244

ON SIMULTANEOUS APPROXIMATION PROBLEMS IN NORMED SPACES WITH APPLICATION TO DIFFERENTIAL EQUATIONS

Thesis submitted by

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for the degree of Ph D

IMPERIAL COLLEGE OF SCIENCE & TECHNOLOGY

July 1977

ACKNOWLEDGEMENTS

I wish to express my sincere gratitude to my supervisor Dr. E. L. Ortiz for making this thesis possible and for his continual guidance and encouragement. I also express my appreciation to Prof. H.W. McLaughlin for enabling me to make a beneficial study visit to RPI.

ABSTRACT

There are two classical problems which fall into the category of Simultaneous Approximation and which are the subject of this thesis. The first is the problem of determining the best approximation to a set of functions belonging to a normed linear space, from a linear subspace or a nonlinear subset, subject to a measure of 'distance' - the norm.

In the Introductory Chapter, we show the progression of a fundamental characterising property in Approximation Theory known as the Kolmogoroff criterion from the uniform norm to its generalistion by the Hahn- Banach theorem to an arbitrary norm.

In Chapter Two, we obtain a unifying theory for the simultaneous approximation problem by a nonlinear subset in an arbitrary norm. The approach is based on a Lemma in [25] developed by Blatt in [6] for the case of the uniform norm. We also consider the characterisation property of an element of V which is only locally a best approximation, illustrated by the family of generalised rationals.

The second problem is that of obtaining a polynomial (or rational) expression in x to approximate a function while at the same time, the derivative of the polynomial approximates the derivative of the function. In the situation that our function is the solution of a linear second order differential equation, we consider instead, determining a pair of polynomials which satisfy exactly a perturbed system. This technique is known as the Lanczos Tau Method. An error analysis is given for a variety of problems treated in the three Parts of Chapter Three.

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CHAPTER ONE

INTRODUCTION AND REVIEW

1. THE APPROXIMATION OF A SINGLE FUNCTION

1.1. Definition of the Problem

Given a normed linear space (N.L.S.) X (see below), V a subset of X with $f \in X$ but $f \notin V^o$, the closure of V.

If there exists a $v_o \in V$ such that

$$||f - v_o|| \leq ||f - v|| \quad \text{for all } v \in V$$

then v_o is said to be a best approximation (b.a.) to f from V.

It is called a local best approximation if it is a best approximation to f in some norm neighbourhood of v_o.
The aspects that concern us in this thesis are the existence, characterisation and uniqueness of v_o.

We shall set $\rho_V(f) := \inf_{v \in V} ||f - v||$

We define the metric projection associated with f to be

$$P_V(f) := \{v \in V : ||f - v|| = \rho_V(f)\}$$

If $P_V(f) \neq \phi$ we shall say V is proximinal.

If $P_V(f)$ contains exactly one element for each $f \in X$ we shall say V is Chebyshev.

The following normed linear spaces are of interest to us:

(i) C(B,H) The space of continuous functions mapping B, a compact Hausdorff space, into an inner product space H over the real or complex numbers.
The inner product induces a norm on $h \in H$ which we write

$$||h||_H := \sqrt{\langle h,h \rangle}$$

We now define the norm of $f \in C(B,H)$ by

$$||f|| := \sup_{x \in B} ||f(x)||_H .$$

(ii) C(B) This is the space C(B,H) with H either the reals R or the complex numbers $\mathbb{C}$.

(iii) $C_o(B)$ The space of real valued continuous functions on B, B a locally compact Hausdorff space, that vanish at infinity. Thus $f \in C_o(B)$ if and only if f is continuous and for each $\varepsilon > 0$,

$$\{x \in B : |f(x)| \geq \varepsilon\} \quad \text{is compact.}$$

The spaces in (ii) and (iii) are assumed endowed with the supremum (or Chebyshev) norm

$$||f|| \ := \ \sup_{x \in B} |f(x)|$$

(iv) L_p (S, Σ, μ), $p \geq 1$ (see Appendix I)

We note that an atom is a set $A \in \Sigma$ with $0 < \mu(A) < \infty$ such that $A' \in \Sigma$, $A' \subset A$ implies that either $\mu(A') = 0$ or $\mu(A') = \mu(A)$. (S, Σ, μ) is called purely atomic if S is the union of atoms.

1.2. V a Linear Subspace

The Classical Existence and Uniqueness Theorem

If V is finite dimensional then V is proximinal.
If, furthermore, X is strictly convex, i.e. $||f|| = ||g|| = 1$ and $f \neq g$ imply $||\frac{1}{2}(f + g)|| < 1$, then V will be Chebyshev.
The normed linear spaces ℓp and $Lp\,[a,b]$, $1 < p < \infty$, are strictly convex but not $C(B)$ nor $L_1[a,b]$.
All these results are proven in the Introduction of [63].

1.2.1. Linear Uniform Approximation on C(B)

The Kolmogoroff (K-) criterion I, formulated in 1948, is the fundamental characterisation of the best approximation, (see [48] Theorem 18) namely v_o is a best approximation to f if and only if

$$\min_{x \in M_{f-v_o}} \operatorname{Re}\{\operatorname{sgn}\overline{(f(x) - v_o(x))}\, v(x)\} \leq 0 \text{ for all } v \in V$$

where $M_{f-v_o} := \{x \in B : f(x) - v_o(x) = \pm\, ||f - v_o||_\infty\}$

and $$\operatorname{sgn}(z) = \begin{cases} \dfrac{z}{|z|} & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$$

Now suppose V is n(finite)-dimensional and satisfies the Haar condition, namely every function in V can have at most n-1 zeros or vanishes identically. Equivalently, for every set of n distinct points $x_1, \ldots, x_n \in B$

$$\det\{\phi_i(x_j)\} \neq 0 \quad \text{where } V = \text{span}\{\phi_1, \ldots, \phi_n\}$$

Then among the properties enjoyed by the best approximation are the following:

(P1) The best approximation is unique;

(P2) The best approximation possesses strong unicity in the sense of Newman and Shapiro [53] , i.e. for each $f \in X$ there exists a number $\gamma = \gamma(f)$ with $0 < \gamma \leq 1$ such that
$||f - v|| \geq ||f - P_V(f)|| + \gamma||P_V(f) - v||$ for every $v \in V$;

(P3) P_V is pointwise Lipschitz continuous, i.e. for each $f \in X$, there exists a number $\lambda(f) > 0$ such that
$||P_V(f) - P_V(g)|| \leq \lambda||f - g||$ for every $g \in X$;

(P4) The best approximation possesses an alternant (or extremal signature) of length n + 1 (the classical equioscillation theorem);

(P5) De la ValleePoussin's Theorem provides a lower bound for $\rho_V(f)$.

1.2.2. Linear Approximation on $L_1(S, \Sigma, \mu)$

When S is the interval [0,1] and μ is the Lebesgue measure, the characterisation of the best approximation has been obtained by the analytic study of the Lebesgue integral (see [62], chapter 4). For a comprehensive paper see [40] .

The Haar condition on V is sufficient to guarantee uniqueness when f is continuous and $V \subset C(B)$. This was first found by Jackson [37] for the real case.

The approach that interests us here was mainly developed by Singer [68] as a unifying theory for all normed spaces. (Other contributions are in [61] and [23] .) It is based on the Hahn-Banach theory that to any element g of an N.L.S. X there exists a complex-valued linear functional L in the dual space X* with

$$\text{Re } L(g) = ||g|| \text{ and } ||L|| = 1.$$

We denote by B^* the set of $\{L \in X^* : ||L|| = 1\}$.

The generalisation of the K-criterion I is the following version II:
v_o is a best approximation to f if and only if

$$\min_{L \in \Sigma_{f-v_o}} \operatorname{Re} L v \leq 0 \text{ for all } v \in V$$

where $\Sigma_{f-v_o} := \{L \in B^* : L(f - v_o) = ||f - v_o||\}$

We remark that Σ_{f-v_o} can be replaced by the extreme points (ext) of that set, denoted by E_{f-v_o} (see [15] , Lemma 2) and by [14], p.30,

$$E_{f-v_o} = \{L \in \operatorname{ext}(B^*) : L(f - v_o) = ||f - v_o||\}$$

We can re-express K-criterion II for an N.L.S. for which $L \in \operatorname{ext}(B^*)$ has an explicit representation, as follows. Assume $f,g,h \in X$.

(i) $C(B)$ and $C_o(B)$

$Lg = \varepsilon(x)\, L_x g$ where $L_x g = g(x)$ is the point evaluation functional at $x \in B$ and $\varepsilon(x) = \pm 1$. In fact $\varepsilon(x) = \operatorname{sgn}\left(\overline{f(x) - v_o(x)}\right)$ in order that $L(f - v_o) = ||f - v_o||_\infty$. Hence we can rederive K-criterion I.

(ii) $L_1(S, \Sigma, \mu)$

(a) If S is the interval $[0,1]$ and μ is the Lebesgue measure, then

$$Lg = \int_0^1 \operatorname{sgn}\left[f(x) - v_o(x)\right] g(x)\, dx + \int_{Z(f-v_o)} \sigma(x) g(x)\, dx$$

where $Z(h) = \{x \in [0,1] : h(x) = 0\}$ and $|\sigma(x)| = 1$ on $Z(f - v_o)$

(b) If $S = \bigcup_{i \in I} A_i$, A_i an atom, I countable, then

$$Lg = \sum_{i \in I} \operatorname{sgn}\left[f(A_i) - v_o(A_i)\right] g(A_i)\mu(A_i) + \sum_{i \in Z(f-v_o)} \sigma(A_i)\, g(A_i)$$

where $Z(h) = \{i \in I, h(A_i) = 0\}$, $|\sigma(A_i)| = 1$

and $g(A_i)$ denotes the constant value which g has a.e. on A_i.

Now suppose V is n-dimensional and is an interpolating subspace, that is for every set of n linearly independent functionals, $L_1, \ldots, L_n$ in ext(B^*)

$$\det [L_i(\phi_j)] \neq 0 \text{ where } V = \text{span } [\phi_1, \ldots, \phi_n]$$

For X = C(B) or C_o(B) this is equivalent to V satisfying the Haar condition (see [3] , Theorem 3.2.)

For X = $L_1(S, \Sigma, \mu)$ we have the result that it contains on interpolating subspace of dimension n > 1, if and only if, S is the union of at least n atoms (see [3] , Theorem 3.3, [59],Section 2).

Now the important consequence is that the best approximation from an interpolating subspace enjoys properties (P1) - (P3) and generalisations of (P4) and (P5). (See [3]).

In Chapter II section 5 , we will find it advantageous to restrict the interpolating condition $\det [L_i(\phi_j)] \neq 0$ to a subset of ext(B^*) of finite cardinality, say m > n. We shall then say V is an interpolating subspace on $\{L_i\}_{i=1}^m$ We may then assert that $\inf |\det [L_i(\phi_j)]| > 0$ where the inf is taken over selections of n linearly independent functionals.

1.3. V a Non-linear Subset

The existence of a best approximation cannot in general be guaranteed and so we assume $P_V(f) \neq 0$.

We now state some useful examples of non-linear subsets which depend on a finite-dimensional parameter set D, i.e. $V \equiv \{v(a) : a \in D\}$.

(i) The generalised rationals

Let $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_m\}$ be fixed sets of linearly independent real-valued continuous functions on B.

Let $P = \text{span}\,\{g_1, \ldots, g_n\}$

$Q = \text{span}\,\{h_1, \ldots, h_m\}$

and $Q^+ = \{q \in Q,\ q(x) > 0 \text{ on } B\}$.

Then we have the following rational families:-

$R_{n,m} := \{p/q : p \in P,\ q \in Q \quad q \not\equiv 0\}$

$\hat{R}_{n,m} := \{p/q : p \in P,\ q \in Q \quad q \geqslant 0\}$, assumed non-empty

$R^+_{n,m} := \{p/q : p \in P,\ q \in Q^+\}$, assumed non-empty.

(ii) The γ-polynomials (see [62] Chapter 8)

Let $\gamma(t,x)$ be a real valued function on $T\times[0,1]$ where T is a subset of $(-\infty,\infty)$. For a fixed positive integer N

$$V = \{\sum_{j=1}^{N} a_j\, \gamma(t_j,x) : a_j \in R;\ t_j \in T; \quad j = 1, \ldots, N\}$$

Illustrations of γ-polynomials

(a) The sums of exponentials

Take $\gamma(t,x) = e^{tx}$ with $T = R$

(b) The sums of elementary rational functions

Take $\gamma(t,x) = \frac{1}{t+x}$ with $T = (0,\infty)$.

1.3.1. Non-linear Uniform Approximation on C(B)

A sufficient condition for v(a) to be a best approximation to f is that it satisfies the K-criterion III

$$\min_{x \in M_{f-v(a)}} \mathrm{Re} \quad \mathrm{sgn}\,\overline{[f(x) - v(b,x)]}\,[v(b,x) - v(a,x)] \leq 0$$

for all $v(a) \in V$

(see [48] Theorem 86)

It was found by Meinardus and Schwedt [49] that if V is asymptotic convex then K-criterion III is also necessary for a best approximation.

Definition: V is asymptotic convex if for each pair (a,b) of elements in D and each real t, $0 \leq t \leq 1$, there exists a parameter $a(t) \in D$ and a continuous real-valued function g(x,t) on B x[0,1] with $g(x,0) > 0$ on B, such that

$$||(1-t\,g(x,t))(v(a,x)) + t\,g(x,t)v(b,x) - v(a(t),x)|| = 0(t)$$

as $t \to 0$.

A certain differentiability property equivalent to asymptotic convexity is shown in [39]. The following are asymptotic convex

i) convex sets

ii) $R^{+}_{n,m}$

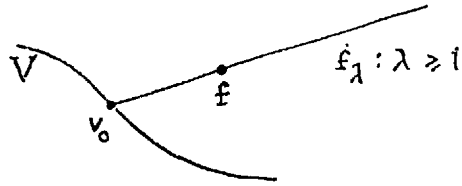
iii) The sums of exponentials.

1.3.2. Suns and Regularity

Suns were introduced by Efimov and Steckin in [30] to assist in characterising Chebyshev sets by geometrical properties. For example, in a finite dimensional normed linear space, every Chebyshev set is a sun. Moreover, using the concept of suns, Brosowski was able to characterise the subsets of C(B) for which the K-criterion III is necessarily satisfied by a best approximation.

Definition. A proximinal subset V of a normed linear space X is a sun (α-sonne in [15] and strict sun in [9]) if for any $f \in X$ and for all $v_o \in P_V(f)$, we have $v_o \in P_V(v_o + \lambda(f-v_o))$, $\lambda \geq 1$, that is, all elements of the form $f_\lambda := v_o+\lambda(f-v_o)$, with $\lambda \geq 1$, have, likewise, v_o as their best approximation from V.

A comprehensive account of properties of suns is in [69] .



We observe that this concept is valid in arbitrary normed spaces.

It is possible to characterise suns by properties not referring to 'best approximation' by using the concept of regularity, first introduced for the space C(B,H), (see [11], [13] and [15]).

Definition. A subset V of C(B,H) is regular I

if (i) for each pair of elements v, $v_o \in V$ and

(ii) for each closed subset $A \subset B$ and

(iii) for each $f \in C(B,H)$ with

(R*) $\quad \text{Re} \langle f(x), v(x) - v_o(x) \rangle > 0$ for $x \in A$,

and (iv) for each real number $\lambda > 0$, we have that there exists an element $v_\lambda \in V$ satisfying the following properties :

(R1) $2 \text{ Re} \langle f(x), v_\lambda(x) - v_o(x) \rangle > ||v_\lambda(x) - v_o(x)||_H^2$ for $x \in A$

(R2) $||v_\lambda - v_o|| < \lambda$

In [12], (ii) is dropped and $A = \{x \in B : ||f(x)||_H = ||f||\}$.

We refer to this definition of regularity as version II.

If $H = R$, then version I can be simplified. (R*) reduces to $|v(x)-v_o(x)| > 0$ for $x \in A$

while (R1) becomes

$$(f(x) - v_o(x)).(v_\lambda(x) - v_o(x)) > 0 \quad \text{for } x \in A.$$

We call this regularity version III.

If we let $K(v_o,f)$ denote the cone

$$\{v \in V : \inf_{L \in E_{f-v_o}} \text{Re } L(v - v_o) > 0\}$$

then version II becomes, for $H = R$, the following version IV, (see [16]):

$K(f, v_o) \cap V \neq 0$ implies that v_o is in the closure of the set $\{v \in V : (f(x) - v_o(x)).(v(x) - v_o(x)) > 0 \text{ for all } x \in M_{f-v_o}\}$.

We can now state that for $X = C(B,H)$ $(C(B,R))$ and V a proximinal subset of X, the following are equivalent ([8] , Theorem 4.1.):

(A) $v_o \in V$ is a best approximation to $f \in X$ implies that v_o satisfies K-criterion III.

(B) V is a sun.

(C) V is regular I or II (III or IV).

(D) Every local best approximation in V is a global best approximation.

A comprehensive survey is in [8] with a section on characterising Chebyshev sets.

1.3.3. Local Linearisation in Uniform Approximation

If D is taken to be an open subset of E^n and V has a Fréchet derivative at each $a \in D$ (see Chapter II, Section 4), further results can be obtained by considering the linear tangent space $\mathcal{L}[v(a)]$, with dimension d(a), at the best approximation v(a).

Non-linear Uniqueness Theorem ([48] , Theorem 90)
The best approximation, v(a) is unique if all the following are satisfied:

(U1) V has a Fréchet derivative at each $a \in D$,

(U2) $\mathcal{L}[v(a)]$ satisfies the Haar condition,

(U3) V has property Z of degree d(a) at a, i.e. for all $b \in D$, $v(a,.) - v(b,.)$ possesses at most $d(a) - 1$ zeros on B or vanishes identically

(U4) V is asymptotic convex.

Tangential characteristics are further discussed in [71] .
We note that the K-criterion I is not necessarily satisfied by a local best approximation (see [14], p.27). However, the K-criterion I is necessarily satisfied on $\mathcal{L}[v(a)]$ by a (local) best approximation even though it is not in general sufficient (see [17], p.374).

Rational approximation by real-valued ordinary polynomials was first considered by Chebyshev and De la Vallée Poussin who obtained existence, uniqueness and (P4),(P5) type results (see [48], Theorem 98).
Further investigations were carried out by Cheney and Loeb [19] . In [18] , chapter 5, properties (P1) - (P5) are derived.

1.3.4. Characterisation of a (local) best approximation in an N.L.S.

Brosowski in [10] and [14] extended K-criterion II and found that a sufficient condition for $v_o \in V$ to be a best approximation to f is that it satisfies the global K-criterion IVa

$$\min_{L \in E_{f-v_o}} \text{Re } L(v - v_o) \leq 0 \text{ for all } v \in V.$$

To obtain the most general necessary condition satisfied by a local best approximation, Brosowski, in [15], developed the following concept . Let $K[v_o, V]$ be the non-empty cone with apex 0 consisting of the set of elements $g \in X$ such that for each neighbourhood U of g and for all $\varepsilon > 0$, there exists a real number η, $0 < \eta < \varepsilon$ and an element $g' \in U$ with $v_o + \eta g' \in V$.

Now if $v_o \in V$ is a (local) b.a. to f, then it satisfies the following local K-criterion IVb

$$\min_{L \in E_{f-v_o}} \text{Re } L(h) \leq 0 \quad \text{for all } h \in K[v_o, V].$$

A full review of these K-criteria and how various Fréchet and Gateaux tangent spaces are included in $K[v_o, V]$ is in [26].

Brosowski, in [15], showed that the K-criterion IVa will be satisfied by the best approximation if V is a sun. He also generalised the definition of regularity I and III to the following version V adopted in Chapter II (with the suffix V ommitted) and showed that if V is regular V than it is a sun.

<u>Definition</u> $V \subset X$ is regular V at a point $v_o \in V$ if for each $v \in V$ and for each closed subset A of B^* with $\text{Re } L(v - v_o) > 0$ for all $L \in A$, and for each real number $\lambda > 0$, there exists a $v_\lambda \in V$ with

(R1) $\quad \text{Re } L\ (v_\lambda - v_o) > 0$ for all $L \in A$

(R2) $\quad \| v_\lambda - v_o \|$

The subset V of X is regular V if it is regular V at every point of V.

Lemma

If V is star-shaped with respect to $v_o \in V$, then V is regular $\underline{V}$ at v_o
This result is given by Brosowski [15] p.155.

PROOF

Let $v \in V$ and $v \neq v_o$.
Furthermore, let A be a ω^* closed subset of B^* with $\mathrm{Re}\, L(v - v_o) > 0$ for all $L \in A$

For $\lambda > 0$ set $v_\lambda = (1 - \frac{\mu}{||v - v_o||}) v_o + \frac{\mu}{||v - v_o||} v$

with $0 < \mu < \min(||v - v_o||, \lambda)$.

Then $v_\lambda \in V$ and (R1) and (R2) are satisfied.

It follows that we have the important cases of linear spaces and convex sets for the approximating family included in the category of regular $\underline{V}$ subsets of X.

A variation is to generalise version IV to the following version VI based on E_{f-v_o} :

A subset V of X is regular VI (or a 'moon' in [2]) if

$$(M) \quad K(v_o, f) \cap V \neq \phi \Rightarrow v_o \in (K(v_o, f) \cap V)^o$$

A sun in any normed linear space satisfies condition (M). Those spaces X for which the condition (M) on V implies that V is a sun are called MS spaces. Examples of MS spaces, given in [2], are (a) $C(B,R)$, (b) $C_o(B)$, (c) $L_1(S, \Sigma, \mu)$ where (S, Σ, μ) is a σ-finite measure space which is purely atomic.
Related results are in [16].
The conditions of regularity $\underline{V}$ can be further modified (see [17]) so that

V is a sun $\Leftrightarrow$ V is 'regular'.

A recent survey on nonlinear approximation in an N.L.S. is in [9].

For $V = R^+_{m,n}$, I have derived, in [32], generalisations of (P1), (P2), (P4), (P5), for a local best approximation from an interpolating subspace.

2. SIMULTANEOUS APPROXIMATION OF A SET F OF FUNCTIONS

2.1. F a Set of Real-valued Functions

2.1.1. V a Linear Subspace of C(B,R)

For f, a real-valued bounded function defined on the closed interval I, and $V = \pi_n$, the polynomials of degree less than or equal to n, Remes showed in 1934, that the determination of $q \in \pi_n$ such that

$$||f - q||_\infty = \rho_{\pi_n}(f)$$

was equivalent to the simultaneous and one-sided approximation of two bounded functions $f_1(x)$ and $f_2(x)$, on I, where

$$f_1(x) := \lim_{y \to x} \sup f(y) \quad \text{or} \quad \inf_{\delta > 0} \sup_{0 \leq |x-y| < \delta} f(y)$$

is the upper envelope of f and is upper semi-continuous (u.s.c.);

$$f_2(x) := \lim_{y \to x} \inf f(y) \quad \text{or} \quad \sup_{\delta > 0} \inf_{0 \leq |x-y| < \delta} f(y)$$

is the lower envelope of f and is lower semi-continuous (l.s.c.).

We recall that $f: I \to R$ is upper (lower) semi-continuous if for each $x \in I$ and each real number c with $c > f(x)$ $(c < f(x))$ there exists an open neighbourhood $U(x) := (y: |y-x| < \delta)$ with $c > f(y)$ $(c < f(y))$ for all $y \in U(x)$.

The above approximation is simultaneous and one-sided in the sense that

$$\rho_{\pi_n}(f) = \inf_{p \in \pi_n} \max \{\max_{x \in I} (f_1(x) - p(x)), \max_{x \in I} (p(x) - f_2(x))\}$$

Dunham [28] extended the scope of this problem to approximating a family F of a finite number of continuous real-valued functions by P_n, an n-dimensional Haar subspace of C(B,R), i.e. determining

$$\rho_{P_n}(F) = \inf_{p \in P_n} \sup_{f \in F} ||f - p||_\infty$$

by reducing it to the simultaneous approximation of the two continuous functions $\max_{f \in F} f(x)$ and $\min_{f \in F} f(x)$.

Finally, Diaz and McLaughlin showed in [24] that the simultaneous approximation of a non-empty family F of real-valued bounded functions from P, a linear subspace of C(B,R), can be reduced to approximating

$$F^+ := \lim_{y\to x} \sup \sup_{f\in F} f(y)$$

and

$$F^- := \lim_{y\to x} \inf \inf_{f\in F} f(y)$$

with $\rho_P(F) = \inf_{p\in P} \max \{||F^+ - p||_\infty , ||F^- - p||_\infty\}$

However, this result can be misleading, since a more careful analysis of [24] reveals that, in fact, the approximation is one-sided and

$$\text{(FM1)} \quad \rho_P(F) = \inf_{p\in P} \max \{\sup_{x\in I} (F^+(x) - p(x)), \sup_{x\in I} (p(x) - F^-(x))\}$$

<u>Definition:</u> $v_o \in V$ is a simultaneous best approximation (s.b.a.) to F if

$$\sup_{f\in F} ||f - v_o|| = \rho_V(F).$$

A practical motivation for determining the s.b.a. to a set of functions arises when we try to approximate a single function that depends on a finite set of parameters; for example $x \to f(\lambda_1,\ldots,\lambda_m; x)$ $x \in B$.

It may be that $\lambda_1,\ldots,\lambda_m$ are not known exactly, for instance if they are obtained from experimental data or from computational programs based on interval arithmetic. However, suppose that upper and lower bounds are available, i.e. $\lambda_i \in [\alpha_i, \beta_i]$ and that the corresponding set of functions F is bounded uniformly on B. It would be reasonable that we should want to obtain one s.b.a. to the whole family F defined by this spread, from the approximating family V.

We shall set $\Delta(v) := \sup_{f\in F} ||f - v||$ and denote by $V^\perp$ the set $\{L \in X^* : Lv = 0 \text{ for all } v \in V\}$.

2.1.2. V a linear subspace of an N.L.S.

The simultaneous approximation of a compact set F by a closed convex subset G of an arbitrary N.L.S., X, was first treated by Laurent and Tuan in [46]. They based their characterisation of the s.b.a. on an expression for the sub-differential of a convex functional, a technique developed by Laurent in [45].

They took K, a symmetric weak* compact and norm-bounded subset of X* and defined a continuous semi-norm p, on X, by

$$p(x) = \max_{k \in K} k(x), \; x \in X$$

They introduced the convex, l.s.c. function d, on X, given by

$$d(g) = \max_{f \in F} p(g - f), \; g \in X$$

and sought the 'solution' $g^* \in G$ such that

$$d(g^*) < d(g) \text{ for all } g \in G.$$

They took H, a weak* compact and norm-bounded subset of X* and a weakly continuous functional ω on H, and defined a convex subset C by

$$C := \{x \in X: \text{for all } h \in H, \; h(x) \leq \omega(h)\}$$

For $G = C \cap V_n$ where V_n is an n-dimensional subspace of X and under the assumption that there exists a $g_0 \in V_n$ such that

$$h(g_0) \leq \omega(h) \text{ for all } h \in H$$

they obtained

(LT) (Theorem 2.1) $g^* \in C \cap V_n$ is a solution if and only if there exist

(i) r functionals $k_1, \ldots, k_r \in K \quad (r \geqslant 1)$,

(ii) r elements $f_1, \ldots, f_r \in F$ (not necessarily distinct), satisfying

$$k_i(g^* - f_i) = p(g^* - f_i) = \max_{f \in F} p(g^* - f_i), \; i = 1,\ldots,r.$$

(iii) s functionals $h_1, \ldots, h_s \in H \quad (s > 0)$, satisfying

$$h_i(g^*) = \omega(h_i), \; i = 1,\ldots,s, \text{ with } r + s \leqslant n + 1.$$

(iv) r + s positive scalars $\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_s$, such that

$$\sum_{i=1}^{r} \lambda_i k_i + \sum_{i=1}^{s} \mu_i h_i \in V^{\perp}.$$

Under the further assumption that K and H are convex and the functional ω concave, the K and H in (LT) can be replaced by their extreme points. In the applications, knowledge of expressions for the extreme points plays an important role.

In particular, when $K = B^*$ then $p(x) = ||x||$, and for $G = V_n$ they obtained

Corollary. $g^* \in V_n$ is a s.b.a. to F if and only if there exist

(i) r functionals $k_1, \ldots, k_r \in \text{ext}(B^*)$

(ii) r elements $f_1, \ldots, f_r \in F$ (not necessarily distinct)

(iii) r positive scalars $\lambda_1, \ldots, \lambda_r$ with $1 \le r \le n + 1$ such that

(1) $k_i(g^* - f_i) = ||g^* - f_i|| = \Delta(g^*)$, $i = 1, \ldots, r$.

