INSTABILITY OF SOME UNSTEADY VISCOUS FLOWS

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ABSTRACT

In the first part the linear stability of viscous flows in a curved channel due to time dependent slowly varying pressure gradients is considered by an approach of the W.K.B. type. The asymptotic behaviour of small perturbation waves is determined, allowing their characteristics (amplitude, transverse structure, amplification rate) to be slowly varying with time. The evolution of such disturbances is followed and the instantaneous marginal states are determined according to a "momentary" criterion for stability which is based on the definition of a "growth rate" of the disturbance. An asymptotic representation for the growth rate is found.

Low frequency modulated basic flows are investigated by using the periodicity criterion to define the marginal state. The modulation is always found to destabilize the mean flow and the "critical" wavenumber is found to decrease from its unmodulated value when the amplitude of the oscillation increases.

Slowly accelerated basic flows are also investigated. The evolution of the linear perturbations is then followed in the weakly non linear regime and the existence of a supercritical equilibrium amplitude solution is proved both in the steady and in the unsteady case.

In the second part the instability of the flow induced by a circular cylinder oscillating in an infinite fluid is investigated. The flow is shown to be unstable to a Taylor vortex mode of instability. A series solution of the partial differential system governing the stability of the flow is obtained.

The method used has several advantages over the numerical methods used by different authors for related problems.

The instability predicted by the theory leads to a flow with no mean velocity component tangential to the cylinders. The disturbance velocity field decays exponentially at the edge of the Stokes layer. The theoretical results are qualitatively confirmed by an experimental investigation of the problem.

Weakly non linear effects are also examined and show the existence of a supercritical equilibrium amplitude solution.

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PART ONE

LINEAR AND WEAKLY NON LINEAR STABILITY OF SLOWLY VARYING FLOWS IN A CURVED CHANNEL

CHAPTER 1

FORMULATION OF THE PROBLEM

1.1 - Introduction

Linear stability theory for steady basic flows has been very successful in providing the explanation for the selective amplification of small disturbances which, eventually, leads to a change in the flow configuration (transition or bifurcation). The theory is consistently based on the asymptotic criterion for stability and allows one to determine in the space of the parameters of the problem a region of "marginal" (or "neutral") stability. This region separates the "stable" configurations (which satisfy the property of being the disturbances asymptotically vanishing in time) from the "unstable" ones.

When the basic flow is unsteady the previous approach applied to the instantaneous velocity profile, "frozen" in time, leads to a "quasi-steady" analysis. This approach is justifiable if the basic flow is slowly varying with time, i.e. if the growth of the perturbations, once started, can be assumed to be much faster than the evolution of the basic flow. However, on a more careful examination this approximation is found unsatisfactory in some respects.

As Shen (1961) has clearly pointed out, the asymptotic criterion for stability, which is adopted in the quasi-steady analysis, is no longer meaningful when the basic flow is time-dependent. Since the disturbances evolve with respect to a configuration which is itself varying with time, Shen argues that what is relevant is some instantaneous measure of the tendency of the flow towards stability or instability. His suggestion is to use a "growth rate" based on the ratio of the disturbance kinetic energy to the basic flow kinetic energy. This "momentary" criterion does not allow one to define in the space of parameters of the problem an equivalent region of marginal stability. This implies that whereas in the steady case a change in configuration may certainly be expected after the disturbances have started to grow, in the time-dependent case the instantaneous tendency to a change in configuration expressed by a positive growth rate may or may not reverse depending on the behaviour of the basic flow. Moreover, linear theory cannot cope with situations where the disturbances sustain periods of growth such that their amplitudes become too large for linearization to be a valid approximation. Non linear effects must be taken into account in order to follow their development in time.

The effect induced by the evolution of the basic flow on the instantaneous properties of the disturbance is neglected by the quasi--steady theory. This paper tries to overcome such a deficiency for the case of basic flows which are slowly varying with time. Under such conditions the problem is amenable to solution by the "W.K.B." technique. This well known method for the solution of differential systems with slowly varying coefficients (which, in this context, is equivalent to the method of "multiple scales") was first suggested for stability problems by Benney & Rosenblat (1964). Rosenblat & Herbert (1970) have employed it to solve the modulated Benard convection problem. In two recent papers Bouthier (1972, 1973) has developed the method for a general steady, spatially dependent shear flow and applied it to the stability of the boundary layer on a flat plate. Drazin (1974) has investigated the stability characteristics for a model of flow in a channel whose width is slowly varying in time or space. Finally, Eagles & Weissmann (1975) have treated the stability of the flow in a slowly-diverging channel.

Essentially, by this method a solution is sought in the form of an asymptotic expansion in terms of the small parameter which characterizes the slow variation of the basic flow. Each coefficient of the expansion has the structure of a wave whose instantaneous properties are allowed to be slowly varying with time. The slowly varying amplification rate can be evaluated as a solution of the instantaneous eigenvalue problem, parametrically dependent on time, which arises at lowest order. The corresponding eigenfunctions define the instantaneous slowly varying transverse structure of the disturbance and are unique only up to an arbitrary multiple of a function of the slow time variable. This "amplitude function" is determined by an "amplitude equation" which arises at higher order from a solvability condition imposed on the higher order system. The three functions amplification rate, transverse structure and amplitude, which are complex in general, reduce to a real form when the principle of exchange of stabilities is assumed to be valid. This perturbation scheme leads to an asymptotic representation of the disturbance velocity field in terms of the small parameter. The determination of the growth rate associated with any quantity describing the evolution of the perturbation follows easily and the stability of the flow can then be discussed according to the "momentary" criterion previously mentioned.

The stability problem studied in this chapter refers to flows in a curved channel. This choice has a physiological motivation. In fact recent velocity measurements within the ascending and upper descending aorta of animals and humans (see, for example, Seed & Wood 1971 and Nerem et al. 1974) have shown highly disturbed waveforms which have been attributed to the presence of turbulence in the flow. The analysis of the stability of curved channel flows may then be considered as a first step in order to understand the more complicated phenomenon occurring in the aorta. The steady case has been investigated theoretically by Dean (1928), Reid (1958) and Hämmerlin (1958). They examined rotationally symmetric disturbances in the small gap limit and found that instability first develops in the form of a set of toroidal vortices of the kind which characterizes circular Couette flow instability. These results have been experimentally confirmed by Brewster, Grosberg & Nissan (1959). Recently Gibson & Cook (1974) have considered the behaviour of asymmetric and mixed modes for finite values of the gap and found that the asymmetric mode becomes the most unstable when the ratio between the gap width and the radii of the cylinders becomes less than 2.179×10^{-5}.

The effect of the time dependence of the basic flow arising from driving pressure gradients which are slowly varying with time is considered here. In the next section the basic flow is derived and the governing linear differential system with slowly varying coefficients for small disturbances to this flow is given. Chapter 2 is devoted to the linear theory. The quasi-steady approach is outlined in § 2.2. In § 2.3 the slowly varying approximation is employed to obtain an asymptotic solution for the disturbance velocity field which leads to an asymptotic representation for the growth rate. The particular case of low frequency modulated basic flows is considered in § 2.4 where, for small amplitudes of the modulation, the equivalence will be shown of the slowly varying approach to the method used by Hall (1975a) for the analogous modulated Couette problem. Various results and some comparison with previous work are presented in § 2.5 for the modulated case and for some slowly accelerated basic flows. Finally some conclusions follow in § 2.6. Chapter 3 is devoted to the weakly non-linear theory. Time dependent basic flows are considered where both the amplitude and the frequency of the unsteady component are "small". The analysis given in § 3.2 follows the line of Di Prima & Stuart (1975) approach. Some results are presented in § 3.3 and discussed in § 3.4.

1.2 - The stability problem

Consider viscous, incompressible flow between concentric cylinders of infinite length and radii R_1, R_2 ($R_2 > R_1$). The difference in radii of the two cylinders d is taken to be small compared with their mean $\bar{R}$ (small gap approximation). Let (r, ϑ, Z) be cylindrical polar coordinates with the axis of the cylinders along the Z axis and let (U , V^*, W^*) be the corresponding velocity vector. Let also p^*, ρ , ν and t^* denote pressure, density, kinematic viscosity and time respectively.

Consider now a driving pressure gradient of the form

$$\frac{1}{\rho}\,\frac{\partial p^*}{\partial \vartheta} = -\,K\,\mathcal{F}(\omega t^*)\;, \qquad (1.1)$$

where ω^{-1} is a characteristic time scale and K is a positive constant whose dimensions are L^2T^{-2}.

A second time scale, d^2/ν, also exists and it represents the time for vorticity to diffuse outwards from the boundary through the characteristic distance d.

The non dimensional parameter

$$\sigma = \frac{\omega d^2}{\nu}\;, \qquad (1.2)$$

is then suitable to describe the time dependence of the basic flow. Small values of σ correspond to "slow" variations, where viscous effects are felt throughout the gap width: the basic flow is here "fully viscous". Large values of σ describe "rapidly" varying basic flows where unsteady viscous effects keep confined within small layers of thicknesses $O(\sqrt{\nu/\omega})$ adjacent to the walls, the core flow being essentially inviscid.

Let us now define dimensionless variables $\zeta, \tau, \mathcal{V}$ by

$$\zeta = \frac{r - R_1}{d} \quad , \quad \tau = \omega t^* \quad , \quad \mathcal{V} = \frac{V^*}{V_m} \quad , \tag{1.3}$$

where the reference velocity V_m is the mean velocity corresponding to the pressure gradient $\mathfrak{F} = 1$ and may be written

$$V_m = \frac{K\, d^2}{12 \nu R_1} \tag{1.4}$$

The basic velocity field is then given by $(0, \mathcal{V}(\zeta,\tau), 0)$ where $\mathcal{V}$ satisfies the differential system

$$\begin{aligned} &\left(\frac{\partial^2}{\partial \zeta^2} - \sigma \frac{\partial}{\partial \tau}\right) = -12\, \mathfrak{F}(\tau) \quad , \\ &\mathcal{V} = 0 \qquad\qquad (\zeta = 0,1) \quad , \end{aligned} \tag{1.5}$$

and terms $O(d/R_1)$ have been neglected.

The solution of (1.5) in the limit $\sigma \to 0$ is

$$\mathcal{V} = \sum_{n=0}^{\infty} \chi_n(\zeta)\, \Phi_n(\tau)\, \sigma^n \, , \tag{1.6}$$

where

$$\begin{aligned} &\chi_0 = 6(\zeta - \zeta^2) \, , \\ &\frac{d^2 \chi_{n+1}}{d\zeta^2} = \chi_n \quad , \\ &\chi_n = 0 \qquad (\zeta = 0,1) \, , \end{aligned} \tag{1.7 a,b,c}$$

and

$$\begin{aligned} &\Phi_0 = \mathfrak{F}(\tau) \quad , \\ &\Phi_{n+1} = \frac{d\,\Phi_n}{d\tau} \, . \qquad (n = 0, 1, 2, \ldots) \end{aligned} \tag{1.8 a,b}$$

Let us now formulate the stability problem. Suppose that the basic flow is disturbed such that the velocity vector is of the form $(u^*, v^* + V^*, w^*)$. The disturbance is assumed to be rotationally symmetric. Asymmetric and mixed modes are not considered since:

(i) asymmetric and mixed modes are known to be more stable in the steady case , except for gap widths which are extremely small compared with the radii of the cylinders (Gibson & Cook (1974)).

(ii) Asymmetric or mixed modes have not been detected in experiments on unsteady Taylor vortex flow (Donnelly (1964), Thompson (1968)).

However a more satisfactory justification of such assumption requires further work.

Let (u,v,w) be the dimensionless disturbance velocity obtained by the scaling which is usual for Taylor vortex flow

$$u^* = \frac{\nu}{2d}\, u(\zeta,z,t) \;;\quad v^* = V_m v(\zeta,z,t) \;;\quad w^* = \frac{\nu}{2d}\, w(\zeta,z,t) , \qquad (1.9)$$

where

$$Z = dz \;;\quad t^* = (\frac{d^2}{\nu})\, t . \qquad (1.10)$$

The differential system which governs its behaviour can be derived from the momentum and continuity equations by the usual manipulations. If terms of $O(d/R_1)$ are neglected we find that

$$\begin{cases} \left(\mathcal{L} - \frac{\partial}{\partial t}\right)\mathcal{L} u + T V \frac{\partial^2 v}{\partial z^2} = \frac{\partial^2}{\partial z^2}(Q_1) - \frac{\partial^2}{\partial z \partial \zeta}(Q_2) - \frac{T}{2}\frac{\partial^2 v^2}{\partial z^2} , \\ \left(\mathcal{L} - \frac{\partial}{\partial t}\right) v - \frac{1}{2} u \frac{\partial V}{\partial \zeta} = Q_3 , \\ \frac{\partial u}{\partial \zeta} + \frac{\partial w}{\partial z} = 0 , \\ u = v = w = 0 \qquad (\zeta = 0,1) , \end{cases} \qquad (1.11)$$

where

$$\mathcal{L} \equiv \frac{\partial^2}{\partial \zeta^2} + \frac{\partial^2}{\partial z^2} \quad , \tag{1.12}$$

$$Q_1 = \frac{1}{2}\left(u \frac{\partial u}{\partial \zeta} + w \frac{\partial u}{\partial z}\right),$$

$$Q_2 = \frac{1}{2}\left(u \frac{\partial w}{\partial \zeta} + w \frac{\partial w}{\partial z}\right), \tag{1.13 a,b,c}$$

$$Q_3 = \frac{1}{2}\left(u \frac{\partial v}{\partial \zeta} + w \frac{\partial v}{\partial z}\right),$$

and we have defined the Taylor number T by

$$T = 4\left(\frac{V_m d}{\nu}\right)^2 \frac{d}{R_1} \quad . \tag{1.14}$$

The differential system (1.11) is strictly valid in the limit $d/R_1 \to 0$ with ζ, z, t, σ, u, v, w fixed.

CHAPTER 2

LINEAR THEORY

2.1 - Linearized problem and quasi-steady approximation

The disturbance is now assumed to be small enough for linearization to be a valid approximations. The differential system which governs its behaviour is obtained in the form

$$\left(\mathcal{L}-\frac{\partial}{\partial t}\right)\mathcal{L}u = -T\,V\,\frac{\partial^2 v}{\partial z^2}\ ,$$
$$\left(\mathcal{L}-\frac{\partial}{\partial t}\right)v = \frac{1}{2}u\,\frac{\partial V}{\partial \zeta}\ . \qquad (2.1)$$

The boundary conditions associated with (2.1) are

$$u = v = \partial u/\partial\zeta = 0 \ , \qquad (2.2)$$

i.e. the no-slip condition at the walls.

The coefficients of the perturbation differential equations are independent of z and vary with ζ , and slowly with t. The quasi-steady theory now ignores this variation and the usual analysis by normal modes follows for the instantaneous configuration at $t = \bar{t}$ by setting

$$\tilde{u} = \int_{-\infty}^{\infty} \tfrac{1}{2}f(\zeta;\bar{t})\left\{\exp\left[i(az-\Omega t)\right]+c.c.\right\}da\ ,$$
$$\tilde{v} = \int_{-\infty}^{\infty} \tfrac{1}{2}g(\zeta;\bar{t})\left\{\exp\left[i(az-\Omega t)\right]+c.c.\right\}da\ , \qquad (2.3\text{ a,b})$$

where

$$(\tilde{u},\tilde{v}) = (u,\left[\phi_0(\sigma\,\bar{t})\right]^{-1} v). \qquad (2.4)$$

Here c.c. denotes complex conjugate, a is the dimensionless (real) wavenumber, Ω is a complex number whose imaginary part gives the amplification rate of the disturbance and (f,g) are functions which describe the transverse structure of the perturbation.

An asymptotic representation of the disturbance velocity field in terms of the small parameter σ can be obtained by expanding in the form

$$\begin{cases} i\Omega(\bar{t}) = i\Omega_0(\bar{t}) + i\Omega_1(\bar{t})\sigma + O(\sigma^2) , \\ f(\zeta;\bar{t}) = f_0(\zeta;\bar{t}) + f_1(\zeta;\bar{t})\sigma + O(\sigma^2) , \\ g(\zeta;\bar{t}) = g_0(\zeta;\bar{t}) + g_1(\zeta;\bar{t})\sigma + O(\sigma^2) . \end{cases} \qquad (2.5)\ \text{a,b,c}$$

By substituting the general mode of (2.3) into (2.1), (2.2) and using (2.5),(2.4) and (1.6) it follows at order σ^0

$$Q \begin{pmatrix} f_0 \\ g_0 \end{pmatrix} = 0 ,$$

$$f_0 = g_0 = \frac{df_0}{d\zeta} = 0 \quad (\zeta = 0,\ 1) . \qquad (2.6)$$

Here Q is the linear operator defined by

$$Q \equiv \begin{pmatrix} \left(\frac{d^2}{d\zeta^2} - a^2 + i\Omega_0\right)\left(\frac{d^2}{d\zeta^2} - a^2\right) & -a^2 \tilde{T}_0 \chi_0 \\ -\frac{1}{2}\frac{d\chi_0}{d\zeta} & \left(\frac{d^2}{d\zeta^2} - a^2 + i\Omega_0\right) \end{pmatrix} , \qquad (2.7)$$

and

$$\tilde{T} = T\left[\phi_0(\sigma\bar{t})\right]^2 . \qquad (2.8)$$

Thus $\tilde{T}_0$ represents the value of T associated with the lowest order approximation for the basic flow at $t = \bar{t}$. The differential system (2.6) defines an eigenvalue problem for $(a, \tilde{T}_0, \Omega_0)$ which leads to an eigen-relation of the form

$$f(a, \tilde{T}_0, \Omega_0) = 0 , \qquad (2.9)$$

identical to the eigenrelation which characterizes a steady flow with $T = \tilde{T}_0$.

If the principle of exchange of stabilities is assumed to be valid (see Chandrasekhar 1961) the structure of (2.9) can easily be determined by means of a numerical procedure.

If we now substitute from (2.3)(2.4)(2.5) into (2.1),(2.2) and equate terms of order σ an inhomogeneous differential system is obtained whose solvability condition leads to the following expression for $i\Omega_1$

$$i\Omega_1 = \frac{\int_0^1 \left(a^2 \tilde{T}_0 f_0^+ \chi_1 g_0 + \tfrac{1}{2} \frac{d\chi_1}{d\zeta} g_0^+ f_0\right) d\zeta}{\int_0^1 \left(f_0^+ N f_0 + g_0^+ g_0\right) d\zeta} \; \frac{\bar{\Phi}_1}{\Phi_0} \,. \quad (2.10)$$

Here $\phi_0 = \phi_0(\sigma\bar{t})$, $\phi_1(\sigma\bar{t})$ and N is the linear operator defined by

$$N = d^2/d\zeta^2 - a^2 \,. \quad (2.11)$$

Furthermore (f_0^+, g_0^+) is the adjoint pair of functions defined by the adjoint differential system of (2.6)

$$Q^+ \cdot \begin{pmatrix} f_0^+ \\ g_0^+ \end{pmatrix} = 0 \,,$$
$$f_0^+ = g_0^+ = \frac{df_0^+}{d\zeta} = 0 \quad (\zeta = 0, 1) \,. \quad (2.12)$$

where Q^+ is the linear operator defined by

$$Q^+ \equiv \begin{pmatrix} \left(\frac{d^2}{d\zeta^2} - a^2 + i\Omega_0\right)\left(\frac{d^2}{d\zeta^2} - a^2\right) & -\frac{1}{2}\frac{d\chi_0}{d\zeta} \\ -a^2\tilde{T}_0\chi_0 & \left(\frac{d^2}{d\zeta^2} - a^2 + i\Omega_0\right) \end{pmatrix} . \quad (2.13)$$

An eigenvalue problem for $(a, \tilde{T}_0, \Omega_0)$ arises from the differential system (2.12) which determines an eigenrelation identical to (2.9). However a glance at the form of the equations reveals that the pair of functions (f_0, g_0) differs from its adjoint (f_0^+, g_0^+). The marginal state for the instantaneous configuration is then determined from (2.5a), (2.9), (2.10) by adopting some criterion for stability (asymptotic or momentary). Such an approach is subject to some criticism as pointed out in §1.1. The slowly varying analysis developed in the next section provides a more rational theory where the quasi-steady approximation appears within the framework of a rigorous perturbation scheme.

2.2 - The slowly varying approach

Since the coefficients of system (2.10) are independent of z and slowly varying with time, a solution can be sought of the form

$$
\begin{aligned}
u(\zeta,z,t) &= \int_{-\infty}^{\infty} \tfrac{1}{2}u_a(\zeta,\tau)\left\{\exp\left[i(az-\theta(t))\right] + c.c.\right\}da ,\\
v(\zeta,z,t) &= \int_{-\infty}^{\infty} \tfrac{1}{2}v_a(\zeta,\tau)\left\{\exp\left[i(az-\theta(t))\right] + c.c.\right\}da ,
\end{aligned}
\qquad (2.14)
$$

where $d\theta/dt$ is expected to be a function of the "slow" time variable

$$\frac{d\theta}{dt} = \Lambda(\tau) .$$

This is equivalent to assuming that the general mode of the disturbance has the structure of a wave whose instantaneous properties are allowed to be slowly varying with time. The function $\theta(t)$ which describes the "fast" variation is obviously related to the "slowly" varying function $\Lambda(\tau)$ by

$$\theta(t) = \frac{\int_0^{\tau}\Lambda(\tau)d\tau}{\sigma}.$$

By substituting the general mode of (2.14) into (2.1),(2.2) the differential system for (u_a, v_a) is obtained

$$
\begin{cases}
\left(M - \sigma\dfrac{\partial}{\partial\tau} + i\Lambda\right)M u_a - a^2 T V v_a = 0 , \\[2mm]
\left(M - \sigma\dfrac{\partial}{\partial\tau} + i\Lambda\right)v_a - \dfrac{1}{2}\dfrac{\partial V}{\partial\zeta}u_a = 0 , \\[2mm]
u_a = v_a = \dfrac{\partial u_a}{\partial\zeta} = 0 \qquad (\zeta = 0,1).
\end{cases}
\qquad (2.15)
$$

where M is the linear operator defined by

$$M = \partial^2/\partial\zeta^2 - a^2 . \qquad (2.16)$$

When $\sigma \to 0$ an asymptotic solution can be obtained in the form

$$(u_a(\zeta,\tau),\ v_a(\zeta,\tau)) = \sum_{n=0}^{\infty} (u_n(\zeta,\tau),\ v_n(\zeta,\tau)\ \phi_0(\tau))\ \sigma^n . \qquad (2.17)$$

If we substitute from (2.17) into (2.15) and equate powers of order σ^0

we obtain for (u_0, v_0) the differential system

$$Q^* \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = 0 \, ,$$
$$u_0 = v_0 = \frac{\partial u_0}{\partial \zeta} = 0 \quad (\zeta = 0,\, 1) \, , \tag{2.18}$$

where Q^* is the linear operator obtained from Q by replacing $\sigma \bar{t}$ by τ, $d/d\zeta$ by $\partial/\partial\zeta$ and $\Omega_0(\tau)$ by $\Lambda(\tau)$. This system now defines an eigenvalue problem parametrically dependent on time whose eigenfunctions are unique but for an arbitrary multiple of a function of the slow variable and can be written in the form

$$(u_0, v_0) = A(\tau)(f_0(\zeta; \Omega_0(\tau), \tilde{T}_0(\tau), a), g_0(\zeta; \Omega_0(\tau), \tilde{T}_0(\tau), a)) \, . \tag{2.19}$$

The solution is normalized in all the calculations such that

$$g_0(\tfrac{1}{2}, \tau) \, \phi_0(\tau) = 1 \, . \tag{2.20}$$

The determination of $A(\tau)$ requires the consideration of the order σ problem, which after substituting from (2.17) into (2.15) and equating terms of order σ is found to be

$$Q^* \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial (M u_0)}{\partial \tau} + a^2 \tilde{T}_0 \, V_1 \, v_0 \, \Phi_0^{-1} \\ \left[\dfrac{\partial (v_0 \Phi_0)}{\partial \tau} + \dfrac{1}{2} \dfrac{\partial V_1}{\partial \zeta} u_0 \right] \Phi_0^{-1} \end{pmatrix} , \tag{2.21}$$
$$u_1 = v_1 = \partial u_1 / \partial \zeta = 0 \qquad (\zeta = 0, 1) \, .$$

It can be shown that the inhomogeneous system (2.21) only has a solution if an orthogonality condition is satisfied. This condition leads to the required "amplitude equation"

$$\frac{dA}{d\tau} + H(\tau) \, A(\tau) = 0 \, , \tag{2.22}$$

with

$$H(\tau) = \frac{\displaystyle\int_0^1 \left\{ f_0^+ \left(a^2 \tilde{T}_0 V_1 g_0 \Phi_0^{-1} + \frac{\partial (M f_0)}{\partial \tau} \right) + g_0^+ \Phi_0^{-1} \left(\frac{1}{2} \frac{\partial V_1}{\partial \zeta} f_0 + \frac{\partial (g_0 \Phi_0)}{\partial \tau} \right) \right\} d\zeta}{\displaystyle\int_0^1 \left(f_0^+ M f_0 + g_0^+ g_0 \right) d\zeta} \, . \tag{2.23}$$

Here (f_0^+, g_0^+) is the instantaneous solution of the adjoint differential system (2.12) with $\sigma\bar{t}$ replaced by τ and $d/d\zeta$ by $\partial/\partial\zeta$.

Once the eigenfunctions of the order σ^0 system and of its adjoint are known for every $\tau \geqslant 0$, $H(\tau)$ can be evaluated and equation (2.22) can be integrated to give the amplitude function $A(\tau)$. The lowest order approximation to the disturbance velocity field is then determined by (2.19). The next order correction to the instantaneous transverse structure of the disturbance could now be calculated from system (2.21). However such a correction will not be derived here since Eagles & Weissman (1975) showed (and it will be demonstrated in the following) that the order σ correction to the growth rate arises from the lowest order approximation for the disturbance.

