THE ASYMPTOTIC BEHAVIOUR OF FUNCTIONS REGULAR IN THE UNIT DISK

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ABSTRACT

In 1955 Hayman (1955a) showed that if $f(z) = z + a_2z^2 + \ldots$ is a circumferentially mean univalent function in $|z| < 1$ then the limit

$$\alpha = \lim_{r\to 1} (1-r)^2 M(r, f)$$

exists, and

$$\lim_{n\to\infty} \frac{|a_n|}{n} = \alpha < 1 \quad ,$$

unless f is a Koebe function. Hence there exists $n_0 = n_0(f)$ such that

(1) $$|a_n| < n \quad , \quad n > n_0(f) \quad ,$$

unless f is a Koebe function.

We show in this thesis that if $f(z) = z + a_2z^2 + \ldots$ is weakly univalent in $|z| < 1$ then the limit

$$\alpha(\theta) = \lim_{r\to 1} (1-r)^2 |f(re^{i\theta})|$$

exists, $0 \leq \theta < 2\pi$; $\alpha(\theta) = 0$ except for a sequence $\theta = \theta_\nu$, and

$$\sum_\nu \alpha(\theta_\nu) < 1 \quad ,$$

unless f is a Koebe function.

We derive the expansion

(2) $$a_n = ne^{i\lambda_1(1-1/n)} \left\{ \sum_\nu \alpha(\theta_\nu) e^{-in\theta_\nu} + o(1) \right\} \quad , \quad (n \to \infty) \quad ,$$

where $\lambda_1(1-1/n) = \arg\{ f [(1-1/n)e^{i\theta_1}] \}$, and show that

$$\overline{\lim_{n\to\infty}} \frac{|a_n|}{n} = \sum_\nu \alpha(\theta_\nu) \quad ,$$

the limit existing if and only if $\alpha(\theta) = 0$ except for a unique θ_o . Hence we have (1) for f weakly univalent. We prove analogous results for non-zero weakly univalent functions and also general weakly p-valent functions.

Hayman (1955b) showed that if a regular function

$$f(z) = \sum_{n=o}^{\infty} a_n z^n$$

omits a sequence of values $\{w_k\}$ which lie on or near the negative real axis and satisfy

$$(3) \qquad |w_{k+1}-w_k| = O(|w_k|^{\frac{1}{2}}) \quad ,$$

then

$$(4) \qquad |a_n| = O(n) \quad , \quad (n\to\infty) \quad .$$

We show that the exponent $\frac{1}{2}$ in (3) may be replaced by any number less than 1 for (4) to remain true. We also show that functions of this form allow an asymptotic expansion of the form (2).

CONTENTS

INTRODUCTION

The thesis is divided into three chapters. In the first chapter we develop some general theory on the asymptotic behaviour of functions regular in $|z| < 1$. We use these results in Chapter 2 to prove results about weakly p-valent functions and in Chapter 3 about functions omitting a sequence of values.

We first give a short history of p-valent functions (§0.1) and functions omitting values (§0.2).

0.1 p-valent functions

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be regular in $|z| < 1$. We define the maximum modulus, $M(r, f)$, and the λ-th integral means $I_\lambda(r, f)$ $(0 < \lambda < \infty)$ as follows:

$$M(r, f) = \max_{|z|=r} |f(z)| \quad , \quad 0 < r < 1 \quad ,$$

$$I_\lambda(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\lambda d\theta \quad , \quad 0 < r < 1 \quad .$$

Let p be a positive integer. f is said to be p-*valent* in $|z| < 1$ if the equation $f(z) = w$ has at most p roots (multiple roots being counted multiply) in $|z| < 1$ for any complex w. Cartwright (1935) proved that if f is p-valent in $|z| < 1$ then

$$(0.1) \qquad M(r, f) \le A(p)\mu_p(1-r)^{-2p} \quad , \quad 0 < r < 1 \quad ,$$

where $A(p)$ is a constant depending only on p and $\mu_p = \max_{0 \leq \nu \leq p} |a_\nu|$.

Biernacki (1936) used Cartwright's result to prove

$$(0.2) \qquad I_1(r, f) \leq A(p)\mu_p(1-r)^{1-2p} \quad , \quad 0 < r < 1 \quad .$$

He then showed that since

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(re^{i\theta})e^{-in\theta}d\theta \quad ,$$

it follows, on writing $r = 1 - 1/n$, that

$$(0.3) \qquad |a_n| < A(p)\mu_p n^{2p-1} \quad .$$

If $p = 1$, then f is univalent (schlicht). In this case (0.1) was first proved by Bieberbach (1916) and (0.2), (0.3) by Littlewood (1924).

Spencer (1942) generalised the class of p-valent functions as follows:

let $W(R)$ denote the area (regions covered multiply being counted multiply) of that portion of the transform of $|z| < 1$ by $w = f(z)$ which lies in the circle $|w| = R$; then if

$$W(R) \leq pR^2$$

for all $R > 0$, where p is a positive number (not necessarily integral), we say that f is *areally mean p-valent* (in the sense of Spencer).

Spencer (1942, 1940) generalised (0.1), (0.2), (0.3) to mean p-valent functions. He proved (0.1) for $p > 0$, (0.2) for $p > \frac{1}{2}$, and (0.3) for $p > \frac{1}{4}$.

Biernacki (1946) introduced a class of functions which were less general than Spencer's:

let $n(w)$ denote the number of roots (multiple roots being counted multiply) in $|z| < 1$ of the equation $f(z) = w$ and let p be a positive number. Write

$$p(R) = p(R, f) = \frac{1}{2\pi} \int_0^{2\pi} n(Re^{i\phi})d\phi \quad .$$

Then we say that f is *circumferentially mean p-valent* (in the sense of Biernacki) if $p(R) \leq p$ for all $R > 0$.

Now if p is a positive integer and f is p-valent, then $n(w) \leq p$ for all complex w and hence $p(R) \leq p$, $(R > 0)$. Thus a p-valent function is circumferentially mean p-valent. On the other hand, if $p(R) \leq p$, $(R > 0)$, then

$$W(R) = \int_0^R p(\rho)d\rho^2 \leq pR^2 \quad , \quad (R > 0) \quad .$$

Thus a circumferentially mean p-valent function is always areally mean p-valent.

Hayman (1955a) (see also 1958) showed that if f is circumferentially mean p-valent then the limit

$$(0.4) \qquad \alpha = \lim_{r\to1} (1-r)^{2p}M(r, f)$$

exists, and that, if $p > \frac{1}{4}$, then

$$(0.5) \qquad \lim_{n\to\infty} \frac{|a_n|}{n^{2p-1}} = \frac{\alpha}{\Gamma(2p)} \quad ,$$

where Γ is the gamma function (see e.g. Titchmarsh (1938)).

Hayman (1958) also obtained sharp bounds for (0.1) in two special cases. Here ω denotes a real constant, $0 \leq \omega < 2\pi$.

(i) if $f(z) = z^p + a_{p+1}z^{p+1} + \ldots$ is circumferentially mean p-valent in $|z| < 1$, where p is a positive integer, then

$$(0.6) \qquad M(r, f) < r^p(1-r)^{-2p} \quad , \quad 0 < r < 1 \quad ,$$

unless $f(z) = z^p(1-ze^{i\omega})^{-2p}$.

We note that if a suitable branch of f is chosen in a cut disk, then (i) holds for all $p > 0$.

(ii) if $f(z) = a_0 + a_1 z + \ldots$ is circumferentially mean p-valent and non-zero in $|z| < 1$ then

$$(0.7) \qquad M(r, f) < |a_0| [(1+r)/(1-r)]^{2p} \quad , \quad 0 < r < 1 \quad ,$$

unless $f(z) = a_0[(1+ze^{i\omega})/(1-ze^{i\omega})]^{2p}$.

To prove (0.5), Hayman showed that f attains its maximal growth along a unique radius, $\theta = \theta_0$, (called a radius of greatest growth) and that $|f|$ is relatively small away from a neighbourhood of θ_0. He defined a major arc, $\gamma = \{\theta : |\theta - \theta_0| < k(1-r)\}$, where k is a large positive constant, and a complementary minor arc, $\gamma^c = [0, 2\pi) \setminus \gamma$, and proved that the contribution of the minor arc to the integral

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta$$

is relatively small compared to the contribution of the major arc. It follows that the asymptotic behaviour of the coefficients is determined by the behaviour of the function on the major arc, as $r \to 1$.

Once bounds for α are obtained, asymptotic bounds for the coefficients follow immediately from (0.5). For the special case $p = 1$, $f(z) = z + a_2 z^2 + \dots$, Hayman (1958) showed that

(0.8) $\qquad \alpha < 1 \quad ,$

unless f is a Koebe function, i.e. $f(z) = z(1-ze^{i\omega})^{-2}$. Thus, for a fixed f, there exists $n_0 = n_0(f)$ such that for $n > n_0$,

(0.9) $\qquad |a_n| < n \quad ,$

unless f is a Koebe function. This is usually referred to as the asymptotic Bieberbach conjecture. Bieberbach (1916) conjectured that if f is univalent in $|z| < 1$ and is normalised, i.e. $f(z) = z + a_2 z^2 + \dots$ then $|a_n| \leq n$, $n \geq 2$, equality holding if and only if f is a Koebe function. The best result to date is that of Horowitz (1977) who proved that

$$|a_n| < n\left(\frac{209}{140}\right)^{1/6} < 1.0691\,n \quad , \quad n \geq 2 \quad .$$

For the case $f(z) \neq 0$, Hayman showed that

(0.10) $\qquad \alpha < 4|a_0| \quad ,$

unless $f(z) = [(1+ze^{i\omega})/(1-ze^{i\omega})]^2$, and hence

(0.11) $\qquad |a_n| < 4|a_0|n \quad , \quad n > n_0(f) \quad ,$

unless $f(z) = [(1+ze^{i\omega})/(1-ze^{i\omega})]^2$.

Eke (1965) extended (0.4), (0.5), (0.8) to the areally mean p-valent case of Spencer.

A generalisation of p-valent functions in a different direction was given by Hayman (1951):

a function f is said to be *weakly p-valent* (p integral) in $|z| < 1$ if, for every $R > 0$ the equation $f(z) = w$ has either

a) exactly p roots in $|z| < 1$ for every w on the circle $|w| = R$, or

b) less than p roots in $|z| < 1$ for some w on the circle $|w| = R$;

multiple roots being counted multiply.

It follows from this definition that a circumferentially mean p-valent function is weakly p-valent. The class of areally mean p-valent functions however neither contains nor is contained in the class of weakly p-valent functions.

Extensions of (0.1), (0.6), (0.7) to the weakly p-valent case were provided by Hayman (1951) and of (0.2), (0.3) by Hayman and Weitsman (1975). Baernstein (1974) obtained sharp bounds for the means for the weakly p-valent version of (i) and (ii) above. In fact he obtained:

(i) if $f(z) = z^p + a_{p+1}z^{p+1} + \ldots$ is weakly p-valent in $|z| < 1$ then

$$I_\lambda(r, f) < \int_0^{2\pi} \frac{r^{p\lambda}}{|1-re^{i\theta}|^{2p\lambda}}\, d\theta \quad , \quad 0 < \lambda < \infty \quad , \quad 0 < r < 1 \quad ,$$

unless $f(z) = z^p(1-ze^{i\omega})^{-2p}$;

(ii) if $f(z) = a_0 + a_1 z + \ldots$ is weakly p-valent and non-zero in $|z| < 1$ then

$$I_\lambda(r, f) < |a_0|^\lambda \int_0^{2\pi} \left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right|^{2p\lambda} d\theta \ , \ 0 < \lambda < \infty \ , \ 0 < r < 1 \ ,$$

unless $f(z) = a_0[(1+ze^{i\omega})/(1-ze^{i\omega})]^{2p}$.

In case (i), $p = 1$, it follows that

$$|a_n| < \tfrac{1}{2}en \quad , \quad n \geq 2 \quad ,$$

and this is the best known estimate for general n.

In Chapter 2 of this thesis we show that instead of having a unique radius of greatest growth as in the mean p-valent functions of Biernacki and Spencer, a weakly p-valent function may have infinitely many. We show that

$$\overline{\lim_{n\to\infty}} \frac{|a_n|}{n^{2p-1}} = \frac{\sum_\nu \alpha(\theta_\nu)}{(2p-1)!} \quad ,$$

where the θ_ν , $\nu = 1,2,\ldots$ are the radii of greatest growth of f , and $\alpha(\theta_\nu) = \lim_{r\to 1} (1-r)^{2p} |f(re^{i\theta_\nu})|$. We also prove that if f has more than one radius of greatest growth then the limit $\lim_{n\to\infty} |a_n|/n^{2p-1}$ does not exist. If however, f has a unique radius of greatest growth θ_0 , then this limit does exist and

$$\lim_{n\to\infty} \frac{|a_n|}{n^{2p-1}} = \frac{\alpha(\theta_0)}{(2p-1)!} \quad .$$

We obtain sharp bounds for the special cases (i), (ii) using the sharp bounds for means obtained by Baernstein (1974). We have for $p = 1$:

(i) $\quad \sum_\nu \alpha(\theta_\nu) < 1$

except when $f(z) = z(1-ze^{i\omega})^{-2}$;

(ii) $\quad \sum_\nu \alpha(\theta_\nu) < 4|a_0|$

except when $f(z) = a_0[(1+ze^{i\omega})/(1-ze^{i\omega})]^2$.

Thus the asymptotic Bieberbach conjecture holds also for weakly univalent functions.

0.2 Functions omitting a sequence of values

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be regular in $|z| < 1$ and let $\{w_k\}$ be a sequence of complex numbers such that $r_k = |w_k|$ is monotonic increasing and $r_k \to \infty$ as $k \to \infty$. If

$$f(z) \neq w_k \quad , \quad k = 0,1,\ldots \quad ,$$

what can we say about the maximum modulus, integral means, and coefficients of f ?

Littlewood (1924) proved that if

$$r_{k+1} \leq Cr_k \quad ,$$

then

$$M(r, f) = O(1-r)^{-A(C)} \quad , \quad r \to 1 \quad ,$$

where $A(C)$ depends on C only.

Cartwright (1935) showed that if

$$\frac{r_{k+1}}{r_k} \to 1 \quad ,$$

then

$$M(r, f) = O(1-r)^{-2-\varepsilon} \quad , \quad r \to 1 \quad ,$$

for every $\varepsilon > 0$.

Baernstein and Rochberg (1977) have recently proved that under the hypothesis of Cartwright

$$I_1(r, f) = O(1-r)^{-(1+\varepsilon)} \quad , \quad |a_n| = O(n^{1+\varepsilon}) \quad ,$$

for every $\varepsilon > 0$.

Hayman (1949) obtained the following result, which is essentially best possible:

if

$$\sum_{k=0}^{\infty} \left\{ \log \left(\frac{r_{k+1}}{r_k} \right) \right\}^2 < \infty \quad ,$$

then

$$M(r, f) = O(1-r)^{-2} \quad , \quad r \to 1 \quad .$$

Littlewood (1924, see also 1944) showed, by considering the elliptic modular function, that if

$$P(z) \neq 0,1 \quad ;$$

$$Q(z) = \log[P(z)] \neq \pm 2\pi i k \quad , \quad k = 0,1,\ldots \quad ;$$

$$R(z) = [Q(z)]^2 \neq -4\pi^2 k^2 \quad , \quad k = 0,1,\ldots \quad ,$$

then

$$M(r,R) = O(1-r)^{-2} \quad ,$$

$$I_1(r,R) = O(1-r)^{-1} \quad , \quad r \to 1 \quad ,$$

and if $R(z) = \sum r_n z^n$, then

$$|r_n| = O(n) \quad , \quad n \to \infty \quad .$$

Hayman (1955b) generalised these results in the following way: suppose that $f(z)$ is regular in $|z| < 1$ and $f(z) \neq w_k$, where $\{w_k\}$ satisfies

(i) $\arg w_k = O(|w_k|^{-\frac{1}{2}})$;

(ii) $|w_{k+1} - w_k| = O(|w_k|^{\frac{1}{2}})$,

then

$$M(r, f) = O(1-r)^{-2} \quad ;$$

$$I_1(r, f) = O(1-r)^{-1} \quad , \quad (r \to 1) \quad ;$$

and if $f(z) = \sum a_n z^n$, then

$$|a_n| = O(n) \quad , \quad n \to \infty \quad .$$

In Chapter 3 we generalise still further. We show that if $2 \leq p < \infty$, and

(i) $\arg w_k = O(|w_k|^{-1/p})$;

(ii) $|w_{k+1} - w_k| = O(|w_k|^{(p-1)/p})$,

then

$$M(r, f) = O(1-r)^{-2} \quad ;$$

$$I_1(r, f) = O(1-r)^{-1} \quad ;$$

$$|a_n| = O(n) \quad .$$

We also apply the theory of Chapter 1 to these functions and obtain

$$\overline{\lim_{n\to\infty}} \frac{|a_n|}{n} = \frac{\sum_{\nu} \alpha(\theta_\nu)}{(2p-1)!} ,$$

where $\alpha(\theta_\nu) = \lim_{r\to 1} (1-r)^2 |f(re^{i\theta})| > 0$.

CHAPTER 1

General Results

1.0 Introduction

Let f be a function regular in $|z| < 1$ and let $p \geq 1$ be a given integer. Suppose that for each θ, $0 \leq \theta < 2\pi$, the limit

$$(1.1) \qquad \alpha(\theta) = \lim_{r \to 1} (1-r)^{2p} |f(re^{i\theta})| \quad \text{exists, with } 0 \leq \alpha(\theta) < \infty .$$

We say that θ is a radius of greatest growth of f if $\alpha(\theta) > 0$. If we are considering θ_ν, $\nu = 1,\ldots,N$ say, then for convenience we write $\alpha_\nu = \alpha(\theta_\nu)$.

We choose a fixed ε, $0 < \varepsilon < \pi/2$. Then if θ_ν is a radius of greatest growth of f we denote by $\Delta_n^{(\nu)} = \Delta_n^{(\nu)}(\varepsilon)$ the domain

$$\{z : \frac{\varepsilon}{n} < |1-ze^{-i\theta_\nu}| < \frac{1}{\varepsilon n} ; \quad |\arg(1-ze^{-i\theta_\nu})| < \frac{\pi}{2} - \varepsilon\} .$$

For $n \geq 1$ we write

$$r_n = 1 - 1/n \quad , \quad z_n = r_n e^{i\theta_\nu} \quad ,$$

$$\alpha_\nu^{(n)} = n^{-2p} f(z_n) \quad ,$$

$$\lambda_\nu(r_n) = \arg \alpha_\nu^{(n)} \quad ,$$

and set

$$f_\nu^{(n)}(z) = \frac{\alpha_\nu^{(n)}}{(1-ze^{-i\theta_\nu})^{2p}} .$$

Thus $\alpha_\nu^{(n)}$, $\lambda_\nu(r_n)$ and $f_\nu^{(n)}(z)$ are defined for $n \geq 1$ and using (1.1) we have that

$$|\alpha_\nu^{(n)}| \to \alpha_\nu \quad , \quad (n \to \infty) .$$

Our first result is

THEOREM (1.1)

Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is regular in $|z| < 1$ *and satisfies* (1.1). *Suppose also that* f *satisfies the following:*

(1.2) *If* θ_ν *is a radius of greatest growth of* f *, then, as* $n \to \infty$ *uniformly for* $z \in \Delta_n^{(\nu)}(\varepsilon)$,

$$f(z) \sim f_\nu^{(n)}(z) \quad ;$$

(1.3) *For some* $\lambda > 1/(2p)$,

$$\overline{\lim_{r\to 1}}\, (1-r)^{2p\lambda-1} I_\lambda(r, f) \leq C ,$$

where C *is a constant.*

Then we have

(1.4) $\alpha(\theta) = 0$, *except for a sequence* $\theta = \theta_\nu$, *and*

(1.5) $$\sum_\nu \alpha_\nu^{\lambda} \leq C_1 = \frac{2\Gamma(\frac{1}{2})\Gamma(p\lambda)C}{\Gamma(\lambda p-\frac{1}{2})} ,$$

Theorem (1.1) says that the set $E = \{\theta : \alpha(\theta) > 0\}$ is countable. Thus, given $\eta > 0$, we may define $N = N(\eta)$ to be the number of radii of greatest growth of f for which $\alpha(\theta) \geq \eta$. It follows that N is finite and increases as η decreases. Let θ_ν , $\nu = 1,\ldots,N$, be such that

$\alpha_1 \geq \alpha_2 \geq \dots \alpha_N \geq \eta$. For $\nu = 1,\dots,N$ and K a large positive constant we define

$$\gamma_\nu = \{\theta : |\theta-\theta_\nu| < K(1-r)\} \quad , \quad \gamma_\nu{}^c = [0,2\pi)\setminus\gamma_\nu \quad ,$$

$$\gamma = \bigcup_{\nu=1}^{N} \gamma_\nu \quad , \quad \gamma^c = [0,2\pi)\setminus \gamma \quad ,$$

and let γ_ν^* be the set of $re^{i\theta}$, for which $\theta \in \gamma_\nu$. We denote the closure of a set E by $(E)'$.

