ERRATA

p. 36 line 11 insert $\exists$ between if and α

p. 51 lines 16 and 19 for X_1 read X_2

p. 52 for y read y* and for Y read Y*

p. 54 line 11 for Z read $Z^{\perp}$

p. 63 line 7 for when read When

p. 72 bottom line for y* read y

p. 81 line 10 for differential read differentiable

p. 83 line 10 for z read Z

p. 89 line 12 for of M read if M

p. 118 line 2 for $\frac{\partial u_o}{\partial n}$ read $\frac{\partial u}{\partial n}$

p. 123 equations (6.3.6) and (6.3.8) insert Ω after d on LHS

p. 126 line 3 for $(6.3.11)_1$, read $(6.3.13)_1$

p. 142 line 4 should be labelled (6.4.22)

p. 153 line 11 for ∇_2 read ∇_1

p. 154 line 20 for $H_o^1(\Omega)$ read $H_o^m(\Omega)$

p. 166 Line 7 for ^{n} read $\mathbb{R}^n$

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VARIATIONAL PRINCIPLES FOR LINEAR AND NON-LINEAR ELLIPTIC EQUATIONS

by

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ABSTRACT

A class of linear elliptic differential equations with homogeneous boundary conditions is considered. These equations are put into an abstract form

$$T^*ETu = f,$$

involving linear operators on Hilbert spaces. Conditions are given for the existence and uniqueness of solutions to this operator equation. A set of variational principles associated with the equation are derived. In the elasticity context these variational principles include the Hu-Washizu principle, the Hellinger-Reissner principle as well as the well known potential energy and complementary energy principles. Conditions are given for the variational problems to have a unique solution which is also the solution of the differential equation. The relationships between the different variational principles are also stressed.

The most general non-linear equations considered are those that can be put into the abstract form

$$T^*E(Tu) + F(u) = 0,$$

where E and F are non-linear operators. Conditions are given under which a unique solution can be shown to exist. The variational problems analogous to those of the linear case are derived and shown to have a unique solution corresponding to the solution of the differential equation. Finally it is shown how more general boundary conditions are incorporated into the theory given.

Throughout the development of the theory, examples are considered to bring out the relationship between the abstract formulation and its practical applications. In particular the operators and Hilbert spaces are given explicitly for both the abstract differential equations and the associated variational problems.

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CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Introduction

The study of elliptic differential equations plays a central role in mathematical physics. In this thesis we shall study an important class of linear elliptic equations and a related class of non-linear equations. The linear equations are of the form

$$T^*ETu = f, \tag{1.1.1}$$

where T, T* and E are linear operators on Hilbert spaces. A large number of problems in mathematical physics can be put into this form by the appropriate choice of the operators and Hilbert spaces. Some examples are: heat conduction in anisotropic media; fully-developed fluid flow in ducts; the torsion of cylindrical bars; the bending of elastic plates. Many other examples can be found in Arthurs [1970] and Oden-Reddy [1974].

The non-linear equations we shall study are of the form

$$T^*E(Tu) + F(u) = 0 \tag{1.1.2}$$

where T and T* are linear operators and E and F are non-linear operators on Hilbert spaces. Some examples of this class of equation are: non-Newtonian viscous fluid flow; charged particles in equilibrium; the theory of colloids. For further examples see Arthurs [1970].

All of the above examples have variational principles, such as the minimum potential energy principle or the minimum complementary energy principle, associated with them. The variational approach to the statement of problems has always played an important part in mathematical physics, indeed many problems are initially formulated in terms of a variational problem. For example the (small) displacements of an elastic solid subject to prescribed displacements and body forces can be found by minimizing the potential energy of the solid amongst all possible displacements, that is, those displacements satisfying the prescribed displacement boundary conditions. A variational principle is a very concise way of formulating a problem, both mathematically and physically. From the mathematical point of view there is only one functional to deal with, which incorporates many of the boundary conditions. Physically, one only needs to know a quantity such as the potential energy of the system to be able to completely specify the problem. Of course, the potential energy is itself a physically important parameter which we might need to compute separately when using any other approach to the problem.

Variational principles not only play an important role in the mathematical statement of physical problems but also in the computation of solutions. The best known computational techniques based on a variational method are of course the Rayleigh-Ritz and Galerkin methods. It is a variant of these methods, the finite element method, which, in the last twenty years, has proved to be a powerful computational method.

In this thesis we study a set of variational principles associated with each of the operator equations (1.1.1) and (1.1.2). Some of these are very well known principles but others have only been discovered in the last few years. However, all can be used as the basis for finite element methods.

1.2 Historical Review

In the eighteenth and early nineteenth centuries the calculus of variations played a central role in mechanics. The developments of that time are typified by the work of the Bernoullis, Euler, Lagrange, Legendre and Jacobi. For a more detailed account of that period see Kline [1972] or Bell [1945]. The developments of that period culminated in the proof that the solution of a variational principle could be found as the solution of a differential equation. Hence from then on the emphasis was on the statement of problems in mechanics in terms of differential equations. We shall give an outline of further developments in variational principles later. First, however, we shall give a brief review of the historical development of the relevant theory of abstract differential operators.

The theory of abstract operators in the form T*T was initiated by von Neumann [1932] and was extended by Murray [1935] and Friedrichs [1939]. They gave the basic structure in terms of adjoint linear operators on Hilbert spaces and proved many theorems on the existence and uniqueness of both operators and solutions to the operator equation. Kato [1953] and Fujita [1955] used the works of von Neumann and Murray to develop approximation methods for operators of the form T*T. They also gave some important examples of equations with this structure.

Implicit in the papers of von Neumann etc. were the concepts of generalised derivative and what are now called Sobolev spaces. These concepts were formalised in the late 1930's by Sobolev [1937]. In 1950 in his book "Theorie des Distributions", Schwartz [1950] gave an alternative basis for these concepts in terms of distributions. These developments lead to a great deal of work in the 1950's and 60's on the abstract theory of differential equations by many authors, see for example, Lions-Magenes [1972] and the references contained therein.

Returning to the development of variational principles, these continued from a more applied viewpoint, particularly in continuum mechanics. This led to the formulation of complementary variational methods by Castigliano [1879] in the late nineteenth century. In the early twentieth century an approximate method based on a variational method was developed by Ritz [1909] (repeating the work of Rayleigh [1871]) for solving problems in elasticity. This is now known as the Rayleigh-Ritz method and can be used for many problems in mechanics. A generalisation of this method was put forward by Galerkin [1915], which is now widely known as Galerkin's method.

An important advance in our knowledge of variational principles occurred in the 1950's, when Reissner [1950] correctly formulated an earlier attempt by Hellinger [1914] at a mixed variational principle. Hu [1955] and Washizu [1955] generalised this to a variational principle which incorporates a general form of constitutive equation. At the same time an approximate method, now known as the finite element method, was proposed, see Turner, et al. [1956]. This method was based on the minimum potential energy principle and was a variant of the Rayleigh-Ritz method. In the mid 1960's Herrmann [1966] proposed the use of a finite element method based on a mixed variational principle for the solution of plate bending problems.

In 1964 Noble [1964] gave a general framework for complementary variational principles. This was further developed by Arthurs [1969], Rall [1966] and Noble-Sewell [1971]. Arthurs developed an extensive range of applications in the area of differential equations, see Arthurs [1970].

In the late 1960's and early 1970's a rigorous attack on the finite element method based on a minimum principle was made by several mathematicians, e.g. Babuska [1971], Ciarlet-Raviart [1972], Strang [1972] and Zlamal [1968]. In 1973, Johnson [1973] gave a rigorous account of a mixed finite element method applied to a problem in plate bending. Brezzi [1974] developed an abstract formulation which can include mixed variational principles. Recently Oden-Reddy [1974] have attempted to put the theory of Noble into a more rigorous setting.

1.3 Thesis Objectives

The major objective of this thesis is to give a rigorous account of the variational principles associated with classes of elliptic differential equations. More precisely, for each class of equations, we

1) show how differential equations may be put into one of the abstract forms considered,
2) give existence and uniqueness theorems for the abstract operator equation,
3) derive the set of variational principles associated with the abstract equation,

and

4) show the relationships between the variational principles.

The problem of putting differential equations into an abstract form was of course considered at the very beginning of the development of the theory of abstract operators. For operators of the form T^*T see von Neumann [1932], Murray [1935] or Friedrichs [1939]. However, in the last twenty five years there have been considerable developments in the treatment of boundary value problems from the abstract point of view, see Lions-Magenes [1972] or Necas [1967]. We adopt an approach originally due to Friedrichs [1939], who considered operators acting from Hilbert spaces. However, we incorporate many of the modern techniques, in particular we think of the Hilbert spaces as Banach spaces. This allows us a greater variety of norms which are essential for identifying the abstract operators with bounded linear differential operators.

The conditions for the existence and uniqueness of solutions of linear operator equations are well known, see Stone [1932]. In this thesis we give a simple existence and uniqueness proof for operators of the form T^*T. This proof forms the basis of the existence theorems we shall give for the more complicated linear and non-linear equations considered.

An extensive set of interrelated variational principles for linear equations have only recently been elaborated by Oden-Reddy [1974]. In this thesis we give conditions under which each variational problem is in fact equivalent to the abstract differential equation. Similar results are derived for some non-linear equations.

Relationships between various variational principles have been suggested by many authors, for example, see Washizu [1975] and Oden-Reddy [1974]. In this thesis we take the approach in which we consider the general variational principle as a starting point. From there, by successive specializing assumptions we derive all the variational principles associated with a particular equation.

1.4 Outline of the Thesis

In Chapter 2 we shall consider some simple problems which give rise to linear or non-linear differential equations. We shall use these examples throughout the thesis to illustrate how the theoretical results can be applied to practical examples. We show how these examples can be put into an abstract setting involving operators on Hilbert spaces. We then consider the simplest abstract problem

$$T^*Tu = f$$

and give conditions on the operators T and T* for a unique solution to exist.

In Chapter 3 we give some theoretical results on saddle functionals, derive the main variational problem associated with the abstract form given in Chapter 2, and show how other variational problems can be derived using the concept of saddle functionals.

Chapter 4 is concerned with the extensions required to the theory given in Chapters 2 and 3 to include the abstract equation

$$T^*ETu = f,$$

where E is a linear operator.

In Chapter 5 we study some non-linear equations, the most general being of the form

$$T^*E(Tu) + F(u) = 0,$$

where E and F are non-linear operators. Existence and uniqueness theorems are given and the variational principles corresponding to those in the linear case are derived.

Chapter 6 shows how more general boundary conditions can be incorporated into the theoretical results given. Non-homogeneous Dirichlet and Neumann type boundary conditions are treated, a mixture of these being the most general form considered.

Finally in Chapter 7 we summarize the material covered and suggest some directions for further work on this subject.

CHAPTER 2

THE ABSTRACT PROBLEM

2.1 Types of Problem Considered

In this section we shall briefly give an example for each class of problem to be considered. In a later section we will show rigorously how these examples can be put into the abstract forms. We shall use these examples throughout the text to demonstrate how the abstract theory can be applied to practical problems. For each class of equations we shall also indicate other important problems that can be put into the abstract form but we shall not give all the details here.

Example 2.1.1

First we shall consider a very simple ordinary differential equation which we shall subsequently study in great detail to bring out the important points of the abstract theory. Consider the equation

$$\begin{aligned} -\frac{d^2u}{d\xi^2} &= f \quad \text{in } \Omega \\ u(a) &= u(b) = 0 \end{aligned} \tag{2.1.1}$$

where f is a given function of ξ and Ω is the interval (a,b). We can rewrite this as the problem of finding the functions u and v such that

$$\begin{aligned} \frac{du}{d\xi} &= v, \\ -\frac{dv}{d\xi} &= f, \\ u(a) &= u(b) = 0 \end{aligned} \tag{2.1.2}$$

This is in the canonical form

$$\tau T u = v$$
$$T^* v = f \qquad (2.1.3)$$

where $T = \frac{d}{d\xi}$, $T^* = -\frac{d}{d\xi}$ and τ is an identity mapping. Obviously this can be written as the single equation

$$T^* \tau T u = f. \qquad (2.1.4)$$

The boundary conditions are included in the abstract form by ensuring that the domain of the operator T consists only of those functions satisfying the boundary conditions, that is, those u such that

$$u(a) = u(b) = 0.$$

We may of course consider other boundary conditions for this problem, for example

$$\frac{du(a)}{d\xi} = \frac{du(b)}{d\xi} = 0.$$

This is equivalent to the conditions

$$v(a) = v(b) = 0 \qquad (2.1.5)$$

and in this case the domain of T^* would have to consist of only those functions satisfying (2.1.5).

Many other differential equations may be put into the form of equation (2.1.4), here we mention two. Firstly, Poisson's equation

$$-\nabla^2 u = f \quad \text{in } \Omega$$
$$u = 0 \text{ on } \Gamma \tag{2.1.6}$$

on a region $\Omega \subset \mathbb{R}^n$ with boundary Γ. Taking T = grad, T^* = - div and $\tau = I$, (2.1.6) is in the form (2.1.4), provided the domain of T is restricted to those functions satisfying the boundary condition $u = 0$ on Γ. Secondly, the biharmonic equation

$$\nabla^4 u = f \text{ in } \Omega$$
$$u = \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma \tag{2.1.7}$$

where n is the outward normal to Γ, the boundary of Ω. Here we take T and T^* to be ∇^2 and τ to be the identity mapping. Then if the domain of T is restricted to those functions satisfying the boundary conditions $u = \frac{\partial u}{\partial n} = 0$ on Γ, this is in the form of (2.1.4).

Example 2.1.2

Next we consider a problem in heat conduction in an anisotropic material, such as for example crystalline substances, sedimentary rocks, wood, asbestos and laminated materials, e.g. transformer cores used in engineering practice.

For the domain Ω let us consider the rectangular region $0 < \xi_1 < a$, $0 < \xi_2 < b$ of the $\mathbb{R}^2$ plane. The boundary Γ is then as shown in figure 2.1.1. The differential equation is

$$-\nabla.(K.\nabla u) = f \text{ in } \Omega$$
$$u = 0 \text{ on } \Gamma. \tag{2.1.8}$$

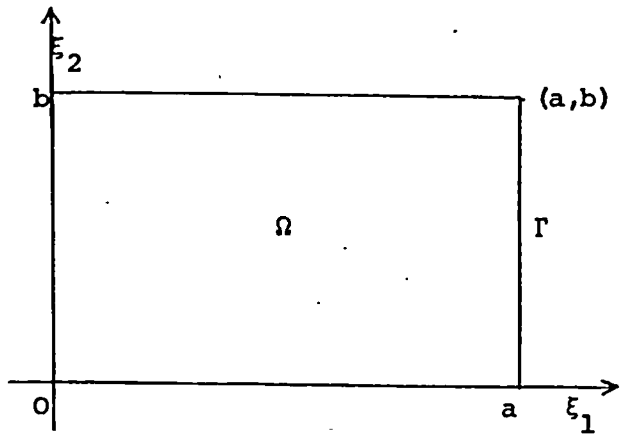

Figure 2.1.1 The domain of Example 2.1.2

Here u is the temperature field on Ω, f is the heat source in Ω and $K = [k_{ij}]$ is a tensor of thermal conductivities. The components of K represent the thermal conductivities in different directions and in practice K is considered to be a symmetric tensor.

Note that it is always possible to transform the anisotropic equations to a new set of axes called the principal axes of conductivity such that the equation is in the same form as for an isotropic material i.e. $k_{11} = k_{22} = k$, $k_{12} = k_{21} = 0$, see Carslaw-Jaeger [1947]. However, this transformation can considerably complicate the boundary conditions and distort the boundary to irregular shapes. Hence it is often easier to treat the material as anisotropic and retain "nice" boundary conditions.

Equation (2.1.8) can be put into the canonical form

$$\begin{aligned} \nabla u &= w \\ v &= Kw \qquad \text{in } \Omega \\ -\nabla . v &= f \end{aligned} \tag{2.1.9}$$

with $u = 0$ on Γ.

Hence setting $T = \nabla$, $T^* = -\nabla$, and $E = K$, this can be put in the form

$$\begin{aligned} Tu &= w \\ v &= Ew \\ T^*v &= f. \end{aligned} \tag{2.1.10}$$

As in the previous example the boundary condition is included by taking the domain of T to be only those functions which satisfy the boundary conditions. Equations (2.1.10) can of course be written as

$$T^*ETu = f. \tag{2.1.11}$$

Another important example of an equation of this form occurs in linear elasticity. The equations for the small displacements of an elastic body with domain $\Omega \subset \mathbb{R}^3$ are

$$\begin{aligned} \frac{1}{2}\left(\frac{\partial u_i}{\partial \xi_j} + \frac{\partial u_j}{\partial \xi_i}\right) &= w_{ij} \\ c_{ijkl}\, w_{kl} &= v_{ij} \qquad \text{in } \Omega \\ -\frac{\partial v_{ij}}{\partial \xi_j} &= f_i \end{aligned} \tag{2.1.12}$$

where u_i is the displacement in direction ξ_i, $i = 1,2,3$, w_{ij} is the strain tensor, c_{ijkl} is the tensor of elastic coefficients, v_{ij} is the stress tensor and f_i is the vector of applied body forces. Then taking T to be the operator defined by the first of equations (2.1.12), T* to be the operator

$$\begin{pmatrix} -\dfrac{\partial}{\partial \xi_1} \\ \\ -\dfrac{\partial}{\partial \xi_2} \\ \\ -\dfrac{\partial}{\partial \xi_3} \end{pmatrix}$$

and E the tensor c_{ijkl}, equations (2.1.12) can be written in the form of (2.1.10).

Example 2.1.3

Here we consider an equation of the form

$$T^*E(Tu) = f \qquad (2.1.13)$$

where E is a non-linear operator. The canonical form of this equation is

$$\begin{aligned} Tu &= w \\ E(w) &= v \\ T^*v &= f\,. \end{aligned} \qquad (2.1.14)$$

The non-linear relationship $E(w) = v$ occurs in many physical situations where it represents the constitutive equations. For example taking w to be the strain and v to be the stress $(2.1.14)_2$ represents a non-linear stress-strain relationship. This is often given by a power law formula of the form

$$v = c_1 w^{c_2} \qquad (2.1.15)$$

where c_1, $c_2 > 0$ are constants (see Reiner [1969]). Another example of this type of constitutive equation is in the magnetic saturation problem, where w represents the magnetic intensity and v is the flux density, (see Cunningham [1967]).

A simple example that we shall consider in more detail arises in the laminar flow of an incompressible non-Newtonian viscous fluid in a circular cylinder of radius a and length l, see Jaeger [1969]. The equations are

$$\frac{d\dot{u}}{dr} = \dot{\gamma}\ , \tag{2.1.16}$$

$$k\dot{\gamma}^{1/3} = \sigma, \tag{2.1.17}$$

$$-\frac{d\sigma}{dr} = \frac{P}{2\ell}\ , \tag{2.1.18}$$

$$\dot{u}(a) = \dot{u}(-a) = 0, \tag{2.1.19}$$

where $\dot{u}$ is the axial velocity, $\dot{\gamma}$ is the rate of strain, $k > 0$ is a constant, σ is the stress and P is the pressure applied at one end of the cylinder. Setting T to $\frac{d}{dr}$ and T* to $-\frac{d}{dr}$, with $E(\dot{\gamma}) = k\dot{\gamma}^{1/3}$ and $f = \frac{P}{2\ell}$, equations (2.1.16)-(2.1.19) are in the form of equations (2.1.14). The boundary conditions (2.1.19) are satisfied by choosing the correct domain for T as in the previous example.

<u>Example 2.1.4</u>

As a final example we consider a non-linear equation which has applications in many fields. It is the Liouville equation

$$\begin{aligned} -\nabla^2 u + ce^u &= 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma \end{aligned} \tag{2.1.20}$$

where c is a positive constant. This equation arises for example in plasma theory. Taking $u = \frac{qV}{kT}$, where q is the charge per particle, V is the electrostatic potential and kT is the Boltzmann energy, equation (2.1.20) gives the potential of charged particles in equilibrium. For more details see Arthurs [1970] or Longmire [1963]. Another example is the case of steady vortex motion of an incompressible fluid. For this problem equation (2.1.20) can be used to find the stream function u, see Davis [1960].

Equation (2.1.20) can be put into the abstract form

$$T^*\tau Tu + F(u) = 0 \tag{2.1.21}$$

where, as in example 2.1.2, T = grad and $T^* = -\text{div}$. The isomorphism τ is the identity mapping and the non-linear term F(u) is ce^u. As in the other examples the boundary condition is incorporated into the domain of the operator T. Equation (2.1.21) can of course be written in the canonical form

$$\begin{aligned} Tu &= w \\ v &= \tau w \\ T^*v + F(u) &= 0 . \end{aligned} \tag{2.1.22}$$

For other examples of equations of this type see the monograph of Arthurs [1970].

2.2 Some Results from Functional Analysis

In this section we introduce some notation and give results from functional analysis which will be needed to put the problems of section 1 into an abstract setting.

Let X be a Banach space. The norm on X will be denoted by $\|\cdot\|_X$, the dual of X by X* and the duality pairing between X and X* by ${}_{X^*}\langle\cdot,\cdot\rangle_X$. If X is a Hilbert space the canonical isomorphism between X and X* is denoted by τ_X and the inner product on X by $(\cdot,\cdot)_X$. When there is no ambiguity, $\|\cdot\|_X$, ${}_{X^*}\langle\cdot,\cdot\rangle_X$, τ_X and $(\cdot,\cdot)_X$ may be written as $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, τ and $(\cdot,\cdot)$ respectively. Let X and Y be Hilbert spaces with X a dense subset of Y. Identifying Y with Y* via the canonical isomorphism τ_Y, we have $X \subset Y \simeq Y^* \subset X^*$. Then if $x^* \in Y^* \subset X^*$ and $x \in X$, we have, see Lions-Magenes [1972],

$$ {}_{X^*}\langle x^*,x\rangle_X = {}_{Y^*}\langle x^*,x\rangle_Y = (\tau_Y^{-1}x^*,x)_Y. $$

A point in $\mathbb{R}^n$ is denoted by $\xi = (\xi_1,\xi_2,\ldots,\xi_n)$. If $\alpha = (\alpha_1,\ldots,\alpha_n)$ is a n-tuple of nonnegative integers α_j, we call α a multi-index and denote by ξ^α the monomial $\xi_1^{\alpha_1}\xi_2^{\alpha_2}\ldots\xi_n^{\alpha_n}$, which has degree $|\alpha| = \sum_{j=1}^{n} \alpha_j$. Similarly, if $D_j = \frac{\partial}{\partial\xi_j}$ for $1 \le j \le n$, then

$$ D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2}\ldots D_n^{\alpha_n} $$

denotes a differential operator of order $|\alpha|$.

Let Ω be an open domain in $\mathbb{R}^n$ with Γ its boundary. We denote $\Omega + \Gamma$ by $\bar{\Omega}$. For any nonnegative integer m let $C^m(\bar{\Omega})$ be the vector space of all functions x which, together with all their partial derivatives $D^\alpha x$ of orders $|\alpha| \le m$, are continuous and bounded on Ω. We let $C^0(\bar{\Omega})$ be denoted by $C(\bar{\Omega})$ and $C^\infty(\bar{\Omega}) = \bigcap_{m=0}^{\infty} C^m(\bar{\Omega})$.

We next define the important concept of a closed operator. This is most easily expressed in terms of the graph of an operator.

Definition 2.2.1

Let T be an operator from a set X to a set Y. Then the graph G of T is the subset of X×Y consisting of the ordered pairs (x,Tx), where x ranges over the domain of T.

Definition 2.2.2

Let X and Y denote Banach spaces, then an operator $T:X \to Y$ is called a closed operator if its graph is a closed subset of X×Y.
An important property of closed operators is that they map closed sets into closed sets. They play an important role in existence theorems which are based on Banach's closed range theorem.

Example 2.2.1

Consider the operator $\frac{d}{d\xi}$ defined on functions belonging to the space $C^1(\overline{\Omega})$, where Ω is an open interval of the real line. Let the norm on $C^1(\overline{\Omega})$ be

$$\|x\|_{C^1(\overline{\Omega})} = \left[\int_\Omega \{x^2 + (\frac{dx}{d\xi})^2\} d\Omega\right]^{1/2} .$$

Then $\frac{d}{d\xi}$ is a continuous linear operator from $C^1(\overline{\Omega})$ to $C(\overline{\Omega})$, with $\|x\|_{C(\overline{\Omega})} = [\int_\Omega x^2 d\Omega]^{1/2}$, since

$$\|\frac{dx}{d\xi}\|_{C(\overline{\Omega})} = \left[\int_\Omega (\frac{dx}{d\xi})^2 d\Omega\right]^{1/2} \leq \|x\|_{C^1(\overline{\Omega})} .$$

It is also a closed operator since for any sequence $\{x_n\} \to x$, with $x_n \in C^1(\overline{\Omega})$ for all n and $x \in C^1(\overline{\Omega})$,

$$\frac{dx_n}{d\xi} \to \frac{dx}{d\xi} \in C(\overline{\Omega}) . \qquad \square$$

This brings out an important point for neither the domain of $\frac{d}{d\xi}$ i.e. $C^1(\overline{\Omega})$, nor the range $C(\overline{\Omega})$ is a closed space with the norms given, although $\frac{d}{d\xi}$ is a closed operator. However, in order to prove existence theorems, it is essential to consider closed operators with closed domains. Hence the question is, can we find some generalisation of the operator $\frac{d}{d\xi}$ which has a closed domain?

We can answer this by considering an extension of the operator.

<u>Definition 2.2.3</u>

Let $T_1:D_1 \subset X \to Y$ and $T_2:D_2 \subset X \to Y$ be operators with graphs G_1 and G_2 respectively. Then T_2 is said to be an <u>extension</u> of T_1 if $G_2 \supset G_1$.

We shall want to find a closed extension of an operator T. The following theorems to be found in Stone [1932] guarantee the existence and uniqueness of a closed linear extension.

<u>Theorem 2.2.1</u>

If the operator T_1 has a closed linear extension, then there exists a unique closed linear operator T with the properties:

(1) T is an extension of T_1,

(2) every closed linear extension of T_1 is also an extension of T.

In essence, T is the smallest closed linear extension of T_1.

<u>Theorem 2.2.2</u>

If T_1 is a continuous linear operator, then the extension T exists and is a continuous operator whose domain is the closure of the domain of T_1; also $\|T_1\| = \|T\|$.

Theorem 2.2.3

If T_1 is a continuous linear operator whose domain is dense in X, then T is defined throughout X and is the only closed linear extension of T_1.

Example 2.2.2

From theorem 2.2.2 a closed extension of $\frac{d}{d\xi}$ exists and its domain X is the closure of $C^1(\overline{\Omega})$. Theorem 2.2.3 guarantees the uniqueness of this extension if $C^1(\overline{\Omega})$ is dense in X. □

We now describe the nature of the extended operator T and its domain X. For this we will introduce the notions of generalised derivatives and Sobolev spaces.

We define the space

$$\mathcal{D}(\Omega) = \{\phi \in C^{\infty}(\overline{\Omega});\quad \phi \text{ has compact support in } \Omega\}.$$

Now suppose x is a locally integrable function on Ω. Then we have the following definition:

Definition 2.2.5

If there exists a locally integrable function y such that

$$\int_{\Omega} x(\xi)D^{\alpha}\phi(\xi)d\Omega = (-1)^{|\alpha|} \int_{\Omega} y(\xi)\phi(\xi)d\Omega,$$

for all $\phi \in \mathcal{D}(\Omega)$, then y is called the <u>generalised derivative</u> of x. □

If x is sufficiently smooth to have a continuous partial derivative $D^{\alpha}x$ in the usual sense, then $D^{\alpha}x$ is also a generalised derivative of x. However, $D^{\alpha}x$ may exist in the generalised sense without existing in the classical sense. When $\Omega \subset \mathbb{R}$ we sometimes use the notation $\frac{d}{d\xi}$ to mean a derivative in the generalised sense.

Next we introduce Sobolev spaces. We restrict ourselves to the family of Sobolev spaces based on the $L_2(\Omega)$ space.

Define the norm

$$\|x\|_{m,\Omega} = \{ \sum_{0\leq|\alpha|\leq m} \|D^\alpha x\|^2\}^{1/2}$$

where $\|\cdot\|$ is the $L_2(\Omega)$ norm, i.e.

$$\|x\| = \{ \int_\Omega x^2 d\Omega\}^{1/2}.$$

Where no confusion can occur we write $\|\cdot\|_{m,\Omega}$ as $\|\cdot\|_m$.

<u>Definition 2.2.6</u>

The <u>Sobolev space</u> $H^m(\Omega)$, $m \geq 0$ is defined as the completion of $C^m(\overline{\Omega})$ with respect to the norm $\|\cdot\|_{m,\Omega}$. □

The space $H^0(\Omega)$ is in fact $L_2(\Omega)$.

As with the space $L_2(\Omega)$, the elements of $H^m(\Omega)$ are equivalence classes of functions.

It can be proved that $H^m(\Omega)$ is a Hilbert space with inner product

$$(x_1,x_2) = \sum_{0\leq|\alpha|\leq m} \int_\Omega D^\alpha x_1 D^\alpha x_2 d\Omega.$$

An equivalent definition of a Sobolev space is as follows:

$$H^m(\Omega) = \{x \in L_2(\Omega);\ D^\alpha x \in L_2(\Omega) \text{ for } 0 \leq |\alpha| \leq m\},$$

where $D^\alpha x$ is a derivative in the generalised sense.

An important subspace of $H^m(\Omega)$ is $H^m_0(\Omega)$, the closure of $\mathcal{D}(\Omega)$ in $H^m(\Omega)$. A more illuminating characterisation of this space is

$$H_0^m(\Omega) = \{x \in H^m(\Omega);\ \frac{\partial^j x}{\partial \nu_j} = 0 \text{ on } \Gamma\ ,\ 0 \leq j \leq m-1\}$$

where $\frac{\partial^j x}{\partial \nu_j}$ is the "j-order normal derivative on Γ". A definition of what we mean by $\frac{\partial^j x}{\partial \nu_j}$ on Γ will be given in a later chapter. We also define the space $H^{-m}(\Omega)$, $m > 0$ as the dual of the space $H_0^m(\Omega)$. See Adams [1975] for these and many other results on Sobolev spaces.

Example 2.2.3

It is now easily shown that the closed extension of $\frac{d}{d\xi}$ discussed in example 2.2.2 is the generalised derivative D , its domain X is $H^1(\Omega)$ and its range is in $L_2(\Omega)$. We can see that D is closed from the following:

$$||Dx|| = \{\int_\Omega (Dx)^2 d\xi\}^{1/2} \leq \{\int_\Omega [x^2 + (Dx)^2] d\xi\}^{1/2} = ||x||_1$$

i.e. D is bounded. Hence the range of D is a closed subspace of $L_2(\Omega)$ and so D is closed. □

Next we give a general theorem which gives a simple condition for an operator to be closed.

Theorem 2.2.4

Let X and Y be Banach spaces and T a bounded linear operator with domain $D(T) \subset X$ and range in Y. If there exists an $\alpha > 0$ such that

$$||Tx||_Y \geq \alpha ||x||_X \qquad \forall x \in D(T), \tag{2.2.6}$$

then T is a closed operator.

Proof Let $\{x_n\}$ be a sequence in D(T) such that $\{Tx_n\}$ converges to $y \in Y$. Then from (2.2.6) $\{x_n\}$ is a Cauchy sequence and converges to $x \in X$. Now by the boundedness of T, $\{Tx_n\}$ converges to $Tx \equiv y$. Hence $x \in D(T)$ and the graph of T is closed. □

Finally we give some important results for linear and non-linear operators.

Theorem 2.2.5 (see e.g. Rudin [1973], p.93)

Suppose X and Y are normed spaces. To each bounded linear operator $T:X \to Y$ there corresponds a unique bounded linear operator $T^*:Y^* \to X^*$ that satisfies

$$<Tx,y^*> = <x,T^*y^*>$$

for all $x \in X$ and all $y^* \in Y^*$.