2.3 - The growth rate of the disturbance

The growth rate associated with any quantity suitable to describe the evolution of the disturbance can now be defined in the form

$$G(\bar{Q}_i) = c_i\, Q_i^{-1} \frac{\partial \bar{Q}_i}{\partial t}, \tag{2.24}$$

where c_i is a numerical factor which can conveniently be chosen so that comparison is enabled between growth rates corresponding to different quantities.

The choice of $\bar{Q}_i$, which mathematically means the choice of a metric in the space of solutions, depends on what property of the disturbance we choose to consider. Since no experimental results are available it would seem useful to determine the local as well as the global behaviour of the perturbation. Thus the following quantities will be considered

$$\bar{Q}_1 = u^{(a)}, \qquad \bar{Q}_2 = v^{(a)},$$

$$\bar{Q}_3 = E(\tau) = \int_0^{2\pi/a} dz \int_0^1 \left(u^{(a)2} + v^{(a)2} + w^{(a)2}\right) d\zeta, \quad \bar{Q}_4 = E(\tau) \Big/ \int_0^{2\pi/a} dz \int_0^1 V^2 d\zeta = e(\tau), \tag{2.25 a,b,c,d}$$

where $(u^{(a)}, v^{(a)}, w^{(a)})$ denotes the general mode of the disturbance

(2.14), $E(\tau)$ is the instantaneous value of the disturbance kinetic energy density evaluated per unit wavelength of the disturbance, and $e(\tau)$ is the previous quantity evaluated relative to the corresponding one for the basic flow.

If we now apply the definition (2.24) (with a convenient choice for the value of c_i) to the quantities (2.25), and take (2.17),(2.19) and (2.14) into account, the following asymptotic representation arises for the growth rates

$$\begin{cases} G_1 = \operatorname{Im}\Omega_0(\tau) + \sigma\left(A^{-1}\dfrac{dA}{d\tau} + f_0^{-1}\dfrac{\partial f_0}{\partial\tau}\right) + O(\sigma^2)\,, \\ G_2 = \operatorname{Im}\Omega_0(\tau) + \sigma\left(A^{-1}\dfrac{dA}{d\tau} + g_0^{-1}\dfrac{\partial g_0}{\partial\tau} + \phi_0^{-1}\dfrac{d\phi_0}{d\tau}\right) + O(\sigma^2)\,, \\ G_3 = \operatorname{Im}\Omega_0(\tau) + \sigma\left(A^{-1}\dfrac{dA}{d\tau} + \dfrac{1}{2}F^{-1}\dfrac{dF}{d\tau}\right) + O(\sigma^2)\,, \\ G_4 = \operatorname{Im}\Omega_0(\tau) + \sigma\left(A^{-1}\dfrac{dA}{d\tau} + \dfrac{1}{2}F^{-1}\dfrac{dF}{d\tau} - \phi_0^{-1}\dfrac{d\phi_0}{d\tau}\right) + O(\sigma^2)\,, \end{cases} \qquad (2.26)^{a,b,c,d}$$

where Im denotes the imaginary part, G_i is the growth rate associated with Q_i and $F(\tau)$ is defined by

$$F(\tau) = \int_0^1 \left[f_0^2 + g_0^2\,\phi_0^2 + \frac{1}{a^2}\left(\frac{\partial f_0}{\partial\zeta}\right)^2 \right] d\zeta. \qquad (2.27)$$

Thus the order σ^0 approximation to the expressions for the growth rates corresponding to every quantity describing the evolution of disturbances reduces to the quasi steady amplification rate of the disturbance velocity field associated with the order σ^0 approximation for the instantaneous basic configuration.

The order σ correction includes terms of different kinds. By using (2.22)(2.23) and (2.10) it can be shown that the coefficient $A^{-1}dA/d\tau$ can be split into two parts: the quasi-steady component $\operatorname{Im}\Omega_1$ due to the order σ correction for the instantaneous basic flow and a slowly varying component arising from the slow variation of the disturbance structure. The remaining terms (apart from $\phi_0^{-1} d\phi_0/d\tau$ which is the growth rate of the basic flow at order σ^0) again account for the effect of the slow variation on the disturbance transverse structure.

An interesting feature is exhibited by the order σ correction: different growth rates are associated with different flow quantities. This confirms Bouthier's (1972, 1973) and Eagles & Weissman's (1975) results and implies that the quasi-steady instants of "momentary neutral stability" (defined as the values of τ when the growth rate is zero) are shifted in time by different amounts depending on the quantity chosen to describe the evolution of disturbances. Furthermore the order σ corrections are slowly varying with time, their behaviour depending on the evolution of the basic flow.

For a better understanding of this behaviour the specification of a particular time dependence is instructive.

2.4 - The modulated case: comparison with Di Prima-Stuart method

Suppose now that the function $\mathfrak{F}$ is given by

$$\mathfrak{F}(\omega t^*) = 1 + \epsilon \cos(\omega t^*) , \qquad (2.28)$$

where $\sigma = \omega d^2/\nu$ is now the small non-dimensional frequency of oscillation. The basic flow associated with this time-dependence is determined by substituting from (2.28) into (1.8).

This is an interesting case to analyse because the behaviour of disturbances is now governed by a linear differential system with periodic coefficients. Under such circumstances Yudovich's (1970) extension of Floquet theory to partial differential systems provides an asymptotic criterion for stability where the marginal state is defined by the periodicity of the solution. This "periodicity criterion" has first been used by Venezian (1969) and Rosenblat & Herbert (1970) who treated the stability of modulated Benard convection. Hall (1975a) has applied it when investigating the stability of modulated Couette flow. His method in the low frequency limit will now be shown to be equivalent, in the limit

$\epsilon \to 0$, to the slowly varying approach.

Following Hall (1975a) a solution for the system (2.1)(2.2) can be sought for ϵ and σ approaching zero in such a way that the time dependences of the right and left hand sides of the two differential equations "balance" in some sense. This condition is satisfied if $\sigma \sim \epsilon$ thus we put $\sigma = \alpha\epsilon$ and let $\epsilon \to 0$ with α fixed. This procedure was first used by Di Prima & Stuart (1972) who considered the stability of the flow between eccentric rotating cylinders.

We assume the disturbances to be periodic in the z-direction and expand their amplitudes and the parameter T in powers of the small parameter ϵ about the marginal state. Thus we write

$$\begin{cases} (u,v) = \frac{1}{2}\int_{-\infty}^{\infty}\left\{\left[\sum_{n=0}^{\infty}(u^{(n)},v^{(n)})\,\epsilon^{n}\right]e^{iaz} + c.c.\right\}da \ , \\ T = T_0 + \epsilon T_1 + \epsilon^2 T_2 + \epsilon^3 T_3 + O(\epsilon^4) \ . \end{cases} \qquad (2.29)\ a,b$$

On substituting from $(2.29)_{a,b}$ into (2.1), (2.2), we obtain at order ϵ^0 a partial differential system for $(u^{(0)}, v^{(0)})$ whose solution can be written in the form

$$(u^{(0)}, v^{(0)}) = B_0(\tau)\ (f_{00}(\zeta),\ g_{00}(\zeta)) \ . \qquad (2.30)$$

Here $(f_{0,0}, g_{0,0})$ are the eigenfuctions of the eigenvalue problem for (a, T_0) defined by the system

$$L\begin{pmatrix} f_{0,0} \\ g_{0,0} \end{pmatrix} = 0 \ ,$$

$$f_{0,0} = g_{0,0} = df_{0,0}/d\zeta = 0 \qquad (\zeta = 0,1) \ . \qquad (2.31)$$

where L is the linear operator obtained from Q by putting $\Omega_0 = 0$ and $\tilde{T}_0 = T_0$. It follows that the eigenrelation defined by the differential system (2.31) is (2.9) with $\Omega_0 = 0$ and $\tilde{T}_0 = T_0$. This is the eigenrelation of the steady marginal state.

The determination of the function $B_0(\tau)$, as yet unknown, requires the consideration of the $O(\epsilon)$ system.

The latter is an in-homogeneous linear partial differential system for $(u^{(1)}, v^{(1)})$ whose solvability implies that the following condition be satisfied

$$\alpha \frac{dB_0}{d\tau} + B_0(\Gamma \cos\tau + a^2 T_1 \Lambda) = 0 \quad , \tag{2.32}$$

with

$$\Gamma = \frac{\int_0^1 (a^2 T_0 \chi_0 \overset{+}{f}_{0,0} g_{0,0} + \frac{1}{2} \frac{d\chi_0}{d\zeta} f_{0,0} \overset{+}{g}_{0,0}) d\zeta}{\int_0^1 (\overset{+}{f}_{0,0} N f_{0,0} + \overset{+}{g}_{0,0} g_{0,0}) d\zeta} \quad , \tag{2.33}$$

$$\Lambda = \frac{\int_0^1 \overset{+}{f}_{0,0} g_{0,0} \chi_0 \, d\zeta}{\int_0^1 (\overset{+}{f}_{0,0} N f_{0,0} + \overset{+}{g}_{0,0} g_{0,0}) d\zeta} \quad . \tag{2.34}$$

The adjoint pair of functions $(\overset{+}{f}_{0,0}, \overset{+}{g}_{0,0})$ is a solution of the system (2.12) with $\Omega_0 = 0$ and $\tilde{T}_0 = T_0$.

The periodicity criterion imposed on the solution of (2.32) gives, as expected

$$T_1 = 0 \quad , \tag{2.35}$$

and

$$B_0 = A \exp\left(-\frac{\Gamma}{\alpha} \sin\tau\right) \quad , \tag{2.36}$$

where A is a constant (which can only be evaluated by considering non linear effects) whose square is negligible.

The solution for $(u^{(1)}, v^{(1)})$ can now be expressed in the form

$$(u^{(1)}, v^{(1)}) = B_1(\tau)\,(f_{0,0}, g_{0,0}) + B_0 \cos\tau\,(f_{0,1}, g_{0,1}) \quad , \tag{2.37}$$

where the pair of functions $(f_{0,1}, g_{0,1})$ is a solution of the differential system

$$\begin{cases} L\begin{pmatrix} f_{01} \\ g_{01} \end{pmatrix} = \begin{pmatrix} -\Gamma N f_{00} + a^2 T_0 \chi_0 g_{00} \\ -\Gamma g_{00} + \frac{1}{2} \frac{d\chi_0}{d\zeta} f_{00} \end{pmatrix} \quad , \\ f_{01} = g_{01} = \frac{df_{01}}{d\zeta} \qquad (\zeta = 0, 1) \quad , \end{cases} \tag{2.38}$$

and $B_1(\tau)$ is an unknown function of τ whose determination requires the consideration of the $O(\epsilon^2)$ system.

The latter is an inhomogeneous linear partial differential system for $(u^{(2)}, v^{(2)})$ whose solvability depends on the following condition

$$0 = \alpha \frac{dB_1}{d\tau} + \Gamma \cos\tau \, B_1 + \Gamma_1 \alpha \sin\tau \, B_0 + \frac{\Gamma_2}{2} B_0 (1+\cos 2\tau) - a^2 B_0 \Gamma_3 T_2, \quad (2.39)$$

where

$$\Gamma_1 = \frac{\int_0^1 \left[f^+_{0,0}\left(a^2 T_0 g_{0,0} \chi_1 - N f_{0,1}\right) + g^+_{0,0}\left(f_{0,0} \tfrac{1}{2} d\chi_1/d\zeta - g_{0,1}\right)\right] d\zeta}{\int_0^1 \left(f^+_{0,0} N f_{0,0} + g^+_{0,0} g_{0,0}\right) d\zeta}, \quad (2.40)$$

$$\Gamma_2 = \frac{\int_0^1 \left[f^+_{0,0}\left(-\Gamma N f_{0,1} + a^2 T_0 \chi_0 g_{01}\right) + g^+_{0,0}\left(\tfrac{1}{2}(d\chi_0/d\zeta) f_{0,1} - \Gamma g_{0,1}\right)\right] d\zeta}{\int_0^1 \left(f^+_{0,0} N f_{0,0} + g^+_{0,0} g_{0,0}\right) d\zeta}, \quad (2.41)$$

$$\Gamma_3 = - \frac{\int_0^1 \chi_0 f^+_{0,0} g_{0,0} d\zeta}{\int_0^1 \left(f^+_{0,0} N f_{0,0} + g^+_{0,0} g_{0,0}\right) d\zeta}. \quad (2.42)$$

The requirement that B_1 be a periodic function of τ leads to the following expression for T_2

$$T_2 = \frac{\Gamma_2}{2 a^2 \Gamma_3} . \quad (2.43)$$

Furthermore the solution for $B_1(\tau)$ is

$$B_1 = B_0(\tau) \left\{ \Gamma_1 \cos\tau - \frac{\Gamma_2}{4\alpha} \sin 2\tau \right\} . \quad (2.44)$$

We need not proceed to higher approximations in order to show the equivalence of the above procedure to the slowly varying approach.

Thus the marginal state is found to be characterized to $O(\epsilon)$ by the following expression for the disturbance

$$(u, v) = B \exp\left(-\frac{\Gamma}{\alpha}\sin\tau\right)\left\{(f_{0,0}, g_{0,0})\left[1+\epsilon\left(\Gamma_1\cos\tau - \frac{\Gamma_2}{4\alpha}\sin 2\tau\right)\right] + \epsilon\,(f_{0,1}, g_{0,1})\cos\tau\right\} + O(\epsilon^2). \quad (2.45)$$

Furthermore the marginal value of T is given by $(2.29)_b$ with T_1 and T_2 defined by (2.35) and (2.43) respectively and T_3 equal to zero.

We now consider the approach developed in § 2.2 in the limit $\epsilon \to 0$. If we expand u_0, v_0, Ω_0, T in powers of ϵ in the form

$$\begin{cases} u_0 = u_{0,0} + u_{0,1}\epsilon + u_{0,2}\epsilon^2 + O(\epsilon^3), \\ v_0 = v_{0,0} + v_{0,1}\epsilon + v_{0,2}\epsilon^2 + O(\epsilon^3), \\ \Omega_0 = \Omega_{0,1}\epsilon + \Omega_{0,2}\epsilon^2 + O(\epsilon^3), \\ T = T_0 + T_1\epsilon + T_2\epsilon^2 + O(\epsilon^3), \end{cases} \quad \begin{matrix}(2.46)\\ a,b,c,d\end{matrix}$$

we can then substitute from (2.46) into (2.18) to obtain a set of differential systems arising at different orders for the coefficients $u_{0,i}$, $v_{0,i}$ $(i = 1,2,\ldots)$.

At order ϵ^0 the following system arises

$$\begin{cases} Q_0^* \begin{pmatrix} u_{0,0} \\ v_{0,0} \end{pmatrix} = 0 , \\ u_{0,0} = v_{0,0} = \dfrac{\partial u_{0,0}}{\partial \zeta} = 0 \quad (\zeta = 0,1) , \end{cases} \quad (2.47)$$

where Q_0^* is the operator obtained from Q^* by putting $\Omega_0 = 0$ and $\tilde{T}_0 = T_0$.

The solution of (2.47) may be written in the form

$$(u_{0,0}, v_{0,0}) = C_0(\tau)\,(f_{0,0}(\zeta), g_{0,0}(\zeta)) , \quad (2.48)$$

where $C_0(\tau)$ is a function whose determination requires the consideration of the order σ problem in the limit $\epsilon \to 0$.

By substituting again from (2.46) into (2.18) and equating terms of order ϵ , it follows

$$\left\{\begin{array}{l} Q_0^* \begin{pmatrix} u_{0,1} \\ v_{0,1} \end{pmatrix} = \begin{pmatrix} -i\Omega_{0,1} C_0 N f_{0,0} + a^2 T_0 \chi_0 g_{0,0} C_0 \cos\tau + a^2 T_1 \chi_0 C_0 g_{0,0} \\ -i\Omega_{0,1} C_0 g_{0,0} + \frac{1}{2}\frac{d\chi_0}{d\zeta} C_0 f_{0,0} \cos\tau \end{pmatrix}, \\ u_{0,1} = v_{0,1} = \frac{\partial u_{0,1}}{\partial \zeta} = 0 \qquad (\zeta = 0,1) . \end{array}\right. \tag{2.49}$$

The solvability condition of (2.49), together with (2.33) gives

$$i\Omega_{0,1} = \Gamma \cos\tau - a^2 T_1 \Gamma_3 . \tag{2.50}$$

The periodicity condition imposed on (2.50) gives as expected

$$T_1 = 0 . \tag{2.51}$$

The solution of (2.49) can now be written in the form

$$(u_{0,1}, v_{0,1}) = C_1(\tau)(f_{0,0}(\zeta), g_{0,0}(\zeta)) + C_0(\tau)(f_{0,1}(\zeta), g_{0,1}(\zeta)) \cos\tau , \tag{2.52}$$

where $C_1(\tau)$ is a function whose determination again requires the consideration of the order σ problem in the limit $\epsilon \to 0$.

Finally at order ϵ^2 the $O(\sigma^0)$ system (2.18) reads

$$\left\{\begin{array}{l} Q_0^* \begin{pmatrix} u_{02} \\ v_{02} \end{pmatrix} \\ = \begin{pmatrix} \left[C_0 \cos^2\tau + \left(-\Gamma N f_{0,1} + a^2 T_0 \chi_0 g_{0,1}\right) + C_1 \cos\tau\left(-\Gamma N f_{0,0} + a^2 T_0 \chi_0 g_{0,0}\right) + \right. \\ \left. - i\Omega_{02} C_0 N f_{0,0} + a^2 T_2 \chi_0 C_0 g_{0,0}\right] \\ \left[C_0 \cos^2\tau \left(-\Gamma g_{0,1} + \frac{1}{2}\frac{d\chi_0}{d\zeta} f_{0,1}\right) + C_1 \cos\tau\left(-\Gamma g_{0,0} + \frac{1}{2}\frac{d\chi_0}{d\zeta} f_{0,0}\right) + \right. \\ \left. - i\Omega_{02} C_0 g_{0,0}\right] \end{pmatrix}, \\ u_{0,2} = v_{0,2} = \frac{\partial u_{0,2}}{\partial \zeta} = 0 \qquad (\zeta = 0,1) , \end{array}\right. \tag{2.53}$$

where (2.48),(2.51),(2.52) have been taken into account. The solvability condition of (2.53) together with (2.41),(2.42) gives

$$i\Omega_{0,2} = \Gamma_2 \cos^2\tau - a^2 T_2 \Gamma_3 \quad . \tag{2.54}$$

The periodicity condition imposed on (2.54) leads to an expression for T_2 which is identical to (2.43).

Let us now consider the $O(\sigma)$ system (2.21) and expand (u_1, v_1) in powers of ϵ in the form

$$\begin{aligned} u_1 &= u_{1,0} + \epsilon u_{1,1} + O(\epsilon^2) \quad , \\ v_1 &= v_{1,0} + \epsilon v_{1,1} + O(\epsilon^2) \quad . \end{aligned} \tag{2.55 a,b}$$

The differential system for $(u_{1,0}, v_{1,0})$ is found to be

$$\begin{cases} Q_0^* \begin{pmatrix} u_{1,0} \\ v_{1,0} \end{pmatrix} = \begin{pmatrix} N f_{0,0} \dfrac{dC_0}{d\tau} \\ g_{0,0} \dfrac{dC_0}{d\tau} \end{pmatrix} \quad , \\ u_{1,0} = v_{1,0} = \dfrac{\partial u_{1,0}}{\partial \zeta} = 0 \qquad (\zeta = 0,1) \, . \end{cases} \tag{2.56 a,b}$$

The solvability condition for such system gives

$$C_0(\tau) = C_0 = \text{cost} \, . \tag{2.57}$$

At $O(\epsilon)$ the system for $(u_{1,1}, v_{1,1})$ is found in the form

$$\begin{cases} Q_0^* \begin{pmatrix} u_{1,1} \\ v_{1,1} \end{pmatrix} = \begin{pmatrix} N f_{0,0} \dfrac{dC_1}{d\tau} - N f_{0,1} \sin\tau \; C_0 + a^2 T_0 \chi_1 g_{0,0} \Phi_1 C_0 \\ g_{0,0} \dfrac{dC_1}{d\tau} - g_{0,1} \sin\tau \; C_0 + \frac{1}{2} f_{0,0} \dfrac{d\chi_1}{d\zeta} \Phi_1 C_0 \end{pmatrix} , \\ u_{1,1} = v_{1,1} = \partial u_{1,1}/\partial\zeta = 0 \qquad (\zeta = 0,1) \, . \end{cases} \tag{2.58}$$

The solvability condition for the previous system provides a differential equation for C_1, which can be solved to give

$$C_1(\tau) = C_0 \Gamma_1 \cos\tau \ . \qquad (2.59)$$

By substituting from (2.48),(2.52) into $(2.46)_{a,b}$ with C_0 and C_1 given by (2.57), (2.59) respectively, (u_0, v_0) are determined to order ϵ. Moreover if (2.50), and (2.54) are taken into account, $i\Omega_0$ is known to order ϵ^2. Thus we are in a position to derive an expression for the disturbance amplitudes which can be shown to reduce to (2.45) to order ϵ if we take $C_0 = A$.

Thus it is seen that Di Prima-Stuart expansion procedure can be interpreted as a limiting form of the WKB procedure.

2.5 - Results

The starting point for the calculation is the determination of the marginal state for the steady case. Dean (1928), Reid (1958) and Hämmerlin (1958) obtained approximate analytical solutions for the differential system (2.6). Gibson & Cook (1974) employed a Chebyshev collocation method. We solved it numerically by means of the Runge-Kutta-Gill procedure of the fourth order (40 steps). The results for the curves of neutral stability are shown in figure 1. The critical values of T_0 and a_0 are given in table 1. The corresponding eigenfunctions $(f_{0,0}, g_{0,0})$ are shown in figures 3 and 4, normalized such that $g_{0,0}(\frac{1}{2})=1$.

The system (2.12) was solved with $i\Omega_0 = 0$ to obtain the pair of adjoint eigenfunctions $(f^+_{0,0}, g^+_{0,0})$ which are shown in figure 5 normalized such that $f^+_{0,0}(\frac{1}{2}) = 1$. The constant Γ was evaluated from (2.33) and the system (2.38) was solved giving the pair of functions $(f_{0,1}, g_{0,1})$. Finally T_2, Γ_3, Γ_2 were determined from (2.43) (2.42) (2.41), respectively. Each calculation was performed for the steady critical configuration $(a_{0,c}, T_{0,c})$. The procedures used to solve the various differential systems have been discussed by Eagles (1971) and will not be described here again. The numerical solutions of the boundary value problems were obtained by the Runge-Kutta-Gill procedure of the fourth order (40 steps). The integrations were performed by using the Simpson rule (40 steps). We obtained

$$\Gamma = -42.87 \quad ,$$
$$\Gamma_2 = -4.505 \quad ,$$
$$\Gamma_3 = 0.2659\times10^{-3} \quad , \qquad (2.60)\ a,b,c,d$$
$$T_2 = -542.59 \quad .$$

We then examined the approach described in § 2.2 for the periodic time dependence expressed by (2.28). For given values of ϵ and σ the marginal state was determined, for every wavenumber, by following in time the perturbation corresponding to different values of T until the periodicity of the disturbance was verified. For a given T the method of solution proceeded as follows. For each τ the instantaneous value of $\tilde{T}_0$ was defined by (2.8). The eigenvalue problem (2.18) was then solved and gave the instantaneous value of Ω_0 and the disturbance transverse structure at order σ^0. After determining the instantaneous adjoint function pair (f_0^+, g_0^+) from (2.12) (with $\sigma\bar{t}$ replaced by τ and $d/d\zeta$ by $\partial/\partial\zeta$), we were able to perform the integrations in (2.23) and obtain the instantaneous value of H. Finally equation (2.22) was solved and gave the instantaneous value of the amplitude function $A(\tau)$ which was normalized such that $A(0) = 1$. This procedure was repeated for discrete values of τ until a period of the basic flow was completed. Furthermore for every τ the expressions (2.26) could be evaluated and provided the instantaneous values of the growth rates. The marginal value of T was obtained by linear interpolation in a neighbourhood of the neutral curve. At any order in σ this value does not depend on the quantity chosen to describe the evolution of the disturbance.

Indeed to order σ the periodicity condition reduces to the relation

$$\int_\tau^{\tau+2\pi} - H(\tau)\, d\tau - \left[\int_\tau^{\tau+2\pi} i\Omega_0(\tau)\,d\tau\right] \Big/ \sigma = 0 \quad . \qquad (2.61)$$

But we have

$$\int_\tau^{\tau+2\pi} H(\tau)\, d\tau = 0 \quad . \qquad (2.62)$$

This follows by showing that $H(\tau)$ satisfies the condition

$$H(\tau_0) = - H(2\pi - \tau_0) \ . \qquad (2.63)$$

In fact systems (2.18), (2.12) clearly show that

$$(f_0, g_0, \overset{+}{f}_0, \overset{+}{g}_0)(\tau_0) = (f_0, g_0, \overset{+}{f}_0, \overset{+}{g}_0)(2\pi - \tau_0) \ ,$$

so that

$$\left(\frac{\partial M f_0}{\partial \tau}, \frac{\partial (g_0\ \phi_0)}{\partial \tau}\right)(\tau_0) = - \left(\frac{\partial M f_0}{\partial \tau}, \frac{\partial (g_0\ \phi_0)}{\partial \tau}\right)(2\pi - \tau_0).$$

The latter behaviour is also exhibited by the functions ϑ_1 and $\partial\vartheta_1/\partial\zeta$. Then (2.62) follows from (2.23). By taking (2.62) into account, the periodicity condition (2.61) is found to be

(i) independent of the quantity chosen to describe the evolution of the disturbance;

(ii) independent of σ.

The latter conclusion confirms the result obtained in the limit conditions $\epsilon \to 0$ and can also be formally derived if we expand the marginal value of T in the form $T = T_0 + \sigma T_1 + \sigma^2 T_2 + \ldots$; then by considering higher order terms in the velocity expansion it can easily be shown that the first non zero correction term to T_0 above is T_2. The solvability condition of the order σ system would now involve T_1, and similar arguments as those given above, along with the condition (2.63) would give

$$T_1 = 0 \ , \qquad (2.64)$$

as expected.

