IMPERIAL COLLEGE OF SCIENCE AND TECHNOLOGY

(University of London)

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THE TERM STRUCTURE OF COMMODITY PRICES:

SOME APPLICATIONS OF STOCHASTIC CALCULUS

by

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A Thesis Presented for the Degree of Doctor of Philosophy

in the University of London

June 1977

CONTENTS

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ABSTRACT

Firstly, the theory of spot price behavior in commodity markets is considered. In this, particular reference is made to:

(a) the effect of 'noise' on stable dynamic relationships between demand, supply and price where 'noise' stands proxy for numerous unspecified variables;

(b) the application to the economic system of the theory of interacting populations.

Secondly, a formulation of the theories of the term structure of interest rates is an operational form and in continuous terms is applied to the term structure of commodity prices.

Thirdly, an analytical relationship is derived (by way of stochastic calculus) between the spot price, forward price and maturity in terms of fixed parameters, using the Black Scholes-Merton formulation of the option pricing model.

Finally, the pricing mechanism for risky assets is considered as a control problem and price trajectories in time are derived as a solution to an optimal control problem.

Acknowledgements

My most sincere thanks are due to my supervisors, Mr. R.J Betts and Mr. G.R. Salkin for their guidance, advice, co-operation - and forbearance - during the course of this work. I also wish to thank all the members of the Department of Management Science in general and my fellow research students for their constructive criticisms and help.

I am also indebted to the Library staff of the Mechanical Engineering Department of Imperial College, the London Business School, the Lyon Playfair, the Geological Museum and the Science Museum for the help and facilities provided to me.

NOTATION EMPLOYED

(In order of their appearance)

CHAPTER 2

P	price
t	time
H	a monotonically increasing function of excess demand
Q	excess demand
H'	dH/dQ
qs	quantity supplied
qD	quantity demanded
P^e	equilibrium price
D(P)	demand function
S(P)	supply function
D_p^e	$\left(\frac{dD}{dP}\right)_{P = P^e}$ = slope of demand curve at equilibrium
S_p^e	$\left(\frac{dS}{dP}\right)_{P = P^e}$ = slope of supply curve at equilibrium
λ	the value of H' at equilibrium
p	the difference between price and its equilibrium value
r	the difference between the slopes of supply and demand curves at equilibrium
i<t>	a Gaussian noise input
P^o	initial price
σ	incremental variance
F	probability density function
a(P)	the mean function in the Fokker-Planck (Chapman-Kolomogoroff) equation

CHAPTER 2

$b(P)$	the variance function in the above equation
k	a dummy variable
y	a dummy variable equal to $\frac{1}{\sigma}\ln(P/P^e)$
y^o	initial value of y
$\hat{a}(y)$	modified a(P)
g	probability density functions
τ	gamma functions
θ	an intermediate variable $= \frac{2\lambda r P^e}{\sigma^2}$
<>	average
aji	a matrix of $(\frac{\partial Qi}{dPj})_{P\partial = P_j^e}$ in a multimarket situation
Ui(t)	a Gaussian noise input
n	number of commodities in the market
m	incremental mean of U
k	a dummy variable

CHAPTER 3

P	price
P^e	equilibrium price
S	supply
S^e	equilibrium supply
D	demand
D^e	equilibrium demand
$\lambda_1, \lambda_2, \lambda_3$	constants
t	time
η	excess supply
ε	excess demand
a,b	constants, b refers to the slope of the linear demand curve

CHAPTER 3

o,k,l,m,	arbitrary constants for the prey-predator scheme
M,α,β, n	arbitrary constants for the epidemic model
A,B	constants

CHAPTER 4

x(t)	spot price
τ(tau)	maturity
P(τ,t)	forward price
α	incremental mean of 'return' due to price change
σ	the variance of the above
E	expectation operator
Z	a Gauss-Wiener input
$y(\frac{\tau}{P})$	yield for maturity τ (risk-free) nominal price
f	margin on forward payment
r(τ)	coupon discount

CHAPTER 5

N	number of instruments
τ_i(tau)	time to maturity of the i'th instrument
$P(\tau_i)$	price of the instrument i
$\bar{P}$	nominal price
r_i	coupon return of i
μ_i	expected mean of the i'th instrument
$\delta^2(i)$	expected variance of the above
$dq(t,\tau_i)$	a standard Gauss-Wiener process for maturity τ_i
ε	expectations
ρij	correlation coefficient
G(i)	capital gain of i

CHAPTER 5

R(i,dt)	total return during the period dt of i
U(R)	utility of R
b	the risk-aversion coefficient in the fn.U.
E	expected return
S	standard derivation
Wi	fraction of wealth invested in i
L	Lagrangian
λ	the Lagrangian coefficient or the certainties equivalent
X(i)	expected return of i
Z(i)	risk premium of i
$\bar{S}$	vector of supply of instruments
$\bar{R}$	vector (ri/Pi)
V	the matrix (Pijδiδj)
$\bar{i}$	vector of unity
$\bar{W}$	vector of wealth fractions
W_o	total wealth
$\bar{k}$	vector of the slopes of supply functions

CHAPTER 6

t	time
τ	maturity
F(t,τ)	forward price
x(t)	spot price
k	margin fraction
R(τ)	interest rate functional (yield curve)
W	effective price of forwards
α(x,t)	instantaneous change in x
σ(x)	the instant standard derivation

CHAPTER 6

dZ	standard Gauss-Wiener process
(dRet)	the instant return on a dummy portfolio
fi^* i=1,3	fractions invested in spot forward and bond

CHAPTER 7

W_o	initial wealth
T	horizon
r	certain interest
$W(t)$	wealth at time t
$C(t)$	consumption at t
t	time
ρ	discount factor
U	utility function
Z	a random variable
$W(t)$	fraction of wealth invested at t in risky assets
J	objective functional
$x(t)$	state variable
$\bar{U}$	control variable
I	intermediate function
P	price
D	physical demand
$S(P,t)$	supply function
R	propensity to consume
Ct^*, W^*	optimal control variables

THE COMMON ONES

$\sum_i$	sum over i
$\leq$	less than or equal to

THE COMMON ONES

$\geqslant$	greater than or equal to
$\frac{d}{dx}$	differential
$\partial/\partial x$	partial differential
$\int$	integral

CHAPTER I

INTRODUCTION

In this research report, we present some theoretical developments we have made on the following topics:

(1) Development of some explanatory models for the behavior of spot prices in commodity markets.

(2) Application of some theories of the term-stucture of interest rates to the problem of the term-structure of commodity prices.

(3) Application of the capital asset pricing theory to the problem of the term-structure of commodity prices.

(4) Forming the basis for developing a sound theory of viewing the financial and/or asset markets as a dynamic control system.

The problem of pricing of commodities touches the heart of many sectors of the economic and the political world. Questions about political autonomy, governmental intervention, industrial relations, the merits and demerits of cartels (either of the consumers or of the producers) and numerous other topics arise in this context. By the word 'commodity', we normally imply any of the soft and hard materials sold by auction in London and other leading centres of world trade. But the theories that are applicable to explain the behavior of the prices of commodities are equally applicable - sometimes with minor modifications - to the pricing of foreign exchange bonds and virtually to the pricing of every financial or physical asset traded by auction.

Any aspect of pricing concerns the micro-economic segment of economic theory. The augmented effects of the behavior of commodity prices, foreign exchanges and the money markets concern the governmental agencies in charge of formulating and regulating the fiscal and monetary policies and thus the economists concerned with the aggregate economy.

The theoretical foundations for the present project are numerous and can be summarised under the following headings:

(1) Applications of stochastic calculus in order to obtain refined formulations of some leading economic theories.

(2) Applications of biological population models to the problem of price behavior.

(3) Developing continuous time versions of some leading theories of the term-structure of interest rates.

(4) Effect of noise - symbolising unspecified information - on the behavior of prices and on the criteria that are normally applied to model a stable behavior of prices.

(5) Derivation of a version of the capital asset pricing theory that relates to the problem of the term-structure of financial and physical asset prices.

(6) Deriving operational theories of term-structure of commodity prices applying some of the above results.

(7) Application of modern control theory in the development of a theory of viewing the market mechanism as a control system.

We believe that the applications of the above to tackling the problems we began with are authentic and complex enough to warrent this doctoral report.

Now, we present in detail the way the thesis develops in the following phases.

First, we consider the behavior of spot prices in commodity markets. Under this topic, we deal with two approaches to the problem. The first, in Chapter (2), analyses the implications of a random noise input - which symbolises the effects of numerous unspecified and unspecifiable variables - for the Walrasian criterion of stable price behavior. Finding the Walrasian dynamics unsuitable under such circumstances, it goes on to examine the effect of such an input on the logistic model of price dynamics. The logistic model with stochasticity leads to reasonable probability density functions. The models are analysed for single and multiple market situations. The second approach, in Chapter (3), examines two dynamic models of spot price behavior in continuous time. Both of these are based on biological models of population changes of interacting species. That based on the prey-predator system leads to an undamped cyclical price behavior. That based on the epidemic model leads to a damped harmonic pattern.

Secondly, we consider the continuous time formulations of some theories of the term-stucture of commodity prices. In Chapter (4), we apply the continuous time versions of the expectations model and the error-learning model of the term-structure of interest rates to the problem of the term-structure of commodity prices. Under the assumption of a random-walk model for the behavior of the spot price, the resultant formulae for forward prices are in terms of observable and historically measurable variables and hence are operational. In Chapter (5), we consider the implications for the term-structure of interest rates of the twin assumptions of a joint random-walk model for the movements of prices and of risk-averting behavior on the part of investors. Such a consideration leads to the Sharpe capital asset pricing model for financial instruments. The results are applicable - with some minor modifications - to the term-structure behavior of commodity prices. The results are also operational and easily testable. We consider also the problem of the control of the supply of financial instruments - a problem faced by any Central (Reserve) Bank and the problem of equilibrium term-structure of commodity prices, assuming the amounts of supply in this case to be speculative.

Thirdly, in Chapter (6), we examine a theory of the term-structure of commodity prices based on some recent work on rational option pricing theory. On the assumption of a general random movement of spot prices and a known term-structure of interest rates, we derive a term-structure of commodity prices, when the forwards are buyable on margin. The result is entirely in terms of observable variables.

Finally, in Chapter (7), we consider the pricing mechanism for risky assets as a control system and derive the price trajectories in time as a solution to an optimal control problem.

Each of the above chapters has a section surveying relevant literature and relevant summary and conclusions.
In a final chapter, we present a general summary of the conclusions of the project and its possible extensions.

Appendices explaining the mathematical content of the above formulations are presented at the end of the report.

Knowledge obtained through this report has many useful extensions. The applications of biological models to the economic system - which is a new extension in itself - can be developed to obtain a dynamic equilibrium model of the total economic system.

Following the progress made in this project on the topic of formulating continuous time versions of some of the theories of the term-structure of interest rates and their applications to the problem of the term-structure of commodity prices, we can envisage the application of similar ideas to other leading theories of the term-structure and the possibility of formulating unitary models of term-structure that relate to all financial and asset markets.

Extending the simple beginning made in the project, it is possible to develop a general and sound theory of viewing the financial markets as a control system. It is also possible to apply the continuous-time capital asset pricing theory to the determination of the risk-structure of the assets and liabilities of any concern or financial project and thus to the development of a capital budgeting theory leading to a healthy matching of risk, return and time between assets and liabilities.

CHAPTER 2

STOCHASTICITY AND THE STABILITY OF PRICE CHANGE

1. Introduction

The movement of the price of any commodity in a complex economy is governed by a host of factors - some of which affected by the very movement they cause. A complete model of the movement would be possible only on the basis of complete knowledge of the causal mechanism, interactions of the forces and their future variability. Such a model is untenable because of

(a) the vastness of the information required

(b) the rapidity of change in the economic system, which makes any model, once put together, out of date very soon and hence in need of updating, and

(c) the unpredictability of some of the variables.

2. Survey of Literature

2.1 Economic Models:

Barring such a complete model, one could, as a second-best attempt, try to formulate an approximate model of price behavior. One could, by using what could be termed as econometric methods, try to estimate the importance of certain variables with respect to the behavior of price. There one does not assess all the relevant variables but only those considered relevant after a prior economic reasoning - and thus the choice of the variables and hence the model is open to criticism as every line of economic reasoning has its critics.

In a single linear equation model, the variables chosen must be relatively 'independent' and must be individually predictable one period ahead or else be lagged one or more periods behind the price series. Thus, in order to explain and predict price behavior, we must be able to predict several other variables. Such models are to be developed for each particular commodity. If one wants a model for commodity price movements as a whole, one has to solve the problems of weighting, aggregation and indexation. The econometric way deals with the particular and not with the 'general'. The coefficients of linear regression are estimated according to the method of least squares and greater independence among the explanatory variables ensures that the estimates will be unbiased. Shisko (62), Hieronymus (30), Shepherd (61) and Fox (25) have published results of using such models for agricultural commodities.

The above could be extended to simultaneous equation systems. Here the direction of causation is not unilateral. But, unless the results prove consistently superior to those obtainable by the single equation approach, the extra effort and expense involved in such a construct would hardly be justifiable. Witherell (74) has published results of such a model for the wool market, Weymour (72) for cocoa and Houck (31) for the soyabean-soyabean oil-soyabean meal market.

One could include adaptive models - exponential smoothing, Box-Jenkins models etc. - as belonging to the econometric group of models. Such adaptive models are - unlike those above - purely predictive, since they do not use explanatory variables. They are attempts at forecasting the future levels of a price series on the basis of an understanding of how it had behaved in the past. Empirical work on the application of such models in the area of commodity price forecasting has been minimal. (See Granger and Labys (28) for an exhaustive coverage of the above topic.)

2.2 General Models:

We now turn to consider models that are general since they stem from consideration of the presumed speculative nature of all commodity markets. By definition, speculative markets cannot produce a price series having predictable patterns, for if they did, they would have been arbitraged out of existence by the numerous keenly anticipative participants that such markets ought to have if they are to live up to the economist's definition of them. Thus a random movement in prices is the theoretically expected price series in such markets. Samuelson (51) derives such a result from a general stochastic formulation of anticipated behavior. Bachelier (8) derives a similar result as early as 1900, using stochastic calculus. Working (77) compares a randomly generated price series with several actual ones and found the similarity rather remarkable. But, in commodity markets, the random walk hypothesis has not met with the universal acclaim it enjoys in the stock market. But, then again, the criticisms are not consistent in themselves either. Working (78) reports a tendency for serial correlation in prices for Chicago corn, wheat and rye futures. Larson (39) suggests the existence of some linear dependence in the futures prices of corn. Both Houthakker (32) and Schmidt (60) have investigated the use of simple filter trading rules and because of the apparent profitability of such rules, suggest that the random-walk model is incorrect. In a first study, using a so-called filter rule, Alexander (2) suggested that the random walk model was incorrect. But in light of subsequent criticism about the practicability of his filter rule, withdrew most of his conclusions (3). Finally, in a very complete and careful study, Fama and Blume (21) reach the conclusion that the results obtained from the particular class of trading rules that they used do not in any way contradict the random-walk model. In view of the mixedness of opinions, obviously further empirical work is needed to clarify the position of the random-walk model. But, perhaps the theoretical requirement for such a result to hold, i.e., numerous keenly anticipative participants, does not hold for commodity markets. It may be that, as Houthakker argues, price deviations are watched in commodity markets by relatively few, rather old-fashioned traders and thus the markets are too unsophisticated for all predictable patterns to be completely eliminated by active arbitrage(32).

In this part we consider a method of approach to the problem of price behavior. We start from the premise that the function of price is to equilibrate supply and demand, which are necessary functions of price and of other variables. The effect of other variables on price will be through the twin scissors of supply and demand. At a given form of the supply/demand twin functional, there is only one clearing price. The change of this price occurs because either the demand function or the supply function or both change due to change in the value or the importance of one or more of the parameters. There must be a new equilibrium price and this price will have to be finite, i.e., the change in price cannot be explosive, after a while it will settle down to the new equilibrium value - for prices are not seen to explode to infinity or zoom down to negative values. This kind of response of price to a step change in parameters is considered by stability theory.

There are several stable relationships in economic theory (51) which relate the rate of price change to the discrepancy between supply and demand. We will choose the simplest of them called the Walrasian criterion. It is algebraically similar to the others. Our interest is not so much in itself but in the way such a criterion - and hence such criteria - respond to a constantly changing environment. Though considered stable for a step change in parameters, we are interested in finding out how such a criterion behaves under constantly changing differential between supply and demand - and a change taking place in such a manner that is not predictable, (if it were, one could build a purely deterministic dynamic model - which, as we saw - was impossible even on the basis of complete knowledge). We shall assume that this change occurs in the form of new information arriving on the scene and we consider this input, to simplify analysis, as a Brownian input. (59) (see the mathematical appendix for guidance through the following.)

3. <u>The Simple Walrasian Criterion</u>: A simple criterion for stable price adjustment is the Walrasian one of linear dependence of rate of price change on excess demand.

$$\frac{dP}{dt} = H(q_D - q_S) \equiv H(Q) \qquad (1)$$

where P : price

t : time

H : a monotonically increasing function with $H(O) = O$; $H' > O$

Q : quantity in excess demand

and q_S : quantity supplied

q_D : quantity demanded

Let q_S and q_D be entirely functions of price, i.e., there are no shift variables. (This is a simple assumption).

$$\left.\begin{aligned} q_D &= D(P) \\ q_S &= S(P) \end{aligned}\right\} \qquad (2)$$

Equation 1 becomes

$$\frac{dP}{dt} = H(D(P) - S(P)) \qquad (3)$$

For the given twin functional (2), there is only one equilibrium price at which $D(P) = S(P)$ and hence

$$\frac{dP}{dt} = H(O) = O$$

Call the equilibrium price P^e. Expanding (3) in Taylor series around

P^e, and omitting terms of orders higher than the

$$\frac{dP}{dt} = \lambda(D_P^e - S_P^e)(P - P^e) \qquad (4)$$

where

$$D_P^e = (\frac{dD}{dP})_{P=P^e}; \quad S_P^e = (\frac{dS}{dP})_{P=P^e}; \quad \text{and } \lambda = (\frac{dH}{dQ})_{Q=0} > 0$$

and this has the solution

$$P(t) = P^o + (P^e - P^o)\, e^{\lambda\,(Dp^e - Sp^e)t} \qquad (5)$$

which will be stable if and only if

$$(Dp^e - Sp^e) < 0 \qquad (6)$$

which will be true for any simple demand-supply functions - negative sloping demand curve and upward sloping supply curve. So let us assume this the validity of the above inequality and thus the stability of equation (3). We will now see how the dynamic behaves under an additive Gaussian input.