Definition 2.2.7

The operator T^* defined in theorem 2.2.5 is called the adjoint of the operator T.

Definition 2.2.8 (Vainberg [1973], p.10)

Suppose X is a Banach space. Then an operator $T:X \to X^*$ is said to be positive if

$$<Tx,x> \geq 0, \quad \forall x \in X.$$

Definition 2.2.9

Suppose X is a Banach space. Then the operator $T:X \to X^*$ is said to be symmetric if

$$<Tx_1,x_2> = <Tx_2,x_1> \quad \forall x_1,x_2 \in X. \qquad \square$$

We now give some results on the differentiation of operators. For these and other results see Vainberg [1973] or Tapia [1971]. Let X and Y be Banach spaces and $T:X \to Y$ an operator which need not be linear.

Definition 2.2.10

Let x and h belong to X and t be a real scalar and suppose

$$DT(x,h) = \lim_{t \to 0} \left\{ \frac{T(x+th) - T(x)}{t} \right\}$$

exists. Then $DT(x,h) \in Y$ is called the Gateaux derivative or G-derivative of T at x in the direction h. □

In the case where Y is the real line, the mapping T is a functional on X. An alternative definition can then be given.

Definition 2.2.11

If f is a functional on X, the Gateaux derivative of f, if it exists, is

$$Df(x,h) = \left[\frac{d}{dt} f(x + th)\right]_{t=0} .$$

Definition 2.2.12

A functional f on X is said to be G-differentiable if Df(x,h) exists for all $x \in X$ and all $h \in X$. □

Note that for each fixed $x \in X$, Df(x,h) is a functional with respect to the variable $h \in X$.

Definition 2.2.13

Suppose f is a functional on X, such that Df(x,h) is a bounded linear functional. Then the unique element $\nabla f(x) \in X^*$ such that

$$Df(x,h) = \langle \nabla f(x), h \rangle, \qquad h \in X$$

is called the gradient of f at x.

Further, if X is a product space, i.e. $X = X_1 \times X_2$, then with $x = (x_1,x_2)$, $h = (h_1,h_2)$ we can define $\nabla_1 f(x_1,x_2) \in X_1^*$ and $\nabla_2 f(x_1,x_2) \in X_2^*$ by

$$\begin{aligned} Df(x,h) &= \langle \nabla f(x),h \rangle \\ &= {}_{X_1^*}\langle \nabla_1 f(x_1,x_2),h_1 \rangle_{X_1} + {}_{X_2^*}\langle \nabla_2 f(x_1,x_2),h_2 \rangle_{X_2} . \end{aligned}$$

Then $\nabla_1 f(x_1,x_2)$ and $\nabla_2 f(x_1,x_2)$ are called the partial gradients of f. The extension to a product of more than two spaces is easily deduced from this.

2.3 The Problems in an Abstract Setting

In this section we shall show how the problems of section 2.1 may be put into an abstract form. All of the problems can be written in terms of a pair of adjoint operators T and T^*. The main idea is to define T (or T^*) as the closed linear extension of one of the differential operators in the problem. We then formally define the adjoint operator T^* (or T). Then we give conditions under which the abstract equation involving T and T^* is equivalent to our original problem. This approach is a generalisation of that of Friedrichs [1939] to Banach space adjoint operators.

Consider the first example of section 2.1. Equations (2.1.2) for this problem are

$$\begin{aligned} \frac{du}{d\xi} &= v \\ -\frac{dv}{d\xi} &= f \end{aligned} \qquad (2.3.1)$$

$$u(a) = u(b) = 0.$$

We recall from section 2.2 that the space $H_0^1(\Omega)$ is the closure of the space of smooth functions satisfying the boundary conditions $(2.3.1)_3$. Hence we can extend the operator $\frac{d}{d\xi}$ to the closed linear operator D defined on $H_0^1(\Omega)$. Therefore we define T to be the generalised derivative D with domain $H_0^1(\Omega)$ and range in $L_2(\Omega)$.

We can formally define the adjoint operator $T^*: (L_2(\Omega))^* \to H^{-1}(\Omega)$ by

$$(Tx,y) = \langle x,T^*y^*\rangle \qquad \forall x \in H_0^1(\Omega), \quad \forall y^* \in (L_2(\Omega))^* \tag{2.3.2}$$

with $\quad y^* = \tau y$,

where $\langle\cdot,\cdot\rangle$ is the duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Then consider the problem

$$\begin{aligned} \tau T u &= v^* \\ T^* v^* &= f^* \end{aligned} \tag{2.3.3}$$

where $f^* \in H^{-1}(\Omega)$ is given. We want to show that equations (2.3.1) and equations (2.3.3) are equivalent in some sense.

Recall that the integration by parts formula is valid for generalised derivatives, see Necas [1967]. Hence

$$\int_\Omega Dx\ y\ d\Omega = -\int_\Omega x\ Dy\ d\Omega, \quad \forall x \in H_0^1(\Omega), \forall y \in H^1(\Omega). \tag{2.3.4}$$

Comparing with (2.3.2) we see that, with $y^* = \tau y$,

$$-\int_\Omega x\,Dy\,d\Omega = {}_{H_0^1(\Omega)}\langle x,T^*y^*\rangle_{H^{-1}(\Omega)}\ ,\quad \forall x \in H_0^{\,1}(\Omega),\ \ \forall y \in H^1(\Omega)$$

$$= {}_{L_2(\Omega)}\langle x,T^*y^*\rangle_{(L_2(\Omega))^*}\ , \text{ from section 2.2,}$$

$$= (x,\ \tau^{-1}T^*\tau y).$$

Hence for all $y \in H^1(\Omega)$ we may identify $\tau^{-1}T^*\tau y$ with $-Dy$. Now let us assume that (2.3.1) has a solution (u,v) and further, let us assume that $f \in L_2(\Omega)$. Then we certainly have that $v \in H^1(\Omega)$ and so $\tau^{-1}T^*\tau v$ can be identified with $-Dv$. Let $v^* = \tau v$ and $f^* = \tau f$, then equation $(2.3.3)_2$ can be identified with equation $(2.3.1)_2$. Hence any solution (u,v) of (2.3.1) gives a solution $(u,v^* = \tau v)$ of (2.3.3).

Let us now consider the problem with the boundary conditions (2.1.5), that is,

$$\begin{aligned} \frac{du^*}{d\xi} &= v^* \\ -\frac{dv^*}{d\xi} &= f^* \\ v^*(a) = v^*(b) &= 0 \end{aligned} \tag{2.3.5}$$

In this case we define T^* to be the extension of $-\frac{d}{d\xi}$, i.e. $-D$, the generalised derivative with domain $H_0^{\,1}(\Omega)$ and range in $L_2(\Omega)$. Then $(2.3.5)_2$ and the boundary conditions $(2.3.5)_3$ are given by $T^*v^* = f^*$.

We can formally define the operator $T : (L_2(\Omega))^* \to H^{-1}(\Omega)$ as the adjoint of T^*, that is,

$$_{H^{-1}(\Omega)}\langle Tx,y^*\rangle_{H_0^{\,1}(\Omega)} = (x^*,T^*y^*),\quad \forall x^* \in L_2(\Omega),\ \ \forall y^* \in H_0^{\,1}(\Omega) \tag{2.3.6}$$

with $x^* = \tau x$.

Then consider the problem

$$\begin{aligned} \tau T u &= v^* \\ T^* v^* &= f^* \end{aligned} \tag{2.3.7}$$

where $f^* \in L_2(\Omega)$ is given. We shall show that equations (2.3.5) and (2.3.7) are equivalent in some sense.

The integration by parts formula for generalised derivatives is

$$\int_\Omega Dx^*y^*d\Omega = -\int_\Omega x^*Dy^*d\Omega, \quad \forall x^* \in H^1(\Omega), \quad \forall y^* \in H_0^{\,1}(\Omega).$$

Comparing with (2.3.6) we have, with $x^* = \tau x$,

$$\begin{aligned} \int Dx^*y^*d\Omega &= {}_{H^{-1}(\Omega)}\langle Tx, y^* \rangle_{H_0^{\,1}(\Omega)}, \quad \forall x \in H^1(\Omega), \ \forall y^* \in H_0^{\,1}(\Omega) \\ &= {}_{(L_2(\Omega))^*}\langle Tx, y^* \rangle_{L_2(\Omega)} \\ &= (\tau T \tau^{-1} x^*, y^*). \end{aligned}$$

Hence for all $x^* \in H^1(\Omega)$ we identify $\tau T \tau^{-1} x^*$ with Dx^*. Let us assume that (2.3.5) has a solution (u^*, v^*). We also assume that $f^* \in L_2(\Omega)$, in which case we certainly have that $u^* \in H^1(\Omega)$. Hence $\tau T \tau^{-1} u^*$ can be identified with Du^*. Let $\tau u = u^*$, then equation (2.3.5) can be identified with equation $(2.3.7)_1$. Hence any solution (u^*, v^*) of (2.3.5) gives a solution $(u = \tau^{-1} u^*, v^*)$ of (2.3.7).

All of the examples in section 2.1 can be put into an abstract form as we have just done for the first example. Here we briefly state more precisely the operators and spaces involved in examples 2, 3 and 4.

In Example 2 the equations are

$$\begin{aligned} \operatorname{grad} u &= w \\ v &= Kw \\ -\operatorname{div} v &= f. \end{aligned}$$

We may take the operator T to be grad in a generalised sense with domain $X = H_0^{\,1}(\Omega)$ and range $Y = (L_2(\Omega))^2$. Then provided $f \in L_2(\Omega)$ we can interpret $T^*: [(L_2(\Omega))^2]^* \to H^{-1}(\Omega)$ as -div, at least at the solution. The operator $E: (L_2(\Omega))^2 \to [(L_2(\Omega))^2]^*$ is the tensor of heat transfer coefficients K.

In example 3 the equations are

$$\frac{d\hat{u}}{dr} = \dot{\gamma}$$

$$\sigma = k\dot{\gamma}^{\,1/3}$$

$$-\frac{d\sigma}{dr} = \frac{P}{2\ell} .$$

Provided $\frac{P}{2\ell} \in L_2(\Omega)$, the operators T and T* and the spaces X and Y are taken as in example 1, i.e. equation (2.3.1). The operator $E: L_2(\Omega) \to (L_2(\Omega))^*$ is defined by

$$E(\dot{\gamma}) = k\dot{\gamma}^{\,1/3} .$$

For the fourth example

$$\begin{aligned} \operatorname{grad} u &= w \\ v &= \tau w \\ -\operatorname{div} v + ce^{u} &= 0 , \end{aligned}$$

we take T and T* and X and Y as in example 2. The operator $F:H_0^1(\Omega) \to H^{-1}(\Omega)$ is a non-linear operator given by

$$F(u) = ce^u$$

for which we certainly have the range of F, $R(F) \subset L_2(\Omega)$.

2.4 The Simplest Abstract Problem

The main result of this section is a proof of the existence and uniqueness of a solution to the abstract problem:

given $f \in X^*$, find $(u,v) \in X \times Y^*$ such that

$$\begin{aligned} \tau T u &= v \\ T^* v &= f, \end{aligned} \tag{2.4.1}$$

where X and Y are Hilbert spaces with duals X* and Y* respectively and where $T:X \to Y$ and $T^*:Y^* \to X^*$ are linear adjoint operators and τ is the canonical isomorphism from Y to Y*. The result is an adaptation of a result of Brezzi [1974] to the abstract form we are considering here. First we state some standard theorems that will be needed for this proof.

Let $Z = N(T^*)$, the null space of T*, and define the annihilator Z^0 of Z as follows:

$$Z^0 = \{y \in Y; \quad \langle y,z \rangle = 0, \quad \forall z \in Z\}. \tag{2.4.2}$$

See figure 2.4.1 for the relationship between the spaces and operator. Then we have

<u>Theorem 2.4.1</u> (Banach's Closed Range theorem; see Yosida [1965], p.205)

Let X and Y be Banach spaces, and T a closed linear operator from X into Y such that D(T) is dense in X. Then the following propositions are all equivalent:

$R(T)$ is closed in Y,

$R(T^*)$ is closed in X^*,

$R(T) \;=\; Z^{O}$,

$R(T^*) \;=\; N(T)^{O}$. □

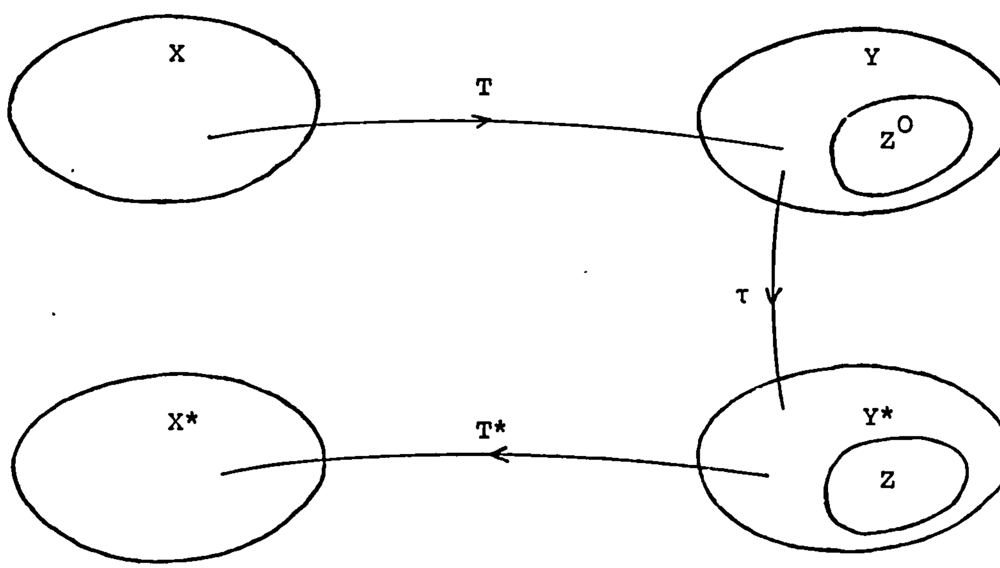

Figure 2.4.1 Spaces and operators for the abstract problem

We now characterise the dual of Z^{O}.

<u>Lemma 2.4.1</u> (c.f. Brezzi [1974])

$$(Z^{O})^* \;=\; Z^{\perp}$$

<u>Proof</u>

$$\begin{aligned}(Z^{O})^* &= \{z^* \in Y^*;\;\; z^* = \tau y \text{ for some } y \in Z^{O}\} \\ &= \{z^* \in Y^*;\;\; z^* = \tau y \text{ for some } y \text{ such that } \\ &\qquad\qquad\qquad \langle y,z\rangle = 0,\ \forall z \in Z\}.\end{aligned}$$

However $\langle y,z\rangle = (\tau y,z)$.

Hence $\quad (Z^{O})^* = \{z^* \in Y^*;\ (z^*,z) = 0,\ \forall z \in Z\}$

i.e. $\quad (Z^{O})^* = Z^{\perp}$. □

The next theorem gives conditions under which the adjoint operator T^* is a bijective operator.

<u>Theorem 2.4.2</u> (see for example Rudin [1973], p.94)

Let X and Y be Banach spaces and $T:X \to Y$ a bounded linear operator. Then $R(T)$ is dense in Y iff T^* is a bijective mapping. □

We are now in a position to give the main result of this section.

<u>Theorem 2.4.3</u>

Problem (2.4.1) has a unique solution if $\alpha > 0$ such that

$$\|Tx\|_Y \geq \alpha\|x\|_X \ , \quad \forall x \in X . \tag{2.4.3}$$

<u>Proof</u> We divide the proof into three parts, the first two of which we shall make use of in subsequent theorems.

(1) From theorem 2.2.4 and inequality (2.4.3), T is a closed operator and since the domain of T is closed, R(T) is closed. Hence from Banach's closed range theorem 2.4.1 we have $R(T) = Z^{O}$. Therefore T is a surjective operator onto Z^{O} and from (2.4.3), T is injective. Hence T is an isomorphism from X to Z^{O}.

(2) Now from lemma 2.4.1, $(Z^{O})^* = Z^{\perp}$ and so using theorem 2.4.2, replacing Y by Z^{O}, we have T^* is a bijective map and hence an isomorphism from $Z^{\perp}$ to X^*.

(3) Hence, given an $f \in X^*$, (2) shows that there exists a unique $v \in Z^{\perp} \subset Y^*$ satisfying

$$T^*v = f .$$

From lemma 2.4.1 $\tau^{-1}v \equiv w \in Z^o$. Hence (1) shows there exists a unique $u \in X$ such that

$$Tu = w,$$

see figure 2.4.2. Therefore there exists a unique $(u,v) \in X \times Y^*$ satisfying problem (2.4.1).

Example 2.4.1

Recall the first example of section 2.1 which we put into an abstract form in section 2.3. The operator T is the generalised derivative $\frac{d}{d\xi}$ with domain $X = H_o^1(\Omega)$ and range in $Y = L_2(\Omega)$. To prove the existence and uniqueness of a solution to this problem using theorem 2.4.3, we need only show that there exists $\alpha > 0$ such that

$$\left\|\frac{dx}{d\xi}\right\|_{L_2(\Omega)} \geq \alpha \|x\|_{H^1(\Omega)}, \quad \forall x \in H_o^1(\Omega).$$

This inequality follows from the Poincare-Friedrichs inequality, see, e.g. Necas [1967]. This can be written as

$$\int_\Omega \left(\frac{dx}{d\xi}\right)^2 d\Omega \geq c^2 \int_\Omega x^2 d\Omega, \qquad \forall x \in H_o^1(\Omega)$$

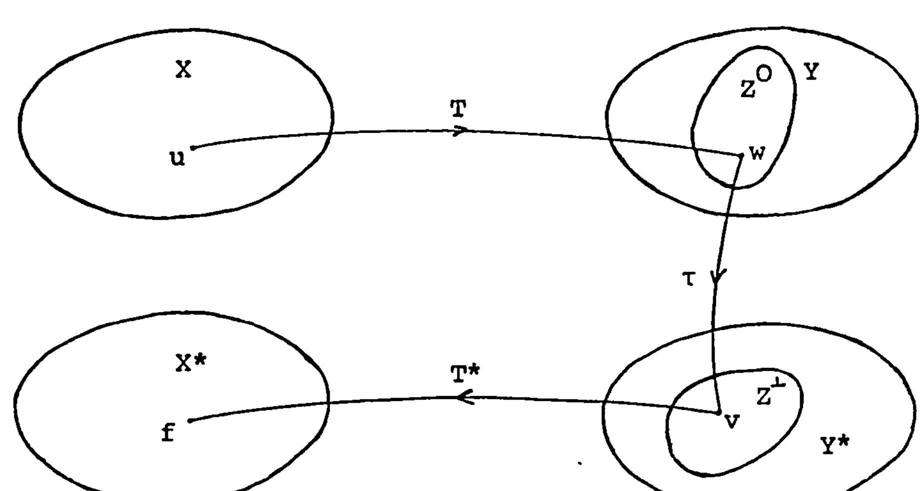

Figure 2.4.2 The solution of the abstract problem

where c is a constant. From this we see that for $0 < \beta < 1$,

$$\left\| \frac{dx}{d\xi} \right\|^2_{L_2(\Omega)} \geq \beta c^2 \int_\Omega x^2 dx + (1 - \beta) \int_\Omega \left(\frac{dx}{d\xi}\right)^2 d\Omega$$

$$\geq \alpha^2 \| x \|^2_{H^1(\Omega)} , \quad \forall x \in H_0^1(\Omega)$$

where $\alpha^2 = \max(\beta c^2, 1 - \beta)$. Hence example 2.1.1 has a unique solution u in the space $H_0^1(\Omega)$. □

By adapting a theorem of Brezzi [1974] we have shown that problem (2.4.1) has a unique solution provided inequality (2.4.3) is satisfied. In later chapters we shall greatly extend this theorem to more complex linear problems and to non-linear problems.

CHAPTER 3

VARIATIONAL PRINCIPLES

3.1 Saddle Functionals

The concept of a saddle functional provides an intuitive way of looking at the variational principles we shall derive for the abstract problem of Chapter 2. In particular the relationships between the variational principles can be seen in a very graphical way using this concept. In this section, we define and give some relevant results regarding convex, concave and saddle functionals.

Definition 3.1.1 (Luenberger [1969], p.190)

A functional $F(x)$ defined on a convex subset X of a linear vector space is said to be convex if

$$F(\alpha x_1 + (1-\alpha)x_2) \leq \alpha F(x_1) + (1-\alpha)F(x_2) \qquad (3.1.1)$$

for all x_1, $x_2 \in X$ and all α, $0 < \alpha < 1$. F is said to be strictly convex if the inequality of (3.1.1) is strict for all x_1, $x_2 \in X$, $x_1 \neq x_2$. $\square$

We usually deal with G-differentiable functionals and in this case we have an equivalent definition of a convex functional.

Lemma 3.1.1

If $F : X \to \mathbb{R}$ is G-differentiable in X then the following are equivalent statements:

(i) F is (strictly) convex in X

(ii) $F(x_1) - F(x_2) - \langle \nabla F(x_2), x_1 - x_2 \rangle \geq (>) 0, \quad \forall x_1, x_2 \in X.$ (3.1.2)

<u>Proof</u> (i) => (ii). From (3.1.1) we have

$$F(\alpha x_1 + (1-\alpha)x_2) - F(x_2) \leq (<)\alpha(F(x_1) - F(x_2)),$$

i.e. $$\frac{F(x_2 + \alpha(x_1-x_2)) - F(x_2)}{\alpha} \leq (<)F(x_1) - F(x_2).$$

Taking the limit as $\alpha \to 0$ we have

$$<\nabla F(x_2),\ x_1-x_2> \leq (<)F(x_1) - F(x_2).$$

(ii) => (i). From (3.1.2) we have

$$F(x_2) - F(x_2+\alpha(x_1-x_2)) - <\nabla F(x_2+\alpha(x_1-x_2)),x_2 - x_2 - \alpha(x_1-x_2)> \geq (>)0,$$

i.e. $$F(x_2) - F(x_2+\alpha(x_1-x_2)) + \alpha<\nabla F(x_2+\alpha(x_1-x_2)),x_1-x_2> \geq (>)0. \qquad (3.1.3)$$

Also from (3.1.2) we get

$$F(x_1) - F(x_2+\alpha(x_1-x_2)) - <\nabla F(x_2+\alpha(x_1-x_2)),x_1-x_2-\alpha(x_1-x_2)> \geq (>)0,$$

i.e. $$F(x_1) - F(x_2+\alpha(x_1-x_2)) - (1-\alpha)<\nabla F(x_2+\alpha(x_1-x_2)),\ x_1-x_2> \geq (>)0. \qquad (3.1.4)$$

Then $(1-\alpha)$ times (3.1.3) plus α times (3.1.4) gives

$$(1-\alpha)F(x_2) - (1-\alpha)F(x_2+\alpha(x_1-x_2)) + \alpha F(x_1) - \alpha F(x_2+\alpha(x_1-x_2)) \geq (>)0,$$

i.e. $$\alpha F(x_1) + (1-\alpha)F(x_2) \geq (>)F(\alpha x_1 + (1-\alpha)x_2).$$

Definition 3.1.2

A functional $F : X \to \mathbb{R}$ is said to be (strictly) concave iff $-F$ is (strictly) convex on X.

Lemma 3.1.2

If $F : X \to \mathbb{R}$ is G-differentiable in X then the following are equivalent statements:

(i) F is (strictly) concave in X,

(ii) $F(x_1) - F(x_2) - \langle\nabla F(x_1), x_1-x_2\rangle \geq (>) 0$. (3.1.5)

Proof From lemma 3.1.1 and definition 3.1.2

$$\text{(i)} \iff -F(x_1) + F(x_2) - \langle\nabla(-F(x_2)), x_1-x_2\rangle \geq (>)0$$

$$\iff F(x_2) + F(x_1) - \langle\nabla F(x_2), x_2-x_1\rangle \geq (>)0$$

$$\iff F(x_1) + F(x_2) - \langle\nabla F(x_1), x_1-x_2\rangle \geq (>)0$$

since x_1 and x_2 are arbitrary elements of X.

Theorem 3.1.1

If F is G-differentiable and strictly convex (concave) on X, then F has a unique minimum (maximum) point u, i.e.

$$F(u) < (>)F(x), \quad \forall x \in X, \ x \neq u.$$

Proof Let F be a strictly convex functional and suppose F has two minimum points u_1 and u_2, i.e.

$$F(u_1) = F(u_2) < F(x), \ \forall x \in X, \ x \neq u_1, \ x \neq u_2.$$

Now from the definition of strict convexity, definition 3.1.1, with $x = \frac{1}{2}u_1 + \frac{1}{2}u_2$, we have

$$F(x) < \frac{1}{2}F(u_1) + \frac{1}{2}F(u_2) = F(u_1) = F(u_2).$$

Hence we have a contradiction and so there exists only one minimum u of F(x). Similarly when F is strictly concave there is a unique maximum. □

We now define a saddle functional and give some of its properties. For a wider discussion of saddle functionals as well as convex and concave functionals see Noble-Sewell [1971].

Definition 3.1.3

A functional $L : X \times Y \to \mathbb{R}$ is called a convex-concave saddle functional if $L(x,y)$ is convex in x for every $y \in Y$ and concave in y for every $x \in X$. See figure 3.1.1.

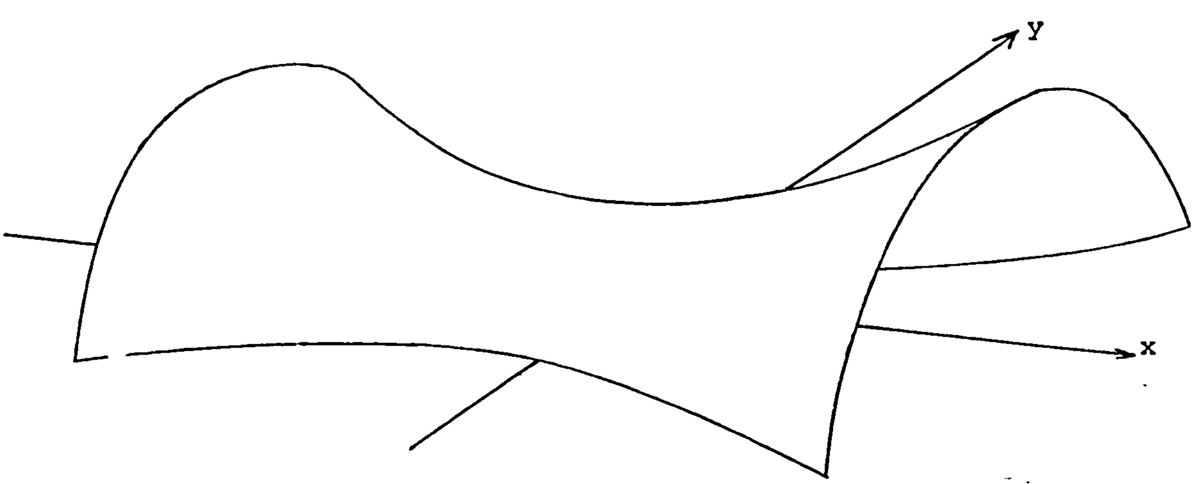

Figure 3.1.1 A convex-concave saddle functional

Remark 3.1.1 We can similarly define a concave-convex functional in the obvious manner. When it is not relevant to distinguish between the two we drop the prefix and just call the functional a saddle functional.

Lemma 3.1.3

If $L : X \times Y \to \mathbb{R}$ is G-differentiable then the following are equivalent statements:

(i) $L(x,y)$ is a convex-concave saddle functional on $X \times Y$,

(ii)
$$L(x_1,y_1) - L(x_2,y_2) - \langle \nabla_1 L(x_2,y_2),\ x_1 - x_2 \rangle - \langle \nabla_2 L(x_1,y_1), y_1 - y_2 \rangle \geq 0,\ \forall x_1,x_2 \in X,\ \forall y_1,y_2 \in Y. \tag{3.1.6}$$

Proof (i) => (ii). From lemma 3.1.1 we have for $y_2 \in Y$,

$$L(x_1,y_2) - L(x_2,y_2) - \langle \nabla_1 L(x_2,y_2),\ x_1 - x_2 \rangle \geq 0\ \forall x_1,x_2, \in X. \tag{3.1.7}$$

From lemma 3.1.2 we have for $x_1 \in X$

$$L(x_1,y_1) - L(x_1,y_2) - \langle \nabla_2 L(x_1,y_1),\ y_1 - y_2 \rangle \geq 0\ \forall y_1,y_2 \in X. \tag{3.1.8}$$

Adding (3.1.7) and (3.1.8) we have

$$L(x_1,y_1) - L(x_2,y_2) - \langle \nabla_1 L(x_2,y_2),\ x_1 - x_2 \rangle - \langle \nabla_2 L(x_1,y_1),\ y_1 - y_2 \rangle \geq 0\quad \forall x_1,x_2 \in X,\ \forall y_1,y_2 \in Y.$$

(ii) => (i). Let $y = y_1 = y_2 \in Y$ in (3.1.6), then

$$L(x_1,y) - L(x_2,y) - \langle \nabla_1 L(x_2,y),\ x_1 - x_2 \rangle \geq 0 \quad \forall x_1,x_2 \in X,$$

i.e. $L(x,y)$ is convex in x for any $y \in Y$.

Let $x = x_1 = x_2 \in X$ in (3.1.6), then

$$L(x,y_1) - L(x,y_2) - \langle \nabla_2 L(x,y_1), y_1 - y_2 \rangle \geq 0 \quad \forall y_1,y_2 \in Y,$$

i.e. $L(x,y)$ is concave in y for any $x \in X$.

Definition 3.1.4

$L(x,y)$ is called a strict saddle functional if

$$L(x_1,y_1) - L(x_2,y_2) - \langle \nabla_1 L(x_2,y_2),\ x_1 - x_2 \rangle$$
$$- \langle \nabla_2 L(x_1,y_1),\ y_1 - y_2 \rangle > 0$$

for all $x_1,x_2 \in X$, $x_1 \neq x_2$, $\forall y_1,y_2 \in Y$, $y_1 \neq y_2$.

Definition 3.1.5 (Cea [1971], p.196)

A point $(u,v) \in X \times Y$ satisfying

$$L(u,y) \leq L(u,v) \leq L(x,v) \quad \forall x \in X,\ \forall y \in Y, \tag{3.1.9}$$

is called a saddle point of $L(x,y)$.

Remark 3.1.2 A functional having a saddle point need not be a saddle functional. Conversely we have the following lemma.

Lemma 3.1.4

If $L : X \times Y \to \mathbb{R}$ is a saddle functional then any stationary point of $L(x,y)$ is a saddle point.

Proof Suppose $L(x,y)$ has a stationary point at $(u,v) \in X \times Y$, i.e.

$$\nabla_1 L(u,v) = 0 \tag{3.1.10}$$

and

$$\nabla_2 L(u,v) = 0. \tag{3.1.11}$$

Let $x_2 = u$ and $y_1 = y_2 = v$ in (3.1.6), then using (3.1.10) we get

$$L(x_1,v) - L(u,v) \geq 0 \qquad \forall x_1 \in X. \tag{3.1.12}$$

Now let $x_1 = x_2 = u$ and $y_1 = v$ in (3.1.6), then using (3.1.11) we get

$$L(u,v) - L(u,y_2) \geq 0 \qquad \forall y_2 \in Y. \tag{3.1.13}$$

Then (3.1.12) and (3.1.13) imply (u,v) is a saddle point of $L(x,y)$.

3.2 A Variational Principle

In this section we derive the most general variational principle associated with the abstract problem given in Chapter 2, i.e.

$$\begin{aligned} \tau T u &= v \\ T^* v &= f . \end{aligned} \tag{3.2.1}$$

The main result is theorem 3.2.1 which shows that any solution of the abstract problem (3.2.1) is a solution of the following variational problem:

find $(u,v) \in X \times Y^*$ saddle point of

$$L(x,y^*) = \langle Tx,y^* \rangle - \frac{1}{2} \langle \tau^{-1} y^*,y^* \rangle - \langle x,f \rangle, \quad x \in X,\ y^* \in Y^*. \tag{3.2.2}$$

This is related to the Hellinger-Reisner principle of linear elasticity.