If higher order terms are taken into account it can be shown that (i) can be verified to any order of approximation whereas (ii) is only true to order σ.

Neutral stability curves were determined by this procedure for the cases $\epsilon = 0.4$, $\epsilon = 0.9$ and are shown in figure 2. The evolution in time of the disturbance transverse structure is shown in figure 3 and 4 for the critical state corresponding to $\epsilon = 0.9$. For the same configuration figure 6 shows the behaviour of the growth rate G_4 (associated with the energy of the disturbance) compared with the quasi-steady correction term (of order σ) $i\Omega_1(\tau)$ and with the function $H(\tau)$.

The behaviour of the growth rates G_1, G_2, G_3 is very close to the one obtained for G_4 and is not shown. Finally figure 7 shows the amplitude function $A(\tau)$ for the previous critical state.

Checks on the computation were provided

(i) by comparing the structure of the function $i\Omega_0(\tau)$ as given by (2.46c) (2.50) (2.51) (2.54) (2.41) (2.43) and (2.60) with the instantaneous eigenvalues of the system (2.18).

(ii) by the following relation, which can be derived algebraically

$$\Gamma_3 = -\frac{\Gamma}{2a_{0,c}^2 \, T_{0,c}} \quad ;$$

(iii) by comparing the results obtained by means of Di Prima-Stuart approach with the results given by the method employed in this work. This will be described in the next section.

Slowly accelerated case

We then examined slowly accelerated basic flows characterized by the time dependence expressed by

$$\zeta(\omega t^*) = \epsilon(1 + \tanh\tau). \tag{2.65}$$

The corresponding basic velocity field can be obtained by substituting from (2.65) into (1.8). The evolution in time of the perturbation is then determined by a boundary value problem at any instant in time similar to the one previously outlined for the modulated case. The solution was obtained by a similar numerical procedure and allowed us to determine the instants of momentary neutral stability $\tau_M^{(Q_i)}$ and the corresponding values $\tilde{T}_M^{(Q_i)}$ associated with the quantity Q_i. These are defined as the values of τ and $\tilde{T}$ when the growth rate G_i vanishes. For the basic flows we are examining, G_i vanishes at most once in the interval $-\infty<\tau<\infty$. We have

$$\begin{aligned} \tau_M^{(Q_i)} &= \tau_M^{(Q_i)}(\tilde{T}_f, a, \sigma), \\ \tilde{T}_M^{Q_i} &= \tilde{T}_M^{(Q_i)}(\tilde{T}_f, a, \sigma), \end{aligned} \tag{2.66}$$

where $\tilde{T}_f = [\tilde{T}]_{\tau\to\infty}$. The curves of momentary marginal stability corresponding to some given values for $\tilde{T}_f$ are shown in figures 8 and 9 for Q_4 and $\sigma = 0.3$. For given values of σ and $\tilde{T}_f$ the function $\tau_M^{(Q_i)}$ has a minimum $\tau_c^{(Q_i)}$ for a value of a ($a_c^{(Q_i)}$) which determines the critical conditions for momentary neutral stability. The variation of the critical wavenumber with $\tilde{T}_f$ is fairly small and exhibits a slight dependence on σ as shown in table 2.

The behaviour of the growth rate of the relative energy of the perturbation for particular values of the parameters is shown in figure 10 along with the quasi-steady and the slowly varying corrections of order σ . The amplitude function corresponding to the same values of the parameters is shown in figure 11.

2.6 - Discussion

We first note the remarkable agreement of Dean's (1928) and Gibson & Cook's (1974) results compared with the results obtained in the present work for the steady case. The first approximate method used by Reid (1958), though some-what simpler, is less accurate as he pointed out. A similar argument applies to Hämmerlin's (1958) method.

We now discuss the modulated case. We have shown in § 2.4 that, in the limit $\epsilon \to 0$, the perturbation velocity field arising from the approach used in this work reduces to the one obtained by Di Prima--Stuart method. Moreover from the latter approach it follows that the critical value of the parameter T at which instability first occurs is given by $(2.29)_b$ evaluated at $a = a_{0,c}$. Indeed, Venezian (1969) showed for the analogous modulated Benard problem that the critical value for the wavenumber is given by

$$a_c = a_{0,c} + \epsilon^2 a_{2,c} + O(\epsilon^4) , \qquad (2.67)$$

where

$$a_{2,c} = - \frac{(\partial T_2/\partial a)_{a=a_{0,c}}}{2(\partial^2 T_0/\partial a^2)_{a=a_{0,c}}} . \qquad (2.68)$$

The correction term $\epsilon^2 a_{2,c}$ affects the relation $(2.29)_b$ only at order ϵ^4 as Hall (1973) showed by proceeding to the higher approximations in the analysis of modulated Couette flow. Thus such an effect is fairly small within Hall's scheme. The present approach is not subject to the condition $\epsilon \ll 1$ and this allowed us to evaluate the previous effect quantitatively. In fact, by quadratic interpolation in a neighbourhood of the critical configuration, we were able to find the values of a_c and T_c which are shown in table 3. It appears that a_c decreases sensibly when ϵ increases with σ fixed. By evaluating $(2.29)_b$ and (2.68) numerically for $a = a_{0,c}$, taking into account (2.35) and (2.43) we could determine the degree of accuracy given by Di Prima-Stuart method at order ϵ^2. The results are shown in table 3. The agreement is very satissfactory for $\epsilon = 0.15$ and still fairly good for $\epsilon = 0.4$ even though only terms of order ϵ^2 have been retained in $(2.29)_b$ so that the dependence of a_c on ϵ has been neglected. The results are no longer comparable for $\epsilon = 0.9$.

In the low frequency limit it seems that the dominant effect of modulation is to destabilize the flow, the degree of destabilization increasing as ϵ is increased from zero. Such a conclusion confirms Hall's (1975a) results for modulated Couette flow in the limit ϵ and σ tending to zero. On the contrary experimental work done by Donnelly (1964) on modulated Couette flow showed that modulation enhanced the stability of the flow, the degree of enhancement being maximum, for all ϵ, for a certain value ($\simeq 0.27$) of the frequency parameter σ. In view of the analogy between the mechanism of instability considered here and the nature of Couette flow instability we can conclude that the linear theory developed in this paper is not able to explain Donnelly's results.

Let us now discuss the results obtained for the slowly accelerated basic flows satisfying (2.65). It is interesting to examine the nature of the correction terms of order σ which appear in the expression (2.26) for the growth rates of the disturbance.

Figure 10 shows that the global effect to the order σ correction for the growth rate G_4 (there denoted by G_{41}) is stabilizing. In fact the main contribution to this correction is due to the term $A^{-1}dA/d\tau = -H(\tau)$. As mentioned in § 2.3 this can be split into two parts: the quasi-steady term $\mathrm{Im}\,\Omega_1(\tau)$ and a slowly varying component arising from the variation with time of the disturbance transverse structure. Both the components exhibit a stabilizing effect as shown in figure 10. Further stabilization is associated, as expected, with measuring the energy relative to the corresponding value for the basic flow. In fact the growth rate, $\phi_0^{-1}d\phi_0/d\tau$, of the basic flow, is positive for $0 \leq \tau < \infty$ due to the acceleration and this induces a negative contribution into the order σ correction for G_4. The only term which exhibits a destabilizing effect is the one which directly accounts for the growth rate of the disturbance transverse structure, namely $\frac{1}{2}F^{-1}dF/d\tau$. This term is not explicity shown in figure 10, but its behaviour can be inferred and is qualitatively similar to the one shown for $[\zeta_0]^{-1}_{\eta=0.5}\,[\partial\zeta_0/\partial\tau]_{\eta=0.5}$.

Figure 10 shows also that the order σ correction for G_4 is rather small compared with the leading term Im $\Omega_0(\tau)$. This is due to the characteristic scale of instability, which, unlike in Eagles & Weissman's (1975) work, is here much bigger than the "slow" scale of the basic state. This argument also explains the apparently odd decreasing of $A(\tau)$ when τ increases as shown in figure 11.

In fact the dominant τ dependence in the disturbance velocity field is associated with the "W.K.B." exponent $\int^{\tau}(\mathrm{Im}\,\Omega_0(\tau)/\sigma)d\tau$ rather more than with the exponent $-\int^{\tau}H(\tau)d\tau$ on which the "amplitude" function $A(\tau)$ depends. Under these conditions the quasi-steady approximation may be considered as a fairly accurate approach. This also implies that the curves of "momentary" neutral stability shown in figures 8 and 9 do not differ appreciably from the ones which are obtained by a quasi-steady analysis.

We should also mention that a variation of $a_c^{(Q.)}$ with $\tilde{T}_f$ had already been obtained by Chen & Kirchner (1971) who studied numerically the stability of time dependent rotational Couette flow when the inner cylinder is impulsively started at $t = 0$ and maintained at a constant speed. They examined high frequency basic configurations and found that $a_c^{(E)}$ increased sensibly when the Reynolds number was increased. A direct comparison between their results and ours is not possible but it seems worthwhile to note that the effect we obtained, even though much slighter, exhibits similar features.

Table 1

	$T_{0,c}$	$a_{0,c}$
Dean	5169.6	3.954
Hämmerlin	5100	4.0
Reid	5277	3.889
Gibson & Cook	5161.9	3.950
Present work	5161.86	3.951

The critical values of the Taylor number and wavenumber in the steady case as obtained by Dean (1928), Hämmerlin (1958), Reid (1958), Gibson & Cook (1974) and the present author.

Table 2

σ \ $\tilde{T}_f$	6252.1	8732.3	11625.8	14932.7	
0.2	3.9535	3.9563	3.9583	3.9598	$a_c^{(e)}$
0.3	3.9544	3.9585	3.9615	3.9638	

The critical values of the wavenumber for the time dependence expressed by (2.65) with $\sigma = 0.2$, 0.3 and some values of $\tilde{T}_f$.

Table 3

		Present approach		Di Prima-Stuart approach	
ϵ	σ	T_c	a_c	T_c	a_c
0.15	0.1	5149.6	3.9497	5149.6	3.944
0.4	0.1	5065.1	3.9053	5075.0	3.900
0.9	0.1	4355.5	3.6555		

Comparison between the critical values for (a,T) as obtained by Di Prima-Stuart approach and by the present method.

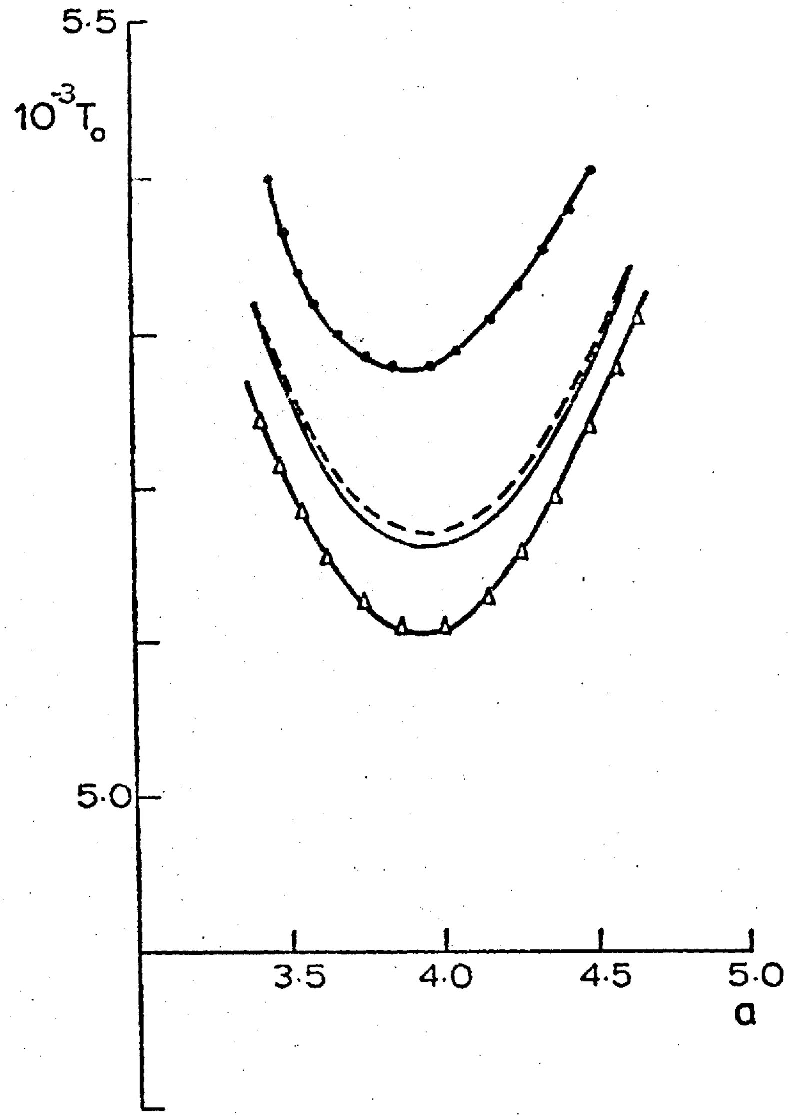


FIG. 1: The curves of neutral stability for the steady case. •—◦—• Reid, — — — Dean, Δ—Δ—Δ Hämmerlin, ——— Present work.

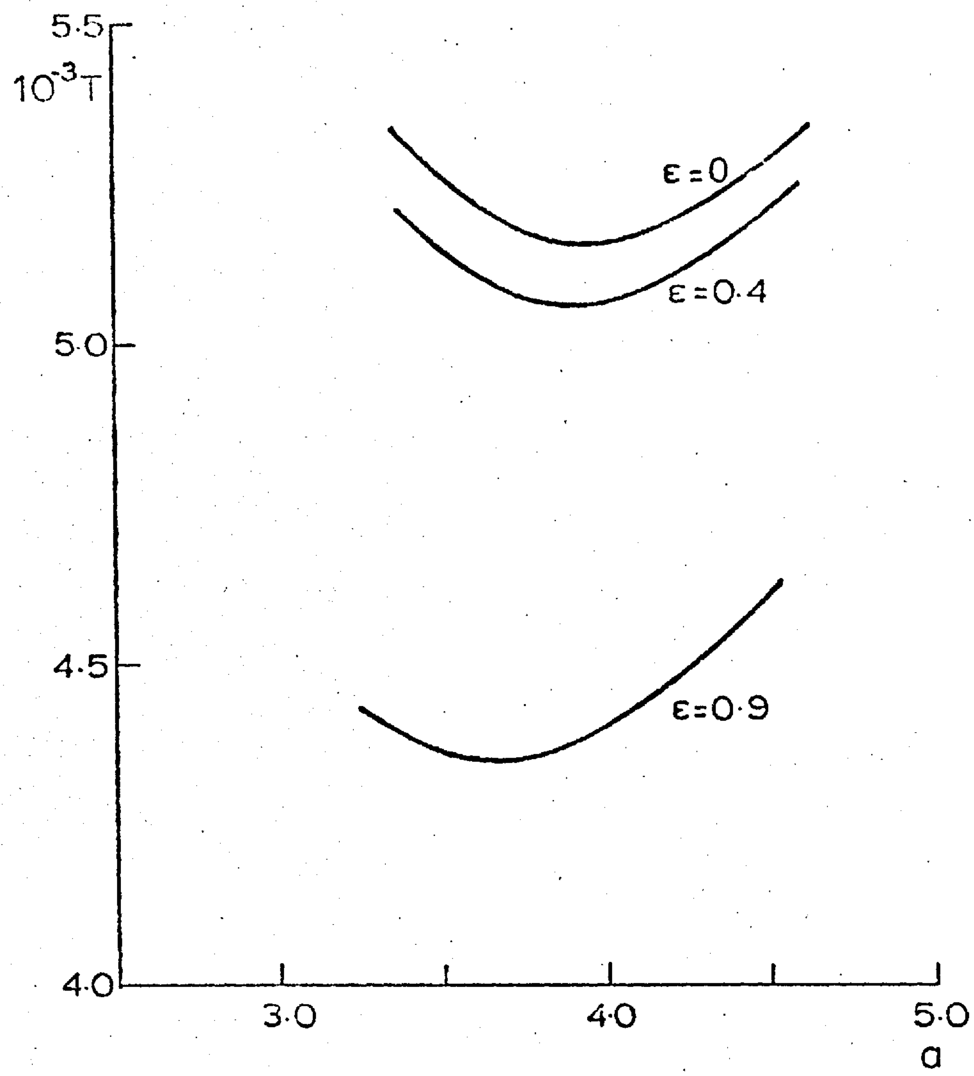


FIG. 2: The curves of neutral stability for the modulated case

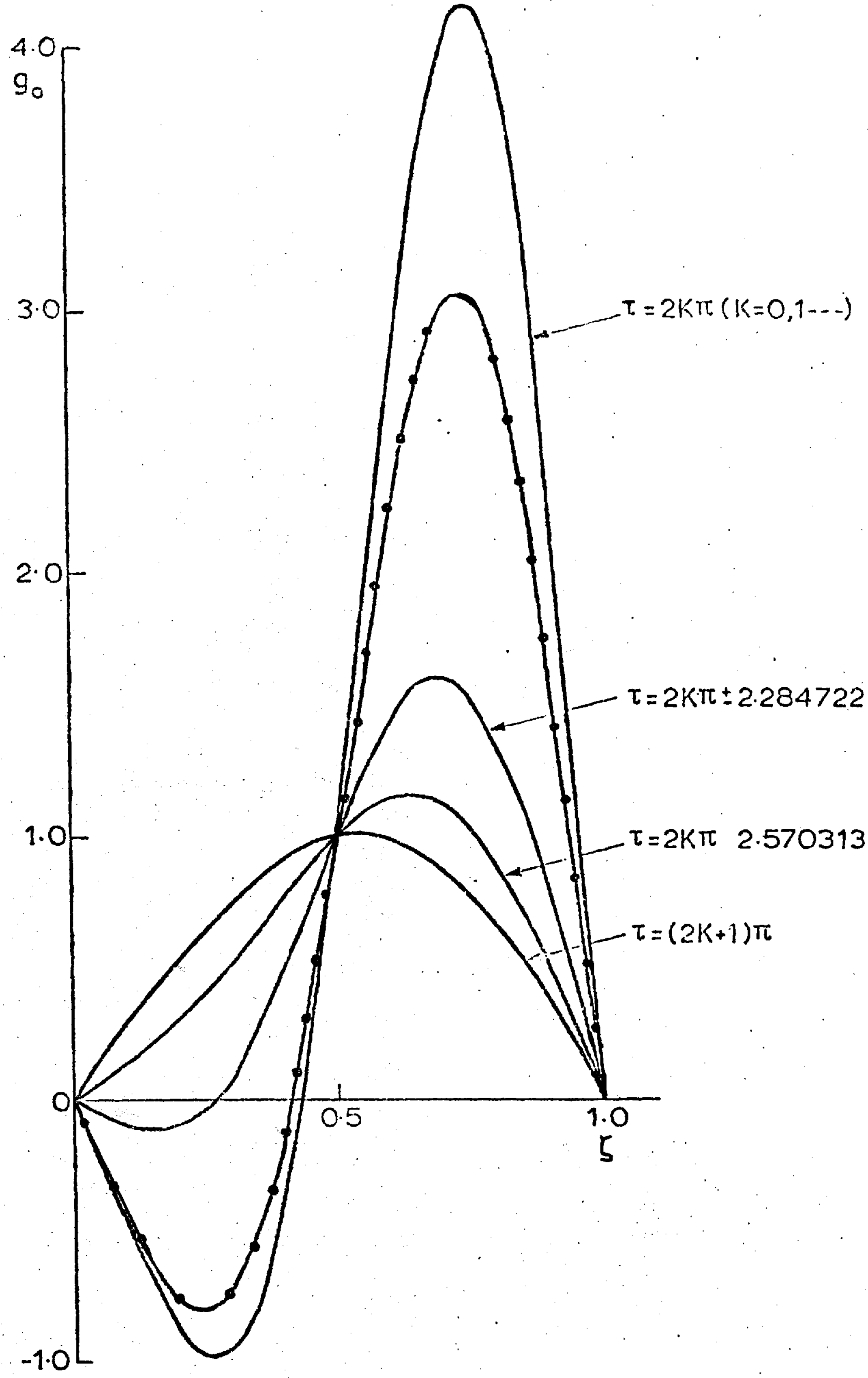


FIG. 3: The evolution in time of the function $g_o(\zeta,\tau)$ for the critical state corresponding to the modulated case with $\varepsilon = 0{\cdot}9$. •—•—• The function $g_{oo}(\zeta)$.

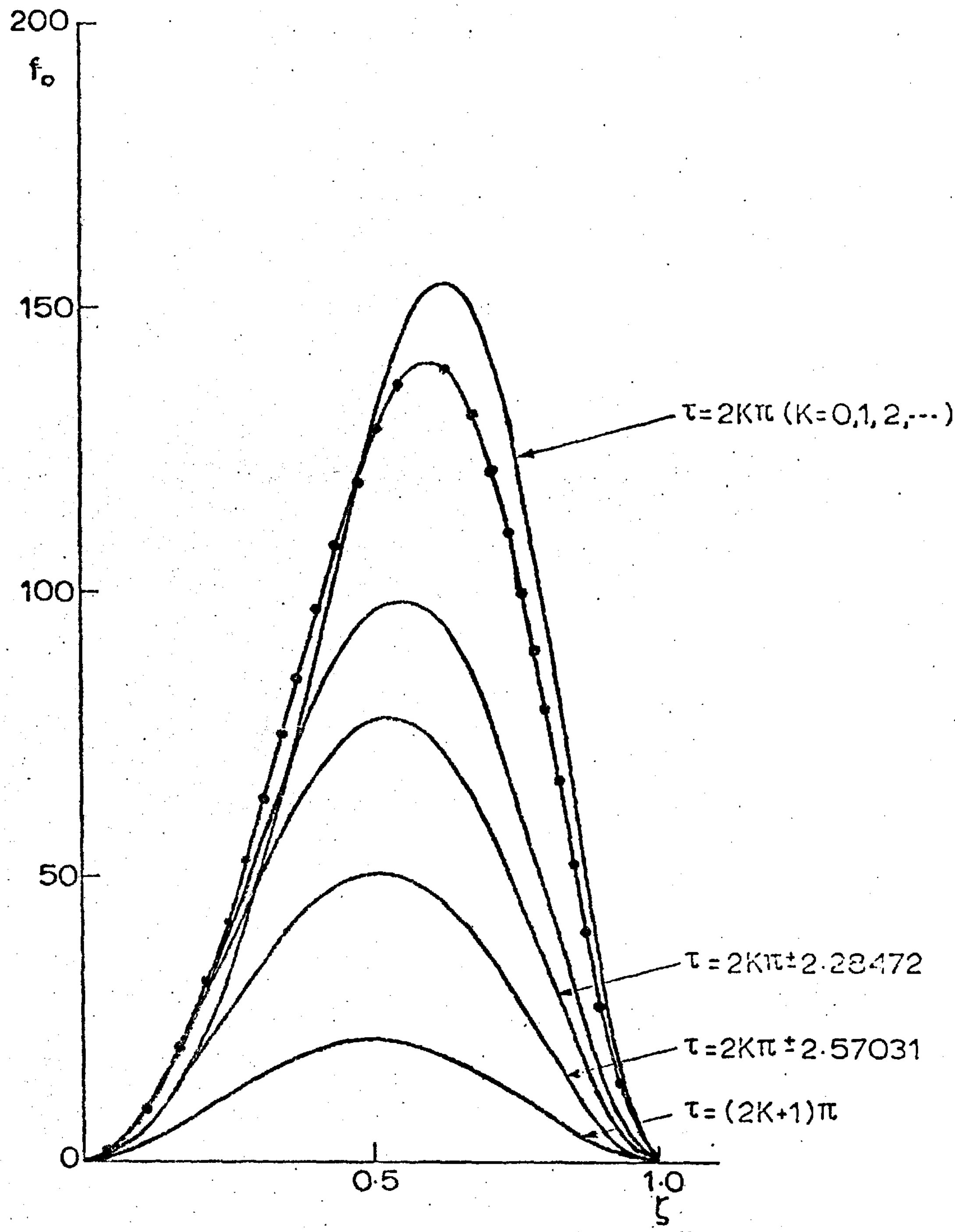


FIG. 4: The evolution in time of the function $f_o(\zeta, \tau)$ for the critical state corresponding to the modulated case with $\varepsilon = 0.9$. ●—●—● The function $f_{oo}(\zeta)$.

