \theta_{\dot a,\mu}\, \theta_{\dot b}{}^{,\mu} + 2(G^{\bar a \dot b}\, J_{A\bar a})_1\, \theta_{\dot b,\mu} W^{A\mu} \\ -(J_{A\bar a} J_B{}^{\bar a})_1 W^{A\mu} W^B{}_\mu - \tfrac{1}{2}(f_{AB})_1\, F^A_{\mu\nu} F^{B\mu\nu} \end{aligned} \tag{3.9}$$

All the massless scalar fields have been removed.

The coefficient of the quadratic term in the vector fields is now a function of the θ's . The mass matrix of the vector fields is now given by :

$$\mu^2_{AB} = \left[(\mathcal{Y}_{A\bar{a}} J_B{}^{\bar{a}})_1 \right]_{min} \tag{3.10}$$

where min denotes the value at a minimum of the "potential" e : Stable constant classical solution for the $\theta_{\dot{a}}$'s (with $W^A_\mu = 0$) - their vacuum expectation value in the quantum theory. Equation gives actually the mass matrix provided that

$$\left[(f_{AB})_1 \right]_{min} , \tag{3.11}$$

has been transformed beforehand to the unit matrix by a reparametrization (2..7(b)). If an analogous reparametrization is used to set also

$$\left[(G^{\dot{a}\dot{b}})_1 \right]_{min} = \delta^{\dot{a}\dot{b}} \tag{3.12}$$

then the mass matrix of the remaining scalar fields $\theta_{\dot{a}}$ is :

$$(m^2)^{\dot{a}\dot{b}} = \frac{1}{2} \left[\frac{\partial^2 e}{\partial\theta_{\dot{a}} \partial\theta_{\dot{b}}} \right]_{min} \tag{3.13}$$

In the case that the potential function e has many minima there are correspondingly many mass matrices defined.

The comments made at the end of the treatment of the transitive case concerning reduced gauge symmetry, apply also here with respect to minimal invariant varieties (with n replaced by m).

3. One-dimensional space of scalar fields

A one-dimensional space can admit only a one-parameter group of motions whose killing vector can always be brought to the form

$$X = (1)$$

The metric of the space $\mathcal{K}$ is given by

$$dL^2[\varphi(^3x)] = d\varphi^2$$

and $f_{AB} = f_{II} =$ is a positive constant.

After fixing the value of φ at $\varphi = 0$ we obtain the Lagrangian

$$\mathcal{L} = KR - \lambda - W^{\mu} W_{\mu} - \frac{1}{2} c f^{\mu\nu} f_{\mu\nu}$$

describing a massive spin-one field in G.R.

4. Two dimensional space of scalar fields.

A two-dimensional space can admit one, two, and three parameter groups of motions. There are two two-parameter groups, the abelian, (corresponding space flat) and the non-abelian (corresponding space of constant negative curvature) both of which are transitive. The three-parameter groups are of course the complete groups of motions of a two-dimensional space of constant curvature.

In the following use will be made of a represantation of the killing vectors and a coordinate system of the $\mathcal{K}$-space such as to make our study of the three-dimensional $\mathcal{K}$-spaces admitting an intransitive group of motions easier.

4.1 One-parameter group

The killing vector can be taken in the form

$$X = (0,1)$$

and the metric is given by

$$dL^2[\varphi(^3x)] = d\theta^2 + b(\theta)\,d\psi^2 \qquad b(\theta) > 0$$

$f_{AB}=f_{II}=f$ is now a positive function of θ .
Using the generator X we set ψ =0 and our Lagrangian becomes

$$\mathcal{L} = h(\theta)\,R - e(\theta) - \theta_{,\mu}\theta_{,}^{\mu} - b(\theta)W^{\mu}W_{\mu} - \tfrac{1}{2}c(\theta)f^{\mu\nu}f_{\mu\nu}$$

describing a spin-0 field which acts as a scalar gravitational field in interaction with a massive spin-I field.

4.2 Two-parameter groups

a) Abelian

The killing vectors commute

$$[X_1, X_2] = 0 \qquad\qquad X_1 = (1,0) \quad X_2 = (0,1)$$

The metric of $\mathcal{K}$ is

$$dL^2 = b^{11}\,d\varphi_1^2 + 2b^{12}\,d\varphi_1\,d\varphi_2 + b^{22}\,d\varphi_2^2$$

where the constant matrix (b) must be chosen positive definite. The gauge group is simply the group of translations of a flat 2-dimensional space.

f_{AB} is a positive definite constant matrix:

$f_{11} = c_{11} \qquad f_{12} = c_{12} \qquad f_{22} = c_{22}$

Using X_1, X_2, we may fix $\varphi_1 = \varphi_2 = 0$ and our Lagrangian assumes the form

$$\mathcal{L} = KR - \lambda - b^{AB} W^{A\mu} W_\mu^B - c_{AB} f^{A\mu\nu} f_{\mu\nu}^B \qquad (A,B = 1,2)$$

which describes two uncoupled fields of spin-1 in G.R.

b) Non abelian

The commutator of X_1, X_2 can be brought to the form

$$[X_1, X_2] = X_2 \qquad\qquad X_1 = (1,0) \quad X_2 = (0, e^{\varphi_1})$$

and the metric is

$$dL^2[\varphi(^3x)] = (b^{11} - 2b^{12}\varphi_2 + b^{22}\varphi_2^2)\,d\varphi_1^2 + 2(b^{12} - b^{22}\varphi_2)\,d\varphi_1\,d\varphi_2 + b^{22}\,d\varphi_2^2$$

The matrix (b) must be chosen again positive definite.

Integrating eqs.(2.101) we obtain for f_{AB} the following

$$f_{11} = c_{11} - 2c_{12}\varphi_2 + c_{22}\varphi_2^2$$

$$f_{12} = (c_{12} - c_{22}\varphi_2)\,e^{\varphi_1}$$

$$f_{22} = c_{22}\,e^{2\varphi_1}$$

where the constant matrix (C) should be taken to be positive definite. After fixing $\varphi_1 = \varphi_2 = 0$ we obtain the Lagrangian

$$\mathcal{L} = KR - \lambda - b^{AB} W^{A\mu} W^{B}_{\mu} - \frac{1}{2} C_{AB} F^{A\mu\nu} F^{B}_{\mu\nu} \qquad (A,B=1,2)$$

where: $F^{1}_{\mu\nu} = f^{1}_{\mu\nu}$, $\quad F^{2}_{\mu\nu} = f^{2}_{\mu\nu} - (W^{1}_{\mu} W^{2}_{\nu} - W^{1}_{\nu} W^{2}_{\mu})$

Evidently this Lagrangian describes the interaction of two massive spin-I fields.

4.3 Three-parameter groups

a) Euclidean

This is the well known complete group of motions of a two dimensional Euclidean space $E_2 = SO(2) \otimes_s T(2)$

The killing vectors are

$$X_1 = (1,0) \qquad X_2 = (0,1) \qquad X_3 = (-\phi_2, \phi_1)$$

and they obey the following commutation relations:

$$[X_1, X_2] = 0 \qquad [X_3, X_1] = -X_2 \qquad [X_2, X_3] = -X_1$$

The metric is of course flat

$$dL^2 [\varphi(^3x)] = b(d\varphi_1^2 + d\varphi_2^2)$$

where b is a positive constant. Eqs.(2.101) give

$$f_{11} = f_{22} = \alpha_2 \qquad f_{33} = \alpha_1 + \alpha_2 (\varphi_1^2 + \varphi_2^2)$$

$$f_{12} = 0 \qquad f_{13} = -\alpha_2 \varphi_2 \qquad f_{23} = \alpha_2 \varphi_1$$

and to ensure positive definiteness we must take $\alpha_1, \alpha_2 > 0$.

Using the translations we may fix $\varphi_1 = \varphi_2 = 0$.

Defining then

$$W_\mu = W^1_\mu + i\,W^2_\mu \quad , \quad A_\mu = W^3_\mu$$

the Lagrangian after the fixation becomes:

$$\mathcal{L} = KR - \lambda - bW^\mu W^*_\mu - \frac{\alpha_2}{2} F^{\mu\nu} F^*_{\mu\nu} - \frac{\alpha_1}{2} H^{\mu\nu} H_{\mu\nu}$$

$$F_{\mu\nu} = W_{\mu,\nu} - W_{\nu,\mu} - i(W_\mu A_\nu - W_\nu A_\mu) \qquad H_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$$

This Lagrangian describes the electromagnetic interaction of a spin-I field.

b) Spherical

This is the complete group of motions of a two dimensional sphere $(SO(3))$ with the following killing vectors

$$X_1 = (\cos\varphi_2, -\cot\varphi_1 \sin\varphi_2) \;,\; X_2 = (\sin\varphi_2, \cot\varphi_1 \cos\varphi_2)$$
$$X_3 = (0,1)$$

The commutation relations are

$$[X_1, X_2] = -X_3 \qquad [X_2, X_3] = -X_1 \qquad [X_3, X_1] = -X_2$$

and the metric is given by

$$dL^2[\varphi(^3x)] = b\,(d\varphi_1^2 + \sin^2\varphi_1\, d\varphi_2^2)$$

where b is a positive constant. Integrating eqs.(2.101) we obtain

$$f_{11} = \alpha_1 - \alpha_2 \sin^2\varphi_1 \sin^2\varphi_2 \qquad f_{12} = \alpha_2 \sin^2\varphi_1 \sin\varphi_2 \cos\varphi_2$$
$$f_{22} = \alpha_1 - \alpha_2 \sin^2\varphi_1 \cos^2\varphi_2 \qquad f_{13} = -\alpha_2 \sin\varphi_1 \cos\varphi_1 \sin\varphi_2$$
$$f_{33} = \alpha_1 - \alpha_2 \cos^2\varphi_1 \qquad f_{23} = \alpha_2 \sin\varphi_1 \cos\varphi_1 \cos\varphi_2$$

The requirement of positive definiteness is satisfied if $\alpha_1 > 0, \alpha_2 > \alpha_1$

If we now fix $\varphi_1 = \frac{\pi}{2}$ $\quad \varphi_2 = 0$ and define

$$W_\mu = W^1_\mu + i\,W^3_\mu \qquad A_\mu = W^2_\mu$$

the Lagrangian becomes

$$\mathcal{L} = KR - \lambda - bW^{\mu}W^{*}_{\mu} - \tfrac{1}{2}\alpha_1 F^{\mu\nu}F^{*}_{\mu\nu} - \tfrac{1}{2}(\alpha_1 - \alpha_2)H^{\mu\nu}H_{\mu\nu}$$

where

$$F_{\mu\nu} = W_{\mu,\nu} - W_{\nu,\mu} - i\,(W_\mu A_\nu - W_\nu A_\mu)$$

$$H_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} + \tfrac{i}{2}\,(W_\mu W_\nu^{*} - W_\nu W_\mu^{*})$$

It describes the interaction between a massive complex spin-I field a massless spin-I field

c) Hyperbolic

The group is $SO(2,1)$ with killing vectors

$$X_1 = (\cos\varphi_2, -\operatorname{cth}\varphi_1 \sin\varphi_2)$$

$$X_2 = (\sin\varphi_2, \operatorname{cth}\varphi_1 \cos\varphi_2) \qquad X_3 = (0, 1)$$

which obey the following commutation relations

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = -X_1, \quad [X_3, X_1] = -X_2$$

Themetric is

$$dL^2[\varphi(^3x)] = b(d\varphi_1^2 + \operatorname{sh}^2\varphi_1\, d\varphi_2^2) \quad (b = \text{const.} > 0)$$

Eqs. (2.101) give

$$f_{11} = \alpha_1 + \alpha_2 \operatorname{sh}^2\varphi_1 \sin^2\varphi_2 \qquad f_{12} = -\alpha_2 \operatorname{sh}^2\varphi_1 \sin\varphi_2 \cos\varphi_2$$

$$f_{22} = \alpha_1 + \alpha_2 \operatorname{sh}^2\varphi_1 \cos^2\varphi_2 \qquad f_{13} = -\alpha_2 \operatorname{sh}\varphi_1 \operatorname{ch}\varphi_1 \sin\varphi_2$$

$$f_{33} = \alpha_2 \cosh^2\varphi_1 - \alpha_1 \qquad f_{23} = -\alpha_2 \operatorname{sh}^2\varphi_1 \sin\varphi_2 \cos\varphi_2$$

and this is positive definite for $\alpha_1 > 0$, $\alpha_2 > \alpha_1$.

