GAUGE THEORIES, SUPERSYMMETRIES AND ASYMPTOTIC FREEDOM

by

EUSTACHE THEOTOKOGLOU

A thesis presented for the Degree of Doctor of Philosophy of the University of London and the Diploma of Membership of Imperial College.

Department of Theoretical Physics
Imperial College of Science and Technology
London SW7

July, 1975

ABSTRACT

This thesis is in two parts: Chapters One and Two.

In Chapter One

We investigate to all orders in perturbation theory certain eigenvalue conditions for asymptotic freedom obtained from the renormalization group equations for any non-Abelian gauge theory and called the Chang eigenvalue conditions.

It is shown formally that, for a non-Abelian gauge theory which is known to be supersymmetric for some choice of coupling constants, the supersymmetric theory will correspond to a solution of the Chang eigenvalue conditions for the theory. The conjecture is also made that the Lagrangians corresponding to some particular solutions of the Chang eigenvalue conditions will possess a higher symmetry.

It is also found that the Chang eigenvalue conditions are satisfied to all orders in perturbation theory provided that the coupling constants can be made to satisfy, (a) the Chang-conditions to the one loop approximation and (b) a new system of "existence" conditions which are easily derived from the one loop computations.

The Chang eigenvalue conditions are computed explicitly, to the order of one loop, for two non-Abelian gauge theories known to be supersymmetric for certain choices of coupling constants. The supersymmetric solutions are found not to be unique. We also consider the one loop renormalizability of the supersymmetric theories. It is found that the supersymmetric constraints on the masses and coupling constants are preserved by the renormalization.

In Chapter Two

The "strong corrections" to the weak (V-A) current for hadronic strangeness conserving processes, as expressed in terms of the p and n fundamental quark-fields, are worked out perturbatively to one loop within the context of a unified model for strong, weak and electromagnetic interactions. It is found that the vector part of the current remains unrenormalized and is equal to its "bare" value.

PREFACE

The work described in this thesis was carried out under the supervision of Professor Abdus Salam in the Department of Physics, Imperial College, London, between October 1972 and March 1975. Except where otherwise stated, the material presented is the original work of the author, and has not been submitted for a degree in this or any other University.

The author wishes to thank his supervisor for suggesting the investigation of the problems on which this thesis is based, and for his advice and encouragement during the completion of this work. He also wishes to acknowledge, with thanks, many useful discussions with Drs R. Delbourgo, C. Nash, D.A. Ross, and other members of the Theoretical Group at Imperial College.

Dedicated

to my parents

Nicholaos and Sophia Theotokoglou

CONTENTS

CHAPTER ONE

EIGENVALUE CONDITIONS FOR ASYMPTOTIC FREEDOM AND SUPERSYMMETRIES

SECTION I

INTRODUCTION

The motivation for the work described in this Chapter, Chapter One, has been a paper by N.P. Chang[1] on the problem of asymptotic freedom for gauge theories involving Higgs scalar fields. As it will become clearer shortly, such theories are asymptotically free if the response of the renormalized coupling constants, for the Yukawa and self-quartic Higgs scalar vertices, to a change of the "subtraction point" can be described using only the corresponding change of the gauge coupling constant of the theory, and that the gauge coupling constant does vanish asymptotically[2]. The observation made by N.P. Chang[1] is that such a "description" for the "effective" Yukawa and self-quartic scalar coupling constant is possible, i.e. being explicit functions only in the "effective" gauge coupling constant, provided that certain eigenvalue conditions, obtained from the renormalization group equations of the theory, are satisfied.

Before we proceed to give more precise meaning to these general ideas, it would be in order to consider briefly the renormalization group techniques[3-7] and the concept of asymptotic freedom. We shall subsequently see that the aforementioned system of eigenvalue conditions is essentially an infinite system - though finite when considered to any finite order in the gauge coupling constant of the theory. The problem of existence of solutions for the Yukawa and self-quartic scalar couplings, for any gauge theory, such that the eigenvalue conditions for the theory are satisfied to all orders is considered in section II.

In this introductory section we also give a brief review of the "Supersymmetry-Formalism" and, in particular, of the construction of supersymmetric Yang-Mills Lagrangian theories[8-11]. The two supersymmetric and SU(2) gauge invariant Lagrangians[10,11] that are studied in sections III and IV, in connection to the "Chang-problem", are given in a manifestly supersymmetric form and also, for a special gauge[8], in component form[10,11]

(a) The Renormalization Group and Asymptotic Freedom

A powerful tool for studying the asymptotic properties of renormalizable field theories is the "renormalization group techniques" developed by various authors[3-6]: in each case[3-6] there is a definite equation which governs the behaviour of the renormalized Green's functions of the theory under scaling of the external momenta.

Any renormalizable field theory contains two types of parameters; masses or coupling constants with positive dimensions of mass, "the generalized mass terms", and secondly dimensionless coupling constants. When we consider a Green's function for large space-like momenta, so as to exclude any Landau singularities, we would expect that the "masses" can be neglected and the leading asymptotic behaviour of the Green's function, therefore, to be the same as that of the massless theory. This can be shown to any finite order using Weinberg's theorem. Furthermore - if we pursuit this argument intuitively - since the massless theory contains no dimensional parameters to set the scale momenta, the asymptotic behaviour of the Green's function would be expected to be determined by dimensional analysis. This is called canonical scaling, but it does not occur in practise because renormalizable field theories contain a "hidden" dimensional parameter μ which sets the scale. This parameter μ is arbitrary and is the point at which the subtractions are performed to make the theory finite or alternatively, when the n-dimensional regularization method[12] is used, μ is defined[6,13] as the "unit of mass" for the renormalized theory.

The renormalization group equations can be obtained by exploiting the fact that the unrenormalized Green's functions are independent of the arbitrary parameter μ, and reflect the fact that any change of the point μ can be compensated by a corresponding change of the charges, the scale of the fields, and the gauge parameter (for gauge theories) - and also of the "masses" (if present) in the techniques developed by Weinberg[5] and t'Hooft[6].

We consider the renormalization group equations[3-6] for a renormalized, connected, and amputated Green's function $\Gamma_{ren}(\lambda p_1, \lambda p_n, g_i, \mu)$ at some large unexceptional space-like n-momenta $\lambda p_1 .. \lambda p_n$, where $\lambda (\lambda \geq 1)$ is a parameter whose limit $\lambda \to \infty$ is studied, and the "masses" of the theory are neglected. This equation, obtained simply by considering the massless theory as outlined above, assumes a simple form in the Landau gauge (for gauge theories) and is given by

$$\left[\mu \frac{\partial}{\partial \mu} + \beta_K(g_j) \frac{\partial}{\partial g_K} - \gamma(g_j) \right] \Gamma_{ren}(\lambda p_1 \ldots \lambda p_n ; g_i ; \mu) = 0 \qquad (1.1)$$

where g_i, $i = 1,2....$ are the dimensionless renormalized coupling constants of the theory, and

$$\beta_K(g_{i..}) \equiv \mu \frac{\partial g_K}{\partial \mu} \quad , \qquad \gamma(g_{i..}) \equiv \mu \frac{\partial \ln Z}{\partial \mu}$$

with the bare parameters being kept constant, and Z being the overall multiplicative renormalization factor of the Green's function; i.e. $\Gamma_{ren}(\lambda p\, ; g_i, \mu) = Z\ \Gamma_{unren}(\lambda p\, ; g_i^{\circ})$

The general solution of the renormalization group equation (1.1) is expressed in terms of the so called effective coupling constants $g_K^{eff}(t, g_K)$, where $t = \ln\lambda$, which can be understood physically to be the renormalized coupling constants of the theory corresponding to the "subtraction point" $\mu' = \mu\lambda = \mu e^{t}$. The effective coupling constants satisfy the linear differential equations

$$\frac{d g_K^{eff}}{dt} = \beta_K(g_{i,..}^{eff}) \qquad K = 1,2\ldots \tag{1.2}$$

with the boundary condition $g_K^{eff}(0, g_K) = g_K$, and the general solution to the renormalization group equation (1.1) is given by

$$\Gamma_{ren}(\lambda p_1, \ldots \lambda p_n\, ; g_i, \mu) = \lambda^{D_\Gamma}\, \Gamma_{ren}(p_1, \ldots p_n\, ; g_i^{eff}(t), \mu) \left\{ -\int_0^t \gamma(g^{eff}(x))\, dx \right\} \tag{1.3}$$

where D_Γ gives the dimensions of the Green's function Γ in units of mass.

The ultraviolet behaviour of the Green's function $\Gamma_{ren}(\lambda p\, ; g_i, \mu)$ as $\lambda \to \infty$, can be studied by considering the behaviour of the expression on the left hand side of (1.3) as $t \to \infty$. This is basically a study of the effective coupling constants to this limit. If $\lim_{t\to\infty} g_k^{eff}(t, g_k)$ exists and equals some given set denoted by g_k^*, called an ultraviolet stable point, then the theory at this limit is determined by the coupling constants g_k^* and all information of the initial values (t = o) of the coupling constants is "lost". It is due to this consequence that this approach enable us to handle the dynamics of strongly interacting field theories as it has been stressed in the literature(7).

A renormalizable field theory is said to be asymptotically free if $g_i^* = 0$, i.e. $\lim_{t\to\infty} g_i^{eff}(t, g_i) = 0$ for all i. The asymptotic behaviour of amplitudes of asymptotically free theories is calculable by ordinary perturbation theory. The anomalous dimensions of the Green's functions $\gamma(g_i^*)$ are zero when the theory is asymptotically free and canonical scaling is obtained up to calculable logarithmic corrections. Asymptotically free theories offer an explanation for the observed scaling in deep inelastic lepton-hadron scattering. It has been shown(14) that Bjorken scaling can be obtained only if the strong interactions are described by asymptotically free theories.

The existence of an ultraviolet stable point other than the origin in the space of coupling constants of the theory is only a conjecture. The stability or otherwise for the origin, though, may be determined by investigating the system (1.2) perturbatively in a small region near the origin.

Coleman and Gross have shown(15) that no renormalizable field theory without non-abelian gauge fields can be asymptotically free. The asymptotic properties of non-abelian gauge theories have been studied extensively by a number of authors[2,16,17]. In general any non-abelian gauge theory contains three types of dimensionless parameters: the gauge coupling constant g; the Yukawa-type of coupling constants h_i, i = 1,2..p; and the self-quartic scalar coupling constants λ_α, α = 1,2...q -with the system (1.2) for the effective coupling constants of the theory, where the t-dependence is understood and we dispense with the superscript (eff).

$$\frac{dg}{dt} = \beta_g(g,h,\lambda) \quad , \quad \frac{dh_i}{dt} = \beta_{h_i}(g,h,\lambda) \quad , \quad \frac{d\lambda_\alpha}{dt} = \beta_{\lambda_\alpha}(g,h,\lambda) \tag{1.4}$$

for i = 1,2..p and α = 1,2..q, and with the notation that h and λ stand for the sets $\{h_i ; i=1,2\ldots p\}$ and $\{\lambda_\alpha ; \alpha=1,2\ldots q\}$ respectively.

In a small region near the origin the β-functions of (1.4) can be approximated by their lowest order terms in the effective coupling constants of the theory. It is found that[2,16,17] $\beta_g(g,h,\lambda) = -b_1 g^3(t)$ + h. orders. Hence $g(t) \to 0$ as $t \to 0$ provided that the constant b_1, whose sign depends on the overall number of matter fields of the theory, is positive (and having assumed that the "higher order terms" neglected in β_g vanish asymptotically faster than $g^3(t)$ - which is found to be true if $h_i(t)$ and $\lambda_\alpha(t) \to 0$ as $t \to 0$). The condition $b_1 > 0$ is found[16,17] not to be restrictive in constructing "realistic models" for the strong interactions.

Similarly the functions β_{h_i} and β_{λ_α} for i = 1,2..p and α = 1,2..q are also computed only to the one loop approximation. The analysis of the system of coupled differential equations (1.4) obtained to this order may be simplified by factoring out their dependence on g(t) by defining a new set of variables $\bar{h}_i(t) = h_i(t)/g(t)$ and $\bar{\lambda}_\alpha(t) = \lambda_\alpha(t)/g^2(t)$ for all $i = 1,2..p$ and $\alpha = 1,2..q$. The system of coupled equations (1.4), to this order, in terms of the new variables have the form (t-dependence understood).

$$\frac{1}{g^2(t)} \frac{d\bar{h}_i}{dt} = Y_i(\bar{h}) \quad \& \quad \frac{1}{g^2(t)} \frac{d\bar{\lambda}_\alpha}{dt} = Q_\alpha(\bar{h}, \bar{\lambda}) \tag{1.5}$$

for $i = 1,2..p$ and $\alpha = 1,2..q$, and with similar notation as previously that $\bar{h}$ stands for $\{\bar{h}_i \; ; \; i = 1,2...p\}$ and $\bar{\lambda}$ for $\{\bar{\lambda}_\alpha \; ; \; \alpha = 1,2...q\}$.

We summarize the general conclusions obtained[16,17] for the asymptotic behaviour of the coupling constants $h_i(t)$ and $\lambda_\alpha(t)$.

(a) For $h_i(t) \to 0$ as $t \to \infty$ for all $i = 1,2..p$ we have the condition[17] that there must exist a point $\bar{h}_i^*$ such that:

(i) $\left. Y_i(\bar{h}) \right|_{\bar{h}_j(t) = \bar{h}_j^*} = 0$ for all $i = 1,2...p$, and

(ii) all the eigenvalues of the matrix defined below have negative real parts.

$$\left. \left\{ \frac{\partial}{\partial \bar{h}_j} Y_i(\bar{h}) \right\} \right|_{\bar{h}_K(t) = \bar{h}_K^*}$$

It is found[17] that the effective Yukawa coupling constants do vanish asymptotically for most (gauge) theories of interest, but they vanish faster than the gauge coupling constant, i.e.

$\bar{h}_i^* = 0$ for $i = 1,2..p$, and hence their contributions to the coupled equations for the effective self-quartic scalar coupling constants are neglected.

(b) Similarly the system of coupled equations in $\bar{\lambda}_\alpha(t)$ (1.5) is investigated along parallel lines; having set $\bar{h}(t) = 0$ in the functions $Q_\alpha(\bar{h}.\bar{\lambda})$. It is found(16,17) that the effective self-quartic scalar coupling constants vanish asymptotically only if the number of Higgs scalar multiplets in the theory is smaller than some critical number which depends on the dimensions (no. of generators) of the particular non-abelian gauge group considered. Furthermore this critical number is found(16,17) to be much smaller than that needed to break the gauge symmetry spontaneously and generate masses for all the gauge fields according to the Higgs-Kibble mechanism(18). The conclusion was drawn(16,17) that in order to construct realistic asymptotically free models for the strong interactions the masses of the gauge fields would have to be generated by dynamical means(19).

N.P. Chang(1) has pointed out recently that solutions of the form $h_i(t) = \sum_{K=1}^{\infty} \bar{h}_i^{(K)} g^K(t)$ exist to the renormalization group equation, for the general non-abelian gauge theory considered above, provided that certain eigenvalue conditions are satisfied. The coefficients $\bar{h}_i^{(1)}$, $\bar{h}_i^{(2)}$ ... are constants <u>independent</u> of t and the eigenvalue conditions are obtained by substituting "these solutions" in the equations (1.4) and requiring consistency of the system; i.e. denoting this substitution into any function $F(g,h,\lambda)$ by $F(g,h,\lambda)/_{E.C.}$ we obtain from (1.4) that

$$\frac{dg}{dt} = \beta_g(g,h,\lambda)\Big|_{E.C} \qquad \left\{\sum_{K=1}^{\infty} K\,\bar{h}_i^{(K)}\, g^{K-1}\right\}\frac{dg}{dt} = \beta_{h_i}(g,h,\lambda)\Big|_{E.C}$$

and for consistency

$$\left\{\sum_{K=1}^{\infty} K\,\bar{h}_i^{(K)}\, g^{K-1}\right\}\ \beta_g(g,h,\lambda)\Big|_{E.C} \equiv \beta_{h_i}(g,h,\lambda)\Big|_{E.C}$$

for all $i = 1,2..p$, where the constants $\bar{h}_i^{(1)}$, $\bar{h}_i^{(2)}$ .. must satisfy these conditions to all orders in g and λ . The basic idea is that, if such a solution exists, the Yukawa coupling constants vanish asymptotically like g(t), i.e. for $\bar{h}_i^{(1)} \neq 0$, $i = 1,2..p$, and their non-trivial contributions to the renormalization group equation for the self-quartic scalar coupling constants cannot be neglected. This is found[1] to improve "the critical number of Higgs scalar multiplets" for a non-abelian gauge theory to be asymptotically free.

The self-quartic scalar coupling constants can also be treated in a similar way and eigenvalue conditions obtained for which $\lambda_\alpha(t) = \bar{\lambda}_\alpha^{(1)} g^2(t) + \bar{\lambda}_\alpha^{(2)} g^4(t) +$ h. orders in g(t), where $\bar{\lambda}_\alpha^{(1)}$, $\bar{\lambda}_\alpha^{(2)}$ ..... are constants independent of t. We shall refer to the total system of eigenvalue conditions obtained this way as the Chang eigenvalue conditions. We note that, for any given gauge theory, if a solution exists satisfying the Chang eigenvalue conditions of the theory to all orders in g(t) then one of the following two cases is true:

(1) either $\lim_{t\to\infty} g(t) = 0$; the origin is an ultraviolet stable point and the theory asymptotically free

(2) or, $\lim_{t\to -\infty} g(t) = 0$; the origin is an infrared stable point.

(b) Supersymmetric and non-Abelian Gauge Invariant Theories

The word Supersymmetry is used to describe the fundamental global Fermi-Bose symmetry firstly introduced by Wess and Zumino(20). These authors constructed a Lagrangian theory which is invariant up to a total 4-divergence under certain transformations which mix Bose and Fermi fields.

Salam and Strathdee have shown that these transformations may be viewed(21) as the realization of the "Supersymmetry group" on some generalized fields, called superfields and defined over an 8-dimensional space whose points are labelled by (x_μ, θ_α); where X_μ denotes the ordinary space-time coordinate and θ_α is an anticommuting Majorana spinor (see Appendix I).

The anticommutativity of the Majorana spinors implies that any superfield $\Phi(x, \theta)$ is a polynomial in θ_α and is fully specified by sixteen ordinary functions of space-time which are the coefficients in its expansion in powers of θ. The transformation properties of these coefficients or components under the action of the Poincare or the Supersymmetry group can be determined from those of the superfield; i.e. for a scalar superfield, by definition, $\Phi'(x', \theta') = \Phi(x, \theta)$ where $(x_\mu, \theta_\alpha) \to (x'_\mu, \theta'_\alpha)$ is given by(11)

$$x'_\mu = \Lambda_{\mu\nu} x_\nu + b_\mu \quad , \qquad \theta'_\alpha = a_\alpha^{\beta}(\Lambda)\, \theta_\beta \qquad \text{for the Poincare group} \tag{1.6}$$

$$x'_\mu = x_\mu + \frac{i}{2}\bar{\epsilon}\gamma_\mu\theta \quad , \quad \theta'_\alpha = \theta_\alpha + \epsilon_\alpha \qquad \text{for the Supersymmetry group}$$

(1.7)

where the matrix $\alpha_\alpha^\beta(\Lambda)$ denotes the Dirac spinor representation of the homogeneous Lorentz transformation Λ and the parameter ϵ_α is an anticommuting Majorana spinor.

In the construction of supersymmetric Lagrangians one uses the so called chiral superfields[11,22] $\Phi_+(x,\theta)$ and $\Phi_-(x,\theta)$ rather than the general 16-component superfield $\Phi(x,\theta)$. They are defined as the general solutions to the following two linear differential equations

$$\left[\frac{1-i\gamma_5}{2}.D\right]\Phi_+(x,\theta) = 0 \quad , \quad \left[\frac{1+i\gamma_5}{2}.D\right]\Phi_-(x,\theta) = 0 \tag{1.8}$$

where $$D_\alpha \equiv \frac{\partial}{\partial\bar{\theta}_\alpha} - \frac{i}{2}(\gamma_\mu\theta)_\alpha\frac{\partial}{\partial x_\mu} \tag{1.9}$$

and may be given in powers of θ by

$$\Phi_\pm(x,\theta) = A_\pm(x) + \bar{\theta}\Psi_\pm(x) + \frac{1}{4}\bar{\theta}\theta F_\pm(x) + \frac{1}{4}\bar{\theta}\gamma_5\theta\, i F_\pm(x) +$$

$$+\frac{1}{4}\bar{\theta}i\gamma^\nu\gamma_5\theta(\pm i\partial_\nu A_\pm(x)) + \frac{1}{4}\bar{\theta}\theta\bar{\theta}(-i\partial\!\!\!/\Psi_\pm(x)) + \frac{1}{32}(\bar{\theta}\theta)^2(-\partial^2 A_\pm(x)) \tag{1.10}$$

where $A_\pm$ (x) and $F_\pm$ (x) are complex boson fields, Ψ_- and Ψ_+ are right- and left-handed Dirac spinors respectively; $\Psi_\pm = \frac{1\mp i\gamma_5}{2}\Psi$. It is possible to identify Φ_- with the complex conjugate of Φ_+. If this is done Ψ_+ and Ψ_-

are identified as the left and right handed components of a Majorana spinor.

When products of superfields are considered the following multiplication laws are obtained[11] (in obvious notation).

$$\Phi_{1+}.\Phi_{2+} = \Phi_{3+} \quad \text{(chiral)}, \qquad \Phi_{1-}.\Phi_{2-} = \Phi_{3-} \quad \text{(chiral)} \tag{1.11}$$

$$\Phi_{1+}.\Phi_{2-} = \Phi_3 \quad \text{(general superfield)} \tag{1.12}$$

The construction of supersymmetric Lagrangians in this formalism is rather simple. It is found that the action integral of a Lagrangian density $\mathcal{L}(\Phi_+, \Phi_-)$ is invariant under the transformations (1.7) if every θ-dependent term in $\mathcal{L}(\Phi_+, \Phi_-)$ has the form of spacetime divergence; i.e.

$$\delta \int d^4x\, \mathcal{L}(\Phi_+, \Phi_-) = \int d^4x\, \bar{\epsilon}\left[\frac{\partial}{\partial\bar{\theta}} + \frac{i}{2}(\gamma_\mu\theta)\frac{\partial}{\partial x_\mu}\right]\mathcal{L}(\Phi_+, \Phi_-) \quad \text{by (1.7)}$$

$$= \bar{\epsilon}\,\frac{\partial}{\partial\bar{\theta}} \int d^4x\, \mathcal{L}(\Phi_+, \Phi_-) + \text{surface terms} = 0$$

Salam and Strathdee have shown that it is always possible to construct Lagrangians having any θ-dependent terms as a total divergence. We list their conclusions[11]:

For chiral superfields

$$-(\bar{D}.D)\,\Phi_\pm = F_\pm \text{ (the component of } \Phi_\pm\text{) + total 4-divergence} \tag{1.13}$$

For any general superfield

$$\frac{1}{64}(\bar{D}.D)^2 \Phi(x,\theta) = \text{coeff. of term } (\bar{\theta}\theta)^2 \text{ for } \Phi + \text{total 4-divergence} \tag{1.14}$$

The operator ($\bar{D}D$) is invariant under the transformations (1.6) and (1.7), where $\bar{D}^{\alpha} \equiv (C^{-1})^{\alpha}{}_{\beta} D_{\beta}$, C being the charge conjugation matrix.

It has been shown that supersymmetric Lagrangians invariant under local abelian(8) or non-abelian(9-11) group transformations can be constructed. The two supersymmetric and SU(2)-gauge invariant Lagrangians which we consider in Sections III & IV are given by:

Model I,(11) for matter superfields $\Phi_{\pm}(x,\theta)$ $\{\Phi_{+} \neq \Phi_{-}^{*}\}$ transforming as doublets of SU(2); i.e. $\Phi_{\pm}(x,\theta) \rightarrow \Omega_{\pm}(x,\theta)\,\Phi_{\pm}(x,\theta)$ (1.15)

$$\mathcal{L}_{SI} = \mathcal{L}_{\Psi} + \frac{1}{8}(\bar{D}D)^2[\Phi_{+}^{\dagger} e^{g\Psi}\Phi_{+} + \Phi_{-}^{\dagger} e^{-g\Psi}\Phi_{-}] - \frac{M}{2}(\bar{D}D)[\Phi_{-}^{\dagger}\Phi_{+} + \Phi_{+}^{\dagger}\Phi_{-}]$$

Model II,(10,11) for matter superfields $\Phi_{\pm}$ defined by $\Phi_{\pm} = \Phi_{\pm}^{K}\tau^{K}$ where τ^{K} are the Pauli matrices and $\Phi_{\pm}^{K}$ $\{\Phi_{\pm}^{K} = \Phi_{\mp}^{K*}\}$ transform as triplets under SU(2); i.e.

$$\Phi_{\pm}(x,\theta) \rightarrow \Omega_{\pm}(x,\theta)\,\Phi_{\pm}(x,\theta)\,\Omega_{\pm}(x,\theta) \tag{1.16}$$

$$\mathcal{L}_{SII} = \mathcal{L}_{\Psi} + \frac{1}{16}(\bar{D}D)^2\,\mathrm{Tr}\left[\Phi_{-} e^{g\Psi}\Phi_{+} e^{-g\Psi}\right]$$

The Lagrangian $\mathcal{L}_{\Psi}$ for the gauge superfield $\Psi(x,\theta)$ a general hermitian and pseudascalar superfield, will be considered shortly. The remaining terms of Models I and II are supersymmetric by (1.11 - 1.14) and invariant under the transformations (1.15) and (1.16), respectively, for $e^{g\Psi}$ transforming according to (g is just a constant)

$$e^{g\Psi} \rightarrow \Omega_- e^{g\Psi} \Omega_+^{-1} \tag{1.17}$$

with the condition

$$\Omega_+^\dagger = (\Omega_-)^{-1}$$

The Lagrangian $\mathcal{L}_\Psi$ is constructed by defining the superfield V_μ by(9)

$$V_\mu = -\frac{1}{g}\left[C^{-1}\gamma_\mu \frac{1+i\gamma_5}{2}\right]^{\alpha\beta} D_\alpha\left[e^{-g\Psi} D_\beta\, e^{g\Psi}\right]$$

using the eqns (1.8) that $\Omega_\pm$ satisfy, it has been shown(9) that V_μ and V_μ^+ transform under the gauge transformation (1.17) according to

$$V_\mu \rightarrow \Omega_+ V_\mu \Omega_+^{-1} + \frac{2i}{g}\Omega_+ \partial_\mu \Omega_+^{-1} \quad , \quad V_\mu^+ \rightarrow \Omega_- V_\mu^+ \Omega_-^{-1} + \frac{2i}{g}\Omega_- \partial_\mu \Omega_-^{-1}$$

and furthermore that

$$\left[\frac{1-i\gamma_5}{2} D\right]_\alpha V_\mu \qquad \& \qquad \left[\frac{1+i\gamma_5}{2} D\right]_\alpha V_\mu^+$$

transform homogeneously under gauge transformations and are chiral, hence

$$\mathcal{L}_\Psi = -\frac{1}{32}\,\bar{D}.D\;\mathrm{Tr}\left\{\left(C^{-1}\frac{1-i\gamma_5}{2}\right)^{\alpha\beta}(D_\alpha V_\mu)(D_\beta V_\mu) + \left(C^{-1}\frac{1+i\gamma_5}{2}\right)^{\alpha\beta}(D_\alpha V_\mu^+)(D_\beta V_\mu^+)\right\}$$

$$= \frac{1}{128}\,(\bar{D}.D)^2\;\mathrm{Tr}\left[V_\mu V_\mu + V_\mu^+ V_\mu^+\right] + \text{total 4-divergence} \tag{1.18}$$

is gauge invariant and supersymmetric by (1.11) and (1.13).

We shall now give the models I and II in component form using (1.10) for $\Phi_\pm$, the components being doublets and triplets of SU(2) respectively, and a special gauge[8,10] for Ψ for which

$$\Psi = \frac{1}{4}\bar{\Theta} i\gamma^\nu\gamma_5\Theta\, W_\nu + \frac{1}{2\sqrt{2}}\,\bar{\Theta}\Theta\,\bar{\Theta}\gamma_5\chi + \frac{1}{16}(\bar{\Theta}\Theta)^2 D_5$$

where $W_\mu \equiv W_\mu^k \tau^k$, $\chi \equiv \chi^k\tau^k$ & $D_5 \equiv D_5^k\tau^k$ and W_μ is transverse.

$$\mathcal{L}_{SI} = -\frac{1}{4}F^i_{\mu\nu}F^i_{\mu\nu} + \frac{i}{2}\bar{\chi}^k\gamma^\mu(\partial_\mu\chi^k + g\epsilon^{kij}W^i_\mu\chi^j) + (D_\mu A)^+(D_\mu A) +$$

$$+(D_\mu B)^+(D_\mu B) + i\bar{\Psi}\gamma^\mu D_\mu\Psi - M\bar{\Psi}\Psi - M^2A^+A - M^2B^+B$$

$$+\frac{ig}{2}A^+\bar{\chi}^i\tau^i\Psi - \frac{ig}{2}\bar{\Psi}\tau^i\chi^iA + \frac{ig}{2}B^+\bar{\chi}^i\tau^i\gamma_5\Psi - \frac{ig}{2}\bar{\Psi}\tau^i\gamma_5\chi^iB$$

$$-\frac{g^2}{2}(A^+A)(B^+B) + \frac{g^2}{4}(A^+B)(B^+A) + \frac{g^2}{8}(B^+A)^2 + \frac{g^2}{8}(A^+B)^2$$

where

$$F^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g\epsilon^{ijk}W^j_\mu W^k_\nu \quad , \qquad D_\mu \equiv \partial_\mu - \frac{i}{2}g\,\tau^i W^i_\mu$$

$A_\pm = (A \pm iB)/\sqrt{2}$, both A & B being complex fields, and $\Psi = \Psi_+ + \Psi_-$ (a Dirac spinor)

$$\mathcal{L}_{SII} = -\frac{1}{4}F^i_{\mu\nu}F^i_{\mu\nu} + i\bar{\Psi}^i\gamma^\mu(\partial_\mu\Psi^i + g\epsilon^{ijk}W^j_\mu\Psi^k) +$$

$$+\frac{1}{2}(\partial_\mu A^i + g\epsilon^{ijk}W^j_\mu A^k)^2 + \frac{1}{2}(\partial_\mu B^i + g\epsilon^{ijk}W^j_\mu B^k)^2$$

$$+ig\epsilon^{ijk}(A^i\bar{\Psi}^j\Psi^k + B^i\bar{\Psi}^j\gamma_5\Psi^k) - \frac{g^2}{2}(\epsilon^{ijk}A^jB^k)(\epsilon^{ilm}A^lB^m)$$

where $A^i_\pm = (A^i \pm i B^i)/\sqrt{2}$ and $\Psi^i = (\chi^i + i\lambda^i)/\sqrt{2}$ where χ^i & λ^i are the Majorana spinors of the superfields $\Psi(x,\theta)$ and $\Phi_\pm(x,\theta)$ respectively.

Finally we point out that the fields D_5 for $\Psi(x,\theta)$ and $F_\pm$ for $\Phi_\pm(x,\theta)$ of Models I and II are "auxiliary" and have been eliminated from the Lagrangians using their field equations.

SECTION II

THE CHANG EIGENVALUE CONDITIONS AND SUPERSYMMETRIES

In section I.a the Chang eigenvalue problem(1) was formulated. It was pointed out that if certain eigenvalue conditions are satisfied, for any non-abelian gauge theory, we obtain the solutions

$$h_i(t) = \bar{h}_i^{(1)} g(t) + \text{h. orders in } g(t)$$
$$\lambda_\alpha(t) = \bar{\lambda}_\alpha^{(1)} g^2(t) + \text{h. orders in } g(t) \qquad (2.1)$$

for $i = 1,2..p$ and $\alpha = 1,2..q$, and the theory is asymptotically free if $\lim_{t\to\infty} g(t) = 0$; we use for convenience the same definition of coupling constants as that given in section I.a.

On the other hand it has been shown that internal local symmetries are compatible with supersymmetries[9-11]. The main feature of the supersymmetric and non-abelian gauge invariant Lagrangians that have been constructed[10,11] is that the only independent coupling constants of these theories are the gauge coupling constants (see section I.b). This class of Lagrangians are asymptotically free provided that the renormalizability preserves their supersymmetric nature and that $\lim_{t\to\infty} g(t) = 0$. Slavnov[23] has shown very recently that the renormalization of the supersymmetric and gauge invariant Lagrangians <u>does</u> preserve the nature of these theories.

The problem that presents itself is whether the supersymmetric Lagrangians can be obtained as solutions of the eigenvalue conditions obtained by Chang; as it would be expected if everything is consistent[24]. Furthermore we would expect that a solution corresponding to some supersymmetric theory will belong to a special subclass of solutions of the Chang eigenvalue problem. This special subclass, by definition, consists of solutions of the form $h_i(t) \equiv \bar{h}_i^{(1)} g(t)$ and $\lambda_\alpha(t) \equiv \bar{\lambda}_\alpha^{(1)} g^2(t)$ satisfying the Chang eigenvalue conditions of the theory to all orders, i.e. no higher order corrections in g(t) are obtained. This is basically a problem to all orders of perturbation theory in the effective coupling constants of the theory.

This section is divided into three parts: in part (a) we study to the one loop approximation the general form of the bare Lagrangian which "corresponds" to a renormalized theory with its Yukawa and self-quartic scalar coupling constants satisfying the Chang eigenvalue conditions to this order; i.e. given by (2.1).

In part (b) we investigate what the necessary and sufficient conditions are for the existence of the aforementioned special subclass of solutions to the Chang eigenvalue conditions. It is found that these conditions are satisfied by any choice of coupling constants for which the Lagrangian is known to be supersymmetric.

Finally in part (c) we consider the Chang eigenvalue conditions to any finite order in perturbation theory for solutions of the general form (2.1). We obtain the system of simultaneous equations for the unknown coefficients, of the power series in g(t) solutions (2.1), and study the existence of the higher order terms for any given lowest order solution $\bar{h}_i^{(1)}$ and $\bar{\lambda}_\alpha^{(1)}$. Sufficient (but not necessary) conditions are found for the Chang eigenvalue conditions to be satisfied to all orders, for any given lowest order solution obtained to the Chang-problem.