We first show that $L(x,y^*)$ is a convex-concave saddle functional.

Lemma 3.2.1

$L(x,y^*)$ is a convex-concave saddle functional.

Proof Let x_1, $x_2 \in X$ and y_1^*, $y_2^* \in Y^*$, then using lemma 3.1.3 we have

$$\begin{aligned} L(x_1,y_1^*) - L(x_2,y_2^*) - \langle \nabla_1 L(x_2,y_2^*),\ x_1 - x_2 \rangle & \\ - \langle \nabla_2 L(x_1,y_1^*),\ y_1^* - y_2^* \rangle & \\ = \langle Tx_1,y_1^* \rangle - \frac{1}{2} \langle \tau^{-1} y_1^*,\ y_1^* \rangle - \langle x_1,f \rangle & \\ - \langle Tx_2,y_2^* \rangle - \frac{1}{2} \langle \tau^{-1} y_2^*,y_2^* \rangle - \langle x_2,f \rangle & \\ - \langle Tx_1,y_2^* \rangle + \langle Tx_2,y_2^* \rangle + \langle x_1,f \rangle - \langle x_2,f \rangle & \\ - \langle Tx_1,y_1^* \rangle + \langle Tx_1,y_2^* \rangle + \langle \tau^{-1} y_1^*,y_1^* \rangle - \langle \tau^{-1} y_1^*,y_2^* \rangle & \\ = \frac{1}{2} \langle \tau^{-1} y_1^*,y_1^* \rangle - \langle \tau^{-1} y_1^*,y_2 \rangle + \frac{1}{2} \langle \tau^{-1} y_2^*,y_2^* \rangle & \\ = \frac{1}{2} \langle \tau^{-1} (y_1^*-y_2^*),\ y_1^* - y_2^* \rangle = \frac{1}{2} (y_1^*-y_2^*,y_1^*-y_2^*) \geq 0. & \end{aligned}$$

Hence $L(x,y^*)$ is a convex-concave saddle functional.

Remark 3.2.1 Note that the strict inequality holds for all y_1^*, $y_2^* \in Y^*$, $y_1^* \neq y_2^*$. Hence from definition 3.1.4, $L(x,y^*)$ is a strict saddle functional.

We now state the relationship between problem (3.2.1) and problem (3.2.2).

Theorem 3.2.1

Assume the hypothesis of theorem 2.4.3. Then $L(x,y^*)$ has a unique saddle point $(u,v) \in X \times Y^*$. Moreover (u,v) is the solution of problem (3.2.1).

Proof We compute the Gateaux derivatives of $L(x,y^*)$ at (u,v) and deduce that

(1) $$<\nabla_1 L(u,v), x> = <Tx,v> - <x,f>$$

$$= <x,T^*v> - <x,f> = 0 \quad \forall x \in X \qquad (3.2.3)$$

iff $T^*v = f$.

(2) $$<\nabla_2 L(u,v),y^*> = <Tu,y^*> - <\tau^{-1}v,y^*> = 0 \qquad \forall y^* \in Y^* \qquad (3.2.4)$$

iff $\tau^{-1}v = Tu$

i.e. iff $v = \tau Tu$.

Hence (u,v) is a stationary point of $L(x,y^*)$ iff $T^*v = f$ and $v = \tau Tu$.

Therefore from theorem 2.4.3 there exists a unique stationary point (u,v) of $L(x,y^*)$ where (u,v) is the solution of problem (3.2.1). Further as $L(x,y^*)$ is a saddle functional (lemma 3.2.1), (u,v) is a saddle point of $L(x,y^*)$ by lemma 3.1.4.

Example 3.2.1

The variational problem for the first example of Chapter 2, equations (2.3.1), is: find $(u,v) \in H_0^1(\Omega) \times L_2(\Omega)$, saddle point of

$$L(x,y^*) = \int_\Omega \left(\frac{dx}{d\xi}\, y^* - \frac{1}{2}\, y^{*2} - xf\right) d\Omega, \quad x \in H_0^1(\Omega), \quad y \in L_2(\Omega).$$

Note that the last term can be written as

$$\int xf \, d\Omega$$

since we have assumed that $f \in L_2(\Omega)$, see section 2.3.

3.3 Complementary Variational Principles

In this section we show that any saddle functional with a saddle point can give rise to a pair of complementary extremum principles. The essential idea is to restrict the domain of the saddle functional in such a way that it still contains the saddle point but provides us with a simpler functional.

We restrict the domain of $L(x,y)$ to those points which satisfy the constraint $\nabla_1 L(x,y) = 0$. Let us denote the restricted domain by

$$\mathcal{D}_1 = \{ (x,y) \in X \times Y; \quad \nabla_1 L(x,y) = 0 \} . \tag{3.3.1}$$

Remark 3.3.1 A saddle point (u,v) of $L(x,y)$ belongs to $\mathcal{D}_1$ since $\nabla_1 L(u,v) = 0$.

Theorem 3.3.1

Any saddle point (u,v) of $L(x,y)$ on $X \times Y$ is a maximum point of $L(x,y)$ on $\mathcal{D}_1$.

Proof From (3.1.6) we have

$$L(x_1,y_1) - L(x_2,y_2) - \langle \nabla_2 L(x_1,y_1), y_1 - y_2 \rangle \geq 0 \tag{3.3.2}$$

for all (x_1,y_1), $(x_2,y_2) \in \mathcal{D}_1$.

Now let (x_1,y_1) be the saddle point (u,v), then

$$L(u,v) - L(x_2,y_2) \geq 0, \qquad \forall (x_2,y_2) \in \mathcal{D}_1 .$$

Hence (u,v) is a maximum point of $L(x,y)$ on $\mathcal{D}_1$. □

In most of the applications it will be the case that $L : \mathcal{D}_1 \to \mathbb{R}$ can be denoted by a functional of the single variable y, i.e. as $K : Y_1 \to \mathbb{R}$, where

$$Y_1 = \{y \in Y;\ \exists\, x \in X \text{ s.t. } (x,y) \in \mathcal{D}_1\} .$$

This may arise because the constraint $\nabla_1 L(x,y) = 0$ provides an explicit relationship for x in terms of y. Also if $L(x,y)$ is linear in x then $\nabla_1 L(x,y)$ is "the coefficient" of x. Hence on $\mathcal{D}_1$ the coefficient of x in $L(x,y)$ will be zero and so we can set $L(x,y) = K(y)$.

<u>Lemma 3.3.1</u>

If we can write $L : \mathcal{D}_1 \to \mathbb{R}$ as $K : Y_1 \to \mathbb{R}$, then $K(y)$ is a concave functional on Y_1. Further, if $L(x,y)$ is strictly saddle cn $\mathcal{D}_1$ then $K(y)$ has a unique maximum point.

<u>Proof</u> From (3.3.2) we have

$$K(y_1) - K(y_2) - \langle \nabla K(y_1),\ y_1 - y_2 \rangle \geq 0 \qquad \forall y_1, y_2 \in Y_1 .$$

Hence from (3.1.5), K is a concave functional. When $L(x,y)$ is a strict saddle functional on $\mathcal{D}_1$ we have

$$K(y_1) - K(y_2) - \langle \nabla K(y_1), y_1 - y_2 \rangle > 0 \qquad \forall y_1, y_2 \in Y_1,\ y_1 \neq y_2 .$$

Hence K is strictly concave and from theorem 3.1.1 has a unique maximum. □

The domain of $L(x,y)$ can be restricted by another constraint. We let

$$\mathcal{D}_2 = \{(x,y) \in X \times Y; \quad \nabla_2 L(x,y) = 0\}. \tag{3.3.3}$$

Remark 3.3.2 A saddle point (u,v) of $L(x,y)$ belongs to $\mathcal{D}_2$ since $\nabla_2 L(u,v) = 0$.

Theorem 3.3.2

Any saddle point (u,v) of $L(x,y)$ on $X \times Y$ is a minimum point of $L(x,y)$ on $\mathcal{D}_2$.

Proof From (3.1.6) we have

$$L(x_1,y_1) - L(x_2,y_2) - \langle \nabla_1 L(x_2,y_2), x_1 - x_2 \rangle \geq 0 \tag{3.3.4}$$

$$\forall (x_1 y_1), \ (x_2,y_2) \in \mathcal{D}_2 \ .$$

Now let (x_2,y_2) be the saddle point (u,v), then

$$L(x_1,y_1) - L(u,v) \geq 0 \qquad \forall (x_1,y_1) \in \mathcal{D}_2.$$

Hence (u,v) is a minimum point of $L(x,y)$ on $\mathcal{D}_2$. □

As before we may be able to write $L : \mathcal{D}_2 \to \mathbb{R}$ as a functional of a single variable x, i.e. as $J : X_2 \to \mathbb{R}$, where

$$X_2 = \{x \in X; \ \exists \ y \in Y \text{ s.t. } (x,y) \in \mathcal{D}_2\}.$$

Lemma 3.3.2

If we can write $L : \mathcal{D}_2 \to \mathbb{R}$ as $J : X_2 \to \mathbb{R}$, then $J(x)$ is a convex functional on X_2. Further, if $L(x,y)$ is strictly saddle on $\mathscr{D}_2$, then $J(x)$ has a unique minimum point.

Proof From (3.3.4) we have

$$J(x_1) - J(x_2) - \langle \nabla J(x_2), x_1 - x_2 \rangle \geq 0 \qquad \forall x_1, x_2 \in X_2.$$

Hence from (3.1.2), J is a convex functional. When $L(x,y)$ is a strict saddle functional on $\mathscr{D}_2$ we have

$$J(x_1) - J(x_2) - \langle \nabla J(x_2), x_1 - x_2 \rangle > 0 \qquad \forall x_1, x_2 \in X, \; x_1 \neq x_2.$$

Hence $J(x)$ is strictly convex and from theorem 3.1.1 has a unique minimum point. □

Therefore from the saddle point problem (3.2.2) based on the saddle functional $L(x,y)$ we have derived

(i) a maximum problem:

$$\text{find } v \in Y_1 \quad \text{s.t.} \quad K(v) = \max_{y \in Y_1} K(y),$$

which is based on the concave functional $K(y)$.

(ii) a minimum problem:

$$\text{find } u \in X_1 \quad \text{s.t. } J(u) = \operatorname*{Min}_{x \in X_1} J(x),$$

which is based on the convex functional $J(x)$. Hence (i) and (ii) can be regarded as complementary extremum principles in the sense that

$$\operatorname*{Min}_{x \in X_1} J(x) = J(u) = L(u,v) = K(v) = \operatorname*{Max}_{y \in Y_1} K(y),$$

see figure 3.3.1.

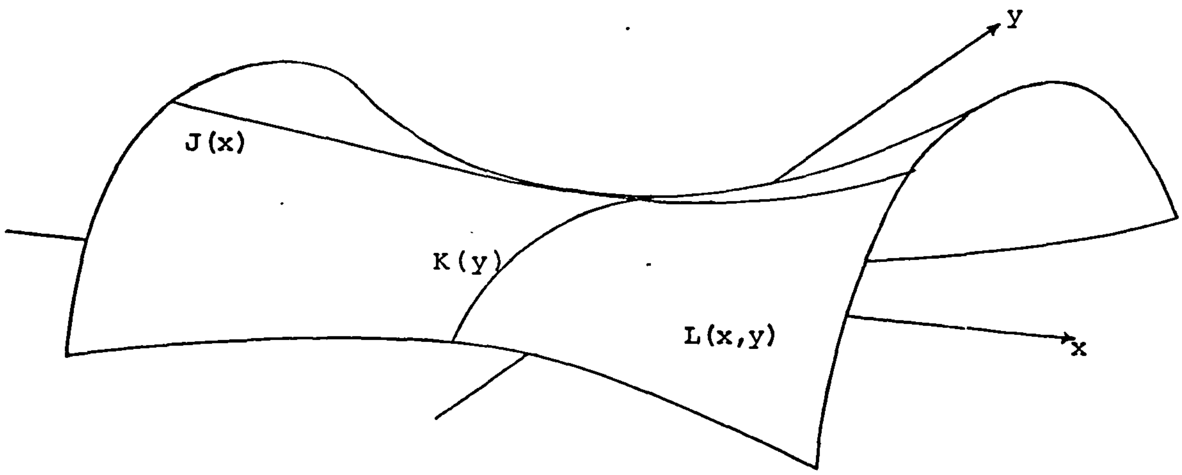

Figure 3.3.1 Complementary extremum principles

3.4 The Minimum Potential Energy and Complementary Energy Methods

We apply the theory of section 3.3 to the saddle functional (3.2.2). Restrict the domain $X \times Y^*$ of $L(x,y^*)$ to those (x,y^*) satisfying $\nabla_1 L(x,y^*) = 0$, i.e. $\mathcal{D}_1$. From (3.2.3) $\mathcal{D}_1$ is $(x,y^*) \in X \times Y^*$ such that

$$T^* y = f.$$

Hence we take $\mathcal{D}_1 = X_1 \times Y_1$ with $X_1 = X$ and

$$Y_1 = Z_f = \{y \in Y;\ T^*y = f\}.$$

Remark 3.4.1 Z_f is a linear variety and when $f = 0$ we have the linear space $Z_0 = Z = \{y \in Y;\ T^*y = 0\}$. □

Now from (3.2.2) and theorem 2.2.5 we have

$$\begin{aligned} L(x,y^*) &= \langle x,T^*y\rangle - \tfrac{1}{2}\langle \tau^{-1}y^*,y^*\rangle - \langle x,f\rangle \qquad \forall x \in X,\ \forall y^* \in Z_f \\ &= -\tfrac{1}{2}\langle \tau^{-1}y^*,y^*\rangle \equiv K(y^*). \qquad (3.4.1) \end{aligned}$$

We see that $L(x,y^*)$ is a functional of the variable y only, in this case because $L(x,y^*)$ on $X \times Y^*$ is linear in x. Then we have the following theorem:

Theorem 3.4.1

The concave functional $K : Z_f \to \mathbb{R}$ defined by

$$K(y^*) = -\frac{1}{2} \langle \tau^{-1} y^*, y^* \rangle \tag{3.4.2}$$

has a unique maximum at $y^* = v$, where (u,v) is the solution of (3.2.2).

Proof From theorem 3.3.1, $K(y^*)$ has a maximum point v, where (u,v) is the saddle point of (3.2.2). From lemma 3.3.1 and remark 3.2.1, $K(y^*)$ is a concave functional with a unique maximum.

Remark 3.4.2 In the context of elasticity the problem of finding the maximum of $K(y)$ is called the complementary energy principle.

Example 3.4.1

The saddle functional for example 1 of Chapter 2 is, see example 3.2.1,

$$L(x,y^*) = \int_\Omega \left(\frac{dx}{d\xi} y^* - \frac{1}{2} y^{*2} - xf\right) d\Omega \tag{3.4.3}$$

with domain $H_0^1(\Omega) \times L_2(\Omega)$. We restrict the domain to those y^* satisfying

$$-\frac{dy^*}{d\xi} = f.$$

Hence $\mathfrak{D}_1 = H_0^1(\Omega) \times Z_f$ where

$$Z_f = \{y^* \in L_2(\Omega); \; -\frac{dy^*}{d\xi} = f\}.$$

Now using the adjoint relationship (2.3.4) we see that on $\mathcal{D}_1$, $L(x,y^*)$ reduces to

$$K(y^*) = -\frac{1}{2}\int_\Omega y^{*2}d\Omega, \quad y^* \in Z_f.$$

Theorem 3.4.1 shows that v is the unique maximum point of this functional. □

Now restrict the domain $X \times Y^*$ of $L(x,y^*)$ to those $(x,y^*) \in X \times Y^*$ such that $\nabla_2 L(x,y^*) = 0$, i.e., such that

$$\tau^{-1}y^* = Tx$$

$$\text{or} \qquad y^* = \tau Tx. \tag{3.4.4}$$

Hence we take $\mathcal{D}_2 = X_2 \times Y_2$, where $X_2 = X$ and $Y_2 = Z^\perp$ from part (2) of theorem 2.4.3. Therefore from (3.2.2) we have

$$\begin{aligned} L(x,y^*) &= \langle Tx,y^*\rangle - \frac{1}{2}\langle \tau^{-1}y^*,y^*\rangle - \langle x,f\rangle, \quad \forall x \in X,\ \forall y^* \in Z \\ &= \frac{1}{2}\langle Tx,\tau Tx\rangle - \langle x,f\rangle. \end{aligned}$$

We see that $L(x,y^*)$ is a functional of the variable x only since (3.4.4) provided us with an explicit relationship for y^* in terms of x.

Theorem 3.4.2

The convex functional $J : X \to \mathbb{R}$ defined by

$$J(x) = \frac{1}{2}\langle Tx,\tau Tx\rangle - \langle x,f\rangle \tag{3.4.5}$$

has a unique minimum at $x = u$, where $(u, v = \tau Tu)$ is the solution of (3.2.2).

Proof From theorem 3.3.2, $J(x)$ has a minimum point u where (u,v) is the solution of (3.2.2). From lemma 3.3.2 and remark 3.2.1, $J(x)$ is a convex functional with a unique minimum. □

See figure 3.4.1 which shows the functionals with the assumptions relating each functional.

Remark 3.4.3 In elasticity theory the problem of finding the minimum of $J(x)$ is called the principle of minimum potential energy.

Example 3.4.2

We restrict the saddle functional (3.4.3) to those $(x,y^*) \in H_0^{\,1}(\Omega) \times L_2(\Omega)$ satisfying

$$\frac{dx}{d\xi} = y^*.$$

Hence (3.4.3) can be written as

$$J(x) = \frac{1}{2}\int_\Omega \left[\left(\frac{dx}{d\xi}\right)^2 - xf\right] d\Omega \,, \quad x \in H_0^{\,1}(\Omega).$$

Theorem 3.4.2 shows that u is the unique minimum point of this functional.

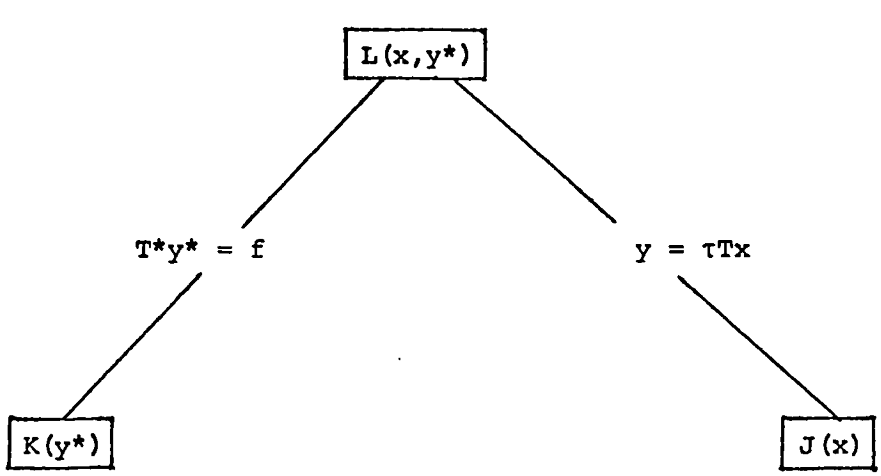

Figure 3.4.1 The functionals for the abstract problem.

CHAPTER 4

THE EXTENDED ABSTRACT PROBLEM

4.1 The Extended Abstract Problem

In this chapter we introduce an extension of the abstract problem of Chapter 2 so that we can easily deal with problems having a constitutive relation which is not the identity mapping, e.g. example 2 of section 2.1. We introduce a linear operator $E: Y \to Y^*$ and consider the problem:

given $f \in X^*$, find $(u,w,v) \in X \times Y \times Y^*$ such that

$$\begin{aligned} Tu &= w \\ Ew &= v \\ T^*v &= f, \end{aligned} \tag{4.1.1}$$

where the spaces X, Y, X^*, Y^* and operators T, T^* are as defined in section 2.4. The second of equations (4.1.1) represents the constitutive relation we wish to introduce. In this section we shall prove an existence and uniqueness theorem for problem (4.1.1). The rest of the chapter is devoted to the derivation of all the variational principles associated with this extended abstract problem.

The proof of the existence and uniqueness of a solution to this problem closely follows theorem 2.4.3. Therefore we shall only prove the extension that is needed to theorem 2.4.3 to cope with problem 4.1.1. First we state some theorems we shall need.

Theorem 4.1.1 (Lax-Milgram [1954])

Given a continuous bilinear form $a(x_1,x_2): X \times X \to \mathbb{R}$ for which there exists $\alpha > 0$ such that

$$a(x,x) \geq \alpha \| x \|_X^2 \quad \forall x \in X, \tag{4.1.2}$$

then the problem:

given $f \in X^*$, find $u \in X$ such that

$$a(u,x) = \langle f,x \rangle \quad \forall x \in X, \tag{4.1.3}$$

has a unique solution.

<u>Remark 4.1.1</u> Condition (4.1.2) is sometimes called X-ellipticity.

<u>Lemma 4.1.1</u> Let Z and Z^O be defined as in section 2.4, then

$$Z^O = (Z^*)^{\perp}.$$

<u>Proof</u> Recall that

$$Z = \{y^* \in Y^*;\ T^*y^* = 0\}$$

Now

$$Z^* = \{z^* \in Y;\ z^* = \tau^{-1}z \text{ for some } z \in Z\}$$

Hence

$$\begin{aligned}(Z^*)^{\perp} &= \{y \in Y;\ (z^*,y)_Y = 0 \quad \forall z^* \in Z^*\} \\ &= \{y \in Y;\ \langle z,y \rangle = 0 \quad \forall z \in Z\} \\ &= Z^O.\end{aligned}$$

<u>Theorem 4.1.2</u> (Yosida [1965], p.43)

A linear operator $E:X \to Y$ admits a bounded linear inverse $E^{-1}: R(E) \subset Y \to X$ iff there exists $\alpha > 0$ such that

$$\| Ex \|_Y \geq \alpha \| x \|_X \quad \forall x \in X. \tag{4.1.4}$$ □

We are now in a position to prove the extension required to theorem 2.4.3. In this we proved (1) that T is an isomorphism from X onto Z^{O} and (2) that T^* is an isomorphism from $Z^{\perp}$ onto X^*, see figure 4.1.1. Hence we concluded that there exists a unique $\tilde{v} \in Z^{\perp}$ such that

$$T^*\tilde{v} = f. \tag{4.1.5}$$

The next stage of the proof is to find a unique $w \in Z^{O}$ such that $Ew = \tilde{v} + \tilde{z}$ where $\tilde{z}$ is any element of Z.

Theorem 4.1.3

If there exists $\alpha_1, \alpha_2 > 0$ such that

$$\|Ey\|_{Y^*} \geq \alpha_1 \|y\|_Y \qquad \forall y \in D(E), \tag{4.1.6}$$

$$\langle E^{-1}z, z\rangle \geq \alpha_2 \|z\|^2_{Y^*} \qquad \forall z \in Z, \tag{4.1.7}$$

and $R(E) = Y^*$, then for any $\tilde{v} \in Z^{\perp}$ we can find a unique $w \in Z^{O}$ such that $v \equiv \tilde{v} + \tilde{z} = Ew$, where $\tilde{z}$ is an element of Z.

Proof From theorem 4.1.2 and (4.1.6) E^{-1} is a bounded linear operator. Hence $E^{-1}\tilde{v}$ is a unique element of Y. From lemma 4.1.1 we can represent $E^{-1}\tilde{v}$ by

$$E^{-1}\tilde{v} = z^{O} + z^* \qquad \text{where } z^{O} \in Z^{O}, \quad z^* \in Z^*.$$

Since E^{-1} is bounded, $\langle E^{-1}y_1,y_2\rangle: Y^* \times Y^* \to \mathbb{R}$ is a continuous bilinear form. Also (4.1.7) is satisfied, so from theorem 4.1.1, given $z^* \in Z^*$ there exists a unique $\tilde{z} \in Z$ such that

$$\langle E^{-1}\tilde{z},z\rangle = -\langle z^*,z\rangle \qquad \forall z \in Z,$$

see figure 4.1.1. Therefore there exists a unique $w = E^{-1}(\tilde{v} + \tilde{z})$ for which

$$\begin{aligned}\langle w,z\rangle &= \langle z^0 + z^* - z^*,z\rangle \qquad \forall z \in Z \\ &= \langle z^0,z\rangle = 0,\end{aligned}$$

since $z^0 \in Z^0$. Hence $w \in Z^0$. □

Therefore we have the following theorem concerning problem (4.1.1).

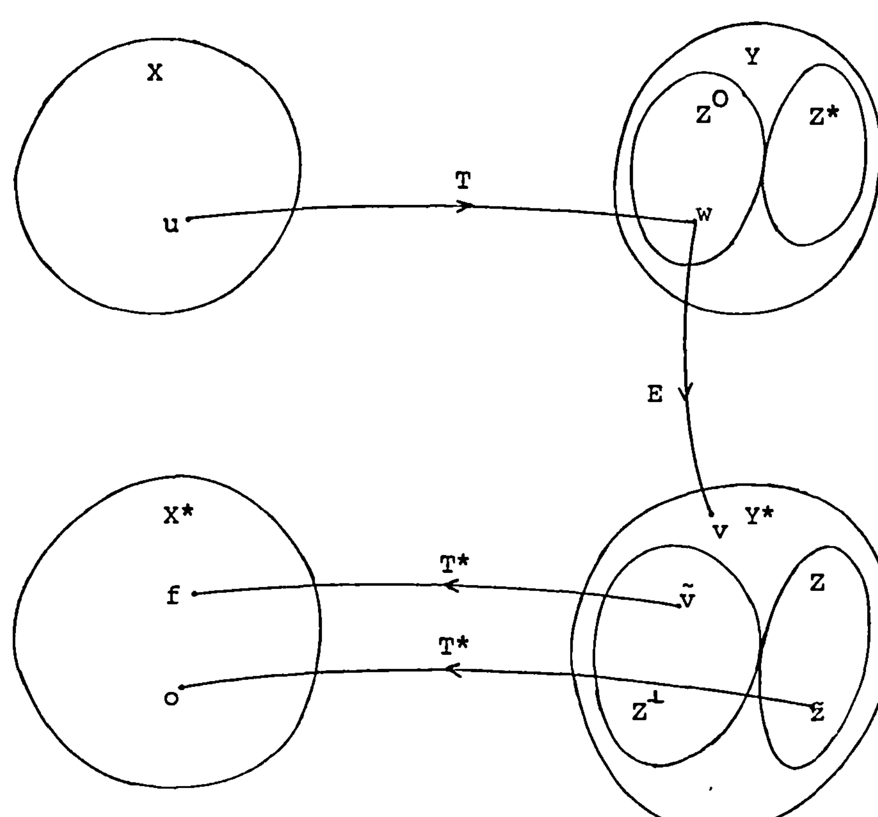

Figure 4.1.1 The spaces in the extended abstract problem

Theorem 4.1.4

Problem (4.1.1) has a unique solution if there exists $\alpha_1, \alpha_2, \alpha_3 > 0$ such that

(i) $\|Tx\|_Y \geq \alpha_1 \|x\|_X \quad \forall x \in X$

(ii) $\|Ey\|_{Y^*} \geq \alpha_2 \|y\|_Y \quad \forall y \in D(E)$

(iii) $\langle E^{-1}z, z\rangle \geq \alpha_3 \|z\|^2_{Y^*} \quad \forall z \in Z,$

and $R(E) = Y^*$.

Proof. From theorem 2.4.3 and theorem 4.1.3.

Example 4.1.1

Recall example 2 of section 2.1. The equations are

$$\begin{aligned} \text{grad } u &= w \\ Kw &= v \qquad\qquad \text{in } \Omega \\ -\text{div } v &= f \end{aligned}$$

with $u = 0$ on Γ.

Setting $T = \text{grad}$, $T^* = -\text{div}$ and $E = K$, this can be put into the form (4.1.1) when we take $X = H_0^1(\Omega)$ and $Y = (L_2(\Omega))^2$. Then $X^* = H^{-1}(\Omega)$ and $Y^* = (L_2(\Omega))^2$. T and T^* are of course considered as extensions of the operators grad and -div as in example 2.4.1.

To prove the existence and uniqueness of a solution to this problem we have to show that the conditions of theorem 4.1.4 are satisfied. Condition (i) is satisfied for

$$\|Tx\|^2_{(L_2(\Omega))^2} = \int_\Omega \left[\left(\frac{\partial x}{\partial \xi_1}\right)^2 + \left(\frac{\partial x}{\partial \xi_2}\right)^2\right] d\Omega \geq \alpha \|x\|_{H^1(\Omega)} \tag{4.1.8}$$

for all $x \in H_0^1(\Omega)$ using the Poincare-Friedrichs inequality as in example 2.4.1. To prove condition (ii) we assume that the tensor K is symmetric and positive definite. Then we have

$$\|Ky\|_{(L^2(\Omega))^2} = \sup_{y^* \in (L_2(\Omega))^2} \frac{(Ky,y^*)}{\|y^*\|_{(L_2(\Omega))^2}} \geq \frac{\langle Ky,y\rangle}{\|y\|_{(L_2(\Omega))^2}} . \qquad (4.1.9)$$

However,

$$\begin{aligned} \langle Ky,y\rangle &= \int_\Omega (k_{11}y_1^2 + 2k_{12}y_1y_2 + k_{22}y_2^2)d\Omega \\ &\geq \int_\Omega [k_{11}y_1^2 + k_{22}y_2^2 - k_{12}(y_1^2 + y_2^2)]d\Omega \\ &\geq \alpha\|y\|^2_{(L_2(\Omega))^2} , \end{aligned} \qquad (4.1.10)$$

where $y = (y_1,y_2)$ and $\alpha = \min[k_{11}-k_{12}, k_{22} - k_{21}]$. From (4.1.9) and (4.1.10) we have

$$\|Ky\|_{(L_2(\Omega))^2} \geq \alpha\|y\|_{(L_2(\Omega))^2} , \qquad \forall y \in (L_2(\Omega))^2 .$$

Hence if K is a symmetric and positive definite tensor, we have $\alpha > 0$ and so condition (ii) is satisfied. The inverse K^{-1} of K is also symmetric and positive definite, hence the inequality (4.1.10) will apply to K^{-1} and so condition (iii) is also satisfied. Therefore example 2.1.2 has a unique solution if K is symmetric positive definite.

4.2 More on Saddle Functionals

In this section we show how the concepts of convexity and saddle functionals can apply to functionals whose domains are product spaces.

Let $W = X \times Y$ be a real Hilbert space with dual $W^* = X^* \times Y^*$. Then we have

$$\langle w^*, w\rangle = \langle x^*, x\rangle + \langle y^*, y\rangle, \quad \forall w \in W,\ w^* \in W^*,$$

where $w = (x,y)$ and $w^* = (x^*,y^*)$, $x \in X$, $y \in Y$, $x^* \in X^*$ and $y^* \in Y^*$. From definition 3.1.1, $F: W \to \mathbb{R}$ is a convex functional if $F(\alpha w_1 + (1-\alpha)w_2) \leq \alpha F(w_1) + (1-\alpha)F(w_2)$, $\forall w_1, w_2 \in W$. Then if $w_1 = (x_1,y_1)$ and $w_2 = (x_2,y_2)$ this definition becomes

$$F(\alpha x_1 + (1-\alpha)x_2,\ \alpha y_1 + (1-\alpha)y_2) \leq \alpha F(x_1,y_1) + (1-\alpha)F(x_2,y_2),\ \forall x_1,x_2 \in X,\ y_1,y_2 \in Y,\ 0 < \alpha < 1. \tag{4.2.1}$$

Now if F is G-differentiable on W then

$$\langle \nabla F(w_1), w_2\rangle = \langle \nabla_1 F(x_1,y_1), x_2\rangle + \langle \nabla_2 F(x_1,y_1), y_2\rangle$$

where $\nabla_i F$, $i = 1,2$ are the partial G-derivatives of F. Hence from lemma 3.1.1 the convexity of F on $X \times Y$ is equivalent to

$$F(x_1,y_1) - F(x_2,y_2) - \langle \nabla_1 F(x_2,y_2),\ x_1-x_2\rangle - \langle \nabla_2 F(x_2,y_2), y_1-y_2\rangle \geq 0,\quad \forall x_1,x_2 \in X,\ y_1,y_2 \in Y. \tag{4.2.2}$$

We can of course define concave functionals on product spaces in a similar way. Also the extension to the product of n spaces, $n > 2$ is obvious. However, as we shall not be using products of more than two spaces, we do not give the more general form.