We now fix $\varphi_1 = \varphi_2 = 0$ and define

$$W_\mu = W_\mu^1 + i\,W_\mu^2 \qquad A_\mu = W_\mu^3$$

after which our Lagrangian becomes

$$\mathcal{L} = KR - \lambda - bW^{\mu}W_{\mu}^{*} - \frac{1}{2}\alpha_1 F^{\mu\nu}F_{\mu\nu}^{*} - \frac{1}{2}(\alpha_2 - \alpha_1)H^{\mu\nu}H_{\mu\nu}$$

$$F_{\mu\nu} = W_{\mu,\nu} - W_{\nu,\mu} - i(W_{\mu}A_{\nu} - W_{\nu}A_{\mu})$$

$$H_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} - \frac{i}{2}(W_{\mu}W_{\nu}^{*} - W_{\nu}W_{\mu}^{*})$$

5. Three-dimensional space of scalar fields.

A three-dimensional space can admit I,2,3,4 and 6-parameter groups of motions. In the following we shall destinguish between the transitive and intransitive 3-parameter groups. Also since a 4-parameter group contains necessarily a 3-parameter one we shall examine separately those containing a 3-parameter transitive from those containing an intransitive one.

5.1 One-parameter group

The killing vector is

$$X = (0, 0, 1)$$

and the metric

$$dL^2[\varphi({}^3x)] = b^{\alpha b}\, d\varphi_\alpha\, d\varphi_b$$

where $b=b(\varphi_1, \varphi_2)$ is a positive definite matrix.

f_{AB}= f, h, and e will now be functions of φ_1, φ_2 and f must be chosen positive.

If we fix $\varphi_3=0$ we get the Lagrangian

$$\mathcal{L} = h(\varphi_1, \varphi_2)R - e(\varphi_1, \varphi_2) - b^{33}W_{\mu\nu}{}^{\mu} - b^{11}\varphi_{1,\mu}\varphi_{1,}{}^{\mu} -$$

$$b^{22}\varphi_{2,\mu}\varphi_{2,}{}^{\mu}+2b^{13}W_{\mu}\varphi_{1,}{}^{\mu}+2b^{23}W_{\mu}\varphi_{2,}{}^{\mu}-2b^{12}\varphi_{1,\mu}\varphi_{2,}{}^{\mu}-\tfrac{1}{2}c(\varphi_1,\varphi_2)f_{\mu\nu}f^{\mu\nu}$$

5.2 Two-parameter groups

Since two generators cannot have the same paths, the minimum invariant varieties of a two parameter group are surfaces geodetically parallel and of constant curvature (negative or zero). Taking these surfaces for θ = constant and their orthogonal trajectories along the lines (θ) with parameter θ their arc-length, the metric will assume the geodetic form

$$dL^2\left[\varphi({}^3x)\right]=d\theta^2+b_{11}\,d\varphi_1^2+2b_{12}\,d\varphi_1\,d\varphi_2+b_{22}\,d\varphi_2^2$$

where the matrix (b) depends on $\theta,\varphi_1,\varphi_2$ and is positive definite.

With such a choice of coordinates the action of the group reduces to its transitive action over the two dimensional hypersurfaces defined here for θ = constant. The groups are of course the same as in the two-dimensional case. The metric on the surfaces θ = constant has obviously the same φ_1,φ_2 dependence as before. However there is an extra θ-dependence, that is the matrix (b) will now be θ-dependent. Since the eqs. (2.101) for f_{AB} are the same as regards the φ_1,φ_2 variables we get the same solutions as previously where the constants C_{AB} will now be arbitrary functions of θ subject to the positive definiteness requirement. h and e will now evidently be functions of θ.

a) Abelian

Metric:

$$dL^2\left[\varphi({}^3x)\right] = d\theta^2 + b^{11}\,d\varphi_1^2 + 2b^{12}\,d\varphi_1\,d\varphi_2 + b^{22}\,d\varphi_2^2$$

where (b) is a positive definite Θ-dependent matrix.

f_{AB} is now a positive definite Θ-dependent matrix

$$f_{11} = C_{11}(\theta) \qquad f_{22} = C_{22}(\theta) \qquad f_{12} = C_{12}(\theta)$$

After making the fixation $\varphi_1 = \varphi_2 = 0$ we obtain the Lagrangian

$$\mathcal{L} = h(\theta)R - e(\theta) - \theta_{,\mu}\,\theta_{,}^{\mu} - b^{AB}(\theta)\,W^{A\mu}W_{\mu}^{B} - C_{AB}(\theta)\,F^{A\mu\nu}F_{\mu\nu}^{B} \quad (A,B=1,2)$$

describing the interaction of a spin-0 field with two massive spin I fields.

b) Non abelian

The metric now is

$$dL^2[\varphi({}^3x)] = d\theta^2 + (b^{11} - 2b^{12}\varphi_2 + b^{22}\varphi_2^2)\,d\varphi_1^2 + 2(b^{12} - b^{22}\varphi_2)\,d\varphi_1\,d\varphi_2 + b_{22}\,d\varphi_2^2$$

where $b=b(\Theta)$ and positive definite.

For f_{AB} we have

$$f_{11} = C_{11} - 2C_{12}\varphi_2 + C_{22}\varphi_2^2 \qquad f_{12} = (C_{12} - C_{22}\varphi_2)e^{\varphi_1} \qquad f_{22} = C_{22}\,e^{2\varphi_1}$$

and at $\varphi_1 = \varphi = 0$ the lagrangian becomes

$$\mathcal{L} = h(\theta)R - e(\theta) - b^{AB}(\theta)\,W^{A\mu}W_{\mu}^{B} - C_{AB}(\theta)\,F_{\mu\nu}^{A}F^{B\mu\nu}$$

where

$$F_{\mu\nu}^{1} = f_{\mu\nu}^{1}, \qquad F_{\mu\nu}^{2} = f_{\mu\nu}^{2} - (W_{\mu}^{1}W_{\nu}^{2} - W_{\nu}^{1}W_{\mu}^{2})$$

5.3 Three-parameter intransitive groups

The minimum invariant varieties are again surfaces so we shall also here use the same coordinate system as previously. This brings us to exactly the same situation as before the only change being that the similarity is now with the two-dimensional scalar spaces admitting a three-parameter group.

a) Euclidean

The metric is now given by

$$dL^2[\varphi(^3x)] = d\theta^2 + b(\theta)\,(d\varphi_1^2 + d\varphi_2^2) \qquad b(\theta) > 0$$

For f_{AB} we obtain

$$f_{11} = f_{22} = \alpha_2(\theta) \qquad f_{33} = \alpha_1(\theta) + \alpha_2(\theta)\,(\varphi_1^2 + \varphi_2^2)$$

$$f_{12} = 0 \qquad f_{13} = -\alpha_2(\theta)\,\varphi_2 \qquad f_{23} = \alpha_2(\theta)\,\varphi_1$$

where $\alpha_1(\theta),\ \alpha_2(\theta) > 0$.

After fixing $\varphi_1 = \varphi_2 = 0$ and defining

$$W_\mu = W_\mu^1 + i\,W_\mu^2 \qquad A_\mu = W_\mu^3$$

we get

$$\mathcal{L} = h(\theta)R - e(\theta) - \theta_{,\mu}\theta_{,}^{\mu} - b(\theta)W_\mu W^{*\mu} - \frac{\alpha_2(\theta)}{2}F^{\mu\nu}F^{*}_{\mu\nu} - \frac{\alpha_1(\theta)}{2}H^{\mu\nu}H_{\mu\nu}$$

where $F^{\mu\nu}$, $H^{\mu\nu}$ are the same as in case 4.3 a)

b) Spherical

The metric is

$$dL^2[\varphi(^3x)] = d\theta^2 + b(\theta)\,(d\varphi_1^2 + \sin^2\varphi_1\, d\varphi_2^2) \qquad b(\theta) > 0$$

and f_{AB} has exactly the same form as in case 4.3 b) where α_1, α_2 are now functions of θ and $\alpha_1(\theta) > 0$ $\alpha_2(\theta) < \alpha_1(\theta)$.

Fixing $\varphi_1 = \frac{\pi}{2}$ $\varphi_2 = 0$ and defining as before,

$$W_\mu = W_\mu^1 + i\, W_\mu^3 \qquad A_\mu = W_\mu^2$$

we obtain the Lagrangian

$$\mathcal{L} = h(\theta) R - e(\theta) - \theta_{,\mu}\theta_{,}^{\mu} - b(\theta) W^\mu W_\mu^* - \frac{\alpha_1(\theta)}{2} F^{\mu\nu} F_{\mu\nu}^* - \frac{(\alpha_2(\theta) - \alpha_1(\theta))}{2} H^{\mu\nu} H$$

where $F^{\mu\nu}$, $H^{\mu\nu}$ are the same as in 4.3 b)

c) Hyperbolic

$$dL^2[\varphi(^3x)] = d\theta^2 + b(\theta)\left(d\varphi_1^2 + sh^2\varphi_1\, d\varphi_2^2\right) \qquad b(\theta) > 0$$

and f_{AB} is the one given in 4.3 c) with α_1, α_2 now functions of θ and $\alpha_1(\theta) > 0$ $\alpha_2(\theta) > \alpha_1(\theta)$.

Fixing $\varphi_1 = \varphi_2 = 0$ and defining

$$W_\mu = W_\mu^1 + i\, W_\mu^2 \qquad A_\mu = W_\mu^3$$

we obtain

$$\mathcal{L} = h(\theta) R - e(\theta) - \theta_{,\mu}\theta_{,}^{\mu} - b(\theta) W^\mu W_\mu^* - \frac{\alpha_1(\theta)}{2} F^{\mu\nu} F_{\mu\nu}^* - \frac{(\alpha_2(\theta) - \alpha_1(\theta))}{2} H^{\mu\nu} H_{\mu\nu}$$

where $F^{\mu\nu}$, $H^{\mu\nu}$ are the same as in 4.3 c).

5.4 Three-parameter transitive groups

These are the nine groups that have been labeled by the latin numbers I, II, ... IX by Bianchi[16] who has also obtained the corresponding three dimensional spaces which admit them as groups of motions. The type VIII and IX groups are simple whereas all the others are integrable (non-simple). In the following we shall make extensive use of Bianchi's work in the determination of the Lagrangians. The fixation of $\varphi_1, \varphi_2, \varphi_3$ will always be done at the origin $\varphi_1 = \varphi_2 = \varphi_3 = 0$.

The solution of eqs.(2.101) will now depend on six arbitrary constants C_{AB}. This matrix (C) must always be taken positive definite in the following so that the positive definiteness of f_{AB} is ensured.

a) Type I

This is the abelian group in three dimensions

$$X_1 = (1,0,0)\ ,\quad X_2 = (0,1,0)\ ,\quad X_3 = (0,0,1) \qquad [X_A, X_B] = 0\ .$$

The space is of course flat

$$dL^2[\varphi(^3x)] = d\varphi_1^2 + d\varphi_2^2 + d\varphi_3^2$$

and f_{AB} is a constant matrix.