(a) <u>The Chang-Problem and the One-Loop Renormalization of the Theory</u>

The general solution of the type (2.1), an infinite series in g(t), does not hold for (t = 0) if the renormalized gauge coupling constant of the theory, g(t=0), is a "strong coupling constant". In this case the renormalized coupling constants for the Yukawa and self-quartic scalar vertices are given functions of the renormalized gauge coupling constant determined by analytically continuing the solutions (2.1) beyond the given radius of convergence for their validity.

We shall assume that the renormalized gauge coupling constant, g(t = 0), is a weak one or that the solutions of the type (2.1) are finite series in g(t) so that their validity extends to (t = 0). Given the renormalized Lagrangian corresponding to the solution (2.1) for t = 0, we can raise the following question: What, if any, are the relations among the bare coupling constants for the theory to be finite?

We shall only attempt to answer this question to the one-loop approximation. It will follow naturally that the supersymmetric Lagrangians can be obtained as solutions to the Chang eigenvalue conditions considered to lowest order.

We use the regularization method of analytically continuing the dimensions of space-time(12), denoted by the letter n. All the subtractions that need be performed to render the vertices finite, to the one loop approximation, appear as simple poles at n = 4. The coupling constants of the bare Lagrangian $\{ g^{o}, h_i^{o}, \lambda_\alpha^{o} \}$ have non-zero dimensions for $n \neq 4$. Their general forms, in terms of the renormalized dimensionless coupling constants $\{ g, h_i, \lambda_\alpha \}$, are given to this approximation by:

$$g^{o} = \mu^{2-n/2} \left(g + \frac{b_1^{g} \, g^{3}}{n-4} \right) \tag{2.2}$$

$$h_i^{o} = \mu^{2-n/2} \left(h_i + \frac{b_1^{h_i}(h, g)}{n-4} \right) \quad \text{for } i = 1, 2 \ldots p \tag{2.3}$$

$$\lambda_\alpha^{o} = \mu^{4-n} \left(\lambda_\alpha + \frac{b_1^{\lambda_\alpha}(\lambda, h, g)}{n-4} \right) \quad \text{for } \alpha = 1, 2 \ldots q \tag{2.4}$$

Where μ is the arbitrary unit of mass introduced in defining the dimensionless renormalized coupling constants of the theory. We have used the coupling constant symbols as superscripts on the residues of the poles at n = 4 in an obvious notation; i.e. one to one correspondence with the bare coupling constants of the theory. The dependence of the residues on the renormalized coupling constants has also been explicitly shown to the approximation considered, with the shorthand notation that h stands for the set $\{h_i ; i=1,2\ldots p\}$ and λ for $\{\lambda_\alpha ; \alpha=1,2\ldots q\}$. We also observe that the residues $b_1^{h_i}(h,g)$ are homogeneous in g and $\{h\}$ and of order 3, while $b_1^{\lambda_\alpha}(\lambda,h,g)$ are homogeneous in $\{\lambda\}$, $\{h^2\}$ and g^2 and of order 2.

The differential equations for the effective coupling constants of the theory can be easily computed following the work by 't Hooft(6,25). Using the expressions (2.2), (2.3) and (2.4) we obtain

$$\frac{d}{dt} g = - b_1^g \, g^3 \tag{2.5}$$

$$\frac{d}{dt} h_i = - b_1^{h_i}(h,g) \qquad \text{for } i=1,2\ldots p \tag{2.6}$$

$$\frac{d}{dt} \lambda_\alpha = - b_1^{\lambda_\alpha}(\lambda,h,g) \qquad \text{for } \alpha=1,2\ldots q \tag{2.7}$$

The Chang eigenvalue conditions considered to the one loop approximation determine only the coefficients $\bar{h}_i^{(1)}$ and $\bar{\lambda}_\alpha^{(1)}$ of the solution (2.1). We substitute in the equations (2.6) and (2.7) the forms

$$h_i(t) = \bar{h}_i^{(1)} g(t) \qquad \& \qquad \lambda_\alpha(t) = \bar{\lambda}_\alpha^{(1)} g^2(t) \tag{2.8}$$

and obtain

$$\bar{h}_i^{(1)} \frac{d}{dt} g = - b_1^{h_i}(\bar{h}^{(1)} g, g) \qquad \text{for } i=1,2\ldots p \tag{2.9}$$

$$2g\bar{\lambda}_{\alpha}^{(1)} \frac{dg}{dt} = -b_1^{\lambda_\alpha}(\bar{\lambda}^{(1)}g^2, \bar{h}^{(1)}g, g) \qquad \text{for } \alpha = 1,2\ldots q \qquad (2.10)$$

The constant coefficients $\bar{h}_i^{(1)}$ and $\bar{\lambda}_\alpha^{(1)}$ are determined from the system of simultaneous equations obtained by substituting the equation (2.5) in the equations (2.9) and (2.10). The dependence on g can be factored out and the Chang eigenvalue conditions are, in obvious notation.

$$\bar{h}_i^{(1)}\, b_1^g = b_1^{h_i}(\bar{h}^{(1)}, 1) \qquad \text{for } i = 1,2\ldots p \qquad (2.11)$$

$$2\,\bar{\lambda}_\alpha^{(1)}\, b_1^g = b_1^{\lambda\alpha}(\bar{\lambda}^{(1)}, \bar{h}^{(1)}, 1) \qquad \text{for } \alpha = 1,2\ldots q \qquad (2.12)$$

We shall assume that a non-trivial physically acceptable solution exists to the system of equations (2.11) and (2.12). The solutions (2.8) for t = 0 can be substituted in the expressions for the bare coupling constants (2.3) and (2.4), we obtain

$$h_i^0 = \mu^{2-n/2}\left\{\bar{h}_i^{(1)}g + \frac{b_1^{h_i}(\bar{h}^{(1)}g, g)}{(n-4)}\right\} \qquad \text{for } i = 1,2..p \qquad (2.13)$$

$$\lambda_\alpha^0 = \mu^{4-n}\left\{\bar{\lambda}_\alpha^{(1)}g^2 + \frac{b_1^{\lambda_\alpha}(\bar{\lambda}^{(1)}g^2, \bar{h}^{(1)}g, g)}{(n-4)}\right\} \qquad \text{for } \alpha = 1,2..q \qquad (2.14)$$

Substituting (2.11) in (2.13) and using (2.2) we obtain

$$h_i^0 = \mu^{2-n/2}\left\{\bar{h}_i^{(1)}g + \frac{\bar{h}_i^{(1)}b_1^g g^3}{(n-4)}\right\} = \bar{h}_i^{(1)}g^0 \qquad (2.15)$$

$$\text{for } i = 1,2..p$$

Similarly substituting (2.12) in (2.14) and using (2.2) we obtain

$$\lambda_\alpha^o = \mu^{4-n}\left\{\bar{\lambda}_\alpha^{(1)} g^2 + \frac{2\bar{\lambda}_\alpha^{(1)} b_1^g g^4}{(n-4)}\right\} = \bar{\lambda}_\alpha^{(1)} g_o^2 + \text{h. orders in } g^o$$

for $\alpha = 1,2\ldots q$ (2.16)

We conclude that a solution to the Chang eigenvalue problem of the form (2.1), valid for $t = 0$, implies the relations for the unrenormalized coupling constants given by, to lowest order in g_o

$$h_i^o = \bar{h}_i^{(1)} g_o \quad \& \quad \lambda_\alpha^o = \bar{\lambda}_\alpha^{(1)} g_o^2$$

for $i = 1,2..p$ & $\alpha = 1,2..q$ (2.18)

The implications of this result are obvious. We observe that all the relevant vertices of the Lagrangian corresponding to the Chang solution (2.1) are rendered finite, to the one loop approximation, by a single arbitrary subtraction, i.e. $\delta g = g_o - g$.

It also follows simply that the supersymmetric Lagrangians can be obtained as solutions of the Chang eigenvalue problem. That is, given a non-abelian gauge theory known to reduce to a supersymmetric one for a given choice of coupling constants, then the coupling constants corresponding to the supersymmetric theory satisfy the Chang eigenvalue conditions considered to lowest order. The proof will be given for completeness sake.

Let us assume, for convenience, that the Lagrangian considered reduces to a supersymmetric one for the choice of coupling constants given by (2.18). Renormalization preserves the supersymmetric nature of the theory[23], so the coupling constants are given by (2.8) after the one loop renormalization of the supersymmetric theory. We substitute (2.8) and (2.18) in the expressions (2.3) and (2.4) and obtain

$$\bar{h}_i^{(1)} g^o = \mu^{2-n/2} \left\{ \bar{h}_i^{(1)} g + \frac{b_1^{h_i}(\bar{h}^{(1)} g, g)}{(n-4)} \right\}$$

$$\text{for } i = 1,2..p \qquad (2.19)$$

and

$$\bar{\lambda}_\alpha^{(1)} g^{o2} = \mu^{4-n} \left\{ \bar{\lambda}_\alpha^{(1)} g^2 + \frac{b_1^{\lambda_\alpha}(\bar{\lambda}^{(1)} g^2, \bar{h}^{(1)} g, g)}{(n-4)} \right\}$$

$$\text{for } \alpha = 1,2..q \qquad (2.20)$$

We stress that the consistency of the expressions (2.2), (2.19) and (2.20) follows from the fact renormalization preserves the constraints on the coupling constants of the supersymmetric theory[23]. Substituting (2.2) in (2.19) and (2.20) we obtain that the Chang eigenvalue conditions (2.11) and (2.12) are satisfied, Q.E.D..

(b) The Chang-Problem and Solutions of Higher-Symmetries

It has been pointed out that within the general class of solutions (2.1) to the Chang eigenvalue problem there may exist a special subclass which is of particular interest; i.e. the solutions

$$h_i(t) = \bar{h}_i\, g(t) \qquad \text{and} \qquad \lambda_\alpha(t) = \bar{\lambda}_\alpha\, g^2(t) \qquad (2.21)$$

$$\text{for } i = 1,2...p \ \& \ \alpha = 1,2..q$$

to all orders of perturbation theory. We shall now proceed to investigate what the necessary and sufficient conditions are for the existence of these special solutions (2.21).

The differential equations for the effective coupling constants of the theory can be worked out formally to all orders of perturbation theory. Using this system of differential equations, we can calculate the Chang eigenvalue conditions corresponding to the special type of solutions (2.21). An infinite system of simultaneous equations in $\{\bar{h}_i ; i=1,2..p\}$ and $\{\bar{\lambda}_\alpha ; \alpha=1,2..q\}$ is obtained. The problem is to find the necessary and sufficient conditions for the existence of a solution to this infinite system of simultaneous equations.

Our analysis of the Chang eigenvalue conditions to the one loop approximation is very suggestive to what these conditions should be. If we assume that a special solution exists, then it satisfies the Chang eigenvalue conditions to all orders, order by order; i.e. to the m$^{\text{th}}$ order $h_i^{(m)} = \bar{h}_i g^{(m)}$ and $\lambda_\alpha^{(m)} = \bar{\lambda}_\alpha (g^{(m)})^2$. It has been shown that the relevant vertices of the theory corresponding to this solution are rendered finite, to the one loop approximation, by a single arbitrary subtraction. Hence, "naively", this is a property that should hold to all orders or equivalently, it is a necessary condition for the existence of special solutions to the Chang eigenvalue problem; i.e. the (m+1)$^{\text{th}}$ order counterterms for the Yukawa and self-quartic scalar vertices are given by

$$h_i^{(m+1)} - h_i^{(m)} = \bar{h}_i \{ g^{(m+1)} - g^{(m)} \} \quad \& \quad \lambda_\alpha^{(m+1)} - \lambda_\alpha^{(m)} = \bar{\lambda}_\alpha \{ (g^{(m+1)})^2 - (g^{(m)})^2 \}$$

for $i=1,2..p$ and $\alpha = 1,2..q$

It is fairly obvious that this necessary condition should also be sufficient.

We shall, therefore, formulate "the single coupling constant renormalization" problem. This problem can be stated simply in notational form: we make the assumption that the bare and renormalized coupling constants of the theory may be given by

$$\frac{h_i^o}{g_o} = \frac{h_i}{g} = x_i \qquad \& \qquad \frac{\lambda_\alpha^o}{g_o^2} = \frac{\lambda_\alpha}{g^2} = y_\alpha \tag{2.22}$$

for $i = 1,2..p$ and $\alpha = 1,2..q$

where $\{x_i \,;\, i=1,2..p\}$ and $\{y_\alpha \,;\, \alpha=1,2..q\}$ are unknown finite constants. Given the stringent assumption (2.22), all the relevant vertices must be rendered finite to all orders, order by order, by a single arbitrary subtraction. Hence a system of simultaneous equations is obtained for the unknowns x_i, $i=1,2..p$ and y_α, $\alpha = 1,2..q$. It will be shown that this system of simultaneous equations is <u>exactly</u> equivalent to the Chang eigenvalue problem obtained for the solutions of the special form (2.21).

The bare coupling constants of the non-abelian gauge theory, considered to all orders in the renormalized set of coupling constants, are given by[6]

$$g_o = \mu^{2-n/2}\left\{ g + \sum_{\nu=1}^{\infty} \frac{\alpha_\nu^g(g,h,\lambda)}{(n-4)^\nu} \right\} \tag{2.23}$$

$$h_i^o = \mu^{2-n/2}\left\{ h_i + \sum_{\nu=1}^{\infty} \frac{\alpha_\nu^{h_i}(g,h,\lambda)}{(n-4)^\nu} \right\} \tag{2.24}$$

for $i = 1,2..p$

$$\lambda_\alpha^o = \mu^{4-n} \left\{ \lambda_\alpha + \sum_{\nu=1}^{\infty} \frac{a_\nu^{\lambda_\alpha}(g,h,\lambda)}{(n-4)^\nu} \right\} \tag{2.25}$$

for $\alpha = 1,2\ldots q$

Subtractions of higher order poles at $n = 4$ are generally needed to render a theory finite beyond the one loop approximation. The sum over all such subtractions, for the relevant vertices considered, are given in the expressions (2.23 - 2.25); where the letter ν denotes the order of the poles and is used as a postscript for the corresponding residues $a_\nu(g,h,\lambda)$. We use the same notation as in the one loop problem for the explicit dependence of the relevant residues on the coupling constants of the theory. We note that there is no dependence on the coupling constants of the self-cubic scalar vertices of the theory or masses(6,13).

The differential equations for the effective coupling constants of the theory are given in terms of the residues of the simple poles by(6).

$$\frac{dg}{dt} = \frac{1}{2} a_1^g(g,h,\lambda) - \left\{ \frac{g}{2}\frac{d}{dg} + \sum_{i=1}^{p} \frac{h_i}{2}\frac{d}{dh_i} + \sum_{\alpha=1}^{q} \lambda_\alpha \frac{d}{d\lambda_\alpha} \right\} a_1^g(g,h,\lambda) \tag{2.26}$$

$$\frac{dh_i}{dt} = \frac{1}{2} a_1^{h_i}(g,h,\lambda) - \left\{ \frac{g}{2}\frac{d}{dg} + \sum_{i=1}^{p} \frac{h_i}{2}\frac{d}{dh_i} + \sum_{\alpha=1}^{q} \lambda_\alpha \frac{d}{d\lambda_\alpha} \right\} a_1^{h_i}(g,h,\lambda) \tag{2.27}$$

for $i = 1,2\ldots p$

$$\frac{d\lambda_\alpha}{dt} = a_1^{\lambda_\alpha}(g,h,\lambda) - \left\{ \frac{g}{2}\frac{d}{dg} + \sum_{i=1}^{p} \frac{h_i}{2}\frac{d}{dh_i} + \sum_{\alpha=1}^{q} \lambda_\alpha \frac{d}{d\lambda_\alpha} \right\} a_1^{\lambda_\alpha}(g,h,\lambda) \tag{2.28}$$

for $\alpha = 1,2\ldots q$

The residues of the simple poles are essentially polynomials in the coupling constants of the theory and may, therefore, be redefined as

$$a_1^{g} \equiv \sum_{W=1}^{\infty} b_W^{g}(g,h,\lambda) \tag{2.29}$$

$$a_1^{h_i} \equiv \sum_{W=1}^{\infty} b_W^{h_i}(g,h,\lambda) \qquad \text{for } i = 1,2\ldots p \tag{2.30}$$

$$a_1^{\lambda_\alpha} \equiv \sum_{W=1}^{\infty} b_W^{\lambda_\alpha}(g,h,\lambda) \qquad \text{for } \alpha = 1,2\ldots q \tag{2.31}$$

where $b_W^{g}(g,h,\lambda)$ and $b_W^{h_i}(g,h,\lambda)$ are homogeneous functions of g, h and " $\lambda^{\frac{1}{2}}$" and of order (2W+1). Similarly $b_W^{\lambda_\alpha}(g,h,\lambda)$ are homogeneous in g^2, h^2 and λ and of order (W+1).

We substitute (2.29 - 2.31), correspondingly, in the differential equations (2.26 - 2.28) and obtain

$$\frac{dg}{dt} = -\sum_{W=1}^{\infty} W\, b_W^{g}(g,h,\lambda) \tag{2.32}$$

$$\frac{dh_i}{dt} = -\sum_{W=1}^{\infty} W\, b_W^{h_i}(g,h,\lambda) \tag{2.33}$$

for $i = 1,2\ldots p$

$$\frac{d\lambda_\alpha}{dt} = -\sum_{w=1}^{\infty} w\, b_w^{\lambda_\alpha}(g, h, \lambda) \tag{2.34}$$

for $\alpha = 1,2\ldots q$

The Chang eigenvalue conditions for the special subclass of solutions (2.21) can be easily derived from the renormalization group equations of the theory (2.32 - 2.34). We substitute (2.21) in the system (2.32 - 2.34) and obtain for consistency:

From (2.32) and (2.33)

$$\bar{h}_i \sum_{w=1}^{\infty} w\, b_w^{g}(g, \bar{h}g, \bar{\lambda}g^2) \equiv \sum_{w=1}^{\infty} w\, b_w^{h_i}(g, \bar{h}g, \bar{\lambda}g^2) \tag{2.35}$$

for $i = 1,2\ldots p$

Similarly from (2.32) and (2.34)

$$2g\bar{\lambda}_\alpha \sum_{w=1}^{\infty} w\, b_w^{g}(g, \bar{h}g, \bar{\lambda}g^2) \equiv \sum_{w=1}^{\infty} w\, b_w^{\lambda_\alpha}(g, \bar{h}g, \bar{\lambda}g^2) \tag{2.36}$$

for $\alpha = 1,2\ldots q$

Equating the coefficients of terms of the same order in g of the left- and right-hand sides, respectively, for each of the identities (2.35) and (2.36) we obtain

$$\bar{h}_i\, b_w^{g}(1, \bar{h}, \bar{\lambda}) = b_w^{h_i}(1, \bar{h}, \bar{\lambda}) \qquad \text{for } i = 1,2\ldots p \tag{2.37}$$

$$2\bar{\lambda}_\alpha\, b_W^{g}(1,\bar{h},\bar{\lambda}) = b_W^{\lambda\alpha}(1,\bar{h},\bar{\lambda}) \quad \text{for } \alpha = 1,2\ldots q \tag{2.38}$$

for all $w = 1,2\ldots\infty$

We have obtained an infinite system of simultaneous equations for the variables $\bar{h}_i$, $i = 1,2..p$ and $\bar{\lambda}_\alpha$, $\alpha = 1,2..q$. The solutions of the special form (2.21) are found, therefore, by solving (?) this infinite system of simultaneous equations. Equivalently, we may say that a special solution exists if the Chang eigenvalue conditions (2.37) and (2.38) are satisfied. Practically speaking all the roots to the Chang eigenvalue problem are found at the one loop approximation, i.e. from the system (2.37) and (2.38) for W = 1 only. Hence a given solution is of the special kind if it is shown to satisfy the conditions (2.37) and (2.38) for W = 2,3,... ∞ . A formidable task to show explicitly, to say the least.

We shall now consider the "single coupling constant renormalization" problem and obtain the system of simultaneous equations for the variables x_i , $i = 1,2..p$ and y_α , $\alpha = 1,2..q$ defined by the relations (2.22). The general principle used is simple and has been already outlined. A set of consistency conditions is obtained from the expressions for the bare coupling constants (2.23), (2.24) and (2.25) when the constraints on the coupling constants (2.22) are assumed. We obtain from (2.23), (2.24) and (2.22)

$$x_i\left\{g + \sum_{\nu=1}^{\infty} \frac{a_\nu^{g}(g, xg, yg^2)}{(n-4)^\nu}\right\} \equiv \left\{x_i g + \sum_{\nu=1}^{\infty} \frac{a_\nu^{h_i}(g, xg, yg^2)}{(n-4)^\nu}\right\} \tag{2.39}$$

for $i = 1,2\ldots p$

Similarly from (2.23), (2.25) and (2.22)

$$y_\alpha \left\{ g + \sum_{\nu=1}^{\infty} \frac{a_\nu^g(g, xg, yg^2)}{(n-4)^\nu} \right\}^2 \equiv \left\{ y_\alpha g^2 + \sum_{\nu=1}^{\infty} \frac{a_\nu^{\lambda_\alpha}(g, xg, yg^2)}{(n-4)^\nu} \right\} \tag{2.40}$$

for all $\alpha = 1,2...q$

with the notation that x stands for the set $\{x_i\,;\ i = 1,2..p\}$ and y for $\{y_\alpha\,;\ \alpha = 1,2..q\}$.

The set of relations (2.39) and (2.40) are identities and should hold for any arbitrary dimensions of space-time (n) and gauge coupling constant g. A system of conditions for the variables $\{x_i\}$ and $\{y_\alpha\}$ is obtained by equating the same order residues of the left- and right-hand sides, respectively, of the identity (2.39), and (2.40). All these conditions though are not independent. It is known from the work by 't Hooft that the residues of the poles at n = 4, needed to render the theory finite, are not all independent[6]. In particular the residues of all higher order poles can be determined from the residues of the simple poles. It follows, therefore, that the non-trivial independent conditions for the existence of a solution to the single coupling constant renormalization problem are obtained by equating the residues for the simple poles <u>only</u>. We refer the reader to the Appendix II for further details on this point.

From (2.39), for $\nu = 1$

$$x_i \sum_{w=1}^{\infty} b_w^g(g, xg, yg^2) \equiv \sum_{w=1}^{\infty} b_w^{h_i}(g, xg, yg^2) \tag{2.41}$$

for $i = 1,2...p$

Similarly from (2.40), for $\nu = 1$

$$2 y_\alpha g \sum_{W=1}^{\infty} b_W^{g}(g, xg, yg^2) \equiv \sum_{W=1}^{\infty} b_W^{\lambda_\alpha}(g, xg, yg^2) \tag{2.42}$$

for $\alpha = 1,2\ldots q$

where we have used the expressions (2.29 - 2.31) for the residues of the simple poles. We equate the coefficients of terms of the same order in g for the identity (2.41) and, similarly, for (2.42) and obtain

$$x_i\, b_W^{g}(1, x, y) = b_W^{h_i}(1, x, y) \tag{2.43}$$

$$2 y_\alpha\, b_W^{g}(1, x, y) = b_W^{\lambda_\alpha}(1, x, y) \tag{2.44}$$

for all $W = 1, 2\ldots\infty$, $i = 1,2\ldots p$ and $\alpha = 1,2\ldots q$.

The system of conditions (2.43) and (2.44) for the single coupling constant renormalization problem is exactly equivalent to the system (2.37) and (2.38) obtained for the special solutions (2.21) to the Chang eigenvalue problem; i.e. $x_i = \bar{h}_i$ and $y_\alpha = \bar{\lambda}_\alpha$ for all $i = 1,2\ldots p$ and $\alpha = 1,2\ldots q$. In other words the necessary and sufficient condition for the existence of a special solution (2.21) to the Chang eigenvalue problem is that the Lagrangian theory corresponding to this solution be renormalizable; the renormalizability of the theory preserving the given relations for the coupling constants to all orders in perturbation theory.

We have seen that this result had been expected on the grounds of the so called naive argument used at the beginning of this subsection, but it is worth stressing further the significance of this analysis. What we mean is that the valid idea proposed by Chang and qualified to the special solutions (2.21) had to be shown rigorously to "tie up" with the renormalizability of such theories, as it has been explained. We wish to stress that it is due to the powerful methods of the renormalization group techniques for the regularization method of analytically continuing the space-time dimensions of the theory developed by 't Hooft[6] that we were able to prove the aforementioned equivalence. In particular the recurssion formula obtained by t'Hooft for the residues of the poles at $n = 4$ and its consequence that only the simple pole residues are independent was essential in the proof, see Appendix II. In our view this is a "kind of confirmation" for the validity of this rather intriguing recurssion formula or, in a stricter sense, simply a consistency test.

The conjecture is made that if a special solution exists to the Chang eigenvalue problem for a given gauge theory, then the theory corresponding to this solution will possess a higher symmetry. This higher symmetry is associated with the Ward-identities needed to guarantee that the single coupling constant renormalization conditions are satisfied to all orders.

Lastly, it follows that the supersymmetric theories can be "obtained" as solutions of the Chang eigenvalue problem, considered to all orders in perturbation theory, if the constraints on the coupling constants of the supersymmetric theory are preserved by the renormalization of the theory to all orders. From the recent work by Slavnov[23] it appears that this is indeed the case.

Let us also consider now, in more detail, the condition, given the existence of a Chang solution for any non-abelian gauge theory, that the theory be asymptotically free, i.e. $\lim_{t\to\infty} g(t) = 0$.

In section I.a this condition was discussed briefly. In particular the differential equation for the effective gauge coupling constant (1.4) was considered to lowest order, i.e. eqn (2.5), and the condition obtained for $\lim_{t\to\infty} g(t) = 0$ is that the constant b_1^g be greater than zero; if b_1^g is zero, then we need to compute the lowest non-vanishing term of the function β_g in order to determine the asymptotic properties of the theory.

A closed expression has been computed[16,17] for the constant b_1^g for any non-abelian gauge theory: if we let C_{abc} be the antisymmetric structure constants of the group considered and σ^a the representation matrix of the generators on some basis, b_1^g is given by

$$b_1^g = \frac{1}{16\pi^2}\left\{ \frac{11}{3} C_2(G) - \frac{2}{3}\sum T(M) - \frac{4}{3}\sum T(D) - \right.$$

$$\left. -\frac{1}{3}\sum T(S_c) - \frac{1}{6}\sum T(S_R) \right\} \tag{2.45}$$

where

$$C_{acd}\, C_{bcd} = C_2(G)\, \delta_{ab}$$

$$\mathrm{Trace}(\sigma^a \sigma^b) = T(R)\, \delta_{ab}$$

where R stands (notationally) for M, D, S_C and S_R according to whether the representation matrices σ^a are defined on a basis formed by a Majorana-, Dirac-, complex scalar-, or real scalar- multiplet, respectively, and the letter Σ in (2.45) denotes the sum of the contributions from all multiplets of the theory of a given type.

Hence, from the expression (2.45), we may determine the maximum number of matter fields for the theory to be asymptotically free provided a Chang solution does exist. This general analysis, therefore, is of particular interest only when the theory considered is known to correspond to a supersymmetric one, for a given choice of coupling constants, so that at least one solution is known to exist to the Chang-conditions. On the other hand, if that is the case, the maximum number of "matter fields" for the theory to be asymptotically free can also be given in terms of the corresponding number (M) of matter superfields.

We consider this maximum number for a general (renormalizable) SU(N) gauge Lagrangian known to describe a supersymmetric theory[9-11] for a given choice of coupling constants. The theory would, therefore, contain at least one Majorana spinor transforming according to the adjoint representation of the gauge group, which is associated with the gauge field of the supersymmetric theory; i.e. a component of the gauge superfield. Furthermore, we assume that the remaining matter fields of the theory are the components of M chiral superfields Φ_+ and Φ_- of the corresponding supersymmetric theory. For each chiral superfield Φ_+ that can be and is identified with the complex conjugate of (its parity conjugate) Φ_- superfield, the component matter fields obtained are two real scalar fields (a scalar and a pseudoscalar) and a Majorana spinor. While, if we do not identify Φ_+ with Φ_-^*, we have two complex scalar fields and a Dirac spinor.

The two cases that we consider explicitly are:

Case 1, all M matter superfields transform according to the vector representation of the gauge group (i.e. $\Phi_+ \neq \Phi_-^*$). Using that T(vector R) = ½ and T(adjoint R) = N = $C_2(N)$, we obtain from (2.45) for b_1^g

$$(16\pi^2)\, b_1^g = \left\{ \frac{11}{3} N - \frac{2}{3} N - \frac{4}{3} M\left(\frac{1}{2}\right) - \frac{1}{3}\, 2M\left(\frac{1}{2}\right) \right\}$$

hence, we have the condition for the theory to be asymptotically free that M < 3N; for M = 3N we need to compute the next order contribution to the β_g-function.

Case 2, all M matter superfields transform according to the adjoint representation of the gauge group with $\Phi_+ = \Phi_-^*$. From (2.45) we obtain for b_1^g

$$(16\pi^2)\, b_1^g = \left\{ \frac{11}{3} N - \frac{2}{3} N - \frac{2}{3} M(N) - \frac{1}{6}\, 2M(N) \right\}$$

and for the theory to be asymptotically free we have the condition[10,11] that M < 3. For M = 3, the next order term of the β_g-function has been computed and the theory found not to be asymptotically free[26].

Likewise the maximum number of matter superfields transforming according to different (plus higher) representations of the gauge group can be determined for the theory to be asymptotically free. We note that the two supersymmetric Lagrangians[10,11], considered in section I.b, are asymptotically free. In contrast, in the "conventional analysis for asymptotic freedom", a SU(N) gauge theory containing two scalar multiplets, transforming according to the vector representation of the gauge group, is asymptotically free[16,17] for $N \geqslant 4$ and similarly, for only one scalar multiplet transforming according to the adjoint representation, for $N \geqslant 6$.

(c) <u>The Problem of Existence of the General Chang Solutions</u>

There is no definite criterion for knowing whether physically acceptable solutions exist to the Chang eigenvalue conditions for any gauge theory that we may consider; unless, of course, we possess the foreknowledge that there exists a choice of coupling constants for which the theory is supersymmetric, as it has been explained.

On the other hand, it is in practise impossible to check explicitly that the Chang eigenvalue conditions, for any gauge theory, can be satisfied order by order to all orders in g - equivalently, that a (general) Chang solution does exist - though it is a relatively simple problem to check these conditions to lowest order, as explained in part (a) of this section.

The question that we shall investigate is whether the Chang eigenvalue conditions can be satisfied to all orders in perturbation theory by the general assumed Chang solution

$$h_i(t) = \sum_{r=1}^{\infty} \bar{h}_i^{(r)} g_{(t)}^{2r-1} \qquad i = 1,2\ldots p$$

$$\lambda_\alpha(t) = \sum_{r=1}^{\infty} \bar{\lambda}_\alpha^{(r)} g^{2r}(t) \qquad \alpha = 1,2\ldots q \tag{2.46}$$

if they are satisfied to lowest order by a given set of numbers $\bar{h}_i^{(1)}$ and $\bar{\lambda}_\alpha^{(1)}$; it will become clear later on why $h_i(t)$ and $\lambda_\alpha(t)$ are odd and even powers in $g(t)$, respectively.

This is essentially a qualified statement for the existence of general solutions to the Chang eigenvalue conditions for any gauge theory, given the set of lowest order solutions to this problem.

We proceed by assuming that the Chang eigenvalue conditions are satisfied up to some finite order (N-1), the coefficients $\bar{h}_i^{(1)}, \bar{h}_i^{(2)}\ldots\bar{h}_i^{(N-1)}$ and $\bar{\lambda}_\alpha^{(1)}\ldots\bar{\lambda}_\alpha^{(N-1)}$ having been determined, and consider whether the Chang conditions can be satisfied to the next order. That is, we shall consider whether the simultaneous equations obtained for the coefficients of the next order terms of (2.46), which we shall represent from now on by $X_i^{(N)}$ and $Z_\alpha^{(N)}$, is a consistent system of equations so that the Chang eigenvalue conditions for the Yukawa and self-quartic scalar coupling constants are satisfied up to the orders g^{2N+1} and g^{2N+2}, respectively, and the solutions obtained are given by (the t dependence of the coupling constants understood)

$$h_i = \sum_{r=1}^{N-1} \bar{h}_i^{(r)} g^{2r-1} + X_i^{(N)} g^{2N-1}$$

$$\lambda_\alpha = \sum_{r=1}^{N-1} \lambda_\alpha g^{2r} + Z_\alpha^{(N)} g^{2N} \tag{2.47}$$

for all $i = 1,2\ldots p$ and $\alpha = 1,2\ldots q$.

The Chang eigenvalue conditions up to the order N (i.e. to the orders g^{2N+1} for Yukawa and g^{2N+2} for the self-quartic scalar couplings) are obtained by substituting the tentative solution (2.47) in the differential equations (2.32) to (2.34) - which need be considered only up to the order of N loops (W = 1,2..N) - and we obtain for consistency

$$\left\{ \sum_{r=1}^{N-1} (2r-1)\, \bar{h}_i^{(r)}\, g^{2r-2} + (2N-1)\, X_i^{(N)}\, g^{2N-2} \right\} \times \left\{ \sum_{W=1}^{N} W\, b_W^{g}(g,h,\lambda)\Big|_{E.C} \right\} = \left\{ \sum_{W=1}^{N} W\, b_W^{h_i}(g,h,\lambda)\Big|_{E.C} \right\} \tag{2.48}$$

$$\left\{ \sum_{r=1}^{N-1} 2r\, \bar{\lambda}_\alpha^{(r)}\, g^{2r-1} + 2N\, Z_\alpha^{(N)}\, g^{2N-1} \right\} \times \left\{ \sum_{W=1}^{N} W\, b_W^{g}(g,h,\lambda)\Big|_{E.C} \right\} = \left\{ \sum_{W=1}^{N} W\, b_W^{\lambda_\alpha}(g,h,\lambda)\Big|_{E.C} \right\} \tag{2.49}$$

for all i = 1,2..p and α = 1,2..q; where (2.48) is an identity up to the order g^{2N+1} and (2.49) up to the order g^{2N+2}. The notation used is that for any function of the coupling constants $F(g,h,\lambda)$, $F(g,h,\lambda)/_{EC}$ means that we have substituted for h_i and λ_α the tentative solutions given by (2.47).