(2) $\sum_{i=1}^{r} \lambda_i k_i \in V^{\perp}$.

They gave applications with numerical illustrations employing Remes' algorithm provided $K = B^*$ and $V = \pi_n$, for the cases when X was

(i) $C(B,R)$ with the added constraint $H = \{$positive point evaluation functionals on a compact subset U of $B\}$.

(ii) $C_1[a,b]$ the space of real differentiable functions on $[a,b]$ with the norm

$$||f||' = \max_{t \in [a,b]} |f(t)| + \nu \max_{t \in [a,b]} |f'(t)| \; ; \; \nu > 0$$

(iii) $L_1(S, \Sigma, \mu)$ with $S = [0,1]$ and μ, the Lebesgue measure.

Freilich and McLaughlin [33] suggest a direct approach for the problem of simultaneous approximation in an N.L.S. by P, a linear subspace, encompassing the special cases of F being compact and F being only norm bounded.

We take K, a subset of B^*, such that (1) K is weak* compact and (2) for every $f \in F$ and $p \in$ ˉ. there exists an $L \in K$ with $L(f - p) = ||f-p||$.

We define $U_F(L) := \sup_{f \in F} Lf$ for $L \in K$ and $P(L) := Lp$.

We find

(FM2) $$\Delta(p) = \sup_{L \in K} [U_F(L) - p(L)].$$

We now obtain the upper envelope of $U_F(L)$ by taking the collection $\eta(L)$, of all ω^* open neighbourhoods in K of L and setting

$$U_F^+(L) := \inf_{W \in \eta(L)} \sup_{\ell \in W} U_F(\ell).$$

We obtain the following results:

(FM3) $U_F^+(L)$ is u.s.c. in the ω^* topology on K and

$$\Delta(p) = \sup_{L \in K} [U_F^+(L) - p(L)]$$

(FM4) $p_o \in P$ is a s.b.a. to F if and only if for each $p \in P$ there exists an $L \in K$ such that (1) $U_F^+(L) - p_o(L) = \Delta(p_o)$ and

(2) $p(L) \leq 0$.

(FM5) If F is a compact subset of X then

(i) $U_F(L)$ is ω^* continuous on B^* and $U_F(L) \equiv U_F^+(L)$,

(ii) for every $L \in B^*$, there exists an $f \in F$ such that $U_F(L) = Lf$,

(iii) $U_F(L)$ is a convex function on B^*.

(FM6) $p_o \in P$ is a s.b.a. to F, a compact subset of X, if and only if for each $p \in P$, there exists an $L \in \text{ext}(B^*)$ and an $f \in F$ such that

(1) $L(f - p_o) = \Delta(p_o)$

(2) $Lp \leq 0$.

When $P = V_n$ we obtain an equivalent result to the corollary of Laurent and Tuan.

(FM7) If F is a bounded subset of $C(B,R)$ and $K = ext(B^*)$ then

$$U_F^+(L_x) = F^+(x) \qquad x \in B$$

$$U_F^+(-L_x) = -F^-(x) \qquad x \in B$$

where L_x is the point evaluation functional at $x \in B$.

Hence from (FM4), we obtain the following characterisation of (FM1).

(FM8) $p_o \in P$ is a s.b.a. to F, a bounded subset of $C(B,R)$ if and only if for each $p \in P$, there exists an $x \in B$ such that

either $\quad F^+(x) - p_o(x) = \Delta(p_o)$

and $\quad p(x) \leq 0$

or $\quad p_o(x) - F^-(x) = \Delta(p_o)$

and $\quad p(x) \geq 0.$

This is a reduced version of Theorem 3.1. in [25].

2.2 F a set of Complex-Valued Functions

2.2.1. V a linear Subspace of C(B)

When the functions in F are complex-valued and bounded on B, which is now a compact metric space containing at least n points, and $V = P_n$, an n-dimensional Haar subspace of C(B), Diaz and McLaughlin, in [25], transformed the problem into the 'approximation' of a set-valued function h*.

They defined $h(x) := \{ z \in \mathbb{C} \mid f(x) = z,\ f \in F \} \quad x \in B$

and $$h^*(x) := \bigcap_{\varepsilon > 0} \Big(\bigcup_{|x-y|<\varepsilon} h(y) \Big)^o \quad x \in B$$

h* is u.s.c. on B and $h^* : B \to K(\mathbb{C})$, the non-empty compact subsets of $\mathbb{C}$.

We recall that for topological spaces X and Y, a set-valued function $f : X \to A(Y)$, the non-empty subsets of Y, is u.s.c. on B if for each $x \in B$ and for every open set $G \subset Y$, with $f(x) \subset G$, there exists an open neighbourhood U(x) with $f(U(x)) \subset G$.

The following lemma played a fundamental role

(DM1) (Lemma 1.1.) Let $x \in B$. Then $z \in h^*(x)$ if and only if there exists a sequence of ordered pairs $\{(x_n, z_n)\}$ such that

(1) $x_n \in B$, (2) $x_n \to x$ as $n \to \infty$, (3) $z_n \in h(x_n)$, (4) $z_n \to z$ as $n \to \infty$.

Setting $$\Delta(p) := \sup_{f \in F} \|f - p\|$$

and $$D[h^*, p] := \{(x,z) \in B \times \mathbb{C} \mid z \in h^*(x) \mid p(x) - z| = \Delta(p)\}$$

they showed

(DM2) (Lemma 1.3) $$\Delta(p) = \sup_{x \in B} \sup_{z \in h^*(x)} |p(x) - z|$$

(DM3) (Theorems 2.1, 2.3) $q \in P_n$ is a s.b.a. to F if and only if for each $p \in P_n$ there exists an $(x,z) \in D[h^*, q]$ satisfying

$$\mathrm{Re}\, \{(q(x) - z)\, \bar{p}(x)\} \geqslant 0.$$

(DM4) (Theorem 2.2) If $q \in P_n$ is a s.b.a. to F and if
(DM*) for every two points (x,z) (x,z') in $D[h^*,q]$ one has

$$\text{Re}\ \{(q(x) - z)\ \overline{(q(x) - z')}\} \geq 0;$$

then q is unique.

2.2.2. V a Non-Linear Subset of C(B)

Blatt in [6] extended the approach of [25] to nonlinear subsets.
Letting $\bar{d}(A,b) := \sup_{a \in A} |a - b|$

he set $g_v(x) := \bar{d}(h^*(x), v(x))$

and $M(v) := \{x \in B : g_v(x) = \Delta(v)\}$.

He obtained the following results

(B1) (Lemma 2.3.) g_v is u.s.c. on B.

(B2) (Lemmas 2.4, 2.5) M(v) and $D[h^*,v]$ are non-empty and compact.

(B3) (Theorem 3.2) A sufficient condition for $v_o \in V$ to be a s.b.a. to F or equivalently a b.a. to an u.s.c. $h^*:B \to K(\mathbb{C})$ is the following K-criterion $\underline{V}a$ on $D[h^*,v_o]$.

$$\min_{(x,z) \in D[h^*,v_o]} \text{Re}\ \{\overline{(z - v_o(x))}(v(x) - v_o(x))\} \leq 0 \text{ for all } v \in V$$

Blatt now defined V to be strongly regular when

(i) for each pair of elements v, $v_o \in V$ and

(ii) for each compact subset $\tilde{B} \subset B \times \mathbb{C}$ with

(R*) $\text{Re}\ [\overline{(z - v_o(x))}(v(x) - v_o(x))] > 0$ for $(x,z) \in \tilde{B}$, and

(iii) for each real number $\lambda > 0$; there exists a $v_\lambda \in V$ satisfying

(R1) $2\ \text{Re}\ \{\overline{(z - v_o(x))}(v_\lambda(x) - v_o(x))\} > |v_\lambda(x) - v_o(x)|^2$ for $(x,z) \in \tilde{B}$

(R2) $||v_\lambda - v_o|| < \lambda$.

(B4) (Theorems 3.5, 3.6) The following are equivalent

(E) $v_o \in V$ is a b.a. to an u.s.c. $h^* : B \to K(\mathbb{C})$ implies that v_o satisfies K-criterion $\overline{V}a$ on $D[h^*,v_o]$

(F) V is strongly regular.

__Definition.__ A compact subset Σ of $B\times\mathbb{C}$ is an extremal set for v_o if K-criterion $\overline{V}a$ is satisfied on Σ .

(B5) (Theorem 3.13) If V is stronly regular and $v_o \in V$ is a b.a. to an u.s.c. $h^* : B \to K(\mathbb{C})$, then v_o is unique if $v(x) = v_o(x)$ on Σ, an extremal set for v_o with $\Sigma \subset D[h^*,v_o]$ implies that $v(x) \equiv v_o(x)$ on B.

Blatt proceeded to consider Fréchet differentiable V.

(B6) (Theorem 3.9) If $v(a) \in V$ is a b.a. to an u.s.c. $h^* : B \to K(\mathbb{C})$ then v(a) satisfies the following K-criterion $\overline{V}b$

$$\min_{(x,z)\in D[h^*,v(a)]} \text{Re } \{\overline{(z - v(a,x))}v'[b,a]\ (x)\} \le 0 \text{ for all } b \in E$$

where $v'[\cdot,a]$ is the Fréchet derivative of v(a) at a. (see Chapter II, Section 4.)

Under further conditions on V, he found K-criterion $\overline{V}b$ to be sufficient for $v(a) \in V$ to be a b.a. to h^*.

(B7) (Theorem 3.15) Extension of the Nonlinear Uniqueness Theorem.
If $v(a) \in V$ is a b.a. to an u.s.c. $h^* : B \to K(\mathbb{C})$ then v(a) is unique if

(i) (U1) and (U2) hold
(ii) (DM*) holds with q replaced by v(a)
(iii) V has property Z of degree $d[a] + 1$ at a,
(iv) V is strongly regular.

Blatt gave a further characterisation and illustrations of strong regularity.

In particular the asymptotic convex sets are strongly regular (Example 3.2). Furthermore, in C(B), if V is strongly regular then it is regular I, while in C(B,R), if V is regular III then it is strongly regular (Theorems 3.3, 3.4).

2.3 Continuation of this Section to Chapter Two

In chapter II, we produce a unifying characterisation theory for the non-linear simultaneous approximation problem in an arbitrary N.L.S., X. The main aim is to extend the K-criterion IV to characterise the s.b.a. when F is a bounded subset of X. We achieve this aim by first modifying the definition of h* for the envelope of F so that it is u.s.c. on the dual space.
We proceed to obtain (almost) equivalent results to (B1) - (B6) with an appropriate concept of regularity (Definition 1.7).

Furthermore, in Section 2, Lemma 2.7, we relate the two definitions for the envelopes of F when X is a real valued N.L.S. We can thereby derive the characterisation result (FM4) as a subcase of this unifying theory. This resolves the question inherent in [25] section 3, case 3.

Finally we develop the characterisation of a local s.b.a. or, equivalently, a local best approximation to h*. This has application for the case $V = R^{+}_{m,n}$ which we treat in Section 5.

Generalisations of (P1), (P2) and (P4) for a local s.b.a. from an interpolating subspace are again readily obtained.

We remark that the characterisation theorems of Chapter II are valid for functions defined in more than one variable (see [62], Chapter 12).

It is envisaged that the results of Chapter II would be instrumental in the development of Remes type algorithms for determining an s.b.a.

The theory could also be developed on similar lines in the case when we require our s.b.a. to satisfy constraints.

2.4. Existence of the s.b.a. in an N.L.S.

We now give a brief review on this important aspect.

We shall assume throughout that $\rho_V(F) < \infty$, otherwise any $v \in V$ is a s.b.a.

2.4.1 Some Classical Existence Results

The first existence result for the simultaneous approximation problem was obtained by Diaz and McLaughlin in [25] Theorem 1.1. Thus:

Theorem 2.1. For $X = C(B)$, $V = P$, a finite-dimensional subspace of X, and F, a uniformly bounded subset of X we have that $\rho_P(F) \neq \phi$.
The proof, in outline, is as follows:

(i) We can take a sequence $\{p_n\} \subset P$ such that

$$\lim_{n\to\infty} \Delta(p_n) = \rho_P(F)$$

(ii) $\{p_n\}$ is uniformly bounded and hence contains a convergent subsequence with limit $q \in P$

(iii) $\Delta(q) = \rho_P(F)$

and therefore q is a s.b.a.

We note that the proof is still valid for X, an arbitrary N.L.S.

Furthermore, we can extend the scope of Theorem 2.1. to the following. Let us say V is simultaneous approximatively compact (s.a.c.) in an N.L.S. if to every sequence $\{v_j\}$ in V with

$$\lim_{j\to\infty} \Delta(v_j) = \rho_V(F)$$

there exists a subsequence converging in norm to some element of V. We can now assert

Theorem 2.2. If F is a norm-bounded subset of X, an N.L.S., and V is s.a.c., then $\rho_V(F) \neq \phi$.

Illustration. Take $X = L_p[0,1]$ $1 < p < \infty$ and $V = \hat{R}_{m,n}$. Then V is s.a.c. see [7].

The following two theorems were obtained in [34] for $F = \{f_1, f_2\}$ and are stated in [36] for F, a compact subset of X.

Theorem 2.3. If V is a finite dimensional subspace of X then $\rho_V(F) \neq \phi$ and if X is strictly convex the s.b.a. is unique.

Theorem 2.4. If V is a closed convex subset of a uniformly convex Banach space X, then $\rho_V(F)$ is a singleton.

We can extend the existence proofs devised for F, a singleton, for the following non-linear approximating families on $L_p(S,\Sigma,\mu)$, $1 \leq p < \infty$ when $S = [0,1]$ and μ is the Lebesgue measure.

2.4.2. The Existence of the s.b.a. from the Rationals

Here we can adapt the technique developed by Dunham in [29]. He framed the existence problem in the more general setting of a generalised integral norm which includes all L_p norms $0 < p < \infty$.

Let τ be a non-negative continuous function such that $\tau(t) \to \infty$ as $t \to \infty$. Let $\int$ denote the Lebesgue integral on $[0,1]$ and define $'||g||' = \int\tau(g)$ when g is measurable on $[0,1]$.

In particular, $\tau(t) = |t|^p$, $0 < p < \infty$, relates to L_p norms.

We introduce the following parametrisation for $R_{n,m}$.

Let $A: = \{\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_m\} \in E^{n+m}$

and

$$R(A, x): = \frac{\sum_{i=1}^{n} \alpha_i g_i(x)}{\sum_{i=1}^{n} \beta_i h_i(x)} \in R_{n,m}$$

Without loss of generality we can introduce the following normalisation for R(A,x)

$$\sum_{i=1}^{m} |\beta_i| = 1.$$

However, by $||A||$ we understand $\max\{|a_i|: 1 \leq i \leq n\}$.

We further make the assumption that

$$Q: = \text{span } \{h_1, \ldots, h_m\}$$

has the zero-measure property , namely

$$\text{meas } [Z(q)] = 0 \text{ for all } q \in Q, \ q \not\equiv 0.$$

Theorem 2.5. If F is a set of bounded measurable functions, then under the above assumptions, there exists a s.b.a. to F from $R_{n,m}$.

We shall require the following Lemma proven in [29].

Lemma 2.1. If $\{||A^k||\} \to \infty$, then there exists a non degenerate closed interval I such that

$$M = \inf \{|f(x) - R(A^k, x)| : x \in I\} \to \infty \quad \text{as } k \to \infty$$

Proof of Theorem 2.5.

Let $R(A^k,.)$ be a sequence in $R_{n,m}$ with

$$\lim_{k\to\infty} \sup_{f\in F} ||f - R(A^k,.)|| = \rho_{R_{n,m}}(F)$$

If $\{||A^k||_\infty\}$ is unbounded, we have for each $f \in F$

$$\int \tau(f - R(A^k, .)) > \int_I \tau(f - R(A^k, .)) > \int_I \min[\tau(f(x) - R(A^k, x)]$$

where I is as in the Lemma.

The extreme right side tends to infinity as $k \to \infty$.

It follows that $\sup_{f\in F} ||f - R(A^k, .)|| \to \infty$, giving a contradiction.

Hence $\{||A^k||_\infty\}$ is bounded and $\{A^k\}$ has a limit point A.

Then $\{R(A^k, .)\}$ converges to $R(A,.) \in R_{n,m}$ except on $Z(Q(A,.))$.

Hence for each $f \in F$, $\tau(f - R(A^k, .))$ converges pointwise to $\tau(f - R(A,.))$ except on $Z(Q(A,.))$ which has zero measure.

Applying Fatou's Theorem ([60], p.28) we have for each $f \in F$

$$\begin{aligned} \int \tau(f - R(A, .) &\leq \lim_{k\to\infty} \int \tau(f - R(A^k, .)) \\ &= \lim_{k\to\infty} ||f - R(A^k, .)|| \\ &\leq \lim_{k\to\infty} \sup_{f\in F} ||f - R(A^k, .)|| \end{aligned}$$

Therefore $\sup_{f\in F} ||f - R(A, .)|| \leq \rho_{R_{n,m}}(F)$

The proof is valid for $\hat{R}_{n,m}$.
Furthermore it can be shown, as in [29] section 4, that there exists a s.b.a. from $R^+_{n,m}$ in the L_p norms $1 \leq p < \infty$ when P and Q are the ordinary polynomials.

2.4.3. The existence of the s.b.a. from the γ-polynomials

here we can adapt the technique developed by Barrar and Loeb in [4]. $\gamma(t,x)$ must satisfy certain assumptions (BL) stated in [4], which for the examples (a) and (b) of section 1.3(ii) is the case. We must also allow the best approximation to be in the closure of the approximating family.

Theorem 2.6. Under the assumptions (BL), each set $F \subset L_p[0,1]$ has a s.b.a. in V^o.

We shall require the following Lemma proven in [4].

Lemma 2.2. Let $\{v_k\} \subset V$ be bounded in the Lp norm.
Then under the assumptions (BL) there exists a $v \in V^o$ and a sequence of closed sets $\{U_j\}$ so that

(a) $U_j \subset U_{j+1} \subset [0,1]$ and $\bigcup_j U_j$ differs from $[0,1]$ on a set of measure zero.

(b) For each U_j, some subsequence of $\{v_k\}$ converges to v in the L_p norm restricted to U_j i.e. $||v_{k_i} - v||_j \to 0$.

Proof of Theorem 2.6.

We can choose a sequence $\{v_k\} \subset V$ with

$$\lim_{k\to\infty} \sup_{f\in F} ||f - v_k|| = \rho_{Vo}(F)$$

By the Lemma part (b), for each j

$$\sup_{f\in F} ||f - v||_j = \lim_{i\to\infty} \sup_{f\in F} ||f - v_{k_i}||_j$$

$$\leq \lim_{i\to\infty} \sup_{f\in F} ||f - v_{k_i}||$$

$$= \rho_{Vo}(F)$$

Therefore $\sup_{f\in F} ||f - v|| \leq \rho_{Vo}(F)$

and so v is a s.b.a. to F.

SUMMARY

There are still cases of nonlinear approximations in an N.L.S. where no proof has been provided to guarantee the existence of a best approximation for a single function.
In particular for $L_p(S,\Sigma,\mu)$, where S is the union of the atoms and μ is the counting measure. *

however it would appear from the above Theorems that where there is such a proof, one can generally formulate an existence proof for the corresponding simultaneous problem.

It can still occur that an s.b.a. happens to exist for a particular set F and approximating family V without a priori guarantee and a characterisation of the s.b.a. as derived in Chapter II, will still be valid.

* Wolfe in [70] treats the existence problem of the best approximation to $f \in S(B)$, the linear space of real-valued functions defined on the finite set B, $B: = \{x_1, \ldots, x_N\} \subset [a,b]$ and endowed with an ℓ_p norm, $1 \leq p < \infty$, from the approximating family $R_{n,m}$ and its pointwise closure in $S(B)$, denoted $[R_{n,m}(B)]^o$.

By making the additional requirement that P and Q are Haar subspaces of $C[a,b]$ of dimension n and m respectively with $N \geq n + m + 1$, he obtains an explicit representation of $[R_{n,m}(B)]^o$.

3. SIMULTANEOUS APPROXIMATION OF A FUNCTION AND ITS DERIVATIVE IN THE UNIFORM NORM.

3.1. The Approximation Theory Approach

Let $C^1[I]$ be the space of continuous real-valued differentiable functions on the closed interval I and $V = V_n$. The norm of an element $f \in C^1[I]$ is defined to be the double sup norm -

$$||f|| := \max\left[\max_{x\in I} |f(x)| , \max_{x\in I} |f'(x)|\right]$$

The element $v_o \in V$ which is a best approximation to f enjoys the additional property that its derivative is approximating the derivative of f.

In order to develop a Kolmogoroff characterisation into practical application, we need to know the extremal functionals of the unit ball of $C^1[I]$. These have been found by P. J. Laurent in [44] to be of the form

$$\varepsilon f(x_1) \text{ or } \varepsilon' f(x_2) \text{ where } x_1, x_2 \in I \text{ and } \varepsilon, \varepsilon' \text{ are } \pm 1.$$

In [44], interest centres on the norm

$$||f||' := \max_{x\in I} |f(x)| + \max_{x\in I} |f'(x)|$$

and an extension of Remes' algorithm for determining the b.a. with this norm is given there, while in [46], it is applied to the simultaneous approximation problem.

Aspects of unicity for this type of b.a. with the double sup norm are considered in [52].

Theoretical considerations in L_p norms $p > 1$ are in [51], where constraints are added.

3.2. Tau Method Solution of First Order Differential Equations

3.2.1. Introduction

Consider the first order differential equation

$$Dy = 0 \text{ where } Dy := \frac{dy}{dx} - y$$

subject to the single initial value condition, $y(o) = 1$. The solution is of course $y(x) = \exp(x)$.

In the Tau method, we obtain the exact polynomial solution y_n of

$$Dy_n = \tau^{(n)} H_n$$

where H_n is a pre-determined polynomial of degree n the simplest case being x^n.

We usually want to find a good approximation to y over an interval J. In [41], Lanczos suggested to use for H_n the Chebyshev polynomials of the first kind shifted to J, as these give the best polynomial approximations to zero. If y_n^* satisfies $Dy_n^* \doteq \tau^{(n)} T_n^*$, then $D(y_n^* - y)$ has the equi-oscillation property and the error in the image of D is more evenly distributed on J. This is not to say that y_n^* is then a best uniform approximation to y from π_n, however, it can be shown that y_n^* is asymptotically of the same order of approximation as the best. The technique, devised by Meinardus and Strauer cf. [50], is basically to invert the operator D using the Green's function. This was further refined in [66] by performing an integration by parts.

3.2.2. Validation of the use of Chebyshev Polynomials

To justify the use of Chebyshev perturbations, Rivlin and Weiss in [66] argued as follows. Suppose we seek a polynomial approximation $y_n^* \in \pi_n$ to $\exp(x)$ on $J \equiv [-1,1]$ such that $y_n^*(0) = 1$

and $\| Dy_n^* \|_\infty \leq \| Dy_n \|_\infty$

for all $y_n \in \pi_n$ with $y_n(0) = 1$.

Let $v_n(x) = b_o + b_1 x + \ldots + b_n x^n$

and $y_n(x) = a_o + a_1 x + \ldots + a_n x^n$

with $y_n(0) = 1$ and $Dy_n = v_n$.

Comparing coefficients in the differential equation, we obtain

$$a_j = \frac{1}{j!} \sum_{i=j}^{n} i!\, b_i$$

In particular, since $a_o = 1$ we have

$$L_1 v_n := \sum_{i=0}^{n} i!\, b_i = 1.$$

We have reduced our problem to finding $\bar{v}_n$ such that $||\bar{v}_n||_\infty \leq ||v_n||_\infty$ among all "admissable" v_n satisfying the constraint $L_1 v_n = 1$.

We sketch one approach adopted in [66] for which the following two Lemmas were required.

Lemma 3.1. Suppose the roots, $x_1, \ldots, x_n$ of v_n, $n \geq 1$ are all real and satisfy $x_j \leq 1$, $j = 1, \ldots, n$ with strict inequality holding for at least one j. If $b_n = 1$ then $L_1 v_n > 0$.

Lemma 3.2. Let L be any linear functional on V, a k-dimensional subspace of C(I). Then there exist points $x_1, \ldots, x_r \in I$, $r \leq k$ and non-zero constants $\alpha_1, \ldots, \alpha_r$ such that for any $v \in V$

$$Lv = \sum_{j=1}^{r} \alpha_j\, v(x_j)$$

$$\text{and } ||L|| = \sum_{j=1}^{r} |\alpha_j| .$$

If we take $V = \pi_n$ then $k = n+1$ in Lemma 3.2.

Lemma 3.2 appears in [65] Corollary 3 and in [64] Theorem 2.13 .

We can apply Lemma 3.2 to our functional L_1. For $v = (x-x_1)\dots(x-x_r)$ we have by Lemma 3.1. that r cannot be less then n, so $r = n + 1$. But this implies there exists a $v_n^* \in \pi_n$ with $||v_n^*||_\infty = 1$, $|L_1 v_n^*| = ||L||$ and $|v_n^*(x_j)| = 1$, for $j = 1, \dots, n+1$; see [65] Remark 2, p.676. This can only be if $v_n^* = \pm T_n$. (see [65] Lemma 2). Thus

$||L_1|| = |L_1 T_n| \geq |L_1 v_n|$ for all $v_n \in \pi_n$ with $||v_n||_\infty = 1$, and hence $||\tau^{(n)} T_n||_\infty \leq ||v_n||_\infty$ for all v_n satisfying $L_1 v_n = 1$, when $\tau^{(n)} = \dfrac{1}{L_1 T_n}$.

In [67] Rivlin extended this reasoning to the differential operator

$$Dy := (A + Bx)\, y' + Cy = 0$$

with the boundary condition $y(0) = K$ and showed that for a large class of intervals and values for K,

$$\min_{\substack{y_n \in \pi_n \\ y_n(0) = K}} ||Dy_n||_\infty = ||Dy_n^*||_\infty$$

where y_n^* is the solution of $Dy_n^* = \tau^{(n)} T_n$.

The case of D being the second order differential operator

$$Dy = y'' + c^2 y$$

with the single initial condition given either by $y(0) = 1$ or $y'(0) = 1$, was also considered in [66]. The use of Chebyshev perturbations was justified and error bounds were found that were again asymptotically best possible.

3.2.3. The Canonical Polynomials and the Tau Solution

For an efficient way of determining y_n^*, there is the method of canonical polynomials introduced by Lanczos, see [42], and developed by Ortiz, see [56].

Let D be any linear differential operator with polynomial coefficients. Then $D : \pi_n \to \pi_m$ where $m \geq n$. The canonical polynomials $\{Q_n(x)\}$ for D can be defined by

$$D\, Q_n(x) = x^n \text{ for all } n \in N, \text{ the non-negative integers.}$$

If we use the Chebyshev perturbation and set

$$T_n^*(x) = \sum_{k=0}^{n} c_k^{(n)} x^k$$

then
$$y_n^*(x) = \tau^{(n)} \sum_{k=0}^{n} c_k^{(n)} Q_k(x).$$

We determine $\tau^{(n)}$ from the initial condition $y(0) = \sigma$.

Thus
$$\tau^{(n)} = \sigma / \sum_{k=0}^{n} c_k^{(n)} Q_k(0).$$

$\{Q_n(x)\}$ can generally be found by a recursive technique, when D is a linear operator, as follows. Suppose we know $Q_r(x)$ for $r < m$, and that $D\, x^n = \sum_{r=0}^{m} a_r^{(n)} x^r$.

Then
$$D\left[\frac{1}{a_m^{(n)}} \left(x^n - \sum_{r=0}^{m-1} a_r^{(n)} Q_r(x) \right)\right] = x^m$$

yielding an expression for $Q_m(x)$.

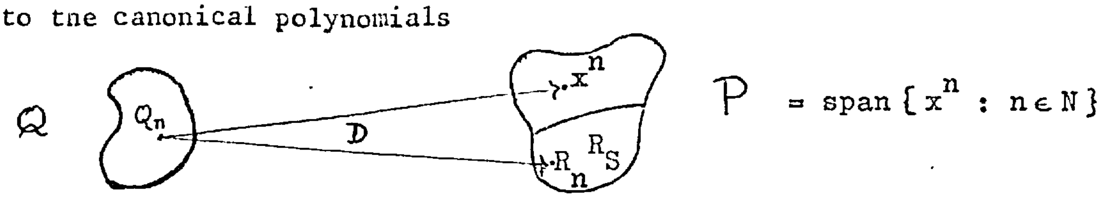
Special attention has to be paid to situations where m is greater than n and where $a_m^{(n)}$ is zero for some values of n. In both these cases there will be, in general, gaps in the sequence $\{Q_n\}$, i.e. there will be an index set S of undefined canonical polynomials such that no Q_ν is known to generate x^ν, $\nu \in S$, in the expansion of $T_n^*(x)$. We remark that S has finite cardinality, see [54]. We set

$$R_S = \text{span}\,\{x^\nu : \nu \in S\}$$

and redefine $\{Q_n(x)\}$ by

$$D\, Q_n = x^n + R_n(x) \quad \text{where } R_n \in R_S.$$

We now have the following diagrammatic representation of the application of D to the canonical polynomials



A further point is that for some n, there may be several Q_{n_i} for which $D\,Q_{n_i} = x^n + R_{n_i}$. However it is readily shown that in such cases

$Q_{n_i} - Q_{n_j} \in U_D$, the kernel of the operator D.