With the above notation we have

THEOREM (1.2)

Suppose that f satisfies the hypotheses of Theorem (1.1) ; (1.3) *being satisfied for some $\lambda < 1$, and also the following:*

(1.6) *Given $\eta > 0$, there exist constants N , K , r_o such that*

$$|f(re^{i\theta})| < \eta(1-r)^{-2p} \quad , \quad r_o < r < 1 \quad , \quad \theta \in \gamma^c \quad ;$$

(1.7) *If θ_j, θ_k are any two radii of greatest growth of f , then*

$$\lambda_j(r_n) = \lambda_k(r_n) + o(1) \quad , \quad (n \to \infty) \quad .$$

We then have

(1.8) $$\overline{\lim_{n\to\infty}} \frac{|a_n|}{n^{2p-1}} = \frac{\sum_\nu \alpha(\theta_\nu)}{(2p-1)!}$$

Further, if f has more than one radius of greatest growth we have

(1.9) $$\underline{\lim_{n\to\infty}} \frac{|a_n|}{n^{2p-1}} \leq \frac{(\sum_\nu \alpha(\theta_\nu)^2)^{\frac{1}{2}}}{(2p-1)!} \quad ,$$

and so $\lim_{n\to\infty} |a_n|/n^{2p-1}$ *does not exist.*

The proof of Theorem (1.2) is based on the proof of the corresponding result for circumferentially mean p-valent functions by Hayman (1958, Chapter 5), the essential difference between the two proofs being that a circumferentially mean p-valent function has a unique radius of greatest growth whereas a function which satisfies the hypotheses of Theorem (1.2) may have infinitely many.

Chapter 1 is in four sections.

In the first section we prove Theorem (1.1). We show in the second section that the contribution of the "minor arc", γ^c, is relatively small compared to the "major arc", γ, and then prove Theorem (1.2) for the case when f has no radii of greatest growth, i.e. $\alpha(\theta) = 0$ for all θ in $[0,2\pi)$. In §1.3 we derive the formula

$$a_n = \frac{n^{2p-1}}{(2p-1)!} e^{i\lambda_1(r_n)} \left\{ \sum_\nu \alpha_\nu e^{-in\theta_\nu} + o(1) \right\} , \quad (n\to\infty) .$$

In §1.4 we apply a result of Diophantine approximation theory to

$$c(n) = \sum_\nu \alpha_\nu e^{-in\theta_\nu}$$

and show that

$$\overline{\lim_{n\to\infty}} |c(n)| = c(o) = \sum_\nu \alpha_\nu ,$$

$$\underline{\lim_{n\to\infty}} |c(n)| = \inf_{n\geq 0} |c(n)| \leq (\sum_\nu \alpha_\nu^2)^{\frac{1}{2}} .$$

We then conclude the proof of Theorem (1.2).

1.1 Proof of Theorem (1.1)

We need the following

LEMMA (1.1)

For $\frac{1}{2} < \lambda < \infty$ *and* $\varepsilon_1 > 0$, *define*

(1.10) $$J_0 = J_0(\varepsilon_1,\lambda) = \{(5^\lambda.2)/[\varepsilon_1(2\lambda-1)]\}^{1/(2\lambda-1)} .$$

Then if $K \geq J_0$ *and* $\frac{1}{2} < r < 1$, *we have*

$$\int_{K(1-r)\leq|\theta|\leq\pi} \frac{d\theta}{|1-re^{i\theta}|^{2\lambda}} < \frac{\varepsilon_1}{(1-r)^{2\lambda-1}} .$$

Now $|1-re^{i\theta}|^{2\lambda} = [(1-r)^2 + 4r\sin^2(\frac{1}{2}\theta)]^\lambda$, and for $\frac{1}{2} < r < 1$, $0 \leq \theta \leq \pi$, we have

(1.11) $$[(1-r)^2 + 4r\sin^2(\tfrac{1}{2}\theta)]^\lambda \geq \left(\frac{4r\theta^2}{\pi^2}\right)^\lambda \geq \left(\frac{\theta^2}{5}\right)^\lambda .$$

Hence, with J_0 defined by (1.10), we have

$$\int_{K(1-r)}^{\pi} \frac{d\theta}{|1-re^{i\theta}|^{2\lambda}} \leq 5^\lambda \int_{K(1-r)}^{\pi} \frac{d\theta}{\theta^{2\lambda}} \qquad (\tfrac{1}{2} < r < 1)$$

$$< 5^\lambda \int_{K(1-r)}^{\infty} \frac{d\theta}{\theta^{2\lambda}}$$

$$= \frac{5^\lambda}{(2\lambda-1)[K(1-r)]^{2\lambda-1}} \leq \frac{\varepsilon_1}{2(1-r)^{2\lambda-1}} ,$$

for $K \geq J_0$. Similarly we have

$$\int_{-\pi}^{-K(1-r)} \frac{d\theta}{|1-re^{i\theta}|^{2\lambda}} < \frac{\varepsilon_1}{(1-r)^{2\lambda-1}} , \quad \tfrac{1}{2} < r < 1 , \quad K \geq J_0 .$$

This proves Lemma (1.1).

Let θ_ν , $\nu = 1,\dots,N_0$, be N_0 radii of greatest growth of f and fix λ with $1/(2p) < \lambda < \infty$. Then given $\varepsilon_1 > 0$ we choose $J_0 = J_0(\varepsilon_1, p\lambda)$ as in (1.10) and with $K \geq J_0$, $\frac{1}{2} < r < 1$, we choose $\varepsilon = \varepsilon(K)$ so that

$$(\gamma_\nu^*)' \subset \Delta_n^{(\nu)}(\varepsilon) \subset \{z : |z| < 1\}$$

for all large enough n and $r_n < r < r_{2n}$, where γ_ν^* , $\Delta_n^{(\nu)}(\varepsilon)$, r_n are defined in §1.1.

From (1.2) we have

$$f(z) \sim f_\nu^{(n)}(z) = \frac{\alpha_\nu^{(n)}}{(1-ze^{-i\theta_\nu})^{2p}} \quad , \quad (n\to\infty) \quad , \; z \in \Delta_n^{(\nu)}(\varepsilon) \quad ,$$

and hence, given $\varepsilon_2 > 0$, we can choose $n_0 = n_0(\varepsilon_2, \lambda)$ such that for $n > n_0$,

$$|f(z)|^\lambda > (1-\varepsilon_2)|f_\nu^{(n)}(z)|^\lambda \quad , \quad 1 \leq \nu \leq N_0 \quad .$$

We can also choose $n_1 = n_1(\varepsilon_2, \lambda) > n_0$ so that for $n > n_1$

$$|\alpha_\nu^{(n)}|^\lambda > (1-\varepsilon_2)\alpha_\nu^\lambda \quad , \quad 1 \leq \nu \leq N_0 \quad .$$

We now choose r so near 1 , $r > r_1 > \frac{1}{2}$ say, that the γ_ν are disjoint. We then have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^\lambda d\theta \geq \frac{1}{2\pi}\sum_{\nu=1}^{N_0}\int_{\gamma_\nu} |f(re^{i\theta})|^\lambda d\theta$$

$$> \frac{(1-\varepsilon_2)^2}{2\pi}\sum_{\nu=1}^{N_0}\alpha_\nu^\lambda\int_{-K(1-r)}^{K(1-r)} \frac{d\theta}{|1-re^{i\theta}|^{2p\lambda}}$$

$$= \frac{(1-\varepsilon_2)^2}{2\pi} \sum_{\nu=1}^{N_0} \alpha_\nu{}^\lambda \left[\left\{ \int_{-\pi}^{\pi} - \int_{K(1-r)\le|\theta|\le\pi} \right\} \frac{d\theta}{|1-re^{i\theta}|^{2p\lambda}} \right] ,$$

provided that $n > n_1$.

So from Lemma (1.1) we have for $K \geq J_0$, $\frac{1}{2} < r_1 < r < 1$,

(1.11) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} | f(re^{i\theta})|^\lambda d\theta >$$

$$\frac{(1-\varepsilon_2)^2}{2\pi} \sum_{\nu=1}^{N_0} \alpha_\nu{}^\lambda \left\{ \int_{-\pi}^{\pi} \frac{d\theta}{|1-re^{i\theta}|^{2p\lambda}} - \frac{\varepsilon_1}{(1-r)^{2p\lambda-1}} \right\} .$$

We have from Hayman (1955a, page 280) that

(1.12) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\theta}{|1-re^{i\theta}|^{2p\lambda}} \sim \frac{\Gamma(\lambda p-\frac{1}{2})(1-r)^{1-2p\lambda}}{2\Gamma(\frac{1}{2})\Gamma(p\lambda)} \quad , \quad r \to 1 \quad ,$$

and hence, given $\varepsilon_3 > 0$ there exists $r_2 > r_1$ such that for $r_2 < r < 1$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\theta}{|1-re^{i\theta}|^{2p\lambda}} > \frac{(1-\varepsilon_3)\Gamma(\lambda p-\frac{1}{2})(1-r)^{1-2p\lambda}}{2\Gamma(\frac{1}{2})\Gamma(p\lambda)} .$$

Since ε_1 , ε_2 , ε_3 are arbitrary, we have, using (1.3)

$$\sum_\nu \alpha_\nu{}^\lambda \leq C_1 = C \cdot \frac{2\Gamma(\frac{1}{2})\Gamma(p\lambda)}{\Gamma(\lambda p-\frac{1}{2})} .$$

Since N_0 is arbitrary we deduce the countability of the set $E = \{\theta:\alpha(\theta) > 0\}$ and hence

$$\sum_E \alpha(\theta)^\lambda \leq C_1$$

by a standard argument.

This completes the proof of Theorem (1.1).

1.2 The minor arc, γ^c

If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad ,$$

then we have, on writing $z = re^{i\theta}$,

$$2\pi r^n a_n = \int_{-\pi}^{\pi} f(re^{i\theta})e^{-in\theta}d\theta \quad .$$

In this section we use (1.6) to show that the contribution of the minor arc, γ^c , to the above integral is relatively small compared to the contribution of the major arc γ . We then prove (1.8) for the case when f has no radii of greatest growth, i.e. we prove

$$|a_n| = o(n^{2p-1}) \quad , \quad (n \to \infty) \quad .$$

We prove first

LEMMA (1.2)

Suppose that $f(z)$ *is regular in* $|z| < 1$ *and satisfies* (1.1), (1.3) *and* (1.6). *Then, given* η, λ, so that $0 < \eta < 1$, $1/(2p) < \lambda < 1$, we can choose constants K, r_1 such that for $r_1 < r < 1$

$$\int_{\gamma^c} |f(re^{i\theta})|d\theta < 4\pi\eta^{1-\lambda}C(1-r)^{1-2p} \quad .$$

We fix λ , $1/(2p) < \lambda < 1$. Then it follows from (1.6) that given $\eta > 0$ there exist constants N , K , r_0 such that for $\theta \in \gamma^c$,

$$|f(re^{i\theta})| < \frac{\eta}{(1-r)^{2p}} \quad , \quad r_0 < r < 1 \quad .$$

Also, using (1.3), we can choose $r_1 > r_0$ such that

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^{\lambda} d\theta < \frac{4\pi C}{(1-r)^{2p\lambda-1}} \quad , \quad r_1 < r < 1 \quad .$$

Now

$$\int_{\gamma^c} |f(re^{i\theta})| d\theta = \int_{\gamma^c} |f(re^{i\theta})|^{1-\lambda} |f(re^{i\theta})|^{\lambda} d\theta$$

$$< \left\{\frac{\eta}{(1-r)^{2p}}\right\}^{1-\lambda} \int_{\gamma^c} |f(re^{i\theta})|^{\lambda} d\theta$$

$$< \frac{\eta^{1-\lambda}}{(1-r)^{2p(1-\lambda)}} \cdot \frac{4\pi C}{(1-r)^{2p\lambda-1}}$$

$$= \frac{4\pi C\eta^{1-\lambda}}{(1-r)^{2p-1}} \quad .$$

We thus have

$$\int_{\gamma^c} |f(re^{i\theta})| d\theta < 4\pi C\eta^{1-\lambda}(1-r)^{1-2p} \quad , \quad r_1 < r < 1 \quad ,$$

as required.

If $f(z)$ has no radii of greatest growth, then $\alpha(\theta) = 0$ for all θ and so $\gamma^c = [0,2\pi)$. Using (1.6) we see that given $\eta > 0$, there exists r_0 with

$$|f(re^{i\theta})| < \eta(1-r)^{-2p} \quad , \quad (r_0 < r < 1 \; ; \; 0 \leq \theta < 2\pi) \quad .$$

From Lemma (1.2) we have

$$2\pi r^n |a_n| \leq \int_{-\pi}^{\pi} |f(re^{i\theta})| d\theta < 4\pi\eta^{1+\lambda} C(1-r)^{1-2p} \quad , \quad r_0 < r < 1 \quad .$$

We put $r = 1 - 1/n$ and since $r^n = (1-1/n)^n \geq \frac{1}{4}$ for $n \geq 2$ we have

$$|a_n| = o(n^{2p-1}) \quad , \quad (n \to \infty) \quad ,$$

as required.

1.3 An asymptotic formula for a_n

THEOREM (1.3)

Suppose that f *satisfies the conditions of Theorem* (1.2). *Then*

$$a_n = \frac{n^{2p-1}}{(2p-1)!} e^{i\lambda_1(r_n)} \left\{ \sum_{\nu} \alpha_\nu e^{-in\theta_\nu} + o(1) \right\} \quad , \quad (n \to \infty) \quad .$$

We assume that there are an infinite number of radii of greatest growth. If there are only a finite number the discussion is simpler. Given $\eta > 0$ we choose $N = N(\eta)$ as in §1.0. Then for $\nu = 1,\dots,N$ we have

$$(1-ze^{-i\theta_\nu})^{-2p} = \sum_{m=0}^{\infty} c_m e^{-im\theta_\nu} z^m \quad ,$$

where

$$c_m = \frac{(m+2p-1)!}{m!(2p-1)!} \sim \frac{m^{2p-1}}{(2p-1)!} \quad , \quad (m \to \infty) \quad .$$

With the notation of §1.0 we have

$$f_\nu^{(n)}(z) = \alpha_\nu^{(n)} \sum_{m=0}^{\infty} c_m e^{-im\theta_\nu} z^m \quad ,$$

and it follows that

$$c_n \alpha_\nu^{(n)} e^{-in\theta_\nu} = \frac{1}{2\pi i} \int_{|z|=r} \frac{f_\nu^{(n)}(z)dz}{z^{n+1}} \quad .$$

Thus

$$2\pi r^n c_n \alpha_\nu^{(n)} e^{-in\theta_\nu} = \int_{-\pi}^{\pi} f_\nu^{(n)}(re^{i\theta}) e^{-in\theta} d\theta \quad , \quad \nu = 1,\dots,N \quad .$$

Now

$$2\pi r^n a_n = \int_{-\pi}^{\pi} f(re^{i\theta}) e^{-in\theta} d\theta \quad ,$$

so we have

$$(1.13) \qquad 2\pi r^n [a_n - c_n \sum_{\nu=1}^{N} \alpha_\nu^{(n)} e^{-in\theta_\nu}] = \int_{-\pi}^{\pi} [f(re^{i\theta}) - \sum_{\nu=1}^{N} f_\nu^{(n)}(re^{i\theta})] e^{-in\theta} d\theta \quad .$$

Let K , r_0 be the constants of (1.6). Then with γ_ν , γ_ν^c , γ , γ^c defined as in §1.0 we have

$$\left| \int_{-\pi}^{\pi} [f(re^{i\theta}) - \sum_{\nu=1}^{N} f_\nu^{(n)}(re^{i\theta})] e^{-in\theta} d\theta \right|$$

$$\leq \sum_{\nu=1}^{N} \int_{\gamma_\nu} |f(re^{i\theta}) - f_\nu^{(n)}(re^{i\theta})| d\theta$$

$$+ \int_{\gamma^c} |f(re^{i\theta})| d\theta + \sum_{\nu=1}^{N} \int_{\gamma_\nu^c} |f_\nu^{(n)}(re^{i\theta})| d\theta$$

$$= I_1 + I_2 + I_3 \quad .$$

Now

$$I_3 = \sum_{\nu=1}^{N} |\alpha_\nu^{(n)}| \int_{K(1-r)\le|\theta|\le\pi} \frac{d\theta}{|1-re^{i\theta}|^{2p}} ,$$

and $|\alpha_\nu^{(n)}| \to \alpha_\nu$ as $n \to \infty$, $\nu = 1,\ldots,N$, so we choose n_0 such that for $n > n_0$

$$|\alpha_\nu^{(n)}| < 2\alpha_\nu \qquad \nu = 1,\ldots,N .$$

With $\lambda = p$ and $\varepsilon_1 = \eta$ in Lemma (1.1) we have

$$\int_{K(1-r)\le|\theta|\le\pi} \frac{d\theta}{|1-re^{i\theta}|^{2p}} < \frac{\eta}{(1-r)^{2p-1}} , \quad \tfrac{1}{2} < r < 1 .$$

We take $\sum_\nu \alpha_\nu = a$. Then $a < \infty$ in view of (1.5) with $\lambda = 1$. Thus for $n > n_0$ and $\frac{1}{2} < r < 1$

$$(1.14) \qquad I_3 < 2\sum_{\nu=1}^{N} \alpha_\nu . \eta(1-r)^{1-2p}$$

$$\le 2a\eta(1-r)^{1-2p} .$$

Next, we have from Lemma (1.2) that we can choose $r_0 = r_0(\eta) > \frac{1}{2}$ such that for $r_0 < r < 1$ and $n > n_0$

$$(1.15) \qquad \int_{\gamma^c} |f(re^{i\theta})| d\theta < 4\pi C\eta^{1-\lambda}(1-r)^{1-2p} ,$$

where C is the constant of (1.3).

We now choose $\varepsilon = \varepsilon(K) > 0$ so that for $\nu = 1,\ldots,N$ $(\gamma_\nu^*)' \subset \Delta_n^{(\nu)}(\varepsilon)$ for large n . Then (1.3) shows that

$$f(z) \sim f_\nu^{(n)}(z) , \quad (n \to \infty) , \quad \theta \in \gamma_\nu .$$

This means that given $\xi > 0$, there exists $n_1 = n_1(\xi)$ such that for $n > n_1$

$$|f(z) - f_\nu^{(n)}(z)| < \xi |f_\nu^{(n)}(z)| \quad , \quad \theta \in \gamma_\nu \quad .$$

Hence

$$I_1 = \sum_{\nu=1}^{N} \int_{\gamma_\nu} |f(re^{i\theta}) - f_\nu^{(n)}(re^{i\theta})| d\theta < \xi \sum_{\nu=1}^{N} \int_{\gamma_\nu} |f_\nu^{(n)}(re^{i\theta})| d\theta$$

$$= \xi \sum_{\nu=1}^{N} |\alpha_\nu^{(n)}| \int_{-K(1-r)}^{K(1-r)} \frac{d\theta}{|1-re^{i\theta}|^{2p}} < \xi \sum_{\nu=1}^{N} |\alpha_\nu^{(n)}| \int_{-\pi}^{\pi} \frac{d\theta}{|1-re^{i\theta}|^{2p}} \quad .$$

It follows from (1.12) that given $\varepsilon^* > 0$ we can choose $r_1 = r_1(\varepsilon^*) > r_0$ such that for $r_1 < r < 1$

$$\int_{-\pi}^{\pi} \frac{d\theta}{|1-re^{i\theta}|^{2p}} < \frac{(1+\varepsilon^*)2\pi\Gamma(p-\frac{1}{2})(1-r)^{1-2p}}{\Gamma(\frac{1}{2})\Gamma(p)} \quad .$$

Since ξ is arbitrary we deduce that

$$(1.16) \qquad I_1 = o(1-r)^{1-2p} \quad , \quad r \to 1 \quad .$$

Combining (1.13), (1.14), (1.15), (1.16) and putting $r = 1 - 1/n$ we see that

$$r^n |a_n - c_n \sum_{\nu=1}^{N} \alpha_\nu^{(n)} e^{-in\theta_\nu}| < \{2a\eta + 4\pi C\eta^{\frac{1}{2}} + o(1)\} n^{2p-1} \quad ,$$

as $n \to \infty$.