It is now straightforward to define a convex-concave saddle functional $L: (X \times Y) \times Z \to \mathbb{R}$ which is convex on $X \times Y$ and concave on Z. when L is G-differentiable we have

$$L((x_1,y_1),z_1) - L((x_2,y_2),z_2) - \langle \nabla_1 L(x_2,y_2,z_2),\ x_1 - x_2 \rangle$$

$$- \langle \nabla_2 L(x_2,y_2,z_2),\ y_1 - y_2 \rangle - \langle \nabla_3 L(x_1,y_1,z_1),\ z_1 - z_2 \rangle \geq 0 \qquad (4.2.3)$$

for all $x_1,x_2 \in X$, $y_1,y_2 \in Y$, $z_1,z_2 \in Z$.
Note that where no confusion arises we write $L((x,y),z)$ as $L(x,y,z)$. From definition 3.1.5 a saddle point $((u,w),v) \in (X \times Y) \times Z$ of $L(x,y,z)$ satisfies

$$L(u,w,z) \leq L(u,w,v) \leq L(x,y,v) \qquad (4.2.4)$$

for all $x \in X$, $y \in Y$, $z \in Z$.

4.3 The Extended Variational Principle

The main result of this section is to give conditions under which we can derive a variational principle for problem (4.1.1). We shall show that any solution of problem (4.1.1) is a solution of the problem:

$$\begin{aligned} &\text{find } ((u,w),v) \in (X \times Y) \times Y^*, \text{ saddle point of} \\ &L(x,y,y^*) = \langle Tx,y^* \rangle - \langle y,y^* \rangle + \frac{1}{2} \langle y,Ey \rangle - \langle x,f \rangle , \end{aligned} \qquad (4.3.1)$$

where the spaces and operators are as in problem (4.1.1).

Throughout the rest of the chapter we assume that $D(E) = Y$. However, all the results are valid for $D(E)$ a dense subset of Y.

Remark 4.3.1 In the context of elasticity this variational principle is sometimes called the Hu-Washizu principle.

Lemma 4.3.1

(u,w,v) is a stationary point of $L(x,y,y^*)$ iff

$$T^*v = f,$$

$$v = Ew,$$

$$Tu = w.$$

Proof $\quad \nabla_1 L(u,w,v) = 0$

$$\Leftrightarrow \quad \langle \nabla_1 L(u,w,v), x\rangle = \langle x, T^*v\rangle - \langle x, f\rangle = 0 \qquad \forall x \in X$$

$$\Leftrightarrow \quad T^*v = f.$$

$$\nabla_2 L(u,w,v) = 0$$

$$\Leftrightarrow \quad \langle \nabla_2 L(u,w,v), y\rangle = -\langle y, v\rangle + \langle y, Ew\rangle = 0, \qquad \forall y \in Y$$

$$\Leftrightarrow \quad v = Ew.$$

$$\nabla_3 L(u,w,v) = 0$$

$$\Leftrightarrow \quad \langle \nabla_3 L(u,w,v), y^*\rangle = \langle Tu, y^*\rangle - \langle w, y^*\rangle = 0, \quad \forall y^* \in Y^*$$

$$\Leftrightarrow \quad Tu = w.$$

Lemma 4.3.2

Suppose E is positive and symmetric. Then $L(x,y,y^*)$ is a saddle functional, convex in $X \times Y$ and concave in Y^*.

Proof Inequality (4.2.3) is satisfied as

$$<Tx_1,y_1^*> - <y_1,y_1^*> + \frac{1}{2} <y_1,Ey_1> - <x_1,f>$$

$$-<Tx_2,y_2^*> + <y_2,y_2^*> - \frac{1}{2} <y_2,Ey_2> + <x_2,f>$$

$$-<Tx_1,y_2^*> + <Tx_2,y_2^*> + <x_1,f> - <x_2,f>$$

$$+<y_1,y_2^*> - <y_2,y_2^*> - <y_1,Ey_2> + <y_2,Ey_2>$$

$$-<Tx_1,y_1^*> + <Tx_1,y_2^*> + <y_1,y_1^*> - <y_1,y_2^*>$$

$$= \frac{1}{2}<y_1,Ey_1> - <y_1,Ey_2> + \frac{1}{2}<y_2,Ey_2>$$

$$= \frac{1}{2}<y_1-y_2,E(y_1-y_2)> \quad \text{since E is symmetric}$$

$$\geq 0 \text{ since E is positive.} \quad \square$$

The next theorem shows the relationship between the extended variational problem (4.3.1) and the extended abstract problem (4.1.1).

Theorem 4.3.1

Assume the hypotheses of theorem 4.1.4 are satisfied. Assume in addition that E is symmetric and positive. Then $L(x,y,y^*)$ has a unique saddle point $((u,w),v)$, which is the solution of problem (4.1.1).

Proof From lemma 4.3.1, (u,w,v) is a stationary point of $L(x,y,y^*)$ iff $T^*v = f$, $v = Ew$ and $Tu = w$. Hence from theorem 4.1.4 (u,w,v) is a unique stationary point of $L(x,y,y^*)$. Now from lemma 4.3.2 and lemma 3.1.4 $((u, w),v)$ is a saddle point of $L(x,y,y^*)$.

Example 4.3.1

Recall example 4.1.1, where we proved that example 2 of section 2.1 has a unique solution provided K is a symmetric positive definite tensor.

From theorem 4.3.1 we have that under these conditions the main variational problem associated with example 2.1.2 is:

find $(u,w,v) \in H_0^{\,1}(\Omega) \times (L_2(\Omega))^2 \times (L_2(\Omega))^2$, saddle point of

$$L(x,y,y^*) \;=\; \int_\Omega (\text{grad } x.y^* - y.y^* + \tfrac{1}{2}\, y.Ky - xf)\,d\Omega, \tag{4.3.2}$$

$x \in H_0^{\,1}(\Omega)$, $y \in (L_2(\Omega))^2$, $y^* \in (L_2(\Omega))^2$, provided $f \in L_2(\Omega)$.

4.4 Further Variational Principles

In this section we show that several other variational principles can be derived from the main variational principle of the previous section. We apply the same technique used in section 3.4 of restricting the domain of the saddle functional to give us further functionals associated with the problem. Recall that $L{:}X{\times}Y{\times}Y^* \to \mathbb{R}$ is given by

$$L(x,y,y^*) \;=\; \langle Tx,y^*\rangle - \langle y,y^*\rangle + \tfrac{1}{2}\langle y,Ey\rangle - \langle x,f\rangle\,. \tag{4.4.1}$$

We shall restrict the domain $X{\times}Y{\times}Y^*$ to $\mathcal{D}_1$, i.e. those (x,y,y^*) satisfying $\nabla_1 L(x,y,y^*) = 0$. From the first part of the proof of lemma 4.3.1 we have

$$\mathcal{D}_1 \;=\; \{(x,y,y^*) \in X{\times}Y{\times}Y^*;\; T^*y^* = f\}\,.$$

Let us write $\mathcal{D}_1 = X_1 \times Y_1 \times Y_1^{\,*}$ where $X_1 = X$, $Y_1 = Y$ and $Y_1^{\,*} = Z_f$. Then using theorem 2.2.5, (4.4.1) becomes

$$L(x,y,y^*) \;=\; -\langle y,y^*\rangle + \tfrac{1}{2}\langle y,Ey\rangle \equiv M(y,y^*) \tag{4.4.2}$$

$$\forall y \in Y,\ \forall y^* \in Z_f.$$

Theorem 4.4.1

The convex-concave saddle functional $M(y,y^*)$ has a unique saddle point (w,v), where $Tu = w$ and $((u,w),v)$ is the saddle point of $L(x,y,y^*)$.

Proof Since $\mathcal{D}_1$ contains the saddle point $((u,w),v)$ we have from (4.2.4)

$$M(w,y^*) \leq M(w,v) \leq M(y,v) \qquad \forall y \in Y,\ \forall y^* \in Z_f.$$

Hence (w,v) is a saddle point of $M(y,y^*)$. Now

$$M(y_1,y_1^*) - M(y_2,y_2^*) - \langle\nabla_1 M(y_2,y_2^*),y_1-y_2\rangle - \langle\nabla_2 M(y_1,y_1^*),y_1^*-y_2^*\rangle$$

$$= \quad -\langle y_1,y_1^*\rangle + \frac{1}{2}\langle y_1,Ey_1\rangle + \langle y_2,y_2^*\rangle - \frac{1}{2}\langle y_2,Ey_2\rangle$$

$$+\langle y_1,y_2^*\rangle - \langle y_2,y_2^*\rangle - \langle y_2,Ey_1\rangle + \langle y_2,Ey_2\rangle$$

$$+\langle y_1,y_1^*\rangle - \langle y_1,y_2^*\rangle$$

$$= \quad \frac{1}{2}\langle y_1,Ey_1\rangle - \langle y_2,Ey_1\rangle + \frac{1}{2}\langle y_2,Ey_2\rangle$$

$$= \quad \frac{1}{2}\langle y_1-y_2,\ E(y_1-y_2)\rangle \geq 0 \quad \text{since } E \text{ is positive.}$$

Hence from (3.1.6), $M(y,y^*)$ is a convex-concave saddle functional.

Now

$$\langle\nabla_1 M(w,v),y\rangle \;=\; -\langle y,v\rangle + \langle y,Ew\rangle \;=\; 0, \qquad \forall y \in Y$$

$$\text{iff} \quad v = Ew.$$

Also

$$\langle\nabla_2 M(w,v),z\rangle \;=\; -\langle w,z\rangle \;=\; 0, \quad \forall z \in Z$$

$$\text{iff} \quad w \in Z^0,$$

i.e. $\qquad \text{iff} \quad w = Tu, \qquad u \in X.$

Hence from theorem 4.1.4 (u,w,v) is a unique solution of problem 4.1.1 and so (w,v) is a unique saddle point of $M(y,y^*)$.

Example 4.4.1

The domain of the functional of example 4.3.1 can be restricted to those y^* satisfying

$$-\operatorname{div} y^* = f.$$

Hence defining

$$Z_f = \{y^* \in (L_2(\Omega))^2;\ -\operatorname{div} y^* = f\}$$

and using the adjoint relationship we have that (4.3.2) can be written as:

$$\begin{aligned} &\text{find } (w,v) \in (L_2(\Omega))^2 \times Z_f, \text{ saddle point of} \\ &M(y,y^*) = \int_\Omega (-y.y^* + \tfrac{1}{2}\, y.Ky)\,d\Omega,\ y \in (L_2(\Omega))^2,\ y^* \in Z_f. \end{aligned} \tag{4.4.3}$$

□

Let us restrict the domain $X \times Y \times Y^*$ of $L(x,y,y^*)$ to $\mathcal{D}_2$, i.e. those (x,y,y^*) satisfying $\nabla_2 L(x,y,y^*) = 0$. From lemma 4.3.1 we have

$$\mathcal{D}_2 = \{(x,y,y^*) \in X \times Y \times Y^*;\quad y^* = Ey\}.$$

Let us write $\mathcal{D}_2 = X_2 \times Y_2 \times Y_2^*$ where $X_2 = X$, $Y_2 = Y$ and $Y_2^* = Y^*$. Then from (4.4.1)

$$L(x,y,y^*) = \langle Tx,Ey\rangle - \frac{1}{2}\langle y,Ey\rangle - \langle x,f\rangle \equiv G(x,y), \qquad \forall x \in X,\ \forall y \in Y. \tag{4.4.4}$$

Theorem 4.4.2

The convex-concave saddle functional $G(x,y)$ has a unique saddle point (u,w) where $((u,w),v)$ is the saddle point of $L(x,y,y^*)$.

Proof Since $\mathcal{D}_2$ contains the saddle point $((u,w),v)$ we have from (4.2.4), remembering that $y^* = Ey$,

$$G(u,w) \le G(x,w) \qquad \forall x \in X.$$

Now let $y = w + \tilde{y}$ where $\tilde{y} \in Y$, then

$$\begin{aligned} G(u,y) &= G(u,w+\tilde{y}) = \langle Tu, E(w+\tilde{y})\rangle - \frac{1}{2}\langle w+\tilde{y}, E(w+\tilde{y})\rangle - \langle u,f\rangle \\ &= G(u,w) + \langle Tu, E\tilde{y}\rangle - \langle w, E\tilde{y}\rangle - \frac{1}{2}\langle \tilde{y}, E\tilde{y}\rangle. \end{aligned}$$

However

$$\langle Tu, E\tilde{y}\rangle - \langle w, E\tilde{y}\rangle = \langle Tu-w, E\tilde{y}\rangle = 0$$

and

$$\langle \tilde{y}, E\tilde{y}\rangle \ge 0 \qquad \forall \tilde{y} \in Y.$$

Hence

$$G(u,y) \le G(u,w) \qquad \forall y \in Y.$$

Therefore (u,w) is a saddle point of $G(x,y)$.

From (3.1.6) we can show that $G(x,y)$ is a convex-concave saddle functional.

We also have

$$\langle \nabla_1 G(u,w), x\rangle = \langle Tx, Ew\rangle - \langle x, f\rangle = 0, \quad \forall x \in X$$

$$\text{iff} \quad T^*Ew = f,$$

and

$$\langle \nabla_2 G(u,w), y\rangle = \langle Tu, Ey\rangle - \langle w, Ey\rangle = 0, \quad \forall y \in Y$$

$$\text{iff} \quad Tu = w.$$

Hence from theorem 4.1.4 the saddle point (u,w) is unique. □

Now suppose that E is an isomorphism from Y to Y*. Then we can write (4.4.4) as

$$H(x,y^*) = \langle Tx,y^*\rangle - \frac{1}{2}\langle E^{-1}y^*,y^*\rangle - \langle x,f\rangle, \qquad (4.4.5)$$

$$\forall x \in X, \quad \forall y^* \in Y^*.$$

Then we have

Theorem 4.4.3

The convex-concave saddle functional $H(x,y^*)$ has a unique saddle point (u,v) where $((u,w),v)$ is the saddle point of $L(x,y,y^*)$.

Proof From (4.2.4) we have

$$H(u,v) \leq H(x,v) \qquad \forall x \in X.$$

Now let $y^* = v + \tilde{y}^*$, $\tilde{y}^* \in Y^*$. Then

$$H(u,y^*) = H(u,v+\tilde{y}^*) = \langle Tu,v+\tilde{y}^*\rangle - \frac{1}{2}\langle E^{-1}(v+\tilde{y}^*),\, v + \tilde{y}^*\rangle - \langle u,f\rangle$$

$$= H(u,v) + \langle Tu,\tilde{y}^*\rangle - \langle E^{-1}v,\tilde{y}^*\rangle - \frac{1}{2}\langle E^{-1}\tilde{y}^*,\tilde{y}^*\rangle.$$

However

$$\langle Tu,\tilde{y}^*\rangle - \langle E^{-1}v,y^*\rangle = 0$$

and

$$\langle E^{-1}\tilde{y}^*,\tilde{y}^*\rangle \geq 0 \qquad \forall \tilde{y}^* \in Y^*.$$

Hence

$$H(u,y^*) \leq H(u,v) \qquad \forall y^* \in Y^*.$$

Therefore (u,v) is a saddle point of $H(x,y^*)$. From (3.1.6) we can show that $H(x,y^*)$ is a convex-concave saddle functional. Taking Gateaux derivatives of $H(x,y^*)$ at (u,v) and using theorem 4.1.4 we find that (u,v) is unique.

Remark 4.4.1 In the context of elastomechanics the variational principle based on functional (4.4.5) is called the Hellinger-Reisner principle. Note its resemblance to the variational principle of section 3.2.

Example 4.4.2

From example 4.3.1 we can restrict the domain of $L(x,y,y^*)$ to those y and y^* satisfying

$$y^* = Ky.$$

This gives rise to two problems:

$$\begin{aligned} &\text{find } (u,w) \in H_0^{\,1}(\Omega) \times (L_2(\Omega))^2, \text{ saddle point of} \\ &G(x,y) = \int_\Omega \left(\text{grad } x.Ky - \tfrac{1}{2}\, yKy - xf\right) d\Omega, \quad x \in H_0^{\,1}(\Omega), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad y \in (L_2(\Omega))^2 \end{aligned} \tag{4.4.6}$$

and

$$\begin{aligned} &\text{find } (u,v) \in H_0^{\,1}(\Omega) \times (L_2(\Omega))^2, \text{ saddle point of} \\ &H(x,y^*) = \int_\Omega \left(\text{grad } x.y^* - \tfrac{1}{2}\, y^*.K^{-1}y^* - xf\right) d\Omega, \ x \in H_0^{\,1}(\Omega), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad y \in (L_2(\Omega))^2 . \quad \square \end{aligned} \tag{4.4.7}$$

Finally, let us restrict the domain of $L(x,y,y^*)$ to $\mathcal{D}_3$, i.e. those $(x,y,y^*) \in X \times Y \times Y^*$ satisfying $\nabla_3 L(x,y,y^*) = 0$. From lemma 4.3.1 we have

$$\mathcal{D}_3 = \{(x,y,y^*) \in X \times Y \times Y^*;\ y = Tx\}.$$

Let us write $\mathcal{D}_3 = X_3 \times Y_3 \times Y_3^*$ where $X_3 = X$, $Y_3 = Z^o$, $Y_3^* = Y^*$. Then from (4.4.1) we get

$$L\ (x,y,y^*) \ = \ \frac{1}{2} \langle Tx,ETx\rangle - \langle x,f\rangle \equiv J(x), \quad \forall x \in X. \tag{4.4.8}$$

Theorem 4.4.4

The convex functional $J(x)$ has a unique minimum point u where $((u,w = Tu),\ v = Ew)$ is the saddle point of $L(x,y,y^*)$.

Proof $\mathcal{D}_3$ contains the saddle point $((u,w),v)$, hence from (4.2.4) we have

$$J(u) \ \leq \ J(x) \qquad \forall x \in X.$$

Now

$$J(x_1) - J(x_2) - \langle \nabla J(x_2),\ x_1 - x_2\rangle$$

$$= \frac{1}{2}\langle Tx_1,ETx_1\rangle - \langle x_1,f\rangle - \frac{1}{2}\langle Tx_2,ETx_2\rangle + \langle x_2,f\rangle$$

$$-\langle Tx_2,ETx_1\rangle + \langle Tx_2,ETx_2\rangle + \langle x_1,f\rangle - \langle x_2,f\rangle$$

$$= \frac{1}{2}\langle Tx_1,ETx_1\rangle - \langle Tx_2,ETx_1\rangle + \frac{1}{2}\langle Tx_2,ETx_2\rangle$$

$$= \frac{1}{2}\langle T(x_1-x_2),\ ET(x_1-x_2)\rangle > 0 \qquad \forall x_1,x_2 \in X, \quad x_1 \neq x_2.$$

Hence $J(x)$ is a strictly convex function and so the minimum point u is unique.

Remark 4.4.2 $J(x)$ is the functional of the minimum potential energy principle. Compare with functional (3.4.5).

Example 4.4.3

Constraining the domain of $L(x,y,y^*)$ in example 4.3.1 to those y^* satisfying

$$y^* \ = \ Tx,$$

we can write (4.3.2) as

$$\text{find } u \in H_0^{\,1}(\Omega), \text{ minimum point of}$$
$$J(x) \;=\; \int_\Omega (\tfrac{1}{2}\,\text{grad } x.K \text{ grad } x - xf)d\Omega, \quad x \in H_0^{\,1}(\Omega). \qquad \square \tag{4.4.9}$$

Hence from the saddle functional $L(x,y,y^*)$ we have generated two further saddle functionals $M(y,y^*)$ and $H(x,y^*)$ and a convex functional $J(x)$. (We consider the functionals $G(x,y)$ and $H(x,y^*)$ to be essentially the same.) $M(y,y^*)$ and $H(x,y^*)$ lead to complementary variational principles as follows.

Restrict the domain $Y \times Z_f$ of $M(y,y^*)$ to

$$\mathcal{D}_{11} = \{(y,y^*) \in Y \times Z_f;\ \ y^* = Ey\}.$$

Then with $\mathcal{D}_{11} = Y_{11} \times Z_{f11}$ where $Y_{11} = \{y \in Y;\ Ey \in Z_f\}$ and $Z_{f11} = Z_f$, we can write

$$M(y,y^*) \;=\; -\tfrac{1}{2}\langle y,Ey\rangle \equiv I(y), \quad \forall y \in Y_{11}. \tag{4.4.10}$$

Letting $y = E^{-1}y^*$ we also have

$$M(y,y^*) \;=\; -\tfrac{1}{2}\langle E^{-1}y^*,y^*\rangle \equiv K(y^*), \quad \forall y^* \in Z_f. \tag{4.4.11}$$

<u>Theorem 4.4.5</u>

The concave functional $K(y^*)$ defined by (4.4.11) has a unique maximum at $y^* = v$.

<u>Proof</u> From theorem 3.3.1 $K(y^*)$ has a maximum at $y^* = v$. From lemma 3.3.1 this maximum is unique and $K(y^*)$ is a concave functional.

Remark 4.4.3 $K(y^*)$ is the functional of the complementary potential energy principle. Compare with functional (3.4.2).

Remark 4.4.4 Since we can write $I(y) = K(E^{-1}y)$, we can deduce from theorem 4.4.5 that $I(y)$ is a concave functional with a unique maximum at $y = w$.

Example 4.4.4

From example 4.4.1, defining the space

$$Y_{11} = \{y \in (L_2(\Omega))^2; \quad Ky \in Z_f\}$$

we can substitute $y^* = Ky$ into (4.4.3) to give the problem:

$$\begin{aligned} &\text{find } w \in Y_{11}, \text{ maximum point of} \\ &I(y) = -\frac{1}{2}\int_\Omega y\,Ky\,d\Omega, \quad y \in Y_{11}\,. \end{aligned} \tag{4.4.12}$$

Substituting instead, $y = K^{-1}y^*$ we get:

$$\begin{aligned} &\text{find } v \in Z_f \text{ maximum point of} \\ &K(y^*) = -\frac{1}{2}\int_\Omega y^*K^{-1}y^*d\Omega, \quad y^* \in Z_f. \qquad \square \end{aligned} \tag{4.4.13}$$

We can also restrict the domain of $M(y,y^*)$ to

$$\mathcal{D}_{12} = \{(y,y^*) \in Y \times Z_f; \quad y = Tx\}.$$

Then with $\mathcal{D}_{12} = Y_{12} \times Z_{f12}$ where $Y_{12} = Z^O$, $Z_{f12} = Z_f$

we have

$$M(y,y^*) = -\langle Tx,y^*\rangle + \frac{1}{2}\langle Tx,ETx\rangle$$

$$= \frac{1}{2}\langle Tx,ETx\rangle - \langle x,f\rangle \equiv J(x), \qquad \forall x \in X, \tag{4.4.14}$$

using theorem 2.2.5. From theorem 4.4.4, $J(x)$ is a convex functional with a unique minimum at $x = u$.

Example 4.4.5

We can easily see that substituting $y = Tx$ in (4.4.3) and using the adjoint relationship and the already assumed relationship $-\mathrm{div}\; y^* = f$, we get the functional

$$J(x) = \int_\Omega \left(\frac{1}{2}\,\mathrm{grad}\; x.K\;\mathrm{grad}\; x - xf\right)d\Omega$$

from $M(y,y^*)$. $\square$

Hence from the functional $M(y,y^*)$ we have derived complementary variational principles based on $K(y^*)$ and $J(x)$ since

$$\underset{x\in X}{\mathrm{Min}}\; J(x) = J(u) = M(w,v) = K(v) = \underset{y^*\in Z_f}{\mathrm{Max}}\; K(y^*)\;.$$

$H(x,y^*)$ also leads to complementary principles. Restrict the domain $X \times Y^*$ to

$$\mathcal{D}_{21} = \{(x,y^*) \in X \times Y^*;\;\; T^*y^* = f\}.$$

Set $\mathcal{D}_{21} = X_{21} \times Y^*_{21}$, where $X_{21} = X$ and $Y^*_{21} = Z_f$, then

$$H(x,y^*) = -\frac{1}{2}\langle E^{-1}y^*,y^*\rangle \equiv K(y^*), \qquad \forall y^* \in Z_f. \tag{4.4.15}$$

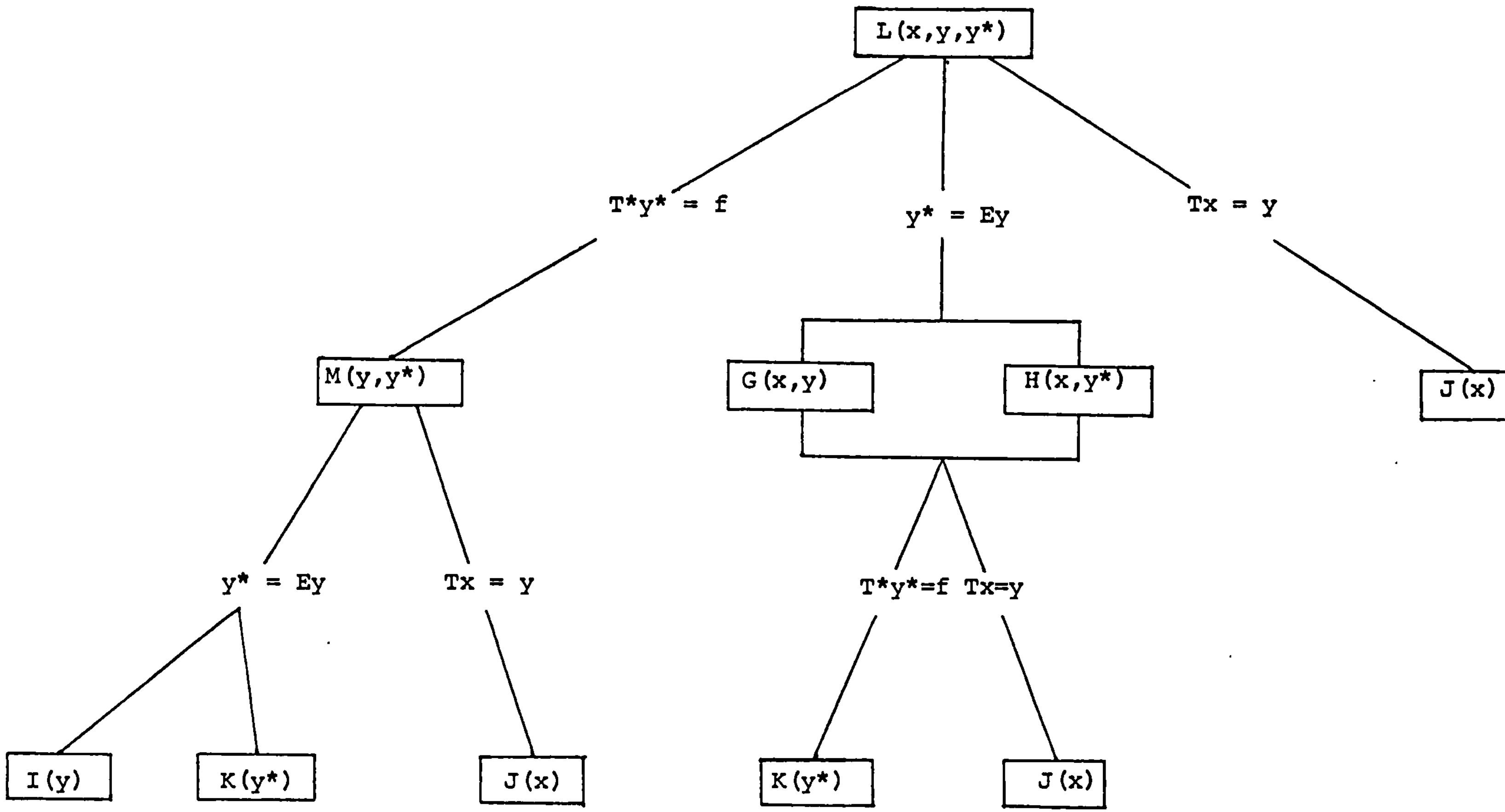

Fig. 4.4.1. Variational principles derived from L(x,y,y*).

Now restricting the domain $X \times Y$ of $G(x,y)$ to

$$\mathcal{D}_{22} = \{(x,y) \in X \times Y; \; y = Tx\},$$

we have

$$G(x,y) = \frac{1}{2}\langle Tx, ETx\rangle - \langle x, f\rangle \equiv J(x), \qquad \forall x \in X. \tag{4.4.16}$$

<u>Example 4.4.6</u>

Using the adjoint relationship and $-\text{div } y^* = f$ in (4.4.7) gives the variational problem (4.4.13). Also substituting $y = \text{grad } x$ in (4.4.6) we get the variational problem (4.4.9). □

Hence the complementary functionals $J(x)$ and $K(y^*)$ can also be derived from $H(x,y^*)$ (and $G(x,y) = H(x,Ey)$). The relationships between all the functionals are given in Figure 4.4.1.

CHAPTER 5

NON-LINEAR EQUATIONS

5.1 Results from Non-Linear Operator Theory

In this section we give some results in the theory of non-linear operators which will allow us to extend the variational principles already discussed to include non-linear problems. The basic concept is that of a monotone operator. We show how this is related to convex functionals and give some existence and uniqueness theorems of monotone operator equations.

Definition 5.1.1 (Vainberg [1973], p.46)

An operator $F(x)$ from $A \subset X$ to X^* is said to be potential if there exists a functional $f(x)$, defined on X, such that for all $x \in A$

$$F(x) = \nabla f(x). \tag{5.1.1}$$

The functional $f(x)$ is called the potential of the operator $F(x)$. □

The next theorem gives a very general condition for an operator $F(x)$ to be a potential operator.

Theorem 5.1.1 (Vainberg [1973], p.56)

Let $F(x)$ be a continuous operator from an open convex set $A \subset X$ to X^*. Then $F(x)$ is potential if and only if, for any polygonal line $\ell \subset A$, the line integral

$$\int_\ell \langle F(x), dx\rangle$$

is independent of the path of integration. □

If the operator is G-differentiable we have the following condition for it to be potential.

Theorem 5.1.2 (Vainberg [1973], p.59)

Let $F(x)$ be an operator from X to X^* which is G-differentiable at every point of an open convex set $A \subset X$, the Gateaux differential $DF(x,h)$ being continuous in x. Then $F(x)$ is potential in A if and only if the bilinear functional $\langle DF(x,h_1),h_2\rangle$ is symmetric, i.e.

$$\langle DF(x,h_1),h_2\rangle = \langle DF(x,h_2),h_1\rangle. \tag{5.1.2}$$

Remark 5.1.1 (Vainberg [1973], p.56)

Under the hypotheses of theorem 5.1.1 or theorem 5.1.2 the potential $f(x)$ of the operator $F(x)$ has the form

$$f(x) = f_0 + \int_0^1 \langle F(x_0 + t(x-x_0)), x - x_0\rangle dt, \tag{5.1.3}$$

where f_0 is a constant. □

Many of the operators we shall consider will be positive-homogeneous operators. In this case the potential takes a much simpler form.

Definition 5.1.2

$F(x)$ is a positive-homogeneous operator of degree $k > 0$ if

$$F(tx) = t^k F(x) \qquad \text{for } t > 0.$$

Remark 5.1.2 (Vainberg [1973], p.59)

Let $F(x)$ be a positive-homogeneous potential operator of degree $k > 0$. Then from (5.1.3), its potential is of the form

$$f(x) = f_0 + \frac{1}{k+1} \langle F(x), x \rangle.$$

<u>Lemma 5.1.1</u>

If $F(x)$ is a positive-homogeneous operator of degree $k > 0$ and F^{-1} exists, F^{-1} is a positive-homogeneous operator of degree $1/k$.

