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PROBLEM EQUIVALENCE AND NECESSARY CONDITIONS OF RELAXED DYNAMIC PROGRAMMING TYPE IN OPTIMAL CONTROL

BY

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ABSTRACT

Certain relaxed optimal control problems are shown to be equivalent to mathematical programs defined on constrained sets of Radon measures.This is re-expressed as a representation theorem for the measures,reflecting the derivation of the programs.The class of problems for which the equivalence is valid is exhaustively examined. Just as relaxed controls may be viewed as introducing convexity into the control and velocity sets,our measures achieve convexity in the set of admissible trajectory-control pairs.

Application of duality results to the convex program produces a new necessary and sufficient condition for optimality in the control problem,in the form of a weakened,or relaxed,version of a previously known sufficiency criterion.Similar results had been achieved earlier only under the assumption of a regular feedback form for the optimal control.Finally,the relationship of the new condition to the Maximum Principle and its utility as a practicable means of solving control problems are discussed and an illustrative example is presented.

TO JANICE

ACKNOWLEDGEMENTS AND A STATEMENT OF ORIGINAL CONTRIBUTIONS

I am indebted to my supervisor , Dr.R.B.Vinter , for suggesting this area of research.Much of the work is the result of collaboration between us and is substantially original in concept,all other material being thoroughly referenced in the text. The initial development was due to Dr.Vinter , my own contributions including the definitions and techniques for dealing with problems having costs directly control dependent , the incorporation of further constraints and the development of equivalence for variable end-point problems , together with the alternative statement to equivalence , a measure representation theorem . The new necessary and sufficient condition has a similar history - first proposed by Dr.Vinter , it was extended to increasingly wider classes of problem by both of us simultaneously as the corresponding equivalence results became available .

I am grateful to my colleagues Mssrs. Georges Stassinopoulos and Jose-Luis Farah for many illuminating discussions of technical points in functional analysis and measure theory.

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THE STRUCTURE OF THE THESIS

The contents of this thesis divide naturally into two parts. The first,the derivation of a convex program related to an optimal control problem and the proof of their equivalence,is independent of the second.The development of the new necessary and sufficient condition is equally self-contained,apart from the use of the central equivalence result of part one.In view of this and the bewildering nature of existing results related to the new condition, the two parts are presented separately,with individual introductions. All appendices,however,are collected together after part two.

NOTATION

Let X be a topological space. We denote the Banach space of continuous functions from X to $\mathbb{R}$, with norm $\| f \| = \sup_{x \in X} | f(x) |$, by $C(X)$. In every instance we deal with X is compact, in fact $X \subset \mathbb{R}^n$, for some $1 \leq n < \infty$. In this case the space of restrictions of differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$, to X, is denoted $C^1(X)$. The positive cone in $C(X)$ is defined to be :

$$P(X) \triangleq \{ f \in C(X) : f(x) \geq 0 \ \forall \ x \in X \}$$

$C^*(X)$, the dual space of $C(X)$, is isomorphic to and identified with the space of finite Radon measures on X, $\mathrm{frm}(X)$. The standard norm here is $\| \mu \| = \sup \{ |\int f d\mu| : f \in C(X) , \|f\| \leq 1 \}$, which generates the strong topology on $C^*(X)$. Other norms are introduced in the text. The positive cone in $C^*(X)$ is :

$$P^{\oplus}(X) \triangleq \{ \mu \in C^*(X) : \int f d\mu \geq 0 \ \forall \ f \in P(X) \} .$$

For any $A \subset X$, $\mathrm{co}(A)$ denotes the convex hull of A in X, $\mathrm{co}(A) = \{ \sum_{i=1}^{n} \alpha_i x_i : x_i \in A, \alpha_i \geq 0 , \sum_{i=1}^{n} \alpha_i = 1 \}$. $\overline{\mathrm{co}}(A)$ denotes the closure of $\mathrm{co}(A)$. If X is finite dimensional and A is closed, $\overline{\mathrm{co}}(A) = \mathrm{co}(A)$.

$N(A,\varepsilon)$ denotes the ε-neighbourhood of A in X, $\{ x \in X : \|x-y\| < \varepsilon$ for some $y \in A \}$, the topology on X being generated by the norm $\|\cdot\|$.

If μ is a measure on X, the support of μ is the smallest element of the Borel field on X such that the restriction of μ to its complement is trivial. When μ is a Radon measure on a topological space, $\mathrm{supp}\{\mu\}$ can be defined to be closed, i.e. a smallest closed set satisfying the above exists. (See §4.1)

When A and B are two sets, $\mathcal{C}(B)$ designates the complement of B and $A \backslash B$ denotes $A \cap (\mathcal{C}B)$.

Other frequently used symbols and abbreviations are :

$\forall$	for all
$\exists$	there exist(s)
a.a	almost all
a.c.	absolutely continuous
a.e.	almost every (t in $[t_0,t_1]$) or almost everywhere (in $[t_0,t_1]$) , depending upon the context.
argmax[f]	the argument of f maximizing f (over a specified set)
argmin[f]	the argument of f minimizing f (over a specified set)
cl(A)	the closure of A
diam(A)	the diameter of A i.e.the maximum separation (in a metric space distance) of two elements in A.
dist(A,B)	Hausdorff distance between two sets (in a metric space); if $A = \{x\}$ we write dist(x,B) .
iff	if and only if
inf[f]	the infimum of f (not necessarily attained)
l.s.c.	lower-semi-continuous
max[f]	maximum of f (attained)
min[f]	minimum of f (attained)
M.P.	Maximum Principle
p.d.e.	partial differential equation
p.d.i.	partial differential inequality
sgn(s)	the signum function $\operatorname{sgn}(s) \triangleq \begin{cases} 1 & \text{if } s \geq 0 \\ -1 & \text{if } s < 0 \end{cases}$
span[G]	the vector space spanned by G i.e. the space of all linear combinations of elements in G
s.t.	such that
sup[f]	the supremum of f (not necessarily attained)
u.s.c.	upper-semi-continuous
w.r.t.	with respect to

PART I

THE EQUIVALENCE OF STRONG AND WEAK FORMULATIONS OF CERTAIN OPTIMAL CONTROL PROBLEMS.

Chapter 0

INTRODUCTION

Over the past twenty years a large body of literature concerned with the optimal control of systems described by ordinary differential equations has accumulated.By optimality is meant the minimization of a 'cost' functional associated with the admissible trajectory-control pairs of the system.Three themes have arisen ; the question of whether there exists an admissible pair achieving the minimum ; characterization of such a pair by means of necessary and/or sufficient conditions for optimality ; and the approximation of a minimizing pair,usually numerically,in the case where the exact solution is difficult to determine. Existence theory is dominated by convexity - to guarantee the existence of a minimand the velocity set is convexified by admitting relaxed controls and the associated generalized trajectories. In the weak formulation of the control problem we take this one stage further,effectively introducing convexity into the combined trajectory-control set.

The Maximum Principle is established as the most useful necessary condition,while the method of Dynamic Programming provides a verification condition.These are discussed in the second part of this thesis,where duality theory is applied to the weak problem to prove the necessity,as well as sufficiency, of a weakened version of the verification condition.The possibility of using this result to construct algorithms for the generation of sequences of pairs yielding at least decreasing cost,at best convergence to a minimizing pair,is included there.

The plan of part I is as follows.Chapter one concerns the fundamental form of the control problem to be studied,that of Lagrange,in which the 'cost' is an integral functional.The initial discussion centres around a restricted version called the 'Strong Problem' ,with fixed boundary conditions,satisfying a hypothesis due to McShane which guarantees the existence of a minimizing pair.These pairs define continuous linear functionals on a Banach space of continuous functions containing the cost integrand. The class of all such functionals is isomorphic to a space of Radon measures - in chapter two we use the essential features of admissible pairs to define a constraint set on this space, thus imbedding the control problem in one posed over a subset of Radon measures,the 'Weak Problem'.

In turn this is embedded in a parametric problem in which the dynamics are an explicit side-constraint.We recognize this as a generalized problem in the Calculus of Variations, for which L.C.Young has derived powerful approximation and representation theorems which enable us to show that the parametric problem has a generalized curve solution (chapters three and four).

Equivalence between weak and strong problems is concluded by constructing an admissible pair from the generalized curve, such that the value of the cost is preserved.

A different approach may be taken,where approximations leading to this pair are derived directly from the weak problem. However, Young's results are not available in this non-parametric case and it is the existence of this developed theory of boundary conditions that allows us to extend our class of control problems to include those with variable end-points and additional constraints.

Systems described by differential inclusions are also treated - it is noticed that the construction of an admissible pair is not affected by these generalizations.

The final chapter deals with the difficulties encountered when the aforementioned hypothesis is relaxed.We conjecture that if the control problem has the property Q defined by Cesari then it is equivalent to a properly posed weak problem.The possibility of an equivalence theory for infinite dimensional systems is considered.

Chapter 1

THE LAGRANGE CONTROL PROBLEM - GENERALIZED CURVES

Our basic problem is the following:

$$
(\mathrm{L})\quad \begin{cases}
\text{minimize} \displaystyle\int_{t_0}^{t_1} \ell(x(t),t,u(t))\,dt & \text{(L1) subject to} \\[2ex]
\dot{x}(t) = f(x(t),t,u(t)) \text{ a.e. in } [t_0,t_1] & \\
 & \text{(L2)} \\
x(t_0) = x_0 \qquad x(t_1) = x_1 & \\[2ex]
u(t) \in \Omega \subset R^m \text{ a.e. in } [t_0,t_1] & \text{(L3)}
\end{cases}
$$

where $\ell : R^n \times R \times R^m \to R$ and $f : R^n \times R \times R^m \to R^n$ are continuous functions, $t_0 \in R$, $t_1 \in R$, $t_0 < t_1$, $x_0 \in R^n$, $x_1 \in R^n$ are given points and Ω is a compact subset of R^m. An ordinary admissible trajectory-control pair is an absolutely continuous function $x(\cdot) : [t_0,t_1] \to R^n$, the state, which together with the control, a measurable function $u(\cdot) : [t_0,t_1] \to R^m$ satisfies (L2) and (L3). The name Lagrange Control Problem is given to (L) specifically in the case where the cost functional has the integral form (L1) and is dealt with in detail by Berkovitz [Ber.].

It is well documented that, even when a plentiful supply of admissible ordinary pairs exists there need not be one among them which solves (L), unless $f(x,t,\Omega)$ is a convex set in R^n for all x, t and $\ell(x,t,u)$ is convex in u, $u \in \Omega$. See, for example, [Ber. p. 42, 2.2(b)].

The natural remedy is to admit relaxed controls and their corresponding generalized trajectories, concepts first introduced by Young [You 2] into the calculus of variations for the same purpose. A relaxed control is a

$C^*(\Omega)$ valued function on $[t_0,t_1]$ $u^*: t \to \mu_t$, where

$$\left.\begin{array}{l} \mu_t \in P^{\oplus}(\Omega) \text{ a.e. in } [t_0,t_1] \\ \\ |\mu_t| = 1 \quad \text{a.e. in } [t_0,t_1] \end{array}\right\} \qquad (S3)$$

and $\Phi : t \to \int_\Omega \phi(t,u)d\mu_t(u)$ is Lebesgue measurable for all $\phi \in C([t_0,t_1]\times\Omega)$, i.e. a relaxed control is a probability measure on Ω for almost every $t \in [t_0,t_1]$, satisfying the measurability criterion. An admissible trajectory control pair is now an a.c. function $x(\cdot) : [t_0,t_1] \to R^n$ satisfying

$$\left.\begin{array}{l} \dot{x}(t) = \int_\Omega f(x(t),t,u)d\mu_t(u) \text{ a.e. in } [t_0,t_1] \\ \\ x(t_0) = x_0 \qquad x(t_1) = x_1 \end{array}\right\} \qquad (S2)$$

together with the corresponding relaxed control $u : t \to \mu_t$ (satisfying (S3)).

What we shall call the Strong Control Problem is :

$$(S) \quad \left\{\begin{array}{ll} \text{minimize} \int_{t_0}^{t_1} \int_\Omega \ell(x(t),t,u)\,d\mu_t(u)\,dt & \qquad (S1) \\ \\ \text{subject to (S2) and (S3)} & \end{array}\right.$$

The set of admissible pairs will be denoted by S and the value of (S), that is $\inf\{\int_{t_0}^{t_1} \int_\Omega \ell d\mu_t dt : (S2),(S3)\}$, by $\eta(S)$. Let us put forward the hypothesis (H).

(H) $\left\{ \begin{array}{l} \text{There exists a minimizing admissible sequence } \{x_i(\cdot),u_i\} \\ \text{i.e. } \int_{t_0}^{t_1} \int_\Omega \ell(x_i(t),t,u)d\mu_{it}(u)dt \to \eta(S) \text{ such that} \\ x_i(t) \in A \subset R^n, \text{ A compact, for all } t \in [t_0,t_1], \text{ for i} \\ \text{sufficiently large.} \end{array} \right.$

This hypothesis, naturally subsuming that of the existence of at least one admissible pair, is precisely that which McShane uses to prove that there exists a pair $\{x_0(\cdot),u_0\} \in S$ such that

$\eta(S) = \int_{t_0}^{t_1} \int_\Omega \ell(x_0(t),t,u)d\mu_{0t}(u)dt$ [McS., thm. 2.7]. He also offers conditions on f, ℓ under which (H) will be valid.

Henceforth we restrict attention to admissible pairs where the trajectory is contained in A, i.e. we deal with the restrictions of f, ℓ to the compact set $A \times [t_0,t_1] \times \Omega$. This implies that admissible trajectories are Lipshitz continuous - continuity of f together with compactness and the nature of relaxed controls imply that $\dot{x}(\cdot)$ is uniformly bounded almost everywhere on $[t_0,t_1]$.

It is pertinent to remark here that McShane's existence theorem (2.7) is derived for far more general boundary conditions than $x(t_0) = x_0$ $x(t_1) = x_1$. These are used here to avoid obscuring the essentials of our proofs by unnecessary detail and more general conditions will be introduced at a later stage.

Chapter 2

THE WEAK FORMULATION OF THE OPTIMAL CONTROL PROBLEM

Define $\underline{A} \triangleq A \times [t_0,t_1]$ and take any $g \in C(\underline{A}\times\Omega)$. Let $\{x(\cdot),u\} \in S$ be an admissible pair then clearly we can regard it as an element of $C^*(\underline{A}\times\Omega)$ for the mapping

$$g \to \int_{t_0}^{t_1} \int_{\Omega} g(x(t),t,u)\,d\mu_t(u)\,dt \qquad (2.1)$$

is well-defined (by virtue of the measurability properties of $t \to \mu_t$), linear in g and bounded (by compactness of $\underline{A}\times\Omega$) hence continuous. Let us write the mapping as:

$$g \to \int_{\underline{A}\times\Omega} g\,d\mu \qquad \mu \in \text{frm }(\underline{A}\times\Omega) \triangleq C^*(\underline{A}\times\Omega)$$

so that the strong problem can be thought of as:

minimize $\int_{\underline{A}\times\Omega} \ell\,d\mu$ over $\mu \in C^*(\underline{A}\times\Omega)$ where μ has representation (2.1) for some $\{x(\cdot),u\} \in S$.

Examining the properties of such a measure μ derived from an element of S it is noticed that:

(a) $\mu \in P^*(\underline{A}\times\Omega)$

(b) $||\mu|| = \int_{t_1}^{t_0} 1\,dt = t_1 - t_0$

(c) for any $\phi \in C^1(\underline{A})$

$$\int_{\underline{A}\times\Omega} \phi_t(x,t) + \phi_x(x,t)\cdot f(x,t,u)\,d\mu = \phi(x_1,t_1) - \phi(x_0,t_0) \triangleq \Delta\phi$$

since the integrand is just $\frac{d}{dt}[\phi(x(t),t)]$.

A new optimization program can be introduced over the space of Radon measures :

$$
(W) \begin{cases} \text{minimize} \int_{\underline{A}\times\Omega} \ell(x,t,u)\,d\mu & \text{(W1)} \quad \text{subject to} \\ \mu \in P^{\oplus}(\underline{A}\times\Omega), \ ||\mu|| \leq t_1 - t_0 & \text{(W2)} \\ \int_{\underline{A}\times\Omega} \phi_t(x,t) + \phi_x(x,t)\,f(x,t,u)\,d\mu = \Delta\phi \ \text{for all} \ \phi \in C^1(\underline{A}) & \\ & \text{(W3)} \end{cases}
$$

This is called the Weak Problem. If the set of $\mu \in C^*(\underline{A}\times\Omega)$ satisfying (W2) and (W3) is denoted by $\mathcal{W}$, $\mathcal{W}$ is non-empty since it contains $\mathcal{S}$ and, by definition, it is convex. (W) is consequently a convex program ($\ell \to \int \ell d\mu$ being linear in μ) and can be subjected to the methods of convex analysis , as will be done in part 2 .

(W) has been constructed to contain (S) i.e. $\mathcal{W} \supset \mathcal{S}$, so that $\eta(W) \leq \eta(S)$. The remainder of this part of the thesis is devoted to proving :

<u>Theorem (2.1)</u>

The problems (W) and (S) both have solutions and $\eta(W) = \eta(S)$. Furthermore, if $\{x(\cdot),u\} \in \mathcal{S}$ and solves (S) then $\mu \in \mathcal{W}$ defined from $\{x(\cdot),u\}$ by (2.1) solves (W) . □

It is not claimed that any solution to (W) solves (S) and in fact this is untrue if the solution to either problem is not unique. For, let $\{x_i(\cdot),u_i\} \in \mathcal{S}$ $i = 1,2$ both solve (S) then for $0 < \alpha < 1$, $\{ \alpha x_1(\cdot)+(1-\alpha)x_2(\cdot), \alpha u_1+(1-\alpha)u_2 \} \notin \mathcal{S}$ but the corresponding $\alpha\mu_1 + (1-\alpha)\mu_2 \in \mathcal{W}$ and solves (W), the addition being in $C^*(\underline{A}\times\Omega)$.

It is equally true that unless $\mathcal{S}$ consists of just one element, $\mathcal{S} \neq \mathcal{W}$. However a consequence of theorem (2.1) is the structural result :

Theorem (2.2)

The extreme points of the weak * compact convex set $\mathcal{W}$ are contained in S. Thus $\mathcal{W} = \overline{\text{co}}S$.

Notes

(i) In the above we have used u alternately to denote a relaxed control function $u : t \to \mu_t$, for example when writing $\{x(\cdot),u\} \in S$, and to denote points in Ω , the original control constraint set. It should be clear from the context which is meant , for example in $\int_\Omega h(t,u)d\mu_t(u)$, u is the dummy variable in Ω . The reason that $u(\cdot)$ is not used to denote the control functions is that we wish to emphasize that they are not ordinary R^m valued functions but probability measure valued functions.

(ii) In the following section , §3.1 , we have also used u to mean a point in Ω , or its generalization , a probability measure . Since the material in this section appears in isolation from the ideas of trajectories and controls , again no confusion should arise .

Chapter 3

THE OPTIMAL CONTROL PROBLEM AS A PROBLEM IN THE CALCULUS OF VARIATIONS WITH SIDE-CONSTRAINTS. PARAMETRIZATION

Motivated by a desire to apply Young's density results [You 1] directly, the weak formulation is rewritten as a parametric problem in the calculus of variations. This procedure is comprised of two stages; firstly the variational problem will be derived in non-parametric form; it will subsequently be parametrized.

§3.1 The Weak-Variational Problem

The central difficulty is defining a cost functional integrand in terms of the variables $(x,t,\dot{x})$ instead of (x,t,u). Since $\underline{A}$, Ω are compact, co $f(x,t,\Omega)$ is compact for each $(x,t) \in \underline{A}$ and there exists a compact $F \subset R^n$ such that co $f(x,t,\Omega) \subset F$ for all $(x,t) \in \underline{A}$. In particular we shall choose a convex set F. For each $\{x(\cdot),u\} \in S$ and almost every $t \in [t_0,t_1]$ $\dot{x}(t) \in$ co $f(x(t),t,\Omega) \subset F$. It should therefore be sufficient for our purposes to define the new integrand over $\underline{A}$xF.

Definition

Take $(x,t) \in \underline{A}$, $\dot{x} \in$ co $f(x,t,\Omega)$ and define $\underline{\ell}(x,t,\dot{x})$ by

$$\underline{\ell}(x,t,\dot{x}) \triangleq \min_{u} \{\ell(x,t,u):\dot{x}=f(x,t,u)\} \tag{3.1}$$

When $(x,t) \in \underline{A}$, $\dot{x} \in F\backslash$co $f(x,t,\Omega)$ take

$$\begin{aligned}&\underline{\ell}(x,t,\dot{x}) \triangleq \underline{\ell}(x,t,\dot{y}) \text{ where } \dot{y} \in \text{co } f(x,t,\Omega) \text{ and}\\ &\text{is the minimand of } \operatorname{dist}(\dot{x}, \text{co } f(x,t,\Omega))\end{aligned} \tag{3.2}$$

Remarks

(i) In (3.1) we interpret u as a probability measure on Ω rather than a point in Ω. The set of u satisfying $\dot{x} = f(x,t,u)$ is closed and continuity of ℓ justifies writing min in (3.1).

(ii) Convexity of co $f(x,t,\Omega)$ ensures that $\dot{y}$ in (3.2) is unique i.e. $\underline{\ell}$ is well defined on $\underline{A}xF$.

(iii) For all $(x,t,\dot{x}) \in \underline{A}xF$, $\underline{\ell}(x,t,\dot{x}) \in$ co $\ell(x,t,\Omega)$, i.e. $\underline{\ell}(x,t,F) \subset$ co $\ell(x,t,\Omega)$. Consequently $\underline{\ell}$ is bounded.

Lemma (3.3)

The function $\underline{\ell}$ defined by (3.1) and (3.2) above is lower semi-continuous on $\underline{A}xF$.

Proof

Take any sequence $\{(x_i,t_i,\dot{x}_i)\} \subset \underline{A}xF$ converging to $(x,t,\dot{x})$. Assume that $\dot{x}_i \in F\backslash$co $f(x_i,t_i,\Omega)$ for all i or $\dot{x}_i \in$ co $f(x_i,t_i,\Omega)$ for all i since if this is not the case we can select two distinct subsequences where it is and deal with them separately.

(a) $\dot{x}_i \in F\backslash$co $f(x_i,t_i,\Omega)$. Let $\dot{y}_i$ be defined by (3.2). By Hausdorff continuity of co $f(x,t,\Omega)$ on $\underline{A}$, $\{\dot{y}_i\}$ converges to $\dot{y}$ and $\dot{y} \in$ co $f(x,t,\Omega)$ is the minimand of dist $(\dot{x},\text{co } f(x,t,\Omega))$. This case is therefore reduced to that of (b) below. Note that $\dot{x} = \dot{y}$ is not excluded here.

(b) $\dot{x}_i \in$ co $f(x_i,t_i,\Omega)$. Again by Hausdorff continuity of co $f(x,t,\Omega)$ we must have $\dot{x} \in$ co $f(x,t,\Omega)$. By remark (i) one can select u_i as interpreted there such that $\underline{\ell}(x_i,t_i,\dot{x}_i) = \ell(x_i,t_i,u_i)$, for every i. The probability measures μ_i on Ω representing u_i lie in the unit ball of $C^*(\Omega)$, which is w^* compact (Alaoglu's theorem) therefore we can extract at least one w^* convergent subsequence. Let $\{u_k\}$ be any such sequence, $u_k \overset{w^*}{\to} u$. Then $f(x_k,t_k,u_k) \to \dot{x}$ and $f(x_k,t_k,u_k) \to f(x,t,u)$, that is $\dot{x} = f(x,t,u)$. Thus

$$\underline{\ell}(x,t,\dot{x}) \leq \ell(x,t,u) = \lim_{k\to\infty} \ell(x_k,t_k,u_k)$$

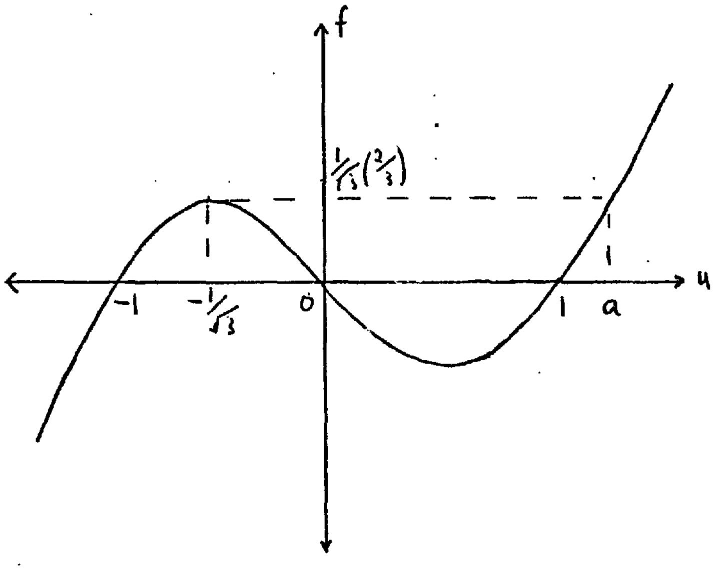
This applies equally to all subsequences, so that

$$\underline{\ell}(x,t,\dot{x}) \leq \liminf_{i\to\infty} \ell(x_i,t_i,u_i) = \liminf_{i\to\infty} \underline{\ell}(x_i,t_i,\dot{x}_i)$$

□

A simple example shows that $\underline{\ell}$ need not be continuous.

$f(x,t,u) = u^3 - u \qquad \ell(x,t,u) = u$



If $\dot{x}_i$ is any sequence converging to $\frac{1}{\sqrt{3}}\left(\frac{2}{3}\right)$ from above,

$\liminf_i \ell(x_i,t_i,\dot{x}_i) = a \approx 1,15$ but

$$\underline{\ell}\left(x,t,\frac{1}{\sqrt{3}}\left(\frac{2}{3}\right)\right) = -\frac{1}{\sqrt{3}} \approx -0,58$$

$\underline{\ell}$ is defined over $\underline{A}$xF and the non parametric "Weak Variational" problem is posed over Radon measures on $\underline{A}$xF.

$$
(WV)\quad \begin{cases}
\text{minimize} \int_{\underline{A}xF} \underline{\ell}(x,t,\dot{x})\,d\mu & \text{(WV1)} \quad \text{subject to} \\[2ex]
\mu \in P^{\oplus}(\underline{A}xF),\ ||\mu|| \leq t_1 - t_0 & \text{(WV2)} \\[2ex]
\int_{\underline{A}xF} \phi_t(x,t) + \phi_x(x,t)\dot{x}\,d\mu = \Delta\phi \text{ for all } \phi \in C^1(\underline{A}) & \\
 & \text{(WV3)} \\[2ex]
\int_{\underline{A}xF} \text{dist } (\dot{x}, f(x,t,\Omega))\,d\mu = 0 & \text{(WV4)}
\end{cases}
$$

Lower semi-continuity of $\underline{\ell}$ implies that (WV) is well-posed, since any lower semi-continuous function is the pointwise limit of a monotone non-decreasing sequence of continuous functions, $\{\ell^i\}$ say. For any admissible μ these are μ-integrable and the boundedness of $\underline{\ell}$ enables us to bound $\{\ell^i\}$. Lebesgue's dominated convergence theorem implies $\underline{\ell}$ is μ-integrable and

$$\int_{\underline{A}xF} \underline{\ell}\,d\mu = \lim_{i\to\infty} \int_{\underline{A}xF} \ell^i\,d\mu$$

for all admissible μ.

Denoting this set of admissible measures satisfying (WV2), (WV3) and (WV4) by $\mathcal{WV}$ we see that the admissible elements $\mathcal{W}$ for (W) can be 'embedded' in $\mathcal{WV}$.

<u>Proposition (3.4)</u>

If $\mu \in \mathcal{W}$ then μ defines an element $\bar{\mu} \in \mathcal{WV}$ such that

---

$$\int_{\underline{A}xF} \underline{\ell}\,d\bar{\mu} \leq \int_{\underline{A}x\Omega} \ell\,d\mu$$

Hence $\eta(WV) \leq \eta(W)$.

Proof

Let $\mu \in \mathcal{W}$, $g \in C(\underline{A}xF)$. Define $\int_{\underline{A}xF} g d\bar{\mu}$ by:

$$\int_{\underline{A}xF} g(x,t,\dot{x})d\bar{\mu}(x,t,\dot{x}) = \int_{\underline{A}x\Omega} g(x,t,f(x,t,u))d\mu(x,t,u)$$

$g(x,t,f(x,t,u))$ is continuous therefore $\bar{\mu}$ is well defined and (W2) $\Rightarrow$ (WV2), (W3) $\Rightarrow$ (WV3) while (WV4) is obvious. By definition of $\underline{\ell}$,

$$\int_{\underline{A}xF} \underline{\ell}(x,t,\dot{x})d\bar{\mu}(x,t,\dot{x}) = \int_{\underline{A}x\Omega} \underline{\ell}(x,t,f(x,t,u))d\mu(x,t,u)$$

$$\leq \int_{\underline{A}x\Omega} \ell(x,t,u)d\mu(x,t,u) \qquad \square$$

Remarks

(i) $\mathcal{WV}$ is non-empty.

(ii) (WV4) is the 'side-constraint' corresponding loosely to the differential equation constraint in the strong optimal control problem.

(iii) Functions $\underline{\ell}$ to replace ℓ have been defined in various ways by different authors, a common one being (3.1) with $\underline{\ell}(x,t,\dot{x}) = +\infty$ if $\dot{x} \in F\backslash \text{co } f(x,t,\Omega)$. For this function (WV4) would be omitted because it is implicit in the requirement $\int_{\underline{A}xF} \underline{\ell} d\mu < \infty$. Contrary to the present trend in optimization the explicit constraint formulation is more useful here.

§3.2 Parametrization

An absolutely continuous change of variable $t \to \sigma$, namely one such that $\dot{t}(\sigma) = \frac{dt(\sigma)}{d\sigma}$ is integrable, in the strong control problem necessitates the following replacements: $\dot{x}(t) \to \frac{\dot{x}(\sigma)}{\dot{t}(\sigma)}$ and for $g \in C(\underline{A}xF)$

$$\int_{t_0}^{t_1} g(x(t),t,\dot{x}(t))dt \to \int_{\sigma(t_0)}^{\sigma(t_1)} g(x(\sigma),t(\sigma),\frac{\dot{x}(\sigma)}{\dot{t}(\sigma)})\,|\dot{t}(\sigma)|\,d\sigma$$

A convenient parameter is $\sigma(t) = \int_{t_0}^{t} (|\dot{x}(\tau)|^2+1)^{\frac{1}{2}} d\tau$, the arc length along the curve $(x(t),t)$.

Note:

(i) In the above the dot "." denotes differentiation with respect to the argument of the function over which it appears. When no argument is given $\dot{q}$ is a point in the same space as q, viz R^n when $q = x$, R when $q = t$.

(ii) In the sequel points in $\underline{A} \subset R^{n+1}$ will be denoted interchangably by (x,t) and y as will points $\dot{y} = (\dot{x},\dot{t})'$ in (another copy of) R^{n+1}.

Mimicry of these replacements in the weak variational problem produces the parametric problem. Firstly let us define on $\underline{A}xR^{n+1}$

$$L(y,\dot{y}) = \begin{cases} \underline{\ell}(x,t,\frac{\dot{x}}{\dot{t}})\,|\dot{t}| & \dot{t} \neq 0 \\ 0 & \dot{t} = 0 \end{cases}$$

$$M(y,\dot{y}) = \max[-\dot{t},0] \tag{3.5}$$

$$D(y,\dot{y}) = \begin{cases} \text{dist }(\frac{\dot{x}}{\dot{t}},f(x,t,\Omega))\,|\dot{t}| & \dot{t} \neq 0 \\ ||\dot{x}|| & \dot{t} = 0 \end{cases}$$

We notice that L, M, D and $\phi_y\dot{y} = \phi_t(x,t)\dot{t} + \phi_x(x,t)\dot{x}$, $\phi \in C^1(\underline{A})$ are all homogeneous in $\dot{y}$: for any $\alpha \geq 0$, $L(y,\alpha\dot{y}) = \alpha L(y,\dot{y})$ etc. and their values on $\underline{A}xR^{n+1}$ are captured in $\underline{A}x\underline{B}$, $\underline{B}$ the unit sphere $\{\dot{y}:||\dot{y}||=1\}$ of R^{n+1}.

Lemma (3.6)

L, M and D are well defined in the sense that they are single valued functions and their integrands w.r.t. any finite Radon measure μ on $\underline{AxB}$ exist and are finite.

Proof

See Appendix 1. □

The parametric problem is therefore well posed:

$$
(P)\quad\begin{cases}
\text{minimize } \int_{\underline{A}\times\underline{B}} L(y,\dot{y})\,d\mu \text{ over } \mu \in P^{\oplus}(\underline{A}\times\underline{B}) & \text{(P1) subject to} \\[2ex]
\int_{\underline{A}\times\underline{B}} \phi_y(y)\dot{y}\,d\mu = \Delta\phi \text{ for all } \phi \in C^1(\underline{A}) & \text{(P2)} \\[2ex]
\int_{\underline{A}\times\underline{B}} D(y,\dot{y})\,d\mu = 0 & \text{(P3)} \\[2ex]
\int_{\underline{A}\times\underline{B}} M(y,\dot{y})\,d\mu = 0 & \text{(P4)}
\end{cases}
$$

The set of measures admissible for (P) is denoted by $\mathcal{P}$.