Hence we are lead to define classes of equivalence

$\{\mathcal{L}_n\}$ $n \in N-S$, modular U_D.

If we let E denote this equivalence relationship then we have a quotient set

$\{\mathcal{L}_n\} = \{ Q_{n_i} \} / E$ $\qquad n \in N-S.$

There is now a bijection between

$L \equiv \{\mathcal{L}_n\}$ with $n \in N-S$ and $P - R_S$,

as well as a unique correspondence between L and D, see [54].
In the numerical computation of y_n^* it will be necessary to eliminate any contribution from R_S. This is achieved in practice by having one free parameter for each $\nu \in S$ to match the coefficient of the weighted sum of residuals, with the coefficient of x^ν in H_n. This eliminates the component of Q_ν. For example, if card(S) = 1, we could let $H_n = \tau_0^{(n)} T_n^* + \tau_1^{(n)} T_{n-1}^*$. The same device is employed if there is an extra constraint to be satisfied. In either case, the equi-oscillation property and the argument of 3.2.2. is lost, although good results are nevertheless obtainable.

In [55], Ortiz treats the evaluation of the coefficients of the expansion of y_n^* in an arbitrary system of polynomials which span π_n.
In [56], he discusses the direct generation of the canonical polynomials for a Chebyshev perturbation.

3.3. Continuation of this Section to Chapter Three

In chapter III, we suppose the function y we wish to approximate satisfies a linear second order differential equation with variable rational coefficients and is subject to two initial conditions $y(0) = \sigma$, $y'(0) = \rho$. We treat this subject in three parts. In Parts I and II, we shall be interested in determining two polynomials $(y_n, z_n) \in \pi_n \times \pi_n$ with y_n approximating y and $-z_n$ approximating y' on $J = [0,1]$. (y_n, z_n) is a feasible solution if $y_n(0) = \sigma$, $z_n(0) = -\rho$.

We note that this method requires that we store coefficients of two polynomials, but that there is little disadvantage from this.

According to 3.1, we ought to be minimising $\max[||y - y_n||_\infty, ||y' + z_n||_\infty]$ over all feasible solutions. However, since y and y' are known only implicitly through

$$D \begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where D is now a pair of first order linear differential equations, we search for a feasible solution $(y_n{}^*, z_n{}^*)$ satisfying

$$D \begin{pmatrix} y_n{}^* \\ z_n{}^* \end{pmatrix} = \begin{pmatrix} \tau_1{}^{(n)} T_n{}^* \\ \tau_2{}^{(n)} T_n{}^* \end{pmatrix}$$

with the object of minimising the error in the image of D. In Part I,

We extend 3.2.2. to validate the use of shifted Chebyshev perturbations for the simple case $y'' + y = 0$ subject to $y(0) = 1$ and $y'(0) = 0$. We also introduce the vectorial form of canonical polynomials to aid the construction of the solution $(y_n{}^*, z_n{}^*)$, and then perform an error analysis. In Part II, we cosider applications of some of these ideas to a variety of linear second order differential equations.

In Part III, we consider the use of Legendre perturbations for the case of $y'' + y = 0$ with the modification that we produce a pair of rational forms as our approximation. The error analysis demonstrates an improvement in accuracy for $n \geq 4$, though at extra expense.

CHAPTER TWO

SIMULTANEOUS APPROXIMATION OF A SET OF BOUNDED COMPLEX-VALUED FUNCTIONS

1. INTRODUCTION OF PROBLEM WITH BASIC DEFINITIONS AND RESULTS

Let B be a compact space and S(B) be the linear space of complex-valued functions endowed with a norm $||.||$.

For α, a positive real number, denote by $F(= F(\alpha))$ a nonempty class of complex-valued functions defined on B such that if $f \in F$ then $||f|| \leq \alpha$.

Let C(B) be the set of complex-valued continuous functions defined on B and V(B) be a non-linear subset of C(B).

We wish to characterise the best approximation v_0 from V to F, if it exists,

given by $$\sup_{f \in F} ||f-v_0|| = \inf_{v \in V} \sup_{f \in F} ||f-v||$$

The case of the uniform norm has been treated in [6].

In section 2, we show that this problem is equivalent to finding the best one-sided approximation from V to a ω^* upper semi-continuous function h^* (Definition 1.4) where h^* and V are defined now to be on a ω^* compact subset of the dual space and h^* is set-valued.

In section 3, we obtain a sufficient condition that v_0 satisfies by generalising the Kolmogoroff criterion. Furthermore, by imposing on V that it is regular (Definition 1.7), the Kolmogoroff criterion is found to be a necessary condition for a global best approximation and we can further deduce a uniqueness result.

In section 4, we develop the characterisation of a local best approximation for approximating families which depend on a parameter, with respect to which they have a Fréchet derivative.

This includes the case when V is a set of generalized rational polynomials.

For this example, we develop our results further in section 5, to show that, under appropriate conditions, a local best approximation is (i) locally unique (ii) locally strongly unique and (iii) characterised by a generalised "alternation" theorem .

Notation

Let R, C be the fields of real, complex numbers respectively endowed with the usual metric topologies given by $d(x,y) = |x - y|$.

Let X and Y be topological spaces, X* the dual of X, i.e. the set of complex-valued bounded linear functionals $X \to C$

Let $\quad A(Y) := [E \subset Y \mid E \neq \phi]$

and $\quad \mathcal{K}(Y) := [E \subset Y \mid E$ compact in the topology on Y and $E \neq \phi]$.

E^o denotes the closure of E, $\mathcal{C}(E)$ the complement of E and co(E) the convex hull or cover of E. $W(L,\theta,\varepsilon)$ is a ω^* open neighbourhood (nbhd) of L

i.e. $W(L,\theta,\varepsilon) := \{ \ell \in X^* : |(\ell - L)x| < \varepsilon$ for all $x \in \theta$;

where θ is some finite subset of X and $\varepsilon > 0\}$

Where there is no loss of clarity we abbreviate $W(L,\theta,\varepsilon)$ by $W(L)$ or W.

Definition 1.1.

$f: X^* \to A(Y)$ is ω^* upper semi-continuous (u.s.c.) at $L \in X^*$ if to every open set G with $f(L) \subset G$ there exists a ω^* open nbhd $W(L)$ such that $f(W(L)) \subset G$.

Definition 1.2.

$f: X^* \to R$ is ω^* u.s.c. at $L \in X^*$ if to every real number $c > f(L)$ there exists a ω^* open nbhd $W(L)$ with $f(\ell) < c$ for all $\ell \in W(L)$.

The following Theorems can be obtained by generalisations of standard topological arguments [35] :

Theorem 1.1. If $E \subset X^*$ is ω^* compact and $f: E \to \mathcal{K}(Y)$ is ω^* u.s.c. on E then $f(E)$ is compact in Y.

Theorem 1.2. If $E \subset X^*$ is ω^* compact and $f: E \to R$ is ω^* u.s.c. on E then there exists an $L_o \in E$ such that

$$f(L_o) = \sup_{L \in E} f(L).$$

We recall that to each $x \in X$ we can associate the evaluation $\hat{x}: X^* \to R$ given by $\hat{x}(L) \equiv Lx$. We remark that $\hat{x}$ is continuous. We shall omit the cap in the sequel when portraying x as a function on a subset of X*.

Definition 1.3.

Let K be a subset of B*, the unit ball in X*, satisfying

(i) K is ω^* closed

(ii) For every $f \in F$ and $v \in V$, there exists an $L \in K$ with $\text{Re}L(f - v) = ||f - v||$

Remark: The existence of L in B* is guaranteed by the Hahn Banach Theorem. We shall henceforth take all neighbourhoods of L to be in K.

We understand by $\ell_n \overset{\omega}{\underset{\theta}{\to}} L$ that for this θ and any $\varepsilon > 0$, there exists an $n_0 \equiv n_0(\theta, \varepsilon)$ such that $\ell_n \in W(L, \theta, \varepsilon)$ for all $n \geq n_0$.

The following definitions are generalisations of corresponding ones in [25].

Definition 1.4

Let $h(L) := \{z \in \mathbb{C} \mid \text{there exists an } f \in F \text{ with } f(L) \equiv Lf = z\}$ for $L \in K$

Define $h^*(L) := \bigcap_{\theta, \varepsilon > 0} \left(\bigcup_{\ell \in W(L,\theta,\varepsilon)} h(\ell) \right)^{o}$ for $L \in K$

$h^*(L)$ is a set-valued mapping from K into $A(\mathbb{C})$.

Theorem 1.3.

$h^*(L) = \{ z \in \mathbb{C} \mid$ for any θ there exists a sequence $\{(\ell_n, z_n)\}$

such that (1) $\ell_n \in K$, (2) $\ell_n \overset{\omega}{\underset{\theta}{\to}} L$

(3) $z_n \in h(\ell_n)$ (4) $z_n \to z$. $\}$

PROOF

By definition, $z \in h^*(L)$ implies $z \in \left(\bigcup_{\ell \in W(L,\theta,\varepsilon)} h(\ell) \right)^{o}$ for all $\varepsilon > 0$, and all θ. For each θ then, we have $z \in \left(\bigcup_{\ell \in W(L,\theta,\frac{1}{n})} h(\ell) \right)^{o}$ and so there exists a sequence $\{(\ell_n, z_n)\}$ depending possibly on θ, with $|z - z_n| < \frac{1}{n}$ and $z_n \in h(\ell_n)$ where $\ell_n \in K$ and $\ell_n \in W(L, \theta, \frac{1}{n})$. Conversely, if for each θ and $\varepsilon > 0$, there exists a sequence $\{(\ell_n, z_n)\}$ satisfying the four conditions, then there exists an n_0 such that for $n \geq n_0$, $\ell_n \in W(L, \theta, \varepsilon)$ and by (3)

$z_n \in h(\ell_n) \subset \bigcup_{\ell \in W(L,\theta,\varepsilon)} h(\ell)$.

Now $z = \lim z_n$, therefore $z \in \left(\bigcup_{\ell \in W(L,\theta,\varepsilon)} h(\ell)\right)^o$.

Since the arbitrary intersection of closed sets is again closed,
$z \in \bigcap_{\varepsilon>0} \left(\bigcup_{\ell \in W(L,\theta,\varepsilon)} h(\ell)\right)^o$ and finally, since this is true for each finite θ, the intersection may be taken over all such θ.

<u>Corollary</u>. If $L = \lambda L_1 + (1 - \lambda)L_2$ where $L, L_1, L_2 \in K$ and $0 < \lambda < 1$

$$\text{then} \quad h^*(L) \subseteq \lambda h^*(L_1) + (1 - \lambda)h^*(L_2)$$

PROOF

For any θ and $\varepsilon > 0$, let $W^{(i)}(L_i, \theta, \varepsilon)$ be a ω^* open nbhd of L_i, i=1, 2.
Then $\lambda W^{(1)} + (1 - \lambda) W^{(2)}$ is a ω^* open nbhd of $\lambda L_1 + (1 - \lambda) L_2$

$$\text{setting} \quad \overline{h}(L) := \bigcap_{\theta,\varepsilon>0} \left(\bigcup_{\ell \in \lambda W^{(1)} + (1-\lambda)W^{(2)}} h(\ell)\right)^o$$

it is obvious that $h^*(L) \subset \overline{h}(L)$.

Now $\overline{h}(L) = \{z \in \mathbb{C} \mid$ for any θ, there exists a sequence $\{(\ell_n, z_n)\}$ satisfying (1) - (4) where $\ell_n = \lambda p_n + (1-\lambda) q_n, p_n \in W^{(1)}, q_n \in W^{(2)}\}$

Furthermore $p_n \overset{\omega}{\underset{\theta}{\to}} L_1$, $q_n \overset{\omega}{\underset{\theta}{\to}} L_2$ and since there exists an $f \in F$ with $\ell_n f = z_n$, we have $v_n \in h(p_n)$ and $w_n \in h(q_n)$
and $z_n = \lambda v_n + (1-\lambda) w_n$ with $\lim z_n = z$.
Extracting a subsequence if necessary, we are assured the existence of $v \in h^*(L_1)$ and $w \in h^*(L_2)$ such that

$$z = \lambda v + (1-\lambda) w$$

and
$$\overline{h}(L) \subset \lambda h^*(L_1) + (1 - \lambda) h^*(L_2)$$

Definition 1.5

A non-void subset of $Y \subset X$ is an extremal subset of X if a proper convex combination $\lambda x_1 + (1 - \lambda)x_2$, $0 < \lambda < 1$, of two points $x_1, x_2 \in X$, is in Y only if both x_1 and x_2 are in Y.

An extremal subset of X consisting of just one point is called an extremal point of X.
The collection of extremal points of X is denoted by ext (X).

Lemma 1.1.

If C is a convex and compact subset in R^n then C = co(ext (C).
(See e.g. [60] p.232).

Lemma 1.2

Let ϕ be a continuous linear mapping of E_1 into E_2 (two Hausdorff locally convex topological spaces) and M be a compact subset of E_1. Then for every extremal point e_2 of $\phi(M)$ there exists at least one extremal point e_1 of M such that $\phi(e_1) = e_2$. (See [38], p.333)

Definition 1.6.

A non-empty subset Σ of B^* is sign-extremal for $v_o \in V \subset X$

if $\min_{L \in \Sigma} \operatorname{Re} L(v - v_o) \leq 0$ for all $v \in V$.

Lemma 1.3.

If Σ is a ω^* closed subset of B^* then Σ is sign-extremal for $v_o \in V$ if and only if ext(Σ) is sign-extremal for v_o.
The proof is given in [15] Lemma 2.

We define regular subsets of X in the sense of Brosowski.

Definition 1.7

$V \subset X$ is regular at a point $v_o \in V$ if for each $v \in V$ and each real number $\lambda > 0$ and for each ω^* closed subset A of B^* satisfying $\operatorname{Re} L(v - v_o) > 0$ for all $L \in A$, there exists a $v_\lambda \in V$ with

(R1) $\operatorname{Re} L\,(v_\lambda - v_o) > 0$ for all $L \in A$

(R2) $\| v_\lambda - v_o \| < \lambda$

The subset V of X is regular if it is regular at every point of V.

2. CONVERSION OF PROBLEM TO APPROXIMATION OF h^*

We first deduce a basic property of h^*.

<u>Lemma 2.1.</u>

$h^*(L)$ is ω^* u.s.c. on K and $h^*:K \to \mathcal{K}(\mathbb{C})$.

PROOF

Suppose at $L_o \in K$ it is not ω^* u.s.c.

Then there exists an open neighbourhood G of $h^*(L_o)$ such that for every ω^* open neighbourhood $U(L_o)$ there exists at least one $\ell \in U(L_o)$ with $h^*(\ell) \nsubseteq G$.

For any θ, let $\{U_n\}$ be a neighbourhood basis for L_o

i.e. $U_1 \supset U_2 \supset \ldots \supset U_n \supset \ldots$

with $\ell_n \in U_n$ but $h^*(\ell_n) \nsubseteq G$

i.e. there exists a $z_n \in h^*(\ell_n)$ but $z_n \notin G$.

For $n \to \infty$, $\ell_n \underset{\theta}{\overset{\omega}{\to}} L_o$ and z_n has a cluster point z_o since it is a bounded sequence.

Now $z_n \in h^*(\ell_n)$ implies that there exists a sequence $\{(q_k^{(n)}, \eta_k^{(n)})\}$ with (1) $q_k^{(n)} \in K$, (2) $q_k^{(n)} \underset{\theta}{\overset{\omega}{\to}} \ell_n$, (3) $\eta_k^{(n)} \in h(q_k^{(n)})$, (4) $\eta_k^{(n)} \to z_n$.

For (z_n, ℓ_n) choose k_n such that (i) $q_{k_n}^{(n)} \in U_n$ and (ii) $|\eta_{k_n}^{(n)} - z_n| < \frac{1}{n}$.

Hence for $\{(q_{k_n}^{(n)}, \eta_{k_n}^{(n)})\}$ we have (1') $q_{k_n}^{(n)} \in K$

(2') $q_{k_n}^{(n)} \underset{\theta}{\overset{\omega}{\to}} L_o$ (3') $\eta_{k_n}^{(n)} \in h(q_{k_n}^{(n)})$ (4') $\eta_{k_n}^{(n)} \to z_o$

(1') - (4') imply $z_o \in h^*(L_o) \subset G$.

But $z_n \in \complement(G)$ implies that $z_o \in \complement(G)$, hence a contradiction.

The proof that $h^*(L_o)$ is a closed set is similar and is omitted.

Furthermore, $h^*(L_o)$ is bounded since $||f|| \leq \alpha$ for all $f \in F$ and the neighbourhoods of L_o are subsets of K.

It follows that $h^*(L_o)$ is compact.

The following "distance" function is most suitable for our problem.

Definition 2.1.

$$\hat{d}(A,b) = \sup_{a\in A} \operatorname{Re}(a - b)$$

We are now able to take the first step towards an equivalent formulation of our original problem.

Lemma 2.2.

$$\sup_{f\in F} \|f - v\| = \sup_{L\in K} \hat{d}(h^*(L), v(L)).$$

PROOF

For any $f \in F$ and $L \in K$, $\operatorname{Re}L(f - v) \leq \sup_{z\in h(L)} \operatorname{Re}(z - Lv) \leq \sup_{z\in h^*(L)} \operatorname{Re}(z - Lv)$

But there exists an $L \in K$ such that $\operatorname{Re}L(f - v) = \|f - v\|$

Therefore $\|f - v\| \leq \hat{d}(h^*(L), v(L)) \leq \sup_{L\in K} \hat{d}(h^*(L),v(L))$

The right hand bound is independent of f.

Therefore $\sup_{f\in F} \|f - v\| \leq \sup_{L\in K} \hat{d}(h^*(L), v(L))$.

On the other hand, consider the sequence $\{L_n, z_n\}$ with $L_n \in K$ and $z_n \in h^*(L_n)$ and

$$\lim_{n\to\infty} \operatorname{Re}(z_n - L_n v) = \sup_{L\in K} \sup_{z\in h^*(L)} \operatorname{Re}(z - Lv)$$

By Theorem 1.3. for $\delta = v$, there exists a sequence $\{q_k^{(n)}, \eta_k^{(n)}\}$ with (1) $q_k^{(n)} \in K$, (2) $q_k^{(n)} \overset{\omega}{\underset{v}{\to}} L_n$, (3) $\eta_k^{(n)} \in h(q_k^{(n)})$, (4) $\eta_k^{(n)} \to z_n$.

Choose k_n so that (i) $|\eta_{k_n}^{(n)} - z_n| < \frac{1}{n}$ and (ii) $|q_{k_n}^{(n)} v - L_n v| < \frac{1}{n}$

Then $\|f_{k_n} - v\| \geq \operatorname{Re} q_{k_n}^{(n)} (f_{k_n} - v) = \operatorname{Re}(\eta_{k_n}^{(n)} - q_{k_n}^{(n)} v)$

$$\geq \operatorname{Re}(-L_n v + z_n + L_n v - q_{k_n}^{(n)} v + \eta_{k_n}^{(n)} - z_n)$$

$$\geq \operatorname{Re}(z_n - L_n v) - |\operatorname{Re}(L_n - q_{k_n}^{(n)})v| - |\operatorname{Re}(\eta_{k_n}^{(n)} - z_n)|$$

$$\geq \operatorname{Re}(z_n - L_n v) - \frac{2}{n}$$

Therefore $\sup_{f\in F} \|f - v\| \geq \lim_{n\to\infty} \|f_{k_n} - v\| \geq \lim_{n\to\infty} \operatorname{Re}(z_n - L_n v) = \sup_{L\in K} \hat{d}(h^*(L),v(L))$

A consequence of Lemma 2.2. is that we can reformulate our problem as that of finding the best approximation from V to h* on K using the distance function $\hat{d}$ on $\mathbb{C}$ for approximating a set valued function.
It is desirable to investigate further the function on the right hand side of Lemma 2.2.

Lemma 2.3

Set $g_v(L) := \hat{d}(h^*(L), v(L))$ for $L \in K$

Then g_v is a mapping of K into R and $g_v(L)$ is ω^* u.s.c. on K for each v.

PROOF

Let $L_o \in K$ and $\beta > g_v(L_o)$ with $\varepsilon \equiv \dfrac{\beta - g_v(L_o)}{2}$ and

$$\mathcal{O} \equiv \bigcup_{z \in h^*(L_o)} \mathcal{O}_\varepsilon(z) \quad \text{where } \mathcal{O}_\varepsilon(z) \equiv \{w : |w - z| < \varepsilon\}$$

$\mathcal{O}$ is ω^* open and $h^*(L_o) \subset \mathcal{O}$.

By Lemma 2.1, h* is ω^* u.s.c. at L_o. Hence there exists a ω^* open nbhd $W_1(L_o)$ such that for all $\ell \in W_1(L_o)$, $h^*(\ell) \subset \mathcal{O}$.

But for each $\eta \in h^*(\ell)$ where $\ell \in W_1(L_o)$, there exists a $z_\eta \in h^*(L_o)$ such that $|\eta - z_\eta| < \varepsilon$ by definition of $\mathcal{O}$.

Therefore for $\ell \in W_1(L_o)$, $\hat{d}(h^*(\ell), v(L_o)) = \sup\limits_{\eta \in h^*(\ell)} \text{Re}(\eta - v(L_o))$

$$\leq \sup_{\eta \in h^*(\ell)} \text{Re}\,\{(z_\eta - v(L_o)) - (z_\eta - \eta)\}$$

$$\leq \sup_{z \in h^*(L_o)} \text{Re}\,(z - v(L_o)) + \varepsilon$$

$$= g_v(L_o) + \varepsilon$$

Now $g_v(\ell) = \sup\limits_{z \in h^*(\ell)} \text{Re}(z - v(\ell)) \leq \sup\limits_{z \in h^*(\ell)} \text{Re}(z - v(L_o)) + |v(\ell) - v(L_o)|$

$$= |v(\ell) - v(L_o)| + \hat{d}(h^*(\ell), v(L_o)).$$

Take a ω^* open nbhd $W_1(L_o)$ such that $|\ell v - L_o v| < \varepsilon$ for all $\ell \in W_2(L_o)$

Then for all $\ell \in W_1(L_o) \cap W_2(L_o)$

$$g_v(\ell) < g_v(L_o) + 2\varepsilon = \beta$$

which completes the proof.

We remark that by Theorem 1.2, g_v attains its supremum on K.

Lemma 2.4

$g_v(L)$ is a convex functional on K in the following sense. Suppose $L = \lambda L_1 + (1-\lambda) L_2$ where $L, L_1, L_2 \in K$ and $0 \leq \lambda \leq 1$. Then

$$g_v(L) \leq \lambda g_v(L_1) + (1-\lambda) g_v(L_2).$$

The proof follows from considering $\sup_{z \in h^*(L)} \mathrm{Re}\, z$ and applying the corollary to Theorem 1.3. We now restate our problem as that of finding $\inf_{v \in V} \sup_{L \in K} g_v(L)$ and for convenience introduce the following notations:

$$\Delta(v) \equiv \sup_{L \in K} g_v(L), \quad \rho_v(h^*) = \inf_{v \in V} \Delta(v)$$

Furthermore, we set
$$M(v) \equiv [L \in K \mid g_v(L) = \Delta(v)]$$
$$D[h^*, v] \equiv [(L,z) \in K \times C \mid z \in h^*(L), \ \mathrm{Re}(z - Lv) = \Delta(v)]$$
$$\eta[h^*, v, L] \equiv [z \in h^*(L) \mid \mathrm{Re}(z - Lv) = g_v(L)]$$

Since K and $h^*(L)$ are compact, $M(v)$, $D[h^*, v]$ and $\eta[h^*, v, L]$ are non-empty.

We observe
$$\{(L,z) \mid L \in M(v),\ z \in \eta[h^*, v, L]\} = D[h^*, v]$$

Lemma 2.5

$M(v)$ is ω^* compact in K.

PROOF

If $L \in \complement M(v)$ then $g_v(L) < \Delta(v)$.

Since, however, g_v is ω^* u.s.c. on K, there exists a ω^* open nbhd $U(L)$ such that
$$g_v(\ell) < \Delta(v) \quad \text{for all } \ell \in U(L)$$
hence $\complement M(v)$ is ω^* open and therefore $M(v)$ is ω^* closed and the result follows.

Lemma 2.6

$\mathrm{ext}(M(v)) \subset \mathrm{ext}(K)$

PROOF

Suppose to the contrary, there exists an $L \in \mathrm{ext}(M(v))$ and $L \notin \mathrm{ext}\, K$.

Then there exists $L_1, L_2 \in K$ and λ, $0 < \lambda < 1$ with $L = \lambda L_1 + (1-\lambda) L_2$.

Hence $g_v(L) = \Delta(v) \leq \lambda g_v(L_1) + (1-\lambda) g_v(L_2)$ by Lemma 2.4.

But $g_v(L) \leq \Delta(v)$ for all $L \in K$

Therefore $g_v(L_1) = g_v(L_2) = \Delta(v)$, i.e. $L_1, L_2 \in M(v)$, which contradicts $L \in \mathrm{ext}(M(v))$.

We now consider relating two separate approaches to describing the envelope of F.

First we define $F^+(L) := \sup_{z \in h^*(L)} \text{Re } z.$

Since $g_v(L) = F^+(L) - \text{Re } v(L)$ we have that $F^+(L)$ is ω^* u.s.c. on K.

Now define $U_F(L) = \sup_{f \in F} \text{Re } Lf.$

Let $\eta(L)$ denote the collection of all ω^* open nbhds in K of L.

Let $U_F^+(L) := \inf_{W \in \eta(L)} \sup_{\ell \in W} U_F(\ell).$

The characterisation of the s.b.a. from a linear subspace has been obtained in [33] in terms of $U_F^+(L)$. It is now obtainable from the results in section 3 by employing the following Lemma.

<u>Lemma 2.7.</u> $U_F^+(L)$ is identical to $F^+(L)$ on K.

<u>Proof</u>

Suppose to the contrary there exists an $L \in K$ with $F^+(L) = a$ and $a > U_F^+(L)$. Then there exists a $W \in \eta(L)$ with

$$a > \sup_{\ell \in W} \sup_{f \in F} \text{Re } \ell f$$

$$= \sup_{\ell \in W} \sup_{z \in h(\ell)} \text{Re } z$$

$$= \sup \{\text{Re } z : z \in [\bigcup_{\ell \in W} h(\ell)]^o\}$$

On the other hand

$$a \leqslant \sup \{\text{Re } z : z \in \bigcap_{W \in \eta(L)} [\bigcup_{\ell \in W} h(\ell)]^o\}$$

$$\leqslant \sup \{\text{Re } z : z \in [\bigcup_{\ell \in W} h(\ell)]^o\}$$

leading to a contradiction.

Now suppose there exists an $L \in K$ with $F^+(L) = r$ and $r < U_F^+(L)$. Since F^+ is ω^* u.s.c. on K, there exists a $W \in \eta(L)$ such that

$$F^+(\ell) < r \text{ for all } \ell \in W$$

but $h(\ell) \subset h^*(\ell)$ for all $\ell \in W$.

Therefore $\sup_{z \in h(\ell)} \text{Re } z \leqslant \sup_{z \in h^*(\ell)} \text{Re } z = F^+(\ell)$ for all $\ell \in W$

and $\sup \{\text{Re } z : z \in [\bigcup_{\ell \in W(L)} h(\ell)]^o \leqslant r \}.$

However, $\sup_{\ell \in W} \sup_{f \in F} \text{Re } \ell f > r$, leading to a contradiction.

3. CHARACTERISATION OF THE BEST APPROXIMATION TO h^*

We first find circumstances under which $\rho_V(h^*)$ is bounded between two real numbers.

Theorem 3.1.