For $\nu = 1,\ldots,N$

$$\lambda_\nu(r_n) = \lambda_1(r_n) + o(1)$$

and

$$|\alpha_\nu^{(n)}| = \alpha_\nu + o(1) \quad , \quad (n \to \infty) \quad ,$$

in view of (1.7) and (1.1) respectively, so we have

$$\sum_{\nu=1}^{N} \alpha_\nu^{(n)} e^{-in\theta_\nu} = e^{i\lambda_1(r_n)} \sum_{\nu=1}^{N} \alpha_\nu e^{-in\theta_\nu} + o(1) \quad , \quad (n \to \infty) \quad .$$

Since

$$c_n = \left\{ \frac{1}{(2p-1)!} + o(1) \right\} n^{2p-1}$$

as $n \to \infty$ it follows that

$$r^n |a_n - c_n e^{i\lambda_1(r_n)} \sum_{\nu=1}^{\infty} \alpha_\nu e^{-in\theta_\nu}|$$

$$< r^n |a_n - c_n e^{i\lambda_1(r_n)} \sum_{\nu=1}^{N} \alpha_\nu e^{-in\theta_\nu}| + r^n |c_n e^{i\lambda_1(r_n)} \sum_{\nu=N+1}^{\infty} \alpha_\nu e^{-in\theta_\nu}|$$

$$< \left\{ 2a\eta + 4\pi C\eta^{\frac{1}{2}} + o(1) + \frac{r^n}{(2p-1)!} \left| \sum_{\nu=N+1}^{\infty} \alpha_\nu e^{-in\theta_\nu} \right| \right\} n^{2p-1} \quad .$$

$$< \left\{ 2a\eta + 4\pi C\eta^{\frac{1}{2}} + \frac{r^n}{(2p-1)!} \sum_{\nu=N+1}^{\infty} \alpha_\nu + o(1) \right\} n^{2p-1} \quad , \quad (n \to \infty) \quad .$$

Since $a < \infty$ we have

$$\rho_N = \sum_{\nu=N+1}^{\infty} \alpha_\nu \to 0 \quad \text{as} \quad N \to \infty \quad .$$

Now $r^n = (1-1/n)^n \geq \frac{1}{4}$, $(n \geq 2)$, and η is arbitrarily small; so on letting η tend to zero we obtain

$$a_n = \frac{n^{2p-1}}{(2p-1)!} e^{i\lambda_1(r_n)} \left\{ \sum_{\nu=1}^{\infty} \alpha_\nu e^{-in\theta_\nu} + o(1) \right\} \quad , \quad (n \to \infty) \quad .$$

This completes the proof of Theorem (1.3).

1.4 Proof of Theorem (1.2)

We need the following result, due to Dirichlet, which can be found e.g. in Niven (1963, page 13).

LEMMA (1.3)

Given any N *real numbers* $\theta_1,\ldots,\theta_N$ *there exist infinitely many sets of integers* $(a_1,\ldots,a_N,n)$ *with* n *positive such that*

$$|n\theta_\nu - 2\pi a_\nu| < n^{-1/N} \quad , \quad \nu = 1,\ldots,N \quad .$$

We can now prove

LEMMA (1.4)

For $\nu = 1,2,\ldots$ *let* θ_ν *satisfy* $0 \le \theta < 2\pi$ *and let* $\alpha_\nu > 0$ *be such that* $\alpha_\nu \geq \alpha_{\nu+1}$ *and* $\sum\alpha_\nu = a < \infty$. *Then if*

$$c(n) = \sum_\nu \alpha_\nu e^{-in\theta_\nu}$$

we have

$$\overline{\lim_{n\to\infty}} |c(n)| = a$$

and

$$\underline{\lim_{n\to\infty}} |c(n)| = \inf_{m \in N} |c(m)| \le \left(\sum_\nu \alpha_\nu{}^2 \right)^{\frac{1}{2}} \quad ,$$

where N *is the set of non-negative integers.*

We again assume that there are an infinite number of radii of greatest growth. If there are only a finite number the discussion is simpler.

Given $\varepsilon > 0$ we choose N so large that

$$\sum_{\nu=N+1}^{\infty} \alpha_\nu < \varepsilon \quad .$$

From Lemma (1.3) there exist integers $(a_1,\ldots,a_N,n)$ such that

$$|n\theta_\nu - 2\pi a_\nu| < n^{-1/N} \quad , \quad \nu = 1,\ldots,N \quad .$$

Having fixed N , we choose n so large that $n^{-1/N} < \varepsilon$. We thus have

$$|n\theta_\nu - 2\pi a_\nu| < \varepsilon \quad , \quad \nu = 1,\ldots,N \quad ,$$

from which it follows that

$$|e^{-in\theta_\nu}-1| = |e^{-i(n\theta_\nu-2\pi a_\nu)}-1| < |e^{i\varepsilon}-1| = 2\sin(\varepsilon/2) < \varepsilon \quad ,$$

and so for any non-negative integer n_0

$$\left|\sum_{\nu=1}^{N} \alpha_\nu e^{-i(n+n_0)\theta_\nu} - \sum_{\nu=1}^{N} \alpha_\nu e^{-in_0\theta_\nu}\right| < \varepsilon \sum_{\nu=1}^{N} \alpha_\nu \leq \varepsilon a \quad .$$

We thus have

$$\left|\sum_{\nu=1}^{\infty} \alpha_\nu e^{-i(n+n_0)\theta_\nu} - \sum_{\nu=1}^{\infty} \alpha_\nu e^{-in_0\theta_\nu}\right|$$

$$\leq \left|\sum_{\nu=1}^{N} \alpha_\nu e^{-i(n+n_0)\theta_\nu} - \sum_{\nu=1}^{N} \alpha_\nu e^{-in_0\theta_\nu}\right| + 2\sum_{\nu=N+1}^{\infty} \alpha_\nu$$

$$< \varepsilon(a + 2) \quad .$$

Hence, given any $\varepsilon > 0$ and n_0 we can choose n arbitrarily large such that

$$(1.17) \qquad |c(n+n_0)-c(n_0)| < \varepsilon(a+2) \quad .$$

If $n_0 = 0$ this gives

$$|c(n)| > c(0) - \varepsilon(a+2)$$

for arbitrarily large n, while $|c(n)| \leq c(0)$ for all n. Thus

$$\overline{\lim_{n\to\infty}} |c(n)| = c(0) \quad .$$

We will now prove that

$$(1.18) \qquad \alpha = \underline{\lim}_{n\to\infty} |c(n)| = \inf_{n\geq 0} |c(n)| = \beta \quad .$$

Evidently $\beta \leq \alpha$. On the other hand, given $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$ such that

$$|c(n_0)| < \beta + \varepsilon \quad ,$$

and so by (1.17) we can choose n arbitrarily large such that

$$|c(n+n_0)| < |c(n_0)| + \varepsilon(a+2) < \beta + \varepsilon(a+3) \quad .$$

We thus have $\alpha \leq \beta + \varepsilon(a+3)$. Since ε is arbitrary we deduce $\alpha \leq \beta$ and so $\alpha = \beta$ as required.

To obtain a bound for α we obtain a bound for the average of $|c(n)|$ over all non-negative integers. We prove first that

$$(1.19) \qquad \psi(K) = \frac{1}{K}\sum_{n=0}^{K-1}\left\{\sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^{\infty} \alpha_\nu\alpha_\mu \cos n(\theta_\mu-\theta_\nu)\right\} \to 0 \quad \text{as} \quad K\to\infty \quad .$$

Given $\varepsilon > 0$ we choose $N = N(\varepsilon)$ so large that

$$\sum_{\nu=N+1}^{\infty} \alpha_\nu < \varepsilon \quad .$$

We write

$$\sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^{\infty} \alpha_\mu\alpha_\nu \cos n(\theta_\mu-\theta_\nu) = \sum\nolimits_1 + \sum\nolimits_2 \quad .$$

where $\sum_1$ is taken over all terms with $\mu \leq N$ and $\nu \leq N$ and $\sum_2$ over the remainder. Then

$$\left|\sum\nolimits_2\right| \leq \sum_{\mu=1}^{N}\alpha_\mu \sum_{\nu=N+1}^{\infty}\alpha_\nu + \sum_{\mu=N+1}^{\infty}\alpha_\mu \sum_{\nu=1}^{N}\alpha_\nu + \sum_{\mu=N+1}^{\infty}\alpha_\mu \sum_{\nu=N+1}^{\infty}\alpha_\nu$$

$$< \varepsilon(2a+\varepsilon) \quad .$$

Now

$$\frac{1}{K}\sum_{n=0}^{K-1}\sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^{N}\alpha_\mu\alpha_\nu \cos n(\theta_\mu-\theta_\nu) = \frac{1}{K}\sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^{N}\alpha_\mu\alpha_\nu\sum_{n=0}^{K-1}\cos n(\theta_\mu-\theta_\nu)$$

$$= \frac{1}{K}\sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^{N}\alpha_\mu\alpha_\nu \operatorname{Re}\left\{\sum_{n=0}^{K-1} e^{in(\theta_\mu-\theta_\nu)}\right\}$$

$$= \frac{1}{2K}\sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^{N}\alpha_\mu\alpha_\nu\left\{\frac{\sin(K-\frac{1}{2})(\theta_\mu-\theta_\nu)}{\sin\frac{1}{2}(\theta_\mu-\theta_\nu)} + 1\right\} \quad ,$$

so we have

$$|\psi(K)| = \left|\frac{1}{K}\sum_{n=0}^{K-1}\left\{\Sigma_1 + \Sigma_2\right\}\right|$$

$$\leq \frac{1}{2K}\sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^{N}\alpha_\mu\alpha_\nu\left\{\frac{|\sin(K-\frac{1}{2})(\theta_\mu-\theta_\nu)|}{|\sin\frac{1}{2}(\theta_\mu-\theta_\nu)|} + 1\right\} + \varepsilon(2a+\varepsilon) \quad .$$

Each term in the sum is bounded and there are only a finite number of terms; so we choose $K_0 = K_0(\varepsilon)$ so large that for $K > K_0$, the sum is less than ε. We thus have for $K > K_0$

$$\psi(K) < \varepsilon(2a+\varepsilon+1) \quad .$$

Since ε is arbitrary we deduce (1.19).

Now for any $n \geq 0$

$$|c(n)| = \left\{\sum_{\nu=1}^{\infty}\alpha_\nu^2 + \sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^{\infty}\alpha_\mu\alpha_\nu\cos n(\theta_\mu-\theta_\nu)\right\}^{\frac{1}{2}}$$

From Cauchy's inequality

$$\left(\frac{1}{K}\sum_{n=0}^{K-1}|c(n)|\right)^2 \leq \frac{1}{K}\sum_{n=0}^{K-1}|c(n)|^2 \quad ,$$

so we have

$$\frac{1}{K}\sum_{n=0}^{K-1}|c(n)| \leq \left(\frac{1}{K}\sum_{n=0}^{K-1}\left\{\sum_{\nu=1}^{\infty}\alpha_\nu^2 + \sum_{\substack{\mu,\nu=1\\ \mu\neq\nu}}^{\infty}\alpha_\mu\alpha_\nu\cos n(\theta_\mu-\theta_\nu)\right\}\right)^{\frac{1}{2}}$$

$$= \left(\sum_{\nu=1}^{\infty}\alpha_\nu^2 + \psi(K)\right)^{\frac{1}{2}} \quad .$$

Now $\psi(K) \to 0$ as $K \to \infty$ from (1.19) so we have

$$\inf_{n\geq 0}|c(n)| \leq \overline{\lim_{K\to\infty}} \frac{1}{K}\sum_{n=0}^{K-1}|c(n)| \leq \left(\sum_{\nu=1}^{\infty}\alpha_\nu^2\right)^{\frac{1}{2}} \quad .$$

This completes the proof of Lemma (1.4).

Theorem (1.2) now follows from Theorem (1.3) and Lemma (1.4).

For

$$\overline{\lim_{n\to\infty}} \frac{|a_n|}{n^{2p-1}} = \overline{\lim_{n\to\infty}} \frac{\left|\sum_{\nu=1}^{\infty}\alpha_\nu e^{-in\theta_\nu}\right|}{(2p-1)!}$$

$$= \overline{\lim_{n\to\infty}} \frac{|c(n)|}{(2p-1)!} = \frac{\sum_{\nu=1}^{\infty}\alpha_\nu}{(2p-1)!} \quad .$$

Also, if there are N radii of greatest growth, with $2 \leq N \leq \infty$,

$$\underline{\lim_{n\to\infty}} \frac{|a_n|}{n^{2p-1}} = \underline{\lim_{n\to\infty}} \frac{|c(n)|}{(2p-1)!} \leq \frac{1}{(2p-1)!}\left(\sum_{\nu=1}^{N}\alpha_\nu^2\right)^{\frac{1}{2}}$$

$$< \frac{1}{(2p-1)!}\sum_{\nu=1}^{N}\alpha_\nu = \overline{\lim_{n\to\infty}} \frac{|c(n)|}{(2p-1)!} = \overline{\lim_{n\to\infty}} \frac{|a_n|}{n^{2p-1}} \quad ,$$

and so $\lim_{n\to\infty}|a_n|/n^{2p-1}$ does not exist.

CHAPTER 2

Weakly p-valent Functions

2.0 Introduction

We define a function $f(z)$ regular in $|z| < 1$ to be weakly p-valent (Hayman [1951]) there if for every $R > 0$ the equation $f(z) = w$ has either

(i) exactly p roots in $|z| < 1$ for every w on the circle $|w| = R$ or

(ii) less than p roots in $|z| < 1$ for some w on the circle $|w| = R$; multiple roots being counted multiply.

In this chapter we show that a weakly p-valent function satisfies the hypotheses of Theorems (1.1) and (1.2) and hence the conclusions of those theorems. We have in fact

THEOREM (2.1)

Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is weakly p-valent in $|z| < 1$ *and has* q *zeros there, where* $0 \leq q \leq p$. *Then for each* θ , $0 \leq \theta \leq 2\pi$, *the limit*

$$\alpha(\theta) = \lim_{r \to 1} (1-r)^{2p} |f(re^{i\theta})|$$

exists. Further, $\alpha(\theta) = 0$, *except for a sequence* $\theta = \theta_\nu$, *and*

$$\sum_{\nu} \alpha(\theta_\nu)^{1/(2p)} \leq \pi B(q) \mu_q^{1/(2p)} ,$$

where $\mu_q = \max_{0 \leq \nu \leq q} |a_\nu|$ *and* $B(q)$ *is a constant depending only on* q .

The condition $0 \leq q \leq p$ is a consequence of weak p-valence (see e.g. Hayman and Weitsman [1975, page 131]).

THEOREM (2.2)

With the above hypotheses

$$\overline{\lim_{n\to\infty}} \frac{|a_n|}{n^{2p-1}} = \frac{\sum_{\nu} \alpha(\theta_\nu)}{(2p-1)!} .$$

Further, if f *has more than one radius of greatest growth*

$$\underline{\lim_{n\to\infty}} \frac{|a_n|}{n^{2p-1}} \leq \frac{\left(\sum_{\nu} \alpha(\theta_\nu)^2\right)^{\frac{1}{2}}}{(2p-1)!} ,$$

and so $\lim_{n\to\infty} |a_n|/n^{2p-1}$ *does not exist.*

Equality need not hold in the second part of Theorem (2.2), since the function

$$f(z) = [(1+z^2)/(1-z^2)]^{2p}$$

shows that the left-hand side may be zero.

We also consider special cases of weakly p-valent functions. We have

THEOREM (2.3)

Suppose that

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$$

is weakly p-valent in $|z| < 1$ *and has* p *zeros at the origin. Then*

$$\sum_{\nu} \alpha(\theta_\nu)^{1/(2p)} \leq 1 .$$

Further, $\sum \alpha(\theta_\nu) < 1$ *, except when* $f(z) = z^p(1-ze^{i\omega})^{-2p}$.

It follows from Theorem (2.2) that for a function f satisfying the hypotheses of Theorem (2.3) there exists $n_0 = n_0(f)$ such that for $n > n_0$,

$$|a_n| \leq \frac{n(n^2-1)(n^2-2^2)\ldots(n^2-(p-1)^2)}{(2p-1)!} ,$$

with equality if and only if $f(z) = z^p(1-ze^{i\omega})^{-2p}$. In particular, if $p = 1$ we obtain $|a_n| < n$ for all $n > n_0(f)$ unless f is a Koebe function.

THEOREM (2.4)

Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is weakly p-valent and non-zero in $|z| < 1$. *Then*

$$\sum_\nu \alpha(\theta_\nu)^{1/(2p)} \leq (4^p|a_0|)^{1/(2p)} .$$

Further, $\sum \alpha(\theta_\nu) < 4^p|a_0|$ *, except when* $f(z) = [(1+ze^{i\omega})/(1-ze^{i\omega})]^{2p}$.

Thus, if we write

$$F_p(z) = [(1+z)/(1-z)]^{2p} = \sum_{n=0}^{\infty} A_{n,p} z^n ,$$

we have that for a non-zero weakly p-valent function f there exists $n_0 = n_0(f)$ such that for $n > n_0(f)$

$$|a_n| \leq |a_0| A_{n,p} ,$$

with equality if and only if $f(z) = a_0 F_p(ze^{i\omega})$. In particular, if $p = 1$, we obtain $|a_n| < 4|a_0|n$ for all $n > n_0(f)$ unless $f(z) = a_0 F_1(ze^{i\omega})$.

Chapter 2 is divided into five sections. In the first section we prove Theorem (2.1). In the second and third sections we show that (1.7) and (1.6) hold for weakly p-valent functions so that Theorem (2.2) follows from Theorem (1.2). In §2.4 we prove the sharp Theorems (2.3) and (2.4) using recent results of Baernstein (1974). In the last section we look at some examples of weakly univalent functions. The first example shows that the constant $1/(2p)$ in the inequality (Theorem (2.4))

$$\sum_{\nu} \alpha(\theta_\nu)^{1/(2p)} \leq (4^p|a_0|)^{1/(2p)}$$

for non-zero weakly univalent functions is best possible. The second function is an example of a weakly univalent function which has an infinite number of radii of greatest growth.

2.1 Proof of Theorem (2.1)

We first show that $\alpha(\theta)$ exists. We need the following

LEMMA (2.1)

Suppose that $f(z)$ *is weakly* p-*valent in* $|z| < 1$ *and* $f(z) \neq 0$ *for* $1 - 2\delta < |z| < 1$, *where* $0 < \delta < \frac{1}{2}$. *Then for* $0 \leq \theta < 2\pi$ *and* $1 - \delta < |z| < 1$ *we have that*

$$\left(\frac{1-r}{r+2\delta-1}\right)^{2p} |f(re^{i\theta})|$$

decreases with increasing r .

The proof of Lemma (2.1) for circumferentially mean p-valent functions given in Hayman (1958, page 95) is based on the inequality

$$(2.1) \qquad \left|\frac{f'(z)}{f(z)}\right| \le \frac{4p}{1-|z|^2} \quad .$$

Since this inequality also holds for weakly p-valent functions (Hayman 1951, page 174) the argument for Lemma (2.1) is identical to that for the circumferentially mean p-valent case.

We now take δ , $0 < \delta < \frac{1}{2}$, so that $f(z) \neq 0$ in $1 - 2\delta < |z| < 1$. It follows immediately that $\alpha(\theta)$ exists, since

$$\left(\frac{1-r}{r+2\delta-1}\right)^{2p} |f(re^{i\theta})| + \frac{\alpha(\theta)}{(2\delta)^{2p}} \quad ,$$

as $r \to 1$.

With the notation of §1.0 we have

LEMMA (2.2)

Suppose that f *is weakly* p-*valent in* $|z| < 1$ *and that* θ_ν *is a radius of greatest growth of* f . *Then, as* $n \to \infty$, *uniformly for* $z \in \Delta_n^{(\nu)}(\varepsilon)$,

$$f(z) \sim f_\nu^{(n)}(z) \quad .$$

The proof of Lemma (2.2) for circumferentially mean p-valent functions is given in Hayman (1958, page 108). Since it is also based on the inequality (2.1) the proof extends to weakly p-valent functions.

We now quote the following result of Hayman and Weitsman (1975, Theorem 6).

LEMMA (2.3)

Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is weakly p*-valent in* $|z| < 1$ *and that* $f(z)$ *has* q *zeros there, so that* $0 \leq q \leq p$. *Then for* $\lambda > 1/(2p)$ *and* $0 \leq r < 1$ *we have*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^{\lambda} d\theta < \left(\frac{2p\lambda}{2p\lambda-1}\right) B(q)^{2p\lambda} \mu_q^{\lambda} [\pi^2 p_2 (1+2p)^2]^{\lambda-1/(2p)} (1-r)^{1-2p\lambda} ,$$

where $B(q)$ *is a constant depending only on* q *and* $\mu_q = \max_{0\leq\nu\leq q} |a_\nu|$.

We now prove the second part of Theorem (2.1).