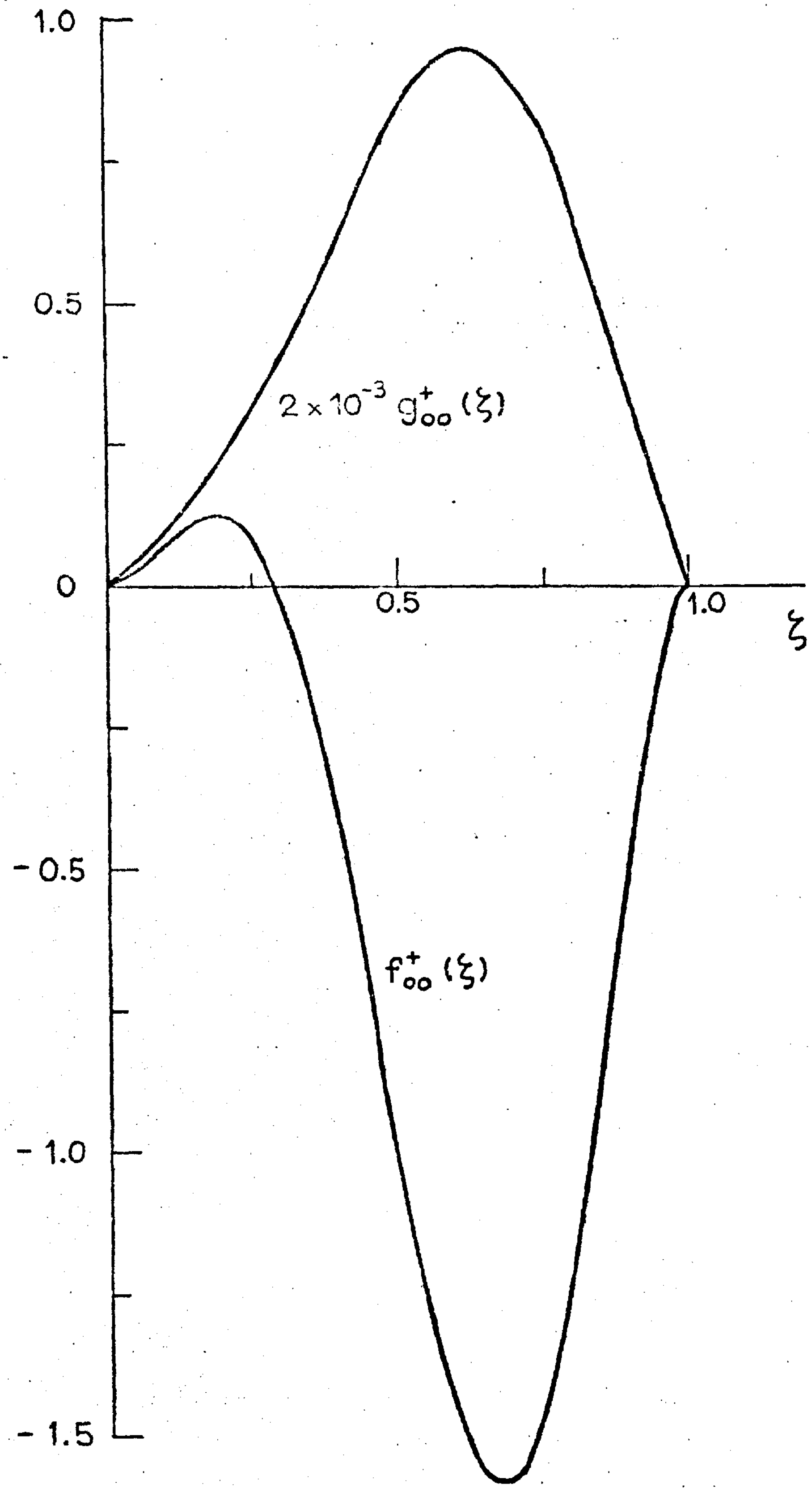


FIG. 5 - The adjoint eigenfunctions $f^{+}_{oo}(\xi)$, $g^{+}_{oo}(\xi)$ normalized such that $\dot{f}^{+}_{oo}(1/2) = 1$.

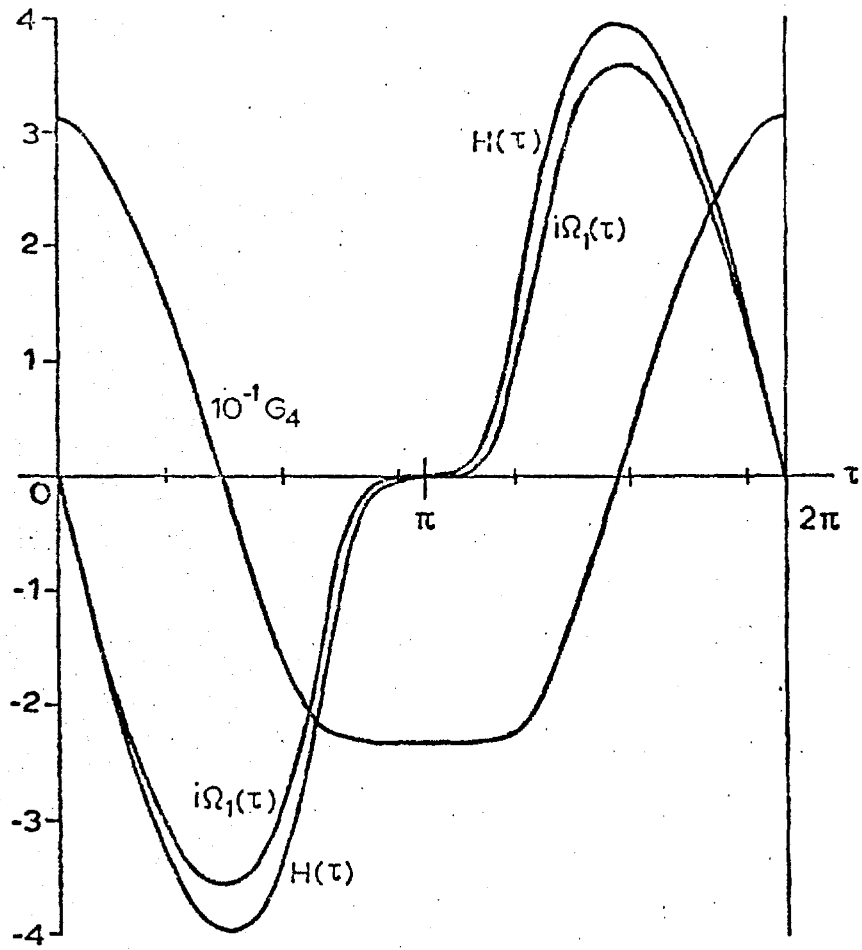


FIG.6: The growth rate of the relative energy of the disturbance (G_4) compared with the quasi-steady correction term of order σ ($i\Omega_1(\tau)$) and the slowly varying term ($H(\tau)$) for the critical state corresponding to the modulated case with $\varepsilon = 0{\cdot}9$.

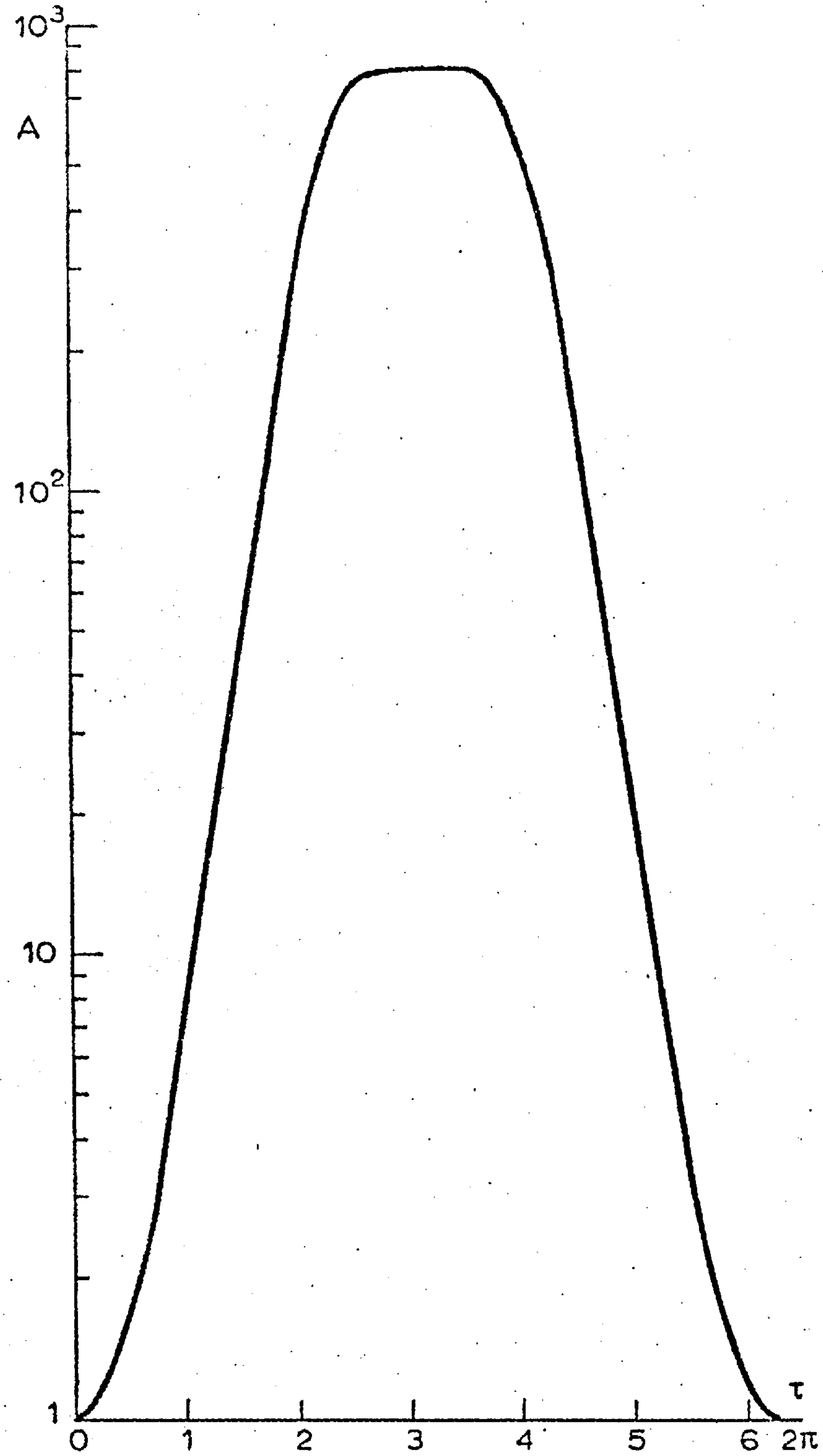


FIG. 7: The amplitude function $A(\tau)$ for the critical state corresponding to the modulated case with $\varepsilon = 0{\cdot}9$.

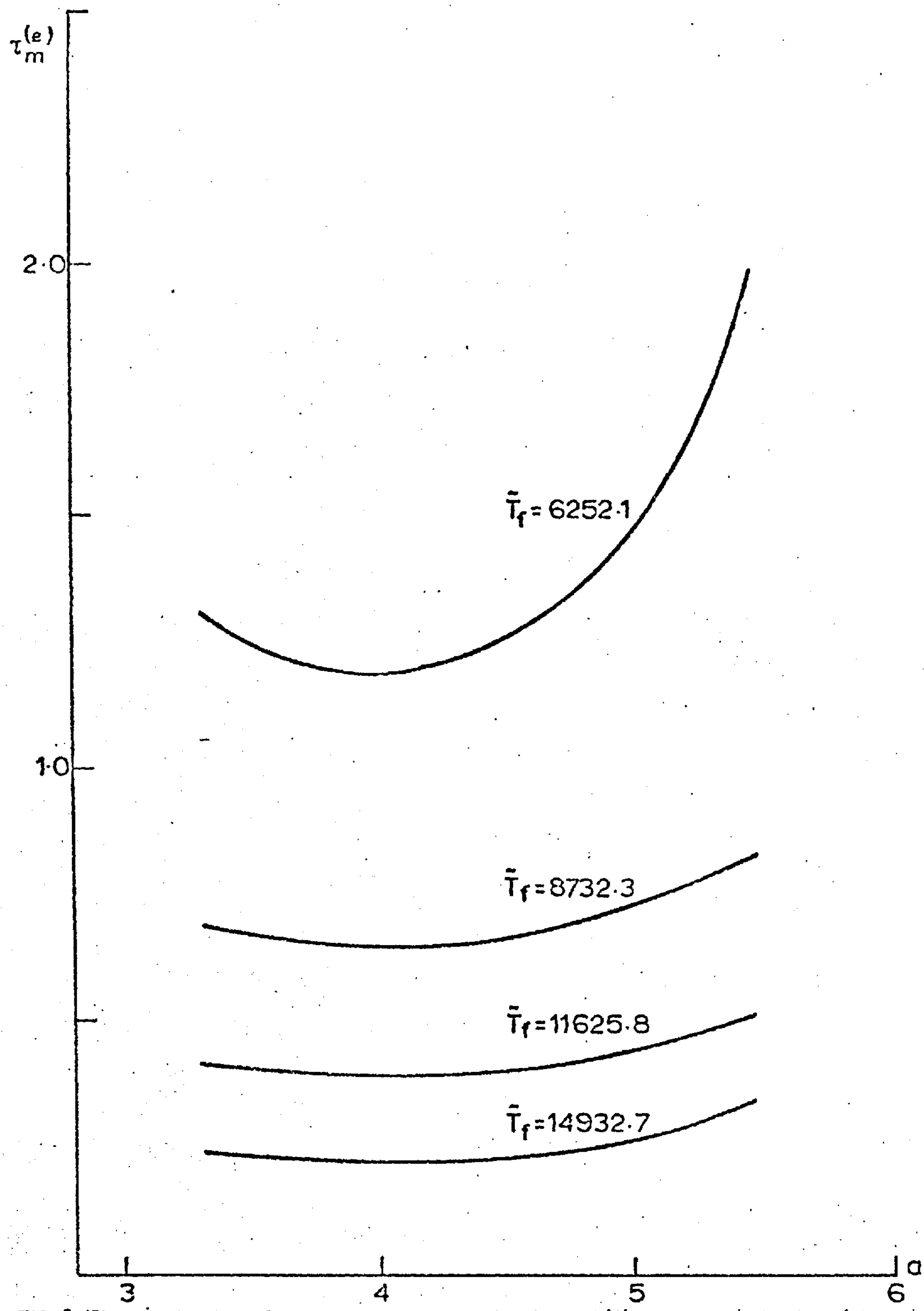


FIG. 8: The instants of momentary neutral stability associated with the relative energy of the disturbance for the time dependence expressed by (2.65) with $\sigma = 0{\cdot}3$.

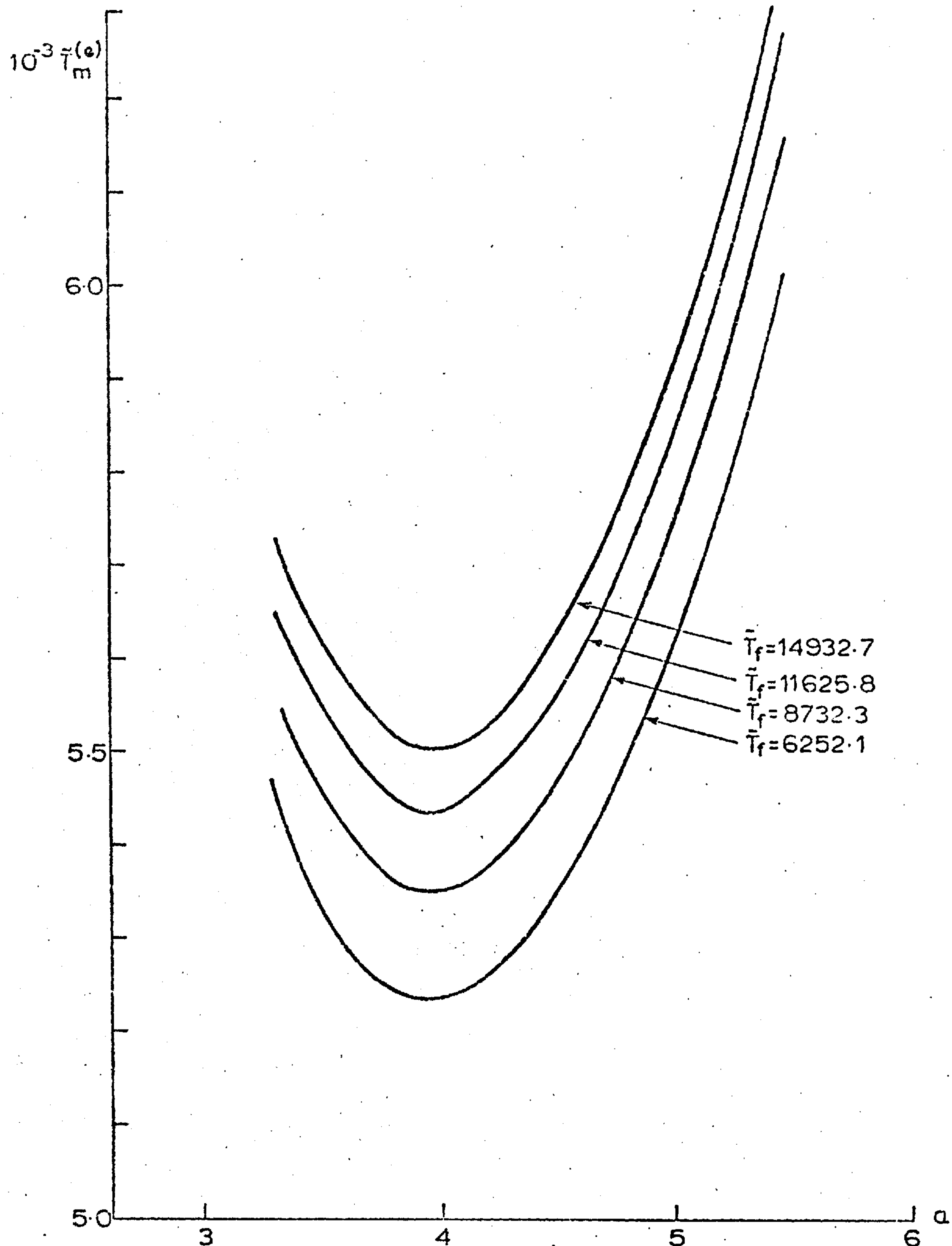


FIG.9: The configurations of momentary neutral stability associated with the relative energy of the disturbance for the time dependence expressed by (2.65) with $\sigma = 0{\cdot}3$.

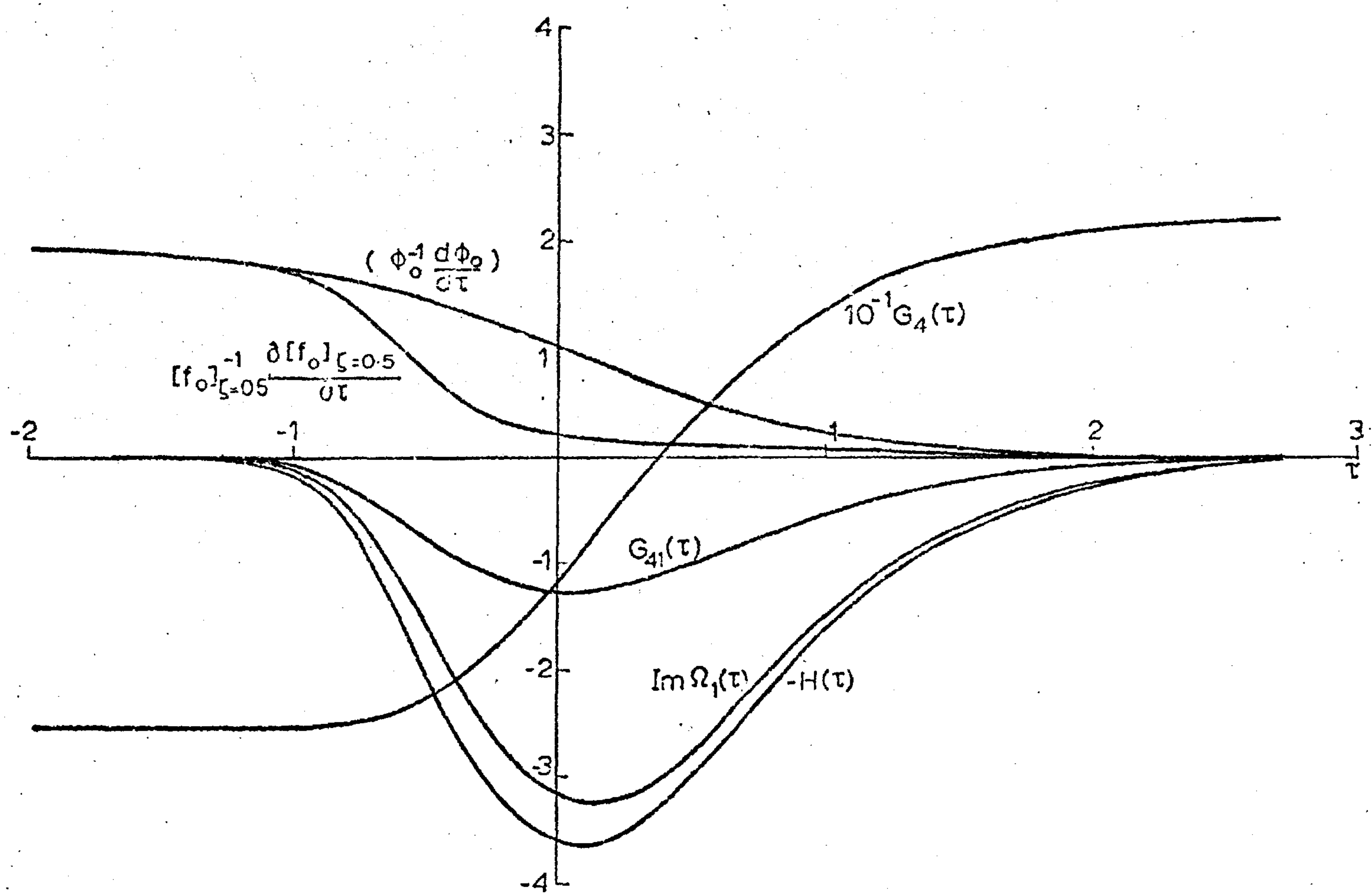


FIG.10: The growth rate $G_4(\tau)$ compared with the order σ correction (denoted by G_{41}) for the time dependence expressed by (2.65) with $\sigma = 0{\cdot}3$ and $\tilde{T}_f = 11625{\cdot}8$. Some single components of the order σ corrections for the various growth rates are also shown.

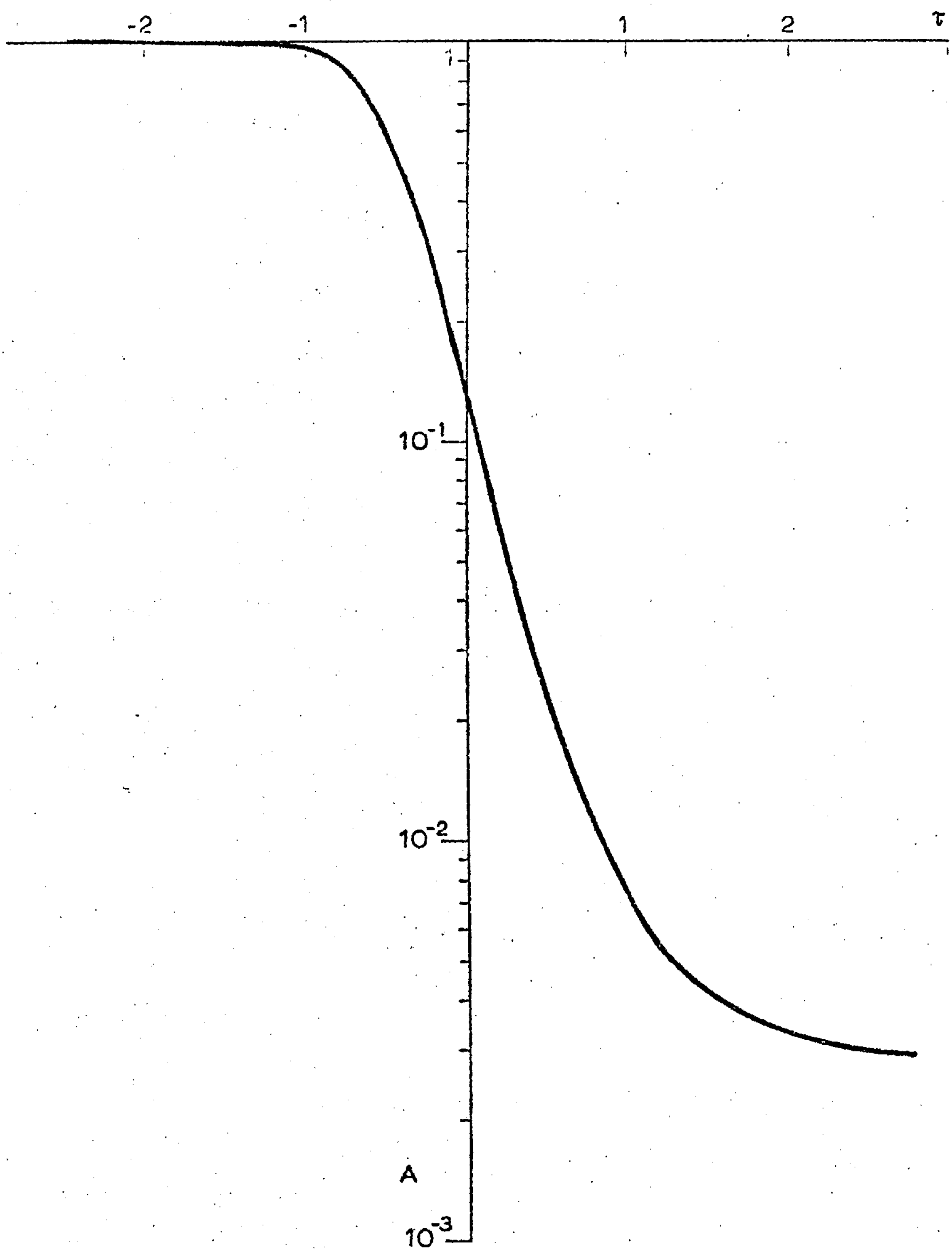


FIG.11: The amplitude function $A(\tau)$ for the time dependence expressed by (2.65) with $a = a_{oc}$, $\sigma = 0{\cdot}3$, $\tilde{T}_f = 11625{\cdot}8$.

CHAPTER 3

WEAKLY NON LINEAR THEORY

3.1 - Introduction

In this chapter we develop a perturbation expansion in order to determine the weakly non linear growth and equilibrium state of the disturbance both in the steady and in some unsteady cases. The analysis follows the line of Stewartson & Stuart's (1971) work on non linear parallel instability.

Time dependent basic flows are considered such that

$$\underset{\sim}{V}_0(\tau) = 1 + \epsilon f(\tau) \quad , \tag{3.1}$$

where both the amplitude ϵ and the frequency σ of the unsteady component are "small". Under such conditions the problem can be treated following the procedure used by Hall (1975a) for the analogous unsteady Taylor problem and first suggested by Di Prima & Stuart (1975) when studying the stability of the flow between eccentric rotating cylinders. The idea (which was already mentioned in § 2.4) is to let σ and ϵ tend to zero keeping their ratio σ/ϵ fixed. Furthermore the growth rate of the disturbance is of order ϵ , within a small neighbourhood of the marginal configuration. Thus expansions of the type used by Stewartson-Stuart can be set up for the various harmonics in terms of the small parameter ϵ . An analysis of the differential problems obtained for the coefficients of such expansions at the various orders of approximation, shows that the solution depends on an "amplitude function" $A(\tau)$ which, as expected, is found to satisfy an amplitude equation of Bernoulli type identical to that found by Hall (1975a). A discussion of such equation similar to that given by Hall (1975a) shows that:

(i) an equilibrium amplitude solution exists in the supercritical regime in the steady case;

(ii) when $f(\tau) = \tanh(\tau)$ an equilibrium amplitude solution exists as $\tau \to \infty$ in the supercritical regime; such solution is just the equilibrium amplitude solution for the steady problem with the Taylor number based on the final average speed of the basic flow;

(iii) when $f(\tau) = \cos\tau$ the amplitude solution is periodic in τ . However by taking further limits it is shown that the effect of modulation tends to disappear as T_1/T_o tends to infinity with α fixed and as $\alpha \to \infty$ with T_1/T_o fixed.