The identities (2.48) and (2.49) are satisfied up to the orders g^{2N-3} and g^{2N-2}, respectively, since by assumption

(2.47) with X = Z = 0 is a solution to the Chang eigenvalue problem up to the order (N-1). The system of simultaneous equations for the variables $X_i^{(N)}$ and $Z_\alpha^{(N)}$ are obtained by differentiating (2.48) 2N+1 times and (2.49) 2N+2 times with respect to g and then setting g = 0. We obtain

From (2.48)

$$\left\{ \frac{1}{(2N+1)!} \frac{d^{2N+1}}{dg^{2N+1}} \left[(2N-1)\, X_i^{(N)}\, g^{2N-2}\, b_1^{g}(g,h,\lambda)\Big|_{E.C} - \right.\right.$$

$$\left.\left. - b_1^{h_i}(g,h,\lambda)\Big|_{E.C} \right] \right\}\Bigg|_{g=0} =$$

$$= \left\{ \frac{1}{(2N+1)!} \frac{d^{2N+1}}{dg^{2N+1}} \sum_{W=2}^{N} W\, b_W^{h_i}(g,h,\lambda)\Big|_{E.C} \right\}\Bigg|_{g=0} -$$

$$- \left\{ \frac{1}{(2N+1)!} \frac{d^{2N+1}}{dg^{2N+1}} \left[\left(\sum_{t=1}^{N-1} (2t-1)\, \bar{h}_i^{(t)}\, g^{2t-2} \right) \sum_{W=1}^{N} W\, b_W^{g}(g,h,\lambda)\Big|_{E.C} \right] \right\}\Bigg|_{g=0}$$

for all $i = 1,2\ldots p$ (2.50)

From (2.49)

$$\left\{ \frac{1}{(2N+2)!} \frac{d^{2N+2}}{dg^{2N+2}} \left[2N\, Z_\alpha^{(N)}\, g^{2N-1}\, b_1^{g}(g,h,\lambda)\Big|_{E.C} - b_1^{\lambda_\alpha}(g,h,\lambda)\Big|_{E.C} \right] \right\}\Bigg|_{g=0}$$

$$= \left\{ \frac{1}{(2N+2)!} \frac{d^{2N+2}}{dg^{2N+2}} \sum_{W=2}^{N} W\, b_W^{\lambda_\alpha}(g,h,\lambda)\Big|_{E.C} \right\}\Big|_{g=0} -$$

$$- \left\{ \frac{1}{(2N+2)!} \frac{d^{2N+2}}{dg^{2N+2}} \left[\left(\sum_{r=1}^{N-1} 2r\, \bar{\lambda}_\alpha^{(r)}\, g^{2r-1} \right) \sum_{W=1}^{N} W\, b_W^{g}(g,h,\lambda)\Big|_{E.C} \right] \right\}\Big|_{g=0}$$

for all $\alpha = 1,2\ldots q$ (2.51)

since

$$\left\{ \frac{d^{2N+1}}{dg^{2N+1}} \left[g^{2N-2} \sum_{W=2}^{N} W\, b_W^{g}(g,h,\lambda)\Big|_{E.C} \right] \right\}\Big|_{g=0} = 0$$

We shall now consider the $X_i^{(N)}$ and $Z_\alpha^{(N)}$ dependence of the equations (2.50) and (2.51). The equations (2.50) are expressed in terms of the functions $b_W^g(g,h,\lambda)$ and $b_W^{h_i}(g,h,\lambda)$ for $1 \leq W \leq N$. They are homogeneous functions in the coupling constants with a typical term being $g^s\, h^m\, \lambda^n$ where $s+m+2n \equiv 2W+1$. When we substitute the tentative solution (2.47) we obtain

$g^s\, h^m\, \lambda^n \rightarrow$ (2.47)

$$\rightarrow g^s\, g^m\, g^{2n} \left[\sum_{r=1}^{N-1} \bar{h}_i^{(r)}\, g^{2r-2} + X_i^{(N)}\, g^{2N-2} \right]^m \left[\sum_{r=1}^{N-1} \bar{\lambda}_\alpha^{(r)}\, g^{2r-2} + Z_\alpha^{(N)}\, g^{2N-2} \right]^n$$

hence, for $s+m+2n = 2W+1$

$$\left(g^s\, h^m\, \lambda^n\right)\Big|_{E.C} = g^{2W+1} \left\{ \left[\left(\sum_{r=1}^{N-1} \bar{h}_i^{(r)}\, g^{2r-2} \right)^m + \right.\right.$$

$$+ m\, X_i^{(N)}\, g^{2N-2}\left(\sum_{r=1}^{N-1} \bar{h}_i^{(r)}\, g^{2r-2}\right)^{m-1} + O(g^{4N-4}) + .\Bigg] \times \Bigg[\left(\sum_{r=1}^{N-1} \bar{\lambda}_\alpha^{(r)}\, g^{2r-2}\right)^{n} +$$

$$+ n\, Z_\alpha^{(N)}\, g^{2N-2}\left(\sum_{r=1}^{N-1} \bar{\lambda}_\alpha^{(r)}\, g^{2r-2}\right)^{n-1} + O(g^{4N-4}) + ..\Bigg]\ \Bigg\} \tag{2.52}$$

We are interested in terms of order g^{2N+1} hence we observe that the only terms in $X_i^{(N)}$ and $Z^{(N)}$ that contribute to this order are linear in X or Z and only for W = 1. That is only the <u>one loop</u> contributions to equation (2.50) have X and Z dependence. Furthermore for W = N the only contribution of interest arises for the value of the brackets given by r = 1, i.e. b_N^g and $b_N^{h_i}$ are evaluated effectively at $h_i = \bar{h}_i^{(1)}\, g$ and $\lambda_\alpha = \bar{\lambda}_\alpha^{(1)}\, g^2$.

When we consider $b_W^{\lambda_\alpha}(g,h,\lambda)/_{E.C.}$ we obtain similar results but for the terms of order g^{2N+2} since, in this case, $r+m+2n \equiv 2W+2$ for the terms $g^r\, h^m\, \lambda^n$.

We can also justify now why the general solutions (2.46) for the Yukawa and scalar quartic coupling constants are odd and even power series in g respectively. We conclude from (2.52) that both the left and right hand sides of equation (2.48) are odd power series in g so we obtain a consistent system for the assumed solutions (2.46). Similarly we can conclude that both the left and right hand sides of (2.49) are even power series in g. In fact a more detailed study yields that these are the most general solutions and only in exceptional cases can we possibly "accomodate" a general power series in g for the scalar quartic coupling constants.

It is known from the one loop studies of any gauge theory that:

$$b_1^g(g,h,\lambda) \equiv b_1^g\, g^3 \quad (b_1^g \text{ just a constant})$$

and

$$b_1^{h_i}(g,h,\lambda) \equiv b_1^{h_i}(g,h) \quad (\text{independent of } \lambda)$$

hence we observe that (2.50) is of the form

$$A_{ij}^{\underline{Yu}}\, X_j^{(N)} = C_i^{\underline{Yu}} \tag{2.53}$$

where

$$A_{ij}^{\underline{Yu}} \equiv (2N-1)\, b_1^g\, \delta_{ij} - \left\{ \frac{\partial}{\partial X_j^{(N)}} \left[\frac{1}{(2N+1)!} \frac{d^{2N+1}}{dg^{2N+1}}\, b_1^{h_i}(g,h)\Big|_{E.C} \right] \right\}\Bigg|_{\substack{X^{(N)}=0\\ g=0}}$$

and

$$C_i^{\underline{Yu}} \equiv N \left\{ b_N^{h_i}(1,\bar{h}^{(1)},\bar{\lambda}^{(1)}) - \bar{h}_i^{(1)}\, b_N^g(1,\bar{h}^{(1)},\bar{\lambda}^{(1)}) \right\} +$$

$$+ \left\{ \frac{1}{(2N+1)!} \frac{d^{2N+1}}{dg^{2N+1}}\, b_1^{h_i}(g,h)\Big|_{E.C} \right\}\Bigg|_{\substack{X^{(N)}=0\\ g=0}} +$$

$$+ \left\{ \frac{1}{(2N+1)!} \frac{d^{2N+1}}{dg^{2N+1}} \sum_{W=2}^{N-1} w\, b_w^{h_i}(g,h,\lambda)\Big|_{E.C} \right\}\Bigg|_{g=0} -$$

$$- \left\{ \frac{1}{(2N+1)!} \frac{d^{2N+1}}{dg^{2N+1}} \left[\left(\sum_{\tau=1}^{N-1} (2\tau-1) \bar{h}_i^{(\tau)} g^{2\tau-2} \right) \sum_{W=2}^{N-1} w\, b_w^{g}(g,h,\lambda)\Big|_{E.C} \right] \right\} \Big|_{g=0}$$

Similarly (2.51) is of the form

$$B_{\alpha\beta}^{\underline{sc}} z_{\beta}^{(N)} + B_{\alpha j}^{\underline{Yu}} x_j^{(N)} = C_{\alpha}^{\underline{sc}} \qquad (2.54)$$

where

$$B_{\alpha\beta}^{\underline{sc}} \equiv 2N b_1^{g} \delta_{\alpha\beta} - \left\{ \frac{\partial}{\partial z_{\beta}^{(N)}} \left[\frac{1}{(2N+2)!} \frac{d^{2N+2}}{dg^{2N+2}} b_1^{\lambda\alpha}(g,h,\lambda)\Big|_{E.C} \right] \right\}_{g=0}$$

$$B_{\alpha j}^{\underline{Yu}} \equiv - \left\{ \frac{\partial}{\partial x_j^{(N)}} \left[\frac{1}{(2N+2)!} \frac{d^{2N+2}}{dg^{2N+2}} b_1^{\lambda\alpha}(g,h,\lambda)\Big|_{E.C} \right] \right\}_{g=0}$$

and

$$C_{\alpha}^{\underline{sc}} \equiv N \left\{ b_N^{\lambda\alpha}(1,\bar{h}^{(1)}, \bar{\lambda}^{(1)}) - 2 \bar{\lambda}_{\alpha}^{(1)} b_N^{g}(1,\bar{h}^{(1)}, \bar{\lambda}^{(1)}) \right\} +$$

$$+ \left\{ \frac{1}{(2N+2)!} \frac{d^{2N+2}}{dg^{2N+2}} b_1^{\lambda\alpha}(g,h,\lambda)\Big|_{E.C} \right\} \Bigg|_{\substack{z^{(N)}=0 \\ x^{(N)}=0, \ \& \ g=0}} +$$

$$+ \left\{ \frac{1}{(2N+2)!} \frac{d^{2N+2}}{dg^{2N+2}} \sum_{W=2}^{N-1} w\, b_w^{\lambda\alpha}(g,h,\lambda)\Big|_{E.C} \right\}_{g=0} +$$

$$- \left\{ \frac{1}{(2N+2)!} \frac{d^{2N+2}}{dg^{2N+2}} \left[\left(\sum_{\tau=1}^{N-1} (2\tau-1) \bar{h}_i^{(\tau)} g^{2\tau-2} \right) \sum_{W=2}^{N-1} w\, b_w^{g}(g,h,\lambda)\Big|_{E.C} \right] \right\}_{g=0}$$

Finally we conclude that the Chang eigenvalue conditions are satisfied up to the order N if solutions exist to the system of linear simultaneous equations (2.53) and (2.54).

If C_i^{Yu} and C_α^{sc} are equal to zero for all $i = 1,2..p$ and $\alpha = 1,2..q$ then the Chang eigenvalue conditions are satisfied trivially to this order. On the other hand if C_i^{Yu} and $\{ C_\alpha^{sc} - B_{\alpha j}^{Yu} X_j^{(N)} \}$ are different from zero, where the variables $X_j^{(N)}$ are determined from equation (2.53), then we obtain the conditions

$$\det A^{Yu} \neq 0 \quad \text{and} \quad \det B^{sc} \neq 0 \tag{2.55}$$

for the Chang eigenvalue conditions to be satisfied to this order.

The conditions (2.55) are very useful practically because the matrices A_{ij}^{Yu} and $B_{\alpha\beta}^{sc}$ can be determined from the one loop contributions to the differential equations for the effective coupling constants of the theory. Furthermore, they are simple functions of N, the number of loops considered, and we may check easily whether the conditions (2.55) are satisfied to all orders and if not: what orders may present problems and must be considered explicitly in order to determine that the system of simultaneous linear equations (2.53) and (2.54) is consistent.

In fact - although in general the conditions (2.55) have to be checked out explicitly for each lowest order solution obtained - a closer look at the matrix A_{ij}^{Yu} and from the standard results of the conventional analysis for asymptotic freedom[17] we find that, when we consider a solution such that <u>all</u> the Yukawa couplings vanish asymptotically like the gauge coupling constant, i.e. $\bar{h}_i^{(1)} \neq 0$ for all $i = 1,2..p$, then the determinant of the matrix A_{ij}^{Yu} is different from zero for all $N = 2,3..\infty$.

This may be seen by noting, as it can be easily deduced from the expression (2.52), that the matrix A_{ij}^{Yu} can be written in the following equivalent form

$$A_{ij}^{Yu} = 2(N-1)\, b_1^g \delta_{ij} + \left\{ b_1^g \delta_{ij} - \frac{\partial}{\partial \bar{h}_j^{(1)}} b_1^{h_i}(1, \bar{h}^{(1)}) \right\} \tag{2.56}$$

For the effective gauge coupling constant to vanish asymptotically ($t \to \infty$) we have the condition that $b_1^g > 0$. Hence a sufficient condition for $\det A^{Yu} \neq 0$ for all $N = 2,3..\infty$ is that the eigenvalues of the matrix given within the brackets of expression (2.56) are all positive. Furthermore, from the previous work on the subject[17] it follows that this condition is satisfied when $\bar{h}_i^{(1)} \neq 0$ for all $i = 1,2..p$ which is the type of solution of particular interest[1], if it exists ($\bar{h}_i^{(1)}$ real), to the eigenvalue conditions for the Yukawa couplings.

An argument along parallel lines may also be given for the matrix B^{Sc} of equation (2.54) but it is essentially of no practical use because the lowest order eigenvalue conditions for the scalar quartic couplings and solutions $\bar{\lambda}_\alpha^{(1)}$ lack the simplicity of those for the Yukawa couplings. Even so, it is worth writing the matrix B^{sc} in the equivalent form, as it can also be easily deduced from the expression (2.52),

$$B^{sc}_{\alpha\beta} = 2(N-1)\, b_1^{g}\, \delta_{\alpha\beta} + \left\{ 2\, b_1^{g}\, \delta_{\alpha\beta} - \frac{\partial}{\partial \bar{\lambda}^{(1)}_{\beta}}\, b_1^{\lambda_\alpha}(1, \bar{h}^{(1)}, \bar{\lambda}^{(1)}) \right\} \tag{2.57}$$

And in similar way we obtain that for det $B^{sc} \neq 0$, for all $N = 2,3..\infty$, a sufficient condition is that the real eigenvalues of the matrix within the brackets of expression (2.57), are all positive. What is rather interesting is to compare these sufficient conditions for the determinants of the matrices A^{Yu} and B^{sc}, for any given lowest order solution $(\bar{h}_i^{(1)}, \bar{\lambda}_\alpha^{(1)})$, to be different from zero to all orders N, with the conditions obtained in the conventional analysis for asymptotic freedom[17]: which state that a theory is asymptotically free if a solution exists $(\bar{h}_i^{(1)}, \bar{\lambda}_\alpha^{(1)})$ such that all the eigenvalues of the matrices "within the brackets" of expressions (2.56) and (2.57) have negative real parts.

In most cases, therefore, we would only need to check whether the determinant of the matrix $B^{sc}_{\alpha\beta}$ is different from zero to all orders N, for every solution $\bar{\lambda}_\alpha^{(1)}$, $\bar{h}_i^{(1)}$ obtained from the Chang eigenvalue conditions of the theory considered to lowest order - having that $\bar{h}_i^{(1)} \neq 0$ for all $i = 1,2..p$, as often needs to be the case[1].

If this sufficient condition is satisfied, then the higher order terms to these solutions are obtained formally from the equations (2.53) and (2.54), and the Chang eigenvalue conditions are satisfied to all orders.

On the other hand, if some lowest order solution does not satisfy this condition for at least one value of N, then the status of this lowest order solution is not clear. It would be necessary to consider that given order N explicitly to determine whether the coefficient matrix $B^{sc}_{\alpha\beta}$ and augmented matrix ($B^{sc}_{\alpha\beta}$, $C^{sc}_{\alpha} - B^{Yu}_{\alpha j} X^{(N)}_j$) of the equation (2.54), where the variables $X^{(N)}_j$ have been determined from equation (2.53), have the same rank so that the system of simultaneous equations (2.54) is consistent. Otherwise that particular lowest order solution does not correspond to a solution satisfying the Chang eigenvalue conditions to all orders and must be rejected. The only cases known for which $C^{sc}_{\alpha} = 0 = C^{Yu}_{i}$, being non-incidental and to all orders, are the solutions corresponding to supersymmetric theories and no problems are envisaged for these solutions. We shall return to this problem for a specific Lagrangian model considered in section IV.

SECTION III

MODEL I - A ONE LOOP STUDY

In section II it was shown formally that the Chang eigenvalue conditions, for a given Lagrangian, are satisfied by any choice of coupling constants known to be such that the corresponding Lagrangian describes a supersymmetric theory[24].

In this section we shall calculate to the order of one loop the β-functions of the differential equations for the effective coupling constants of a particular SU(2)-gauge Lagrangian theory. This Lagrangian describes for a given choice of coupling constants the supersymmetric model I, given in section I.b by the Lagrangian $\mathcal{L}_{SI}$ in the special gauge for the superfield ψ.

We wish to show here explicitly that the Chang eigenvalue conditions for the theory are satisfied to the one loop approximation by the given supersymmetric choice of coupling constants.

We have seen that this is equivalent to showing that the one loop infinite contributions of all the vertices of the supersymmetric theory can be absorbed into a single redefinition for the gauge coupling constant of the theory only. We shall naturally extend, therefore, our analysis to the complete study of the one-loop renormalizability of the supersymmetric theory.

This section is divided into two parts:

In part (a) we consider the one loop renormalizability of the supersymmetric theory and show that the supersymmetric constraints on the masses and coupling constants of the theory are preserved by the renormalization.

In part (b) we raise the question of uniqueness of the supersymmetric case as a solution to the Chang eigenvalue conditions for the theory and investigate whether any other physically acceptable solutions exist.

(a) <u>The One-Loop Renormalizability of Supersymmetric Model I</u>

The particular SU(2)-gauge invariant Lagrangian that we consider is constructed using the same "matter" multiplets as for the Lagrangian $\mathcal{L}_{IS}$

$$\begin{aligned}\mathcal{L}_I = & -\frac{1}{4}\left(\partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu + g\,\vec{W}_\mu \wedge \vec{W}_\nu\right)^2 + i\,\bar{\Psi}\,\gamma^\mu D_\mu \Psi + \\ & + \frac{i}{2}\,\bar{\chi}^i \gamma^\mu\left(\partial_\mu \chi^i + g\,\epsilon^{ijk} W^j_\mu \chi^k\right) + (D_\mu A)^\dagger (D^\mu A) + \\ & + (D_\mu B)^\dagger (D^\mu B) - M\,\bar{\Psi}\Psi - M^2 A^\dagger A - M^2 B^\dagger B \\ & + i h_1 A^\dagger \bar{\chi}^i \tau^i \Psi - i h_1 \bar{\Psi}\tau^i \chi^i A + i h_2 B^\dagger \bar{\chi}^i \tau^i \gamma_5 \Psi - i h_2 \bar{\Psi}\tau^i \gamma_5 \chi^i B \\ & - \lambda_1 (A^\dagger . A)^2 - \lambda_2 (B^\dagger . B)^2 - \lambda_3 (A^\dagger . A)(B^\dagger . B) \\ & - \lambda_4 (A^\dagger . B)(B^\dagger . A) - \lambda_5 (B^\dagger . A)^2 - \lambda_5 (A^\dagger . B)^2\end{aligned}$$

where $$D_\mu = \partial_\mu - i g \frac{\tau^i}{2} W^i_\mu$$

Apart from the supersymmetric constraints on the masses, this is the most general renormalizable SU(2)-gauge invariant Lagrangian that can be constructed using the given set of matter fields; parity being conserved. Furthermore this Lagrangian is identical to the supersymmetric Lagrangian $\mathcal{L}_{SI}$, by construction, for the following choice of coupling constants

$$h_1 = h_2 = g \quad , \quad \lambda_1 = \lambda_2 = 0 \quad , \quad \lambda_3 = \frac{g^2}{2} \quad , \quad \lambda_4 = 2\lambda_5 = -\frac{g^2}{4} \qquad (3.1)$$

We consider here the one loop renormalizability of the supersymmetric theory, $\mathcal{L}_{SI}$. Our calculations will be carried out though in terms of the arbitrary coupling constants of the Lagrangian $\mathcal{L}_I$ and the counterterms obtained that render $\mathcal{L}_I$ finite will be subsequently considered at the supersymmetric point (3.1), in the space of coupling constants. We follow this procedure rather than considering $\mathcal{L}_{SI}$ directly so that we can determine simply afterwards the Chang eigenvalue conditions for the Lagrangian $\mathcal{L}_I$.

A gauge fixing term is added to the Lagrangian $\mathcal{L}_I$ and also the effective Lagrangian for the compensating ghost fields obtained in the usual way[27,28]

$$\mathcal{L}_G = -\frac{1}{2\alpha}\left(\partial_\mu \vec{W}_\mu\right)^2 - \xi^i \Box \eta^i - \epsilon^{ijk} g\, \xi^i \partial_\mu \left(W_\mu^j \eta^k\right)$$

The counterterms that must be added to the renormalized Lagrangian $\mathcal{L}_I + \mathcal{L}_G$ are obtained in the standard way by

$$\delta \mathcal{L}_c = \mathcal{L}_I^{\circ} + \mathcal{L}_G^{\circ} - \left(\mathcal{L}_I + \mathcal{L}_G\right)$$

where $\mathcal{L}_I^{o} + \mathcal{L}_G^{o}$ is the bare Lagrangian of the theory, with the unrenormalized parameters distinguished from the renormalized by the superscript (0). The notation we use is:

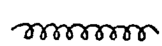
(1) for the wavefunction renormalizations $\phi = (1+Z_2^{\phi})^{1/2}\,\phi^{\circ}$, where ϕ stands for any field of the theory

(2) the subtraction of the proper vertex diagrams for any given vertex of interaction-strength χ is represented by χZ_1^{χ}, where χ stands for any coupling constant of the theory, and

(3) multiplicative renormalization for the mass given by $M_o = (1+Z_M)M$. The mass counterterms obtained this way will not render the masses finite for the Lagrangian $\mathcal{L}_I$. This is immaterial though because we are interested in the mass counterterms only at the supersymmetric point (3.1).

We use the regularization method of analytically continuing the dimensions of space-time(12). The infinite subtraction constants are determined by the requirement that the coupling constants, masses and wavefunctions of the Lagrangian $\mathcal{L}_I + \mathcal{L}_G + \delta\mathcal{L}_C$ be finite in 4-dimensions. A general list of the momentum integrals that are met in our calculations are given in the Appendix III.

We shall work in the Landau gauge given by $\alpha = 0$, and use the following graphical representation for the fields of the theory.

W-field:		ψ-field:	————————
Ghost-field:		A-field:	---------
χ-field:		B-field:	=========

The Slavnov-Taylor(29) identities for the theory guarantee that a single redefinition of the bare gauge coupling constant g^o renders finite all the vertices of interaction-strength g^o of the Lagrangian $\mathcal{L}_I^o + \mathcal{L}_G^o$. We may, therefore, consider the renormalization of the gauge coupling constant for the interaction vertex $\bar{\psi}$-ψ-W only. The contributing diagrams to the order of one loop are given in Figure I.

It is found that the infinite contributions to the gauge coupling constant renormalization from the diagrams in group (b), and similarly for group (c), of Fig. I add up to zero. This is a general feature (and necessary for the renormalization) of gauge theories which we have already used in section II.

We obtain that for the gauge vertices to be rendered finite to this order the unrenormalized gauge coupling constant is given in terms of the renormalized by

$$g^o = \left(g + \frac{5\, g^3}{16\pi^2(n-4)} \right) \tag{3.2}$$

The relevant diagrams for the renormalization of the coupling constants of the Yukawa and self-quartic scalar vertices to the order of one loop are given in Figures II to VII. We shall firstly consider, though, the wavefunction and mass renormalizations for the fields of the theory to this order.

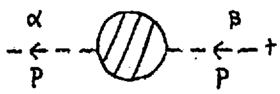
<u>The self-energy parts of the fields</u>

We observe that the supersymmetric Lagrangian $\mathcal{L}_{SI}$ is invariant under the discreet transformation

$$\chi \rightarrow \gamma_5 \chi \ , \qquad A \rightarrow B \quad \& \quad B \rightarrow -A \tag{3.3}$$

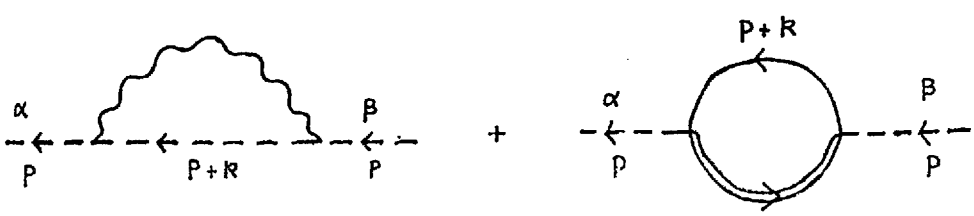
This guarantees that the Majorana field χ remains massless to all orders and also that the masses and wavefunction renormalizations of the scalar and pseudoscalar fields A and B are equal to all orders.

The wavefunction and mass renormalizations of the A-field are defined by



$$+\, i\, Z_2^A\,(p^2 - M^2)\,\delta_{\alpha\beta} - 2i\, Z_M\, M^2\,\delta_{\alpha\beta} = \text{finite at } (n=4)$$

and we need to calculate only the infinite contributions of the one loop self-energy diagrams of the A-field. Using the Feyman rules of the theory we have



$D_A(1)$ $\qquad\qquad$ $D_A(2)$

$$D_A(1) = -g^2 \left(\frac{\tau^i\tau^i}{4}\right)_{\alpha\beta} \int d^n k \; \frac{(2P+k)_\mu\,(2P+k)_\nu}{[\,(p+k)^2 - M^2\,]} \; \frac{(g_{\mu\nu} - k_\mu k_\nu/k^2)}{k^2}$$

$$= -g^2 \left(\frac{\tau^i\tau^i}{4}\right)_{\alpha\beta} \int_0^1 dx \int d^n k \left\{ \frac{4P^2}{[\,k^2 + 2x\,p.k + x\,p^2 - x M^2\,]^2} - \frac{4(P.k)^2\,(1-x)\,\Gamma(3)}{[\,k^2 + 2x\,p.k + x\,p^2 - x M^2\,]^3} \right\}$$

performing the integrations and neglecting terms finite at $n = 4$.

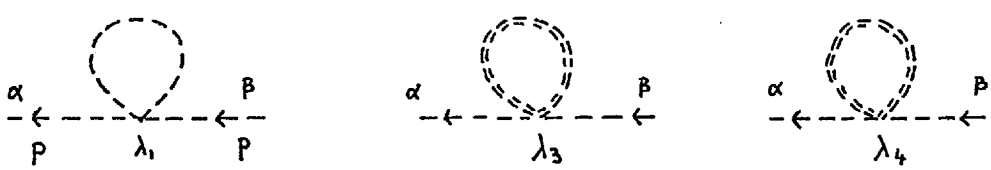
$$\text{Inf.}\, D_A(1) = \frac{6\,i\,g^2}{16\,\pi^2\,(n-4)} \left(\frac{\tau^i\tau^i}{4}\right)_{\alpha\beta} P^2 = \frac{i\,9\,g^2\,\delta_{\alpha\beta}}{32\,\pi^2\,(n-4)} \left\{ (p^2 - M^2) + M^2 \right\}$$

$$D_A(2) = -h_1^2\,(\tau^i\tau^i)_{\alpha\beta}\int d^n k\;\frac{Tr[(\not p+\not k)\not k]}{k^2\,[(p+k)^2-M^2]}$$

$$Inf.\,D_A(2) = \frac{8\,i\,h_1^2}{16\pi^2(n-4)}\,(\tau^i\tau^i)_{\alpha\beta}\left\{-\frac{p^2}{2}+M^2\right\} =$$

$$= \frac{24\,i\,h_1^2\,\delta_{\alpha\beta}}{16\pi^2(n-4)}\left\{-\frac{1}{2}(p^2-M^2)+\frac{M^2}{2}\right\}$$

plus the diagrams



Contributing =

$$\{6\lambda_1 + 2\lambda_3 + \lambda_4\}\int d^n k\;\frac{\delta_{\alpha\beta}}{[k^2-M^2]}$$

performing the loop integration and neglecting finite terms at n = 4, we obtain the infinite contribution

$$\frac{-2i\,\{6\lambda_1+2\lambda_3+\lambda_4\}}{16\,\pi^2\,(n-4)}\;M^2\,\delta_{\alpha\beta}$$

Adding up all the infinite self-energy contributions for the A-field we obtain that

$$Z_2^A = \frac{1}{16\pi^2(n-4)}\left\{-\frac{9}{2}g^2 + 12\,h_1^2\right\} \qquad (3.4a)$$

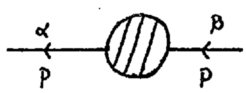
$$2 Z_M = \frac{1}{16\pi^2 (n-4)} \left\{ \frac{9}{2} g^2 + 12 h_1^2 - 2(6\lambda_1 + 2\lambda_3 + \lambda_4) \right\} \qquad (3.5a)$$

The wavefunction and mass renormalization for the B-field are determined by a similar set of diagrams to those of the A-field. Furthermore it is easily deduced that the self-energy infinite contributions for the B-field may be obtained simply from those determined for the A-field by the substitution $\lambda_1 \rightarrow \lambda_2$ and $h_1 \rightarrow h_2$ (note that $\gamma_5^2 = -1$ for the convention used). We obtain

$$Z_2^B = \frac{1}{16\pi^2 (n-4)} \left\{ -\frac{9}{2} g^2 + 12 h_2^2 \right\} \qquad (3.4b)$$

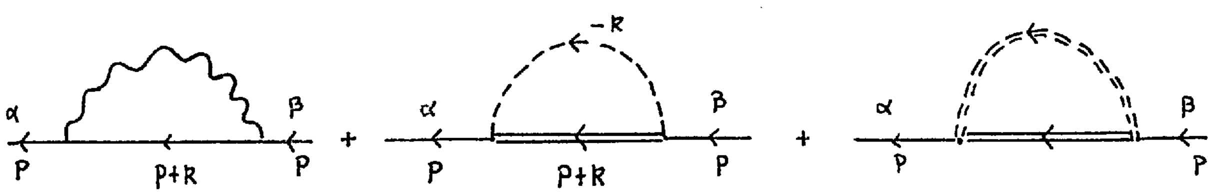
$$2 Z_M = \frac{1}{16\pi^2 (n-4)} \left\{ \frac{9}{2} g^2 + 12 h_2^2 - 2(6\lambda_2 + 2\lambda_3 + \lambda_4) \right\} \qquad (3.5b)$$

The wavefunction and mass renormalization of the Dirac spinor ψ are defined by



$+ i Z_2^{\psi} (\not{p} - M) \delta_{\alpha\beta} - i Z_M M \delta_{\alpha\beta}$ = finite at (n = 4)

where the one loop self-energy diagrams contributing are



$D_\psi(1)$ $\qquad$ $D_\psi(2)$ $\qquad$ $D_\psi(3)$

$$D_\psi(1) = -g^2\left(\frac{\tau^i\tau^i}{4}\right)_{\alpha\beta}\int d^n k\ \gamma^\mu \frac{(\not p+\not k+M)}{[(p+k)^2-M^2]}\gamma^\nu\ \frac{(g_{\mu\nu}-k_\mu k_\nu/k^2)}{k^2}$$

$$= -g^2\left(\frac{\tau^i\tau^i}{4}\right)_{\alpha\beta}\int_0^1 dx\int d^n k\left\{\frac{(n-1)M+(3-n)\not p+(1-n)\not k}{[k^2+2x\,p.k+x\,p^2-x\,M^2]^2} - \frac{2\not k\,(k.p)\,(1-x)\,\Gamma(3)}{[k^2+2x\,p.k+x\,p^2-x\,M^2]^3}\right\}$$

performing the momentum integrations and neglecting terms finite at n = 4 we obtain

$$\text{Inf.}\,D_\psi(1) = \frac{6\,i\,g^2}{16\pi^2(n-4)}\left(\frac{\tau^i\tau^i}{4}\right)_{\alpha\beta} M = \frac{9\,i\,g^2}{32\pi^2(n-4)}\delta_{\alpha\beta}\,M$$

$$D_\psi(2) = h_1^2\,(\tau^i\tau^i)_{\alpha\beta}\int d^n k\ \frac{\not p+\not k}{(p+k)^2\,(k^2-M^2)}$$

$$\text{Inf.}\,D_\psi(2) = \frac{-i\,h_1^2}{16\pi^2(n-4)}(\tau^i\tau^i)_{\alpha\beta}\,\not p = \frac{-3\,i\,h_1^2\,\delta_{\alpha\beta}}{16\pi^2(n-4)}\{(\not p-M)+M\}$$

The contribution of the third diagram, $D_\psi(3)$, can be obtained from that of $D_\psi(2)$ by the substitution $h_1 \to h_2$. Adding up all the infinite contributions of the above three diagrams we obtain that

$$Z_2^\psi = \frac{3}{16\pi^2(n-4)}\{h_1^2+h_2^2\} \qquad (3.4c)$$

$$Z_M = \frac{1}{16\pi^2(n-4)}\left\{\frac{9}{2}g^2 - 3(h_1^2+h_2^2)\right\} \qquad (3.5c)$$

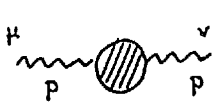
It has been pointed out that the Majorana field χ remains massless in the supersymmetric theory, to all orders, because of its invariance under the discrete transformation (3.3). Hence we consider only the infinite wavefunction renormalization for the χ-field of the Lagrangian $\mathcal{L}_I$, determined by a similar set of diagrams to those considered for the Dirac spinor ψ.