Let $p = P - P^e$, and $r = (S_P^e - D_P^e) > 0$. Then (4) becomes

$$\frac{dp}{dt} = -\lambda rp \qquad (7)$$

But p cannot be less than $(-P^e)$, in which case P, the price, will have to be negative. Thus (7) cannot accommodate an additive Gaussian input, which will make it

$$\frac{dp}{dt} = -\lambda rp + i\langle t\rangle \qquad (8)$$

where $i\langle t\rangle$ is a Gaussian noise input - and thus can range from $-\infty$ to ∞, which p cannot. Thus we cannot superimpose a random input into the Walrasian model. (56)

4. A Logistic Model

So as a simple alternative to the Walrasian dynamic, let us consider the following model of proportional price change.

$$\frac{\left(\frac{dP}{P}\right)}{dt} = \frac{d\ln P}{dt} = H(q_D - q_S) \qquad (9)$$

where t, H, q_D, q_S, P are as defined earlier. Now if P^e is the equilibrium price, expanding (9) around P^e in Taylor series as before

$$\frac{d\ln P}{dt} = \lambda(D_P^e - S_P^e)(P - P^e) \qquad (10)$$

where λ, D_P^e, S_P^e are defined as earlier.

Let $r = S_P^e - D_P^e$.

Equation (10) becomes

$$\frac{dP}{dt} = PP^e\lambda r(1 - \frac{P}{P^e}) \qquad \lambda > 0 \qquad (11)$$

Equation 11 is a Verhulst-type logistic equation. It solution with and initial price P^o, is

$$P(t) = P^o\left(\frac{P^o}{P^e} + (1 - \frac{P^o}{P^e})\, e^{-\lambda r P^e t}\right)^{-1} \qquad (12)$$

If in equation (12), as $t \to \infty$, P(t) will tend to P^e if $r > 0$. Thus the stability criterion for the logistic model is the same as that of the Walrasian.

5. Logistic Model with Stochasticity

Let us assume that $r > 0$. Let us now consider the effect of a simple independently additive random input with incremental mean zero and incremental variance σ^2 on the logistic model. This can be represented as (27).

$$\frac{d\ln P}{dt} = \lambda r(P^e - P) + \sigma\, i\langle t\rangle \qquad (13)$$

where $i\langle t\rangle$ is a standard Brownian noise,

or

$$\frac{dP}{dt} = \lambda r P P^e(1 - \frac{P}{P^e}) + \sigma P i\langle t\rangle \qquad (14)$$

Through mathematical analysis (see ref:), eg. 14 can be and it can be shown that as $t \to \infty$, the limit distribution for (14) can be evaluated. And hence the limit distribution for P is,

$$F(\frac{P}{P^o}, \infty) = \frac{1}{P\gamma(\Theta)} \left[\Theta\left(\frac{P}{P^e}\right)\right]^{\Theta} \exp(-\Theta \frac{P}{P^e})) \qquad (15)$$

where $\Theta = \frac{2\lambda r P^e}{\sigma^2}$, and γ is the gamma function. (15) has a probability density for P entirely on the positive real line. $(0 \leq P \geq \infty)$. And the most probable price at $t = \infty$, it can be shown, will be lower than P^e. Thus the logistic model behaves reasonably under a simple Gaussian input.

6. Time Density Functions

If one, then, infers that (14) represents the way in which prices do change, then it will be interesting to study, instead of the steady state density functions of (15), the time- dependent density functions of P. The latter may be useful in understanding the term-structure of commodity prices.

The general time-dependent density solution to (14) is unavailable. Let us consider, instead, two 'special' cases.

6.1 (a) Price P far below P^e: The Random Walk Model

Let $P^e \to \infty$; then in (15), $a(P) = kP + \frac{\sigma^2 P}{2}$, and $b(P) = \sigma^2 P^2$.

The assumption is quite preposterous, but the conclusion is interesting. Hence in the transformed variable y, we get $\hat{a}(y) = \frac{k}{\sigma}$, and equation (16) becomes

$$\frac{\partial g}{\partial t} = - \frac{\partial}{\partial y}(\frac{k}{\sigma} g) + \frac{1}{2} \frac{\partial^2 g}{\partial y^2} \qquad (17)$$

Equation (17) is the diffusion equation for a simple random walk. Its solution, when transformed back into P is

$$F(\frac{P}{P^o}, t) = \frac{1}{P(2\pi t\sigma^2)^{\frac{1}{2}}} \exp\left\{-\left(\ln\left(\frac{P}{P^o}\right)e^{-kt}\right)^2/2\sigma^2 t\right\} \qquad (18)$$

$$O \leq P \geq \infty \qquad (19)$$

The mean of P at t is

$$<P(t)> = P^o \exp\{(k + \frac{\sigma^2}{2})t\} \qquad (20)$$

This is the distribution of the so-called random-walk theory of price movement. Thus, from what we have done so far, we derive a famous theory of price movement.

6.2 (b) P fluctuating around P^e

Let $\frac{P}{P^e} \simeq 1$. Then, the transformed variable y, ($y = \frac{1}{\sigma}\ln(\frac{P}{P^e})$) is small enough for $e^{\sigma y} = (1 + \sigma y)$. Then, $\hat{a}(y) = -ky$ and equation (17) becomes

$$\frac{\partial g}{\partial t} = \frac{\partial}{\partial y}(kgy) + \frac{1}{2}\frac{\partial^2 g}{\partial y^2} \qquad (21)$$

(21) describes an Ornstein-Uhlenbeck process in y. The solution of (21) in the original variable P is

$$F(\frac{P}{P^o}, t) = \left\{\frac{P^e}{\pi\sigma^2 P^2(1 - e^{-2kt})}\right\}^{\frac{1}{2}} \exp\left\{\frac{k(\ln u)^2}{\sigma^2(1 - e^{-2kt})}\right\} \qquad (22)$$

where

$$u = \left(\frac{P}{P^e}\right)\left(\frac{P^e}{P^o}\right) e^{-kt}$$

The mean of P at time t is

$$<P(t)> = P^e \left(\frac{P^o}{P^e}\right)^{\exp(-kt)} \exp\left\{\left(\frac{\sigma^2}{4k}\right)(1 - e^{-2kt})\right\} \text{ which as } t \to \infty = P^e \exp\frac{\sigma^2}{4k}$$

Which of the 'special' cases is more 'correct' is an important question. Assuming the pure expectational theory of forward pricing to be valid, one could use the term-structure of commodity markets to answer that question.

7. The Multimarket Situation

In a many-commodity situation, equation (9) becomes

$$\frac{\left(\frac{dP_i}{P_i}\right)}{dt} = \frac{d\ln P_i}{dt} = H_i(Q_i(P_1, P_2, \ldots, P_n)) \qquad (24)$$

with $H_i' > 0$, $H_i(0) = 0$ and Q_i being the excess demand function for commodity i in terms of all prices. If the vector $|P^e|$ represents equilibrium, expanding (24) around equilibrium

$$\frac{d\ln P_i}{dt} = H_{i_e}' \sum_{j=1}^{n} a_{ij}{}^e (P_j - P_j^e) \qquad (25)$$

where $\lambda_i = H_{i_e}' > 0$, and $a_{ij}^e = \left(\frac{\partial Q_i}{\partial P_j}\right)_{P_j = P_j^e}$

The equation system (25) will be stable if the real parts of the roots of the characteristic equation

$$|a - \bar{\lambda} I| = 0 \qquad (26)$$

are negative. (75)

As before, let us assume the stability conditions. Following the single market case, let

$$\left(\frac{\partial Q_i}{\partial P_i}\right) = -r < 0$$

Let $p_j = P_j - P_j^e$. Equation (25) becomes

$$\frac{d\ln P_i}{dt} = \lambda_i rP_i^e - \lambda_i rP + \lambda_i \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}^e p_j \qquad (27)$$

or

$$\frac{dP_i}{dt} = \lambda_i rP_i^e P_i \left(1 - \frac{P_i}{P_i^e}\right) + \lambda_i P_i \sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}^e p_j \qquad (28)$$

Equation (28) resembles the Volterra-Lotka equation for the growth in populations of n interacting species. If the a_{ij}^e matrix were antisymmetric, it would correspond to a prey-predator system. But without detailed knowledge about the a^e_{ij} matrix, we could only say that (28) represents an interacting system.

As before, let us represent the influence of unspecified variables by adding a noise term to the right hand side of the dynamic equation. Equation (28) becomes

$$\frac{dP_i}{dt} = \lambda_i r P^e_i P_i \left(1 - \frac{P_i}{P^e_i}\right) + P_i \left\{ \lambda_i \sum_{\substack{j=1 \\ j \neq i}}^{n} a^e_{ij} P_j + U_i(t) \right\} \quad (29)$$

where $U_i(t)$ represents a Gaussian noise input.

When n is large and when each P_j and hence P_j is influenced by a differential equation such as (29) with an independent noise input, then, by central limit theorem, one can represent the terms within square brackets on the right hand side of equation (29) by a single noise term with incremental mean m and incremental variance σ^2. (26) Dropping subscripts, equation (29) becomes

$$\frac{dP}{dt} = \lambda r P^e P \left(1 - \frac{P}{P^e}\right) + P(m + \sigma i(t)) \quad (30)$$

where i(t) is a standard Brownian noise.

If we let

$$k = \lambda r P^e + m$$

and

$$K = P^e\left(1 + \frac{m}{\lambda r P^e}\right)$$

$$= \frac{P^e k}{\lambda r P^e}$$

equation (30) becomes

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) + \sigma P i(t) \quad (31)$$

which is exactly similar to equation (14) considered above. Thus a similar discussion to the one made above could be made with similar conclusions.

The multi-market analysis leads to some equations of interdependent price behaviour that resembles some equations arrived at in biology for representing price behaviour. Our next chapter will have more to say on biological population models and some of their applications to studying price behaviour.

8. Summary

The above analysis suggests that the logistic equation of price dynamics may be a more suitable description of price dynamics - as it is able to sustain unspecifiable and random influences and yet exhibit reasonable price behaviour. The behaviour of prices in multiple market situations leads to results similar to the single market case.

The way the above model considers price changes relates closely to the geometrical propagation of biological populations - as a result of natural birth rates and death rates. Stocks of goods are similarly consumed and grown and hence price - determined by the prices of supply (birth) and demand (death) thus exhibits the cyclical movement of numbers observable in my normal species. The interdependence of the market place may also be seen as similar to the interdependence of the species in biology. The progress in this manner of viewing populations made in mathematical biology could be related with benefit to the examination of prices.

CHAPTER 3

SOME APPLICATIONS OF BIOLOGICAL POPULATION MODELS TO THE PROBLEM OF PRICE BEHAVIOR

1. Introduction

In this chapter, we examine certain applications of dynamic biological population models to the problem of spot price behavior. Some applications of ecological models to economic problems have been witnessed recently in economic literature. Samuelson (58) has applied a mathematical model of biological population behavior developed by Lotka (40) and Volterra (72) to derive a universal mechanism for explaining the business cycle. Conservative oscillations are deduced by posititing competition and ignoring limitations imposed by scarcity of resources, and by applying a prey-predator model developed by Volterra and Lotka. Generalisations to more than two species are capable of solutions through applications of the mathematical language of classical statitistical mechanics. However, a recognition of limited space and inorganic matter makes such generalisations inapplicable. (57) Introduction of simple diminishing returns leads to damped motions that are kept cyclically alive by shocks of the weather and exogeneous elements. Applying to the stock market, Kerner's (35) treatment - using the thermodynamic constructs of canonical ensembles - of the Volterra-Lotka system of equations for describing the movements of interactive populations, La Violette (38) proves a strong evidence of predator-prey behavior in stock-market trend-fluctuations.

In the following pages, we will be developing, on the basis of certain economic assumptions, models of price behavior. The equations that stem from the assumptions resemble the equations that describe population behavior in an interactive ecological system. It is clear that such an analogy may seem unwarranted. But we feel that, starting from this simple one market analogy, one can develop a more detailed dynamic equilibrium model of a complex, interactive economic system and the strides made in the ecological field can be of great help in this development; moreover, the biological models are particularly useful in analysing situations where unforseeable variables - which could be considered random - play an important role.

2. The Prey-predator system:

There is a system of equations of the Volterra-Lotka kind (16) in biology known as the pre-predator system. The equations are

$$\frac{ds}{dt} = jS - kSD$$

$$\frac{dD}{dt} = DS - mD \qquad (1)$$

In (1), S represents the population of the prey species, which in time dt gains a number jsdt in its population due to birth - but loses due to predation by the predator species D, a number kSDdt in the same duration. The predator, D, in the same period gains an amount proportional to the loss of S it predates upon i.e. DSdt but loses due to death the amount mDdt. The system has an equilibrium point, where,

$$\frac{ds}{dt} = \frac{dD}{dt} = 0$$

and from (1),

$$D^e = j/k \quad ; \quad S^e = m/ \quad .$$

where D^e and S^e are the equilibrium populations.

If we let,

$\varepsilon = D - D^e$ and $\eta = S - S^e$, the deviations from the equilibrium, the dynamics of population change as represented in (1) can now take the form ,

$$\frac{d\eta}{dt} = -k\,(S^e + \eta)\,\varepsilon$$

and $$\frac{d\varepsilon}{dt} = (D^e + \varepsilon)\,\eta \qquad (2)$$

If we consider points close to equilibrium only and neglect the non-linear terms, we get,

$$\frac{d\eta}{dt} = -k\,\frac{m}{\ell}\,\varepsilon$$

and $$\frac{d\varepsilon}{dt} = \frac{\ell j}{k}\,\eta \qquad (3)$$

3. <u>Economic model 1:</u>

Assume the following -

(1) At any time, the producer suppliers of a commodity adjust the intended output of a commodity at a rate proportional to the difference between prevailing price and felt equilibrium price i.e. the price at which, etc. If S is the supply, S^e is equilibrium supply, P is the price and P^e the equilibrium price, a mathematical representation of this assumption,

$$\frac{dS}{dt} = \lambda_1 (P - P^e) \quad \lambda_1 > 0 \qquad (4)$$

Let η denote $(S - S^e)$; then

$$\frac{d\eta}{dt} = \lambda_1 (P - P^e)$$

(2) At any time, the rate of change of price is proportional to the negation of stress supply i.e.

$$\frac{dP}{dt} = \frac{d\ (P - P^e)}{dt} = -\lambda_2\ (S - S^e) \quad \eta_2 > o \tag{5}$$

$$= -\lambda_2 \eta$$

(3) At all times, demand D is a linear dealing function of price,

$$\text{i.e.} \quad D = a - bP \qquad b > o \tag{6}$$

or ε = excess demand = $D - D^e = -b\ (P - P^e)$

$$\text{and hence,} \quad \frac{d\varepsilon}{dt} = -b\,\frac{(P - P^e)}{dt} \tag{7}$$

4. <u>Simplification</u> :- Model (1) is a complete system. From that, the trajectories of S, D and P are solvable. We can also make the following modification of the model in order that it resembles the prey-predator system.

$$\frac{d\eta}{dt} = \lambda_1 (P - P^e) = \frac{-\lambda_1}{b}\,\varepsilon \tag{8}$$

$$\text{and since} \quad \frac{d(P - P^e)}{dt} = -\lambda_2 \eta\ ,$$

$$\text{therefore,} \quad \frac{d\varepsilon}{dt} = \lambda_2 b \eta \ . \tag{9}$$

Equation (7) and (8) form a simple differential equation system.

5. Interpretation.:

Model (1) is analogous to the prey-predator system considered above. With our economic axioms, it would be tantemount to viewing the market place as a struggle between the supply or producer side of the scenario - similar in its endeavour to the prey or victim of the jungle, and the demand - a consumer side of the market as similar in being to the predator in the jungle. In our symbolid notation in both models, S refers to the Supply and the prey; P refers to the demand and the predator. Close to equilibrium, both can be viewed as a relationship between the excess functions ε and η . The net growth of excess of the supply or prey species, η , is negatively proportional to the excess of the demand-predator species, as is both economically and biologically logical. The net growth of the demand or predator species ε is proportional to the excess of the supply species η .

The role played by price is obvious. In the animal kingdom, the prey-predator relationship is obvious. In economics, price hides the neatly viewable direct relationship between consumption and supply. The higher the supply with any given demand schedule the lower the price and hence the greater the quantity demanded. The higher the demand with any given supply schedule the higher the price and therefore the greater the quantity supplied.

6. Solution:

Differentiating (a),

$$\frac{d^2\varepsilon}{dt^2} = \lambda_1 b \frac{d\eta}{dt} = \lambda_1 \lambda_2 \varepsilon . \quad (10)$$

(10) represents a simple harmonic motion in ε . Assume that $\varepsilon = 0$ at $t = 0$. The solution to (10) is,

$$\varepsilon = A \sin \lambda_1 \lambda_2 t. \qquad (11)$$

Because y the way price relates to excess demand, the significance to price trajectory of the above is that,

$$P(t) = P^e - k \sin \sqrt{\lambda_1 \lambda_2}\, t \qquad (12)$$

where k is a constant.