Using the generators of the group we move to the origin and our Lagrangian assumes the form

$$\mathcal{L} = KR - \lambda - W^1_\mu W^{1\mu} - W^2_\mu W^{2\mu} - W^3_\mu W^{3\mu} - C_{AB} f^A_{\mu\nu} f^{B\mu\nu}$$

describing three uncoupled massive spin-I fields .

b) Type II

This group has the structure

$$(X_1, X_2) = (X_1, X_3) = 0 \qquad (X_2, X_3) = X_1$$

$$X_1 = (0,1,0)\ , \quad X_2 = (0,0,1)\ , \quad X_3 = (-1, \varphi_3, 0)$$

The metric is

$$dL^2[\varphi(^3x)] = d\varphi_1^2 + d\varphi_2^2 + 2\varphi_1 d\varphi_2 d\varphi_3 + (1+\varphi_1^2)\, d\varphi_3^2$$

Solving eqs.(2.101) for f_{AB} we obtain

$$f_{11} = C_{11}\ , \quad f_{22} = C_{22} + 2C_{12}\varphi_1 + C_{11}\varphi_1^2\ , \quad f_{33} = C_{33} + 2C_{13}\varphi_3 + C_{11}\varphi_3^2$$

$$f_{12} = C_{12} + C_{11}\varphi_1\ , \quad f_{13} = C_{13} + C_{11}\varphi_3\ , \quad f_{23} = C_{23} + C_{11}\varphi_1\varphi_3 + C_{12}\varphi_3 + C_{13}\varphi_1$$

where C_{AB} as has already been said must be chosen positive definite.

After the fixation we obtain

$$\mathcal{L} = KR - \lambda - W^1_\mu W^{1\mu} - W^2_\mu W^{2\mu} - W^3_\mu W^{3\mu} - C_{AB}\, f^A_{\mu\nu} f^{B\mu\nu}$$

where

$$F^1_{\mu\nu} = f^1_{\mu\nu} - (W^2_\mu W^3_\nu - W^2_\nu W^3_\mu)\,.$$

c) Type III

The killing vectors are

$$X_1 = (0,1,0)\ , \quad X_2 = (0,0,1)\ , \quad X_3 = (1, -\varphi_2, 0)$$

and they obey the commutation relations

$$(X_1, X_2) = (X_2, X_3) = 0 \qquad (X_3, X_1) = X_1 .$$

The corresponding space has metric

$$dL^2[\varphi({}^3x)] = d\varphi_1^2 + e^{2\varphi_1} d\varphi_2^2 + 2n e^{\varphi_1} d\varphi_2 d\varphi_3 + d\varphi_3^2$$

where n is an essential constant (that is for different values of n the type of the space is different) and $n^2 < 1$.

From eqs. (2.101) we get

$$f_{11} = c_{11} e^{2\varphi_1} , \quad f_{22} = c_{22} , \quad f_{33} = c_{33} + c_{11} \varphi_2^2 e^{2\varphi_1} - 2 c_{13} \varphi_2 e^{\varphi_1}$$

$$f_{12} = c_{12} e^{\varphi_1} , \quad f_{13} = c_{13} e^{\varphi_1} - c_{11} \varphi_2 e^{2\varphi_1} , \quad f_{23} = c_{23} - c_{12} \varphi_2 e^{\varphi_1}$$

The Lagrangian after the fixation is

$$\mathcal{L} = KR - \lambda - W^1_\mu W^{1\mu} - W^2_\mu W^{2\mu} - W^3_\mu W^{3\mu} - 2n W^1_\mu W^{2\mu}$$

$$-\tfrac{1}{2} c_{11} F^1_{\mu\nu} F^{1\mu\nu} - \tfrac{1}{2} c_{22} f^2_{\mu\nu} f^{2\mu\nu} - \tfrac{1}{2} c_{33} f^3_{\mu\nu} f^{3\mu\nu} - c_{12} F^1_{\mu\nu} f^{2\mu\nu} - c_{13} F^1_{\mu\nu} f^{3\mu\nu} - c_{23} f^2_{\mu\nu} f^{3\mu\nu}$$

where

$$F^1_{\mu\nu} = f^1_{\mu\nu} - (W^3_\mu W^1_\nu - W^1_\mu W^3_\nu)$$

d) Type IV

$$(X_1, X_2) = 0 \qquad (X_1, X_3) = X_1 \qquad (X_2, X_3) = X_1 + X_2$$

$$X_1 = (0,2,0) , \quad X_2 = (0,0,1) , \quad X_3 = (-2, \varphi_2 + 2\varphi_3 , \varphi_3)$$

The metric of $\mathcal{K}$ is

$$dL^2[\varphi({}^3x)] = d\varphi_1^2 + e^{\varphi_1}\left(d\varphi_2^2 + 2\varphi_1 d\varphi_2 d\varphi_3 + (\varphi_1^2 + n^2) d\varphi_3^2\right)$$

where n is a constant (essential).

For f_{AB} we obtain

$$f_{11} = C_{11}\, e^{\varphi_1} \qquad f_{22} = \left(C_{22} + C_{12}\varphi_1 + \tfrac{1}{4} C_{11}\varphi_1^2\right) e^{\varphi_1}$$

$$f_{33} = C_{33} + \tfrac{1}{4} C_{11}\varphi_2^2 e^{\varphi_1} + \left(\tfrac{1}{2} C_{11}\varphi_1 + C_{11} + C_{12}\right)\varphi_2\varphi_3 e^{\varphi_1} + \left(C_{13}\varphi_2 + 2C_{13}\varphi_3 + C_{13}\varphi_1\varphi_3 + 2C_{23}\varphi_3\right) e^{\frac{1}{2}\varphi_1} + \left(C_{11}\varphi_1 + \tfrac{1}{4} C_{11}\varphi_1^2 + C_{12}\varphi_1 + C_{11} + C_{22}\right)\varphi_3^2 e^{\varphi_1}$$

$$f_{12} = \left(C_{12} + \tfrac{1}{2} C_{11}\varphi_1\right) e^{\varphi_1}, \quad f_{13} = C_{13} e^{\frac{1}{2}\varphi_1} + \left(\tfrac{1}{2} C_{11}\varphi_2 + C_{11}\varphi_3 + \tfrac{1}{2} C_{11}\varphi_1\varphi_3 + C_{12}\varphi_3\right) e^{\varphi_1}$$

$$f_{23} = \left(C_{23} + \tfrac{1}{2} C_{13}\varphi_1\right) e^{\frac{1}{2}\varphi_1} + \left(\tfrac{1}{4} C_{11}\varphi_1\varphi_2 + \tfrac{1}{2} C_{11}\varphi_1\varphi_3 + C_{12}\varphi_1\varphi_3 + \tfrac{1}{2} C_{12}\varphi_2 + C_{12}\varphi_3 + C_{22}\varphi_3 + \tfrac{1}{4} C_{11}\varphi_1^2\varphi_3\right) e^{\varphi_1}.$$

After moving to the origin we have

$$\mathcal{L} = KR - \lambda - 4 W^1_\mu W^{1\mu} - n^2 W^2_\mu W^{2\mu} - 4 W^3_\mu W^{3\mu} - \tfrac{1}{2} C_{11} F^1_{\mu\nu} F^{1\mu\nu} - \tfrac{1}{2} C_{22} F^2_{\mu\nu} F^{2\mu\nu} - \tfrac{1}{2} C_{33} f^3_{\mu\nu} f^{3\mu\nu}$$

where

$$F^1_{\mu\nu} = f^1_{\mu\nu} - \left(W^1_\mu W^3_\nu - W^3_\mu W^1_\nu\right) - \left(W^2_\mu W^3_\nu - W^3_\mu W^2_\nu\right)$$

$$F^2_{\mu\nu} = f^2_{\mu\nu} - \left(W^2_\mu W^3_\nu - W^3_\mu W^2_\nu\right)$$

e) Type **V**

$$(X_1, X_2) = 0, \quad (X_1, X_3) = X_1, \quad (X_2, X_3) = X_2$$

$$X_1 = (0,1,0), \quad X_2 = (0,0,1), \quad X_3 = \left(-\tfrac{1}{n}, \varphi_2, \varphi_3\right)$$

$$dL^2[\varphi(^3x)] = d\varphi_1^2 + e^{2n\varphi_1}\left(d\varphi_2^2 + d\varphi_3^2\right)$$

This is the hyperbolic three-dimensional space (constant negative curvature).

Eqs.(2.101)for f_{AB} give

$$f_{11}=C_{11}e^{2n\varphi_1},\quad f_{22}=C_{22}e^{2n\varphi_1},\quad f_{33}=C_{33}+(C_{11}\varphi_2^2+C_{22}\varphi_3^2+2C_{12}\varphi_2\varphi_3)e^{2n\varphi_1}+$$
$$2(C_{13}\varphi_2+C_{23}\varphi_3)e^{n\varphi_1},\quad f_{12}=C_{12}e^{2n\varphi_1},\quad f_{13}=C_{13}e^{n\varphi_1}+(C_{11}\varphi_2+C_{12}\varphi_3)e^{2n\varphi_1}$$
$$f_{23}=C_{23}e^{n\varphi_1}+(C_{12}\varphi_2+C_{22}\varphi_3)e^{2n\varphi_1}$$

At $\varphi_1=\varphi_2=\varphi_3=0$ we have

$$\mathcal{L}=KR-\lambda-W_\mu^1W^{1\mu}-W_\mu^2W^{2\mu}-\frac{1}{n^2}W_\mu^3W^{3\mu}-\frac{1}{2}C_{11}F_{\mu\nu}^1F^{1\mu\nu}-\frac{1}{2}C_{22}F_{\mu\nu}^2F^{2\mu\nu}-$$
$$\frac{1}{2}C_{33}f_{\mu\nu}^3f^{3\mu\nu}-C_{12}F_{\mu\nu}^1F^{2\mu\nu}-C_{13}F_{\mu\nu}^1f^{3\mu\nu}-C_{23}F_{\mu\nu}^2f^{3\mu\nu}$$

where

$$F_{\mu\nu}^1=f_{\mu\nu}^1-(W_\mu^1W_\nu^3-W_\mu^3W_\nu^1),\quad F_{\mu\nu}^2=f_{\mu\nu}^2-(W_\mu^2W_\nu^3-W_\mu^3W_\nu^2)$$

f) Type VI

$$(X_1,X_2)=0,\quad (X_1,X_3)=X_1,\quad (X_2,X_3)=\ell X_2$$

$$X_1=(0,1,0),\quad X_2=(0,0,1),\quad X_3=(-1,\varphi_2,\ell\varphi_3)$$

$$dL^2[\varphi(^3x)]=d\varphi_1^2+e^{2\varphi_1}d\varphi_2^2+2ne^{(\ell+1)\varphi_1}d\varphi_2\,d\varphi_3+e^{2\ell\varphi_1}d\varphi_3^2$$

where l, n, essential constants and $l\neq 0$, $0<n^2<1$

$$f_{11}=C_{11}e^{2\varphi_1}\qquad f_{22}=C_{22}e^{2\ell\varphi_1}$$
$$f_{33}=C_{33}+C_{11}\varphi_2^2e^{2\varphi_1}+\ell^2C_{22}\varphi_3^2e^{2\ell\varphi_1}+4\ell C_{12}\varphi_2\varphi_3e^{(\ell+1)\varphi_1}+2C_{13}\varphi_2e^{\varphi_1}+2\ell C_{23}\varphi_3e^{\ell\varphi_1}$$
$$f_{12}=C_{12}e^{(\ell+1)\varphi_1}\qquad f_{13}=C_{13}e^{\varphi_1}+C_{11}\varphi_2e^{2\varphi_1}+\ell C_{12}\varphi_3e^{(\ell+1)\varphi_1}$$
$$f_{23}=C_{23}e^{\ell\varphi_1}+C_{12}\varphi_2e^{(\ell+1)\varphi_1}+\ell C_{22}\varphi_3e^{2\ell\varphi_1}$$