<u>Proposition (3.7)</u>

If $\mu \in \mathcal{WV}$ then μ defines an element $\bar{\mu} \in \mathcal{P}$ and

$$\int_{\underline{A}\times\underline{B}} L(y,\dot{y})\,d\bar{\mu} = \int_{\underline{A}\times F} \ell(x,t,\dot{x})\,d\mu$$

Consequently $\eta(P) \leq \eta(WV)$.

<u>Proof</u>

Take $\mu \in \mathcal{WV}$, $G(y,\dot{y}) = g(x,t,\dot{x},\dot{t}) \in C(\underline{A}\times\underline{B})$. Define

$$\int_{\underline{A}\times\underline{B}} G(y,\dot{y})\,d\bar{\mu} = \int_{\underline{A}\times F} g\left(x,t,\frac{\dot{x}}{(|\dot{x}|^2+1)^{\frac{1}{2}}},\frac{1}{(|\dot{x}|^2+1)^{\frac{1}{2}}}\right)(|\dot{x}|^2+1)^{\frac{1}{2}}\,d\mu$$

Then (WV3) $\Rightarrow$ (P2) and (WV4) $\Rightarrow$ (P3) for:

$$\int_{\underline{A}\times\underline{B}} \phi_y \dot{y}\,d\bar{\mu} = \int_{\underline{A}\times\underline{B}} \phi_t \dot{t} + \phi_x \dot{x}\,d\bar{\mu} = \int_{\underline{A}\times F} \phi_t . 1 + \phi_x \dot{x}\,d\mu = \Delta\phi \qquad \text{(P2)}$$

Similarly for (P3). Furthermore

$$\int_{\underline{A}\times\underline{B}} M(y,\dot{y})\,d\bar{\mu} = \int_{\underline{A}\times F} \max\left[-\frac{1}{(|\dot{x}|^2+1)^{\frac{1}{2}}},0\right](|\dot{x}|^2+1)^{\frac{1}{2}}d\mu = \int_{\underline{A}\times F} 0\,d\mu = 0 \tag{P4}$$

and

$$\int_{\underline{A}\times\underline{B}} L(y,\dot{y})\,d\bar{\mu} = \int_{\underline{A}\times\underline{B}} \underline{\ell}(x,t,\frac{\dot{x}}{\dot{t}})\,|\dot{t}|\,d\bar{\mu} = \int_{\underline{A}\times F} \underline{\ell}(x,t,\dot{x})\,d\mu$$

□

We have therefore established $\eta(P) \leq \eta(WV) \leq \eta(W) \leq \eta(S)$. Theorem (2.1) will be proved by showing that $\eta(P) = \eta(S)$.

Chapter 4

SOLUTION OF THE PARAMETRIC PROBLEM

§4.1 Properties of the Admissible Set P

The support of any measure μ on a topological space X is defined as

Defn. 4.1

$$\text{supp}\{\mu\} = \mathcal{C}\{V\subset X : V \text{ open and } \mu(U)=0,\ U \text{ open} \Rightarrow U\subset V\}$$

i.e. the support of a measure is the complement of the largest open μ-negligable set in X. Evidently it is closed, which in our case where $X = \underline{A}\times\underline{B}$ is compact means that every $\mu \in P$ has compact support.

An interesting example of a non-trivial measure on a compact space, which does not have a support, i.e. the support is trivial, is to be found in [Sch1 p. 45]. However, it is well known [Fre. Ch. 7] that a Radon measure is specified by its values on open sets (or closed or compact sets) in the topology of X. Consequently a non-trivial Radon measure always has a non-trivial support. (The measure in the above mentioned example is Borel, it is specified only when its value on each Borel set is known.)

Using the constraints (P2), (P3) and (P4) we shall now prove three results, the first concerning a restriction on the supports of measures in P, the second on their norms, the last on the weak * closedness of P.

Lemma (4.2)

For all $\mu \in P$, $\text{supp}\{\mu\} \subset \{(y,\dot{y})\in\underline{A}\times\underline{B} : \dot{t}\geq 0, \dot{x}\in\dot{t}f(x,t,\Omega)\}$

Proof

See Appendix 2. □

Proposition (4.3)

There exists $K < \infty$ such that $||\mu|| \leq K$ for all $\mu \in P$.

Proof

Take any $\mu \in \mathcal{P}$ then $\mu \in P^{\oplus}(\underline{A}\times\underline{B})$ implies $||\mu|| = \int_{\underline{A}\times\underline{B}} 1 d\mu$. From (4.2) :

$$\int_{\underline{A}\times\underline{B}} |\dot{t}|d\mu = \int_{\underline{A}\times\underline{B}} \dot{t}d\mu = \int_{\underline{A}\times\underline{B}} \phi_t \dot{t} + \phi_x \dot{x} d\mu \quad \text{for } \phi(x,t) = t$$

therefore

$$\int_{\underline{A}\times\underline{B}} |\dot{t}|d\mu = \Delta\phi = t_1 - t_0 \quad \text{by (P2)}$$

Further from (4.2) $\text{supp}\{\mu\} \subset \{(y,\dot{y}) \in \underline{A}\times\underline{B} : ||\dot{x}|| \leq k|\dot{t}|\}$ where $k = \max_{\underline{A}\times\Omega} ||f(x,t,u)|| < \infty$. Therefore

$$||\mu|| = \int_{\underline{A}\times\underline{B}} 1 d\mu = \int_{\underline{A}\times\underline{B}} (||\dot{x}||^2 + |\dot{t}|^2)^{\frac{1}{2}} d\mu \,,\quad (||\dot{x}||^2 + |\dot{t}|^2)^{\frac{1}{2}} = 1 \text{ in } \underline{B}.$$

$$\leq \int_{\underline{A}\times\underline{B}} ||\dot{x}|| + |\dot{t}| d\mu \leq \int_{\underline{A}\times\underline{B}} (k+1)|\dot{t}| d\mu = (k+1)(t_1 - t_0)$$

Putting $K = (k+1)(t_1 - t_0)$ gives the result. □

Proposition (4.4)

$\mathcal{P}$ is weak-star closed.

Proof

Let $\{\mu_i\} \subset \mathcal{P}$, $\mu_i \xrightarrow{w^*} \mu$. In the proof of lemma (3.6) in Appendix 1 it is shown that $D(y,\dot{y}) \in C(\underline{A}\times\underline{B})$, similarly $M(y,\dot{y})$ and $\phi_y \dot{y}$ for any $\phi \in C^1(\underline{A})$. So $\int_{\underline{A}\times\underline{B}} D(y,\dot{y}) d\mu = \lim_{i\to\infty} \int_{\underline{A}\times\underline{B}} D(y,\dot{y}) d\mu_i = 0$ and $\int_{\underline{A}\times\underline{B}} M d\mu = 0$, $\int_{\underline{A}\times\underline{B}} {}_y \dot{y} d\mu = \Delta\phi$; therefore $\mu \in \mathcal{P}$. Thus $\mathcal{P}$ is sequentially w* closed - since $C(\underline{A}\times\underline{B})$ is separable , $\mathcal{P}$ is w* closed . □

We have shown that $\mathcal{P}$ is norm-bounded and weak-star closed. By Alaoglu's theorem , then :

Theorem (4.5)

$\mathcal{P}$ is weak * compact. □

Remark

The 'embedding' of $\mathcal{S}$ in $\mathcal{W}$, $\mathcal{W}$ in $\mathcal{WV}$ and $\mathcal{WV}$ in $\mathcal{P}$ and the hypothesis H on $\mathcal{S}$ imply that $\mathcal{P}$ is non-empty.

§4.2 The Existence of a Solution to (P)

We now know that we are seeking the minimum of $\int_{\underline{A}\times\underline{B}} L d\mu$ over a weak * compact set of measures μ. A solution to this problem therefore exists if the map $\mu \to \int_{\underline{A}\times\underline{B}} L d\mu$ can be shown to be weak * lower semi-continuous on $\mathcal{P}$. This is shown to be a consequence of the lower semi-continuity and boundedness of L.

Theorem (4.6)

Let $\{\mu_n\} \subset \mathcal{P}$, $\mu_n \xrightarrow{w^*} \mu$. For any F lower semi-continuous and bounded on $\underline{A}\times\underline{B}$, $\int_{\underline{A}\times\underline{B}} F d\mu \leq \liminf_{n \to \infty} \int_{\underline{A}\times\underline{B}} F d\mu_n$.

Proof

The proof follows directly from lemma (A2.2) in Appendix 2. □

Theorem (4.7)

The parametric problem has a solution; there exists $\mu_0 \in \mathcal{P}$ such that $\int_{\underline{A}\times\underline{B}} L d\mu_0 \leq \int_{\underline{A}\times\underline{B}} L d\nu$ for all $\nu \in \mathcal{P}$.

Proof

A lower semi-continuous function on a compact subset of a topological space achieves its minimum there [Sch.1 Ch. 4]. In the weak * topology on $C^*(\underline{A}\times\underline{B})$, $\mu \to \int_{\underline{A}\times\underline{B}} L d\mu$ is l.s.c. by (4.6) and $\mathcal{P}$ is compact (4.5). □

Remarks

(a) It is established in corollary (A2.6) that $\mu \to \int_{\underline{A}\times\underline{B}} d\mu$ is l.s.c. even in the case where L is not bounded. This has been done to show that the alternative definition of L yields existence just as above.

(b) Proofs of lower semi-continuity are an important part of existence theory for optimal controls. Their increasingly dominant role can be seen in [JOTA].

§ 4.3 A Generalized Curve Solution to (P)

What does μ_0 look like? In the light of Young's work on the solutions of parametric variational problems without side constraints and the (P3) and (P4) of (P), we expect to be able to select a generalized curve which does the job. Our expectations are fulfilled by proposition (4.13). A brief explanation of the ideas and notation in [You. 1], statements (and in one case an extended proof) of relevant theorems and a translation into our notation are to be found in Appendix 3.

Firstly, a definition of and representation theorem for generalized curves in R^{n+1}.

Definition (4.8)

A generalized curve is a weak-star limit of a sequence of ordinary curves. 'Ordinary curves' may here be taken to mean piece-wise differentiable R^{n+1} valued functions.

Representation Theorem (4.9)

An element $\mu \in P^{\oplus}(\underline{A}x\underline{B})$ is a generalized curve iff there exists a Lipshitz continuous function $y(\sigma)$; $0 \le \sigma \le 1$ taking values in $\underline{A}$ and a collection $\{\mu_\sigma \in P^{\oplus}(\underline{B})\}$ defined for almost all σ with $||\mu_\sigma||$ uniformly bounded satisfying (i) for every $h \in C([0,1]x\underline{B})$, $\sigma \to \int_{\underline{B}} h(\sigma,\dot{y})d\mu_\sigma(\dot{y})$ is (Lebesgue) measurable (ii) $\int_{\underline{A}x\underline{B}} G d\mu = \int_0^1 \int_{\underline{B}} G(y(\sigma),\dot{y})d\mu_\sigma(\dot{y})d\sigma$ for all $G \in C(\underline{A}x\underline{B})$ and (iii) $\dot{y}(\sigma) = \int_B \dot{y} d\mu_\sigma(\dot{y})$ ae $\sigma \in [0,1]$.

The proof of this is to be found in [You 1 , pp 171-178] □

<u>First Approximation Theorem for $\mathcal{P}$ (4.10)</u>

Suppose $\mu \in P^{\oplus}(\underline{A}x\underline{B})$ satisfies $\int_{\underline{A}x\underline{B}} \phi_y \dot{y} d\mu = \phi(y_1) - \phi(y_0)\ \forall\ \phi \in C^1(\underline{A})$, then μ is the weak * limit of sequence $\{\mu_i\}$ in $P^{\oplus}(\underline{A}x\underline{B})$ where

$$\mu_i = \sum_{j=1}^{n_i} \alpha_j^i \mu_j^i + \sum_{j=1}^{m_i} \beta_j^i \nu_j^i \quad \text{with} \quad \sum_{j=1}^{n_i} \alpha_j^i = 1,\ \alpha_j^i,\ \beta_j^i > 0,$$

and each μ_j^i has representation $G \to \int_0^1 G(y(\sigma), \frac{\dot{y}(\sigma)}{|y(\sigma)|}) |\dot{y}(\sigma)| d\sigma\ \forall\ G \in C(\underline{A}x\underline{B})$ (4.11)

for some continuous, piece-wise linear function $y(\sigma)$ $0 \le \sigma \le 1$ such that $y(0) = y_0$, $y(1) = y_1$. Each ν_j^i has representation (4.11) with a similar $y(\cdot)$ but with $y(0) = y(1)$.

<u>Proof</u>

See Appendix 3. □

<u>Second Approximation Theorem for $\mathcal{P}$ (4.12)</u>

If in the above we put $\rho_i \triangleq \sum_{j=1}^{m_i} \beta_j^i \nu_j^i$ then $||\rho_i|| \to 0$.

<u>Proof</u>

Let $\mu \in \mathcal{P}$ as in (4.10) and take V open in $\underline{A}x\underline{B}$, $V \supset \text{supp}\ \mu$, $\text{dist}(V, \text{supp}\ \mu) < \varepsilon$ for some $\varepsilon > 0$. Compactness of $\text{supp}\{\mu\}$ ensures the existence of V. Putting $U = \underline{A}x\underline{B} \backslash V$ corollary (A2.4) of Appendix 2 leads to $\mu_i(U) < \varepsilon$ for i sufficiently large. Positivity of ρ_i implies $\rho_i(U) < \varepsilon$.

$\text{supp}\{\mu\} \subset \{(y,\dot{y}) : \dot{t} \geq 0, ||\dot{x}|| \leq k|\dot{t}|\}$ (Lemma (4.2)) and $V \subset N(\text{supp}\{\mu\}, \varepsilon)$ gives $V \subset \{(y,\dot{y}) : |\dot{t}| < \dot{t} + 2\varepsilon, ||\dot{x}|| < k|\dot{t}| + (k+1)\varepsilon\}$. Therefore

$$\int_{\underline{A}x\underline{B}} |\dot{t}| d\rho_i = \int_V |\dot{t}| d\rho_i + \int_U |\dot{t}| d\rho_i$$

$$< \int_V \dot{t} + 2\varepsilon d\rho_i + \int_U |\dot{t}| d\rho_i$$

$$\leq \int_{\underline{A}x\underline{B}} \dot{t} d\rho_i + \int_U |\dot{t}| - \dot{t} d\rho_i + 2\varepsilon \int_{\underline{A}x\underline{B}} 1 d\rho_i$$

$$< 0 + 2\varepsilon + 2\varepsilon||\rho_i||$$

because ρ_i is closed, $0 \leq |\dot{t}| - \dot{t} \lesssim 2$.

Similarly

$$\int_{\underline{A}\times\underline{B}} ||\dot{x}||d\rho_i = \int_V ||\dot{x}||d\rho_i + \int_U ||\dot{x}||d\rho_i$$

$$< \int_V k|\dot{t}|d\rho_i + \int_V (k+1)\varepsilon d\rho_i + \varepsilon \qquad \cdot||\dot{x}|| \leq 1$$

$$< 2k\varepsilon(1+||\rho_i||) + (k+1)\varepsilon||\rho_i|| + \varepsilon$$

But $||\rho_i|| \leq ||\mu_i|| \to ||\mu|| \leq K$ so that:

$$||\rho_i|| = \int_{\underline{A}\times\underline{B}} 1d\rho_i = \int_{\underline{A}\times\underline{B}} (||\dot{x}||^2+|\dot{t}|^2)d\rho_i \leq \int_{\underline{A}\times\underline{B}} ||\dot{x}|| + |\dot{t}|d\rho_i$$

$$< (2(k+1)(1+K) + (k+1)K+1)\varepsilon \qquad \square$$

Out intuition confirms this: the constraint $\int M d\mu = 0$ was designed to contain the support of μ in $\{\dot{t} \geq 0\}$. A non-trivial closed curve cannot be so contained - P consists of weak star limits of structurally simple measures.

The above development is based upon the relatively straightforward theory of real valued Radon measures on a compact subset of a finite dimensional space. Regrettably, at present it appears necessary to invoke a powerful theorem based upon vector valued measures on infinite dimensional spaces, to present our conclusions in their fullest generality.

Theorem (4.13)

Every member of P is a mixture of bounded generalized curves; that is each $\mu \in P$ can be represented as $\int \mu d\Lambda(\mu)$ where Λ is a positive vector valued Radon measure on the space of generalized curves, with support in the subset of bounded curves.

Proof

This appears as 'every generalized flow with a simplicial boundary is a mixture of bounded streams (generalized curves)' in [You 1, Thm(89.1) (ii) pp. 209-12]. The relationship of the terms in the two statements is explained in Appendix 3. □

Because (4.13) has been presented out of context an alternative proof valid where L is continuous and not dependent upon (4.13) will also be presented.

Proposition (4.14)

The parametric problem (P) admits a generalized curve solution.

Proof

Let μ_0 be the optimal measure guaranteed by (4.7), with representation $\mu_0 = \int \mu d\Lambda(\mu)$ as above, with supp$\{\Lambda\}$ denoted by λ. λ consists of generalized curves $\bar{\mu}$, Λ almost all of which satisfy: $\int D d\bar{\mu} = \int M d\bar{\mu} = 0$. For, supposing the contrary, there exists a set of curves $\lambda_1 \subset \lambda$, $\int D d\bar{\mu} \geq c > 0$ for $\bar{\mu} \in \lambda_1$, $\int_{\lambda_1} 1 d\Lambda(\bar{\mu}) > 0$ which leads to

$\int D d\mu_0 \geq \int_{\lambda_1}\int D d\bar{\mu} d\Lambda(\bar{\mu}) \geq \int_{\lambda_1} c.1 d\Lambda(\bar{\mu}) > 0$, a contradiction. {1 is the unit constant function on the set of gen. curves.}

Further, Λ almost all curves in λ have endpoints y_0, y_1. Again, taking λ_1 of positive Λ measure to be a set of curves in λ violating this at y_1, find $\phi \in C^1(\underline{A})$ with $\phi = 0$ outside an arbitrarily small ε neighbourhood of y_1, $\phi = 1$ at y_1 then:

$$1 = \Delta\phi = \int \phi_y \dot{y} d\mu_0 = \int\int \phi_y \dot{y} d\mu d\Lambda(\mu)$$

$$= \int_{\lambda_1} \int \phi_y \dot{y} d\mu d\Lambda(\mu) + \int_{\lambda-\lambda_1} \int \phi_y \dot{y} d\mu d\Lambda(\mu)$$

$$= \int_{\lambda_1} 0 d\Lambda(\mu) + \int_{\lambda-\lambda_1} 1 d\Lambda(\mu) = \Lambda(\lambda-\lambda_1) < 1$$

Λ being a unit mixture (measure) is derived from its being a limit of the sequence of unit measures $\sum_{j=1}^{n_i} \alpha_j^i$. (This last result is where (4.13) is most useful for if λ contained elements other than generalized curves we could not assert that all elements of λ have endpoints.)

Lastly, using a similar contradiction, it is evident that the set of measures $\lambda_2 = \{\bar{\mu}\epsilon\lambda: \int L d\bar{\mu} \leq \int L d\mu_0\}$ has positive Λ measure. Since Λ almost all elements in λ_2 belong to P, there exists at least one $\bar{\mu} \in P$, a generalized curve, such that $\int L d\bar{\mu} \leq \int L d\mu_0$. □

Proposition (4.15)

In the case L continuous, (P) admits a generalized curve solution.

Proof

Take μ_0 as before, $\{\mu_i\}$ an approximating sequence as in (4.10), (4.12): $\bar{\mu}_i \triangleq \mu_i - \rho_i = \sum_{i=1}^{n_i} \alpha_j^i \mu_j^i$. Positivity of $\bar{\mu}_i$, M and D mean that $\int D d\bar{\mu}_i \to 0$, $\int M d\bar{\mu}_i \to 0$, while L bounded, $||\rho_i|| \to 0$ implies

$\lim_i \int L d\bar{\mu}_i = \lim_i \int L d\mu_i = \int L d\mu_0$. (*)

The 4 vector $a_i \triangleq \begin{bmatrix} \int 1 d\bar{\mu}_i \\ \int L d\bar{\mu}_i \\ \int D d\bar{\mu}_i \\ \int M d\bar{\mu}_i \end{bmatrix} \in R^4$ is a convex combination $\sum_{j=1}^{n_i} \alpha_j^i b_j^i$

of vectors $b_j^i = \begin{bmatrix} \int 1 d\mu_j^i \\ \int L d\mu_j^i \\ \int D d\mu_j^i \\ \int M d\mu_j^i \end{bmatrix}$. Caratheodory's theorem tells us that [Roc 1, p. 153]

we can write $a_i = \sum_{j=1}^{5} \bar{\alpha}_j^i b_j^i$ where $\bar{\alpha}_j^i > 0$, $\sum_{j=1}^{5} \bar{\alpha}_j^i = 1$. Because of positivity

$\bar{\alpha}_j^i \int D d\mu_j^i \to 0$, $\bar{\alpha}_j^i \int M d\mu_j^i \to 0$ and, choosing a subsequence if necessary, it can be arranged that $\lim_i \bar{\alpha}_j^i = \bar{\alpha}_j$, $\lim_i \bar{\alpha}_j^i \int 1 d\mu_j^i$, $\lim_i \bar{\alpha}_j^i \int L d\mu_j^i$ exist for all $j = 1, \ldots 5$. Discarding terms $\bar{\alpha}_j = 0$, assume $\bar{\alpha}_j > 0$ then

(a) $\int D d\mu_j^i \to 0$ (b) $\int M d\mu_j^i \to 0$ and (c) each sequence $\{\mu_j^i\}$ is norm-bounded.

$$\text{So} \quad \int L d\mu_0 = \lim_i \int L d\mu_i = \lim_i \sum_{j=1}^{5} \bar{\alpha}_j^i \int L d\mu_j^i$$

$$= \sum_{j=1}^{5} \bar{\alpha}_j \lim_i \int L d\mu_j^i$$

$$\geq \sum_{j=1}^{5} \bar{\alpha}_j \lim_i \int L d\mu_k^i \quad \text{where } k \text{ minimizes } \lim_i \int L d\mu_j^i$$

$$= \lim_i \int L d\mu_k^i \quad \text{for} \quad \sum_{j=1}^{5} \bar{\alpha}_j = \lim_i \sum_{j=1}^{5} \alpha_j^i = 1$$

From (c), there exists a weak * convergent subsequence of μ_k^i with limit $\bar{\mu}$ say. By definition $\bar{\mu}$ is a generalized curve: $\bar{\mu} \in P$ for $\int D d\bar{\mu} = \int M d\bar{\mu} = 0$, $\int \phi_y \dot{y} d\bar{\mu} = \lim_i \int \phi_y \dot{y} d\mu_k^i = \Delta\phi$. Finally $\int L d\mu_0 \geq \int L d\bar{\mu}$. □

Comments

The proof of (4.15) may fail when L is only l.s.c. It is possible that $\liminf_i \int L d\bar{\mu}_i = \liminf_i \int L d\mu_i > \int L d\mu_0$ instead of (*). Ignoring this we could still select a generalized curve $\bar{\mu}$ as follows: proceed as above until (c) is reached, then choose a subsequence along which each $\{\mu_j^i\}$ $j = 1, \ldots 5$ is W* convergent, $\mu_j^i \xrightarrow{W^*} \bar{\mu}_j$. Choose $\bar{\mu}$ to minimize $L d\bar{\mu}_j$ over j.

$$\int L d\bar{\mu} \leq \sum_{j=1}^{5} \alpha_j \int L d\bar{\mu}_j \leq \liminf_i \int L d\bar{\mu}_i. \qquad (**)$$

We expect a strict inequality in (*) to imply an equal inequality in (**) but a proof has not been possible to date and it would appear that it can only be done using a result similar to (4.13).

Nevertheless, (4.15) covers a large number of interesting problems; indeed it may be argued that discontinuities in L are evidence of an ill-posed control problem. They are caused by there being identical trajectories generated by distinct controls with different associated costs, so stipulating that each trajectory have unique cost, though not necessarily control, eliminates the possibility.

Chapter 5

RECONSTRUCTION OF A SOLUTION TO THE OPTIMAL CONTROL PROBLEM

§5.1 Properties of a Generalized Curve Solving (P)

It has now been established that (P) has a solution of the following form: $\{y(\sigma),\mu_\sigma : 0\leq\sigma\leq 1\}$ (4.9) where:

(i) $y(\sigma)$ is a Lipshitz continuous R^{n+1} valued fn. of σ, $y(0) = y_0$ $y(1) = y_1$.

(ii) μ_σ is a uniformly bounded collection of positive Radon measures on $\underline{B}$, measurable in the sense of (4.9) (i) and (ii).

(iii) $\dot{y}(\sigma) = \int_{\underline{B}} \dot{y}\, d\mu_\sigma(\dot{y})$ a.e. in [0,1].

(iv) $\eta(P) = \int_0^1 \int_{\underline{B}} L(y(\sigma),\dot{y})\, d\mu_\sigma(\dot{y})\, d\sigma$.

(v) $\text{supp}\{\mu_\sigma\} \subset \{(\dot{x},\dot{t}) : \dot{t}\geq 0, \dot{x}\in\dot{t} f(x(\sigma),t(\sigma),\Omega)\}$ a.e. in [0,1].

The last property is a consequence of the identical support constraint on all $\mu \in \mathcal{P}$. Writing down the integrals in (P) will facilitate our development:

$$\int_0^1 \int_{\underline{B}} \mathcal{L}(x(\sigma),t(\sigma),\dot{x}/\dot{t})\,|\dot{t}|\, d\mu_\sigma(\dot{x},\dot{t})\, d\sigma \qquad \text{(P1)}$$

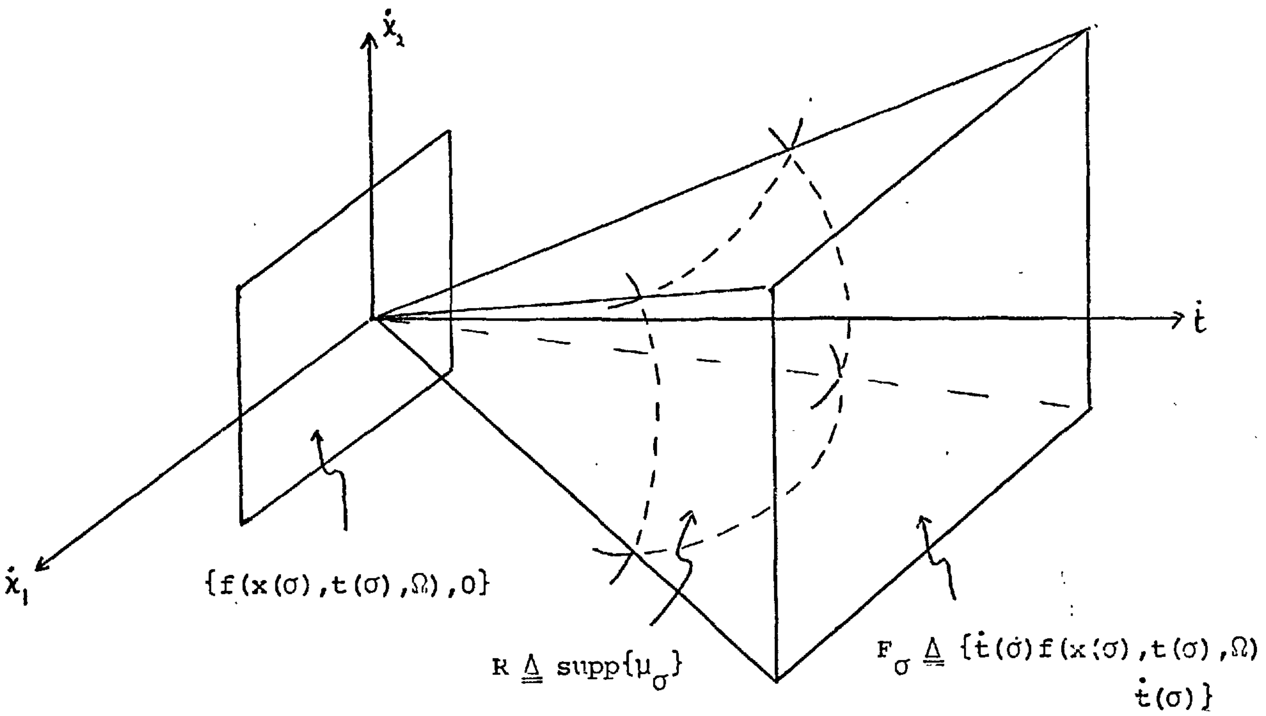
$$\int_0^1 \phi_x(x(\sigma),t(\sigma)) \int_{\underline{B}} \dot{x}\, d\mu_\sigma(\dot{x},\dot{t})\, d\sigma + \int_0^1 \phi_t(x(\sigma),t(\sigma)) \int_{\underline{B}} \dot{t}\, d\mu_\sigma(\dot{x},\dot{t})\, d\sigma \quad (=\Delta\phi) \qquad \text{(P2)}$$

$$\int_0^1 \int_{\underline{B}} \text{dist}(\dot{x}/\dot{t}, f(x(\sigma),t(\sigma),\Omega))\,|\dot{t}|\, d\mu_\sigma(\dot{x},\dot{t})\, d\sigma \quad (=0) \qquad \text{(P3)}$$

$$\int_0^1 \int_{\underline{B}} \max[-\dot{t},0]\, d\mu_\sigma(\dot{x},\dot{t})\, d\sigma \quad (=0) \qquad \text{(P4)}$$

The integrands are uniformly bounded in $\dot{y} = (\dot{x},\dot{t})$ so we may safely assume that (v) holds for $\sigma \in [0,1]$ and $\text{supp}\{\mu_\sigma\}$ is contained in a subset of $\underline{B}$ satisfying $\dot{t} = (|\dot{t}|^2)^{1/2} \geq (1/1+k^2)^{1/2} > 0$, k being the sup of $\|f(x,t,u)\|$ on $\underline{A}\times\Omega$. The difficulties associated with $\dot{t} = 0$ in the definitions of L and D are overcome, if $\dot{t}(\sigma) = \int_{\underline{B}} \dot{t}\,d\mu_\sigma = 0$, $\dot{\mu}_\sigma = \Theta$, the zero measure.

A more useful representation of the generalized curve will now be given, assisted by a picture for the case n+1 = 3



<u>A Homeomorphism from R to F_σ</u> (As defined in the above diagram)

For $(\dot{x},\dot{t}) \in R$ the ray ρ is defined as $\rho = \{(\dot{z},\dot{s}) \in \mathcal{R}^{n+1} : \dot{z}/\dot{s} = \dot{x}/\dot{t}\}$, $\rho \cap F_\sigma = \{(\dot{z},\dot{t}(\sigma)) : \dot{z} = \dot{t}(\sigma)\dot{x}/\dot{t}\} = \{(\dot{t}(\sigma)\dot{x}/\dot{t},\dot{t}(\sigma))\}$. Define $\rho : R \to F_\sigma$: $(\dot{x},\dot{t}) \to (\dot{t}(\sigma)/\dot{t})(\dot{x},\dot{t})$, then ρ is onto, one-to-one (since R is contained in a hemisphere of $\underline{\dot{B}}$) and continuous (because $k < \infty$). ρ is a homeomorphism from R to F_σ and has a continuous inverse $\rho^{-1} : F_\sigma \to R$. If $\mathcal{R}$ denotes the family of Borel sets on R, $\mathcal{F}$ the Borel sets on F_σ, then $\rho(\mathcal{R}) = \mathcal{F}$ and $\rho^{-1}(\mathcal{F}) = \mathcal{R}$. Furthermore, the space of Radon measures on $(F_\sigma,\mathcal{F})$ is exhausted by $\{\nu = \mu\rho^{-1} : \mu$ a Radon measure on $(R,\mathcal{R})\}$ [Sch. 1, p. 37].

Take $\mu_{\sigma N} \triangleq \begin{cases} \dot{t}/\dot{t}(\sigma)\mu_\sigma & \dot{t}(\sigma) > 0 \\ \Theta & \dot{t}(\sigma) = 0 \end{cases}$, the product of a measurable function and a Radon measure. $\mu_{\sigma N}$ is itself a positive Radon measure with support in R and when $\dot{t}(\sigma) > 0$, $||\mu_{\sigma N}|| = 1$ i.e. $\mu_{\sigma N}$ is a unit measure.