<u>Proof</u> Let $F(x) = x^*$, then from definition 5.1.2

$$tF^{-1}(x^*) = F^{-1}(t^k x^*).$$

Let $s = t^k$, then

$$s^{1/k}F^{-1}(x^*) = F^{-1}(sx^*),$$

i.e. F^{-1} is positive-homogeneous of degree $1/k$.

<u>Definition 5.1.3</u> (Vainberg [1973], p.10)

An operator $F : X \to X^*$ is said to be <u>monotone</u> if

$$\langle F(x_1) - F(x_2), x_1 - x_2 \rangle \geq 0 \qquad \forall x_1, x_2 \in D(F).$$

It is strictly monotone if equality can hold only when $x_1 = x_2$. □

The next theorem gives the relationship between monotone operators and convex functionals.

<u>Theorem 5.1.3</u> (Vainberg [1973], p.51)

A potential operator $F(x)$ defined on an open convex set A of X is monotone (strictly monotone) if and only if its potential $f(x)$ is a convex (strictly convex) functional on A. □

We now give two theorems on the existence and uniqueness of the solution of non-linear operator equations.

Theorem 5.1.4 (Vainberg [1973], p.79 and p.96)

Let $f(x)$ be a convex lower semicontinuous functional defined in a reflexive Banach space X, satisfying the condition

$$\lim_{\|x\| \to +\infty} f(x) = +\infty.$$

Then $f(x)$ has an absolute minimum point. Further if $f(x)$ is strictly convex then the minimum point is unique.

Theorem 5.1.5 (Vainberg [1973], p.97)

Let $f(x)$ be a G-differential functional defined on X and such that $F(x) = \nabla f(x)$ is a continuous monotone operator satisfying

$$\lim_{\|x\| \to \infty} \frac{\langle F(x), x\rangle}{\|x\|} = +\infty.$$

Then $f(x)$ has a minimum point x_0 and $F(x_0) = 0$. If F is strictly monotone, then the minimum point of the functional is unique and $f(x)$ has an absolute minimum there. $\square$

We now give a stronger definition of a monotone operator.

Definition 5.1.4 (Vainberg [1973], p.232)

An operator $F : X \to X^*$ is said to be strongly monotone if

$$\langle F(x_1) - F(x_2), x_1 - x_2\rangle \geq \|x_1 - x_2\| \gamma(\|x_1 - x_2\|)$$

where $\gamma(t)$ is an increasing function such that $\gamma(0) = 0$ and $\gamma(t) \to \infty$ as $t \to \infty$.

Theorem 5.1.6 (Vainberg [1973], p.232)

Let $F : X \to X^*$ be a continuous strongly monotone operator. Then F is a homeomorphism of X onto X^*.

5.2 Some Non-Linear Equations

We consider some non-linear equations related to the linear equation studied previously, i.e.

$$T^*ETu = f.$$

We shall introduce the non-linearity in two ways, first by taking the operator E to be non-linear and secondly by replacing the function f by a non-linear function $F : X \to X^*$.

In this section we shall prove some existence and uniqueness theorems for the non-linear equations and in the next section we shall develop the associated variational principles.

First we consider the equations

$$\begin{aligned} Tu &= w \\ E(w) &= v \\ T^*v &= f, \end{aligned} \tag{5.2.1}$$

where as before X and Y are Hilbert spaces, $T : X \to Y$ is a linear operator with adjoint $T^* : Y^* \to X^*$. $E : Y \to Y^*$ we now consider to be a non-linear operator. Obviously $u \in X$ and $f \in X^*$.

In section 4.1 we extended theorem 2.4.3 to prove the existence and uniqueness of a solution to problem (4.1.1). Here we shall adopt a similar method to show that equation (5.2.1) has a unique solution. Recall that theorem 2.4.3 established that T^* is an isomorphism from $Z^{\perp}$ to X^* and T is an isomorphism from X to Z^{O}, see figure 2.4.2. The next theorem is analogous to theorem 4.1.3 where E is a linear operator.

Theorem 5.2.1

Suppose E is a homeomorphism and that for any $\tilde{v} \in Z^{\perp}$ we have $G_{\tilde{v}}(z) \equiv E^{-1}(\tilde{v} + z)$ is a strictly monotone operator on Z. Also suppose

$$\lim_{\|z\| \to \infty} \frac{\langle G_{\tilde{v}}(z), z\rangle}{\|z\|} = +\infty. \tag{5.2.2}$$

Then given a $\tilde{v} \in Z^{\perp}$, there exists a unique $w \in Z^{O}$ such that

$$E(w) = \tilde{v} + \tilde{z}, \tag{5.2.3}$$

where $\tilde{z}$ is an element of z.

Proof

From theorem 5.1.5 there exists a unique $\tilde{z} \in Z$ such that

$$\langle G_{\tilde{v}}(\tilde{z}), z\rangle = 0 \qquad \forall z \in Z. \tag{5.2.4}$$

Hence given $\tilde{v} \in Z^{\perp}$ we can define

$$w = G_{\tilde{v}}(\tilde{z}) = E^{-1}(\tilde{v} + \tilde{z}),$$

and from (5.2.4) $w \in Z^{O}$. $\square$

The next theorem achieves the same result but with assumptions which are more easily proved in the applications.

Theorem 5.2.2

Suppose E is a continuous strictly monotone operator with $R(E) = Y^*$ and

$$\lim_{\|y\|\to\infty} \frac{\langle E(y), y\rangle}{\|y\|} = +\infty. \tag{5.2.5}$$

Then given $\tilde{v} \in Z^{\perp}$ there exists a unique $w \in Z^{O}$ such that

$$E(w) = \tilde{v} + \tilde{z},$$

where $\tilde{z}$ is an element of Z.

Proof

Since E is strictly monotone it is one-to-one and as $R(E) = Y^*$, E is a homeomorphism. Now since E is strictly monotone, it follows easily that E^{-1} is strictly monotone and hence $G_{\tilde{v}}(z) = E^{-1}(\tilde{v} + z)$ is strictly monotone on Z for any $\tilde{v} \in Z^{\perp}$. Also E^{-1} and hence G satisfies the growth property (5.2.2). Hence the result follows from theorem 5.2.1.

Remark 5.2.1 We may replace condition (5.2.5) by the conditions that $e(y)$, the potential of $E(y)$, is strictly convex and satisfies

$$\lim_{\|y\|\to\infty} e(y) = +\infty.$$

Then we can use theorem 5.1.4 instead of theorem 5.1.5. □

The existence and uniqueness of a solution to equations (5.2.1) now comes from theorem 2.4.3 and either theorem 5.2.1 or theorem 5.2.2. We state this as the next theorem.

Theorem 5.2.3

Let the spaces X and Y and operators T, T* and E be defined as in equation (5.2.1). Then suppose there exists $\alpha > 0$ such that

$$\|Tx\|_Y \geq \alpha \|x\|_X, \qquad \forall x \in X.$$

Suppose also that the conditions of theorem 5.2.1 or of theorem 5.2.2 are satisfied. Then equations (5.2.1) have a unique solution (u,w,v).

Example 5.2.1

Recall example 2.1.3. The equations are

$$\begin{aligned} \frac{d\dot{u}}{dr} &= \dot{\gamma} \\ k\dot{\gamma}^{1/3} &= \sigma \\ -\frac{d\sigma}{dr} &= \frac{P}{2\ell} \end{aligned} \tag{5.2.6}$$

with $\quad \dot{u}(a) = \dot{u}(-a) = 0.$

$T : H_0^1(\Omega) \to L_2(\Omega)$ is the generalised derivative $\frac{d}{dr}$. $T^* : L_2(\Omega) \to H^{-1}(\Omega)$ is the adjoint of $\frac{d}{dr}$, that is, T^* is an extension of $-\frac{d}{dr}$ as we have seen before.

The operator E is given by

$$E(y) = ky^{1/3}.$$

We prove the existence and uniqueness of a solution to this problem by verifying the hypotheses of theorem 5.2.3. The inequality

$$\left\|\frac{dx}{dr}\right\|_{L_2(\Omega)} \geq \alpha\|x\|_{H^1(\Omega)}, \quad \forall x \in H_0^1(\Omega)$$

we have proved before, see example 2.4.1. The operator $E(y)$ is a positive-homogeneous operator of degree 1/3 and is easily shown to be potential from theorem 5.1.2. Hence from remark 5.1.2 we have that the potential of $E(y)$ is

$$e(y) = \frac{3k}{4}\langle y^{1/3},y\rangle = \frac{3}{4}k\int_{-a}^{a} y^{4/3}d\Omega = \frac{3}{4}k\|y^{2/3}\|^2_{L_2(\Omega)}. \tag{5.2.7}$$

Using remark 5.1.1, we need to show that $e(y)$ is a strictly convex functional satisfying

$$\lim_{\|y\|\to\infty} e(y) = +\infty. \tag{5.2.8}$$

Now from theorem 5.1.3 $e(y)$ is strictly convex if $E(y)$ is strictly monotone. This is easily shown as

$$\langle E(y_1) - E(y_2), y_1 - y_2\rangle = \int_{-a}^{a} (y_1^{1/3}-y_2^{1/3})(y_1 - y_2)d\Omega$$

$$> \frac{3}{4}\int_{-a}^{a} (y_1^{2/3}-y_2^{2/3})^2 d\Omega > 0 \quad \forall y_1,y_2 \in Y, \quad y_1 \neq y_2.$$

(5.2.8) is also satisfied, hence a unique solution of problem (5.2.6) exists. □

We now introduce the second type of non-linearity we shall be considering. As before let X and Y be Hilbert spaces and $T : X \to Y$ a linear operator with adjoint $T^* : Y^* \to X^*$ and let $\tau : Y \to Y^*$ be the canonical isomorphism from Y to Y^*. Then we introduce the non-linear operator $F : X \to X^*$ and consider the equations

$$\begin{aligned} \tau T u &= v \\ T^* v + F(u) &= 0 \end{aligned} \tag{5.2.12}$$

We can easily prove the existence and uniqueness of a solution to these equations under certain ass'umptions on the operator F. These results are given in the next theorem.

Theorem 5.2.4

Let all spaces and operators be defined as in equations (5.2.12). Then suppose there exists $\alpha > 0$ such that

$$\|Tx\|_Y \geq \alpha \|x\|_X, \ \forall x \in X. \tag{5.2.13}$$

Suppose also that F is a monotone operator satisfying

$$\liminf_{\|x\|\to\infty} \frac{\langle F(x), x\rangle}{\|x\|} > -\infty. \tag{5.2.14}$$

Then equations (5.2.12) have a unique solution (u,v).

Proof

Let $G(x) = T^*\tau Tx + F(x)$, then we only need to prove that $G(u) = 0$ has a unique solution. Now

$$\begin{aligned} &\langle G(x_1) - G(x_2), x_1 - x_2\rangle \\ &= \langle T^*\tau T(x_1-x_2), x_1 - x_2\rangle + \langle F(x_1) - F(x_2), x_1 - x_2\rangle \\ &= \langle \tau T(x_1-x_2), T(x_1-x_2)\rangle + \langle F(x_1) - F(x_2), x_1 - x_2\rangle > 0 \end{aligned}$$

for all $x_1, x_2 \in X$, $x_1 \neq x_2$, from (5.2.13) and the monotonicity of F. Hence

G is a strictly monotone operator. Also

$$\frac{\langle G(x),x\rangle}{\|x\|} = \frac{\langle T^*\tau Tx + F(x),x\rangle}{\|x\|}$$

$$= \frac{\langle \tau Tx,Tx\rangle}{\|x\|} + \frac{\langle F(x),x\rangle}{\|x\|}$$

$$\geq \alpha\|x\| + \frac{\langle F(x),x\rangle}{\|x\|} .$$

Hence as F satisfies (5.2.14) we have

$$\lim_{\|x\|\to\infty} \frac{\langle G(x),x\rangle}{\|x\|} = +\infty .$$

Therefore from theorem 5.1.5, equations (5.2.12) have a unique solution $(u,w' = \tau Tu)$.

Example 5.2.2

Recall example 2.1.4, that is,

$$\begin{aligned} \text{grad } u &= v \\ -\text{div } v + ce^u &= 0 \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \Gamma . \end{aligned} \tag{5.2.15}$$

We take $T : H_0^1(\Omega) \to (L_2(\Omega))^2$ to be the generalised grad operator and T^* is defined through the adjoint relation as an extension of -div. The operator F is given by

$$F(x) = ce^x.$$

We prove the existence and uniqueness of a solution to this problem by verifying the hypotheses of theorem 5.2.4. Inequality (5.2.13) has already been proved in example 4.1.1. Hence we only need to show that F is a monotone operator satisfying (5.2.14). For any point $\xi \in \Omega$ we have

$$(e^{x_1(\xi)} - e^{x_2(\xi)})(x_1(\xi) - x_2(\xi)) \geq 0, \quad \forall x_1, x_2 \in H_0^{\,1}(\Omega).$$

Hence

$$\langle F(x_1) - F(x_2), x_1 - x_2 \rangle \geq 0, \quad \forall x_1, x_2 \in H_0^{\,1}(\Omega),$$

i.e. F is a monotone operator. Condition (5.2.14) is

$$\liminf_{||x|| \to \infty} c \frac{\int_\Omega e^x x d\Omega}{||x||} > -\infty.$$

Now at any point $\xi \in \Omega$ we have

$$e^{x(\xi)} x(\xi) \geq A$$

where $A = -e^{-1}$. Hence of M is the measure of Ω we have

$$\int_\Omega e^x x d\Omega \geq MA,$$

and therefore

$$\liminf_{||x|| \to \infty} \frac{\int_\Omega e^x x d\Omega}{||x||} \geq 0.$$

Hence from theorem 5.2.4 a unique solution of problem (5.2.15) exists. □

Finally we combine the two forms of non-linearity we have considered. Let X and Y be Hilbert spaces and $T : X \to Y$ a linear operator with adjoint $T^* : Y^* \to X^*$. Let $E : Y \to Y^*$ and $F : X \to X^*$ be non-linear operators. Then consider the equations

$$\begin{aligned} Tu &= w \\ E(w) &= v \\ T^*v + F(u) &= 0. \end{aligned} \tag{5.2.16}$$

The following theorem gives the conditions on E and F for these equations to have a unique solution.

<u>Theorem 5.2.5</u>

Suppose there exists $\alpha > 0$ such that

$$\|Tx\| \geq \alpha \|x\|, \quad \forall x \in X.$$

Suppose that E is strictly monotone on Z^0 and F is monotone on X. Further suppose that either

(i) $$\lim_{\|y\| \to \infty} \frac{\langle E(y), y \rangle}{\|y\|} = +\infty \qquad y \in Z^0,$$

and $$\liminf_{\|x\| \to \infty} \frac{\langle F(x), x \rangle}{\|x\|} > -\infty \qquad x \in X,$$

or (ii) $$\liminf_{\|y\| \to \infty} \frac{\langle E(y), y \rangle}{\|y\|} > -\infty \qquad y \in Z^0,$$

and $$\lim_{\|x\| \to \infty} \frac{\langle F(x), x \rangle}{\|x\|} = +\infty \qquad x \in X.$$

Then equations (5.2.16) have a unique solution (u,w,v).

Proof

Let $G(x) = T^*E(Tx) + F(x)$.

Then

$$\langle G(x_1) - G(x_2), x_1 - x_2 \rangle$$

$$= \langle T^*E(Tx_1) - T^*E(Tx_2), x_1 - x_2 \rangle + \langle F(x_1) - F(x_2), x_1 - x_2 \rangle$$

$$= \langle E(Tx_1) - E(Tx_2), Tx_1 - Tx_2 \rangle + \langle F(x_1) - F(x_2), x_1 - x_2 \rangle > 0$$

for $x_1, x_2 \in X$, $x_1 \neq x_2$, since E is strictly monotone on Z^O and F is monotone on X. Now

$$\frac{\langle G(x),x \rangle}{\|x\|} = \frac{\langle E(Tx),Tx \rangle}{\|x\|} + \frac{\langle F(x),x \rangle}{\|x\|}$$

$$\geq \alpha \frac{\langle E(Tx),Tx \rangle}{\|Tx\|} + \frac{\langle F(x),x \rangle}{\|x\|} .$$

Hence

$$\lim_{\|x\| \to \infty} \frac{\langle G(x),x \rangle}{\|x\|} = +\infty$$

is satisfied if either (i) or (ii) are satisfied. Therefore from theorem 5.1.5, the equation $G(u) = O$ has a unique solution u. Now since E is strictly monotone on Z^O, E is a one-to-one mapping of Z^O to Y*. Hence there exists a unique $(u, w = Tu, v = E(w))$ satisfying equations (5.2.16).

5.3 Variational Principles for Non-Linear Equations

In this section we develop variational principles associated with the non-linear equations given in section 5.2. The development will be similar to that of sections 4.3 and 4.4 where we described variational principles for linear equations. We shall consider only the most general non-linear equation of section 5.2, that is,

$$T^*E(Tu) + F(u) = 0 , \tag{5.3.1}$$

where the spaces and operators are as defined in equations (5.2.16). We can rewrite this as

$$\begin{aligned} T^*v + F(u) &= 0, \\ v &= E(w), \\ w &= Tu . \end{aligned} \tag{5.3.2}$$

Then if (u,w,v) is a solution of (5.3.2) u is a solution of (5.3.1).

We shall prove the following result (theorem 5.3.1): assume the hypotheses of theorem 5.2.5 are satisfied and in addition that the operators E and F are potential operators, then the solution (u,w,v) of equations (5.3.2) is also a solution of the variational problem,

$$\begin{aligned} &\text{find } ((u,w),v) \in (X \times Y) \times Y^* \text{ a saddle point of} \\ &L(x,y,y^*) = \langle Tx,y^*\rangle - \langle y,y^*\rangle + e(y) + f(x) . \end{aligned} \tag{5.3.3}$$

The functionals $e(y)$, and $f(x)$ are derived from (5.1.3) and are

$$e(y) = \int_0^1 \langle E(ty),y\rangle \, dt, \tag{5.3.4}$$

and $$f(x) = \int_0^1 \langle F(tx), x \rangle \, dt. \tag{5.3.5}$$

Then we have

<u>Lemma 5.3.1</u> Suppose E and F are potential operators, then (u,w,v) is a stationary point of $L(x,y,y^*)$ iff equations (5.3.2) are satisfied.

<u>Proof</u>

$$\langle \nabla_1 L(u,w,v), x\rangle = \langle x, T^*v\rangle + \langle F(u), x\rangle = 0 \qquad \forall x \in X$$

$$\text{iff} \qquad T^*v + F(u) = 0.$$

$$\langle \nabla_2 L(u,w,v) y\rangle = -\langle y,v\rangle + \langle E(w), y\rangle = 0 \quad \forall y \in Y$$

$$\text{iff} \qquad v = E(w).$$

$$\langle \nabla_3 L(u,w,v), y^*\rangle = \langle Tu, y^*\rangle - \langle w, y^*\rangle = 0 \quad \forall y^* \in Y^*$$

$$\text{iff} \qquad Tu = w.$$

<u>Lemma 5.3.2</u>

Suppose E and F are potential monotone operators, then $L(x,y,y^*)$ is a saddle functional, convex in $X \times Y$ and concave in Y^*.

<u>Proof</u>

From inequality (5.2.3) and theorem 5.1.3. □

Hence we can now prove the existence and uniqueness of a saddle point of $L(x,y,y^*)$.

<u>Theorem 5.3.1</u>

Suppose the conditions of theorem 5.2.5 are satisfied, that is, there exists $\alpha > 0$ such that

$$\|Tx\| \geq \alpha\|x\|, \qquad \forall x \in X,$$

E is strictly monotone on Z^O, F is monotone on X and either

$$\lim_{\|y\|\to\infty} \frac{\langle E(y),y\rangle}{\|y\|} = +\infty, \quad y \in Z^O,$$

and $$\lim_{\|x\|\to\infty} \frac{\langle F(x),x\rangle}{\|x\|} > -\infty \quad x \in X,$$

are satisfied or

$$\lim_{\|y\|\to\infty} \frac{\langle E(y),y\rangle}{\|y\|} > -\infty \quad y \in Z^O,$$

and $$\lim_{\|x\|\to\infty} \frac{\langle F(x),x\rangle}{\|x\|} = +\infty \quad x \in X.$$

Suppose in addition that E and F are potential operators. Then L(x,y,y*) has a unique saddle point ((u,w),v) which is also the solution of (5.3.2).

Proof

From lemma 5.3.1 (u,w,v) is a stationary point of L(x,y,y*) if and only if (5.3.2) is satisfied. Hence from theorem 5.2.5, (u,w,v) is a unique stationary point of L(x,y,y*). Now from lemma 5.3.2 and lemma 3.1.4 (u,w,v) is a saddle point of L(x,y,y*).

Example 5.3.1

Consider the problem of example 5.2.1 in which the non-linear operator E is given by

$$E(y) = ky^{1/3} .$$

This is a positive-homogeneous operator of degree 1/3, see definition 5.1.2, and is easily shown to be a potential operator from theorem 5.1.2.

Hence from remark 5.1.2, the functional e(y) is given by

$$e(y) = \frac{3}{4} k \int_{-a}^{a} y^{4/3} dr.$$

Therefore the variational form of this problem is:

find $((\dot{u},\dot{\gamma}),\sigma) \in (H_0^1(\Omega) \times L_2(\Omega)) \times L_2(\Omega)$, saddle point of

$$L(x,y,y^*) = \int_{-a}^{a} \left(\frac{dx}{dr} y^* - yy^* + \frac{3k}{4} y^{4/3} - \frac{Px}{2\ell}\right) dr. \tag{5.3.6}$$

This has a unique solution from theorem 5.3.1 which is also the solution of equations (5.2.6).

Example 5.3.2

Consider the problem of example 5.2.2 for which the operator E is just the isometric isomorphism τ. Hence

$$e(y) = \frac{1}{2} \langle y, \tau y \rangle = \frac{1}{2} \langle \tau^{-1} y^*, y^* \rangle.$$

The non-linear operator F is given by

$$F(x) = ce^x .$$

Hence

$$f(x) = \int_0^1 \langle ce^{tx}, x \rangle dt$$

$$= c \langle \frac{e^x - 1}{x}, x \rangle$$

$$= c \int_\Omega (e^x - 1) d\Omega.$$

Therefore the functional $L(x,y,y^*)$ is given by

$$L(x,y,y^*) = \int_\Omega \text{grad } x.y^* d\Omega - \langle y,y^*\rangle + \frac{1}{2}\langle \tau^{-1}y^*,y^*\rangle + c\int_\Omega (e^x - 1)d\Omega$$
$$= \int_\Omega (\text{grad } x.y^* - \frac{1}{2}\, y^*.y^* + c(e^x - 1))d\Omega.$$

Hence the variational problem is:

$$\text{find } (u,v) \in H_0^1(\Omega) \times (L_2(\Omega))^2, \text{ saddle point of}$$
$$L(x,y^*) = \int_\Omega \{\text{grad } x.y^* - \frac{1}{2}\, y^*.y^* + c(e^x - 1)\}d\Omega.$$

Theorem 5.3.1 shows this has a unique solution which is also the solution of equations (5.2.15). $\square$

As in section 4.4 we can derive other variational principles associated with equations (5.3.2) by restricting the domain $X \times Y \times Y^*$ of $L(x,y,y^*)$. Let us restrict $X \times Y \times Y^*$ to $\mathcal{D}_1$, i.e. those (x,y,y^*) satisfying $\nabla_1 L(x,y,y^*) = 0$. From the proof of lemma 5.3.2 this is

$$\mathcal{D}_1 = \{(x,y,y^*) \in X \times Y \times Y^*;\ \ T^*y^* + Fx = 0\}.$$

Now if we assume that F is a bijective map from X to X^* we can express any $x \in X$ by

$$x = F^{-1}(-T^*y^*), \qquad y^* \in Y^*,$$

as T^* is a one-to-one map from $Z^\perp \subset Y^*$ to X^*. Hence the restriction of $L(x,y,y^*)$ to $\mathcal{D}_1$ can be written as

$$M(y,y^*) = \langle F^{-1}(-T^*y^*), T^*y^*\rangle - \langle y,y^*\rangle + e(y)$$
$$+ f(F^{-1}(-T^*y^*)) \qquad \forall y \in Y, \ \ \forall y^* \in Y^*. \tag{5.3.7}$$

Then we have the following theorem.

Theorem 5.3.2

Suppose the conditions of theorem 5.2.5 are satisfied and in addition that E and F are potential operators with F also bijective, then the convex-concave saddle functional $M(y,y^*)$ has a unique saddle point (w,v) where $((u,w),v)$ is the saddle point of $L(x,y,y^*)$.

Proof

$$\langle \nabla_1 M(w,v), y\rangle = -\langle y,v\rangle + \langle y, E(w)\rangle = 0$$

$$\text{iff} \qquad v = E(w)$$

$$\langle \nabla_2 M(w,v), y^*\rangle = \langle DF^{-1}(-T^*v,-T^*v), T^*y^*\rangle + \langle F^{-1}(-T^*v), T^*y^*\rangle$$

$$- \langle y^*, w\rangle + \langle DF^{-1}(-T^*v,-T^*v), -T^*y^*\rangle = 0$$

$$\Longleftrightarrow \quad \langle F^{-1}(-T^*v), T^*y^*\rangle - \langle y^*, w\rangle = 0$$

$$\text{i.e.} \qquad w = T(F^{-1}(-T^*v))$$

Hence $w \in R(T) = Z^0$ and since T is an isomorphism from X to Z^0, there exists a unique $u \in X$ such that

$$Tu = w.$$

Therefore any stationary point of $M(y,y^*)$ satisfies (5.3.2) and so under the conditions of theorem 5.2.5, this stationary point is unique. Now $M(y,y^*)$ is a convex-concave saddle functional since the term $\langle F^{-1}(-T^*y^*), T^*y^*\rangle + f(F^{-1}(-T^*y^*))$ is concave in y^*. To show this we have

$$\langle F^{-1}(-T^*y_1^*), T^*y_1^*\rangle + f(F^{-1}(-T^*y_1^*)) - \langle F^{-1}(-T^*y_2^*), T^*y_2^*\rangle - f(F^{-1}(-T^*y_2^*))$$

$$- \langle DF^{-1}(-T^*y_1^*, -T^*y_1^*), T^*(y_1^*-y_2^*)\rangle - \langle F^{-1}(-T^*y_1^*), T^*(y_1^*-y_2^*)\rangle$$

$$- \langle DF^{-1}(-T^*y_1^*, -T^*y_1^*), -T^*(y_1^*-y_2^*)\rangle$$

$$= \; f(F^{-1}(-T^*y_1^*)) - f(F^{-1}(-T^*y_2^*))$$

$$- \langle -T^*y_2^*, F^{-1}(-T^*y_1^*) - F^{-1}(-T^*y_2^*)\rangle \geq 0$$

as f is convex.

Example 5.3.3

From problem (5.3.6) of example 5.3.1 we can easily derive the problem

find $(w,v) \in L_2(\Omega) \times Z_f$, saddle point of

$$M(y,y^*) \;=\; -\int_{-a}^{a} (yy^* - \tfrac{3}{4}\,ky^{4/3})dr, \; y \in L_2(\Omega), \quad y^* \in Z_f.$$

where $\quad Z_f \;=\; \{y^* \in L_2(\Omega);\; -\frac{dy^*}{dr} \;=\; \frac{P}{2\ell}\}.$ □

Let us restrict the domain of $L(x,y,y^*)$ to $\mathcal{D}_2$, i.e. those (x,y,y^*) satisfying $\nabla_2 L(x,y,y^*) = 0$. This is given by

$$\mathcal{D}_2 \;=\; \{(x,y,y^*) \in X \times Y \times Y^*;\;\; y^* = E(y)\}.$$

Provided E and F are potential operators this leads to two functionals $G(x,y)$ and $H(x,y^*)$, in this case given by

$$G(x,y) \;=\; \langle Tx, E(y)\rangle - \langle y, E(y)\rangle + e(y) + f(x), \tag{5.3.8}$$

$$H(x,y^*) \;=\; \langle Tx, y^*\rangle - \langle E^{-1}(y^*), y^*\rangle + e(E^{-1}(y^*)) + f(x), \tag{5.3.9}$$

where the functionals $e(y)$ and $f(x)$ are given by (5.3.4) and (5.3.5) respectively.

However, let us suppose that $E(y)$ is a positive homogeneous potential operator of degree $k > 0$, see definition 5.1.2. Then from remark 5.1.2 we have

$$e(y) = \frac{1}{k+1} \langle y, E(y) \rangle .$$

Hence
$$e(y) - \langle y, E(y) \rangle = -\frac{k}{k+1} \langle y, E(y) \rangle . \tag{5.3.10}$$

Now if $y^* = E(y)$ then

$$e(y) - \langle y, E(y) \rangle = -\frac{k}{k+1} \langle E^{-1}(y^*), y^* \rangle . \tag{5.3.11}$$

But as E^{-1} is positive-homogeneous of degree $\frac{1}{k}$, see lemma 5.1.1, we can define

$$e^*(y^*) \equiv \int_0^1 \langle E^{-1}(ty^*), y^* \rangle dt = \frac{k}{k+1} \langle E^{-1}(y^*), y^* \rangle . \tag{5.3.12}$$

Hence we have

$$e(y) - \langle y, y^* \rangle = -e^*(y^*) . \tag{5.3.13}$$

Therefore we can rewrite $H(x,y^*)$ as

$$H(x,y^*) = \langle Tx, y^* \rangle - e^*(y^*) + f(x) . \tag{5.3.14}$$

Then we have the following theorem

Theorem 5.3.3

Assume the conditions of theorem 5.2.5 hold and that E and F are potential operators with E a positive homogeneous operator of degree $k > 0$. Then $H(x,y^*)$ is a convex-concave saddle functional with a unique saddle point (u,v), where $((u,w),v)$ is the saddle point of $L(x,y,y^*)$.

Proof

$$\langle \nabla_1 H(u,v),x\rangle = \langle Tx,v\rangle + \langle F(u),x\rangle = 0, \quad \forall x \in X,$$

$$\text{iff} \qquad T^*v + F(u) = 0.$$

$$\langle \nabla_2 H(u,v),y^*\rangle = \langle Tu,y^*\rangle - \langle E^{-1}(v),y^*\rangle = 0, \; \forall y^* \in Y^*,$$

$$\text{iff} \qquad Tu = E^{-1}(v) = w.$$

Hence any stationary point of $H(x,y^*)$ satisfies equations (5.3.2) and therefore it is a unique stationary point. Now since E^{-1} is monotone we can use theorem 5.1.3 to show that $-e(y^*)$ is concave. Also $f(x)$ is a convex functional and so $H(x,y^*)$ is a convex-concave saddle functional. Hence using lemma 3.1.4, the theorem is proved.

Example 5.3.4

The following variational problems are easily derived from problem (5.3.6) of example 5.3.1:

find $(u,w) \in H_0^1(\Omega) \times L_2(\Omega)$, saddle point of

$$G(x,y) = \int_{-a}^{a} \left(\frac{dx}{dr}\, ky^{1/3} - ky^{4/3} + \frac{3k}{4}\, y^{4/3} - \frac{Px}{2\ell}\right) dr, \quad x \in H_0^1(\Omega),\; y \in L_2(\Omega),$$

find $(u,v) \in H_0^1(\Omega) \times L_2(\Omega)$, saddle point of

$$H(x,y^*) = \int_{-a}^{a} \left(\frac{dx}{dr}\, y^* - \frac{1}{4}\left(\frac{y^*}{k}\right)^3 y^* - \frac{xP}{2\ell}\right) dr, \; x \in H_0^1(\Omega),\; y^* \in L_2(\Omega). \qquad (5.3.15) \quad \square$$

Finally we can restrict the domain of $L(x,y,y^*)$ to $\mathcal{D}_3$, i.e. those (x,y,y^*) satisfying $\nabla_3 L(x,y,y^*) = 0$. This is

$$\mathcal{D}_3 = \{(x,y,y^*) \in X \times Y \times Y^*; \quad y = Tx\}$$

On $\mathcal{D}_3$ we can write $L(x,y,y^*)$ as

$$J(x) = e(Tx) + f(x), \qquad \forall x \in X . \tag{5.3.16}$$

Then we have

Theorem 5.3.4

Provided the conditions of theorem 5.2.5 are satisfied and provided E and F are potential operators, the convex functional $J(x)$ has a unique minimum point u where $((u,v),v)$ is the saddle point of $L(x,y,y^*)$.