As in the proof of Theorem (1.1) we have from (1.11) and (1.12) that given positive constants ε_1 , ε_2 , ε_3 there exists r_2 such that for $r_2 < r < 1$, $\lambda > 1/(2p)$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^{\lambda} d\theta > (1-\varepsilon_2)^2 \sum_{\nu=1}^{N_0} \alpha_\nu^{\lambda} \frac{(1-\varepsilon_3)\Gamma(\lambda p-\frac{1}{2})}{2\Gamma(\frac{1}{2})\Gamma(\lambda p)} - \frac{\varepsilon_1}{2\pi} (1-r)^{1-2p\lambda} .$$

Now $\Gamma(\lambda p-\frac{1}{2}) = \Gamma(\lambda p+\frac{1}{2})/(\lambda p-\frac{1}{2})$ so we have from Lemma (2.3), since ε_1 , ε_2 , ε_3 are arbitrary, that

$$\sum_{\nu=1}^{N_0} \alpha_\nu^{\lambda} \leq \frac{2p\lambda B(q)^{2p\lambda} \mu_q^{\lambda} [\pi^2 p_2 (1+2p)^2]^{\lambda-1/(2p)} \Gamma(\frac{1}{2})\Gamma(\lambda p)}{\Gamma(\lambda p+\frac{1}{2})} .$$

This is true for $1/(2p) < \lambda < \infty$ and so by continuity it is true for $\lambda = 1/(2p)$. We thus have

$$\sum_{\nu=1}^{N_0} \alpha_\nu^{1/(2p)} \leq \pi B(q) \mu_q^{1/(2p)} ,$$

and since N_0 is arbitrary we deduce the countability of E and

$$\sum_{E} \alpha(\theta)^{1/(2p)} \leq \pi B(q) \mu_q^{1/(2p)}$$

by a standard argument. This proves Theorem (2.1).

2.2 The argument of $\alpha_\nu^{(n)}$

LEMMA (2.4)

Let θ_j , θ_k *be any two radii of greatest growth of a function* $f(z)$ *weakly p-valent in* $|z| < 1$. *Then if* $r_n = 1 - 1/n$, *and*

$$\arg \alpha_\nu^{(n)} = \lambda_\nu(r_n) \quad , \quad \nu = j,k \quad ,$$

we have

$$\lambda_j(r_n) = \lambda_k(r_n) + o(1) \quad , \quad (n \to \infty) \quad .$$

We take δ , $0 < \delta < \frac{1}{2}$, so that $f(z) \neq 0$ in $1 - 2\delta < |z| < 1$. We choose θ_0 so that $\alpha(\theta_0) = 0$ and cut the annulus $1 - 2\delta < |z| < 1$ from $(1 - 2\delta)e^{i\theta_0}$ to $e^{i\theta_0}$. Denote the annulus minus the cut by Δ . Then $[f(z)]^{1/p}$ is regular in Δ and so weakly 1-valent there. We restrict ourselves in the proof to $1 - 2\delta < r < 1$ and prove the lemma for $p = 1$, since otherwise we may consider $[f(z)]^{1/p}$ instead of $f(z)$.

Assume $0 < \alpha_k \leq \alpha_j$. Given $\eta > 0$ satisfying $0 < \eta < \frac{1}{8}(\alpha_k/\alpha_j)$ choose $n_0 = n_0(\eta)$ so large that for $\nu = j,k$ and $n > n_0$ $\Delta_n^{(\nu)}(\frac{1}{2}\eta) \subset \{|z| < 1\}$. For $\nu = j,k$ let

$$z = \ell_n^{(\nu)}(Z) \equiv e^{i\theta_\nu}\{r_n + \frac{1}{n} Ze^{-i\theta_\nu}\} \quad ,$$

so that

$$1 - ze^{-i\theta_\nu} = \frac{1}{n}(1 - Ze^{-i\theta_\nu}) \quad ;$$

then $\Delta_n^{(\nu)}(\eta)$ in the z-plane corresponds in the Z-plane to

$$\Delta_*^{(\nu)}(\eta) = \{Z : \eta < |1-Ze^{-i\theta_\nu}| < \eta^{-1} ; |\arg(1-Ze^{-i\theta_\nu})| < \pi/2-\eta\} \quad .$$

Thus, for $Z \in \Delta_*^{(\nu)}(\tfrac{1}{2}\eta)$, $f[\ell_n^{(\nu)}(Z)]$ and $f_\nu^{(n)}[\ell_n^{(\nu)}(Z)]$ are defined for $n > n_0$.

Now $w_n^{(\nu)} = f_\nu^{(n)}[\ell_n^{(\nu)}(Z)]$ maps $\Delta_*^{(\nu)}(\eta)$ 1-1 and conformally onto the domain

$$D_n^{(\nu)}(\eta) = \{w_n^{(\nu)} : |\alpha_\nu^{(n)}|\eta^2 n^2 < |w_n^{(\nu)}| < |\alpha_\nu^{(n)}|\eta^{-2} n^2 ;$$

$$|\arg w_n^{(\nu)} - \arg\alpha_\nu^{(n)}| < \pi - 2\eta\} \quad .$$

Choose $n_1 = n_1(\eta) > n_0$ so that for $n > n_1$

$$\tfrac{1}{2}\alpha_\nu < |\alpha_\nu^{(n)}| < 2\alpha_\nu \quad , \quad \nu = j,k \quad .$$

Then for $n > n_1$

$$(2\eta)^4 < 2\eta < \frac{\alpha_k}{4\alpha_j} < \left|\frac{\alpha_k^{(n)}}{\alpha_j^{(n)}}\right| < \frac{4\alpha_k}{\alpha_j} \le 4 < \frac{1}{2\eta} < \left(\frac{1}{2\eta}\right)^4$$

so that the regions $D_n^{(j)}(2\eta)$ and $D_n^{(k)}(2\eta)$ overlap and meet the circle $|w| = R$, where

$$R = |\alpha_j^{(n)}\alpha_k^{(n)}|^{1/2} n^2 \quad .$$

For $\nu = j,k$ let $\partial\Delta_*^{(\nu)}(\tfrac{1}{2}\eta)$ denote the boundary of $\Delta_*^{(\nu)}(\tfrac{1}{2}\eta)$. Then $f[\ell_n^{(\nu)}(Z)]$ maps $\partial\Delta_*^{(\nu)}(\tfrac{1}{2}\eta)$ onto a closed curve $\Gamma_n^{(\nu)}(\tfrac{1}{2}\eta)$ in the $w_n^{(\nu)}$-plane. We show first of all that we can choose $n_2 = n_2(\eta) > n_1$ such that for $n > n_2$, $\Gamma_n^{(\nu)}(\tfrac{1}{2}\eta)$ does not meet $D_n^{(\nu)}(\eta)$. Let the shortest distance from $(D_n^{(\nu)}(\eta))'$ (the closure of $D_n^{(\nu)}(\eta)$), to the complement of $D_n^{(\nu)}(\tfrac{1}{2}\eta)$ be $d_n^{(\nu)}(\eta)$. Then

$$d_n^{(\nu)}(\eta) = |\alpha_\nu^{(n)}|n^2\eta^2 . \min(\sin\eta, \tfrac{3}{4})$$

$$> \frac{2}{\pi}|\alpha_\nu^{(n)}|n^2\eta^3 \quad ,$$

since $\eta < 1$. Put $\varepsilon = \frac{1}{8}\eta^5$. In view of Lemma (2.2) there exists $n_3 = n_3(\varepsilon) > n_2$ such that for $Z \in \partial\Delta_*^{(\nu)}(\frac{1}{2}\eta)$ and $n > n_3$

$$|f[\ell_n^{(\nu)}(Z)] - f_\nu^{(n)}[\ell_n^{(\nu)}(Z)]| < \varepsilon|f_\nu^{(n)}[\ell_n^{(\nu)}(Z)]|$$

$$\leq \tfrac{1}{8}\eta^5|\alpha_\nu^{(n)}|n^2(\tfrac{1}{2}\eta)^{-2}$$

$$= \tfrac{1}{2}\eta^3|\alpha_\nu^{(n)}|n^2$$

$$< d_n^{(\nu)}(\eta) \quad .$$

Next let Z_0 be any point of $\Delta_*^{(\nu)}(2\eta)$. Let $2\delta_1$ be the distance of $(\Delta_*^{(\nu)}(2\eta))'$ from the complement of $\Delta_*^{(\nu)}(\eta)$ and define $\gamma = \{Z : |Z-Z_0| = \delta_1\}$. Then

$$\eta^2 < 2\eta \cdot \frac{2}{\pi} \cdot \eta \leq 2\delta_1 = 2\eta\sin\eta < 2\eta^2$$

and for $Z \in \gamma$

$$|(1-Ze^{-i\theta_\nu})+(1-Z_0e^{-i\theta_\nu})| = |2(1-Z_0e^{-i\theta_\nu})+e^{-i\theta_\nu}(Z_0-Z)|$$

$$> 4\eta - \eta^2 = \eta(4-\eta) \quad .$$

Thus for $Z \in \gamma$ and $n > n_3$

$$|f_\nu^{(n)}[\ell_n^{(\nu)}(Z)] - f_\nu^{(n)}[\ell_n^{(\nu)}(Z_0)]| = |f(r_ne^{i\theta_\nu})|\left|\frac{1}{(1-Ze^{-i\theta_\nu})^2} - \frac{1}{(1-Z_0e^{-i\theta_\nu})^2}\right|$$

$$= |f(r_ne^{i\theta_\nu}) \frac{|Z-Z_0||2-(Z+Z_0)e^{-i\theta_\nu}|}{|1-Ze^{-\theta_\nu}|^2|1-Z_0e^{-i\theta_\nu}|^2}$$

$$> \frac{|f(r_ne^{i\theta_\nu})|\delta_1\eta(4-\eta)}{|1-Ze^{-i\theta_\nu}|^2(2\eta)^{-2}}$$

$$> |f_\nu^{(n)}[\ell_n^{(\nu)}(Z)]|2\eta^5(4-\eta) \quad .$$

Again from Lemma (2.2) we can choose $n_4 = n_4(\eta) > n_3$ such that for all $n > n_4$ and any $Z \in \gamma$

$$|f[\ell_n^{(\nu)}(Z)]-f_\nu^{(n)}[\ell_n^{(\nu)}(Z)]| < 2\eta^5(4-\eta)|f_\nu^{(n)}[\ell_n^{(\nu)}(Z)]|$$

$$< |f_\nu^{(n)}[\ell_n^{(\nu)}(Z)]-f_\nu^{(n)}[\ell_n^{(\nu)}(Z_0)]| \quad .$$

We apply Rouché's Theorem in the form given e.g. in Ahlfors (1953, page 152) to γ, $f[\ell_n^{(\nu)}(Z)] - f_\nu^{(n)}[\ell_n^{(\nu)}(Z_0)]$ and $f_\nu^{(n)}[\ell_n^{(\nu)}(Z)] - f_\nu^{(n)}[\ell_n^{(\nu)}(Z_0)]$ and conclude that $f[\ell_n^{(\nu)}(Z)]$ takes the value $f_\nu^{(n)}[\ell_n^{(\nu)}(Z_0)]$ at least once in $\Delta_*^{(\nu)}(\eta)$. This holds for every $Z_0 \in \Delta_\nu^*(2\eta)$ and $n > n_4$ where n_4 depends on η only and not on Z_0. Since $f_\nu^{(n)}[\ell_n^{(\nu)}(Z)]$ maps $\Delta_*^{(\nu)}(2\eta)$ 1-1 and conformally onto $D_n^{(\nu)}(2\eta)$ we have that for all $n > n_4$, $f[\ell_n^{(\nu)}(Z)]$ takes every value in $D_n^{(\nu)}(2\eta)$ at least once in $\Delta_*^{(\nu)}(\eta)$. It follows from Hayman (1951, Lemma 4) that there exists $R_0 = R_0(f)$ such that for $R > R_0$ f omits at least one value $w_R(|w_R|=R)$ in $|z| < 1$. Thus, if w_R satisfies

$$|\alpha_\nu^{(n)}|n^2(2\eta)^2 < R < |\alpha_\nu^{(n)}|n^2(2\eta)^{-2} \quad ,$$

it must satisfy

$$|\arg w_R - \arg\alpha_\nu^{(n)} - \pi| < 4\eta \quad .$$

Putting $R = |\alpha_j^{(n)}\alpha_k^{(n)}|^{\frac{1}{2}}n^2$ we have for $n > n_4$

$$|\arg w_R - \arg\alpha_\nu^{(n)} - \pi| < 4\eta \quad , \quad \nu = j,k \quad ,$$

and so

$$|\arg\alpha_j^{(n)}-\arg\alpha_k^{(n)}| = |\lambda_j(r_n)-\lambda_k(r_n)| < 8\eta \quad , \quad n > n_4 \quad .$$

Since η can be chosen as small as we please, we deduce Lemma (2.4).

2.3 The minor arc

We now show that (1.6) holds if f is weakly p-valent.

LEMMA (2.5)

Suppose that $f(z)$ *is weakly* p-*valent in* $|z| < 1$. *Then, given* $\eta > 0$ *there exist constants* K, r_0 *such that for* $r_0 < r < 1$ *and* $\theta \in \gamma^c$

$$|f(re^{i\theta})| < \frac{\eta}{(1-r)^{2p}} \quad ,$$

where γ^c *is defined as in* §1.0.

Choose δ , $0 < \delta < \frac{1}{2}$, such that $f(z) \neq 0$ for $1 - 2\delta < |z| < 1$. Given $\eta > 0$, choose $N = N(\eta)$ as in §1.0. We consider two cases:

I. $\theta : K(1-r) \leq |\theta-\theta_\nu| < K(1-R) = \delta_2 \quad (1 \leq \nu \leq N)$ where R,K will be fixed below;

II. $\theta^{(\nu)} + \delta_2 \leq \theta \leq \theta^{(\nu+1)} - \delta_2 \quad (1 \leq \nu \leq N-1)$,
$\theta^{(N)} + \delta_2 \leq \theta \leq \theta^{(1)} + 2\pi - \delta_2$,
where the $\theta^{(\nu)}$ $(1 \leq \nu \leq N)$ are the θ_ν $(1 \leq \nu \leq N)$ with $\theta^{(1)} < \theta^{(2)} < \ldots < \theta^{(N)}$.

<u>Case I</u>.

We choose $K = K(\eta)$ so that

$$\frac{10^{2p} \cdot 2\alpha_1}{K^{2p}} < \eta \quad .$$

As in Lemma (1.1) we have (see (1.11))

$$|1-re^{i(\theta-\theta_\nu)}|^2 \geq \left(\frac{\theta-\theta_\nu}{5}\right)^2$$

and so for $1-\delta < r < 1$, $|\theta-\theta_\nu| \geq K(1-r)$, $(1 \leq \nu \leq N)$ we get

$$\text{(2.2)} \qquad \frac{2\alpha_\nu}{|1-re^{i(\theta-\theta_\nu)}|^{2p}} \leq \frac{5^{2p} \cdot 2\alpha_\nu}{K^{2p}(1-r)^{2p}}$$

$$< \frac{5^{2p} \cdot 2\alpha_\nu}{K^{2p}(1-r)^{2p}} \cdot 2^{2p}\left(\frac{r+2\delta-1}{2\delta}\right)^{2p}$$

$$< \eta\left\{\frac{r+2\delta-1}{2\delta(1-r)}\right\}^{2p} ,$$

since $\alpha_\nu \leq \alpha_1$.

Having fixed K we choose $\varepsilon = \varepsilon(K)$ so that

$$(\gamma_\nu^*)' \subset \Delta_n^{(\nu)}(\varepsilon) \quad (1 \leq \nu \leq N)$$

for all large n and $r_n < r < r_{2n}$. Then, by Lemma (2.2) we have, as $n \to \infty$, uniformly for $z \in (\gamma_\nu^*)'$

$$|1-ze^{-i\theta_\nu}|^{2p}|f(z)| \to \alpha_\nu .$$

Hence we can find n_0 such that for $n > n_0$ and $re^{i\theta} \in (\gamma_\nu^*)'$, $(1 \leq \nu \leq N)$,

$$\text{(2.3)} \qquad |f(re^{i\theta})| < \frac{2\alpha_\nu}{|1-re^{i(\theta-\theta_\nu)}|^{2p}} .$$

Taking $R = r_{2n_0}$ and θ such that $|\theta-\theta_\nu| = K(1-R)$, we have that $Re^{i\theta}$ satisfies (2.2), (2.3) and obtain

$$\left(\frac{1-R}{R+2\delta-1}\right)^{2p} |f(Re^{i\theta})| < \frac{\eta}{(2\delta)^{2p}} \quad .$$

From Lemma (2.1) we have for $R \leq r < 1$,

$$(1-r)^{2p}|f(re^{i\theta})| \leq (r+2\delta-1)^{2p}\left(\frac{1-R}{R+2\delta-1}\right)^{2p}|f(Re^{i\theta})|$$

$$< \eta\left(\frac{r+2\delta-1}{2\delta}\right)^{2p} < \eta \quad ,$$

as required, since we shall later fix r_0 with $R \leq r_0 < 1$.

<u>Case II</u>.

For the purposes of this argument we let $\theta^{(N+1)} = \theta^{(1)} + 2\pi$ and consider the set

$$\Gamma_\nu = \{\theta : \theta^{(\nu)} + \delta_2 \leq \theta \leq \theta^{(\nu+1)} - \delta_2\} \quad .$$

On Γ_ν we have

$$(2.4) \qquad \lim_{\rho\to 1}\left(\frac{1-\rho}{\rho+2\delta-1}\right)^{2p}|f(\rho e^{i\theta})| < \frac{\eta}{(2\delta)^{2p}} \quad .$$

For $\mu > 1/\delta$ let $\rho_\mu = 1 - 1/\mu$. If for some μ we have for all $\theta \in \Gamma_\nu$

$$\left(\frac{1-\rho_\mu}{\rho_\mu+2\delta-1}\right)^{2p}|f(\rho_\mu e^{i\theta})| < \frac{\eta}{(2\delta)^{2p}} \quad ,$$

then from Lemma (2.1)

$$\left(\frac{1-\rho}{\rho+2\delta-1}\right)^{2p}|f(\rho e^{i\theta})| < \left(\frac{1-\rho_\mu}{\rho_\mu+2\delta-1}\right)^{2p}|f(\rho_\mu e^{i\theta})|$$

for $\rho_\mu < \rho < 1$, $\theta \in \Gamma_\nu$ and so

$$(1-\rho)^{2p}|f(\rho e^{i\theta})| < \eta$$

for $\rho_\mu < \rho < 1$, $\theta \in \Gamma_\nu$, which is our result.

We thus suppose contrary to this that for each μ there exists $\phi_\mu \in \Gamma_\nu$ such that

$$(2.5) \qquad \left(\frac{1-\rho_\mu}{\rho_\mu+2\delta-1}\right)^{2p}|f(\rho_\mu e^{i\phi_\mu})| \geq \frac{\eta}{(2\delta)^{2p}} \quad .$$

Let $\phi^{(\nu)}$ be a limit point of the sequence $\{\phi_\mu\}$. It follows from (2.5) and Lemma (2.1) that for each fixed μ

$$\left(\frac{1-\rho}{\rho+2\delta-1}\right)^{2p}|f(\rho e^{i\phi_\mu})| \geq \frac{\eta}{(2\delta)^{2p}} \quad , \quad 1-\delta < \rho < \rho_\mu \quad .$$

If ρ is fixed the result holds for all large μ and by letting $\mu \to \infty$ we have $\phi_\mu \to \phi^{(\nu)}$ and

$$\left(\frac{1-\rho}{\rho+2\delta-1}\right)^{2p}|f(\rho e^{i\phi^{(\nu)}})| \geq \frac{\eta}{(2\delta)^{2p}} \quad , \quad 1-\delta < \rho < 1 \quad ,$$

which contradicts (2.4). So (2.5) is false for $1 - 1/\mu = \rho_\mu < \rho < 1$ and some $\mu > 1/\delta$.

Thus, given $\eta > 0$, there exists $\mu_\nu = \mu_\nu(\eta)$ such that for $1 - 1/\mu_\nu < \rho < 1$ and $\theta \in \Gamma_\nu$ we have

$$(1-\rho)^{2p}|f(\rho e^{i\theta})| < \eta \quad .$$

This is true for $\nu = 1,\ldots,N$, so we let $r_0 = \max\{R,\rho_{\mu_1},\ldots,\rho_{\mu_N}\}$. Then for $r_0 < r < 1$ and $\theta \in \gamma^c$ we have

$$|f(re^{i\theta})| < \frac{\eta}{(1-r)^{2p}} \quad .$$

This completes the proof of Lemma (2.5).