Viewing the market place as a prey-predator system between S and D leads to C simple harmonic motion of price. i.e. to a cyclical price behaviour. But, to quote Samuelson, (there is at least one serious objection to a non-damped system. If on them were superimposed random influences - due to unspecified variables - the price vice explode!

A nearly cyclical price behavior is observed. By viewing the market as a prey-predator relationship in S and D, we do get an exact cycle in price. Many other explanations of cyclical price behaviour expectational and others - are possible. Ours is a new one - bringing in a biological analogy and trying to verify it by reference to economic theory. But like all results that predict a non-damped exact cycle, it is incapable of adjustment to noise. In the next model, by bringing in a simple-delay, we hope to arrive at a new hypothesis - which resembles an epidemic relationship in S and D - and thus leads to a damped harmonic in price - which is capable of adjustment to noise and still be reasonably stable. This is then a more exact model of the market mechanism.

7. Model (2):

If we alter the assumption (1) of the previous model to include a lagged response of the rate of change of supply i.e.

$$\frac{dS}{dt} = \lambda_1(P - P^e) + \lambda_3 \frac{d(P - P^e)}{dt} \qquad \lambda_1, \lambda_3 > 0 \tag{13}$$

then equations (13) and (5) represent a complete model.

In terms of η and ε , they become, after manipulation,

$$\frac{d\eta}{dt} = \frac{\lambda_1}{b} - \lambda_3\lambda_2\eta \tag{14}$$

and
$$\frac{d\varepsilon}{dt} = \lambda_2 b\eta$$

8. The Epidemic Model and its interpretation.

In biology there exists a second volterra-Lotke type relationship between two species S and D which leads to a similar set of relationships between the excess functions η and ε to that in (14). This is the "epidemic" model.

Here the growth functions of S and D are,

$$\frac{dS}{dt} = \mu - \alpha SD \tag{15}$$

and

$$\frac{dD}{dt} = BSD - rD$$

The set of equations (15) describe a two-species interaction system where S, the strek of susceptibles in an epidemic (or the supply ineconomics) is being continuously added to at a constant rate and consumed, due to attack by the infective species D (or the consumer species D), at a rate proportional to SD. The growth function of D is as in the last model.

To see how the equations (15) relate to equations (14), we need to consider the behaviour of (15) - which represent an epidemic relationship beween supply and demand, near equilibrium.

Let $\eta = S - S^e$ and $\varepsilon = D - D^e$, the general excess functions, (15) becomes,

$$\frac{d\eta}{ldt} = \alpha(D_e^2 - (D^e + \varepsilon) \quad (S^e + \eta)$$

and (16)

$$\frac{d\varepsilon}{dt} = \beta(D^e + \varepsilon)\eta$$

which near equilibrium and without the nonlinear terms, will lead to

$$\frac{d\eta}{dt} = \alpha D^{\ell} \eta - \alpha S^e \varepsilon$$

and (17)

$$\frac{d\varepsilon}{dt} = \beta D^e \eta$$

Equations (17) correspond exactly to the equations (14) model (2) - which was a result of a modified version of the set of axioms of model (1). Thus the epidemic model could be arrived at both via sound economic reasoning or via a straight postulate of an epidemic interdependence of S and D.

9. Solution:

Differentiating the second equation of (17),

$$\frac{d^2\varepsilon}{dt^2} = \lambda_2 b \frac{d\eta}{dt}$$

$$= \lambda_2 b \left(- \frac{\lambda_1}{b} \varepsilon - \lambda_3 \lambda_2 \eta \right)$$

But, $\eta = \frac{1}{\lambda_2 b} \frac{d\varepsilon}{dt}$

Therefore $$\frac{d^2\varepsilon}{dt^2} + \lambda_3\lambda_2 \frac{d\varepsilon}{dt} + \lambda_1\lambda_2\varepsilon = 0. \qquad (18)$$

(18) is a homogeneous, linear second order differential equation of the type,

$$\frac{d^2\varepsilon}{dt^2} + a\frac{d\varepsilon}{dt} + R\varepsilon = 0 \qquad (19)$$

which has its solution dependent on whether

$$a^2 \gtreqless 4R \text{ i.e. if } \lambda_2 \gtreqless \frac{4\lambda_1}{\lambda_3^2}$$

This is an empirical question. But the exponential solution (when a^2 4R) and the cyclical solution (when $a^2 = 4R$) could be ruled out a priori-because of their illogicality as models of price behaviour in that they would imply etc., so we could assert that the solution to (19) will take the form,

$$\varepsilon = A \quad e^{-\frac{at}{2}} \quad \frac{Ra^2}{2} t \qquad (20)$$

which represents a damped harmonic motion in ε and thus a damped sinusoidal motion in price. Thus the epidemic model of S and D interaction - which could also be derived as shown through economic reasoning.

This system is stable because both ε and $\eta \to$ zero and $P \to P^e$ ceteris paritons, as $t \to \infty$ and is thus capable of non explosive price behaviour. If extraneous influences like weather impinge on the market as random inputs, this system is capable of some behaviour. Such lag or dissipative damping is needed to ensue an ergodic state under shock effects.

10. Summary.

In this chapter, we understand that it is possible to explain cyclical price behavior observed in the market place through a biological interpretation. The analogy of a predator-prey system representing supply-demand interaction leads to an undamped cyclical price behavior (which is untenable). The introduction of economic lags leads to the possibility of an analogy to the epidemic model studied by biologists and leads to a damped harmonic motion of price. This is also tractable with stochastic noise input representing unspecified and unforeseeable influences. These can be extended to multiple market situations and to an understanding of the trade cycle and its possible correctors. More empirical study is needed.

CHAPTER 4

SOME APPLICATIONS OF THE TERM-STRUCTURE OF INTEREST RATES TO THE TERM-STRUCTURE OF COMMODITY PRICES (RISKS NEUTRALITY).

1. Introduction

In this chapter, we wish to discuss the term structure of commodity prices. For this purpose, we will be applying certain theories of the term structure of interest rates.

Let the price (spot) of a commodity at any time t be $x(t)$ and the price of a contract promising delivery of one unit of the same commodity τ units of time ahead be $P(\tau,t)$. The movement of $x(t)$ in the future is uncertain and hence the value of one unit of the commodity at time $(t + \tau)$ is uncertain as well.

2. Pure expectations hypothesis:

A simple model of the relationship between $x(t)$ and $P(\tau,t)$ would be to say that $P(\tau,t)$ represents the expected value at time $(t + \tau)$ of the random variable $x(t)$. A similar theory was proposed for the term structure of interest rates by I. Fisher (23) and was developed further by F. Lutz (41). The difficulty in testing this hypothesis is that the expectations of market participants are not directly observable. One either sees if the forward rates (prices) lead future short rates as T.J. Sargent (50) and others have done, or one uses the actual (ex-post) short rates for a given period in place of the rate expected (ex-ante) at some time earlier to prevail during the present period (as Culbertson (17) and others have done) in order to test the hypothesis. Neither is a correct test for the hypothesis itself.

To state the theory operationally, we need to state it entirely in terms of observable variables. It is clear that if one posits a stochastic dynamic model for x(t), then one can state the expected value of x(t+τ) in terms of x(t), an observable. For this purpose, we propose to use the well-accepted random walk hypothesis of price movement, which says that the generating process for x(t) is governed by the stochastic differential equation

$$dx/x = d\ln x = \alpha dt + \sigma dz \qquad (1)$$

where α is the incremental mean of the 'return' due to price change and and σ is the incremental variance of the same. Z is a process of independent Gaussian increments, i.e., $E(dz(t)dz(s)) = 0$ if $t \neq s$ $E(dz(t))^2 = dt$.

It can be shown that with the above dynamics, the expectation of the spot price τ periods from the present time in terms of the present value of x is (16)

$$E(x(t+\tau)) = x(t)e(\alpha+ \frac{\sigma^2}{2})\tau \qquad (2)$$

Applying the pure expectations hypothesis, it leads to the operationally varifiable equation that

$$P(\tau,t) = x(t)\ e^{(\alpha+ \frac{\sigma^2}{2})\tau} \qquad (3)$$

The truth of the random-walk hypothesis is independently verifiable and if true, α and σ estimatable from historic data. Then equation (3) is entirely in terms of measurables, as P(τ,t) and x(t) are directly observable at time t.

3. Yield and payment schemes:

But the above is a naive hypothesis. It does not take into account the payment and yield structures of the forward markets. We posit that as a market competing for funds with other commodity and financial forward markets, its profit or yield structure is identical with - (in the absence of risk, that is) - that of the riskless yield structure for capital - which is knowable at any time in the shape of the yield curve of the gilt-edged bond market. Let $y(\tau)$ be the 'yield' for a maturity of τ time units. This represents a relationship between cash flows in the present and in the future up to τ time units for the present, e.g., if $P(\tau,t)$ is the price of a bond at time t that promises to pay a coupon at the rate of $r(\tau)$ per period until τ periods from t and at time $(t+\tau)$ - the time of maturity - yields the nominal value of the bond $\bar{P}$, the yield $y(\tau)$ of the above payment scheme is given by the relationship (73)

$$P(\tau,t) = \int_0^\tau e^{-y(\tau)t} r(\tau)dt + e^{-y(\tau)\tau}\bar{P} \qquad (4)$$

The commodity forward market - by our assumption of riskless perfect competition - has the same yield structure as the gilt-edged market but a different payment scheme. If $P(\tau,t)$ is the price of a contract promising to deliver one unit of the commodity τ periods from t, the following payment scheme applies

1. A proportion f of $P(\tau,t)$ is payable at t.
2. There are no coupon payments
3. At time $(t+\tau)$ the remainder of the contracted price, i.e. $(1-f)$ $P(\tau,t)$ is payable and one unit of commodity of value $x(t+\tau)$ is receivable and immediately saleable at the same price. (We are assuming that there are no transaction costs, tax considerations, or differentials between buying and selling prices).

Now $x(t+\tau)$ is an uncertain variable at time t. But if the above payment scheme and the riskless yield for maturity , i.e., $y(\tau)$, is taken to be consistent with the expected spot price at $(t+\tau)$ - (one version of the expectations hypothesis), then, from equation (4)

$$f(P(\tau,t) = e^{-y(\tau)\tau}(E(x(t+\tau)) - (1-f)P(\tau,t)) \qquad (5)$$

or

$$P(\tau,t) = \frac{e^{-y(\tau)\tau}\,E(x(t+\tau))}{(f+(1-f)e^{-y(\tau)\tau})} \qquad (6)$$

Again, to make the expectations term in equation (6) operational, if we fall back on the dynamics of the random-walk, we can apply equation (2) to get

$$P(\tau,t) = \frac{x(t)e^{(\alpha+\frac{\sigma^2}{2} - y(\tau))\tau}}{(f + (1-f)e^{-y(\tau)\tau})} \qquad (7)$$

Once again, equation (7) is entirely in terms of measurables and observables - and is therefore testable.

4. The error learning model:

Next we wish to look at a modified form of the error-learning hypothesis, originated with respect to the term structure of interest rates, by Meiselman. (44)

At any time t, there is the spot price $x(t)$ and a forward price for delivery of goods τ time periods from t, i.e., $P(\tau,t)$. Now, at the next time unit, there will be the spot price $x(t+1)$ and a price for delivery of goods at $(t+\tau)$, a forward price $P((\tau-1),t+1)$. At time $(t+\tau)$, $P(0,t+\tau) = x(t+\tau)$. In the original form, the error-learning hypothesis has it that the forward price each period gets adjusted by a factor proportional to the error in forecasting the spot price. The model was in time-series form and its statement in the context of commoditiy pricing would be

$$P(\tau-1,t+1) - P(\tau,t) = \beta'(x(t+1) - x_1(t)) + \varepsilon_t \qquad (8)$$

where

β' : proportionally constant

ε_t : error-term; $E(\varepsilon_t) = 0$, $E(\varepsilon_t \varepsilon_s) = 0$ (if $t \neq s$) or $(\sigma')^2$ (if $t = s$) and $x_1(t)$ is the forecast of of $x(t+1)$ made at time t.

Meiselman made the above model a test of the unbiased expectations hypothesis. He posited that in so far as there was no constant term in addition to the terms already on the right hand side of the equation (8), proving the validity of the above model was tantamount to proving the unbiased expectations hypothesis. The truth or the fallacy of this statement (for a rebuff, see Wood (76)) or the empirical valididty of equation (8) (for results of tests, se Van Horne (71) and Grant (29)) does not concern us here - for we are interested in quite a different statement of the above hypothesis. We make the following modifications:

1. We posit that learning takes place in continuous time.
2. Since the forward rate $P(\tau,t)$ tends to $x(t)$ as $\tau \to 0$, (and to make the hypothesis independent of the nature of the forecasting procedure), we state that changes in forward price are proportional to changes in spot price. In Meiseman's model in continuous time, this really implies that the best estimate for next moment's price is this moment's price. (49)
3. We introduce a discount factor dependent on the maturity τ as the proportionality factor β' - i.e., the discounted absolute value of changes of price for all maturities are the same, and equal to the change in spot price, i.e., $\beta' = e^{r\tau}$ where r is the discount factor.

The hypothesis can now be stated as

$$dP = e^{r\tau}\, dx + \sigma'\, dz' \qquad (9)$$

where dz' is similar to dz defined earlier.

Now instead of resorting to forecasting and the errors in it, to get the r.h.s. in equation 9, we define the dynamics of dx as before, (the random-walk assumption in equation (1)), i.e.

$$dx/x = d\ln x = \propto dt + \sigma dz \qquad (1)$$

Because of equation (9) and because of the terminal condition $P(\tau=0,t) = x(t)$, we see that P is FUNCTIONALLY dependent only on x and τ. The other parameters in the relationship are constants like r, $\propto$ or σ. From stochastic calculus, if P is a function of two variables, x and τ(symbolisable as $P(x,\tau)$), and if one of them, in this case, x, is stochastic, with its generating equation as given in equation (1), then the total differential of P is given by (34)

$$dP = \frac{(\partial P)}{\partial x}\, dx - \frac{(\partial P)}{\partial \tau}\, dt + \frac{1}{2}\, \sigma^2 x^2\, \frac{(\partial^2 P)}{\partial x^2}\, dt \qquad (10)$$

Equation (10) is so because (a) $d\tau = -dt$ and (b) $(dz)^2$ has the dimension dt. (37)

Substituting for dx in equations (9) and (10) and replacing equation (9) in equation (10) in place of dP, we find that on taking expectations on both sides, we get

$$\frac{(\partial P)}{\partial \tau} = \frac{(\partial P)}{\partial x} \propto x + \frac{1}{2}\, \sigma^2 x^2\, \frac{\partial^2 P}{\partial x^2} - \propto e^{r\tau} x \qquad (11)$$

The initial condition for equation (11), which is a parabolic 2nd order partial differential equation is, as we saw earlier,

$$P(x, \tau = 0) = x \qquad 0 \leq x \leq \infty \tag{12}$$

The solution of the above is (79)

$$P(\tau,t) = x(t) \frac{e^{\alpha r}}{(\alpha + r)} (2\alpha + r - \alpha e^{-(\alpha + r)\tau}) \tag{13}$$

5. Conclusions.

Here we have derived some directly testable statements using continuous time analysis of three well known term structure theories and developed them in the context of commodity markets using stochastic calculus. The theories are the simple expectations hypothesis, with the yield and payment scheme of the market place and the error - learning hypothesis. We have not reported here the similar approach to the fourth - the preferred habitat scheme, as this has been insufficiently developed here. Both the stochastic analysis and the continuous time perspective are original contributions. With the random walk assumption of spot price behaviour, the model is directly operational and testable. The models as stated so far in the low market context (84) have been period studies and econometric in nature. The nature of all econometric analysis is their imprecise statements and lack of direct verifiability; secondly they demand 'expectations' data which are not satisfactorally obtainable. Our approach avoids this difficulty and states the theorem in purely observable variables - as price data or such historically measurable and averageable quantities as the random walk coefficients. We have not reproduced our empirical results - which were fruitless. However our contribution in these areas has been the stochastic formulation with continuous time analysis which merely puts the theories described into firm, directly testable forms.

CHAPTER 5

THE TERM STRUCTURE QUESTION UNDER RISK AVERSION

1. Introduction.

The purpose of this chapter is to put forward a new view about two standard hypotheses about the term structure of commodity prices.

2. Assumptions.

1. There are no taxes, transaction costs or problems with indivisibilities.
2. Buying and selling prices are the same (this might in fact follow from (1), i.e., absence of transaction costs).
3. Trading takes place in continuous time.

3. Purpose.

The two theories we will be looking at are :

1. pure expectations hypothesis;
2. risk-premium hypothesis.

Our purpose is one of putting these in an easily testable form and (a) to derive control parameters by which the central supplier can decide on the amount of supply of different instruments to get the desired yield curve, and (b) to get equilibrium curves - with a speculative supply of instruments. Actually, we will study these supply/control ideas mainly with respect to the second (risk premium) hypothesis - as it is the commonest and strive merely to put the first one in a testable format.

4. Characteristics of the market.

At any time t, there are N instruments i, i = 1 to N; each has a time to maturity τ_i, and without loss of generality $\tau_{i+1} \geq \tau_i$, for i = 1 to N - 1. Each has a price P_i or $P(\tau_i)$; interest payments are at the rate r_i or $r(\tau_i)$ per unit per stock until maturity. The current price is P.