Furthermore, as T_1/T_o tends to zero with α also tending to zero the τ dependence of A becomes that of $B_o(\tau)$ found in § 2.4.

3.2 - Analysis for slightly unsteady slowly varying basic flows

Let us consider basic flows such that $\mathfrak{Z}$ is given by (3.1) with

$$\epsilon \ll 1 \quad , \quad \sigma \ll 1 \quad , \tag{3.2}$$

The basic velocity field is given by (1.6),(1.7),(1.8) with $\mathfrak{Z}$ defined by (3.1).

Under such conditions the analysis given in § 2.4 (see eqn $(2.46)_c$) suggests that $-i\Omega \sim O(\epsilon)$. Thus the growth (or decay) rate of the disturbance is of order ϵ and an approach of the type used by Stewartson and Stúart (1971) can be employed using ϵ as a small parameter.

Thus we expand (u,v,w,T) in the form

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \epsilon^{1/2} \begin{pmatrix} f_{10} \cos az \\ g_{10} \cos az \\ h_{10} \sin az \end{pmatrix} + \epsilon \left[\begin{pmatrix} f_{01} \\ g_{01} \\ h_{01} \end{pmatrix} + \begin{pmatrix} f_{20} \cos 2az \\ g_{20} \cos 2az \\ h_{20} \sin 2az \end{pmatrix} \right] + \epsilon^{3/2} \left[\begin{pmatrix} f_{11} \cos az \\ g_{11} \cos az \\ h_{11} \sin az \end{pmatrix} + \begin{pmatrix} f_{30} \cos 3az \\ g_{30} \cos 3az \\ h_{30} \sin 3az \end{pmatrix} \right] + O(\epsilon^2), \tag{3.3 a,b,c}$$

$$T = T_o + \epsilon T_1 + O(\epsilon^2) \quad , \tag{3.4}$$

and require (f_{10}, g_{10}, h_{10}) to behave like the linear solution for (u,v,w) as $\tau \to -\infty$.

Furthermore we let σ and ϵ tend to zero keeping σ/ϵ fixed and equal to α say.

Such a procedure was used by Hall (1975a)for the analysis of the non linear stability of unsteady cylinder flows. Di Prima & Stuart (1975) had developed the method when studying the non linear stability of the flow between eccentric rotating cylinders.

On substituting from (3.3)(3.4) into (1.11) and equating terms of order $\epsilon^{1/2}$ we obtain the following differential system for the fundamental component of the disturbance

$$L^{(1)}\begin{pmatrix} f_{10} \\ g_{10} \end{pmatrix} = 0 \ , \qquad (3.5)_{a,b}$$

$$f_{10} = g_{10} = \frac{\partial f_{10}}{\partial \zeta} = 0 \quad (\zeta = 0,1),$$

where

$$L^{(n)} = \begin{pmatrix} (\frac{\partial^2}{\partial \zeta^2} - a^2 n^2)^2 & - a^2 n^2 T_o \chi_o \\ - \frac{1}{2} \chi_o^I & (\frac{\partial^2}{\partial \zeta^2} - a^2 n^2) \end{pmatrix} . \qquad (3.6)$$

The similarity between (3.5) and (2.47) suggests that we take

$$(f_{10}, g_{10}) = A(\tau) \ (F_{10}(\zeta), G_{10}(\zeta)) \ , \qquad (3.7)$$

where $(F_{10}, G_{10}) = (f_{00}, g_{00})$ is the pair of eigenfunctions associated with the differential system (2.31) and $A(\tau)$ is an amplitude function which is expected to behave like $B\exp(- \frac{\Gamma}{\alpha} \sin\tau)$ as $\tau \rightarrow -\infty$.

If we now substitute from (3.3)(3.4) into (1.11) equate terms of order ϵ and take (3.7) into account, we can show that the pair of functions (f_{20}, g_{20}) is the solution of a linear non homogeneous partial differential system parametrically dependent on time with coefficients which are proportional to $A^2(\tau)$. Thus we write

$$(f_{20}, g_{20}) = A^2(\tau) \ (F_{20}(\zeta), G_{20}(\zeta)) \ , \qquad (3.8)$$

where

$$\begin{cases} N^{(2)}\begin{pmatrix} F_{20} \\ G_{20} \end{pmatrix} = \begin{pmatrix} a^2 T_o G_{10}^2 - \frac{F_{10}^I F_{10}^{II}}{2} + \frac{F_{10} F_{10}^{III}}{2} \\ \frac{1}{4}\left(F_{10} G_{10}^I - G_{10} F_{10}^I\right) \end{pmatrix} , \\ F_{20} = F_{20}^I = G_{20} = 0 \qquad (\zeta = 0,1) \end{cases} . \qquad (3.9)$$

and $N^{(n)}$ is the operator defined by (3.6) where $\frac{\partial}{\partial \zeta}$ be replaced by $d/d\zeta$.

Furthermore it can be shown that the function $g_{01}(\zeta,\tau)$ which represents the non linear distortion of the basic flow can also be written in the form

$$g_{01} = A^2(\tau)\, G_{01}(\zeta), \tag{3.10}$$

where $G_{01}(\zeta)$ satisfies the following differential equation

$$\begin{cases} \dfrac{d^2}{d\zeta^2} G_{01} = \dfrac{1}{4}\left(F_{10} G_{10}^{I} + F_{10}^{I} G_{10}\right), \\[2ex] G_{01} = 0 \qquad\qquad (\zeta = 0,1). \end{cases} \tag{3.11}$$

The distortion that the fundamental component of the disturbance undergoes in the non linear regime is described by the pair of functions $(f_{11}(\zeta,\tau);\ g_{11}(\zeta,\tau))$. By taking (3.7),(3.8) and (3.10) into account, we can write

$$\begin{cases} L^{(1)}\begin{pmatrix} f_{11} \\ g_{11} \end{pmatrix} = \begin{pmatrix} \alpha \dfrac{dA}{d\tau} N F_{10} + a^2 \chi_0\, G_{10}\, A\left(T_1 + T_0\, \varrho(\tau)\right) + A^3 \mathcal{F}_1(\zeta) \\[2ex] \alpha \dfrac{dA}{d\tau} G_{10} + A^3 \mathcal{G}_1(\zeta) + \dfrac{1}{2}\chi_0'\, F_{10}\, A\, \varrho(\tau) \end{pmatrix}, \\[4ex] f_{11} = g_{11} = \dfrac{\partial f_{11}}{\partial \zeta} = 0 \qquad\qquad (\zeta = 0,1), \end{cases} \tag{3.12}$$

where

$$\mathcal{F}_1(\zeta) = -\frac{3}{4} a^2\left(\frac{1}{2} F_{10} F_{20}^{I} + F_{10}^{I} F_{20}\right) + a^2 T_0\, G_{10}\left(G_{01} + \frac{G_{20}}{2}\right) + \frac{1}{4}\left(\frac{1}{2} F_{20}^{III} F_{10} - \frac{1}{2} F_{20}^{I} F_{10}^{II} - F_{10}^{III} F_{20} + F_{10}^{I} F_{20}^{II}\right), \tag{3.13}$$

$$\mathcal{G}_1(\zeta) = \frac{1}{4}\left(F_{20} G_{10}^{I} + F_{10} G_{20}^{I} + 2 G_{20} F_{10}^{I} + \frac{1}{2} G_{10} F_{20}^{I} + 2 F_{10} G_{01}^{II}\right). \tag{3.14}$$

The differential system (3.12) only has a solution if its non homogeneous part satisfies a certain orthogonality condition. By using (2.33) this condition can be written in the form

$$\alpha \frac{dA}{d\tau} = -\left[\frac{T_1}{T_0} + f(\tau)\right] \Gamma A + a_1 A^3, \tag{3.15}$$

where

$$a_1 = -\frac{\int_0^1 \left[F_{10}^{+} \mathcal{F}_1(\zeta) + G_{10}^{+} \mathcal{G}_1(\zeta)\right] d\zeta}{\int_0^1 \left(F_{10}^{+} N F_{10} + G_{10}^{+} G_{10}\right) d\zeta}, \tag{3.16}$$

and (F_{10}^{+}, G_{10}^{+}) is the solution of the adjoint differential system (2.12) with $i\Omega_0 = 0$.

3.3 - Results

The aim of the computation was the determination of the constant a_1 which appears in the amplitude equation (3.15).

The expression (3.16) shows that the knowledge of the pair of functions (F_{20}, G_{20}) and of the function G_{01} is needed in order to perform such computations. Thus the differential systems (3.9) and (3.11) were solved numerically by means of usual procedure. (The solution of a non homogeneous boundary value problem is found as a linear combination of a suitable number of independent solutions of the homogeneous initial value problem plus a particular solution of the non homogeneous initial value problem). The numerical integrations were performed by means of the Runge--Kutta-Gill procedure of the IV order with 40 steps. The solutions are shown in figures 12, 13.

By using the values for the pairs of functions $(F_{1,0}, G_{1,0}) = (f_{00}, g_{00})$ and $(F^+_{10}, G^+_{10}) = (f^+_{00}, g^+_{00})$ obtained in chapter 2, the quadratures present in (3.16) could be performed numerically by means of Simpson rule with steplength 0.1. We obtain

$$a_1 = -34.613 . \qquad (3.17)$$

3.4 - Discussion

Equation (3.15) is a "Bernoulli" type equation and is identical to that obtained by Hall (1975a). Much of the discussion will then be to repeat Hall's (1975a) considerations.

Equation (3.15) can be solved by substitution of variables by using $A^{-2} \exp\left[-\phi(\tau)\right]$ as a variable, where

$$\phi(x) = \frac{\Gamma}{\alpha}\left\{2\left[\int^{x} f(y)\,dy + \frac{T_1}{T_0}x\right]\right\}. \qquad (3.18)$$

We obtain

$$A^{-2}\exp\left[-\phi(x)\right]_0^{\tau} = -\frac{2a_1}{\alpha}\int_0^{\tau}\exp\left[-\phi(x)\right]. \qquad (3.19)$$

Let us first consider the steady case where

$$f(\tau) = 0 . \qquad (3.20)$$

We can see from (3.19) that the amplitude equation admits an equilibrium amplitude solution A_e as $\tau \to \infty$ in the supercritical regime ($T_1 > 0$). We have

$$A_e = \left(\frac{\Gamma T_1}{a_1 T_0}\right)^{\frac{1}{2}} . \qquad (3.21)$$

Such a result completes the analogy between the instability characteristics of curved channel flow and circular Couette flow. Moreover it agrees with Brewster Grosberg and Nissan's (1959) experimental findings.

Let us now examine the case

$$f(\tau) = \tanh \tau \ . \qquad (3.22)$$

As in Hall (1975a) one can show from (3.18),(3.19) that an equilibrium amplitude solution still exists as $\tau \to \infty$. That is just the equilibrium amplitude solution for the steady problem with the Taylor number based on the final average speed of the flow.

Let us finally study the periodic case where

$$f(\tau) = \cos \tau \ . \qquad (3.23)$$

The periodicity condition imposed on (3.19) determines A(0). Thus we can write A^2 in the form

$$A^2(\tau) = -\frac{\alpha}{2a_1} \frac{\exp(-\psi(\tau)) \left[\exp\left(-\psi(2\pi)\right) - 1\right]}{\int_0^{2\pi} \exp(-\psi(x))dx + \left[\exp\left(-\psi(2\pi)\right) - 1\right] \int_0^{\tau} \exp(-\psi(x))\,dx} , \qquad (3.24)$$

where

$$\psi(x) = \frac{\Gamma}{\alpha}\left\{2\left[\sin x + \frac{T_1}{T_0} x\right]\right\} . \qquad (3.25)$$

Further informations on the behaviour of $A(\tau)$ can be obtained by considering some special limits.

(i) If we let T_1/T_0 tend to infinity with α fixed, we can show from (3.25) that

$$A(\tau) \sim A_e \left\{1 + 0\,(T_1/T_0)^{-1}\right\}. \qquad (3.26)$$

Thus the effect of modulation on the equilibrium amplitude tends to vanish as the flow becomes more and more supercritical.

(ii) If we let $\alpha \to \infty$ with T_1/T_0 fixed we obtain

$$A(\tau) \sim A_e \left\{1 + 0\,(\alpha^{-1})\right\}. \qquad (3.27)$$

Thus the effect of modulation is also negligible when the ratio σ/ϵ

is large, although ϵ and σ are both small.

(iii) Let us now take the limit $T_1/T_0 \to 0$ with α fixed.

We obtain

$$A^{-2}(\tau) \sim \frac{a_1 T_0}{\pi \Gamma T_1} \exp\left(2 \frac{\Gamma \sin\tau}{\alpha}\right) I_0\left(-2\frac{\Gamma}{\alpha}\right)\left\{1+O\left(\frac{T_1}{T_0}\right)\right\}. \quad (3.28)$$

where I_0 is a modified Bessel function of zero order.

By letting α tend to zero in (3.28) the following expression is obtained for the amplitude function

$$A(\tau) \sim \left[\frac{-T_1 \pi^{3/2} (-2\Gamma)^{1/2}}{a_1 T_0}\right]^{1/2} \alpha^{-1/4} \exp \frac{\Gamma}{\alpha}(1-\sin\tau) \left\{1+O\left(\frac{T_1}{T_0},\alpha\right)\right\}. \quad (3.29)$$

The time dependence shown by (3.29) is similar to that obtained for $B_0(\tau)$ in § 2.4. The function $A(\tau)$ is plotted in fig. 14, where appears that the maximum of $A(\tau)$ is attained at $\tau = \frac{1}{2}\pi$.

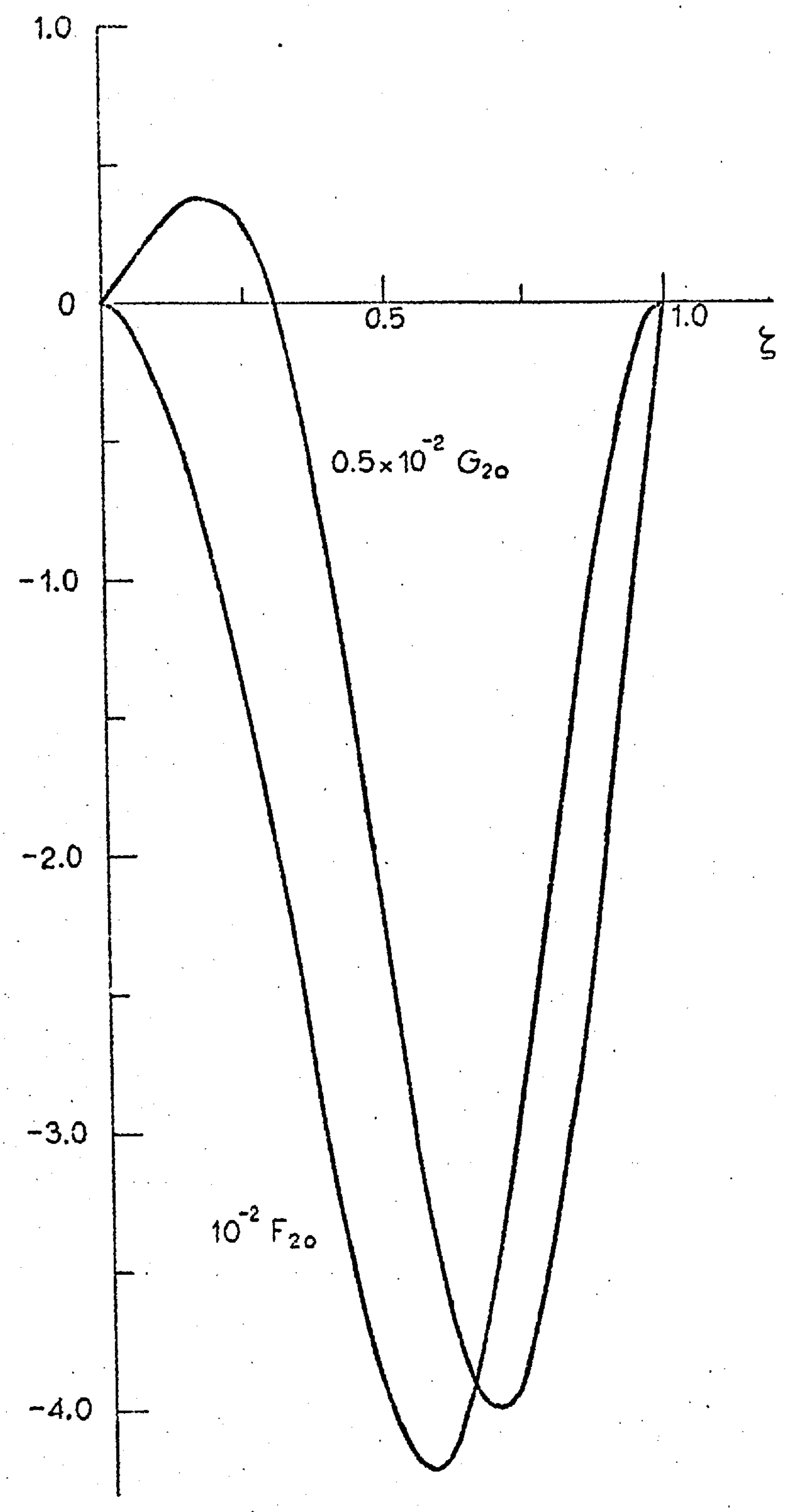


FIG. 12 - The transverse structure of the radial (F_{2o}) and azimuthal (G_{2o}) components of the first harmonic.

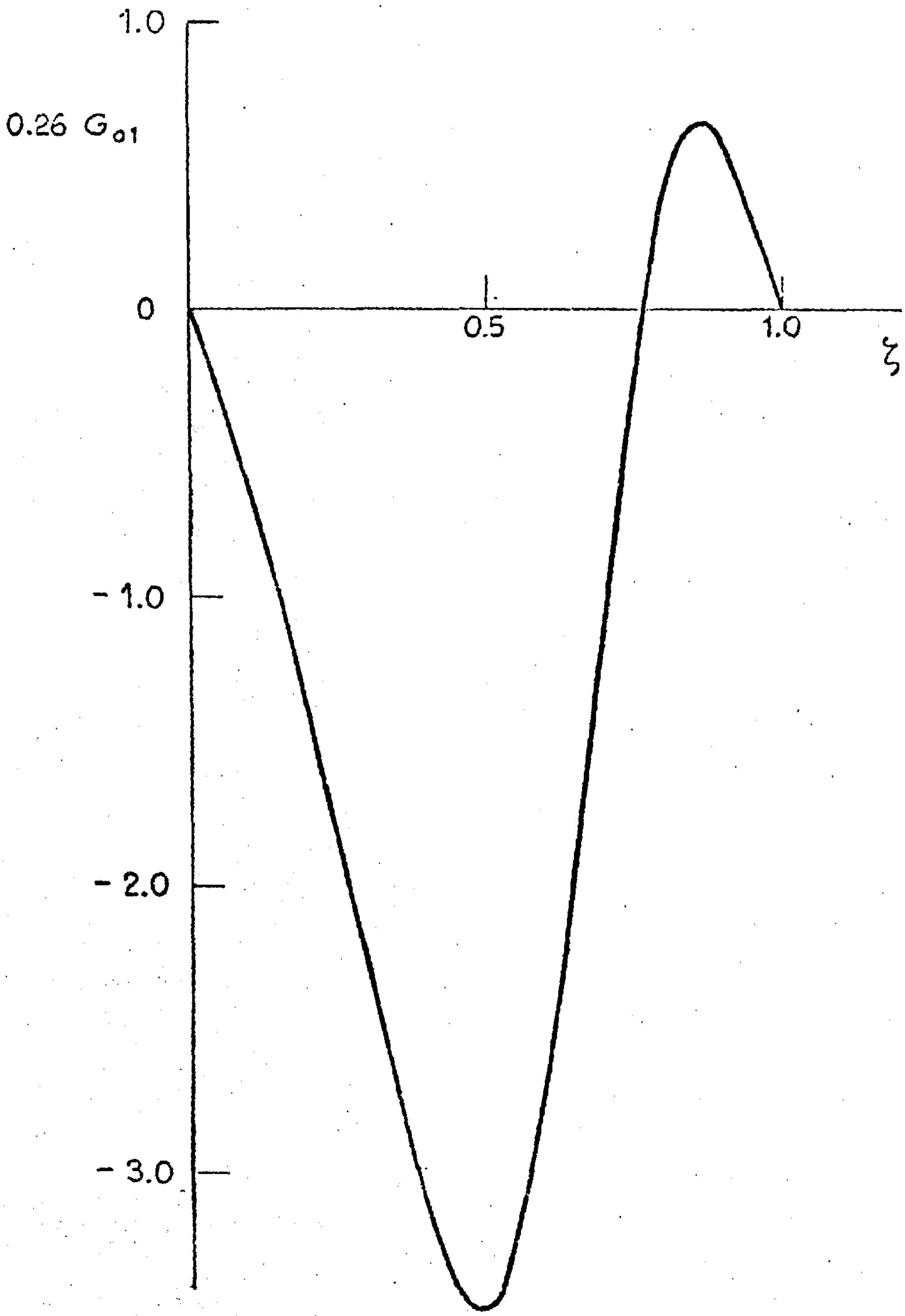


FIG. 13 - The transverse structure of the non linear distortion of the basic flow.

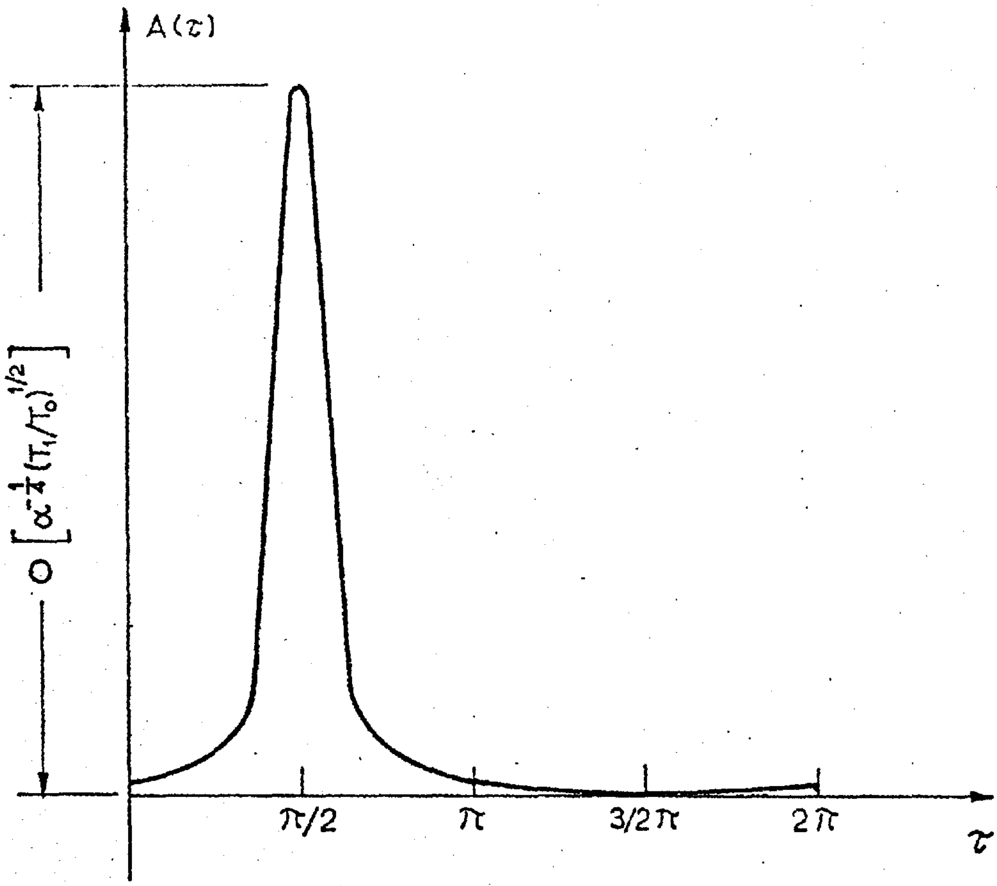


FIG. 14 - The amplitude function $A(\tau)$ in the weakly non linear regime with T_1/T_0 and α both tending to zero.

PART TWO

LINEAR AND WEAKLY NON LINEAR STABILITY OF THE STOKES LAYER ON A CIRCULAR CYLINDER

INTRODUCTION

The second part of this thesis deals with an example of unsteady centrifugal instability where the basic flow is modulated about a zero mean.

However the stability of purely oscillatory laminar flows has not received much attention. The problem is no longer amenable to the type of asymptotic methods used by Hall (1975a). Moreover a "slowly varying" approach of the kind developed in part 1 would be restricted to high values of the relevant stability parameter. Except for the inviscid analysis of Rosenblat (1968), in which the problem is simplified considerably, the above difficulties have only been overcome by approximate techniques suitable to numerical work.