On F_σ let $\nu_\sigma \triangleq \mu_{\sigma N}\rho^{-1}$. For integrands H homogeneous in R^{n+1} the change of variable law [KT, §6.5] gives:

$$\int_{F_\sigma} H(\dot{y})d\nu_\sigma = \int_R H(\rho(\dot{y}))d\mu_{\sigma N} = \int_R H(\frac{\dot{t}(\sigma)}{\dot{t}}(\dot{y}))\dot{t}/\dot{t}(\sigma)d\mu_\sigma$$

i.e.

$$\int_{F_\sigma} H(\dot{y})d\nu_\sigma = \int_R H(\dot{y})d\mu_\sigma \tag{5.1}$$

Further

$$||\nu_\sigma|| = \int_{F_\sigma} 1d\nu_\sigma = \int_R 1d\mu_{\sigma N} = ||\mu_{\sigma N}|| = 1 \text{ when } \dot{t}(\sigma) > 0 \tag{5.2}$$

<u>Remarks</u>

(i) If $\mu_\sigma = \Theta$, $\nu_\sigma = \Theta$.

(ii) $\rho(R) = F_\sigma$, $\text{supp}\{\nu_\sigma\} = \text{supp}\{\mu_{\sigma N}\rho^{-1}\} \subset \rho\{\text{supp}\{\mu_{\sigma N}\}\}$

i.e. $\text{supp}\{\nu_\sigma\} \subset F_\sigma$ which is established by lemma A4.1.

The localization lemma [You 1 (88.5)] tells us that any Radon measure ν_σ on a compact space F_σ can be written down as a finite sum of its restrictions to a finite number of measurable subsets of the space, whose union is the whole space. Taking these subsets sufficiently small i.e. contained in small R^{n+1} balls, we see that ν_σ is the w* limit of a sequence of measures with support in a finite no. of points, $\sum_{j=1}^{n_i}\nu_\sigma^j$ say, where each ν_σ^j attaches weight $\nu_\sigma(U^j)$ to a point $\dot{y}_j \in U^j$, with U^j measurable and $\bigcup_j U^j = F_\sigma$. Therefore

$$\dot{x}(\sigma) = \int_{F_\sigma} \dot{x}d\nu_\sigma(\dot{x},\dot{t}) = \lim_i \sum_{j=1}^{n_i}\nu_\sigma^j\dot{x}_j$$

$$= \lim_i \sum_{j=1}^{n_i} \nu_\sigma^j[\dot{t}(\sigma)f(x(\sigma),t(\sigma),u_j)] \text{ for some } \{u_j\} \subset \Omega$$

$$= \dot{t}(\sigma) \lim_i \sum_{j=1}^{n_i}\nu_\sigma^j f(x(\sigma),t(\sigma),u_j)$$

But $\sum_{j=1}^{n} {}^{i}\nu_{\sigma}^{j} = ||\nu_{\sigma}|| = 1$. Therefore

$$\dot{x}(\sigma) \in \dot{t}(\sigma) \text{ co } f(x(\sigma),t(\sigma),\Omega) \tag{5.3}$$

The generalized curve solving (P) can be represented as $\{y(\sigma),\nu_{\sigma}:0\le\sigma\le 1\}$ where ν_{σ} is a unit measure with support in a hyperplane in R^{n+1} orthogonal to the $\dot{t}$ axis.

§5.2 A Non-Parametric Curve Solving the Weak Variational Problem (WV)

The function $t : [0,1] \to [t_0,t_1]$ is Lipshitz continuous and satisfies the conditions of the following change of variable lemma [You 1, pp. 180-181].

Lemma (5.4)

Let $t(\sigma)$ $0 \le \sigma \le 1$ be Lipshitzian ,nondecreasing, $t(0) = t_0$, $t(1) = t_1$ and let Σ_0 denote the set of σ where $\dot{t}(\sigma) = 0$. Then there exists a Borel set Σ, $m(\Sigma) = m([0,1]\backslash\Sigma_0)$ and a set $T \subset [t_0,t_1]$ such that:

(a) T is Borel, $m(T) = t_0 - t_1$, $t : \Sigma \to T$ is one-to-one.

(b) In Σ and T respectively, $\frac{dt}{d\sigma}$ and $\frac{d\sigma}{dt}$, the derivatives of $t(\sigma)$ and the inverse function $\sigma(t)$ exist and their product is unity.

(c) If $\phi(\sigma)$ is Borel measurable, bounded and vanishes in Σ_0 and if in T $\psi(t) \triangleq \phi(\sigma(t))\frac{d\sigma}{dt}$, then $\int_{t_0}^{t_1}\psi(t)dt = \int_0^1\phi(\sigma)d\sigma$. □

Now take the function $t \to \bar{x}(t)$ given by $\bar{x}(t) = x(\sigma(t))$ defined on $[t_0,t_1]$.

Lemma (5.5)

$\bar{x}(\cdot)$ is well defined, single valued and Lipshitz continuous on $[t_0,t_1]$.

Proof

$\sigma \to t(\sigma)$ is continuous on $[0,1]$ so assumes all values in $[t_0,t_1]$. Consequently $\sigma(t)$ is non empty for all t and $\bar{x}(t)$ is defined. Suppose $\sigma_2 > \sigma_1$ then $||x(\sigma_2)-x(\sigma_1)|| \leq \int_{\sigma_1}^{\sigma_2}||\dot{x}(\sigma)||d\sigma \leq k\int_{\sigma_1}^{\sigma_2}\dot{t}(\sigma)d\sigma = k(t(\sigma_2)-t(\sigma_1))$ i.e. if $t_1 = t(\sigma_1)$, $t_2 = t(\sigma_2)$ $||x(t_2)-x(t_1)|| \leq k(t_2-t_1)$. □

With Σ and T as in (5.4) define a family of mappings τ_σ, $\sigma \in \Sigma$

$$\tau_\sigma : R^n \times \{\dot{t}(\sigma)\} \to R^n : (\dot{x},\dot{t}(\sigma)) \to \frac{d\sigma}{dt}(\dot{x}) = \dot{x}/\dot{t}(\sigma) \tag{5.6}$$

For each $t \in T$ define a Radon measure on R^n by:

$$\bar{\mu}_t = \nu_{\sigma(t)}\tau^{-1}_{\sigma(t)} \tag{5.7}$$

while for $t \in T^c$ set $\bar{\mu}_t = \Theta$.

Proposition (5.8)

(a) $\text{supp}\{\bar{\mu}_t\} \subset f(\bar{x}(t),t,\Omega)$ for all $t \in T$. Otherwise $\text{supp}\{\bar{\mu}_t\} = \phi$.

(b) Let $h \in C(\underline{A}\times F)$ (F as in Ch. 3, not F_σ) then $H(t) \triangleq \int_F h(\bar{x}(t),t,\dot{x})d\bar{\mu}_t(\dot{x})$ is measurable.

(c) $\dot{\bar{x}}(t) = \int_F \dot{x}d\bar{\mu}_t$ a.e.

Proof

(a) By lemma A4.1, $\text{supp}\{\bar{\mu}_t\} \subset \tau_{\sigma(t)}(\text{supp}\{\nu_{\sigma(t)}\})$ for $t \in T$.

$\tau_{\sigma(t)}(\text{supp}\{\nu_{\sigma(t)}\}) \subset \tau_{\sigma(t)}[F_{\sigma(t)}] = f(x(\sigma(t)),t(\sigma(t)),\Omega)$ by definition.

When $t \in T_c$, $\bar{\mu}_t = \Theta$ so $\text{supp}\{\bar{\mu}_t\} = \phi$. □

(N.B. the purpose of §(5.1) was to make the mapping τ_σ as simple as possible.)

(b) Let M be any measurable set (in the σ-field of R^* the extended real numbers). We have to prove $H^{-1}(M) \in$ σ-field of $[t_0,t_1]$ i.e. measurable

$$H^{-1}(M) = \{t : H(t) \in M\} = \{t : \int h(\bar{x}(t), t, \dot{x})\, d\bar{\mu}_t(\dot{x}) \in M\}$$

$$= \{t \in T : \int h(\tau_{\sigma(t)})\, d\nu_{\sigma(t)} \in M\} \cup \{t \in T^c : 0 \in M\}$$

$$= t(\{\sigma \in \Sigma : \int h(x(\sigma), t(\sigma), \dot{x}/\dot{t}(\sigma))\, d\nu_\sigma \in M\}) \cup T_2$$

$$= t(\{\sigma \in \Sigma : 1/\dot{t}(\sigma) \int h(\cdot)\, \dot{t}(\sigma)\, d\nu_\sigma \in M\}) \cup T_2$$

$$= t(\{\sigma \in \Sigma : 1/\dot{t}(\sigma) \int h(x(\sigma), t(\sigma), \dot{x}/\dot{t})\, \dot{t}\, d\mu_\sigma \in M\}) \cup T_2$$

$$= t(\Sigma_1) \cup T_2 = T_1 \cup T_2$$

where

$$T_2 = \begin{cases} T^c & \text{if } 0 \in M \\ \phi & \text{otherwise} \end{cases}$$

Σ_1 is measurable because μ_σ is measurable in the sense of (4.9) (i) and (ii). Thus T_1 and T_2 are measurable, hence $H^{-1}(M)$. □

(c) For almost all $t \in T^c$, $\dot{\bar{x}}(t) = \dot{x}(\sigma(t)) = \dot{t}(\sigma(t)) = 0$. When $t \in T$, $\dot{\bar{x}}(t) = \dot{x}(\sigma(t)) = \dot{x}(\sigma)/\dot{t}(\sigma)\big|_{\sigma=\sigma(t)}$ and $\int_F \dot{x}\, d\bar{\mu}_t \triangleq \int_{F_{\sigma(t)}} \dot{x}/\dot{t}(\sigma)\, d\nu_{\sigma(t)} = \dot{x}(\sigma)/\dot{t}(\sigma)\big|_{\sigma=\sigma(t)}$. □

Note

In (b) and (c) the convention adopted in §(3.2) is adhered to.

We have arrived at a generalized curve $\{\bar{x}(t), \bar{\mu}_t : t_0 \le t \le t_1\}$ which by (a) and (c) above and because $\bar{\mu}_t$ is a unit measure a.e. in $[t_0, t_1]$, satisfies $\dot{\bar{x}}(t) \in \text{co } f(\bar{x}(t), t, \Omega)$ for almost every t. It remains to show that the cost functional evaluated along this curve yields $\eta(P)$.

Proposition (5.9)

$$\int_{t_0}^{t_1} \int_F \underline{\ell}(\bar{x}(t),t,\dot{x})d\bar{\mu}_t(\dot{x})dt = \eta(P).$$

Proof

Lemmas A4.2 and A4.3 show that the change of variable lemma (5.4) remains true when ϕ is Lebesgue measurable and not necessarily Borel measurable.

Let $\phi = \phi(\sigma) \triangleq \int_{\underline{B}} \underline{\ell}(x(\sigma),t(\sigma),\dot{x}/\dot{t})\dot{t}d\mu_\sigma(\dot{x},\dot{t})$ then

(i) Let $\underline{\ell}$ be the pointwise limit of the sequence $\{\ell^i\}$ as in chapter 3. For all $\sigma \in [0,1]$, $\phi(\sigma)$ is the (non-uniform) limit of $\phi^i(\sigma) \triangleq \int_{\underline{B}} \ell^i(x(\sigma),t(\sigma),\dot{x}/\dot{t})\dot{t}d\mu_\sigma(\dot{x},\dot{t})$ so by (4.9) (i) and the dominated convergence theorem, $\phi(\sigma)$ is Lebesgue measurable.

(ii) $\mu_\sigma = \Theta$ in $\Sigma_0 \Rightarrow \phi(\sigma) = 0$ in Σ_0.

ϕ satisfies the conditions of the lemma and in T

$$\psi(t) = \phi(\sigma(t))\frac{d\sigma}{dt} = \frac{d\sigma}{dt}[\int_{\underline{B}} \underline{\ell}(x(\sigma),t(\sigma)\dot{x}/\dot{t})\dot{t}d\mu_\sigma(\dot{x},\dot{t})]\Big|_{\sigma=\sigma(t)}$$

$$= \frac{d\sigma}{dt}[\int_{F_\sigma} \underline{\ell}(x(\sigma),t(\sigma),\dot{x}/\dot{t}(\sigma))\dot{t}(\sigma)d\nu_\sigma(\dot{x},\dot{t})]\Big|_{\sigma=\sigma(t)}$$

$$= \int_F \underline{\ell}(\bar{x}(t),t,\dot{x})d\bar{\mu}_t(\dot{x}) \qquad \text{a.e. in } [t_0,t_1]$$

The result follows by applying (5.4). □

We conclude from (5.8) and (5.9) that the generalized curve $@ \triangleq \{\bar{x}(t),\bar{\mu}(t):t_0 \leq t \leq t_1\}$ is admissible for the weak variational problem and $\int_{t_0}^{t_1} \int_F \underline{\ell}(\bar{x}(t),t,\dot{x})d\bar{\mu}_t(\dot{x})dt = \eta(P) \leq \eta(WV)$. @ is a solution to (WV) and we have shown that the weak-variational and parametric problems are equivalent.

§ 5.3 Completion of the Proof of Equivalence

Put $x_0(t) = \int_{t_0}^{t} \int_F \underline{\ell}(\bar{x}(s),s,\dot{x})d\bar{\mu}_s ds$ then

$$\dot{x}_0(t) = \int_F \underline{\ell}(\bar{x}(t),t,\dot{x})d\bar{\mu}_t \quad \text{a.e. in } [t_0,t_1]$$

Consider the augmented state variable $\tilde{x}(t) = \begin{bmatrix} x_0(t) \\ \bar{x}(t) \end{bmatrix}$ satisfying

$$\dot{\tilde{x}}(t) = \int_F g(\bar{x}(t),t,\dot{x})d\bar{\mu}_t \text{ a.e. in } [t_0,t_1] \tag{5.10}$$

where $g(x,t,\dot{x}) \triangleq \begin{bmatrix} \underline{\ell}(x,t,\dot{x}) \\ \dot{x} \end{bmatrix}$. Let T be a set of full measure in $[t_0,t_1]$ such that (5.10) holds for all $t \in T$.

Take $t \in T$ and suppose $\bar{\mu}_t$ has finite support in F, i.e. $\bar{\mu}_t = \sum_{i=1}^{N} \alpha_i \delta(\dot{x}_i)$ where $\sum_{i=1}^{N} \alpha_i = 1$ and $\dot{x}_i \in \text{supp}\{\bar{\mu}_t\} \subset f(\bar{x}(t),t,\Omega)$. We can therefore select $u_i \in \Omega$ such that $\dot{x}_i = f(\bar{x}(t),t,u_i)$ and $\underline{\ell}(\bar{x}(t),t,\dot{x}_i) = \ell(\bar{x}(t),t,u_i)$.

Setting $\tilde{f}(x,t,u) = \begin{bmatrix} \ell(x,t,u) \\ f(x,t,u) \end{bmatrix}$, $\tilde{f}$ is continuous and we find:

$$\dot{\tilde{x}}(t) = \sum_{i=1}^{N} \alpha_i g(\bar{x}(t),t,\dot{x}_i)$$

$$= \sum_{i=1}^{N} \alpha_i \tilde{f}(\bar{x}(t),t,u_i)$$

so

$$\dot{\tilde{x}}(t) \in \text{co } \tilde{f}(\bar{x}(t),t,\Omega) \tag{5.11}$$

Lemma (5.12)

Given any l.s.c. function $\underline{\ell}$ and a finite positive measure $\bar{\mu}_t$ with compact support, there exists a sequence of measures $\{\mu_i\}$ with finite support in $\text{supp}\{\bar{\mu}_t\}$ with:

$$\mu_i \xrightarrow{w^*} \bar{\mu}_t \text{ and } \int \ell d\mu_i \to \int \ell d\bar{\mu}_t$$

Proof

A slight modification of the standard construction suffices. Take a sequence of finite subcovers of $\text{supp}\{\bar{\mu}_t\}$ by balls of diminishing radius, $\{U_j^i\}_{j=1}^{N_i}$, $i = 1, 2, \ldots$ The collection of differences $A_j^i = U_j^i \setminus \cup_{k \neq j} U_k^i$ is a collection of Borel sets covering $\text{supp}\{\bar{\mu}_t\}$.

Let $\alpha_j^i = \bar{\mu}_t(A_j^i)$ and choose $\dot{x}_j^i \in \text{cl}A_j^i$, a minimand of $\underline{\ell}(\dot{x})$ over $\text{cl}A_j^i$. Such $\dot{x}_j^i$ exist because $\underline{\ell}$ is l.s.c. Define $\mu_i = \sum_{j=1}^{N_i} \alpha_j^i \delta(\dot{x}_j^i)$. Since A_j^i, $j = 1, \ldots N_i$ are disjoint and $\text{diam}(A_j^i) \to 0$ as $i \to \infty$,

$$\mu_i \xrightarrow{w^*} \bar{\mu}_t$$

Further:

$$\underline{\ell} d\mu_i = \sum_{j=1}^{N_i} \alpha_j^i \underline{\ell}(\dot{x}_j^i) = \sum_{j=1}^{N_i} \int_{A_j^i} \underline{\ell}(\dot{x}_j^i) d\bar{\mu}_t$$

$$\leq \sum_{j=1}^{N_i} \int_{A_j^i} \underline{\ell}(\dot{x}) d\bar{\mu}_t = \int \underline{\ell} d\bar{\mu}_t$$

By (A2.2) $\liminf_{i \to \infty} \int \underline{\ell} d\mu_i \geq \int \underline{\ell} d\bar{\mu}_t$ hence $\lim_{i \to \infty} \int \underline{\ell} d\mu_i$ exists and equals $\int \underline{\ell} d\bar{\mu}_t$. □

Note

(5.12) is not true in general when more than one l.s.c. function is involved.

For any $t \in T$ select $\{\mu_i\}$ according to (5.12) so that $\dot{\tilde{x}}(t) = \lim_i \dot{\tilde{x}}_i \triangleq \lim_i \int [\frac{\underline{\ell}(\bar{x}(t), t, \dot{x})}{\dot{x}}] d\mu_i(\dot{x})$. For each i, since μ_i has finite support:

$$\dot{\tilde{x}}_i \in \text{co } \tilde{f}(\bar{x}(t), t, \Omega)$$

Therefore

$$\dot{\tilde{x}}(t) \in \overline{co}\, \tilde{f}(\bar{x}(t),t,\Omega)$$

i.e.

$$\dot{\tilde{x}}(t) \in co\, \tilde{f}(\bar{x}(t),t,\Omega) \qquad (5.13)$$

since $\tilde{f}$ is continuous and Ω is compact.

Invoking Fillipov's lemma [You 1 p. 297] there exists a relaxed control $\bar{\nu}_t$ satisfying the differential equation

$$(\dot{x}_0(t),\dot{\bar{x}}(t)) = \left(\int_\Omega \ell(\bar{x}(t),t,u)\,d\bar{\nu}_t(u),\ \int_\Omega f(\bar{x}(t),t,u)\,d\bar{\nu}_t(u)\right)$$

$$\text{a.e. in } [t_0,t_1] \qquad (5.14)$$

The generalized curve $\{\bar{x}(t),\bar{\nu}_t : t_0 \leq t \leq t_1\}$ (still called @) satisfies the differential equation and endpoint constraints (S2) with relaxed control $u : t \to \bar{\nu}_t$, $\text{supp}\{\bar{\nu}_t\} \subset \Omega$, satisfying (S3). @ is admissible for the Strong Control Problem. By design:

$$\begin{aligned}\eta(S) &\leq \int_{t_0}^{t_1} \int_\Omega \ell(\bar{x}(t),t,u)\,d\bar{\nu}_t(u)\,dt \\ &= x_0(t_1) \\ &= \int_{t_0}^{t_1} \int_F \ell(\bar{x}(t),t,\dot{x})\,d\bar{\mu}_t(\dot{x})\,dt \\ &= \eta(P)\end{aligned}$$

@ solves (S) and the Strong Problem is equivalent to the Parametric one:

$$\eta(S) \geq \eta(W) \geq \eta(WV) \geq \eta(P) = \eta(S)$$

@ with suitable measure, solves every problem.

§5.4 Realities Behind the Equivalence

We began by regarding curves @ $\in \mathcal{S}$ as linear functionals on $C(\underline{A}\times\Omega)$ (our attention is restricted to @ with trajectories contained in $\underline{A}$) and then devised a set of Radon measures on $\underline{A}\times\Omega$, namely $\mathcal{W}$, which is convex. In the weak * topology on $C^*(\underline{A}\times\Omega)$, $\mathcal{W}$ is closed and $\overline{\overline{co}}\ \mathcal{S} \subset \mathcal{W}$ where $\overline{\overline{co}}$ denotes the closure of the convex hull. Replacing ℓ with any other $q \in C(\underline{A}\times\Omega)$ take $\mu \in \mathcal{W}$ and $\bar{\mu} \in \mathcal{P}$ the induced measure given in proposition (3.7). The structure theorems of chapters 4 and 5 apply equally to $\bar{\mu}$ as to μ_0, so that the machinery of the previous sections will turn out two elements $@_1$, $@_2 \in \mathcal{S}$ with $@_1 q \leq \mu q$ and $@_2 q \geq \mu q$. If there is equality in either of these, we may have $@_1 = @_2$, otherwise they are distinct.

N.B.

Two elements emerge because $\bar{\mu}$ is not necessarily the minimand of $\bar{\mu}Q$ over $\mathcal{P}$. (Q is the homogeneous integrand on $\underline{A}\times\underline{B}$ corresponding to q on $\underline{A}\times\Omega$.) Thus the single element $@_1$ produced exactly as in chapters 4 and 5 may satisfy $@_1 q < \mu q$ but then it is easy to see that there must be an $@_2 \in \mathcal{S}$ with $\mu q < @_2 q$.

Proposition (5.15)

For any $q \in C(\underline{A}\times\Omega)$ and $\mu \in \mathcal{W}$ there exist ν_1, $\nu_2 \in \overline{\overline{co}}\ \mathcal{S}$ such that $\nu_1 q \leq \mu q$ and $\nu_2 \geq \mu q$. □

Structure Theorem for the Weak Control Problem (5.16)

$$\overline{\mathrm{co}}(S) = \mathcal{W}$$

Proof

Since $\overline{\mathrm{co}}(S) \subset \mathcal{W}$ suppose $\mu \in \mathcal{W}$ but $\mu \notin \overline{\mathrm{co}}\ S$. $\{\mu\}$ is compact in $C^*(\underline{A}\times\Omega)$ and $\overline{\mathrm{co}}(S)$ is closed and convex. The very general separation theorem [RS, p. 130] states the existence of a separating hyperplane $q \in C(\underline{A}\times\Omega)$, the predual, $\mu q > 0$, $\nu q \leq 0$ for all $\nu \in \overline{\mathrm{co}}\ S$. This is absurd in view of (5.15) so $\mu \in \overline{\mathrm{co}}(S)$. □

The measures in $\mathcal{W}$ are just positive unit mixtures of generalized curves in S, with none more complex nor exciting.

Comments

(i) The route by which we have arrived at this is similar to that travelled by Young in deriving (4.10). While a more direct derivation would be welcome, it may not be possible.

(ii) In the light of theorem (4.13) which is virtually the above in parametric form, this structure may not appear so remarkable. However one should point out again that for a wide class of f (the r.h.s. of the differential equation) (4.13) is unnecessary, we get by with the more elementary (4.10).

We can now complete the proof of theorem (2.2). The extreme points of the convex set $\mathcal{W}$ are those points $\mu \in \mathcal{W}$ not contained on any segment $\{\alpha\mu_1+(1-\alpha)\mu_2 : 0\leq\alpha\leq 1\}$ defined by distinct $\mu_1, \mu_2 \in \mathcal{W}$. When $\mathcal{W}$ is closed such points always exist. Since S and $\mathcal{W} = \overline{\mathrm{co}}(S)$ are both weak-star compact in $C^*(\underline{A}\times\Omega)$, which with the corresponding weak topology is locally convex, applying the following theorem proves (2.2).

Theorem (5.17)

If A is a compact subset of a locally convex space , such that $\overline{\text{co}}(A)$ is compact , then each extreme point of $\overline{\text{co}}(A)$ is an element of A .

Proof

See [Shf , corollary 10.5] □

Thus : All extreme points of $\mathcal{W}$ are in $\mathcal{S}$. (2.2)

Notes

The extreme points of $\mathcal{W}$ are central to Rubio's approach to equivalence [Ru 1] . The minimum of $\int \ell d\mu$ over $\mathcal{W}$ can be taken at an extreme point . $\mathcal{W}$ is approximated by means of $\mathcal{W}_k$ where $\mathcal{W}_k$ involves k constraints of the form $\int \phi^i_t + \phi^i_x f \, d\mu = \Delta\phi^i \quad 1 \leq i \leq k$, where $\{\phi^i : i=1,2,\ldots\}$ is dense in $C^1(\underline{A})$. The extreme points of $\mathcal{W}_k$ are measures with support in a finite number $n_k \leq k+1$ of points in $\underline{A}\times\Omega$.

It is then to be shown that these points approximate a polygonal arc and that the sequence of arcs generated as k=1,2,... contains a subsequence convergent in some sense to a generalized curve solving the weak problem .

Chapter 6

GENERAL BOUNDARY CONDITIONS

The development so far has been for ordinary differential systems within a reasonably wide class, namely the dynamic and cost functions f and ℓ are required to be continuous. Largely to ease presentation the boundary conditions are given by two fixed points in R^{n+1}, (x_0,t_0) and (x_1,t_1). We shall now investigate the formulation of weak and parametric problems when the final time condition for the control problem, $x(t_1) = x_1$, is replaced by $(x(t_1),t_1) \in \Gamma$ where Γ is a closed subset of R^{n+1}. Proofs of equivalence are extended to this case, which includes problems such as minimum-time control.

A simultaneous relaxation of the initial condition is also considered.

§ 6.1 Posing a Convex Problem

We replace the problem (S) with the following (for convenience also labelled (S)):

$$
(S)\quad \left\{
\begin{array}{ll}
\text{minimize } \displaystyle\int_{t_0}^{t_1} \int_\Omega \ell(x(t),t,u)\,d\mu_t(u)\,dt & \text{(S1) subject to} \\[2ex]
\left.\begin{array}{l}
\dot{x}(t) = \displaystyle\int_\Omega f(x(t),t,u)\,d\mu_t(u) \text{ a.e. in } [t_0,t_1] \\[2ex]
(x_0,t_0) \text{ fixed, } (x(t_1),t_1) \in \Gamma
\end{array}\right\} & \text{(S2)} \\[2ex]
u : t \to \mu_t \text{ a relaxed control satisfying} & \text{(S3)}
\end{array}
\right.
$$

ℓ and f as before. Γ is the target, a closed subset of R^{n+1}; for definiteness take $(x_0,t_0) \notin \Gamma$, otherwise in case $\ell \geq 0$ the optimal solution is trivial. Assume at least one trajectory connects (x_0,t_0) and Γ.

Hypothesis H, of the existence of a bounded minimizing sequence of curves contained in $\underline{A}$ (now possibly more general than $A\times[t_0,t_1]$), a compact subset of R^{n+1}, permits us to restrict our attention to a compact subset of Γ (still labelled Γ) contained in the cone $(x_0,t_0) + P$, $P = \{(x,t) \in R^{n+1} : t \geq 0, \ ||x|| \leq kt\}$ where $k = \sup_{\underline{A}\times\Omega}||f(x,t,u)|| < \infty$. Points outside $(x_0,t_0) + P$ are not reachable, i.e. for all admissible trajectory-control pairs $(x(\cdot),u(\cdot)) \in S$ $(x(t),t) \in (x_0,t_0) + P$, for all $t \geq t_0$.

<u>Note</u>

Solving (S) implicitly involves determination of the final time t_1. The class of problems of form (S) includes minimum time problems, $\ell = 1$.

Encouraged by the achievements in the fixed end-point case, let us retrace the course of Chapter 2. Regarding an admissible element $(x(\cdot),u(\cdot)) \in S$ as an element $\mu \in C^*(\underline{A}\times\Omega)$ once again we see that

(a) $\quad \mu \in P^{\oplus}(\underline{A}\times\Omega)$

(b) $\quad ||\mu|| = \int_{t_1}^{t_0} 1dt = t_1 - t_0$

(c) $\quad$ for any $\phi \in C^1(\underline{A})$

$$\int_{\underline{A}\times\Omega} \phi_t(x,t) + \phi_x(x,t)f(x,t,u)d\mu = \phi(x_1,t_1) - \phi(x_0,t_0) \quad \text{where}$$

$(x_1,t_1) \in \Gamma$ is the endpoint of the augmented trajectory $(x(\cdot),\cdot)$.

Unless the endpoints for (S) are essentially fixed, i.e. there is only one reachable point in Γ, the class of $\mu \in P^{\oplus}(\underline{A}\times\Omega)$ satisfying (c) is not convex. For if μ_1, μ_2 satisfy (c) with $(x_1,t_1) \neq (x_2,t_2) \in \Gamma$, respectively, then $\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ does not satisfy (c) for any $(x,t) \in \Gamma$,

$$\int_{\underline{A}\times\Omega} \phi_t + \phi_x f d\mu = \tfrac{1}{2}\phi(x_1,t_1) + \tfrac{1}{2}\phi(x_2,t_2) - \phi(x_0,t_0) \qquad \text{(d)}$$

To obtain a convex weak program, enlargement of the class of admissible measures is necessary. (d) suggests convexification of the boundary condition, to give the following:

$$\text{(W)} \begin{cases} \text{minimize} \int_{\underline{A}\times\Omega} \ell(x,t,u)d\mu & \text{(W1) subject to} \\ \mu \in P^{\oplus}(\underline{A}\times\Omega),\ \|\mu\| \le T_1 - t_0 & \text{(W2)} \\ \left.\begin{array}{l} \exists \text{ a probability measure } \beta \text{ on } \Gamma \text{ such that } \forall\ \phi \in C^1(\underline{A}) \\ \int_{\underline{A}\times\Omega} \phi_t(x,t) + \phi_x(x,t)f(x,t,u)d\mu = \int_{\Gamma}\phi(x,t)d\beta - \phi(x_0,t_0) \end{array}\right\} & \text{(W3)} \end{cases}$$

where $T_1 = \max \{t_1 : (x_1,t_1) \in \Gamma\} < \infty$

The space of probability measures is closed under the formation of convex sums, hence convexity of (W).

By construction $\mathcal{S} \subset \mathcal{W}$, so $\eta(S) \geq \eta(W)$. The question is again whether $\mathcal{W}$ is essentially larger than $\mathcal{S}$. We expect a negative answer, for if the first stage of convexification carried out in the preceeding chapters did not lower the problem value, the second stage should be similarly well behaved. This confidence is borne out below.

Remark

Convexity is essential to the analysis in part 2. Here, if (W) were posed with the restriction that the β be measures concentrated at single points in Γ i.e. μ satisfy (c) above, compactness of Γ implies the existence of a solution to (W), whence a generalized curve solution can be constructed exactly as before. The equivalence of (W) and (S) for this restricted (W) is a trivial consequence of previous results.

§6.2 The Parametric Problem

Neither the well-posedness of the parametric problem (P) nor the 'imbedding' of (W) in (P) [props. (3.4) and (3.7)] are affected by the generalized boundary condition.

$$
(\mathrm{P}) \quad \begin{cases}
\min \int_{\underline{A}\mathrm{x}\underline{B}} L(y,\dot{y})\,d\mu \text{ over } \mu \in P\,(\underline{A}\mathrm{x}\underline{B}) & \text{(P1) subject to} \\[2ex]
\exists\, \beta \text{ st } \int_{\underline{A}\mathrm{x}\underline{B}} \phi_y(y)\dot{y}\,d\mu = \int_{\Gamma} \phi(y)\,d\beta - \phi(x_0,t_0) \ \forall\, \phi \in C^1(\underline{A}) & \text{(P2)} \\[2ex]
\int_{\underline{A}\mathrm{x}\underline{B}} D(y,\dot{y})\,d\mu = 0 & \text{(P3)} \\[2ex]
\int_{\underline{A}\mathrm{x}\underline{B}} M(y,\dot{y})\,d\mu = 0 & \text{(P4)}
\end{cases}
$$

Notation

If a positive measure g (generalized flow in Young's terminology) solves (P2) for a probability measure β, we denote the boundary of g by $\partial g = \beta$ because (x_0,t_0) is fixed.

Properties of the admissible set $\mathcal{P}$, given in §s (4.1) and (4.2) are unchanged. Recall:

Lemma (4.2)

For all $\mu \in \mathcal{P}$ $\text{supp}\{\mu\} \subset \{(y,\dot{y}) \in \underline{A}\mathrm{x}\underline{B} : t \geq 0, \dot{x} \in \dot{t}f(x,t,\Omega)\}$.

Proposition (4.3)

There exists $K < \infty$ such that $||\mu|| \leq K$ for all $\mu \in \mathcal{P}$.

Theorem (4.7)

The parametric problem has a solution, μ_0 say. □

Previously we applied Young's boundary value theory to the generalized flow μ_o satisfying (P2), (4.2) and (4.3) to obtain various useful representations for μ_0, (4.10), (4.12) and (4.13). We are, with some effort, able to extend these to the present case. Derivations of the following theorems are contained in Appendix 5.