5. Price Dynamics:

Out starting point will be an assumption whose ultimate validity must be based on empirical evidence. We hypothesise that the prices of all instruments follow an N-dimensional log normal random-walk in continuous time - i.e., the dynamics of each P_i in time is:

$$d\ell n\ P_i = dP_i/P_i = \mu_i\ dt + \delta(\tau_i)\ dq\ (t,\tau_i) \qquad (1)$$

where $\mu(\tau_i)$ or μ_i is the instantaneous expected mean, $\delta^2(\tau_i)$ is the expected variance and $dq(t,\tau_i)$ is a standard Gauss-Wiener process for maturity τ_i. The multi-dimensionality comes from positing that the dq's for different maturities at any time are correlated with

$$\varepsilon(dq(t,\tau_i)dq(t,\tau_j)) = \rho_{ij}dt \quad (\text{with } \rho_{ii} = 1) \qquad (2)$$

ε : expectation operator

It is clear from the above that dq has a dimension of dt in the mean square. This is a characteristic of the Gauss-Wiener process (a characteristic which we will soon exploit). However, we assume that there is no serial correslation among the unanticipated returns (dp/P) of any of the assets, i.e.,

$$\left.\begin{array}{llll} \text{(a)} & dq(s,\tau_i)\ dq(t,\tau_j) = 0 & \text{for } s \neq t & \text{for all } i,j \\ \text{(b)} & dq(s,\tau_i)\ dz(t) = 0 & \text{for } s \neq t & \text{for all } i \end{array}\right\} \quad (3)$$

where dz is a std. Gaussian noise. Equations (3) follow from the several efficient market hypotheses of Fama (20) and Samuelson (52). If τ_i is continuous, $P(\tau = 0) = \bar{P}$ with no uncertainty, i.e. $\delta(\tau = 0) = 0$. Thus $\delta(\tau)$ depends on τ. But otherwise δ is a nonstochastic function of τ only (i.e., is independent of P).

6. Certainty Model (aside)

Before we go head long into our different hypothesis, let us first consider what happens under certainty; assume too that there are no risks. Then, under our assumptions, the return over any interval must be identical for all instruments. Take an interval dt; without risk the only return over dt is via capital gains. Let G(i) be the return by capital gain alone with instrument i over dt

$$\begin{aligned} G(i) &= dP_i/P_i \\ &= \mu_i dt \text{ (from 1)} \\ &= -\mu_i d\tau \text{ (for } dt = -d\tau) \end{aligned} \qquad (4)$$

which is the same for all i.

i.e., μ_i must be the same for all i; let $\mu(\tau_i) = R$ for all i. Then from equation (1)

$$\begin{aligned} dP(\tau)/P(\tau) &= Rdt \\ &= -Rd\tau \end{aligned} \qquad (5)$$

Integrating

$$\int_{\bar{P}}^{P(\tau)} dP(\tau)/P(\tau) \quad = \quad -R\int_{o}^{\tau} d\tau$$

(6)

or $$P(\tau) \quad = \quad e^{-R\tau}$$, where is the spot price.

will be the term structure that follows from certainty (and no coupons) and this is a '<u>flat</u>' yield curve. For, if $y(\tau)$ is the yield for maturity τ, $f(\tau)$ are coupons, y related to P by

$$P(\tau) \quad = \quad \int_{o}^{\tau} e^{-y(\tau)t}\, r(\tau)\ dt + e^{-y(\tau)\tau}\bar{P} \qquad (7)$$

In the above case, $r(\tau) = 0$; that $y(\tau) = R$ for all is the result of certainty.

<u>Market homogeneity</u>

We posit that the market has one common view point for the values of $_i$ and $_i$ for all i and for all $_{ij}$. We can rationalize this as the point of view of one rational representative investor. This saves us aggregation problems.

7. <u>Hypothesis No. One - Pure Expectation again</u>

This holds that with coupon payments and uncertainty of prices, the certainty model provides a reasonable approximation <u>if</u> it is modified now to say that the <u>expected</u> return over any interval is identical for all instruments. Let us see what this leads to. Over a period dt, any instrument i can now have a coupon return and capital gain summing to R(i,dt), equalling

$$R(i,dt) = dP_i/P_i + \frac{r(\tau_i)}{P(\tau_i)}\, dt \qquad (8)$$

and from (1)

$$\varepsilon(R(i,dt)) = \mu_i dt + \frac{r(\tau_i)dt}{P(\tau_i)}$$

$$= (\mu_i + \frac{r(\tau_i)}{P(\tau_i)})dt \qquad (9)$$

where (ε : expectation operation).

Thus the hypothesis has it that the sum $(\mu_i + \frac{r(\tau_i)}{P(\tau_i)})$ must be equal at any time for all i. At any time, all P's and r's are observable. μ_i are known from history. Thus testing the constancy of

$$(\mu_i + \frac{r(\tau_i)}{P(\tau_i)})$$

is easy enough at any time, the test gaining more significance the larger the number of instruments. Thus we have here an easily testable version of the expectations hypothesis.

8. Hypothesis No. Two - Risk Aversion

We mean to derive a version of the Sharpe risk premium model of capital asset pricing - but by a different route. (5)

Assume that an investor faced with an uncertain return R over a period of time has the following utility function (Tobin (68) and Markowitz (43)).

$$u(R) = R - bR^2 \qquad (10)$$

where U(R) is the utility of return R.

His expectation of u(R) is,

$$\varepsilon(\mu(R)) = \varepsilon(R) - b\varepsilon(R^2) \qquad (11)$$

Let $E = \varepsilon(R)$

and $S^2 = \varepsilon(R - E)^2$

$= \varepsilon(R^2) - E^2$

Then

$$\varepsilon\,(u(R)) = \varepsilon(R) - b\varepsilon(R^2)$$

$$= E - b(E^2 + S^2) \qquad (12)$$

But our investor has a choice of N instruments and his problem is one of division of his wealth between these. Let us say that he devotes a fraction Wi of his wealth to instrument i, s.t. $\sum_i W_i = 1$. Now, over period dt, the return on instrument i, is, as we saw from equation (8)

$$R(i,dt) = dP_i/P_i + r_i/P_i$$

$$= (\mu_i + \frac{r_i}{P_i})\ dt + \delta_i\ dq(t,\tau_i) \qquad (13)$$

and the investor's net return R over dt is,

$$R(dt) = \sum_i W_i\ R(i,dt) \qquad (14)$$

Let us apply equation (12) to equation (14)

$$E = \varepsilon(R(dt)) = \varepsilon \sum_i W_i (R(i,dt)$$

$$= \sum_i W_i \varepsilon(R(i,dt)$$

$$= \sum_i W_i (\mu_i + \frac{r_i}{P_i})\ dt, \text{ (from equation (13)} \qquad (15)$$

$$E^2 = \sum_i \sum_j W_i W_j (\mu_i + \frac{r_i}{P_i})(\mu_j + \frac{r_j}{P_j})\ (dt)^2 \text{ (from equation (15))} \qquad (16)$$

$$S^2 = \varepsilon (R - E)^2$$

from equations (13), (14), and (15)

$$= \varepsilon \sum_i \sum_j W_i W_j \ (R(i,dt) - \varepsilon(R(i,dt)))$$

$$= \sum_i \sum_j W_i W_j \delta_i \delta_j \ dq\ (t,\tau_i)\ dq(t,\tau_j)$$

which, from equation (2) (Gauss-Wiener theory)

$$= \sum_i \sum_j W_i W_j \rho_{ij} \delta_i \delta_j \ dt \qquad (17)$$

Thus,

$$\varepsilon(u(R)) = E - b(E^2 + S^2)$$

$$= \sum_i W_i \ (\mu_i + \frac{r_i}{P_i})\ dt$$

$$- b \sum_i \sum_i W_i W_j \ (\mu_i + \frac{r_i}{P_i})(\mu_j + \frac{r_i}{P_j})(dt)^2$$

$$- b \sum_i \sum_j W_i W_j \ \rho_{ij} \delta_i \delta_j \ dt \qquad (18)$$

Thus (u(R)) is a function of the period over which R is reckoned. Let us posit that the investor wishes to maximise $\varepsilon(u(R))$ over an infinitesimal dt - or which is the same thing - he wishes to <u>maximise</u>

$\frac{d(\varepsilon(u(R))}{dt}$ or the rate of change $\varepsilon(u(R))$. Now,

$$\frac{d(\varepsilon(u(R))}{dt} = \sum_i W_i \left(\mu_i + \frac{r_i}{P_i}\right)$$

$$- b \sum_i \sum_j W_i W_j \delta_i \delta_j \rho_{ij} \quad (19)$$

(The second term containing $(dt)^2$ drops out).

Thus, although $\varepsilon(u(R)) = E - b(E^2 + S^2)$

$$\frac{d\varepsilon(u(R))}{dt} = \frac{dE}{dt} - b \frac{dS^2}{dt} \quad (20)$$

$$\text{Thus max. } \frac{d\varepsilon(u(R))}{dt} = \text{max. } dE/dt - b\, dS^2/dt \quad (21)$$

$$= \text{max. } \sum_i W_i \left(\mu_i + \frac{r_i}{P_i}\right)$$

$$- b \sum_i \sum_j W_i W_j \mu_{ij} \delta_i \delta_j \quad (22)$$

subject to $\sum_i W_i = 1$

In the lagrangian form equation (22) becomes,

$$\text{Max.}_{W_i} L = \sum_i W_i \left(\mu_i + \frac{r_i}{P_i}\right) - b \sum_i \sum_j W_i W_j \rho_{ij} \delta_i \delta_j \quad (23)$$

$$+ \lambda\left(\sum_i W_i - 1\right)$$

Taking the partial derivative of L, w.r.t. W_i in equation (23), we get

$$\partial L/\partial W_i = \left(\mu_i + \frac{r_i}{P_i}\right) - b \sum_j W_j \rho_{ij} \delta(i) \delta(j) \quad (24)$$

$$+\lambda = 0$$

As we saw earlier, the expected rate of return for instrument i over dt is

$$X(i) = (\mu_i + \frac{r_i}{P_i}) \qquad (25)$$

Let us posit that the risk premium for instrument i is denoted by Z(i). Equation (24) implies that

$$Z(i) = \sum_j W_j \rho_{ij} \delta_i \delta_j \qquad (26)$$

Thus $$X(i) - b Z_i = -\lambda \qquad (27)$$

= certainty equivalent yield for all i.

This is the interpretation of the Lagrange multiple; thus equation (27) is a statement of the Sharpe capital asset pricing model.

Testing equation (27) is now a simple matter. We do not have the problems of White (73) of estimating P_i^E, (In his notion, P_i^E is the expected value of P_i) with complicated risk-premia. In our case, at any time, W_j's, r_i's and P_i's are known and $\mu_i, \delta_i, \rho_{ij}$ are estimated from history. Once we calibrate for b and λ, over two instruments, we can check their validity for the rest at any time.

9. Control: of the supply of instruments

Equation (23) can be used for 'control' as well. Let us first put it in a 'vector' form.

Let $\bar{S}$ denote the vector of supply of instruments

Let $\bar{P}$ denote the vector of prices

Let $\bar{W}$ denote the vector wealth-fraction

$$W_j = P_j S_j / \sum_j P_j S_j$$

Let $\bar{\mu}$ denote the vector of returns

Let $\bar{r}$ denote the vector of coupons

Let $\bar{i}$ denote the vector of unity

Let $\bar{R}$ denote the vector (r_i/P_i)

Let V denote the matrix $(\rho_{ij}\delta_i\delta_j)$

Equation (23) becomes, with the above notation

$$\text{Max } L = (\bar{\mu} + \bar{R})' \bar{W} - b \bar{W}' V \bar{W} + \lambda(\bar{i}' \bar{W} - 1) \qquad (25)$$

Differentiating w.r.t. $\bar{W}$ and λ, we get

$$(\bar{\mu} + \bar{R}) - 2b V \bar{W} + \lambda \bar{i} = 0$$

and

$$\bar{i}'\bar{W} - 1 = 0 \qquad (26)$$

Solving for $\bar{W}$, we get

$$\bar{W} = \frac{1}{2b} \{V^{-1} - \frac{V^{-1} - \bar{i}'\bar{i}\, V^{-1}}{\bar{i}'V^{-1}\,\bar{i}}\} \quad (\bar{\mu} + \bar{R}) + \frac{V^{-1}}{\bar{i}'\, V^{-1}\, \bar{i}} \qquad (27)$$

Thus given any desired term structure, the only unknown vector on the right hand side of equation (27), i.e. $\bar{P}$ becomes known and thus the vector $\bar{W}$ becomes known. But $\bar{W} = \frac{1}{W_o}\{\text{diag } \bar{P}\}\bar{S}$, where W_o is the total invested wealth. Thus

$$\bar{S} = W_o \{\text{Diag } \bar{P}\}^{-1} \bar{W} \qquad (28)$$

Thus the central supplier or state knows the correct amounts of supply of the instruments to achieve desired $\bar{P}$ (given that these are market clearing prices) given a knowledge of the total investible money supply.

10. Equilibrium term structure under speculative supply

If, instead, as normally in commodity markets, supply of instruments is fixed by speculation and

$$\bar{S} = \{k_i P_i\} \text{ as well, where } \bar{k} \text{ is the vector of slopes of supply}$$

$$= \text{Diag } \{\bar{P}\}\bar{k} \qquad (29)$$

Then from equation (28),

$$\{\text{Diag } \bar{P}\}\, \bar{k} = W_o \{\text{Diag } \bar{P}\}^{-1} \{\frac{1}{2b}(V^{-1} - \frac{V^{-1} - \bar{i}'\bar{i}V^{-1}}{\bar{i}'\, V^{-1}\, \bar{i}})\ (\bar{\mu} + \bar{R})$$

$$+ V^{-1}/\bar{i}'\, V^{-1}\bar{i}\} \qquad (30)$$

Thus, the $\bar{P}$ inherent in the solution of equation (30) represents the equilibrium price structure.

11. Summary.

The above have self evident implications. The combined assumptions of a joint random walk of prices and risk aversion lead easily to the Sharpe capital asset pricing model; what is more, under our analysis, it lends itself in an easily verifiable form. The second important consequence of the above - and this depends on the empirical success of the first - is that it enables the centre to control the supply of instruments to get the derived term structure effect with greater precision. As to the application for commodity prices, it perhaps leads to a formation for the first time, of the theoretically 'correct' term structure under speculative supply and this is in a 'testable' form. The amount of data required precluded one form conducting the verification that in all markets, supply is a result of a central authority plus speculation, could be investigated as a combination of the two.

In this chapter, we proceeded to understand the problem of the term structure of commodity prices based on the assumption of a representative investor, with a quadratic, risk-averting utility function, a speculative supply function and a random movement of prices - as may be expected in a healthy auction market. We derive both the risk premium - certainty equivalent and expected return form of the Sharpe Capital asset pricing model but also, based on a speculative supply of instruments - an equilibrium term structure of prices.

The test ability of the term structure model is different. Although testing has not been attempted here, the model is in a testable format.

The quadratic ability function as inherent in our approach leads to a model which depends on particular levels of wealth. If we employ the Merton-Samuelson approach (83) we could derive a model which is free of time. There, the particular fraction of wealth invested in risking assets is independent of the level of wealth. The testability of their model is free of the burden of averaging information over time - which we need to have.

Merton (82) in a recent work develops his ideas further to produce an inter-temporal, continuous time asset pricing model. He also uses a random walk of price behaviour too. His analysis is based on continuous time analysis, stochastic calculus - and their application which is different from the simpler maximisation of expected utility and the further Lagrangian analysis we have employed.

The approach to the term structure of commodity prices here is new. The application of the CAPM to this problem too is new.

The CAPM rules out short sale. In our approach - this problem does not arise. Short sale is allowable in commodity markets. In the Merton approach, there is no restriction on short sale too.

CHAPTER 6

THE RISK-STRUCTURE OF COMMODITY PRICES UNDER RATIONALITY

1. Introduction

There have been so far a number of theories of the term structure of commodity prices. The purpose of this chapter is to present what may be called a theory of the risk structure of commodity prices.

In a seminal paper, Black and Scholes (15) presented a complete general equilibrium theory of option pricing which is particularly attractive because the final formula is entirely a function of observable variables. Therefore, the model is subject to direct empirical tests, which is then (14)performed with some success. Merton (45) clarified and extended the Black-Scholes model. While options are highly specialised and relatively unimportant financial instruments, both Black and Scholes (15) and Merton (46) recognised that the same basic approach could be applied for assets in general. Black (13), in a recent paper, has applied the option pricing principles of his original paper with Scholes (15), to the problem of commodity option pricing. He also derives formulae for the values of forward contracts in terms of future prices and other variables. (See also (10))

We are concerned with arriving at a testable theory of the term structure of commodity prices based on some of the above work.

Consider the forward price quotation at time t for actual delivery of goods at time $(t+\tau)$. Let us denote the forward price by $F(t,\tau)$. Let the spot price at time t be denoted by $x(t)$. At the next period, we will have a new spot price $x(t+1)$ and for the same goods to be delivered at time $(t+\tau)$, a new forward price $F(t+1, r-1)$. Thus we will have a series of spot prices $x(t)$, - - $x(t+n)$ - - $x(t+\tau)$ and a series of forward prices $(F(t,r), -- F(t+n, t-n), -- F(t+\tau,0)$. At time $(t+\tau)$, by definition, $F(t+\tau,0) = x(t+\tau)$. In this paper, we wish to derive a general relationship between these two series which converge at maturity (53). (also (12))

2. Assumptions

1. Trading takes place in continuous time.
2. There are no restrictions on short sale.
3. There are no transaction costs and buying and selling prices are identical.
4. The term structure of interest rates is known with certainty as a function of maturity τ; let the certain interest rate functional be $R(\tau)$.
5. The forward goods are buyable on margin. I.e., if the forward price at t for maturity τ is $F(t,\tau)$, it requires a down-payment at t of kF, $(0 < k < 1)$, with the remainder payable on maturity at $(t+\tau)$. It is clear that with continuous discounting, the effective price of the forward goods at t is $kF + (1-k)Fe^{-R(\tau)\tau}$ where R is the interest rate. Let us denote this effective price by W.