In the Landau gauge, only the two self-energy diagrams for the field χ similar to $D_\psi(2)$ and $D_\psi(3)$ contribute to Z_2^χ and we note that these contributions may be obtained from those of $D_\psi(2)$ and $D_\psi(3)$ by the substitution $(\tau^i\tau^i)_{\alpha\beta} \to 2\,\mathrm{Tr}(\tau^i\tau^i)$ The factor 2 is the combinatorial factor for the two ways of "attaching" the external χ-lines. We obtain that

$$Z_2^\chi = \frac{4}{16\pi^2(n-4)}\left\{ h_1^2 + h_2^2 \right\} \tag{3.4d}$$

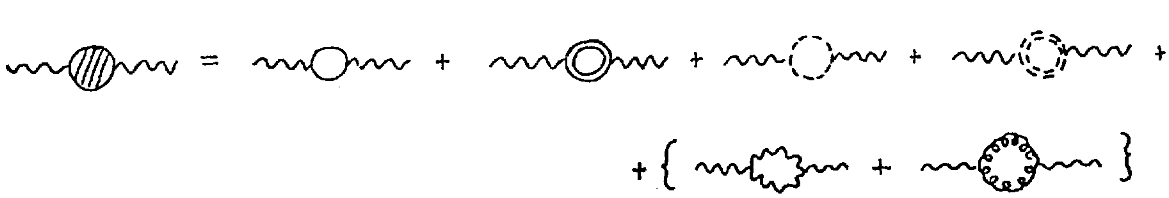
We also give for "completeness" the wavefunction renormalization of the gauge field W .

The gauge field remains massless to all orders by the gauge invariance of the theory and in particular only the transverse part of the propagator is renormalized. The infinite wavefunction renormalization is determined by



$$- i\, Z_2^W \left(g^{\mu\nu} p^2 - p^\mu p^\nu \right) = \text{finite at } (n=4)$$

where



and hence (in the same order as the contributing diagrams)

$$Z_2^W = \frac{1}{16\pi^2(n-4)}\left\{\frac{4}{3}g^2 + \frac{8}{3}g^2 + \frac{1}{3}g^2 + \frac{1}{3}g^2 - \frac{26}{3}g^2\right\} = \frac{-4g^2}{16\pi^2(n-4)} \qquad (3.4e)$$

We may now evaluate the mass and wavefunction renormalizations obtained at the supersymmetric point (3.1). We find that the mass renormalization Z_M, determined separately for each of the fields A, B and ψ (3.5a - 3.5c), is consistently given by

$$Z_M = \frac{3g^2}{16\pi^2(n-4)}$$

We conclude, therefore, that a single mass renormalization renders all masses of the supersymmetric theory finite to the order of one loop. That is the constraints on the masses of the supersymmetric theory are preserved by the one loop renormalizability of the theory.

On the other hand we obtain for the wavefunction renormalizations of the fields (3.4a - 3.4e), at the supersymmetric point (3.1), that

$$Z_2^B = Z_2^A \neq Z_2^\psi \qquad \text{and} \qquad Z_2^\chi \neq Z_2^W$$

Hence these cannot be absorbed into two overall wavefunction renormalizations, one for each superfield of the theory. However since the Lagrangian $\mathcal{L}_{SI}$ in the special gauge used for the gauge superfield ψ is not "manifestly supersymmetric", being invariant only under ordinary gauge transformations[8,10], this presents no problems. We shall also see shortly that the supersymmetric constraints on the coupling constants of the theory are preserved by the renormalization. This is a similar set of results to what has been obtained by Wess and Zumino[8] for a different supersymmetric Yang Mills theory and, hence, not completely unexpected.

<u>The coupling constant renormalizations</u>

We shall now determine the bare coupling constants, in terms of the renormalized, so that all the vertices of the Lagrangian $\mathcal{L}_I$ are finite to the one loop approximation.

We have already determined the wavefunction renormalizations of the fields and the bare gauge coupling constant in terms of the renormalized one. We only need therefore to compute the subtraction constants Z_1's which render finite the proper vertex diagrams of the Yukawa and scalar quartic vertices. It is a simple matter then to obtain the corresponding bare coupling constants and consider whether these are consistent with the renormalization of the gauge coupling constant (3.2) at the supersymmetric point (3.1).

The masses of the fields are neglected, for convenience, in our calculations since the coupling constant infinite renormalizations are independent of the masses of the theory[6,13].

The proper vertex diagrams for the Yukawa vertices are given by the groups (a) of Fig. II and III. Using the Feyman rules of the theory, we obtain for the group (a) of Fig. II

For diagram 1:

$$\Gamma_{h_1}(1) = \frac{g^2 h_1}{2} (\tau^\ell \tau^m)_{\alpha\beta}\, \epsilon^{m\ell i} \int d^n k\ \gamma^\mu \frac{(\not p_2 + \not k)}{(p_2+k)^2} \frac{(\not p_2 - \not p_1 + \not k)}{(p_2 - p_1 + k)^2} \gamma^\nu \frac{(g_{\mu\nu} - k_\mu k_\nu / k^2)}{k^2}$$

$$= \frac{g^2 h_1}{4} [\tau^\ell, \tau^m]_{\alpha\beta}\, \epsilon^{m\ell i} \left\{ \int_0^1 dx \int d^n k \frac{n-1}{[k^2 - 2x p\cdot k + x p^2]^2} + \text{finite contrib} \right\}$$

performing the integrations and neglecting terms finite at $n = 4$.

$$\text{Infin.}\,\Gamma_{h_1}(1) = \frac{-6 g^2 h_1\ \tau^i_{\alpha\beta}}{16\pi^2\ (n-4)}$$

For diagram 2:

$$\Gamma_{h_1}(2) = i\, h_1^3 (\tau^m \tau^i . \tau^m)_{\alpha\beta} \int d^n k \left\{ \frac{\not k C}{k^2} \frac{(\not k + \not p_1)^T}{(k+p_1)^2} C^{-1^T} \frac{1}{(k+p_2)^2} \right\}$$

$$= -i\, h_1^3\, \tau^i_{\alpha\beta} \left\{ \int_0^1 dx \int d^n k \frac{1}{[k^2 + 2x p_1 \cdot k + x p_1^2]^2} + \text{finite contribution} \right\}$$

where we have used that $C \gamma_\mu^T C^{-1^T} = \gamma_\mu$. Performing the integrations and neglecting terms finite at $n = 4$ we obtain

$$\text{Infin.}\,\Gamma_{h_1}(2) = \frac{-2 h_1^3\ \tau^i_{\alpha\beta}}{16\pi^2 (n-4)}$$

The contribution of the third diagram of this group (a) may be obtained from that of the second diagram by the substitution

$h_1^3 \to -h_1 h_2^2$. This is easily derived using the properties of the γ_5-matrix; $[C, \gamma_5]_- = 0$, $\gamma_5^T = \gamma_5$

$$\text{Infin.}\ \Gamma_{h_1}(3) = \frac{2\, h_1 h_2^2\, I^i_{\alpha\beta}}{16\pi^2\,(n-4)}$$

Adding up all the infinite contributions computed we obtain that the subtraction $Z_1^{h_1}$ is given by

$$Z_1^{h_1} = \frac{1}{16\pi^2(n-4)} \left\{ 6\, g^2 + 2\, h_1^2 - 2\, h_2^2 \right\}$$

The subtraction constant $Z_1^{h_2}$ for the proper vertex diagrams of Fig. III can be determined in the same way. We note after some simple algebra that $Z_1^{h_2}$ can be obtained from $Z_1^{h_1}$ by the interchange $h_1 \leftrightarrow h_2$.

The bare coupling constant for the vertex $\bar{\psi}$-λ-A to the order of one loop is obtained by substituting the relevant computed subtraction constants in the expression

$$h_1^\circ = \frac{h_1\,(1 + Z_1^{h_1})}{(1+Z_2^A)^{1/2}\,(1+Z_2^\psi)^{1/2}\,(1+Z_2^\chi)^{1/2}}$$

we obtain that

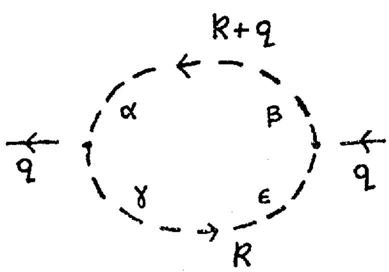
$$h_1^\circ = h_1 + \frac{1}{16\pi^2(n-4)} \left\{ \frac{33}{4}\, g^2 h_1 - \frac{15}{2}\, h_1^3 - \frac{11}{2}\, h_1 h_2^2 \right\} \qquad (3.6a)$$

and similarly that

$$h_2^o = h_2 + \frac{1}{16\pi^2(n-4)}\left\{\frac{33}{4} g^2 h_2 - \frac{15}{2} h_2^3 - \frac{11}{2} h_2 h_1^2\right\} \tag{3.6b}$$

The proper vertex diagrams of the self-quartic scalar vertices are given in Fig. IV to VII, and have been classified into the three distinct groups (a), (b) and (c) distinguished by the type of field that forms the closed loop.

We note, therefore, that the same loop momentum integral to be computed is obtained for all the diagrams given by the groups (a) of Fig. IV-VII. Assuming that the net momentum flow into the "loop" is given by q, we have



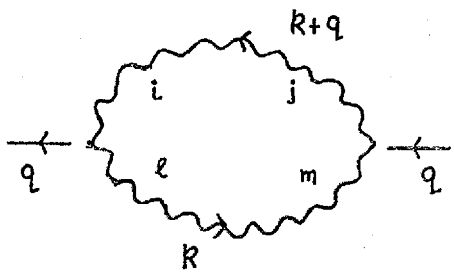
where $\alpha,\beta,\gamma,\varepsilon = 1,2$ are the SU(2)-indices of the scalar fields forming the loop, and the loop integral I_s obtained is

$$I_s = \delta_{\alpha\beta}\,\delta_{\gamma\varepsilon}\int d^n k \,\frac{i}{k^2}\,\frac{i}{(k+q)^2} = \delta_{\alpha\beta}\,\delta_{\gamma\varepsilon}\int_0^1 dx \int d^n k \,\frac{-1}{[k^2+2xq.k+xq^2]^2}$$

performing the integrations and neglecting terms finite at $n = 4$ we obtain

$$\text{Infin. } I_s = \frac{2i\,\delta_{\alpha\beta}\,\delta_{\gamma\varepsilon}}{16\pi^2(n-4)}$$

Similarly, we obtain the same loop momentum integral to be computed for all the diagrams in the groups (b) of Fig. IV to VI. This is given by



where $i,j,\ell,m = 1,2,3$ are the SU(2)-indices of the gauge field forming the closed loop, and the loop integral I_v obtained is

$$I_v = \delta_{ij}\,\delta_{\ell m} \int d^n k \;\frac{-i\left(g^{\mu\rho} - k^\mu k^\rho / k^2\right)}{k^2}\;\frac{-i\left(g^{\nu\sigma} - (k+q)^\nu (k+q)^\sigma\right)}{(k+q)^2}\; g_{\mu\nu}\, g_{\sigma\rho}$$

$$= -\delta_{ij}\,\delta_{\ell m} \int d^n k \;\frac{2 + [k.(k+q)]^2 / k^2 (k+q)^2}{k^2\,(k+q)^2}$$

$$= -\delta_{ij}\,\delta_{\ell m}\left\{ \int_0^1 dx \int d^n k \;\frac{3}{[k^2 + 2x\,k.q + x q^2]^2} + \text{finite contributions} \right\}$$

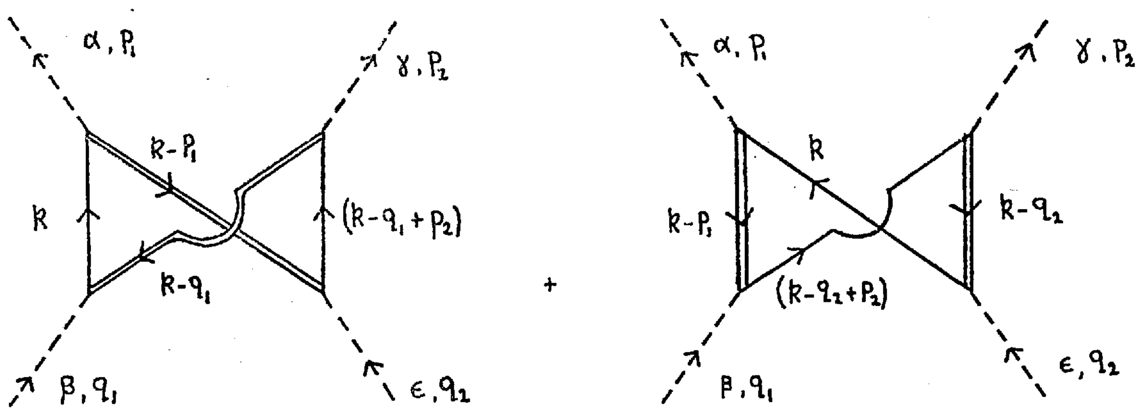
performing the integrations and neglecting terms finite at $n = 4$ we obtain

$$\text{Infin. } I_v = \frac{6\,i\,\delta_{ij}\,\delta_{\ell m}}{16\pi^2\,(n-4)}$$

It is mainly a problem in combinatorics to obtain, using these results, the contributions of the relevant diagrams to the subtraction constants $Z_1^{\lambda\alpha}$ $\alpha = 1,2,3$ and 4. The contributions obtained will not be given here explicitly since they can be easily identified in the final expressions for the bare self-quartic couplings given at the end of this subsection.

Finally we consider the contributions to the subtractions $Z_1^{\lambda\alpha}$ due to the Yukawa vertices of the theory, given by the groups (c) of Figs. IV to VII.

The overall momentum integrals obtained for these diagrams are found to be essentially all the same but due care must be taken regarding their overall signs. We calculate explicitly the contributions of these diagrams for the Fig. IV and infer all others from these.



(1) (2)

The spinor loop contributes an overall factor of (-1) for these diagrams like for quantum electrodynamics; with the momentum flow as indicated for the Majorana spinors. Using the Feyman rules of the theory we obtain for Diagram (1)

$$B(1) = -h_1^4\,(\tau^i\tau^i)_{\alpha\beta}\,(\tau^j\tau^i)_{\gamma\epsilon}$$

$$\times \int d^n k\ \mathrm{Tr}\left[\frac{i\not k}{k^2}\cdot\frac{-i(\not k-\not q_1)C}{(k-q_1)^2}\cdot C^{-1}\cdot\frac{i(\not k-\not q_1+\not p_2)}{(k-q_1+p_2)^2}\cdot\frac{-i(\not k-\not p_1)C}{(k-p_1)^2}\cdot C^{-1}\right]$$

$$= -h_1^4\,(\tau^i\tau^j)_{\alpha\beta}(\tau^j\tau^i)_{\gamma\epsilon}\left\{\int_0^1 dx\int d^n k\,\frac{4}{\left[k^2-2x\,k.p_1+x\,p_1^2\right]^2} + \text{finite contributions}\right\}$$

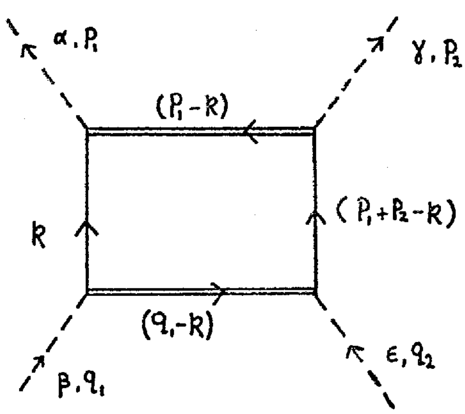
performing the integrations and neglecting finite terms at $n = 4$ we obtain

$$\text{Inf. } B(1) = \frac{8 i h_1^4}{16\pi^2 (n-4)} (\tau^i \tau^j)_{\alpha\beta} (\tau^j \tau^i)_{\gamma\epsilon} = \frac{8 i h_1^4}{16\pi^2 (n-4)} (4 \delta_{\alpha\epsilon} \delta_{\gamma\beta} + \delta_{\alpha\beta} \delta_{\gamma\epsilon})$$

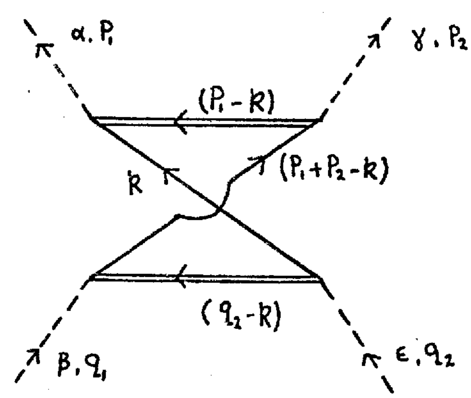
We note that the contribution of diagram 2, B(2), may be obtained from that of diagram 1, B(1), by the following interchange of the SU(2)-indices and momenta $(\beta, q_1) \leftrightarrow (\epsilon, q_2)$. We obtain therefore

$$\text{Inf. } B(2) = \frac{8 i h_1^4}{16\pi^2 (n-4)} (\tau^i \tau^j)_{\alpha\epsilon} (\tau^j \tau^i)_{\gamma\beta} = \frac{8 i h_1^4}{16\pi^2 (n-4)} (4 \delta_{\alpha\beta} \delta_{\gamma\epsilon} + \delta_{\alpha\epsilon} \delta_{\gamma\beta})$$

The remaining two diagrams in groups of Fig. IV are



(3)



(4)

We observe that the diagrams (3) and (4), with the momentum flow as indicated, differ from the diagrams (1) and (2), respectively, by the exchange of two Majorana χ-lines. So the spinor loops contribute an overall factor of (+1) for these diagrams, and we obtain for diagram (3)

$$B(3) = h_1^4\,(\tau^i\tau^j)_{\alpha\beta}(\tau^i\tau^j)_{\gamma\epsilon}$$

$$\times \int d^n k\ \mathrm{Tr}\left[\frac{-i(\not p_1-\not k)C}{(p_1-k)^2}.C^{-1}.\frac{i(\not p_1+\not p_2-\not k)}{(p_1+p_2-k)^2}.\frac{-i(\not q_1-\not k)C}{(q_1-k)^2}.\frac{(C^{-1}i\not k)^T}{k^2}\right]$$

since $(C^{-1}\gamma_\mu)^T = (C^{-1}\gamma_\mu)$ we note that the infinite contribution of the above integral is minus that obtained for diagram (1) and hence

$$\mathrm{Inf.}\,B(3) = \frac{8\,i\,h_1^4}{16\pi^2(n-4)}(\tau^i\tau^j)_{\alpha\beta}(\tau^i\tau^j)_{\gamma\epsilon} = \frac{8\,i\,h_1^4}{16\pi^2(n-4)}(5\,\delta_{\alpha\epsilon}\delta_{\gamma\beta}-4\,\delta_{\alpha\beta}\delta_{\gamma\epsilon})$$

Similarly the infinite contribution of diagram (4) obtained is

$$\mathrm{Inf.}\,B(4) = \frac{8\,i\,h_1^4}{16\pi^2(n-4)}(\tau^i\tau^j)_{\alpha\epsilon}(\tau^i\tau^j)_{\gamma\beta} = \frac{8\,i\,h_1^4}{16\pi^2(n-4)}(5\,\delta_{\alpha\beta}\delta_{\gamma\epsilon}-4\,\delta_{\alpha\epsilon}\delta_{\gamma\beta})$$

Adding up all the infinite contributions computed for the diagrams in group C of Fig. IV we obtain

$$\mathrm{Inf.}\,[\mathrm{Fig\ IV.c}] = \left\{\frac{-24\,h_1^4}{16\pi^2(n-4)}\right\}\ \left\{-2i\,(\delta_{\alpha\beta}\delta_{\gamma\epsilon}+\delta_{\alpha\epsilon}\delta_{\gamma\beta})\right\} \tag{3.7}$$

And the term in the subtraction constant $\lambda_1\, Z_1^{\lambda_1}$ which renders finite these diagrams equals minus the expression in the first bracket, on the left hand side of (3.7).

The infinite contributions of each of the diagrams in group C of Figs. V to VII may be obtained by comparing these with the corresponding diagrams of Fig. IV that have been computed. We note that these may differ only by an overall factor of (-1) from the corresponding ones in Fig. IV - having made the appropriate substitutions for the coupling constants - due to the presence of the γ_5 matrices in the traces obtained for the spinor loops of these diagrams. We find that only the contributions of the diagrams 2 of Fig. VI and 3 and 4 of Fig. VII have an overall (-1) factor relatively to the corresponding diagrams of Fig. IV. Hence the infinite contribution of the diagrams in group C of Fig. V may be obtained from (3.7) by the substitution $h_1^4 \to h_2^4$, while the corresponding infinite contributions of Figs. VI and VII are given respectively by

$$\mathrm{Inf.}[\mathrm{Fig\ VI.c}] = \frac{-112\, h_1^2 h_2^2}{16\pi^2(n-4)} (-i\,\delta_{\alpha\beta}\delta_{\gamma\epsilon}) + \frac{32\, h_1^2 h_2^2}{16\pi^2(n-4)} (-i\,\delta_{\alpha\epsilon}\delta_{\gamma\beta})$$

$$\mathrm{Inf.}[\mathrm{Fig.VII.c}] = \frac{16\, h_1^2 h_2^2}{16\pi^2(n-4)} \{-2i(\delta_{\alpha\beta}\delta_{\gamma\epsilon} + \delta_{\alpha\epsilon}\delta_{\gamma\beta})\}$$

Finally, having computed all the infinite contributions of the diagrams in Figs. IV to VII as explained, we obtain that, for the self-quartic scalar vertices of the Lagrangian $\mathcal{L}_1$ to be rendered finite to the one loop, the corresponding bare coupling constants are given in terms of the renormalized by

$$\lambda_1^0 = \lambda_1 - \frac{1}{16\pi^2(n-4)} \{ 24\lambda_1^2 + 2\lambda_3^2 + 2\lambda_3\lambda_4 + \lambda_4^2 + 4\lambda_5^2 - 24 h_1^4 + \frac{9}{8} g^4 - 9 g^2 \lambda_1 + 24 h_1^2 \lambda_1 \} \tag{3.8a}$$

$$\lambda_2^o = \lambda_2 - \frac{1}{16\pi^2(n-4)} \{ 24\lambda_2^2 + 2\lambda_3^2 + 2\lambda_3\lambda_4 + \lambda_4^2 + 4\lambda_5^2$$
$$-24 h_2^4 + \frac{9}{8} g^4 - 9 g^2 \lambda_2 + 24 h_2^2 \lambda_2 \}$$

(3.8b)

$$\lambda_3^o = \lambda_3 - \frac{1}{16\pi^2(n-4)} \{ 12\lambda_3(\lambda_1+\lambda_2) + 4\lambda_4(\lambda_1+\lambda_2) + 4\lambda_3^2 + 2\lambda_4^2$$
$$+8\lambda_5^2 - 112 h_1^2 h_2^2 + \frac{9}{4} g^4 - 9 g^2 \lambda_3 + 12(h_1^2+h_2^2)\lambda_3 \}$$

(3.8c)

$$\lambda_4^o = \lambda_4 - \frac{1}{16\pi^2(n-4)} \{ 4\lambda_4(\lambda_1+\lambda_2) + 4\lambda_4^2 + 8\lambda_3\lambda_4 + 32\lambda_5^2$$
$$+32 h_1^2 h_2^2 - 9 g^2 \lambda_4 + 12(h_1^2+h_2^2)\lambda_4 \}$$

(3.8d)

$$\lambda_5^o = \lambda_5 - \frac{1}{16\pi^2(n-4)} \{ 4\lambda_5(\lambda_1+\lambda_2) + 8\lambda_3\lambda_5 + 12\lambda_4\lambda_5$$
$$+16 h_1^2 h_2^2 - 9 g^2 \lambda_5 + 12(h_1^2+h_2^2)\lambda_5 \}$$

(3.8e)

It is now a simple matter to check whether the renormalizability of the Yukawa and self-quartic scalar vertices of the supersymmetric Lagrangian $\mathcal{L}_{SI}$ is consistent with that of the gauge coupling constant of the theory.

We substitute for the renormalized coupling constants in the expressions (3.6) and (3.8) the supersymmetric values

(3.1) and, using (3.2), we find that the bare coupling constants of the theory also satisfy the supersymmetric constraints (3.1).

Alternatively, substituting in the expressions (3.6) and (3.8) the supersymmetric values for both sets of bare and renormalized coupling constants of the theory we find that the resulting expressions for the bare gauge coupling constant of the theory are consistent with that of (3.2).

We conclude therefore that the supersymmetric constraints on the coupling constants of the theory are preserved by the renormalization to the order of one loop.

It also follows, as explained in section II, that the Chang eigenvalue conditions for the Lagrangian $\mathcal{L}_1$ are satisfied by the supersymmetric values for the Yukawa and self-quartic scalar coupling constants given by (3.1).

(b) <u>The Chang Eigenvalue Conditions for Model I</u>

An interesting question that we have already raised is whether the supersymmetric solution is a unique solution to the Chang eigenvalue conditions for the theory; which we shall now proceed to investigate.

There is, of course, no "hidden" implication in this question that in general the solutions found to the Chang eigenvalue conditions for any gauge theory correspond to supersymmetric theories only. Indeed we have seen in section II how more restrictive the Chang-conditions are for the solutions of the special kind compared to the general ones.

The problem of existence, though, of lowest order solutions to the Chang eigenvalue conditions for any gauge theory is still an open question, whether these correspond to general or special solutions. Furthermore we would expect that the existence of lowest order solutions depends, in general, critically on the interplay between the dimensions of the non-abelian gauge group and the number of Higgs multiplets for the theory considered; not unlike the results obtained in the conventional analysis for asymptotic freedom[16,17]

The model that we consider here is the smallest SU-gauge group with two scalar vector multiplets which is supersymmetric when the coupling constants are given by (3.1). The idea of uniqueness is, therefore, a question of whether the hidden supersymmetric invariance guarantees the existence of the supersymmetric solution while (somehow) no other solutions exist to the Chang eigenvalue conditions for this theory.

The differential equations for the effective coupling constants of model I can be easily computed[6] from the expressions (3.2), (3.6) and (3.8) for the bare coupling constants of the theory.

$$16\pi^2 \frac{d}{dt} g = -5 g^3 \tag{3.9}$$

$$16\pi^2 \frac{d}{dt} h_1 = \frac{15}{2} h_1^3 + \frac{11}{2} h_1 h_2^2 - \frac{33}{4} g^2 h_1 \tag{3.10a}$$

$$16\pi^2 \frac{d}{dt} h_2 = \frac{15}{2} h_2^3 + \frac{11}{2} h_2 h_1^2 - \frac{33}{4} g^2 h_2 \qquad (3.10b)$$

$$16\pi^2 \frac{d}{dt} \lambda_\alpha = - b^{\lambda_\alpha}(g, h, \lambda) \qquad (3.11)$$

where (in order to avoid repetition of long expressions) $b^{\lambda\alpha}(g,h,\lambda)$ for $\alpha = 1,2,3$ and 4 are the residues of the poles at $n = 4$ of the corresponding bare coupling constants λ_α^o of the theory, given by the expressions (3.8).

We consider now the Chang eigenvalue conditions of the theory to lowest order. We define a new set of variables given by

$$\frac{h_i}{g} = \bar{h}_i \quad \text{for } i = 1,2 \quad \text{and} \quad \frac{\lambda_\alpha}{g^2} = \bar{\lambda}_\alpha \quad \text{for } \alpha = 1,2..5$$

and look for solutions to the system of differential equations (3.9) to (3.11) for which the barred variables are constants independent of t.

From equations (3.9) and (3.10) we obtain for $\frac{d\bar{h}_i}{dt} = 0$, $i = 1,2$

$$\bar{h}_1 \left(-\frac{13}{4} + \frac{15}{2}\bar{h}_1^2 + \frac{11}{2}\bar{h}_2^2\right) = 0 \qquad (3.12a)$$

$$\bar{h}_2 \left(-\frac{13}{4} + \frac{15}{2}\bar{h}_2^2 + \frac{11}{2}\bar{h}_1^2\right) = 0 \qquad (3.12b)$$

It is found that the solutions $\bar{h}_1 = \bar{h}_2 = 0$; $\bar{h}_1^2 = \frac{13}{20}$, $\bar{h}_2 = 0$; (or $\bar{h}_1 = 0$, $\bar{h}_2 = \frac{13}{20}$) of the equations (3.12) do not correspond to any Chang-solutions for the total system of equations (3.9) to (3.11). For example, for $h_2=0$ from equations (3.9) and (3.11) for the coupling λ_2 we obtain

$$(\bar{\lambda}_2 + 1/48)^2 + (\bar{\lambda}_3 + \bar{\lambda}_4)^2 + \bar{\lambda}_3^2 + 4\bar{\lambda}_5^2 = -107/96$$

We now consider the only solution to the equations (3.12) such that both Yukawa coupling constants vanish asymptotically like the gauge coupling constant of the theory. This solution of interest is

$$\bar{h}_1^2 = \bar{h}_2^2 = 1/4 \tag{3.13}$$

From the differential equations (3.9) and (3.11) we obtain the following system of simultaneous equations in the variables $\bar{\lambda}_\alpha$ for $\frac{d\bar{\lambda}_\alpha}{dt} = 0$, $\alpha = 1,2..5$, and $\bar{h}_1^2 = \bar{h}_2^2 = 1/4$

$$24\bar{\lambda}_1^2 + 2\bar{\lambda}_3^2 + 2\bar{\lambda}_3\bar{\lambda}_4 + \bar{\lambda}_4^2 + 4\bar{\lambda}_5^2 + 7\bar{\lambda}_1 = 3/8 \tag{3.14}$$

$$24\bar{\lambda}_2^2 + 2\bar{\lambda}_3^2 + 2\bar{\lambda}_3\bar{\lambda}_4 + \bar{\lambda}_4^2 + 4\bar{\lambda}_5^2 + 7\bar{\lambda}_2 = 3/8 \tag{3.15}$$

$$12\bar{\lambda}_3(\bar{\lambda}_1 + \bar{\lambda}_2) + 4\bar{\lambda}_4(\bar{\lambda}_1 + \bar{\lambda}_2) + 4\bar{\lambda}_3^2 + 2\bar{\lambda}_4^2 + 8\bar{\lambda}_5^2 + 7\bar{\lambda}_3 = 19/4 \tag{3.16}$$

$$4\bar{\lambda}_4(\bar{\lambda}_1 + \bar{\lambda}_2) + 4\bar{\lambda}_4^2 + 8\bar{\lambda}_3\bar{\lambda}_4 + 32\bar{\lambda}_5^2 + 7\bar{\lambda}_4 = -2 \tag{3.17}$$

$$4\bar{\lambda}_5(\bar{\lambda}_1 + \bar{\lambda}_2) + 8\bar{\lambda}_3\bar{\lambda}_5 + 12\bar{\lambda}_4\bar{\lambda}_5 + 7\bar{\lambda}_5 = -1 \tag{3.18}$$

Eliminating the variables $(\bar{\lambda}_1+\bar{\lambda}_2)$ and $\bar{\lambda}_3$ from the equation (3.17) using equation (3.18), we obtain the equation

$$(2\bar{\lambda}_5 - \bar{\lambda}_4) + 8\bar{\lambda}_5(4\bar{\lambda}_5^2 - \bar{\lambda}_4^2) = 0$$

This equation factorises and using equation (3.17) once more we observe two cases

<u>Case 1</u> $\quad 2\bar{\lambda}_5 = \bar{\lambda}_4$

<u>Case 2</u> $\quad 16\bar{\lambda}_5 = 4(\bar{\lambda}_1 + \bar{\lambda}_2) + 4\bar{\lambda}_4 + 8\bar{\lambda}_3 + 7$

The system of simultaneous equations (3.14) to (3.18) is symmetric in the variables $\bar{\lambda}_1$ and $\bar{\lambda}_2$. From the equations (3.14) and (3.15) we obtain that, for all real solutions to this system of simultaneous equations, $\bar{\lambda}_\alpha$ for $\alpha = 1,2$. satisfy the inequality

$$3/8 - 24\bar{\lambda}_\alpha^2 - 7\bar{\lambda}_\alpha > 0 \quad \text{or} \quad -0{\cdot}34 < \bar{\lambda}_\alpha < 0{\cdot}05 \quad \text{for} \quad \alpha = 1,2$$

and furthermore that $\bar{\lambda}_1 = \bar{\lambda}_2$ and/or $\bar{\lambda}_1 + \bar{\lambda}_2 = -7/24$.

We may now consider each of the two cases separately, reduce the number of unknown variables and proceed to solve the system of equations (3.14) to (3.18) completely for each case. This is a rather long winded and tedious business so we shall just list the solutions found.

<u>Case 1</u>

$$\bar{\lambda}_1 = \bar{\lambda}_2 = 0, \quad \bar{\lambda}_3 = \tfrac{1}{2}, \quad 2\bar{\lambda}_5 = \bar{\lambda}_4 = -\tfrac{1}{4} \tag{3.19a}$$

$$\bar{\lambda}_1 = \bar{\lambda}_2 \simeq -0.231, \quad \bar{\lambda}_3 \simeq 0.634, \quad 2\bar{\lambda}_5 = \bar{\lambda}_4 \simeq -0.548 \tag{3.19b}$$

$$\bar{\lambda}_1 = \bar{\lambda}_2 \simeq -0.018, \quad \bar{\lambda}_3 \simeq 0.421, \quad 2\bar{\lambda}_5 = \bar{\lambda}_4 \simeq -0.547 \tag{3.19c}$$

$$\bar{\lambda}_1 \simeq -0.03,\ \bar{\lambda}_2 \simeq -0.261,\ \bar{\lambda}_3 \simeq 0.536,\ 2\bar{\lambda}_5 = \bar{\lambda}_4 \simeq -0.527 \quad (3.19e)$$

$$\bar{\lambda}_1 \simeq -0.261,\ \bar{\lambda}_2 \simeq -0.03,\ \bar{\lambda}_3 \simeq 0.536,\ 2\bar{\lambda}_5 = \bar{\lambda}_4 \simeq -0.527 \quad (3.19f)$$

$$\bar{\lambda}_1 \simeq -0.062,\ \bar{\lambda}_2 \simeq -0.229,\ \bar{\lambda}_3 \simeq 0.682,\ 2\bar{\lambda}_5 = \bar{\lambda}_4 \simeq -0.237 \quad (3.19g)$$

$$\bar{\lambda}_1 \simeq -0.229,\ \bar{\lambda}_2 \simeq -0.062,\ \bar{\lambda}_3 \simeq 0.682,\ 2\bar{\lambda}_5 = \bar{\lambda}_4 \simeq -0.237 \quad (3.19h)$$

Case 2

$$\bar{\lambda}_1 = \bar{\lambda}_2 = -1/8,\ \bar{\lambda}_3 = 1/4,\ \bar{\lambda}_4 = -1,\ \bar{\lambda}_5 = 1/4 \quad (3.20a)$$

$$\bar{\lambda}_1 = \bar{\lambda}_2 = -0.163,\ \bar{\lambda}_3 \simeq 0.272,\ \bar{\lambda}_4 \simeq -1.001,\ \bar{\lambda}_5 \simeq 0.242 \quad (3.20b)$$

$$\bar{\lambda}_1 \simeq -0.11,\ \bar{\lambda}_2 \simeq -0.181,\ \bar{\lambda}_3 \simeq 0.249,\ \bar{\lambda}_4 \simeq -1.00,\ \bar{\lambda}_5 \simeq 0.239 \quad (3.20c)$$

$$\lambda_1 \simeq -0.181,\ \bar{\lambda}_2 \simeq -0.11,\ \bar{\lambda}_3 \simeq 0.249,\ \bar{\lambda}_4 \simeq -1.001,\ \bar{\lambda}_5 \simeq 0.239 \quad (3.20d)$$

It is seen that there is a rather large number of solutions to the Chang eigenvalue conditions for the theory considered to lowest order. This is rather interesting, and illustrates the point raised by N.P. Chang[1] on the significance of the contributions of the Yukawa vertices to the coupled equations for the effective self-quartic scalar coupling constants of the theory. We note that, for the model considered here, both Yukawa coupling constants had to be taken to vanish asymptotically like the gauge coupling constant of the theory, i.e. solution (3.13), in order to find any solution to the eigenvalue conditions for the scalar quartic coupling constants.