Approximation Theorem for P (6.1)

Each $\mu \in P$ is the weak * limit of a sequence $\{\mu_i\}$ in $P^{\oplus}(\underline{A}x\underline{B})$ where $\mu_i = \sum_{j=1}^{n_i} \alpha_i^j \mu_i^j$ with $\sum_{j=1}^{n_i} \alpha_i^j = 1$, $\alpha_i^j > 0$ and each μ_i^j has representation

$$G \to \int_0^1 G(y(\sigma), \frac{\dot{y}(\sigma)}{|\dot{y}(\sigma)|}) |\dot{y}(\sigma)| d\sigma \quad \forall\, G \in C(\underline{A}x\underline{B}) \qquad (6.2)$$

for some continuous, piece-wise linear function $y(\sigma)$ $0 \leq \sigma \leq 1$ st. $y(0) = (x_0, t_0)$ and $y(1) = y_i^j \in \Gamma$.

Proof

Appendix 5, thm. (A5.5) □

Representation Theorem for P (6.2)

Each $\mu \in P$ is a unit mixture of jets, almost all of which are bounded generalized curves.

Proof

Appendix 5 thm. (A5.8); also for defn. of jet. □

The effort of course goes into (6.2) and if one is prepared to restrict equivalence proofs to problems where L is continuous, it need not be made. Then however, (6.1) would pose the following dilemma: it appears to imply (6.2), yet (6.2) is not true for all consistent flows (notes and comments, p. A5.8 et seq). Proving (6.2) removes the uneasy feeling.

Proposition (6.3)

The parametric problem (P) admits a generalized curve solution.

Proof

In case L is continuous, using (6.1) proceed along the lines of Prop. (4.15). Since W* convergence of a sequence of generalized curves implies convergence of their endpoints, the extracted $\bar{\mu}$ is admissible and $\int L d\mu_0 \geq \int L d\bar{\mu}$.

Otherwise (6.2) can be employed in a proof similar to that of Prop. (4.14). □

§ 6.3 Reconstruction of a Solution to the Optimal Control Problem

The generalized curve $\bar{\mu}$ solving (P) has the same properties as that solving the fixed end point program and proof of equivalence is completed as before.

So:

$$\eta(P) = \eta(W) = \eta(S)$$

Convexification has been completed without reducing the value of the problem.

Control problems arise where both initial and terminal points are constrained to lie in closed sets Γ_0 and Γ_1. The generalized boundary condition on the flow in $\mathcal{U}$ is then:

$\exists$ prob. measures β_0 and β_1 on Γ_0 and Γ_1 respectively such that

$$\int_{\underline{A}\times\Omega} \phi_t + \phi_x f d\mu = \int_{\Gamma_1} \phi d\beta_1 - \int_{\Gamma_0} \phi d\beta_0$$

The likelihood of equivalence being true is discussed at the end of Appendix 5.

Chapter 7

EXTENSIONS TO A WIDER CLASS OF DYNAMIC CONSTRAINTS

The power of our approach is demonstrated by the immediate extension of the conclusions (of chapters 4, 5, and 6) to any case of additional constraints which can be expressed as $\int Q d\mu = 0$ for some $Q \geq 0$ in the weak and parametric forms. State constraints are a typical example.

Control problems associated with differential inclusions $\dot{x} \in F(x,t)$, F a set valued function, are also treated.

§ 7.1 Additional Constraints

In this and the following section fixed endpoint problems are discussed since the dynamic constraints are not as important as boundary conditions in proving equivalence, i.e. any constraint change leaving (4.14) valid also does not affect the validity of (6.3).

In the definition of P there are two constraints (P3) and (P4), each being a statement that the integral w.r.t. μ of particular homogeneous positive continuous integrands $D(y,\dot{y}) = \mathrm{dist}(\dot{x}/\dot{t}, f(x,t,\Omega))|\dot{t}|$ and $M(y,\dot{y}) = \max[-\dot{t},0]$ be zero. Take (P5) to be any similar constraint

$$\int_{\underline{A}\times\underline{B}} G(y,\dot{y})\,d\mu = 0 \qquad \text{(P5)}$$

then, assuming that the constraint in (S) which leads to (P5) yields a non-empty admissible set, the results of chapters 4 and 5 go through unaltered.

(a) Existence: (P5) does not affect the closedness of P. Minimization once again takes place over a w* compact set, guaranteed non-empty by the above assumption.

(b) Generalized Curve Solution to (P): The proof of (4.14) as before, with the inclusion of the (obvious) fact that Λ almost all curves $\bar{\mu}$ in any admissible mixture satisfy $\int G d\bar{\mu} = 0$. Alternatively (4.15) goes through by considering the 5-vector $a_i = [\int 1 d\bar{\mu}_i, \int L d\bar{\mu}_i, \int D d\bar{\mu}_i, \int M d\bar{\mu}_i, \int G d\bar{\mu}_i]$

(c) Reconstruction can only be studied when the original constraint is given. A state constraint gives an illustrative example. Suppose in addition to the constraints (S2) and (S3) the generalized curves in S are required to satisfy:

$$g(x(t),t) \leq 0 \text{ for all } t \in [t_0,t_1] \tag{S4}$$

This is equivalent to

$$\int_{t_0}^{t_1} \max[g(x(t),t),0]dt = 0$$

which becomes

$$\int_{\underline{A}\times\Omega} \max[g(x,t),0]d\mu = 0 \tag{W4}$$

or

$$\int_{\underline{A}\times F} \max[g(x,t),0]d\mu = 0 \tag{WV5}$$

or

$$\int_{\underline{A}\times\underline{B}} \max[g(x,t),0]|\dot{t}|d\mu = 0 \tag{P5}$$

Taking $G(y,\dot{y}) = \max[g(x,t),0]|\dot{t}|$ let $\bar{\mu}$ be the optimal generalized curve found in (b), i.e. $\bar{\mu} = \{y(\sigma),\mu_\sigma : 0 \leq \sigma \leq 1\}$.

$$\int_0^1 \max[g(x(\sigma),t(\sigma)),0]\int_{\underline{B}} \dot{t}\, d\mu_\sigma d\sigma = 0$$

It is easy to see that this goes to

$$\int_{t_0}^{t_1} \max[g(\bar{x}(t),t)]dt = 0$$

upon a change of variable, which completes the demonstration in this case.

Notes

(i) We have studiously avoided a combined state and control constraint of the form $\int_\Omega g(x(t),t,u)d\mu_t(u) \leq 0$ a.e. $t \in [t_0,t_1]$. Here one would define $\bar{g}(x,t,\dot{x}) = \sup\{g(x,t,u): \dot{x}=f(x,t,u)\}$ which might only be upper semi-continuous, in which case so would $G(y,\dot{y})$ and P may not be weak star closed. When $\bar{g}$ is continuous we are able to proceed as above but taking care that in the change of variable $\sigma \to t$ and the use of Fillipov's lemma, $\bar{g}$ is treated like $\underline{\ell}$ is.

(ii) Any finite number of constraints similar to (S4) and (P5) can be dealt with in the same way.

§ 7.2 More General Control Constraint Sets

Suppose that instead of $u(t) \in \Omega$, a compact subset of R^n, we are given $u(t) \in \Omega(x(t),t)$ where each $\Omega(x,t)$ is compact and there exists a compact $\Omega \supset \bigcup_{(x,t)\in \underline{A}} \Omega(x,t)$. What conditions on $\Omega(\cdot,\cdot)$ ensure that our programs are well defined?

Hausdorff continuity in (x,t) is sufficient in all cases. More interestingly if $\Omega(\cdot,\cdot)$ is upper semi continuous on $\underline{A}$, i.e. for all $(x_0,t_0) \in \underline{A}$, given $\varepsilon > 0$, $\exists\, \delta > 0 : \Omega(x,t) \subset N[\Omega(x_0,t_0),\varepsilon]$ for $(x,t) \in N[(x_0,t_0),\delta]$, then $\mathrm{dist}(\dot{x},f(x,t,\Omega(x,t)))$ and $\underline{\ell}(x,t,\dot{x})$ are l.s.c. , which is sufficient for the results of Chapters 2 to 6 to be valid .

This is generally the condition used in existence theorems in optimal control for it implies that (in our case) the set valued function

$$Q(x,t) \triangleq \text{co } f(x,t,\Omega(x,t))$$

has property (Q) [C.S.]

Comment

If the alternative definition of $\underline{\ell}$ with $\underline{\ell} = \infty$ on $F \backslash Q(x,t)$ is used, upper semi-continuity of $\Omega(x,t)$ is sufficient for $\underline{\ell}$ to be l.s.c.

§ 7.3 Differential Inclusions

A differential inclusion is an extension of the notion of an ordinary differential equation, in which the r.h.s. of the equation becomes a set valued function [Bri. 1].

$$\dot{x}(t) = f(x(t),t) \rightarrow \dot{x}(t) \in E(x(t),t) \text{ where } E : R^{n+1} \rightarrow K^n \tag{7.4}$$

K^n some subset of the power set of R^n. We shall take the collection of compact subsets of R^n for K^n.

An ordinary solution to (7.4) is any absolutely continuous R^n valued function $x(\cdot)$ such that $\dot{x}(t) \in E(x(t),t)$ a.e. $t \in [t_0,t_1]$. A generalized solution will satisfy $\dot{x}(t) = \int_{R^n} \dot{x}\, d\mu_t(\dot{x})$ a.e. $t \in [t_0,t_1]$ where $\mu : t \rightarrow \mu_t$ is measurable and each μ_t is a probability measure with $\text{supp}\{\mu_t\} \subset E(x(t),t)$.

Let us take $E(\cdot,\cdot)$ Hausdorff continuous and uniformly bounded on $\underline{A}$, then analogous to the strong control problem we can pose :

$$
\text{(I)}\quad\left\{\begin{array}{ll}
\text{minimize} \displaystyle\int_{t_0}^{t_1} \int_{R^n} \ell(x(t),t,\dot{x})\,d\mu_t(\dot{x})\,dt & \text{(I1) subject to} \\[2ex]
\dot{x}(t) = \displaystyle\int_{R^n} \dot{x}\,d\mu_t(\dot{x}) \text{ a.e. } t \in [t_0,t_1], x(t_0) = x_0,\ x(t_1) = x_1 & \text{(I2)} \\[2ex]
\left.\begin{array}{l}\text{for some measurable } \mu : t \to \mu_t,\ \mu_t \text{ a probability} \\[2ex] \text{measure, } \operatorname{supp}\{\mu_t\} \subset E(x(t),t)\end{array}\right\} & \text{(I3)}
\end{array}\right.
$$

where ℓ is continuous : $R^{n+1} \times R^n \to R^1$, t_0, t_1, x_0, x_1 are fixed.

Notes

(i) Any differential equation can be regarded as a differential inclusion.

(ii) The essential difference from a control point of view is that whereas the derivative $\dot{x} = f(x,t,u)$ is continuously parametrized by u, no such parameter exists for $\dot{x} \in E(x,t)$ in general.

The weak problem has no counterpart here. We go on to:

$$
\text{(WI)}\quad\left\{\begin{array}{ll}
\text{minimize} \displaystyle\int_{\underline{A}\times E} \ell(x,t,\dot{x})\,d\mu & \text{(WI1) subject to} \\[2ex]
\mu \in P^{\oplus}(\underline{A}\times E),\ ||\mu|| \le t_1 - t_0 & \text{(WI2)} \\[2ex]
\displaystyle\int_{\underline{A}\times E} \phi_t(x,t) + \phi_x(x,t)\dot{x}\,d\mu = \Delta\phi \text{ for all } \phi \in C'(\underline{A}) & \text{(WI3)} \\[2ex]
\displaystyle\int_{\underline{A}\times E} \operatorname{dist}(\dot{x},E(x,t))\,d\mu = 0 & \text{(WI4)}
\end{array}\right.
$$

Here $\underline{A}$ arises from an hypothesis H and E is any compact set in R^n containing $E(\underline{A})$. Admissibility of the integrand $\operatorname{dist}(\dot{x},E(x,t))$ can be proved along the same lines as before (lemma (3.6)(b)).

The similarity of (WI) and (WV) is so complete we hardly need write down the parametric problem corresponding to (I), nor need we justify the following:

Theorem (7.5) $\eta(I) = \eta(WI)$.

Chapter 4 can be written with just one amendment, $E(x,t)$ replaces $f(x,t,\Omega)$. From chapter 5 we need only sections (5.1) and (5.2).

Notes

(i) The admissible sets $\mathcal{I}$ and $\mathcal{WI}$ for (I) and (WI) respectively are related by $\mathcal{WI} = \overline{\text{co}}\ \mathcal{I}$

(ii) More general endpoint constraints and further dynamic constraints may be incorporated into (I) as in the preceeding sections.

(iii) Among the interesting dynamical systems which may be cast as differential inclusions are those described by inequalities

$$f_1(x,t) \leq \dot{x} \leq f_2(x,t)$$

Chapter 8

DISCUSSION AND FUTURE DIRECTIONS

§8.1 Simplifying the Proofs

Much of the material in the previous chapters has a more simple derivation, if one is satisfied with the restriction $\ell \geq \varepsilon > 0$. In case the end times are fixed, this is no loss of generality - if ℓ is any integrand,

$$\min \int_{t_0}^{t_1} \ell dt \quad \text{and} \quad \min \int_{t_0}^{t_1} \ell + k dt = \min\{\int_{t_0}^{t_1} \ell dt + k(t_1 - t_0)\}$$

produce the same answer.

The details of the structure of $\mathcal{P}$ are no longer relevant to the equivalence proofs, for example instead of showing that the elements of $\mathcal{P}$ contain no non-trivial closed curves, one can merely ignore their contribution to $\int_{\underline{A \times B}} L d\mu_0 = \int_{\underline{A \times B}} L d\bar{\mu} + \int_{\underline{A \times B}} L d\rho \geq \int_{\underline{A \times B}} L d\bar{\mu}$, ρ being the closed curves, because it is positive.

This applies equally to both proofs of the existence of a generalized curve solving (P).

Unfortunately this sacrifices being able to prove $\mathcal{W} = \overline{co}(S)$ for by neglecting part of μ we cannot ensure the existence of ν_2 as in proposition (5.15).

§8.2 Difficulties Posed By Unbounded Control Constraint Sets and Infinite Time Problems

It is crucial for the existence of a solution to the parametric problem (P) that we are able to show that the admissible set $\mathcal{P}$ is w* compact or at least that in performing the minimization we can restrict attention to a w* compact subset of P. In chapter 4 this is done by showing that for any $\mu \in P$ $\mathrm{supp}\{\mu\} \subset \{(y,\dot{y}):\dot{t}\geq 0, \dot{x}\in \dot{t}f(x,t,\Omega)\} \subset \{(y,\dot{y}):\dot{t}\geq 0, ||\dot{x}||\leq k|\dot{t}|\}$ where $k = \max_{\underline{A}\times\Omega}||f(x,t,u)|| < \infty$, then $\int|\dot{t}|d\mu = \int\dot{t}d\mu = t_1 - t_0$ and $\int|\dot{x}|d\mu \leq k(t_1-t_0)$ to give $||\mu|| = \int 1 d\mu \leq (k+1)(t_1-t_0) < \infty$.

For the infinite time case $t_1 = \infty$ and w* compactness is irretrievably lost. Indeed in the study of the existence of solutions to infinite time control problems, the weak topology (or w* if generalized curves are viewed as linear functionals) is found to be too strong and is usually replaced by a 'weak compact' topology i.e. $\mu_n \overset{wc}{\to} \mu$ if $\mu_u f \to \mu f$ for all $f \in C(\underline{A}\times\Omega)$ with compact support. For us this would mean a complete reworking so it will not be pursued further. See [Ba].

Compactness can also be lost when Ω, the control constraint set, is unbounded, for then even if $k = \sup_{\underline{A}\times\Omega}||f(x,t,u)|| < \infty$, $f(x,t,\Omega)$ need not be closed i.e. compact. A solution is not impossible however: in many cases we can call upon lower closure results developed for the control existence problem by Cesari and others [CS], to demonstrate that our admissible sets are closed. Combined with growth conditions of the form $\ell \geq 0$, $||f(x,t,u)||^2 \leq k\ell(x,t,u) + \psi(t)$ for all $(x,t,u) \in \underline{A}\times\Omega$, some $k < \infty$ and summable ψ, and the assumption that at least one control-trajectory pair exists which has finite cost, we can restrict attention to a compact subset of P. For if this cost is η_0 then the non-empty subset of P, $\{\mu\in P: \int_{\underline{A}\times\underline{B}} Ld\mu\leq\eta_0\}$ is closed and norm bounded i.e. weak * compact.

For $\mu \in P$ and $(x,t,\dot{x},\dot{t}) \in \mathrm{supp}\{\mu\}$, $\dot{x} \in \dot{t}f(x,t,\Omega)$ so $||\dot{x}||^2 \leq |\dot{t}|^2\, ||f(x,t,u)||^2$ for all u st $\dot{x}/\dot{t} = f(x,t,u)$. Therefore $||\dot{x}||^2 \leq |\dot{t}|^2(k\ell(x,t,u)+\psi(t))$ for all such u and by definition $||\dot{x}||^2 \leq |\dot{t}|^2(k\underline{\ell}(x,t,\dot{x}/\dot{t})+\psi(t))$ so

$$\int_{\underline{A}x\underline{B}} ||\dot{x}||d\mu \leq \left(\int_{\underline{A}x\underline{B}} ||\dot{x}||^2 d\mu\right)^{\frac{1}{2}}$$

$$\leq \left(\int_{\underline{A}x\underline{B}} |\dot{t}|kL(y,\dot{y})d\mu + \int_{\underline{A}x\underline{B}} |\dot{t}|^2\psi(t)d\mu\right)^{\frac{1}{2}}$$

$$\leq \left(k\int_{\underline{A}x\underline{B}} L(y,\dot{y})d\mu + \int_{\underline{A}x\underline{B}} \psi(t)d\mu\right)^{\frac{1}{2}} \text{ since } |\dot{t}|^2 \leq |\dot{t}| \leq 1 \text{ on } \underline{B}$$

$$\leq (k\eta_0+\Psi)^{\frac{1}{2}} \qquad \Psi = \int_{t_0}^{t_1} \psi(t)dt = \int_{\underline{A}x\underline{B}} \psi d\mu$$

Further $\int_{\underline{A}x\underline{B}} |\dot{t}|d\mu = t_1 - t_0$ and

$||\mu|| = \int 1d\mu \leq \int||\dot{x}|| + |\dot{t}|d\mu = (k\eta_0+\Psi)^{\frac{1}{2}} + t_1 - t_0 < \infty$. The parametric problem therefore has a solution, which can be taken to be a generalized curve, $\{y(\sigma),\mu_\sigma : 0\leq\sigma\leq 1\}$. We are now faced by the fact that points $(\dot{x},0)$ may lie in $\text{supp}\{\mu_\sigma\}$ even when $\mu_\sigma \neq \Theta$: previously such a possibility was automatically excluded.

Let U be an ε neighbourhood of the subset $\{(\dot{x},0):||\dot{x}||=1\}$ in $\underline{B}$

$$\eta_0 \geq \int_0^1 \int_{\underline{B}} L(y(\sigma),\dot{y})d\mu_\sigma(\dot{y})d\sigma$$

$$\geq \int_0^1 \int_U L(y(\sigma),\dot{y})d\mu_\sigma(\dot{y})d\sigma \qquad \text{since } L \geq 0$$

Now $L(y(\sigma),\dot{y}) = \underline{\ell}(x(\sigma),t(\sigma),\dot{x}/\dot{t})|\dot{t}| \geq |\dot{t}|(||\dot{x}||^2/|\dot{t}|^2 - \psi(t(\sigma)))1/k$, therefore

$$\eta_0 \geq 1/k \int_0^1 \int_U [||\dot{x}||^2/|\dot{t}| - |\dot{t}|\psi(t(\sigma))]d\mu_\sigma(\dot{x},\dot{t})d\sigma$$

$$\geq 1/k \int_0^1 (1-\varepsilon)^2/\varepsilon \; \mu_\sigma(U)d\sigma - \varepsilon\Psi$$

So given any $\delta > 0$ we can choose ε sufficiently small that $\mu_\sigma(U) < \delta$ in $L^1[0,1]$.

$\mathrm{supp}\{\mu_\sigma\} \subset \{(\dot{x},\dot{t})\in \underline{B}:\dot{t}\geq 0\}$ and $\dot{t}(\sigma) = \int_{\underline{B}} \dot{t}\,d\mu_\sigma = 0$ therefore imply $\mathrm{supp}\{\mu_\sigma\} \subset \{(\dot{x},0)\} \subset U$ for all $\varepsilon > 0$, i.e. $\mu_\sigma(U) < \delta$ or $\dot{x}(\sigma) = 0$.

This is what we require for the reconstruction of solutions to the optimal control problem - the trajectory $x(t(\sigma))$ is absolutely continuous, there being no discontinuities caused by impulse control measures.

Remarks

(i) It may be necessary to define $\mu_\sigma \underline{\underline{\Delta}}\ \Theta$ whenever $\dot{t}(\sigma) = 0$, before proceeding with the construction of solutions to the weak variational problem.

(ii) A solution to (S) can then be obtained from the above mentioned lower closure results which are generalizations of Fillipov's lemma.

(iii) Among others, linear quadratic problems satisfy the growth conditions.

An alternative approach to these difficulties has been taken by Rubio, for aproblem in the calculus of variations. He replaces the original problem by a sequence of problems (W_k) say wherein the derivative has constraint $||\dot{x}|| \leq k$ $k = 1, 2, \ldots$ A growth condition enables him to show that the solutions of the (W_k) converge in the appropriate sense to a 'solution' to the original problem. Unfortunately this 'solution' does not necessarily satisfy the boundary conditions. See [Ru 2].

§8.3 Future Extensions

The equivalence of control problems for systems with dynamics described by ordinary differential equations or inclusions with convex programs over Radon measures has been given. What other dynamics can be considered?

Central to our development is Young's theory of generalized flows and their boundaries. It applies to flows on $\underline{A}\mathrm{x}\underline{B}$ where $\underline{A}$ and $\underline{B}$ are the unit cube and unit sphere of R^{n+1} and compels us to consider only systems where the state $x(t)$ and its derivative $\dot{x}(t)$ are points in R^n, or perhaps sets of points in R^n. An example of the latter are set-valued differential equations.

Consider the differential inclusion $\dot{x}(t) \in E(x(t),t)$. If α is the set of all absolutely continuous trajectories $x(\cdot)$ emanating from Γ_0 at t_0 and ending in Γ_1 at t_1 and solving the differential inclusion, define $S(t) = \{x(t):x(\cdot)\epsilon\alpha\}$. Then $S(\cdot)$ is a solution of the set-valued d.e.

$$\dot{S}(t) = E(S(t)) \qquad S(t_0) = \Gamma_0 \qquad S(t_1) = \Gamma_1 \tag{8.1}$$

Note that the above merely provides an interpretation of (7.4), c.f. [Bri 1].

The limitations of set valued differential equations are immediately apparent. There is no interaction between trajectories in α i.e. no mixing or diffusion and set valued d.e.'s play no role in either stochastic system theory or partial differential equations. For these and many other system varieties a theory of generalized flows on function spaces is awaited.

§ 8.4 Conclusions

In chapters 2 to 5 we proved the equivalence of the value of a problem of optimal control with that of a convex program on a set of measures. The merits of our roundabout approach became apparent when we extended these results to problems with general boundary conditions (chapter 6) and dynamics (chapter 7). In doing so we discovered the relationships $\mathcal{W} = \text{co}\ (\mathcal{S})$ between admissible sets. The limits of applicability were explored in the opening sections of this chapter and were found to coincide roughly with the limits of existence theory for optimal controls.

Apart from the intrinsic interest of the relationships between the measures and trajectories, the convexity of the weak problem renders it a suitable object for the application of duality theory. The value if the dual program will be the same as that of the weak one, hence also the control problem. This is the subject of Part II, and leads to quite general necessary and sufficient conditions for optimality.

PART II

NECESSARY CONDITIONS OF RELAXED DYNAMIC PROGRAMMING FORM

While the notion of problem equivalence is quite recent, characterization of necessary and sufficient conditions for optimality has produced a very full literature. Much of the material on Dynamic Programming is derived heuristically, problems are often ill-posed and results imprecisely stated, therefore a detailed introduction is given to avoid this and to properly distinguish the different concepts.

Chapter 9

INTRODUCTION

We want to characterize optimal solutions to control problems involving the differential equation :

$$\dot{x}(t) = f(x(t),t,u(t)) \qquad u(t) \in \Omega \tag{9.1}$$

and cost functional :

$$J(u(\cdot)) = \int \ell(x(t),t,u(t))dt \tag{9.2}$$

Relaxed controls $u(\cdot)$ are admitted in (9.1) , where the relations are to hold for almost all t in an interval $[t_0,t_1]$. In various particular problems $[t_0,t_1]$ may be fixed a priori or determined as part of the solution , through satisfaction of boundary conditions on the state x .

Our aim is to minimize $J(u(\cdot)) = \int_{t_0}^{t_1} \ell(x(t),t,u(t))dt$ over all admissible controls $u(\cdot)$. Of course admissibility of $u(\cdot)$ involves admissibility of the associated trajectory $x(\cdot)$.

9.1 The Bellman Equation

In [Be] Bellman considers the following class of problems :

for each $(x_0,t_0) \in Q = R^n x(-\infty,T]$,

minimize $\int_{t_0}^{t_1} \ell(x(t),t,u(t))dt$ subject to (9.1) with initial condition $x(t_0) = x_0$. The cost function is :

$$W(x_0,t_0) \triangleq \inf \{ \int_{t_0}^{t_1} \ell(x(t),t,u(t))dt \} \tag{9.3}$$

Note .

T is as before . We are assuming that from the (x_0,t_0) considered there exists a trajectory intersecting Γ at time $t_1 \leqq T$; clearly t_1 will be a function of (x_0,t_0).For other (x_0,t_0) put $W = \infty$.

For the case $W \in C^1(Q)$ it is easy to show heuristically that

$$W_t(x,t) + W_x(x,t)f(x,t,u) + \ell(x,t,u) \geq 0 \text{ on } (\text{int } Q)\times\Omega \quad (9.4)$$

Equality holds for almost all t along any optimal trajectory i.e.

$$W_t(x,t) + \min_{u\in\Omega}[W_x(x,t)f(x,t,u)+\ell(x,t,u)] = 0 \text{ on int } Q \quad (9.5)$$

which is known as the Bellman or Dynamic Programming equation.

In the presence of terminal constraints Q should be defined as $Q \triangleq \{(x,t)\in R^{n+1}: \exists$ an admissible trajectory from x at time t, to $\Gamma\}$ $(x(t_1),t_1) \in \Gamma$ being the constraint . Although in general $W \notin C^0(Q)$ classes of problems where (9.5) is justifiable in some sense have been isolated.

Theorem (9.6)

For the family of fixed end time, free end point problems with f and ℓ Lipschitz continuous in x, W is locally Lipschitz on $Q = R^n \times [T_0,T]$ for any T_0. By Rademacher's theorem it is differentiable except on a set of R^{n+1} Lebesgue measure zero. [FR p. 85].

Statements of this nature can be obtained in the general case only by complex constructional hypotheses.

Definition

A feedback control is a function $u(\cdot,\cdot):Q^0 \to \Omega$, $Q^0 \subset Q$ such that for each $(y,s) \in Q^0$ there is a unique solution to $\dot{x} = f(x,t,u(x,t))$ with $x(s) = y$. $u(\cdot,\cdot)$ must be measurable or smoother.

We require an optimal feedback control, a feedback control such that for every $(y,s) \in Q^0$ the above trajectory is optimal with optimal control $u(x(t),t)$, $t \in [s,t_1]$. Thus

$$W(y,s) = \int_s^{t_1} \ell(x(t),t,u(x(t),t))dt$$

Differentiability of W depends upon differentiability of t_1 and x w.r.t. initial conditions (y,s), $\ell(\cdot)$ w.r.t. (x,u) and $u(\cdot)$ w.r.t. (x,t). The last is the most difficult to check.

The support function U(p,x,t) is defined for $p \in R^n$ by

$$U(p,x,t) = \arg \sup_{u \in \Omega}[pf(x,t,u)+\ell(x,t,u)] \qquad (9.7)$$

Suppose $U \in C^1(R^n\backslash\{0\}xQ)$ then if the partial differential equation

$$\left.\begin{array}{l} V_t(x,t) + V_x(x,t)f(x,t,U(V_x,x,t))+\ell(x,t,U(V_x,x,t))=0 \\ (x,t) \in \text{int } Q, \quad V(x,t) = 0 \text{ on } \Gamma \end{array}\right\} \qquad (9.8)$$

has non-characteristic initial data (final data) it has a solution on Q and $U(V_x(x,t),x,t)$ is an optimal feedback control. We notice that $V = W \in C^1(Q)$ even though $U(V_x,x,t)$ is differentiable only when $V \in C^2(Q)$.

Determination of U is generally known as synthesis. Ignoring ℓ (e.g. in time optimal case $\ell = 1$) a necessary and sufficient condition for $U \in C^1(R^n\backslash\{0\}xQ)$ is that $f(x,t,\Omega)$ be convex with strictly positive Gaussian curvature. In [H] Hermes shows that when $f(x,t,\Omega)$ is just convex the system dynamics can be approximated arbitrarily closely by h(x,t,u) with $h(x,t,\Omega)$ satisfying the above condition. The relationship between this approximation, the corresponding solutions of (9.8) and our work is detailed in a later chapter.

Boltyanskii has developed the notion of a feedback control with an admissible set of discontinuities. He hypothesizes the existence of sets

$S_0 \subset S_1 \subset \ldots \subset S_n \subset Q$, S_i i-dimensional and relatively closed in R^{n+1}. Some may be empty. The components (disjoint subsets) of $S_i \backslash S_{i-1}$ and $Q - S_n$ are called cells and are required to be relatively open subsets of smooth manifolds. The feedback control u(x,t) has discontinuities relative to S_i contained in S_{i-1} and gives rise to unique trajectories which pass through a only a finite number of cells before reaching the smooth terminal manifold $\Gamma \subset Q - S_n$, their transitions from cell to cell being governed by various requirements according to the types of cell entered and left ([FR] Ch. 4 §6). When an optimal feedback control verifies this array of assumptions, the cost functional W is of class C^1 on $Q - S_n$ i.e. except on a smooth manifold of dimension < n+1.

It is evident that these conditions can only be verified for each example separately; in practice the maximum principle is used to construct a suspected optimal feedback control and associated S_i i = 0, ..., n. Unsatisfactory as this may be, it is similar to the extension of Caratheodory's method Young uses to get a least time sufficient condition [You 3] and may be the best that can be done. A point of interest is that Caratheodory's method, an extension of the classical field of extremals, has been used by Osborn to validate dynamic programming in the calculus of variations [Os]. It should follow that an admissible optimal feedback control exists wherever the extended method applies but this is not the goal fo our work.

Confronted by these difficulties Pronozin investigated necessary conditions for the smoothness of W, for minimum-time transfer to the origin controllers. His results have been used to construct examples having discontinuous W and as a guide to setting up sufficient criteria for our results to go through. They are related to the reachability properties of the system.

Theorem (9.9) ([Pr])

Let $f = f(x,u)$ be Lipschitz in x, continuous in u and suppose a solution to the control problem exists for each initial point in P, the projection of Q onto x space R^n. Then if $n \geq 2$, $W \in C(P)$, $C^1(P\setminus\{0\})$ implies $f(0,\Omega)$ has no corner points and $0 \in \mathrm{co}\, f(0,\Omega)$. If $f(0,\Omega)$ has a corner at u^* (a) $f(0,u^*) = 0 \Rightarrow W$ discontinuous at $x = 0$ (b) $f(0,u^*) \neq 0$ and $W \in C(P) \Rightarrow$ along each optimal trajectory there are points at which W is non-differentiable.

Theorem (9.10)

W is Lipshitz on P iff $0 \in \mathrm{int\ co}\, f(0,\Omega)$.

Such is the current state of the art in dynamic programming.

§ (9.2) Verification Theorems

Fix the initial point (x_0,t_0) in the control problem for (9.1) and (9.2). An admissible pair $(x(\cdot),u(\cdot))$ is a solution of (9.1) satisfying $x(t_0) = x_0$ $(x(t_1),t_1) \in \Gamma$ for some $t_1 > t_0$. Take $\underline{A}$ any closed region in R^{n+1} containing the trajectory points $\{(x(t),t) : t_0 \leq t \leq t_1\}$. Motivated by (9.4) is:

Theorem (9.11)

If $\phi \in C^1(\underline{A})$ satisfies the partial differential inequality

$$\phi_t(x,t) + \phi_x(x,t)f(x,t,u) - \ell(x,t,u) \leq 0 \text{ on } \underline{A}\times\Omega, \quad \phi(x,t) \geq 0 \text{ on } \Gamma \cap \underline{A} \tag{9.12}$$

whilst $\phi(x(t_1),t_1) = 0$, $\phi_t(x(t),t)+\phi_x(x(t),t)f(x(t),t,u(t)-\ell(x(t),t,u(t)) = 0$ (9.13)

almost everywhere in $[t_0,t_1]$, for the admissible pair $(x(\cdot),u(\cdot))$, then $(x(\cdot),u(\cdot))$ is optimal with respect to all admissible pairs with trajectories contained in $\underline{A}$.