3. The Derivation of an analytical term-structure

Let us posit that the spot prices are generated by the general stochastic differential equation

$$dx = \alpha(x,t)\,dt + \sigma(x)\,dz \qquad (1)$$

where $\alpha(x,t)$ is the instantaneous change in x as a function of x and t; $(\sigma(x))^2$ is the instantaneous variance of the change as a function of x; dz is the standard Gauss-Wiener process. We wish to derive a relation between $F(t,\tau)$ and $x(t)$. Let us posit this functional dependence we are seeking by denoting $F(t,\tau)$ as $F(x,\tau)$. Thus F is functionally dependent on the variables x and τ - and other parameters presumed to be constant. With this functional representation, from Ito's lemma, we can write the stochastic differential equation for F as (34)

$$dF = F_x dx + \tfrac{1}{2} F_{xx} (dx)^2 + F_\tau d_\tau \qquad (2)$$

which, from (1) and from the fact that $dt = -d\tau$ and $(dz)^2 = dt$ and after omitting terms with powers of dt higher than one, becomes

$$dF = \{F_x \alpha(x,t) + \tfrac{1}{2} F_{xx} \sigma^2(x) - F_\tau\} dt + F_x \sigma(x) dz \qquad (3)$$

and since the effective price of forward holdings , W, is

$$W = F(k + (1 - k)e^{-R(\tau)\tau}) \qquad (4)$$

then

$$dW = (k + (1 - k)e^{-R(\tau)\tau}) dF + (1 - k)e^{-R(\tau)\tau} F(R(\tau) + \tau R'(\tau)) dt \qquad (5)$$

Now consider forming a portfolio consisting of spot, forward and riskless bond holdings, such that the net aggregate investment in the portfolio is zero. This is possible by financing long positions through short sale and borrowings. Let f_1 be the amount of investment in spot, f_2 in forward and f_3 in riskless bond holdings, with $f_3 = -(f_1 + f_2)$. Let (dRet) be the instantaneous return to the portfolio. Then

$$(dRet) = f_1 \frac{dx}{x} + f_2 \frac{dW}{W} + f_3 R(\tau) dt$$

$$= f_1 \left(\frac{dx}{x} - R(\tau) dt\right) + f_2 \left(\frac{dF}{F} + \frac{(1 - k)e^{-R(\tau)\tau}(R(\tau) + \tau \frac{dR}{d\tau}) dt}{(k + (1 - k)e^{-R(\tau)\tau})} - R(\tau) dt\right) \qquad (6)$$

$$= f_1\left(\frac{dx}{x} - R(\tau)dt\right) \qquad (7)$$

$$+ f_2\left(\frac{dF}{F} + \frac{((1-k)e^{-R(\tau)\tau}\tau R'(\tau) - kR(\tau))\,dt}{(k + (1-k)e^{-R(\tau)\tau})}\right)$$

which, from equations (1) and (3),

$$= \left\{\frac{f_1}{x}\ (\alpha(x,t) - R(\tau)x) + \frac{f_2}{F}(F_x\alpha(x,t) + \tfrac{1}{2}\sigma^2(x)F_{xx} - F_\tau\right.$$

$$\left. + \frac{((1-k)e^{-R(\tau)\tau}\tau R'(\tau) - kR(\tau))F}{(k + (1-k)e^{-R(\tau)\tau})}\right\} dt \qquad (8)$$

$$+ \left\{\frac{f_1}{x}\sigma(x) + \frac{f_2}{F}F_x\sigma(x)\right\}\ dz$$

Suppose the portfolio strategy $f_i = f_i^*$ is chosen such that the coefficient of dz is always zero and thus the return is non-stochastic. But, due to arbitrage, this certain return on a portfolio with zero net investment, can only be zero. i.e.

$$f_1^*\frac{\sigma(x)}{x} + \frac{f_2^*}{F}F_x\sigma(x) = 0 \qquad \text{(no risk)} \qquad (9)$$

$$\frac{f_1^*}{x}(\alpha(x,t) - R(\tau)x) + \frac{f_2^*}{F}\ (F_x\alpha(x,t) + \tfrac{1}{2}\alpha^2(x)F_{xx} - F_\tau$$

$$+ \frac{((1-k)e^{-R(\tau)\tau}\tau R'(\tau) - k\dot{R}(\tau))F}{(k + (1-k)e^{-R(\tau)\tau})}) = 0 \qquad (10)$$

(no certain return)

From (9), it follows that,

$$f_1^* = -\frac{f_2^*}{F}(xF_x) \qquad (11)$$

and, incorporating in equation (10) and simplifying,

$$\frac{f_2^*}{F} \quad (\tfrac{1}{2}\sigma^2(x)F_{xx} + R(\tau)x\,F_x + \frac{((1-k)e^{-R(\tau)\tau}\tau R'(\tau) - kR(\tau))F}{(k + (1-k)e^{-R(\tau)\tau})} - F_\tau) \qquad (12)$$

If we are to have a nontrivial solution wherein not all f_i^* are zero, then it follows that the expression in brackets in equation (12) equals zero. I.e.,

$$F_\tau = \tfrac{1}{2}\sigma^2(x)F_{xx} + R(\tau)xF_x + \frac{((1-k)e^{-R(\tau)\tau}\tau R'(\tau) - kR(\tau))F}{(k + (1-k)e^{-R(\tau)\tau})} \qquad (13)$$

Equation (13) is a second order parabolic differential equation of the function $F(x,\tau)$, with the initial condition, as we saw earlier,

$$F(x,\tau = 0) = x \qquad (14)$$

By transforming the spot-price variable into the new variable y, where $y = \ln x$ and seeing if the transformed function $F(y,\tau)$ is seperable in y and τ, i.e., of the form $F(y,\tau) = Y(y)T(\tau)$, (which it is), one finds that the solution to the differential equation (13) with the boundary condition (14) is (79)

$$F(x,\tau) = \frac{x}{(k + (1-k)e^{-R(\tau)\tau})} \qquad (15)$$

Of course, the above result could have been more easily arrived at from a simpler assumption that the forward and spot markets are priced in such a manner that the returns in all maturities are the same. Then

$$\frac{dx}{x} = \frac{dW}{W}$$

$$= \frac{dF}{F} - \frac{(1-k)e^{-R(\tau)\tau}(R(\tau) + \tau R'(\tau))d\tau}{k + (1-k)e^{-R(\tau)}} \qquad (16)$$

or

$$d\ln(F/x) = -d\ln\ (k + (1 - k)e^{-R(\tau)\tau}) \qquad (17)$$

At $\tau = 0$, $F/x = 1$. Integrating (17) from 0 to τ,

$$F(x,\tau) = \frac{x}{(k + (1 - k)e^{-R(\tau)\tau})} \qquad (18)$$

which is the same as the result (15).

4. Summary.

Under the 'rationality' assumption of pure hedging, the risk terms disappear completely and we have, hence, a pure relationship between spot and forward prices. This would be valid if markets were perfect with numerous keenly anticipative participants and the investors were rational, risk-averting and logical mathematicians. These assumptions are never upheld in commodity markets with their sparesness of enthusiasm and their element of irrational, speculative, risk preferring conditions.

The rationality approach to hedging was first studied with regard to options by Black and Scholes. The forward contract is viewable as a call option (i.e. an option to buy ahead with a known price). Alexander (2) and Malkiel and Quandt (81) express in matrix form the content of a forward or a call option contract. Their approach to representing the logic of risk-bearing of the contract mechanism is beautiful. Call options - like forward Contracts - eliminate risk. As we pointed out, we borrowed our analysis from studies of the call option.

CHAPTER 7

Risky Asset Pricing Mechanism as a Control System

1. Introduction

In this chapter, we postulate that the price mechanism in risky asset markets can be looked upon as a control system and the price trajectories in time as a solution to an optimal control problem.

Let us first present the hypothesis graphically before proceeding with the mathematics. We hypothesise that a security market can be visualised as below:

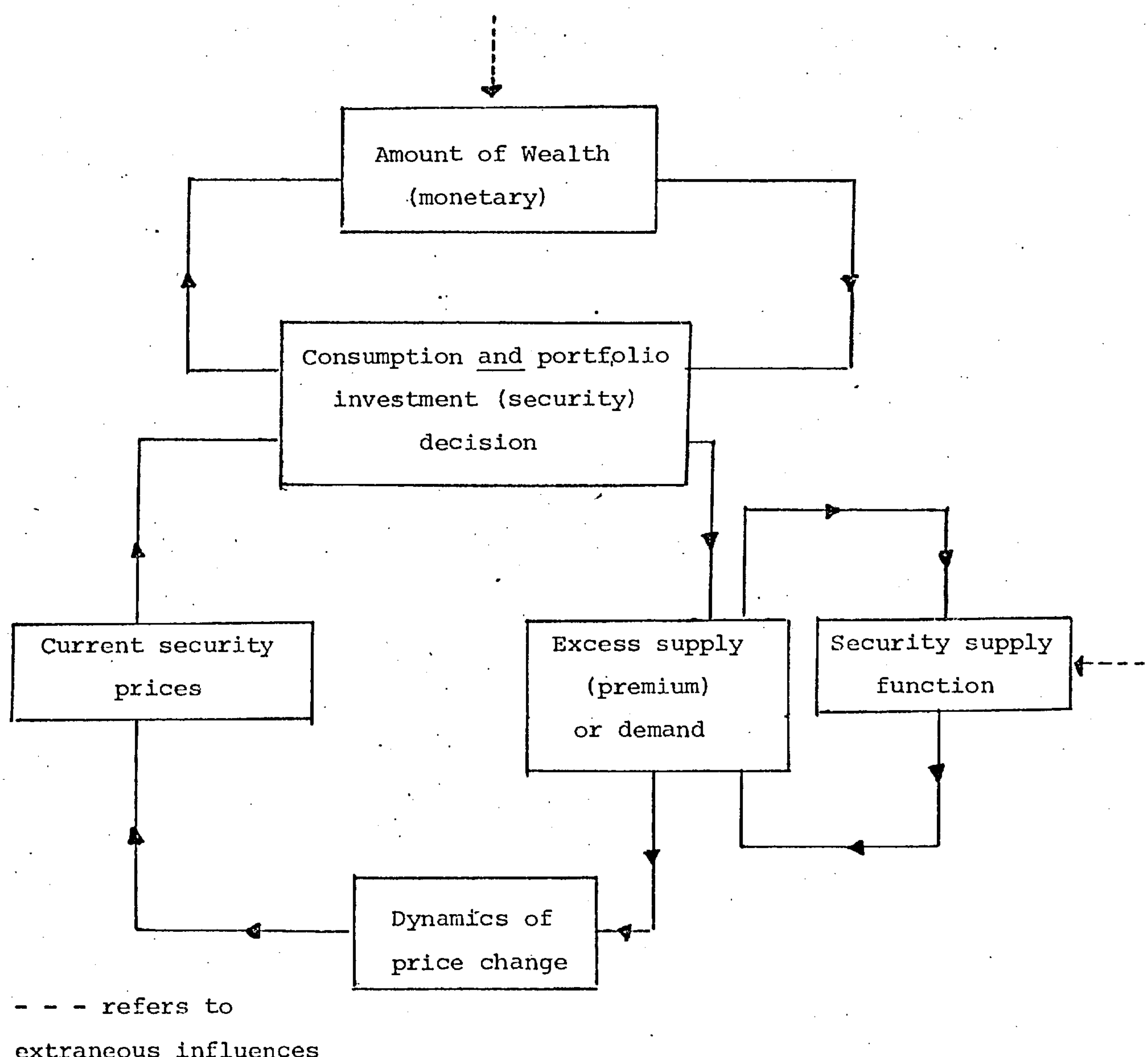

Fig. 1

2. Assumptions

1. Let us ignore the dotted lines and say that (a) the initial wealth W_o is known and wealth, at any time thereafter is governed entirely by the consumption decisions and the fortunes of the investment (in securities) decisions taken up to then from the time of beginning, and, (b) the supply function is known as a function of price and time. (We will release these assumptions later as a result of the nature of the solution we get for the simpler problem).

2. Consumption/savings and investment decisions are taken as to maximise (discounted, expected) utility over a horizon T (which can be ∞; from Bellman's optimality theorem, a subtrajectory of an optimal trajectory is still optimal over its horizon). The u tility function is a (ny) function of consumption. (11)

3. The dynamics of price change are a Walrarian function of excess physical demand. (This assumption too can be altered).

3. Objective Function

The variable we are interested in is price. Let us view this as the state variable. What is (are) the control variable(s)? As a simple beginning, let us say that there is only one investible security and it yields a certain interest, r. Then there are two uses only to money:

(a) consumption, and,

(b) investment in the safe security (return yielding).

If wealth at any time is W(t) and consumption C(t), then, because we assume that the investor is at all times constrained by the amount of wealth he began with at time t_o, his changes in wealth, consumption and return are at any time governed by the relation, (a tautology). (54)

$$dW/dt = W(t)\, r - C(t) \qquad (1)$$

(which in discrete time will be

$$C(t) = W(t) - \frac{W(t + 1)}{(1 + r)} \qquad (2)$$

With this constraint on C(t), the investor maximises the function

$$J = \text{Max}_{C(t)} \int_0^T e^{-\rho t}\, U(C(t))\, dt \qquad (3)$$

or incorporating the constraint

$$J = \text{Max}_{W(t)} \int_0^T e^{-\rho t}\, U(rW - dW/dt)\, dt \qquad (4)$$

where U : utility function

ρ : discount factor

T : horizon time

Equation (4) is a standard calculus of Variations problem and with any specific utility function we can work out the optimal trajectory for C(t) as a function of W(t), which in turn can be a function of Wo and t, (because of the certainty of return on the security). Thus, the control variable in this case is C(t) which could be worked out given the utility fn, (W – C) at any time going to buy the safe asset.

4. Risky Assets

Now, in addition to the safe asset, let us introduce another asset that makes every £1 invested in it at t return as £Z at (t + 1), where Zt is a random variable whose distribution

$$P(Zt \leq z) \quad = \quad P(z) \qquad (5)$$

is known.

Now the investor has another (control) decision. He must, as before, make the consumption/savings decision at any time. Now, in addition, he must also decide on the fraction wt of every £1 of investment (savings) that he should put in the risky asset, (1 - wt) going to buy the safe asset. The consumption constraint is now

$$C_t \;=\; W(t)\,(wt(Zt - 1) \;+\; (1 - wt)r) \;-\; \frac{dW(t)}{dt} \qquad (6)$$

$$0 \leqslant wt \leqslant 1 \quad \text{(no short sale)}$$

Again, the constraint could be introduced in the objective function J, which is now an expected (not absolute) maximum because of the stochasticity of Zt, as

$$J \;=\; \text{Max } (Ct, wt)\; E \int_0^T \;(1 + \rho)^{-t}\; U(C(t)) \qquad (7)$$

C(t) constrained by (6). Thus the maximum must be over two control variables C(t) and ω(t). Solution of (7) will lend to the trajectories (in terms of W(t)) of C(t) and W(t) . (6)

5. Control Theory

Before proceeding with the solution, it is perhaps appropriate to make clear the control terminology that we are using. In any optimal control problem, we have the following: (30)

1. Initial value of state variable (call it x) at to. I.e., x(to) = xo (known). x can be a vector.

2. Let $\bar{u}$ be the vector of control variables. The feasible region of $\bar{u}$ at any time must be known (and compact) and out of this region, we must choose one set of values.
3. Given x and $\bar{u}$ at any time, the rate of change of x, i.e., dx/dt must be calculable, i.e., $dx/dt = f(x, \bar{u}, t)$ where f is a known function. (f can be stochastic).
4. $\bar{u}$ at any time must be chosen (out of feasible values) so as to maximise an objective functional over a horizon T.

$$J = \text{Man } \bar{u}(t) \int_{to}^{T} I(x,\bar{u},t)\,dt \qquad (8)$$

where I is known as the intermediate function. (See (17))

6. Analogy to the market

Now, let us see if Fig. 1 represents a control problem.

(a) We know the initial conditions of price (of risky security) and Wo. Thus condition (1) is met. (Denote price by P).

(b) The feasible values of C(t) are given by (6) and for w(t) we have $0 \leq wt \leq 1$. Thus condition (2) is met for our $\bar{u}$ is (Ct,wt).

(c) The dynamics of price change are given by the Walrasian. This needs to be elaborated. Assume that as a solution to (7), we get C(t) as a function of (Wt,t) and wt as a function of (Wt,t). Then, wt(Wt - Ct) represents the amount of money that goes into the purchase of risky security. Let D be the physical demand; the PxD represents the money value of risky securities. Then

$$PxD = wt(Wt - Ct)$$

or $$D = wt/P\ (Wt - Ct) \qquad (9)$$

Let S(P,t) be the supply function. Then, from the Walrasian,

$$dP/dt = \lambda(D - S)$$

$$= \lambda(\frac{wt}{P}(Wt - Ct) - S(t,P))\ (\lambda\ 0) \qquad (10)$$

Since Ct and wt are known as functions of Wt, knowing Wo, the r.h.s. in (10) will be a random (because the risk of the return) function of Wo and t and P. Thus the dynamics are known and condition (3) is met.

(d) The control variables Ct and Wt are chosen so as to maximise (7) subject to (6). i.e., Our intermediate function is the utility function.

Thus in Fig. 1. , we have an optimal control problem.

7. Solution

The proper solution to (7) is to be found in the reference.(54) Here, we will merely indicate the process of solution. In discrete time, (6) becomes

$$Ct = Wt - \frac{W(t+1)}{(1 - w_t)(1 + r) + w_t Zt} \qquad (11)$$

where the objective functional

$$Jt(Wo) = \text{Max}_{(Ct, w_t)}\ E \sum_{t=0}^{T} (1+\rho)^{-t}\ U(Ct) \qquad (12)$$

This is a stochastic dynamic programming problem. Assuming that Zt's are independent, i.e.

$$E\,F(Zt) = \int_{-\infty}^{\infty} F(Zt)\ dP\ (Zt \mid Zt-1, \text{-----}\ Zo)$$

$$= \int_{-\infty}^{\infty} F(Zt)\ dP\ (Zt) \qquad (13)$$

one can solve (12) subject to (11) through Bellman's recursive equations to the relations

$$\left.\begin{aligned} Ct^* &= g_{T-t}\ (Wt) \\ \text{and} \qquad wt^* &= f_{T-t}\ (Wt) \end{aligned}\right\} \qquad (14)$$

which are the optimal consumption and portfolio decision (whose functional nature depends upon the nature of the utility function).