Proof

$$\langle \nabla J(u), x\rangle = \langle E(Tu), Tx\rangle + \langle F(u), x\rangle = 0 \qquad \forall x \in X$$

$$\text{iff} \qquad T^*E(Tu) + F(u) = 0.$$

Hence from theorem 5.2.5, $J(x)$ has a unique stationary point u. Since $e(y)$ and $f(x)$ are convex functionals $J(x)$ is convex and hence the stationary point is a minimum point of $J(x)$.

Example 5.3.5

From example 5.3.1 writing $y = \frac{dx}{dr}$ we get the problem

find $u \in H_0^{\,1}(\Omega)$, minimum point of

$$J(x) = \int_{-a}^{a} \left\{\frac{3k}{4}\left(\frac{dx}{dr}\right)^{4/3} - \frac{xP}{2\ell}\right\} dr, \qquad x \in H_0^{\,1}(\Omega).$$

Example 5.3.6

From example 5.3.2 writing $y^* = \text{grad } x$ we get the problem

find $u \in H_0^1(\Omega)$, minimum point of

$$J(x) = \int_\Omega \{ \tfrac{1}{2} \text{grad } x . \text{grad } x + c(e^x - 1)\} \, d\Omega, \quad x \in H_0^1(\Omega) . \quad \square$$

In the same way we can derive all the functionals shown in figure 4.4.1. Here we show only how $K(y^*)$ is derived from $H(x,y^*)$. We restrict the domain $X \times Y^*$ of $H(x,y^*)$ to

$$\mathcal{D}_{21} = \{(x,y^*) \in X \times Y^*; \quad T^*y^* + Fx = 0\}.$$

Hence we can write $H(x,y^*)$ as

$$K(y^*) = \langle F^{-1}(-T^*y^*), T^*y^*\rangle - e^*(y^*) + f(F^{-1}(-T^*y^*)). \qquad (5.3.17)$$

Then we have the following theorem

Theorem 5.3.5

Under the conditions of theorem 5.3.3, the concave functional $K(y^*)$ has a unique maximum point v, where $((u,w),v)$ is the saddle point of $L(x,y,y^*)$.

Proof

$$\langle \nabla K(v), y^*\rangle = \langle DF^{-1}(-T^*v, -T^*v), T^*y^*\rangle + \langle F^{-1}(-T^*v), T^*y^*\rangle$$

$$- \langle E^{-1}(v), y^*\rangle + \langle DF^{-1}(-T^*v, -T^*v), -T^*y^*\rangle = 0$$

$$\Leftrightarrow \quad \langle F^{-1}(-T^*v), T^*y^*\rangle - \langle E^{-1}(v), y^*\rangle = 0, \quad \forall y^* \in Y^*,$$

$$\text{iff} \quad E^{-1}(v) = T(F^{-1}(-T^*v)).$$

However $E^{-1}(v) = w$ and so $w \in R(T) = Z^o$. Hence there exists a unique $u \in X$ such that

$$Tu = w.$$

Therefore equations (5.3.2) are satisfied and so from theorem 5.2.5 the stationary point of $K(y^*)$ is unique. From theorem 3.3.1 and lemma 3.3.1, $K(y^*)$ is a concave functional and v is a unique maximum point.

Example 5.3.7

From problem (5.3.15) of example 5.3.4 using the adjoint relationship and the constraint

$$-\frac{dy^*}{dr} = \frac{P}{2\ell}$$

we get the variational problem

find $v \in Z_f$, maximum point of

$$K(y^*) = -\int_{-a}^{a} \frac{1}{4}\left(\frac{y^*}{k}\right)^3 y^* dr, \quad y^* \in Z_f$$

where $$Z_f = \{y^* \in L_2(\Omega);\ -\frac{dy^*}{dr} = \frac{P}{2\ell}\}.$$

Example 5.3.8

From example 5.3.2 we can derive the problem

find $v \in (L_2(\Omega))^2$, maximum point of

$$K(y^*) = \int_\Omega \{-\ln(\operatorname{div} y^*)\operatorname{div} y^* + c_1 \operatorname{div} y^* - \frac{1}{2} y^* y^* - c\}\, d\Omega, \quad y^* \in (L_2(\Omega))^2,$$

where $c_1 = \ln c + 1$. □

As in the linear case the functionals $J(x)$ and $K(y^*)$ are complementary for

$$J(u) = e(Tu) + f(u)$$

and

$$\begin{aligned} K(v) &= \langle F^{-1}(-T^*v), T^*v\rangle - e^*(v) + f(F^{-1}(-T^*v)) \\ &= \langle u, T^*v\rangle - e^*(v) + f(u) \\ &= \langle w, v\rangle - e^*(v) + f(u) \\ &= e(w) + f(u). \end{aligned}$$

Hence

$$\min_{x \in X} J(x) = J(u) = K(v) = \max_{y^* \in Y^*} K(y^*).$$

In this chapter we have achieved the same results for the non-linear problems as we gave for the linear problems in Chapter 4. To summarise these results, we have

(1) given conditions on the operators T, T^*, E and F for the operator equation

$$T^*E(Tu) + F(u) = 0$$

to have a unique solution,

(2) shown that under further conditions on the operators E and F, the operator equation is equivalent in some sense to the variational problem

find $((u,w),v) \in (X \times Y) \times Y^*$ saddle point of

$$L(x,y,y^*) = \langle Tx,y^* \rangle - \langle y,y^* \rangle + e(y) + f(x),$$

(3) shown that from this variational problem we can derive an inter-related set of variational problems, two of which are complementary extremum problems,

(4) given examples of how the abstract theory is applied to boundary value problems of mathematical physics.

In applying these results to simple problems we have always considered homogeneous boundary value problems. The abstract theory can, however, be applied to non-homogeneous boundary value problems and it is this application of the theory that we consider in the next chapter.

CHAPTER 6

NON-HOMOGENEOUS BOUNDARY VALUE PROBLEMS

6.1 Theoretical Results on Elliptic Boundary Value Problems

In this chapter we show how boundary conditions, other than the homogeneous Dirichlet conditions we have considered previously, are incorporated into the abstract formulation. To do this we first need to ensure that the problem is "well posed" and that it has a Green's formula associated with it. We shall study the class of problems called regular elliptic problems which are extensively studied by Lions-Magenes [1972]. The major result that we shall use is stated in theorem 6.1.4 which shows that a regular elliptic problem has a Green's formula. In this section we shall briefly state the major concepts involved in the study of regular elliptic problems using Lions-Magenes [1972] as our main source. First we define precisely what is meant by a function having a value on the boundary Γ of a region Ω.

Let Ω be an open bounded set of $\mathbb{R}^n$ with boundary Γ. Then we wish to define, in some sense, the values of a function x on the boundary Γ. For $x \in C^m(\overline{\Omega})$ we can define a trace operator γ_0 such that $\gamma_0 x$ is the value of x on Γ. We may also define a trace operator γ_j such that

$$\gamma_j x = \frac{\partial^j x}{\partial n^j} \quad \text{on } \Gamma, \qquad 0 \leq j \leq m-1 ,$$

where $\dfrac{\partial^j}{\partial n^j}$ is the j^{th} order outward normal derivative on Γ. Then the most general trace operator we can define for $x \in C^m(\overline{\Omega})$ is

$$\gamma x = \{\gamma_0 x, \ldots, \gamma_{m-1} x\}.$$

Now can we extend this concept to functions $x \in H^m(\Omega)$?

For simplicity we shall make the following assumptions about Γ:

the boundary Γ of Ω is an (n-1) dimensional
infinitely differentiable variety, Ω being (6.1.1)
locally on one side of Γ.

Definition 6.1.1

For Ω a domain satisfying (6.1.1), $H^m(\Gamma)$ is the space of functions for which all generalised derivatives of order $\leq m$ on Γ belong to $L_2(\Gamma)$.

Remark 6.1.1. The space $H^m(\Gamma)$ for m not an integer can be defined by interpolation, see Necas [1967] or Lions-Magenes [1972]. □

Then we have the following trace theorem

Theorem 6.1.1 (Lions-Magenes [1972], p.39).

Let Ω be a bounded domain satisfying (6.1.1). The mapping

$$x \to \{\gamma_j x; \quad j = 0,\ldots,m-1\}$$

of $C^\infty(\overline{\Omega}) \to (\mathcal{D}(\Gamma))^m$ extends by continuity to a continuous linear mapping, still denoted

$$x \to \{\gamma_j x; \quad j = 0,\ldots,m-1\}$$

of $H^m(\Omega) \to \prod_{j=0}^{m-1} H^{m-j-1/2}(\Gamma)$.

Corollary 6.1.1 (Necas [1967], p.99)

For Ω a bounded domain satisfying (6.1.1) and $x \in H^m(\Omega)$, there exists constants $\alpha_j > 0$ such that

$$\|\gamma_j x\|_{H^{n-j-1/2}(\Gamma)} \leq \alpha_j \|x\|_{H^m(\Omega)}, \quad 0 \leq j \leq m-1. \qquad (6.1.2) \ \square$$

In the applications we shall frequently use spaces such as

$$H(\mathrm{div},\Omega) = \{x \in (L_2(\Omega))^n; \ \ \mathrm{div}\, x \in L_2(\Omega)\}$$

with norm

$$\|x\|_{H(\mathrm{div}\ \Omega)} = \{\|x\|^2_{L_2(\Omega)} + \|\mathrm{div}\, x\|^2_{L_2(\Omega)}\}^{1/2}.$$

For this space we have the following trace theorem, see Raviart-Thomas [to appear].

Theorem 6.1.2

Let Ω be a bounded domain satisfying (6.1.1). Then the mapping $x \to x.n$ of $H(\mathrm{div},\Omega) \to H^{-1/2}(\Gamma)$, where n is the unit outward normal to Γ, is such that there exists a constant $\alpha > 0$ such that

$$\|x.n\|_{H^{-1/2}(\Gamma)} \leq \alpha \|x\|_{H(\mathrm{div},\Omega)}. \qquad \square$$

To simplify the notation we generalise the operator γ discussed above and denote $x.n$ by $\gamma_0 x$.

We also have the following theorem which will be needed in the applications for a fourth order problem.

Theorem 6.1.3 (Necas [1967], p.21)

Let Ω be a bounded domain satisfying (6.1.1). Then for $x \in H^2(\Omega)$ we have

$$\|x\|_{H^2(\Omega)} \leq c\{\int_\Gamma |x|^2 d\Gamma + \int_\Omega \sum_{|i|=2} |D^i x|^2 d\Omega\}^{1/2}$$

where c is a constant.

Remark 6.1.2. If $x \in H^2(\Omega) \cap H_0^1(\Omega)$, then since $\gamma_0 x = 0$ on Γ, we have,

$$\|x\|_{H^2(\Omega)} \leq c\{\int_\Omega \sum_{|i|=2} |D^i x|^2 d\Omega\}^{1/2}. \qquad \square$$

We now give some results on elliptic boundary value problems which are given in Lions-Magenes [1972] ch.2. Let

$$Ax = \sum_{|p|,|q|\leq m} (-1)^{|p|} D^p (a_{pq}(\xi) D^q x) \tag{6.1.3}$$

be a linear differential operator of order 2m with infinitely differentiable coefficients $a_{pq}(\xi)$. We associate with it the polynomial

$$A_0(\xi,\zeta) = \sum_{|p|,|q|=m} (-1)^m a_{pq}(\xi) \zeta^{p+q} \tag{6.1.4}$$

which is the characteristic form of A.

Definition 6.1.1

The operator A is said to be elliptic if

$$A_0(\xi,\zeta) \neq 0, \quad \forall \zeta \in \mathbb{R}^n, \quad \zeta \neq 0,$$

for all $\xi \in \bar{\Omega}$.

The operators we shall be studying belong to the class of strongly elliptic operators.

Definition 6.1.3

The operator A defined by (6.1.3) is said to be strongly elliptic if there exists a constant $\alpha > 0$ such that

$$\pm A_0(\xi,\zeta) \geq \alpha|\zeta|^{2m}, \quad \forall\zeta \in \mathbb{R}^n,$$

for all $\xi \in \overline{\Omega}$. □

The problem we shall be considering is of the form

$$\begin{aligned} Au &= f \text{ in } \Omega, \\ B_j u &= g_j \text{ on } \Gamma, \end{aligned} \tag{6.1.5}$$

where A is an elliptic operator, the B_j are certain differential boundary operators and f and g_j are given.

However, we know that we cannot arbitrarily choose the operators B_j and obtain a well posed problem. We must introduce some restrictions on the number and type of the boundary operators B_j. Let the operator B_j be given by

$$B_j x = \sum_{|h|=m_j} b_{jh}(\xi)\gamma_0(D^h x),$$

where m_j is the order of B_j and the coefficients b_{jh} are infinitely differentiable on Γ. Then we have the following definitions:

Definition 6.1.4

The system of operators $\{B_j;\ 0 \leq j \leq \nu-1\}$ is a normal system on Γ if

(a) $\sum_{|h|=m_j} b_{jh}(\xi)\zeta^h \neq 0$, $\forall \xi \in \Gamma$ and $\forall \zeta \neq 0$ which are normal to Γ at ξ,

(b) $m_j \neq m_i$ for $j \neq i$.

Definition 6.1.5

The system $\{B_j,\ 0 \leq j \leq m-1\}$ covers the operator A on Γ if for all $\xi \in \Gamma$, all $\zeta \in \mathbb{R}^n$ not equal to zero and tangent to Γ at ξ, and all $\zeta' \in \mathbb{R}^n$, not equal to zero and normal to Γ at ξ, the polynomials in the complex variable η: $\sum_{|h|=m_j} b_{jh}(\xi)(\zeta + \eta\zeta')$, $j = 0,\ldots,m-1$, are linearly independent modulo the polynomial $\prod_{i=1}^{m} (\eta - \eta_i^+(\xi,\zeta,\zeta'))$, where $\eta_i^+(\xi,\zeta,\zeta')$ are the roots of the polynomial $A_0(\xi,\zeta + \eta\zeta')$ with positive imaginary part.

Definition 6.1.6

Problem (6.1.5) is called a regular elliptic problem if the following hypotheses are satisfied:

(1) the operator A is strongly elliptic in $\overline{\Omega}$ and has infinitely differentiable coefficients in $\overline{\Omega}$,

(2) there are m operators B_j,

(3) the coefficients of B_j are infinitely differentiable on Γ,

(4) the system $\{B_j;\ 0 \leq j \leq m-1\}$ is normal on Γ,

(5) the system $\{B_j;\ 0 \leq j \leq m-1\}$ covers the operator A on Γ,

(6) the order m_j of B_j is $\leq 2m-1$.

Remark 6.1.2. Among the systems of operators $\{B_j\}$ which satisfy hypotheses (1),...,(6) for every strongly elliptic operator A, there is the system of Dirichlet conditions

$$B_j = \gamma_j, \qquad 0 \leq j \leq m-1.$$

With these boundary conditions problem (6.1.5) is called the Dirichlet problem for the operator A.

Definition 6.1.7

The system $\{B_j;\ 0 \leq j \leq \nu-1\}$ is a Dirichlet system of order ν on Γ if it is normal on Γ and if the orders m_j run through exactly the set $0,1,\ldots,\nu-1$, when j goes from 0 to $\nu-1$. □

With the elliptic operator A given by (6.1.3) we may associate the form

$$a(\tilde{x},x) = \int_\Omega \sum_{|p|,|q|\leq m} a_{pq}(\xi)D^p\tilde{x}D^q x\,d\Omega.$$

Then we have the following Green's theorem.

Theorem 6.1.4

Let $\{F_j;\ 0 \leq j \leq m-1\}$ be a Dirichlet system of order m, with infinitely differentiable coefficients on Γ. Then there exists a system $\{\phi_j;\ 0 \leq j \leq m-1\}$ which is normal on Γ, has infinitely differentiable coefficients with: order of F_j + order of ϕ_j = 2m-1, such that

$$a(\tilde{x},x) = \int_\Omega (A\tilde{x})x\,d\Omega - \sum_{j=0}^{m-1} \int_\Gamma \phi_j\tilde{x}\,F_j x\,d\Gamma. \qquad (6.1.6)$$

Remark 6.1.4. Equation (6.1.6) is also valid when the derivatives are interpreted in the generalised sense. In this case we must have $\tilde{x} \in H^{2m}(\Omega)$ and $x \in H^m(\Omega)$.

6.2 Non-Homogeneous Boundary Value Problems

We shall be studying the class of problems given by

$$\begin{aligned} S^* G \ S\tilde{u} &= \tilde{f} \quad \text{in } \Omega \\ B_j\tilde{u} &= \tilde{g}_j \text{ on } \Gamma, \quad 0 \leq j \leq m-1, \end{aligned} \tag{6.2.1}$$

where S and S* are formally adjoint differential operators of order m and G is an operator of order 0. S maps the function x into the vector $y = (y_{q_1}, \ldots, y_{q_r})$ with r components $y_q = D^q x$, $|q| = m$. S* maps the vector $y^* = (y^*_{p_1}, \ldots, y^*_{p_r})$ into the scalar $\sum_{|p|=m} (-1)^{|p|} D^p y^*$. The operator G can be represented by the matrix with components $a_{pq}(\xi)$, c.f. equation (6.1.3). The B_j are boundary differential operators of order $m_j \leq 2m-1$.

In this section we shall show how (6.2.1) can be put into an intermediate canonical form given by equations (6.2.3). At this stage we do not concern ourselves with the precise definition of the spaces involved in this classical formulation of boundary value problems. However, in section 6.3 we extend the operator S to a generalised differential operator and show how the intermediate canonical form can be related to an abstract formulation

$$\begin{aligned} Tu &= w \\ Ew &= v \\ T^*v &= f, \end{aligned}$$

in the sense that the abstract problem is an extension of the intermediate canonical problem. This involves the precise definition of the spaces and operators and hence in a natural way we obtain restrictions on the class of functions in which the solution lies.

Section 6.4 is concerned with a second method of extending the intermediate canonical form (6.2.3) to the abstract form by extending the operator S* to a generalised differential operator. First, however, we introduce the intermediate canonical form.

Suppose that in (6.2.1) the $\tilde{g}_j$ are sufficiently smooth so that there exists a u_0 satisfying

$$B_j u_0 = \tilde{g}_j \quad \text{on} \quad \Gamma, \quad 0 \leq j \leq m-1.$$

Then $u = \tilde{u} - u_0$ satisfies

$$\begin{aligned} S^* G Su &= \tilde{f} - S^* G Su_0 \text{ in } \Omega \\ B_j u &= 0 \quad \text{on} \quad \Gamma, \quad 0 \leq j \leq m-1. \end{aligned}$$

Hence we have the homogeneous boundary value problem

$$\begin{aligned} S^* G Su &= f \text{ in } \Omega \\ B_j u &= 0 \quad \text{on } \Gamma, \quad 0 \leq j \leq m-1. \end{aligned} \qquad (6.2.2)$$

We assume that (6.2.2) is a regular elliptic problem, see definition 6.1.6, with ν of the boundary operators B_j of order $m_j < m$. Reorder the B_j so that these are the first ν operators, i.e. $m_j < m$, $0 \leq j \leq \nu-1$. Then provided G has an inverse with components a^{-1}_{pq}, we define the operators C_j, $\nu \leq j \leq m-1$, by

$$C_j y = \sum_{|h|=m} b_{jh} \gamma_0 \Big(\sum_{|p|,|q|=m} D^{h-q} a^{-1}_{pq} y_p \Big),$$

so that C_j satisfies

$$C_j\, G\, Sx = B_j x, \quad \nu \leq j \leq m-1,$$

where C_j is of order $n_j = m_j - m$. Hence equations (6.2.2) can be written as

$$
\begin{aligned}
Su &= w \quad \text{in } \Omega, \\
B_j u &= 0 \quad \text{on } \Gamma, \quad 0 \le j \le \nu-1, \\
Gw &= v \quad \text{in } \Omega, \\
S^* v &= f \quad \text{in } \Omega, \\
C_j v &= 0 \quad \text{on } \Gamma, \quad \nu \le j \le m-1.
\end{aligned}
\tag{6.2.3}
$$

Remark 6.2.1. The terminology for boundary value problems of the form (6.2.3) is not universal. We adopt the following classification (c.f. Necas [1967]). If there are m boundary operators of order $<m$, i.e. $\nu = m$, then there are no equations involving the operators C_j and we call this a Dirichlet problem. If there are no boundary operators of order $<m$, i.e. $\nu = 0$, then there are no equations involving the operators B_j and we call this a Neumann problem. If there are some boundary operators of order $<m$ and others of order $\ge m$, then we call the problem an intermediate problem. □

We now give some examples of regular elliptic problems of these types.

Example 6.2.1

Consider the problem

$$
\begin{aligned}
-\nabla^2 \tilde{u} &= \tilde{f} \quad \text{in } \Omega \in \mathbb{R}^2 \\
\tilde{u} &= \tilde{g} \quad \text{on } \Gamma,
\end{aligned}
\tag{6.2.4}
$$

where the domain Ω satisfies condition (6.1.1). The operator $-\nabla^2$ is of order 2, i.e. $m = 1$. The boundary condition is to be interpreted in the sense

$$\tilde{u}|_\Gamma = \gamma_0\tilde{u} = \tilde{g},$$

and so $B_0 = \gamma_0$ is of order 0, i.e. $m_0 = 0$. As $m_0 < m$, this boundary condition is a Dirichlet type condition and as there are no other boundary conditions, $\nu = m=1$ and this is a Dirichlet problem.

Now suppose that $\tilde{g}$ is sufficiently smooth such that there exists a u_0 satisfying

$$\gamma_0 u_0 = \tilde{g} \quad \text{on } \Gamma,$$

then $u = \tilde{u} - u_0$ satisfies

$$\begin{aligned} -\nabla^2 u &= f \equiv \tilde{f} + \nabla^2 u_0 \text{ in } \Omega \\ \gamma_0 u &= 0 \quad \text{on } \Gamma . \end{aligned} \tag{6.2.5}$$

We shall verify that (6.2.5) is a regular elliptic problem, i.e. it satisfies the conditions of definition 6.1.6.

(1) The characteristic form of $-\nabla^2$ is

$$A_0(\xi,\zeta) = -(\zeta_1{}^2 + \zeta_2{}^2) .$$

Then taking $\alpha = 1$ in definition 6.1.3 we have

$$-A_0(\xi,\zeta) \geq \alpha|\zeta|^2, \ \forall\zeta \in \mathbb{R}^2$$

and so $-\nabla^2$ is a strongly elliptic operator.

(2) and (3) are obviously satisfied.

(4) $B_0 = \gamma_0$ is trivially normal on Γ.

(5) B_0 trivially covers the operator $-\nabla^2$.

(6) is obviously satisfied.

Hence problem (6.2.5) is a regular elliptic problem.

As $\nu = m$, there are no boundary conditions involving the operators C_j, hence we can write (6.2.5) as

$$\begin{aligned} \text{grad } u &= w \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma, \\ w &= v \quad \text{in } \Omega, \\ -\text{div } v &= f \quad \text{in } \Omega, \end{aligned} \tag{6.2.6}$$

where w and v are vectors with $r = 2$ components.

Example 6.2.2

Consider the problem

$$\begin{aligned} -\nabla^2 \tilde{u} &= \tilde{f} \quad \text{in } \Omega \in \mathbb{R}^2, \\ \frac{\partial \tilde{u}}{\partial n} &= \tilde{g} \quad \text{on } \Gamma, \end{aligned} \tag{6.2.7}$$

where $\frac{\partial}{\partial n}$ is the outward normal derivative to the boundary Γ which we assume satisfies condition (6.1.1). As in the previous example $m = 1$. The boundary condition $B_0 = \frac{\partial}{\partial n} = \gamma_1$ is of order 1, i.e. $m_0 = 1$. This is a Neumann type boundary condition and as there are no other boundary conditions, $\nu = 0$ and this is a Neumann problem.

Suppose that there exists a u_0 satisfying

$$\frac{\partial u_0}{\partial n} = \tilde{g} \quad \text{on } \Gamma,$$

then $u = \tilde{u} - u_0$ satisfies

$$-\nabla^2 u = f \equiv \tilde{f} + \nabla^2 u_0 \text{ in } \Omega; \qquad \frac{\partial u_0}{\partial n} = 0 \quad \text{on } \Gamma. \tag{6.2.8}$$

As the operator $-\nabla^2$ is strongly elliptic (see the previous example) we need to show that the boundary operator $B_0 = \frac{\partial}{\partial n}$ is normal on Γ and covers the operator $-\nabla^2$ for (6.2.8) to be a regular elliptic problem. We shall not prove this here, but refer to Kellog [1972] for the proof.

As $\nu = 0$ there are no boundary conditions involving the operators B_j, hence we can write (6.2.8) as

$$\begin{aligned} \text{grad } u &= w \quad \text{in } \Omega, \\ w &= v \quad \text{in } \Omega, \\ -\text{div } v &= f \quad \text{in } \Omega, \\ \gamma_0 v &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{6.2.9}$$

as $\gamma_0 v = \gamma_0(\text{grad } u) = \frac{\partial u}{\partial n}$ on Γ.

<u>Example 6.2.3</u>

Next we consider a problem which arises in the theory of flat elastic plates (see, for example, Duvaut-Lions [1976] chapter 4 for a brief description of the theory of flat plates). The equations are

$$\begin{aligned} \nabla^4 \tilde{u} &= \tilde{f} \quad \text{in } \Omega \in \mathbb{R}^2, \\ \tilde{u} &= \tilde{g} \quad \text{on } \Gamma, \\ M\tilde{u} &= \tilde{h} \quad \text{on } \Gamma, \end{aligned} \tag{6.2.10}$$

where

$$M\tilde{u} = \sigma \nabla^2 \tilde{u} + (1-\sigma)\left(\frac{\partial^2 \tilde{u}}{\partial \xi_1^2} n_1^2 + 2\,\frac{\partial^2 \tilde{u}}{\partial \xi_1 \partial \xi_2} n_1 n_2 + \frac{\partial^2 \tilde{u}}{\partial \xi_2^2} n_2^2\right)$$

is the moment about the tangent at Γ, with σ being Poisson's ratio ($0 < \sigma < \frac{1}{2}$) and $n = (n_1, n_2)$ the unit exterior normal to Γ. $\tilde{u}(\xi_1, \xi_2)$ represents the displacement of the plate from its initial plane, $\tilde{f}(\xi_1, \xi_2)$ represents the body forces acting on the plate, e.g. gravity, $\tilde{g}$ is the prescribed displacement of Γ and $\tilde{h}$ is the prescribed tangential moment at Γ. We assume Γ satisfies condition (6.1.1).

The problem is of order 4, i.e. $m = 2$. The first boundary operator $B_0 = \gamma_0$ is of order $m_0 = 0$ and the second boundary operator $B_1 = \gamma_0 M$ is of order $m_1 = 2$. Only one of the boundary operators is of order $<m$ and so we set $\nu = 1$. Hence this is an example of an intermediate problem.

Suppose there exists u_0 satisfying

$$\gamma_0 u_0 = \tilde{g} \quad \text{on } \Gamma$$

$$\text{and} \quad \gamma_0 M u_0 = \tilde{h} \quad \text{on } \Gamma,$$

then $u = \tilde{u} - u_0$ satisfies

$$\begin{aligned} \nabla^4 u &= f \equiv \tilde{f} - \nabla^4 u_0 \text{ in } \Omega \\ u &= 0 \text{ on } \Gamma, \\ Mu &= 0 \text{ on } \Gamma. \end{aligned} \tag{6.2.11}$$

Problem (6.2.11) satisfies the hypotheses of definition 6.1.6, i.e. it is a regular elliptic problem, since:

(1) The characteristic form of ∇^4 is

$$A_0(\xi, \zeta) = (\zeta_1^2 + \zeta_2^2)^2$$

and obviously ∇^4 is strongly elliptic (see definition 6.1.3).

(2) and (3) are obviously satisfied.

(4) and (5) are satisfied, see Kellogg [1972] where all the sets of boundary operators which are normal and cover ∇^4 are given.

(6) is obviously satisfied.

Defining the operators S, S* and G by

$$Sx = \left(\frac{\partial^2 x}{\partial \xi_1{}^2}, \frac{\partial^2 x}{\partial \xi_1 \partial \xi_2}, \frac{\partial^2 x}{\partial \xi_2{}^2}\right), \tag{6.2.12}$$

$$Gy = \begin{pmatrix} 1 & 0 & \sigma \\ 0 & 2(1-\sigma) & 0 \\ \sigma & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \tag{6.2.13}$$

and $$S^*y^* = \frac{\partial^2 y_1{}^*}{\partial \xi_1{}^2} + \frac{\partial^2 y_2{}^*}{\partial \xi_1 \partial \xi_2} + \frac{\partial^2 y_3{}^*}{\partial \xi_2{}^2}, \tag{6.2.14}$$

where $y = (y_1, y_2, y_3)$ and $y^* = (y_1{}^*, y_2{}^*, y_3{}^*)$ are vectors with $r = 3$ components, we can show that

$$S^* \, G \, Su = \nabla^4 u.$$

We define the operator C_1 by

$$C_1 y^* = n_1{}^2 y_1{}^* + n_1 n_2 y_2{}^* + n_2{}^2 y_3{}^* \tag{6.2.15}$$

and after some manipulation we get

$$C_1 \, G \, Sx = Mx,$$

where $y^* = GSx$. C_1 is of order $n_1 = 0$.

We can now write (6.2.11) as

$$\begin{aligned} Su &= w \quad \text{in } \Omega \\ \gamma_0 u &= 0 \quad \text{on } \Gamma \\ Gw &= v \quad \text{in } \Omega \\ S^* v &= f \quad \text{in } \Omega \\ C_1 v &= 0 \quad \text{on } \Gamma, \end{aligned} \tag{6.2.16}$$

where S, G and S* are given by (6.2.12), (6.2.13) and (6.2.14) respectively and C_1 is given by (6.2.15).

6.3 The First Abstract Formulation

Having put the non-homogeneous boundary value problem

$$\begin{aligned} S^* \, G \, S\tilde{u} &= \tilde{f} \text{ in } \Omega \\ B_j \tilde{u} &= \tilde{g}_j \quad \text{on } \Gamma, \quad 0 \le j \le m-1, \end{aligned}$$

into the intermediate canonical form

$$\begin{aligned} Su &= w \quad \text{in } \Omega, \\ B_j u &= 0 \quad \text{on } \Gamma, \quad 0 \le j \le \nu-1 \\ Gw &= v \quad \text{in } \Omega, \\ S^* v &= f \quad \text{in } \Omega, \\ C_j v &= 0 \quad \text{on } \Gamma, \qquad \nu \le j \le m-1, \end{aligned} \tag{6.3.1}$$

we now want to extend equations (6.3.1) in such a way that they can be related to an abstract form

$$\begin{aligned} Tu &= w \\ Ew &= v \\ T^*v &= f\ , \end{aligned} \tag{6.3.2}$$

in the sense that any solution of (6.3.1) is a solution of (6.3.2).

To achieve this we shall introduce two further restrictions on the boundary operators B_j and C_j. Then we shall define the abstract problem in one of two ways. The first method is to define the operator T as a suitable extension of the differential operator S and then define T* as the adjoint of T. The second approach is to define T* as a suitable extension of the differential operator S* and then define T as the adjoint of T*. This second approach is described in the next section, here we concentrate on the first approach.