Proof

For any admissible pair considered, $(\bar{x}(\cdot),\bar{u}(\cdot))$,

$$\int_{t_0}^{\bar{t}_1} \ell(\bar{x}(t),t,\bar{u}(t))dt \geq \int_{t_0}^{\bar{t}} \phi_t+\phi_x f dt = \phi(\bar{x}(t_1),t_1)-\phi(x_0,t_0)$$

$$\geq -\phi(x_0,t_0)$$

For the particular pair $\int_{t_0}^{t_1} \ell(x(t),t,u(t))dt = -\phi(x_0,t_0)$. □

Theorems such as (9.11) are known generally as verification theorems - $(x(\cdot),u(\cdot))$ is suspected of being optimal, if ϕ satisfying (9.12) and (9.13) exists, it is. Such a crude result (there is not even the hypothesis that an optimal pair exists - it does because we admit relaxed controls) inevitably flounders upon its assumption, does such a $\phi \in C^1(\underline{A})$ exist? Before answering this question the following must be made clear.

(9.14) There is no a priori or necessary relationship between any verification function ϕ and Bellman's function W, though if ϕ verifies the optimality of $(x(\cdot),u(\cdot))$ then $W(x(t),t) = -\phi(x(t),t)$, $t \in [t_0,t_1]$. If the initial point (x_0,t_0) of the problem is altered, a different ϕ may be needed or may cease to exist i.e. $W(x_0,t_0) = -\phi_{(x_0,t_0)}(x_0,t_0)$ at best. Existence of ϕ can depend upon choice of $\underline{A}$; if $\underline{A}$ contains no trajectories other than $x(\cdot)$, (9.11) is trivial.

Despite this most attempts to guarantee the existence of verifying functions ϕ have looked to the Bellman function. Indeed sufficient conditions like (9.11) have been proved for problems with admissible feedback controls, [FR, p. 97].

We escape these restrictions by demonstrating the existence of a sequence $\{\phi^i\}$ each ϕ^i satisfying (9.12) and with (9.13) satisfied in the limit as $i \to \infty$. Moreover the existence of such a sequence is *necessary* for optimality as well as sufficient. Unified necessary and sufficient conditions are the ultimate aim of all optimality theory - ours apply more widely than any others.

§9.3 Structure of the Remainder

Chapter 10 describes the control problem of interest and its equivalence with a weak problem, proved in Part 1. Then in chapter 11 we reformulate the weak problem as a convex problem. This has a dual program for which we evaluate the functionals, the conjugates of those in the original program. Since the conjugates satisfy the conditions of a thoerem of Rockafellar , the dual has the same value as its dual which is of course the original program. The structure of the conjugate functionals yields our results. Some extensions and examples occupy chapters 12 and 13. Chapter 14 is devoted to a discussion of alternative derivations of the results and their relationship with the better known necessary condition, the maximum principle. Included here is a discussion of the computational potential of the new conditions. The final chapter presents conclusions for the thesis as a whole, indicating where future research may lead.

Chapter 10

THE CONTROL PROBLEM OF LAGRANGE

§10.1 Formulation

The notation used in the following is that of part I. In contrast to the equivalence theorems, for which the statements for general problems are immediatly apparent, application of duality theory to simple problems does not reveal the most general structure. Consequently we adopt as a model the Lagrange problem with 'free' terminal point (by which we mean the constraint Γ includes more than a single point) and a state constraint. To avoid convexity assumptions relaxed controls are admitted.

$$(S)\quad \begin{cases} \min \displaystyle\int_{t_0}^{t_1} \int_{\Omega} \ell(x(t),t,u)\,d\mu_t(u)\,dt & \text{(S1) subject to} \\ \dot{x}(t) = \displaystyle\int_{\Omega} f(x(t),t,u)\,d\mu_t(u) \quad \text{a.e. in } [t_0,t_1] & \\ & \text{(S2)} \\ x(t_0) = x_0 \quad (x(t_1),t_1) \in \Gamma & \\ u(\cdot) : t \to \mu_t \quad \text{a relaxed control satisfying} & \\ & \text{(S3)} \\ \mu_t \in P^{\oplus}(\Omega), \quad |\mu_t| = 1 \text{ a.e. in } [t_0,t_1] & \\ g(x(t),t) \leq 0 \text{ for all } t \text{ in } [t_0,t_1] & \text{(S4)} \end{cases}$$

Without loss of generality we can suppose $g(y,s) \leq 0$ on Γ. Invoking the hypothesis H as modified for the free end point, there exists a compact set $\underline{A} \in R^{n+1}$ containing a minimizing sequence of augmented trajectories, $\{(x^i(t),t):t_0\leq t\leq t_1^i\}$. Implicit in this is the existence of an admissible control-trajectory pair $(x(\cdot),\cdot,u(\cdot))$ (all trajectories are now considered augmented) and the boundedness of t_1^*, the terminal time for any optimal pair. $\underline{A}$ can be enlarged to a cube, to satisfy the derivation of equivalence, containing as large a compact subset of Γ as desired. If Γ is compact or has compact projection onto the t-axis, the original hypothesis suffices since t_1 is automatically bounded.

We remark again that H permits the existence of an optimal pair to be demonstrated.

§ 10.2 The Equivalent Weak Problem

Let us recapitulate the pertinent definitions and conclusion of part I. There (chapter 2) the strong optimal control problem (S) was imbedded in an optimization program over the space of Radon measures on $\underline{A}x\Omega$ (we agree not to distinguish between these measures and the linear functionals on $C(\underline{A}x\Omega)$ which they represent), called the Weak Problem, (W). The set of admissible elements for (S), generalized curves, and for (W), measures, are denoted by $\mathcal{S}$ and $\mathcal{W}$ respectively.

$$
(W)\quad \begin{cases}
\text{minimize } \int_{\underline{A}x\Omega} \ell(x,t,u)\,d\mu & \text{(W1) subject to} \\
\mu \in P^{\oplus}(\underline{A}x\Omega),\quad \|\mu\| \leq T - t_0 & \text{(W2)} \\
\left.\begin{array}{l}\int_{\underline{A}x\Omega} \phi_t(x,t)+\phi_x(x,t)f(x,t,u)\,d\mu = \int_{\Gamma}\phi(x,t)\,d\beta(x,t)-\phi(x_0,t_0) \\ \text{for some prob. meas. } \beta \text{ on } \Gamma \text{ corr to } \mu,\ \forall\, \phi \in C^1(\underline{A})\end{array}\right\} & \text{(W3)} \\
\int_{\underline{A}x\Omega} \max[g(x,t),0]\,d\mu = 0 & \text{(W4)}
\end{cases}
$$

$\mathcal{S} \subset \mathcal{W}$ so $\eta(W) \leq \eta(S)$. Remarkably, the constraint (W2) and the boundary condition (W3) restrict elements of $\mathcal{W}$ to being unit mixtures of curves in $\mathcal{S}$, i.e. $\mathcal{W} = \overline{\overline{co}}(\mathcal{S})$. Now $\eta(W) = \eta(S)$ and we can use characterizations of solutions to (W) to characterize those of (S).

<u>N.B.</u> T in (W2) can be defined as $\max\limits_{(x,t)\in \underline{A}}[t]$ or $\max\limits_{(x,t)\in\Gamma}[t]$.

Chapter 11

A FENCHEL PROGRAM AND ITS DUAL

§11.1 Recasting the Weak Problem as a Fenchel Program

$\mathcal{W} \subset C^*(\underline{A}\times\Omega)$ is a convex set defined by three constraints, (W2), (W3) and (W4), and can be written as $\mathcal{W}_2 \cap \mathcal{W}_{3,4}$ where

$$\mathcal{W}_2 \triangleq \{\mu\in C^*(\underline{A}\times\Omega):\mu\in P^{\oplus}, ||\mu||\leq T-t_0\}$$

$$\mathcal{W}_{3,4} \triangleq \{\mu\in C^*(\underline{A}\times\Omega):\mu \text{ satisfies (W3) and (W4)}\}$$

On $C^*(\underline{A}\times\Omega)$ with the weak * topology, define $p : C^* \to R \cup \{+\infty\}$ and $q : C^* \to \{-\infty\} \cup R$ by

$$p(\mu) = \begin{cases} \int_{\underline{A}\times\Omega} \ell d\mu & \text{if } \mu \in \mathcal{W}_2 \\ +\infty & \text{if otherwise} \end{cases}$$

$$q(\mu) = \begin{cases} 0 & \text{if } \mu \in \mathcal{W}_{3,4} \\ -\infty & \text{if otherwise} \end{cases}$$

Proposition (11.1)

p is lower semi-continuous and convex and q is upper semi-continuous and concave. The set $\{\mu\in C^*:p(\mu)\neq+\infty, q(\mu)\neq-\infty\}$ is non-empty.

Proof

$\mathcal{W}_2$ and $\mathcal{W}_{3,4}$ are closed convex subsets of C^*. We prove that $\mathcal{W}_{3,4}$ is closed - the remaining properties are obvious. Take a generalized sequence $\{\mu_p:p\in D$, D a partially ordered set$\}$ in $\mathcal{W}_{3,4}$ converging weakly * to $\mu_0 \in C^*(\underline{A}\times\Omega)$. To each μ_p, by the axiom of choice, corresponds a probability measure β_p on Γ, the boundary of μ_p. The unit sphere in $C^*(\Gamma)$

contains $\{\beta_p : p \in D\}$ and is weak * compact hence $\{\beta_p : p \in D\}$ has a weak * accumulation point β.

For any $\phi \in C^1(\underline{A})$, $\varepsilon > 0$ there is a $p \in D$ such that $|\int \phi_t + \phi_x f d(\mu_p - \mu_0)| < \varepsilon/2$, $|\int \phi d(\beta_p - \beta_0)| < \varepsilon/2$ so $|\int \phi_t + \phi_x f d\mu_0 - [\int \phi d\beta_0 - \phi(x_0, t_0)]|$ $\leq |\int \phi_t + \phi_x f d\mu_p - [\int \phi d\beta_p - \phi(x_0, t_0)]| + \varepsilon/2 + \varepsilon/2 = \varepsilon$.

So $\int \phi_t + \phi_x f d\mu_0 = \int \phi d\beta_0 - \phi(x_0, t_0)$ i.e. $\mu_0 \in \mathcal{W}_{3,4}$.

The restrictions of p and q to their effective domains $\mathcal{W}_2$ and $\mathcal{W}_{3,4}$ are linear and continuous; the semi-continuity, convexity and concavity properties follow from the definitions and the closedness and convexity of these domains.

The set is $\mathcal{W}_2 \cap \mathcal{W}_{3,4} = \mathcal{W}$, the non-empty set of feasible elements for (W). □

<u>N.B.</u> It is apparent that:

$$p(\mu) - q(\mu) = \begin{cases} \int_{\underline{A}x\Omega} \ell d\mu & \text{if } \mu \in \mathcal{W} \\ \infty & \text{if otherwise} \end{cases}$$

and the weak problem can be put into Fenchel program form,

(W) minimize $\{p(\mu) - q(\mu)\}$ over $\mu \in C^*(\underline{A}x\Omega)$.

<u>Remark:</u>

We have followed the usual approach of writing convex constraints into semi-continuous functionals with convex effective domains $\mathcal{W}_2$ and $\mathcal{W}_{3,4}$. Our choice of these two sets is determined by the utility of the corresponding dual program, for if p is defined on C*, q on $\mathcal{W}$, the dual is trivial, the same as (W).

§ 11.2 <u>Evaluation of Conjugate Functionals</u>

The pairing of $C^*(\underline{A}x\Omega)$ with w* topology and $C(\underline{A}x\Omega)$ with uniform topology as topological spaces in duality makes it natural to consider the conjugate (dual) functionals of p and q.

$$p^*(\xi) \triangleq \sup\{\int_{\underline{A}\times\Omega} \xi d\mu - p(\mu) : \mu \in C^*(\underline{A}\times\Omega)\} \tag{11.2}$$

$$q^*(\xi) \triangleq \inf\{\int_{\underline{A}\times\Omega} \xi d\mu - q(\mu) : \mu \in C^*(\underline{A}\times\Omega)\} \tag{11.3}$$

where $\xi \in C(\underline{A}\times\Omega)$.

The structures of $\mathcal{W}_2$ and $\mathcal{W}_{3,4}$ can be used to find explicit evaluations of p* and q*.

Proposition (11.4)

$p^*(\xi) = \max\{[\xi(x,t,u) - \ell(x,t,u)]^+ : (x,t,u) \in \underline{A}\times\Omega\}.(T-t_0)$ for all $\xi \in C(\underline{A}\times\Omega)$, where $h^+ \triangleq \begin{cases} h & h \geq 0 \\ 0 & h < 0 \end{cases}$ for any $h \in C(\underline{A}\times\Omega)$.

Proof

$$p^*(\xi) = \sup\{\int \xi - \ell d\mu : \mu \in \mathcal{W}_2\}$$

$$= \sup\{\int (\xi-\ell)^+ d\mu - \int (\xi-\ell)^- d\mu : \mu \in P^{\oplus}, ||\mu|| \leq T-t_0\}$$

Positivity of μ implies $\int (\xi-\ell)^- d\mu \geq 0$. Choosing μ with support in $\{(x,t,u) \in \underline{A}\times\Omega : (\xi-\ell)(x,t,u) \geq 0\}$

$$p^*(\xi) = \sup\{\int (\xi-\ell)^+ d\mu : \mu \in P^{\oplus}, ||\mu|| \leq T-t_0\}$$

$$\leq \max\{[\xi(x,t,u) - \ell(x,t,u)]^+ : (x,t,u) \in \underline{A}\times\Omega\}.(T-t_0)$$

Equality is achieved by taking μ with support in $\arg\max[\xi-\ell]^+ \neq \phi$ with norm $(T-t_0)$ □

A corollary to equivalence in part I is $\mathcal{W} = \overline{\overline{co}}(S)$, a strong statement about the structure of measures $\mu \in \mathcal{W}$.

$$\mathcal{W}_{3,4} \triangleq \{\mu \in C^*(\underline{A}\times\Omega):$$

$$\text{(i)} \int_{\underline{A}\times\Omega} \phi_t + \phi_x f d\mu = \int_\Gamma \phi d\beta - \phi(x_0, t_0) \text{ for}$$

some prob. meas. β on Γ corr. to μ, $\forall\ \phi \in C^1(\underline{A})$ and

$$\text{(ii)} \int_{\underline{A}\times\Omega} \max[g(x,t),0]d\mu = 0\}.$$

When $\mu \in \mathcal{W}$, the probability measure β has support in the reachable subset $R(x_0,t_0) \cap \Gamma$,

$$R(x_0,t_0) \triangleq \{(x,t)\in\underline{A}: \exists \text{ a trajectory-control pair } (\bar{x}(\cdot),\cdot,\bar{u}(\cdot))\in S$$

$$\text{with } \bar{x}(t)=x\}.$$

The various notions of reachability associated with the measures in $\mathcal{W}_{3,4}$ are discussed in appendix 6, where the following is proved.

<u>Theorem (11.5)</u>

In the absence of state constraints and assuming there is a fixed neighbourhood $N(0,\varepsilon)$ of zero in R^n, $N(0,\varepsilon) \subset$ co $f(x,t,\Omega)$ uniformly in $(x,t) \in \underline{A}$ then to any probability measure β on Γ there corresponds a $\mu \in \mathcal{W}_{3,4}$.

Henceforth it is assumed that Γ is reachable in the original sense or that we are dealing only with its reachable subset.

Define $\mathcal{P} = \{\xi\in C(\underline{A}\times\Omega): \xi=\phi_t+\phi_x f \text{ for some } \phi \in C^1(A)\}$.

Proposition (11.6)

(a) $q^*(\xi) > -\infty$ on $\overline{\text{span}}[\Phi, G(x,t)]$ $\quad$ ($G(x,t) = \max[g(x,t),0]$)

(b) $q^*(\xi) = -\infty$ otherwise

(c) if $\xi = \lim\limits_{i\to\infty}[\phi_t^i + \phi_x^i f + \alpha^i G]$ in $C(\underline{A}\times\Omega)$ then

$$q^*(\xi) = \lim_{i\to\infty}[\min_{(x,t)\in\Gamma} \phi^i(x,t) - \phi^i(x_0,t_0)].$$

Proof

(a) $\xi = \phi_t + \phi_x f + \alpha G \in \text{span}[\Phi,G]$ for some $\alpha \in R$

$$q^*(\xi) \triangleq \inf[\int_{\underline{A}\times\Omega} \xi d\mu : \mu \in \mathcal{W}_{3,4}]$$

$$= \inf[\int_\Gamma \phi(x,t)d\beta - \phi(x_0,t_0) : \beta \text{ any prob. meas. on } \Gamma]$$

$$= \min_{(x,t)\in\Gamma} \phi(x,t) - \phi(x_0,t_0)$$

$\xi \in \overline{\text{span}}[\Phi,G]$ is dealt with in (c).

(b) $\xi \notin \overline{\text{span}}[\Phi,G]$. Since $\overline{\text{span}}[\Phi,G]$ is a linear subspace of $C(\underline{A}\times\Omega)$ $\exists$ a hyperplane separating it from ξ i.e. $\bar\mu \in C^*(\underline{A}\times\Omega)$ s.t.

$$\int \zeta d\bar\mu = 0 \text{ for } \zeta \in \overline{\text{span}}[\Phi,G]$$

$$\int \xi d\bar\mu < 0 \qquad \text{(see for example [Tay].)}$$

So for any $\mu \in \mathcal{W}_{3,4}$, $\beta > 0$, $\mu + \beta\bar\mu \in \mathcal{W}_{3,4}$ and

$$q^*(\xi) = \inf[\int\xi d\mu : \mu\in W_{3,4}]$$

$$= \inf[\beta\int\xi d\bar{\mu} : \beta>0]$$

$$q^*(\xi) = -\infty$$

(c) Take $\xi = \lim_{i\to\infty} \xi^i = \lim_{i\to\infty} \phi_t^i + \phi_x^i f + \alpha^i G$ strongly in $C(\underline{A}\times\Omega)$

As the conjugate of an upper-semi-continuous function not everywhere equal to $-\infty$, q^* is itself u.s.c., i.e.

$$q^*(\xi) \geq \limsup_i q^*(\xi^i) = \limsup_i [\min_{(x,t)\in\Gamma} \phi^i(x,t) - \phi^i(x_0,t_0)]$$

But Γ is reachable, so if (x^i,t^i) is a minimand of $\phi^i(x,t)$ over Γ, $\exists\, \mu^i \in W \subset W_{3,4}$ such that $\int\phi_t + \phi_x f d\mu^i = \phi(x^i,t^i) - \phi(x_0,t_0)\ \forall\ \phi$. Thus

$$q^*(\xi) = \inf[\int\xi d\mu : \mu\in W_{3,4}]$$

$$\leq \lim_i \int\xi d\mu^i = \lim_i \int\xi^i d\mu^i$$

i.e.

$$q^*(\xi) \leq \lim_i [\min_{(x,t)\in\Gamma} \phi^i(x,t) - \phi^i(x_0,t_0)].$$

Combining the two inequalities completes the proof. □

Remarks

(1) In general $\bar{\mu}$ in (b) is not positive, for example when ξ itself is positive. So $\mu + \beta\bar{\mu} \notin W$, which is why we wrote the Fenchel program in terms of W_2 and $W_{3,4}$.

(2) The role of reachability is seen in (c). The definitions of measure reachability in appendix I are designed to let (c) go through without the original reachability assumption.

§ 11.3 Dual Program

The dual functionals p^* and q^* give rise to a mathematical program on $C(\underline{A}\times\Omega)$.

$$\text{(D)} \qquad \text{maximize } \{q^*(\xi)-p^*(\xi)\} \text{ over } \xi \in C(\underline{A}\times\Omega)$$

The study of such dual programs was initiated by Fenchel for the case where the original, and hence the dual program, is defined on a finite dimensional space. For any such pair of convex programs,

$$q^*(\xi)-p^*(\xi) \leq \int\xi d\mu - q(\mu) - \int\xi d\mu + p(\mu) = p(\mu)-q(\mu)$$

i.e.

$$\eta(D) \leq \eta(W)$$

Fenchel showed that if the relative interiors of the effective domains of p and q have a common point then there is no 'duality gap' i.e.

$$\eta(D) = \eta(W) \tag{11.7}$$

Though this is not true in arbitrary topological spaces, Rockafellar has given a version which is: [Roc. 2, Thm. 1].

Theorem (11.8)

If either p or q is continuous at some point where both functions are finite, then (11.7) is true.

Further, if the underlying spaces are in duality , the theorem is true with p^* and q^* replacing p and q in the statement [Roc. 2, cor. 2].

In our original problem the effective domains of p and q are $\mathcal{W}_2$ and

$W_{3,4}$ respectively; in the w* topology neither of these sets have interior points i.e. p and q are nowhere continuous. However, p* is continuous on $C(\underline{A}\times\Omega)$ and q^* is not everywhere equal to $-\infty$. The dual of $C^*(\underline{A}\times\Omega)$ with weak * topology is $C(\underline{A}\times\Omega)$ i.e. they are in duality and the theorem applies: $\eta(D) = \eta(W)$.

Note: In the application of duality theory (D) has been taken as the primal problem, with dual (W). (D) does not necessarily have a solution but solutions to (W) have already been constructed.

§ 11.4 The Value of the Weak Problem

The value of (D) is computed using (11.4) and (11.6).

$$\eta(D) = \sup\{q^*(\xi)-p^*(\xi):\xi\in C(\underline{A}\times\Omega)\}$$

$$= \sup\{q^*(\xi)-p^*(\xi):\xi\in \overline{\text{span}}[\Phi,G(x,t)]\}$$

$$= \sup\{\lim_{i\to\infty}[\min_{(x,t)\in\Gamma} \phi^i(x,t)-\phi^i(x_0,t_0)] - (T-t_0).$$

$$\max_{(x,t,u)\in\underline{A}\times\Omega} [\lim_{i\to\infty} \phi^i_t(x,t)+\phi^i_x(x,t)f(x,t,u)+\alpha^i G(x,t)-\ell(x,t,u)]^+\}$$

the supremum being taken over sequences $\{\phi^i\}\subset C^1(\underline{A})$ and $\{\alpha^i\}\subset R$.

Continuity of $\max_{(x,t,u)} [h(x,t,u)]^+$ w.r.t. h enables us to interchange $\max_{(x,t,u)}$ and $\lim_{i\to\infty}$ above. Therefore

$$\eta(D) = \sup\{\bar{z}(\phi,\alpha):\phi\in C^1(\underline{A})\alpha\in R\} \tag{11.9}$$

where

$$\bar{z}(\phi,\alpha) \triangleq \min_{(x,t)\in\Gamma} [\phi(x,t)-\phi(x_0,t_0)] - (T-t_0).$$

$$\max_{(x,t,u)\in \underline{A}x\Omega} [\phi_t(x,t)+\phi_x(x,t)f(x,t,u)+\alpha G(x,t)-\ell(x,t,u)]^+$$

Positivity of G makes $\bar{z}(\phi,\alpha)$ a monotone non-increasing function of α for fixed ϕ. Let $G^+ = \{(x,t)\in\underline{A}:G(x,t)>0\}$ and $G^0 = \underline{A}\backslash G^+ = \{(x,t)\in\underline{A}:G(x,t)\underline{<}0\}$. If $(x,t) \in G^+$, for any ϕ, decreasing α sufficiently

$$\phi_t(x,t) + \phi_x(x,t)f(x,t,u) + \alpha G(x,t) - \ell(x,t,u) \leq 0$$

$\forall$ $(x,t,u) \in \underline{A}x\Omega$, because the function on the l.h.s. is uniformly continuous (its domain $\underline{A}x\Omega$ is compact).

So

$$\eta(D) = \sup\{z(\phi):\phi\in C^1(\underline{A})\} \tag{11.10}$$

where

$$z(\phi) \triangleq \min_{(x,t)\in\Gamma} [\phi(x,t)-\phi(x_0,t_0)] - \max_{(x,t,u)\in G^0x\Omega} [\phi_t+\phi_x f-\ell]^+ (T-t_0)$$

Consider the set $\Psi \triangleq \{\phi\in C^1(\underline{A}):\phi(x,t)\underline{>}0$ on Γ, $\phi_t+\phi_x f-\ell\underline{<}0$ on $G^0x\Omega\}$. If $\phi \in \Psi$ and $\phi(x,t) \geq b > 0$ on Γ then $\phi-b \in \Psi$ and we therefore always assume $\phi(x,t) = 0$ for some $(x,t) \in \Gamma$. Note that $-[\phi(x_0,t_0)-b] > -\phi(x_0,t_0)$. When $\phi \in \Psi$ therefore

$$\eta(D) \geq z(\phi) = \min_{(x,t,)\in\Gamma} [\phi(x,t)-\phi(x_0,t_0)] = -\phi(x_0,t_0) \tag{11.11}$$

On the other hand, if $\phi \in C^1(\underline{A})$, defining

$$k \triangleq \max_{(x,t,u)\in G^0x\Omega} [\phi_t(x,t)+\phi_x(x,t)f(x,t,u)-\ell(x,t,u)]^+ \geq 0$$

and

$$\tilde{\phi}(x,t) \triangleq \phi(x,t) - \min_{(y,s)\in\Gamma} [\phi(y,s)] + k(T-t), \quad \tilde{\phi} \in C^1(\underline{A})$$

then

$$\tilde{\phi}(x,t) \geq k(T-t) \geq 0 \ \forall \ (x,t) \in \Gamma$$

$$\tilde{\phi}_t + \tilde{\phi}_x f - \ell = \phi_t + \phi_x f - \ell - k \leq 0 \text{ on } G^0 x\Omega.$$

So $\tilde{\phi} \in \Psi$, while

$$z(\tilde{\phi}) = \min_{(x,t)\in\Gamma} [\tilde{\phi}(x,t) - \tilde{\phi}(x_0,t_0)]$$

$$= \min_{(x,t)\in\Gamma} [\phi(x,t)+k(T-t)] - \phi(x_0,t_0) - k(T-t_0)$$

$$\geq \min_{(x,t)\in\Gamma} [\phi(x,t)] - \phi(x_0,t_0) - k(T-t_0) = z(\phi) \qquad (11.12)$$

Combining (11.11) and (11.12)

$$\eta(D) = \sup\{z(\phi):\phi\in\Psi\} = \sup\{-\phi(x_0,t_0):\phi\in\Psi\}.$$

Finally, $\eta(D) = \eta(W) = \eta(S)$ ((S) the control problem) yields:

Theorem (11.13)

The value of the relaxed control problem (S) is equal to

$$\sup\{-\phi(x_0,t_0):\phi\in C^1(\underline{A}), \phi\geq 0 \text{ on } \Gamma, \ \phi_t+\phi_x f-\ell\leq 0 \text{ on } G^0 x\Omega\}.$$

Comments:

(1) Theorem (11.13) does not assert the existence of a $\phi \in \Psi$ solving $\eta(S) = -\phi(x_0,t_0)$, just a sequence $\{\phi^i\} \subset \Psi$ such that $\eta(S) = \lim_{i\to\infty} -\phi^i(x_0, t_0)$. Even if a solution ξ to (D) exists, one can only claim that ξ is

continuous. More generally it will be discontinuous. Of course if $\eta(S) = -\phi(x_0,t_0)$ then $\xi = \phi_t + \phi_x f$ does solve (D).

(2) For problems without state constraints the same result obtains with G^0 replaced by $\underline{A}$. Inclusion of a constraint widens the class of admissible $\phi \in \Psi$ since the defining inequalities are required to hold over smaller sets in $\underline{A}\times\Omega$. The obvious conclusion that constrained problems have higher values emerges.

(3) Multiple state constraints $g_i(x(t),t) \leq 0$ $i = 1, \ldots M$ are easily handled, replacing G^0 with $\bigcap_{i=1}^{M} G_i^0$. Other constraints which fit into the framework of part I, even involving the control variable u, are similarly dealt with.

Note

The hypothesis ensuring the existence of at least one admissible control-trajectory pair also ensures that G^0 is non-empty.

$G^0 = \{(x,t) \in R^{n+1} : \max[0,g(x,t)] = 0\}$ i.e. G^0 is the inverse image of the closed set $\{0\}$ under the continuous map $(x,t) \to \max[0,g(x,t)]$. Therefore G^0 is closed, hence compact as it is contained in $\underline{A}$.

Chapter 12

NEW NECESSARY AND SUFFICIENT CONDITIONS FOR OPTIMALITY

We now have sufficiently sharp tools for a complete characterization of solutions to the strong control problem. That a solution exists is an essential part of the equivalence proofs (part I, §5.3); together with the value theorem (11.13) this enables us to assert the existence of:

(12.1) an admissible control-trajectory pair $(\bar{u}(\cdot),\cdot,\bar{x}(\cdot))$ satisfying (S2), (S3) and S(4), with terminal time $\bar{t}_1$, $\bar{u} : t \to \bar{\mu}_t$.

(12.2) A sequence $\{\phi^i\} \subset \Psi = \{\phi \in C^1(\underline{A}) : \phi \geq 0 \text{ on } \Gamma,\ \phi_t + \phi_x f - \ell \leq 0 \text{ on } G^0 \times \Omega\}$ such that

$$(12.3) \qquad \int_{t_0}^{\bar{t}_1} \int_\Omega \ell(\bar{x}(t),t,u)\,d\bar{\mu}_t(u)\,dt = \eta(S) = \lim_{i\to\infty}\{-\phi^i(x_0,t_0)\}.$$

<u>Theorem (12.4)</u>

An admissible pair $(u(\cdot),\cdot,x(\cdot))$ with terminal time t_1 solves the control problem if and only if

$$\lim_{i\to\infty}\{t \to \phi_t^i(x(t),t) + \int_\Omega [\phi_x^i(x(t),t) f(x(t),t,u) - \ell(x(t),t,u)]\,d\mu_t(u)\} = 0 \qquad (12.4)\,(a)$$

strongly in $L^1[t_0,t_1]$ and $\lim_{i\to\infty}[\phi^i(x(t_1),t_1)] = 0$ (12.4) (b)

for some sequence $\{\phi^i\} \in \Psi$.

<u>Proof</u>

(a) Sufficiency is an obvious extension of the verification theorem (9.11).

(b) Necessity. Take $(\bar{u}(\cdot),\cdot,\bar{x}(\cdot))$ an optimal pair and $\{\phi^i\} \subset \Psi$ a sequence, as in (12.1) and (12.2). Then for each $i = 1, 2, \ldots$

$$\int_{t_0}^{\bar{t}_1}[\phi^i_t(\bar{x}(t),t)+\int_\Omega \phi^i_x(\bar{x}(t),t)f(\bar{x}(t),t,u)d\bar{\mu}_t(u)]dt = \phi^i(\bar{x}(\bar{t}_1),\bar{t}_1)-\phi^i(x_0,t_0)$$

i.e.

$$\lim_i \int_{t_0}^{\bar{t}_1}[\phi^i_t+\int_\Omega \phi^i_x f d\bar{\mu}_t]dt = \lim_i[\phi^i(\bar{x}(\bar{t}_1),\bar{t}_1)] - \lim_i[\phi^i(x_0,t_0)]$$

$$= \lim_i[\phi^i(\bar{x}(\bar{t}_1),\bar{t}_1)] + \int_{t_0}^{\bar{t}_1}\int_\Omega \ell(\bar{x}(t),t,u)d\bar{\mu}_t dt$$

(by (12.3))

therefore

$$\lim_i \int_{t_0}^{\bar{t}_1}[\phi^i_t+\int_\Omega (\phi^i_x f-\ell)d\bar{\mu}_t]dt = \lim_i[\phi^i(\bar{x}(\bar{t}_1),\bar{t}_1)]$$

But $\phi^i_t + \phi^i_x f - \ell \leq 0$ and $\phi^i(\bar{x}(\bar{t}_1),\bar{t}_1) \geq 0$ therefore $\lim_i[\phi^i(\bar{x}(\bar{t}_1),\bar{t}_1)] = 0$

and

$$\lim_i[\int_{t_0}^{\bar{t}_1}[\phi^i_t+\int_\Omega (\phi^i_x f-\ell)d\bar{\mu}_t]dt = 0$$

Non-positivity of the integrand implies that the sequence of functions $\{t\to[\phi^i_t\int_\Omega \phi^i_x f-\ell d\bar{\mu}_t]|_{(\bar{x}(t),t)}\}$ converges strongly to the zero function in $L^1[t_0,\bar{t}_1]$. □

Note .

The essential point is that a verification sequence $\{\phi^i\}$ exists, so that the verification condition becomes necessary for optimality .

Chapter 13

DYNAMIC PROGRAMMING FOR A WIDER CLASS OF PROBLEMS

In this chapter we shall extend the results of chapters 11 and 12.