Suppose $v_o \in V$ and Ω a subset of K have the following properties:

(i) $\mathrm{Re}(z - Lv_o) \neq 0$ for all $L \in \Omega$ and $z \in \eta[h^*, v_o, L.]$

(ii) For no v in V do we have the inequality $\mathrm{Re}\, L(v-v_o) > 0$ satisfied for all $L \in \Omega$.

Then $\inf\limits_{L\in\Omega} \hat{d}(h^*(L),\, v_o(L)) \leq \rho_V(h^*) < \Delta(v_o)$.

PROOF

Suppose $\rho_V(h^*) < \inf\limits_{L\in\Omega} \hat{d}(h^*(L),\, v_o(L))$.

Then there exists a $v \in V$ with $\rho_V(h^*) \leq \Delta(v) < \inf\limits_{L\in\Omega} \hat{d}(h^*(L), v_o(L))$.

hence for every $L \in \Omega$, $\hat{d}(h^*(L),\, v(L)) < \hat{d}(h^*(L),\, v_o(L))$.

Therefore for all $L \in \Omega$ and $z \in \eta[h^*,\, v_o,\, L]$

$$\mathrm{Re}(z - v(L)) \leq \sup_{z\in h^*(L)} \mathrm{Re}(z - v(L)) < \mathrm{Re}(z - v_o(L)).$$

Hence $0 < \mathrm{Re}[v(L) - v_o(L)]$ contradicting (ii).

We are now in a position to generalise the global Kolmogoroff criterion for a sufficient condition for the best approximation from V.

Theorem 3.2

$v_o \in V$ is a best approximation to h^* if for all $v \in V$

$$\min_{L \in M(v_o)} \operatorname{Re} L(v - v_o) \leq 0.$$

PROOF

Take $\Omega = M(v_o)$ in Theorem 3.1.

If there exists a $(L,z) \in D[h^*, v_o]$, such that $z - Lv_o = 0$, then obviously v_o is a best approximation.

If for all $(L, z) \in D[h^*, v_o]$, $\operatorname{Re}(z - Lv_o) \neq 0$,

then by Theorem 3.1.

$$\Delta(v_o) = \inf_{L \in M(v_o)} \hat{d}(h^*(L), v_o(L)) \leq \rho_V(h^*) \leq \Delta(v_o)$$

and hence v_o is a best approximation.

The condition of Theorem 3.2. is not always necessarily satisfied by a best approximation from V.

However, if V is regular, we can prove the following.

Theorem 3.3

If $V \subset X$ is regular at v_o then v_o is a best approximation to h^* if and only if for all $v \in V$ $\min_{L \in M(v_o)} \text{Re } L(v - v_o) \leqslant 0$

PROOF

The sufficiency of the condition follows from Theorem 3.2. It remains to show the necessity.

Suppose there exists a $v \in V$ with $\min_{L \in M(v_o)} \text{Re } L(v - v_o) = a > 0$.

Set $U: = \{L \in K \mid \text{Re } L(v - v_o) > \frac{a}{2}\}$

U is ω^* open in K and contains $M(v_o)$. For all $L \in U^o$, $\text{Re } L(v - v_o) \geqslant \frac{a}{2}$

by the regularity of V at v_o, for all real $\lambda > 0$, there exists a $v_\lambda \in V$

with $\text{Re } L(v_\lambda - v_o) > 0$ for all $L \in U^o$

and $||v_\lambda - v_o|| < \lambda$.

For $L \in U$ and $z \in h^*(L)$,

$$\text{Re}(z - Lv_\lambda) = \text{Re}(z - Lv_o) + \text{Re}(Lv_o - Lv_\lambda) < \text{Re}(z - Lv_o)$$

Since $h^*(L)$ is compact for each $L \in U$, $\hat{d}(h^*(L), Lv_\lambda) < \hat{d}(h^*(L), Lv_o)$.

On the other hand, $K \setminus U$ is weak* compact and is disjoint from $M(v_o)$.

Therefore $\sup_{L \in K \setminus U} \hat{d}(h^*(L), v_o(L)) = E^* < \Delta(v_o)$

If we set $\lambda: = \Delta(v_o) - E^*$ then for $z \in h^*(L)$ we have

$$\text{Re } (z - Lv_\lambda) = \text{Re}(z - Lv_o) + \text{Re } (Lv_o - Lv_\lambda) < \Delta(v_o)$$

hence $\hat{d}(h^*(L), v_\lambda(L)) < \Delta(v_o)$

and $\Delta(v_\lambda) = \sup_{L \in K} \hat{d}(h^*(L), v_\lambda(L)) < \Delta(v_o)$

We now formulate a uniqueness result for the best approximation, analogous to Theorem 3.13 in [6].

Theorem 3.4

If $V \subset X$ is regular and v_o is a best approximation to h^* from V, then the best approximation is unique, in the case that $\text{Re}\, L(v - v_o) = 0$ on a subset of $M(v_o)$ which is sign-extremal for v_o implies $v = v_o$ on K.

PROOF

Suppose v_1 is another best approximation to h^*.
For any $(L,z) \in D[h^*, v_o]$

$$\begin{aligned} \text{Re}\,(z - Lv_1) &= \text{Re}(z - Lv_o) + \text{Re}\, L(v_o - v_1) \\ &\leq \hat{d}\,(h^*(L), v_1(L)) \\ &\leq d\,(h^*(L), v_o(L)) \\ &= \text{Re}(z - Lv_o) \end{aligned}$$

Therefore $\text{Re}\, L(v_1 - v_o) \geq 0$ for all $L \in M(v_o)$

But by Theorem 3.3. $\min_{L \in M(v_o)} \text{Re}\, L(v_1 - v_o) \leq 0$

Hence $\Sigma' := \{L \in M(v_o) \mid \text{Re}\, L(v_1 - v_o) = 0\} \neq \emptyset$

Assume $\Sigma' \neq M(v_o)$, otherwise the result follows trivially.

It follows by Lemmas 8 and 9 in [15] that Σ' is sign-extremal and by the condition of our theorem $v_1 = v_o$ on K.

4. APPROXIMATING FUNCTIONS WITH A FRÉCHET DERIVATIVE

Let D be an open subset of a Banach space E with norm $||\cdot||_E$.

Let V be the set of elements $v(a) \in X$ which depend on the parameter $a \in D$.

i.e. $V:D \to X$ and $V = \{v(a) \in X, a \in D\}$.

We shall henceforth assume that v(a) has a Fréchet derivative with respect to a for each $a \in D$.

i.e. for any $b \in E$ there exists a linear bounded mapping $v'_a : E \to X$ which we denote by $v'[b,a]$ with

$$||v(a + b) - v(a) - v'[b,a]|| = o(||b||_E) \text{ as } ||b||_E \to 0.$$

Let $\mathcal{L}[a]$ denote the linear subspace of X consisting of all elements $v'[b,a]$ $b \in E$.

Let N be the dimension of $\mathcal{L}[a]$ and $\Phi_1, \ldots \Phi_N$ be a basis for $\mathcal{L}[a]$.

We observe that if v(a) has a Fréchet derivative at a, then

$$||v(a + tb) - v(a)|| = O(t) \text{ for any } b \in E.$$

We can therefore say that for $0 < t \leq t_o$, $v(a + tb)$ lies in the ε-locality of v(a) defined by the norm sphere $S(v(a),\varepsilon)$ for some $\varepsilon > 0$.

v(a), then, is a local best approximation to h^* when $\Delta(v(a)) \leq \Delta(v(c))$, for all $v(c) \in V$ and in an ε-locality of v(a) for some $\varepsilon > 0$.

Theorem 4.1.

v(a) is a (local) best approximation to h* implies that for all $b \in E$

$$\min_{L \in M(v(a))} \text{Re } L\, v'[b,a] \leq 0$$

PROOF

Suppose to the contrary, there exists a $b \in E$ with $\min_{L \in M(v(a))} \text{Re } Lv'[b,a] > 0$.

We show there exists a better approximation to h* than v(a).

Let U be the set of $L \in K$ for which

$$\text{Re } Lv'[b,a] \geq 2\sigma > 0.$$

Since D is an open set in E, there exists a $t_o > 0$ for all t in $0 < t < t_o$

$a + tb \in D$ $(v(a + tb)$ lies in an ε-locality of $v(a))$.

For $L \in U$

$$\text{Re } L[v(a + tb) - v(a)] = \text{Re } L[v'[tb,a]] + \text{Re } L[v(a + tb) - v(a) - v'[tb,a]]$$
$$\geq 2\sigma\, t - O(t).$$

Hence there exists a t_1 with $0 < t_1 \leq t_o$ such that for all t, $0 \leq t \leq t_1$ and $L \in U$

$$\text{Re } L[v(a + tb) - v(a)] \geq \sigma t > 0$$

and therefore

$$\text{Re } [z - Lv(a + tb)] = \text{Re } [z - Lv(a)] + \text{Re } [L(v(a) - v(a + tb))]$$
$$< \text{Re } [z - Lv(a)]$$

Therefore $\hat{d}(h^*(L), v(a+tb)(L)) < \Delta(v(a))$ for all $L \in U$.

We observe here that

$$\|v(a + tb) - v(a)\| \leq \|v'[tb,a]\| + \|v(a + tb) - v(a) - v'[tb,a]\|$$
$$= t\,\|v'[b,a]\| + O(t)..$$

Hence there exists a t_2, $0 < t_2 \leq t_1$ such that for all t in $0 \leq t \leq t_2$

$$\|v(a + tb) - v(a)\| \leq 2t\,\|v'[b,a]\|$$

We now consider the set $W = K \setminus U$.

This is weak* compact and does not contain any member of $M(v(a))$.

Therefore $\sup_{L \in W} \hat{d}(h^*(L), v(a)(L)) = E^* < \Delta(v(a))$

Let τ be such that $0 < \tau < \min\left(t_2, \frac{\Delta(v(a)) - E^*}{2||v'[b,a]||}\right)$

For $L \in W$, $z \in h^*(L)$

$$\begin{aligned} \mathrm{Re}\left[z - Lv(a + \tau b)\right] &\leq \mathrm{Re}\left[z - Lv(a)\right] + \mathrm{Re}\left[L(v(a) - v(a + \tau b))\right] \\ &\leq \sup_{z \in h^*(L)} \mathrm{Re}\left[z - Lv(a)\right] + ||v(a) - v(a + \tau b)|| \\ &< E^* + 2\tau\, ||v'[b,a]|| \end{aligned}$$

Therefore $\hat{d}(h^*(L), v(a + \tau b)(L)) < \Delta(v(a))$ for all $L \in W$

Hence $\Delta(v(a + \tau b)) < \Delta(v(a))$.

We remark that in this theorem, we can replace $M(v(a))$ by its extremal points, denoted by $E_o(M)$, by applying Lemma 1.3. Likewise, we have the following equivalence of two convex hulls, relating to the sequel.

Let $[M,\Phi]$ denote $[(L\Phi_1, \ldots, L\Phi_N)^T$ over all $L \in M(v(a))$]. This is a compact set in Euclidean N-space as is its convex hull [18] p. 18. Now $\mathrm{ext}[\mathrm{co}[M,\Phi]] \subset \mathrm{ext}[M,\Phi] \subset [E_o(M),\Phi]$ by Lemma 1.2. Applying Lemma 1.1, $\mathrm{co}[M,\Phi] = \mathrm{co}[\mathrm{ext}(\mathrm{co}[M,\Phi])] \subset \mathrm{co}[E_o(M),\Phi]$. Obviously, $\mathrm{co}[E_o(M),\Phi] \subset \mathrm{co}[M,\Phi]$ and hence the two are identical.

<u>Corollary 4.1.</u>

If $v(a)$ is a (local) best approximation to h^* from V, then

$$\underline{0} \in \mathrm{co}\left[(L\Phi_1, \ldots, L\Phi_N)^T \text{ over all } L \in M(v(a))\right]$$

PROOF

Suppose to the contrary that $\underline{0}$ does not belong to the convex hull. Since $[(L\Phi_1, \ldots, L\Phi_N)^T$ over all $L \in M(v(a))]$ is a compact set in Euclidean N-space, there exists an N-dimensional vector $\underline{c} \in E$ so that

$$\mathrm{Re}\left(\sum_{i=1}^{N} c_i L\Phi_i\right) > 0 \text{ for all } L \in M(v(a))$$

But $\sum_{i=1}^{N} c_i \phi_i \in \mathcal{L}[a]$

and $\text{Re } L\left(\sum_{i=1}^{N} c_i \phi_i\right) > 0$ for all $L \in M(v(a))$

would imply that $v(a)$ could not have been a (local) best approximation by the previous theorem.

For any $b \in E$, let $a + tb$ be represented by $a(t)$ with $a(0) = a$.

Suppose $v(a(t))$ satisfies now a further condition (T) namely that $\dfrac{v(a(t)) - v(a)}{t}$ is in the linear span of $\{\phi_i(a(t))\}_{i=1}^{N}$ where

$$||\phi_i(a(t)) - \phi_i(a)|| = O(1) \text{ as } t \to 0 \text{ for } i = 1, \ldots, N.$$

Theorem 4.2

If $v(a(t))$ satisfies (T), then a sufficient condition for $v(a)$ to be a local best approximation to h^* from V is that

$$\underline{0} \in \text{interior co } \left[(L\phi_1(a), \ldots, L\phi_N(a))^T \text{ over all } L \in E_o(M)\right]$$

PROOF

By the assumed condition and the Appendix II for any $b \in E$, there exists an $\varepsilon_o > 0$ with $\underline{0} \in \text{co } \left[(L\phi_1(a(t)), \ldots, L\phi_N(a(t)))^T \text{ over all } L \in E_o(M)\right]$

for $0 \leq t \leq \varepsilon_o$.

Suppose to the contrary $v(a)$ is not a local best approximation to h^*. Then for all $\varepsilon > 0$, there exists a t, $0 < t \leq \varepsilon$ and $b \in E$ such that $a(t) \in D$

and $\rho_V(h^*) \leq \Delta(v(a(t))) < \Delta(v(a))$

i.e. for all $L \in K$

$$\hat{d}(h^*(L), v(a(t))(L)) < \sup_{L \in K} \hat{d}(h^*(L), v(a)(L))$$

hence for all $L \in E_o(M)$ and $z \in h^*(L)$

$$\text{Re}(z - v(a(t))(L)) < \text{Re}(z - v(a)(L))$$

i.e. $\text{Re}\left[L(v(a(t)) - v(a))\right] < 0$ for all $L \in E_o(M)$

Dividing through by t, we find

$$\underline{0} \notin \text{co } \left[(L\phi_1(a(t)), \ldots, L\phi_N(a(t)))^T \text{ over all } L \in E_o(M)\right]$$

Hence a contradiction follows by taking $\varepsilon = \varepsilon_o$

5. APPROXIMATION OF REAL-VALUED FUNCTIONS BY GENERALISED RATIONALS IN INTERPOLATING SUBSPACES OF L_1

We may relate the results of section 4 to the following setting. Suppose we are working in the space $L_1(B, \Sigma, \mu)$ abbreviated $L_1(\mu)$, where B, with an appropriate topology is a compact Hausdorff space, and μ is a σ-finite measure (see Appendix I) . If we further assume that B is the union of at most countably many atoms, say $B = \bigcup_{i \in I} A_i$

then it can be shown that ext(B*) is weak * closed and that each $L \in \text{ext}(B^*)$ has the representation

$$L(f) = \sum_{i \in I} f(A_i)\sigma(A_i)\mu(A_i) \qquad f \in L_1(\mu)$$

where $|\sigma(A_i)| = 1$ and $f(A_i)$ denotes the constant value of f a.e. on A_i .

The relevance of these points is immediate if we take K in section 4, to be B* or ext(B*) and recall Lemma 2.6 that $E_0(M) \subset \text{ext}(K)$ i.e. the above representation is valid for $E_0(M)$. Furthermore, the presence of atoms enables us to use the concept of interpolating subspaces (see p.9).
We remark that in computational work with the L_1 norm, we are obliged to discretise and hence our setting is a practical one.

Suppose we are given a set of real-valued functions $F \subset L_1(\mu)$ and we wish to characterise local best approximations from $V = R^+_{n,m}$ (see p. 10). To recall, let $g_1, \ldots, g_n; h_1, \ldots, h_m$ belong to the subspace of $L_1(\mu)$ consisting of real-valued continuous functions.

Let $D: = \{(\alpha_1,\dots,\alpha_n;\beta_1,\dots,\beta_m)\in E^{n+m}, \sum_{i=1}^{m}\beta_i h_i(x) > 0 \text{ on } B\}$

For $(a_1,\dots,a_n;b_1,\dots,b_m)\in D$

$(c_1,\dots,c_n;d_1,\dots,d_m)\in E^{n+m}$ and real λ

set
$$r_\lambda(x): = \frac{\sum_{i=1}^{n}(a_i+\lambda c_i)g_i(x)}{\sum_{i=1}^{m}(b_i-\lambda d_i)h_i(x)}.$$

Then
$$r_0(x): = \frac{\sum_{i=1}^{n}a_i g_i(x)}{\sum_{i=1}^{m}b_i h_i(x)} \in R^+_{n,m}$$

For any $\underline{d} = (d_1,\dots,d_m)$ we can always find a $\hat{\lambda} = \hat{\lambda}(\underline{d}) > 0$ and a $\lambda^* = \lambda^*(\underline{c},\underline{d})$, $0<\lambda^*\le\hat{\lambda}$ such that

$$|\hat{\lambda}\sum_{i=1}^{m}d_i h_i(x)| < \sum_{i=1}^{m}b_i h_i(x) \text{ on } B$$

and r_λ belongs to an ε - locality of r_0 for $|\lambda|\le\lambda^*$

We shall use the following abbreviations.

$q_m(\lambda,\underline{d},x): = \sum_{i=1}^{m}(b_i-\lambda d_i)h_i(x)$ and $q_m(x): = \sum_{i=1}^{m}b_i h_i(x)$.

We can present a simplification of our problem to that of approximating a single valued w^* u.s.c. function F^+

$F^+: K\to R$ defined by $F^+(L) = \max_{z\in h^*(L)} z$ (see p.47 top)

For now we have

$$\Delta(r_0) = \sup_{L\in K} g_{r_0}(L) = \sup_{L\in K}[F^+(L) - r_0(L)]$$

THEOREM 5.1

Let $p_n \quad \varepsilon P \equiv \text{span }[g_1,\dots,g_n]$

and $q_m \quad \varepsilon Q \equiv \text{span }[h_1,\dots,h_m]$

and suppose $r_0 := \dfrac{p_n}{q_m} \quad \varepsilon R^+_{n,m}$

If (a) r_0 is a locally best L_1 approximation to F^+

and (b)

$\mathcal{L}[r_0] = \dfrac{P}{q_m} + r_0 \dfrac{Q}{q_m}$ is an N-dimensional interpolating subspace of $L_1(\mu)$, with basis $\phi_1,\dots,\phi_N$.

Then

(i) There exist N + 1 independent functionals

$L_1,\dots,L_{N+1}$ in ext ($M(r_0)$), abbreviated $E_0(M)$, such that

$$\underline{0} \in \text{interior co } [(L_i\,\phi_1\,,\dots,L_i\,\phi_N\,)^T \quad i=1,\dots,N+1] .$$

(ii) 0 is the only element ϕ of $\dfrac{P}{q_m} + r_0 \dfrac{Q}{q_m}$

having the property $L_i\phi \geq 0$ for $i=1,\dots,N+1$, L_i as in (i).

(iii) $\exists\sigma \quad 0 < \sigma \leq \lambda^*$ such that $\forall\, \lambda$, $|\lambda| \leq \sigma$

$$\frac{P}{q_m(\lambda,\underline{d})} + r_0 \frac{Q}{q_m(\lambda,\underline{d})}$$ is an interpolating subspace on $\{L_i\}_{i=1}^{N+1}$

(iv) r_0 is a unique locally best approximation in the ε- locality of r_0 restricted to $|\lambda| \leq \sigma$ and denoted by $U(r_0,\sigma)$

PROOF. (i)

By Corollary 4.1, the origin of N space lies in the convex hull of the set

$$[(L_i \phi_1, \ldots, L_i \phi_N)^T \quad \text{for} \quad i=1,\ldots,k] .$$

By Caratheodory's Theorem, (cf. [64] p.58) $k \leq N+1$.
Now for each j, $0 = \sum_{i=1}^{k} \theta_i L_i \phi_j$ with $\theta_i \geq 0$.
Hence, by the interpolating condition,
$k \geq N+1$ and so $k = N+1$.

Furthermore, the origin cannot lie on the boundary, for then k would be equal to N.

Hence the origin of N space lies in the interior of the convex hull of the set

$$[(L_i \phi_1, \ldots, L_i \phi_N)^T \quad \text{for} \quad i=1,\ldots,N+1] .$$

Finally we remark that it also follows that this convex hull does not lie in a plane, and hence is a body in Euclidean N space.

(ii) Suppose ϕ is a non-zero element of $\frac{P}{q_m} + r_0 \frac{Q}{q_m}$

$$\phi = \sum_{j=1}^{N} a_j \phi_j$$

$$L_i \phi = \sum_{j=1}^{N} a_j L_i(\phi_j)$$

Now $0 = \sum_{i=1}^{N+1} \theta_i L_i(\phi_j)$

and multiplying this equation by a_j and summing over j

$$0 = \sum_{i=1}^{N+1} \theta_i \sum_{j=1}^{N} a_j [L_i(\phi_j)]$$

$$0 = \sum_{i=1}^{N+1} \theta_i L_i \phi \quad .$$

By the interpolating condition at most $N - 1$ of the numbers $L_i \phi$ can vanish. Hence at least one of the $L_i \phi$ is positive and at least one is negative.

Hence ϕ is zero.

(iii) Let λ be sufficiently small

Then $\tilde{\phi}_i(\lambda,\underline{d}) \equiv \dfrac{q_m}{q_m(\lambda,\underline{d})}\,\phi_i \qquad i=1,\ldots,N$

is a basis for $\dfrac{P}{q_m(\lambda,\underline{d})} + r_0 \dfrac{Q}{q_m(\lambda,\underline{d})}$.

By continuity of determinants, we have

$\dfrac{P}{q_m(\lambda,\underline{d})} + r_0 \dfrac{Q}{q_m(\lambda,\underline{d})}$ is an interpolating subspace on $\{L_i\}_{i=1}^{N+1}$ since $\inf |\det[L_i\tilde{\phi}_j(\lambda,\underline{d})]| > 0$

We note that the argument in (ii) is valid for this subspace also, since $\underline{0}$ belongs to the perturbed convex hull, by (i) and APPENDIX II.

(iv) Let $r_\lambda(x) \in U(r_0,\sigma)$ be another locally best L_1 approximation to F^+ in the vicinity of r_0 .

Take $\phi := r_0 - r_\lambda \in \dfrac{P}{q_m(\lambda,\underline{d})} + r_0 \dfrac{Q}{q_m(\lambda,\underline{d})}$

and $L_i(r_0-r_\lambda) = (F^+(L_i) - r_\lambda(L_i)) - (F^+(L_i) - r_0(L_i))$

$$\leq 0 \qquad i=1,\ldots,N+1 \;, \quad L_i \text{ as in (i)}.$$

But from (i) and the Appendix II

$$\underline{0} \in \text{convex hull } [(L_i\tilde{\phi}_1(\lambda,\underline{d}),\ldots,L_i\tilde{\phi}_N(\lambda,\underline{d}))^T$$

$$i=1,\ldots,N+1] \;.$$

Hence by the note to (iii) ,

$$r_0 \equiv r_\lambda$$

We now strengthen (iv) of Theorem 5.1. and show that under suitable conditions there is local strong unicity in the sense of Newman and Shapiro.

We will need the following lemma adapted from [18] p.162.

Lemma 5.1.

If $r_o := \frac{p_n}{q_m} \in R^+_{n,m}$ such that

(c) $\dim\left(\frac{P}{q_m} + r_o\frac{Q}{q_m}\right) = \dim\left(\frac{P}{q_m}\right) + \dim\left(\frac{Q}{q_m}\right) - 1$

and if $p \in P$, $q \in Q$ satisfy

(i) $||q|| = ||q_m||$

(ii) $p = r_o q$

(iii) $q(x) \geq 0$ on B.

Then $p = p_n$, $q = q_m$.

THEOREM 5.2.

Under conditions (a) and (b) of Theorem 5.1 and (c) of Lemma 5.1. there exists a constant $\gamma > 0$ such that

for all $r_\lambda(x) \in U(r_o,\sigma)$

$$\Delta(r_\lambda) \geq \Delta(r_o) + \gamma||r_\lambda - r_o||$$

PROOF

For $0 < |\lambda| \leq \sigma$, define for the set $U(r_o,\sigma)$

$$\gamma(r_\lambda) = \frac{\Delta(r_\lambda) - \Delta(r_o)}{||r_\lambda - r_o||}$$

and suppose to the contrary, there exists a sequence $\{r_{\lambda_k}\} \in U(r_o,\sigma)$ $r_{\lambda_k} \neq r_o$ and $\gamma(r_{\lambda_k}) \to 0$.

We may suppose $\gamma(r_{\lambda_k}) < \frac{1}{2}$ for $k \geq n_o$.

Then we can show $0 < ||r_{\lambda_k} - r_o|| < \infty$, $k \geq n_o$.

For take any $f \in F$

$$||r_{\lambda_k} - r_o|| \leq ||r_{\lambda_k} - f|| + ||r_o - f||$$

$$\leq \sup_{f \in F} ||r_{\lambda_k} - f|| + \sup_{f \in F} ||r_o - f||$$

$$\leq \Delta(r_{\lambda_k}) + \Delta(r_o)$$

$$\leq 2\Delta(r_o) + \frac{1}{2}||r_{\lambda_k} - r_o|| \text{ for } k \geq n_o \text{ by our supposition.}$$

Therefore $||r_{\lambda_k} - r_o|| \leq 4\Delta(r_o) < 4\alpha \qquad k \geq n_o$,

Next we show there exists a sequence of r_{λ_k} relabelled the same, such that

$$\lim_{k \to \infty} r_{\lambda_k} = r_o$$

Since $0 < |\lambda_k| \leq \sigma$, either $\lim_{k \to \infty} \lambda_k = 0$ for every subsequence in which

case $\lim_{k \to \infty} r_{\lambda_k} = r_o$, or there exists a subsequence relabelled the same with

$\lim_{k \to \infty} \lambda_k = \lambda_o$ where $0 < \lambda_o \leq \sigma$.

Assume the latter to be the case.

Now $q_m(\lambda_k, \underline{d}_k) = \sum_{i=1}^{m} \beta_i^{(k)} h_i(x)$ where $\sum_{i=1}^{m} |\beta_i^{(k)}| = 1$ by our normalisation

convention of chapter 1, section 2.4.2.

Hence for each i, $1 \leq i \leq m$, and for all k we have

$|\beta_i^{(k)}| < 1$ and therefore $b_i - 1 \leq \lambda_k d_i^{(k)} \leq b_i + 1$.

It follows that for each i, $\{d_i^{(k)}\}$ is a bounded sequence and we can extract

a convergent subsequence such that $\lim_{k \to \infty} d_i^{(k)} = d_i^{(o)}$ and hence

$\lim_{k \to \infty} q_m(\lambda_k, \underline{d}_k) = q_m(\lambda_o, \underline{d}_o)$

By definition

$$\gamma(r_{\lambda_k}) \, ||r_{\lambda_k} - r_o|| = \Delta(r_{\lambda_k}) - \Delta(r_o)$$

$$\geq \max_{j=1,..,N+1} L_j(r_o - r_{\lambda_k})$$

With $k \to \infty$ and our knowledge concerning the left hand side we apply the note to (iii) of Theorem 5.1, to obtain

$$\lim r_{\lambda_k} = r_o.$$

Now by Lemma 5.1,

$$q_m(\lambda_o, \underline{d}_o) = q_m.$$

Consequently as $k \to \infty$,

$$\frac{P}{q_m(\lambda_k, \underline{d}_k)} + r_o \frac{Q}{q_m(\lambda_k, \underline{d}_k)} \to \frac{P}{q_m} + r_o \frac{Q}{q_m}.$$

Finally we reason as follows.

For $L_j \in E_o(M)$ and $\phi \in \dfrac{P}{q_m(\lambda_k, \underline{d}_k)} + r_o \dfrac{Q}{q_m(\lambda_k, \underline{d}_k)}$

we have by virtue of results (iii) and (ii) of Theorem 5.1 that for all k, including the limiting case,

$$c_k = \min_{||\phi||_1 = 1} \max_{j=1,..,N+1} L_j \phi > 0.$$

But, $\gamma(r_{\lambda_k}) \, ||r_{\lambda_k} - r_o||_1 \geq \max_{j=1,..,N+1} L_j(r_o - r_{\lambda_k})$

and $\dfrac{r_o - r_{\lambda_k}}{||r_o - r_{\lambda_k}||_1} \in \dfrac{P}{q_m(\lambda_k, \underline{d}_k)} + r_o \dfrac{Q}{q_m(\lambda_k, \underline{d}_k)}$ and is of norm one.