Rosenblat (1968) showed that any purely oscillatory cylinder flow (apart from rigid body rotation) is centrifugally unstable in the inviscid limit. Instability is associated with a phase lag between velocity and vorticity. Riley and Lawrence (1976) tackled the concentric cylinder problem when the inner cylinder performs harmonic oscillations. They examined the behaviour of rotationally symmetric disturbances whose time dependent structure was assumed according to Floquet's theory after first approximating the radial dependence by means of a Galerkin expansion in terms of Jacobi polynomials. The critical values of the relevant stability parameters were determined numerically by following the evolution of the disturbances in time and imposing the periodicity criterion. The flow was found to be unstable for large enough speeds at all frequencies. The critical parameters become independent of gap width at high frequencies in which case the flow reduces to a Stokes layer confined to a region adjacent to the inner cylinder. The above method requires a great deal of computation and does not provide a deep understanding of the physical and mathematical features of the problem. Furthermore a Galerkin expansion procedure only applies to functions defined in a finite interval. Serious doubts about the validity of the results arise when flows of the Stokes layer type are treated by this method which requires the introduction of a second boundary. After introducing this second boundary the Galerkin method can be used to approximate the disturbance flow between the two boundaries. The thickness of the Stokes layer compared to the separation of the boundaries must then be allowed to tend to zero in order to infer results for the infinite Stokes layer. Thus the Galerkin method is particularly unsuitable since, in this limit, the disturbance velocity , which will be non zero only in the Stokes layer, must be approximated by functions defined in the finite interval between the boundaries. In order to obtain sensible results the number of terms of the Galerkin expansion needed will rapidly increase.

Such an argument applies to Kerczek and Davis (1974) who investigated the linear stability of Stokes layers on a flat plate. A "finite Stokes layer" is considered with a stationary infinite plate set parallel to the first. The stability of this flow is studied by a method similar to the one previously described. The flow is found to be stable for Reynolds numbers less than about 750 and wavenumbers in the range .3, 1.3. These results were obtained for a maximum separation distance d eight times bigger than the characteristic thickness of the Stokes layer. To what extent they can be extrapolated to the case of an "infinite" Stokes layer is questionable. Such uncertainty particularly refers to disturbances of "small" wave number. In fact it appears unlikely that a disturbance whose wavelength is O(d) does not interact with the stationary wall. If the effect of such an interaction is appreciable, as one might reasonably expect, the results obtained for wavenumbers O(1) cannot be considered representative of the behaviour of an "infinite" Stokes layer.

Finally we mention Kuwabara & Takaki's (1975) work on secondary flow around a circular cylinder in rotatory oscillation. Such work was not known to the present author till this thesis was completed. Kuwabara & Takaki (1975) examine the possibility of occurrence of small, unsteady axisymmetric disturbances superimposed on the basic flow originated by a circular cylinder performing rotatory oscillations about its axis. Their analysis will be more extensively discussed in § 5.5. At this stage we notice that they treated the radial dependence of the perturbations by means of a Galerkin expansion in terms of Laguerre polynomials. The disturbance time dependence was assumed on the basis of some experimental observations by Taneda (1971). The perturbation was not allowed for any growth or decay and was assumed to be synchronous with the basic flow. Thus such study cannot be considered as a proper linear stability analysis. Furthermore some inconsistencies are present in the perturbation procedure which will be discussed in § 5.5. Finally the number of terms retained in the Galerkin expansion and in approximating the time dependence of the perturbation was too small for reasonable accuracy to be achieved. Thus, as will be seen in § 5.5, Kuwabara & Takaki's (1975) results for the critical values of the parameters above which secondary flow may occur, disagree considerably with those obtained by the present author.

In view of the above difficulties, a more reliable approach appears to be needed. A thorough understanding of the stability mechanism of Stokes layers is relevant to the analysis of a wider class of external and internal oscillatory incompressible flows. In fact viscous boundary layers of the Stokes type are developed near the walls of any such flow at high frequencies of the oscillation. Important examples are: the viscous layer at the bottom of a channel over which a gravity wave propagates (Longuet-Higgins, 1953); the viscous layer generated close to the wall of a cylindrical body oscillating along a diameter (Stuart,1963, 1966); the flow near the walls of a straight or curved rigid pipe under the action of an oscillatory pressure gradient (Sexl, 1930; Lyne, 1971). We emphasize that the occurrence of such flows in practical problems is of considerable importance. For example the flow regime at the bottom of a water wave controls the sedimentation process. Similarly the flow field close to the arterial wall appears to be associated with the uptake of lipoproteins which is thought to be responsible for the onset of atheroma (Caro, 1973). The velocity fields in the above flows are generally more complicated than that of a simple Stokes layer. A normal as well as a tangential component of velocity is present. Furthermore the flow is not always confined to the Stokes layer; a steady streaming sometimes persists away from the layer. These additional features have some influence on the stability characteristics of these flows. However it is of fundamental interest to investigate first the instability process for the simplest of them, a Stokes layer.

We will consider the flow generated by an infinite cylinder which oscillates harmonically about its axis in an unbounded viscous incompressible fluid. For high values of a suitable frequency parameter (see §4.1) the flow is confined to a thin boundary layer adjacent to the wall, which reduces to a Stokes layer at the lowest order of approximation. Thus, at this order, centripetal forces associated with the curvature of the streamlines have negligible effect on the basic flow. However, as in a steady laminar flow with curved streamlines, centripetal forces may be expected to play a major role in controlling the stability of the flow.

In chapter 4 we consider a "small" periodic disturbance of the Taylor vortex type (rotationally symmetric and axially periodic) and assume its tangential velocity to contain a harmonic component with the same frequency ω as the basic flow. Then, from the interaction between such disturbance and the basic flow a convective force arises in the azimuthal direction. This force contains a steady part plus a harmonic component of frequency 2ω . For a balance to be possible between such inertial force and the azimuthal viscous force, a radial velocity with the same time dependence is required. A similar dynamic balance between radial viscous force and centripetal force can be imposed. For this to be possible a higher harmonic (frequency 3ω) is needed for the azimuthal velocity. The above balances are coupled, thus giving rise to the production of higher and higher harmonics in a cascade process.

A more general structure for the disturbance can be envisaged where an exponential time factor, with complex exponent in general, is introduced in order to account for: (i) damping or growth of the perturbation away from the neutral state; (ii) possible subharmonic responses of the type characteristic of the solutions of Mathieu equation. Such a structure may be anticipated on the basis of Floquet's theory.

The solution of the stability problem then arises from solving an eigenvalue problem for the growth rate in terms of the stability parameters. This is obtained by substituting the above form of the solution into the governing differential system. It is apparent that such a method is only effective if the contributions associated with the higher harmonics of the perturbation tend to become increasingly negligible. Such a condition is expected to be satisfied in the present problem, since the partial differential system which describes the behaviour of the perturbations has a time dependence of the Hill's type. The method of solution outlined above is in fact just an extension of the procedure given by McLachlan (1947) for the solution of Hill's equation.

In the present case the absolute convergence of the series expansions defining the disturbance velocity can be proved. The coefficients of these series are the eigenfunctions associated with the eigenvalue problem mentioned above, which essentially consists of solving an infinite system of homogeneous coupled differential equations with homogeneous boundary conditions. An analytic solution is obtained which is thought to be the general solution of the eigenvalue problem. Comparison with the results obtained by a numerical approach strongly supports this conclusion. The solution automatically satisfies the boundary conditions at infinity and is given in terms of an infinite set of unknown constants to be determined. This is accomplished by imposing that the no-slip condition at the wall should also be satisfied. An infinite set of linear homogeneous, algebraic equations is then obtained for the above constants. The infinite determinant associated with the algebraic system must vanish if a non trivial solution is to exist. This leads to the required eigenrelation between the stability parameters. The marginal state is obtained by imposing that the growth rate of the perturbation should vanish.

On carrying out the above procedure the flow is found to be unstable to rotationally symmetric disturbances characterized by dominant steady radial and axial velocity components and smaller unsteady components in all the directions. The relevant stability parameter is not surprisingly a Taylor number, T, based on the thickness of the Stokes layer. The critical value of T was found to be $T_c = 232.52$, and corresponds to a critical wavenumber $a_c = 0.85852$.

Chapter 6 is devoted to the problem of the weakly non-linear development of the linear solution. The analysis follows the lines of Stewartson and Stuart (1971) approach. Supercritical equilibrium amplitude solutions are constructed. The results agree with the analysis of Joseph (1972). By means of a Poincaré-Linstedt type perturbation procedure Joseph (1972) showed that forced periodic basic solutions of the Navier Stokes equations bifurcate to supercritical stable and subcritical unstable periodic solutions with the same frequency as the basic flow, provided the Floquet exponent of the linear solution is zero at criticality (as is the case in the present problem).

These results were qualitatively confirmed by some experimental observations carried out on the apparatus described in chapter 7. A Taylor vortex type flow of the kind suggested by the theoretical analysis was observed to develop when T exceeded a critical value $T_c \simeq 210$ with $a_c^{(1)} \simeq 0.88$. However at values of T higher than $T_c^{(2)} \simeq 260$ a second stage of instability was observed, which gave rise to a set of bigger vortices ($a_c^{(2)} \simeq 0.17$) characterized by steady velocity components in all the directions. The linear and weakly non linear analyses developed in this work cannot provide an explanation for the occurrence of this second stage of instability. No rotationally symmetric mode with steady tangential velocity was found unstable with respect to the basic flow. However, this type of disturbance might give rise to instability when interacting with the first mode. Only a non linear theory of the kind developed by Davey, Di Prima and Stuart (1968) can provide a complete understanding of the whole process. Finally Taneda's (1971) experimental observations could not be made available to the present author. However the experimental results plotted in Kuwabara & Takaki's (1975) paper suggest that the time dependence of the disturbances observed by Taneda (1971) is similar to that obtained in the present work. Furthermore the critical value of the Taylor number observed by Taneda (1971) seems to be qualitatively in agreement with the above results.

CHAPTER 4

FORMULATION OF THE PROBLEM

4.1 - Basic Flow

Let us consider a circular cylinder of infinite length and radius R in an unbounded viscous incompressible fluid. Let (r, ϑ, Z) be cylindrical polar coordinates with the axis of the cylinder taken as the Z axis and let (U^*, V^*, W^*) be the corresponding velocity vector (the star denotes dimensional quantities). Furthermore ν and t^* define kinematic viscosity and time respectively.

The cylinder is assumed to rotate about its axis with an angular velocity $\frac{\Delta\omega}{R} \cos \omega t^*$ where $2\pi\Delta$ is the tangential displacement of the cylinder in centimetres from its mean position and ω is the frequency of the oscillation in rad/s . A purely tangential basic flow may then be induced in the surrounding fluid, where viscous forces are balanced by the inertial forces associated with the local acceleration and the centripetal forces due to the curvature of the streamlines.

The fluid motion is confined to a region of thickness $O(\sqrt{2\nu/\omega})$ adjacent the cylinder. The relative importance of the centripetal forces then tends to vanish when the value of the parameter $(\sqrt{2\nu/\omega})/R$ tends to zero. In this limit the basic flow at the lowest order of approximation is a "Stokes layer". The velocity field is given by $(0, \mathcal{V}(\eta, t), 0)$ where η, t, $\mathcal{V}$ are dimensionless quantities defined by

$$\eta = \frac{r - R}{\sqrt{2\nu/\omega}}, \quad t = \omega t^*, \quad \mathcal{V} = \frac{V^*}{\Delta\omega} , \qquad \text{(4.1 a,b,c)}$$

and

$$\mathcal{V} = \frac{e^{-\eta - \eta i + it} + \text{c.c.}}{2} . \qquad \text{(4.2)}$$

Here c.c. denotes "complex conjugate".

4.2 - The equations for rotationally symmetric disturbances

A disturbed flow is considered such that the disturbed velocity field is of the form $(u^*, v^* + (\Delta\omega)\mathcal{V}, w^*)$. The vector (u^*, v^*, w^*) is rescaled by writing

$$(u^*, v^*, w^*) = (\sqrt{2\nu\omega}\,\tilde{u},\ (\Delta\omega)\,\tilde{v},\ \sqrt{2\nu\omega}\,\tilde{w}). \qquad (4.3)$$

Furthermore $(\tilde{u},\tilde{v},\tilde{w})$ is assumed to be independent of ϑ. This assumption (rotationally symmetric disturbances) in supported by the experimental observations reported in chapter 7, suggesting that this type of disturbances is the most unstable one. However, as found in the classical Taylor instability, asymmetric and mixed modes might become important in the further stages of the bifurcation process (see chapter 7). An understanding of their role requires further work.

The differential system which governs the behaviour of the disturbance velocity can be derived from the momentum and continuity equations and the condition of zero relative velocity at the surface of the cylinder. We obtain

$$\begin{cases} \left(M - 2\frac{\partial}{\partial t}\right) M\tilde{u} + T V \dfrac{\partial^2 \tilde{v}}{\partial z^2} = -\dfrac{T}{2}\dfrac{\partial^2 \tilde{v}^2}{\partial z^2} + \dfrac{\partial^2}{\partial z^2}(Q_1) - \dfrac{\partial^2}{\partial\eta\,\partial z}(Q_2), \\ \left(M - 2\frac{\partial}{\partial t}\right)\tilde{v} - 2\tilde{u}\dfrac{\partial V}{\partial\eta} = Q_3, \\ \dfrac{\partial \tilde{u}}{\partial\eta} + \dfrac{\partial \tilde{w}}{\partial z} = 0, \\ \tilde{u} = \tilde{v} = \tilde{w} = 0 \qquad (\eta = 0), \\ \tilde{u},\ \tilde{v},\ \tilde{w} \to 0 \qquad (\eta \to \infty), \end{cases} \qquad \begin{matrix}(4.4)\\ a,b,c,d,e\end{matrix}$$

where

$$z = (\sqrt{2\nu/\omega})^{-1} Z \quad ; \quad T = 2\frac{\Delta^2}{R}\sqrt{\frac{2\omega}{\nu}}, \qquad (4.5)_{a,b}$$

$$M = \frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial z^2}, \qquad (4.6)$$

$$Q_1 = 2\left(\tilde{u}\frac{\partial\tilde{u}}{\partial\eta} + \tilde{w}\frac{\partial\tilde{u}}{\partial z}\right), \qquad (4.7)$$

$$Q_2 = 2\left(\tilde{u}\frac{\partial\tilde{w}}{\partial\eta} - \tilde{w}\frac{\partial\tilde{u}}{\partial\eta}\right), \qquad (4.8)$$

$$Q_3 = 2\left(\tilde{u}\frac{\partial\tilde{v}}{\partial\eta} + \tilde{w}\frac{\partial\tilde{v}}{\partial z}\right). \qquad (4.9)$$

The parameter T is recognized as a Taylor number based on the Stokes layer thickness. The boundary conditions $(4.4)_e$ require the disturbance velocity to vanish at infinity. Finally the solution $(\tilde{u},\tilde{v},\tilde{w})$ has been expanded in powers of the small parameter $\sqrt{2\nu/\omega}/R$ and the system (4.4) is obtained at lowest order. This system may be expected to provide a good approximation for the solution in the limit $\sqrt{2\nu/\omega}/R \to 0$ with η, z, t, $\tilde{u}$, $\tilde{v}$, $\tilde{w}$, T fixed.

CHAPTER 5

LINEAR THEORY

5.1 - Linearized problem

The disturbance is now assumed to be small enough for linearization to be a valid approximation (strictly infinitesimal). Moreover the coefficients of the system (4.4) are independent of z, so a Fourier analysis is convenient in the z direction. Thus we can write

$$(\tilde{u}, \tilde{v}, \tilde{w}) = \frac{\epsilon}{2} \int_{-\infty}^{+\infty} \left\{ (u, v, w)\, e^{iaz} + \text{c.c.} \right\} da \,, \qquad (5.1)$$

with ϵ small.

On substituting from (5.1) into the system (4.4) the following linear partial differential system is obtained for (u,v)

$$\begin{cases} \left(\mathcal{L} - 2\frac{\partial}{\partial t}\right)\mathcal{L}\, u - a^2 T\, V v = 0 \,, \\ \left(\mathcal{L} - 2\frac{\partial}{\partial t}\right) v - 2\frac{\partial V}{\partial \eta}\, u = 0 \,, \\ u = v = \frac{\partial u}{\partial \eta} = 0 \qquad (\eta = 0) \,, \\ u \,,\; v \,,\; \frac{\partial u}{\partial \eta} \rightarrow 0 \qquad (\eta \rightarrow \infty) \,, \end{cases} \qquad (5.2)\ \text{a,b,c,d}$$

where

$$\mathcal{L} \equiv \partial^2/\partial\eta^2 - a^2 \,. \qquad (5.2)_e$$

The previous system is of second order in time and has coefficients periodic in time. Such time dependence can be suitably modelled by considering an ordinary differential equation of the type

$$\frac{d^2 y}{dt^2} + \psi(2t)\, y = 0 \,, \qquad (5.3)$$

where $\psi(2t)$ is a periodic function.

The equation (5.3) is the well known Hill equation which has been extensively studied (see Mc Lachlan (1947) particularly with reference to the stability of its solution. On the basis of such knowledge a solution for (u,v) can be sought in the form

$$(u(\eta, t; a,T), v(\eta, t; a,T)) = K e^{-i\Omega t} \sum_{-\infty}^{\infty} \left(u_p(\eta;a,T), v_p(\eta;a,T)\right) e^{ipt}. \tag{5.4}$$

Here K is an unknown constant, $-i\Omega$ is a complex number whose real part gives the growth rate of the disturbance while the imaginary part allows for the possibility of subharmonic responses, and (u_p, v_p) are functions which account for the η dependence of the differential system (5.2).

On substituting from (5.4) into (5.2) - an infinite system of differential equations is obtained for the functions (u_n, v_n). This can be shown to consist of 2 independent systems, each of which is of the form

$$\begin{cases} \left[N + 2i\Omega - 2i(2n)\right] N u_{2n} - \dfrac{a^2 T}{2}\left[e^{-\eta(1+i)} v_{2n-1} + e^{-\eta(1-i)} v_{2n+1}\right] = 0 , \\ \left[N + 2i\Omega - 2i(2n-1)\right] v_{2n-1} + \left[(1+i)\, e^{-\eta(1+i)} u_{2n-2} + (1-i)\, e^{-\eta(1-i)} u_{2n}\right] = 0, \\ u_{2n} = v_{2n-1} = \dot{u}_{2n} = 0 \qquad (\eta = 0) , \\ u_{2n} = v_{2n-1}, \quad \dot{u}_{2n} \to 0 \qquad (\eta \to \infty), \end{cases} \tag{5.5 a,b,c,d}$$

where

$$N \equiv d^2/d\eta^2 - a^2 . \tag{5.6}$$

The other system can be shown to be identical to (5.5) by redefining Ω.

Thus, an eigenvalue problem for the parameters $(a, T, -i\Omega)$ is obtained which defines an eigenrelation of the type

$$f(a, T, \Omega) = 0 . \tag{5.7}$$

The configurations of neutral stability are then defined by (5.7) with the growth rate (Ω_i) equal to zero. Various "modes" may arise with different values of Ω_r.

5.2 - An analytic approach for the solution of the eigenvalue problem

Suppose $\underset{\sim}{V}$ is defined as $\underset{\sim}{V} = \{V_m\}$ where $V_{2j} = u_{2j}$, $V_{2j+1} = v_{2j+1}$ and j is any integer. Then the expression

$$V_m = \sum_{M=-\infty}^{\infty} \sum_{n=0}^{\infty} \left\{ \alpha_{m,n}^{(M)} \, e^{-\left\{(\sigma_r^{(M)} + 2n + |M-m|) + i\,(\sigma_i^{(M)} - M + m)\right\}\eta} + \beta_{m,n}^{(2M)} \, e^{-\left\{(a + 2n + |2M-m|) + i\,(m - 2M)\right\}\eta} \right\},$$

$$\text{(5.8)}_{a,b}$$

where

$$\sigma^{(M)} = \sigma_r^{(M)} + i\,\sigma_i^{(M)} = \sqrt{a^2 - 2i\Omega + 2Mi}\ ,$$

can be shown to satisfy the differential system (5.5) with $\alpha_{m,n}^{(M)}, \beta_{m,n}^{(2M)}$ constants to be determined.

Let us discuss briefly the construction of the solution (5.8). Consider equation (5.5)a with n = 0. The solution for u_0 will depend on each of the V_m's through the coupling terms arising from the interaction between the disturbance and the basic flow. In the absence of such an interaction, (5.5)a (n = 0) admits a particular solution of the form $u_{0,0}^{(0)} = \alpha_{0,0}^{(0)} \exp\left\{-\left[\sigma_r^{(0)} + i\,\sigma_i^{(0)}\right]\eta\right\}$. Then $u_{0,0}^{0}$ interacts with the basic flow in (5.5)b (n = +1) and a term arises which can only be balanced by the contribution, $v_{1,0}^{(0)}$ from v_1. Thus $v_{1,0}^{(0)} = \alpha_{1,0}^{(0)} \exp\left\{-\left[(\sigma_r^{(0)} + 1) + i(\sigma_i^{(0)} + 1)\right]\eta\right\}$.

It should be noticed that a similar balance could not involve the term proportional to u_2 since this would produce an increasing power of η in the solution for u_2 and, in a successive step, would lead to functions not bounded at infinity. Similar arguments in (5.5)b (n = -1) give $v_{-1,0}^{(0)} = \alpha_{-1,0}^{(0)} \exp\left\{-\left[(\sigma_r^{(0)} + 1) + i\,(\sigma_i^{(0)} + 1)\right]\eta\right\}$.

The process continues through the feedback in (5.5)a (n = 0) where the interaction involving $v_{1,0}^{(0)}$ and $v_{-1,0}^{(0)}$ leads to a new term for u_0, $u_{0,1}^{(0)} = \alpha_{0,1}^{(0)} \exp\left\{-\left[\sigma_r^{(0)} + 2 + i\sigma_i^{(0)}\right]\eta\right\}$, and so on. In such a way series expansions of the form

$$\begin{cases} u_0^{(0)} = \sum_{n=0}^{\infty} \alpha_{0,n}^{(0)} \exp\left\{-\left[(\sigma_r^{(0)} + 2n) + i\,\sigma_i^{(0)}\right]\eta\right\}, \\ \left(V_1^{(0)},\, V_{-1}^{(0)}\right) = \sum_{n=0}^{\infty} \exp\left\{-\left[(\sigma_r^{(0)} + 2n + 1) + i\,\sigma_i^{(0)}\right]\eta\right\}\left(\alpha_{1,n}^{(0)} e^{-i\eta};\ \alpha_{-1,n}^{(0)} e^{i\eta}\right), \end{cases} \quad \text{(5.9)}_{a,b,c}$$

are generated for u_0, v_1 and v_{-1} .

Such solutions do not apparently account for the influence of higher harmonics. In fact the cascade process proceeds involving the higher order equations. Now $v_1^{(0)}$ and $v_{-1}^{(0)}$ interact in (5.5)a (n = 1, n = -1) to give terms which require a balance from the terms associated with u_2 and u_{-2}. These, in turn, will produce new contributions for v_1 and v_{-1} in (5.5)b (n = 0). Fortunately the "loop is closed", i.e. it can be shown that the new series expansions obtained for v_1 and v_{-1} are of the same form as (5.9)b,c with initial values n = 1. They may then be absorbed into (5.9)b,c . This is the main feature of the problem: the feedback effect coming from higher harmonics simply reinforces part of the solution obtained at lower order.

The structure of the solution (5.8) becomes clear when it is recognized that a series expansion of the type (5.9) for all the V_n's can be similarly constructed starting from a particular solution of each of the equations (5.5)a,b in the absence of the coupling terms. The double series expansion in (5.8) is thus obtained.

By substituting from (5.8) into (5.5)a,b the constants $\alpha_{m,n}^{(M)}$, $\beta_{m,n}^{(2M)}$ appearing in (5.8) can be determined in terms of $\alpha_{M,0}^{(M)}$, $\beta_{2M,0}^{(2M)}$. The following relationships are obtained for $\alpha_{2m,n}^{(M)}$

$$\alpha_{2m,n}^{(M)} = \frac{a^2 T}{2} A_{2m,n}^{(M)} \cdot \left\{\left[\sigma^{(M)} + 2n + |M-2m| + (2m-M)i\right]^2 - \left[\sigma^{(2m)}\right]^2\right\}^{-1} \times \left\{\left[\sigma^{(M)} + 2n + |M-2m| + (2m-M)i\right]^2 - a^2\right\}^{-1}, \tag{5.10}$$

where

$$A^{(M)}_{2m,n} = \begin{cases} \left(\alpha^{(M)}_{2m-1,n-1} + \alpha^{(M)}_{2m+1,n-1}\right) & (M = 2m,\ n \neq 0), \\ \left(\alpha^{(M)}_{2m-1,n} + \alpha^{(M)}_{2m+1,n-1}\right) & (M < 2m,\ n \neq 0), \\ \left(\alpha^{(M)}_{2m-1,n-1} + \alpha^{(M)}_{2m+1,n}\right) & (M > 2m,\ n \neq 0), \\ \left(\alpha^{(M)}_{2m-1,0}\right) & (M < 2m,\ n = 0), \\ \left(\alpha^{(M)}_{2m+1,0}\right) & (M > 2m,\ n = 0), \end{cases}$$

(5.11)

and

$$\alpha^{(M)}_{2m+1,n} = - A^{(M)}_{2m+1,n} \ast \left\{ \left[\sigma^{(M)} + 2n + |M-(2m+1)| + i\,(-M + 2m+1) \right]^2 - \left[\sigma^{(2m+1)}\right]^2 \right\}^{-1},$$

(5.12)

where

$$A^{(M)}_{2m+1,n} = \begin{cases} \left((1+i)\,\alpha^{(M)}_{2m,n-1} + (1-i)\,\alpha^{(M)}_{2m+2,n-1}\right) & (M = 2m+1,\ n \neq 0), \\ \left((1+i)\,\alpha^{(M)}_{2m,n} + (1-i)\,\alpha^{(M)}_{2m+2,n-1}\right) & (M < 2m+1,\ n \neq 0), \\ \left((1+i)\,\alpha^{(M)}_{2m,n-1} + (1-i)\,\alpha^{(M)}_{2m+2,n}\right) & (M > 2m+1,\ n \neq 0), \\ (1+i)\,\alpha^{(M)}_{2m,0} & (M < 2m+1,\ n = 0), \\ (1-i)\,\alpha^{(M)}_{2m+2,0} & (M > 2m+1,\ n = 0). \end{cases}$$

(5.13)

($n = 0, 1, 2, \ldots; m = 0, \pm 1, \pm 2, \ldots; M = 0, \pm 1, \pm 2, \ldots$).