The solution (3.19a) is known to correspond to a theory that is supersymmetric and satisfies therefore the Chang eigenvalue conditions to all orders. All other solutions found must be assumed to be of the general Chang-type with higher order corrections needed in order to satisfy the Chang eigenvalue conditions to all orders, as it was explained in section II.

There are two questions that have to be considered though, (a) whether higher order terms can be found for each of the lowest order solutions obtained such that the Chang eigenvalue conditions are satisfied to all orders, and (b) whether all such solutions found are physically acceptable, that is they correspond to well defined Lagrangian theories.

The first question raised will not be considered here in great detail. As it was pointed out in section II.c the answer is affirmative for each lowest order solution found when the determinant of the corresponding matrix $B^{SC}_{\alpha\beta}$ of equation (2.54) is different from zero for all $N = 2,3..\infty$. We have only checked this condition for the lowest order solution (3.20a), which is exact, and found that it is satisfied for all N. The application and use of these conditions will be considered in more detail in section IV, for the solutions of model II which are more interesting.

We shall now consider the second question raised. We have for example the conditions that the self-quartic scalar coupling constants λ_1 and λ_2 of the Lagrangian theory I must be greater than or equal to zero. This is obtained from the requirement that the potential of the theory be bounded from below. The potential is defined, in the conventional way, with an overall minus sign to that given explicitly in the Lagrangian $\mathcal{L}_I$.

The Chang-solutions must be checked therefore against such requirements as that given above, and only solutions which describe theories with potentials bounded from below being physically acceptable. The difficulty is though that these conditions on the effective self-quartic scalar coupling constants are valid for $t = 0$. This implies that in general such solutions must be known exactly to all orders in the effective gauge coupling constant of the theory in order that we can determine their values at $t = 0$.

On the other hand we have obtained only the lowest order terms of the general Chang-solutions for model I, hence we cannot check whether these conditions on the self-quartic scalar coupling constants hold for an arbitrary gauge coupling strength. However when we consider a weak gauge coupling constant for the theory so that the infinite series Chang-solutions are valid for $t = 0$ and the lowest order terms are the leading terms of these solutions, then the physical conditions for the potential of the theory to be bounded from below must be satisfied by the lowest order terms of the Chang-solutions.

We observe that there exist no lowest order solutions to the Chang eigenvalue conditions for the theory with λ_1 and λ_2

greater than zero, and the only solution with $\lambda_1 = \lambda_2 = 0$ is the one that corresponds to the supersymmetric theory. We conclude that the supersymmetric solution is the only physically acceptable solution when we consider a weak gauge coupling constant for the theory.

SECTION IV

THE CHANG EIGENVALUE CONDITIONS FOR MODEL II

In section III it was shown explicitly that the Chang eigenvalue conditions, for the particular theory considered, are satisfied to lowest order by the choice of coupling constants for which the theory is supersymmetric. This was a direct consequence of the fact that all the vertices of the supersymmetric theory are rendered finite to the one loop by a redefinition of the gauge coupling constant only. The question, though, of uniqueness of the supersymmetric case as a solution of the Chang eigenvalue conditions was not resolved in a completely satisfactory manner: the problem being that no definite conclusion can be drawn for the lowest order solutions obtained, when the gauge coupling constant of the theory is taken to be a "strong one".

In this section we consider the Chang eigenvalue conditions to the one loop approximation for a general renormalizable SU(2) gauge invariant Lagrangian, with the same "matter fields" as the supersymmetric model II given in section I.b. It is shown that, apart from the solution corresponding to the supersymmetric theory, there exists another solution which corresponds to a theory with a potential bounded from below, when we consider a weak gauge coupling constant for the theory.

Another interesting feature of the model considered here is that a solution exists to the Chang eigenvalue conditions for the theory, considered to all orders, whose higher order terms are not determinded uniquely in terms of the gauge coupling constant of the theory. Furthermore the lowest order terms of the Yukawa and self-quartic scalar coupling constants for this solution are given exactly by the supersymmetric set of coupling constant "values" (for model II), and the supersymmetric theory is only a particular case of the Lagrangian theory corresponding to this solution.

The model that we consider is given by the Lagrangian

$$\begin{aligned}\mathcal{L}_{II} = & -\frac{1}{4}\left(\partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu + g\, \vec{W}_\mu \wedge \vec{W}_\nu\right)^2 + \frac{1}{2}\left(\partial_\mu \vec{A} + g\, \vec{W}_\mu \wedge \vec{A}\right)^2 \\ & + \frac{1}{2}\left(\partial_\mu \vec{B} + g\, \vec{W}_\mu \wedge \vec{B}\right)^2 + i\, \bar{\Psi}^i \gamma^\mu \left(\partial_\mu \Psi^i + g\, \epsilon^{ijk} W^j_\mu \Psi^k\right) \\ & + i\, h_1\, \epsilon^{ijk} A^i \bar{\Psi}^j \Psi^k \quad + i\, h_2\, \epsilon^{ijk} B^i \bar{\Psi}^j \gamma_5 \Psi^k \\ & - \lambda_1 (\vec{A}.\vec{A})^2 - \lambda_2 (\vec{B}.\vec{B})^2 - \lambda_3 (\vec{A}.\vec{A})(\vec{B}.\vec{B}) - \lambda_4 (\vec{A}\wedge\vec{B})^2\end{aligned}$$

The supersymmetric model II, in the special gauge[8,10] for the gauge superfield ψ, corresponds to this Lagrangian, by construction, for the choice of coupling constants given by

$$h_1 = h_2 = g \quad , \quad \lambda_1 = \lambda_2 = \lambda_3 = 0 \quad , \quad \lambda_4 = g^2/2 \tag{4.1}$$

The asymptotic properties of the Lagrangian $\mathcal{L}_{II}$ were firstly studied by M. Suzuki[30]. He pointed out in his work that the differential equations for the effective coupling constants of the theory are satisfied "trivially" by the supersymmetric choice of coupling constants; i.e. a lowest order solution to the Chang eigenvalue conditions for the theory. This implies, from the work on section II, that all the vertices of the supersymmetric model II are rendered finite to the order of one loop by a redefinition of the gauge coupling constant of the theory only.

We shall now proceed to determine all other solutions to the Chang eigenvalue conditions of the theory considered to the one loop approximation. The differential equations for the effective coupling constants of the theory are given by

$$\frac{d}{dt} g = - b_1^g \, g^3 = - \frac{4}{16\pi^2} g^3 \tag{4.2}$$

$$\frac{d}{dt} h_1 = - b_1^{h_1}(g,h) = \frac{1}{16\pi^2} \{ 8 h_1^3 - 12 h_1 g^2 \} \tag{4.3}$$

$$\frac{d}{dt} h_2 = - b_1^{h_2}(g,h) = \frac{1}{16\pi^2} \{ 8 h_2^3 - 12 h_2 g^2 \} \tag{4.4}$$

$$\frac{d}{dt} \lambda_1 = - b_1^{\lambda_1}(g,h,\lambda) = \frac{1}{16\pi^2} \{ 88 \lambda_1^2 + 6 \lambda_3^2 + 4 \lambda_4^2 + 8 \lambda_3 \lambda_4 - 24 \lambda_1 g^2 + 16 \lambda_1 h_1^2 - 4 h_1^4 + 3 g^4 \} \tag{4.5}$$

$$\frac{d}{dt} \lambda_2 = - b_1^{\lambda_2}(g,h,\lambda) = \frac{1}{16\pi^2} \{ 88 \lambda_2^2 + 6 \lambda_3^2 + 4 \lambda_4^2 + 8 \lambda_3 \lambda_4 - 24 \lambda_2 g^2 + 16 \lambda_2 h_2^2 - 4 h_2^4 + 3 g^4 \} \tag{4.6}$$

$$\frac{d}{dt} \lambda_3 = - b_1^{\lambda_3}(g,h,\lambda) = \frac{1}{16\pi^2} \{ 40 \lambda_3 (\lambda_1 + \lambda_2) + 16 \lambda_3^2 + 16 \lambda_4 (\lambda_1 + \lambda_2) + 8 \lambda_4^2 - 24 \lambda_3 g^2 + 8 \lambda_3 (h_1^2 + h_2^2) - 8 h_1^2 h_2^2 + 6 g^4 \} \tag{4.7}$$

$$\frac{d}{dt} \lambda_4 = - b_1^{\lambda_4}(g,h,\lambda) = \frac{1}{16\pi^2} \{ 16 \lambda_4 (\lambda_1 + \lambda_2) + 32 \lambda_3 \lambda_4 + 12 \lambda_4^2 - 24 \lambda_4 g^2 + 8 (h_1^2 + h_2^2) \lambda_4 - 3 g^4 \} \tag{4.8}$$

We now define a new set of variables by

$$\bar{h}_i = \frac{h_i}{g} \text{ for } i = 1,2 \quad \text{and} \quad \bar{\lambda}_\alpha = \frac{\lambda_\alpha}{g^2} \text{ for } \alpha = 1,2,3,4 \tag{4.9}$$

and look for solutions to the system of equations (4.2) to (4.8) for which the barred variables are constants, independent of t.

From the equations (4.2) to (4.4) we obtain for $\frac{d\bar{h}_i}{dt} = 0$

$$\bar{h}_1(\bar{h}_1^2 - 1) = 0 \tag{4.10}$$

$$\bar{h}_2(\bar{h}_2^2 - 1) = 0 \tag{4.11}$$

Any solution of the equations (4.10) and (4.11) for which either/or both the "barred Yukawa couplings" are zero does not correspond to a Chang-solution for the total system of equations (4.2) to (4.8). This is a similar result to that obtained for model I of section III and may be easily deduced for, say, $\bar{h}_1 = 0$ from the equations (4.2) and (4.5).

We now consider the solution of equations (4.10) and (4.11) given by

$$\bar{h}_1^2 = \bar{h}_2^2 = 1 \tag{4.12}$$

From the equations (4.2) and (4.5) to (4.8) we obtain the following system of simultaneous equations in the variables $\bar{\lambda}_\alpha$ (4.9) for $\frac{d\bar{\lambda}_\alpha}{dt} = 0$, $\alpha = 1,2,3,4$.

$$88\,\bar{\lambda}_1^2 + 6\,\bar{\lambda}_3^2 + 4\,\bar{\lambda}_4^2 + 8\,\bar{\lambda}_3\,\bar{\lambda}_4 = 1 \tag{4.13}$$

$$88\,\bar{\lambda}_2^2 + 6\,\bar{\lambda}_3^2 + 4\,\bar{\lambda}_4^2 + 8\,\bar{\lambda}_3\,\bar{\lambda}_4 = 1 \tag{4.14}$$

$$20\,\bar\lambda_3(\bar\lambda_1+\bar\lambda_2) + 8\,\bar\lambda_3^2 + 8\,\bar\lambda_4(\bar\lambda_1+\bar\lambda_2) + 4\,\bar\lambda_4^2 = 1 \quad (4.15)$$

$$16\,\bar\lambda_4(\bar\lambda_1+\bar\lambda_2) + 32\,\bar\lambda_3\,\bar\lambda_4 + 12\,\bar\lambda_4^2 = 3 \quad (4.16)$$

where we have substituted for $\bar h_1^2$ and $\bar h_2^2$ their values given by (4.12).

The equations (4.13) to (4.15) are symmetrical under the interchange $\bar\lambda_1 \leftrightarrow \bar\lambda_2$ and we observe two cases

<u>Case 1</u> $\quad \bar\lambda_1 = -\bar\lambda_2$

<u>Case 2</u> $\quad \bar\lambda_1 = \bar\lambda_2$

The case 1 would not lead to any solutions that we would be interested in since either λ_1 or λ_2 would be negative unless $\bar\lambda_1 = \bar\lambda_2 = 0$. In fact the only solution obtained for case 1, and for real $\bar\lambda_\alpha$, is the supersymmetric solution (4.1).

We now consider case 2 in more detail. We eliminate the variable $\bar\lambda_2$ and consider an equivalent system of simultaneous equations in the variables $\bar\lambda_1$, $\bar\lambda_3$ and $\bar\lambda_4$ by taking the following linear combinations of the equations (4.13) to (4.16).

<u>Eqn. (4.13) - Eqn. (4.15)</u>:

$$44\,\bar\lambda_1^2 - \bar\lambda_3^2 + 4\,\bar\lambda_3\,\bar\lambda_4 - 20\,\bar\lambda_3\,\bar\lambda_1 - 8\,\bar\lambda_4\,\bar\lambda_1 = 0 \quad (4.17)$$

<u>3xEqn. (4.13) + 3xEqn. (4.15) - 2xEqn. (4.16)</u>:

$$132\,\bar\lambda_1^2 + 21\,\bar\lambda_3^2 - 20\,\bar\lambda_3\,\bar\lambda_4 + 60\,\bar\lambda_3\,\bar\lambda_1 - 8\,\bar\lambda_4\,\bar\lambda_1 = 0 \quad (4.18)$$

<u>plus equation (4.16)</u>; for $\bar\lambda_2 = \bar\lambda_1$

$$32\,\bar\lambda_1\,\bar\lambda_4 + 32\,\bar\lambda_3\,\bar\lambda_4 + 12\,\bar\lambda_4^2 = 3 \quad (4.19)$$

The equation (4.17) factorises and we obtain the two solutions

$$2 \bar{\lambda}_1 = \bar{\lambda}_3 \tag{4.17a}$$

$$22 \bar{\lambda}_1 + \bar{\lambda}_3 - 4 \bar{\lambda}_4 = 0 \tag{4.17b}$$

Substituting for the variable $\bar{\lambda}_3$, given by (4.17b), in the equation (4.18) and completing the squares we obtain

$$\frac{183}{64} \bar{\lambda}_1^2 + \left(\frac{189}{8} \bar{\lambda}_1 - 4 \bar{\lambda}_4\right)^2 = 0$$

The only real solution of this equation is $\bar{\lambda}_1 = \bar{\lambda}_4 = 0$, but $\bar{\lambda}_4 \neq 0$ by equation (4.19), hence the solution (4.17b) is rejected.

We consider now the solution (4.17a). We substitute for $\bar{\lambda}_3$, given by (4.17a), in the equations (4.18) and (4.19) respectively and obtain

$$7 \bar{\lambda}_1^2 - \bar{\lambda}_4 \bar{\lambda}_1 = 0 \tag{4.20}$$

$$4 \bar{\lambda}_4^2 + 32 \bar{\lambda}_4 \bar{\lambda}_1 = 1 \tag{4.21}$$

The equations (4.20) and (4.21) can be solved trivially. All the solutions obtained to the Chang eigenvalue problem of model II are

$$h_1^2 = h_2^2 = g^2, \quad \lambda_1 = \lambda_2 = \lambda_3 = 0 \quad \& \quad \lambda_4 = g^2/2 \tag{4.22a}$$

$$h_1^2 = h_2^2 = g^2, \quad \lambda_1 = \lambda_2 = \lambda_3 = 0 \quad \& \quad \lambda_4 = -g^2/2 \tag{4.22b}$$

$$h_1^2 = h_2^2 = g^2, \quad 2\lambda_1 = 2\lambda_2 = \lambda_3 = \frac{g^2}{2\sqrt{105}} \ \& \ \lambda_4 = \frac{-7 g^2}{2\sqrt{105}} \tag{4.23a}$$

$$h_1^2 = h_2^2 = g^2, \quad 2\lambda_1 = 2\lambda_2 = \lambda_3 = \frac{-g^2}{2\sqrt{105}} \ \& \ \lambda_4 = \frac{-7\ g^2}{2\sqrt{105}} \quad (4.23b)$$

Unlike the previous model that we have considered in section III, we observe that there are two physically acceptable lowest order solutions if we consider a weak gauge coupling constant for the theory. That is, either of the solutions (4.22a), the supersymmetric, or (4.23a) correspond to theories with a potential bounded from below; the potential being defined, in the conventional manner, as minus that given in the Lagrangian $\mathcal{L}_{II}$.

The solution (4.23a) is not known to correspond to any supersymmetric theory and it must be assumed that it corresponds to some general Chang solution. The question that must be investigated though is whether a general Chang solution with its lowest order terms given by the solution (4.23a) exists.

In section II.c we found certain sufficient conditions, but not necessary, for the existence of a general Chang solution given its lowest order terms. We shall now proceed to investigate whether they are satisfied by the solution (4.23a) and if not, what orders of the Chang eigenvalue conditions must be checked out explicitly.

We consider firstly the determinant of the coefficient matrix A_{ij}^{Yu} (for i,j = 1,2) of equation (2.53) which is used to determine the higher order corrections to the lowest order solutions for the Yukawa couplings (4.12), for each of the Chang solutions obtained (4.22) and (4.23). As explained in section II.c, the determinant of the matrix A_{ij}^{Yu} for each of the solutions (4.12), i.e. $\bar{h}_i^{(1)} \neq 0$ for i = 1,2, is different from

from zero for all $N = 2,3..\infty$. This may be easily shown explicitly to be the case, using the convenient form for the coefficient matrix A^{Yu}_{ij} given by (2.56) and substituting in it for the relevant computed functions for the theory considered, given by the equations (4.2) to (4.4). For completeness, the coefficient matrix A^{Yu}_{ij} evaluated at the point (4.2) is given - which for the model considered turns out to be diagonal and (hence) independent of the relative sign of the values of $\bar{h}_1$ and $\bar{h}_2$ for the solutions (4.12). We find that

$$(16\pi^2)\; A_{ij} = (8N+12)\; \delta_{ij} \qquad i,j = 1,2$$

and hence its determinant is different from zero to all orders N. No problems are, therefore, encountered in satisfying the Chang eigenvalue conditions for the Yukawa couplings to all orders in perturbation theory, for any of the lowest order Chang solutions obtained for the Lagrangian model II.

We shall now consider in more detail the determinant of the coefficient matrix $B^{sc}_{\alpha\beta}$ defined by the equation (2.54), which determines the higher order terms to the general Chang solutions for the self-quartic scalar coupling constants. We use the convenient form for the matrix $B^{SC}_{\alpha\beta}$ (2.57) given by

$$B^{sc}_{\alpha\beta} = 2N\, b^{g}_{1}\, \delta_{\alpha\beta} - \frac{\partial}{\partial\bar{\lambda}_\beta}\, b^{\lambda\alpha}_{1}(1, \bar{h}, \bar{\lambda})$$

where for the model studied $\alpha,\beta = 1,2,3,4$, $b^{g}_{1} = 4/(16\pi^2)$ and the functions $b^{\lambda\alpha}_{1}(1,\bar{h},\bar{\lambda})$ are given, in terms of the "unbarred variables", by the equations (4.5) to (4.8). Hence, for the model considered we obtain the explicit form for the matrix $B^{sc}_{\alpha\beta}$

$$B^{sc}_{\alpha\beta} = \frac{1}{16\pi^2}\left\{ 8(N-1)\delta_{\alpha\beta} + Q_{\alpha\beta}\right\} \tag{4.24}$$

where

$$Q_{\alpha\beta} = \begin{pmatrix} 176\bar{\lambda}_1 & 0 & 12\bar{\lambda}_3 + 8\bar{\lambda}_4 & 8(\bar{\lambda}_4+\bar{\lambda}_3) \\ 0 & 176\bar{\lambda}_2 & 12\bar{\lambda}_3 + 8\bar{\lambda}_4 & 8(\bar{\lambda}_4+\bar{\lambda}_3) \\ 40\bar{\lambda}_3 + 16\bar{\lambda}_4 & 40\bar{\lambda}_3 + 16\bar{\lambda}_4 & 32\bar{\lambda}_3 + 40(\bar{\lambda}_1+\bar{\lambda}_2) & 16(\bar{\lambda}_1+\bar{\lambda}_2+\bar{\lambda}_4) \\ 16\bar{\lambda}_4 & 16\bar{\lambda}_4 & 32\bar{\lambda}_4 & 16(\bar{\lambda}_1+\bar{\lambda}_2+2\bar{\lambda}_3)+24\bar{\lambda}_4 \end{pmatrix}$$

and having evaluated the matrix $B^{sc}_{\alpha\beta}$ at the point $\bar{h}_1^2 = \bar{h}_2^2 = 1$.

We now substitute for the barred variables in the matrix $B^{sc}_{\alpha\beta}$ the values obtained for the solution (4.23a) and compute its determinant. We obtain that

$$\det B^{sc} = \frac{128}{(16\pi^2)^4}\left(K + \frac{11}{\sqrt{105}}\right)\{2K(K^2 - \frac{356}{105}) + \frac{1}{\sqrt{105}}(94K^2 - \frac{1344}{105})$$

where $K \equiv 2(N-1)$. We observe that the determinant of the matrix $B^{sc}_{\alpha\beta}$, for the solution (4.23a), is different from zero to all orders N, $N = 2,3..\infty$. It is concluded therefore, as explained in section II.c, that the lowest order solution (4.23a) <u>does</u> correspond to a definite Chang solution satisfying the eigenvalue conditions to all orders in perturbation theory.

We shall now consider a rather interesting case that arises for the solution (4.22a) of model II, which corresponds to the supersymmetric theory. If we compute the determinant of the matrix $B^{sc}_{\alpha\beta}$ with the barred variables evaluated at their supersymmetric values, "determined" by the solution (4.22a), we obtain that it vanishes for $N = 1$, $N = 2$ and is different from zero for all other positive integer values of N.

We believe that this property of the supersymmetric solution for model II is incidental, without any underlying physical significance. The determinant of the coefficient matrix for model I has been checked for the corresponding supersymmetric solution and it has been found that it is different from zero for all positive integer values of N.

The fact that the determinant of the matrix B^{sc} vanishes for N = 1 is irrelevant in so far as the Chang eigenvalue conditions are concerned, but this is not so for the root N = 2. In section II.c it was explained that, for any other lowest order solution but the supersymmetric, it would have been necessary in this case to compute and verify explicitly that the Chang eigenvalue conditions can be satisfied to two loops. For the supersymmetric solution no problems are encountered because no higher order corrections are needed for such solutions in order to satisfy the Chang eigenvalue conditions to all orders. This has been shown in section II.b to be closely related to the fact that the relations of the coupling constants of the supersymmetric theory are preserved by the renormalization of the theory.

On the other hand, strictly speaking, the supersymmetric solution in this case is only a particular solution of a class of solutions to the Chang eigenvalue conditions for model II whose lowest order terms are given by (4.22a). This is a simple consequence of the fact that the determinant of the matrix B^{sc} vanishes at N = 2 for the solution (4.22a), which we shall discuss in more detail.

We consider the equations (2.53) and (2.54) to two loops, for the next order terms to the Yukawa and self-quartic scalar couplings $x_i^{(2)}\ g^3$, $i = 1,2$ and $z_\alpha^{(2)}\ g^4$, $\alpha = 1,2,3,4$, respectively, when their lowest order terms are given by the solution (4.22a). Using the same notation as in section II.c, where for the solution (4.22a) $\bar{h}_i^{(1)} = 1$ for $i = 1,2$, $\bar{\lambda}_\alpha^{(1)} = 0$ for $\alpha = 1,2,3$ and $\bar{\lambda}_4^{(1)} = \frac{1}{2}$, we obtain the equations

$$A_{ij}^{Yu}\, x_j^{(2)} = 2\left\{ b_2^{h_i}(1,\bar{h}^{(1)},\bar{\lambda}^{(1)}) - \bar{h}_i^{(1)}\ b_2^{g}(1,\bar{h}^{(1)},\bar{\lambda}^{(1)}) \right\} \tag{4.25}$$

$$B_{\alpha\beta}^{sc}\, z_\beta^{(2)} + B_{\alpha j}^{Yu}\, x_j^{(2)} = 2\left\{ b_2^{\lambda_\alpha}(1,\bar{h}^{(1)},\bar{\lambda}^{(1)}) - 2\,\bar{\lambda}_\alpha^{(1)}\ b_2^{g}(1,\bar{h}^{(1)},\bar{\lambda}^{(1)}) \right\} \tag{4.26}$$

and for the vertices of the supersymmetric theory to be rendered finite to any order N by a redefinition of the gauge coupling constant only, we have the identities (for the simple pole residues at n = 4 of the bare coupling constants of the theory)

$$b_N^{h_i}(1,\bar{h}^{(1)},\bar{\lambda}^{(1)}) \equiv \bar{h}_i^{(1)}\ b_N^{g}(1,\bar{h}^{(1)},\bar{\lambda}^{(1)}) \tag{4.27}$$

$$b_N^{\lambda_\alpha}(1,\bar{h}^{(1)},\bar{\lambda}^{(1)}) \equiv 2\,\bar{\lambda}_\alpha^{(1)}\ b_N^{g}(1,\bar{h}^{(1)},\bar{\lambda}^{(1)}) \tag{4.28}$$

for all $N = 1,2,..\infty$; $i = 1,2$ and $\alpha = 1,2,3$ & 4. All the matrices of the equations (4.25) and (4.26), $B_{\alpha\beta}^{sc}$ given explicitly for the model by (4.24), have also been evaluated at the point given by the solution (4.22a) and for N = 2.

The left hand sides of the equations (4.25) and (4.26) are equal to zero by the identities (4.27) and (4.28), respectively, for N = 2. From the equation (4.25) we obtain that $x_i^{(2)} = 0$ for $i = 1,2$, since $\det A^{Yu} \neq 0$. The determinant, though, of the matrix $B^{sc}_{\alpha\beta}$ of equation (4.25) is equal to zero and a non-trivial solution exists for this equation.

The matrix B^{sc} and (hence) the solution to the equation (4.25) are given by

$$(16\pi^2)\ B^{sc}_{\alpha\beta} = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 4 & 5 \end{pmatrix} \quad \rightarrow \quad z^{(2)}_{\alpha} = \ell \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ -3/2 \\ 1 \end{pmatrix}$$

where ℓ is an undetermined parameter.

The general solution to the Chang eigenvalue conditions for model II corresponding to the lowest order solution (4.22a) is of the form

$$\begin{aligned} h_1 &= g + H_1(\ell,g) , & h_2 &= g + H_2(\ell,g) \\ \lambda_1 &= \frac{\ell}{4} g^4 + \Lambda_1(\ell,g) , & \lambda_2 &= \frac{\ell}{4} g^4 + \Lambda_2(\ell,g) \\ \lambda_3 &= -\frac{3\ell}{2} g^4 + \Lambda_3(\ell,g) , & \lambda_4 &= \frac{1}{2} g^2 + \ell g^4 + \Lambda_4(\ell,g) \end{aligned} \tag{4.29}$$

where the functions $H_i(\ell,g)$ $i = 1,2$ and $\Lambda_\alpha(\ell,g)$ $\alpha = 1,2,3,4$ are the higher order contributions to the solution obtained to two loops. These are determined uniquely as functions of ℓ by the equations (2.53) and (2.54) since, for the solution (4.22a), $\det B^{sc} \neq 0$ for $N = 3,4..\infty$. Furthermore, using the identities (4.27) and (4.28), we observe that the constants C_i $i = 1,2.$ and C_α $\alpha = 1,2,3,4.$ of the equations (2.53) and (2.54) for $N = 3,4..\infty$, determined to each order by the known form of the solution (4.29) to that order, vanish for $\ell = 0$. Hence we obtain that

$$H_i(\ell,g)\Big|_{\ell=o} = 0 \quad \text{and} \quad \Lambda_\alpha(\ell,g)\Big|_{\ell=o} = 0$$

for all $i = 1,2.$ and $\alpha = 1,2,3,4$. The supersymmetric theory is, therefore, given by the solution (4.29) for $\ell = 0$, as it would have been expected, and it is only a particular case of the Lagrangian theory corresponding to the solution (4.29).

In conclusion, we have obtained for model II that (a) a general Chang solution exists whose lowest order terms (4.23a) correspond to a theory with a potential bounded from below, and (b) the Chang eigenvalue conditions do not fix uniquely the higher order terms of the lowest order solution (4.22a) - corresponding to the supersymmetric theory - to be zero.

The existence of the solution (4.29) is rather interesting. It illustrates that the Chang eigenvalue conditions may be satisfied to all orders by solutions which are not completely fixed in terms of the gauge coupling constant of the theory. In such cases, the Chang eigenvalue conditions fail to eliminate completely the arbitrariness of the Yukawa and scalar quartic coupling constants of the initial theory.

On the other hand, on theoretical grounds, it is a desirable feature that the requirement of a gauge theory to be asymptotically free should eliminate "all" arbitrariness in the Yukawa and self-quartic scalar coupling constants[1]. It is more likely than not, though, that if in general the determinant of the coefficient matrix B^{sc} of equation (2.54), for some given lowest order Chang-solution, vanishes at some order

N, then the Chang eigenvalue conditions cannot be satisfied at that order. This is because, in general, it is highly improbable that the "constants" $\left(C_\alpha - B^{sc}_{\alpha j}\, x^{(N)}_j\right)$ of equation (2.54), where $x^{(N)}_j$ are determined by the equation (2.53), are accidentally zero at that order so that we obtain a consistent system of equations - strictly speaking the system of simultaneous linear equations (2.54) would be consistent if the coefficient matrix B^{sc} and the augmented matrix, for this system, have the same rank.

It is also of interest to note that the infinite contributions of the proper vertex diagrams for the self-quartic scalar vertices of the supersymmetric model II add up to zero, i.e. only finite contributions are obtained. This may be seen by considering the right hand sides of the differential equations (4.5) to (4.8) from which the simple pole residue contributions of the relevant proper vertex diagrams can be easily identified. Furthermore it is easily seen that this is related to/or implied by the fact that (a) the simple pole residue at n = 4 of the bare gauge coupling constant of the theory equals minus the residue of the wavefunction renormalization of the scalar or pseudoscalar field for $\bar{h}_1^2 = \bar{h}_2^2 = 1$, and (b) that all the vertices of the supersymmetric theory are rendered finite to one loop by a redefinition of the gauge coupling constant, only.

Since the above argument holds for all the solutions found to the Chang eigenvalue conditions for the theory considered to lowest order, it follows that the sum of the proper vertex diagrams for the self-quartic scalar vertices of the Lagrangian corresponding to the solution (4.23a) are also finite, as it may be seen directly.

Essentially these relations for the various subtractions constants of the theory have been implicit in our calculations. In fact the vanishing of the determinant of the matrix $B^{sc}_{\alpha\beta}$ (4.24) for the solution (4.22a) when N = 1 can be understood along these lines; i.e. for N = 1 and $\bar{\lambda}_1 = \bar{\lambda}_2 = 0$ the matrix B^{sc} (4.24) has two identical rows. That the elements of the diagonal matrix in (4.24) are proportional to (N-1), rather than some other function of N, follows from the aforementioned observation (a) on the relations obtained for the subtraction constants of the theory.

CHAPTER TWO

ONE LOOP "STRONG CORRECTIONS" TO THE HADRONIC WEAK CURRENT

Introduction

The local current-current Lagrangian for weak interactions[31] gives a very good description, in lowest order, of all known leptonic and semileptonic processes. But being a non-renormalizable Lagrangian and with bad high energy behaviour[31], it may be viewed only as a phenomenological theory for weak interactions.

In recent years there has been great progress in the field of weak interactions[32]. Weak and electromagnetic interactions share the common property of being universal and attempts to unify these theories seemed natural. On the other hand, in contrast to the photon, the weak charged boson mediating the weak interactions is expected to be massive so that an effective current-current point interaction is obtained for small momentum transfer weak processes.

In 1967 a unified model for the weak and electromagnetic interactions was constructed[33] based on the gauge group SU(2) x U(1). The photon and weak charged boson were identified with certain linear combinations of the gauge fields of the theory with appropriate masses being generated by the Higgs-Kibble mechanism[18]. One byproduct of this scheme was the appearance of a neutral heavy intermediate boson, orthogonal to the massless linear combination of gauge fields identified as the photon.

It was shown later on that gauge theories with a spontaneously broken symmetry, due to the non-vanishing of the vacuum expectation value of the Higgs scalar fields[18], are renormalizable[28,34]. About the same time, a large number of models for the weak and electromagnetic interactions appeared in the literature based on different gauge groups but essentially with the same ingredients as the initial Weinberg-Salam model[33]. In extending the unified models to the description of semileptonic as well as leptonic processes, with a Cabbibo form for the hadronic weak current, a problem was to ensure the absence of strangeness changing neutral currents. Such currents are known experimentally to be greatly suppressed relatively to the charged ones, but seem inevitable in model building if only the usual three fundamental quarks p, n and λ are used. For the Weinberg-Salam type of model, strangeness changing neutral currents can be avoided by assuming the existence of an additional fourth quark χ carrying a new quantum number called charm[35] - with the quark fields p & n, and χ & λ introduced in a parallel way to the leptonic multiplets of the gauge group. The strangeness conserving neutral currents are a feature of most unified models and have been found experimentally to exist.