The Lagrangian after the fixation is given by

$$\mathcal{L} = KR - \lambda - W^1_\mu W^{1\mu} - W^2_\mu W^{2\mu} - W^3_\mu W^{3\mu} - 2n W^1_\mu W^{2\mu} - \tfrac{1}{2} C_{11} F^1_{\mu\nu} F^{1\mu\nu} -$$
$$- \tfrac{1}{2} C_{22} F^2_{\mu\nu} F^{2\mu\nu} - f^3_{\mu\nu} f^{3\mu\nu} - C_{12} F^1_{\mu\nu} F^{2\mu\nu} - C_{13} F^1_{\mu\nu} f^{3\mu\nu} - C_{23} F^2_{\mu\nu} f^{3\mu\nu} .$$

$$F^1_{\mu\nu} = f^1_{\mu\nu} - (W^1_\mu W^3_\nu - W^3_\mu W^1_\nu) , \quad F^2_{\mu\nu} = f^2_{\mu\nu} - \ell (W^2_\mu W^3_\nu - W^3_\mu W^2_\nu) .$$

g) Type VII

$$(X_1, X_2) = 0 , \quad (X_1, X_3) = X_2 , \quad (X_2, X_3) = -X_1 + \ell X_2$$
$$X_1 = (0,1,0) , \quad X_2 = (0,0,1) , \quad X_3 = (1, -\varphi_3 , \varphi_2 + \ell \varphi_3)$$
$$dL^2[\varphi({}^3x)] = d\varphi_1^2 + e^{-\ell\varphi_1}\{(n + \cos\omega\varphi_1) d\varphi_2^2 + (\ell \cos\omega\varphi_1 + \omega \sin\omega\varphi_1 +$$
$$+ \ell n) d\varphi_2 d\varphi_3 + \left(\frac{\ell^2 - 2}{2} \cos\omega\varphi_1 + \frac{\ell\omega}{2} \sin\omega\varphi_1 + n\right) d\varphi_3^2 \}$$

where l , n , essential constants, $0 \le l^2 < 4$ $\omega = \sqrt{4 - l^2}$, $n^2 > 1$

Eqs.(2.101) for f_{AB} give

$$f_{11} = \left[\frac{2C_{11} - 2(\ell C_{12} - C_{22})}{b^2} + \frac{2(\ell C_{12} - C_{22}) + (2 - \ell^2) C_{11}}{b^2} \cos b\varphi_1 - \frac{2C_{12} - \ell C_{11}}{b} \sin b\varphi_1\right] e^{-\ell\varphi_1}$$
$$f_{22} = \left[\frac{2C_{22} - 2(\ell C_{12} - C_{11})}{b^2} + \frac{2(\ell C_{12} - C_{11}) + (2 - \ell^2) C_{22}}{b^2} \cos b\varphi_1 - \frac{\ell C_{22} - 2C_{12}}{b} \sin b\varphi_1\right] e^{-\ell\varphi_1}$$
$$f_{12} = \left[\frac{\ell(C_{11} + C_{22}) - \ell^2 C_{12}}{b^2} + \frac{4C_{12} - \ell(C_{11} + C_{22})}{b^2} \cos b\varphi_1 - \frac{C_{22} - C_{11}}{b} \sin b\varphi_1\right] e^{-\ell\varphi_1}$$
$$f_{33} = C_{33} + 2\rho(\varphi_2 + \ell\varphi_3) - 2\mu\varphi_3 + f_{22}\varphi_2^2 + 2(\ell f_{22} - f_{12})\varphi_2\varphi_3 + (\ell^2 f_{22} - 2\ell f_{12} + f_{11})\varphi_3^2$$
$$f_{13} = \mu + f_{12}\varphi_2 + (\ell f_{12} - f_{11})\varphi_3$$
$$f_{23} = \rho + f_{22}\varphi_2 + (\ell f_{22} - f_{12})\varphi_3$$

where

$$\mu=\left[C_{13}\cos\frac{b}{2}\varphi_1-\frac{(2C_{23}-\ell C_{13})}{b}\sin\frac{b}{2}\varphi_1\right]e^{-\frac{\ell}{2}\varphi_1}$$

$$\rho=\left[C_{23}\cos\frac{b}{2}\varphi_1+\frac{(2C_{13}-\ell C_{23})}{b}\sin\frac{b}{2}\varphi_1\right]e^{-\frac{\ell}{2}\varphi_1}$$

This is a rather complicated Lagrangian. However after fixation we have

$$\mathcal{L}=KR-\lambda-(n+1)W^1_\mu W^{1\mu}-\left(\frac{\ell^2-2}{2}+n\right)W^2_\mu W^{2\mu}-W^3_\mu W^{3\mu}-\ell(1+n)W^1_\mu W^{2\mu}-$$

$$-\frac{1}{2}C_{11}F^1_{\mu\nu}F^{1\mu\nu}-\frac{1}{2}C_{22}F^2_{\mu\nu}F^{2\mu\nu}-\frac{1}{2}C_{33}F^3_{\mu\nu}F^{3\mu\nu}-C_{12}F^1_{\mu\nu}F^{2\mu\nu}-C_{13}F^1_{\mu\nu}F^{3\mu\nu}-$$

$$-C_{23}F^2_{\mu\nu}F^{3\mu\nu}$$

where

$$F^1_{\mu\nu}=f^1_{\mu\nu}-(W^3_\mu W^2_\nu-W^2_\mu W^3_\nu),\quad F^2_{\mu\nu}=f^2_{\mu\nu}-(W^1_\mu W^3_\nu-W^3_\mu W^1_\nu)$$

$$F^3_{\mu\nu}=f^3_{\mu\nu}-\ell(W^2_\mu W^3_\nu-W^3_\mu W^2_\nu)$$

h) Type VIII

This is the first of the two simple three-parameter transitive groups of motions. It has the structure

$$(X_1,X_2)=X_1\ ,\quad (X_1,X_3)=2X_2\ ,\quad (X_2,X_3)=X_3$$

$$X_1=(e^{-\varphi_3},-\varphi_2^2e^{-\varphi_3},-2\varphi_2e^{-\varphi_3}),\quad X_2=(0,0,1),\quad X_3=(0,e^{\varphi_3},0)$$

The metric of the correspoding space is given by

$$dL^2[\varphi(^3x)]=b^{ij}d\varphi_i\,d\varphi_j\qquad i,j=1,2,3$$

$$b^{11}=\frac{Q''''(x_1)}{24},\quad b^{22}=Q(x_1),\quad b^{33}=\left(Q(x_1)x_2^2-\frac{1}{2}Q'(x_1)x_2+\frac{1}{2}Q''(x_1)-\frac{n}{2}\right)$$

$$b^{12}=\frac{Q''(x_1)}{12}+n,\quad b^{13}=\frac{Q'''(x_1)}{24}-\left(\frac{Q''(x_1)}{12}+n\right)x_2,\quad b^{23}=\frac{Q'(x_1)}{4}-Q(x_1)x_2$$

$$Q(x_1)=q_1x_1^4+q_2x_1^3+q_3x_1^2+q_4x_1+q_0\quad (q_1,q_2,q_3,q_4,q_0,n,\text{const.})$$

For the f_{AB} we have

$$f_{11}=(y_{11}-4\varphi_2 y_{12}+4\varphi_2^2 y_{22}+2\varphi_2^2 y_{13}-4\varphi_2^3 y_{23}+\varphi_2^4 y_{33})e^{-2\varphi_3}$$

$$f_{22}=y_{22}-2\varphi_2 y_{23}+\varphi_2^2 y_{33}\ ,\quad f_{33}=y_{33}\,e^{2\varphi_3}$$

$$f_{12}=(y_{12}-2\varphi_2 y_{22}-\varphi_2 y_{13}+3\varphi_2^2 y_{23}-\varphi_2^3 y_{33})e^{-\varphi_3}$$

$$f_{13}=y_{13}-2\varphi_2 y_{23}+\varphi_2^2 y_{33}\ ,\quad f_{23}=(y_{23}-\varphi_2 y_{33})e^{\varphi_3}$$

where

$$y_{11}=C_{11}\qquad y_{22}=C_{22}+2C_{12}\varphi_1+C_{11}\varphi_1^2$$

$$y_{33}=C_{33}+4C_{23}\varphi_1+2(2C_{22}+C_{13})\varphi_1^2+4C_{12}\varphi_1^3+C_{11}\varphi_1^4$$

$$y_{12}=C_{12}+C_{11}\varphi_1\ ,\quad y_{13}=C_{13}+2C_{12}\varphi_1+C_{11}\varphi_1^2$$

$$y_{23}=C_{23}+(2C_{22}+C_{13})\varphi_1+3C_{12}\varphi_1^2+C_{11}\varphi_1^3$$

After fixation we obtain

$$\mathcal{L}=KR-\lambda-q_1 W^1_\mu W^{1\mu}-\left(q_3-\frac{n}{2}\right)W^2_\mu W^{2\mu}-q_0 W^3_\mu W^{3\mu}-$$
$$-\frac{q_2}{3}W^1_\mu W^{2\mu}-2\left(\frac{q_3}{6}+n\right)W^1_\mu W^{3\mu}-\frac{q_4}{2}W^2_\mu W^{3\mu}-\frac{1}{2}C_{AB}F^A_{\mu\nu}F^{B\mu\nu}$$

i) Type IX

This is the familiar $SO(3)$ group whose structure is

$$(X_1,X_2)=X_3\ ,\quad (X_2,X_3)=X_1\ ,\quad (X_3,X_1)=X_2$$

The Killing vectors are given by

$$X_1 = (0,1,0), \quad X_2 = \left(\cos\varphi_2, -\cot\varphi_1 \sin\varphi_2, \frac{\sin\varphi_2}{\sin\varphi_1}\right)$$

$$X_3 = -\sin\varphi_2, -\cot\varphi_1 \cos\varphi_2, \frac{\cos\varphi_2}{\sin\varphi_1})$$

The metric b^{ij} where

$$dL^2[\varphi({}^3x)] = b^{ij}\, d\varphi_i\, d\varphi_j \qquad i,j = 1,2,3$$

is

$$b^{11} = 2e\cos\frac{\varphi_3}{2} + 2f\sin\frac{\varphi_3}{2} + \frac{\alpha^2+d}{2}$$

$$b^{22} = 2\sin\varphi_1\cos\varphi_1(b\sin\varphi_3 - c\cos\varphi_3) - 2b^{11}\sin^2\varphi_1 + \alpha^2 + d\sin^2\varphi_1$$

$$b^{33} = \alpha^2$$

$$b^{12} = \cos\varphi_1(b\cos\varphi_3 + c\sin\varphi_3) + \frac{1}{2}\sin\varphi_1\left(e\sin\frac{\varphi_3}{2} - f\cos\frac{\varphi_3}{2}\right)$$

$$b^{13} = b\cos\varphi_3 + c\sin\varphi_3$$

$$b^{23} = \alpha^2\cos\varphi_1 + \sin\varphi_1(b\sin\varphi_3 - c\cos\varphi_3)$$

f_{AB} is now given by

$$f_{11} = Z_{11}, \quad f_{22} = \frac{1}{2}(Z_{22}+Z_{33}) + \frac{1}{2}(Z_{22}-Z_{33})\cos 2\varphi_2 + Z_{23}\sin 2\varphi_2$$

$$f_{33} = \frac{1}{2}(Z_{22}+Z_{33}) - \frac{1}{2}(Z_{22}-Z_{33})\cos 2\varphi_2 - Z_{23}\sin 2\varphi_2$$

$$f_{12} = Z_{12}\cos\varphi_2 + Z_{13}\sin\varphi_2$$

$$f_{13} = Z_{13}\cos\varphi_2 - Z_{12}\sin\varphi_2$$

$$f_{23} = Z_{23}\cos 2\varphi_2 - \frac{1}{2}(Z_{22}-Z_{33})\sin 2\varphi_2$$

where $Z_{11} = \frac{1}{2}(y_{11} + y_{33}) - \frac{1}{2}(y_{33} - y_{11}) \cos 2\varphi_1 - y_{13} \sin 2\varphi_1$

$$Z_{22} = y_{22} \ , \quad Z_{33} = \frac{1}{2}(y_{11} + y_{33}) + \frac{1}{2}(y_{33} - y_{11}) \cos 2\varphi_1 + y_{13} \sin 2\varphi_1$$

$$Z_{12} = y_{12} \cos\varphi_1 - y_{23} \sin\varphi_1 \ , \quad Z_{13} = y_{13} \cos 2\varphi_1 - \frac{1}{2}(y_{33} - y_{11}) \sin 2\varphi_1$$

$$Z_{23} = y_{23} \cos\varphi_1 + y_{12} \sin\varphi_1$$

and

$$y_{11} = C_{11} \ , \quad y_{22} = \frac{1}{2}(C_{22} + C_{33}) + \frac{1}{2}(C_{22} - C_{33}) \cos 2\varphi_3 + C_{23} \sin 2\varphi_3$$

$$y_{33} = \frac{1}{2}(C_{22} + C_{33}) - \frac{1}{2}(C_{22} - C_{33}) \cos 2\varphi_3 - C_{23} \sin 2\varphi_3$$

$$y_{12} = C_{12} \cos\varphi_3 + C_{13} \sin\varphi_3 \qquad y_{13} = C_{13} \cos\varphi_3 - C_{12} \sin\varphi_3$$

$$y_{23} = C_{23} \cos 2\varphi_3 - \frac{1}{2}(C_{22} - C_{33}) \sin 2\varphi_3$$

where C_{AB} must be chosen positive definite.