There are two further assumptions that have to be made about the boundary differential operators of (6.3.1). First, let $\{F_j,\ 0 \le j \le m-1\}$ be a set of boundary operators given by

$$F_j = \begin{cases} B_j, & 0 \le j \le \nu-1 \\ \gamma_{p_j}, & \nu \le j \le m-1 \end{cases} \tag{6.3.3}$$

where the set of integers $\{p_j,\ \nu \le j \le m-1\}$ is such that $\{m_0,\ldots,m_{\nu-1},\ p_\nu,\ldots p_{m-1}\}$ takes all values between 0 and m-1. Then we assume that:

$$\{F_j,\ 0 \le j \le m-1\} \quad \text{is a Dirichlet system of order } m \text{ on } \Gamma. \tag{6.3.4}$$

Then from theorem 6.1.4 there exists a normal system $\{\phi_j,\ 0 \le j \le m-1\}$ of boundary operators such that

$$a(\tilde{x},x) \quad = \quad \int_\Omega (S^*GS\tilde{x})x\,d\Omega - \sum_{j=0}^{m-1} \int_\Gamma \phi_j\tilde{x}F_jx\,d\Gamma, \qquad (6.3.5)$$

where $a(\tilde{x},x) = \int_\Omega GS\tilde{x}\ Sx\ d\Omega$, and the order of ϕ_j = 2m-1-order of F_j. As $\{F_j\}$ is a Dirichlet system of order m, the order of $F_j \le m-1$. Hence order of $\phi_j \ge m$. Therefore as we saw in section 6.2, provided G has an inverse, we may define the operators ψ_j by

$$\psi_j GS\tilde{x} \quad = \quad \phi_j\tilde{x}, \qquad 0 \le j \le m-1.$$

Then with $y^* = GS\tilde{x}$, (6.3.5) becomes

$$\int_\Omega y^*Sxd \quad = \quad \int_\Omega (S^*y^*)x\,d\Omega - \sum_{j=0}^{m-1} \int_\Gamma \psi_j y^*F_jx\,d\Gamma. \qquad (6.3.6)$$

The second assumption we make is that the C_j can be permuted such that

$$\psi_j \quad = \quad (-1)^{n_j+1}\ C_j, \qquad \nu \le j \le m-1. \qquad (6.3.7)$$

Then (6.3.6) can be written as

$$\int_\Omega y^*Sxd \ + \sum_{j=0}^{\nu-1} \int_\Gamma \psi_j y^*B_jx\,d\Gamma \quad =$$

$$\int_\Omega (S^*y^*)x\,d\Omega - \sum_{j=\nu}^{m-1} (-1)^{n_j+1} \int_\Gamma C_j y^*\gamma_{p_j} x\,d\Gamma, \qquad (6.3.8)$$

which is a general integration by parts formula for the differential operator S.

<u>Remark 6.3.1</u>. The assumption (6.3.7) is equivalent to the assumption

$$\phi_j \quad = \quad B_j, \qquad \nu \le j \le m-1 .$$

Remark 6.3.2. The assumption (6.3.7) imposes restrictions on the boundary operators $\{C_j, \nu \le j \le m-1\}$ and hence on the operators $\{B_j, \nu \le j \le m-1\}$. The order of $\phi_j = 2m-1-p_j$, $\nu \le j \le m-1$ and so, from remark 6.3.1, the order of the boundary operators B_j are restricted by

$$m_j = 2m-1-p_j, \qquad \nu \le j \le m-1.$$

This is equivalent to the restriction that

$$n_j = m-1-p_j, \qquad \nu \le j \le m-1. \qquad \square$$

Now we are in a position to define the abstract equations (6.3.2) and show they are an extension of the classical differential equations (6.3.1). Let us define the spaces

$$X = \{\bar{x} = (x, x_{p_\nu}, \ldots, x_{p_{m-1}}) \in H^m(\Omega) \times H^{m-p_\nu-1/2}(\Gamma) \times \ldots \times H^{m-p_{m-1}-1/2}(\Gamma);$$

$$B_j x = 0, \quad 0 \le j \le \nu-1; \; x_{p_\nu} = \gamma_{p_\nu} x, \ldots, x_{p_{m-1}} = \gamma_{p_{m-1}} x\}$$

and $Y = (L_2(\Omega))^r$, where r is the number of components of a vector in the range of S.

Remark 6.3.3. The space X may be identified with the space $\{x \in H^m(\Omega);$ $B_j x = 0,\ 0 \le j \le \nu-1\}$. $\square$

We define the operator $T:X \to Y$ by

$$T\bar{x} = Sx, \tag{6.3.9}$$

where $\bar{x} = (x, x_{p_\nu}, \ldots, x_{p_{n-1}}) \in X$ and the operator S is interpreted in the generalised sense.

The adjoint operator T^* is defined via the adjoint relationship

$$\langle T\bar{x}, y^* \rangle = \langle \bar{x}, T^*y^* \rangle, \quad \forall \bar{x} \in X, \; y^* \in Y^*. \tag{6.3.10}$$

To identify the operator T^* we shall use the integration by parts formula (6.3.8). This is valid in the generalised sense for all $x \in H^m(\Omega)$ and all $y^* \in H(S^*;\Omega)$, where

$$H(S^*;\Omega) = \{y^* \in (L_2(\Omega))^r; \quad S^*y^* \in L_2(\Omega)\}.$$

Then since $B_j x = 0$, $0 \leq j \leq \nu-1$ when $\bar{x} = (x, x_{p_\nu}, \ldots, x_{p_{m-1}}) \in X$, the left hand side of (6.3.8) is identical to the left hand side of (6.3.10). Hence we have

$$\langle \bar{x}, T^*y^* \rangle = \int_\Omega (S^*y^*) x \, d\Omega - \sum_{j=\nu}^{m-1} (-1)^{n_j+1} \int_\Gamma C_j y^* \gamma_{p_j} x \, d\Gamma, \tag{6.3.11}$$

when $\bar{x} \in X$ and $y^* \in H(S^*,\Omega)$. Then we can say that, at least for all $y^* \in H(S^*,\Omega)$,

$$T^*y^* = (S^*y^*, (-1)^{n_\nu+1} C_\nu y^*, \ldots, (-1)^{n_{m-1}+1} C_{m-1} y^*). \tag{6.3.12}$$

<u>Remark 6.3.4</u>. For $y^* \notin H(S^*;\Omega)$ we cannot characterise T^* simply by (6.3.12). Therefore in general T^* is an extension of the right hand side of (6.3.12). □

Hence the abstract problem

$$\begin{aligned} T\bar{u} &= w \\ Ew &= v \\ T^*v &= \bar{f}, \end{aligned} \tag{6.3.13}$$

where $T:X \to Y$ is given by (6.3.9), $T^*:Y^* \to X^*$ is given by (6.3.10), $E:Y \to Y^*$ is given by $Ey = Gy$, $\forall y \in Y$, $\bar{u} = (u, \gamma_{p_\nu} u, \ldots, \gamma_{p_{m-1}} u) \in X$ and

$\overline{f} = (f,0,\ldots,0) \in X^*$, can be considered an extension of equations (6.3.1) which reduce essentially to (6.3.1) when $v \in H(S^*;\Omega)$, that is, when $f \in L_2(\Omega)$. Note that $(6.3.11)_1$ gives rise to the equations

$$\begin{aligned} Su &= w \quad \text{in } \Omega \\ B_j u &= 0 \quad \text{on } \Gamma,\ 0 \leq j \leq \nu-1, \end{aligned}$$

since $\overline{u} \in X$ implies $B_j u = 0$, $0 \leq j \leq \nu-1$. Provided $v \in H(S^*;\Omega)$ we see from (6.3.12) that $(6.3.13)_3$ gives rise to the equations

$$\begin{aligned} S^*v &= f \quad \text{in } \Omega \\ C_j v &= 0 \quad \text{on } \Gamma, \quad \nu \leq j \leq m-1. \end{aligned}$$

Equations (6.3.13) are in the form of the abstract problem of Chapter 4. Hence the conditions for the existence and uniqueness of a solution to this problem are given by theorem 4.1.4. Variational principles associated with this problem are also given in Chapter 4, the most general being (4.3.1).

Example 6.3.1

Consider the problem

$$\begin{aligned} -\nabla^2\tilde{u} &= \tilde{f} \quad \text{in } \Omega \in \mathbb{R}^2 \\ \tilde{u} &= \tilde{g} \quad \text{on } \Gamma. \end{aligned} \tag{6.3.14}$$

In example 6.2.1 we showed that this is a regular elliptic problem which can be put into the form

$$\begin{aligned} \text{grad } u &= w \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \Gamma \\ w &= v \quad \text{in } \Omega \\ -\text{div } v &= f \quad \text{in } \Omega, \end{aligned} \tag{6.3.15}$$

where $u = \tilde{u}-u_o$, u_o satisfying $\gamma_o u_o = \tilde{g}$ on Γ. Since $\nu = m = 1$, the boundary operator

$$F_o = B_o = \gamma_o .$$

Hence condition (6.3.4) is satisfied as γ_o is a Dirichlet system of order 1 on Γ. Theorem 6.1.4 shows that Green's formula is valid, that is, there exists a ϕ_o such that

$$\int_\Omega \text{grad } \tilde{x}. \text{ grad } x \, d\Omega = - \int_\Omega (\nabla^2 \tilde{x}) x d\Omega - \int_\Gamma \phi_o \tilde{x} \gamma_o x d\Gamma, \tag{6.3.16}$$

with ϕ_o of order 1. As G is the identity map in this problem, it has an inverse and hence as in section 6.2 we can define the operator ψ_o by

$$\psi_o \text{ grad } \tilde{x} = \phi_o \tilde{x}$$

and letting $y^* = \text{grad } \tilde{x}$, (6.3.16) becomes

$$\int_\Omega y^*.\text{grad } x \, d\Omega = -\int_\Omega (\text{div } y^*) x d\Omega - \int_\Gamma \psi_o y^* \gamma_o x d\Gamma. \tag{6.3.17}$$

Assumption (6.3.7) is trivially satisfied as there are no C_j operators in this problem.

Hence we define the spaces

$$X = \{x \in H^m(\Omega);\ \gamma_o x = 0 \text{ on } \Gamma\} \equiv H_o^1(\Omega)$$

$$Y = (L_2(\Omega))^2.$$

We define the operator $T:X \to Y$ by

$$Tx = \text{grad } x$$

where grad is interpreted in the generalised sense. The adjoint operator is defined via the adjoint relationship

$$\langle Tx,y^*\rangle = \langle x,T^*y^*\rangle, \quad \forall x \in H_0^1(\Omega),\ y^* \in (L_2(\Omega))^2 .$$

Comparing this with (6.3.17), which is valid in the generalised sense for all $x \in H^1(\Omega)$ and all $y^* \in H(\text{div};\Omega)$, we have

$$\langle x,T^*y^*\rangle = -\int_\Omega x \text{ div } y^* d\Omega, \ \forall x \in H_0^1(\Omega), \quad y^* \in H(\text{div};\Omega).$$

Hence we can identify T^*y^* with $-\text{div } y^*$, for all $y^* \in H(\text{div};\Omega)$. Now we can define the problem

$$Tu = w$$

$$\tau w = v$$

$$T^*v = f,$$

where $T:X \to Y$ and $T^*:Y^* \to X^*$ are as given above and τ is the identity mapping. This is an extension of equations (6.3.15) provided $f \in L_2(\Omega)$, since the equation $Tu = w$ implies

$$\text{grad } u = w \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \Gamma,$$

and $T^*v = f$ implies

$$-\text{div } v = f \quad \text{in } \Omega$$

provided $v \in H(\mathrm{div};\Omega)$, i.e. provided $f \in L_2(\Omega)$. This abstract problem is of the type studied in Chapters 2 and 3, and all the results given there are valid for this example.

Example 6.3.2

Recall the example from the theory of flat plates, example 6.2.3. This was reduced to the set of equations

$$\begin{aligned} Su &= w \text{ in } \Omega \\ \gamma_0 u &= 0 \text{ on } \Gamma \\ Gw &= v \text{ in } \Omega \\ S^*v &= f \text{ in } \Omega \\ C_1 v &= 0 \text{ on } \Gamma, \end{aligned} \tag{6.3.18}$$

where

$$Sx = \left(\frac{\partial^2 x}{\partial \xi_1^{\,2}}, \frac{\partial^2 x}{\partial \xi_1 \partial \xi_2}, \frac{\partial^2 x}{\partial \xi_2^{\,2}}\right) \tag{6.3.19}$$

$$Gy = \begin{pmatrix} 1 & 0 & \sigma \\ 0 & 2(1-\sigma) & 0 \\ \sigma & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \tag{6.3.20}$$

$$S^*y^* = \frac{\partial^2 y_1^*}{\partial \xi_1^{\,2}} + \frac{\partial^2 y_2^*}{\partial \xi_1 \partial \xi_2} + \frac{\partial^2 y_3^*}{\partial \xi_2^{\,2}} \tag{6.3.21}$$

and $$C_1 y^* = n_1^{\,2} y_1^* + n_1 n_2 y_2^* + n_2^{\,2} y_3^*. \tag{6.3.22}$$

In example 6.2.3 we proved that this is a regular elliptic problem. We now show that it satisfies the two further assumptions (6.3.4) and (6.3.7).

The system of boundary operators $\{F_j, 0 \le j \le 1\}$, see (6.3.3), is given by

$$F_j = \begin{matrix} \gamma_0, & j = 0 \\ \gamma_1, & j = 1. \end{matrix}$$

This is easily shown to be a Dirichlet system of order 2 on Γ, definition 6.1.7, and so assumption (6.3.4) is satisfied. Then from theorem 6.1.4, there exists a normal system of operators $\{\phi_j, 0 \le j \le 1\}$ such that

$$\int_\Omega GS\tilde{x}Sxd\Omega = \int_\Omega (S^*GS\tilde{x})xd\Omega - \sum_{j=0}^{1} \int_\Gamma \phi_j\tilde{x}\gamma_j xd\Gamma \qquad (6.3.23)$$

with ϕ_0 of order 3 and ϕ_1 of order 2. As G is non-singular we can define the operators ψ_j by

$$\psi_j GS\tilde{x} = \phi_j\tilde{x}, \qquad 0 \le j \le 1,$$

and letting $y^* = GS\tilde{x}$, (6.3.23) becomes

$$\int_\Omega y^*Sxd\Omega = \int_\Omega (S^*y^*)xd\Omega - \sum_{j=0}^{1} \int_\Gamma \psi_j y^*\gamma_j xd\Gamma. \qquad (6.3.24)$$

However integration by parts gives us the relationship

$$\int_\Omega y^*Sxd\Omega = \int_\Omega (S^*y^*)xd\Omega - \int_\Gamma (n_1{}^2 \frac{\partial y_1^*}{\partial n} + n_1 n_2 \frac{\partial y_2^*}{\partial n} + n_2{}^2 \frac{\partial y_3^*}{\partial n})\gamma_0 xd\Gamma$$

$$+ \int_\Gamma C_1 y^*\gamma_1 xd\Gamma. \qquad (6.3.25)$$

Hence from (6.3.24) and (6.3.25) we see that assumption (6.3.7) is satisfied, i.e.

$$\psi_1 = -C_1 .$$

Now define the spaces

$$X = \{\bar{x} = (x, x_1) \in H^2(\Omega) \times H^{1/2}(\Gamma);\ \gamma_0 x = 0 \text{ on } \Gamma \text{ and } x_1 = \gamma_1 x\}$$

$$Y = (L_2(\Omega))^3.$$

We define the operator T by

$$T\bar{x} = \left(\frac{\partial^2 x}{\partial \xi_1^2}, \frac{\partial^2 x}{\partial \xi_1 \partial \xi_2}, \frac{\partial^2 x}{\partial \xi_2^2}\right)$$

where $\bar{x} = (x, \gamma_1 x) \in X$ and the derivatives are interpreted in the generalised sense. The adjoint operator T* is defined by the adjoint relationship

$$\langle T\bar{x}, y^* \rangle = \langle \bar{x}, T^* y^* \rangle \qquad \forall \bar{x} \in X,\ y^* \in (L_2(\Omega))^3.$$

Comparing with (6.3.24), which is valid in the generalised sense for all $x \in H^2(\Omega)$ and all $y^* \in H(S^*;\Omega)$, we see that for $y^* \in H(S^*;\Omega)$, T^* can be defined by

$$T^* y^* = (S^* y^*, -C_1 y^*).$$

Hence the problem

$$\begin{aligned} T\bar{u} &= w \\ Ew &= v \\ T^* v &= \bar{f}, \end{aligned} \tag{6.3.26}$$

where $\bar{u} = (u, \gamma_1 u) \in X$, $\bar{f} = (f, 0) \in L_2(\Omega) \times L_2(\Gamma)$ and E is the operator G, is an extension of problem (6.3.18).

Equations (6.3.26) are in the form of the abstract problem of Chapter 4 and we shall prove that it has a unique solution by verifying the hypotheses of theorem 4.1.4. As $0 < \sigma < \frac{1}{2}$ the operator E is a one-to-one mapping with an inverse E^{-1}. Hence a unique solution of this problem exists if we can prove that there exists $\alpha > 0$ such that

$$\| T\bar{x}\|_{(L_2(\Omega))^3} \equiv \{ \int_\Omega (\frac{\partial^2 x}{\partial \xi_1{}^2})^2 + (\frac{\partial^2 x}{\partial \xi_1 \partial \xi_2})^2 + (\frac{\partial^2 x}{\partial \xi_2{}^2})^2 d\Omega\}^{1/2}$$

$$\geq \alpha\{ \| x\|^2_{H^2(\Omega)} + \| \gamma_1 x\|^2_{H^{1/2}(\Omega)} \}^{1/2}$$

$$= \alpha \| \bar{x}\|_X , \ \forall \tilde{x} \in X.$$

This is easily shown from remark 6.1.1 and the trace inequality (6.1.2).

The most general variational principle considered in Chapter 4, see (4.3.1), can, for this problem, be written as:

find $(\bar{u},w,v) \in X \times (L_2(\Omega))^3 \times (L_2(\Omega))^3$ saddle point of

$$L(\bar{x},y,y^*) = \int_\Omega (\frac{\partial^2 x}{\partial \xi_1{}^2} y_1^* + \frac{\partial^2 x}{\partial \xi_1 \partial \xi_2} y_2^* + \frac{\partial^2 x}{\partial \xi_2{}^2} y_3^*) \, d\Omega$$

$$- \int_\Omega (y_1 y_1^* + y_2 y_2^* + y_3 y_3^*) d\Omega$$

$$+ \frac{1}{2} \int_\Omega \{y_1(y_1 + \sigma y_2) + 2(1 - \sigma) y_2^2 + y_3(\sigma y_1 + y_3)\} d\Omega$$

$$- \int_\Omega xf \, d\Omega,$$

where $\bar{x} = (x,\gamma_1 x) \in X$, $y = (y_1,y_2,y_3) \in (L_2(\Omega))^3$ and $y^* = (y_1^*, y_2^*, y_3^*) \in (L_2(\Omega))^3$. This is a generalisation of the Hu-Washizu principle for flat plates, see Washizu [1975].

Remark 6.3.5. Note that although the abstract problem (6.3.26) involves very general operators, e.g. T^*, the variational problem can be expressed in terms of relatively straightforward operators such as generalised derivatives on Sobolev spaces.

6.4 The Second Abstract Formulation

Returning to the intermediate canonical problem

$$\begin{aligned} Su &= w \text{ in } \Omega \\ B_j u &= 0 \text{ on } \Gamma, \quad 0 \le j \le \nu-1 \\ Gw &= v \text{ in } \Omega \\ S^*v &= f \text{ in } \Omega \\ C_j v &= 0 \text{ on } \Gamma, \quad \nu \le j \le m-1 , \end{aligned}$$

we shall show in this section that this classical problem can be related to an abstract problem

$$\begin{aligned} Tu &= w \\ Ew &= v \\ T^*v &= f, \end{aligned}$$

as in the previous section. Here, however, we take T^* as the primary abstract operator and define T as its adjoint. First, as in section 6.3, we introduce two restrictions on the boundary operators B_j and C_j of the intermediate canonical problem.

We know that there exist boundary operators $\{B_j', \ \nu \le j \le m-1\}$ which define a set

$$F_j = \begin{cases} B_j\ , & 0 \leq j \leq \nu-1 \\ B_j', & \nu \leq j \leq m-1 \end{cases} \tag{6.4.1}$$

of boundary operators such that $\{F_j,\ 0 \leq j \leq m-1\}$ is a Dirichlet system of order m. Then from theorem 6.1.4, there exists a normal system $\{\phi_j,\ 0 \leq j \leq m-1\}$ of boundary operators such that

$$a(\tilde{x},x) = \int_\Omega (S^*GS\tilde{x})x d\Omega - \sum_{j=0}^{m-1} \int_\Gamma \phi_j \tilde{x} F_j x d\Gamma \tag{6.4.2}$$

with order of ϕ_j = 2m-1-order of F_j. As in the previous section we may define the operators ψ_j by

$$\psi_j GS\tilde{x} = \phi_j \tilde{x}, \quad 0 \leq j \leq m-1 .$$

Then (6.4.2) becomes

$$\int_\Omega y^* Sx d\Omega = \int_\Omega (S^*y^*)x d\Omega - \sum_{j=0}^{m-1} \int_\Gamma \psi_j y^* F_j x d\Gamma \tag{6.4.3}$$

where $y^* = GS\tilde{x}$.

Now we make two assumptions. Let $\{p_j,\ 0 \leq j \leq \nu-1\}$ be a set of integers such that the set $\{p_0,\dots,p_{\nu-1},n_\nu,\dots,n_{m-1}\}$ take all the values from 0 to m-1. Then we assume that

$$\psi_j = (-1)^{p_j+1} \gamma_{p_j}\ , \quad 0 \leq j \leq \nu-1. \tag{6.4.4}$$

Secondly we assume that the C_j can be permuted such that

$$\psi_j = (-1)^{n_j+1} C_j, \quad \nu \leq j \leq m-1. \tag{6.4.5}$$

With these assumptions (6.4.3) can be written as

$$\int_\Omega y^*Sx\,d\Omega + \sum_{j=0}^{\nu-1} (-1)^{p_j+1} \int_\Gamma \gamma_{p_j} y^* B_j x\,d\Gamma =$$

$$\int_\Omega (S^*y^*)x\,d\Omega - \sum_{j=\nu}^{m-1} (-1)^{n_j+1} \int_\Gamma C_j y^* B_j' x\,d\Gamma. \qquad (6.4.6)$$

Remark 6.4.1. As in section 6.3, the assumption (6.4.5) imposes restrictions on the boundary operators $\{C_j,\ \nu \le j \le m-1\}$, which must be of order

$$n_j = m-1-p_j, \qquad \nu \le j \le m-1. \qquad \square$$

We are now in a position to define the abstract equations and to show they are an extension of the classical equations of the intermediate canonical form. Let us define the spaces

$$Y^* = \{\bar{y}^* = (y^*, y^*_{p_0}, \ldots, y^*_{p_{\nu-1}}) \in (L_2(\Omega))^r \times H^{-p_0-1/2}(\Gamma) \times \ \ldots \times H^{-p_{\nu-1}-1/2}(\Gamma);$$

$$S^*y^* \in L_2(\Omega);\ C_j y^* = 0,\ \nu \le j \le m-1;\ y^*_{p_0} = \gamma_{p_0} y^*, \ldots, y^*_{p_{\nu-1}} = \gamma_{p_{\nu-1}} y^*\},$$

where r is the number of components of a vector in the domain of S^*, and

$$X^* = L_2(\Omega).$$

Here S^* is considered as an operator in the generalised sense.

Remark 6.4.2. The space Y^* may be identified with the space

$$\{y^* \in (L_2(\Omega))^r;\ S^*y^* \in L_2(\Omega);\ C_j y^* = 0,\ \nu \le j \le m-1\}. \qquad \square$$

We shall define $T^*:Y^* \to X^*$ by

$$T^*\overline{y}^* = S^*y^*, \tag{6.4.7}$$

where $\overline{y}^* = (y^*, y^*_{p_0}, \ldots, y^*_{p_{\nu-1}}) \in Y^*$ and the operator S^* is interpreted in the generalised sense. The adjoint operator T is defined via the adjoint relationship

$$\langle Tx, \overline{y}^*\rangle = \langle x, T^*\overline{y}^*\rangle \qquad \forall x \in X,\ \overline{y}^* \in Y^*. \tag{6.4.8}$$

We shall use (6.4.6) to identify the operator T. Note that (6.4.6) is valid in the generalised sense for all $x \in H^m(\Omega)$ and all $y^* \in H(S^*;\Omega)$. The right hand side of (6.4.8) is identical to the right hand side of (6.4.6) since $C_jy^* = 0$, $\nu \le j \le m-1$, for $\overline{y}^* \in Y^*$. Hence

$$\langle Tx, \overline{y}^*\rangle = \int_\Omega y^*Sx\,d\Omega + \sum_{j=0}^{\nu-1} (-1)^{p_j+1} \int_\Gamma \gamma_{p_j} y^*B_jx\,d\Gamma, \tag{6.4.9}$$

for all $x \in H^m(\Omega)$ and $\overline{y}^* \in Y^*$. Hence, at least for $x \in H^m(\Omega)$, we can identify T by

$$Tx = (Sx, (-1)^{p_0+1} B_0x, \ldots, (-1)^{p_\nu+1} B_{\nu-1}x). \tag{6.4.10}$$

Now define a linear operator E with domain

$$D(E) = \{\overline{y} \in Y;\ \overline{y} = (y, 0, .., 0)\}$$

such that

$$E\overline{y} = (Gy, \gamma_{p_0}Gy, \ldots, \gamma_{p_{\nu-1}}Gy).$$

Then we define the abstract problem

$$\begin{aligned} Tu &= \bar{w} \\ E\bar{w} &= \bar{v} \\ T^*\bar{v} &= f \end{aligned} \qquad (6.4.11)$$

where $T: X \to Y$ is defined by (6.4.8), $T^*: Y^* \to X^*$ is defined by (6.4.7), $\bar{w} = (w, 0, \ldots, 0)$ and $\bar{v} = (v, \gamma_{p_0} v, \ldots, \gamma_{p_{\nu-1}} v)$. We can easily show that this is an extension of the intermediate canonical form (6.3.1), provided $u \in H^m(\Omega)$. For if this is the case we can use (6.4.10) to show that $(6.4.11)_1$ gives rise to the equations

$$\begin{aligned} Su &= w \text{ in } \Omega \\ B_j u &= 0 \text{ on } \Gamma, \quad 0 \le j \le \nu-1 . \end{aligned}$$

Also $(6.4.11)_2$ implies

$$Gw = v \quad \text{in } \Omega$$

and $(6.4.11)_3$ leads to the equations

$$\begin{aligned} S^*v &= f \text{ in } \Omega \\ C_j v &= 0 \text{ on } \Gamma, \quad \nu \le j \le m-1 , \end{aligned}$$

since $\bar{v} \in Y^*$ implies $C_j v = 0$, $\nu \le j \le m-1$.

The assumption that $u \in H^m(\Omega)$ is easily shown to be valid in the applications considered. In fact if the problem can be put into the first abstract formulation given in section 6.3, then we must have $u \in H^m(\Omega)$.

Conditions for existence and uniqueness of an abstract form of problem (6.4.11) are given in Chapter 4, theorem 4.1.4. The variational principles for this problem are also given in Chapter 4.

Example 6.4.1

Consider the problem

$$\begin{aligned} -\nabla^2\tilde{u} &= \tilde{f} \text{ in } \Omega \in \mathbb{R}^2 \\ \tilde{u} &= \tilde{g} \text{ on } \Gamma. \end{aligned} \tag{6.4.12}$$

This is a regular elliptic problem, see example 6.2.1, which can be put into the form

$$\begin{aligned} \text{grad } u &= w \text{ in } \Omega \\ u &= 0 \text{ on } \Gamma \\ w &= v \text{ in } \Omega \\ -\text{div } v &= f \text{ in } \Omega, \end{aligned} \tag{6.4.13}$$

where $u = \tilde{u}-u_0$, u_0 satisfying $\gamma_0 u_0 = \tilde{g}$ on Γ. Since $\nu = m = 1$, the boundary operator F_0 is given by

$$F_0 = \gamma_0.$$

This is a Dirichlet system and so from theorem 6.1.4 there exists a normal operator ϕ_0, such that

$$\int_\Omega \text{grad } \tilde{x}.\text{grad } x \, d\Omega = -\int_\Omega (\nabla^2\tilde{x})x d\Omega - \int_\Gamma \phi_0\tilde{x}\gamma_0 x d\Gamma, \tag{6.4.14}$$

with ϕ_0 of order 1. The operator ψ_0 is defined by

$$\psi_0 \text{ grad } \tilde{x} = \phi_0\tilde{x},$$

and (6.4.14) becomes

$$\int_\Omega y^*.\text{grad } x \, d\Omega = -\int_\Omega (\text{div } y^*) x d\Omega - \int_\Gamma \psi_0 y^* \gamma_0 x d\Gamma, \tag{6.4.15}$$

where $y^* = \text{grad } \hat{x}$. However, integration by parts gives

$$\int_\Omega y^*.\text{grad } x \, d\Omega = -\int_\Omega (\text{div } y^*) x d\Omega + \int_\Gamma \gamma_0 y^* \gamma_0 x d\Gamma. \tag{6.4.16}$$

Comparing this with (6.4.15) we see that assumption (6.4.4), i.e.

$$\psi_0 = -\gamma_0 ,$$

is satisfied. The second assumption (6.4.5) is trivially satisfied as there are no C_j operators for this problem. Hence (6.4.15) can be written as

$$\int_\Omega y^*.\text{grad } x \, d\Omega - \int_\Gamma \gamma_0 y^* \gamma_0 x d\Gamma = -\int_\Omega (\text{div } y^*) x d\Omega. \tag{6.4.17}$$

We define the spaces

$$Y^* = \{\bar{y} = (y, y_0) \in (L_2(\Omega))^2 \times H^{-1/2}(\Gamma); \; \text{div } y \in L_2(\Omega); \; y_0{}^* = \gamma_0 y^*\}$$

and

$$X^* = L_2(\Omega).$$

Then we define $T^*: Y^* \to X^*$ by

$$T^*\bar{y}^* = -\text{div } y^*$$

where $\bar{y}^* = (y^*, y_0^*)$. Then comparing the adjoint relationship

$$\langle Tx, \bar{y}^* \rangle = \langle x, T^*\bar{y}^* \rangle, \quad \forall x \in X, \; y^* \in Y^*,$$

with (6.4.17) we see that

$$<Tx,\overline{y^*}> \;=\; \int_\Omega y^*.\text{grad } x \, d\Omega - \int_\Gamma \gamma_O y^* \gamma_O x d\Gamma,$$

for $x \in H^1(\Omega)$, $\overline{y^*} \in Y^*$, since (6.4.17) is valid in the generalised sense for all $x \in H^1(\Omega)$ and all $y^* \in Y^*$. Then, provided $x \in H^1(\Omega)$, we can identify Tx with (grad x, $-\gamma_O x$).

Now define the linear operator E with domain

$$D(E) \;=\; \{\overline{y} \in Y;\; \overline{y} = (y,0)\}$$

and such that

$$E\overline{y} \;=\; (\tau y, \gamma_O \tau y).$$

Then we define the problem

$$\begin{aligned} Tu &= \overline{w} \\ E\overline{w} &= \overline{v} \qquad\qquad (6.4.17) \\ T^*\overline{v} &= f, \end{aligned}$$

where T and T* are defined as above, $\overline{w} = (w,0)$ and $\overline{v} = (v,\gamma_O v)$. Then if $u \in H^1(\Omega)$ we can identify T as above and hence (6.4.17) reduces to (6.4.13). The fact that $u \in H^1(\Omega)$ for the generalised form of (6.4.13) can be seen from example 6.3.1.