The introduction of strong interactions for the p,n,λ and χ fundamental quark fields into one unified theory for all, strong weak and electromagnetic interactions was just the next step to be taken. Experimentally it is known that for small momentum transfer semileptonic processes, like the β-decay, the strong interactions leave the vector part of the Cabbibo current unrenormalized. In this Chapter we consider the effects of the strong interactions on the effective semileptonic strangeness conserving interaction obtained within the framework of a unified model for the strong, weak and electromagnetic interactions[36]. In particular we calculate the one loop "strong corrections" to the semileptonic process ($n\nu_e \to pe^-$) and consider whether the vector part of the hadronic current remains unrenormalized.

The Model

In a paper by Pati and Salam[36] it was proposed that all the fundamental hadrons and leptons belong to the same irreducible representation $(4,4^*)$ of the global symmetry group SU(4') x SU(4") for all matter. In scheme A of reference (36), the basic matter fields of the representation $(4,4^*)$ are interpreted to be the known physical leptons, nine Han-Nambu type quarks and three charmed quarks.

The unified model for the strong, weak and electromagnetic interactions that we consider is the first of two proposed by these authors[36,37]. It is obtained[36] by gauging the subgroup $SU(2')_L \times U(1) \times SU(3'')$ of the global symmetry group, and may be viewed effectively as an extension of the Weinberg Salam model.

The $(4,4^*)$ fundamental matter multiplet is

$$\left(\begin{array}{ccc|c} p_\alpha^0 & p_b^+ & p_c^+ & \nu_e \\ n_\alpha^- & n_b^0 & n_c^0 & e^- \\ \hline \lambda_\alpha^- & \lambda_b^0 & \lambda_c^0 & \mu^- \\ \chi_\alpha^0 & \chi_b^+ & \chi_c^+ & \nu_\mu \end{array}\right) \equiv \left(\begin{array}{c|c} \Psi_{p(\alpha,i)} & \Psi_e \\ \hline \tau_1)_{\alpha\beta}\,\Psi_{\chi(\beta,i)} & \Psi_\mu \end{array}\right)$$

Where i = a,b,c corresponds to the SU(3") gauge group index (colour), α to the $SU(2')_L$-index and τ_1 is the conventional Pauli matrix. The superscript on the elements of the (4x4) array denotes the corresponding charge of the particles (in obvious notation).

Three Higgs scalar multiplets are also introduced and appropriate masses for the gauge bosons and fermion fields generated by the Higgs-Kibble mechanism. Two of these multiplets, represented by $(\sigma_u)^i_\alpha$ and $(\sigma_\ell)^i_\alpha$, each consists of six complex components transforming as $(2,1,3^*)$ under $SU(2')_L \times U(1) \times SU(3'')$. The third is represented by ϕ_α, it consists of two complex components transforming as (2,1,1) under the gauge group.

The eight gauge bosons associated with the SU(3") gauge group for the strong interactions (coupling constant f) are represented by V^m_μ m = 1,2..8. Similarly, the gauge bosons associated with the SU(2) and U(1) gauge groups are represented by W^i_μ i = 1,2,3 and U_μ respectively, they couple to matter with strengths g and g' respectively.

The unified model proposed(36) for the strong, weak and electromagnetic interactions is described by the $SU(2')_L \times U(1) \times SU(3'')$ gauge invariant Lagrangian

$$\mathcal{L}_1 = -\frac{1}{4} \vec{G}_{\mu\nu} . \vec{G}^{\mu\nu} - \frac{1}{4} \vec{\Xi}_{\mu\nu} . \vec{\Xi}^{\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} +$$

$$+ i \bar{\Psi}_{p_L(\alpha,i)} \gamma^\mu \Big[\delta_{ij} \delta_{\alpha\beta} \partial_\mu - i f \big(\frac{\lambda_m}{2} V^m_\mu \big)_{ij} \delta_{\alpha\beta} - i g \big(\frac{\vec{\tau}}{2} \vec{W}_\mu \big)_{\alpha\beta} \delta_{ij}$$

$$- \frac{i}{6} g' \delta_{\alpha\beta} \delta_{ij} U_\mu \Big] \Psi_{p_L(\beta,j)} +$$

$$+ i\, \overline{\Psi}_{p_R(\alpha,i)}\, \gamma^\mu \left[\delta_{\alpha\beta}\delta_{ij}\, \partial_\mu - i f \left(\frac{\lambda_m}{2} V_\mu^m\right)_{ij} \delta_{\alpha\beta} - i g' \left(\tilde{Y}_R^h\, U_\mu\right)_{(\alpha,\beta)(i,j)} \right] \Psi_{p_R(\beta,j)}$$

$$+ i\, \overline{\Psi}_{e_L}\, \gamma^\mu \left[\partial_\mu - i g \left(\frac{\vec{\tau}}{2}.\vec{W}_\mu\right) + \frac{i}{2} g' U_\mu \right] \Psi_{e_L} \quad +$$

$$+ i\, \overline{\Psi}_{e_R}\, \gamma^\mu \left[\partial_\mu - i g' \left(\tilde{Y}_R^\ell\, U_\mu\right) \right] \Psi_{e_R} \quad +$$

$$+ \sum_{\alpha=1,2} \left| \partial_\mu \Phi_\alpha - i g \left(\frac{\vec{\tau}}{2}.\vec{W}_\mu\right)_{\alpha\beta} \Phi_\beta + \frac{i}{2} g' U_\mu \Phi_\alpha \right|^2$$

$$+ \sum_{\substack{i=1,2,3\\ \alpha=1,2}} \left| \partial_\mu \sigma_u)^i_\alpha - i f \left(\frac{\lambda_m}{2} V_\mu^m\right)_{ij} \sigma_u)^j_\alpha - i g \left(\frac{\vec{\tau}}{2}.\vec{W}_\mu\right)_{\alpha\beta} \sigma_u)^i_\beta - \frac{i}{6} g' U_\mu\, \sigma_u)^i_\alpha \right|^2$$

+ terms obtained by the substitutions

$$\Psi_p \to \Psi_x \quad , \quad \Psi_e \to \Psi_\mu \quad , \quad \sigma_u)^i_\alpha \to \sigma_\ell)^i_\alpha$$

$$+ \sum_{i=a,b,c} \left\{ \left[G_p\, \overline{p}_{i_R}\, \Phi^+ + G_n\, \overline{n}_{i_R}\, \Phi^T\!.\, B + \Delta_{n\lambda}\, \overline{\lambda}_{i_R}\, \Phi^T\!.\, B \right] \begin{pmatrix} p_i \\ n_i \end{pmatrix}_L \right. +$$

$$\left. + \left[\Delta_{n\lambda}\, \overline{n}_{i_R}\, \Phi^T\!.\, B + G_\lambda\, \overline{\lambda}_{i_R}\, \Phi^T\!.\, B + G_x\, \overline{x}_{i_R}\, \Phi^+ \right] \begin{pmatrix} x_i \\ \lambda_i \end{pmatrix}_L \right\} + h.c.$$

$$+ G_e\, \overline{e}_R\, \Phi^T\!.\, B.\Psi_e \quad + \quad G_\mu\, \overline{\mu}_R\, \Phi^T\!.\, B.\Psi_\mu \quad + \quad h.c.$$

$$- \quad V(\Phi^+, \Phi, \sigma_u^+, \sigma_u, \sigma_\ell^+, \sigma_\ell)$$

Where:

The postscript L,R on any fermion field denotes the left and right hand components respectively of the field[38], i.e.

$$\Psi_L = \frac{1-\gamma_5}{2}\Psi \quad \& \quad \Psi_R = \frac{1+\gamma_5}{2}\Psi$$

$$\vec{G}_{\mu\nu} = \partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu + g\, \vec{W}_\mu \wedge \vec{W}_\nu$$

$$\Xi^m_{\mu\nu} = \partial_\mu V_\nu^m - \partial_\nu V_\mu^m + f\, f_{mk\ell}\, V_\mu^k\, V_\nu^\ell$$

where $f_{mk\ell}$ are the structure constants for SU(3)

$\lambda^m, m = 1,2..8$ are the Gell-Mann matrices for the 3^* representation of SU(3).
$\tau^i, i = 1,2,3$ are the Pauli matrices.

$$(\tilde{Y}_R^h)_{(\alpha,\beta)(i,j)} = \frac{1}{6}(3\tau_3 + \mathbb{1})_{\alpha\beta}\,\delta_{ij} \quad , \quad (\tilde{Y}_R^\ell)_{\alpha\beta} = \begin{pmatrix} 0 & \\ & -1 \end{pmatrix}$$

and the matrix $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
The gauge invariant potential $V(\phi,\sigma)$ is constructed by including all gauge invariant terms up to the fourth degree in the Higgs fields. It has been assumed[36] that for appropriate interactions among the scalar fields the non-zero vacuum expectation values of the Higgs multiplets are given by

$$<\sigma_u> = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \end{pmatrix}, \qquad <\sigma_\ell> = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma \end{pmatrix} \qquad <\phi> = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}$$

where σ and λ are real and σ/λ is small of the order 10^{-2} to to 10^{-3}. The Higgs fields are redefined by $\sigma_u = <\sigma_u> + h_u$, $\sigma_\ell = <\sigma_\ell> + h_e$ and $\phi = <\phi> + \varphi$ so that φ, h_u and h_e have zero vacuum expectation values. On substituting the redefined Higgs fields in the Lagrangian $\mathcal{L}_1$, the following eigenstates and masses are obtained[36] for the gauge and fermion fields

$$V^\pm_{K^*} = \frac{V^4 \mp iV^5}{\sqrt{2}} \quad ; \quad M(V^\pm_{K^*}) = f\sigma$$

$$(V^0_{K^*}, \bar{V}^0_{K^*}) = \frac{V^6 \mp iV^7}{\sqrt{2}} \quad ; \quad M(V^0_{K^*}) = f\sigma$$

$$V^\pm_{\tilde{\rho}} = V^\pm_\rho \cos\delta + W^\pm \sin\delta \quad ; \quad M(V^\pm_{\tilde{\rho}}) = f\sigma + O(\sigma/\lambda)$$

$$\tilde{W}^\pm = W^\pm \cos\delta - V^\pm_\rho \sin\delta \quad ; \quad M(\tilde{W}^\pm) = \frac{g\lambda}{\sqrt{2}} + O(\sigma/\lambda)$$

where $W^{\pm} = \frac{W_1 \mp i W_2}{\sqrt{2}}$, $V_\rho^{\pm} = \frac{V_1 \mp i V_2}{\sqrt{2}}$ and

$$\tan 2\delta = -\frac{4 f \sigma^2}{g \lambda^2} \left[1 - \frac{2 f^2 \sigma^2}{g^2 \lambda^2} + \frac{3\sigma^2}{\lambda^2} \right] \quad \text{very small.}$$

The electromagnetic field A (massless) and heavy neutral boson Z^o will not be exhibited here. The neutral bosons orthogonal to these which interact strongly are

$$V_{X^o} = \frac{V_3 - \sqrt{3}\, V_8}{2} \quad , \quad M(V_{X^o}) = f\sigma$$

$$V_{Y^o} = \frac{g g'(g' W_3 + g U) - (3/4) f (g^2 + g'^2)(V_3 + V_8/\sqrt{3}) + O(\sigma/\lambda)}{\{ g^2 g'^2 (g^2 + g'^2) + \frac{3}{4} f^2 (g^2 + g'^2)^2 \}^{1/2}}$$

$$M(V_{Y^o}) = f\sigma + O(\sigma/\lambda)$$

Similarly for the fermions we have that the two neutrinos remain massless and $m_p = \lambda G_p$, $m_x = \lambda G_x$, $m_e = \lambda G_e$, $m_\mu = \lambda G_\mu$, and

$$\tilde{n}_i = \cos\theta\, n_i - \sin\theta\, \lambda_i \quad , \quad m_{\tilde{n}} = \lambda [G_n + G_\lambda - 2 \operatorname{cosec} 2\theta\, \Delta_{n\lambda}] \equiv \lambda G_{\tilde{n}}$$

$$\tilde{\lambda}_i = \cos\theta\, \lambda_i + \sin\theta\, n_i \quad , \quad m_{\tilde{\lambda}} = \lambda [G_n + G_\lambda + 2 \operatorname{cosec} 2\theta\, \Delta_{n\lambda}] \equiv \lambda G_{\tilde{\lambda}}$$

where θ is the Cabbibo angle, $\tan 2\theta = \frac{2\Delta_{n\lambda}}{G_\lambda - G_n}$

The gauge fixing Lagrangian $\mathcal{L}_2$ that is added to $\mathcal{L}_1$ is selected so that the bilinear terms in the gauge and Higgs fields of $\mathcal{L}_1$, which give rise to "mixed propagators", are cancelled[28].

The Lagrangian $\mathcal{L}_2$ must be supplemented by an effective Lagrangian $\mathcal{L}_3$ which describes the closed loop contributions of the ghost fields and can be obtained(27,28) by performing an infinitesimal gauge transformation on $\mathcal{L}_2$.

$$
\begin{aligned}
\mathcal{L}_2 = & -\xi \left| \partial_\mu V^{+}_{\tilde{\rho}\cdot\mu} + i \frac{M_{\tilde{\rho}}}{\xi} B^{+}_{\tilde{\rho}} \right|^2 - \xi \left| \partial_\mu \tilde{W}^{+}_\mu + i \frac{M_{\tilde{W}}}{\xi} B^{+}_{\tilde{W}} \right|^2 \\
& - \xi \left| \partial_\mu V^{+}_{K^*\mu} + i \frac{M_{K^*}}{\xi} B_{K^+} \right|^2 - \xi \left| \partial_\mu V^{0}_{K^*\mu} + i \frac{M_{K^0}}{\xi} B_{K^0} \right|^2 \\
& - \frac{\xi}{2} \left(\partial_\mu Z^0_\mu + \frac{M_{Z^0}}{\xi} B_{Z^0} \right)^2 - \frac{\xi}{2} \left(\partial_\mu V_{Y^0\mu} + \frac{M_{Y^0}}{\xi} B_{Y^0} \right)^2 \\
& - \frac{\xi}{2} \left(\partial_\mu V_{X^0\mu} + \frac{M_{X^0}}{\xi} B_{X^0} \right)^2 - \frac{\xi}{2} \left(\partial_\mu A_\mu \right)^2
\end{aligned}
$$

where the letter B with subscript of a given gauge boson denotes the corresponding Goldstone boson of the field. For example, for $B_{\tilde{W}}$ and $B_{\tilde{\rho}}$ we have

$$
B^{+}_{\tilde{W}} = \frac{1}{M_{\tilde{W}}} \left\{ \left[f\sigma \sin\delta - g\sigma\cos\delta \right] \frac{\left[(h_u)^2_1 - (h^{+}_u)^1_2 \right]}{\sqrt{2}} - g \frac{\cos\delta}{\sqrt{2}} \left[(\sigma h_e)^3_2 - \lambda \varphi^{+}_2 \right] \right\}
$$

$$
B^{+}_{\tilde{\rho}} = \frac{1}{M_{\tilde{\rho}}} \left\{ \left[f\sigma\cos\delta + g\sigma\sin\delta \right] \frac{\left[(h_u)^2_1 - (h^{+}_u)^1_2 \right]}{\sqrt{2}} + g \frac{\sin\delta}{\sqrt{2}} \left[(\sigma h_e)^3_2 - \lambda \varphi^{+}_2 \right] \right\}
$$

We let $\mathcal{L}_1$ represent the renormalized Lagrangian of the theory. The bare Lagrangian $\mathcal{L}^B_1$ has the same structure as $\mathcal{L}_1$, and we distinguish the unrenormalized parameters and fields by the superscript (B).

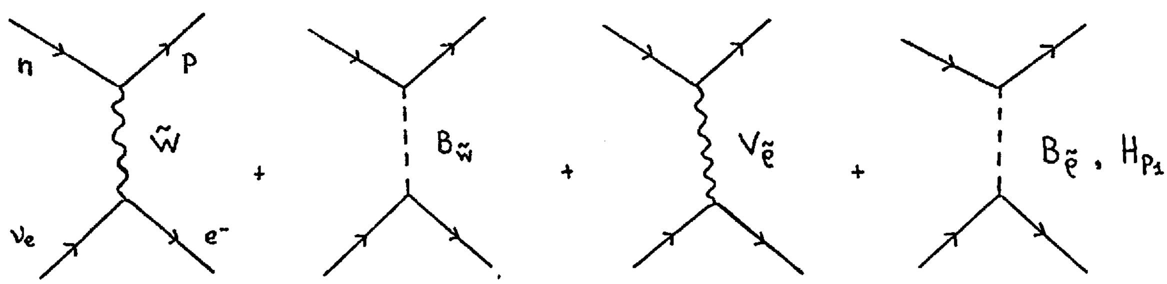
It has been shown that the gauge fixing terms remain unrenormalized[34], and hence the same gauge fixing Lagrangian $\mathcal{L}_2$ may be used for both the bare and renormalized theories. The only difference being that the mixing of the gauge fields with their corresponding Goldstone bosons for the bare Lagrangian do not cancel. On the other hand the effective Lagrangian for the ghost contributions is renormalized, the renormalized and bare Lagrangians being invariant under different set of gauge transformations[34], but this will not interest us here.

The counterterms of the theory are obtained by $\Delta\mathcal{L}_c = (\mathcal{L}_1^B + \mathcal{L}_3^B) - (\mathcal{L}_1 + \mathcal{L}_3)$ according to the usual prescription[34]. That is, there is one overall wavefunction renormalization for each multiplet. Similarly, all other subtractions for the coupling constants, masses and vacuum expectation values must be carried out in a consistent way with the gauge invariance and overall structure of the theory. So, in general, the renormalized quantities may differ from the physical ones by finite overall renormalizations. We shall firstly, though, consider the diagrams of order $f^2\ g^2$ contributing to the process $n\ \nu_e \to pe^-$ and then determine the counterterms needed according to the usual prescription.

The Feyman rules of the theory are manifestly renormalizable for $\xi \neq 0$ while for $\xi = 0$ all the unphysical scalar fields are transformed away and we obtain the unitary gauge. We shall work within the general renormalizable ξ-gauge formulation of the theory and consider the gauge independence of the matrix element for $n\nu_e \to pe^-$.

The Effective Semileptonic $\Delta S=0$ Interaction

The lowest order diagrams contributing to the semi-leptonic strangeness conserving ($\Delta S = 0$) process $n\ \nu_e \rightarrow p\ \ e^-$ are



D(1) D(2) D(3) D(4)

If we consider the effective coupling strengths of each of the above diagrams for small momentum transfer (Q^2) compared to the masses of the gauge fields, we obtain that:

Diagram (1) leads to an effective (V-A) current-current point interaction of strength ~ $g^2/M^2_{\tilde{W}}$ and also to (scalar+pseudoscalar) current-current type of point interaction suppressed by a factor ~ $1/M^2_{\tilde{W}}$ ($M_{\tilde{W}}$ ~ large) relatively to the (V-A) interaction.

Diagram (2) leads only to the (scalar+pseudoscalar) type of interaction suppressed also by a factor $1/M^2_{\tilde{W}}$ relative to the (V-A) interaction of D.1. The contribution of the Goldstone boson $B_{\tilde{W}}$ though cancels the term of the $\tilde{W}$-propagator having a negative metric scalar boson pole at $Q^2 = M^2/\xi$ (see relevant propagators in Fig. VIII). So although the $B_{\tilde{W}}$-contributions are suppressed, they will be retained because we shall be interested in the gauge independence of the matrix element (and as a check of our calculations).

Essentially D.4 represents two diagrams obtained by the exchange of (a) the Goldstone boson $B_{\tilde{\rho}}$ and (b) the physical charged scalar field Hp_1 which is orthogonal to the fields $B_{\tilde{w}}$ and $B_{\tilde{\rho}}$, and couples to fermions with strength $\sim g\sigma m_f/\lambda\ M_{\tilde{w}}$. ($m_f$ ~fermion mass).

The diagrams D.3 and D.4 arise only due to the small mixing of the fields $W^{\pm}$ and $V_{\rho}^{\pm}$, i.e. the small breaking of the SU(3") group for the strong interactions. They are suppressed by a factor $\sim \sin^2\delta\ M_w^2/M_v^2 \sim (\sigma^2/\lambda^2)$ relatively to D.1 and D.2 and are therefore neglected.

The Fermi coupling constant G_F to this approximation is

$$G_F/\sqrt{2} = g^2/8M_{\tilde{w}}^2 \quad \text{with } \cos\delta = 1$$

We now consider the diagrams of order f^2g^2 contributing to the above process. The Feyman rules of the theory are numerous but we find that most diagrams of the order f^2g^2 are suppressed by a factor σ^2/λ^2 and can be neglected. The only diagrams that need be considered are obtained from the diagrams D.1 and D.2 by

(a) an insertion of a self-energy part of order f^2 for the p and n quark fields, and

(b) the exchange of a gauge meson across the weak vertices for the n and p quark fields.

We also consider briefly the type of diagrams that have been neglected, i.e. suppressed at least by a factor σ^2/λ^2. These are

(c) the diagrams obtained from D.1 and D.2 by an insertion of a self-energy part, of order f^2, for the $\tilde{W}^+$ and $B_{\tilde{W}}$ propagators respectively, and

(d) diagrams obtained from the "3-point vertices" of the gauge fields, Higgs scalar fields and mixed (gauge and Higgs fields) of order f coupled directly to the "fermion lines" $n \to p$ and $\nu_e \to e^-$ (to the order fg^2), plus

(e) all possible f^2-corrections to the diagrams D.3 and D.4 and the diagrams obtained from further mixing of the $\tilde{W}^{\pm}$ and $V^{\pm}_{\tilde{\rho}}$ fields.

We have also neglected the f^2-corrections to the vacuum expectation value $\lambda(\simeq \lambda^B)$ due to the small mixing of the Higgs multiplet φ with the multiplets $\mathcal{h}$, since ϕ is an SU(3") singlet, and hence the corresponding f^2-corrections to the masses of the fermions.

The approximations are also made that the masses of the gauge mesons are all equal, represented by M_V, and similarly that the masses of the quarks p and $\tilde{n}$ are also equal.

The relevant counterterms needed to render finite the diagrams of order f^2g^2 retained are very few and simple. The infinite parts of the renormalizations of the parameters of the theory which are multiplied by the "suppression factor" σ^2/λ^2 will be effectively ignored.

Strong interactions conserve parity and the wavefunction renormalizations of the left and right hand fermion multiplets are equal. Also, no further mixing is obtained to the order f^2 for the $\tilde{n}$ and $\tilde{\lambda}$ quark fields and we may choose, for convenience, the same wavefunction renormalization $(1+Z_2^p)^{\frac{1}{2}}$ for both fermion multiplets ψ_p and ψ_χ.

The wavefunction renormalization $(1+Z_2^p)^{\frac{1}{2}}$ is defined so that the renormalized p-quark field has unit matrix element between the vacuum and the one particle p-state. In the approximation that we use $m_p = m_{\tilde{n}}$, this is also true for the corresponding matrix element of the renormalized ñ-quark field.

The mass-subtractions for the p and ñ quark-fields are also performed on the mass-shell. If we let $\delta m_{\tilde{n}} = m_{\tilde{n}} - m_{\tilde{n}}^B$ and $\delta m_p = m_p - m_p^B$ then from the bare and renormalized Lagrangians of the theory we have

$$\delta m_p = \lambda \delta G_p + G_p \, \delta\lambda \quad \& \quad \delta m_n = \lambda \delta G_{\tilde{n}} + G_{\tilde{n}} \, \delta\lambda$$

where $\delta G_p = G_p - G_p^B$, $\delta G_{\tilde{n}} = G_{\tilde{n}} - G_{\tilde{n}}^B$ and $\delta\lambda = \lambda - \lambda^B$. Hence for $\delta\lambda = 0$, δG_p and $\delta G_{\tilde{n}}$ (equal in this case for $m_p = m_{\tilde{n}}$) are determined once the mass counterterms have been computed.

Finally we consider the renormalization of the weak gauge coupling constant which can be determined, up to a finite constant, by considering any of the gauge interaction vertices of the theory. The effective semileptonic interaction to the order f^2g^2 to be computed will be expressed in terms of the physical coupling strength g for the vertex $\nu_e - e^- - \tilde{W}$. The strong corrections to the weak vertex $\nu_e - e^- - \tilde{W}$ are suppressed by a factor σ^2/λ^2 and to this approximation the weak gauge coupling constant remains "unrenormalized". We also note that this choice for the renormalized weak coupling constant is consistent with the constraint $\delta M_W = \delta(g\lambda)$ to the approximation considered.

The f^2 self-energy contributions to the p_a or $\tilde{n}_a$ quark fields are obtained from the diagrams

$$\Sigma^{g^2}(q) =$$ 

and performing the subtractions on the mass-shell (with $m_p = m_{\tilde{n}} \equiv m$), the mass and wavefunction counterterms are determined by

$$\Sigma^{g^2}(q)\Big|_{\not q = m} = -i\,\delta m \quad , \quad \frac{\partial \Sigma^{g^2}(q)}{\partial \not q}\Big|_{\not q = m} = -i\, Z_2^P \tag{1}$$

The relevant Feyman rules of the theory are given in Fig. VIII, we have (with cos δ = 1)

$$\Sigma^{g^2}(q) = g^2\left(1+\frac{\chi^2}{4}\right)\int d^n k \left\{ \gamma_\mu \frac{(-1)}{(\not k + \not q - m)} \gamma_\nu \frac{[g^{\mu\nu} - (1-1/\xi)\, k^\mu k^\nu/(k^2 - M_V^2/\xi)]}{(k^2 - M_V^2)} \right\}$$

After some algebra and using the Feyman parametrization we obtain

$$= g^2\left(1+\frac{\chi^2}{4}\right)\left[\int_0^1 dx \int d^n k \left\{ \frac{-[(2-n)\not q + nm + (2-n)\not k]}{[k^2 + 2x\, q.k + x(q^2-m^2) - (1-x)M_V^2]^2} - \frac{(1-1/\xi)\,(\not q - m)}{[k^2 - xM_V^2 - (1-x)\frac{M_V^2}{\xi}]^2} \right\} \right.$$

$$\left. + \int_0^1 dx \int d^n k \frac{(1-1/\xi)\,\not k}{[k^2 - xM_V^2 - (1-x)\frac{M_V^2}{\xi}]^2} + \int d^n k\, (1-1/\xi) \frac{[(q^2-m^2)(\not q - m) + (\not q - m)\not k(\not q - m)]}{[(k+q)^2 - m^2][k^2 - M_V^2]\,[k^2 - \frac{M_V^2}{\xi}]} \right]$$

The last term is neglected as it does not contribute to the subtractions performed on the mass-shell, while the third term is zero on integration. The integrations over the loop momenta are given in Appendix III, and neglecting terms that vanish at n = 4 we obtain

$$\Sigma^{g^2}(q) = g^2(1+\frac{\chi^2}{4})\frac{i}{16\pi^2}\Big\{\frac{6m\Gamma(3-n/2)}{(n-4)} - \frac{2(\not{q}-m)\Gamma(3-n/2)}{\xi\,(n-4)} - (\not{q}-m) + m +$$

$$+(4m-2\not{q})\int_0^1 \ln[xm^2+(1-x)M_v^2+x(x-1)q^2]\,dx + 2\not{q}\int_0^1 x\ln[xm^2+(1-x)M_v^2+x(x-1)q^2]\,dx$$

$$+(1-1/\xi)(\not{q}-m)\int_0^1 \ln[xM_v^2+(1-x)\frac{M_v^2}{\xi}]\,dx + O(q-m)^2\ldots\Big\}$$

Hence the counterterms defined by (1) are

$$\delta m = -\frac{g^2(1+\chi^2/4)m}{16\pi^2}\Big\{\frac{6\Gamma(3-n/2)}{(n-4)} + 1 + 2\int_0^1(1+x)\ln[(1-x)M_v^2+x^2m^2]\,dx\Big\} \tag{2}$$

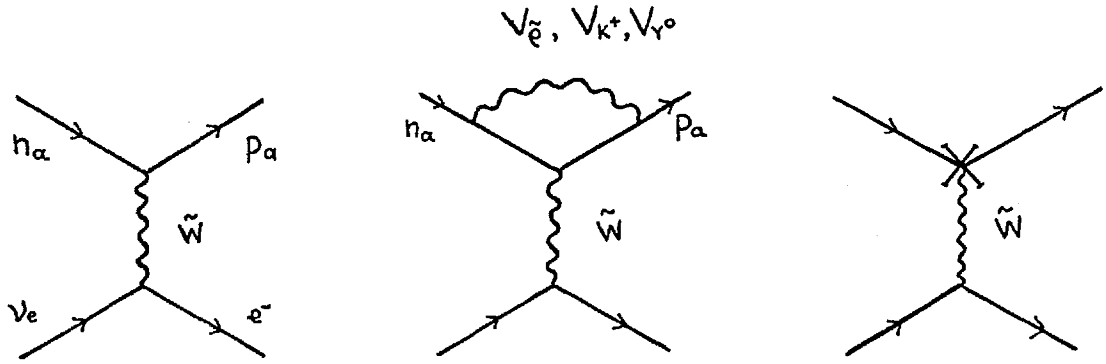
$$Z_2^P = \frac{g^2(1+\chi^2/4)}{16\pi^2}\Big\{\frac{2\Gamma(3-n/2)}{\xi\,(n-4)} + 1 - (1-\frac{1}{\xi})\int_0^1 \ln[xM_v^2+(1-x)\frac{M_v^2}{\xi}]\,dx +$$

$$+2\int_0^1(1-x)\ln[x^2m^2+(1-x)M_v^2]\,dx + 4m^2\int_0^1\frac{x(1-x^2)\,dx}{[x^2m^2+(1-x)M_v^2]}\Big\} \tag{3}$$

This completes the discussion on the subtractions that need be determined to the approximation considered. The counter-terms obtained from $\mathcal{L}_1^B - \mathcal{L}_1$ for the vertices $p_a - \tilde{n}_a - \tilde{W}$ and $p_a - \tilde{n}_a - B_{\tilde{w}}$ are given respectively by (with cos δ = 1 and $m_p = m_{\tilde{n}} \equiv m$)

$$\frac{i g\cos\theta}{2\sqrt{2}} Z_2^P \gamma^\mu(1-\gamma_5) \quad \text{and} \quad \frac{-i g\cos\theta}{\sqrt{2}\,M_{\tilde{w}}}(mZ_2^P - \delta m)\gamma_5$$

We now consider the effective semileptonic $\Delta S = 0$ interaction to the order $f^2 g^2$. The diagrams that have been retained to this order for the process $n_a(q_1)\ \nu_e(p_1) \rightarrow p_a(q_2)\ e^-(p_2)$ are



plus the corresponding set of diagrams obtained from the exchange of the Goldstone boson $B_{\tilde{W}}$ instead of the weak boson $\tilde{W}$. From the Feyman rules of the theory we have, with $Q_\mu \equiv (q_2 - q_1)_\mu$

$$A_{SL} = i\,\frac{g^2\cos\theta}{8}\,.\,\bar{u}(p_2)\gamma^\nu(1-\gamma_5)u(p_1)\,.\left[\frac{(g_{\nu\mu} - Q_\nu Q_\mu / M_{\tilde{W}}^2)}{(Q^2 - M_{\tilde{W}}^2)} + \frac{Q_\nu Q_\mu}{M_{\tilde{W}}^2\,(Q^2 - \frac{M_{\tilde{W}}^2}{3})}\right].$$

$$.\ \bar{u}(q_2)\left\{\gamma^\mu(1-\gamma_5) + I^\mu_{\tilde{W}} + \gamma^\mu(1-\gamma_5)Z_2^P\right\}u(q_1)\ +$$

$$-\,i\,\frac{g^2\cos\theta}{8}\,.\,\frac{m_e}{M_{\tilde{W}}^2}\,.\ \bar{u}(p_2)(1-\gamma_5)u(p_1)\,.\ \frac{1}{(Q^2 - M_{\tilde{W}}^2/3)}\ .$$

$$.\ \bar{u}(q_2)\left\{2m\gamma_5 + I_{B_{\tilde{W}}} + 2m\gamma_5 Z_2^P - 2\,\delta m\,\gamma_5\right\}u(q_1) \qquad (4)$$

or in terms of the weak current form factors[31] to be computed, and for small momentum transfer, $Q^2/m_W^2 \sim 0$

$$A_{SL} = i \frac{G_F}{\sqrt{2}} \cos\theta \,.\, \bar{u}(p_2)\gamma^\mu(1-\gamma_5)u(p_1) \,.\, \bar{u}(q_2)\Big\{ g_V(Q^2)\gamma_\mu -$$

$$- g_A(Q^2)\gamma_\mu\gamma_5 + i f_V(Q^2)\sigma_{\mu\nu}Q^\nu + h_V(Q^2)Q_\mu + f_A(Q^2)\gamma_5 Q_\mu +$$

$$+ i h_A(Q^2)\gamma_5\sigma_{\mu\nu}Q^\nu \Big\} u(q_1) + O\Big(\frac{Q^2}{M_{\tilde{W}}^2}, \frac{m\, m_e}{M_{\tilde{W}}^2}\Big) \tag{5}$$

Where $I^\mu_{\tilde{W}}$ and $I_{B_{\tilde{W}}}$ denote, apart from a numerical factor, the f^2 corrections to the vertices p_a-$\tilde{n}_a$-$\tilde{W}$ and p_a-$\tilde{n}_a$-$B_{\tilde{W}}$, respectively, which we shall now proceed to calculate. We consider firstly the f^2-contributions to the process denoted by $I^\mu_{\tilde{W}}$; which is understood to be "sandwiched" between the two spinors for the $\tilde{n}_a$ and p_a quark fields.

$$I^\mu_{\tilde{W}} = -i f^2 (1+\frac{\chi^2}{4}) \int d^n k \Big\{ \gamma^\rho \frac{1}{\not{q}_2+\not{k}-m} \gamma^\mu(1-\gamma_5) \frac{1}{\not{q}_1+\not{k}-m} \gamma^\sigma \,.$$

$$. \frac{[g_{\rho\sigma} - (1-1/\xi) k_\rho k_\sigma/(k^2 - M_V^2/\xi)]}{(k^2 - M_V^2)} \Big\}$$

After some rearrangement of terms and using the equations of motion for the spinors, the integral simplifies to

$$I^\mu_{\tilde{W}} = -i f^2 (1+\frac{\chi^2}{4}) \int d^n k \Big\{ \frac{(2q_2^\sigma + \gamma^\sigma \not{k})\gamma^\mu(1-\gamma_5)(2q_{1\sigma} + \not{k}\gamma_\sigma)}{(k^2+2k.q_2)(k^2+2k.q_1)(k^2-M_V^2)} -$$

$$- \gamma^\mu(1-\gamma_5) \frac{(1-1/\xi)}{(k^2-M_V^2)(k^2-M_V^2/\xi)} \Big\}$$

and after some algebra of the γ-matrices, the integral may be cast into the following form

$$I^\mu_{\tilde{w}} = -i g^2 \left(1+\frac{\chi^2}{4}\right) \int d^n k \Bigg\{ \frac{[\{4(q_1.q_2)+2k.(q_1+q_2)+(n-4)k^2\}\gamma^\mu + 2(2-n)k^\mu \not{k}](1-\gamma_5)}{(k^2+2k.q_2)\;(k^2+2k.q_1)\;(k^2-M_v^2)} +$$

$$+\gamma^\mu(1-\gamma_5)\left[\frac{1}{(k^2+2k.q_1)}+\frac{1}{(k^2+2k.q_2)}\right]\frac{1}{(k^2-M_v^2)} +$$

$$+\frac{2m\{2k^\mu-(\gamma^\mu \not{k} - \not{k}\gamma^\mu)\gamma_5\} - 4(q_1+q_2)^\mu \not{k}(1-\gamma_5)}{(k^2+2k.q_2)\;(k^2+2k.q_1)\;(k^2-M_v^2)} +$$

$$-\gamma^\mu(1-\gamma_5)(1-1/\xi)\frac{1}{(k^2-M_v^2)\;(k^2-M_v^2/\xi)}\Bigg\}$$

We now combine the denominators by the standard Feyman parameter technique and perform the integrations. Using the results of Appendix III we obtain

$$I^\mu_{\tilde{w}} = g^2\left(1+\frac{\chi^2}{4}\right)\frac{\pi^{n/2}}{(2\pi)^n}\Bigg\{\int_0^1 dx_1\int_0^{x_1}dx_2\,(-1)\,\gamma^\mu(1-\gamma_5)\Bigg[\frac{\{4(q_1.q_2)-2(q_1+q_2).(x_2q_1+(x_1-x_2)q_2)\}\,\Gamma(3-n/2)}{D^{3-n/2}}$$

$$+(n-4)\left\{\frac{(x_2q_1+(x_1-x_2)q_2)^2\,\Gamma(3-n/2)}{D^{3-n/2}} - \frac{n}{2}\,\frac{\Gamma(2-n/2)}{D^{2-n/2}}\right\}\Bigg] +$$

$$-2(2-n)\int_0^1 dx_1\int_0^{x_1}dx_2\left[\frac{(x_2q_1^\mu+(x_1-x_2)q_2^\mu)(x_2\not{q}_1+(x_1-x_2)\not{q}_2)\,\Gamma(3-n/2)}{D^{3-n/2}} - \frac{\gamma^\mu}{2}\,\frac{\Gamma(2-n/2)}{D^{2-n/2}}\right](1-\gamma_5) +$$

$$+\gamma^\mu(1-\gamma_5)\,\Gamma(2-n/2)\int_0^1 dx\left[\frac{1}{[(1-x)M_v^2+x^2q_2^2]^{2-n/2}}+\frac{1}{[(1-x)M_v^2+x^2q_1^2]^{2-n/2}}\right] +$$

$$+2m\,\Gamma(3-n/2)\int_0^1 dx_1\int_0^{x_1}dx_2\;\frac{\{2(x_2q_1^\mu+(x_1-x_2)q_2^\mu)-[\gamma^\mu,\gamma^\nu](x_2q_1+(x_1-x_2)q_2)_\nu\,\gamma_5\}}{D^{3-n/2}}$$

$$-4\,(q_1+q_2)^\mu\int_0^1 dx_1\int_0^{x_1}dx_2\;\frac{(x_2\not{q}_1+(x_1-x_2)\not{q}_2)\,(1-\gamma_5)}{D^{3-n/2}}$$

$$-\,(1-\tfrac{1}{\xi})\,\gamma^\mu(1-\gamma_5)\,\Gamma(2-n/2)\int_0^1\frac{dx}{[\,xM_V^2+(1-x)M_V^2/\xi\,]^{2-n/2}}\quad\Big\}$$

where $D=[(1-x_1)M_V^2+(x_2q_1+(x_1-x_2)q_2)^2]$

$$=[(1-x_1)M_V^2+x_1x_2m_{\tilde n}^2+(x_1^2-x_1x_2)m_p^2+x_2(x_2-x_1)Q^2]$$

We note that to the approximation $m_{\tilde n}=m_p$, the integrals over the variable X_2 whose numerators are odd power functions in $(2X_2-X_1)$ are zero. Hence it is found that to this approximation no contributions are obtained for the weak form factors $h_V(Q^2)$ and $h_A(Q^2)$ associated with second class currents(31).