§13.1 The Problems of Mayer and Bolza

If instead of the integral cost functional $\int_{t_0}^{t_1} \ell(x(t),t,u(t))dt$ there is a terminal cost, $\chi(x(t_1),t_1)$ the control problem is said to be in Mayer form

$$\text{(S)} \quad \begin{aligned} &\min \chi(x(t_1),t_1) && \text{(S1)'} \\ &\text{subject to (S2) and (S3)} && \text{(omitting state constraints)} \end{aligned}$$

When $\chi \in C^1(\underline{A})$, (S) can be written in Lagrange form,

$$\text{(S)} \quad \begin{aligned} &\min \int_{t_0}^{t_1} \int_{\Omega} [\chi_t(x(t),t)+\chi_x(x(t),t)f(x(t),t,u)]d\mu_t(u)dt && \text{(S1)} \\ &\text{subject to (S2) and (S3)} \end{aligned}$$

Note:

Since (x_0,t_0) is fixed we can put $\chi(x_0,t_0) = 0$ therefore

$$\eta(S) = \sup\{-\phi(x_0,t_0):\phi\in\Psi\}.$$

Here $\quad \Psi = \{\phi\in C^1(\underline{A}):\phi\geq 0 \text{ on } \Gamma,\ \phi_t+\phi_x f-\chi_t-\chi_x f\leq 0 \text{ on } \underline{A}\times\Omega\}$

Put

$$\Phi = \{\psi \epsilon C^1(A) : \psi \geq -\chi \text{ on } \Gamma,\ \psi_t + \psi_x f \leq 0 \text{ on } \underline{A} x \Omega\} \qquad (13.1)(a)$$

Then

$$\Phi = \Psi + \chi \text{ since } \psi \in \Phi \Leftrightarrow \psi + \chi \in \Psi$$

$$\chi(x_0, t_0) = 0 \quad \text{implies } \eta(S) = \sup\{-\psi(x_0, t_0) : \psi \epsilon \Phi\} \qquad (13.1)(b)$$

(13.1)(a) and (b) constitute the expected generalization of the well known verification theorem for Mayer problems but are valid only in case $\chi \in C^1(\underline{A})$. Suppose $\chi \in C(\underline{A})$, define Φ as in (13.1)(a) and

$$\eta \triangleq \sup\{ -\psi(x_0, t_0) : \psi \in \Phi \}$$

For any admissible pair, $\psi \in \Phi$

$$\int_{t_0}^{t_1} \psi_t + \psi_x f dt = \psi(x(t_1), t_1) - \psi(x_0, t_0) \leq 0$$

therefore

$$\chi(x(t_1), t_1) \geq -\psi(x(t_1), t_1) \geq -\psi(x_0, t_0)$$

$$\eta(S) \geq \eta \qquad (13.2)$$

Given $\varepsilon > 0$ we can obtain $\bar{\chi} \in C^\infty(\underline{A}) \subset C^1(\underline{A})$ approximating χ uniformly on $\underline{A}$

i.e. $\quad |\bar{\chi}(x,t) - \chi(x,t)| < \varepsilon$ for all $(x,t) \in \underline{A}$.

Let $(\bar{S})$ be the Mayer problem with $\bar{\chi}$ replacing χ. By (13.1)(b)

$$\eta(\bar{S}) = \sup\{-\bar{\psi}(x_0, t_0) : \bar{\psi} \epsilon \bar{\Phi}\}$$

$$\bar{\Phi} \triangleq \{\bar{\psi}\in C^1(\underline{A}):\bar{\psi}\geq -\bar{\chi} \text{ on } \Gamma',\ \bar{\psi}_t+\bar{\psi}_x f\leq 0 \text{ on } \underline{A}\times\Omega\}$$

Uniformity of approximation implies that if $\bar{\psi}\in\bar{\Phi}$, $\bar{\psi}+\varepsilon \in \Phi$, so

$$\eta \geq \sup\{-\bar{\psi}(x_0,t_0)-\varepsilon:\bar{\psi}\in\bar{\Phi}\} = \eta(\bar{S})-\varepsilon$$

and

$$\eta(\bar{S}) \geq \eta(S)-\varepsilon, \text{ hence } \eta \geq \eta(S) - 2\varepsilon \tag{13.3}$$

(13.2) and (13.3) prove the value theorem for the Mayer problem with arbitrary cost function $\chi \in C(\underline{A})$.

The most general cost functional is $\int_{t_0}^{t_1}\ell(x(t),t,u(t))dt + \chi(x(t_1),t_1)$ in which case (S) is known as a Bolza problem, for which the value is

$$\eta(S) = \sup\{-\phi(x_0,t_0):\phi\in\Psi\}$$

$$\Psi \triangleq \{\phi\in C^1(\underline{A}):\phi\geq -\chi \text{ on } \Gamma,\ \phi_t+\phi_x f-\ell\leq 0 \text{ on } \underline{A}\times\Omega\}.$$

This value theorem leads immediately to necessary and sufficient conditions for optimality, as before.

Remark: Existence results for Mayer problems are to be found in [McS].

§ 13.2 Differential Inclusions

Recall the control problem where the dynamical system is described by a differential inclusion, (I), and the suitably posed weak problem (WI). In §(6.3) we showed $\eta(I) = \eta(WI)$. By writing (WI) as a Fenchel program and applying the duality results, the value of (I) is obtained.

$$\eta(I) = \sup\{z(\phi,\alpha):\phi\in C^1(\underline{A}),\alpha\in R\}$$

$$z(\phi,\alpha) \triangleq \min_{(x,t)\in\Gamma} [\phi(x,t)-\phi(x_0,t_0)] - (T-t_0)\cdot$$

$$\max_{(x,t,\dot{x})\in \underline{A}\times E} [\phi_t(x,t)+\phi_x(x,t)\dot{x}+\alpha \operatorname{dist}[\dot{x},E(x,t)]-\ell(x,t,\dot{x})]^+$$

The dynamic constraint $\operatorname{dist}[\dot{x},E(x,t)]$ can be treated similarly to the state constraint in §(11.4).

$$\eta(I) = \sup\{-\phi(x_0,t_0):\phi\in\Psi\}$$

$$\Psi \triangleq \{\phi\in C^1(\underline{A}):\phi\geq 0 \text{ on } \Gamma,\quad \phi_t(x,t)+\phi_x(x,t)\dot{x}-\ell(x,t,\dot{x})\leq 0 \ \forall\ (x,t)\in\underline{A},\ \dot{x}\in E(x,t)\}$$

The partial differential inequality can be written formally as

$$\phi_t(x,t)+\phi_x(x,t)E(x,t)-\ell(x,t,E(x,t)) \leq 0 \ \forall\ (x,t) \in \underline{A}$$

§13.3 Alternative Forms When State Constraints are Present

In (S) the state constraint is $g(x(t),t) \leq 0 \ \forall\ t \in [t_0,t_1]$. Suppose g is differentiable, then another state variable x_{n+1} can be considered,

$$\dot{x}_{n+1}(t) = g_t(x(t),t) + \int_\Omega g_x(x(t),t)f(x(t),t,u)d\mu_t(u) \text{ a.e. in } [t_0,t_1]$$

$$x_{n+1}(t_0) = g(x_0,t_0) = x_{n+1,0}$$

Denote the augmented state (x,x_{n+1}) by $\tilde{x}$ etc. The alternative form of (S) is

$$
(S)\quad \begin{cases}
\min \int_{t_0}^{t_1} \int_{\Omega} \ell(x(t),t,u)\,d\mu_t(u)\,dt & \text{(S1) subject to} \\
\dot{\tilde{x}} = \int_{\Omega} \tilde{f}(x(t),t,u)\,d\mu_t(u) \text{ a.e. in } [t_0,t_1] & \\
 & \text{(S2)} \\
\tilde{x}(t_0) = \tilde{x}_0 \qquad (\tilde{x}(t_1),t_1) \in \tilde{\Gamma} & \\
u \text{ a relaxed control} & \text{(S3)} \\
x_{n+1}(t) \leq 0 \text{ for all } t \text{ in } [t_0,t_1] & \text{(S4)}
\end{cases}
$$

$\tilde{\Gamma} = \{(x,x_{n+1},t):(x,t)\in\Gamma,\ x_{n+1}\leq 0\}$. Take $\underline{\tilde{A}} \subset R^{n+2}$ a suitable set as before. The value theorem appears as:

$$\eta(S) = \sup\{-\tilde{\phi}(\tilde{x}_0,t_0):\tilde{\phi}\in\tilde{\Psi}\}$$

$$\tilde{\Psi} \triangleq \{\tilde{\phi}\in C^1(\underline{\tilde{A}}):\tilde{\phi}\geq 0 \text{ on } \tilde{\Gamma},\ \tilde{\phi}_t+\tilde{\phi}_{\tilde{x}}\tilde{f}-\ell\leq 0 \text{ on } \tilde{G}^0 x\Omega\} \tag{13.4}$$

$$\tilde{G}^0 = \underline{\tilde{A}} \cap \{x_{n+1}\leq 0\}$$

Compare (13.4) with (11.13). To any $\tilde{\phi} \in \tilde{\Psi}$ one can define:

$$\phi(x,t) = \tilde{\phi}(x,g(x,t),t) \in \Psi, \text{ for } \phi \geq 0 \text{ on } \Gamma$$

and

$$\phi_t + \phi_x f - \ell = \tilde{\phi}_t + \tilde{\phi}_{x_{n+1}} g_t + \tilde{\phi}_x f + \tilde{\phi}_{x_{n+1}} g_x f - \ell$$

$$= \tilde{\phi}_t + \tilde{\phi}_{\tilde{x}}\tilde{f} - \ell \leq 0 \text{ on } G^0 x\Omega$$

Further,

$$-\phi(x_0,t_0) = -\tilde{\phi}(x_0,g(x_0,t_0),t_0) = -\tilde{\phi}(\tilde{x}_0,t_0)$$

However there is no obvious means of constructing $\tilde{\phi} \in \tilde{\Psi}$ from any given $\phi \in \Psi$. Thus the class $\tilde{\Psi}$ is essentially no larger than Ψ and characterizations of optimality by $\tilde{\Psi}$ may yield more information than the original theorem (12.4).

In establishing the value theorem it was argued that at any point $(x,t) \in G^+$, taking α sufficiently negative, $\phi_t + \phi_x f + \alpha G(x,t) - \ell \leq 0$, for any ϕ. For the same reason, the inequalities

$$\phi_t + \phi_x f - \ell \leq 0 \text{ on } G^0 x\Omega, \ \phi \in C^1(\underline{A})$$

$$\phi_t + \phi_x f - \ell \leq \gamma \max[0, g(x,t)] \text{ on } \underline{A}x\Omega \text{ for some } \gamma \qquad (13.5)$$

are equivalent. The same applies to the p.d.i. defining $\tilde{\Psi}$. (13.5) is useful in that it indicates the relationship of our necessary condition to the known one, the Maximum Principle (see appendix 7).

Chapter 14

THE NEW NECESSARY CONDITION

We shall briefly examine the relationship of the new necessary condition with the Maximum Principle, including derivation of results similar to ours which may prove more useful in this direction, and the application of dynamic programming conditions to designing optimal control algorithms.

§ 14.1 The Maximum Principle

For ease of presentation let us consider control problems (S) where ℓ and f are continuously differentiable with respect to x and which has ordinary optimal trajectory-control pairs $(x(\cdot),u(\cdot))$. Further, assume the Fenchel dual problem (D) has a solution $\phi \in C^2(\underline{A})$, so that (3.4) holds with $\{\phi\}$ replacing $\{\phi^i\}$. The strength of this assumption is apparent from the introduction.

$$\phi_t(x(t),t) + \phi_x(x(t),t)f(x(t),t,u(t)) - \ell(x(t),t,u(t)) = 0 \quad \text{a.e. in } [t_0,t_1]$$

and

$$\phi_t(x,t) + \phi_x(x,t)f(x,t,u) - \ell(x,t,u) \leq 0 \quad \text{on } \underline{A}\times\Omega$$

imply

$$\frac{\partial}{\partial x}\{[\phi_t+\phi_x f-\ell]\big|_{(x(t),u(t))}\} = 0 \quad \text{a.e. in } [t_0,t_1]$$

i.e.

$$[\phi_{tx}+\phi_{xx}f+\phi_x f_x-\ell_x]\big|_{(x(t),u(t))} = 0$$

Defining $p(t) \triangleq \phi_x(x(t),t)$, $p(\cdot)$ is absolutely continuous on $[t_0,t_1]$ and satisfies

$$\dot{p}(t) = \phi_{xt}(x(t),t) + \phi_{xx}(x(t),t)f(x(t),t,u(t))$$

i.e. $$\dot{p}(t) = -p(t)f_x(x(t),t,u(t)) + \ell_x(x(t),t,u(t)) \text{ a.e. in } [t_0,t_1]$$

$$\phi(x(t_1),t_1) = 0, \quad \phi(x,t) \geq 0 \text{ on } \Gamma \text{ imply that}$$

$$p(t_1)[(x(t_1),t_1)-(y,s)] \leq 0 \;\forall\; (y,s) \in \Gamma$$

Set

$$H(x,t,u,p) \triangleq pf(x,t,u) - \ell(x,t,u)$$

Theorem (14.1)

Suppose $(x(\cdot),u(\cdot))$ is admissible for (S). A necessary condition for optimality of $(x(\cdot),u(\cdot))$ is the existence of $p : [t_0,t_1] \to R^n$, absolutely continuous, satisfying:

$$\dot{p}(t) = -H_x(x(t),t,u(t),p(t)) \text{ a.e. in } [t_0,t_1] \qquad (14.2)$$

$$p(t_1)[(x(t_1),t_1)-(y,s)] \leq 0 \;\forall\; (y,s) \in \Gamma \qquad (14.3)$$

$$H(x(t),t,u(t),p(t)) \geq H(x(t),t,v,p(t)) \text{ for all } v \in \Omega$$
$$\text{and almost all } t \in [t_0,t_1] \qquad (14.4)$$

(14.1) is generally known as the Maximum Principle (or Pontryagin's Principle) [McS] while (14.2) and (14.3) are referred to as the adjoint equation and transversality conditions respectively. The principle is true for a wider range of problems than the assumptions necessary for the formal derivation above and an important question is whether or not (12.4) contains the M.P. in any generality.

Little is known about the convergence of the maximizing sequence $\{\phi^i\}$ except along the optimal trajectory where the limit is absolutely continuous,

$$\lim_{i\to\infty} \dot{\phi}^i(x(t),t) = \lim_{i\to\infty}[\phi^i_t+\phi^i_x f]_{(x(t),u(t))} = \mathcal{L}(x(t),t,u(t))$$

Taking the value function of (S) as a guide to the sort of limit function to expect, examples indicate that even along the optimal trajectory $\lim_{i\to\infty} \phi^i_x$ need not exist, i.e. the trajectory runs along a discontinuity of the value function (this example is presented in §(14.4)).

Let ℓ and f be continuously differentiable w.r.t. x and, if necessary, replace the sequence $\{\phi^i\}$ with another, such that $\phi^i \in C^2(\underline{A})$. Given $\varepsilon > 0$ choose i sufficiently large that along the optimal trajectory $(x(\cdot),u(\cdot))$

$$0 \geq [\phi^i_t+\phi^i_x f-\ell]|_{(x(t),u(t))} > -\varepsilon \text{ except in } T \subset [t_0,t_1],\ m(T) \leq \varepsilon$$

Comparing x(t) with $x(t\pm\delta)$, δ sufficiently small

$$[\phi^i_t+\phi^i_x f-\ell]|_{(x(t),u(t))} - [\phi^i_t+\phi^i_x f-\ell]|_{(x(t\pm\delta),u(t))} > -\varepsilon$$

If $u(\cdot)$ is a.e. continuous and t is a point of continuity

$$x(t\pm\delta) = x(t) \pm \delta f(x(t),t,u(t)) + O(\delta) \text{ where } \lim_{\delta\to\infty} \frac{O(\delta)}{\delta} \to 0.$$

Formally, we obtain,

$$\pm\, \delta[(\phi^i_{tx}+\phi^i_{xx}f+\phi^i_x f_x-\ell_x)f]|_{(x(t),u(t))} + O(\delta) > -\varepsilon$$

So if $p^i(t) \triangleq \phi_x^i(t)$ in the limit as $i \to \infty$,

$$\lim_{i\to\infty}[(\dot{p}^i+p^i f_x-\ell_x)f]\Big|_{(x(t),u(t))} = 0 \text{ a.e. in } [t_0,t_1] \qquad (14.5)$$

When each component of $f(x(t),u(t))$ is strictly bounded away from zero, uniformly in t, (14.5) yields the adjoint equation. At present this unsatisfactory ad hoc assumption seems unavoidable, when deriving the M.P. from our necessary condition (12.4).

§ 14.2 Hermes Approximation for Time Optimal Control

From an entirely different approach to ours, Hermes has arrived at a very similar conclusion, specifically for time optimal control, where $\ell = 1$. Briefly, the line of thought in [He] is that given a 'convex' dynamical system,

$$\dot{x}(t) = f(x(t),t,u(t)) \quad u(t) \in \Omega \text{ compact} \qquad (S1)$$

$R(x,t) \triangleq \{f(x,t,u):u\in\Omega\}$ a compact convex subset of R^n, satisfying the Fillipov conditions [He p. 415] (subsumed by hypothesis H in part I) there exists an ε *approximate equivalent* system (defined below).

$$\dot{x}(t) = h^{\varepsilon}(x(t),t,v(t)) \quad v(t) \in V^{\varepsilon}. \qquad (S1\varepsilon)$$

such that the support function $U^{\varepsilon} \in C^1(R^n\backslash\{0\}x\underline{A})$

$$U^{\varepsilon}(p,x,t) \triangleq \arg\sup_{v\in V^{\varepsilon}}[p.h^{\varepsilon}(x,t,v)+1] \qquad (9.7\varepsilon)$$

(S1ε) is said to be ε approximately equivalent to (S1) if

$R^{\varepsilon}(x,t) \triangleq \{h^{\varepsilon}(x,t,v) : v \in V^{\varepsilon}\} \supset R(x,t)$ and in the Hausdorff sense

$$\mathrm{dist}(R^{\varepsilon}(x,t), R(x,t)) < \varepsilon \text{ for all } (x,t) \in \underline{A}$$

For $r^*(p) \triangleq \arg\sup_{r \in R}[p.r]$ to be continuously differentiable on $R^n \backslash \{0\}$ it is necessary and sufficient that R be compact, with a smooth boundary having strictly positive Gaussian curvature at all points (this implies strict convexity of R). The desired approximation is therefore found in proving:

Theorem (14.6)

For any $\varepsilon > 0$ there exists an ε approximate system equivalent to (S1) with dynamics $\dot{x} \in h^{\varepsilon}(x,t,v)$ $v \in V(\varepsilon)$ where:

(i) $V(\varepsilon)$ is the unit ball in R^n, B^n.

(ii) h^{ε} is C^{∞} on $\underline{A} \times B^n$ and $h(x,t,\cdot)$ is 1-1 $\forall$ $(x,t) \in \underline{A}$.

(iii) $R^{\varepsilon}(x,t)$ has smooth boundary with positive Gaussian curvature. □

Hermes concludes with

Theorem (14.7)

Let $\varepsilon(k) \to 0$ as $k \to \infty$. If $x^k(\cdot)$ is the time optimal trajectory when (S1) is replaced by (S1$\varepsilon(k)$) then $\{x^k(\cdot)\}$ is equicontinuous and has a uniformly convergent subsequence, converging to $x(\cdot)$, the time optimal trajectory for the original problem. □

Denote $h^{\varepsilon(k)}$ by h^k etc. Assuming the final data are non-characteristic, uniformly in k, the p.d.e.

$$\left.\begin{aligned} &V_t^k(x,t) + V_x^k(x,t)h^k(x,t,U^k(V_x^k,x,t)) + 1 = 0 \\ &(x,t) \in \mathrm{int}\,\underline{A}, \quad V^k(x,t) = 0 \text{ on } \Gamma \end{aligned}\right\} \quad (14.8)$$

has a solution $V^k \in C^1(\underline{A})$ for each $k = 1, 2, \ldots$ and the cost of the k^{th} problem is $V^k(x_0,t_0)$. Taking the subsequence in (14.7),

$$\eta(S) = \lim_{k\to\infty} V^k(x_0,t_0)$$

Thus V^k corresponds to $-\phi^k$ determined from our dual (D).

Further by definition of U^k and h^k

$$V_t^k(x,t) + V_x^k(x,t)h^k(x,t,v) + 1 \leq 0 \ \forall \ (x,t,v) \in \underline{A}xV(k)$$

and

$$V_t^k(x,t) + V_x^k(x,t)f(x,t,u) + 1 \leq 0 \ \ \forall \ (x,t,u) \in \underline{A}x\Omega$$

The ε-approximate equivalence technique provides an alternative proof of the relaxed dynamic programming principle in the case of time optimal control of a convex system. Bridgland [Bri 2] has extended the principle to the more general 'minimum miss-distance problem' and there can be little doubt that the general Lagrange problem for non-convex systems admitting relaxed controls is amenable to the same treatment. This proof is in many ways superior to ours, for it offers the possibility of constructing the sequence $\{V^k\}$, once a general method for constructing h^k is found. (See [He pp. 424-6] for $f = g(x,t) + h(x,t,)u$.) These V^k are exact for the $\varepsilon(k)$ approximate-equivalent problem,

$$[V_t^k+V_x^kh^k-1]\Big|_{(x^k(t),U^k(V^k(x^k(t),t),x^k(t,t))} = 0 \text{ on } [t_0,t_1^k]$$

and therefore $p^k(t) \triangleq V_x^k(x^k(t),t)$ solves the approximate adjoint equation

$$\dot{p}^k(t) = -p^k(t)h_x^k(x^k(t),u^k(t))$$

Derivation of a maximum principle depends upon the convergence of the r.h.s. in the appropriate topology.

Despite this, the ε approximate technique will remain a theoretical tool as numerical implementation of any algorithm constructing h^k, U^k will be costly and numerically unstable as $\varepsilon \to 0$.

§ 14.3 Lower Bounds

A major difficulty in the design of any sequential algorithm for optimization is the choice of a stopping criterion. For the program

$$(R) \qquad \min\{R(x)=x\in \mathcal{R}\}$$

an algorithm may generate a sequence of points $\{x^i : i = 0,1,2...\}$ x^0 given, with $x^i \in \mathcal{R}$ or 'nearly' in $\mathcal{R}$ and, frequently, $\{R(x^i)\}$ a monotone decreasing sequence. The design objective is $\lim_{i\to\infty} R(x^i) = \eta(R)$ but of course in practice the algorithm must be stopped at a finite value of i, chosen such that $|R(x^i)-\eta(R)|$ is small. The difficulty is that $\eta(R)$ is not known until (R) is solved, in which case the algorithm is superfluous. Stopping must be judged from convergence of the values $R(x^i)$ alone and care taken to avoid stopping at spurious, far from optimal points.

Alternatively, $\eta(R)$ can be replaced by a lower bound $\eta < \eta(R)$, provided $|\eta-\eta(R)|$ is small. For the strong control problem (S), theorem (11.13) presents such a possibility:

$$\eta(S) = \sup\{-\phi(x_0,t_0):\phi\in C^1(\underline{A}),\phi\geq 0 \text{ on } \Gamma,\ \phi_t+\phi_x f-\ell\leq 0 \text{ on } G^0\times\Omega\}$$

Ostensibly we do not have to solve (S) to determine $\eta(S)$ or a 'good' lower bound η and in a few simple problems this may be so, when some guesswork produces a suitable ϕ. Otherwise ϕ is constructed, using another algorithm; from an isolated perspective the vicious circle is complete, our bootstrap snaps.

The seeds of duality germinate only if both programs are present. Let the original algorithm generate $\{x^i(\cdot),u^i(\cdot)\}$, admissible pairs for (S) and the other, $\{\phi^j\}$, admissible for the dual (D). For all i, j:

$$\int_{t_0}^{t_1^i}\int_\Omega \ell(x^i(t),t,u)\,d\mu_t^i(u) \geq \eta(S) \geq -\phi^j(x_0,t_0)$$

The obvious stopping criterion for both algorithms is nearness of the outer terms. Conceptually satisfactory, this scheme is inefficient even for programs where dual algorithms are known; to date the seeds have not borne fruit.

The theory of inequalities remains in the 3rd world of mathematics, underdeveloped and exploited principally for the enrichment of other areas, as Hölders inequality is used to demonstrate the duality $(L^p)^* = L^q$, $\frac{1}{p}+\frac{1}{q} = 1$, $p \geq 1$. For partial differential inequalities, Lakshmikantham and Leela [LL] and Plis [P] have proved some comparison theorems with the objective of bounding the solutions of the corresponding p.d.e.'s. Applying these to $\phi_t + \phi_x f - \ell \leq 0$ produces crude results: $-\phi(x_0,t_0) \leq \int_{t_0}^{t_f}\ell(t)\,dt$ where $\ell(t)$ is $\ell(x,t,u)$ evaluated along any trajectory. The state of the art does not admit anything better.

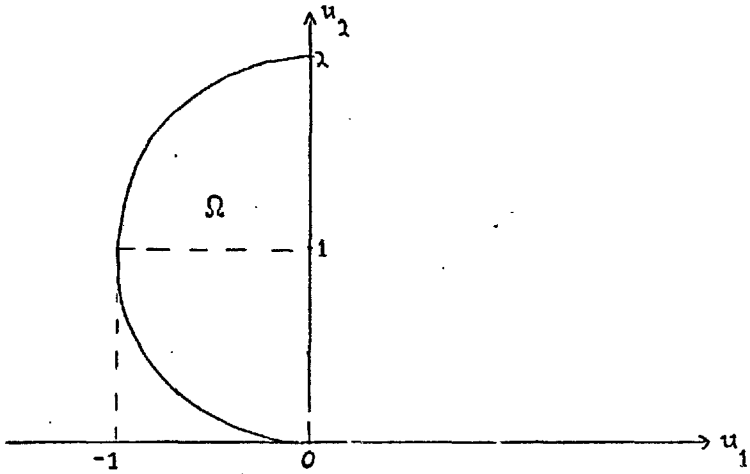
§ 14.4 An Example

The following time optimal control problem satisfies the conditions of theorem (9.9) and has a discontinuous Bellman function .

$$\dot{x}_1 = x_2 + u_1 \qquad x_1(0) = x_{10}$$

$$\dot{x}_2 = -x_1 + u_2 \qquad x_2(0) = x_{20}$$

$\min\{t_1 : x_1(t_1)=x_2(t_1)=0\}$, admissible controls being measurable functions $(u_1(\cdot),u_2(\cdot))$ with $(u_1(t),u_2(t)) \in \Omega$ a.e. t, Ω as indicated:



Optimal controls will be computed using the Maximum Principle.

$$H = \lambda_1(x_2+u_1) + \lambda_2(-x_1+u_2) - 1$$

$$\dot{\lambda}_1 = -H_{x_1} = \lambda_2 \qquad \lambda_1 = a \cos t + b \sin t$$

so

$$\lambda_2 = -H_{x_2} = -\lambda_1 \qquad \lambda_2 = b \cos t - a \sin t$$

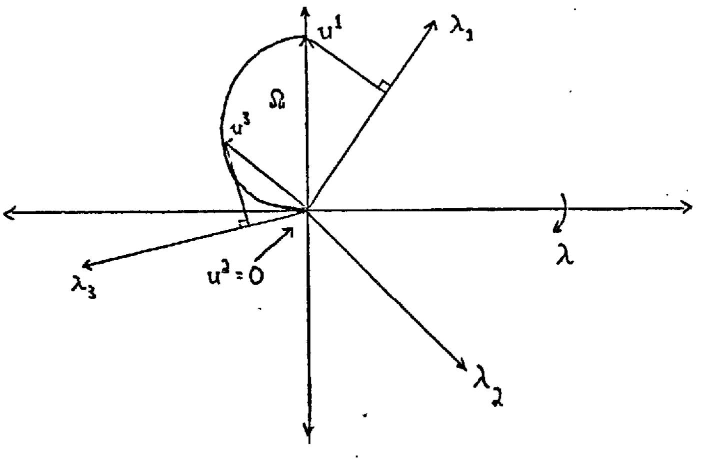
The optimal controls satisfy $u = \arg\max_{v \in \Omega} \lambda^T v$

Since λ rotates clockwise, taking λ initially at $(0,\sqrt{a^2+b^2})$ the control is:

$u = (0,2)$ $\qquad t \in [0,\pi/2]$ $\qquad$ 1

$u = (0,0)$ $\qquad t \in [\pi/2,\pi]$ $\qquad$ 2

$u = (\sin t,\ \cos t + 1)$ $\qquad t \in (\pi,2\pi]$ $\qquad$ 3



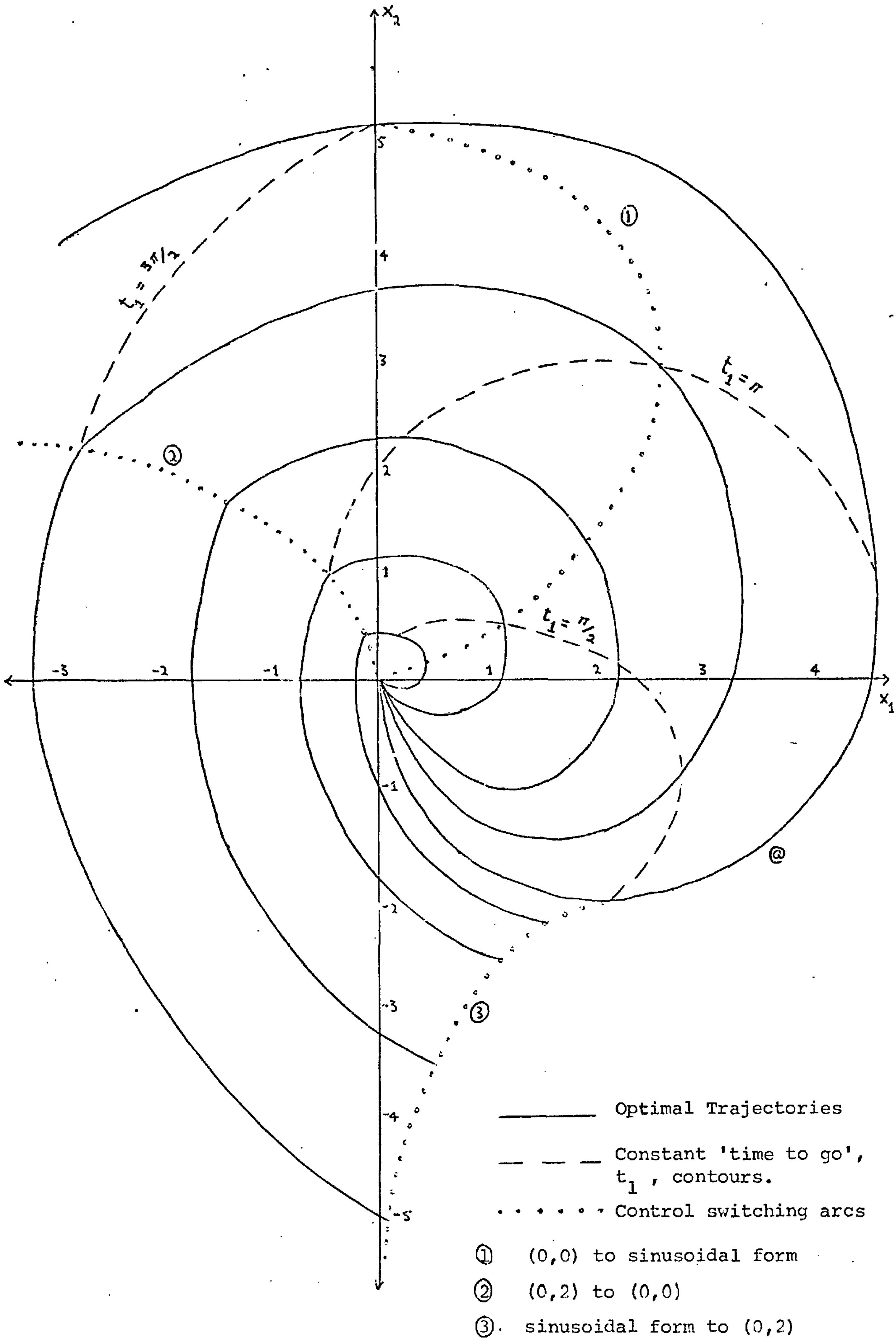
The controls have two switching points every period (2π). The origin is reachable only from the first and fourth quadrants since

in the 3rd quadrant $x_1 \leq 0$ $\qquad \dot{x}_1 = x_2 + u_1 \leq 0$

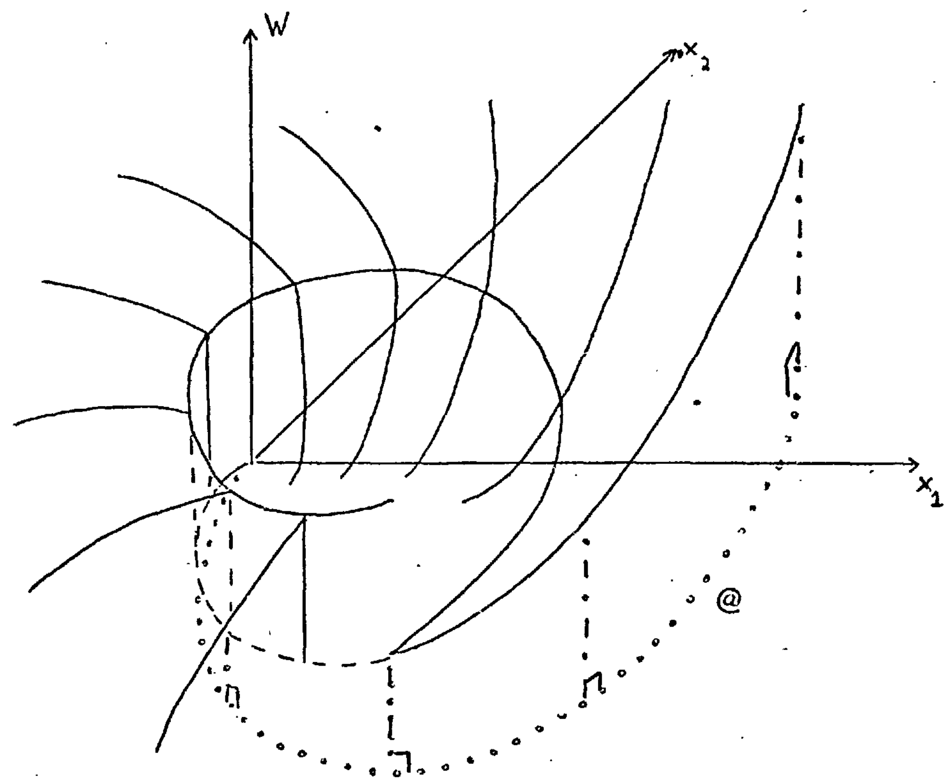
in the 2nd quadrant $x_2 \geq 0$ $\qquad \dot{x}_2 = -x_1 + u_2 \geq 0$

This is the cause of the discontinuity in the 'cost to go' function, i.e. even in a neighbourhood of the target set $\{(0,0)\}$ there are (x,t) such that $(0,0) \notin \operatorname{int} f(x,t,\Omega)$. This indicates a relationship between measure reachability and continuity of the 'cost to go' function though none has been established as yet.