The Lagrangian after fixation is

$$\begin{aligned}\mathcal{L} = {} & KR - \lambda - \alpha^2 W^1 W^1 - \left(2e + \frac{\alpha^2 + d}{2}\right) W^2 W^2 + 4e W^3 W^3 - 2b W^1 W^2 \\ & - 2c W^1 W^3 - f W^2 W^3 - \frac{1}{2} C_{11} F^1_{\mu\nu} F^{1\mu\nu} - \frac{1}{2} C_{22} F^2_{\mu\nu} F^{2\mu\nu} - \frac{1}{2} C_{33} F^3_{\mu\nu} F^{3\mu\nu} \\ & - C_{12} F^1_{\mu\nu} F^{2\mu\nu} - C_{13} F^1_{\mu\nu} F^{3\mu\nu} - C_{23} F^2_{\mu\nu} F^{3\mu\nu}\end{aligned}$$

$$F^1_{\mu\nu} = f^1_{\mu\nu} - (W^2_\mu W^3_\nu - W^3_\mu W^2_\nu)$$

$$F^2_{\mu\nu} = f^2_{\mu\nu} - (W^3_\mu W^1_\nu - W^1_\mu W^3_\nu)$$

$$F^3_{\mu\nu} = f^3_{\mu\nu} - (W^1_\mu W^2_\nu - W^2_\mu W^1_\nu)$$

5.5 Four-parameter groups containing a three-parameter intransitive.

There are two four-parameter groups of motions containing a three-parameter intransitive. These are a) the one containing $SO(3)$ and b) the one containing $SO(2,1)$. The corresponding spaces will of course constitute particular cases of the spaces 5.3 b) and 5.3 c).

a)

The Killing vectors and their commutators are

$$X_1=(0,\cos\varphi_3,-\cot\varphi_2\sin\varphi_3)\,,\quad X_2=(0,\sin\varphi_3,\cot\varphi_2\cos\varphi_3)$$

$$X_3=(0,0,1)\,,\qquad X_4=(1,0,0)$$

$$[X_1,X_2]=-X_3\,,\ [X_2,X_3]=-X_1\,,\ [X_3,X_1]=-X_2\,,\ [X_A,X_4]=0$$

The metric of the φ -space is given by

$$dL^2[\varphi(^3x)]=d\varphi_1^2+d\varphi_2^2+\sin^2\varphi_2\,d\varphi_3^2$$

Eqs. (2.101) for f_{AB} give

$$f_{11}=\alpha_1-\alpha_2\sin^2\varphi_2\sin^2\varphi_3\,,\quad f_{22}=\alpha_1-\alpha_2\sin^2\varphi_2\cos^2\varphi_3$$

$$f_{33}=\alpha_1-\alpha_2\cos^2\varphi_2\,,\quad f_{44}=\beta$$

$$f_{12}=\alpha_2\sin^2\varphi_2\sin\varphi_3\cos\varphi_3\,,\quad f_{13}=-\alpha_2\sin\varphi_2\cos\varphi_2\sin\varphi_3$$

$$f_{14}=-\beta_2\sin\varphi_2\sin\varphi_3\,,\quad f_{23}=\alpha_2\sin\varphi_2\cos\varphi_2\cos\varphi_3$$

$$f_{24}=\beta_2\sin\varphi_2\cos\varphi_3\,,\quad f_{34}=\beta_2\cos\varphi_2$$

where $\alpha_1, \beta_1, \alpha_2, \beta_2$ constants which must satisfy

$$\alpha_1 > 0 \,, \quad \alpha_1 > \alpha_2 \,, \quad \beta_1 > 0 \,, \quad \left((\alpha_1 - \alpha_2)\beta_1 - \beta_2^2 \right) > 0$$

in order that f_{AB} is positive definite.

After the fixation we have

$$\mathcal{L} = KR - \lambda - W^1_\mu W^{1\mu} - W^2_\mu W^{2\mu} - W^4_\mu W^{4\mu} - \tfrac{1}{2}\alpha_1 F^1_{\mu\nu} F^{1\mu\nu} -$$
$$- \tfrac{1}{2}\alpha_2 F^2_{\mu\nu} F^{2\mu\nu} - \tfrac{1}{2}(\alpha_1 - \alpha_2) F^3_{\mu\nu} F^{3\mu\nu} - \tfrac{1}{2}\beta_1 f^4_{\mu\nu} f^{4\mu\nu} -$$
$$- \beta_2 F^3_{\mu\nu} f^{4\mu\nu}$$

$$F^1_{\mu\nu} = f^1_{\mu\nu} - (W^3_\mu W^2_\nu - W^2_\mu W^3_\nu)$$
$$F^2_{\mu\nu} = f^2_{\mu\nu} - (W^1_\mu W^3_\nu - W^3_\mu W^1_\nu)$$
$$F^3_{\mu\nu} = f^3_{\mu\nu} - (W^2_\mu W^1_\nu - W^1_\mu W^2_\nu)$$

b)

$$X_1 = (0, \cos\varphi_3, -\operatorname{cth}\varphi_2 \sin\varphi_3) \,, \quad X_2 = (0, \sin\varphi_3, \operatorname{cth}\varphi_2 \cos\varphi_3)$$
$$X_3 = (0,0,1) \,, \quad X_4 = (1,0,0)$$
$$[X_1, X_2] = X_3 \,, \quad [X_2, X_3] = -X_1 \,, \quad [X_3, X_1] = -X_2 \,, \quad [X_A, X_4] = 0$$
$$dL^2[\varphi(^3x)] = d\varphi_1^2 + d\varphi_2^2 + \operatorname{sh}^2\varphi_2 \, d\varphi_3^2$$
$$f_{11} = \alpha_1 + \alpha_2 \operatorname{sh}^2\varphi_2 \sin^2\varphi_3 \,, \quad f_{22} = \alpha_1 + \alpha_2 \operatorname{sh}^2\varphi_2 \cos^2\varphi_3$$

$$f_{33} = \alpha_2 \, ch^2\varphi_2 - \alpha_1 \quad , \quad f_{44} = \beta_1$$
$$f_{12} = -\alpha_2 \, sh^2\varphi_2 \sin\varphi_3 \cos\varphi_3 \, , \quad f_{13} = -\alpha_2 \, sh\varphi_2 \, ch\varphi_2 \sin\varphi_3$$
$$f_{14} = -\beta_2 \, sh\varphi_2 \sin\varphi_3 \quad , \quad f_{23} = \alpha_2 \, sh\varphi_2 \, ch\varphi_2 \cos\varphi_3$$
$$f_{24} = \beta_2 \, sh\varphi_2 \cos\varphi_3 \quad , \quad f_{34} = \beta_2 \, ch\varphi_2$$

$$\alpha_1, \alpha_2 > 0 \quad , \quad \alpha_2 > \alpha_1 \, , \quad \beta_1 > 0 \, , \left((\alpha_2 - \alpha_1)\beta_1 - \beta_2^2\right) > 0$$

After fixation we get

$$\mathcal{L} = KR - \lambda - W^1_\mu W^{1\mu} - W^2_\mu W^{2\mu} - W^4_\mu W^{4\mu} - \tfrac{1}{2}\alpha_1 F^1_{\mu\nu} F^{1\mu\nu} -$$
$$\tfrac{1}{2}\alpha_1 F^2_{\mu\nu} F^{2\mu\nu} - \tfrac{1}{2}(\alpha_2 - \alpha_1) F^3_{\mu\nu} F^{3\mu\nu} - \tfrac{1}{2}\beta_1 f^4_{\mu\nu} f^{4\mu\nu} - \beta_2 F^3_{\mu\nu} f^{4\mu\nu} .$$

5.6 Four-parameter groups containing a three-parameter transitive.

There are five four parameter groups of motions that contain a three-parameter transitive one. These are:

a) The one containing the type-I group

b) " " " " type-II "

c) " " " " type-III "

d) " " " " type-V "

e) " " " " type-IX " .

a)

This is the group $SO(2) \otimes_2 T(3)$ with Killing vectors

$$X_1=(1,0,0),\ X_2=(0,1,0),\ X_3=(0,0,1),\ X_4=(\varphi_2,-\varphi_1,0)$$

that satisfy

$$[X_1,X_2]=[X_1,X_3]=[X_2,X_3]=0$$
$$[X_1,X_4]=-X_2 \quad , \quad [X_2,X_4]=X_1$$

The metric is of course flat

$$dL^2[\varphi(^3x)]=d\varphi_1^2+d\varphi_2^2+d\varphi_3^2$$

and f_{AB} is given by

$$f_{11}=f_{22}=\alpha \quad , \quad f_{33}=\beta \quad , \quad f_{12}=f_{13}=f_{23}=0$$
$$f_{44}=\gamma+\alpha(\varphi_1^2+\varphi_2^2)\ ,\ f_{14}=\alpha\varphi_2 \ ,\ f_{24}=-\alpha\varphi_1\ ,\ f_{34}=\delta$$

where $\alpha,\beta,\gamma,\delta$ constants and

$$\alpha,\beta,\gamma>0 \qquad \beta\gamma-\delta^2>0$$

After fixation we obtain the Lagrangian

$$\mathcal{L}=KR-\lambda-W^1_\mu W^{1\mu}-W^2_\mu W^{2\mu}-W^3_\mu W^{3\mu}-\tfrac{1}{2}\alpha F^1_{\mu\nu}F^{1\mu\nu}-$$
$$-\tfrac{1}{2}\alpha F^2_{\mu\nu}F^{2\mu\nu}-\tfrac{1}{2}\beta f^3_{\mu\nu}f^{3\mu\nu}-\tfrac{1}{2}\gamma f^4_{\mu\nu}f^{4\mu\nu}-\delta f^3_{\mu\nu}f^{4\mu\nu}$$

where

$$F^1_{\mu\nu} = f^1_{\mu\nu} - (W^2_\mu W^4_\nu - W^4_\mu W^2_\nu)$$

$$F^2_{\mu\nu} = f^2_{\mu\nu} - (W^4_\mu W^1_\nu - W^1_\mu W^4_\nu)$$

b)

$$X_1 = (0,1,0), \quad X_2 = (0,0,1), \quad X_3 = (-1, \varphi_3, 0), \quad X_4 = (\varphi_3, \tfrac{1}{2}(\varphi_1^2 - \varphi_3^2), -\varphi_1)$$

$$[X_1, X_2] = [X_1, X_3] = [X_1, X_4] = 0, \ [X_2, X_3] = X_1, \ [X_2, X_4] = -X_3, \ [X_3, X_4] = X_2$$

$$dL^2[\varphi(3x)] = d\varphi_1^2 + d\varphi_2^2 + 2\varphi_1 \, d\varphi_2 \, d\varphi_3 + (1+\varphi_1^2)\, d\varphi_3^2$$

$$f_{11} = \alpha, \quad f_{22} = \beta + \alpha\varphi_1^2, \quad f_{33} = \beta + \alpha\varphi_3^2, \quad f_{12} = \alpha\varphi_1, \quad f_{13} = \alpha\varphi_3$$

$$f_{23} = \alpha\varphi_1\varphi_3, \quad f_{44} = \gamma + \tfrac{1}{2}\alpha\varphi_1^2\varphi_3^2 + \tfrac{1}{4}e(\varphi_1^4 + \varphi_3^4) - \tfrac{1}{2}(\delta-\beta)(\varphi_1^2 + \varphi_3^2)$$