2.4 Proof of Theorems (2.3) and (2.4)

It follows from Hayman (1951, Lemma 3) that if f satisfies the hypotheses of either Theorem (2.3) or (2.4) then $[f(z)]^{1/p}$ is weakly 1-valent and so we may prove both theorems for $p = 1$, since otherwise we may consider a branch of $[f(z)]^{1/p}$ instead of $f(z)$. We use an argument similar to the proof of Theorem (2.1) but instead of using Lemma (2.3) we use sharp bounds for the means which were obtained by Baernstein (1974).

We prove Theorem (2.3) first. We need

LEMMA (2.6) [Baernstein, 1974, Theorem 4]

Suppose that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

is weakly univalent in $|z| < 1$. *Then for* $0 < \lambda < \infty$

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^{\lambda} d\theta \leq \int_{-\pi}^{\pi} \frac{r^{\lambda} d\theta}{|1-re^{i\theta}|^{2\lambda}} \quad , \quad (0 < r < 1) \quad .$$

As in the proof of Theorem (1.1) (equation (1.11)) we have for N_0 radii of greatest growth and a fixed $\lambda > \frac{1}{2}$ that given $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ there exist $J_0 = J_0(\varepsilon_1,\lambda)$ and $r_0 = r_0(\varepsilon_2,\lambda)$ such that for $K \geq J_0$, $r_0 \leq r < 1$,

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^\lambda d\theta > (1-\varepsilon_2)^2 \sum_{\nu=1}^{N_0} \alpha_\nu^\lambda \left\{ \int_{-\pi}^{\pi} \frac{d\theta}{|1-re^{i\theta}|^{2\lambda}} - \frac{\varepsilon_1}{(1-r)^{2\lambda-1}} \right\} .$$

Since $r < 1$ we have from Lemma (2.6)

$$\int_{-\pi}^{\pi} \frac{d\theta}{|1-re^{i\theta}|^{2\lambda}} > (1-\varepsilon_2)^2 \sum_{\nu=1}^{N_0} \alpha_\nu^\lambda \left\{ \int_{-\pi}^{\pi} \frac{d\theta}{|1-re^{i\theta}|^{2\lambda}} - \frac{\varepsilon_1}{(1-r)^{2\lambda-1}} \right\} .$$

Since ε_1 , ε_2 are arbitrary we have

(2.6) $$\sum_{\nu=1}^{N_0} \alpha_\nu^\lambda \leq 1 .$$

This is true for $\frac{1}{2} < \lambda < \infty$ and so by continuity it is true for $\lambda = \frac{1}{2}$. Now if

$$\sum_{\nu=1}^{N_0} \alpha_\nu^\lambda = 1$$

for any $\lambda > \frac{1}{2}$, we must have $N_0 = 1$. For if $N_0 > 1$ we choose λ' satisfying $\frac{1}{2} < \lambda' < \lambda$ and obtain

$$\sum_{\nu=1}^{N_0} \alpha_\nu^{\lambda'} > \sum_{\nu=1}^{N_0} \alpha_\nu^\lambda = 1 ,$$

which contradicts (2.6). It is an open question whether

$$\sum_{\nu=1}^{N_0} \alpha_\nu^{\frac{1}{2}} = 1$$

implies $N_0 = 1$.

It follows from Hayman (1951, Theorem X) that weakly univalent functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

with one radius of greatest growth and

$$\lim_{r\to 1} (1-r)^2 M(r, f) = 1$$

are Koebe's functions. Since E is countable and (2.6) holds for any N_0 Theorem (2.3) follows.

To prove Theorem (2.4) we need the following result again due to Baernstein (1974, Theorem 6).

LEMMA 2.7

Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is weakly univalent in $|z| < 1$ *and* $f(z) \neq 0$ *there. Then for* $0 < \lambda < \infty$

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^{\lambda} d\theta \leq |a_0|^{\lambda} \int_{-\pi}^{\pi} \left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right|^{2\lambda} d\theta \quad , \quad (0 < r < 1) \quad .$$

Again we have (equation (1.11)) for N_0 radii of greatest growth and a fixed $\lambda > \frac{1}{2}$ that given $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, there exist $J_0 = J_0(\varepsilon_1, \lambda)$ and $r_0 = r_0(\varepsilon_2, \lambda)$ such that for $K \geq J_0$, $r_0 < r < 1$,

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^{\lambda} d\theta > (1-\varepsilon_2)^2 \sum_{\nu=1}^{N_0} \alpha_\nu^{\lambda} \left\{ \int_{-\pi}^{\pi} \frac{d\theta}{|1-re^{i\theta}|^{2\lambda}} - \frac{\varepsilon_1}{(1-r)^{2\lambda-1}} \right\} .$$

Now $|1+re^{i\theta}|^{2\lambda} \leq 4^{\lambda}$ for $r < 1$ and so we have from Lemma (2.7)

$$(4|a_0|)^{\lambda} \int_{-\pi}^{\pi} \frac{d\theta}{|1-re^{i\theta}|^{2\lambda}} > (1-\varepsilon_2)^2 \sum_{\nu=1}^{N_0} \alpha_\nu^{\lambda} \left\{ \int_{-\pi}^{\pi} \frac{d\theta}{|1-re^{i\theta}|^{2\lambda}} - \frac{\varepsilon_1}{(1-r)^{2\lambda-1}} \right\} .$$

Since ε_1 , ε_2 are arbitrary we have

$$\sum_{\nu=1}^{N_0} \left(\frac{\alpha_\nu}{4|a_0|} \right)^{\lambda} \leq 1 .$$

Arguing as in the proof of Theorem (2.3) we have

$$\sum_{\nu} \left(\frac{\alpha_\nu}{4|a_0|} \right)^{\frac{1}{2}} \leq 1$$

and

$$\sum_{\nu} \frac{\alpha_\nu}{4|a_0|} < 1$$

except when $f(z) = a_0[(1+ze^{i\omega})/(1-ze^{i\omega})]^2$.

In this case

$$\sum_{\nu=1}^{N_0} \left(\frac{\alpha_\nu}{4|a_0|} \right)^{\frac{1}{2}} = 1$$

does not imply $N_0 = 1$ as the first example of §2.5 shows. This completes the proof of Theorem (2.4).

2.6 Some examples of weakly univalent functions

Any function $f(z)$ subordinate to $[(1+z)/(1-z)]^2$ in $|z| < 1$ is a non-zero weakly univalent function in $|z| < 1$. Thus if $w(z)$ satisfies the conditions of Schwarz's Lemma, then

$$f(z) = \left\{\frac{1+w(z)}{1-w(z)}\right\}^2$$

is a function which is weakly univalent and non-zero in $|z| < 1$.

Example 1.

Suppose that $w(z) = z^n$, $n \geq 2$. Then

$$f(z) = \left\{\frac{1+z^n}{1-z^n}\right\}^2$$

has n radii of greatest growth and $\theta_\nu = 2\nu\pi/n$, $\nu = 0,1,\ldots,n-1$, are such that

$$\alpha(\theta_\nu) = \lim_{r\to 1} (1-r)^2 |f(re^{i\theta_\nu})| > 0 \quad .$$

For $\nu = 0,1,\ldots,n-1$ we have

$$\alpha(\theta_\nu) = \lim_{r\to 1}\left\{\frac{1+r^n}{1+r+\ldots+r^{n-1}}\right\}^2 = \frac{4}{n^2} \quad .$$

Thus

$$\sum_{\nu=1}^{n} \alpha(\theta_\nu)^{\frac{1}{2}} = 2 \quad .$$

This example shows that in the inequality of Theorem (2.4)

$$\sum_{\nu}\left(\frac{\alpha_\nu}{4|a_0|}\right)^{1/(2p)} \leq 1 \quad ,$$

the exponent $1/(2p)$ cannot be replaced by any smaller quantity.

<u>Example 2</u>.

Suppose that

$$w(z) = \frac{e^{(z+1)/(z-1)} - e^{-1}}{1 - e^{-1}e^{(z+1)/(z-1)}} \quad .$$

Then $f(z) = [(1+w(z))/(1-w(z))]^2$ has an infinite number of radii of greatest growth and

$$\theta_\nu = \arg\left(\frac{2\nu\pi i+1}{2\nu\pi i-1}\right) \quad , \quad \nu = 0,\pm 1,\ldots$$

are such that

$$\alpha(\theta_\nu) = \lim_{r\to 1}(1-r)^2|f(re^{i\theta_\nu})| > 0 \quad , \quad \nu = 0,\pm 1,\ldots \quad .$$

Now

$$\alpha(\theta_\nu) = \lim_{r\to 1}\left|\frac{(1-r)^2(1+w(re^{i\theta_\nu}))^2}{(1-w(re^{i\theta_\nu}))^2}\right|$$

$$= \lim_{r\to 1}\left|\frac{1 + w(re^{i\theta_\nu})}{w'(re^{i\theta_\nu})}\right|^2 \quad ,$$

where

$$w'(re^{i\theta_\nu}) = \frac{\partial w(re^{i\theta_\nu})}{\partial r} \quad ;$$

$$\lim_{r\to 1} w(re^{i\theta_\nu}) = 1$$

and

$$\lim_{r\to 1} |w'(re^{i\theta_\nu})|^2 = 4\left(\frac{1+e^{-1}}{1-e^{-1}}\right)^2 |1-e^{-i\theta_\nu}|^{-4} \quad ,$$

so

$$\alpha(\theta_\nu) = \left(\frac{1-e^{-1}}{1+e^{-1}}\right)^2 |1-e^{-i\theta_\nu}|^4 \quad , \quad \nu = 0,\pm 1,\ldots \quad .$$

Also

$$\theta_\nu = 2\tan^{-1}(2\pi\nu) - \pi \sim -1/(\pi\nu) \quad , \quad \nu \to \infty \quad ,$$

and

$$|1-e^{-i\theta_\nu}| = |2\sin(\tfrac{1}{2}\theta_\nu)| \sim \theta_\nu \quad , \quad \nu \to \infty \quad ,$$

so

$$|1-e^{-i\theta_\nu}|^4 \sim (\pi^4\nu^4)^{-1} \quad , \quad \nu \to \infty \quad .$$

Thus

$$\alpha(\theta_\nu) \sim \frac{(1-e^{-1})^2}{\pi^4(1+e^{-1})^2} \cdot \nu^{-4} \quad , \quad \nu \to \infty \quad ,$$

and $\sum_\nu \alpha(\theta_\nu)^{\frac{1}{2}}$ converges.

We see from this example that $\sum\alpha(\theta_\nu)^\lambda$ diverges for $\lambda = \frac{1}{4}$, so that the exponent in Theorem (2.1) cannot at any rate be replaced by $1/(4p)$.

An example of a weakly univalent function which is not derived from subordination is the function which maps the unit disk onto the infinite covering surface of the plane slit from 0 to -1 and from $-i$ to ∞ along a ray of argument $-\pi/2$.

CHAPTER 3

Functions omitting a sequence of values

3.0 Introduction

A family F of functions regular in $|z| < 1$ is called normal [Montel (1927)] if, given any sequence $f_n \in F$, either $f_n(z) \to \infty$ at each point z in $|z| < 1$ or else there exists a subsequence f_{n_p} such that $f_{n_p}(z)$ converges uniformly in $|z| \leq \rho$, for every ρ, $0 < \rho < 1$.

It follows that if F is normal, then there exists a constant $B = B(F) \geq 0$ such that the following holds:

(3.1) if $f \in F$ and $|f(o)| \leq 1$, then $|f(z)| < e^B$ for $|z| < \frac{1}{2}$.

For otherwise we could find a sequence $f_n \in F$ such that $|f_n(o)| \leq 1$, but

$$M_n = \sup_{|z|<\frac{1}{2}} |f_n(z)| \to \infty \quad .$$

Thus no subsequence $f_{n_p}(z)$ is uniformly bounded, nor, à fortiori, uniformly convergent in $|z| \leq \frac{1}{2}$.

A family F of functions is said to be invariant in $|z| < 1$ if

(3.2) $f(z) \in F$, $|z_0| < 1$, λ real implies

$$f\left[\left(\frac{z + z_0}{1+\bar{z}_0 z}\right) e^{i\lambda}\right] \in F \quad .$$

Conditions (3.1) and (3.2) together give

(3.3) $f(z) \in F$, $|z_1| < 1$, $|z_2| < 1$, $|f(z_1)| \leq 1$, $|f(z_2)| \geq e^B$

implies

$$\left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right| \geq \tfrac{1}{2} \quad .$$

For if $f(z) \in F$, then

$$g(z) = f\left(\frac{z_1-z}{1-\bar{z}_1 z}\right) \in F$$

by (3.2). Hence if $|g(o)| = |f(z_1)| \leq 1$, it follows from (3.1) that $|g(z)| < e^B$ for $|z| < \tfrac{1}{2}$. Thus, if

$$|g(z_2)| = \left|f\left(\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right)\right| \geq e^B$$

we must have

$$\left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right| \geq \tfrac{1}{2} \quad .$$

Following Hayman (1955b) we call a family F of functions f satisfying (3.3) a uniformly normal or normal invariant family.

If $f \in F$ and $f(z) \neq 0$ for $|z| < 1$ we write

$$g(z) = \log f(z) = \sum_{n=0}^{\infty} g_n z^n \quad ;$$

$$h(z) = [g(z)]^2 = \sum_{n=0}^{\infty} h_n z^n \quad .$$

Thus $g(z)$, $h(z)$ are defined for $|z| < 1$.

Hayman (1955b) shows that

$$|g(re^{i\theta})| = O(1)/(1-r) \quad , \quad |h(re^{i\theta})| = O(1)/(1-r)^2 \quad ,$$
$$r \uparrow 1 \quad ,$$

uniformly in θ , $0 \leq \theta < 2\pi$, and that

$$I_\lambda(r,g) \leq \frac{B(g_0,\lambda)}{(1-r)^{\lambda-1}} \quad , \quad \lambda > 1 \quad ,$$

$$I_\lambda(r,h) \leq \frac{B(h_0,\lambda)}{(1-r)^{2\lambda-1}} \quad , \quad \lambda > \tfrac{1}{2} \quad .$$

We show (Theorem (3.3)) that the limits

$$\beta(\theta) = \lim_{r\to1}(1-r)|g(re^{i\theta})| \ , \quad \beta^*(\theta) = \lim_{r\to1}(1-r)|u(re^{i\theta})| \quad ,$$

where $g = u+iv$, and $\alpha(\theta) = \lim_{r\to1}(1-r)^2|h(re^{i\theta})|$ exist, $0 \leq \theta < 2\pi$.
If θ is a radius of greatest growth of h we show (Theorem (3.4)) that

$$\arg h(re^{i\theta}) = o(1)(\mathrm{mod}\,2\pi) \quad , \quad r \to 1 \quad . \tag{3.4}$$

The analogous result for g is that

$$\arg g(re^{i\theta}) = o(1)(\mathrm{mod}\,2\pi) \qquad \text{or}$$
$$\pi + o(1)(\mathrm{mod}\,2\pi)$$

depending on θ . The behaviour of g is thus similar to the behaviour of functions with positive real part. We quote a result of Hayman (1961) to show this.

THEOREM A

Suppose that

$$\Psi(z) = 1 + \sum_{n=1}^{\infty} b_n z^n = u + iv$$

is regular in $|z| < 1$ *and satisfies* $u > 0$ *there. Then the limit*

$$\alpha(\theta) = \lim_{r\to 1} \frac{1-r}{1+r} \Psi(re^{i\theta})$$

exists. The set of distinct values $\theta = \theta_\nu$ *in* $0 \leq \theta < 2\pi$ *for which* $\alpha_\nu = \alpha(\theta_\nu) \neq 0$ *is countable and* $\alpha_\nu > 0$, $\sum\alpha_\nu \leq 1$. *Further, we have, as* $r \to 1$,

$$M(r,\Psi) = 2[\max(\alpha_\nu)+o(1)](1-r)^{-1} \quad ,$$

and for $\lambda > 1$,

$$I_\lambda(r,\Psi) = C_\lambda[\sum\alpha_\nu{}^\lambda+o(1)](1-r)^{1-\lambda} \quad ,$$

where

$$C_\lambda = 2^{\lambda-1}\Gamma[\tfrac{1}{2}(\lambda-1)]/\{\Gamma(\tfrac{1}{2})\Gamma(\tfrac{1}{2}\lambda)\} \quad .$$

(3.4) is a much stronger result than the one for weakly p-valent functions where, if f is weakly p-valent, $\alpha(\theta) > 0$, and $\arg f(re^{i\theta}) = \lambda(r)$, then $\lambda(r)$ is slowly varying as $r \to 1$. This is proved for circumferentially mean p-valent functions in Hayman (1958). The result follows for weakly p-valent functions in the same way.

This enables us to dispense with the $\alpha_\nu{}^{(n)}$ and define

$$h_\nu(z) = \frac{\alpha_\nu}{(1-ze^{-i\theta_\nu})^2}$$

where $\alpha(\theta_\nu) > 0$. We show (Lemma (3.9)) that

$$h(z) \sim h_\nu(z) \quad , \quad z \to e^{i\theta_\nu} \quad ,$$

uniformly in each angle $|\arg(1-ze^{-i\theta_\nu})| < \pi/2 - \varepsilon$, $(\varepsilon > 0)$, and hence, using Theorem (1.1) prove the following:

THEOREM (3.1)

Suppose that $f \in F$, *where* F *is a normal invariant family, and that* $f(z) \neq 0$ *in* $|z| < 1$. *Then if* $h(z) = [\log f(z)]^2$, *we have that the limit*

$$\alpha(\theta) = \lim_{r \to 1} (1-r)^2 |h(re^{i\theta})|$$

exists, $0 \leq \theta < 2\pi$. *Further,* $\alpha(\theta) \equiv 0$, *except for a sequence* $\theta = \theta_\nu$, *and*

$$\sum_\nu \alpha(\theta_\nu)^\lambda \leq B(h_0, \lambda) \quad , \quad \tfrac{1}{2} < \lambda < \infty \quad .$$

We then show that (1.6) [the minor arc] holds for h and use Theorem (1.2) to prove

THEOREM (3.2)

With the hypotheses of Theorem (3.1) *we have that if*

$$h(z) = \sum_{n=0}^{\infty} h_n z^n \quad ,$$

then

$$\overline{\lim_{n \to \infty}} \frac{|h_n|}{n} = \sum_\nu \alpha(\theta_\nu) \quad .$$

Further, if h *has more than one radius of greatest growth*

$$\varlimsup_{n\to\infty} \frac{|h_n|}{n} \le \left(\sum_{\nu} \alpha(\theta_\nu)^2 \right)^{\frac{1}{2}}$$

and so $\lim_{n\to\infty} |h_n|/n$ *does not exist.*

Hayman (1955b) also proved the following:

THEOREM B.

Let $\{R\}$ *be a class of Riemann surfaces spread over the* w*-plane satisfying:*

(3.5) R *is simply connected.*

Let F *be the family of functions* $f(z) = e^{g(z)}$ *, where* $w = g(z)$ *maps* $|z| < 1$ *one-to-one conformally onto one of the surfaces* R *. Then* F *is a normal invariant family if and only if there exists a number* A *such that the following holds:*

(3.7) *if* iv *is any point in the* w*-plane,* $w = u + iv$ *, then the radius,* $d(v)$ *, of any schlicht circle centre* iv *contained in* R *satisfies*

$$d(v) < A \quad .$$

A natural question is the following:

if the schlicht disks of Rg , centre the imaginary axis, are not bounded, but do not tend to infinity "too rapidly" as $v \to \infty$, $w = u + iv$, what can we say about the asymptotic behaviour of g and hence h ?

To answer this question we define a new family of functions $G = G(A,\mu)$ in the following way:

Let A,μ be constants satisfying

(3.8) $\quad A \geq 0 \;;\; 0 \leq \mu < 1$,

and let $\{R\} = \{R(A,\mu)\}$ be the class of Riemann surfaces spread over the w-plane satisfying (3.5) and the following:

(3.9) if iv is any point on the imaginary axis in the w-plane, then the radius, $d(v)$, of any schlicht circle centre iv contained in R satisfies

$$d(v) < A(1 + |v|^{\mu}) \quad .$$

We then define $G = G(A,\mu)$ to be the family of functions g, where $w = g(z)$ maps $|z| < 1$ one-to-one conformally onto one of the surfaces R. We also define $H = H(A,\mu)$ to be the family of functions h, where $h = g^2$ and $g \in G(A,\mu)$.