The existence and uniqueness of a solution of (6.4.17) is guaranteed if the conditions of theorem 4.1.4 are satisfied, i.e. if there exists $\alpha_1, \alpha_2, \alpha_3 > 0$ such that

$$\|Tx\|_Y \geq \alpha_1 \|x\|_X , \quad \forall x \in X, \qquad (6.4.18)$$

$$\|E\bar{y}\|_{Y*} \geq \alpha_2 \|\bar{y}\|_{Y}, \quad \forall \bar{y} \in D(E), \tag{6.4.19}$$

$$<E^{-1}\bar{z},\bar{z}> \geq \alpha_3 \|\bar{z}\|^2_{Y*}, \quad \forall \bar{z} \in Z. \tag{6.4.20}$$

To prove (6.4.18) we adopt an approach of Raviart-Thomas [to appear]. Let h be a solution of the problem:

find $h \in H^2(\Omega) \cap H_0^{\ 1}(\Omega)$ such that

$$-\nabla^2 h = x \text{ in } \Omega$$

$$\gamma_0 h = 0 \text{ on } \Gamma.$$

We know a unique solution $h \in H_0^{\ 1}(\Omega)$ exists for this problem when $x \in L_2(\Omega)$, see example 6.3.1. Let $k^* = \text{grad } h$. Then

$$<Tx,\bar{k}^*> = <x,T^*\bar{k}^*>$$

$$= <x,x> = \|x\|^2_{L_2(\Omega)}, \tag{6.4.21}$$

where $\bar{k}^* = (k^*,\gamma_0 k^*) \in Y^*$. Also

$$\|\bar{k}^*\|^2_{Y*} = \|k^*\|^2_{H(\text{div},\Omega)} + \|\gamma_0 k^*\|_{H^{-1/2}(\Gamma)}$$

$$\leq (1 + \alpha)\|k^*\|^2_{H(\text{div},\Omega)}$$

using the trace inequality of theorem 6.1.2.

Therefore

$$\|\bar{k}^*\|^2_{Y^*} \leq (1+\alpha)\{\|\text{grad } h\|^2_{L_2(\Omega)} + \|\text{div } k^*\|^2_{L_2(\Omega)}\}$$

$$= (1+\alpha)\{\|h\|^2_{H^1(\Omega)} + \|\nabla^2 h\|^2_{L_2(\Omega)}\}$$

$$\leq c\|x\|^2_{L_2(\Omega)} ,$$

where c and α are constants > 0, hence

$$\|Tx\|_Y = \sup_{\bar{y}^* \in Y^* - \{0\}} \frac{\langle Tx, \bar{y}^* \rangle}{\|\bar{y}^*\|_{Y^*}}$$

$$\geq \frac{\langle Tx, \bar{k}^* \rangle}{\|\bar{k}^*\|_{Y^*}} \geq \alpha_1 \|x\|_{L_2(\Omega)}$$

using (6.4.21) and (6.4.22).

To prove (6.4.19), note that

$$\|E\bar{y}\|^2_{Y^*} = \|\tau y\|^2_{H(\text{div},\Omega)} + \|\gamma_0 \tau y\|^2_{H^{-1/2}(\Gamma)} .$$

However

$$\|\bar{y}\|_Y = \|y\|_{H(\text{div},\Omega)^*} = \|\tau y\|_{H(\text{div},\Omega)} .$$

Hence $\|E\bar{y}\|_{Y^*} \geq \alpha_2 \|\bar{y}\|_Y$.

Finally (6.4.20) is easily proved as

$$\langle E^{-1}\bar{z}, \bar{z} \rangle = \langle \tau^{-1} z, z \rangle = \|z\|^2_{H(\text{div},\Omega)}$$

$$\geq \beta \|z\|^2_{H(\text{div},\Omega)} + \alpha(1-\beta)\|\gamma_0 z\|^2_{H^{-1/2}(\Gamma)}$$

for $0 < \beta < 1$ and using theorem 6.1.2. Hence

$$\langle E^{-1}\bar{z},\bar{z}\rangle \geq \alpha_3 \|\bar{z}\|_{Y^*} .$$

Therefore problem (6.4.17) has a unique solution

$$(u,\bar{w},\bar{v}) \in L_2(\Omega) \times Y \times Y^*.$$

The most general of the abstract variational principles studied in Chapter 4 is then applicable to this problem, and is:

find $(u,\bar{w},\bar{v}) \in L_2(\Omega) \times Y \times Y^*$, saddle point of

$$L(x,\bar{y},\bar{y}^*) = -\int_\Omega x \operatorname{div} y^* d\Omega - \frac{1}{2}\int_\Omega y\, y^* d\Omega - \int_\Omega x\, f\, d\Omega,$$

where $\bar{y} = (y,0) \in Y^*$, $\bar{y}^* = (y^*,\gamma_0 y^*) \in Y^*$. This can be regarded as a generalisation of the Hu-Washizu principle applied to this problem.

CHAPTER 7

SUMMARY AND CONCLUSIONS

7.1 Summary

In this section we briefly summarize all the material covered in this thesis before giving a summary in the next section of the contributions this thesis has made to the subject.

We started by considering a simple abstract linear equation

$$T^*\tau Tu = f \tag{7.1.1}$$

where $T : X \to Y$ and $T^* : Y^* \to X^*$ are adjoint operators, X, Y and their duals X^*, Y^* are Hilbert spaces and τ is the canonical isomorphism from Y to Y^*. In Chapter 2 we gave conditions on T necessary to guarantee the existence and uniqueness of the solution $u \in X$ of (7.1.1) for any $f \in X^*$. This abstract problem can be identified with a differential equation in one of two ways. First we may define the operator T as a differential operator in the generalised sense and T^* as its adjoint. Secondly we may define T^* as a differential operator in the generalised sense and T as its adjoint. In the applications we always regard the spaces X and Y as Sobolev spaces and in particular the sub-class of Sobolev spaces based on the $L_2(\Omega)$ space.

In Chapter 3 we showed that the abstract problem (7.1.1) is related to the mixed variational problem:

find $(u,v) \in X \times Y^*$, saddle point of

$$L(x,y^*) = \langle Tx,y^*\rangle - \frac{1}{2}\langle \tau^{-1}y^*,y^*\rangle - \langle x,f\rangle \quad x \in X,\ y^* \in Y^*, \tag{7.1.2}$$

in the sense that, if u is a solution of (7.1.1) then $(u,v = \tau Tu)$ is a solution of (7.1.2) and conversely if (u,v) is a solution of (7.1.2) then u is a solution of (7.1.1).

From (7.1.2) we derived two further variational problems:

find $u \in X$, minimum point of

$$J(x) = \frac{1}{2}\langle Tx,\tau Tx\rangle - \langle x,f\rangle, \quad x \in X \tag{7.1.3}$$

and

find $v \in Y^*$, maximum point of

$$K(y^*) = -\frac{1}{2}\langle \tau^{-1}y^*,y^*\rangle, \quad y^* \in Z_f, \tag{7.1.4}$$

where $Z_f = \{y^* \in y^*;\ T^*y^* = f\}$.
We proved that these two variational problems each have a unique solution provided (7.1.2) has a unique solution. Further we showed that the solutions are related in the sense that, if (u,v) is a solution of (7.1.2) then u is a solution of (7.1.3) and v is a solution of (7.1.4), and that these two variational problems are in fact complementary extremum problems, that is,

$$\underset{x \in X}{\text{Min}}\ J(x) = J(u) = L(u,v) = Y(v) = \underset{y^* \in Z_f}{\text{Max}}\ Y(y^*).$$

In a more general setting we showed that from a saddle functional, such as $L(x,y^*)$, we can obtain another functional, such as $J(x)$, by restricting the domain of the saddle functional to a linear variety.

We can achieve this for any saddle functional $L(x,y^*)$ for which the y^* variable is either linear or such that the relation $\nabla_2 L(x,y^*) = 0$ provides an explicit relationship for y^* in terms of x. We also showed that, for any saddle functional containing a saddle point which satisfies the above conditions for both the x and y^* variables, we can choose two linear varieties containing the saddle point, such that the restriction of the saddle functional to the linear varieties gives complementary extremum functionals.

In Chapter 4 we extended the abstract linear equation to the form

$$T^*ETu = f, \tag{7.1.5}$$

where $E : Y \to Y^*$ is a linear operator. We gave conditions on the operator E for (7.1.5) to have a unique solution and we proved that it is related to the variational problem

$$\begin{aligned} &\text{find } ((u,w),v) \in (X \times Y) \times Y^*, \text{ saddle point of} \\ &L(x,y,y^*) = \langle Tx,y^*\rangle - \langle y,y^*\rangle + \frac{1}{2}\langle y,Ey\rangle - \langle x,f\rangle, \end{aligned} \tag{7.1.6}$$

$x \in X$, $y \in Y$, $y^* \in Y^*$, and that if u is a solution of (7.1.5), $(u, w = Tu,\ v = ETu)$ is a solution of (7.1.6) and conversely if (u,w,v) is a solution of (7.1.6) then u is a solution of (7.1.5). In the theory of elasticity a particular case of this variational principle is known as the Hu-Washizu principle. By the same technique as in Chapter 3 we were able to derive many other variational principles associated with equation (7.1.5). These include generalisations of the Hellinger-Reisner principle, the minimum potential energy principle and the minimum complementary energy principle. We gave the precise relationship between each of the principles and showed that each of them can be obtained by restricting the domain of the functional $L(x,y,y^*)$ in (7.1.6).

In Chapter 5 we studied some non-linear equations, the most general of which is

$$T^*E(Tu) + F(u) = 0 \tag{7.1.7}$$

where $T : X \to Y$ and $T^* : Y^* \to X^*$ are linear adjoint operators and $E : Y \to Y^*$ and $F : X \to X^*$ are non-linear operators. We gave sufficient conditions on the non-linear operators E and F such that the existence and uniqueness theorem for the linear equation could be extended to the non-linear equation (7.1.7). The main variational problem associated with (7.1.7) is

$$\left.\begin{array}{l} \text{find } ((u,w),v) \in X \times Y \times Y^*, \text{ saddle point of} \\ L(x,y,y^*) = \langle Tx,y^*\rangle - \langle y,y^*\rangle + e(y) + f(x), \end{array}\right. \tag{7.1.8}$$

$x \in X$, $y \in Y$, $y^* \in Y^*$. We then proved that, provided the non-linear operators E and F are potential operators with potentials $e(y)$ and $f(x)$ respectively, this problem has a unique solution (u,w,v) where u is the solution of (7.1.7), $w = Tu$ and $v = E(Tu)$.

From the variational problem (7.1.8) we derived a set of variational problems associated with equation (7.1.7). These are the non-linear versions of those derived for the linear problem (7.1.5) and we gave the relationships between them as in the linear case.

Finally in Chapter 6 we showed in detail how to incorporate more general boundary value problems into the abstract formulation we had previously studied. In particular we considered the class of regular elliptic problems of order 2m; e.g.

$$S^* G Su = f \text{ in } \Omega$$
$$B_j u = g_j,\ 0 \leq j \leq m-1 \text{ on } \Gamma, \tag{7.1.9}$$

where S and S* are differential operators of order m, G is of order O and the B_j are boundary differential operators. We saw how this could be put into the abstract form

$$T^* E Tu = f$$

by associating the adjoint relationship for the operators T and T* with the Green's formula for problem (7.1.9). This led to two possible formulations for the problem. One in which the Dirichlet boundary conditions are incorporated into the domain of T and the Neumann boundary conditions are incorporated into the operator T*. The other in which the Dirichlet boundary conditions are incorporated into the operator T and the Neumann boundary conditions incorporated into the domain of T*.

Throughout the text we illustrated the theoretical concepts by applying them to simple problems for which we specified precisely the operators and spaces involved.

7.2 Contributions to the Subject and Relations to Other Work

In this section we briefly show how the material in this thesis relates to work done by other authors. First we show how the abstract formulation we have considered is related to the abstract formulation of Noble-Sewell [1971] and of the French school, e.g. Lions-Magenes [1972], Brezzi [1974] and Raviart-Thomas [to appear].

Considering first the French school, a typical abstract problem from the work of Lions-Magenes might be find $u \in X$ such that

$$a(u,x) = \langle f,x\rangle, \quad \forall x \in X, \tag{7.2.1}$$

where $a(u,x)$ is a continuous bilinear form from $X \times X$ to $\mathbb{R}$ and $f \in X^*$. A sufficient condition for this problem to have a unique solution is that $a(u,x)$ is X-elliptic, that is, there exists $\alpha > 0$ such that

$$a(x,x) \geq \alpha\|x\|^2, \qquad \forall x \in X. \tag{7.2.2}$$

When $a(u,x)$ is symmetric, it is easily shown that problem (7.2.1) is equivalent to the variational problem

$$\begin{aligned}&\text{find } u \in X \text{ minimum point of}\\ &J(x) = \frac{1}{2}a(x,x) - \langle f,x\rangle.\end{aligned} \tag{7.2.3}$$

Problem (7.2.1) can be related to the simplest abstract equation we have studied

$$T^*\tau Tu = f, \tag{7.2.4}$$

by the relation

$$a(u,x) = \langle \tau Tu,Tx\rangle. \tag{7.2.5}$$

Obviously the right hand side of (7.2.5) is bilinear and it is easily shown to be continuous. Using (7.2.5) in (7.2.1), we get

$$\langle \tau Tu,Tx\rangle = \langle f,x\rangle, \qquad \forall x \in X$$

and using the adjoint relation gives

$$a(u,x) = \langle T^*\tau Tu,x\rangle = \langle f,x\rangle, \quad \forall x \in X,$$

which is a weak form of equation (7.2.4).

Using (7.2.5) in inequality (7.2.2), a sufficient condition for existence and uniqueness of a solution is that there exists $\alpha > 0$ such that

$$\langle \tau Tx,Tx\rangle \geq \alpha\|x\|^2, \qquad \forall x \in X.$$

However

$$\langle \tau Tx,Tx\rangle = (Tx,Tx) = \|Tx\|^2$$

and so the condition becomes

$$\|Tx\|^2 \geq \alpha\|x\|^2, \qquad \forall x \in X,$$

which is the inequality we have shown to be sufficient for the existence of a unique solution of equation (7.2.4).

As $\langle \tau Tu,Tx\rangle$ is symmetric we can use the relation (7.2.5) in the variational problem (7.2.3) which becomes

find $u \in X$ minimum point of

$$J(x) = \frac{1}{2}\langle \tau Tx,Tx\rangle - \langle f,x\rangle,$$

which is the abstract minimum potential energy problem, see section 3.4, related to the abstract equation (7.2.4).

The treatment of mixed methods in the French school comes from the work of Brezzi [1974], see e.g. Raviart-Thomas [to appear]. The abstract problem considered by Brezzi can be written as

$$\begin{aligned} &\text{find } (u,v) \in X \times Y^* \text{ such that} \\ &a(v,y^*) + b(u,y^*) = 0, \quad \forall y^* \in Y^*, \\ &\qquad\qquad b(x,v) = \langle x,f\rangle, \quad \forall x \in X, \end{aligned} \tag{7.2.6}$$

where X and Y^* are Hilbert spaces and $a : Y^* \times Y^* \to \mathbb{R}$ and $b : Y^* \times X \to \mathbb{R}$ are bilinear forms. If $a(v,y^*)$ is symmetric and X-elliptic, then (7.2.6) is equivalent to the variational problem

$$\begin{aligned} &\text{find } (u,v) \in X \times Y^* \text{ saddle point of} \\ &H(x,y^*) = b(x,y^*) + \frac{1}{2}a(y^*,y^*) - \langle f,x\rangle. \end{aligned} \tag{7.2.7}$$

Problem (7.2.6) can be related to the abstract equation

$$T^* E\, Tu = f \tag{7.2.8}$$

as follows: Suppose the operator E has an inverse E^{-1}, then let

$$a(y_1^*,y_2^*) = -\langle E^{-1}y_1^*,y_2^*\rangle, \quad \forall y_1^*,y_2^* \in Y^* \tag{7.2.9}$$

and $b(x,y^*) = \langle Tx,y^*\rangle$

$$= \langle x,T^*y^*\rangle, \quad \forall y^* \in Y^*,\ x \in X. \tag{7.2.10}$$

Using these relations in equations (7.2.6) we get

$$-\langle E^{-1}v, y^*\rangle + \langle Tu, y^*\rangle = 0, \quad \forall y^* \in Y^*,$$
$$\langle x, T^*v\rangle = \langle x, f\rangle, \quad \forall x \in X. \qquad (7.2.11)$$

This is a weak form of the equations

$$E^{-1}v = Tu$$

and

$$T^*v = f,$$

that is, equation (7.2.8).

Hence a solution (u,v) of (7.2.6) gives a solution u of (7.2.8) when related by (7.2.9) and (7.2.10).

The variational formulation (7.2.7) is transformed by the relations (7.2.9) and (7.2.10) into the abstract Hellinger-Reissner variational problem, see equation (4.4.5), that is

find $(u,v) \in X \times Y^*$ saddle point of

$$H(x,y^*) = \langle Tx, y^*\rangle - \frac{1}{2}\langle E^{-1}y^*, y^*\rangle - \langle f, x\rangle.$$

Turning to the work of Noble-Sewell [1971], we can express the abstract problem they considered as

$$Tu = \nabla_2 W(u,v)$$
$$T^*v = \nabla_1 W(u,v), \qquad (7.2.12)$$

where T is a linear operator from a Hilbert space X to a Hilbert space Y with adjoint T* and $W : X \times Y^* \to \mathbb{R}$ is a given functional. This is related to the non-linear abstract equation

$$T^*E(Tu) + F(u) = 0 \tag{7.2.13}$$

as follows: Assume the operator E has an inverse E^{-1}, then (7.2.13) can be written as

$$\begin{aligned} Tu &= E^{-1}(v) \\ T^*v &= -F(u). \end{aligned}$$

We can see that this is a particular case of (7.2.12) in which

$$\begin{aligned} \nabla_2 W(u,v) &= E^{-1}(v), \\ \nabla_2 W(u,v) &= -F(u). \end{aligned}$$

Therefore, assuming E^{-1} and F are potential operators, the abstract equation (7.2.13) can be considered as a special case of (7.2.12) where the functional W is given by

$$\begin{aligned} W(x,y^*) = W_0 &+ \int_0^1 \langle E^{-1}(y_0^* + t(y^*-y_0^*)), y^*-y_0^*\rangle dt \\ &- \int_0^1 \langle F(x_0 + t(x-x_0)), x-x_0\rangle dt, \end{aligned}$$

see section 5.1.

Hence we see that the abstract form studied in this thesis is closely related to the abstract problems of both the French school and the work of Noble-Sewell. The approach of the French school has the advantage of great precision in the definition of operators and spaces, and the availability of existence and uniqueness theorems. The Noble-Sewell approach is far more general, including non-linear problems, but does not give any existence results. In this thesis we have given an abstract formulation in sufficient precision for us to prove existence and uniqueness theorems. Also this abstract formulation, although not as general as the Noble-Sewell form, can be applied to some non-linear as well as linear problems.

One of the more difficult aspects of studying differential equations from an abstract point of view is to relate the abstract equation to a specific differential equation. This requires a precise definition of the operators and spaces used in the abstract formulation. To simplify this identification we have restricted ourselves to an abstract formulation in which the linear adjoint operators T and T* act as generalisations of differential operators. For the spaces that form the domains and ranges of these operators we have restricted ourselves to the subclass of Sobolev spaces which are Hilbert spaces, that is, those Sobolev spaces based on the $L_2(\Omega)$ space.

The most common approach for differential equations of order 2m is to use the space $H^m(\Omega)$ or a subspace, e.g. $H_0^1(\Omega)$. These spaces are widely used for the formulation of problems in the weak form or in terms of variational problems of the minimum potential energy type. This approach can be extended to the formulation of problems in terms of a mixed variational principle in which we use the spaces $H^m(\Omega)$ and $(L_2(\Omega))^q$, q an integer, see Oden-Reddy [1976]. Another approach given recently by Raviart-Thomas [to appear] for the mixed method for the second order problem is to use the spaces $L_2(\Omega)$ and $H(\mathrm{div};\Omega)$.

In this thesis we have shown that these two approaches occur quite naturally for operators of the form T*T as suggested by Friedrichs [1939].

The first arises from defining the operator T as an extension of a differential operator of order m and T* is its adjoint. The second arises from defining T* as an extension of a differential operator of order m and T as its adjoint. We have also shown how various boundary conditions are incorporated into the abstract form. In the first approach the Dirichlet boundary conditions are incorporated into the domain of the operator T, i.e. they are essential boundary conditions. The Neumann boundary conditions are incorporated into the operator T*, i.e. they are natural boundary conditions. Conversely in the second approach the Dirichlet boundary conditions are incorporated into the operator T, i.e. they are natural boundary conditions. The Neumann boundary conditions are incorporated into the domain of the operator T*, i.e. they are essential boundary conditions.

We may study the existence and uniqueness of the solution of variational problems in two different ways. Firstly we could show that a unique solution exists by proving that the functional has a unique stationary point. Secondly we could show that a set of abstract equations are satisfied at a stationary point of the functional. Then we would only need to prove that the abstract equations have a unique solution to ensure that the variational problem has a unique solution. For non-linear problems of a very general type involving two variables Noble-Sewell [1971] took the first approach. They proved the uniqueness, but not existence, of the saddle point provided the functional is strictly convex in one variable and strictly concave in the other. However, all the variational problems we have studied are linear in one of the variables and so we are not able to use the Noble-Sewell approach.

The other approach was taken by Brezzi [1974] for linear problems based on a different abstract form which we discussed in the previous section, equations (7.2.6). We have essentially used the approach of Brezzi to prove the existence and uniqueness of a solution to the simplest abstract form we considered, that is,

$$\text{find } (u,v) \in X \times Y^* \text{ such that}$$
$$\tau T u = v$$
$$T^* v = f,$$

where X and Y* are Hilbert spaces and $T : X \to Y$ and $T^* : Y^* \to X^*$ are adjoint linear operators. We extended this result to the more complex linear problem

$$\text{find } (u,w,v) \in X \times Y \times Y^* \text{ such that}$$
$$Tu = w$$
$$Ew = v$$
$$T^* v = f,$$

and to the non-linear problem

$$\text{find } (u,w,v) \in X \times Y \times Y^* \text{ such that}$$
$$Tu = w$$
$$E(w) = v$$
$$T^* v + F(u) = 0,$$

where E and F are non-linear operators in this case.

There are many variational principles associated with an abstract equation of the type we have been considering. We have given a set of principles all of which are known, at least for the linear problems, see Oden-Reddy [1974]. Some are straightforward generalizations of classical variational principles, for example the minimum potential energy principle, others are of more recent origin, e.g. the Hu-Washizu principle. We have shown that each of these has a unique solution which is related to the solution of the abstract equation. We have also stated precisely the domain of the functional involved and shown that for boundary value problems there is a choice of generalizations leading to different functionals defined on different domains. Hence, for example, the mixed variational principle for the harmonic equation in $\mathbb{R}^2$ with homogeneous Dirichlet boundary conditions can have either the domain $H_0^1(\Omega) \times (L_2(\Omega))^2$ or the domain $L_2(\Omega) \times H(\mathrm{div};\Omega)$. It is interesting to note in this case that when the function $f \in L_2(\Omega)$ the solution of the two generalizations is the same. Whether a result of this type can be put into a more general framework remains to be shown.

For either the linear or non-linear abstract equation the set of associated variational principles are interrelated. These relationships may be viewed in several ways, all of which are mathematically equivalent but conceptually rather different. We can view the extremum principles, i.e. the minimum potential energy and minimum complementary energy principles, as fundamental and derive the other variational principles by using Lagrange multipliers, see Washizu [1975]. Alternatively we may view the most general principle involving three variables as fundamental and regard all other principles as derivable by specialising assumptions from the most general principle, see Fraeijs de Veubeke [1974] or Oden-Reddy [1974]. We have taken the latter view, but stressed that the specialising assumptions can be viewed as a restriction of the domain of the functional of the general variational principle. For example, the functional

$$J(x) = \frac{1}{2} <Tx, \tau Tx> - <x,f>, \qquad x \in X$$

of the minimum potential energy principle can be derived from the functional

$$L(x,y^*) = <Tx,y^*> - \frac{1}{2} <\tau^{-1} y^*, y^*> - <x,f>, \qquad x \in X,\ y^* \in Y^*$$

of the Hellinger-Reissner principle by restricting the domain of $L(x,y^*)$ to those (x,y^*) satisfying $y^* = \tau Tx$.

To recapitulate the contributions made in this thesis, we have shown that an elliptic boundary value problem may, by a suitable extension, be put into one of the abstract forms based on the operators T and T*. This may be done in two ways, one in which T is the primary operator and one in which T* is the primary operator. We have given conditions on the operators for a unique solution of the equation to exist and for there to be a set of variational problems associated with the equation. We have shown that under these conditions, the variational problems each have a unique solution which is related to the solution of the abstract equation. We have also shown how these variational problems are interrelated.

An important area of research which follows naturally from these results is the study of approximate methods associated with each of the variational principles to determine which, if any, provides the "best" way of computing solutions to elliptic differential equations. We have not pursued this question in this thesis but in the next section we give a brief review of the questions involved in this extension of the material in this thesis.

7.3 Methods of Approximation

All of the variational principles we have discussed in this thesis may be used as the basis of an approximate method, such as the finite element method, for the solution of differential equations. To do this we need to define an approximate variational principle with a finite dimensional domain. This approximate variational principle will often be the exact variational principle defined over a finite dimensional subspace of its domain. This is sometimes called an interior approximation. We shall not consider the case of exterior approximation, when the finite dimensional domain is not a subspace of the infinite dimensional domain. Further we need to prove that the solution of the approximate variational principle converges to the solution of the exact variational principle, additionally it is useful if the rate of convergence and some estimate of the error can be found. There has been considerable advance in the last ten years in proving some of these results for the finite element method based on some of the variational principles we have discussed. We briefly summarize the results presently known to the author.

The first results along these lines were obtained for the minimum potential energy principle. See, for example, Zlamal [1968] where various piecewise polynomial subspaces were used as the domain of the minimum potential energy principle for second order and fourth order problems. Results for the complementary energy principle for second order problems have only recently been obtained, see Thomas [to appear]. For mixed methods, e.g. the Hellinger-Reisner principle, results have been given by Johnson [1973] for the biharmonic equation and by Raviart-Thomas [to appear] for second order equations. The convergence of approximate methods based on the more general three field principles, e.g. the Hu-Washizu principle, have, apparently, not yet been studied.

For those spaces which give a solution converging to the correct solution for each variational principle we also have to compare the practical difficulties of implementing such schemes on a computer. A major problem is the difficulty of automatically generating the elements of the finite dimensional spaces involved. For example, let us consider Poisson's equation with homogeneous Dirichlet boundary conditions. The minimum potential energy principle uses the space $H_0^1(\Omega)$. It has proved fairly easy to construct approximation spaces for this which can be generated automatically, see George [1971]. The complementary energy principle uses the space $\{y \in (L_2(\Omega))^n;\ -\nabla y = f\}$ for which the construction of approximate spaces presents difficulties, Thomas [to appear]. See also Fraeijs de Veubeke [1965].

The first formulation of the mixed method uses the spaces $H_0^1(\Omega)$ and $(L_2(\Omega))^n$, Oden-Reddy [1976]. This seems to offer no advantage over the minimum potential energy principle unless the second function v of the solution (u,v) is specifically required. However, this is quite often the case, for example in elasticity the v's may represent the stresses. In this case the mixed method may produce more accurate results for v. In the second formulation the mixed method uses the spaces $L_2(\Omega)$ and $H(\mathrm{div};\ \Omega)$, Raviart-Thomas [to appear]. The elements of these spaces require very little continuity and it would presumably be easy to generate elements of a finite dimensional subspace.

The three field principles appear to offer no advantage over the two field mixed methods for linear problems. However, for problems with non-linear constitutive equations it may well be useful to consider these more general principles. There are no results known to the author on the implementation of three field principles for non-linear problems of this type.

7.4 Structure of the Mathematical Analysis

In this section we briefly outline the structure of the existence and uniqueness proofs for the abstract problem and the associated variational problem. In this way we hope to lay bare the methodology used and hence clarify the way in which it could be used for other formulations.

The simplest abstract form we considered, see Chapter 2, was

$$\text{find } (u,v) \in X \times Y^* \text{ such that}$$
$$\begin{aligned} \tau Tu &= v \\ T^*v &= f, \end{aligned} \tag{7.4.1}$$

where X and Y* are Hilbert spaces and $T : X \to Y$ and $T^* : Y^* \to X^*$ are adjoint linear operators. Under the assumption that there exists $\alpha > 0$ such that

$$\|Tx\| \geq \alpha \|x\|, \qquad \forall x \in X,$$

we proved that T* is an isomorphism from a subspace $Z^\perp \equiv [N(T^*)]^\perp \subset Y^*$ onto X* and that τT is an isomorphism from X onto $Z^\perp$. Hence given any $f \in X^*$ there exists a unique v satisfying $(7.4.1)_2$ and also a unique u satisfying $(7.4.1)_1$.

In Chapter 4 we extended this result to the more complex linear problem

$$\text{find } (u,w,v) \in X \times Y \times Y^* \text{ such that}$$
$$\begin{aligned} Tu &= w \\ Ew &= v \\ T^*v &= f. \end{aligned} \tag{7.4.2}$$

For this we only needed to extend the proof to deal with the linear equation $Ew = v$ rather than the isomorphism $\tau w = v$, where $w = Tu$, encountered in the simplest abstract problem. We showed that if there exists $\alpha_1, \alpha_2 > 0$ such that

$$\|Ey\| \geq \alpha_1 \|y\|, \quad \forall y \in Y$$

and

$$\langle E^{-1}z, z\rangle \geq \alpha_2 \|z\|^2, \qquad \forall z \in Z \equiv N(T^*),$$

then for any $\tilde{v} \in Z^{\perp}$ there exists a unique $w \in Z^{0} \equiv (Z^{\perp})^*$ such that

$$v \equiv \tilde{v} + \tilde{z} = Ew$$

where $\tilde{z}$ is an element of the null space of T^*, i.e. Z. We were thus able to use the results of the simplest abstract problem, which showed that T^* is an isomorphism from $Z^{\perp}$ to X^* and T is an isomorphism from X to Z^{0}, to deduce that problem (7.4.2) has a unique solution under the conditions given.

Using some elements of non-linear operator theory, we were able to further extend the existence and uniqueness theorem to the non-linear problem

$$\begin{aligned} &\text{find } (u,w,v) \in X \times Y \times Y^* \text{ such that} \\ &\qquad Tu = w \\ &\qquad E(w) = v \\ &\qquad T^*v + F(u) = 0, \end{aligned}$$

where E and F are non-linear operators. The non-linear operator E was incorporated into the existence and uniqueness proof in essentially the same way as the linear case. Here, however, we used monotone operator theory to establish that for any $\tilde{v} \in Z^{\perp}$ there exists a unique $w \in Z^{o}$ such that

$$v \equiv \tilde{v} + \tilde{z} = E(w),$$

provided E is a continuous strictly monotone operator satisfying

$$\lim_{\|y\| \to \infty} \frac{\langle E(y), y \rangle}{\|y\|} = +\infty.$$

Then as in the linear case we used the results of the simplest abstract problem relating to T and T* to complete the proof.

To incorporate the non-linear operator F instead of the function f of the linear case, we have applied the theory of monotone operators to the whole equation to show that a unique solution exists provided the conditions of the simplest abstract problem are satisfied and F is a monotone operator satisfying

$$\liminf_{\|x\| \to \infty} \frac{\langle F(x), x \rangle}{\|x\|} > -\infty.$$

Hence from the existence and uniqueness theorem of the simplest abstract problem we have been able to build up proofs for more complex linear problems as well as non-linear problems.

These results have also been used as the basis for existence and uniqueness theorems for the variational problems we have studied. The proofs for all the variational problems, both linear and non-linear, have the same general structure. We first show that any stationary point of the functional of the variational problem is also a solution of the related abstract equation. Then we use the existence result for the abstract equation to show that the functional has a unique stationary point. Finally we show that the functional is a saddle functional and hence the stationary point is in fact a saddle point.

7.5 Conclusions

The abstract form we have studied in this thesis has some advantages over the approaches of the French school and the work of Noble-Sewell. The form we have given is a more natural generalization of a differential equation than the abstract form associated with the French school but still retains the preciseness of their approach. We were also able to easily extend the abstract form to include some types of non-linear equation. The Noble-Sewell approach, although including more general non-linear problems, does not include any existence and uniqueness theorems.

We have been able to show that with each of the abstract forms considered there are associated a set of interrelated variational problems with solutions corresponding to the solution of the abstract problem.

In applying the abstract formulation to differential equations we have been able to unify the approaches of Oden-Reddy and Raviart-Thomas. We have seen that these two approaches occur quite naturally for problems in the T*T form, one by taking T as the primary operator and the other by taking T* as the primary operator.

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