The integral $I^\mu_{\tilde w}$ is expanded about $n=4$ and terms that vanish at $n=4$ are neglected. We also cast $I^\mu_{\tilde w}$ in the standard form of independent weak form factors using the identities

$$\bar u(q_2)\,\{(q_1+q_2)^\mu=2m\gamma^\mu-i\sigma^{\mu\nu}Q_\nu\}\,u(q_1)$$

$$\bar u(q_2)\,\{Q^\mu\gamma_5\;=2m\gamma^\mu\gamma_5-i\sigma^{\mu\nu}(q_2+q_1)_\nu\gamma_5\}\,u(q_1)$$

$$I^\mu_{\tilde w}=\frac{g^2(1+X^2/4)}{16\pi^2}\Big\{-\frac{2\gamma^\mu(1-\gamma_5)\Gamma(3-n/2)}{\xi\;(n-4)}+(1-\tfrac{1}{\xi})\gamma^\mu(1-\gamma_5)\int_0^1 dx\,\ln[xM_V^2+(1-x)\frac{M_V^2}{\xi}]\;+$$

$$-2\gamma^\mu(1-\gamma_5)\int_0^1 dx\ \ln[(1-x)M_v^2+x^2m^2] -2(2m^2-Q^2)\gamma^\mu(1-\gamma_5)\int_0^1 dx_1\int_0^{x_1}\frac{dx_2}{D} +$$

$$-\gamma^\mu(1-\gamma_5)+\gamma^\mu(1-\gamma_5)\int_0^1 dx_1\int_0^{x_1}dx_2\left[\frac{-Q^2x_1}{D}+2\ln D\right]+4m^2\gamma^\mu\int_0^1 dx_1\int_0^{x_1}dx_2\ \frac{x_1^2}{D} +$$

$$+2im\sigma^{\mu\nu}Q_\nu\int_0^1 dx_1\int_0^{x_1}dx_2\ \frac{x_1(1-x_1)}{D} -2mQ^\mu\gamma_5\int_0^1 dx_1\int_0^{x_1}dx_2\ \frac{x_1+(2x_2-x_1)^2}{D}\ \Big\}$$

Hence, using the expression (3) for Z_2^P we obtain that

$$I_{\tilde{W}}^\mu+\gamma^\mu(1-\gamma_5)Z_2^P = \frac{g^2(1+\chi^2/4)}{16\pi^2}\Big\{\gamma^\mu(1-\gamma_5)\Big[\int_0^1 dx_1\int_0^{x_1}dx_2\Big\{\frac{Q^2(2-x_1)-4m^2}{D}+2\ln D\Big\} +$$

$$-2\int_0^1 x\ln[x^2m^2+(1-x)M_v^2]\,dx+4m^2\int_0^1\frac{x(1-x^2)\,dx}{[x^2m^2+(1-x)M_v^2]}\Big]+\gamma^\mu\int_0^1 dx_1\int_0^{x_1}dx_2\ \frac{4m^2x_1^2}{D}$$

$$+2im\sigma^{\mu\nu}Q_\nu\int_0^1 dx_1\int_0^{x_1}dx_2\ \frac{x_1(1-x_1)}{D} -2mQ^\mu\gamma_5\int_0^1 dx_1\int_0^{x_1}dx_2\ \frac{x_1+(2x_2-x_1)^2}{D}\Big\} \tag{6}$$

We observe that (6) is finite and gauge independent. For gauge independence of the matrix element for $n_a\,\nu_e \to p_a\,e^-$ we must have, from (4)

$$\bar{u}(q_2)\Big\{Q_\mu[I_{\tilde{W}}^\mu+\gamma^\mu(1-\gamma_5)Z_2^P]+[I_{B\tilde{W}}+2\gamma_5(mZ_2^P-\delta m)]\Big\}u(q_1)=0 \tag{7}$$

From (6) we obtain that

$$Q_\mu[I_{\tilde{W}}^\mu+\gamma^\mu(1-\gamma_5)Z_2^P]=g^2\frac{(1+\chi^2/4)}{16\pi^2}\ 2m\gamma_5\Big\{\int_0^1 dx_1\int_0^{x_1}dx_2\Big[\frac{4m^2-2Q^2}{D}+2\ln D\Big] +$$

$$+2\int_0^1 x\,\ln[x^2m^2+(1-x)M_v^2]\,dx - 4m^2\int_0^1 \frac{x(1-x)\,dx}{[x^2m^2+(1-x)M_v^2]} - Q^2\int_0^1 dx_1\int_0^{x_1} dx_2\,\frac{(2x_2-x_1)^2}{D}\ \Big\} \tag{8}$$

In order to check the condition (7) we now calculate the contributions denoted by $I_{B_{\tilde{W}}}$

$$I_{B_{\tilde{W}}} = -i f^2\left(1+\frac{X^2}{4}\right)\int d^nk\,\Big\{\ \gamma^\rho\,\frac{1}{(\not q_2+\not k-m)}\,2m\gamma_5\,\frac{1}{(\not q_1+\not k-m)}\,\gamma^\sigma\ \times$$

$$\frac{[g_{\rho\sigma}-(1-1/\xi)\,k_\rho k_\sigma/(k^2-M_v^2/\xi)]}{(k^2-M_v^2)}\ \Big\}$$

After some algebra and using the equations of motion for the spinors, the integral may be cast in the following form

$$I_{B_{\tilde{W}}} = -i f^2\left(1+\frac{X^2}{4}\right)\ 2m\gamma_5\int d^nk\,\Big\{\ \frac{4(q_1\cdot q_2)+(n-4)\,k^2}{(k^2+2k\cdot q_2)\,(k^2+2k\cdot q_1)\,(k^2-M_v^2)}\ +$$

$$+\ \frac{2}{(k^2+2k\cdot q_1)(k^2-M_v^2)} + \frac{2}{(k^2+2k\cdot q_2)(k^2-M_v^2)} - \frac{(1-1/\xi)}{(k^2-M_v^2)(k^2-M_v^2/\xi)}\ \Big\}$$

Upon integrating this and neglecting terms that vanish at $n = 4$ we obtain

$$I_{B_{\tilde{W}}} = \frac{f^2(1+X^2/4)}{16\pi^2}\ 2m\gamma_5\ \Big\{\ \frac{-2(3+1/\xi)\,\Gamma(3-n/2)}{(n-4)} - 2 + \left(1-\frac{1}{\xi}\right)\int_0^1 dx\,\left[xM_v^2+(1-x)\frac{M_v^2}{\xi}\right]\ +$$

$$-\ 4\int_0^1 dx\,\ln[x^2m^2+(1-x)M_v^2] - (2m^2-Q^2)\int_0^1 dx_1\int_0^{x_1} dx_2\,\frac{1}{D}\ \Big\}$$

Hence we obtain that, using (2) and (3)

$$\{ I_{B\tilde{W}} + 2\gamma_5(m Z_2^P - \delta m)\} = \frac{f^2(1+\kappa^2/4)}{16\pi^2}\, 2m\gamma_5 \left\{ -2(2m^2 - Q^2)\int_0^1 dx_1 \int_0^{x_1} dx_2 \frac{1}{D} + \right.$$

$$\left. + 4m^2 \int_0^1 \frac{x(1-x^2)\,dx}{[x^2 m^2 + (1-x)M_V^2]} \right\} \qquad (9)$$

Comparing (8) with (9), and after a simple integration by parts for (8), we obtain that the condition (7), for gauge independence of the matrix element (4), is satisfied.

The weak current form factors to the order f^2 for the hadronic $\Delta S = 0$ weak current may be easily obtained by comparing (4) with (5), where $(I_w^\mu + \gamma^\mu(1-\gamma_5)Z_2^P)$ has been determined to this order and is given by (6). For very small or at zero momentum transfer only the vector and axial vector parts of the hadronic weak current contribute to the effective semileptonic interaction considered. Setting the momentum transfer equal to zero ($Q^2 = 0$) in (6) we note that the "denominator" $D/_{Q^2=0} = \left((1-X_1)M_V^2 + X_1^2 m^2\right)$, and it is easily seen that

$$g_V(Q^2=0) = 1 \quad , \qquad g_A(Q^2=0) = 1 - \frac{f^2(1+\kappa^2/4)}{16\pi^2}\int_0^1 \frac{4m^2 x^3\,dx}{[(1-x)M_V^2 + x^2 m^2]}$$

Hence, to the approximations that have been made, the vector part of the current remains unrenormalized while the axial vector part is renormalized. Furthermore we note that the finite renormalization to the axial vector part has a negative overall sign, integral positive, so that $g_A(Q^2=0) < 1$ and also, since the mass of the gauge mesons M_V is assumed[36] to be only a few GeV {3-4}, "the correction" is large for large f. So we face the problem of having considered a perturbation expansion with respect to the strong coupling constant. On the other hand the result obtained for the vector part of the current, which is equivalent to the Ward identity $Z_1 = Z_2$ for quantum electrodynamics, is expected to hold approximately to all orders in f, on the following qualitative argument[39]. We note that the groups SU(2') and SU(3") for the weak and strong interactions commute. Hence if there was no mixing between the gauge fields W_μ^i and V_μ^m, the strongly interacting gauge mesons (linear combinations of V_μ^m only) would behave as a set of U(1) bosons in the SU(2')-space (i.e. singlets of SU(2)) and the standard result for Q.E.D. $Z_1 = Z_2$ would apply to them to all orders in f; for $m_p = m_n$. Mixing between the two sets of gauge fields does occur, though, but since the mixing obtained is small - only the photon spans all spaces "equally" - the above argument still holds but only approximately.

APPENDIX I

In this Appendix we define the notation we use for the Dirac matrices and give a few supplementary details for handling products of superfields. We refer the reader to the reference (11) for a very comprehensive list of the properties of superfields - our notational conventions are the same.

The metric tensor $g^{\mu\nu} = g_{\mu\nu} = \text{diag.}\ (+1, -1, -1, -1)$

The Dirac matrices satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

and the matrix γ_5 is defined by $\gamma_5 = \gamma_0\ \gamma_1\ \gamma_2\ \gamma_3$

The Γ-matrices γ_0, $\gamma_0\ \gamma_5$, $\gamma_0\ \gamma_\mu$, $i\ \gamma^o\ \gamma_\mu\ \gamma_5$ and $\gamma_o\ \sigma_{\mu\nu} = \frac{i}{2}\ \gamma_0\left(\gamma_\mu, \gamma_\nu\right)$ are hermitian. The adjoint spinors are defined by $\bar{\psi} = \psi^+\gamma_0$ and the charge conjugate of ψ defined by $\psi^c = C\ \bar{\psi}^T$ where $C^T = -C$ and $C^{-1}\ \gamma_\mu\ C = -\gamma_\mu{}^T$; and the superscript (T) denotes the transpose of a matrix. We note that if ψ and χ are anticommuting Majorana spinors then $\bar{\psi}\ \chi = \chi\ \bar{\psi}$, $\bar{\psi}\ \gamma_5\ \chi = \bar{\chi}\ \gamma_5\ \psi$, $\bar{\psi}\ \gamma_\mu\ \chi = -\bar{\chi}\ \gamma_\mu\ \psi$, $\bar{\psi}\ i\gamma_\mu\ \gamma_5\ \chi = \bar{\chi}\ i\gamma_\mu\ \gamma_5\ \psi$ and $\bar{\psi}\ \sigma_{\mu\nu}\ \chi = -\chi\ \sigma_{\mu\nu}\ \psi$. Using the completeness property of the Γ-matrices and anticommutativity of the Majorana spinor θ_α we obtain

$$\Theta_\alpha \bar{\Theta}_\beta = -\frac{1}{4}\delta_{\alpha\beta}\bar{\Theta}\Theta + \frac{1}{4}(\gamma_5)_{\alpha\beta}\bar{\Theta}\gamma_5\Theta + \frac{1}{4}i(\gamma_\nu\gamma_5)_{\alpha\beta}\bar{\Theta}i\gamma^\nu\gamma_5\Theta$$

This identity may be used to obtain further relations for the Majorana spinor θ_α, which are used in "choosing" an independent basis for expanding the superfields and in considering their products.

Also, for the covariant derivative D_α, given in section I.b, we have the following two properties: (a) distributive law $D_\alpha(\phi_1 \ \phi_2) = (D_\alpha \ \phi_1)\phi_2 \pm \phi_1(D_\alpha \ \phi_2)$ with +(-) sign according to whether ϕ_1 is bosonic (fermionic) superfield, and (b) the covariant derivatives neither commute nor anticommute - their anticommutator is given by

$$\{ D_\alpha , D_\beta \} = - (\gamma^\mu C)_{\alpha\beta} \ i \frac{\partial}{\partial x^\mu}$$

APPENDIX II

In section II it was stated that only the simple pole residues of the bare coupling constants are independent[6]. This was a crucial factor in our proof, on the equivalence of the "single coupling constant renormalization problem" with the Chang eigenvalue conditions for solutions of the special type, and we wish to elaborate the point further. In particular we shall show that given the conditions (2.41) and (2.42) of section II, then, the whole system of conditions obtained from (2.39) and (2.40) is satisfied.

The recursion formula obtained by t'Hooft[6] for the residues of the bare "parameters" λ_B^K of a theory, relative to the poles at n = 4, is given by (in the notation of reference 6)

$$\sum_\ell \alpha^K_{\nu+1,\ell}\, \rho_{(\ell)}\, \lambda^\ell_R - \rho_{(K)}\, \alpha^K_{\nu+1} = \sum_\ell \alpha^K_{\nu,\ell} \Big\{ \sigma_{(\ell)}\, \lambda^\ell_R - \rho_{(\ell)}\, \alpha^\ell_1 + \sum_m \alpha^\ell_{1,m}\, \rho_{(m)}\, \lambda^m_R \Big\} - \sigma_{(K)}\, \alpha^K_\nu \qquad (1)$$

where:

(a) $$\alpha^K_{\nu,\ell} \equiv \frac{\partial \alpha^K_\nu}{\partial \lambda^\ell_R}$$

α^K_ν being the ν^{th}-order residue of the bare parameter λ^K_B denoting a bare coupling constant or mass of the theory, and is a function of the renormalized coupling constants and masses of the theory, λ^K_R for K = 1,2... (to a total given number, characteristic of the theory considered)

(b) $$\sigma_{(K)} + \rho_{(K)}(4-n) \equiv D_{(K)}$$

where $D_{(K)}$ gives the dimensions of the parameter λ_B^K for the theory in units of mass (μ).

For the particular theory considered in section II, we have:

(c) λ_R^ℓ, $\ell = 1,2 \ldots p+q+1, p+q+2 \ldots p+q+1+\beta$; stands for the renormalized set of coupling constants of the theory $\{g;\ h_i,\ i = 1,2..p;\ \ \lambda_\alpha,\ \alpha = 1,2...q\}$ and for a total no. β of masses and coupling constants for the self-cubic scalar vertices of the theory.

(d) $\alpha^K_{\nu,\ell} = 0$ for all $\ell = p+q+2...p+q+1+\beta$ when $K = 1,2..p+q+1.$; i.e. independence of the residues of the dimensionless, in 4-dimensions, bare coupling constants on the masses and coupling constants of the self-cubic scalar vertices of the theory[6,13].

(e) $\sigma_{(K)} = 0$ for the coupling constants g; h_i, $i = 1,2..p$ and λ_α, $\alpha = 1,2..q$

$$\rho_{(K)} = \begin{cases} \frac{1}{2}\,; & \text{for the coupling constants } g \text{ and } h_i,\ i = 1,2..p \\ 1\,; & \text{" \quad " \quad " \quad " } \quad \lambda_\alpha,\ \alpha = 1,2..q \end{cases}$$

The total set of conditions obtained from the identities (2.39) and (2.40) of section II for the single coupling constant renormalization problem is given by

$$\alpha_\nu^{h_i} = x_i \, \alpha_\nu^g \qquad \text{for } i = 1,2..p \qquad (2)$$

$$\alpha_\nu^{\lambda_\alpha} = \sum_{r=0}^{\nu} y_\alpha \, \alpha_r^g \, \alpha_{\nu-r}^g \qquad \text{for } \alpha = 1,2..q \qquad (3)$$

for all $\nu = 1,2,..\infty$, where we have defined for convenience $\alpha_o^g \equiv g$.

We pointed out that as a consequence of the recurssion formula (1) the system of conditions (2) and (3) <u>for $\nu = 1$ only</u> is exactly equivalent to the total set for $\nu = 1,2,..\infty$; i.e. for $\nu = 1$

$$\alpha_1^{h_i} = x_i \, \alpha_1^g \qquad \text{for } i = 1,2..p \qquad (4)$$

$$\alpha_1^{\lambda_\alpha} = 2 \, y_\alpha \, g \, \alpha_1^g \qquad \text{for } \alpha = 1,2..q \qquad (5)$$

and when the conditions (4) and (5) are satisfied,the coupling constants of the Yukawa and self-quartic scalar vertices of the theory are given by

$$h_i = x_i \, g \quad , \quad \lambda_\alpha = y_\alpha \, g^2 \qquad \text{for } i = 1,2..p \text{ and } \alpha = 1,2..q \qquad (6)$$

to all orders, order by order, in perturbation theory including the bare set.

We wish to show using the recurssion formula (1) that, if the conditions (4) and (5) are satisfied, then all higher order residues of the bare coupling constants for the Yukawa and self-quartic scalar vertices are given by the expressions (2) and (3), respectively. The proof given will be by induction.

We shall assume that the expressions (2) and (3) are valid for some $\nu = m$ $(m > 1)$ and show that they hold for $\nu = m+1$. In this case they must hold for any ν since they hold for $\nu = 1$; i.e. the conditions (4) and (5).

The total derivative with respect to g of any function in the coupling constants g, $\{h_i\}$ and $\{\lambda_\alpha\}$, with $\{h_i\}$ and $\{\lambda_\alpha\}$ given by (6), is

$$\frac{d}{dg} = \frac{\partial}{\partial g} + \sum_{i=1}^{p} x_i \frac{\partial}{\partial h_i} + 2g \sum_{\alpha=1}^{q} y_\alpha \frac{\partial}{\partial \lambda_\alpha}$$

Hence we obtain directly, using (1.d), (1.e) and the relations (6)

$$\sum_m a^{\ell}_{\nu,m}\, \rho_{(m)}\, \lambda^m_R = \frac{1}{2} g \frac{d}{dg} a^{\ell}_{\nu} \qquad (7)$$

only for $\ell = 1,2..p+q+1$.

The recurssion formula (1), for $K = 1,2..p+q+1$ only, may be simplified using the conditions (4) and (5), the property (1.d) and the relations for the coupling constants (6). We obtain by (1.d), (1.e) and (7), for $K = 1,2..p+q+1$

$$\frac{1}{2} g \frac{d}{dg} a^{K}_{\nu+1} - \rho_{(K)}\, a^{K}_{\nu+1} = \sum_{\ell=1}^{p+q+1} a^{K}_{\nu,\ell} \left\{ -\rho_{(\ell)}\, a^{\ell}_{1} + \frac{1}{2} g \frac{d}{dg} a^{\ell}_{1} \right\}$$

and using (4) and (5)

$$\frac{1}{2} g \frac{d}{dg} a^{K}_{\nu+1} - \rho_{(K)}\, a^{K}_{\nu+1} = \frac{d}{dg} a^{K}_{\nu} \left\{ -\frac{1}{2} a^{g}_{1} + \frac{1}{2} g \frac{d}{dg} a^{g}_{1} \right\} \qquad (8)$$

We obtain from the recurssion formula (8) that the residues of the gauge coupling constant g satisfy

$$g\frac{d}{dg}\alpha^{g}_{\nu+1} - \alpha^{g}_{\nu+1} = \frac{d}{dg}\alpha^{g}_{\nu}\left\{-\alpha^{g}_{1} + g\frac{d}{dg}\alpha^{g}_{1}\right\} \tag{9}$$

for all $\nu = 1,2..\infty$, and we note that it holds identically for $\nu = 0$, $\alpha^{g}_{o} = g$.

It is now assumed that the residue $\alpha^{h_i}_{m}$ is given by (2), and we obtain from the recurssion formula (8) for $\alpha^{h_i}_{m+1}$

$$g\frac{d}{dg}\alpha^{h_i}_{m+1} - \alpha^{h_i}_{m+1} = x_i\frac{d}{dg}\alpha^{g}_{m}\left\{-\alpha^{g}_{1} + g\frac{d}{dg}\alpha^{g}_{1}\right\} \qquad \text{by (2)}$$

for $i = 1,2..p$, and using the recurssion formula (9)

$$g\frac{d}{dg}\alpha^{h_i}_{m+1} - \alpha^{h_i}_{m+1} = x_i\left\{g\frac{d}{dg}\alpha^{g}_{m+1} - \alpha^{g}_{m+1}\right\} \tag{10}$$

The general solution of the differential equation (10) is given by

$$\alpha^{h_i}_{m+1} - x_i\alpha^{g}_{m+1} = A_i\, g \qquad i = 1,2..p$$

where A_i are the undetermined constants of integration. It is known that even the lowest order subtractions that we need perform to render the vertices of the theory finite are of the order g^3, hence $A_i = 0$. This concludes the proof, by induction, for the residues of the Yukawa coupling constants to be given by the expression (2), provided that it holds for $\nu = 1$; i.e. condition (4).

Similarly we assume that the residue $\alpha^{\lambda\alpha}_{m}$ is given by the expression (3), and obtain from the recurssion formula (8) for $\alpha^{\lambda\alpha}_{m+1}$

$$\frac{1}{2} g \frac{d}{dg} a^{\lambda_\alpha}_{m+1} - a^{\lambda_\alpha}_{m+1} = 2 y_\alpha \sum_{r=0}^{m} \frac{d}{dg} a^g_r a^g_{m-r} \left\{ -\frac{1}{2} a^g_1 + \frac{g}{2} \frac{d}{dg} a^g_1 \right\} \quad \text{by (3)}$$

$$= y_\alpha \sum_{r=0}^{m} \left\{ g \frac{d}{dg} a^g_{r+1} - a^g_{r+1} \right\} a^g_{m-r} \quad \text{by (9)}$$

(letting s=r+1) $$= y_\alpha \sum_{s=1}^{m+1} g \frac{d}{dg} a^g_s a^g_{m+1-s} - y_\alpha \sum_{s=1}^{m+1} a^g_s a^g_{m+1-s}$$

Hence, since by definition $a^g_0 \equiv g$, we obtain

$$\left[\frac{1}{2} g \frac{d}{dg} - 1 \right] \left\{ a^{\lambda_\alpha}_{m+1} - y_\alpha \sum_{s=0}^{m+1} a^g_s a^g_{m+1-s} \right\} = 0 \qquad (11)$$

for all $\alpha = 1,2..q$

The general solution of the differential equation (11) is given by

$$a^{\lambda_\alpha}_{m+1} - y_\alpha \sum_{s=0}^{m+1} a^g_s a^g_{m+1-s} = B_\alpha g^2 \quad , \alpha = 1,2\ldots q$$

where B_α are the undetermined constants of integration. Using similar arguments as in the case for the Yukawa vertices, we find that $B_\alpha = 0$. Hence, by induction, all the residues of the self-quartic scalar coupling constants are given by the expression (3) provided that the condition (5) holds. Q.E.D.

APPENDIX III

The regularization method of analytically continuing the dimensions of space-time[12,32], denoted by the letter n, has been used in all our work. The infinite contributions in four dimensions of the loop-momentum integrals are obtained as poles at n = 4.

In this Appendix we give a list of the loop-momentum integrals encountered in our calculations and the final expressions obtained as functions of n on performing the integrations. We also list some identities for the Dirac γ-matrices in n-dimensions which are used to simplify the numerators of the integrals when spinor fields are involved.

Identities and properties of the γ-matrices

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \mathbb{1} \quad ; \quad g^\mu_\mu = n \quad ; \quad \mathrm{Tr}[\mathbb{1}] = 2^{n/2}$$

$$\gamma^\mu \gamma^\nu \gamma_\mu = (2-n)\gamma^\nu$$

$$\gamma^\mu \gamma^\rho \gamma^\sigma \gamma_\mu = (n-4)\gamma^\rho \gamma^\sigma + 4 g^{\rho\sigma}$$

$$\gamma^\mu \gamma^\rho \gamma^\sigma \gamma^\nu \gamma_\mu = (6-n)\gamma^\rho \gamma^\sigma \gamma^\nu - 4(g^{\sigma\nu}\gamma^\rho + g^{\rho\sigma}\gamma^\nu - g^{\rho\nu}\gamma^\sigma)$$

$$= (n-6)\gamma^\nu \gamma^\sigma \gamma^\rho + 2(4-n)(g^{\sigma\rho}\gamma^\nu + g^{\nu\sigma}\gamma^\rho - g^{\rho\nu}\gamma^\sigma)$$

When the model considered does not have the Bell-Adler-Jackiw triangle anomalies[40], the γ_5 matrix may be taken to anticommute with all the Dirac γ-matrices. This definition has been adopted in our work; for a generalization of the γ_5-matrix in n-dimensions see reference (41).

The loop-momentum integrals in n-dimensions

$$\int d^n k \, \frac{1}{[k^2 + 2p.k - M^2]^\alpha} = \frac{i\,\pi^{n/2}\,(-1)^\alpha\,\Gamma(\alpha - n/2)}{\Gamma(\alpha)\quad (M^2 + p^2)^{\alpha - n/2}}$$

$$\int d^n k \, \frac{k^\mu}{[k^2 + 2p.k - M^2]^\alpha} = (-p^\mu)\;\frac{i\,\pi^{n/2}\,(-1)^\alpha\,\Gamma(\alpha - n/2)}{\Gamma(\alpha)\quad (M^2 + p^2)^{\alpha - n/2}}$$

$$\int d^n k \, \frac{k^\mu k^\nu}{[k^2 + 2p.k - M^2]^\alpha} = \frac{i\,\pi^{n/2}\,(-1)^\alpha}{\Gamma(\alpha)\,(M^2+p^2)^{\alpha - n/2}} \Big\{ p^\mu p^\nu\,\Gamma(\alpha - n/2) - \\ - \tfrac{1}{2} g^{\mu\nu}\,(M^2 + p^2)\,\Gamma(\alpha - 1 - n/2) \Big\}$$

$$\int d^n k \, \frac{k^\mu k^\nu k^\rho}{[k^2 + 2p.k - M^2]^\alpha} = \frac{i\,\pi^{n/2}\,(-1)^\alpha}{\Gamma(\alpha)\,(M^2+p^2)^{\alpha - n/2}} \Big\{ -p^\mu p^\nu p^\rho\,\Gamma(\alpha - n/2) + \\ + \tfrac{1}{2} [g^{\mu\nu} p^\rho + g^{\mu\rho} p^\nu + g^{\nu\rho} p^\mu]\,(M^2 + p^2)\,\Gamma(\alpha - 1 - n/2) \Big\}$$

The Feyman parameter method (for combining r factors $D_1^{\alpha_1}\;D_2^{\alpha_2}\cdot\cdot D_r^{\alpha_r}$ in the denominator, valid when the exponents $\alpha_1,\;\alpha_2\ldots\alpha_r$ are greater than zero):

$$\frac{1}{D_1^{\alpha_1}\cdot\,D_2^{\alpha_2}\ldots D_r^{\alpha_r}} = \frac{\Gamma(\alpha_1 + \alpha_2 + \ldots \alpha_r)}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\ldots\Gamma(\alpha_r)} \int_0^1 dx_1 \int_0^{x_1} dx_2 \ldots \int_0^{x_{r-2}} dx_{r-1}$$

$$\times \; \frac{x_{r-1}^{\alpha_1 - 1}\;(x_{r-2} - x_{r-1})^{\alpha_2 - 1}\;.\;.\;.\;(1 - x_1)^{\alpha_r - 1}}{[\,D_1\,x_{r-1} + D_2\,(x_{r-2} - x_{r-1}) + \ldots D_r\,(1 - x_1)]^{\alpha_1 + \alpha_2 + \ldots \alpha_r}}$$

REFERENCES AND FOOTNOTES

1. Ngee-Pong Chang, Phys. Rev. D10, 2706 (1974).

2. D.J. Gross and F. Wilczek, Phys. Rev. Letters 30, 1343 (1973);

   H.D. Politzer, Phys. Rev. Letters 30, 1346 (1973).

3. M. Gell-Mann and F. Low, Phys. Rev. 95, 1300 (1954).

4. C.G. Callan, Phys. Rev. D2, 1541 (1970).

   K. Symanzik, Commun. Math. Phys. 18, 227 (1970).

5. S. Weinberg, Phys. Rev. D8, 3497 (1973).

6. G. t'Hooft, Nucl. Phys. B61, 455 (1973).

7. H.D. Politzer, Physics Reports Vol. 14C, 129 (1974).

8. J. Wess and B. Zumino, Nucl. Phys. B78, 1 (1974).

9. A. Salam and J. Strathdee, Phys. Letters 51B, 353 (1974).

10. S. Ferrara and B. Zumino, Nucl. Phys. B79, 413 (1974).

11. A. Salam and J. Strathdee, Phys. Rev. D11, 1521 (1975).

12. G. 't Hooft and M. Veltman, Nucl. Phys. B44, 189 (1972).

13. J.C. Collins and A.J. Macfarlane, Phys. Rev. D10, 1201 (1974).

14. C.G. Callan and D.J. Gross, Phys. Rev. D8, 4383 (1973).

15. S. Coleman and D. Gross, Phys. Rev. Letters 31, 851 (1973).

16. D.J. Gross and F. Wilczek, Phys. Rev. D8, 3633 (1973).

17. T.P. Cheng, E. Eichten and Ling-Fong Li, Phys. Rev. D9, 2259 (1974).

18. P.W. Higgs, Phys. Letters 12, 132 (1964); Phys. Rev. Letters 13, 508 (1964); Phys. Rev. 145, 1156 (1966);

    T.W.B. Kibble, Phys. Rev. 155, 1554 (1967).