$$f_{14} = \delta - \tfrac{1}{2}\alpha(\varphi_1^2 + \varphi_3^2), \quad f_{24} = (\delta-\beta)\varphi_1 - \tfrac{1}{2}\alpha\varphi_1^3 - \tfrac{1}{2}\alpha\varphi_1\varphi_3^2$$

$$f_{34} = (\delta-\beta)\varphi_3 - \tfrac{1}{2}\alpha\varphi_3^3 - \tfrac{1}{2}\alpha\varphi_1^2\varphi_3$$

where $\alpha, \beta, \gamma, \delta$ constants and

$$\alpha, \beta, \gamma > 0, \quad \alpha\gamma - \delta^2 > 0$$

After fixation we obtain

$$\mathcal{L} = KR - \lambda - W^1_\mu W^{1\mu} - W^2_\mu W^{2\mu} - W^3_\mu W^{3\mu} - \tfrac{1}{2}\alpha F^1_{\mu\nu} F^{1\mu\nu} -$$

$$-\tfrac{1}{2}\beta F^2_{\mu\nu} F^{2\mu\nu} - \tfrac{1}{2}\beta F^3_{\mu\nu} F^{3\mu\nu} - \tfrac{1}{2}\gamma F^4_{\mu\nu} f^{4\mu\nu} - \delta F^1_{\mu\nu} f^{4\mu\nu}$$

$$F^1_{\mu\nu} = f^1_{\mu\nu} - (W^2_\mu W^3_\nu - W^3_\mu W^2_\nu), \quad F^2_{\mu\nu} = f^2_{\mu\nu} - (W^3_\mu W^4_\nu - W^4_\mu W^3_\nu)$$

$$F^3_{\mu\nu} = f^3_{\mu\nu} - (W^4_\mu W^2_\nu - W^2_\mu W^4_\nu)$$

c)

$$X_1 = (0,1,0), \quad X_2 = (0,0,1), \quad X_3 = (1, -\varphi_2, 0)$$

$$X_4 = \left(\varphi_2, \tfrac{1}{2}\left(\frac{e^{-2\varphi_1}}{1-n^2} - \varphi_2^2\right), -\frac{n e^{-\varphi_1}}{1-n^2}\right)$$

$$[X_1,X_2] = [X_2,X_3] = [X_2,X_4] = 0, \; [X_1,X_3] = -X_1, \; [X_1,X_4] = X_3, \; [X_3,X_4] = -X_4$$

$$dL^2[\varphi(^3x)] = d\varphi_1^2 + e^{2\varphi_1} d\varphi_2^2 + 2n e^{\varphi_1} d\varphi_2 \, d\varphi_3 + d\varphi_3^2 \quad (n^2 < 1)$$

Eqs. (2.101) for the f_{AB} give

$$f_{11} = C_{11} e^{2\varphi_1}, \quad f_{22} = C_{22}, \quad f_{33} = C_{33} + C_{11}\varphi_2^2 e^{2\varphi_1}, \quad f_{12} = C_{12} e^{\varphi_1}$$

$$f_{13} = -C_{11}\varphi_2 e^{2\varphi_1}, \quad f_{23} = -C_{12}\varphi_2 e^{\varphi_1}$$

$$f_{44} = \tfrac{1}{4} C_{11} \frac{e^{-2\varphi_1}}{(1-n^2)^2} + \tfrac{1}{2} C_{11} \frac{\varphi_2^2}{(1-n^2)} + \tfrac{1}{4} C_{11} \varphi_2^4 e^{2\varphi_1}$$

$$f_{14} = C_{33} - \tfrac{1}{2}\frac{C_{11}}{(1-n^2)} - \tfrac{1}{2} C_{11}\varphi_2^2 e^{2\varphi_1}, \quad f_{24} = -\tfrac{1}{2} C_{12} \frac{e^{-\varphi_1}}{(1-n^2)} - \tfrac{1}{2} C_{12}\varphi_2^2 e^{\varphi_1}$$

$$f_{34} = \tfrac{1}{2} C_{11} \varphi_2^3 e^{2\varphi_1} + \tfrac{1}{2} C_{11} \frac{\varphi_2}{(1-n^2)}$$

After fixation we have

$$\mathcal{L} = KR - \lambda - W^1_\mu W^{1\mu} - W^2_\mu W^{2\mu} - W^3_\mu W^{3\mu} - \frac{1}{4(1-n^2)^2} W^4_\mu W^{4\mu} - 2n W^1_\mu W^{2\mu} -$$
$$- 2\left(1 - \frac{1}{2}\left(\frac{1}{(1-n^2)}\right)\right) W^1_\mu W^{4\mu} + \frac{n}{(1-n^2)} W^2_\mu W^{4\mu} - \frac{1}{2} C_{11} F^1_{\mu\nu} F^{1\mu\nu} -$$
$$- \frac{1}{2} C_{22} F^2_{\mu\nu} F^{2\mu\nu} - \frac{1}{2} C_{33} F^3_{\mu\nu} F^{3\mu\nu} - C_{12} F^1_{\mu\nu} F^{2\mu\nu} - \frac{C_{11}}{8(1-n^2)^2} F^4_{\mu\nu} F^{4\mu\nu} -$$
$$- \left(C_{33} - \frac{C_{11}}{2(1-n^2)}\right) F^1_{\mu\nu} F^{4\mu\nu} + \frac{C_{12}}{2(1-n^2)} F^2_{\mu\nu} F^{4\mu\nu}$$

$$F^1_{\mu\nu} = f^1_{\mu\nu} - (W^3_\mu W^1_\nu - W^1_\mu W^3_\nu), \quad F^2_{\mu\nu} = f^2_{\mu\nu}$$
$$F^3_{\mu\nu} = f^3_{\mu\nu} - (W^1_\mu W^4_\nu - W^4_\mu W^1_\nu), \quad F^4_{\mu\nu} = f^4_{\mu\nu} - (W^4_\mu W^3_\nu - W^3_\mu W^4_\nu)$$

d)

$$X_1 = (0,1,0), \quad X_2 = (0,0,1), \quad X_3 = (-\tfrac{1}{n}, \varphi_2, \varphi_3), \quad X_4 = (0, -\varphi_3, \varphi_2)$$
$$[X_1, X_2] = [X_3, X_4] = 0, \ [X_1, X_3] = X_1, \ [X_2, X_3] = X_2, \ [X_1, X_4] = X_2, \ [X_2, X_4] = -X_1$$
$$dL^2[\varphi({}^3x)] = d\varphi_1^2 + e^{2n\varphi_1}(d\varphi_2^2 + d\varphi_3^2)$$

$$f_{11} = f_{22} = \alpha e^{2n\varphi_1}, \quad f_{33} = \beta + \alpha(\varphi_2^2 + \varphi_3^2) e^{2n\varphi_1}$$
$$f_{44} = \gamma + \alpha(\varphi_2^2 + \varphi_3^2) e^{2n\varphi_1}$$
$$f_{12} = 0, \quad f_{13} = \alpha\varphi_2 e^{2n\varphi_1}, \quad f_{14} = -\alpha\varphi_3 e^{2n\varphi_1}$$
$$f_{23} = \alpha\varphi_3 e^{2n\varphi_1}, \quad f_{24} = \alpha\varphi_2 e^{2n\varphi_1}, \quad f_{34} = \delta$$

The Lagrangian after fixation is

$$\mathcal{L} = KR - \lambda - W^1_\mu W^{1\mu} - W^2_\mu W^{2\mu} - \frac{1}{n^2} W^3_\mu W^{3\mu} - \frac{1}{2}\alpha F^1_{\mu\nu} F^{1\mu\nu} -$$

$$- \frac{1}{2}\alpha F^2_{\mu\nu} F^{2\mu\nu} - \frac{1}{2}\beta f^3_{\mu\nu} f^{3\mu\nu} - \frac{1}{2}\gamma f^4_{\mu\nu} f^{4\mu\nu} - \delta f^3_{\mu\nu} f^{4\mu\nu}$$

$$F^1_{\mu\nu} = f^1_{\mu\nu} - (W^1_\mu W^3_\nu - W^3_\mu W^1_\nu) + (W^2_\mu W^4_\nu - W^4_\mu W^2_\nu)$$

$$F^2_{\mu\nu} = f^2_{\mu\nu} - (W^2_\mu W^3_\nu - W^3_\mu W^2_\nu) - (W^1_\mu W^4_\nu - W^4_\mu W^1_\nu)$$

e)

$$X_1 = (0,1,0), \quad X_2 = \left(\cos\varphi_2, -\cot\varphi_1 \sin\varphi_2, \frac{n\sin\varphi_2}{\sin\varphi_1}\right)$$

$$X_3 = \left(-\sin\varphi_2, -\cot\varphi_1\cos\varphi_2, \frac{n\cos\varphi_2}{\sin\varphi_1}\right), \quad X_4 = (0,0,1)$$

$$[X_1, X_2] = X_3, \ [X_2, X_3] = X_1, \ [X_3, X_1] = X_2, \ [X_A, X_4] = 0$$

$$dL^2[\varphi(^3x)] = d\varphi_1^2 + (\sin^2\varphi_1 + n^2\cos^2\varphi_1)\, d\varphi_2^2 + 2n\cos\varphi_1\, d\varphi_2\, d\varphi_3 + d\varphi_3^2$$

$$f_{11} = \alpha + (\beta-\alpha)\sin^2\varphi_1, \quad f_{22} = \beta - (\beta-\alpha)\sin^2\varphi_1 \sin^2\varphi_2$$

$$f_{33} = \beta - (\beta-\alpha)\sin^2\varphi_1 \cos^2\varphi_2, \quad f_{12} = -\frac{1}{2}(\beta-\alpha)\sin^2\varphi_1 \sin\varphi_2$$

$$f_{13} = -\frac{1}{2}(\beta-\alpha)\sin 2\varphi_1 \cos\varphi_2, \quad f_{23} = -\frac{1}{2}(\beta-\alpha)\sin^2\varphi_1 \sin 2\varphi_2$$

$$f_{44} = \gamma, \quad f_{14} = \delta\cos\varphi_1, \quad f_{24} = \delta\sin\varphi_1\cos\varphi_2, \quad f_{34} = \delta\sin\varphi_1\sin\varphi_2$$

$$\alpha, \beta, \gamma > 0 \qquad \beta^2 - \gamma\delta > 0$$

The Lagrangian after the fixation is given by

$$\mathcal{L} = KR - \lambda - n^2 W^1 W^1 - W^2 W^2 - W^3 W^3 - W^4 W^4 - 2n W^1 W^4 - \tfrac{1}{2}\alpha F^1_{\mu\nu} F^{1\mu\nu}$$
$$- \tfrac{1}{2}\beta F^2_{\mu\nu} F^{2\mu\nu} - \tfrac{1}{2}\beta F^3_{\mu\nu} F^{3\mu\nu} - \tfrac{1}{2}\gamma f^4_{\mu\nu} f^{4\mu\nu} - \delta F^1_{\mu\nu} f^{4\mu\nu}$$

$$F^1_{\mu\nu} = f^1_{\mu\nu} - (W^2_\mu W^3_\nu - W^3_\mu W^2_\nu), \quad F^2_{\mu\nu} = f^2_{\mu\nu} - (W^3_\mu W^1_\nu - W^1_\mu W^3_\nu)$$
$$F^3_{\mu\nu} = f^3_{\mu\nu} - (W^1_\mu W^2_\nu - W^2_\mu W^1_\nu)$$

5.7 Six-parameter groups

These are of course the complete groups of motions of a three-dimensional space, namely the E_3, S_3, and H_3.