The expressions for $\beta^{(2M)}_{m,n}$ are obtained from (5.10), (5.12) by replacing $\sigma^{(M)}$ by a. Finally the values of the constants $\alpha^{(M)}_{M,O}$, $\beta^{(2M)}_{2M,O}$ as yet undetermined must be such that the boundary conditions (5.5)c are satisfied. This condition leads to the following infinite, homogeneous system of linear algebraic equations

$$\begin{cases} \sum_{M=-\infty}^{\infty} \sum_{n=0}^{\infty} \left(\alpha^{(M)}_{m,n} + \beta^{(2M)}_{m,n} \right) = 0 \ , & (5.14) \\ \sum_{M=-\infty}^{\infty} \sum_{n=0}^{\infty} \Big\{ \left[\left(\sigma^{(M)}_r + 2n + |M-2m| \right) + i \left(\sigma^{(M)}_i - M + 2m \right) \right] \alpha^{(M)}_{2m,n} \\ \quad + \left[a + 2n + |2M-2m| + i(2m-2M) \right] \beta^{(2M)}_{2m,n} \Big\} = 0, \end{cases}$$

($m = 0, \pm 1, \pm 2, \ldots$)

(5.15)

where $\alpha^{(M)}_{m,n}$, $\beta^{(2M)}_{m,n}$ are given by (5.10), (5.12). If a non trivial solution of such algebraic system is to exist, its infinite determinant must vanish. This leads to an algebraic eigenvalue problem which allows one to determine the eigenrelation (5.7). The structure of the latter has been obtained by means of a numerical procedure after retaining a suitable number of terms in the expansions (5.14), (5.15). The results will be presented in §5.4.

The convergence of the series expansions defining the elements of the determinant is proved in Appendix A.

5.3 - A numerical approach

The solution (5.8) clearly shows that any accuracy can be achieved by retaining a suitable number of terms in the expansions (5.4). However it may be found quite difficult to obtain an analytic solution of the kind (5.8) for more complicated Stokes layer type flows. In such circumstances it is necessary to employ a numerical method suitable to tackle boundary value problems of the type (5.5). This has been done for the present problem and the results of this alternative approach have supported the conclusion that (5.8) is in fact the general solution of the system (5.5).

Boundary value problems of the type (5.5) can be solved by using a method used by many previous authors in the context of hydrodynamic stability theory (see, for example, Krueger, Gross and Di Prima, 1966). By retaining a suitable number of terms in the expansions (5.4), the infinite system (5.5) is reduced to a finite set of equations. By means of some numerical scheme, independent integrals of the differential system are then obtained, each satisfying the boundary conditions at one of the extremes of the interval. A number of independent solutions of this initial value problem are required such that a linear combination satisfies the boundary conditions at the other extreme. This leads to a homogeneous linear algebraic system for the coefficients of the combination. The determinant of this system must vanish if a non trivial solution is to exist. An eigenvalue problem then arises for the parameters $-i\Omega$, a, T, which can be solved numerically by fixing a,T and locating the zeros of the determinant as a function of $-i\Omega$ by means of some root finding routine.

The numerical integration corresponding to each solution of the initial value problem must be carried out starting from infinity, i.e. some "large" value of $\eta(\eta_\infty)$. This is one of the remedies for the well known (e.g. see Fox and Mayers, 1968) form of induced instability which affects the present numerical method when some of the complementary solutions of the differential system increase very rapidly. Also, the value of η_∞ at which the boundary conditions (5.5)d must be imposed is too high in general for the numerical integration to give accurate results in the whole range $(0,\eta_\infty)$. This difficulty can be avoided by following a method originally suggested by Meksyn (1950). A good approximation for the solution of the system (5.5) for "large" values of η is obtained by neglecting the terms with coefficients proportional to $e^{-\eta}$. Comparison with the analytical solution shows that the neglected terms are $O(a^2 T e^{-2\eta_\infty})$ smaller than those retained. Thus, if a^2T is sufficiently small, satisfactory accuracy can be achieved with reasonably "low" values of η_∞. The solution of the reduced system can be shown to be such that the following conditions hold at $\eta = \eta_\infty$

$$\left(\frac{d}{d\xi} - \sigma^{(2m+1)}\right) V_{2m+1} = 0 \quad ,$$

$$\left(\frac{d}{d\xi} - \sigma^{(2m)}\right)\left(\frac{d}{d\xi} - a\right) V_{2m} = 0 \; , \qquad (5.16)\ \text{a,b,c}$$

$$\frac{d}{d\xi}\left(\frac{d}{d\xi} - \sigma^{(2m)}\right)\left(\frac{d}{d\xi} - a\right) V_{2m} = 0 \; ,$$

where $\xi = \eta_\infty - \eta$.

Thus the procedure outlined at the beginning of this paragraph can be carried out starting the numerical integration from $\xi = 0$ (with η_∞ reasonably small) and initial conditions given by (5.16).

5.4 - Results

The aim of the computation was the determination of the eigenrelation (5.7). This was first obtained from the solution given in §5.2 . The results were then checked following the numerical approach described in § 5.3.

The procedure was as follows. For fixed values of a and T and an initial guess for $(-i\Omega)$ the series (5.8) were evaluated for $|m| \leq 4$ in terms of the unknown constants $\alpha^{(M)}_{M,0}$, $\beta^{(2M)}_{2M,0}$ through the recurrence relationships (5.10) -(5.12). Terms for $n > 8$, $|M| > 4$ were neglected. The accuracy thus achieved was of the order of the accuracy provided by the computer (10^{-14}). The infinite matrix (A-1) was then approximated by a finite (14 x 14) matrix whose determinant was evaluated by determining the product of the eigenvalues. This was done numerically by employing NAG subroutines provided by the Imperial College Computer Centre. The matrix was first balanced, then put into Hessenberg form; the eigenvalues were finally obtained by means of the LR algorithm with shift of origin.

A root finding routine using the secant method was employed to determine the zero of the determinant as a function of $(-i\Omega)$. Convergence was very satisfactory when the procedure was started from a reasonably accurate initial guess. The unstable disturbances detected were of the form

$$u = A\, e^{-i\Omega t} \sum_{n=-\infty}^{\infty} u_{2n}\, e^{i\,2nt} \,,$$
$$v = A\, e^{-i\Omega t} \sum_{n=-\infty}^{\infty} v_{2n-1}\, e^{i\,(2n-1)t} \,, \qquad (5.17)\ \text{a,b}$$

with $\text{Real}(\Omega) = 0$ in the marginal state. The corresponding marginal state is shown in Fig. 15. The critical values T_c, a_c were found to be

$$T_c = 232.52 \,,$$
$$a_c = 0.85852 \,. \qquad (5.18)\ \text{a,b}$$

The functions $u_0(\eta)$, $v_1(\eta)$, $u_2(\eta)$, $v_3(\eta)$, $u_4(\eta)$ are shown in Figs. 16-20. The structure of the disturbance (5.17) and the critical values (5.18) as predicted by the theoretical analysis will appear to agree qualitatively with the first mode of instability observed experimentally (see chapter 7).

The results (5.18) were checked by performing a numerical integration of the system (5.5) following the approach outlined in § 5.3. Terms for $|n| > 2$ were neglected in the series expansions (5.17)a,b so that the system of infinite order (5.5) was replaced by a finite set of 28 first order ordinary differential equations. For fixed values of a,T and an initial guess for $(-i\Omega)$, 14 independent solutions of the initial value problem were obtained as described in § 5.3. The numerical integrations were performed by means of the Runge-Kutta-Gill procedure with initial conditions (5.16) $(m = 0, \pm 1, \pm 2)$ and $\eta_\infty = 7$. A step length $h = 0.1$ was used. The optimum choice of η_∞ and h was the result of some numerical experiments. The linear algebraic system obtained on imposing that the no--slip conditions at the wall are to be satisfied by a linear combination of the 14 independent solutions was solved by a numerical procedure similar to that described above. The zeros of the determinant associated with such algebraic system were similarly obtained. The accuracy of this numerical method can be appreciated on comparing the values obtained for the growth rate by the two independent approaches for fixed values of a,T.

These values are shown in Table 4. The agreement is satisfactory. This feature appears to be fairly important in view of the possible extensions of the numerical approach to treat more complicated Stokes layer type flows of the kind mentioned earlier. The rapid convergence of the numerical method, though obvious in view of the solution (5.8), was checked by retaining an increasing number of terms in the expansion (5.17)a,b. Some typical results for different values of n are given in Table 5.

Finally, an extensive search performed in the range of small wave-numbers ($a \simeq 0.1$) did not show the existence of any further unstable mode with rotational symmetry. An explanation for the experimentally observed second stage of instability in terms of an instability of the first mode probably requires the development of a non linear theory of the kind developed by Davey, Di Prima and Stuart (1968).

5.5 - Discussion

The critical values (5.18)a,b are in good qualitative agreement with the results of the calculations performed by Riley and Lawrence (1976). A comparison is possible because the concentric cylinder problem with the inner cylinder modulated with zero mean, approaches the Stokes layer problem under investigation as the frequency of the modulation tends to infinity. From Figs. 1 and 2 of Riley and Lawrence's paper (1976) it was possible to estimate the asymptotic values of the parameters T_c and a_c which were found as follows

$$T_c^{(RL)} \simeq 236 \pm 3 \;, \qquad a_c^{(RL)} \simeq 0.87 \pm 0.01 \;. \tag{5.19 a,b}$$

Kuwabara & Takaki's (1975) results do not agree as well with those obtained in the present work. This seems to be due to some deficiencies present in their approach. They examine the problem studied in this chapter. First they obtain a solution for the basic flow in closed form i.e. for any value of the parameter $\sqrt{\nu/\omega}\,/R$ (there denoted by α^{-1}). However in the actual calculations they use an asymptotic form of the previous solution obtained by assuming $\eta^{(KT)} \sim O(\alpha) \gg 1$ where $\eta^{(KT)}$ is a radial coordinate related to η by

$$\eta^{(KT)} = \sqrt{2}\,\eta + \alpha \;.$$

The asymptotic form of the basic flow is given in the first expression of § 4 of Kuwabara & Takaki's (1975) paper. Such expression is incorrect. In fact:

- the argument of the exponential should be $\left[-\frac{1}{\sqrt{2}}\eta^{(KT)} + O(\eta^{(KT)^{-1}})\right]$ rather than $\left[-\frac{1}{2}\eta^{(KT)} + O(\eta^{(KT)^{-1}})\right]$;

- the argument of the functions sin and cos should be $\left[\frac{\eta^{(KT)}}{\sqrt{2}} + \frac{5}{8}\pi + O(\eta^{(KT)^{-1}})\right]$ rather than $\left[\frac{7}{\sqrt{2}} + \frac{\pi}{8} + O(\eta^{(KT)^{-1}})\right]$ as reported there.

With the above corrections the expression can then be shown to reduce to (4.2) except for a factor $\left[1 - \frac{\sqrt{2}}{2}\alpha^{-1}\eta + O(\alpha^{-2})\right]$. Thus it would seem that Kuwabara & Takaki (1975) wish to retain terms $O(\alpha^{-1})$ in their analysis.

However this is not so when coming to the study of what they call "secondary flow". They seek the marginal configuration of small axisymmetric disturbances whose time dependence is assumed according to some experimental observations by Taneda (1971). Indeed we have shown that such time dependent structure arises from a proper stability analysis. Thus Kuwabara & Takaki (1975) expand the perturbation velocities in series of the form (5.17) with $-i\Omega$ equal to zero and, without further justification, retain only terms corresponding to u_0, v_1, v_{-1}. Such sharp approximation is not consistent with having retained an $O(\alpha^{-1})$ effect in the basic flow. Furthermore it does not seem to be consistent on one hand to retain $O(\alpha^{-2})$ terms in the disturbance equations, on the other approximating the radial dependence of the perturbation by retaining only three terms of suitable expansions in terms of Laguerre polynomials. Indeed the number of terms required is expected to be much higher and strongly dependent on α as discussed in the Introduction.

Finally, Kuwabara & Takaki (1975) apply Galerkin method but do not specify how large is the domain that they consider. Their results written in terms of T and a are shown in Tab. 6. The critical values $T^{(KT)}$ of the Taylor number for different values of α are considerably smaller than that given by $(5.18)_a$. The critical values $a^{(KT)}$ of the wavenumber fall within a range which is not too far from the value given by $(5.18)_b$. However one would expect $a^{(KT)}$ to become closer and closer to the value $(5.18)_b$ as α increases, whereas the opposite behaviour is shown by Tab.6.

This might be the result of the accuracy of the approximation decreasing as α increases.

Finally fig. 2 of Kuwabara & Takaki's (1975) paper would suggest that the flow is stable against disturbances with small wavenumber. This feature was not shown by the present results and appears to be related to the failure of Galerkin's method for large wavelengths of the disturbance as discussed in the Introduction.

Let us now discuss the implications of the present work to that of Hall (1975a). Attention is focused on the results given by Hall for the stability of high frequency modulated circular Couette flow. It was found that, if the speed of the inner cylinder is $S(1 + \epsilon \cos \omega t^*)$, the Taylor number based on S and the gap width d is perturbed only by an amount of order $\epsilon^2 (\nu/\omega d^2)^3$ in the limit $\omega \to \infty$. In this limit the Stokes layer becomes confined to a thin region near the inner cylinder. These results suggest that a Stokes layer is quite stable to centrifugal effects. However the present work shows that this is not the case. The apparent inconsistency is easily explained. The wavelength of any disturbance considered by Hall (1975a) was scaled on the gap width. Hall did not consider the possibility of a mode of instability of the type discussed here, with the disturbance scaled on the Stokes layer thickness. Let us determine which mode is the most dangerous. The steady basic velocity component is assumed not to alter significantly the critical Taylor number for the mode of instability discussed in this paper. (Notice that this will certainly be true in the limit $\epsilon \to \infty$). We define a Taylor number T_s in terms of S, d, R, and ν in the form

$$T_s = \frac{2\,S^2\,d^3}{R\,\nu^2} , \tag{5.20}$$

where d is the separation of the cylinders, assumed small compared to R. By replacing Δ in $(4.5)_b$ by $\epsilon S/\omega$ we can show that the Stokes layer mode of instability occurs when

$$T_s = 164.3\,\epsilon^{-2}\,\sigma^{3/2} , \tag{5.21}$$

where

$$\sigma = \frac{\omega d^2}{\nu} , \tag{5.22}$$

is the frequency parameter as defined by Hall. However the critical value of T_s for the mode discussed by Hall (1975a) was found to be

$$T_s \rightarrow 3390 \quad , \tag{5.23}$$

when $\sigma \rightarrow \infty$. Thus, for fixed ϵ , the latter mode is expected to be the most dangerous when

$$\sigma > \sigma^* = (20.7\,\epsilon^2)^{2/3} , \tag{5.24}$$

i.e. a change in the mode of instability is expected for $\sigma = \sigma^*$. The critical wavenumber will therefore be discontinuous at $\sigma = \sigma^*$. This discontinuity can be seen in the numerical results of Riley and Lawrence. For example, when $\epsilon = 5.0$, Riley and Lawrence found the wavenumber to be discontinuous at $\sigma = 61.6$ whilst the corresponding value of σ^* is 64.3.

The above discussion might also be relevant to the work of Hall (1975b) on the stability of Plane Poiseuille flow modulated at high frequencies which also overlooked the possibility of a disturbance with wavelength based on the Stokes layer thickness. The stability of the Stokes layer on a flat plate to disturbances with relatively "small" wavenumbers needs to be investigated for the latter possibility to be explored. It should be emphasized that such information can be obtained by using an approach similar to that developed in § 5.2.

5.6 - Relevance to flow in the aorta

Finally one of the motivations for the present work is related to its possible relevance to the stability of the flow in the aorta. The aortic arch may be considered as a curved tapering and branching pipe of slightly elliptic cross-section. The arterial walls exhibit some viscoelastic effects. Blood flow is not fully developed. In spite of this quite complicated picture, attempts have been made to investigate the flow pattern established in the aorta by neglecting the least important of these features. Lyne (1971) has studied the fully developed flow in a curved rigid pipe of circular cross-section and small curvature due to a purely oscillatory pressure gradient acting down the pipe. Blennerhasset (1976) and Smith (1975) have considered fully developed pulsatile flow in a curved rigid pipe of circular cross section and small curvature. We shall not discuss the stability of such solutions here. However we notice that Lyne's solution shows in the high frequency limit that viscous effects are confined to a thin layer on the wall, the flow being essentially inviscid elsewhere. A stokes layer type flow with a "small" secondary motion in the cross-section is then established close to the wall. Due to the curvature of the latter the local flow along the outside bend then might be subject to centrifugal instability of the kind studied in this work. Various problems arise when an attempt is made to extend the present analysis in order to cover the effects of the secondary flow and of the curvature characteristics varying round the pipe. In particular it is necessary to investigate how the stability characteristics of the Stokes layer change on proceeding from a locally concave wall (outer bend) to a locally convex wall (inner bend).

In order to obtain some information about this problem let us study the inviscid limit $T \rightarrow \infty$ in both the cases of convex and concave wall. The differential system (5.5) is first considered and an inviscid balance between centrifugal effects and local acceleration is imposed. It can be seen that such a balance requires that

$$\Omega \sim T^{\frac{1}{2}} . \tag{5.25}$$

Thus the following expansions are set up

$$\begin{aligned} u_{2m} &= u_{2m,0} + u_{2m,1} T^{-1} + O(T^{-2}) \ , \\ v_{2m-1} &= \left[v_{2m-1,0} + v_{2m-1,1} T^{-1} + O(T^{-2}) \right] T^{-\frac{1}{2}} , \\ \Omega &= T^{\frac{1}{2}} \left(\Omega_0 + \Omega_1 T^{-1} + O(T^{-2}) \right) \ . \end{aligned} \tag{5.26 a,b,c}$$

By substituting from (5.26) into (5.5) and equating terms of order $O(T^{\frac{1}{2}})$, the following system of coupled equations is obtained

$$\begin{cases} 2i\Omega_0 N u_{2m,0} - \dfrac{a^2}{2}\left[e^{-\eta(1+i)} v_{2m-1,0} + e^{-\eta(1-i)} v_{2m+1,0} \right] = 0 \ , \\ 2i\Omega_0 v_{2m-1,0} + (1+i)\, e^{-\eta(1+i)} u_{2m-2,0} + (1-i)\, e^{-\eta(1-i)} u_{2m,0} = 0 \ , \\ u_{2m,0} = 0 \qquad (\eta = 0) \ , \\ u_{2m,0} \rightarrow 0 \qquad (\eta \rightarrow \infty) \ . \end{cases} \tag{5.27 a,b,c,d}$$

Some information about the qualitative behaviour of the solution of this system can be obtained by considering, at the lowest approximation, the following equation:

$$\begin{cases} \left[\dfrac{d^2}{d\eta^2} - a^2\right] u_{2m,0} + \lambda\, \phi(\eta)\, u_{2m,0} = 0 \,, & \\ u_{2m,0} = 0 \qquad \eta = 0 \,, & \\ u_{2m,0} \quad 0 \qquad \eta \to \infty \,, \quad (m = 0, \pm 1, \pm 2, \ldots\ldots) & \end{cases} \tag{5.28}$$

where

$$\lambda = - a^2/4\, \Omega_0^{-2} \,, \tag{5.29}$$

$$\phi\,(\eta) = e^{-2\eta} \;. \tag{5.30}$$

Thus $u_{2m,0}$ is the eigenfunction associated with an eigenvalue problem of the Sturm-Liouville type for Ω_0. It is a standard result for such problems (when defined in a finite interval) that the characteristic values λ form a denumerable sequence and are all positive when the function $\phi(\eta)$ is positive for every η within the interval of definition (e.g. see Courant and Hilbert, 1953 pg.294). Thus we may infer that Ω_0 is purely imaginary and the flow is inviscidy unstable at the lowest order of approx imation. It does not seem worthwhile to determine higher order corrections to the present result, which is also expected to be valid in view of the full viscous solution presented in the previous paragraphs. It should also be noticed that Rosenblat's (1968) results are also confirmed.

A similar qualitative analysis can be performed for the case of a Stokes layer on a concave wall. It can be shown that in the inviscid limit the relevant equation at the lowest order of approximation is (5.28) with a change of sign in the last term. It follows that a Stokes layer on a concave wall is inviscidy stable at lowest order. The latter result is expected not to be altered by viscous effects.

Thus the Stokes layer type flow occurring near the wall in Lyne's solution appears to be locally unstable along the outside bend and locally stable along the inside bend (°). Further work is required to determine the flow pattern which is established.

(°) - When the flow is driven by a pressure gradient we expect the behaviour obtained for the Stokes layer on a concave wall to occur along a convex wall and viceversa.

Table 4

a	T	Analytical	Numerical
0.85852	233.345	0.002277	0.002278

Comparison between the values obtained for the growth rate by the analytical and numerical approach

Table 5

a	T	n = 1	n = 2	n = 3	n = 4
0.85852	233.345	0.0074	0.002277	0.002264378	0.002264383

Higher approximations for the growth rate of the perturbation for fixed values of a and T (numerical method).

Table 6

α	$T_c^{(KT)}$	$a_c^{(KT)}$
20	79.5	0.849
40	86.7	0.906
60	89.3	0.927
80	90.9	0.939
100	91.8	0.946

The critical values of T and a for different values of α as obtained by Kuwabara & Takaki (1975).