8. Bernoulli Case

Let us now move from the above generalities to a specific utility function. Consider the Bernoulli case, where

$$U = \log C \qquad (15)$$

(The results are interesting and easy to manipulate.)

It can be shown that the general recursive relation in this case is

$$J_{,}(W) = \text{Max}_{C,w}\ \log C + E(1+\rho)^{-1} \log (W - C)$$

$$\{(1 - w)(1+r) + wZ\}$$

$$= \text{Max}\ W \int_0^\infty \log ((1 - w)(1 + r) + wZ)\, dP(Z) \qquad (16)$$

(We restrict Z here to the positive real line since we are using a logarithmic function.)

Differentiating (16) w.r.t C and W, we get

$$0 = 1/C - (1 + \rho)^{-1} (W - C)^{-1}$$

$$\text{or} \qquad C(t) = \{(1 + \rho)/(2 + \rho)\}\ W(t) \qquad (17)$$

$$\text{and} \quad wt = \int_0^\infty (Z - 1 - r)((1 - w)(1 + r) + wZ)^{-1} dP(Z)$$

$$= w^* \text{ independent of } W(t) \text{ and } t \text{ and } C(t) \qquad (18)$$

The above results are interesting.

(a) (17) shows that with the logarithmic utility function, the consumption/savings propensity depends only on the time-preference discount factor ρ and is thus presumably a constant in time, (and average propensity = marginal propensity).

(b) (16) shows that the optimal portfolio decision is independent of time or wealth level or the consumption/savings decision and is a constant w^* (given the distribution $P(Z)$ and the rate of interest r).

9. Price Trajectory

Armed with the above, we can now go back to the dynamic equation (10) which says

$$dP/dt = \lambda\left(\frac{wt}{P}(Wt - Ct) - S(t, P)\right) \qquad (19)$$

$$= \lambda(w^* (1 - R)\, W/P - S(P)) \qquad (20)$$

where R stands for the average (and marginal) propensity to consume.

$$(R = (1 + \rho)/(2 + \rho))$$

10. Extensions

It is a simple matter to extend the investor dynamics to market dynamics and say that R is the market propensity to consume and w^* is the market optimal portfolio decision and W and S are market wealth and supply respectively. (This, indeed, we could have done earlier).

11. Extraneous Influences

Let us look at the consumption function (17) in which consumption is a constant fraction of W(t). Let us forget that it was derived on the basis of a logarithmic utility for with independent returns on the risky security. Let us take the constancy of propensity to consume as an independent hypothesis (it rings - falsely - a Keynesian note). Again, let us take the constancy of the split of every investment £1 into a risky and safe asset, i.e., the constancy of w^*, as an independent hypothesis. Let us consider the dynamic equation (19) then, as our hypothesis where $(\lambda w^*(1 - R))$ can be taken as a constant m.7.0. This enables us to consider extraneous - or, at any rate, - system-independent influences on W and S; or consider them as functions of our own choice to be tested by the price-trajectory they produce. For example:

(1) W can be considered a minimum function of t and S of P.

(2) W can be considered as make up of linear and cyclical functions in t and S a linear one in P and a cyclical one in t.

(3) W can be a product fn of P and t (price level relates to wealth).

(4) W and P can be random functions in t.

In all the above cases, the price-trajectory P(t) is solvable with an initial condition. (In case (4), we can work out the mean and variance). With the solution P(t), we have a means of testing each of the above hypotheses. Thus the control system approach has led us some way towards making our hypothesis operational. (also 55)

12. Multimarket Situations

In the multimarket case with many risky securities, we still have our two control variables (consumption decision and the risky/ safe portfolio decision). In addition, we have a third control vector ($x_1 \ldots x_n$) representing the fraction of each £1 invested in risky assets that goes to buy each risky security. We now will have one more part to our objective function, which should now include a minimum variance criterion to minimise portfolio risk.

13. Conclusions.

We have taken most of our arguments from Samuelson's analysis on lifetime portfolio selection. The main difference is that we employ the language of control theory - namely in viewing the market as a control system. In the control system - which contains a decision-making unit which decides the consumption, investment package bsed on utility preferences of - in our case - a representative investor, a law of price change (The Walranian), a supply functional and a price setting mechanism. All units thus rest on sound economic theory.

We use the Merton-Samuelson approach to the investment decision. They provide facility for any utility function to be employed but develop a simple - linear log utility function to the fullest extent. We have done the same here. For mathematical convenience they and we too, employ continuous time analysis.

All utility analysis is based on individual, rational investors to whom we give dummy motivational properties - as utility preference schemes. This is both for mathematical simplicity and also to save the bother of aggregation of multiple, heterogeneous opinions and data testing.

The simple expectational utility scheme employed here leads to a decision independent of the level of wealth but to a consumption function which resembles a Keynesian function. With a quadratic utility function, as in the case of Sharpe's approach to the position. Selection problem, the decision will depend on the level of wealth - a function of time. We have dealt with this approach in Chapter 5.

Appendix to Chapter 7

A Sequential Approach for Multimarket Situations

That the split between safe and risky assets is a constant fraction of every investment £ is true only of logarithmic utility fn. But Tobin (68) has used the constant split as an axiom in his separation theorem. There are those who disagree with it (48) and say that with a quadratic utility fn, one will only set a fraction dependent on the level of wealth. But proceeding from his axiom, Tobin uses a quadratic utility fn to determine the composition of a multi-asset model.

Let us proceed, however, along this path of sequential decision in spite of the disagreements and assume that a constant fraction m of investment wealth W is at any time invested in risky securities. Let $W^R = mW$. Our problem now is to decide how much of this W^R goes into each of n risky securities.

Let $\bar{P}$ be the vector of the prices of risky assets; $\bar{D}$ the vector of the quantity demanded; and $\bar{X}$ the vector $\begin{bmatrix} P.D. \\ \vdots \\ PnDn \end{bmatrix}$ of monetary demand.

The following relations are apparent:

1. $(\text{Diag. } \bar{P})'\bar{D} = \bar{X}$ (1)

   where diag. $\bar{P}$ is the matrix $\begin{bmatrix} P_1 & O & O & O \\ O & P_2 & O & \\ O & O & O & Pn \end{bmatrix}$

2. $i'\bar{X} = W^R$ (2)

   where i' is the row vector $[1 \; . \; . \; . \; . \; 1]$ of n elements.

Let us now posit that the investor uses a quadratic utility fn of wealth, i.e.,

$$U(W^R) = W^R - b(W^R)^2 \qquad (3)$$

The investor sees wealth invested in risky securities as a random variable and his expected utility is thus

$$\varepsilon(U(W^R)) = \varepsilon(W^R) - b\varepsilon(W^R)^2 \qquad (4)$$

where ε is the expectation operator, if

$$E = \varepsilon(W^R) \text{ and } S^2 = \varepsilon(W^R - E)^2$$

$$\varepsilon(U(W^R)) = E - b(E^2 + S^2) \qquad (5)$$

E:-

Let us assume a particular distribution for W^R that will not lead to negative wealth values; viewing, as White does, the expected wealth as $\bar{X}$ times the return on a unit vector of holdings, let us assume that the safe (yield) part of the outurn as the vector $\bar{r}$. As for capital gains, let us tie this up with the nonnegative criterion and assume for the unit price, the random representation,

$$d\ln \bar{P} = \bar{a}\, dt + \hat{\sigma} dz \qquad (6)$$

where $\bar{P}$ exhibits (or is expected to do so) a log-normal distn. with

$\bar{a}$: mean log $\bar{P}$ (instantaneous)

$\hat{\sigma}^2$: variance-covariance matrix

dz : Gauss-Wiener variable

Thus $E = \bar{X}' (i + \bar{r} + \bar{a});$ (7)

where $i = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ with n entries.

$\underline{S^2}$:-

$$S^2 = \varepsilon(W - E)^2$$

$$= \varepsilon(\bar{X}' \{(i + \bar{a} + r) + \hat{\sigma} \frac{dz}{dt}\} - \bar{X}'(i + a + r))^2$$

$$= \bar{X}' \hat{\sigma}^2 \bar{X} \qquad (8)$$

Thus the vector form in which the investor maximises $\varepsilon(W^R)$ is now clear. From (7) and (8)

$$\underline{\text{Max}} \quad \varepsilon(W^R) = \bar{X}' (i + \bar{r} + \bar{a}) - b(\bar{X}' (i + \bar{r} + \bar{a})'$$

$$(i + \bar{r} + a)\bar{X} + \bar{X}' \hat{\sigma}^2 X)$$

$$\text{s.t.} \quad (2) \quad \text{i.e.,} \ i'\bar{X} = W^R$$

Or, combining, max. the Langrangian

$$L = \bar{X}' (i + \bar{r} + \bar{a}) - b(\bar{X}' (i + r + a)'(i + \bar{r} + a) \bar{X} + \bar{X}'\hat{\sigma}^2 \bar{X})$$

$$+ \lambda(i'\bar{X} - W^R) \qquad (9)$$

Diff. w.r.t. $\bar{X}$ and , we get

$$(i + \bar{r} + \bar{a}) - 2b((i + r + a)'(i + \bar{r} + \bar{a}) + \hat{\sigma}^2) \bar{X}$$

$$+ \lambda i' = 0 \qquad (10)$$

and $i'X - W^R = 0$ (11)

Let $C = (i + \bar{r} + \bar{a})'(i + \bar{r} + \bar{a})$

and $V = \sigma^2 + cI$, where I = identity matrix (12)

Then the solution to (10) and (11) is

$$\bar{X} = \frac{1}{2b} \left(V^{-1} - \frac{V^{-1} - i\, i'\, V^{-1}}{i'\, V^{-1}\, i}\right)(i + \bar{r} + \bar{a})$$

$$+ \left(\frac{V^{-1}\, i}{i' V^{-1} i}\right) W^R \tag{13}$$

But since $(\text{Diag. } \bar{P})^{1}\, \bar{D} = \bar{X}$,

$$\bar{D} = (\text{Diag. } \bar{P})^{-1}\, \bar{X} \tag{14}$$

with $\bar{X}$ given by (13).

(14) constitutes the demand for portfolio in terms of wealth, price and estimatable parameters. While White proceeds to use a similar result to (13), for determining market equilibra, we shall use (14) to analyse price dynamics.

$$\text{Let } \frac{d\bar{P}}{dt} = (\text{Diag. } \bar{\lambda})(\bar{D} - \bar{S}) \tag{15}$$

where $\bar{\lambda}$ is a set of Walrasian coefficients.

(15) represents the expected trajectory for price and the yield curve can be derived from it given W^R and $\bar{S}$ and knowing market prices. With a given supply function (in terms of $\bar{P}$) and with a given initial wealth Wo, (15) is analytically solvable.

CHAPTER 8

Conclusions and Suggestions

1. Summary of Work Done:-

The doctoral work centres on three major topics. They are:

(a) Some approaches to the theory of spot price behavior in commodity markets. These stem from the consideration of the effect of noise-symbolising numerour unspecified economic variables - on stable relationships between supply and demand; from the application of the dynamic theory of interacting populations to the economic system and generally from the applications of the theory of stochastic processes.

(b) The applications of the theory of the term structure of interest rates to the term structure of commodity prices. These include
1. the continuous-time stochastic formulation of some leading term structure theories - the expectation model, risk-premium theory and the error learning model.
2. consideration of the term-structure market as a stochastic control system and deriving the capital asset pricing model as an optimal solution to the control system

(c) Derivation of an analytical relationship between the spot price, forward price and maturity in terms of fixed parameters - using the Black-Scholes-Merton formulation (via stochastic calculus) of the capital asset pricing model.

Testing of some of the above models using metal prices, whose results are appended.

2. Suggestions for extensions

1. Developing a continuous time and/or a multiperiod capital asset pricing model. The model would be in a dynamic programming format in the multiperiod case; it would be in the differential equation format of Black-Scholes-Merton in the continuous time case - but with several (not just one as in their models) boundary points in time. This would be the first step towards developing a sound theory of viewing the financial markets as a dynamic control system. (19)

2. Application of the capital asset pricing theory to the determination of the risk structure of the assets and liabilities of a company and the development of a capital budgeting theory leading to a healthy matching of risk, return and time between assets and liabilities. (69,70)

3. Some applications of stochastic biological models to the analysis of the total economic system. The forces of supply and demand that determine the quantity of goods bought/sold could be likened to the forces of birth and death in an ecological system and the interactions between various goods - one of them being money - to the interactions between population. This could be the basis for developing a dynamic equilibrium model of the total economic system.

4. Developing a time-series version of the mean-variance portfolio selection model.

APPENDIX I

Mathematical Content of biological population models

One approach to biological modelling is to treat time as a continuous variable and assume that the system can be described by a set of random variables which change either discretely or continuously in their state space. In the continuous case, which is what we employ, the system is characterised by a probablility density function that satisfies a second order partial differential equation, the so-called Fokker-Planck or diffusion equation. This equation can be converted into an equation that is very similar to the Schrödinger equation of quantun mechanics. A second approach begins with one or more important elements of a system whose time-path is describable deterministically. One superimposes noise to describe the effect of other variables and converts the deterministic dynamic variables into random variables whose probability density functions satisfy the Fokker-Planck equation. (1)

The diffusion equations are more amenable to analytical analysis than the differential-difference equations, which specify the evolution of the random processes discrete in state space. Therefore, by approximating the differential-difference equation by a partial differential equation in the form of a diffusion equation, a detailed, though approxiamte, knowledge of the behavior of a process with discrete state space can be obtained. The approximation improves as the ratio of distance between the allowed states and the value of the random variable describing the process decreases.

By using a similar limiting procedure on the backward master equation we get another partial differential equation, the so-called backward Kolmogorov diffusion equation,

$$\frac{\partial}{\partial t} P\ (x|y,t) \quad = \quad a(y)\frac{\partial P(x|y,t)}{\partial y} + \tfrac{1}{2}b(y)\frac{\partial^2 P(x|y,t)}{\partial y^2} \qquad (\ 2\ )$$

Equations (1) and (2) are to be solved with the initial condition

$$\lim_{t\to 0} P(x|y,t) \quad = \quad \delta(x-y) \qquad (\ 3\ )$$

which states that initially the random variable had the value y. We note that if, instead of the random variable initially having a definite value y, its initial state is specified by a probability density p(y) [with $\int_\Omega P\ (y)\,dy = 1$, where Ω denotes the state space], then the solution of the diffusion equations is

$$P(x|t) \quad = \quad \int_\Omega P(x|y,t)\ \ (y)\,dy \qquad (\ 4\ )$$

The function a(x) in Eqs. (1) and (2) is the rate of growth of the mean when the process is at x,i.e.

$$a(x) \quad = \quad \lim_{\tau\to 0} \frac{1}{\tau}\int_\Omega (z - x)P(z\ x,\tau)\,dz \qquad (5.1)$$

This can be seen by rewriting the right-hand side of this equation as

$$\int_\Omega (z - x)\frac{\partial P(z|x,0)}{\partial t}\,dz$$

using the differential equation (2) and the intitial condition (3), and carrying out the integration by parts. Similarly, b(x) in Eqs.

(2) is the rate of growth of the variance when the process is at x, i.e.

$$b(x) = \lim_{t \to 0} \frac{1}{\tau} \int_{\Omega} (z - x)^2 P(z|x,\tau)\,dz \qquad (5.2)$$

By similar arguments all the growth rates of the higher moments of the change in the state of the random variable vanish, i.e.

$$\lim_{t \to 0} \frac{1}{\tau} \int_{\Omega} (z - x)^n P(z|x,\tau)\,dz = 0, \qquad n \geqslant 3 \qquad (5.3)$$

We now derive the partial differential equations satisfied by $P(x\ y,t)$ for a Markovian process for which the state space is continuous, and derive the conditions under which these differential equations become diffusion equations Eqs. (1) and (2) . Our starting equation is the so-called Chapman-Kolmogorov equation:

$$P(x|y,t_1 + t_2) = \int_{\Omega} P(z|y,t_1) P(x|z,t_2)\,dz \qquad (6)$$

This equation is a mathematical manifestation of the Markovian,

according to which the behavior of the process in the time interval $(t_1,t_1 + t_2)$ depends on its state at time t_1, and not on its behavior in the previous time interval $(0,t_1)$. Therefore, if the process is in state y at time $t = 0$, the probability that it will be in state x at a later time $t_1 + t_2$, must be equal to the probability that it will be in some state z at time $t_1 + t_2$, summed over all the intermediate states z. In Eq. (6), we set $t_1 = t, t_2 = \tau(\tau \to 0)$, $z = x - \mu$ to get

$$P(x|y,t + \tau) = \int_{\Omega^1} P_1(x - \mu|y,t) P(x|x - \mu,\tau)\,d\mu \qquad (7)$$

where $(-\Omega^1)$ is the state space translated by -x.