Therefore $\gamma(r_{\lambda_k}) \geq c_k > 0$

Furthermore, if we let $c_o = \min_{||\phi||=1} \max_{j=1,..,N+1} L_j\phi \quad \phi \in \mathcal{L}[r_o]$

with $c_o > 0$ as already deduced, we can show that for all ε, $0 < \varepsilon < c_o$, we have that $c_k > c_o - \varepsilon$ for k sufficiently large.

To prove this last conjecture, assume k to be large enough that $q_m(\lambda_k, \underline{d}_k) \doteq q_m$ and hence $\tilde{\phi}_i(\lambda_k, \underline{d}_k) \doteq \phi_i$

Suppose now to the contrary there exists a convergent sequence(in ν)

$$\phi_\nu^{(k)} \in \frac{P}{q_m(\lambda_k, \underline{d}_k)} + r_o \frac{Q}{q_m(\lambda_k, \underline{d}_k)} \qquad \text{with} \quad ||\phi_\nu^{(k)}|| = 1$$

$$\text{and} \quad \lim_{\nu\to\infty} \max_{j=1,..,N+1} L_j\,\phi_\nu^{(k)} = c_k \le c_o - \varepsilon$$

that is there exists an $\tilde{N}(k)$ such that for $\nu \geq \tilde{N}(k)$

$$\max_{j=1,..,N+1} L_j\,\phi_\nu^{(k)} \le c_o - \frac{3}{4}\varepsilon$$

Assume $\nu \geq \tilde{N}(k)$. If we represent $\phi_\nu^{(k)}$ as $\sum_{i=1}^{N} a_i^{(\nu)}\, \tilde{\phi}_i(\lambda_k, \underline{d}_k)$

then $\{a_i^{(\nu)}\}_{i=1}^{N}$ are bounded by our assumption on $\phi_\nu^{(k)}$

and $\hat{\psi}_\nu := \sum_{i=1}^{N} a_i^{(\nu)}\,\phi_i$ satisfies $||\hat{\psi}_\nu - \phi_\nu^{(k)}|| < \frac{\varepsilon}{4}$

by our assumption on k. Hence $1 - \frac{\varepsilon}{4} < ||\hat{\psi}_\nu|| < 1 + \frac{\varepsilon}{4}$

Now $\psi_\nu := \dfrac{\hat{\psi}_\nu}{||\hat{\psi}_\nu||}$ is of norm one, belongs to $\mathcal{L}[r_o]$

$$\begin{aligned} \text{and} \quad ||\psi_\nu - \phi_\nu^{(k)}|| &\le ||\psi_\nu - \hat{\psi}_\nu|| + ||\hat{\psi}_\nu - \phi_\nu^{(k)}|| \\ &< (\,1 - ||\hat{\psi}_\nu||\,) + \frac{\varepsilon}{4} \\ &< \varepsilon/2 \end{aligned}$$

Consequently $\quad -\frac{\varepsilon}{2} < \max L_j\psi_\nu - \max L_j\,\phi_\nu^{(k)} < \frac{\varepsilon}{2} \qquad j=1,..,N+1$

and $\max_{j=1,..,N+1} L_j\psi_\nu < c_o - \frac{\varepsilon}{4}$ which is clearly impossible

Thus we have shown that for k sufficiently large, $\gamma(r_{\lambda_k})$ is bounded away from zero and we have been led to a contradiction.

We now re-formulate Theorems 5.1 and 4.2 in terms of the more familiar "alternation" theorem.

THEOREM 5.3

Suppose $\frac{P}{q_m} + r_0 \frac{Q}{q_m}$ is an N-dimensional interpolating subspace of $L_1(\mu)$ with basis $\phi_1, \ldots, \phi_N$. Let $L_1, \ldots, L_{N+1} \in L_1^*(\mu)$.

Define Δ_i by

$$\Delta_i = \begin{vmatrix} L_1(\phi_1) \ldots & \ldots L_{i-1}(\phi_1) L_{i+1}(\phi_1) \ldots & \ldots L_{N+1}(\phi_1) \\ \vdots & & \vdots \\ L_1(\phi_N) \ldots & \ldots L_{i-1}(\phi_N) L_{i+1}(\phi_N) \ldots & \ldots L_{N+1}(\phi_N) \end{vmatrix}$$

Then r_0 is a locally best L_1 approximation to F^+ if and only if

(a) there exist $N+1$ linearly independent functionals $L_1, \ldots, L_{N+1}$ in $E_0(M)$.

(b) $\Delta_i \Delta_{i+1} < 0$ for $i=1, \ldots, N$.

Note that by the interpolating condition $\Delta_i \neq 0$, $i=1, \ldots, N+1$.

PROOF. For necessity it remains to prove (b).

Since by the Characterization Theorem 5.1

$\underline{0} \in$ interior convex hull $[(L_i\phi_1, \ldots, L_i\phi_N)^T | L_i \in E_0(M)\ i=1, \ldots, N+1]$, there exist positive scalars θ_i, $i=1, \ldots, N+1$ and

$$\sum_{i=1}^{N} \theta_i L_i \phi_k = -\theta_{N+1} L_{N+1} \phi_k \quad \text{for} \quad k=1,\ldots,N .$$

Solving for θ_i by Cramer's rule

$$\theta_i = (-1)^{N-i+1} \frac{\Delta_i}{\Delta_{N+1}} \theta_{N+1}$$

from which the result follows.

Conversely, the system of equations

$$\sum_{i=1}^{N} x_i L_i \phi_k = - L_{N+1} \phi_k \qquad k=1,\ldots,N$$

has a unique solution given by

$$x_i = (-1)^{N-i+1} \frac{\Delta_i}{\Delta_{N+1}}$$

and $\{x_i\}$ are positive $i=1,\ldots,N$.

Hence $\underline{0}$ ε interior convex hull

$$[(L_i \phi_1, \ldots, L_i \phi_N)^T | L_i \varepsilon E_0(M), \quad i=1,\ldots,N+1] .$$

APPENDIX I : Measure Theory and $L_p(S, S_o, \mu)$

A <u>σ-algebra</u> over a set S is a family S_o of subsets of S such that

(i) $\phi, S \in S_o$

(ii) If $A \in S_o$ then $A^c \in S_o$

(iii) If $\{A_n\}$ is a sequence of sets in S_o then $\bigcup_1^\infty A_n \in S_o$

The sets in S_o are called <u>measurable sets</u>

An extended-real-valued function f on S is called (S_o-) <u>measurable</u> if for each real α, the set $\{ s \in S : f(s) > \alpha \}$ is measurable.

A <u>measure</u> on (S, S_o) is a function μ assigning to each $A \in S_o$ an extended real number $\mu(A)$ such that

(i) $\mu(\phi) = 0$

(ii) $\mu(A) \geqslant 0$ for all A

(iii) μ is countably additive on disjoint sets.

μ is called <u>σ-finite</u> if there is a sequence of sets $\{A_n\}$ such that $\mu(A_n) < \infty$ for all n and $\bigcup_n A_n = S$.

<u>Example.</u> Let $S_o \equiv \Sigma_o$, the class of all subsets of a set S and define $\mu(A)$ to be the number of points in A if this is finite and $+\infty$ otherwise. This μ is referred to as the <u>counting measure.</u>
It is σ-finite if and only if S is countable.

The Characteristic Function of a set E, $\Psi_E = \begin{cases} 1 & \text{for } x \in E \\ 0 & \text{for } x \in E \end{cases}$ is a measurable function $\Longleftrightarrow$ E is a measurable set.

A <u>simple function</u> (sf) is a real valued function on (S, S_o) with the canonical representation

$$sf = \sum_{i=1}^{n} c_i \Psi_{E_i} \quad \text{where } E_i = \{ x : sf(x) = c_i \} \quad \text{and } c_i \in R$$

sf is a measurable function $\Longleftrightarrow E_i$ is measurable for all i.

The class N_o is defined to be the (σ- algebra of) subsets $N \in S$ such that $\mu(N) = 0$. Its members are called <u>μ-null</u> sets.

If the set of points in S, for which a property P does not hold true, belongs to N_o we say P holds μ-<u>almost everywhere</u> (μ-a.e.)

For a simple measurable function (s.m.f.) $\Phi \geqslant 0$, the integral of Φ w.r.t. μ is $\int \Phi \, d\mu = \sum_1^n c_i \, \mu(E_i)$ which is also positive but may be infinite.
[This integral has LINEARITY PROPERTIES]

For a real-valued function $f \geqslant 0$ and measurable on S the integral of f w.r.t. μ is

$\int f d\mu = \sup [\int \Phi \, d\mu : \Phi$ s.m.f., $0 \leqslant \Phi \leqslant f$ on S] and if this is finite, f is said to be integrable.

An arbitrary real-valued measurable function f is called integrable if its positive and negative parts have finite integrals.

$\mathcal{L}_p(S, S_o, \mu)$ is the set of all everywhere–finite measurable functions on S such that $|f|^p$ is integrable, where p is real and $\geqslant 1$. It is a linear space. If we set the norm of $f \in \mathcal{L}_p$ to be $\|f\|_p = (\int |f|^p d\mu)^{1/p}$ and count the functions which are equal μ–a.e. as one equivalence class, we obtain the Lebesgue Space $L_p(S, S_o, \mu)$ which is a complete normed space.

SPECIAL CASES OF Lp

(i) $L_p[0,1]$ is the case when μ is Lebesgue measure on the interval [0,1] and S_o are the Lebesgue measurable subsets of [0,1].

(ii) ℓ_p^n is the space $R^n(C^n)$ with $\|x\|_p = (|x_1|^p + \ldots + |x_n|^p)^{1/p}$ $1 \leq p < \infty$
Take S to be the set (1, ..., n) and μ to be counting measure. Identify functions f on S with n-tuples $(f_i, \ldots, f_n)$.
The integral of g on S, g real-valued and positive is just $\sum_{i=1}^n g_i$
Hence $\|f\|_p = (\Sigma |f_i|^p)^{1/p}$

(iii) ℓp is the sequence space $x = (x_1, x_2, \ldots)$ with $\sum_{i=1}^{\infty} |x_i|^p$ converging together with the norm $\|x\|_p$.

Take S = N (the set of natural numbers) and μ the counting measure and employ the (canonical) identification of the previous example.

APPENDIX II

Given $\phi_1,\ldots,\phi_N$ elements of X, an N.L.S. , and M a ω^* closed subset of B^* the unit ball of the dual space.

Let Φ denote $(\phi_1,\ldots,\phi_N)^T$ and $\underline{0}$ the origin of N-space.

Set co $[M, \Phi]$:= convex hull$[\,(L\phi_1,\ldots,L\phi_N)^T \;:\; L \in M\,]$ and suppose this is a body in Euclidean N-space. $||\Phi||$ denotes $\max\{||\phi_i||\}_{i=1}^{N}$

LEMMA; If $\underline{0}$ is an interior point of co$[M,\Phi]$ then there exists an $\varepsilon > 0$ such that for all Φ' satisfying $||\Phi - \Phi'|| < \varepsilon$, $\underline{0} \in$ co$[M,\Phi]$

PROOF

Suppose to the contrary that for every $\varepsilon > 0$, there exists a $\Phi(\varepsilon)$ with $||\Phi - \Phi(\varepsilon)|| < \varepsilon$ and $\underline{0} \notin$ co$[M,\Phi(\varepsilon)]$.

Then since co$[M,\Phi(\varepsilon)]$ is compact there exists a separating hyperplane. That is there exists constants $c_1(\varepsilon), \ldots, c_N(\varepsilon)$ not all zero, and a real number $\gamma(\varepsilon)$ such that Re $\sum_{i=1}^{N} c_i(\varepsilon)L\phi_i(\varepsilon) \geq \gamma(\varepsilon) > 0$ for all $L \in M$. Without loss of generality, we can normalise $c_i(\varepsilon)$ so that $|c_i(\varepsilon)| \leq 1$ for all i

Let $\varepsilon \to 0$. Then $L\phi_i(\varepsilon) \to L\phi_i$ for each i, and we can also extract a subsequence from $c_i(\varepsilon)$ such that $\lim_{\varepsilon \to 0} c_i(\varepsilon) = c_i$ for each i .

Hence we can deduce
$$\inf_{L \in M} \text{Re} \sum_{i=1}^{N} c_i \, L\phi_i \geq 0$$

It follows that co$[M,\Phi]$ lies to one side of this hyperplane. Furthermore, $\underline{0}$ belongs either outside co$[M,\Phi]$ or on a hyperplane supporting co$[M,\Phi]$ at $\underline{0}$.

It could not however be in the interior of the convex hull for then there would be points of the convex hull to either side of this hyperplane.

Hence we have been led to a contradiction ,

SIMULTANEOUS APPROXIMATION OF A FUNCTION AND ITS DERIVATIVE WITH THE TAU METHOD, PART I

FOREWORD: In this part we discuss the extension of the recursive form of the Tau method (cf. Ortiz 54]) to a simple case of a system of two linear differential equations with constant coefficients, perturbed by a linear combination of Chebyshev polynomials. The problem is closely related to that of finding simultaneous approximations of a function and its derivative.

We use duality arguments, introduced into this type of problem by T.J. Rivlin [64 p.98], to show that the Chebyshev polynomials are the only extremals for the functionals associated with our particular perturbation problem. We discuss the effective construction of the approximate solution of the system with the Tau method and find upper and lower bounds for the error. We also show that the best and the Tau approximations are, in the case considered, asymptotically comparable.

1.1. - INTRODUCTION

We consider the second order differential equation

$$y''(x) + y(x) = 0$$

with the initial conditions

$$y(0) = 1; \; y'(0) = 0 \; .$$

which defines the solution $y(x) = \cos x$.

If we let $z \equiv -\frac{dy}{dx}$, the 2nd order differential equation may be reposed as two simultaneous 1st order equations

$$y - \frac{dz}{dx} = 0$$

$$\frac{dy}{dx} + z = 0$$

For conciseness let $\underset{\sim}{y} \equiv [y,z]^T$ and D represent the 2-dimensional operator

$$\begin{pmatrix} 1 & -\frac{d}{dx} \\ \frac{d}{dx} & 1 \end{pmatrix}.$$

We are now looking for the solution $\underset{\sim}{y}^*$, on some compact interval J, to the system

$$D\,\underset{\sim}{y} = \underset{\sim}{0}, \quad \underset{\sim}{y}(0) = [1,0]^T \tag{1}$$

We assume an approximate solution to (1) is sought on J. In the method described here we obtain separate polynomial approximations of degree n to y and its derivative by computing the exact solution $[y_n^*, z_n^*]^T$ or $\underset{\sim}{y}_n^*$ of the perturbed system

$$D\underset{\sim}{y} = \begin{pmatrix} \tau_1^{(n)} T_n^* \\ \tau_2^{(n)} T_n^* \end{pmatrix} \quad \text{where } T_n^*(x) \equiv \cos 2n \arccos x^{\frac{1}{2}} \tag{2}$$

when $J \equiv [0,1]$.

The choice of the shifted Chebyshev polynomial on the right hand side of (2) signifies that the error vector $\underset{\sim}{e}_n^* \equiv \underset{\sim}{y}_n^* - \underset{\sim}{y}^*$ satisfies the equioscillation property on [0,1] in each component, in the image space of the operator D.

The error vector may be measured by any l_p sum of the individual l_p norms of its components, for $1 \le p \le \infty$.

However, our interest lies in the double or vectorial uniform norm

$$||D\,\underset{\sim}{e}_n^*||_\infty = \max\{||\tau_1^{(n)} T_n^*||_\infty, ||\tau_2^{(n)} T_n^*||_\infty\} = 2^{1-2n} \max\{|\tau_1^{(n)}|, |\tau_2^{(n)}|\}$$

In section we show that, with the Chebyshev perturbation, the system (2) satisfies

$$||D\underset{\sim}{e}_n^*||_\infty \le ||D\,\underset{\sim}{e}_n||_\infty \quad \text{for all } \underset{\sim}{y}_n^T \in \pi_n \times \pi_n$$

where π_n is the space of polynomials of degree less than or equal to n.

1.2. THE DUAL PROBLEM AND THE CHEBYSHEV PERTURBATION.

PROBLEM A. Our problem is to find $[y_n^*, z_n^*] \in \pi_n \times \pi_n$ such that

$$\min_{[y_n, z_n]} \left\| D \begin{pmatrix} y_n \\ z_n \end{pmatrix} \right\|_\infty = \left\| D \begin{pmatrix} y_n^* \\ z_n^* \end{pmatrix} \right\|_\infty .$$

over all $[y_n, z_n]$ satisfying $y_n(0) = \sigma$; $z_n(0) = \rho$ ..(a)

PROBLEM B. If we write $D \begin{pmatrix} y_n \\ z_n \end{pmatrix} = \begin{pmatrix} v_n \\ w_n \end{pmatrix}$, $[v_n, w_n] \varepsilon \pi_n \times \pi_n$

then the supplementary conditions (a) become

$$F_1(v_n) = \alpha(w_n) \equiv \alpha$$
$$F_2(w_n) = \beta(v_n) \equiv \beta \qquad \text{... (b)}$$

where F_1, F_2 are linear functionals. We will call Problem B that of finding a vector $[\bar{v}_n, \bar{w}_n]$ such that for any " admissable " ordered pair $[v_n, w_n]$ satisfying (b), we have $\|\bar{v}_n\| \leq \|v_n\|$; $\|\bar{w}_n\| \leq \|w_n\|$.

A solution of (B) leads to a solution of (A) .

For our example, suppose we write

$$y_n = \sum^n a_j x^j \qquad z_n = \sum^n c_j x^j$$

$$v_n = \sum^n b_j x^j \qquad w_n = \sum^n d_j x^j$$

then $y_n + y_n'' = v_n + w_n'$. and $z_n + z_n'' = w_n - v_v'$

$$a_n = b_n \qquad c_n = d_n$$

$$a_{n-1} - nc_n = b_{n-1} \qquad na_n + c_{n-1} = d_{n-1}$$

$$a_j + (j+2)(j+1)a_{j+2} = b_j + (j+1)\, d_{j+1}$$

$$c_j + (j+2)(j+1)c_{j+2} = d_j - (j+1)\, b_{j+1}$$

for $j = n-2, \ldots, 0$.

Let us restrict ourselves to the case of n even, for simplicity. Then we find for $j = 0, 1, \ldots \ldots n/2$, setting $b_{n+1} = d_{n+1} = 0$

$$(n-2j)!\ a_{n-2j} = \sum_{i=0}^{i=j} (-1)^{i+j}(n-2i)!\ b_{n-2i} - \sum_{i=0}^{i=j} (-1)^{i+j}(n-2i+1)!\ d_{n-2i+1}$$

and similarly

$$(n-2j)!\ c_{n-2j} = \sum_{i=0}^{i=j} (-1)^{i+j}(n-2i)!\ d_{n-2i} + \sum_{i=0}^{i=j} (-1)^{i+j}(n-2i+1)!\, b_{n-2i+1}$$

Since $a_o = \sigma$ and $c_o = \rho$ we can write for $j=n/2$

$$F_1(v_n) \equiv \sum_{i=0}^{i=j} (-1)^{i+j}(n-2i)!\ b_{n-2i} = \sigma + \sum_{i=0}^{i=j} (-1)^{i+j}(n-2i+1)!\ d_{n-2i+1} \equiv \alpha$$

$$F_2(w_n) \equiv \sum_{i=0}^{i=j} (-1)^{i+j}(n-2i)!\ d_{n-2i} = \rho - \sum_{i=0}^{i=j} (-1)^{i+j}(n-2i+1)!\ b_{n-2i+1} \equiv \beta$$

We observe that for our example $F_1 = F_2 \equiv F$

Let $S_n \subset \pi_n$ be such that

$$S_n = \{v_n \in \pi_n : ||v_n|| = 1\}$$

<u>Definition</u> $v_n^* \in S_n$ is an extremal element of F_1 if

$|F_1(v_n^*)| = ||F_1||$. Suppose $\text{sign}(F_1(v_n^*)) = \varepsilon$

<u>Lemma 1</u> For all $v_n \in \pi_n$, $F_1 : F_1(v_n) = \alpha$, $\bar{v}_n = \varepsilon\alpha v_n^*/||F_1||$ we have

v_n^* extremal for $F_1 \Rightarrow ||v_n|| \geq ||\bar{v}_n||$

<u>Proof:</u> Assume that v_n^* is an extremal for F_1. If $\alpha = 0$ the result is trivial. If $\alpha \neq 0$

$$|\alpha| = |F_1(v_n)| \leq ||F_1|| \, ||v_n||$$

and $||v_n|| \geq |\alpha|/||F_1||$, from which the result follows.

Conversely, since bounded linear functionals have extremals, it is readily found that if $||\bar{v}_n|| \leq ||v_n||$ for all v_n satisfying $F_1(v_n) = \alpha$ then $\bar{v}_n/||\bar{v}_n||$ is an extremal element of F_1, provided $\alpha \neq 0$.

Remark: In this argument w_n is arbitrary but fixed and $\bar{v}_n$ is irrespectively a multiple of $v_n{}^*$, an extremal element.

A similar result follows for F_2 and the w_n: we set $\bar{w}_n = \beta w_n{}^*/||F_2||$.

Furthermore, it follows by consistency considerations that α and β are non zero for $\sigma = 1$ and $\rho = 0$.

Since we require $\bar{v}_n$, $\bar{w}_n$ to be minimal together, both must be non-zero multiples respectively of v_n^*, w_n^*.

We require the following Lemma for our example.

Lemma 2. If $v_n(x) = \sum_{j=0}^{n} b_j x^j$, with $b_n = 1$, where all the roots x_i are real, contained in $0 \leq x \leq 1$ and not all equal to zero, then $F(v_n) \neq 0$.

Proof. Obviously

$$\prod_{i=1}^{k+2} x_i \leq \prod_{i=1}^{k} x_i$$

and by considering the relationship between the roots of v_n and its coefficients we have

$$(-1)^k (n-k)!\, b_{n-k} > (-1)^k (n-k-2)!\, b_{n-k-2} \quad , \text{ for } k \geq 2 ,$$

$$\sum_{i=0}^{s} (-1)^i (n-2i)!\, b_{n-2i} > 0 \quad , \text{ for } s \text{ odd},$$

and certainly for s even.

With this Lemma we can derive, as in [64], Theorem 2.20 and using the canonical representation of F, that the only extremals of F are $\pm T_n^*$.

1.3. CONSTRUCTION OF THE TAU SOLUTION

Following Ortiz [54], we introduce for the matrix operator D, a sequence of canonical polynomials $Q = \underset{\sim}{Q}_n(x)$ where each element is a vector $\underset{\sim}{Q}_n(x) = (Q_n^{[1]}(x) \, , \, Q_n^{[2]}(x) \,)^T$ such that

$$D\,[Q_n^{[1]}(x)] = \binom{x^n}{0} \quad \text{and} \quad D\,[Q_n^{[2]}(x)] = \binom{0}{x^n}$$

In fact simple recurrence relationships exist for these polynomials.

$$\text{Consider} \quad \begin{pmatrix} 1 & -\frac{d}{dx} \\ \frac{d}{dx} & 1 \end{pmatrix} \begin{pmatrix} x^{n+1} \\ -(n+1)x^n \end{pmatrix} = \begin{pmatrix} x^{n+1} + n(n+1)x^{n-1} \\ 0 \end{pmatrix}$$

$$= D[Q_{n+1}^{[1]}] + n(n+1) \; D\,[Q_{n-1}^{[1]}]$$

$$\therefore \; Q_{n+1}^{[1]} = \begin{pmatrix} x^{n+1} \\ -(n+1)x^n \end{pmatrix} - n(n+1)Q_{n-1}^{[1]}$$

since $Q_0^{[1]} = \binom{1}{0}$ and $Q_1^{[1]} = \binom{x}{-1}$

we have $Q_2^{[1]} = \begin{pmatrix} x^2-2 \\ -2x \end{pmatrix}$ and $Q_3^{[1]} = \begin{pmatrix} x^3-6x \\ -3x^2+6 \end{pmatrix}$

$$Q_4^{[1]} = \begin{pmatrix} x^4-12x^2+24 \\ -4x^3+24x \end{pmatrix} \quad Q_5^{[1]} = \begin{pmatrix} x^5-20x^3+120x \\ -5x^4+60x^2-120 \end{pmatrix} \qquad \text{etc}$$

Likewise we obtain for the second series

$$\begin{pmatrix} 1 & -\frac{d}{dx} \\ \frac{d}{dx} & 1 \end{pmatrix} \begin{pmatrix} (n+1)x^n \\ x^{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ n(n+1)x^{n-1}+x^{n+1} \end{pmatrix}$$

$$= n(n+1)\ D\ [Q_{n-1}^{[2]}] + D\ [Q_{n+1}^{[2]}]$$

$$\therefore\quad Q_{n+1}^{[2]} = \begin{pmatrix} (n+1)x^n \\ x^{n+1} \end{pmatrix} - n(n+1)Q_{n-1}^{[2]}$$

Since $Q_0^{[2]} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $Q_1^{[2]} = \begin{pmatrix} 1 \\ x \end{pmatrix}$

we have $Q_2^{[2]} = \begin{pmatrix} 2x \\ x^2-2 \end{pmatrix}$, $Q_3^{[2]} = \begin{pmatrix} 3x^2-6 \\ x^3-6x \end{pmatrix}$

$$Q_4^{[2]} = \begin{pmatrix} 4x^3-24x \\ x^4-12x^2+24 \end{pmatrix},\quad Q_5^{[2]} = \begin{pmatrix} 5x^4-60x^2+120 \\ x^5-20x^3+120x \end{pmatrix} \quad \text{etc.}$$

Now $T_n^*(x) = \sum_{k=0}^{n} C_k^{(n)} x^k$ where the coefficients C_k are available to us

and hence $\begin{pmatrix} y_n^* \\ z_n^* \end{pmatrix} = \tau_1^{(n)} \sum_{k=0}^{n} C_k^{(n)} Q_k^{[1]} + \tau_2^{(n)} \sum_{k=0}^{n} C_k^{(n)} Q_k^{[2]}$

where $\tau_1^{(n)}$, $\tau_2^{(n)}$ are determined by the initial conditions.

If we set $\underset{\sim}{\tau}^{(n)} = (\tau_1^{(n)}, \tau_2^{(n)})$ we can present the solution in vector notation $\underset{\sim}{y}_n^* = \underset{\sim}{\tau}^{(n)} \sum_{k=0}^{n} C_k^{(n)} \underset{\sim}{Q}_k(x)$

The form of the solution when there are gaps in the sequence Q (i.e. the case of undefined canonical polynomials) follows trivially from the algebraic theory developed by Ortiz in [54]

Example : Let us take $n=4$, then

$$T_4(x) = 128\,x^4 - 256\,x^3 + 160\,x^2 - 32\,x + 1$$

The initial coditions of the problem provide us with a system of linear algebraic equations

$$\begin{pmatrix}1\\0\end{pmatrix} = \tau_1^{(4)} \begin{pmatrix}128\times 24 - 160\times 2 + 1\\ -6\times 256 + 32\end{pmatrix} + \tau_2^{(4)} \begin{pmatrix}6\times 256 - 32\\ 128\times 24 - 2\times 160 + 1\end{pmatrix}$$

giving us $\tau_1^{(4)} = \frac{2753}{D_4}$ and $\tau_2^{(4)} = \frac{1504}{D_4}$ where $D_4 = (2753)^2 + (1504)^2$

$= 9841025$

We find $y_4^* = \frac{1}{D_4}(352384x^4 + 65280x^3 - 4943200x^2 + 1504\,x + D_4)$

and $z_4^* = \frac{1}{D_4}(192512x^4 - 1794560x^3 + 44800x^2 + 9838272x)$

We can as well generate the approximate solution directly in terms of Chebyshev polynomials, or in any other complete system $\mathbb{R}$, by means of a technique described by Ortiz [55] which essentially consists of using canonical polynomials represented in the basis $\mathbb{R}$ which are mapped by D into the generators of the same basis. In our case, we can introduce

$$\underset{\sim}{q}_n(x) = (\, q_n^{[1]}(x) \,,\, q_n^{[2]}(x) \,)^T$$

such that

$$D\,\underset{\sim}{q}_n(x) = \begin{pmatrix}T_n^*(x)\\ T_n^*(x)\end{pmatrix}$$

The approximate solution has then a simple expression

$$\underset{\sim}{y}_n^*(x) = \underset{\sim}{\tau}^{(n)} \cdot \underset{\sim}{q}_n(x).$$

We henceforth ommit the superscript $*$ on $y_n^*(x)$ without loss of clarity.