With these definitions, the case $\mu = 0$ in (3.9) corresponds to condition (3.7) of Theorem B and hence the family of functions $f = e^g$, with $g \in G(A,\mu=0)$ is a normal invariant family. For the rest of this chapter we shall denote by $B, B_1, B_2, \ldots$ constants depending on A,μ only, and by $B(g_0), B(g_0,\lambda), \ldots$, constants depending on A,μ,g_0 ; A,μ,g_0 and λ ; and so on. Our results are

THEOREM (3.3)

Suppose that $h \in H(A,\mu)$, *where* A,μ *satisfy* (3.8). *Then if*

$$h(z) = \sum_{n=0}^{\infty} h_n z^n \quad ,$$

the limit

$$\alpha(\theta) = \lim_{r\to 1} (1-r)^2 |h(re^{i\theta})|$$

exists, $0 \le \theta < 2\pi$; $\alpha(\theta) = 0$ *except for a sequence* $\theta = \theta_\nu$ *, and*

$$\sum_\nu \alpha(\theta_\nu)^\lambda \le B(h_0,\lambda)$$

for all $\lambda > \max\{\frac{1}{2},\mu\}$.

THEOREM (3.4)

With the above hypotheses

$$\overline{\lim_{n\to\infty}} \frac{|h_n|}{n} = \sum_\nu \alpha(\theta_\nu) \quad ,$$

and if h *has more than one radius of greatest growth,*

$$\underline{\lim_{n\to\infty}} \frac{|h_n|}{n} \le \left\{ \sum_\nu \alpha(\theta_\nu)^2 \right\}^{\frac{1}{2}} \quad ,$$

and hence $\lim_{n\to\infty} |h_n|/n$ *does not exist.*

We prove our results for $0 \le \mu < 1$, so that $\mu = 0$ is a special case.

Chapter 3 is divided into eight sections. In the first section we derive a bound for $|g_1|$, where $g(z) = \sum g_n z^n$. In §3.2 we show that

$$M(r,g) \le B(g_0)/(1-r) \quad ; \quad M(r,h) \le B(h_0)/(1-r)^2 \quad .$$

In §3.3 we show that the limits

$$\beta(\theta) = \lim_{r\to 1} (1-r)|g(re^{i\theta})| \quad ,$$

$$\beta^*(\theta) = \lim_{r\to 1} (1-r)|u(re^{i\theta})| \quad ,$$

$$\alpha(\theta) = \lim_{r\to 1} (1-r)^2|h(re^{i\theta})|$$

exist, $0 \leq \theta < 2\pi$, where $g = u + iv$. In §3.4 we prove that if $\alpha(\theta_\nu) > 0$ then $\arg h(re^{i\theta_\nu}) = o(1)(\mathrm{mod}\,2\pi)$, $r \to 1$. Hence, if we define

$$h_\nu(z) = \frac{\alpha_\nu}{(1-ze^{-i\theta_\nu})^2} \quad ,$$

we have

$$h(z) \sim h_\nu(z) \quad , \quad z \to e^{i\theta_\nu} \quad ,$$

uniformly for z in $|\arg(1-ze^{-i\theta_\nu})| < \pi/2 - \varepsilon$. In §3.5 we derive a bound for the means $I_\lambda(r,h)$. We prove that for $\lambda > \max\{\frac{1}{2},\mu\}$

$$I_\lambda(r,h) \leq \frac{B(h_0,\lambda)}{(1-r)^{2\lambda-1}} \quad , \quad 0 \leq r < 1 \quad .$$

In §3.6 we complete the proofs of Theorems (3.1) and (3.3). In §3.7 we show that (1.6) (the minor arc result) holds for h , and then in the last section we complete the proofs of Theorems (3.2) and (3.4).

The generalisation from $e^g \in F$, where the schlicht disks of Rg are bounded, to $g \in G(A,\mu)$, gives the generalisation of Hayman's result on functions omitting values mentioned in §0.2. Thus if $2 \leq p < \infty$ and $h(z) \neq w_k$, where $\{w_k\}$ satisfies

$$\arg w_k = O(|w_k|^{-1/p}) \quad ,$$

and

$$|w_{k+1} - w_k| = O(|w_k|^{(p-1)/p}) \quad ,$$

then since $g = (-h)^{1/2} \in G(A, 1-2/p)$ we have

$$M(r,h) = O(1-r)^{-2} \quad ,$$

$$I_1(r,h) = O(1-r)^{-1} \quad ,$$

and

$$|h_n| = O(n) \quad .$$

3.1 The bound for $|g_1|$

THEOREM (3.5)

Suppose that $g \in G(A,\mu)$, *where* A,μ *satisfy* (3.8). *Then if*

$$g(z) = \sum_{n=0}^{\infty} g_n z^n \quad , \quad g_0 = \alpha + i\beta \quad ,$$

we have

$$|g_1| \leq 2\{|\alpha| + B_1(|g_0|^{\mu} + 1)\} \quad ,$$

where B_1 *is a constant depending only on* A,μ .

Let $w = g(z)$ map $|z| < 1$ one-to-one conformally onto a Riemann surface Rg lying over the w-plane satisfying (3.5), (3.6), (3.9), and let Dg be the domain in the w-plane which is the projection of Rg .

We suppose with no loss in generality that $\alpha \geq 0$, $\beta \geq 0$, for otherwise the proof of Theorem (3.5) proceeds along the same lines.

We let $w = u + iv$ and define

$$\Delta_1 = \{w \in Dg : u = 0\} ,$$

$$\Delta_2 = \{w \in Dg : u = A_1(1+|v|^{\mu})\} ,$$

$$\Delta_3 = \{w \in Dg : u \geq A_1(1+|v|^{\mu})\} ,$$

where A_1 is defined below.

We quote the following result of Hayman (1964, Theorem 6.8)

LEMMA (3.1)

Suppose that Φ *is a normal invariant family of functions regular in* $|z| < 1$. *Then there exists a constant* C *depending only on* Φ *such that for*

$$\phi(z) = \sum_{n=0}^{\infty} \phi_n z^n \in \Phi ,$$

we have

$$|\phi_1| \leq 2\mu_0\{\log\mu_0 + C\} , \tag{3.10}$$

and

$$M(r,\phi) \leq \mu_0^{(1+r)/(1-r)} . \exp\left(\frac{2Cr}{1-r}\right) , \tag{3.11}$$

where $\mu_0 = \max\{1, |\phi_0|\}$.

We now prove

LEMMA (3.2)

There exists a positive constant $A_1 = A_1(A,\mu)$ *such that* $g \in G$, $|z_1| < 1$, $|z_2| < 1$, $g(z_1) \in \Delta_1$, $g(z_2) \in \Delta_3$ *imply*

$$\left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right| \geq \tfrac{1}{2} \quad .$$

We let

$$\hat{g}(z) = g\left(\frac{z_1-z}{1-\bar{z}_1 z}\right) \quad , \quad z_3 = \frac{z_1-z_2}{1-\bar{z}_1 z_2} \quad ,$$

so that $\hat{g} \in G$, $g(z_1) = \hat{g}(o) \in \Delta_1$, and

$$g(z_2) = \hat{g}\left(\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right) = \hat{g}(z_3) \in \Delta_3 \quad .$$

To proceed further we need to apply Ahlfor's Theory of Covering Surfaces (Ahlfors (1934), see also Hayman (1964, Chapter 5)).

Let $\hat{g}(o) = iv_0$ be any point in Δ_1 . We may suppose with no loss in generality that $v_0 \geq 0$, since otherwise the discussion is similar. We choose $v_1 > v_0$ such that

$$\frac{v_1-v_0}{4A(1+v_1{}^{\mu})} = 1 \quad ,$$

and write $d_1 = A(1+v_1{}^{\mu})$. Since $\hat{g} \in G$, $\hat{g}(z)$ has no simple island over $|w-iv_1| < d_1$. Then the mapping

$$w^* = \phi(z) = \frac{\hat{g}(z)-iv_0}{4d_1 i} = \frac{w-iv_0}{4d_1 i}$$

takes

$$w = iv_1 \quad \text{into} \quad w^* = 1 \quad ,$$

$$w = iv_0 \quad \text{into} \quad w^* = 0 \quad ,$$

$$w = i(v_1 - 8d_1) \quad \text{into} \quad w^* = -1 \quad ,$$

and $\phi(z)$ has no simple island over

$$|w^* - n| < \tfrac{1}{4} \quad , \quad n = 0, \pm 1 \quad .$$

This is true since $|v_1 - 8d_1| < v_1$.

Now $\phi(z)$ is regular in $|z| < 1$, and so has no islands at all over $|w^*| > 2$. It follows from Hayman (1964, Theorems (5.5), (6.2) and (6.5)) that such functions ϕ form a normal invariant family. We may thus apply Lemma (3.1).

If z_3 , $|z_3| < 1$, is such that $|\phi(z_3)| \geq \exp(2C)$, where C is the constant of Lemma (3.1), we have from (3.11), since $|\phi(0)| = 0$, that

$$\exp(2C) \leq |\phi(z_3)| \leq M(|z_3|, \phi) \leq \exp\left\{\frac{2C|z_3|}{1-|z_3|}\right\} \quad ,$$

and hence

$$\frac{|z_3|}{1-|z_3|} \geq 1 \quad ,$$

i.e. $|z_3| \geq \frac{1}{2}$.

This is true for any $iv_0 \in \Delta_1$. We now show that we can choose A_1 such that

$$\{w : |w - iv_0| = 4Ae^{2C}(1+|v_0|^{\mu})\}$$

does not meet $\Delta_2 = \{w = u+iv : u = A_1(1+|v|^{\mu})\}$.

Choose v_2 such that $4Ae^{2C}(1+|v|^{\mu}) < \frac{1}{2}|v|, v \geq v_2$. We consider two cases: (i) $|v_0| > |v_2|$, (ii) $|v_0| \leq |v_2|$.

Case (i).

$|v_0| > |v_2|$. In this case we choose A_2 such that

$$A_2(1+|\tfrac{1}{2}v_0|^{\mu}) > 8Ae^{2C}(1+|v_0|^{\mu}) \quad .$$

Then if $w = u + iv \in \Delta_2$, we have

$$|v| > \tfrac{1}{2}|v_0| \quad \text{implies} \quad A_2(1+|v|^{\mu}) > A_2(1+|\tfrac{1}{2}v_0|^{\mu}) > 8Ae^{2C}(1+|v_0|^{\mu}) \quad ;$$

$$|v| \leq \tfrac{1}{2}|v_0| \quad \text{implies} \quad |w-iv_0| > \tfrac{1}{2}|v_0| > 4Ae^{2C}(1+|v_0|^{\mu}) \quad .$$

Case (ii).

$|v_0| \leq |v_2|$. In this case we choose A_3 such that

$$8Ae^{2C}(1+|v_2|^{\mu}) < A_3 \quad .$$

Then for all $w \in \Delta_2$

$$|w-iv_0| > 8Ae^{2C}(1+|v_2|^{\mu}) > 8Ae^{2C}(1+|v_0|^{\mu}) \quad .$$

We choose $A_1 = \max(A_2,A_3)$. Then if $\hat{g}(o) \in \Delta_1$, $\hat{g}(z_3) \in \Delta_3$, we have $|z_3| \geq \frac{1}{2}$.

Translating back to f we have that if $f(z_1) \in \Delta_1$, $f(z_2) \in \Delta_3$, then

$$\left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right| \geq \tfrac{1}{2} \quad ,$$

as required.

We divide the proof of Theorem (3.5) into two cases:
Case (a) $0 \leq \alpha \leq A_1(1+\beta^\mu)$; Case (b) $\alpha > A_1(1+\beta^\mu)$, where A_1 is the constant of Lemma (3.2).

Case (a).

$0 \leq \alpha \leq A_1(1+\beta^\mu)$.

As in the proof of Lemma (3.2), we choose $v_1 > \beta$ such that $v_1 - \beta = 4A(1+v_1{}^\mu)$, write $d_1 = A(1+v_1{}^\mu)$, and apply Lemma (3.1) to

$$\phi(z) = \frac{g(z)-i\beta}{4d_1 i} .$$

Now

$$|\phi(0)| = \frac{\alpha}{4d_1} , \quad |\phi'(0)| = \frac{|g_1|}{4d_1} ,$$

so we have from (3.10),

$$|\phi'(0)| \leq 2\mu_0(\log\mu_0+C) ,$$

where $\mu_0 = \max\{1,|\phi(0)|\}$.

If $\alpha \leq 4d_1$ we have $\mu_0 = 1$ and hence

$$\frac{|g_1|}{4d_1} \leq 2C ,$$

i.e. $$|g_1| \leq 2C(v_1-\beta) .$$

Now $\dfrac{v_1-\beta}{4A(1+\beta^\mu)} \to 1$ as $\beta \to \infty$ so we may choose A_4 such that

$$(v_1-\beta) < A_4(1+\beta^\mu) .$$

We now have

$$|g_1| \leq 2B_1(1+\beta^{\mu}) \quad ,$$

where $B_1 = CA_4$.

If $4d_1 < \alpha \leq A_1(1+\beta^{\mu})$ we have

$$|g_1| \leq 2\alpha \left\{ \log\left(\frac{\alpha}{4d_1}\right) + C \right\}$$

$$\leq 2A_1(1+\beta^{\mu}) \left\{ \log\left(\frac{A_1}{4A}\right) + C \right\} \quad ,$$

since $d_1 \geq A(1+\beta^{\mu})$.

Thus

$$|g_1| \leq 2B_1(1+\beta^{\mu}) \quad ,$$

where $B_1 = A_1 \left\{ \log\left(\frac{A_1}{4A}\right) + C \right\}$.

This completes the proof of case (a) of Theorem (3.5).

Case (b).

$\alpha > A_1(1+\beta^{\mu})$.

We need the following

LEMMA (3.3)

Suppose that $\phi(z) = u_1(z) + iv_1(z)$ *is regular and* $u_1(z) > 0$ *in* $|z| < 1$. *Then if*

$$\phi(z) = \sum_{n=0}^{\infty} \phi_n z^n \quad , \qquad \phi_0 = \alpha + i\beta \quad ,$$

we have

(3.12) $$|\phi_1| \le 2\alpha \quad ,$$

and if $|z| = t < 1$ *, we have*

(3.13) $$\frac{1-t}{1+t} \le \frac{2u_1(z)\alpha}{\alpha^2+u_1(z)^2+[v_1(z)-\beta]^2} \quad .$$

The proof of (3.12) is well known (see e.g. Titchmarsh, 1938, page 194). To prove (3.13) we note that

$$\mu(z) = \frac{\phi(z)-\phi_0}{\phi(z)+\bar{\phi}_0}$$

satisfies the conditions of Schwarz's Lemma and hence

$$\left|\frac{\phi(z)-\phi_0}{\phi(z)+\bar{\phi}_0}\right| \le |z| \quad ,$$

so that

$$\frac{1-|z|}{1+|z|} \le \frac{1-\left|\frac{\phi(z)-\phi_0}{\phi(z)+\bar{\phi}_0}\right|}{1+\left|\frac{\phi(z)-\phi_0}{\phi(z)+\bar{\phi}_0}\right|}$$

$$\le \frac{1-\left|\frac{\phi(z)-\phi_0}{\phi(z)+\bar{\phi}_0}\right|^2}{1+\left|\frac{\phi(z)-\phi_0}{\phi(z)+\bar{\phi}_0}\right|^2}$$

$$= \frac{|\phi(z)+\bar{\phi}_0|^2 - |\phi(z)-\phi_0|^2}{|\phi(z)+\bar{\phi}_0|^2 + |\phi(z)-\phi_0|^2}$$

$$= \frac{2\alpha u_1(z)}{\alpha^2+u_1(z)^2+[v_1(z)-\beta]^2} \quad ,$$

as required.

We now prove case (b) of Theorem (3.5).

Let ρ be the largest positive number such that

$$u(z) > 0 \quad , \quad |z| < \rho \quad ,$$

where $g(z) = u(z) + iv(z)$. If $\rho \geq 1$, Theorem (3.5) follows from Lemma (3.3) with $B_1 = 0$. If $\rho < 1$, let $r < \rho$ be the largest positive number such that

$$u(z) > A_1(1+|v(z)|^{\mu}) \quad , \quad |z| < r \quad .$$

There exists θ , $0 \leq \theta < 2\pi$, such that

$$u(re^{i\theta}) = A_1(1+|v(re^{i\theta})|^{\mu}) \quad ,$$

i.e. $g(re^{i\theta}) \in \Delta_2$. Consider the function

$$\phi(z) = u_1(z)+iv_1(z) = g(\rho z) = u(\rho z)+iv(\rho z) \quad .$$

Now $\phi(0) = g(0) = g_1$ and $u_1(z) > 0$, $|z| < 1$, so we apply Lemma (3.3), with $z = (r/\rho)e^{i\theta}$, and obtain

$$(3.14) \qquad \frac{\rho-r}{\rho+r} \leq \frac{2u(re^{i\theta})\alpha}{\alpha^2+u(re^{i\theta})^2+[v(re^{i\theta})-\beta]^2} \quad .$$

There exists θ_1 such that $u(\rho e^{i\theta_1}) = 0$, and, by hypothesis

$$u(re^{i\theta_1}) \geq A_1(1+|v(re^{i\theta_1})|^{\mu}) \quad ,$$

i.e. $g(re^{i\theta_1}) \in \Delta_3$. We thus have from Lemma (3.2) that

$$(3.15) \qquad \frac{\rho-r}{1-\rho r} \geq \tfrac{1}{2} \quad .$$

Combining (3.14) and (3.15) we obtain

$$1 - \rho \leq \frac{8u(re^{i\theta})\alpha}{\alpha^2+u(re^{i\theta})^2+[v(re^{i\theta})-\beta]^2} \quad ,$$

where $g(re^{i\theta}) \in \Delta_2$.

Also, we have, by Lemma (3.3),

$$|\phi'(0)| = \rho|g'(0)| \leq 2\alpha \quad .$$

Thus

$$|g_1| \leq 2\alpha\left\{1 + \frac{1-\rho}{\rho}\right\} \quad .$$

Using (3.15) we see that

$$\rho \geq \frac{\frac{1}{2}+r}{1+\frac{1}{2}r} = \tfrac{1}{2} + \frac{\frac{3}{4}r}{1+\frac{1}{2}r} \geq \tfrac{1}{2} \quad .$$

Hence

$$(3.16) \qquad |g_1| \leq 2\alpha\{1 + 2(1-\rho)\}$$

$$\leq 2\alpha\left\{1 + \frac{16u(re^{i\theta})\alpha}{\alpha^2+u(re^{i\theta})^2+[v(re^{i\theta})-\beta]^2}\right\} \quad ,$$

where $g(re^{i\theta}) \in \Delta_2$.

We thus have, that given $g_0 = \alpha + i\beta$, with $\alpha > A_1(1+\beta^\mu)$, we need to show that if $w = u + iv$ is any point of Δ_2 , then

$$\frac{16u\alpha^2}{\alpha^2+u^2+(v-\beta)^2} \leq B_1(1+|g_0|^\mu) \qquad ,$$

where B_1 is a constant depending only on A,μ . We shall assume $B_1 \geq 1$.

We may suppose with no loss in generality that $|g_0| = R$ is greater than some absolute constant A_5 . For if $R \leq A_5$ we have

$$\alpha \leq R \leq A_5 \leq A_5(1+\beta^\mu) \qquad ,$$

which is case (a) of Theorem (3.5) with A_5 instead of A_1 .

We suppose first that $|v| \leq 3R$. Then

$$u = A_1(1+|v|^\mu)$$

$$\leq A_1(1+(3R)^\mu) \qquad ,$$

and hence

$$\frac{16u\alpha^2}{\alpha^2+u^2+(v-\beta)^2} \leq 16u \leq 16A_1(1+(3R)^\mu) \qquad .$$

We choose A_6 such that $16A_1(1+(3R)^\mu) \leq A_6(1+|g_0|^\mu)$.

Now suppose that $|v| > 3R$. We choose $B_1 > A_6$ such that

$$\frac{(v-\beta)^2}{16A_1} > \frac{|v|^2}{B_1} \qquad .$$

Then

$$\frac{16u\alpha^2}{\alpha^2+u^2+(v-\beta)^2} \le \frac{B_1\alpha^2(1+|v|^\mu)}{|v|^2} \le B_1\left(1+\frac{\alpha^2}{|v|^{2-\mu}}\right)$$

$$\le B_1\left(1+\frac{\alpha^2}{(3R)^{2-\mu}}\right)$$

$$\le B_1(1+|g_0|^\mu) \quad .$$

This completes the proof of case (b) of Theorem (3.5). Hence Theorem (3.5) is proved in all cases.

3.2 The maximum modulus

Our next result is

THEOREM (3.6)

Suppose that $g = u + iv \in G(A,\mu)$, $h \in H(A,\mu)$ *and*

$$g(z) = \sum_{n=0}^{\infty} g_n z^n \quad , \quad h(z) = \sum_{n=0}^{\infty} h_n z^n \quad .$$

Then we have for $|z_0| = r < 1$,

$$(3.17) \qquad |g'(z_0)| \le \frac{2}{1-r^2}\left\{|u(z_0)| + B_1(|g(z_0)|^\mu + 1)\right\}$$

and

$$(3.18) \qquad M(r,g) \le \frac{B(g_0)}{1-r} \quad ; \quad M(r,h) \le \frac{B(h_0)}{(1-r)^2} \quad ,$$

where B_1 *is the constant of Theorem* (3.5).