The following pictures emerge from the solution of the problem for various initial points.



x_2
x_1
$t_1 = 3\pi/2$
$t_1 = \pi$
$t_1 = \pi/2$
①
②
③
Optimal Trajectories
Constant 'time to go', t_1 , contours.
Control switching arcs
① (0,0) to sinusoidal form
② (0,2) to (0,0)
③. sinusoidal form to (0,2)



It can be seen that there is indeed an optimal trajectory @ lying along the discontinuity of the Bellman function W (see thm. (9.9)).

Consider the initial point (2,-2) on this trajectory. Choose for A the square [0,2] x [-2,0] then $A\times[0,\pi/2]$ contains the augmented optimal trajectory from (2,-2,0), since the origin is reached at $t = \pi/2$.

The function $\phi \triangleq t - k/4[(x_1-2)^2+x_2^2]$ satisfies:

$$\phi_t + \phi_x f - 1 \leq 0 \quad \text{on } A\times[0,\pi/2]\times\Omega$$

with equality along the trajectory and $\Delta\phi = \phi(0,0,\pi/2) - \phi(2,-2,0) = \pi/2$. ϕ is a suitable choice for the verification theorem (9.11).

ϕ is very different from the Bellman function, the reason being that ϕ solves the partial differential inequality only over $A\times[0,\pi/2]\times\Omega$, while the optimal trajectories from initial points not on the curve considered above are not contained in $A\times[0,\pi/2]$. Therefore ϕ will not verify optimality of such trajectories.

This example clearly illustrates the pitfalls of attempting to construct verification functions from Bellman functions.

§14.5 Conclusions

It has been shown how duality theory may be applied to a convex program equivalent to an optimal control problem , to produce new necessary conditions for optimality in the latter , in terms of a sequence of continuously differentiable functions $\{\phi^i\}$. The most important question not resolved in the thesis is that of the convergence of this sequence . The example in the preceeding section shows that the Bellman function offers little guidance here , that in some problems where it is discontinuous , differentiable verification functions exist. Recent unpublished work by the author indicates that a slightly weaker result is true in general , namely that the sequence $\{\phi^i\}$ can be selected to converge to a Lipshitz continuous function ϕ . There are reasons to believe that this is the best possible result, that is , stronger conclusions cannot be valid for the entire range of control problems considered.

Appendix 1

Proof of Lemma (3.6)

(a) $\underline{\ell}(x,t,\dot{x})$ is defined on $\underline{A}xF$, F compact, convex and is lower semi-continuous and bounded there by the values $\ell(x,t,u)$, u a probability measure on Ω. Convexity of F enables us to extend $\underline{\ell}$ to $\underline{A}xR^n$ in the same way as it was extended to $\underline{A}xF$ in (3.2), maintaining lower semi-continuity and the bounds, written $|\underline{\ell}(x,t,\dot{x})| \leq k < \infty$ on $\underline{A}xR^n$. Thus $L(y,\dot{y}) = \underline{\ell}(x,t,\dot{x}/\dot{t})|\dot{t}|$ is defined and l.s.c. on $\underline{A}xR^{n+1}\backslash\{\dot{t}=0\}$. Let $\{(y_i,\dot{y}_i)\}$ be any sequence such that $\dot{t}_i \to 0$ then

$$|L(y,\dot{y})| \leq k|\dot{t}_i| \to 0 \tag{A1.1}$$

By definition $L(y,\dot{y}) = 0$ when $\dot{t} = 0$ so $L(y,\dot{y})$ is defined and l.s.c. on $\underline{A}xR^{n+1}$ hence on $\underline{A}x\underline{B}$. □

Note

If we had used the definition of $\underline{\ell}$ mentioned in remark (iii) following Proposition (3.4), the limit $|L(y_i,\dot{y}_i)|$ would depend upon the behaviour of $\dot{x}_i/\dot{t}_i$. If $(\dot{x}_i,\dot{t}_i) \in \underline{B}$, $\dot{t}_i \to 0 \Rightarrow ||\dot{x}_i/\dot{t}_i|| \to \infty$ and $\underline{\ell}(x_i,t_i\ \dot{x}_i/\dot{t}_i) = \infty$ for large i. Therefore $\lim_i |L(y_i,\dot{y}_i)| = \infty$. However, allowing $(\dot{x}_i,\dot{t}_i)$ not restricted to $\underline{B}$ we can choose $\dot{x}_i$ so that $|\underline{\ell}(x_i,t_i,\dot{x}_i/\dot{t}_i)| \leq k$ and $\lim_i |L(y_i,\dot{y}_i)| = 0$. Therefore L will not be homogeneous unless we agree to the convention $0.\infty = 0$. □

Schwartz [Sch 1 p. 25] gives the following definition:

Definition (A1.2)

Let μ be a Radon measure on a topological space X and $H : X \to Y$ where Y is a Hausdorff topological space. The mapping H is said to be Lusin μ-measurable if for every compact set $K \subset X$ and every $\delta > 0$ there exists a $K_\delta \subset K$ with $\mu(K\backslash K_\delta) \leq \delta$ and such that H restricted to K_δ is continuous.

He proves [Sch. 1, p. 28]:

Theorem (A1.3)

A lower semi continuous function f on X having values in $\bar{R}$ (extended real line) is Lusin μ-measurable for every Radon measure μ. □

These results guarantee the existence and uniqueness of $\int_{\underline{A}\times\underline{B}} L d\mu$ for all $\mu \in C^*(\underline{A}\times\underline{B})$. Boundedness of L on $\underline{A}\times\underline{B}$ implies $\int_{\underline{A}\times\underline{B}} L d\mu < \infty$ for finite μ.

(b) $D(y,\dot{y})$

(i) We show that $\mathrm{dist}(p, f(y,\Omega))$ is continuous on $R^{n+1} \times R^n$. Let $\{(y_i,p_i)\} \to (y,p)$. Compactness of $f(\bar{y},\Omega)$ for each $\bar{y} \in R^{n+1}$ allows us to choose $u_i \in \Omega$ and that $\mathrm{dist}(p_i, f(y_i,\Omega)) = ||p_i - f(y_i,u_i)||$ and to extract a convergent subsequence $\{(y_i,p_i,u_i)\} \to (y,p,u)$, for some $u \in \Omega$. If u_0 is such that $\mathrm{dist}(p,f(y,\Omega)) = ||p-f(y,u_0)||$ then $||p-f(y,u_0)|| \leq ||p-f(y,u)|| = \lim_{i\to\infty} ||p_i - f(y_i,u_i)||$ and $\lim_{i\to\infty} ||p_i-f(y_i,u_i)|| \leq \lim_{i\to\infty} ||p_i - f(y_i,u_0)|| = ||p-f(y,u_0)||$, which follows from continuity of $||\cdot||$ and f. Thus $\lim_{i\to\infty} \mathrm{dist}(p_i, f(y_i,\Omega)) = \mathrm{dist}(p.f(y,\Omega))$. This is independent of the particular subsequence so the original sequence $\{\mathrm{dist}(p_i, f(y_i,\Omega))\}$ is convergent, to $\mathrm{dist}(p,f(y,\Omega))$ as required.

(ii) By (i) $D(y,\dot{y})$ is continuous on $\underline{A}\times\underline{B}\backslash\{\dot{t}=0\}$. Supose now that $\{(y_i,\dot{y}_i)\} \to (y,(\dot{x},0)) \in \underline{A}\times\underline{B}$, choose $u_i \in \Omega$, arbitrarily if $\dot{t}_i = 0$ such that $D(y_i,\dot{y}_i) = ||\dot{x}_i/\dot{t}_i - f(y_i,u_i)||\ |\dot{t}_i| = ||\mathrm{sgn}(\dot{t}_i)\dot{x}_i - f(y_i,u_i)|\dot{t}_i|\ ||$ where $\mathrm{sgn}(s) = \begin{cases} 1 & s \geq 0 \\ -1 & s < 0 \end{cases}$. Let $k < \infty$ be an upper bound for $||f(y,u)||$ on $\underline{A}\times\Omega$, then:

$$||\dot{x}_i|| - k|\dot{t}_i| \leq D(y_i,\dot{y}_i) \leq ||\dot{x}_i|| + k|\dot{t}_i| \tag{A1.4}$$

The definition $D(y(\dot{x},0)) = ||\dot{x}||$ therefore implies that D is continuous at $\dot{t} = 0$, therefore on $\underline{A}\times\underline{B}$. $\int_{\underline{A}\times\underline{B}} D d\mu$ exists and is finite for finite μ.

(c) $M(y,\dot{y})$. Clearly M is continuous on $\underline{A}x\underline{B}$. The result is immediate. □

Notes

(i) The definitions of L, M and D ensure their homogeneity in $\dot{y} \in R^{n+1}$, viz $L(y,\alpha\dot{y}) = \alpha L(y,\dot{y})$ for all $\alpha \geq 0$. We have been careful to cover the case $\alpha = 0$.

(ii) (A1.1) actually gives L l.s.c. on $\underline{A}xR^{n+1}$.

Appendix 2

Proof of Lemma (4.2)

(a) Let $Q_\varepsilon \triangleq \{(y,\dot{y})\in \underline{A}x\underline{B}: M(y,\dot{y})>\varepsilon\}$. We must prove that $\mu(Q_\varepsilon) = 0$, for all $\varepsilon > 0$. Suppose the contrary then there exists $f \in C(\underline{A}x\underline{B})$ such that $\int_{Q_\varepsilon} fd\mu \neq 0$ and, possibly by changing a sign, $\int_{Q_\varepsilon} fd\mu > 0$.
Compactness of $\underline{A}x\underline{B}$ gives for K sufficiently large: $|f(y,\dot{y})| < K\varepsilon$ but:
$0 < \int_{Q_\varepsilon} fd\mu < K\int_{Q_\varepsilon} \varepsilon d\mu < K\int_{Q_\varepsilon} Md\mu \leq K \int_{\underline{A}x\underline{B}} Md\mu = 0.$

The contradiction establishes that

$$\text{supp}\{\mu\} \subset \{(y,\dot{y})\in\underline{A}x\underline{B}: M(y,\dot{y})\leq 0\} = \{(y,\dot{y})\in\underline{A}x\underline{B}: \dot{t}\geq 0\}$$

(b) Repeating this for D (the sets Q_ε are open because M, D are continuous)

$$\text{supp}\{\mu\} \subset \{(y,\dot{y}): D(\dot{y},\dot{y})\leq 0\} = \{(y,\dot{y}): D(y,\dot{y})=0\}$$

which can be expressed as $\text{supp}\{\mu\} \subset \{(y,\dot{y}): \dot{x}\in\dot{t}f(x,t,\Omega)\}$. □

Lemma (A2.1)

Let g be l.s.c. on a metric space X. Then there exists a sequence $\{f_n\}$ of continuous functions on X, $f_n \uparrow g$ i.e. $\{f_n\}$ is monotone non-decreasing and pointwise convergent to g. Futhermore if g is continuous on a compact subset $K_1 \subset X$, $f_n \uparrow g$ uniformly on K_1.

Proof

Ash shows that $f_n(x) \triangleq \inf_{z\in X}\{g(z)+nd(x,z)\}$ is continuous and $f_n \uparrow g$, d being the metric on X. [Ash p. 222-3].

Take K_1 as stated, $g_n \triangleq f_n - g$, a monotone non-decreasing sequence $g_n \uparrow 0$. For any $\varepsilon > 0$, $x \in K_1$, $\exists$ n(x) such that $g_{n(x)}(x) > -\varepsilon/2$. $g_n(x)$ is continuous on K_1 therefore for each x $\exists$ a nbd U_x of x such that for $y \in U_x \cap K_1$, $g_{n(x)}(y) > g_{n(x)}(x) - \varepsilon/2 > -\varepsilon$. $U = \{U_x: x\in K_1\}$ is an open covering of K_1 - choose a finite subcover $\{U_{x_i} : i=1,\ldots M\}$ define

$N = \max_{1\leq i\leq M} n(x_i) < \infty$ then for every $y \in K_1$, $n \geq N$ because of monotonicity of g_n, $g_n(y) \geq g_N(y) \geq g_{n(x_i)}(y)$ where i is such that $y \in U_{x_i}$ i.e. $g_n(y) > -\varepsilon$ and convergence is uniform on K_1. □

Note

The uniform convergence result is Dinis theorem.

Lemma (A2.2)

Let g be l.s.c. and bounded on a compact metric space X, $\{\mu_i\}$ a sequence of finite positive Radon measures on X, $\mu_i \overset{w^*}{\to} \mu$, μ a finite measure. Then $\int_X g d\mu \leq \liminf_i \int_X g d\mu_i$.

Proof

From theorem (A1.3) g is Lusin μ measurable. Since X is compact, identify K in defn (A1.2) with X, and select K_δ as defined there. Take $f_n \uparrow g$ as in (A2.1) with $K_1 = K_\delta$. For all i sufficiently large:

$$\int_X g d\mu_i \geq \int_X f_n d\mu_i \qquad \mu_i \text{ positive}$$

$$> \int_X f_n d\mu - \varepsilon \qquad w^* \text{ convergence of } \mu_i \text{ to } \mu$$

$$> \int_{K_\delta} f_n d\mu - \varepsilon - \delta||f_n|| \quad \text{by construction of } K_\delta$$

$$> \int_{K_\delta} g d\mu - \varepsilon(1+||\mu||) - \delta||f_n|| \quad \text{by (A2.1) suff. large } n$$

$$> \int_X g d\mu - \varepsilon(1+||\mu||) - \delta(||f_n||+||g||)$$

Therefore

$$\liminf_i \int_X g d\mu_i \geq \int_X g d\mu$$ □

Corollary (A2.3)

X, $\{\mu_i\}$, μ as in (A2.2). Take h any upper semi-continuous bounded function on X, A open, B closed, $A,B \subset X$.

(a) $\limsup_i \int_X h d\mu_i \leq \int_X h d\mu$

(b) $\liminf_i \mu_i(A) \geq \mu(A)$

(c) $\limsup_i \mu_i(B) \leq \mu(B)$

Proof

(a) h is u.s.c. iff $-h$ is l.s.c. The result follows from (A2.2).

(b) (c) Denote the characteristic functions of A, B by χ_A, χ_B respectively A open $\Rightarrow \chi_A$ l.s.c., B closed $\Rightarrow \chi_B$ u.s.c. □

Corollary (A2.4)

X, $\{\mu_i\}$, μ as in (A2.2). Let A be any open subset of X containing $\mathrm{supp}\{\mu\}$ then $\lim_i \mu_i(A)$ exists and equals $\mu(A)$.

Proof

For any Borel set A containing $\mathrm{supp}\{\mu\}$:

$$\begin{aligned}\mu(A) &= \int_X 1 d\mu = \lim_i \int_X 1 d\mu_i \\ &\geq \limsup_i \int_A 1 d\mu_i \\ &= \limsup_i \mu_i(A)\end{aligned}$$

When A is open, by (A2.3)(b)

$$\limsup_i \mu_i(A) \geq \liminf_i \mu_i(A) \geq \mu(A) \geq \limsup_i \mu_i(A) \qquad \square$$

For sufficiently large i therefore $\mu_i(X\backslash A) < \varepsilon$.

Corollary (A2.6)

The boundedness condition on the l.s.c. function g in (A2.2) can be omitted.

Proof

X compact, g l.s.c. but not bounded, implies that $g = \infty$ on an open set $A \subset X$ (g is bounded below). Supp$\{\mu\}$ is compact so if $A \cap \text{supp}\{\mu\} = \phi$, g is bounded on supp$\{\mu\}$ and (A2.2) can be applied. Otherwise $\mu(A) = k > 0$ (the case $A \cap \text{supp}\{\mu\} \neq 0$ and $\mu(A) = 0$ is ruled out by defn of support for A open) and for i large $\mu_i(A) \geq k/2$ (A2.3(b)). $\int_X g d\mu_i = \infty = \int_X g d\mu$. □

Comments

(i) An example shows that (A2.2) is the best possible general result. Take

$$g(x) = \begin{cases} 1 & 0 \leq x < x_0 \\ 0 & x_0 \leq x \leq 1 \end{cases} \quad \text{a l.s.c. function}$$

a sequence of Radon measures μ_i defined by $\int_{[0,1]} f d\mu_i = f(x_i)$, μ defined by $\int_{[0,1]} f d\mu = f(x_0)$ where $x_i \to x_0$ and $x_i < x_0$. Therefore

$$\liminf_i \int_{[0,1]} g d\mu_i = \liminf_i g(x_i) = 1 > 0 = g(x_0) = \int_{[0,1]} g d\mu$$

(ii) This example shows that $A \supset \text{supp}\{\mu\}$ is essential to (A2.4) for $A = (0,x_0)$ yields $\mu_i(A) = 1$, $\mu(A) = 0$.

(iii) If g is bounded and continuous μ almost everywhere i.e. g continuous except on a set A with $\mu(A) = 0$ then

$$\liminf_i \int_X g d\mu_i = \limsup_i \int_X g d\mu_i = \int_X g d\mu \qquad \text{[Ash p. 196-8]}$$

The example in (i) illustrates the difference between Lusin μ measurability and μ a.e. continuity. g is Lusin μ measurable, indeed for all compact $K \subset [0,1]$ the set K_δ can be taken to be $\{x_0\}$ then $\mu(K \backslash \{x_0\}) = 0$ and $g|_{\{x_0\}}$ is continuous (any function restricted to a point is continuous). However g is not continuous μ a.e. - this requires continuity at x_0 i.e. in a neighbourhood of x_0.

Lemma (A2.7)

If g is l.s.c. and bounded on a metric space X and μ is a Radon measure absolutely continuous w.r.t. Lebesgue (Borel) measure then $\mu \to \int_X g d\mu$ is weak * continuous at μ.

Proof

Denote Lebesgue (Borel) measure by m. To any $\delta > 0$ we can find B compact, containing discontinuities of g, $m(B) < \delta$ (Lusin m measurability of g). Absolute continuity of μ implies $\mu(B) < \varepsilon$, $\varepsilon > 0$ small if δ is small. Thus g is μ a.e. continuous and comment (iii) yields the result. $\square$

Appendix 3

Introduction to the Theory of Generalized Flows and Boundaries

Take A the unit cube in R^n, B the unit sphere in R^n. Denote points in AxB by $(x,\dot{x})$. The elements of $P^{\oplus}(AxB)$ are called *generalized flows*. A *segment* is the flow corresponding to a directed line segment between two points in A. A *simple polygonal arc* is a sum $\sum_1^N s_i$ of segments whose representative directed line segments join continuously to form a polygonal arc in A. A *simple closed polygon* is the flow corr. to a closed polygon in A. A *polygonal flow* is a sum $\sum_1^N a_i s_i$ of segments with coeffts. $a_i \geq 0$.

The *boundary* of any generalized flow ν is the restriction of ν to exact integrands $\phi_x(x)\dot{x}$, $\phi \in C^1(A)$, and is denoted $\partial\nu$. When ν is a segment from x_0 to x_1, $\int\phi_x(x)\dot{x}d\nu = \phi(x_1) - \phi(x_0)$:- the boundary is represented by the points x_0, x_1. A boundary corresponding to a polygonal flow is called *simplicial*. Evidently a simple closed polygon ν has boundary $\partial\nu = 0$: this is the general definition of a closed flow.

We shall term a mixture any flow which is a sum with positive coefficients of polygonal flows. A mixture can always be regarded as a polygonal flow but it is more useful in many cases to regard them as mixtures of arcs and closed polygons.

Theorem (A3.1)

A polygonal flow p is closed iff it is expressible as a mixture $\sum_{i=1}^N c_i p_i$ with each p_i a digon or simple closed polygon, $c_i > 0$.

(A digon is a closed polygon comprised of two oppositely directed segments.) □

Theorem (A3.2)

Let p be a polygonal flow with the same boundary $\partial p = \partial s_0$ as a segment s_0. Then $p = p' + p''$ where $\partial p'' = 0$ and p' is a mixture $\sum c_i p_i$ of a finite no. of simple polygonal arcs p_i st $\partial p_i = \partial s_0$, $\sum c_i = 1$.

It is not immediately evident from Young's brief indication of a proof that $c_i > 0$ nor that $p'' \geq 0$ i.e. p'' is gen. flow. In fact the contrary might be thought to be indicated so I shall give a proof.

<u>Proof</u>

$p = \sum_{i=1}^{M} a_i s_i \quad \partial p = \partial s_0 \quad s_0$ the directed segment from P_0 to Q_0.
Arrange the segments s_i so that they have no interior points in common and st. $0 < a_1 \leq a_2 \leq \ldots \leq a_M$. Let the endpoints of s_i be P_i, Q_i. Select P_k different from P_0 and Q_0 (if there are no such P_k the theorem is trivially true) and choose a continuously differentiable ϕ such that $\phi(P_k) > 0$ and $\phi(P_i) = \phi(Q_i) = 0$ for all other P_i, Q_i. Then

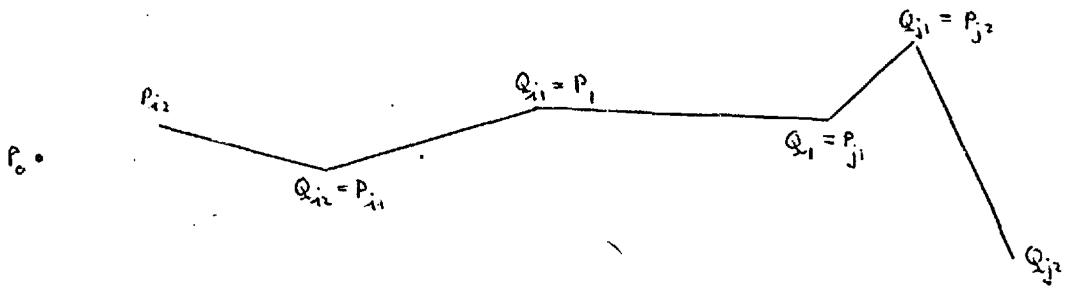
$$p(\phi_x \dot{x}) = \sum_{i \in I} a_i \phi(P_k) - \sum_{j \in J} a_j \phi(P_k) = \phi(Q_0) - \phi(P_0) = 0$$

where I is the index set of segments ending at P_k and where J is the index set of segments starting at P_k, $I \cap J = \phi$. Therefore

$$\sum_{i \in I} a_i - \sum_{j \in J} a_j = 0$$

This means that if a segment ends (begins) at P_k different from P_0, Q_0, at least one other begins (ends) there.

Take s_1. At each end (assuming neither are P_0 or Q_0) take s_i, s_j, resp. such that $Q_{i1} = P_1$, $Q_1 = P_{j1}$. From the 'free' ends P_{i1}, Q_{j1} take s_{i2}, s_{j2} resp. such that $Q_{i2} = P_{i1}$, $Q_{j1} = P_{j2}$.



If we ignore closed polygons this process stops when the endpoints coincide with either or both of P_0, Q_0. The resultant simple polygonal arc r_1 satisfies one of:

$$\partial r_1 = 0, \quad \partial r_1 = \partial s_0, \quad \partial r_1 = -\partial s_0$$

Because we do not allow r_1 to contain (properly) any closed polygons each s_i is used at most once in the construction and $a_1 \leq a_i$, $i = 1, 2, \ldots M$ implies $p = a_1 r_1 + \sum_{i=1}^{M} b_i s_i$ $\quad b_i = a_i - a_1 \geq 0$, i.e. $p = a_1 r_1 + p_1$, p_1 a polygonal flow, $\partial p_1 = \partial p - a_1 \partial r_1 = \alpha \partial s_0$. If $\alpha = 0$ the theorem is proved, for then $a_1 > 0$ implies $\partial r_1 = \partial s_0$ $\quad a_1 = 1$. Otherwise we repeat the above construction with p_1 replacing p, by reordering the b_i st $0 < b_2 \leq b_3 \leq \ldots \leq b_M$ and noting that the boundary $\alpha \partial s_0$, $\alpha \neq 0$, implies the same structure on the segments s_i as ∂s_0 does. Each repetition leaves the remaining term a mixture of at least one fewer segments: we stop when this number reaches 0 or, as above, when $\alpha = 0$. At this stage we have:

$$p = \sum_{j=1}^{N} c_j r_j + p_N$$

p_N a polygonal flow $\partial p_N = 0$, $c_j > 0$ $j = 1, \ldots N$ ($\leq M$) and $\partial r_j = \partial s_0$ or $\partial r_j = -\partial s_0$ $\quad j = 1, \ldots N$. If $\partial r_j = -\partial s_0$ $j = 1, \ldots N$ then $\partial p = -c \partial s_0$, $c = \sum c_i > 0$, a contradiction. Thus if $\partial r_j = -\partial s_0$, $\exists$ k st $\partial r_k = \partial s_0$. Add $d(r_k + r_j)$, $d = \min(c_j, c_k)$, to p_N, subtracting it from $\sum_{j=1}^{N} c_j r_j$, reducing the number of terms in the sum and keeping $\partial p_N = 0$. We can therefore combine all r_j st $\partial r_j = -\partial s_0$ with r_k st $\partial r_k = \partial s_0$ to get

$$p = \sum_{j=1}^{\bar{N}} c_j r_j + p_N \qquad \partial r_j = \partial s_0 \qquad \partial p_N = 0 \qquad \square$$

This proof subsumes that of (A2.1).

Theorem (A3.3)

In order that a generalized flow g_0 posess a simplicial boundary β_0 it is necessary and sufficient that g_0 be the limit of a polygonal flow with the same boundary.

(The proof in [You 1] is based upon an Enlargement Principle - the infimum of pf_0 is the same when p is allowed to be any gen. flow with boundary β_0 as when p is a polygonal flow with the same boundary - and an elementary separation theorem.) □

The proof of theorem (4.9) emerges from relating the boundary conditions on elements of $\mathcal{P}$ to the previous three theorems.

Proof of Theorem (4.10)

(P2) is just $\int_{\underline{A}\times\underline{B}} \phi_y \dot{y} d\mu = \phi(y_1) - \phi(y_0)$ for all $\phi \in C^1(\underline{A})$. $\mathcal{P}$ is comprised of gen. flows ($\mathcal{P} \subset P^{\oplus}(\underline{A}\times\underline{B})$ with this boundary corresponding to a segment. Combining (A3.1), (A3.2) and (A3.3) each $\mu \in \mathcal{P}$ is the weak star limit (for this is the meaning of limit in (A3.3)) of a sequence $\mu_i = \sum_{j=1}^{N_i} \alpha_j^i \mu_j^i + \sum_{j=1}^{M_i} \beta_j^i \nu_j^i$ where $\sum_{j=1}^{N_i} \alpha_j^i = 1$, $\alpha_j^i > 0$ $\beta_j^i > 0$, μ_j^i are simple polygonal arcs with the same boundary as the segment and ν_j^i are simple closed polygons. The structure of μ_j^i, ν_j^i given in (4.10) is now obvious. □

Caution

It may be necessary to redefine $\underline{A}$, since A above is the unit cube. Our purposes are equally well satisfied by any cube containing $\underline{A}$, these exist because $\underline{A}$ is compact and in the event there is no difficulty.

Appendix 4

Lemma (A4.1)

Let $M : X \to Y$ be a continuous, 1-1, onto mapping of a topological space X onto a topological space Y, μ a radon measure on X. Then $\mathrm{supp}(\mu M^{-1}) \subset M\,\mathrm{supp}(\mu)$.

Proof

$A \subset Y$ open, $A \subset \mathcal{C}(M\,\mathrm{supp}(\mu)) \subset M\mathcal{C}(\mathrm{supp}(\mu))$ because M is onto

therefore

$$M^{-1}(A) \subset M^{-1}M\mathcal{C}(\mathrm{supp}(\mu)) = \mathcal{C}(\mathrm{supp}(\mu)) \qquad \text{because } M \text{ is 1-1}$$

Since M is continuous, $M^{-1}(A)$ is open i.e. $\mu(M^{-1}(A)) = 0$

therefore

$$\mu M^{-1}(A) = 0 \text{ i.e. } A \subset \mathcal{C}(\mathrm{supp}(\mu M^{-1}))$$

therefore

$$\mathcal{C}(M\,\mathrm{supp}(\mu)) \subset \mathcal{C}(\mathrm{supp}(\mu M^{-1}))$$

i.e.

$$\mathrm{supp}(\mu M^{-1}) \subset M\,\mathrm{supp}(\mu) \qquad \Box$$

Proof that Lemma (5.4) is True when ϕ is Lebesgue Measurable but not Necessarily Borel Measurable

Lemma (A4.2)

Let f be absolutely continuous, strictly increasing, with an a.c. increasing inverse f^{-1}, g measurable, then $g\circ f \triangleq g(f)$ is measurable.

N.B.

In general the composition is not measurable unless g is Borel measurable, i.e. f continuous, g measurable does not imply gof measurable. (See [Ha, p. 83] for a counter eg.)

Proof

We must show that for any Borel set α, $(g\circ f)^{-1}(\alpha)$ is measurable.

Let $\beta = g^{-1}(\alpha)$ then $(g\circ f)^{-1}(\alpha) = f^{-1}(\beta)$ is measurable (but once again not Borel). Since Lebesgue measure is regular [KT p. 86] $\exists$ a Borel set γ s.t. $\gamma \subset \beta$, $m(\beta\backslash\gamma) = 0$.

$$f^{-1}(\beta) = f^{-1}(\gamma) \cup f^{-1}(\beta\backslash\gamma)$$

$\beta\backslash\gamma$ is measurable hence $\beta\backslash\gamma \subset \bigcup_{i=1}^{N} I_i$ for some collection of intervals I_i satisfying $I_i \cap I_j = \phi$, $i \neq j$, $\sum_{i=1}^{N} m(I_i) < \varepsilon$. Absolute continuity of f^{-1} implies $m[f^{-1}(\bigcup_{i=1}^{N} I_i)] < \delta$ for ε sufficiently small, i.e. $f^{-1}(\beta\backslash\gamma)$ is measurable with measure 0. $f^{-1}(\beta)$ is measurable as γ is Borel, so $f^{-1}(\beta)$ is measurable. □

Proof of (5.4)

We have $\phi : [0,1] \to R^n$ measurable, $\sigma : [t_0,t_1] \to [0,1]$ of bounded variation, strictly increasing. ϕ and σ are adapted in the following sense: $\exists$ Borel sets $T \subset [t_0,t_1]$, $\Sigma \subset [0,1]$ s.t. $m(T) = t_1 - t_0$, σ is absolutely continuous on T, $m(\Sigma) = m([0,1]\backslash\Sigma_0)$ for a measurable set Σ_0 on which ϕ vanishes. Without altering the results we can assume $\Sigma_0 = \mathcal{C}\Sigma$.

Let A be any Borel set in R^n,

$$(\phi\circ\sigma)^{-1}(A) = \sigma^{-1}(\phi^{-1}(A)) = \sigma^{-1}(B_1 \cup B_2) = \sigma^{-1}(B_1) \cup \sigma^{-1}(B_2)$$

where $B_1 \subset \Sigma$, $B_2 = \begin{cases} \phi & \text{if } \Theta \notin A. \\ \Sigma_0 & \text{if } \Theta \in A. \end{cases}$ $\sigma^{-1}(B_2)$ is measurable because ϕ, Σ_0 are Borel. Because σ, σ^{-1} satisfy the conditions of the previous lemma, on Σ, $\sigma^{-1}(B_1)$ is measurable. Thus $(\phi\circ\sigma)(t) = \phi(\sigma(t))$ is measurable. This is all we require. □

Appendix 5

CONSISTENT FLOWS - REPRESENTATIONS AS MIXTURES

§ A5.1 Consistency of Flows in P

The concepts of generalized flow and boundary were introduced in appendix 3. Under addition and multiplication by scalars the class of simplicial boundaries is a vector space, denoted by Σ_0, for which we can define the norm:

$$|\beta| = \inf\{||g|| : g \text{ a gen. flow, } \partial g = \beta\} \qquad \text{(A5.1)}$$

where $||\cdot||$ is the usual norm in $C^*(\underline{A}x\underline{B})$. (Remember that gen. flows are in the positive cone of $C^*(\underline{A}x\underline{B})$, allowing more general elements in (A5.1) may alter the following development considerably.)