1.4. THE TAU SOLUTION

1.4.1 CHEBYSHEV EXPANSIONS OF THE TAU SOLUTION

If we rewrite (2) as
$$y_n - z_n' = \tau_1^{(n)} T_n^*$$
$$y_n' + z_n = \tau_2^{(n)} T_n^*$$

we obtain by differentiating the second equation and adding

$$y_n + y_n'' = \tau_1^{(n)} T_n^* + \tau_2^{(n)} T_n^{*\prime} \tag{3}$$

Repeated differentiation of this equation and alternate subtraction leads us to the following expansion of $y_n(x)$ where we have set $n' = 2[\frac{n}{2}]$

$$y_n(x) = \tau_1^{(n)} [T_n^*(x) - T_n^{*''}(x) + \ldots\ldots + (-1)^{[\frac{n}{2}]} T_n^{*(n')}(x)]$$
$$+ \tau_2^{(n)} [T_n^{*\prime}(x) - T_n^{*'''}(x) + \ldots\ldots + (-1)^{[\frac{n}{2}]} T_n^{*(n'+1)}(x)] \tag{4}$$

The initial condition $y_n(0) = 1$ gives then

$$1 = \tau_1^{(n)} \sum_{r=0}^{\frac{n}{2}} (-1)^r T_n^{*(2r)}(0) + \tau_2^{(n)} \sum_{r=0}^{[\frac{n}{2}]} (-1)^r T_n^{*(2r+1)}(0) \tag{5}$$

On the other hand, differentiating the first equation of (2) and subtracting it from the second, we obtain

$$z_n + z_n'' = \tau_2^{(n)} T_n^* - \tau_1^{(n)} T_n^{*\prime} \tag{6}$$

Applying the same process to this equation leads us to

$$z_n(x) = \tau_2^{(n)} [T_n^*(x) - T_n^{*''}(x) + \ldots\ldots + (-1)^{[\frac{n}{2}]} T_n^{*(n')}(x)]$$
$$- \tau_1^{(n)} [T_n^{*\prime}(x) - T_n^{*'''}(x) + \ldots\ldots + (-1)^{[\frac{n}{2}]} T_n^{*(n'+1)}(x)] \tag{7}$$

and

$$0 = \tau_2^{(n)} \sum_{r=0}^{\frac{n}{2}} (-1)^r T_n^{*(2r)}(0) - \tau_1^{(n)} \sum_{r=0}^{[\frac{n}{2}]} (-1)^r T_n^{*(2r+1)}(0) \tag{8}$$

We shall use the following expression for T_n^* (cf. [3] sect. 3.7)

$$T_n^*(x) = \frac{1}{2}[2^{2n}x^n + \sum_{j=1}^{n-1} (-1)^j \binom{2n-j-1}{j} \frac{2n}{2n-2j} 2^{2n-2j}x^{n-j} + 2(-1)^n] \qquad (9)$$

Hence $T_n^{*(n)}(0) = 2^{2n-1}n!$; $T_n^{*(n-1)}(0) = 2^{2n-1}n!(-\frac{1}{2})$ while

for $2 \leq k \leq n$

$$T_n^{*(n-k)}(0) = 2^{2n-1}n! \left[\frac{(-1)^k 2^{-2k} 2n(2n-k-1)\ldots\ldots(2n-2k+1)}{k!\; n(n-1) \qquad \ldots\ldots(n-k+1)}\right]$$

$$= 2^{2n-1}n! \left[\frac{(-1)^k 2^{-k}(n-\frac{(k+1)}{2}) \;\ldots\ldots\; (n-\frac{(2k-1)}{2})}{k!\;(n-1) \qquad \ldots\ldots\; (n-(k-1))}\right]$$

$$= 2^{2n-1}n! \left[\frac{(-1)^k 2^{-k}}{k!} (1 - \frac{(k-1)}{2(n-1)})(1 - \frac{(k-2)}{2(n-2)})\ldots\ldots(1 - \frac{1}{2}\frac{1}{(n-k+1)})\right] \qquad (10)$$

We observe that we may readily derive the initial conditions for y_n' and z_n'. If we differentiate (4) we obtain

$$y_n'(0) = \tau_1^{(n)} \sum_{r=0}^{[\frac{n}{2}]} (-1)^r T_n^{*(2r+1)}(0) + \tau_2^{(n)} \sum_{r=1}^{[\frac{n}{2}]} (-1)^{r+1} T_n^{*(2r)}(0)$$

and applying (8) we find

$$y_n'(0) = \tau_2^{(n)} T_n^*(0)$$

Similarly differentiating (7) and applying (5) we find

$$z_n'(0) = 1 - \tau_1^{(n)} T_n^*(0)$$

1. 4.2. SOLUTION OF (2) BY GREEN'S FUNCTION

The general solution of (3) is

$$y_n(x) = c_1 \sin x + c_2 \cos x + \int_0^x [\tau_1^{(n)} T_n^*(t) + \tau_2^{(n)} \frac{d}{dt} T_n^*(t)] \sin (x-t) dt$$

which, on a single integration by parts, becomes

$$y_n(x) = (c_1 - \tau_2^{(n)}) \sin x + c_2 \cos x + \int_0^x \tau_1^{(n)} T_n^*(t) \sin(x-t) dt + \int_0^x \tau_2^{(n)} T_n^*(t) \cos(x-t) d$$

Applying the Leibnitz formula for differentiating integrals

$$y_n'(x) = (c_1 - \tau_2^{(n)} T_n^*(0)) \cos x - c_2 \sin x + \tau_1^{(n)} \int_0^x T_n^*(t) \cos(x-t) dt$$
$$- \tau_2^{(n)} \int_0^x T_n^*(t) \sin(x-t) dt + \tau_2^{(n)} \cos(0) T_n^*(x)$$

But for the initial conditions $y_n(0) = 1$, $y_n'(0) = \tau_2^{(n)} T_n^*(0)$ to be satisfied, we must have $c_2 = 1$, $c_1 = \tau_2^{(n)} T_n^*(0)$

$$\therefore \quad y_n(x) - \cos(x) = \int_0^x [\tau_1^{(n)} \sin(x-t) + \tau_2^{(n)} \cos(x-t)] T_n^*(t) dt \qquad (11)$$

The general solution of (6) is

$$z_n(x) = (c_1 + \tau_1^{(n)}) \sin x + c_2 \cos x + \int_0^x [\tau_2^{(n)} \sin(x-t) - \tau_1^{(n)} \cos(x-t)] T_n^*(t) dt$$

Hence

$$z_n'(x) = (c_1 + \tau_1^{(n)} T_n^*(0)) \cos x - c_2 \sin x + \int_0^x [\tau_2^{(n)} \sin(x-t) + \tau_1^{(n)} \cos(x-t)] T_n^*(t) dt$$
$$- \tau_1^{(n)} \cos(0) T_n^*(x)$$

and with the initial conditions $z_n(0) = 0$, $z_n'(0) = 1 - \tau_1^{(n)} T_n^*(0)$ we have $c_2 = 0$, $c_1 + \tau_1^{(n)} T_n^*(0) = 1$

$$\therefore \quad z_n(x) - \sin(x) = \int_0^x [\tau_2^{(n)} \sin(x-t) - \tau_1^{(n)} \cos(x-t)] T_n^*(t)\,dt \qquad (12)$$

1.5. ERROR BOUNDS FOR THE TAU SOLUTION

1.5.1 BOUNDS FOR $\tau_1^{(n)}$, $\tau_2^{(n)}$

If we set $$S_E^{(n)} = \sum_{r=0}^{[\frac{n}{2}]} (-1)^r T_n^{*(2r)}(0)$$

and $$S_O^{(n)} = \sum_{r=0}^{[\frac{n}{2}]} (-1)^r T_n^{*(2r+1)}(0)$$

then, dropping the super-script, we can solve (5) and (8) for $\tau_1^{(n)}$, $\tau_2^{(n)}$

$$\tau_1^{(n)} = \frac{S_E}{S_E^{\,2} + S_O^{\,2}}$$

$$\tau_2^{(n)} = \frac{S_O}{S_E^{\,2} + S_O^{\,2}}$$

We can now apply (10) to deduce that for n even

$$1 - \sum_{r=0}^{\infty} \frac{(\frac{1}{2})^{4r+2}}{(4r+2)!} + \frac{(\frac{1}{2})^4}{4!}\{(1 - \frac{1}{2}\frac{3}{(n-1)})(1 - \frac{1}{(n-2)})(1 - \frac{1}{n-3})\}$$

$$< \frac{(-1)^{[\frac{n}{2}]} S_E}{2^{2n-1} n!} < \sum_{r=0}^{\infty} \frac{(\frac{1}{2})^{4r}}{(4r)!} - \frac{(\frac{1}{2})^2}{2!}(1 - \frac{1}{2(n-1)})$$

and

$$\frac{1}{2} - \sum_{r=1}^{\infty} \frac{(\frac{1}{2})^{4r-1}}{(4r-1)!} < \frac{(-1)^{[\frac{n}{2}]} S_O}{2^{2n-1} n!} < \sum_{r=0}^{\infty} \frac{(\frac{1}{2})^{4r+1}}{(4r+1)!} \frac{-(\frac{1}{2})^3}{3!}[(1 - \frac{2}{2(n-1)})(1 - \frac{1}{2(n-2)})]$$

If we take $n \geq 4$ and make use of expansions for sin(x), sinh(x); cos(x), cosh(x) we obtain

$$1 - \left[\frac{\cosh(\frac{1}{2})-\cos(\frac{1}{2})}{2}\right] + \frac{(\frac{1}{2})^7}{4!} < \frac{(-1)^{\left[\frac{n}{2}\right]} S_E}{2^{2n-1} n!} < \frac{\cosh(\frac{1}{2})+\cos(\frac{1}{2})}{2} - \frac{(\frac{1}{2})^2}{2!} \ \frac{5}{6}$$

$$\frac{1}{2} - \left[\frac{\sinh(\frac{1}{2})-\sin(\frac{1}{2})}{2}\right] < \frac{(-1)^{\left[\frac{n}{2}\right]} S_O}{2^{2n-1} n!} < \frac{\sinh(\frac{1}{2}) + \sin(\frac{1}{2})}{2} - \frac{(\frac{1}{2})^4}{3!}.$$

Hence

$$.875304 < \frac{(-1)^{\left[\frac{n}{2}\right]} S_E}{2^{2n-1} n!} < .898438$$

$$.479165 < \frac{(-1)^{\left[\frac{n}{2}\right]} S_O}{2^{2n-1} n!} < .492448$$

$$1.0497 = (.898438)^2 + (.492448)^2 > \frac{S_E^{\,2} + S_O^{\,2}}{[2^{2n-1} n!]^2} > (.875304)^2 + (.479165)^2 = .995756$$

$$[2^{2n-1} n!]^{-1} (.83386) < |\tau_1^{(n)}| < [2^{2n-1} n!]^{-1} (.902267)$$

$$\ldots\ldots(13)$$

$$[2^{2n-1} n!]^{-1} (.45648) < |\tau_2^{(n)}| < [2^{2n-1} n!]^{-1} (.49455)$$

and $$\frac{\tau_1^{(n)}}{\tau_2^{(n)}} < \frac{.898438}{.479165} = 1.875001$$

For n odd, $\tau_1^{(n)}$ and $\tau_2^{(n)}$ have to be interchanged in (13)

I. 5.2 INTEGRALS OF CHEBYSHEV POLYNOMIALS

We shall require the following result, concerning the integral of a Chebyshev polynomial between two consecutive roots, which is easy to derive.

Lemma 3. Let the roots of $T_n^*(x)$ be x_{n-k} $k = 1,\ldots,n$

where $x_{n-k} = \cos^2 \frac{(n-k+\frac{1}{2})\pi}{2n}$

and $x_n \equiv 0 < x_{n-1} < \ldots\ldots < x_0 < 1 \equiv x_{-1}$.

Then $I_j \equiv \int_{x_j}^{x_{j-1}} T_n^*(t)dt$

$$= (-1)^j \phi(n) \sin \frac{j\pi}{n} \quad \text{for } j = 1,\ldots, n-1$$

$$\text{where } \phi(n) = \frac{n}{n^2-1} \cos \frac{\pi}{2n} \to \frac{1}{n}\left[1 + 0\left(\frac{1}{n^2}\right)\right]$$

$$\text{and } I_j = \frac{(-1)^j}{2(n^2-1)} \left[n \sin \frac{\pi}{2n} - 1\right] \quad \text{for } j = 0 \text{ or } n$$

We shall now restrict ourselves to the case of n being even as the treatment for n odd is similar.

Theorem 1. Let us set $\varepsilon_1^*(x) = y_n(x) - \cos x$ in (11), then $\frac{.08519 \quad \phi^*(n)}{2^{2n} n!} < ||\varepsilon_1(x)|| < \frac{1.5022\,(1+0(\frac{1}{n}))}{2^{2n}\, n!(n-1)}$

where $\phi^*(n) = \phi(n)\,(1 + 0(\frac{1}{n}))$

Proof :

To find an upper bound for $||\varepsilon_1(x)||$ we can proceed as follows

$$\text{Put } I(x) = \int_0^x T_n^*(t)dt = \frac{1}{4}\left[\frac{T_{n+1}^*(x)}{n+1} - \frac{T_{n-1}^*(x)}{n-1}\right] + \frac{1}{2(n^2-1)} \cos (n+1)\pi$$

Then $|I(x)| \leq \frac{1}{2(n-1)}$. Furthermore, integrating by parts in (11)

$$\varepsilon_1(x) = \frac{\tau_2}{4}\left[\frac{T_{n+1}^*(x)}{n+1} - \frac{T_{n-1}^*(x)}{n-1}\right] - \frac{(-1)^n}{2(n^2-1)}(\tau_1 \sin x + \tau_2 \cos x)$$

$$+ \int_0^x I(t)(-\tau_1 \cos (x-t) + \tau_2 \sin (x-t))dt$$

$\therefore$

$$|\varepsilon_1(x)| \leq \frac{\tau_2 n}{2(n^2-1)} + \frac{\tau_1 \sin x+\tau_2 \cos(x)}{2(n^2-1)} + \frac{\tau_2(1-\cos x)+\tau_1 \sin x}{2(n-1)}$$

$$= \frac{\tau_1(n+2) \sin x + \tau_2[(2n+1) - n \cos x)]}{2(n^2-1)}$$

and

$$||\varepsilon_1(x)|| \leq (|\tau_2|(2-\cos(1)) + |\tau_1| \sin(1)) \left[\frac{1}{2(n-1)} + o(\frac{1}{n^2})\right]$$

$$\leq |\tau_2|[1.4597 + 1.87501\times .84147] \left[\frac{1}{2(n-1)} + o(\frac{1}{n^2})\right]$$

$$\leq \frac{1.5022 \ (1 + o(\frac{1}{n}))}{2^{2n} n! (n-1)}$$

To find a lower bound for $||\varepsilon_1(x)||$

consider $\varepsilon_1(x_{\frac{3n}{4}})$ where $x_{\frac{3n}{4}} = \frac{1}{2}[1 + \cos(\frac{3\pi}{4} + \frac{\pi}{2n})] \to \frac{1}{2}[1 - \frac{1}{\sqrt{2}}]$

(If n is not divisible by 4, take $x_{[\frac{3n}{4}]+1}$)

$$\int_0^{x_{\frac{3n}{4}}} \cos(x_{\frac{3n}{4}} - t)T_n^*(t)dt > \cos x_{\frac{3n}{4}} I_n + \cos(x_{\frac{3n}{4}} - x_{n-2})[I_{n-2} - |I_{n-1}|] +$$

$$+ \ldots\ldots + \cos(x_{\frac{3n}{4}} - x_{\frac{3n}{4}-1})[I_{\frac{3n}{4}+1} - |I_{\frac{3n}{4}+2}|]$$

$$> \cos x_{\frac{3n}{4}}[I_n + \phi(n) \sum_{k=1}^{\frac{n}{4}-1} (-1)^k \sin \frac{k\pi}{n}]$$

$$= \cos x_{\frac{3n}{4}}[I_n + 2 \sin \frac{\pi}{2n} \phi(n) \sum_{k=1}^{\frac{n}{8}-1} \cos(2k-\tfrac{1}{2}) \frac{\pi}{n}]$$

$$> \cos x_{\frac{3n}{4}} .\ 2 \sin \frac{\pi}{2n} \phi(n) \sum_{k=1}^{\frac{n}{8}-1} \cos 2k \frac{\pi}{n}$$

But

$$\sum_{k=1}^{\frac{n}{8}-1} \cos 2\frac{k\pi}{n} = \mathrm{Re} \frac{[e^{\frac{2i\pi}{n}} - e^{\frac{i\pi}{4}})(1 - e^{-\frac{2i\pi}{n}})]}{2(1 - \cos \frac{2\pi}{n})}$$

$$= \mathrm{Re} \frac{e^{\frac{2i\pi}{n}} - 1 + e^{i(\frac{\pi}{4} - 2\frac{\pi}{n})} - e^{\frac{i\pi}{4}}}{2(1 - \cos \frac{2\pi}{n})}$$

$$= \frac{\cos\frac{2\pi}{n} - 1}{2(1-\cos \frac{2\pi}{n})} + \frac{2 \sin(\frac{\pi}{4} - \frac{\pi}{n}) \sin \frac{\pi}{n}}{2.2 \sin \frac{\pi}{n} \sin \frac{\pi}{n}}$$

$$\to \frac{n}{2\pi\sqrt{2}} + O(1)$$

For n large $\int_0^{x_{\frac{3n}{4}}} \cos(x_{\frac{3n}{4}} - t) T_n^*(t)\,dt > \dfrac{\cos(\frac{2-\sqrt{2}}{4})\phi^*(n)}{2\sqrt{2}}$

$$\int_0^{x_{\frac{3n}{4}}} \sin(x_{\frac{3n}{4}} - t) T_n^*(t)\,dt > [\sin(x_{\frac{3n}{4}} - x_{n-1}) I_{n-1} + \sin(x_{\frac{3n}{4}} - x_{n-3}) I_{n-2}]$$

$$+ \ldots\ldots + [\sin(x_{\frac{3n}{4}} - x_{n-3}) I_{\frac{3n}{4}+2} + 0.]$$

But $\sin(x_{\frac{3n}{4}} - x_{n-3}) > \sin(x_{\frac{3n}{4}} - x_{n-1}) - (x_{n-3} - x_{n-1})\cos(x_{\frac{3n}{4}} - x_{n-3})$

$I_{n-1} + I_{n-2} > 0$; $x_{n-3} - x_{n-1} < \sin\frac{\pi}{n}$; $\cos(x_{\frac{3n}{4}} - x_{n-3}) < 1$

$$\therefore \int_0^{x_{\frac{3n}{4}}} \sin(x_{\frac{3n}{4}} - t) T_n^*(t)\,dt > -\sin\frac{\pi}{n}\phi(n)\left[\sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \sin\left(\frac{n}{2} - 2\right)\frac{\pi}{n}\right]$$

$$\sum_{k=1}^{\frac{n}{8}-1} \sin\frac{2k\pi}{n} = \text{Imag}\,\frac{\left(e^{\frac{i2\pi}{n}} - e^{\frac{i\pi}{4}}\right)\left(1 - e^{-\frac{2i\pi}{n}}\right)}{2\left(1 - \cos\frac{2\pi}{n}\right)}$$

$$= \frac{\sin\frac{2\pi}{n} + \sin\left(\frac{\pi}{4} - \frac{2\pi}{n}\right) - \sin\frac{\pi}{4}}{2.2\sin^2\frac{\pi}{n}}$$

$$= \frac{\cos\frac{\pi}{n} - \cos\left(\frac{\pi}{4} - \frac{\pi}{n}\right)}{2\sin\frac{\pi}{n}} \to \frac{1 - \frac{1}{\sqrt{2}}}{\frac{2\pi}{n}}$$

$$\therefore \int_0^{x_{\frac{3n}{4}}} \sin(x_{\frac{3n}{4}} - t) T_n^*(t)\,dt > -\phi^*(n)\,\frac{\sqrt{2} - 1}{2\sqrt{2}}$$

$$\therefore \; (-1)^{[\frac{n}{2}]} \varepsilon_1(x_{\frac{3n}{4}}) > \frac{\phi^*(n)}{2\sqrt{2}} \left[|\tau_2| \cos\left(\frac{2-\sqrt{2}}{4}\right) - |\tau_1|(\sqrt{2}-1)\right]$$

$$> \frac{\phi^*(n)|\tau_2|}{2\sqrt{2}} [\cos(.1465) - 1.87501(.414)]$$

$$||\varepsilon_1(x)|| > \frac{\phi^*(n).45648\ (.21303)}{2^{2n-1} n!\ 2.2828}$$

<u>Theorem 2</u> Let us set $\varepsilon_2(x) = z_n^*(x) - \sin x$ in (12)

Then $.49468 \dfrac{\phi^*(n)}{2^{2n} n!} < ||\varepsilon_2(x)|| < 1.76971 \dfrac{(1 + O(\frac{1}{n}))}{2^{2n} n!\ (n-1)}$

<u>Proof:</u> An upper bound for $||\varepsilon_2(x)||$ is obtained as before.

$$\varepsilon_2(x) = \frac{-\tau_1}{4} \left[\frac{T^*_{n+1}(x)}{n+1} - \frac{T^*_{n-1}(x)}{n-1}\right] + \frac{(-1)^n}{2(n^2-1)} [\tau_1 \cos x - \tau_2 \sin x]$$

$$+ \int_0^x I(t) \; [\tau_2 \cos(x-t) - \tau_1 \sin(x-t)]\, dt$$

$$|\varepsilon_2(x)| \le \frac{|\tau_1|[(2n+1) - n \cos x] + |\tau_2|[(n+2) \sin x]}{2(n^2-1)}$$

$$||\varepsilon_2(x)|| \le [|\tau_1|(2 - \cos(1)) + |\tau_2| \sin(1)] \left[\frac{1}{2(n-1)} + O(\frac{1}{n^2})\right]$$

$$\le \frac{1.76971(1 + O(\frac{1}{n}))}{2^{2n} n!\ (n-1)}$$

To find a lower bound for $||\varepsilon_2(x)||$

consider $\varepsilon_2(x_{\frac{n}{2}})$ where $x_{\frac{n}{2}} \to \frac{1}{2}$

This time $$\int_0^{x_{\frac{n}{2}}} \cos(x-t)T_n^*(t)dt > \cos(x_{\frac{n}{2}})2 \sin\frac{\pi}{2n}\ \phi(n)\left(\frac{n}{2} + O(1)\right)$$

$$\to \frac{1}{2}\cos(\tfrac{1}{2})\ \phi^*(n)$$

and $$\int_0^{x_{\frac{n}{2}}} \sin(x-t)T_n^*(t)dt < \sin(x_{\frac{n}{2}})\ 2\sin\frac{\pi}{2n}\phi(n)\ \left(\frac{n}{2} + O(1)\right)$$

$$\to \frac{1}{2}\sin(\tfrac{1}{2})\ \phi^*(n)$$

$$(-1)^{[\frac{n}{2}]}\varepsilon_2(x_{\frac{n}{2}}) < \frac{\phi^*(n)}{2}\left[|\tau_2|\sin(\tfrac{1}{2}) - |\tau_1|\cos(\tfrac{1}{2})\right]$$

$$< \frac{\phi^*(n)}{2^{2n}}\ \frac{[.49455\times.479426 - .83386\times.877582]}{n!}$$

$$\therefore\quad ||\varepsilon_2(x)|| > \frac{\phi^*(n).49468}{2^{2n}\ n!}$$

Remark Since $\frac{1}{n-1} \sim \phi^*(n) \sim \frac{1}{n+1}[1 + O(\frac{1}{n})]$

we see from Theorems 1 and 2 that

$$||\underset{\sim}{e}_n^*||_\infty = \frac{K}{2^{2n}(n+1)!}\ [1 + O(\tfrac{1}{n})]$$

where $.49468 < K < 1.76971$ for n even .

This is comparable to the results of Meinardus in [48] p.80 for the minimal deviation on $[0,1]$ except that then $K = 1$.

SIMULTANEOUS APPROXIMATION OF A FUNCTION AND ITS DERIVATIVE WITH THE TAU METHOD, PART II

FOREWORD: In this part we generalize the Tau method technique developed in Part I for the numerical solution of systems of two first order linear differential equations, to the cases of: (a) the general second order differential equation with constant coefficients, (b) the Euler equation and (c) The Airy equation. In this last example we deal with the case of undefined canonical polynomials.

In each case we give error bounds in the uniform norm for the function and derivative.

Although we have modelled our arguments on systems of order two, the same method could be extended to systems of higher order with corresponding approximations to higher derivatives.

2.1. The General Second Order Linear Differential Equation with Constant Coefficients

In this part we continue the analysis of the numerical solution of systems of linear differential equations with the Tau method, initiated in Part I.

We begin our discussion with the general second order linear differential equation with constant coefficients

$$y''(x) + a_1 y'(x) + a_0 y(x) = 0 \tag{1}$$

subject to the supplementary conditions $y(0) = \alpha$, $y'(0) = \beta$.

As a system of first order linear differential equations, the operator form of (1) is

$$\begin{pmatrix} a_0 & -\frac{d}{dx} & -a_1 \\ \frac{d}{dx} & & 1 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For the perturbed system, we compute the exact solution $[y_n, z_n]^T$ of

$$\begin{aligned} a_0 y_n = z'_n - a_1 z_n &= \tau_1^{(n)} T_n^* \\ y'_n + z &= \tau_2^{(n)} T_n^* \end{aligned} \qquad (2)$$

with the derived initial conditions

$$y_n(0) = \alpha, \qquad \tau_n(0) = -\beta$$

$$y'_n(0) = \tau_2^{(n)} T_n^*(0) + \beta$$

$$z_n'(0) = -\tau_1^{(n)} T_n^*(0) + a_0\alpha + a_1\beta$$

2.1.1 DETERMINATION OF THE TAU

We proceed as in 4.1 of Part I. Combining (2) we obtain

$$y_n'' + a_1 y_n' + a_0 y_n = \tau_1^{(n)} T_n + \tau_2^{(n)} [a_1 T_n^* + T_n^{*\prime}] \quad (3)$$

By repeated differentiation and back substitution, we find assuming $\alpha = 1$, $\beta = 0$ and $n = 4$

$$a_0^3 a_1 = \tau_1^{(4)} [a_0^2 a_1 T_4^*(0) - a_0^2 T_4^{*i}(0) + a_0 T_4^{*iii}(0) - a_1 T_4^{*iv}(0)]$$

$$+ \tau_2^{(4)} [a_0^3 T_4^*(0) - a_0^2 T_4^{*ii}(0) + a_0 a_1 T_4^{*iii}(0) + (a_0 - a_1^2) T_4^{*iv}(0)]$$

Likewise we obtain from

$$z_n^{ii} + a_1 z_n^i + a_0 z_n = \tau_2^{(n)} a_0 T_n^* - \tau_1^{(n)} T_n^* \quad (4)$$

$$(a_0^2 a_1^2 - a_0^3) = \tau_1^{(4)} [(a_0 a_1^2 - a_0^2) T_4^*(0) - a_0 a_1 T_4^{*i}(0) + a_0 T_4^{*ii}(0) = T_4^{*iv}(0)]$$

$$+ \tau_2^{(4)} [a_0^2 a_1 T_4^*(0) - a_0^2 T_4^{*i}(0) + a_0 T_4^{*iii}(0) - a_1 T_4^{*iv}(0)]$$

We solve these two equations for τ_1 and τ_2, and in table I, we illustrate the result for particular values of a_0, a_1.

TABLE I. Evaluation of Tau for certain values of a_0, a_1 when $n = 4$

Ex	a_0	a_1	τ_1	τ_2
1	-1	1	-7.7667×10^{-4}	4.3971×10^{-4}
2	-1	-1	-3.8373×10^{-3}	-2.3499×10^{-3}

For a general prescription for $\tau_1^{(n)}$, $\tau_2^{(n)}$ we take the r-th derivative of equation (3), $r = 0,1,\ldots,n$ and multiply through by $(\frac{1}{\lambda_i})^{r+1}$ $i = 1,2$ where λ_1, λ_2 are the roots of the characteristic equation.

With each system $i = 1,2$, we add all the equations together, making use of the fact that

$$(\frac{1}{\lambda_i})^{r+1}[1 + a_1(\frac{1}{\lambda_i}) + a_0(\frac{1}{\lambda_i})^2] = 0$$

to reduce the system.