We limit the discussion to cases in which the probability distribution $P(x|y,t)$ does not change significantly in the short time τ. Therefore $P(x|y,t+\tau)$ will not differ much from $P(x|y,t)$, and we can expand it in a Taylor's series:

$$P(x|y,t+\tau) = P(x|y,t) + \tau\frac{\partial P}{\partial t}(x|y,t) + \ldots \qquad (8)$$

Since from the initial condition, (3), $P(x|y,0) = \delta(x-y)$, for small τ $P(x|x-\mu,\tau)$ is sharply peaked around $\mu = 0$, and the integral in Eq. (7) need be integrated only in the neighborhood of $\mu = 0$. We can therefore expand the integrand of Eq. (7), regarded as a function of x, in a Taylor's series around the point $x + \mu$ to get

$$\begin{aligned} P(x-\mu|y,t)P(x|x-\mu,\tau) &= P(x|y,t)P(x+\mu|x,\tau) \\ &\quad - \mu\frac{\partial}{\partial x}\left[P(x|y,t)P(x+\mu|x,\tau)\right] \\ &\quad + \tfrac{1}{2}\mu^2\frac{\partial^2}{\partial x^2}\left[P(x|y,t)P(x+\mu|x,\tau)\right] - \ldots \end{aligned} \qquad (9)$$

Substituting this expansion into Eq. (7), carrying out the integration near $\mu = 0$, replacing the left-hand side of (7) by the expansion in (8), and noting that the integral over $P(x+\mu|x,\tau)$ equals 1, we obtain

$$\tau\left[\frac{\partial P(x|y,t)}{\partial t} + O(\tau)\right] = -\int_{\Omega^*}\frac{\partial}{\partial x}\left[P(x|y,t)\,\mu P(x+\mu|x,\tau)\right]d\mu + \tfrac{1}{2}\int_{\Omega^*}\frac{\partial^2}{\partial x^2}\left[P(x|y,t)\,\mu^2 P(x+\mu|x,\tau)\right]d\mu - \ldots \qquad (10)$$

where Ω^* is the neighborhood of $\mu = 0$ in which $P(x|x-\mu,\tau)$ is concentrated. Interchanging the order of integration (with respect to μ) with differentiation (with respect to x), dividing by τ, and taking the limit $\tau \to 0$, we finally arrive at

$$\frac{\partial P(x|y,t)}{\partial t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n} M_n(x) P(x|y,t) \tag{11}$$

where

$$M_n(x) = \lim_{\tau \to 0} \frac{1}{\tau} \int_{\Omega^*} \mu^n P(x+\mu|x,\tau)\,d\mu = \lim_{\tau \to 0} \int_{\Omega} (z-x)^n P(z|x,\tau)\,dz \tag{12}$$

Equation (11) is the required partial differential equation describing probabilistically the evolution of the process. It reduces to the forward diffusion equation (1) if $M_n(x) = 0$ for all $n > 2$, and these are the required conditions (in addition to the Markovian nature of the process) under which the forward diffusion equation describes the evolution of the process. By a similar procedure [which involves the expansion of $P(x|z,t_2)$ in Eq. (6) regarded as a function of z in Taylor's series around the point y] one can show that under the above-stated conditions $P(x|y,t)$ also satisfies the backward diffusion equation (2).

We now introduce a wide class of processes for which $M_n(x) = 0$ for all $n > 2$ and relate the dynamical stochastic equation, obeyed by the processes, with the forward diffusion equation satisfied by the corresponding probability density $P(x|y,t)$.

Consider a process characterized by the stochastic dynamical equation satisfied by the process variable x,

$$dx/dt = h(x) + e(x)i(t) \tag{13}$$

where $i(t)$ is a stochastic memoryless input to the process, $e(x)$ describes the rate of this input and $h(x)$ is the function describing the rate of change of the variable x in the absence of input. Both $e(x)$ and $h(x)$ are assumed to be differentiable functions; $i(t)$ is characterized by two parameters m and σ^2 defined by

$$\langle i(t) \rangle = m \tag{14.1}$$

$$\langle [i(t) - m][i(t + t^1) - m] \rangle = \sigma^2 \delta(t^1) \tag{14.2}$$

with all the correlations of order greater than 3 of $i(t)$ assumed to be zero. The averages are taken over a suitable ensemble, e.g., a number of repetitive observations on the system. Such an $i(t)$ is said to be generated by a Gaussian random process, and σ^2 is known as an incremental variance or intensity of the input. We define a new quantity $F(t)$ by

$$F(t) = \frac{i(t) - m}{\sigma} \tag{15}$$

$$\langle F(t) \rangle = 0 \tag{16}$$

$$\langle F(t)F(t + t^1) \rangle = \delta(t^1) \tag{16.2}$$

with vanishing correlations of order 3 and more of $F(t)$: $F(t)$ so defined is called a white noise. Substituting Eq. (15) into Eq. (14) we get

$$dx/dt = \alpha(x) + \beta(x)F(t) \tag{17}$$

where

$$\alpha(x) = h(x) + me(x) \tag{18.1}$$

$$\beta(x) = \sigma e(x) \tag{18.2}$$

We now show that for the process described by the stochastic dynamical equation (SDE) (13) or equivalently by SDE (17),

$M_n(x) = 0$ for all $n \geqslant 3$,

$$M_1(x) = \propto(x) + \frac{1\partial}{4\partial x}\{\beta(x)\}^2 \equiv a(x) \tag{19.1}$$

$$M_2(x) = \{\beta(x)\}^2 \equiv b(x) \tag{19.2}$$

and the probability density $P(x|y,t)$ satisfies the forward diffusion equation (1).

Dividing Eq. (17) by $\beta(x)$, we obtain

$$dz/dt = \hat{a}(z) + F(t) \tag{20}$$

where

$$dz = dx/\beta(x) \tag{21.1}$$

$$\hat{a}(z) = \propto(x(z))/\beta(x(z)) \tag{21.2}$$

To find $M_1(z)$, we integrate Eq. (20) over the short interval $(t,t+\tau)$ to get

$$\Delta z(t) \equiv z(t+\tau) - z(t) = \hat{a}(z)\tau + \int_t^{t+\tau} d\xi F(\xi) + O(\tau) \tag{22}$$

Therefore, $M_1(z)$, the growth rate of the mean value when the process is at z, is

$$M_1(z) = \lim_{\tau\to 0} \frac{\langle\Delta z\rangle}{\tau} = \hat{a}(z) + \lim_{\tau\to 0}\int_t^{t+\tau} d\xi \langle F(\xi)\rangle = \hat{a}(z) \tag{23}$$

where the last step follows from Eq. (16). Further, from Eq. (22)

$$\langle(\delta z)^2_{\Delta}\rangle = \int_t^{t+\tau}\int_t^{t+\tau} d\xi d\eta \langle F(\xi)F(r)\rangle + O(\tau)$$

and by Eq. (16.2), the double integral in this equation is τ, so that

$$M_2(z) = \lim_{\tau \to 0} \frac{\langle (\Delta z)^2 \rangle}{\tau} = 1 \qquad (24)$$

Since all correlations of F(t) higher than second order vanish, following the same arguement, we obtain

$$M_n(z) = 0 \qquad n \geqslant 3 \qquad (25)$$

From Eqs. (23) - (25), g(z z,t), the probability density that the transformed variable defined by Eqs. (21) has the value z at time t when z_o is its value at t = 0 (i.e., when x = y), satisfies the forward diffusion equation

$$\frac{\partial g}{\partial t} = - \frac{\partial}{\partial z} (\hat{a}(z) g) + \frac{1}{2} \frac{\partial^2 g}{\partial z^2} \qquad (26)$$

Since by the transformation (21.1)

$$\text{Prob}[x_1 \leqslant x \leqslant x_2] = \int_{x_1}^{x_2} P(x|y,t)\,dx = \int_{z_1}^{z_2} P(x(z)|\,y,t)\,\beta(x(z))\,dz$$

$$= \text{prob}[z_1 \leqslant z \leqslant z_2]. = \int_{z_1}^{z_2} g(z|z_o,t)\,dz \qquad (27.1)$$

with $z_i = z(x_i)$, i = 1, 2, $g(z|z_o,t)$ by the relation

$$g(z|z_o,t) = P(x(z)|\,y,t)\,\beta(x(Z)) \qquad (27.2)$$

Differentiating both sides with respect to z, and using Eqs. (21), we obtain

$$\frac{\partial g}{\partial z} = \frac{\partial(\beta P)}{\partial z} = \beta\frac{\partial(\beta P)}{\partial} = \frac{\partial(\beta^2 P)}{\partial x} - \frac{P}{2}\frac{\partial \beta^2}{\partial x},$$

$$\frac{\partial^2 g}{\partial_z{}^2} = \beta \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial z}\right) \qquad (27.3)$$

so that Eq. (26), after the substitutions of Eqs.(27.2), (27.3), and (21.2), becomes

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}(a(x)P) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(b(x)P) \qquad (28)$$

where a(x) and b(x) are given by Eqs. (19.1) and (19.2) respectively. Equation (28) is the same forward diffusion equation as (1). For such an equation, we have already shown earlier in this section tht $M_1(x) = a(x)$, $M_2(x) = b(x)$, and $M_n(x) = 0$, $n \geqslant 3$, which completes the proof of the assertions made above.

To summarize, we have shown that for a process described by the SDE (13) i.e.,

$$dx/dt = h(x) + e(x)i(t) \qquad (29.1)$$

with i(t) a white noise with nonzero mean m and intensity σ^2, the forward diffusion equation or Fokker-Planck (FP) equation is

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left[\left\{h(x) + me(x) + \frac{\sigma^2}{4}\frac{\partial}{\partial x}e^2(x)\right\}P\right] + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\left[e^2(x)P\right] \qquad (29.2)$$

Similarly, the corresponding backward equation is

$$\frac{\partial P}{\partial t} = \left\{h(y) + me(y) + \frac{\sigma^2}{4}\frac{\partial}{\partial y}e^2(y)\right\}\frac{\partial P}{\partial y} + \frac{\sigma^2}{2}e^2(y)\frac{\partial^2 P}{\partial y^2} \qquad (29.1)$$

The results above are very useful in making primitive statistical models of complex biological systems. The deterministic behavior of one (or more) of the components of the complex system can be represented by the dynamical equation dx/dt = h(x), and the remaining unknown fluctuating behavior, due to the presence of other components, can be approximated by e(x)i(t) where i(t) is a white noise. This white noise approximation is a reasonable approximation if the fluctuations occur extremely rapidly on the time scale defining the changes in x. This procedure converts the deterministic dynamical equation into a stochastic dynamical equation and the probabilistic analysis of the process can be carried out by analyzing the diffusion equations (29.1) and (29.2).

It may be noted that although the FP equation for a given SDE is unique, the reverse is not true. In other words, the problem of finding an SDE for a random process satisfying a given FP equation does not have a unique solution. However, the solution will be unique if we restrict ourselves to equations of the type (17) containing a Gaussian delta-correlated random process F(t) with zero mean and unit intensity. With this restriction, the SDE for the FP equation (28) is Eq. (17), or by a simple redefinition, the SDE for the FP equation

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}(a(x)P) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(b(x)P) \qquad (30.1)$$

is

$$\frac{x}{dt} = a(x) - \frac{1}{4}\frac{\partial b(x)}{\partial x} + \{b(x)\}^{1/2}F(t) \qquad (30.2)$$

The formalism presented here for the derivation of an FP equation from a given SDE, follows the Stratonovich rules (65).

There is a controversy in this derivation when the coefficient of the white noise in the SDE [e(x) in Eq. (13)] depends on the process variable x. This controversy arises from the pathalogical nature of the white noise (similar to the pathological nature of the δ function), which is well defined only in terms of its intergral $\int_0^\tau F(\tau)d\tau$. We followed the Stratonovich approach since its rules are the same as ordinary calculus, and transformation of variables as the one carried out in Eqs. (21), is valid. The other approach, taken by Doob (18), and Ito (34), uses the Ito calculus, where differentiation and integration rules differ from the ordinary ones. A discussion of this controversy from the point of view of modeling reality is given by Mortensen (47). He concludes that, since a white noise is only an approximation to the stochastic behavior in nature, the stochastic process, derived by using any of the two approaches, should be checked against reality, and its capability to predict results which are an acceptable approximation to the actual behavior is the only criterion to be considered.

We now discuss the solution of the diffusion equations. Since $P(x|y,t)$ describes the evolution of the process completely, we need solve the forward diffusion equation - the FP equation. From the discussion given above, the FP equation of the type (28) can be transformed into an FP equation of the type (26) with the substitutions

$$dz = [b(x)]^{-1/2}dx, \quad z = z(x) = \int^{x} [b(\xi)]^{-1/2}d\xi \qquad (31.1)$$

$$\hat{a}(z) = \left[a(x) - \frac{1}{4}\frac{db}{dx}\right][b'(x)]^{-1/2} \qquad (31.2)$$

$$g(z|z_o t) = [b(x)]^{1/2}P(x|y,t)\big|_{x=x(z)} \qquad (31.3)$$

and the intial condition

$$g(z|z_o,0) = \delta(z - z_o), \quad z_o = z(y) \qquad (31.4)$$

In Eq. (31.2) the right-hand side is to be expressed in terms of z given by (31.1). Therefore, it is sufficient for us to describe the method for solving Eq. (26) with the boundary conditions which depend on the allowed range of the variable z.

As in the case of discrete processes, the random processes may be basically of two types, one in which there are no forced restrictions on the allowed range of the random variable (we shall call such processes unrestricted), and the other type in which boundary conditions are imposed at one or two points or the state space (such processes will be called restricted). The appropriate boundary conditions imposed on the latter type of processes are given in (27) and the behavior of unrestricted processes near their "built-in" boundaries (finite or infinite) is discussed in (27). In the remaining part of this section we formally indicate a method for solving the FP equation, and in the next two sections, we indicate how to apply this method for various processes.

To solve Eq. (26), we use the standard method of separation of variables.

Taking

$$g(z|z_o,t) = Q(z)e^{-Et/2} \tag{32}$$

as the trial solution, Eq. (26) becomes

$$\frac{d^2Q}{dz^2} - \frac{d}{dz}(\hat{a}(z)Q) + EQ = 0 \tag{33}$$

This equation is to be solved subject to the boundary conditions on Q [implied by the boundary conditions on g through Eq. (32)] i.e., it is an eigenvalue problem. Depending on the form of the function $\hat{a}(z)$ and the boundary conditions, there will be a discrete and/or continuous set of E, with a corresponding set of Q. For a discrete set $\{E_n, Q_n\}$, the solution is

$$g(z|z_o t) = \sum_n \alpha_n Q_n \exp(-E_n t/2) \tag{34.1}$$

and for a continuous set

$$g(z|z_o,t) = \int \alpha(E)Q(E)e^{-Et/2}dE \tag{34.2}$$

where α_n or $\alpha(E)$ is to be evaluated by using the initial condition (31.4).

An equivalent and useful form of Eq. (33) is obtained by making a transformation which converts it into a differential equation free of a first-order partial derivative in z. Such a transformation is

$$Q = \psi(z)\left[\pi(z)\right]^{1/2} \tag{35.1}$$

where

$$\pi(z) = \exp\left\{-2\int^{z} \hat{a}(\xi)\,d\xi\right\} \tag{35.2}$$

On carrying out this transformation, Eq. (33) becomes

$$d^2\psi/dz^2 + \left[E - U(z)\right]\psi(z) = 0 \tag{36}$$

where

$$U(z) = d\hat{a}/dz + \hat{a}^2 \tag{37}$$

The boundary conditions are also transformed into the boundary conditions on ψ. Equation (36) and these boundary conditions once again constitute an eivenvalue problem.

In case z is confined between two finite boundaries and U(z) is finite within these boundaries, the set of eigenvalues and eigenfunctions $\{E_n, \psi_n(z)\}$ for this eigenvalue problem is discrete, and the set of functions $\{\psi_n(z)\}$ is orthonormal (see, e.g. (67))

$$\int_{\Omega} \psi_n(z)\, \psi_m(z)\,dz = \delta_{mn} \tag{38}$$

In view of the transformation (35) the set of functions $\{Q_n(z)\}$ is an orthonormal set with respect to the weight function $\pi(z)$, i.e. $$\int_{\Omega} Q_n(z)Q_m(z)\pi(z)\,dz = \delta_{mn} \qquad (39)$$

Using relation (3 9) and the initial condition (31.4) the coefficients $\{\alpha_n\}$ in Eq. (34.1) can be evaluated by putting $t = 0$ in this equation, multiplying both sides by $\pi(z)Q_n(z)$ and integrating over Ω. The resulting form of α_n is

$$\alpha_n = \int_{\Omega} \pi(z)Q_n(z)\delta(z - z_o)\,dz$$

$$= Q_n(z_o)\pi(z_o)$$

Substituting α_n into Eq. (34.1), we get the simple expression

$$g(z|z_o,t) = \pi(z_o) \sum_{n=0}^{\infty} Q_n(z)Q_n(z_o)\exp(-E_n t/2) \qquad (40)$$

Relation (40) is valid also in case of an infinite state space, if $U(z)$ tends to infinity at the infinite boundaries. Otherwise, the set of eigenvalues $\{E_n\}$ is not discrete anymore, and there is a continuous interval of eigenvalues.

There is an advantage in using the form (36) instead of (33) since this form is very similar to the time-dependent Schrödinger equation of quantum physics

$$\frac{d^2\psi}{dz^2} + \frac{2m}{\hbar^2}\left[E - U(z)\right]\psi(z) = 0$$

which describes the motion, in one dimension, of a particle of mass m moving in a field with potential $U(z)$. E is the energy of the particle, $\hbar = 2\pi h$ is Planck's constant, and $\Psi(z,t) = \psi(z)\exp(-iEt/h)$ is the wave function of the particle and has the physical significance that $|\Psi(z,t)|^2 dz$ is the probability of finding the particle at a point between z and z + dz at time t. Because Schrödinger's equation has been studied quite extensively in mathematical physics, together with many approximation methods for solving it, an extensive literature becomes immediately available for the solution FP equations.

We now make an important observation by writing $\hat{a}(z)$ in the form

$$\hat{a}(z) = \phi'(z)/\phi(z) \qquad (41)$$

Substituting Eq. (41) into (37) we get

$$\phi''(z) - U(z)\,\phi(z) = 0 \qquad (42)$$

Comparing Eq. (36) with this equation, we note that $\phi(z)$ is a solution of (36) when E is taken to be equal to zero. Thus by choosing U(z) such that (36) is analytically solvable for U(z) and U(z) + constant, one can generate a set of $\hat{a}(z)$ [with $\hat{b}(z) = 1$] for which the diffusion equation (26) can be analytically solved. For these $\hat{a}(z)$ and $\hat{b}(z) = 1$, one can calculate a(x) for a given b(x) by using Eqs. (31) or calculate a(x) and b(x) from a given $\beta(x)$ from Eqs. (19) and (21).