The corresponding spaces have constant curvature (zero, positive , and negative respectively).

a) Euclidean $SO(3) \otimes_s T(3)$

$$X_1 = (1,0,0), \quad X_2 = (0,1,0), \quad X_3 = (0,0,1)$$

$$X_4 = (0, -\varphi_3, \varphi_2), \quad X_5 = (\varphi_3, 0, -\varphi_1), \quad X_6 = (-\varphi_2, \varphi_1, 0)$$

$$[X_1, X_2] = [X_1, X_3] = [X_2, X_3] = 0$$
$$[X_1, X_4] = [X_2, X_5] = [X_3, X_6] = 0, \quad [X_2, X_4] = -[X_1, X_5] = X_3$$

$$[X_3,X_5]=-[X_2,X_6]=X_1\ ,\quad [X_1,X_6]=-[X_3,X_4]=X_2$$

$$[X_4,X_5]=-X_6\ ,\ [X_5,X_6]=-X_4\ ,\ [X_6,X_4]=-X_5$$

The metric is of course Euclidean

$$dL^2[\varphi(^3x)]=d\varphi_1^2+d\varphi_2^2+d\varphi_3^2$$

Eqs.(2.101)give

$$f_{11}=f_{22}=f_{33}=\alpha\ ,\quad f_{12}=f_{13}=f_{23}=0$$

$$f_{44}=C_{44}+\alpha(\varphi_2^2+\varphi_3^2),\quad f_{55}=C_{55}+\alpha(\varphi_1^2+\varphi_3^2)$$

$$f_{66}=C_{66}+\alpha(\varphi_1^2+\varphi_2^2)\ ,\quad f_{14}=f_{25}=f_{36}=\beta$$

$$f_{15}=-f_{24}=\alpha\varphi_3\ ,\ f_{16}=-f_{34}=-\alpha\varphi_2\ ,\ f_{26}=-f_{35}=\alpha\varphi_1$$

$$f_{45}=-\alpha\varphi_1\varphi_2\ ,\quad f_{46}=-\alpha\varphi_1\varphi_3\ ,\quad f_{56}=-\alpha\varphi_2\varphi_3$$

After fixing at the origin our Lagrangian is

$$\mathcal{L}=KR-\lambda-W^1_\mu W^{1\mu}-W^2_\mu W^{2\mu}-W^3_\mu W^{3\mu}-\tfrac{1}{2}\alpha F^1_{\mu\nu}F^{1\mu\nu}-$$
$$-\tfrac{1}{2}\alpha F^2_{\mu\nu}F^{2\mu\nu}-\tfrac{1}{2}\alpha F^3_{\mu\nu}F^{3\mu\nu}-\tfrac{1}{2}C_{44}F^4_{\mu\nu}F^{4\mu\nu}-\tfrac{1}{2}C_{55}F^5_{\mu\nu}F^{5\mu\nu}-$$
$$-\tfrac{1}{2}C_{66}F^6_{\mu\nu}F^{6\mu\nu}-\beta F^1_{\mu\nu}F^{4\mu\nu}-\beta F^2_{\mu\nu}F^{5\mu\nu}-\beta F^3_{\mu\nu}F^{6\mu\nu}$$

$$F^1_{\mu\nu}=f^1_{\mu\nu}-(W^6_\mu W^2_\nu-W^2_\mu W^6_\nu)-(W^3_\mu W^5_\nu-W^5_\mu W^3_\nu)$$

$$F^2_{\mu\nu} = f^2_{\mu\nu} - (W^1_\mu W^6_\nu - W^6_\mu W^1_\nu) - (W^4_\mu W^3_\nu - W^3_\mu W^4_\nu)$$

$$F^3_{\mu\nu} = f^3_{\mu\nu} - (W^5_\mu W^1_\nu - W^1_\mu W^5_\nu) - (W^2_\mu W^4_\nu - W^4_\mu W^2_\nu)$$

$$F^4_{\mu\nu} = f^4_{\mu\nu} - (W^6_\mu W^5_\nu - W^5_\mu W^6_\nu); \quad F^5_{\mu\nu} = f^5_{\mu\nu} - (W^4_\mu W^6_\nu - W^6_\mu W^4_\nu)$$

$$F^6_{\mu\nu} = f^6_{\mu\nu} - (W^5_\mu W^4_\nu - W^4_\mu W^5_\nu)$$

b) Spherical $SO(3) \otimes SO(3)$

$$X_1 = (0,1,0), \quad X_2 = \left(\cos\varphi_2, -\cot\varphi_1 \sin\varphi_2, \frac{\sin\varphi_2}{\sin\varphi_1}\right)$$

$$X_3 = \left(-\sin\varphi_2, -\cot\varphi_1 \cos\varphi_2, \frac{\cos\varphi_2}{\sin\varphi_1}\right)$$

$$X_4 = (0,0,1), \quad X_5 = \left(\cos\varphi_3, \frac{\sin\varphi_3}{\sin\varphi_1}, -\cot\varphi_1 \sin\varphi_3\right)$$

$$X_6 = \left(-\sin\varphi_3, \frac{\cos\varphi_3}{\sin\varphi_1}, -\cot\varphi_1 \cos\varphi_3\right)$$

$$[X_1, X_2] = X_3, \ [X_2, X_3] = X_1, \ [X_3, X_1] = X_2, \ [X_4, X_5] = X_6$$

$$[X_5, X_6] = X_4, \ [X_6, X_4] = X_5, \ [X_A, X_B] = 0, \quad A = 1,2,3 \quad B = 4,5,6$$

$$dL^2[\varphi(^3x)] = d\varphi_1^2 + d\varphi_2^2 + d\varphi_3^2 + 2\cos\varphi_1 \, d\varphi_2 \, d\varphi_3$$

For f_{AB} we obtain

$$f_{11} = f_{22} = f_{33} = \alpha \quad , \quad f_{12} = f_{13} = f_{23} = 0$$

$$f_{44} = f_{55} = f_{66} = \beta \quad , \quad f_{45} = f_{46} = f_{56} = 0$$

$\alpha, \beta > 0$, and after the fixation we have

$$\mathcal{L} = KR - \lambda - W^1_\mu W^{1\mu} - W^2_\mu W^{2\mu} - W^3_\mu W^{3\mu} - W^4_\mu W^{4\mu} - W^5_\mu W^{5\mu} - W^6_\mu W^{6\mu} - 2W^1_\mu W^{4\mu} - 2W^2_\mu W^{5\mu} + 2W^3_\mu W^{6\mu} - \frac{1}{2}\alpha F^1_{\mu\nu}F^{1\mu\nu} - \frac{1}{2}\alpha F^2_{\mu\nu}F^{2\mu\nu} - \frac{1}{2}\alpha F^3_{\mu\nu}F^{3\mu\nu} - \frac{1}{2}\beta F^4_{\mu\nu}F^{4\mu\nu} - \frac{1}{2}\beta F^5_{\mu\nu}F^{5\mu\nu} - \frac{1}{2}\beta F^6_{\mu\nu}F^{6\mu\nu}$$

c) Hyperbolic H_3

$$X_1 = (0,1,0),\quad X_2 = (0,0,1),\quad X_3 = (-\tfrac{1}{n}, \varphi_2, \varphi_3),\quad X_4 = (0, -\varphi_3, \varphi_2)$$

$$X_5 = \left(\frac{\varphi_2}{n}, \frac{e^{-2n\varphi_1}}{2n^2} - \frac{1}{2}(\varphi_2^2 - \varphi_3^2), -\varphi_2\varphi_3\right)$$

$$X_6 = \left(\frac{\varphi_3}{n}, -\varphi_2\varphi_3, \frac{e^{-2n\varphi_1}}{2n^2} - \frac{1}{2}(\varphi_3^2 - \varphi_2^2)\right)$$

$$[X_1,X_2] = [X_3,X_4] = [X_5,X_6] = 0,\quad [X_1,X_3] = [X_4,X_2] = X_1$$

$$[X_2,X_3] = X_2,\quad [X_5,X_1] = [X_6,X_2] = X_3,\quad [X_5,X_2] = [X_6,X_1] = X_4$$

$$[X_3,X_5] = [X_4,X_6] = X_5,\quad [X_5,X_4] = [X_3,X_6] = X_6$$

$$dL^2[\varphi({}^3x)] = d\varphi_1^2 + e^{2n\varphi_1}(d\varphi_2^2 + d\varphi_3^2)$$

$$f_{11} = f_{22} = \alpha e^{2n\varphi_1},\quad f_{33} = f_{44} = \frac{\alpha}{2n^2} + \alpha(\varphi_2^2 + \varphi_3^2)e^{2n\varphi_1}$$

$$f_{55} = f_{66} = \frac{\alpha}{4n^4}e^{-2n\varphi_1} + \frac{\alpha}{4}(\varphi_2^2+\varphi_3^2)^2 e^{2n\varphi_1} + \frac{\alpha}{2n^2}(\varphi_2^2 + \varphi_3^2)$$

$$f_{12}=0\,,\quad f_{13}=\alpha\varphi_2\, e^{2n\varphi_1}\,,\quad f_{14}=-\alpha\,\varphi_3\, e^{2n\varphi_1}$$

$$f_{15}=\frac{\alpha}{2}\left(\varphi_3^2-\varphi_2^2\right)e^{2n\varphi_1}\,,\quad f_{16}=-\beta-\alpha\,\varphi_2\,\varphi_3\, e^{2n\varphi_1}$$

$$f_{23}=\alpha\varphi_3\, e^{2n\varphi_1}\,,\quad f_{24}=\alpha\,\varphi_2\, e^{2n\varphi_1}\,,\quad f_{25}=\beta-\alpha\varphi_2\varphi_3\, e^{2n\varphi_1}$$

$$f_{26}=\frac{\alpha}{2}\left(\varphi_2^2-\varphi_3^2\right)e^{2n\varphi_1}\,,\quad f_{34}=\beta$$

$$f_{35}=-\frac{\alpha}{2n^2}\,\varphi_2-\frac{\alpha}{2}\,\varphi_2\left(\varphi_2^2+\varphi_3^2\right)e^{2n\varphi_1}$$

$$f_{36}=-\frac{\alpha}{2n^2}\,\varphi_3-\frac{\alpha}{2}\,\varphi_3\left(\varphi_2^2+\varphi_3^2\right)e^{2n\varphi_1}$$

$$f_{45}=f_{36}\,,\quad f_{46}=-f_{35}\,,\quad f_{56}=0$$

$$\mathcal{L}=KR-\lambda-W^1_\mu W^{1\mu}-W^2_\mu W^{2\mu}-\frac{1}{n^2}W^3_\mu W^{3\mu}-\frac{1}{4n^4}W^5_\mu W^{5\mu}-$$
$$-\frac{1}{4n^4}W^6_\mu W^{6\mu}-\frac{1}{n^2}W^1_\mu W^{5\mu}-\frac{1}{n^2}W^2_\mu W^{6\mu}-\frac{1}{2}\alpha F^1_{\mu\nu}F^{1\mu\nu}-$$
$$-\frac{1}{2}\alpha F^2_{\mu\nu}F^{2\mu\nu}-\frac{\alpha}{4n^2}F^3_{\mu\nu}F^{3\mu\nu}-\frac{\alpha}{4n^2}F^4_{\mu\nu}F^{4\mu\nu}-$$
$$-\frac{\alpha}{8n^4}F^5_{\mu\nu}F^{5\mu\nu}-\frac{\alpha}{8n^4}F^6_{\mu\nu}F^{6\mu\nu}-\beta F^3_{\mu\nu}F^{4\mu\nu}+$$
$$+\beta F^1_{\mu\nu}F^{6\mu\nu}-\beta F^2_{\mu\nu}F^{5\mu\nu}$$

Appendix I

Having determined the form of the Lagrangian, we now give the proof of the existence and uniqueness of the solution of the inhomogeneous Eq.(2.46) for any source term S_A of the form given by Eq.(2.49). So we first look at whether the corresponding homogeneous equation

$$H_{AB}\, n^B = 0 \tag{I.1}$$

possesses non-trivial solutions. Multiplying the above equation by n^A and integrating over the 3-dimensional space manifold we obtain :

$$\int n^A H_{AB}\, n^B \sqrt{g}\, d^3x =$$

$$\int \left\{ f_{AB}\, D^A_m D^{Bm} + G^{ab}\, \mathcal{J}_{Aa} \mathcal{J}_{Bb}\, n^A n^B \right\} \sqrt{g}\, d^3x = 0 \tag{I.2}$$

where

$$D^A_m = D^A{}_{Bm}\, n^B \tag{I.3}$$

Assuming that the matrices f_{AB} and G^{ab} are positive-definite, we realize that Eq.(I.2) compells us to set :

$$D^A_m = n^A{}_{,m} + g^A{}_{BR}\, W^B_m\, n^R = 0 \tag{I.4a}$$

$$\mathcal{J}_{Aa} = 0 \tag{I.4b}$$

The integrability conditions of Eq.(I.4a) are:

$$g^A{}_{BR}\, F^B_{mn} = 0 \tag{I.5}$$

From Eqs.(I.4b) and (I.5) it follows that only at the boundary points of $\bar{S}$ we have non-trivial solutions of the homogeneous equation (I.1). Since then either (if $g^A{}_{BR} \neq 0$) we can set $W^A_m = 0$ in a particular gauge, or we have the Abelian case $g^A{}_{BR} = 0$, the solution of Eq.(I.4a) is

$$n^A = c^A \qquad \text{(constants)}$$

In this case, however, the projection of the source term S_A of the inhomogeneous equation, on the subspace of these solutions vanishes :

$$(S_A, c^A) = \int S_A c^A \sqrt{g}\, d^3x = 0$$

since at the boundary points of $\bar{S}$, S_A is a pure divergence:

$$S_A = -(f_{AB} W^B_m)^{;m}$$

Hence, the operator H_{AB} is invertible, and the solution n^A of the inhomogeneous Eq.(2.46) always exists and is unique, up to a constant c^A which is present only at boundary points, and has no effect on dL^2

The problem of the existence and uniqueness of the solutions to the point correspondence equation is far from trivial, because the equation is not of an elliptic character. This is due to the fact that the metric of Riem($\mathcal{M}$) is not positive definite. This problem is discussed at length in a recent work by D. Christodoulou and M. Francaviglia.[17] They found that the solutions do not always exist and they are not in general unique. However (if a solution exists) the arc length in $\mathcal{S}(\mathcal{M})$ is unique.