Subtracting one system from the other and setting

$$S_O = \sum_{r=0}^{n} [(\frac{1}{\lambda_1})^{r+1} - (\frac{1}{\lambda_2})^{r+1}]T_n^{(r)}(0)$$

$$S_E = \sum_{r=1}^{n} [(\frac{1}{\lambda_1})^{r} - (\frac{1}{\lambda_2})^{r}]T_n^{(r)}(0)$$

we find that

$$(\lambda_2 - \lambda_1)y_n(0) = \tau_1^{(n)}S_O + \tau_2^{(n)}[a_1S_O + S_E]$$

Similarly from (4)

$$(\lambda_2 - \lambda_1)z_n(0) = \tau_2^{(n)}a_0S_O - \tau_1^{(n)}S_E$$

When $\alpha = 1$, $\beta = 0$ we have that

$$\tau_1^{(n)} = \frac{a_0S_O}{(S_E^2 + a_1S_ES_O + a_0S_O^2)(\lambda_2 - \lambda_1)}$$

$$\tau_2^{(n)} = \frac{S_E}{(S_E{}^2 + a_1 S_E S_O + a_O S_O{}^2)(\lambda_2 - \lambda_1)}$$

If

$$\infty > \liminf_{r>0} \left| \frac{[(\frac{1}{\lambda_1})^{r+1} - (\frac{1}{\lambda_2})^{r+1}]}{[(\frac{1}{\lambda_1})^{r} - (\frac{1}{\lambda_2})^{r}]} \right| > 1$$

we may deduce the asymptotic results

$$\lim_{n\to\infty} |\tau_1^{(n)}| = \frac{1 + o(\frac{1}{n})}{|\lambda_1 - \lambda_2| \left| (\frac{1}{\lambda_1})^{n+1} - (\frac{1}{\lambda_2})^{n+1} \right| T_n{}^{*(n)}(0)}$$

$$\lim_{n\to\infty} |\tau_2^{(n)}| = \frac{\left| (\frac{1}{\lambda_1})^{n} - (\frac{1}{\lambda_2})^{n} \right| (1 + o(\frac{1}{n}))}{|\lambda_1 \lambda_2| \left| (\frac{1}{\lambda_1})^{n+1} - (\frac{1}{\lambda_2})^{n+1} \right|^2 T_n{}^{*(n)}(0)}$$

2.1.2. ANALYTIC SOLUTIONS OF THE HOMOGENEOUS EQUATIONS

Both the function y and its derivative -z satisfy

$u''(x) + a_1 u(x) + a_0 u(x) = 0.$

The basis for the solution space of this equation, $\phi_1(x)$, $\phi_2(x)$, depends on the nature of the roots, λ_1, λ_2, of the characteristic equation.

We have

$$y(x) = c_1\phi_1(x) + c_2\phi_2(x)$$
$$z(x) = d_1\phi_1(x) + d_2\phi_2(x)$$

where c_i, d_i, $i = 1,2$; are determined by the initial conditions.

We shall assume the roots are real and distinct. Then

$$\phi_1(x) = e^{\lambda_1 x}, \quad \phi_2(x) = e^{\lambda_2 x}$$

Furthermore, if we set $\kappa(x,t) = \phi_2(x)\phi_1(t) - \phi_1(x)\phi_2(t)$, then

$$\kappa(x,t) = e^{\lambda_2 x+\lambda_1 t} - e^{\lambda_1 x+\lambda_2 t}$$

Now $w(t) = (\lambda_2 - \lambda_1)e^{(\lambda_1+\lambda_2)t}$ is the Wronskian of ϕ_1 and ϕ_2.

$$\frac{\kappa(x,t)}{w(t)} = \frac{e^{\lambda_2(x-t)} - e^{\lambda_1(x-t)}}{\lambda_2 - \lambda_1}$$

and $$\frac{d}{dt}\frac{\kappa(x,t)}{w(t)} = \frac{\lambda_1 e^{\lambda_1(x-t)} - \lambda_2 e^{\lambda_2(x-t)}}{\lambda_2 - \lambda_1}$$

2.1.3 THE TAU SOLUTION BY THE GREEN'S FUNCTION

The solution of (3) is

$$y_n(x) = \hat{c}_1\phi_1(x) + \hat{c}_2\phi_2(x)$$

$$- \tau_2^{(n)} T_n(0) \frac{\kappa(x,0)}{w(0)} + \int_0^x G_1(x,t)\, dt$$

where $G_1(x,t) = [(\tau_1^{(n)} + a_1\tau_2^{(n)})T_n(t)\kappa(x,t) - \tau_2^{(n)} w(t)T_n(t)\frac{d}{dt}\frac{\kappa(x,t)}{w(t)}]/w(t)$

Hence we have $y_n(0) = \hat{c}_1\phi_1(0) + \hat{c}_2\phi_2(0)$.

Moreover from

$$y_n'(x) = \hat{c}_1\phi_1'(x) + \hat{c}_2\phi_2'(x) - \tau_2^{(n)} \frac{T_n(0)}{w(0)} [\phi_2'(x)\phi_1(0) - \phi_1'(x)\phi_2(0)]$$

$$+ G_1(x,x) + \int_0^x \frac{\partial}{\partial x} G_1(x,t)dt$$

we find $y_n'(0) = \hat{c}_1\phi_1'(0) + \hat{c}_2\phi_2'(0)$

since $$\frac{d}{dt}\frac{\kappa(x,t)}{w(t)}\bigg|_{t=x} = -1 .$$

From the initial conditions for y and y_n we obtain

$$\hat{c}_1 - c_1 = - \frac{\tau_2^{(n)} T_n(0)\phi_2(0)}{w(0)} ; \qquad \hat{c}_2 - c_2 = \frac{\tau_2^{(n)} T_n(0)\phi_1(0)}{w(0)}$$

We set $k_1(x) = \int_0^x \frac{\kappa(x,t)}{w(t)} T_n(t)dt$ and $k_2(x) = \int_0^x T_n(t) \frac{d}{dt}\frac{\kappa(x,t)}{w(t)} dt$

Then

$$\varepsilon_1(x) \equiv y_n(x) - y(x) = (\tau_1^{(n)} + a_1\tau_2^{(n)})k_1(x) - \tau_2^{(n)}k_2(x) \tag{5}$$

similarly

$$\varepsilon_2(x) \equiv z_n(x) - z(x) = a_0\tau_2^{(n)}k_1(x) + \tau_1^{(n)}k_2(x) \tag{6}$$

2.1.4 UPPER BOUNDS FOR THE ERROR FUNCTIONS

These follow from (5) and (6) by finding bounds on $k_1(x)$, $k_2(x)$. Performing an integration by parts and using the bounds for $I(x)$, determined in 5.3 of Part I, we obtain

$$|k_1(x)| \leq \frac{1}{2(n-1)}\left\{ \frac{1}{n+1}\left|\frac{\kappa(x,1)}{w(1)}\right| + \int_0^x \left|\frac{d}{dt}\frac{\kappa(x,t)}{w(t)}\right| dt\right\}$$

$$|k_2(x)| \leq \frac{1}{2(n-1)}\left\{\left|\frac{d}{dt}\frac{\kappa(x,t)}{w(t)}_{t=x}\right| + \frac{1}{n+1}\left|\frac{d}{dt}\frac{\kappa(x,t)}{w(t)}_{t=0}\right| + \int_0^x \left|\frac{d^2}{dt^2}\frac{\kappa(x,t)}{w(t)}\right| dt\right\}$$

Now using the positivity of $e^{\lambda t}$, $\lambda^2 e^{\lambda t}$ and convexity of $\lambda e^{\lambda x}$, $\lambda > 0$, we find

$$\max_{0\leq x\leq 1} |k_1(x)| \leq \frac{1}{2(n-1)}\left|\frac{e^{\lambda_1} - e^{\lambda_2}}{\lambda_1 - \lambda_2}\right| \left(1 + \frac{1}{n+1}\right)$$

$$\max_{0\leq x\leq 1} |k_2(x)| \leq \frac{1}{2(n-1)} \left\{ \left(1 + \frac{1}{n+1}\right)\max\left(1,\left|\frac{\lambda_2 e^{\lambda_2} - \lambda_1 e^{\lambda_1}}{\lambda_2 - \lambda_1}\right|\right) + 2 \right\}$$

We illustrate with the same examples as in 2.1.1. with $n = 4$.

TABLE II. UPPER BOUNDS FOR ERROR TERMS

Ex	$\|\|k_1\|\|$	$\|\|k_2\|\|$	$\|\|\varepsilon_1\|\|$	$\|\|\varepsilon_2\|\|$
1	.14821	.46459	2.5423×10^{-4}	4.2601×10^{-4}
2	.40286	1.09298	3.1677×10^{-3}	5.1408×10^{-3}

2.2. THE EULER EQUATION

Suppose $x^2 y''(x) + a_1 x y'(x) + a_0 y(x) = 0$

subject to $y(1) = \alpha$, $y'(1) = \beta$

The operator D is now

$$\begin{pmatrix} a_0 & -x^2 \frac{d}{dx} & -a_1 x \\ \frac{d}{dx} & & 1 \end{pmatrix}$$

For the perturbed system we compute the exact solution $[y_{n+1}, z_n]^T$ of

$$D\,\underline{y} = \begin{pmatrix} \tau_1^{(n)} & \tilde{T}_{n+1} \\ \tau_2^{(n)} & \tilde{T}_n \end{pmatrix} \tag{7}$$

where $\tilde{T}_n$ is the Chebyshev polynomial of degree n shifted to the interval J.

2.2.1 THE CANONICAL POLYNOMIALS

$$\begin{pmatrix} a_0 & -x^2 \frac{d}{dx} - a_1 x \\ \frac{d}{dx} & 1 \end{pmatrix} \begin{pmatrix} x^k \\ -kx^{k-1} \end{pmatrix} = [a_0 + k(k-1) + ka_1] \begin{pmatrix} x^k \\ 0 \end{pmatrix}$$

$\therefore$ for $k \geq 1$ $\quad Q_k^{[1]}(x) = \dfrac{1}{a_0 + k(k-1) + ka_1} \begin{pmatrix} x^k \\ -kx^{k-1} \end{pmatrix}, \quad Q_0^{[1]} = \dfrac{1}{a_0} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

while $\quad Q_k^{[2]}(x) = \dfrac{1}{a_0 + (k+1)(k+a_1)} \begin{pmatrix} (a_1+k)x^{k+1} \\ a_0 x^k \end{pmatrix}$

2.2.2 THE TAU SOLUTION

Setting $\tilde{T}_n(x) = \sum_{k=0}^{n} \tilde{c}_k^{(n)} x^k$ and inverting the operator D, as in section of Part I, we have

$$\begin{pmatrix} Y_{n+1}(x) \\ z_n(x) \end{pmatrix} = \tau_1^{(n)} \sum_{k=0}^{n+1} \tilde{c}_k^{(n+1)} Q_k^{[1]}(x) + \tau_2^{(n)} \sum_{k=0}^{n} \tilde{c}_k^{(n)} Q_k^{[2]}(x)$$

$\tau_1^{(n)}$ and $\tau_2^{(n)}$ can be found by solving this system at $x = 1$.

We shall assume $\alpha = 1$, $\beta = 0$, $J = [1,2]$

$$\tilde{T}_3(x) = 32x^3 - 144x^2 + 210x - 99$$

$$\tilde{T}_4(x) = 128x^4 - 768x^3 + 1696x^2 - 1632\,x + 577$$

Hence, if

$$a_1 = 1, \quad a_0 = -a < 0$$

$$1 = \tau_1^{(3)}\left[\frac{577}{-a} - \frac{1632}{1-a} + \frac{1696}{4-a} - \frac{768}{9-a} + \frac{128}{16-a}\right] + \tau_2^{(3)}\left[\frac{-99}{1-a} + \frac{420}{4-a} - \frac{432}{9-a} + \frac{128}{16-a}\right]$$

$$0 = \tau_1^{(3)}\left[\frac{1632}{1-a} - \frac{3392}{4-a} + \frac{2304}{9-a} - \frac{512}{16-a}\right] + \frac{\tau_2^{(3)}}{a}\left[\frac{99}{1-a} - \frac{210}{4-a} + \frac{144}{9-a} - \frac{32}{16-a}\right]$$

EXAMPLE $a = .25$:

$$\tau_1^{(3)} = -0.259\ 10^{-3}\ ; \qquad \tau_2^{(3)} = 1.08\ 10^{-3}\ .$$

It is clear that for values of 'a' close to r^2, $r = 0, 1,2,3$; the dominating terms in the system are the r-th ones and the tau-s will be found to be small $< 10^{-3}$

The same is true for 'a' close to zero on the negative side, i.e. $-\frac{1}{10} \leq a < 0$, in which case the indicial equation has complex conjugate roots, but not in general, for 'a' negative.

2.2.3 ANALYTIC SOLUTIONS OF THE EULER HOMOGENEOUS EQUATION

$$x^2y'' + a_1 x y' + a_0 y = 0 \qquad y(1) = \alpha,\ y'(1) = \beta$$

We are looking for two linearly independent solutions $\phi_1(x)$, $\phi_2(x)$ such that $y = c_1\phi_1 + c_2\phi_2$ where the constants c_1, c_2 are uniquely determined by the initial conditions.

Suppose $\phi(x) = x^m$ is a solution.

Then m must satisfy the indicial equation $m(m-1) + a_1 m + a_0 = 0$.

The following exhaust all the possibilities:

(i) The indicial equation has 2 distinct real roots m_1, m_2;

(ii) m is a repeated root

Then $\phi_1(x) = x^m$ and $\phi_2(x) = x^m \log x$

(iii) There are 2 complex conjugate roots $m_1 = c + id$, $m_2 = c - id$

Then $\phi_1(x) = x^c \cos d \log x$ and $\phi_2(x) = x^c \sin d \log x$

We set $\kappa_1(x,t) = \phi_2(x)\phi_1(t) - \phi_1(x)\phi_2(t)$

where $\kappa_1(x,x) = 0$ and $w_1(t)$ is the wronskian of ϕ_1 and ϕ_2.

We note that

$$w_1(1) = \begin{cases} m_2 - m_1 & \text{in (i)} \\ 1 & \text{in (ii)} \\ d & \text{in (iii)} \end{cases}$$

For all r,

$$\frac{d}{dt}\ t^r \frac{\kappa_1(x,t)}{w_1(t)} = \frac{-(r+1-m_1)x^{m_1}t^{r-m_1} + (r+1-m_2)x^{m_2}t^{r-m_2}}{m_2 - m_1} \qquad (8)$$

2.2.4 THE TAU SOLUTION BY THE GREEN'S FUNCTION

First, we stipulate that

$$y_{n+1}(1) = \alpha, \quad z_n(1) = -\beta \qquad \text{and hence for consistency with (7)}$$

$$y'_{n+1}(1) = \beta + \tau_2^{(n)}\tilde{T}_n(1).$$

$$z_n'(1) = -\tau_1^{(n)}\tilde{T}_{n+1}(1) + a_0\alpha + a_1\beta$$

From (7) we also find that y_{n+1} satisfies the equation

$$x^2 y''_{n+1}(x) + a_1\, x\, y'_{n+1}(x) + a_0 y_{n+1}(x) = \tau_1^{(n)}\tilde{T}_{n+1}(x) + \tau_2^{(n)}[a_1\, x\, \tilde{T}_n(x) + x^2\tilde{T}'_n(x)]$$

The general solution of this, is

$$y_{n+1}(x) = \hat{c}_1\phi_1(x) + \hat{c}_2\phi_2(x) - \tau_2^{(n)}\tilde{T}_n(1)\,\frac{\kappa_1(x,1)}{w_1(1)}$$

$$+ \int_1^x G_1(x,t)\, dt$$

where $G_1(x,t) = \dfrac{\tau_1^{(n)}\,\tilde{T}_{n+1}(t)\kappa_1(x,t) + \tau_2^{(n)}\tilde{T}_n(t)\left[a_1 t\kappa_1(x,t) - w_1(t)\dfrac{d}{dt}\dfrac{t^2\kappa_1(x,t)}{w_1(t)}\right]}{w_1(t)}$

is obtained after performing an integration by parts on

$$\int_1^x \frac{\kappa_1(x,t)t^2}{w_1(t)}\;\frac{d\tilde{T}_n}{dt}\; dt.$$

Now $G_1(x,x) = \dfrac{d}{dt}\dfrac{t^2\kappa_1(x,t)}{w(t)}\Bigg|_{t=x} = -x^2$ for all cases (i), (ii) and (iii).

Hence we have $y_{n+1}(1) = \hat{c}_1\phi_1(1) + \hat{c}_2\phi_2(1)$

and $y'_{n+1}(x) = \hat{c}_1\phi'_1(x) + \hat{c}_2\phi'_2(x) - \tau_2^{(n)} \frac{\tilde{T}_n(1)}{w_1(1)} [-\phi'_1(x) + \phi'_2(x)]$

$$+ G_1(x,x) + \int_1^x \frac{\partial}{\partial x} G_1(x,t)\, dt$$

and

$$y'_{n+1}(1) = \hat{c}_1\phi'_1(1) + \hat{c}_2\phi'_2(1) \qquad \text{for all cases (i), (ii) and (iii).}$$

It follows, as in 2.1.3 , that

$$y_n(x) - y(x) = \int_1^x G_1(x,t)\,dt \tag{9}$$

2.2.5 UPPER BOUNDS FOR THE ERROR FUNCTIONS IN TERMS OF THE TAU

We restrict ourselves to case (i) and to improve the bounds, assume $a_1 < 2;\ m_1, m_2 < 1$.

From (9) we have

$$|y_{n+1}(x) - y(x)| \le \frac{|\tau_1^{(n)}|}{2n}\{\frac{1}{n+2}\left|\frac{\kappa_1(x,1)}{w_1(1)}\right| + \int_1^x \left|\frac{d}{dt}\frac{\kappa_1(x,t)}{w_1(t)}\right| dt\}$$

$$+ \frac{|\tau_2^{(n)}|}{2(n-1)}\{\left|\frac{-a_1}{n+1}\ \frac{\kappa_1(x,1)}{w_1(1)} \frac{+1}{(n+1)}[\frac{d}{dt}\frac{t^2\kappa_1(x,t)}{w_1(t)}]_{t=1}\right| + x^2$$

$$+ \int_1^x \left|\frac{d}{dt}[a_1 t \frac{\kappa_1(x,t)}{w_1(t)} - \frac{d}{dt}\ t^2 \frac{\kappa_1(x,t)}{w_1(t)}]\right| dt\}$$

Applying (8)

$$||y_{n+1} - y|| \le \frac{|\tau_1^{(n)}|}{2n}\ \frac{|2^{m_1} - 2^{m_2}|}{|m_1 - m_2|}\ (1 + \frac{1}{n+2})$$

$$+ \frac{|\tau_2^{(n)}|}{2(n-1)}\ \{\max(1, \left|\frac{(3-a_1-m_1)2^{m_1} - (3-a_1-m_2)2^{m_2}}{m_1 - m_2}\right|)(1 + \frac{1}{n+1}) + 8\}$$

2.2.6 THE ERROR FUNCTION FOR THE DERIVATIVE AND ITS BOUND

$z_n(x)$ satisfies the equation

$$x^2 z_n''(x) + (2+a_1)\, x\, z_n'(x) + (a_0 + a_1) z_n = -\tau_1^{(n)} \tilde{T}'_{n+1}(x) + a_0 \tau_2^{(n)} \tilde{T}_n(x)$$

Suppose $\psi_1(x)$, $\psi_2(x)$ are a basis for the solution of the homogeneous equation with corresponding $\kappa_2(x,t)$ and $w_2(t)$.
If $\psi(x) = x^{\nu}$ is a solution then ν must satisfy $\nu(\nu-1) + (2+a_1)\nu + (a_0+a_1) = 0$.

We again restrict ourselves to case (i) and assume $\nu_1, \nu_2 < 0$. Then

$$z_n(x) - z(x) = \int_1^x \left[a_0 \tau_2^{(n)} \frac{\tilde{T}_n(t)\kappa_2(x,t)}{w_2(t)} + \tau_1^{(n)} \tilde{T}_{n+1}(t) \frac{d}{dt} \frac{\kappa_2(x,t)}{w_2(t)} \right] dt$$

and

$$||z_n - z|| \le \frac{|a_0 \tau_2^{(n)}|}{2(n-1)} \frac{|2^{\nu_1} - 2^{\nu_2}|}{|\nu_1 - \nu_2|} \left(1 + \frac{1}{n+1}\right)$$

$$+ \frac{|\tau_1^{(n)}|}{2n} \left\{ \left(1 + \frac{1}{n+2}\right) \max\left(1, \left| \frac{(1-\nu_1)2^{\nu_1} - (1-\nu_2)2^{\nu_2}}{\nu_1 - \nu_2} \right| \right) + 2 \right\}$$

2.2.7 EXAMPLE

Referring to our example of section 2.2.2, we find

$$m_1 = \frac{1}{2},\ m_2 = -\frac{1}{2},\ \nu_1 = -\frac{3}{2},\ \nu_2 = -\frac{1}{2}$$

and

$$||y_4 - y|| \le 2.5342 \times 10^{-3}$$

$$||z_3 - z|| \le 1.6797 \times 10^{-4}$$

2.3. THE AIRY EQUATION

We consider the form of Airy equation

$$y''(x) + xy = 1$$

subject to $y(0) = \alpha$, $y'(0) = \beta$

A numerical solution for $y(x)$ with the Tau method can be found in [58]

We compute the exact solution $[y_n, z_n]^T$ of the perturbed system

$$D\begin{pmatrix} y_n \\ z_n \end{pmatrix} \equiv \begin{pmatrix} x & -\frac{d}{dx} \\ \frac{d}{dx} & 1 \end{pmatrix}\begin{pmatrix} y_n \\ z_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \tau_1^{(n)}\begin{pmatrix} T^*_{n+1} \\ 0 \end{pmatrix} + \begin{pmatrix} \tau_2^{(n)} & 0 \\ 0 & \tau_3^{(n)} \end{pmatrix}\begin{pmatrix} T_n^* \\ T_n^* \end{pmatrix} \quad (10)$$

where J is [0,1].

2.3.1 THE CANONICAL POLYNOMIALS

$$Q_{k+2}^{[1]}(x) = \begin{pmatrix} x^{k+1} \\ -(k+1)x^k \end{pmatrix} - k(k+1)\, Q_{k-1}^{[1]}(x)$$

$Q_0^{[1]}$ undefined, $Q_1^{[1]} = \binom{1}{0}$, $Q_2^{[1]}(x) = \binom{x}{-1}$

We note that
$$Q_3^{[1]}(0) = -2Q_0^{[1]}$$
$$Q_4^{[1]}(0) = -6\binom{1}{0}$$

$$Q_{k+1}^{[2]} = \begin{pmatrix} (k+1)x^{k-1} \\ x^{k+1} \end{pmatrix} - (k-1)(k+1)\, Q_{k-2}^{[2]}$$

$Q_0^{[2]} = \binom{0}{1}$, $Q_1^{[2]} = \binom{0}{x} + Q_0^{[1]}$, $Q_2^{[2]} = \begin{pmatrix} 2 \\ x^2 \end{pmatrix}$

Again, we note that
$$Q_3^{[2]}(0) = -3\binom{0}{1}$$
$$Q_4^{[2]}(0) = -8Q_0^{[1]}$$

2.3.2 THE TAU SOLUTION

Inverting the operator D of (10), we have

$$\begin{pmatrix} y_n \\ z_n \end{pmatrix} = Q_0^{[1]} + \tau_1^{(n)} \sum_{j=0}^{n+1} c_j^{(n+1)} Q_j^{[1]} + \tau_2^{(n)} \sum_{j=0}^{n} c_j^{(n)} Q_j^{[1]} + \tau_3^{(n)} \sum_{j=0}^{n} c_j^{(n)} Q_j^{[2]}$$

where $c_j^{(n)}$ are the coefficients of T_n^*.

We take $n = 3$, $\alpha = 3^{-2/3}\Gamma(1/3)$, $\beta = -3^{-1/3}\Gamma(2/3)$

In order for the contribution of the undefined canonical polynomial $Q_0^{[1]}$ to vanish, we have the constraint

$$Q_0^{[1]}\{ 1 + \tau_1^{(3)} [c_0^{(4)} - 2c_3^{(4)}] + \tau_2^{(3)} [c_0^{(3)} - 2c_3^{(3)}] + \tau_3^{(3)} c_1^{(3)} \} = 0$$

From the initial conditions, we have

$$\binom{1.287899}{-.938893} = \tau_1^{(3)} [- 32\binom{1}{0} + 160\binom{0}{-1} + 128\binom{-6}{0}]$$

$$+ \tau_2^{(3)} [18\binom{1}{0} - 48\binom{0}{-1}] + \tau_3^{(3)}[-1\binom{0}{1} - 48\binom{2}{0} + 32\binom{0}{-3}].$$

Hence we find

$\tau_3^{(3)} = .011766 \quad \tau_2^{(3)} = -.006330 \quad \tau_1^{(3)} = -.003164$

2.3.3 ANALYTIC SOLUTION OF THE AIRY EQUATION

$$y'' + xy = 1$$

Two linearly independent solutions of the homogeneous equation are given by

$$\phi_1(x) = \sqrt{x}\, J_{1/3}(\tfrac{2}{3} x^{3/2}), \quad \phi_2(x) = \sqrt{x}\, J_{-1/3}(\tfrac{2}{3} x^{3/2})$$

We set $\kappa(x,t) = \phi_2(x)\phi_1(t) - \phi_1(x)\phi_2(t)$

and $w(t)$ is the Wronskian of ϕ_1 and ϕ_2

From the relationships

$$\frac{d}{d\xi} J_p(\xi) = J_{p-1}(\xi) - \frac{p}{\xi} J_p(\xi)$$

$$\frac{d}{d\xi} J_p(\xi) = \frac{p}{\xi} J_p(\xi) - J_{p+1}(\xi)$$

we obtain respectively

$$\frac{d}{dx}\phi_1(x) = x\, J_{-2/3}(\tfrac{2}{3} x^{3/2}), \quad \frac{d}{dx}\phi_2(x) = -xJ_{2/3}(\tfrac{2}{3} x^{3/2})$$

$$\frac{d^2}{dx^2}\phi_1(x) = -x^{3/2} J_{1/3}(\tfrac{2}{3} x^{3/2}), \quad \frac{d^2}{dx^2}\phi_2(x) = -x^{3/2} J_{1/3}(\tfrac{2}{3} x^{3/2})$$

From the series expansion of the Bessel functions we have

$$\phi_1(0) = 0, \quad \phi_2(0) = 3^{1/3}/\Gamma(\tfrac{2}{3})$$

$$\phi_1'(0) = 3^{2/3}/\Gamma(\tfrac{1}{3}) \quad \phi_2'(0) = 0$$

$$w(0) = -3/[\Gamma(\tfrac{1}{3})\ \Gamma(\tfrac{2}{3})]$$

However $w(t) = w(0)$ for all t, cf [5] p. 29.