To prove (3.17) we apply Theorem (3.5) to

$$g\left(\frac{z_0+z}{1+\bar{z}_0 z}\right) = u(z_0) + iv(z_0) + (1-r^2)g'(z_0)z + \dots \quad ,$$

which also belongs to G , since G is an invariant family. To prove (3.18) we fix θ , $0 \leq \theta < 2\pi$, and put

$$R(r) = |g(re^{i\theta})| + B_1 \quad .$$

Then from (3.17) we have, since $|u(re^{i\theta})| \leq |g(re^{i\theta})|$ and $R(r) \geq B_1 \geq 1$ so that $|g(re^{i\theta})|^{\mu} + 1 \leq 2R(r)^{\mu}$, that

$$(3.19) \qquad R'(r) \leq \frac{2}{1-r^2} \{R(r) + 2B_1 R(r)^{\mu}\} \quad , \quad 0 \leq r < 1 \quad .$$

We integrate this expression from 0 to r and obtain

$$\log\left\{\frac{R(r)^{1-\mu} + 2B_1}{R(0)^{1-\mu} + 2B_1}\right\} \leq (1-\mu)\log\left(\frac{1+r}{1-r}\right) \quad ;$$

so we have for $0 \leq r < 1$,

$$R(r)^{1-\mu} + 2B_1 \leq (R(0)^{1-\mu} + 2B_1)\left(\frac{1+r}{1-r}\right)^{1-\mu} \quad ,$$

and hence, à fortiori

$$R(r) \leq (R(0)^{1-\mu} + 2B_1)^{1/(1-\mu)}\left(\frac{1+r}{1-r}\right) \quad .$$

We thus have, uniformly in θ , $0 \leq \theta < 2\pi$, that

$$|g(re^{i\theta})| \leq B(g_0)/(1-r) \quad , \quad 0 \leq r < 1 \quad ,$$

and hence

$$|h(re^{i\theta})| \leq B(h_0)/(1-r)^2 \quad , \quad 0 \leq r < 1 \quad .$$

This completes the proof of (3.18) and hence Theorem (3.6).

3.3 The existence of $\beta(\theta)$, $\beta^*(\theta)$ and $\alpha(\theta)$

We can now prove

THEOREM (3.7)

Suppose that $g = u + iv \in G(A,\mu)$ *and* $h \in H(A,\mu)$. *Then for each* θ , $0 \le \theta < 2\pi$, *the limits*

$$\beta(\theta) = \lim_{r\to 1} (1-r)|g(re^{i\theta})| \quad , \tag{3.20}$$

$$\beta^*(\theta) = \lim_{r\to 1} (1-r)|u(re^{i\theta})| \quad , \tag{3.21}$$

$$\alpha(\theta) = \lim_{r\to 1} (1-r)^2|h(re^{i\theta})| \tag{3.22}$$

exist.

The proof of Theorem (3.7) depends upon the following.

LEMMA (3.4)

Suppose that $g = u + iv \in G(A,\mu)$. *Then, with* $B_2 = 4B_1/(1-\mu)$, *we have for a fixed* θ , $0 \le \theta < 2\pi$, *that both*

$$\left(\frac{1-r}{1+r}\right) |g(re^{i\theta})| + B_2(1-r)^{1-\mu}$$

and

$$\left(\frac{1-r}{1+r}\right) |u(re^{i\theta})| + B_2(1-r)^{1-\mu}$$

decrease as $r \uparrow 1$, $0 \le r < 1$.

We fix θ , $0 \le \theta < 2\pi$. Then from Theorem (3.6) we have

$$|g'(re^{i\theta})| \leq \frac{2|g(re^{i\theta})|}{1-r^2} + \frac{B_2(1-\mu)}{(1-r)^{1+\mu}} .$$

This yields

$$\frac{\partial}{\partial r}\left\{\left(\frac{1-r}{1+r}\right) |g(re^{i\theta})|\right\} \leq \frac{B_2(1-\mu)}{(1-r)^{\mu}} .$$

We integrate both sides from r_1 to r_2, $0 \leq r_1 < r_2 < 1$ and obtain

$$\left(\frac{1-r_2}{1+r_2}\right) |g(r_2e^{i\theta})| + B_2(1-r_2)^{1-\mu} \leq \left(\frac{1-r_1}{1+r_2}\right) |g(r_1e^{i\theta})| + B_2(1-r_1)^{1-\mu} .$$

This proves the first part of the lemma._

The second part of Lemma (3.4) follows in the same way as part 1, since we have from Theorem (3.6), that

$$|u'(re^{i\theta})| \leq |g'(re^{i\theta})| \leq \frac{2|u(re^{i\theta})|}{1-r^2} + \frac{B_2(1-\mu)}{(1-r)^{1+\mu}} .$$

Theorem (3.7) now follows from Lemma (3.4), since $B_2(1-r)^{1-\mu} \downarrow 0$ as $r \to 1$, and evidently

$$\left(\frac{1-r}{1+r}\right) |g(re^{i\theta})| \to \frac{\beta(\theta)}{2} , \quad r \to 1 ,$$

$$\left(\frac{1-r}{1+r}\right) |u(re^{i\theta})| \to \frac{\beta^*(\theta)}{2} , \quad r \to 1 ,$$

where $0 \leq \beta^*(\theta) \leq \beta(\theta) < \infty$. (3.22) now follows from (3.20).

3.4 The argument of $h(re^{i\theta})$

THEOREM (3.8)

Suppose that $g \in G(A,\mu)$ *and* $h = g^2$. *Then if* θ *is a radius of greatest growth of* h, *we have*

$$\arg h(re^{i\theta}) = \arg [g(re^{i\theta})]^2 = o(1)(\bmod 2\pi) \quad , \quad r \uparrow 1 \quad .$$

To prove Theorem (3.8) we need the following.

LEMMA (3.5)

Suppose that $g = u + iv \in G(A,\mu)$. *Then if* $\beta(\theta) > 0$ *, we have*

$$\left|\frac{v(re^{i\theta})}{u(re^{i\theta})}\right| = o(1) \quad , \quad r \to 1 \quad ,$$

and

$$\beta^*(\theta) = \beta(\theta) \quad , \quad 0 \leq \theta < 2\pi \quad .$$

Clearly $\beta^*(\theta) \leq \beta(\theta)$. Suppose that $0 \leqslant \beta^*(\theta) < \beta(\theta)$.

Choose $\eta = (1-\beta^*(\theta)/\beta(\theta))/3$. Then there exists $r_0 = r_0(\eta)$ such that

$$|u(re^{i\theta})| < (1-2\eta)|g(re^{i\theta})| \quad , \quad r_0 < r < 1 \quad .$$

Now $\mu < 1$, so we choose $r_1 > r_0$ so that

$$B_1(|g(re^{i\theta})|^{\mu}+1) < \eta|g(re^{i\theta})| \quad , \quad r_1 < r < 1 \quad .$$

It follows from (3.17) that

$$|g'(re^{i\theta})| \leq \frac{2(1-\eta)}{1-r^2}|g(re^{i\theta})| \quad , \quad r_1 < r < 1 \quad .$$

Thus

$$\left(\frac{1-r}{1+r}\right)^{1-\eta} |g(re^{i\theta})|$$

decreases, $r_1 < r < 1$, where $0 < \eta \leqslant 1/3$, and hence $\beta(\theta) = 0$, contradicting the hypothesis $\beta(\theta) > 0$. Thus $\beta^*(\theta) = \beta(\theta)$.

Thus if $\beta(\theta) > 0$ we have

$$\frac{u(re^{i\theta})^2}{u(re^{i\theta})^2+v(re^{i\theta})^2} \to 1 \quad , \quad r \to 1 \ ,$$

which shows that

$$\frac{v(re^{i\theta})}{u(re^{i\theta})} = o(1) \quad , \quad r \to 1 \quad .$$

This proves Lemma (3.5).

Theorem (3.8) now follows from Lemma (3.5). For

$$\tan \arg g(re^{i\theta}) = \frac{v(re^{i\theta})}{u(re^{i\theta})} \to 0 \quad \text{as} \quad r \to 1 \quad ,$$

$$\arg h(re^{i\theta}) = \arg[g(re^{i\theta})]^2 = 2\arg g(re^{i\theta}) \quad ,$$

and $\tan\phi = 0$ implies $2\phi = 2n\Pi$, n integral.

This completes the proof of Theorem (3.8).

Now suppose that θ_ν satisfies $\beta(\theta_\nu) > 0$. We define τ_ν in the following way:

$$\tau_\nu = \begin{cases} -1 \ , \text{ if } \lim\limits_{r\to 1} \arg g(re^{i\theta_\nu}) = \pi (\text{mod} 2\pi) \quad , \\ +1 \ , \text{ if } \lim\limits_{r\to 1} \arg g(re^{i\theta_\nu}) = 0 (\text{mod} 2\pi) \quad . \end{cases}$$

We quote the following result (see e.g. Nevanlinna, 1970, page 65):

LEMMA (3.6)

Let the function $w(\xi)$ *be regular and bounded in* $|\arg\xi| < (\pi-\varepsilon)/2$, $(\varepsilon > 0)$, *where the Jordan arc* ℓ *, which ends at the origin* $\xi = 0$ *, is located. Then if* $w(\xi)$ *has a limit* a *as* $\xi \to 0$ *on* ℓ *, then*

$$\lim_{\xi\to 0} w(\xi) = a \quad ,$$

uniformly in each angle $|\arg\xi| < \pi/2 - \varepsilon$.

We define

$$g_\nu(z) = \frac{\tau_\nu\beta_\nu}{(1-ze^{-i\theta_\nu})} \quad ; \quad h_\nu(z) = \frac{\alpha_\nu}{(1-ze^{-i\theta_\nu})^2} \quad ,$$

and prove the following:

LEMMA (3.7)

Suppose that $g \in G(A,\mu)$, $h = g^2$ *and* θ_ν *satisfies* $\alpha(\theta_\nu) > 0$. *Then*

$$g(z) \sim g_\nu(z) \quad , \quad h(z) \sim h_\nu(z) \quad , \quad z \to e^{i\theta_\nu} \quad ,$$

uniformly in each angle $|\arg(1-ze^{-i\theta_\nu})| < \pi/2 - \varepsilon$, $(\varepsilon > 0)$.

We let $\xi = (1-ze^{-i\theta_\nu})/(1+ze^{-i\theta_\nu})$ map $|z| < 1$ onto $\mathrm{Re}\,\xi > 0$, so that $z = e^{i\theta_\nu}$ corresponds to $\xi = 0$ and

$$(1-ze^{-i\theta_\nu})g(z) = \left(\frac{2\xi}{1+\xi}\right) g\left[\left(\frac{1-\xi}{1+\xi}\right) e^{i\theta_\nu}\right] = w(\xi)$$

is regular in $|\arg\xi| < (\pi-\varepsilon)/2$ and $|1-ze^{-i\theta_\nu}| = O(1-r)$ there. Thus in view of (3.18) $w(\xi)$ is bounded in $|\arg\xi| < (\pi-\varepsilon)/2$. Now

$$\lim_{z\to e^{i\theta_\nu}} (1-ze^{-i\theta_\nu})g(z) = \beta(\theta_\nu) \quad , \quad z = re^{i\theta_\nu} \quad ,$$

from Theorems (3.7) and (3.8), so if we apply Lemma (3.6) to

$$w(\xi) = \left(\frac{2\xi}{1+\xi}\right) g\left[\left(\frac{1-\xi}{1+\xi}\right) e^{i\theta_\nu}\right]$$

and

$$\ell = \{\xi : \mathrm{Re}\,\xi > 0, \mathrm{Im}\,\xi = 0\} \quad ,$$

we have that

$$\lim_{\xi \to 0} w(\xi) = \beta(\theta_\nu) \quad ,$$

uniformly in each angle $|\arg\xi| < \pi/2 - \varepsilon$. Translating back to the z-plane we have, as $z \to e^{i\theta_\nu}$,

$$g(z) \sim g_\nu(z)$$

and hence

$$h(z) \sim h_\nu(z) \quad ,$$

uniformly in each angle $|\arg(1-ze^{-i\theta_\nu})| < \pi/2 - \varepsilon$.

This completes the proof of Lemma (3.7).

3.5 The integral means

THEOREM (3.9)

Suppose that $g \in G(A,\mu)$ *, where* $0 \leq \mu < 1$ *, and* $h = g^2$. *Then for* $0 \leq r < 1$,

$$\text{(3.23)} \qquad I_\lambda(r,g) \leq \frac{B[g(0),\lambda]}{(1-r)^{\lambda-1}} \quad , \quad \lambda > \max\{1,2\mu\} \qquad ;$$

(3.24) $$I_\lambda(r,h) \leq \frac{B[h(0),\lambda]}{(1-r)^{2\lambda-1}} \quad , \quad \lambda > \max\{\tfrac{1}{2},\mu\} \quad .$$

For the case $\mu = 0$, functions $f = e^g$, where $g \in G(A,\mu)$ form a normal invariant family. Hayman (1955b, Theorem 5) proved Theorem (3.9) for this case.

We thus prove Theorem (3.9) for the case $0 < \mu < 1$.

(3.24) is an immediate consequence of (3.23). For

$$I_\lambda(r,h) = I_{2\lambda}(r,g) \quad .$$

We may prove (3.23) for some $\lambda < 1 + \mu$. For if

$$I_\lambda(r,g) \leq \frac{B[g(0)]}{(1-r)^{\lambda-1}} \quad , \quad \lambda = \lambda_1 < 1 + \mu \quad ,$$

and $\lambda_2 > \lambda_1$, then

$$I_{\lambda_2}(r,g) \leq M(r,g)^{\lambda_2-\lambda_1} I_{\lambda_1}(r,g) \quad .$$

Now $M(r,g) \leq B(g_0)/(1-r)$ from Theorem (3.6), so we have

$$I_{\lambda_2}(r,g) \leq \{B(g_0)\}^{\lambda_2-\lambda_1} B(g_0)(1-r)^{1-\lambda_2} \quad .$$

The proof of Theorem (3.9) for $0 < \mu < 1$ runs along the same lines as the proof for the case $\mu = 0$. We first quote the following result (see Hayman (1955b), Lemma 3):

LEMMA (3.8)

Suppose that $g = u + iv$ *is regular in* $|z| < 1$. *Then for* $1 < \lambda \leq 2$,

$$I_\lambda(r,g) - |g(0)|^\lambda \le \frac{\lambda}{\lambda-1}\{I_\lambda(r,u)-|u(0)|^\lambda\} \quad , \quad 0 < r < 1 \quad .$$

Now

$$|u|^\lambda < \{\log\surd(1+|e^{2g}|)\}^\lambda + \{\log\surd(1+|e^{-2g}|)\}^\lambda \quad ,$$

so if we define

$$T_\lambda(r,f) = \frac{1}{2\pi}\int_0^{2\pi}\{\log\surd(1+|f(re^{i\theta})|^2\}^\lambda d\theta \quad ,$$

we have

$$I_\lambda(r,u) < T_\lambda(r,e^g) + T_\lambda(r,e^{-g}) \quad .$$

Also, if $g \in G$, then $-g \in G$, so to derive a bound for $I_\lambda(r,g)$ we need to derive a bound for $T_\lambda(r,f)$, where $f = e^g$. To do this, we quote the following special case of a general identity due to Spencer (1943) [see Hayman (1955b), Lemma 2] .

LEMMA (3.9)

Suppose that $\Psi(R) = [\log(1+R^2)]^\lambda$, $1 \le \lambda \le 2$, *and*

$$\Psi(R) = \frac{4\lambda(\lambda-1)R^2}{(1+R^2)^2}[\log(1+R^2)]^{\lambda-2} + \frac{4\lambda}{(1+R^2)^2}[\log(1+R^2)]^{\lambda-1} \quad . \tag{3.25}$$

Then for $0 < r < 1$,

$$r\frac{d}{dr}\int_0^{2\pi}\Psi(|f(re^{i\theta})|)d\theta = \int_0^r t\,dt\int_0^{2\pi}\Psi(|f(te^{i\theta})|)|f'(te^{i\theta})|^2 d\theta \quad .$$

We can now prove

LEMMA (3.10)

Suppose that $g \in G(A,\mu)$ *, where* $0 < \mu < 1$ *, and* $f = e^g$ *. Then for* $\lambda > \max\{1,2\mu\}$ *we have*

$$T_\lambda(r,f) \le \frac{B[g(0),\lambda]}{(1-r)^{\lambda-1}} , \quad 0 \le r < 1 .$$

It follows from Theorem (3.6) that Lemma (3.10) holds for $0 \le r \le 1/e$. Again from Theorem (3.6) we have

$$|f'(te^{i\theta})|^2 \le \frac{4|f(te^{i\theta})|^2}{(1-t^2)^2} \{|u(te^{i\theta})|+B_1(|g(te^{i\theta})|^\mu+1)\}^2 ,$$

where $g = u + iv$, $f = e^g$.

Using Lemma (3.9) we have for $1 \le \lambda \le 2$,

$$\frac{d}{dr} r \frac{d}{dr}\int_0^{2\pi} \Psi(|f(re^{i\theta})|)d\theta \tag{3.26}$$

$$\le \frac{r}{(1-r^2)^2}\int_0^{2\pi} \Psi(|f(re^{i\theta})|)4|f(re^{i\theta})|^2\{|u(re^{i\theta})|+B_1(|g(re^{i\theta})|^\mu+1)\}^2 d\theta ,$$

where $\Psi(R)$ is given by (3.25).

To obtain a bound for the integrand of (3.26) we look at the behaviour of the integrand as $R = |f(re^{i\theta})|$ tends to $0,\infty$. We write

$$I = \left\{\frac{4\lambda(\lambda-1)R^4}{(1+R^2)^2}[\log(1+R^2)]^{\lambda-2} + \frac{4\lambda R^2}{(1+R^2)^2}[\log(1+R^2)]^{\lambda-1}\right\}\{|u|+B_1(|g|^\mu+1)\}^2 .$$

Case (a). $R \to 0$. This implies $e^u \to 0$, i.e. $u \to -\infty$, $|g| \to \infty$. Thus

$$e^{4u}[\log(1+e^{2u})]^{\lambda-2}|u|^2 = O(1) , \quad u \to -\infty ,$$

$$e^{2u}[\log(1+e^{2u})]^{\lambda-1}|u|^2 = O(1) \quad , \quad u \to -\infty \quad ,$$

and so we have

$$I \leq B(\lambda)(|g|^{2\mu}+1) \quad .$$

Case (b). $R \to \infty$. This implies $e^u \to +\infty$, i.e. $u \to +\infty$, $|g| \to \infty$, so we may replace u by $\frac{1}{2}\log(1+R^2)$ and obtain

$$I \leq \lambda(\lambda-1)\{[\log(1+R^2)]^{\lambda}+B(\lambda)[\log(1+R^2)]^{\lambda-1}|g|^{\mu}\} + B(\lambda)|g|^{2\mu} \quad .$$

Now

$$[\log(1+e^{2u})]^{\lambda-1} \leq (2|g|+\log 2)^{\lambda-1} \leq B(\lambda)|g|^{\lambda-1} \quad ,$$

so we have

$$I \leq \lambda(\lambda-1)\{[\log(1+R^2)]^{\lambda}+B(\lambda)|g|^{\lambda+\mu-1}\} + B(\lambda)|g|^{2\mu} \quad .$$

Altogether, we have

$$I \leq \lambda(\lambda-1)\{[\log(1+R^2)]^{\lambda}+B(\lambda)|g|^{\lambda+\mu-1}\} + B(\lambda)(|g|^{2\mu}+1) \quad .$$

We now suppose that $\lambda < 1 + \mu$, so that $\lambda + \mu - 1 < 2\mu$. This yields

$$(3.27) \qquad r\frac{d}{dr}\, r\frac{d}{dr}\int_0^{2\pi} \Psi(|f(re^{i\theta})|)d\theta$$

$$\leq \frac{r^2}{(1-r^2)^2}\left\{\int_0^{2\pi}\gamma(|f(re^{i\theta})|)d\theta+B(\lambda)\int_0^{2\pi}|g(re^{i\theta})|^{2\mu}d\theta\right\}$$

where

$$\gamma(R) = 4\lambda(\lambda-1)[\log(1+R^2)]^{\lambda} + B(\lambda) \quad . \tag{3.28}$$

We now derive a bound for $I_{2\mu}(r,g)$.