On the basis of the literature on Schrödinger's equation on electrostatics and other areas of mathematical physics, we have compliled a list of $\hat{a}(z)$ for which one of the two second-order differential equations (33) and (36) has been solved. We give $\hat{a}(z)$ and the solutions in Table 1. In Table 2 we summarize a list of a(x) and b(x) for which the eigenfunctions of the FP equation (1) are known.

s to the eigenvalue equation $\psi'' + (E-U)\psi = 0$, derived from an FP equation with $b(x) = 1$

	$U(x)$	$h(x)$	$\phi(x)$	References[a]
		$\psi(x) = h(x)\phi(x)$		
$x))$	0	1	$\cos E^{1/2}x$, $\sin E^{1/2}x$	
ιx ιx	α^2	1	$\exp[(\alpha^2-E)^{1/2}x]$, $\exp[-(\alpha^2-E)^{1/2}x]$, $\alpha^2 > E$ $\cos(E-\alpha^2)^{1/2}x$, $\sin(E-\alpha^2)^{1/2}x$, $\alpha^2 < E$	
αx αx	$-\alpha^2$	1	$\cos(E+\alpha^2)^{1/2}x$, $\sin(E+\alpha^2)^{1/2}x$	
	$-\alpha^2 + \dfrac{\alpha(1-\alpha)}{\cos^2 x}$	$(\cos x)^\alpha$	Hypergeometric functions $F(-\nu, \nu+2\alpha; \alpha+\frac{1}{2}; \sin^2(\pi/4 - x/2))$ $\nu(\nu+2\alpha) = E$ ν = integer — Gegenbauer polynomials $T_\nu^{(\alpha)}(\sin x)$	(1)
	$-\alpha^2 - \dfrac{\alpha(1-\alpha)}{\sin^2 x}$	$(\sin x)^\alpha$	Hypergeometric functions $F(-\nu, \nu+2\alpha; \alpha+\frac{1}{2}; \sin^2 x/2)$ $\nu(\nu+2\alpha) = E$ ν = integer — $T_\nu^{(\alpha)}(\cos x)$	(1)
!	$\alpha + (\alpha x+\beta)^2$	$\exp[-(\alpha x+\beta)^2/2\alpha]$	Hermite functions with argument $(\alpha x+\beta)/\alpha^{1/2}$	(1)
β	$\beta^2 + \dfrac{2\alpha\beta}{x} + \dfrac{\alpha(\alpha-1)}{x^2}$	$\exp[-(1-\beta^{-2}E)^{1/2}\beta x]\, x^{1-\alpha}$	Confluent hypergeometric functions $F(1-\alpha+\alpha(1+\beta^{-2}E)^{-1/2}; 2-2\alpha; (1-\beta^{-2}E)^{1/2} 2\beta x)$	(1)
β/x	$\alpha^2x^2 + \alpha(1+2\beta) - \dfrac{1-(2\beta-1)^2}{4x^2}$	$x^{\mu+1/2}\exp(-\lvert\alpha\rvert x^2/2)$ $\mu = \pm(\beta-\frac{1}{2})$	Confluent hypergeometric functions $F(-\lambda; \mu+1; \lvert\alpha\rvert x^2)$, $4\lambda = \dfrac{E}{\lvert\alpha\rvert} - \dfrac{\alpha}{\lvert\alpha\rvert}(1+2\beta) - 2\mu - 2$ $\lambda = n$, an integer — Laguerre polynomials $L_n^{(\mu)}(\lvert\alpha\rvert x^2)$	(1)

E .2

ons to the eigenvalue equation $\dfrac{\partial^2}{\partial x^2}[b(x)Q(x)] - 2\dfrac{\partial}{\partial x}[a(x)Q(x)] + EQ(x) = 0$

$a(x)$	$b(x)$	Eigenfunctions $Q(x)$	References[a]
$\iota x+\beta$	$x(1-x)$	Hypergeometric functions $F(a, b; 2(1-\beta); x)$ $ab = 2\alpha+2-E$, $a+b = 2\alpha+3$	(42)
$\iota x+\beta$	$1-x^2$	Hypergeometric functions $F\left(-\lambda, 2\alpha+\lambda+3; \alpha+\beta+2; \dfrac{1-x}{2}\right)$	
		$\lambda(\lambda+2\alpha+3) = E - 2(\alpha+1)$	
		$\alpha = -1$; $\beta = 0$ Legendre functions $P_\lambda(x)$, $Q_\lambda(x)$	(42)
		$\lambda = n$, an integer, Jacobi polynomials $P_n^{(\alpha+\beta+1,\, \alpha-\beta+1)}(x)$	(42)
		$\beta = 0$, Gegenbauer polynomials $T_n^{(\alpha+3/2)}(x)$	(42)
		$\alpha = -\frac{3}{2}$, Chebycheff polynomials of first kind $T_n(x)$	(42)
		$\alpha = -\frac{1}{2}$, Chebycheff polynomials of second kind $U_n(x)$	(42)
		$\alpha = -1$, Legendre polynomials $P_n(x)$	(42)
$x+\beta$	x	Confluent hypergeometric functions (Kumer's functions) $F(1-E/2\alpha; 2(1-\beta); 2\alpha x)$ $E/2\alpha = n$, an integer, Laguerre polynomials $L_{n-1}^{(1-2\beta)}(2\alpha x)$	
$(1-x^2)$	$1-x^2$	$e^{\alpha x}S(x)$: $S(x)$ oblate spheroidal functions $(1-x^2)S''(x) - 4xS'(x) + \{E-2-\alpha^2+\alpha^2x^2\}S(x) = 0$	(64)
$x(1-x)$	$x(1-x)$	$e^{\alpha x}R(1-2x)$: $R(x)$ oblate spheroidal functions $(1-x^2)R''(x) - 4xR'(x) + \left\{E - 2 - \dfrac{\alpha^2}{4} + \dfrac{\alpha^2}{4}x^2\right\}R(x) = 0$	(64)

When an analytical calculation of $P(x|y,t)$ is somewhat difficult, and if one is interested only in the first few moments of x, one has only to solve a set of differential equations which may be considerably simpler. The moments of x, if they are finite, satisfy ordinary differential equations, which can be derived from the forward diffusion equation (1) by multiplying it by x^n and then integrating over x from A to B, where the allowed range of the variable is $A < x < B$. The resulting equation is

$$\frac{d}{dt}\int_A^B x^n P(x|y,t)\,dx = \frac{d}{dt}\langle x^n\rangle = -\int_A^B x^n \frac{\partial}{\partial x}\left[a(x)P(x|y,t)\right]dx + \tfrac{1}{2}\int_A^B x^n \frac{\partial^2}{\partial x^2}\left[b(x)P(x|y,t)\right]dx$$

Integrating by parts we get

$$\frac{d}{dt}\langle x^n\rangle = n\int_A^B x^{n-1}\, a(x)P(x|y,t)\,dx + \frac{n(n-1)}{2}\int_A^B x^{n-2} b(x)P(x|y,t)\,dx - \left[x^n a(x)P - \frac{x^n}{2}\frac{\partial}{\partial x}\left[b(x)P\right] + \tfrac{1}{2}nx^{n-1}b(x)P\right]_A^B \qquad (43)$$

If the term in square brackets vanishes at $x = A$ and B, (43) reduces to

$$\frac{d}{dt}\langle x^n\rangle = n\langle x^{n-1}a(x)\rangle + \frac{n(n-1)}{2}\langle x^{n-2}b(x)\rangle \qquad (44)$$

Equation (44) has then to be solved subject to the initial conditions

$$\langle x^n\rangle\,(t = 0) = y^n \qquad (45)$$

For $a(x)$ and $b(x)$ polynomials in x, Eq. (44) is a set of coupled linear ordinary differential equations in $\langle x^n\rangle$, $n \geqslant 1$. When the polynomial a(x) is of degree at the most 1, and the polynomial b(x) is of a degree at most 2, the nth equation in the set (44) involves moments up to order n, and hence the equations can be solved successively. For example, if

$$a(x) = a_o + a_1 x, \quad b(x) = b_o + b_1 x + b_2 x^2 \tag{46}$$

Eq. (44) for n = 1 and 2, becomes

$$d\langle x\rangle/dt = \langle a(x)\rangle = a_o + a_1\langle x\rangle \tag{47}$$

$$d\langle x^2\rangle/dt = 2\langle xa(x)\rangle + \langle b(x)\rangle = b_o + (b_1 + 2a_o)\langle x\rangle + (2a_1 + b_2)\langle x^2\rangle \tag{48}$$

Equation (47) is linear in $\langle x\rangle$ and can be easily integrated to give

$$\langle x\rangle = \left(y + \frac{a_o}{a_1}\right)\exp(a_1 t) - \frac{a_o}{a_1} \tag{49}$$

This solution, when substituted in Eq. (48), gives a linear equation in $\langle x^2\rangle$ which can be easily integrated.

When either a(x) is a polynomial of degree greater that 1 or b(x) is of degree greater than 2, the coupled system of equations (44) can only be solved approximately† by using one of the methods presented in (32) in connection with birth and death processes.

In additon to moments of x, there are some other quantities which provide insight into the evolution of the process, and for

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† The solution of Eq. (44) for n = 1 is independent of b(x) as long as a(x) is at most linear, and $\langle x\rangle$ is given by Eq. (49) even when b(x) is of degree greater than 2.

which an analytical expression can be derived. These quantities are the continuous analogs of the quantities defined in (27). Here we list these quantities together with some general results which are independent of the type of process.

(a) The steady-state probability density $P(x|y, \infty)$, which describes the process when it is in some dynamical equilibrium. Such a dynamic equilibrium is reached since $a(x)$ and $b(x)$ do not depend on time.

To calculate $P(x|y,\infty)$, we set $\partial P/\partial t = 0$ in Eq. (2) to get

$$J(x|y, \infty) = \text{constant} \equiv J(\infty) \qquad (50)$$

where

$$J(x|y,t) = a(x)P(x|y,t) - \frac{1}{2}\frac{\partial}{\partial x}\left[b(x)P(x|y,t)\right] \qquad (51)$$

Since in terms of this function J, the FP equation (52) is

$$\partial P/\partial t + \partial J/\partial x = 0 \qquad (52)$$

J can be interpreted as the probability current, and Eq. (52) as the equation of conservation of probability. The general solution of Eq. (51) for $t = \infty$ is obtained by writing

$$P(x|y, \infty) = v(x)/b(x) \qquad (53)$$

so that Eq. (51) becomes

$$\frac{dv}{dx} - 2\frac{a(x)}{b(x)}v = -2J(\infty)$$

This is a linear equation in v, which admits the solution

$$v(x) = -2J(\infty)\int^{x} \exp\left\{2\int_{x'}^{x}\frac{a(\xi)}{b(\xi)}\,d\xi\right\}dx' + C\exp 2\int^{x}\frac{a(\xi)}{b(\xi)}\,d\xi \qquad (54)$$

where C is an arbitrary constant of integration. Substituting this expression for v(x) into Eq. (53) we get the expression for $P(x|y, \infty)$ in terms of two constants $J(\infty)$ and C. The constant C is determined by a normalization condition and $J(\infty)$ depends on the nature of the process. In this appendix, we limit the discussion to processes in which there is no flow of probability into the state space from the outside. Therefore, at the boundaries, $J(x|y,t) \geqslant 0$. In case $J(x|y,t)$ vanishes at both boundaries for all $t \geqslant 0$, $J(\infty) = 0$ by Eq. (51), and the steady-state probability density of the form

$$P(x|y, \infty) = \frac{C}{b(x)} \exp\left\{2 \int^{x} [a(\xi)/b(\xi)] \, d\xi \right\} \qquad (55.1)$$

where C is determined by the condition

$$\int_{\Omega} P(x|y, \infty)\, dx = 1 \qquad (55.2)$$

In case there is a positive flow of probability out from the state space, at least at one of the boundaries, as $t \rightarrow \infty$ all probability is bound to be outside the state space, and the steady-state density is the trivial solution of Eq. (50), i.e.

$$P(x|y, \infty) = 0 \qquad (55.3)$$

(b) The probability R(z y) that the random variable ever takes the value z.

(c) The "first passage time" $T(z|y)$, the time for the process to take the value z for the first time, its probability density function $F(z|y,t)$, and its arbitrary (ith) moment $M_i(z|y,t)$, $i \geqslant 1$.

$F(z|y,t)$ is related to $P(x|y,t)$ through the relation

$$P(x|y,t) = \int_0^t F(z|y,t-\tau)P(x|z,\tau)d\tau , \qquad y \leq z \leq x$$
$$\text{or} \quad x \leq z \leq y \tag{56}$$

This reflects the fact that the random variable can take the value x at time t when initially it has the value y, only if it takes an intermediate value z at some time $t - \tau$, in the time interval (O,t), and then in the remaining time τ its value changes from z to x. An equivalent and simpler form of Eq. (56) is obtained by taking the Laplace transform of Eq. (56). Using the theorem on the Laplace transform of a convolution integral, we get

$$f(z|y,s) = p(x|y,s)\big/p(x|z,s) \tag{57}$$

where $f(z|y,s)$ and $p(x|y,s)$ are the Laplace transforms of $F(z|y,t)$ and $P(x|y,t)$, respectively.

In the derivations of $F(z|y,t)$ and the moments of $T(z|y,t)$ the differential equation satisfied by $F(z|y,t)$ is the starting point, and we will now show that this equation is the backward diffusion equation. Inserting Eq. (56) into the backward diffusion equation (2) we get

$$-F(z|y,O)P(x|y,t) = \int_0^t \left[\frac{\partial}{\partial t} - a(y)\frac{\partial}{\partial y} - \tfrac{1}{2}b(y)\frac{\partial^2}{\partial y^2}\right] F(z|y,t-\tau)P(x|z,\tau)d\tau$$

Since this equation is valid for all t and since, by definition, $F(z|y,O) = O$ for $y = z$, $F(z|y,t)$ has to satisfy the backward equation

$$\frac{\partial F(z|y,t)}{\partial t} = a(y)\frac{\partial F(z|y,t)}{\partial y} + \tfrac{1}{2}b(y)\frac{\partial^2 F(z|y,t)}{\partial y^2} \tag{58}$$

The intial condition follows from the intial condition (3) on $P(x|y,t)$ and from the observation that in Eq. (56), $x = y$ only if $x = y = z$. Therefore

$$F(z|y,0) = \delta(z - y) \qquad (59)$$

One of the two boundary conditions to be imposed on $F(z|y,t)$ follows directly from Eq. (56), i.e.,

$$F(z|z,t) = \rho(t) \qquad (60)$$

while the second condition depends on the nature of the process.

(d) The probability $R(z|y,w)$ that the random variable takes the value z before taking the value w.

(e) The time $T(z|y,w)$ for the random variable to take the value z fore the first time before it takes the value w, its probability density function $F(z|y,w)$, and its arbitrary (ith) moment $M_i(z|y,w)$, $i \geq 1$.

Appendix 2.

The Solution of a 2nd order parabolic p.d.f.

The solution of a partial differential equation of the form

$$\partial F/\partial t \quad = \quad kx^2 \partial^2 F/\partial x^2 + ax\, \partial F/\partial x + b(x,t)$$

$$0 < x < \infty$$

whose solutions we were concerned with in particular situations in the above pages, could be handled in the following manner.

1. First, consider transforming the functional variable x to a variable y whose range covers the entirely entire real time. This is done through the transform

$$y \quad = \quad \ln x$$

With this

$$\partial F/\partial x \quad = \quad \partial F/\partial y \,.\, dy/dx$$

$$= \quad \frac{1}{x}\; \partial F/\partial y$$

$$\partial^2 F/\partial x^2 \quad = \quad \frac{\partial}{\partial x}\; \partial F/\partial x \quad = \quad \frac{\partial}{\partial x}\left(\frac{1}{x}\; \partial F/\partial y\right)$$

$$= \quad \frac{-1}{x^2}\; \partial F/\partial y$$

$$+ \frac{1}{x}\; \frac{1}{\partial y}\, \frac{dy}{dx} \,.\, \partial F/\partial y$$

$$= \quad \frac{1}{x^2}\left(\partial^2 F/\partial y^2 - \partial F/\partial y\right)$$

The p.d.f. reduces now to

$$\partial F/\partial t = k\partial^2 F/\partial y^2 + \partial F/\partial y\ (a - k)$$

$$+ b'(y,t) \qquad -\infty < y < \infty$$

where b'(y,t) is a transformed function.

2. Let us now consider if it is possible to have a solution of F as a product of two separate functions

$$Y(y) \text{ and } T(t)$$

If this were the case,

$$F = Y(y)T(t)$$

$$\partial F/\partial t = Y(y)T'(t)$$

$$\partial F/\partial y = T(t)Y'(y)$$

$$\partial^2 F/\partial y^2 = T(t)Y''(y)$$

The p.d.f. would result as

$$Y(y)T'(t) = kT(t)Y''(y)$$

$$+ (a - k)T(t)Y'(y)$$

$$+ b'(y,t)$$

Boundary Condition

If the boundary condition for F is prescribed at $t = 0$ as a function f(x), then the boundary condition for the above differential equation too is readily available. In the cases we were concerned with this had the effect, due to separabilitiy, of saying,

$$Y(y) = e^y \quad \text{for} \quad F(x,t = 0) = x \, .$$

Then the above reduces to (by dividing throughout by $Y(y)T(t)$, and since

$$Y(y) = Y'(y) = Y''(y) = e^y)$$

$$\frac{T'(t)}{T(t)} = a + \frac{b'(y,t)}{e^y \, . \, T(t)}$$

which is readily solvable over a range of t, say, from 0 to τ.

This is the method that was employed throughout. (63)

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