    G.S. Guralnic, C.R. Hagen and T.W.B. Kibble in Advances in Particle Physics, Vol. 2 (ed. R.L. Cool and R.E. Marshak, Interscience, 1968).

19. S. Coleman and E. Weinberg, Phys. Rev. D7, 1888 (1973); see also the references given in ref. (16, 17).

20. J. Wess and B. Zumino, Nucl. Phys. B70, 39 (1974); Phys. Letters 49B, 52 (1974). (Initially the term Supergauge was used instead of Supersymmetry)

21. A. Salam and J. Strathdee, Nucl. Phys. B76, 477 (1974).

22. S. Ferrara, J. Wess and B. Zumino, Phys. Letters 51B, 239 (1974).

23. A.A. Slavnov, JINR preprint, E2-8449, Dubna 1974 and also independently.

S. Ferrara and O. Piguet, preprint CERN TH 1995.

24. At the time when this work was carried out it was only a conjecture that the renormalizability of supersymmetric Yang-Mills theories preserves their (supersymmetric) nature. The general analysis of section II was carried out with this limitation (i.e., if the renormalizability preserves ...., then .....) and the work, described now in sections III and IV, was an explicit verification of the analysis of section II to lowest order (one loop).

25. Essentially the β-functions of the renormalization group equations - i.e. from eqn (1.1), $\beta g_i = \mu\frac{\partial g_i}{\partial\mu}$ (at $n = 4$) with all bare parameters kept constant - can be obtained from the expressions for the bare parameters, in terms of the renormalized, by noting that βg_i is a regular function in n (no. of dimensions) and can, therefore, be easily determined - for details of this argument see refer. 13.

26. D.R.T. Jones, preprint on "Asymptotic behaviour of super-symmetric Yang-Mills theories in the two loop approx.", Univ. Sussex, 1974.

27. L.D. Faddeev and N.V. Popov, Phys. Letters 25B, 29 (1967).

28. G. 't Hooft, Nucl. Phys. B33, 173 (1971); Nucl. Phys. B35, 167 (1971).

29. A. Slavnov, Theor. and Math. Phys. 10, 99 (1973);
J.C. Taylor, Nucl. Phys. B33, 436 (1971).

30. M. Suzuki, Nucl. Phys. B83, 269 (1974).

31. See, for example, R.E. Marshak, Riazuddin and C.P. Ryan, Theory of Weak Interactions in Particle Physics (Wiley-Interscience, New York, 1969).

32. S. Abers and B. Lee, Physics Reports Vol. 9C, 1 (1973).

33. S. Weinberg, Phys. Rev. Letters 19, 1246 (1967);
A. Salam, in Elementary Particle Physics (ed. N. Svartholm, Almqvist and Niksells, Stockholm, 1968), P. 367.

34. G.'t Hooft and M. Veltman, Nucl. Phys. B50, 318 (1972).
B.W. Lee and J. Zinn-Justin, Phys. Rev. D5, 3137 (1972); D5, 3121 (1972); D5, 3150 (1972); D7, 1049 (1972);
D.A. Ross and J.C. Taylor, Nucl. Phys. B51, 125 (1973).

35. S.L. Glashow, J. Iliopoulos and L. Maiani, Phys. Rev. D2, 1283 (1970).

36. J.C. Pati and A. Salam, Phys. Rev. D8, 1240 (1973).

37. J.C. Pati and A. Salam, Phys. Rev. D10, 275 (1974).

38. Note that we use a hermitian representation for the γ_5-matrix in Chapter Two: it equals $-i\gamma_5$ the one defined in Appendix I.

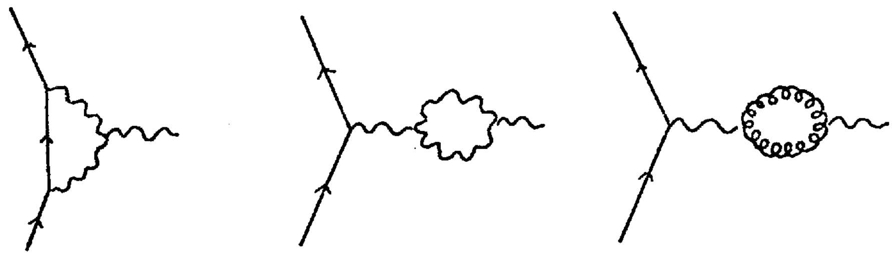
39. D.A. Ross, Phys. Rev. D11, 911 (1975).

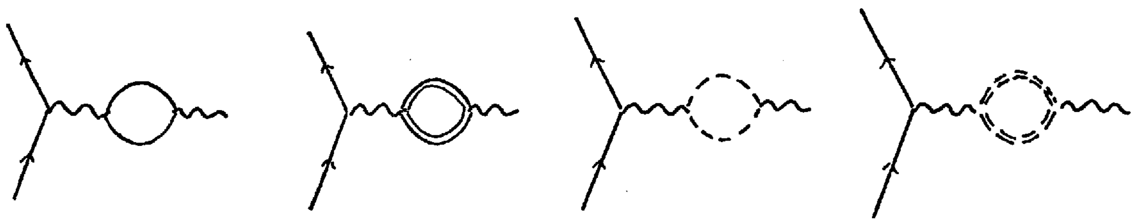
40. See for a review, S.L. Adler in Lectures in Particles and Quantum Field Theory, Vol. 1 (M.I.T. press 1970).

41. D.A. Akyeampong and R. Delbourgo, Nuovo Cim. 17A, 578 (1973).

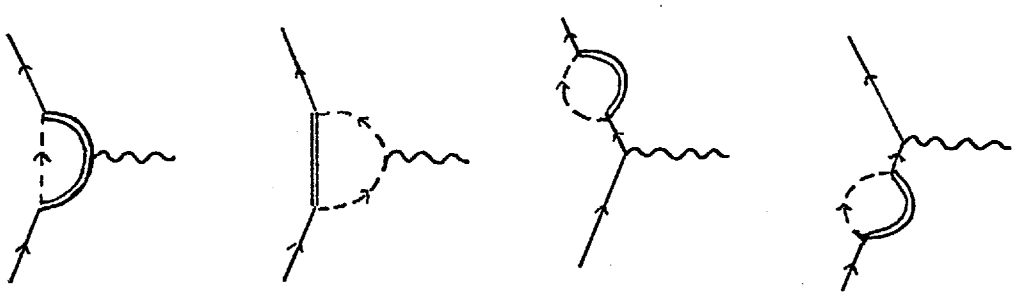
FIGURE I

Group (a)





Group (b)



Group (c)

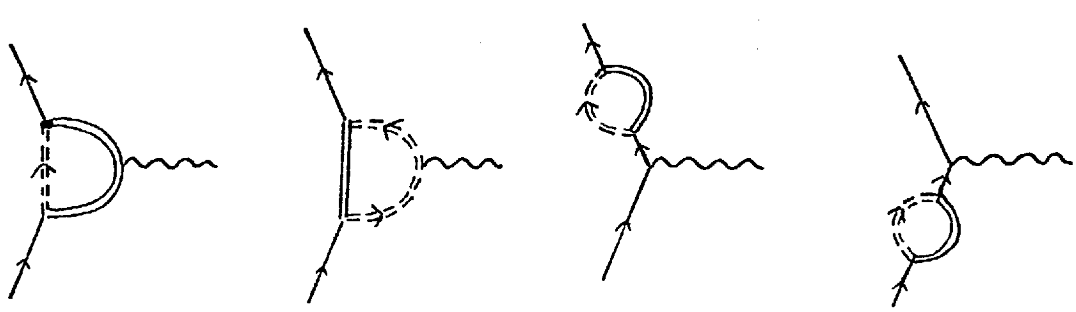
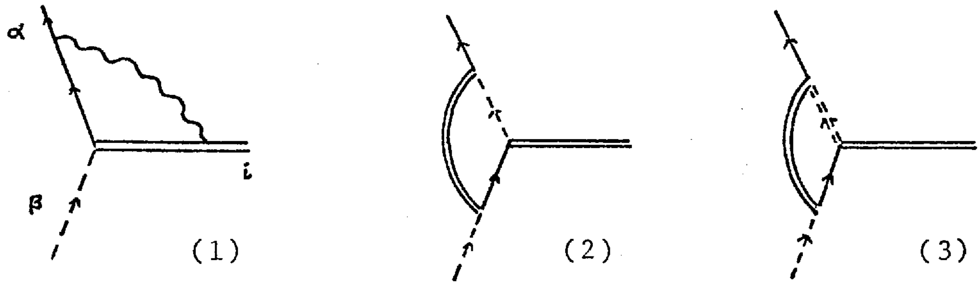


FIGURE II

Group (a)



Group (b)

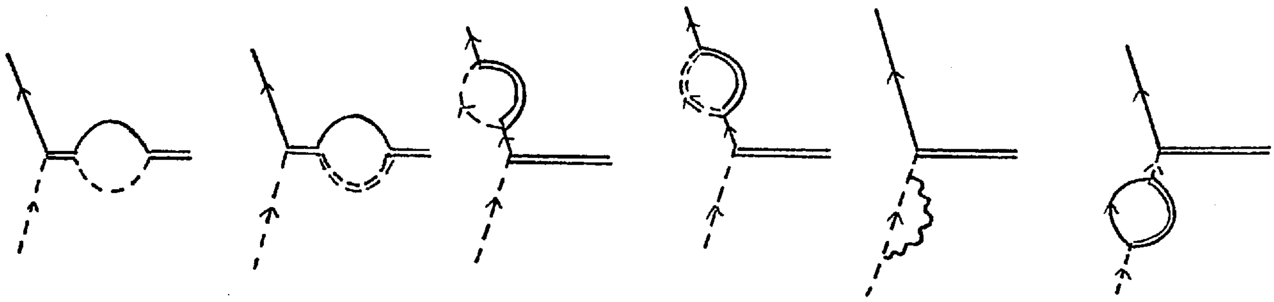
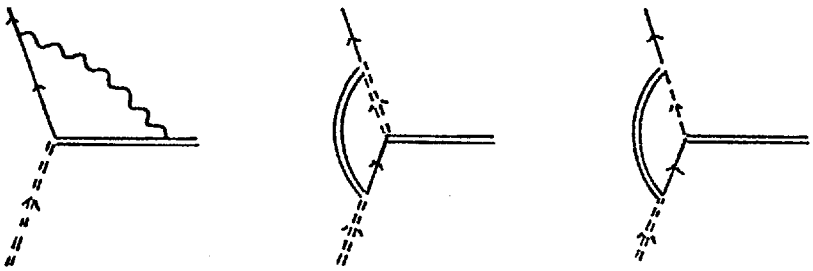
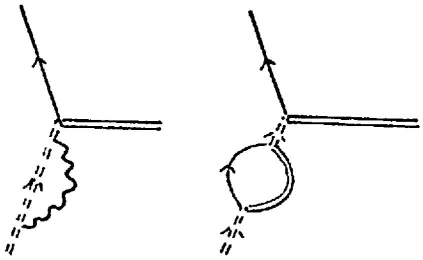


FIGURE III

Group(a)



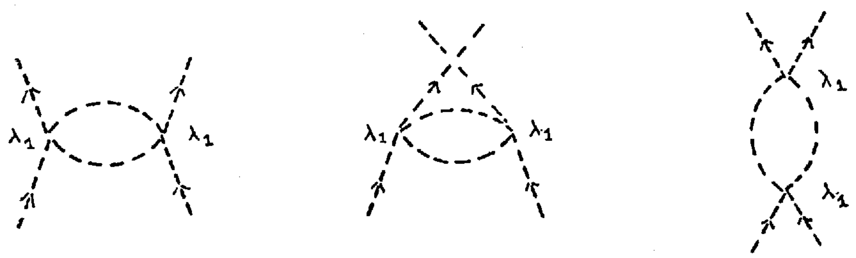
Group (b)

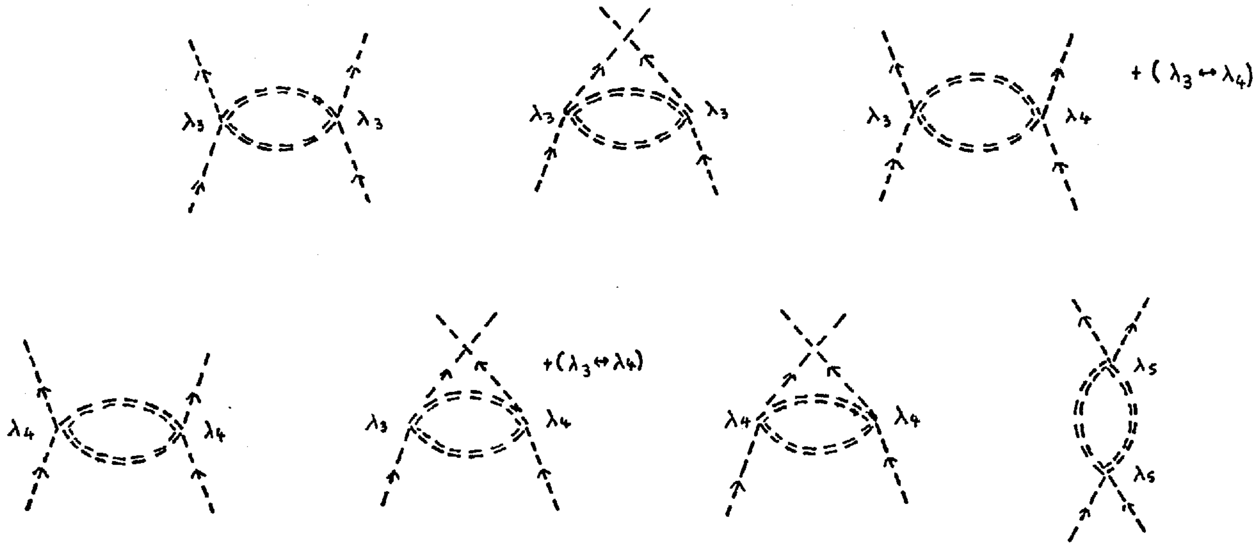


+ 4 other diagrams

FIGURE IV

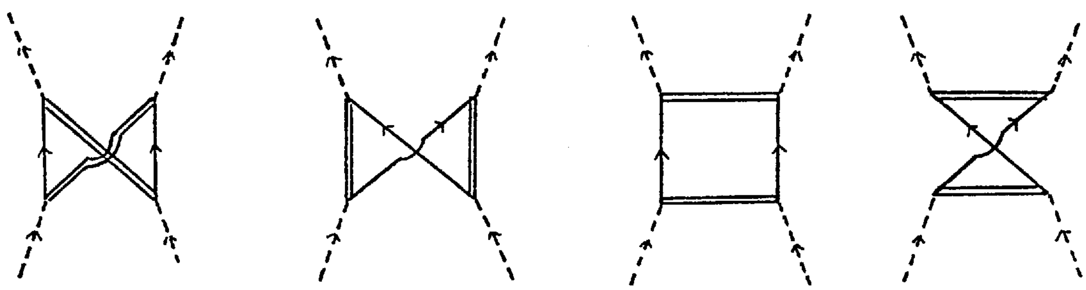
Group (a)



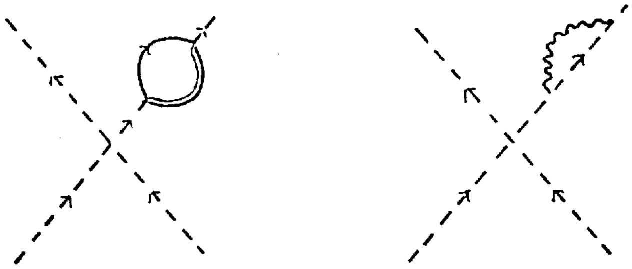


Group (b)

Group (c)



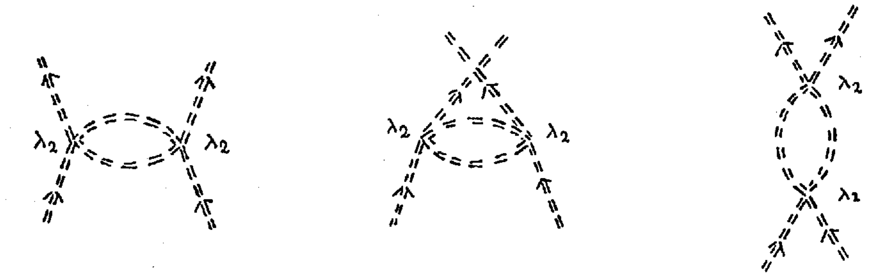
Group (d)

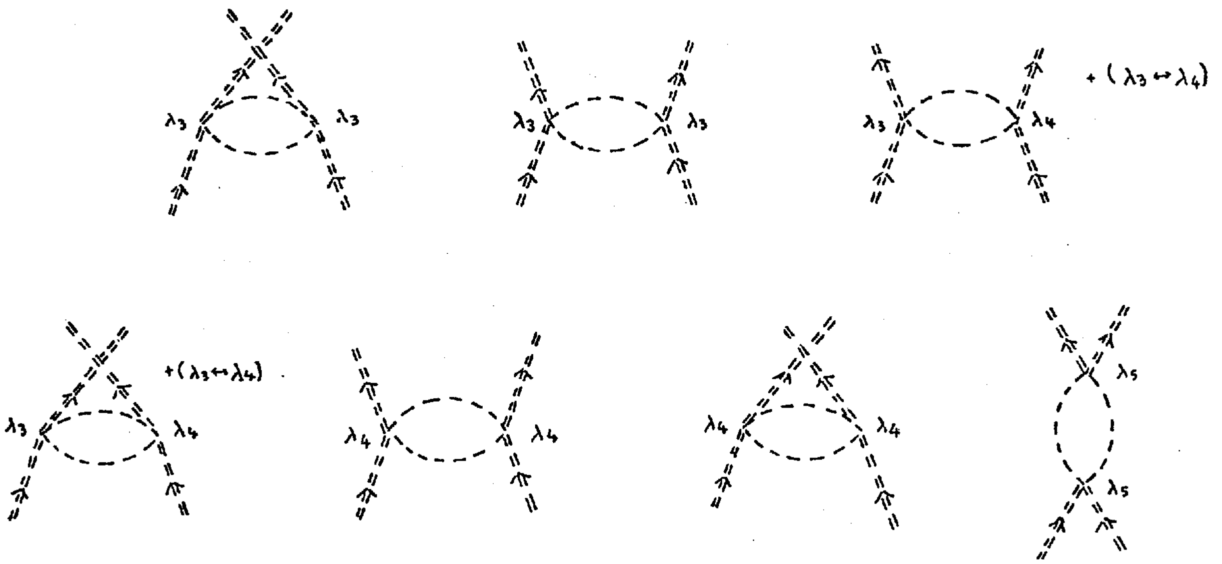


+ 6 other diagrams

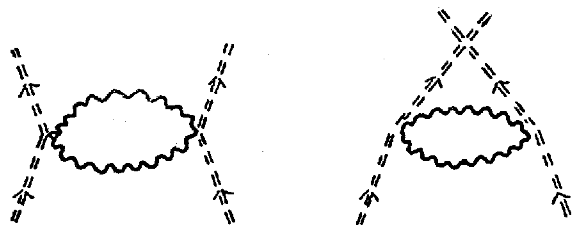
FIGURE V

Group (a)

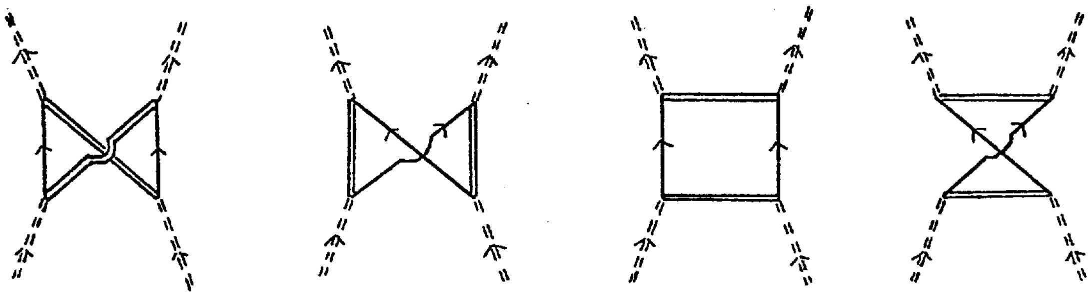




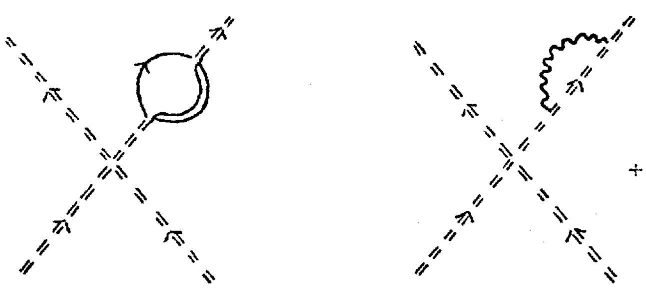
Group (b)



Group (c)



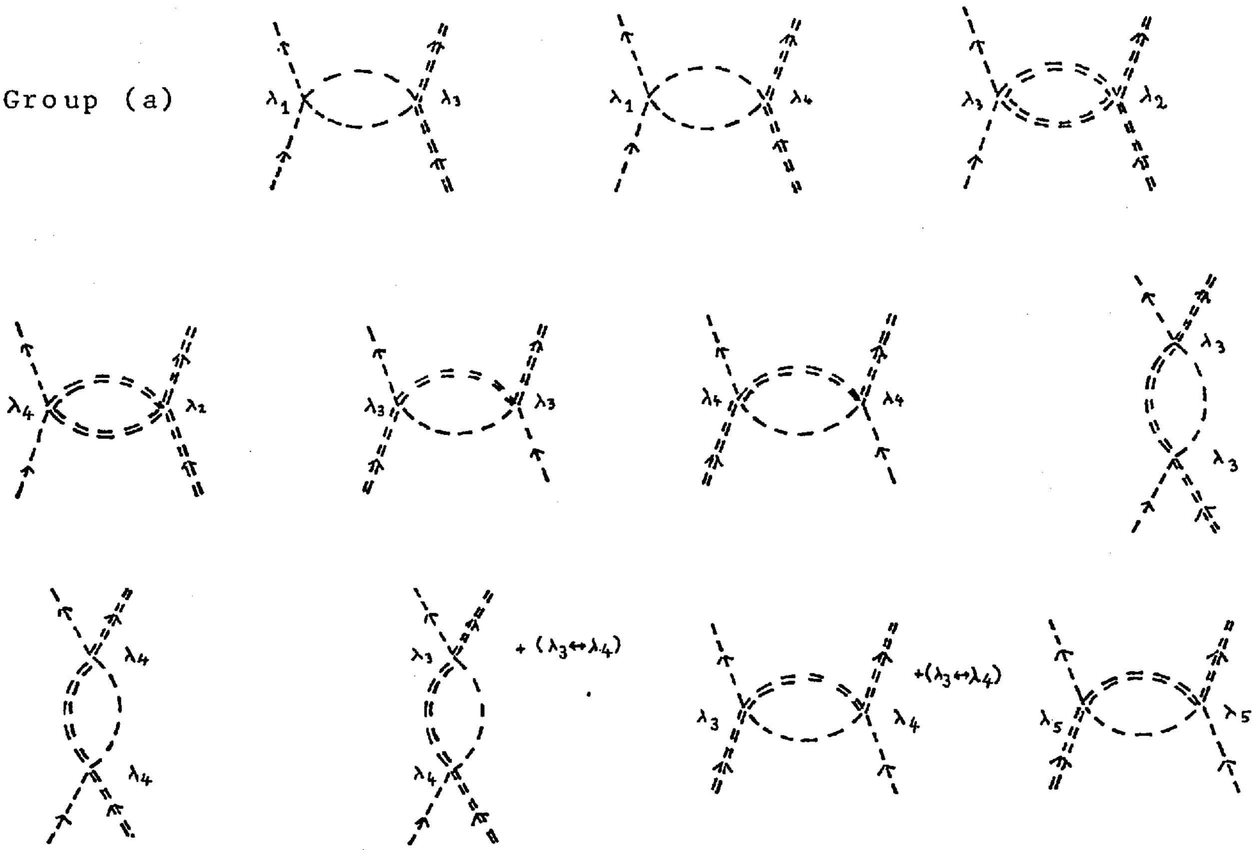
Group (d)



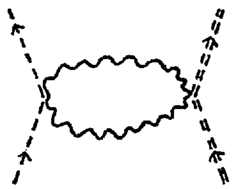
+ 6 other diagrams

FIGURE VI

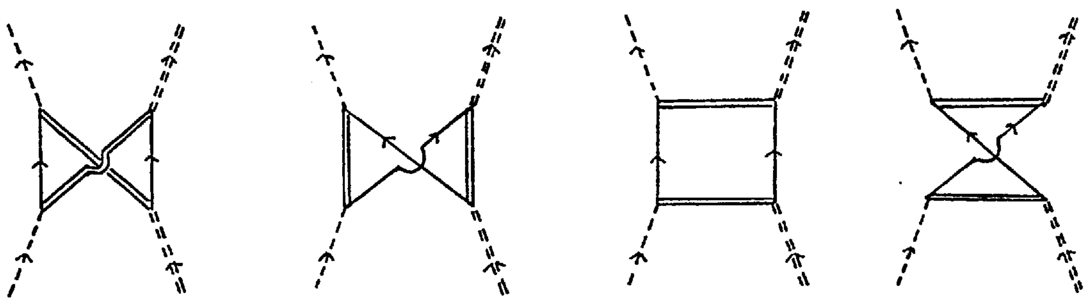
Group (a)



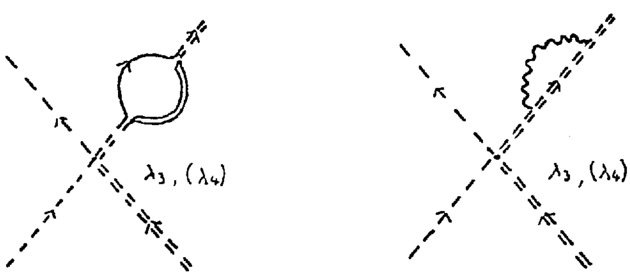
Group (b)



Group (c)



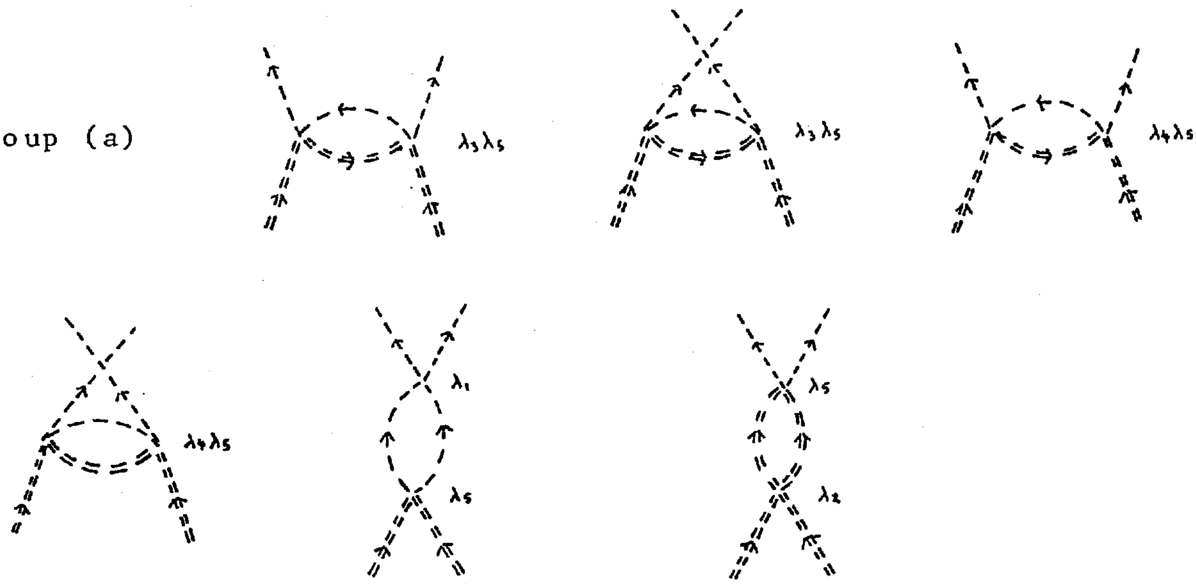
Group (d)



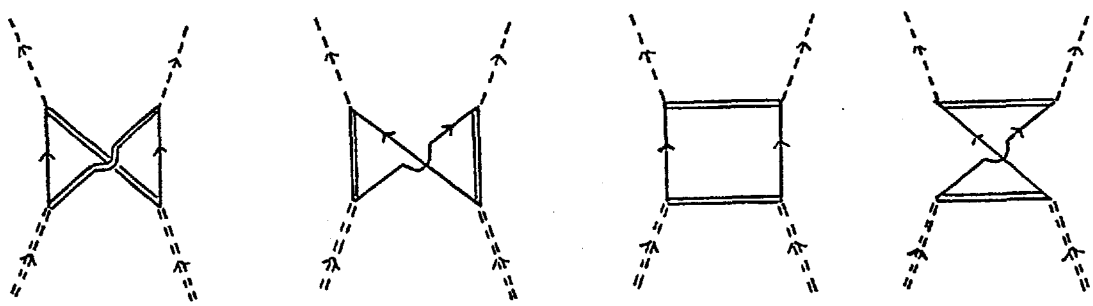
+ 12 other diagrams

FIGURE VII

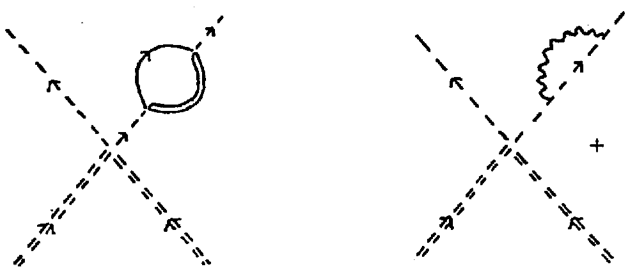
Group (a)



Group (c)



Group (d)



+ 6 other diagrams

FIGURE VIII

The propagator for any massive gauge field, say of mass M, obtained in momentum space and in the general ξ-gauge formulation of the theory is given by

$$-i\{g^{\mu\nu}-(1-\tfrac{1}{\xi})\,p^\mu p^\nu/(p^2-\tfrac{M^2}{\xi})\}\,\frac{1}{(p^2-M^2)}$$

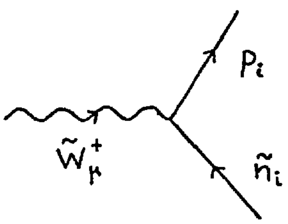
or equivalently

$$-i\{g^{\mu\nu}-p^\mu p^\nu/M^2\}\,\frac{1}{(p^2-M^2)}\;-\;i\,\frac{p^\mu p^\nu/M^2}{(p^2-M^2/\xi)}$$

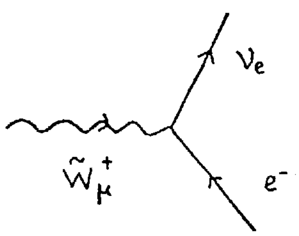
And the propagator of the corresponding Goldstone boson is given by

$$\frac{i}{p^2-M^2/\xi}$$

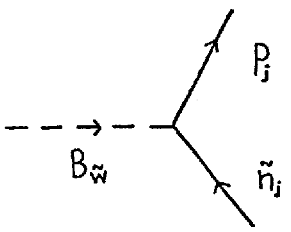
The vertices relevant to the process considered in Chapter Two are



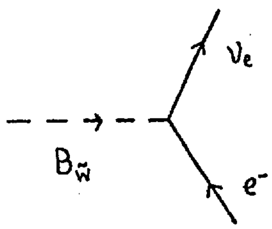
$$\frac{i\,g\,\cos\theta\,\cos\delta}{2\sqrt{2}}\,\gamma^\mu(1-\gamma_5) \qquad i=a,b,c$$



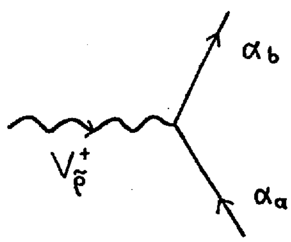
$$\frac{i\,g\,\cos\delta}{2\sqrt{2}}\,\gamma^\mu(1-\gamma_5)$$



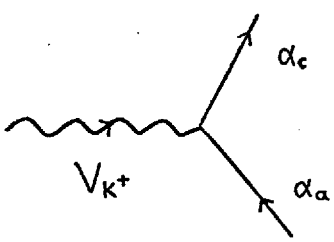
$$-\frac{i g \cos\theta \cos\delta}{2\sqrt{2}\, M_{\tilde w}} \left[m_{\tilde n} - m_p + \gamma_5 (m_{\tilde n} + m_p) \right] , \quad j = a, b, c$$



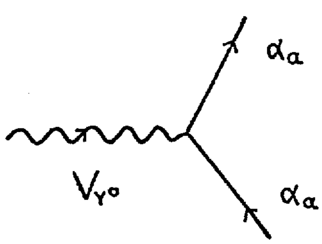
$$-\frac{i g \cos\delta}{2\sqrt{2}\, M_{\tilde w}} \; m_e \, (1 + \gamma_5)$$



$$-\frac{i f \cos\delta}{\sqrt{2}} \, \gamma^\mu \quad , \qquad \alpha = p, \tilde n, \tilde\lambda \ \& \ \chi$$



$$-\frac{i f}{\sqrt{2}} \, \gamma^\mu \qquad \text{"} \quad \text{"} \quad \text{"}$$



$$\frac{i f}{2} \, \varkappa \, \gamma^\mu \qquad \text{"} \quad \text{"} \quad \text{"}$$

$$\varkappa = \frac{f (g^2 + g'^2)}{\left\{ g^2 g'^2 (g^2 + g'^2) + \frac{3}{4} f^2 (g^2 + g'^2)^2 \right\}^{1/2}}$$

$$\simeq \frac{2}{\sqrt{3}} \left\{ 1 - \frac{2}{3} \frac{g^2 g'^2}{f^2 (g^2 + g'^2)} \right\} \simeq \frac{2}{\sqrt{3}} + O(\sim e^2)$$