Let ν satisfy $\max\{1,2\mu\} < \nu \leq \lambda$. Given $\varepsilon > 0$ arbitrarily small there exists $K(\varepsilon)$ such that

$$|v|^{2\mu} < \varepsilon|v|^{\nu} + K(\varepsilon) \quad .$$

We have from Lemma (3.8) that

$$I_{\nu}(r,g) \leq |g(o)|^{\nu} + \frac{\nu}{\nu-1}\{I_{\nu}(r,u)-|u(o)|^{\nu}\} \quad ,$$

and hence

$$I_{2\mu}(r,g) \leq \varepsilon I_{\nu}(r,g) + K(\varepsilon)$$

$$\leq \varepsilon\{|g(o)|^{\nu} + \frac{\nu}{\nu-1} I_{\nu}(r,u)\} + K(\varepsilon)$$

$$\leq B_1(g_0,\nu)\varepsilon I_{\nu}(r,u) + K_1(\varepsilon,g(0)).$$

We thus have, taking $\nu = \lambda$, that

$$r\frac{d}{dr} r \frac{d}{dr}\int_0^{2\pi} \Psi(|f(re^{i\theta})|)d\theta$$

$$\leq \frac{r^2}{(1-r^2)^2}\left\{\int_0^{2\pi}\gamma(|f(re^{i\theta})|)d\theta + B_1(g_0,\lambda)\varepsilon\int_0^{2\pi}|u(re^{i\theta})|^{\lambda}d\theta + K_1(\varepsilon,g(0))\right\}$$

$$\leq \frac{r^2}{(1-r^2)^2}\left\{\int_0^{2\pi}\gamma(|f(re^{i\theta})|)d\theta + B_1(g_0,\lambda)\varepsilon\left[T_\lambda(r,f)+T_\lambda(r,1/f)\right] + K_1(\varepsilon,g(0))\right\}.$$

This inequality holds with $1/f$ instead of f, so if we define

$$T_\lambda^*(r,f) = T_\lambda(r,f) + T_\lambda(r,1/f)$$

we have

$$r\frac{d}{dr}r\frac{d}{dr}T^*_\lambda(r,f) \le \frac{r^2}{(1-r^2)^2}\{[4\lambda(\lambda-1)+\varepsilon B_1(g_0,\lambda)]T^*_\lambda(r,f)+B_2(\lambda,\varepsilon,g(0))\},$$

where $B_2(\lambda,\varepsilon,g(0))=B(\lambda)+K_1(\varepsilon,g(0))$.

Putting $x = \log(1/r)$, $y = T^*_\lambda(r,f)$, we obtain

(3.29) $$\frac{d^2y}{dx^2} \le \{\lambda(\lambda-1)+\varepsilon B_1(g_0,\lambda)\}\frac{y}{x^2} + \frac{B_2(\lambda,\varepsilon,g(0))}{x^2} .$$

We write $\eta = \frac{1}{4}(\lambda - \max\{1, 3\mu\})$ and put $z = x^{\lambda-1+\eta}y$. Then

$$\frac{dz}{dx} = (\lambda-1+\eta)x^{\lambda-2+\eta}y + x^{\lambda-1+\eta}\frac{dy}{dx} ,$$

$$\frac{d}{dx}\left(x^{2-2(\lambda+\eta)}\frac{dz}{dx}\right) = -(\lambda+\eta)(\lambda+\eta-1)x^{-(\lambda+\eta+1)}y + x^{1-\lambda-\eta}\frac{d^2y}{dx^2} .$$

Using the bound for d^2y/dx^2 in (3.29) we get

$$\frac{d}{dx}\left(x^{2-2(\lambda+\eta)}\frac{dz}{dx}\right) \le \{\varepsilon B_1(g_0,\lambda)-\eta[2\lambda-1+\eta]\}\frac{y}{x^{\lambda+\eta+1}} + \frac{B_2(\varepsilon,\lambda,g(0))}{x^{\lambda+\eta+1}}.$$

We choose ε so that $\varepsilon B_1(g_0,\lambda) = \eta[2\lambda-1+\eta]$. We then have

$$\frac{d}{dx}\left(x^{2-2(\lambda+\eta)}\frac{dz}{dx}\right) \le \frac{B_3(\lambda,g(0))}{x^{\lambda+\eta+1}} .$$

Now $x = 1$ corresponds to $r = 1/e$, so z , dz/dx are bounded at $x = 1$ from Theorem (3.6) . We integrate twice from x_0 to 1 , $0 < x_0 < 1$, and deduce

$$z \le B_4(g_0,\lambda) , \quad 0 < x_0 < 1 .$$

Hence

$$y \leq \frac{B_4(g_0,\lambda)}{x^{\lambda-1+\eta}} \quad , \quad 0 < x < 1 \quad .$$

We thus have for $\max\{1,2\mu\} < \lambda < 1 + \mu$ that

$$T^*_\lambda(r,f) < B_4(g_0,\lambda)(1-r)^{1-\lambda-\eta} \quad , \quad 1/e \leq r < 1 \quad ,$$

and hence

$$I_\lambda(r,u) < B_4(g_0,\lambda)(1-r)^{1-\lambda-\eta} \quad , \quad 1/e \leq r < 1 \quad .$$

Using Lemma (3.8) we deduce

$$I_\lambda(r,g) < B_5(g_0,\lambda)(1-r)^{1-\lambda-\eta} \quad , \quad \max\{1,2\mu\} < \lambda < 1 + \mu \quad .$$

Now if $0 < \mu < \frac{1}{2}$, then

$$I_{2\mu}(r,g) \leq I_{1+\eta}(r,g) \leq \frac{B_5(g_0,\lambda)}{(1-r)^{2\eta}} \quad ;$$

and if $\frac{1}{2} \leq \mu < 1$, then

$$I_{2\mu}(r,g) \leq I_{2\mu+\eta}(r,g) \leq \frac{B_5(g_0,\lambda)}{(1-r)^{2\mu-1+2\eta}} \quad .$$

We may thus suppose $\frac{1}{2} \leq \mu < 1$, for if $0 < \mu < \frac{1}{2}$ the argument is similar. It follows that $\eta = \frac{1}{4}(\lambda - 2\mu)$ and then we have

$$I_{2\mu}(r,g) \leq \frac{B(g_0,\lambda)}{(1-r)^{2\mu-1+2\eta}} \quad .$$

We thus have, from (3.27), that for $1 \leq 2\mu < \lambda < 1 + \mu$,

$$r\frac{d}{dr}\,r\frac{d}{dr}\,T_\lambda(r,f)$$

$$\leq \frac{r^2}{(1-r^2)^2}\left\{4\lambda(\lambda-1)T_\lambda(r,f) + \frac{B_5(g_0,\lambda)}{(1-r)^{2\mu-1+2\gamma}}\right\} \quad .$$

Putting $x = \log(1/r)$, $y = T_\lambda(r,f)$ as before, we get

$$(3.30) \qquad \frac{d^2y}{dx^2} \leq \lambda(\lambda-1)\frac{y}{x^2} + \frac{B_5(g_0,\lambda)}{x^{2\mu+1+2\gamma}} \;, \quad 0 < x \leq 1 \quad .$$

We now put $z = x^{\lambda-1}y$ and deduce

$$\frac{d}{dx}\left(x^{2-2\lambda}\frac{dz}{dx}\right) = -\lambda(\lambda-1)x^{-\lambda-1}y + x^{1-\lambda}\frac{d^2y}{dx^2} \quad .$$

Using the bound for d^2y/dx^2 in (3.30) we obtain

$$\frac{d}{dx}\left(x^{2-2\lambda}\frac{dz}{dx}\right) \leq B_5(g_0,\lambda)x^{-(\lambda+2\mu+2\gamma)} \;, \quad 0 < x \leq 1 \quad .$$

Again, z , dz/dx are bounded at $x = 1$ $(r = 1/e)$, from Theorem (3.6), so we integrate first from x_0 to 1 , $0 < x_0 < 1$, and deduce

$$\left.\frac{dz}{dx}\right|_{x=1} - x_0^{2-2\lambda}\frac{dz}{dx_0} \leq \frac{B_5(g_0,\lambda)}{\lambda+3\mu+2\gamma-1}\left\{x_0^{1-(\lambda+2\mu+2\gamma)} - 1\right\} \quad .$$

Replacing x_0 by x we get

$$\frac{dz}{dx} \geq -B_6(g_0,\lambda)\left\{1 + x^{\lambda-(2\mu+2\gamma-1)}\right\} \;, \quad 0 < x \leq 1 \quad .$$

Integrating again from x_0 to 1 , $0 < x_0 < 1$, we get

$$z\Big|_{x=1} - z\Big|_{x=x_0} \geq -B_6(g_0,\lambda)\left\{1-x_0 + \frac{1-x_0^{\lambda-2\mu-2\eta}}{\lambda-2\mu-2\eta}\right\}.$$

Thus, for $\lambda > 2\mu + 2\eta$, we have

$$z \leq B_7(g_0,\lambda) \quad , \quad 0 < x \leq 1 \quad .$$

Now $2\eta = \frac{1}{2}(\lambda-2\mu)$ so this holds for $\lambda > 2\mu$. We thus have for $1 \leq 2\mu < \lambda < 1 + \mu$, that

$$T_\lambda(r,f) \leq \frac{B_7(g_0,\lambda)}{(\log 1/r)^{\lambda-1}} \quad , \quad 1/e \leq r < 1 \quad ,$$

and hence

$$T_\lambda(r,f) \leq \frac{B_7(g_0,\lambda)}{(1-r)^{\lambda-1}} \quad , \quad 1/e \leq r < 1 \quad .$$

Now this is true for $\lambda < 1 + \mu$. If $\lambda_2 \geq 1 + \mu$ we have, with $\lambda_1 < 1 + \mu$

$$\begin{aligned} T_{\lambda_2}(r,f) &\leq \{\log \surd(1+M(r,f)^2)\}^{\lambda_2-\lambda_1} T_{\lambda_1}(r,f) \\ &\leq \left\{\frac{B(g_0)}{1-r}\right\}^{\lambda_2-\lambda_1} . \ T_{\lambda_1}(r,f) \\ &\leq B(g_0)^{\lambda_2-\lambda_1} B_7(g_0,\lambda_1)(1-r)^{1-\lambda_2} \quad , \end{aligned}$$

by Theorem (3.6).

This completes the proof of Lemma (3.10) and hence Theorem (3.9).

3.6 Proof of Theorems (3.1) and (3.3)

Theorem (3.3) contains Theorem (3.1). Theorem (3.3) follows immediately from Theorems (1.1), (3.7) and (3.9), and Lemma (3.7). Theorems (3.7) shows that $\alpha(\theta)$ exists, $0 \leq \theta < 2\pi$; Lemma (3.7) shows that (1.2) holds for $h \in H(A,\mu)$; Theorem (3.9) shows that (1.3) holds, where $p = 1$. Thus the conditions of Theorem (1.1) are satisfied for functions $h \in H(A,\mu)$ and Theorem (3.3) follows.

3.7 The minor arc

We have proved that if $h \in H(A,\mu)$, $0 \leq \mu < 1$, then $E = \{\theta:\alpha(\theta)>0\}$ is countable. Thus, given $\eta > 0$, we define $N = N(\eta)$ to be the number of radii of greatest growth of h for which $\alpha(\theta) \geq \eta^2$, and hence $\beta(\theta) \geq \eta$. Let θ_ν , $\nu = 1,\dots,N$ be such that $\beta_1 \geq \beta_2 \geq \dots \geq \beta_N \geq \eta$. For $\nu = 1,\dots,N$ and K a large positive constant we define

$$\gamma_\nu = \{\theta:|\theta-\theta_\nu|<K(1-r)\} \quad ;$$

$$\gamma = \bigcup_{\nu=1}^{N} \gamma_\nu \quad ;$$

$$\gamma^c = [0,2\pi)\setminus \gamma \quad ,$$

and let γ_ν^* be the set of $re^{i\theta}$ for which $\theta \in \gamma_\nu$. We denote the closure of γ_ν^* by $(\gamma_\nu^*)'$.

We now prove

LEMMA (3.11)

Suppose that $g \in G(A,\mu)$ *and* $h = g^2$. *Then, given* η , $0 < \eta < 1$, *there exist* K , r_0 *such that for* $r_0 < r < 1$ *and* $\theta \in \gamma^c$,

$$|g(re^{i\theta})| < \frac{\eta}{(1-r)} \quad ; \quad |h(re^{i\theta})| < \frac{\eta}{(1-r)^2} \quad .$$

We prove the first inequality, since the second follows immediately. Given η, $0 < \eta < 1$, we define $N = N(\eta)$ as above.

We consider two cases:

I. $\theta : K(1-r) \leq |\theta-\theta_\nu| \leq K(1-R) = \delta_2 (1 \leq \nu \leq N)$, where R,K will be fixed below;

II. $\theta^{(\nu)} + \delta_2 \leq \theta \leq \theta^{(\nu+1)} - \delta_2 \quad (1 \leq \nu \leq N-1)$

$\theta^{(N)} + \delta_2 \leq \theta \leq \theta^{(1)} + 2\pi - \delta_2$,

where the $\theta^{(\nu)}$, $(1 \leq \nu \leq N)$ are the θ_ν, $(1 \leq \nu \leq N)$, with $\theta^{(1)} < \theta^{(2)} < \ldots < \theta^{(N)}$.

Case I.

We choose $K = K(\eta)$ so that

$$\frac{10\beta_1}{K} < \frac{\eta}{4} \quad .$$

As in Lemma (1.1) (see equation (1.11)) we have

$$|1-re^{i(\theta-\theta_\nu)}| \geq \frac{|\theta-\theta_\nu|}{5} \quad ,$$

and so for $\frac{1}{2} < r < 1$, $|\theta-\theta_\nu| \geq K(1-r)$ $(1 \leq \nu \leq N)$ we have

$$(3.31) \qquad \frac{2\beta_\nu}{|1-re^{i(\theta-\theta_\nu)}|} \leq \frac{10\beta_\nu}{K(1-r)}$$

$$< \frac{10\beta_\nu}{K}\left(\frac{1+r}{1-r}\right)$$

$$< \frac{\eta}{4}\left(\frac{1+r}{1-r}\right) \quad ,$$

since $\beta_\nu \leq \beta_1$.

Choose $\varepsilon = \varepsilon(K)$ and $R_0 > \frac{1}{2}$ so that

$$(\gamma_\nu^*)' \subset \{z: |\arg(1-ze^{-i\theta_\nu})| < \pi/2 - \varepsilon\} \subset \{z:|z|<1\} \quad ,$$

$$R_0 < |z| < 1 \quad .$$

We have, from Lemma (3.7), uniformly for $re^{i\theta} \in (\gamma_\nu^*)'$, that

$$|1-re^{i(\theta-\theta_\nu)}||g(re^{i\theta})| \to \beta_\nu \quad .$$

We choose $R > R_0$ so that for $R \leq r < 1$, $re^{i\theta} \in (\gamma_\nu^*)'$,

(3.32) $$|1-re^{i(\theta-\theta_\nu)}||g(re^{i\theta})| < 2\beta_\nu \quad ;$$

$$B_2(1-r)^{1-\mu} < \frac{\eta}{4} \quad .$$

We need to show that if $R < r < 1$, then

$$\theta \in \{\theta: K(1-r) \leq |\theta-\theta_\nu| \leq K(1-R)\} \text{ implies } |g(re^{i\theta})| < \eta/(1-r) \quad .$$

Let r_1,θ satisfy $R_1 \leq r_1 \leq r$, $|\theta-\theta_\nu| = K(1-r_1)$. Then $r_1e^{i\theta}$ satisfies (3.31), (3.32) and we have

$$\left(\frac{1-r_1}{1+r_1}\right) |g(r_1e^{i\theta})| < \frac{\eta}{4} \quad .$$

From Lemma (3.4) we have

$$\left(\frac{1-r}{1+r}\right) |g(re^{i\theta})| + B_2(1-r)^{1-\mu} \leq \left(\frac{1-r_1}{1+r_1}\right) |g(r_ie^{i\theta})| + B_2(1-r_1)^{1-\mu}$$

$$\leq \frac{\eta}{4} + \frac{\eta}{4}$$

$$= \frac{\eta}{2} \quad .$$

Hence, a fortiori

$$\left(\frac{1-r}{1+r}\right) |g(re^{i\theta})| < \tfrac{1}{2}\eta \quad , \quad K(1-r) \leq |\theta-\theta_\nu| \leq K(1-R) \quad ,$$

$$R < r < 1 \quad ,$$

and so

$$|g(re^{i\theta})| < \frac{\eta}{(1-r)} \quad .$$

This completes the proof of Case I of Lemma (3.11).

Case II.

For the purposes of this argument we let $\theta^{(N+1)} = \theta^{(1)} + 2\pi$ and consider the set

$$\Gamma_\nu = \{\theta : \theta^{(\nu)} + \delta_2 \leq \theta \leq \theta^{(\nu+1)} - \delta_2\} \quad .$$

On Γ_ν we have

$$\lim_{\rho\to 1} \left\{ \left(\frac{1-\rho}{1+\rho}\right) |g(\rho e^{i\theta})| + B_2(1-\rho)^{1-\mu} \right\} < \frac{\eta}{2} \quad . \tag{3.33}$$

Let $\rho_\mu = 1 - 1/\mu$, $\mu > 2$. If for some μ we have that for all $\theta \in \Gamma_\nu$

$$\left(\frac{1-\rho_\mu}{1+\rho_\mu}\right) |g(\rho_\mu e^{i\theta})| + B_2(1-\rho_\mu)^{1-\mu} < \frac{\eta}{2}$$

then from Lemma (3.4)

$$\left(\frac{1-\rho}{1+\rho}\right) |g(\rho e^{i\theta})| + B_2(1-\rho)^{1-\mu} \leq \left(\frac{1-\rho_\mu}{1+\rho_\mu}\right) |g(\rho_\mu e^{i\theta})| + B_2(1-\rho_\mu)^{1-\mu}$$

$$< \frac{\eta}{2} \quad ,$$

and hence

$$(1-\rho)|g(\rho e^{i\theta})| < \eta \quad , \quad \rho_\mu < \rho < 1 \quad ,$$

which is our result.

We thus suppose contrary to this that for each μ there exists $\phi_\mu \in \Gamma_\nu$ such that

$$(3.34) \qquad \left(\frac{1-\rho_\mu}{1+\rho_\mu}\right) |g(\rho_\mu e^{i\phi_\mu})| + B_2(1-\rho_\mu)^{1-\mu} \geq \frac{\eta}{2} \quad .$$

Let $\phi^{(\nu)}$ be a limit point of the sequence $\{\phi_\nu\}$.

It follows from (3.34) and Lemma (3.4) that for each fixed μ

$$\left(\frac{1-\rho}{1+\rho}\right) |g(\rho e^{i\phi_\mu})| + B_2(1-\rho)^{1-\mu} \geq \frac{\eta}{2} \quad , \quad \tfrac{1}{2} < \rho < \rho_\mu \quad .$$

If ρ is fixed the result holds for all large μ and by letting $\mu \to \infty$ we have $\phi_\mu \to \phi^{(\nu)}$ and

$$\left(\frac{1-\rho}{1+\rho}\right) |g(\rho e^{i\phi^{(\nu)}})| + B_2(1-\rho)^{1-\mu} \geq \frac{\eta}{2} \quad , \quad \tfrac{1}{2} < \rho < 1 \quad ,$$

which contradicts (3.33). So (3.34) is false for $1 - 1/\mu = \rho_\mu < \rho < 1$ and some $\mu > 2$.

Thus, given $\eta > 0$, there exists $\mu_\nu = \mu_\nu(\eta)$ such that for $1 - 1/\mu_\nu < \rho < 1$ and $\theta \in \Gamma_\nu$ we have

$$\left(\frac{1-\rho}{1+\rho}\right)|g(\rho e^{i\theta})| + B_2(1-\rho)^{1-\mu} < \frac{\eta}{2} \quad ,$$

and hence

$$|g(\rho e^{i\theta})| < \frac{\eta}{(1-\rho)} \quad .$$

This is true for $\nu = 1,\ldots,N$, so we let $r_0 = \max\{R,\rho_{\mu_1},\ldots,\rho_{\mu_N}\}$. Then for $r_0 < r < 1$ and $\theta \in \gamma^c$ we have

$$|g(re^{i\theta})| < \frac{\eta}{(1-r)} \quad .$$

This completes the proof of Lemma (3.11).

3.8 Proof of Theorems (3.2) and (3.4)

Theorem (3.4) contains Theorem (3.2). Theorem (3.4) now follows from Theorems (1.2), (3.8), (3.9) and Lemma (3.11). For Theorem (3.9) shows that (1.3) holds for some $\lambda < 1$ and $h \in H(A,\mu)$, Lemma (3.11) shows that (1.6) holds, and Theorem (3.8) shows that (1.7) holds. Thus the conditions of Theorem (1.2) hold for $h \in H(A,\mu)$ and hence Theorem (3.4) follows from Theorem (1.2).

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