Appendix II

On the integrability of eqs. (2.IOI)

In our study of the above system of partial differential equations we shall examine separately the cases when the group G_R with generators $X_R = \mathfrak{I}_{Ra}\frac{\partial}{\partial\varphi_a}$ is intransitive, simply transitive and multiply transitive.

a) Intransitive group G_R

If the minimum invariant varieties of G_R in $\mathcal{K}$ are of n–q dimensions (i.e. if the rank of the matrix $\mathfrak{I}_{Aa}$ is n–q) we can always write the transfomations of G_R (with a possible change of coordinates) as the transfomations of a transitive group over n–q variables say $\varphi_1, \ldots, \varphi_{n-q}$ only, while the group does not act on the remaining $\varphi_{n-q+1}, \ldots, \varphi_n$ variables (Fubini's theorem). So we have reduced this case to the transitive one.

It is clear that after this reduction the integration of eqs. (2.IOI) will give only the $\varphi_1, \ldots, \varphi_{n-q}$ dependence of f_{AB} while their $\varphi_{n-q+1} \ldots \varphi_n$ dependence will be undetermined. That is the constants of integration of the solution of the system 2.IOI will not actually be constants but functions of $\varphi_{n-q+1}, \ldots \varphi_n$

a) Simply transitive group G_R

This is the case when R= I,...,n and rank($\mathfrak{I}_{Ra}$) =n (a=I,.. ..,n). Then eqs.(2.IOI) constitute actually a system of total differential equations whose explicit form can be obtained by multi-plication by $(J_{Sa})^{-1}$. The integrability conditions are

$$X_R(X_S f_{AB} + f_{AQ}\, g^Q_{BS} + f_{BQ}\, g^Q_{AS}) - X_S(X_R f_{AB} + f_{AQ}\, g^Q_{BR} + f_{BQ}\, g^Q_{AR}) = 0$$

or

$$g^Q_{RS} X_Q f_{AB} + f_{AL}(g^L_{QS} g^Q_{BR} - g^L_{QR} g^Q_{BS}) + f_{BL}(g^L_{QS} g^Q_{AR} - g^L_{QR} g^Q_{AS}) = 0$$

and by using the Jacobi's identity we obtain

$$g^Q_{RS}(X_Q f_{AB} + f_{AL} g^L_{BQ} + f_{BL} g^L_{AQ}) = 0 \qquad (\text{II.I})$$

which is satisfied because of the equations (2.IOI) themselves. In other words our system is completely integrable and so the solutions for f_{AB} will depend on $n(n+I)/2$ arbitrary constants which however in our case are restricted by the positive definiteness of f_{AB}.

c) Multiply transitive G_R $(R = I,2,\ldots,N)$

When G_R is multiply transitive (tha is $R > n$ and rank$(J_{Ra}) = n$ $a = I,\ldots,n$) among its generators $X_I,\ldots,X_R$ there are n linearly independent (with variable coefficients) and we can take them to be the first n

$$X_I,\ X_2,\ X_3,\ldots\ldots,X_n$$

Hence the determinant (J_{Sa})* will be different from zero while the rest of the genarators $X_{n+I},\ldots,X_N$ can be expressed linearly and homogeneously in terms of the first n

* The index S will always be assumed to run from I to n.

$$X_{n+j} = z_j^{\ S} X_S \qquad j = I,\ldots,N-n \tag{II.2}$$

where $z_j^{\ S}$ are functions of $\varphi_1, \ldots\ldots, \varphi_n$.

Since $\det(J_{Sa}) \neq 0$ we can solve the first n of eqs.(2.IOI) for $\partial\varphi_{AB}/\partial\varphi_a$. Introducing then these derivatives in the rest of eqs.(2.IOI) we get some linear and homogeneous relations for the f_{AB}'s. This way we see that we have actually a mixed system and we shall of all find the equations which determine the relatios among the f_{AB}'s. To this end we introduce in the equations

$$X_{n+j} f_{AB} = -f_{AQ} g^{Q}_{\ Bn+j} - f_{BQ} g^{Q}_{\ An+j}$$

the values of $X_{n+j} f_{AB}$ given by II.2

$$X_{n+j} f_{AB} = z_j^{\ S} X_S f_{AB}$$

From the above equations and the first n of eqs.2.IOIit follows easily that

$$z_j^{\ S} f_{AQ} g^{Q}_{\ BS} + z_j^{\ S} f_{BQ} g^{Q}_{\ AS} = f_{AQ} g^{Q}_{\ Bn+j} + f_{BQ} g^{Q}_{\ An+j}$$

or

$$f_{AQ} g^{Q}_{\ Bn+j} + f_{BQ} g^{Q}_{\ An+j} - z_j^{\ S} (f_{AQ} g^{Q}_{\ BS} + f_{BQ} g^{Q}_{\ AS}) = 0 \tag{II.3}$$

Equations II.3 provide the relations which the f_{AB}'s must satisfy and together with the following

$$X_S f_{AB} + f_{AQ} g^{Q}_{\ BS} + f_{BQ} g^{Q}_{\ AS} = 0 \tag{II.4}$$

they constitute a mixed systen of total differential equations.

We shall show in the following that this system is complete. That is, we shall show that a) the integrability conditions for eqs. II.4 are identically satisfied and b) the eqs. obtained when we apply the operator X_S to anyone of the eqs.II.3 and make use of eqs. II.4 are contained in eqs. II.3 . However condition a) has already been proved to be satisfied when we deduced eqs. II.I. We only note here that the calculations done for deducing II.I are also valid and when $R > n$. So we only need to prove b).

We first introduce the notation

$$H_{ABR} = f_{AQ} g^{Q}_{BR} + f_{BQ} g^{Q}_{AR}$$

Then eqs II.3, II.4 are written as

$$H_{ABn+j} - z_j^{\;S} H_{ABS} = 0 \qquad \text{(II.3')}$$

$$X_S f_{AB} + H_{ABS} = 0 \qquad \text{(II.4')}$$

and we must prove that due to these equations themselves

$$X_P (H_{ABn+j} - z_j^{\;S} H_{ABS}) = 0 \qquad P = I, \ldots, n$$

is identically true.

To this end we observe that because of eqs. II.I we have the identity

$$X_K(X_L f_{AB}) - X_L(X_K f_{AB}) + X_K H_{ABL} - X_L H_{ABK} = 0$$

for all values of K, L .In particular

$$X_K(X_{n+j} f_{AB}) - X_{n+j}(X_K f_{AB}) + X_K H_{ABn+j} - X_{n+j} H_{ABK} = 0 \qquad \text{(II.5)}$$

and

$$X_S(X_K f_{AB}) - X_K(X_S f_{AB}) + X_S H_{ABK} - X_K H_{ABS} = 0 \qquad \text{(II.6)}$$

Multipling the last one with $z_j^{\;S}$, summing over S and using II.5 we obtain

$$X_K(X_{n+j} f_{AB}) - z_j^{\;S} X_K(X_S f_{AB}) + X_K H_{ABn+j} - z_j^{\;S} X_K H_{ABS}$$
$$-(X_{n+j} H_{ABK} - z_j^{\;S} X_S H_{ABK}) = 0$$

Because of eq.II.4'it is evident that the following equqtion

$$X_K z_j^{\ S}(X_S f_{AB} + H_{ABS}) = 0$$

holds. Substraction of the above from II.6gives

$$X_K(X_{n+j} f_{AB} - z_j^{\ S} X_S f_{AB}) + X_K(H_{ABn+j} - z_j^{\ S} H_{ABS})$$
$$-(X_{n+j} - z_j^{\ S} X_S) H_{ABK} = 0$$

But because of II.2 the operation $X_{n+j} - z_j^{\ S} X_S$ on any function gives zero and hence we have

$$X_K(H_{ABn+j} - z_j^{\ S} H_{ABS}) = 0$$

which was to be proved.

We conclude that the mixed system of total differential equations II.3, 4 for f_{AB} is in fact a complete system . It will suffice then that the linear homogeneous equatiosII.3 at a fixed (generic) point of the space of the scalar fields $(\varphi_1^0, \dots ; \varphi_n^0)$ are compatible, that is they reduce to a number $q < N(N+I)/2$ of independent equations and the initial values $f_{AB}(\varphi_1^0, \cdot \varphi_n^0)$ can be taken so that f_{AB} is positive definite. This being so the general solution of our system will exist and will depend on $N(N+I)/2 - q$ arbitrary constants.

References

1. P.A.M. Dirac; Can.J.Math. 2, 129 (1950); Proc. Roy.Soc. A246, 333 (1958); Phys. Rev. 114, 924 (1959).
2. B.S. DeWitt; Phys. Rev. 160, 5, 1113 (1967).
3. P.W. Higgs, Phys. Rev. Letters 1, 373 (1958) 3, 66 (1959).
4. R. Arnowitt, S. Deser, C. Misner; Phys. Rev. 116, 5, 1322 (1959); Phys. Rev. 117 6 1595 (1960).
5. J. Wheeler, Geometrodynamics, Academic Press (1962); in Relativity, Groups and Topology. B.S. Dewitt and C. DeWitt, eds. (Blackie London 1963); Batelle Rencontres 1967.
6. A. Fischer, in Relativity, Carmeli, Fickler and Witten, eds., Plenum Press (1970).
7. M. Stern, A.B.,Thesis, Princeton Univ., Princeton (1967).
8. D.G. Ebin; Doctoral Thesis, M.I.T. Cambridge, Mass. (1967); Am.Math. Soc. Bull. 74, 1001 (1968); Proceedings of the 1968 A.M.S. Summer Inst. on Global Analysis.
9. R. Palais, Annals of Math. 73, 295 (1961).
10. B.S. DeWitt, in Relativity, loc. cit.
11. D. Christodoulou, Nuovo Cimento 26B, 67 (1975).
12. S.A. Hojman, K. Kuchar, C. Teitelboim; Nature 245 97 (1973).
13. C.J. Isham, Nuovo Cimento 61A, 188 (1969).
14. P. Jordan, Z. Physik, 157, 112 (1959).
15. C. Brans, R.H. Dicke, Phys. Rev. 124, 3, 925 (1961).
16. L. Bianchi, Lezioni sulla teoria dei gruppi continui finite di transfomazioni, Spoerri, Pisa (1918).
17. D. Christodoulou, M. Francaviglia, in Enrico Fermi School on Isolated Gravitating Systems in Gen. Rel., Varena 1976.