A consequence of (A3.3) is that the infimum may be taken over polygonal flows g when $\beta \in \Sigma_0$. (A5.1) is defined for the boundary of any $g \in C^*(\underline{A}x\underline{B})$, in general $|\partial g| \leq ||g||$. On the elements of $C^*(\underline{A}x\underline{B})$ however, we have been using a norm corresponding to weak * convergence, $|g|'$, so it is necessary to introduce a new norm to connect convergence of measures with that of their boundaries.

$$||g||' \triangleq \max[|g|', |\partial g|] \qquad \text{(A5.2)}$$

is known as the consistent norm of $g \in C^*(\underline{A}x\underline{B})$. Consistent convergence therefore implies w* convergence of the measure and norm convergence of its boundary.

Consistent limits of sequences of polygonal flows will be called consistent flows and their boundaries, consistent boundaries. This class of boundaries, denoted by Σ, is an extremely useful completion of the incomplete space Σ_0 since the class of flows with boundaries in Σ is also complete.

Notes

(i) (A5.1) is well defined. An arbitrary non positive $\mu \in C^*(\underline{A}\times\underline{B})$ can be written $\mu = \mu_1 - \mu_2$, $\mu_1, \mu_2 \in P^{\oplus}(\underline{A}\times\underline{B})$ disjoint, and for any positive measure there is a negative one with the same boundary, hence $\exists\ \mu_3 \in P^{\oplus}$, $\partial\mu_3 = -\partial\mu_2$ so $\partial\mu = \partial(\mu_1+\mu_3)$, $\mu_1 + \mu_3 \in P^{\oplus}$.

(ii) It is obvious from the definition that consistent flows are positive i.e. that they are indeed flows.

Theorem (A5.3)

Take g a generalized flow with $\partial g \in \Sigma$, then g is consistent. [You 1, thm. (86.1) p. 201].

Proof

$\exists\ g_1$ such that $\partial g = \partial g_1$ and g_1 is the consistent limit of $\{p_i\}$, p_i polygonal, i.e. $\partial g_1 = \partial p_i - \partial q_i$, $||q_i||' \to 0$. Thus $\partial g + \partial q_i = \partial p_i$ so $g + q_i$ is the w* limit of polygonal flows $\{r_j\}$ with $\partial r_j = \partial p_i$. Select r_j such that $|g+q_i-r_j|' < ||q_i||'$ then $|g-r_j|' < 2||q_i||'$ and $|\partial g-\partial r_j| = |\partial q_i| \leq ||q_i||'$. □

In chapter 6 a set $\mathcal{P}$ of flows μ with boundaries β (§(6.2)) was introduced. A constructive demonstration of their consistency is given.

Construction (A5.4)

Take $\mu \in \mathcal{P}$ with corresponding boundary β, i.e.

$$\int_{\underline{A}\times\underline{B}} \phi_y \dot{y}\, d\mu = \int_{\Gamma} \phi(y)\, d\beta - \phi(y_0) \quad \forall\ \phi \in C^1(\underline{A})$$

where β is probability measure on Γ.

Let $\{\Gamma_j^i\}_{j=1}^{N_i}$, $i = 1, 2, \ldots$ be a sequence of partitions of Γ into disjoint Borel sets such that $\max\{\mathrm{diam}(\Gamma_j^i)\colon j=1,\ldots N_i\} \to 0$ as $a \to \infty$. (cf. lemma (5.12).) Denote the segment from $\bar{y}$ to y, $\bar{y}$, $y \in \underline{A}$ by $s(\bar{y},y)$. It is clear that the flow p defined by

$$p \triangleq \int_\Gamma s(y_0,y)\,d\beta(y)$$

has boundary β. $[\int g\,dp \triangleq \int_\Gamma\int g\,ds(y_0,y)\,d\beta(y)]$

Put $\alpha_j^i = \int_{\Gamma_j^i} d\beta$ and choose any arbitrary $y_j^i \in \Gamma_j^i$. As in (5.12),

$\sum_{j=1}^{N_i} \alpha_j^i \delta(y_j^i) \overset{w^*}{\to} \beta$ in $C^*(\Gamma)$. Now the map $y \to \int g\,ds(y_0,y)$ is continuous in y for every g, so weak * convergence of $\sum_{j=1}^{N_i} \alpha_j^i \delta(y_j^i)$ to β implies that p_i, defined by $p_i = \sum_{j=1}^{N_i} \alpha_j^i s(y_0,y_j^i)$, converge weak * to p.

Further, the flow $q_i \triangleq \sum_{j=1}^{N_i} \int_{\Gamma_j^i} s(y_j^i,y)\,d\beta(y)$ has boundary $\partial q_i = \partial p - \partial p_i$ therefore:

$$|\partial p-\partial p_i| = |\partial q_i| \le ||q_i||$$

$$= \sum_{j=1}^{N_i} \int_{\Gamma_j^i} \int_{\underline{A}\mathrm{x}\underline{B}} 1\,ds(y_j^i,y)\,d\beta(y)$$

$$\le \max\,\mathrm{diam}\,(\Gamma_j^i) \sum_{j=1}^{N_i} \int_{\Gamma_j^i} d\beta(y).$$

$$= \max\,\mathrm{diam}\,(\Gamma_j^i)$$

i.e. $|\partial p-\partial p_i| \to 0$ and by definition p_i converges consistently to p. p is consistent because p_i are polygonal, so $\partial p = \beta$ is a consistent boundary. Consistency of μ follows from (A5.3). $\square$

§ A5.2 Approximation and Representation

Theorem (A5.5)

Each $\mu \in {}_N P$ is the consistent limit of a sequence of polygonal flows $q_i = \sum_{j=1}^{N_i} \alpha_j^i q_j^i$ with α_j^i as before and each q_j^i is a simply polygonal arc from y_0 to $y_j^i \in \Gamma$ containing no closed proper subarcs.

Proof

Since μ is consistent it is the limit of a sequence of polygonal flows $\{\hat{q}_i\}$. Let $\gamma_i = \partial p_i - \partial \hat{q}_i$, p_i as above, a simplical boundary. Since p_i and $\hat{q}_i$ converge consistently, $|\gamma_i| \to 0$ and there are polygonal flows $\{r_i\}$ with $\partial r_i = \gamma_i$ and $|r_i|' \leq 2|\gamma_i|$. Therefore $\tilde{q}_i = \hat{q}_i + r_i$ satisfies:

$$||\tilde{q}_i - \mu||' \leq ||\hat{q}_i - \mu||' + ||r_i||' \to 0$$

and

$$\partial \tilde{q}_i = \partial p_i$$

Write $\tilde{q}_i = q_i + \rho_i$ where q_i contains no closed proper subarcs and $\partial \rho_i = 0$. The constraints (P3) and (P4) on μ imply that $||\rho_i|| \to 0$ hence $||\rho_i||' \to 0$ ($||\cdot||$ is stronger than $||\cdot||'$). Thus q_i converges consistently to μ and $\partial q_i = \partial p_i$. Since q_i contains no closed proper subarcs it is possible to join the segments comprising q_i into q_j^i satisfying the statement of the theorem. $\square$

Note

(1) The original approximating sequence $\{\hat{q}_i\}$, the elements of which might not even have boundaries admissible for (P), has been replaced by a sequence $\{q_i\}$ in which each element is a convex combination of

simple polygonal arcs.

(2) (A5.5) is stronger that theorem (6.1) in that consistent convergence has been proved.

Recall that in the fixed end point case we had to use a powerful representation theorem to find a generalized curve solving (P) where L was discontinuous, namely that every flow with a simplical boundary is a mixture of generalized curves (theorem (4.13)). Although not all consistent flows are such mixtures, it can be shown that each member of $\mathcal{P}$ is.

Defn. (A5.6)

The measure $g|_E$, the restriction of a generalized curve $g : \{y(\sigma), \mu_\sigma : 0 \leq \sigma \leq 1\}$ to a set E, a countable union of intervals in [0,1], is known as a jet.

The most general representation theorem for consistent flows is the following:

Theorem (A5.7)

Every consistent flow is a Riesz mixture of bounded jets.

Proof

[You 1, Thm 89.1 (iii) pp. 209-12.] □

We improve this to:

Theorem (A5.8)

Every $\mu \in \mathcal{P}$ is a Riesz mixture of bounded jets, almost all of which are generalized curves.

Proof

Take $\mu \in \mathcal{P}$ then by (A5.3) and (A5.7) we can write

$$\mu = \int \gamma_\alpha d\lambda(\alpha)$$

where each γ_α is a jet and λ is a Riesz measure on a parameter set Z. Suppose not almost all γ_α are generalized curves then there is a set of parameters $Y \subset Z$, $\lambda(Y) > 0$ and for all $\alpha \in Y$, γ_α is not a generalized curve.

We have previously obtained

$$\mu = \text{consistent} \lim_{i \to \infty} \mu_i = \text{consistent} \lim_{i \to \infty} \sum_{j=1}^{N_i} \alpha_j^i \mu_j^i$$

where $\sum_{j=1}^{N_i} \alpha_j^i = 1$ and each μ_j^i is a simple polygonal arc from y_0 to $y_j^i \in \Gamma$. μ satisfies $\int M d\mu = \int D d\mu = 0$ where M and D are positive, therefore for any $\varepsilon > 0$, $\delta > 0$ for i sufficiently large

$$\sum_{j \in I_i} \alpha_j^i < \delta, \quad I_i \triangleq \{j \in \{1, \ldots N_i\} : \int M d\mu_j^i, \int D d\mu_j^i > \varepsilon\}$$

The constraints $\int M d\mu_j^i \leq \varepsilon$, $\int D d\mu_j^i \leq \varepsilon$ imply a norm bound on μ_j^i.

$$\int |\dot{t}| d\mu_j^i = \int_{\dot{t} \leq 0} -\dot{t} d\mu_j^i + \int_{\dot{t} > 0} \dot{t} d\mu_j^i$$

$$= 2 \int_{\dot{t} \leq 0} -\dot{t} d\mu_j^i + \int \dot{t} d\mu_j^i$$

$$= 2 \int M d\mu_j^i + t_j^i - t_0$$

$$\leq 2\varepsilon + T - t_0$$

because

$$M = \max[-\dot{t}, 0] = \begin{cases} 0 & \dot{t} > 0 \\ -\dot{t} & \dot{t} \leq 0 \end{cases}$$

With $k \triangleq \max\limits_{(x,t,u)} f(x,t,u)$,

$$\int ||\dot{x}|| d\mu_j^i = \int_{||\dot{x}|| \le k|\dot{t}|} ||\dot{x}|| d\mu_j^i + \int_{||\dot{x}|| > k|\dot{t}|} ||\dot{x}|| d\mu_j^i$$

When $||\dot{x}|| > k|\dot{t}|$

$$D = \mathrm{dist}(\dot{x}/\dot{t}, f(x,t,\Omega))\ \dot{t}$$

$$\ge \mathrm{dist}(\dot{x}/\dot{t}, U_k)|\dot{t}| \qquad U_k \text{ the n-ball of radius k}$$

$$= ||\dot{x}|| - k|\dot{t}| \qquad \text{i.e. } ||\dot{x}|| \le k|\dot{t}| + D$$

Thus

$$\int ||\dot{x}|| d\mu_j^i \le \int_{||\dot{x}|| \le k|\dot{t}|} |\dot{t}| d\mu_j^i + \int_{||\dot{x}|| > k|\dot{t}|} k|\dot{t}| + D d\mu_j^i$$

$$\le \int k|\dot{t}| d\mu_j^i + \int D d\mu_j^i$$

$$\le k(2\varepsilon + T - t_0) + \varepsilon$$

Finally

$$||\mu_j^i|| = \int 1 d\mu_j^i = \int ||\dot{x}||^2 + |\dot{t}|^2 d\mu_j^i$$

$$\le \int ||\dot{x}|| + |\dot{t}| d\mu_j^i \le (1+k)(2\varepsilon + T - t_0) + \varepsilon$$

Let us relabel the index set $\{1 \le j \le N_i : j \notin I_i\}$ as $\{j = 1, 2, \ldots M_i\}$ $M_i \le N_i$. The norm bounds imply that each sequence $\{\bar{\mu}_j^i : i = 1, 2, \ldots\}$ $j = 1, 2, \ldots$ (for different j the sequences may start at different i e.g. the M_i+1^{th} sequence is $\mu_{M_i+1}^{i+1}, \mu_{M_i+2}^{i+2}, \ldots$) has a weak * convergent subsequence. By the usual diagonal procedure we can extract a subsequence (still labelled i for convenience) such that $\{\mu_i \triangleq \sum_{j=1}^{M_i} \alpha_j^i \mu_j^i\}$ and all $\{\mu_j^i\}$ are weak * convergent. The μ_j^i, being simple polygonal arcs, converge to generalized curves and $\bar{\mu}_i$ therefore converges to a mixture of generalized curves, $\bar{\mu}$ say.

It is clear that $\mu = \bar{\mu} + \int_Y \gamma(\alpha)d\lambda(\alpha) + \int_X \gamma(\alpha)d\lambda(\alpha)$ where $X \subset Z$ accounts for the components of the mixture not in Y or in (the representation of) $\bar{\mu}$. But this means that

$$\lambda(Y) \leq \limsup_i \sum_{j\in I_i} \alpha_j^i < \delta$$

which is arbitrarily small. The proof is now complete. □

§ A5.3 Evidence for the Wider Validity of (A5.8)

Associated with any polygonal flow $p = \sum_{i=1}^{N} c_i s_i$ where s_i is a segment from y_{i0} to y_{i1} are two measures β_0 and β_1 on $\underline{A}$ representing the boundary of p. $\beta_0 = \sum_{i=1}^{N} c_i \delta(y_{i0})$, $\beta_1 = \sum_{i=1}^{N} c_i \delta(y_{i1})$ and $\int \phi_y \dot{y} dp = \int \phi d\beta_1 - \int \phi d\beta_0$ for all $\phi \in C^1(\underline{A})$. β_0 and β_1 are positive, have finite support and have the same total variation, $\sum_{i=1}^{N} c_i$, also finite.

Extending the class to consistent flows, the boundary representation remains the same: β_0 and β_1 are positive and have the same total variation but neither this nor their supports need be finite. For example, a jet γ formed by removing a countable number of arcs from a curve g can be written $\gamma = \sum_{i=1}^{\infty} g_i$ where g_i is a subarc from y_{i0} to y_{i1}. Evaluating $\int \phi_y \dot{y} d\gamma = \lim_{N\to\infty} \sum_{i=1}^{N} (\phi(y_{i1}) - \phi(y_{i0})) < \infty$ we see that $\sum_{i=1}^{\infty} |y_{i1}-y_{i0}| < \infty$ is necessary. For a jet this is immediate since $\sum_{i=1}^{\infty} |y_{i1}-y_{i0}| \leq ||g|| < \infty$. Here β_0 and β_1 have countable supports and total variation $\sum_{i=1}^{\infty} 1 = \infty$.

Regarding representation, the class of consistent flows divides into two subclasses, those with boundaries represented by finite measures and those with boundaries represented by infinite measures.

Proposition (A5.9)

A consistent flow can be written as a Riesz mixture of curves (precisely, jets, almost all of which are curves) if and only if its boundary is finitely represented.

Proof

Suppose μ, a consistent flow, has finitely represented boundary. Writing $\mu = \int \gamma_\alpha d\lambda(\alpha)$ as in (A5.3)

$$\int \phi_y \dot{y} d\mu = \int\int \phi_y \dot{y} d\gamma_\alpha d\lambda(\alpha)$$

$$= \int \sum_{i=1}^{\infty} (\phi(y_{i1}^{\alpha}) - \phi(y_{i0}^{\alpha}))d\lambda(\alpha)$$

$$= \int \phi d\beta_1 - \int \phi d\beta_0 \quad \beta_0, \beta_1 \text{ finite}$$

Denoting the total variation of the boundaries associated with γ_α by $TV(\alpha)$ we find $TV(\beta_1) = TV(\beta_0) = \int TV(\alpha) d\lambda(\alpha) < \infty$. If γ_α is a jet $TV(\alpha) = \infty$ so λ almost all γ_α are curves.

Conversely if $TV(\beta_1) = TV(\beta_0) = \infty$ not almost all γ_α can be curves. □

Comments

Unlike (A5.8) the above is achieved without reference to constraints. The necessary properties of admissible elements for the parametric problem and their implications can be separated as follows:

Dynamic Constraints (P3) and (P4) imply norm boundedness of $\mathcal{P}$ hence the existence of a solution to (P).

Boundary Condition (P2) implies representation of solution as a mixture of curves hence the generalized curve solution.

§ A5.4 Problems With Free Initial and Final Times

These were mentioned at the end of chapter 6 and are generalizations of (P) to the case where the boundary condition (P2) is:

$$\int_{\underline{A}\times\underline{B}} \phi_Y \dot{y}\, d\mu = \int_{\Gamma_1} \phi\, d\beta_1 - \int_{\Gamma_0} \phi\, d\beta_0$$

β_0, β_1 probability measures.

It is apparent from the preceeding discussion that any generalized flow with this boundary representation where $TV(\beta_1) = TV(\beta_0) < \infty$ is a consistent flow. $\mathcal{P}$ is still comprised of consistent flows satisfying (P3) and (P4). From (A5.3) and the above comment, free initial and final time problems have generalized curve solutions and equivalence of weak and strong problems is established as before.

Note:

A generalized flow with boundary represented by infinite β_0, β_1 will not be consistent unless the supports of β_0, β_1 satisfy some condition like those of a jet, viz. $\sum_{i=1}^{\infty} |y_{i1}-y_{i0}| < \infty$.

Appendix 6

MEASURE REACHABILITY

In §11.2 we encountered the possibility of a more general definition of reachability than that associated with the control problem. State constraints not being present, the original definition of the reachable set of a control system (S2), (S3), from the initial point (x_0,t_0), is:

$$R(x_0,t_0) \triangleq \{(x,t)\in\underline{A}: \exists \text{ a trajectory control pair } (\bar{x}(\cdot),\cdot,\bar{u}(\cdot))\in S \text{ with } \bar{x}(t)=x\} \tag{A6.1}$$

The reachable set $R(x_0,t_0)$ and its intersection with the target $R(x_0,t_0) \cap \underline{\Gamma}$ are dependent upon (x_0,t_0) except under very special circumstances and their exact evaluation is generally impossible. Their removal from the statements of proposition (11.6) et seq. is therefore desirable.

Recall the proof of (11.6)(c). There, given a probability measure β on Γ, we required the existence of a $\mu \in C^*(\underline{A}\text{x}\Omega)$ such that for all $\phi \in C^1(\underline{A})$

$$\int_{\underline{A}\text{x}\Omega} \phi_t(x,t) + \phi_x(x,t)f(x,t,u)d\mu(x,t,u) = \int_\Gamma \phi(x,t)d\beta(x,t) - \phi(x_0,t_0) \tag{A6.2}$$

μ is not restricted to being positive for if it were, the structural result of part I implies existence of positive μ corresponding to β iff $\text{supp}\{\beta\} \subset R(x_0,t_0) \cap \Gamma$. There are two possible definitions of measure reachability:

Defn. (A6.3)

The (measure) reachable set R_1 is the set of all probability measures β on $\underline{A}$ such that a μ satisfying (A1.2) (with $\underline{A}$ replacing Γ) exists.

Defn. (A6.4)

The (measure) reachable set R_2 is the set of all points $(x_1,t_1) \in \underline{A}$ such that $\mu \in C^*(\underline{A}x\Omega)$ satisfying $\int_{\underline{A}x\Omega} \phi_t + \phi_x f d\mu = \phi(x_1,t_1) - \phi(x_0,t_0)$ exists.

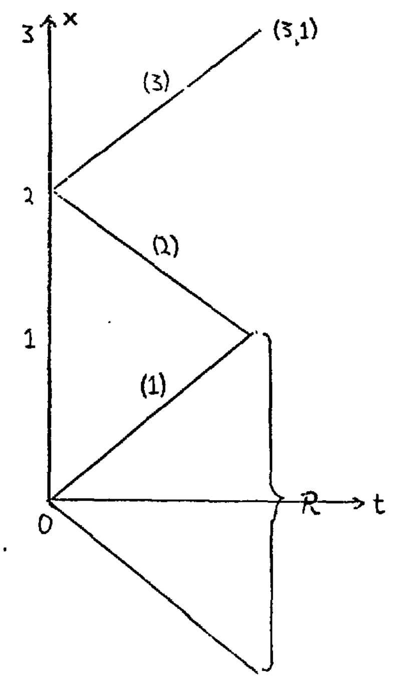
Remark

All prob. measures β on R_2 are contained in R_1 and $R_2 = \underline{A}$ implies R_1 is the set of all prob. measures on $\underline{A}$. Very little is known about the structure of measures μ satisfying (A1.2) for arbitrary β so we adopt definition (A6.4) and look for conditions giving $R_2 = \underline{A}$.

We proceed through an example:

$$\dot{x} = u \qquad |u| \leq 1 \qquad x(0) = 0 \tag{A6.5}$$

Here $R(0,0) = \{(x,t): t \geq 0,\ x \in [-t,t]\}$. In particular at time 1 the possible states of (A6.5) lie in $[-1,1] = \mathcal{R}$ say



(1) $x(t)=t \quad t: 0 \to 1 \quad x=+1$

(2) $x(t)=2-t \quad t: 1 \to 0 \quad x=-1$

(3) $x(t)=2+t \quad t: 0 \to 1 \quad x=+1$

Take the measure formed by integrating along (1), (2) and (3):

i.e.

$$\int \ell \, d\mu = \int_0^1 \ell(t,t,1)\,dt + \int_1^0 \ell(2-t,t,-1)\,dt + \int_0^1 \ell(2+t,t,1)\,dt$$

$$\int \ell \, d\mu = \int_0^1 \ell(t,t,1)\,dt - \int_0^1 \ell(1+s,1-s,-1)\,ds + \int_0^1 \ell(2+t,t,1)\,dt$$

therefore

$$\int \phi_t + \phi_x u \, d\mu = \int_0^1 \phi_t(t,t) + \phi_x(t,t)\,dt - \int_0^1 -\phi_s(1+s,1-s)$$

$$-\phi_x(1+s,1-s)\,ds + \int_0^1 \phi_t(2+t,t) + \phi_x(2+t,t)\,dt$$

$$= \phi(1,1) - \phi(0,0) + \phi(2,0) - \phi(1,1) + \phi(3,1) - \phi(2,0)$$

$$= \phi(3,1) - \phi(0,0)$$

So $(3,1) \in R_2$.

Repetition of this construction indicates that any point (x,t) is reachable. Even points $(x,0) \in R_2$. (A6.6)

<u>Notes</u>

(1) The above μ is not positive. Let

$$\ell \triangleq \begin{cases} 0 & x \leq 1 \quad x \geq 2 \\ x-1 & 1 \leq x < 3/2 \\ 2-x & 3/2 \leq x < 2 \end{cases} \qquad \text{i.e. } \ell \geq 0$$

then

$$\int \ell \, d\mu = -\int_0^{1/2} 1+s-1\,ds - \int_{1/2}^1 2-(1+s)\,ds = -1/4.$$

(2) Arc (2) highlights the restrictiveness of positivity. t decreases along (2) i.e. admitting general measures admits arcs going backwards in time.

Theorem (A6.7)

Assume there is a neighbourhood $N(0,\varepsilon)$ of zero in R^n, $N(0,\varepsilon) \subset \text{co } f(x,t,\Omega)$ uniformly in $(x,t) \in \underline{A}$, then $R_2 = \underline{A}$.

Proof

Select $(\bar{x},\bar{t}) \in \underline{A}$ and $\delta > 0$ suff. small that $\delta(\bar{x}-x_0) \in N(0,\varepsilon)$, i.e. $\pm\delta(\bar{x}-x_0) \in \text{co } f(x,t,\Omega)$ $\forall$ $(x,t) \in \underline{A}$, and $\bar{t} - t_0 = 1/\delta - 2n$ for some integer n. The measure μ is constructed as before, with arcs $@_i$:

$$\dot{x}_i = +\delta(\bar{x}-x_0) \qquad x_i(t) = x_0 + (i-1)\delta(\bar{x}-x_0) + (t-t_0)\delta(\bar{x}-x_0) \quad \text{i odd} \quad t : t_0 \to \bar{t}$$

$$\dot{x}_i = -\delta(\bar{x}-x_0) \qquad x_i(t) = x_0 + i\delta(\bar{x}-x_0) - (t-t_0)\delta(\bar{x}-x_0) \quad \text{i even} \quad t : \bar{t} \to t_0$$

for then

$$x_{2n+1}(\bar{t}) = x_0 + 2n\delta(\bar{x}-x_0) + (\bar{t}-t_0)\delta(\bar{x}-x_0)$$

i.e.

$$x_{2n+1}(\bar{t}) = x_0 + \bar{x} - x_0 = \bar{x}$$

μ consists of integrating along $@_1$, $@_2$, ... $@_{2n+1}$ and $(\bar{x},\bar{t})$ is reached by the non-positive μ. $(\bar{x},\bar{t})$ is arbitrary so the proof is complete. □

(A6.7) implies theorem (11.5).

More specialized results are available when the velocity set does not satisfy the conditions of (A6.7).

Theorem (A6.8)

Suppose $\Gamma = \Sigma \times [t_0,t_1]$ where $\Sigma \subset R^n$ is compact and

(i) t_1 is so large that $(x,t) \in R(x_0,t_0)$ for some $t \leq t_1$, for each $x \in \Sigma$

(ii) $0 \in \text{co } f(x,t,\Omega)$ $\forall$ $x \in \Sigma$, $t \in [t_0,t_1]$.

Then $\Gamma \subset R_2$.

<u>Proof</u>

Let $(x,t) \in \Gamma$. By (i) $\exists$ an admissible curve $@_1$ from (x_0,t_0) to (x,t_f) for some $t_f \leq t_1$.

If $t_f \leq t$ by (ii) there is an admissible curve $@_2$ ($\dot{x} = 0$ on $[t_f,t]$) so $(x,t) \in R(x_0,t_0) \subset R_2$.

Suppose $t_f > t$. Let $@_3$ be the time reversing arc $\dot{x} = 0$ from t_f to t. Then (x,t) is reached by the measure consisting of $@_1$ and $@_3$. $\square$

(A6.8) applies to time optimal problems where the target, e.g. the origin, is to be reached in the shortest possible time, for then we can always pose the problem over a time interval sufficiently long for (i) to hold and (ii) is true if the problem is to be meaningful.

Appendix 7

For state constrained control problems the value theorem can be written in several forms, the original (11.13), (13.4) or (13.5). Combining the idea behind (13.5) with (13.4)

$$\eta(S) = \sup\{-\tilde{\phi}(\tilde{x}_0,t_0):\tilde{\phi}\in\tilde{\Psi}\} \tag{A7.1}$$

where

$$\tilde{\Psi} \triangleq \{\tilde{\phi}\in C^1(\underline{\tilde{A}}):\tilde{\phi}\geq 0 \text{ on } \tilde{\Gamma},\ \tilde{\phi}_t+\tilde{\phi}_{\tilde{x}}\tilde{f}-\ell\leq\gamma \max[0,x_{n+1}] \text{ on } \underline{A}\times\Omega, \text{ some } \gamma\}$$

Suppose $\exists\ \tilde{\phi}\in C^2(\underline{\tilde{A}})$ and a corr. seq. $\{\gamma^i\}$ solving (2.1). Adding constraints increases the value of a problem so we can assume $\gamma^i \geq 0$. If $\{\gamma^i\}$ is bounded, replace it with the limit of any convergent subsequence, γ say. Let $(\tilde{x}(\cdot),u(\cdot))$ be an optimal trajectory control pair and put $\tilde{\psi}(t) \triangleq \tilde{\phi}_{\tilde{x}}(\tilde{x}(t),t)$. The trajectory is comprised of two kinds of arc, interior arcs $x_{n+1}(t) < 0$ and boundary arcs $x_{n+1}(t) = 0$.

(i) Interior arcs: by (12.4) and defn of $\tilde{\Psi}$:

$$\tilde{\phi}_{t\tilde{x}} + \tilde{\phi}_{\tilde{x}\tilde{x}}\tilde{f} + \tilde{\phi}_{\tilde{x}}\tilde{f}_{\tilde{x}} - \ell_{\tilde{x}} = 0 \text{ along } (\tilde{x}(\cdot),u(\cdot))$$

i.e.

$$\dot{\tilde{\psi}}(t) = \tilde{\phi}_{t\tilde{x}} + \tilde{\phi}_{\tilde{x}\tilde{x}}\tilde{f} = -\tilde{\psi}\tilde{f}_{\tilde{x}} + \ell_x$$

Define $\psi(t) \triangleq [\tilde{\psi}_1(t),\ldots,\tilde{\psi}_n(t)] = \tilde{\phi}_x$, $\lambda(t) \triangleq \tilde{\psi}_{n+1}(t) = \tilde{\phi}_{x_{n+1}}$. Then if $p(x,t,u) \triangleq g_t(x,t) + g_x(x,t)f(x,t,u)$

and

$$\left.\begin{aligned} \dot{\psi}(t) &= -\tilde{\psi}\tilde{f}_x + \ell_x = -\psi(t)f_x - \lambda(t)p_x + \ell_x \\ \dot{\lambda}(t) &= -\tilde{\psi}\tilde{f}_{x_{n+1}} + \ell_{x_{n+1}} = 0 \end{aligned}\right\} \tag{A7.2}$$

the rhs. being evaluated along $(x(\cdot),u(\cdot))$.

(ii) Boundary Arcs: for finite γ the subdifferential of $\gamma \max[0,x_{n+1}]$ at $x_{n+1} = 0$ is the set $\partial m = \{(\Theta^n,\beta): 0 \le \beta \le \gamma, \Theta^n = [0,\ldots,0] \subset R^n\} \subset R^{n+1}$. When $\{\gamma^i\}$ above is unbounded, put $\partial m = \{(\Theta^n,\beta): 0 \le \beta \le \infty\}$. Now

$$\tilde{\phi}_{t\tilde{x}} + \tilde{\phi}_{\tilde{x}\tilde{x}}\tilde{f} + \tilde{\phi}_{\tilde{x}}\tilde{f}_{\tilde{x}} - \ell_{\tilde{x}} \in \partial m, \quad \text{becoming}$$

$$\dot{\psi}(t) = -\psi(t) f_x - \lambda(t) p_x + \ell_x$$

$$\dot{\lambda}(t) \in [0,\gamma] \text{ or } [0,\infty] \text{ as the case may be} \tag{A7.3}$$

We have formally arrived at the Maximum Principle for state constrained problems.

<u>Theorem (A7.4)</u>

Suppose $(x(\cdot),u(\cdot))$ is admissible for (S). A necessary condition for optimality of $(x(\cdot),u(\cdot))$ is the existence of $\psi : [t_0,t_1] \to R^n$, absolutely continuous, and $\lambda(\cdot) : [t_0,t_1] \to R$ of bounded variation, satisfying:

$$\dot{\psi}(t) = [-\psi(t) f_x - \lambda(t) p_x + \ell_x]\big|_{(x(t),u(t))}$$

while for all $v \in \Omega$

$$[\psi(t) + \lambda(t) g_x(x(t),t)][f(x(t),t,u(t)) - f(x(t),t,v)]$$
$$- [\ell(x(t),t,u(t)) - \ell(x(t),t,v)] \ge 0$$

Further $\lambda(\cdot)$ is monotone increasing and constant along interior arcs. □

Remarks

(1) The transversality conditions have been omitted from the statement since they are quite complicated and of little interest to us.

(2) When γ is finite, λ is absolutely continuous.

(3) If λ does have discontinuities, the formalism breaks down because $\lambda(t) = \tilde{\phi}_{x_{n+1}}(\tilde{x}(t),t)$ so $\tilde{\phi} \notin C^2(\underline{\tilde{A}})$. That λ is of bounded variation is not really an immediate consequence of (A7.3) either.

(4) Nevertheless the form $\tilde{\phi}_{x_{n+1}}(\tilde{x}(t),t)$ for $\lambda(t)$ may be significant, for the lack of a characterization of $\lambda(t)$ has greatly impeded the solution of state constrained problems until now.

(5) Since (A7.1) is equivalent to (13.4) we do not have to find γ to get $\tilde{\phi}$, which need only satisfy: $\tilde{\phi}_t + \tilde{\phi}_{\tilde{x}}\tilde{f}_{\tilde{x}} - \ell_{\tilde{x}} \leq 0$ on $G^0 x\Omega$ $G^0 = \underline{A} \cap \{x_{n+1} \leq 0\}$.

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