2.3.4 THE TAU SOLUTION BY THE GREENS FUNCTION

The derived initial conditions for the perturbed system (9) are

$$y_n(0) = \alpha, \quad z_n(0) = -\beta$$

and for consistency

$$y_n'(0) = \beta + \tau_3^{(n)} T_n^*(0)$$

$$z_n'(0) = -1 - \tau_1^{(n)} T_{n+1}^*(0) - \tau_2^{(n)} T_n^*(0)$$

y_n, z_n are the solutions respectively of

$$y_n'' + xy_n = 1 + \tau_1^{(n)} T_{n+1}^*(x) + \tau_2^{(n)} T_n^*(x) + \tau_3^{(n)} T_n^{*\prime}(x) \qquad (11)$$

$$z_n' = xy_n - 1 - \tau_1^{(n)} T_{n+1}^*(x) - \tau_2^{(n)} T_n^*(x) \qquad (12)$$

The solution of (11) is

$$y_n(x) = \hat{c}_1\phi_1(x) + \hat{c}_2\phi_2(x) - \tau_3^{(n)} T_n^*(0)\,\frac{\kappa(x,0)}{w(0)} + \int_0^x \frac{G(x,t)}{w(0)}\,dt$$

where $G(x,t) = [1 + \tau_1^{(n)} T_{n+1}^*(t) + \tau_2^{(n)} T_n^*(t)]\kappa(x,t) - \tau_3^{(n)} T_n^*(t)\,\frac{d}{dt}\,\kappa(x,t)$

We deduce, as in 2.1.3 that

$$y_n(x) - y(x) = \int_0^x \frac{[G(x,t) - \kappa(x,t)]}{w(0)}\,dt$$

On the other hand, one readily obtains from (12)

$$z_n(x) - z(x) = \int_0^x t[y_n(t) - y(t)]dt - \int_0^x [\tau_1^{(n)} T_{n+1}^*(t) + \tau_2^{(n)} T_n^*(t)]dt$$

2.3.5 UPPER BOUNDS FOR THE ERROR FUNCTIONS IN TERMS OF THE TAU

From the expansion

$$J_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)} \quad {}_0F_1(\nu+1;\ -z^2/4)$$

we obtain for ξ real $\nu > -1$

$$|J_\nu(\xi)| \leq \frac{|\frac{\xi}{2}|^\nu}{\Gamma(\nu+1)}$$

Hence for $0 \leq x \leq 1$

$$|\phi_1(x)| \leq \frac{(\frac{1}{3})^{1/3}}{\Gamma(\frac{4}{3})} \quad , \quad |\phi_2(x)| \leq \frac{(\frac{1}{3})^{-1/3}}{\Gamma(\frac{2}{3})}$$

$$|\phi_1'(x)| \leq \frac{(\frac{1}{3})^{-2/3}}{\Gamma(\frac{1}{3})} \quad , \quad |\phi_2'(x)| \leq \frac{(\frac{1}{3})^{2/3}}{\Gamma(\frac{5}{3})}$$

$$|\kappa(x,t)| \leq \frac{2}{\Gamma(\frac{2}{3})\Gamma(\frac{4}{3})} = 2|w(0)|$$

$$\left|\frac{d}{dt}\kappa(x,t)\right| \leq \frac{9}{2\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} = \frac{3}{2}|w(0)|$$

$$\left|\frac{d^2}{dt^2}\kappa(x,t)\right| \leq 2|w(0)|$$

$$||y_n - y|| \leq \sum_{i=1}^{3} m_i^{(1)} |\tau_i^{(n)}| \quad \text{where } m_1^{(1)} = \frac{1}{2n}\left[1.5 + \frac{1}{n+2}\right],$$

$$m_2^{(1)} = \frac{1}{2(n-1)}\left[\frac{1}{n+1} + 1.5\right], \quad m_3^{(1)} = \frac{1}{2(n-1)}\left[3 + \frac{1}{n+1}\right]$$

$$||z_n - z|| \leq \sum_{i=1}^{3} m_i^{(2)} |\tau_i| \quad \text{where} \quad m_1^{(2)} = m_1^{(1)} + \frac{1}{2n},$$

$$m_2^{(2)} = m_2^{(1)} + \frac{1}{2(n-1)}; \qquad m_3^{(2)} = m_3^{(1)}.$$

Hence we find, for $n=3$, the following upper bounds for the error functions:

$$\|y_3 - y\| < .01323 \quad ; \quad \|z_3 - z\| < .01415$$

For this value of n, these bounds compare favourably with the standard Tau method solution to this problem adopted in [58].

SIMULTANEOUS APPROXIMATION OF A FUNCTION AND ITS DERIVATIVE WITH THE TAU METHOD, PART III

3.1. INTRODUCTION : LEGENDRE PERTURBATIONS AND RATIONAL APPROXIMATIONS

It has been found in practice, see [21] and [57], that the Tau approximations generated with a Legendre perturbation P_n, have the advantage that they provide more accurate end-point estimations than those generated with a Chebyshev perturbation T_n . This feature is of consequence in designing a suitable step-by-step Tau solution of a problem over a segmented range, see [57] for a detailed description.

A heuristic proof to justify the use of Legendre perturbations was first given by Lanczos in 1962 and is to be found in [43].

It was based on the approximation by the Tau method of the Green's function $G(x,t)$ associated with the differential operator at $x = 1$.

We further support this view in the sequel by arguing on the following lines.

Suppose we write $y - y_n = \varepsilon_n = \tau^{(n)} \int_0^x P_n(t)\, G(x,t)\, dt$.

Now by Rodrigues' definition of the Legendre Polynomial shifted to $[0,1]$

$$P_n(t) = \frac{(-1)^n}{n!} \omega^{(n)}(t) \quad \text{where} \quad \omega = t^n(1-t)^n$$

the superscript (n) denotes the n-th derivative and $\omega^{(i)}(0) = \omega^{(i)}(1) = 0$ for $i = 0, 1, \ldots, n-1$.

Hence by performing repeated integration by parts, we can reduce the integral, in the expression for $\varepsilon_n(1)$, to the integral of a smooth function $\frac{d^n}{dt^n} G(x,t)$ against $\omega(t)$ which behaves increasingly like a delta function as n increases. Consequently the error function diminishes rapidly with n and precise bounds can be found depending on $\frac{d^n}{dt^n} G(x,t)$.

Luke (see [47] Chapter X and also [21]) extended the error analysis of the Tau solution of the simple differential equation

$$y' - y = 0 \text{ with } y(0) = 1$$

to the solution of

$$y_n' - y_n = \tau^{(n)} \phi_n^{(\alpha,\beta)} \left(\frac{x}{\gamma}\right) \qquad (1)$$

on $[0,\gamma]$ with $y_n(0) = 1$.

$\phi_n^{(\alpha,\beta)}$ (t) is a shifted Jacobi polynomial on [0,1] which has the hyper-geometric form

$${}_2F_1\left(\begin{matrix}-n, & n+\lambda \\ & \beta+1\end{matrix}\,\middle|\, t\right) \text{ where } \lambda = \alpha + \beta + 1$$

We note that $\phi_n^{(-\frac{1}{2},\, -\frac{1}{2})}(t) = (-1)^n T_n^*(t)$

while $\phi_n^{(0,0)}(t) = (-1)^n P_n(t)$.

For a given value of γ, the solution of (1) is a polynomial of degree n in x/γ. When the initial conditions are accounted for, the solution y_n takes the form of a rational function. If x is now set equal to γ, then y_n can be expressed as the quotient of two polynomials of degree n in x (see [43] p. 195).

By this means a rational approximation for exp(x) was obtained with little more effort than before and yet with considerably improved accuracy.

In the foregoing we extend these ideas to a particular simple second-order linear differential equation to obtain approximations to both the function and its derivative and which basically are the quotient of two polynomials of degree n in x^2.

3.2. THE PROBLEM AND ITS SOLUTION USING THE GREEN'S FUNCTION

Let us consider the second order differential equation

$$y''(x) + y(x) = 0, \quad x \in [0,1]$$

with the initial conditions $y(0) = 1$, $y'(0) = 0$ which we write as the system

$$\begin{pmatrix} 1 & \frac{-d}{dx} \\ \frac{d}{dx} & 1 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2}$$

with the conditions $y(0) = 1$, $z(0) = 0$.

We now look for an approximate solution of the system (2) in the interval $[0,\gamma]$, $\gamma \leq 1$, by means of the Tau method. We shall use as a perturbation term for y_n and z_n, the Legendre polynomials $P_n(x/\gamma)$ defined in the interval $[0,\gamma]$. Thus $[y_n, z_n]^T$ is the exact solution of

$$\begin{pmatrix} 1 & \frac{-d}{dx} \\ \frac{d}{dx} & 1 \end{pmatrix} \begin{pmatrix} y_n \\ z_n \end{pmatrix} = \begin{pmatrix} \tau_o^{(n,\gamma)} P_n(x/\gamma) \\ \tau_1^{(n,\gamma)} P_n(x/\gamma) \end{pmatrix} \tag{3}$$

The error functions for y and z are defined respectively as

$$\varepsilon_n(x,\gamma) = y_n(x,\gamma) - y(x,\gamma)$$

$$\delta_n(x,\gamma) = z_n(x,\gamma) - z(x,\gamma)$$

From (2) and (3)

$$\frac{d^2}{dx^2} \varepsilon_n(x,\gamma) + \varepsilon_n(x,\gamma) = \tau_o^{(n,\gamma)} P_n(x/\gamma) + \tau_1^{(n,\gamma)} \frac{d}{dx} P_n(x/\gamma) \tag{4-a}$$

$$\frac{d^2}{dx^2} \delta_n(x,\gamma) + \delta_n(x,\gamma) = -\tau_o^{(n,\gamma)} \frac{d}{dx} P_n(x/\gamma) + \tau_1^{(n,\gamma)} P_n(x/\gamma) \tag{4-b}$$

The solution of (4-a) in terms of its Green's function is

$$\varepsilon_n(x,\gamma) = \tau_o^{(n,\gamma)} \int_o^x \sin(x-u)\, P_n(u/\gamma)\, du + \tau_1^{(n,\gamma)} \int_o^x \cos(x-u)\frac{d}{dx} P_n(u/\gamma)\, du$$

$$= \gamma \int_o^{x/\gamma} \left[\tau_o^{(n,\gamma)} \sin(x-\gamma t) + \tau_1^{(n,\gamma)} \cos(x-\gamma t)\right] P_n(t)\, dt \qquad (5\text{-a})$$

Similarly,

$$\delta_n(x,\gamma) = \gamma \int_o^{x/\gamma} \left[-\tau_o^{(n,\gamma)} \cos(x-\gamma t) + \tau_1^{(n,\gamma)} \sin(x-\gamma t)\right] P_n(t)\, dt \qquad (5\text{-b})$$

Following Luke (Vol. 1, p. 281), the repeated integrals of $P_n(t)$ are denoted by

$$P_{n,r}(x) = \frac{1}{(r-1)!} \int_o^x (x-t)^{r-1} P_n(t)\, dt$$

with $P_{n,o} \equiv P_n$. It follows that

$$P_{n,r}(0) = P_{n,r}(1) = 0 \qquad \text{for } r = 1\ (1)\ n\ .$$

Furthermore, using Rodrigues' formula, $P_{n,n}(t) = \frac{(-1)^n}{n!} t^n (1-t)^n$.

Repeated integration by parts of (5-a) yields

$$\varepsilon_n(x,\gamma) = \sum_{k=0}^{r-1} \gamma^{k+1} \Big\{ \left[\tau_o^{(n,\gamma)} \sin\frac{k\pi}{2} + \tau_1^{(n,\gamma)} \cos\frac{k\pi}{2}\right] P_{n,k}(x/\gamma)$$

$$- \left[\tau_o^{(n,\gamma)} \sin\left(x + \frac{k\pi}{2}\right) + \tau_1^{(n,\gamma)} \cos\left(x + \frac{k\pi}{2}\right)\right] P_{n,k+1}(0) \Big\}$$

$$+ \gamma^{r+1} \int_o^{x/\gamma} \left[\tau_o^{(n,\gamma)} \sin\left(x + \frac{r\pi}{2} - \gamma t\right) + \tau_1^{(n,\gamma)} \cos\left(x + \frac{r\pi}{2} - \gamma t\right)\right] P_{n,r}(t)\, dt.$$

If γ is identified with the current point $x \in [0,1]$ and $r = n$, we have that

$$\varepsilon_n(x,x) = x^{n+1} \int_o^1 \{\tau_o^{(n,x)} \sin\left[\frac{n\pi}{2} + x(1-t)\right] + \tau_1^{(n,x)} \cos\left[\frac{n\pi}{2} + x(1-t)\right]\} P_{n,n}(t)\, dt$$

We set $A_{\alpha,n}(x) \equiv \int_o^1 \sin\left[\alpha + x(1-t)\right] t^n (1-t)^n\, dt$

$$= \sin(\alpha + x)\int_o^1 \cos(xt)\, t^n (1-t)^n dt - \cos(\alpha + x)\int_o^1 \sin(xt)\, t^n (1-t)^n dt.$$

3.3 A CLOSED FORM FOR THE ERROR TERMS

Since

$$\int_0^1 e^{ixt}\, t^n(1-t)^n dt = \frac{\sqrt{\pi}\, e^{ix/2} n!}{(ix)^{n+\frac{1}{2}}}\; I_{n+\frac{1}{2}}\left(\frac{ix}{2}\right)$$

where

$$I_\nu\,(ix) = i^{-\nu} J_\nu(-x) = (-1)^\nu J_\nu(x), \qquad (6)$$

we have

$$A_{\alpha,n}(x) = \sqrt{\pi}\, n! \left\{ \sin(\alpha+x) \left[e^{ix/2} \frac{I_{n+\frac{1}{2}}\left(\frac{ix}{2}\right)}{(ix)^{n+\frac{1}{2}}} + e^{-ix/2} \frac{I_{n+\frac{1}{2}}\left(\frac{-ix}{2}\right)}{(-ix)^{n+\frac{1}{2}}} \right] \right.$$

$$\left. + \, i\cos(\alpha+x) \left[e^{ix/2} \frac{I_{n+\frac{1}{2}}\left(\frac{ix}{2}\right)}{(ix)^{n+\frac{1}{2}}} - e^{-ix/2} \frac{I_{n+\frac{1}{2}}\left(\frac{-ix}{2}\right)}{(-ix)^{n+\frac{1}{2}}} \right] \right\}$$

$$= \sqrt{\pi}\, n!\, i \left[\frac{e^{-i(\alpha+\frac{x}{2})} I_{n+\frac{1}{2}}\left(\frac{ix}{2}\right)}{(ix)^{n+\frac{1}{2}}} - \frac{e^{i(\alpha+\frac{x}{2})} I_{n+\frac{1}{2}}\left(\frac{-ix}{2}\right)}{(-ix)^{n+\frac{1}{2}}} \right]$$

Applying (6)

$$A_{\alpha,n}(x) = \frac{i\sqrt{\pi}n!}{x^{n+\frac{1}{2}}} \left[\frac{(-1)^n}{i}\; e^{-i(\alpha+\frac{x}{2})} J_{n+\frac{1}{2}}\left(-\frac{x}{2}\right) - e^{i(\alpha+\frac{x}{2})} J_{n+\frac{1}{2}}\left(\frac{x}{2}\right) \right]$$

$$= \frac{\sqrt{\pi}n!}{x^{n+\frac{1}{2}}} \sin\left(\alpha+\frac{x}{2}\right) J_{n+\frac{1}{2}}\left(\frac{x}{2}\right).$$

Finally,

$$\varepsilon_n(x,x) = (-1)^n \sqrt{\pi x}\, J_{n+\frac{1}{2}}\left(\frac{x}{2}\right) \left[\tau_0^{(n,x)} \sin\tfrac{1}{2}(n\pi+x) + \tau_1^{(n,x)} \cos\tfrac{1}{2}(n\pi+x) \right] \qquad (7\text{-a})$$

Similarly,

$$\delta_n(x,x) = (-1)^n \sqrt{\pi x}\, J_{n+\frac{1}{2}}\left(\frac{x}{2}\right) \left[\tau_1^{(n,x)} \sin\tfrac{1}{2}(n\pi+x) - \tau_0^{(n,x)} \cos\tfrac{1}{2}(n\pi+x) \right] \qquad (7\text{-b})$$

Repeated differentiation of (4-a) and alternate subtraction leads to

$$y_n(x) = \tau_o^{(n,\gamma)} \left[P_n(x/\gamma) - P_n''(x/\gamma) + \ldots + (-1)^\theta P_n^{(2\theta)}(x/\gamma)\right]$$
$$+ \tau_1^{(n,\gamma)} \left[P_n'(x/\gamma) - P_n'''(x/\gamma) + \ldots + (-1)^\theta P_n^{(2\theta+1)}(x/\gamma)\right] \qquad (8)$$

where $\theta = [n/2]$. A similar expression is found for $z_n(x)$.

If we now impose on $y_n(x)$, $z_n(x)$ the conditions of (2) remembering that

$$\frac{d^r}{dx^r} P_n(x/\gamma)\bigg|_{x=0} = \gamma^{-r} \frac{d^r}{dt^r} P_n(t)\bigg|_{t=0},$$

we obtain

$$1 = \tau_o^{(n,\gamma)} \sum_{r=0}^{\theta} (-1)^r \gamma^{-2r} P_n^{(2r)}(0) + \tau_1^{(n,\gamma)} \sum_{r=0}^{\theta} (-1)^r \gamma^{-(2r+1)} P_n^{(2r+1)}(0)$$
$$\qquad (9)$$
$$0 = \tau_1^{(n,\gamma)} \sum_{r=0}^{\theta} (-1)^r \gamma^{-2r} P_n^{(2r)}(0) - \tau_o^{(n,\gamma)} \sum_{r=0}^{\theta} (-1)^r \gamma^{-(2r+1)} P_n^{(2r+1)}(0).$$

We remark that the series expansion of $P_n(t)$ in [0,1],

$$P_n(t) = \sum_{k=0}^{n} \binom{n}{k}^2 (t-1)^{n-k} t^k,$$

is the Bernstein polynomial of order k of a function f which takes the values

$$f_k = \binom{n}{k} (-1)^{n-k} \quad \text{for } k = 0(1)n.$$

Thus,

$$P_n(t) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k f_o \, t^k$$

with $P_n(0) = (-1)^n$ and

$$\frac{d^r}{dt^r} P_n(t)\bigg|_{t=0} = r! \binom{n}{r} \Delta^r f_o$$

$$= (-1)^{n+r} r! \sum_{j=0}^{r} \binom{n}{j}\binom{r}{j} = \frac{(-1)^{n+r}}{r!} \frac{(n+r)!}{(n-r)!} \qquad (10)$$

Furthermore, if we set

$$S_E \equiv S_E^{(n,\gamma)} = \gamma^{-2\theta} (-1)^{\theta} \sum_{r=0}^{\theta} (-1)^r \gamma^{2r} P_n^{(2\theta-2r)}(0) \tag{11}$$

$$S_0 \equiv S_0^{(n,\gamma)} = \gamma^{-(2\theta+1)} (-1)^{\theta} \sum_{r=0}^{\theta} (-1)^r \gamma^{2r} P_n^{(2\theta-2r+1)}(0) \tag{12}$$

the solution of (9) can be expressed in terms of S_E, S_0 as

$$\tau_o^{(n,\gamma)} = \frac{S_E}{S_E^2+S_0^2}, \quad \tau_1^{(n,\gamma)} = \frac{S_0}{S_E^2+S_0^2} \tag{13}$$

We shall see, in (16-c) below, that $S_E^2 + S_0^2 \neq 0$ for $0 < \gamma \leqslant 1$.

Consequently, $\tau_o^{(n,\gamma)}$, $\tau_1^{(n,\gamma)}$ are both well defined in that range.

3.4 Error Bounds in the Range $0 < \gamma \leqslant 1$

Taking into account (10) we obtain for (11)

$$S_E = (-1)^{n/2} \gamma^{-n} \sum_{r=0}^{n/2} \frac{(-1)^r [2(n-r)]! \, \gamma^{2r}}{(n-2r)! \, (2r)!}$$

Thus

$$1 - \frac{\gamma^2}{8} + \frac{\gamma^4}{1680} < \frac{(-1)^{n/2} n! \, \gamma^n}{(2n)!} S_E < 1 - \frac{3\gamma^2}{28} + \frac{\gamma^4}{384} \tag{14}$$

From(12) $$S_0 = (-1)^{n/2} \gamma^{-n-1} \sum_{r=1}^{n/2} \frac{(-1)^{r+1} (2n-2r+1)! \, \gamma^{2r}}{(n-2r+1)!(2r-1)!}$$

and $$\gamma - \frac{\gamma^3}{24} < \frac{(-1)^{n/2} 2n! \, \gamma^n}{(2n)!} S_0 < \gamma - \frac{\gamma^3}{42} \tag{15}$$

For the range $0 < \gamma \leq 1$, (14) - (15) give us the following bounds for S_E and S_0 rounded to 6D:

$$.875395 < \frac{(-1)^{n/2}\, n!\, \gamma^n}{(2n)!} S_E < 1.000000 \qquad (16\text{-a})$$

$$.000000 < \frac{(-1)^{n/2}\, n!\, \gamma^n}{(2n)!} S_O < 0.488095 \qquad (16\text{-b})$$

and

$$0.766667 < \left[\frac{n!\, \gamma^n}{(2n)!}\right]^2 (S_E{}^2 + S_O{}^2) < 1.238237 \qquad (16\text{-c})$$

We cân now reconsider (7-a) and (7-b).

Since
$$(-1)^{n/2}\, \tau_o^{(n,\gamma)} < \max_{0<\gamma\leq 1} \frac{(-1)^{n/2}}{S_E^{(n,\gamma)}}$$

we derive, on applying (16-a) - (16-c), the following bounds on the interval $0 < x \leq 1$:

$$0 < (-1)^{n/2} \frac{(2n)!}{\gamma^n n!}\left[\tau_o^{(n,\gamma)} \sin\frac{x}{2} + \tau_1^{(n,\gamma)} \cos\frac{x}{2}\right] < 1.1842$$

$$-1.142080 < (-1)^{n/2} \frac{(2n)!}{\gamma^n n!}\left[\tau_1^{(n,\gamma)} \sin\frac{x}{2} - \tau_o^{(n,\gamma)} \cos\frac{x}{2}\right] < -0.315340.$$

Furthermore, since

$$\frac{J_{n+\frac{1}{2}}(\frac{x}{2})\sqrt{\pi}}{x^{n+\frac{1}{2}}} < \frac{n!}{(2n+1)!}\left[1 + O(x^2)\right]$$

we obtain for n even, $n \geq 4$ and $0 < x \leq 1$

$$0 < \frac{(2n+1)!\,(2n)!}{(n!)^2} \varepsilon_n(x,x) < 1.1842\, x^{2n+1} \left[1 + O(x^2)\right]$$

$$-1.1421\, x^{2n+1}\left[1 + O(x^2)\right] < \frac{(2n+1)!\,(2n)!}{(n!)^2} \delta_n(x,x) < -0.31534\, x^{2n+1}\left[1+O(x^2)\right]$$

3.5 ERROR BOUNDS AT THE END POINT OF THE INTERVAL [0,1]

We now come back to consider the problem of the error at the matching points of our segmented Tau approximation. For $\Upsilon = 1$ and taking the case n = 4 for our upper bound, we derive from (14) and (15)

$$0.875595 < \frac{(-1)^{n/2}\, n!}{(2n)!} S_E^{(n,1)} < 0.893452$$

$$0.479166 < \frac{(-1)^{n/2}\, n!}{(2n)!} S_O^{(n,1)} < 0.488095$$

and

$$0.996266 < \left[\frac{n!}{(2n)!}\right]^2 \left(\left[S_E^{(n,1)}\right]^2 + \left[S_O^{(n,1)}\right]^2\right) < 1.036492$$

hence

$$0.462295 < \frac{(-1)^{n/2}(2n)!}{n!} \tau_1^{(n,1)} < 0.489924$$

and

$$1.793902 < \frac{\tau_o^{(n,1)}}{\tau_1^{(n,1)}} < 1.864598$$

From (7-a) : $\varepsilon_n(1,1) = (-1)^{n/2} J_{n+\frac{1}{2}}(\tfrac{1}{2})\sqrt{\pi}\, \tau_1^{(n,1)} \left[\frac{\tau_o^{(n,1)}}{\tau_1^{(n,1)}} \sin(\tfrac{1}{2}) + \cos(\tfrac{1}{2})\right]$

which yields, on applying the above bounds,

$$(1 - \frac{2}{16x11})\, 0.80329 < \frac{(2n+1)!(2n)!}{(n!)^2} \varepsilon_n(1,1) < 0.86791$$

From (7-b): $\delta_n(1,1) = (-1)^{n/2} J_{n+\frac{1}{2}}(\tfrac{1}{2})\sqrt{\pi}\, \tau_1^{(n,1)} \left[\sin(\tfrac{1}{2}) - \frac{\tau_o^{(n,1)}}{\tau_1^{(n,1)}} \cos(\tfrac{1}{2})\right]$

from which we find

$$-0.56679 < \frac{(2n+1)!(2n)!}{(n!)^2} \delta_n(1,1) < -0.50615\,(1 - \frac{2}{16x11})$$

3.6 WORKED EXAMPLE

We consider first the end point estimation for the case when the interval of approximation is $[0, \frac{\pi}{8}]$. We make use of the tabulated values of

$j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+\frac{1}{2}}(x)$ to be found in [1] and replace $\sqrt{\pi x}\, J_{n+\frac{1}{2}}(\frac{x}{2})$ by

$x\, j_n(\frac{x}{2})$ in (7-a) and (7-b). Hence for n even,

$$\varepsilon_n(\frac{\pi}{8}, \frac{\pi}{8}) = (-1)^{n/2} \frac{\pi}{8} j_n(\frac{\pi}{16}) \left[\tau_o^{(n,\ \pi/8)} \sin\frac{\pi}{16} + \tau_1^{(n,\pi/8)} \cos\frac{\pi}{16}\right]$$

$$\delta_n(\frac{\pi}{8}, \frac{\pi}{8}) = (-1)^{n/2} \frac{\pi}{8} j_n(\frac{\pi}{16}) \left[-\tau_o^{(n,\pi/8)} \cos\frac{\pi}{16} + \tau_1^{(n,\pi/8)} \sin\frac{\pi}{16}\right]$$

We employ the half angle formula

$$\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} (\frac{\pi}{16}) = \sqrt{\frac{1 \mp \cos(\frac{\pi}{8})}{2}} = \left\{\begin{matrix}.1950903200\\ .9807852807\end{matrix}\right.$$

$j_n(\frac{\pi}{16})$ was determined by a finite difference interpolation formula based on the points $x_r = 0, .1, .2, .3, .4$.

$\tau_o^{(n,\pi/8)}$, $\tau_1^{(n,\pi/8)}$ were determined directly from equation (13).

The results are compared with the nodal error at $\frac{\pi}{8}$ for the function and its derivative in the standard two τ-term Legendre perturbation obtained in [57] and designated ε_n^s δ_n^s respectively

n	τ_o	τ_1	j_n	ε_n	δ_n	ε_n^s	δ_n^s
2	-1.252 E-5	-2.491 E-3	2.5631394E-3	4.918E-5	-1.187E-4	9.463E-4	-1.753E-4
4	1.384546E-5	2.754034E-6	1.56997 E-6	3.33E-12	-8.04E-12	2.719E-9	5.321E-10

To evaluate, for example, y_2 and z_2, we proceed as follows.

From $P_2(t) = 6t^2 - 6t + 1$ we derive

$$S_E^{(2,\gamma)} = 1 - \frac{12}{\gamma^2} \quad ; \quad S_0^{(2,\gamma)} = -\frac{6}{\gamma}.$$

Hence from (8) and (13) we obtain, after scaling

$$y_2(x^2, x^2) = \frac{144 - 60x^2 + x^4}{144 + 12x^2 + x^4}$$

$$z_2(x^2, x^2) = \frac{x\,(144 - 12x^2)}{144 + 12x^2 + x^4}$$

3.7. CONCLUDING REMARKS

For higher values of n, one need not determine explicitly the rational expressions for y_n and z_n. It is more economical to use the terms $S_E^{(n,x)}$ and $S_0^{(n,x)}$ as follows.

From our knowledge of Legendre polynomials on $[-1, 1]$ in particular,

$$P_n(-x) = (-1)^n P_n(x) \quad \text{and}$$

$$P_n'(x) = (2n-1)P_{n-1}(x) + (2n-5)P_{n-3}(x) + \ldots, \quad n = 1,2,\ldots$$

we obtain $P_n^{(k)}(-1) = (-1)^{n-k} P_n^{(k)}(1) \quad k \geq 0$

hence using the same symbol for the Legendre polynomial shifted to $[0,1]$

$$P_n^{(k)}(0) = (-1)^{n-k} P_n^{(k)}(1) \qquad k \geq 0$$

Thus recalling (11) and (12)

$$\sum_{r=0}^{\theta} (-1)^r x^{-2r} P_n^{(2r)}(1) = (-1)^n S_E^{(n,x)} \qquad \text{and}$$

$$\sum_{r=0}^{\theta} (-1)^r x^{-(2r+1)} P_n^{(2r+1)}(1) = (-1)^{n-1} S_0^{(n,x)}$$

Inserting these expressions into (8) and employing (13)

$$y_n(x^2, x^2) = (-1)^n \frac{\{[S_E^{(n,x)}]^2 - [S_O^{(n,x)}]^2\}}{[S_E^{(n,x)}]^2 + [S_O^{(n,x)}]^2}$$

and similarly

$$z_n(x^2, x^2) = \frac{(-1)^n \ 2\, S_E^{(n,x)} \ S_O^{(n,x)}}{[S_E^{(n,x)}]^2 + [S_O^{(n,x)}]^2}$$

We can make the following observations.

S_E and S_O require together $O(n)$ multiplications for their evaluation at $U = x^2$. Hence this is the order of multiplications to evaluate both y_n and z_n at $U = x^2$ as opposed to the direct method requiring $O(3n)$.

We further observe that our solution satisfies $y_n^2 + z_n^2 = 1$.

The same technique, used along the lines of PartIIcan be extended to more general cases, to provide easily computed approximations for both the function and its derivative, with appropriate error bounds.

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