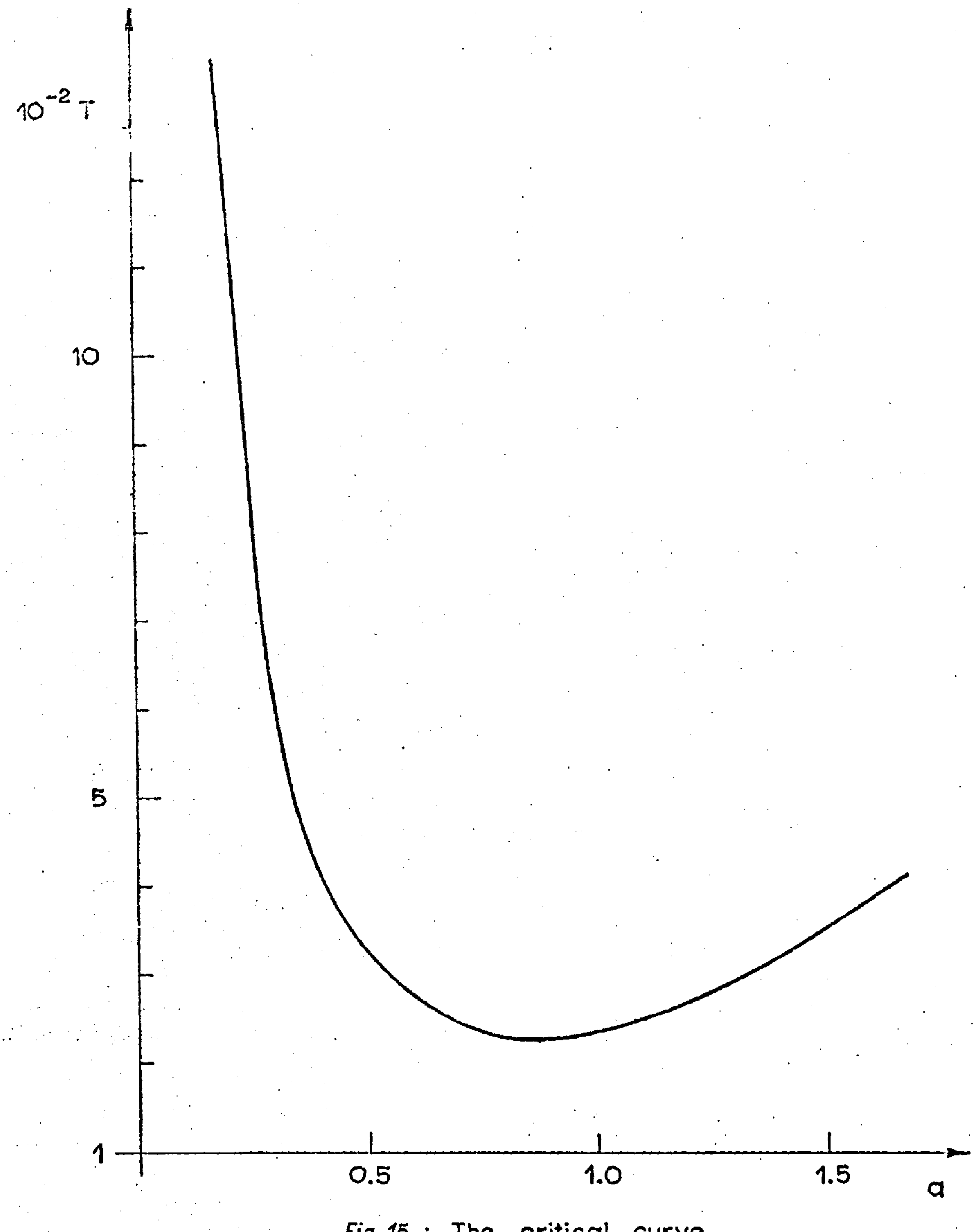


Fig. 15 : The critical curve

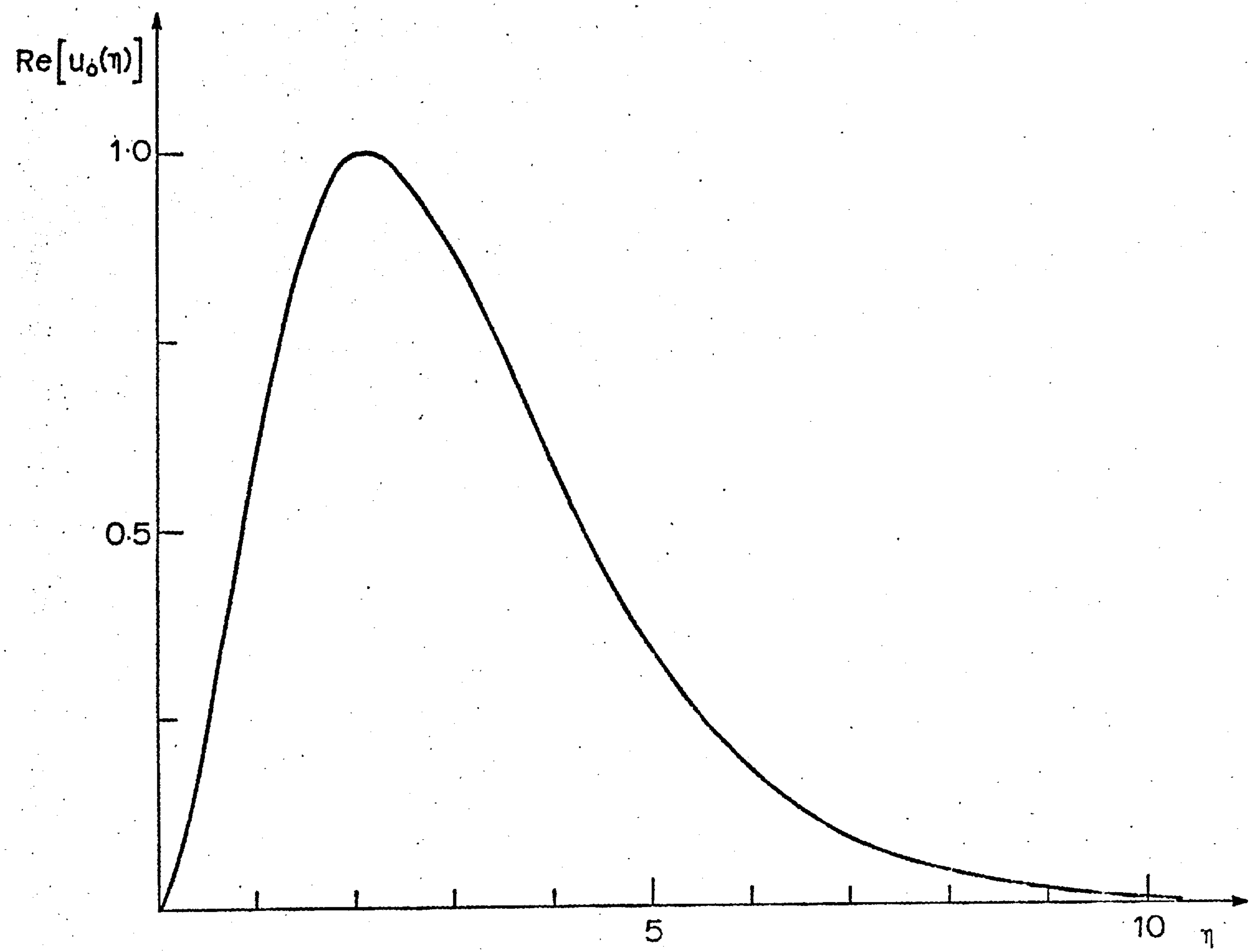


Fig.16: The function $u_o(\eta)$

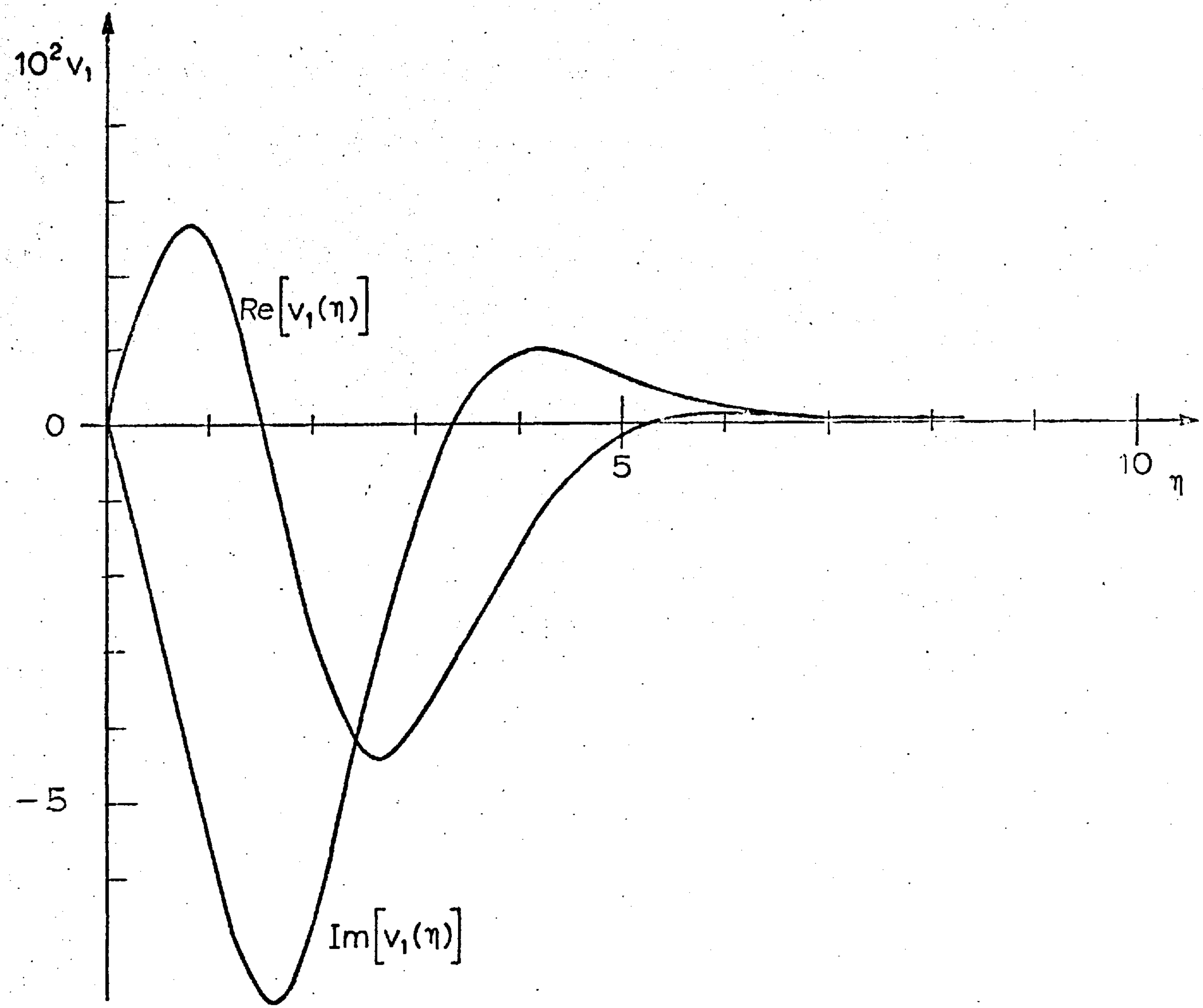


Fig.17: The functions $Re[v_1(\eta)]$, $Im[v_1(\eta)]$

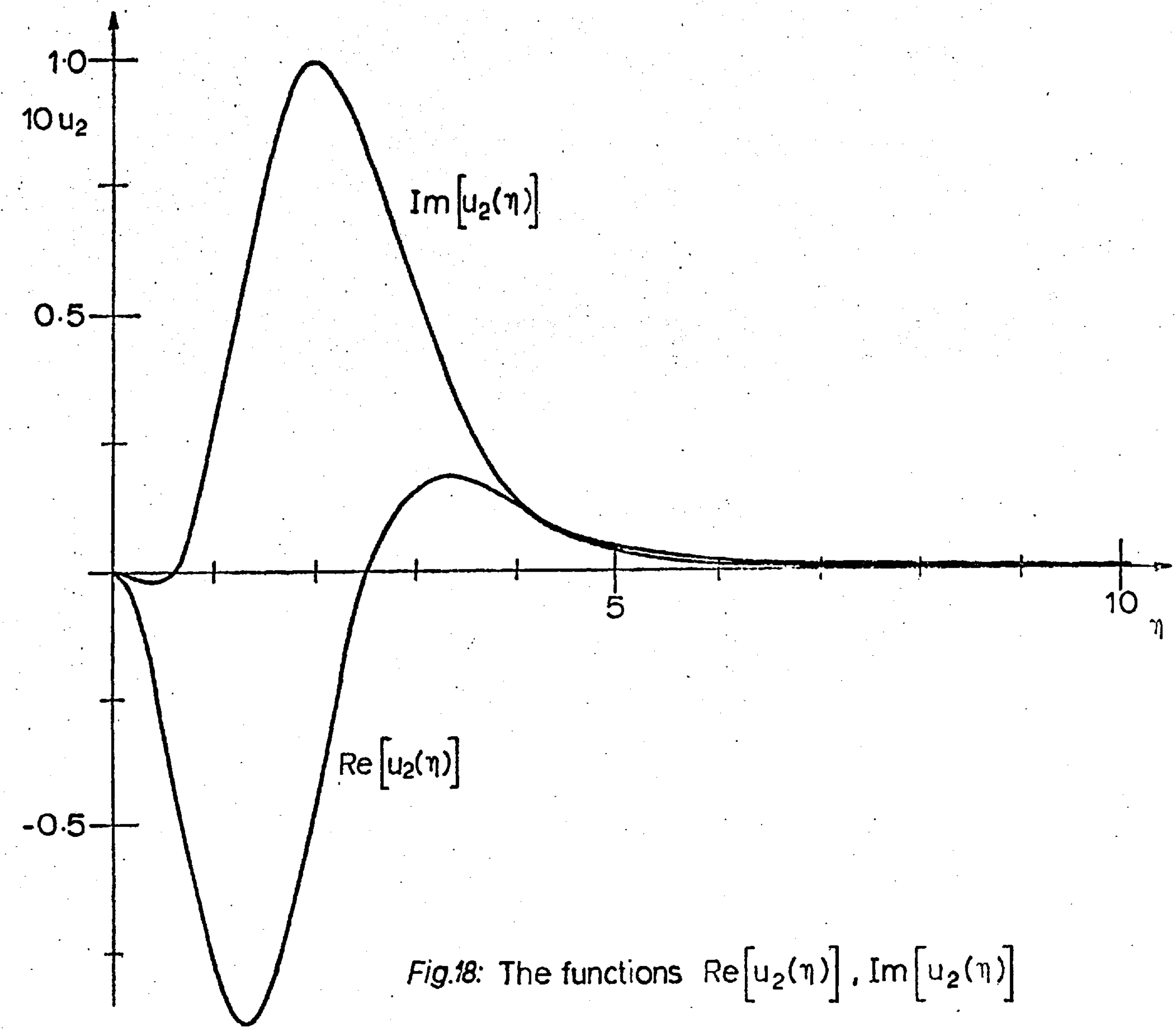


Fig.18: The functions $\mathrm{Re}[u_2(\eta)]$, $\mathrm{Im}[u_2(\eta)]$

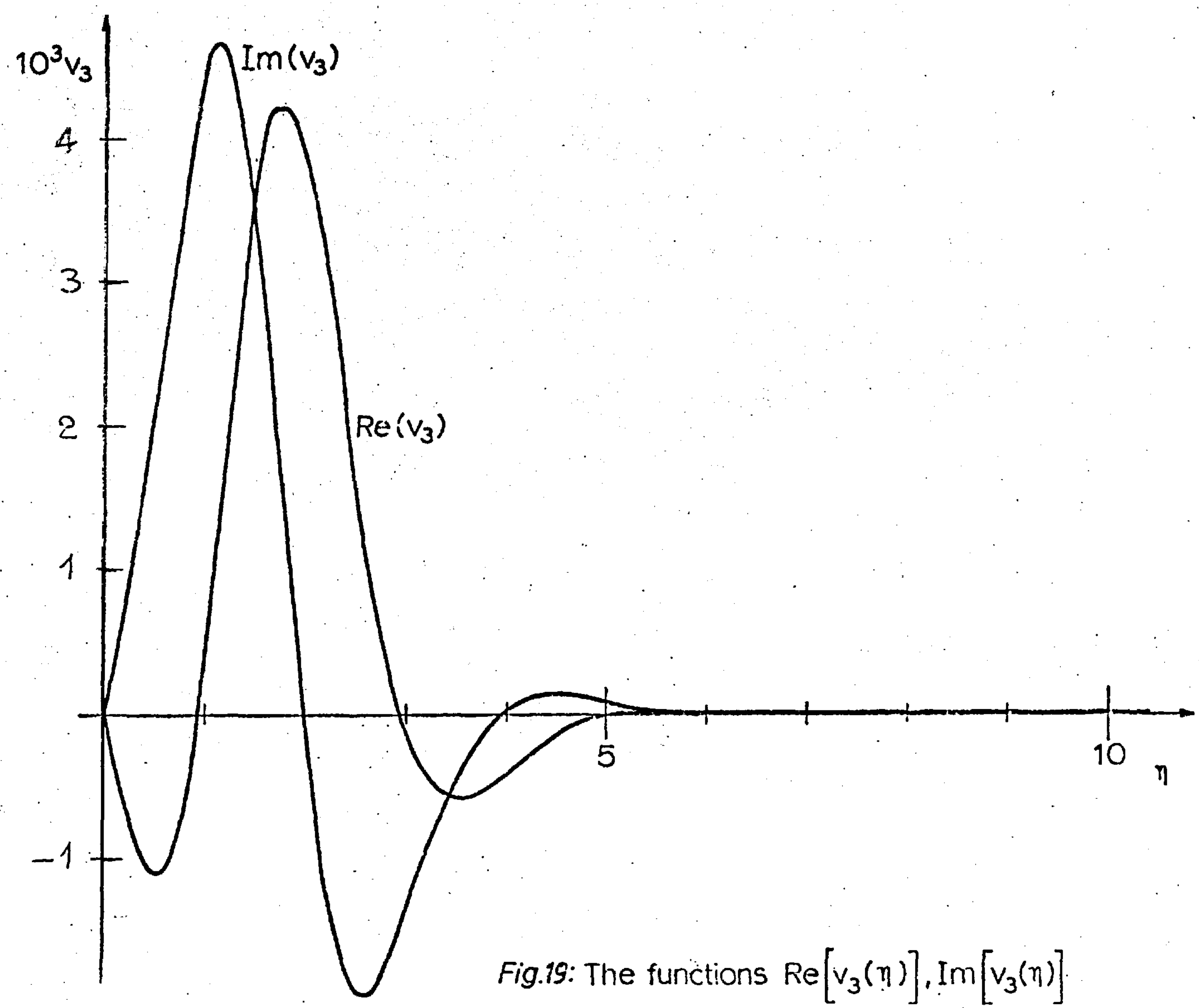


Fig.19: The functions $Re[v_3(\eta)], Im[v_3(\eta)]$

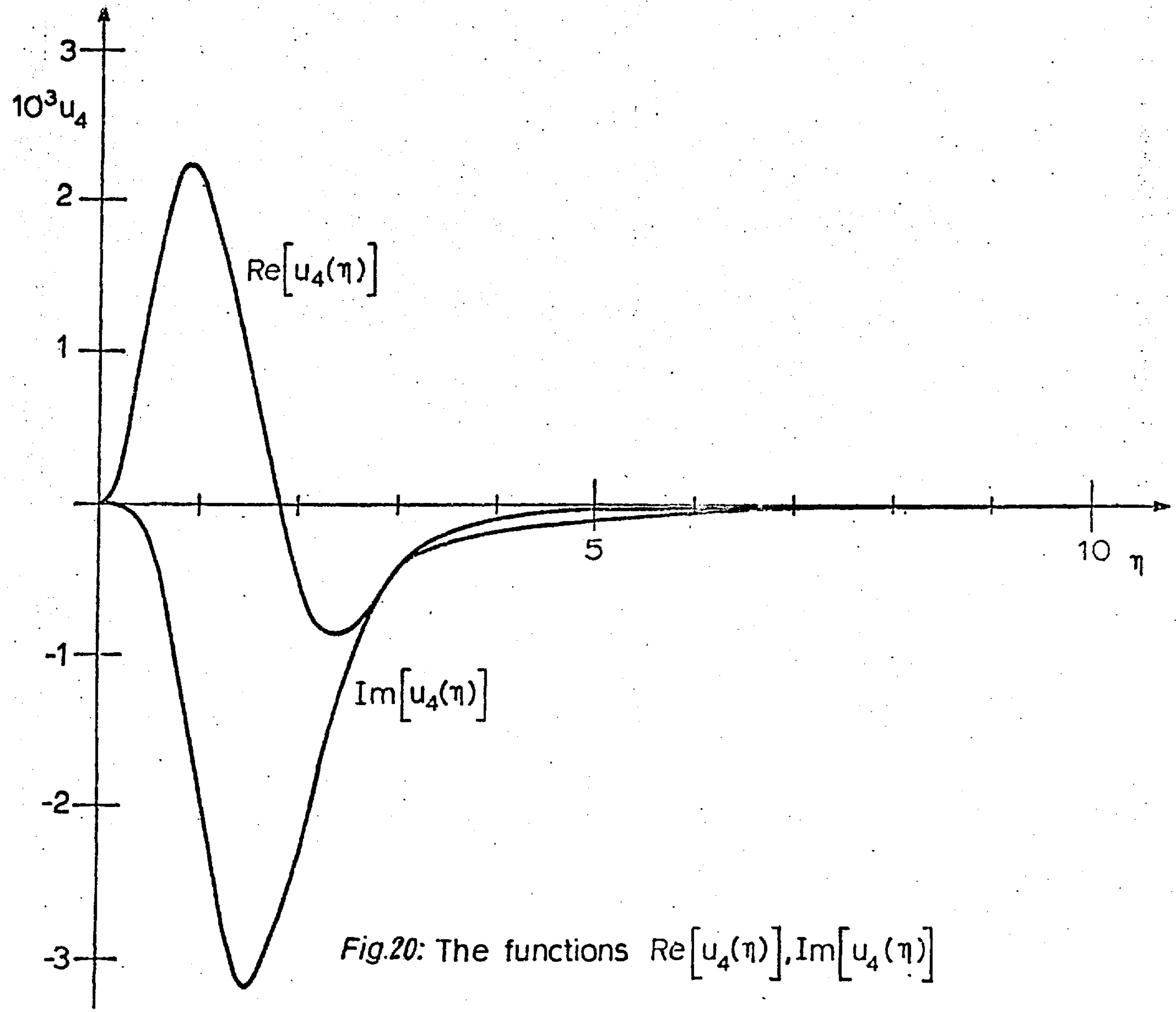


Fig.20: The functions $Re[u_4(\eta)], Im[u_4(\eta)]$

CHAPTER 6

WEAKLY NON LINEAR THEORY

6.1 - Introduction

Our aim in this chapter is to obtain a perturbation solution of (4.4) which is centred round the neutral configuration of the most unstable linear mode. We consider the non linear development of a monochromatic disturbance with axial wavenumber a under slightly supercritical ($T > T_0$) or subcritical ($T < T_0$) conditions. The analysis follows the method of Stewartson-Stuart (1971) with obvious variations needed in order to account for the periodic time dependence of the basic flow.

The inverse time scale for growth in the linear regime is $Im(\Omega)=\Omega_i$ and may be assumed to be small within a small neighbourhood of the neutral configuration. This is discussed in § 6.2. Using the method of multiple scales a new (slow) variable τ is defined by

$$\tau = \epsilon t \quad , \tag{6.1}$$

and a suitable expansion is set up for the disturbance velocity (§ 6.3). An analysis of the differential problems obtained for the coefficients of such expansions at the various orders of approximation, shows that the solution depends on an "amplitude function" $A(\tau)$. This function is found to satisfy an amplitude equation of Bernoulli type which allows for equilibrium amplitude solutions in the supercritical regime.

Such findings are in agreement with the results of Joseph (1972). By using a generalization of Poincaré-Linstedt perturbation procedure, Joseph (1972) treated the problem of bifurcation of quasi-periodic solutions which bifurcate from periodic solutions of the Navier-Stokes equations. After assuming that the Floquet exponents ($(-i\Omega)$ in our case) are simple eigenvalues of the spectral problem, the formal construction gives two bifurcating solutions of the same frequency as the basic flow when the Floquet exponent is zero at criticality. The small amplitude solutions which bifurcate supercritically are stable, subcritical solutions with small amplitude are unstable.

The existence of bifurcating solutions had been proved by Sattinger (1971), Yudovich (1970) and Iooss (1972).

6.2 - The time scale for growth

The inverse time scale of the disturbance in the linear regime is $\mathrm{Im}(\Omega) = \Omega_i$. Following Stuart (1960) we may anticipate that $(\Omega_i)^{-1}$ is also the time scale for the non linear growth under investigation.

Thus, before proceeding in our discussion of the non linear theory it is convenient to investigate the behaviour of the growthrate $(-i\Omega)$ within a small neighbourhood of the neutral configuration. Let us set up series expansions for $(-i\Omega, u_{2m}, v_{2m+1})$ in powers of $(T-T_0)$. We write

$$((-i\Omega), u_{2m}, v_{2m-1}) = (0, u^{(0)}_{2m}, v^{(0)}_{2m-1}) + (d_1, u^{(1)}_{2m}, v^{(1)}_{2m-1})(T-T_0) + \ldots, \quad (6.2)$$

$$(m = 0, \pm 1, \ldots)$$

where a is held fixed and $(T-T_0)$ is assumed to be small.

On substituting from (6.2) into (5.5) and equating terms O(1) an infinite system of ordinary differential equations is obtained for $(u^{(0)}_{2m}, v^{(0)}_{2m-1})$ which is identical to (5.5) with $(T, -i\Omega)$ replaced by their neutral values $(T_0, 0)$ corresponding to each value of a.

At $O(T - T_0)$ the following system is obtained

$$\begin{cases} [N - 2i(2m)]\, N u^{(1)}_{2m} - \dfrac{a^2 T_0}{2}\left[e^{-(1+i)\eta} v^{(1)}_{2m-1} + e^{-(1-i)\eta} v^{(1)}_{2m+1}\right] = \\ \qquad = 2 d_1 N u^{(0)}_{2m} + \dfrac{a^2}{2}\left[e^{-(1+i)\eta} v^{(0)}_{2m-1} + e^{-(1-i)\eta} v^{(0)}_{2m+1}\right], \\ [N - 2i(2m-1)]\, v^{(1)}_{2m-1} + \left[(1+i) e^{-(1+i)\eta} u^{(1)}_{2m-2} + (1-i) e^{-(1-i)\eta} u^{(1)}_{2m+2}\right] \\ \qquad = 2 d_1 v^{(0)}_{2m-1}, \\ u^{(1)}_{2m} = v^{(1)}_{2m-1} = d u^{(1)}_{2m}/d\eta = 0, \qquad (\eta = 0) \\ u^{(1)}_{2m},\; d u^{(1)}_{2m}/d\eta,\; v^{(1)}_{2m-1} \to 0, \qquad (\eta \to \infty) \end{cases} \quad (6.3\,a,b,c,d)$$

where N is the operator defined by (5.6).

It can be shown that the condition that the linear non-homogeneous differential system (6.3) has a solution defines d_1 in the form

$$d_1=\sum_{m=-\infty}^{\infty}\frac{\int_0^{\infty} a^2\left[e^{-(1+i)\eta}\, v^{(0)}_{2m-1}+e^{-(1-i)\eta}\, v^{(0)}_{2m+1}\right]F^+_{2m}\, d\eta}{4\int_0^{\infty}\left[N\, U^{(0)}_{2m}\cdot F^+_{2m}+v^{(0)}_{2m+1}\; G^+_{2m+1}\right]d\eta} \qquad (6.4)$$

Here the infinite set of functions denoted by (F^+_{2m}, G^+_{2m+1}) with m equal to 0, ± 1, $\pm$ 2,..... is a solution of the following linear system of ordinary differential equations

$$\begin{cases}\left[N+2i(2m)\right]N\, F^+_{2m}+\left[(1-i)\, e^{-(1-i)\eta}\, G^+_{2m+1}+(1+i)\, e^{-(1+i)\eta}\, G^+_{2m-1}\right]=0,\\ \left[N+2i(2m+1)\right]G^+_{2m+1}-\frac{a^2T_0}{2}\left[e^{-(1+i)\eta}\, F^+_{2m}+e^{-(1-i)\eta}\, F^+_{2m+2}\right]=0,\\ F^+_{2m}=dF^+_{2m}/d\eta=G^+_{2m+1}=0\,,\\ F^+_{2m}\,,\ dF^+_{2m}/d\eta\,,\ G^+_{2m+1}\rightarrow 0.\end{cases} \qquad \begin{matrix}(6.5)\\ a,b,c,d\end{matrix}$$

The differential system (6.5) is adjoint to the system (5.5) where $(T, -i\Omega)$ be replaced by the neutral values $(T_0, 0)$.

We intend to expand the solution of (4.4) in terms of the small parameter ϵ defined by

$$\epsilon = d_1\left|T - T_0\right| \qquad (6.6)$$

6.3 - Analysis in the limit of small amplification

Let us introduce the "slow" variable τ defined as in (6.1), so that

$$\frac{\partial}{\partial t}\rightarrow\frac{\partial}{\partial t}+\epsilon\frac{\partial}{\partial \tau}\,, \qquad (6.7)$$

Following the lead of Stewartson and Stuart (1971) we seek a perturbation solution of the differential system (4.4) which represents a small finite disturbance whose amplitude grows with the time scale discussed in § 6.2 in a neighbourhood of the neutral configuration. The scaling follows from the usual argument that the amplitude of the fundamental component of the disturbance is of order $\left|T - T_0\right|^{\frac{1}{2}}$ within such neighbourhood. Thus we expand (u, v, w, T) in the form

$$\begin{cases} \begin{pmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{pmatrix} = \epsilon^{1/2} \begin{pmatrix} f_{10} \cos az \\ g_{10} \cos az \\ h_{10} \sin az \end{pmatrix} + \epsilon \left[\begin{pmatrix} f_{01} \\ g_{01} \\ h_{01} \end{pmatrix} + \begin{pmatrix} f_{20} \cos 2az \\ g_{20} \cos 2az \\ h_{20} \sin 2az \end{pmatrix} \right] \\ \qquad + \epsilon^{3/2} \left[\begin{pmatrix} f_{11} \cos az \\ g_{11} \cos az \\ h_{11} \sin az \end{pmatrix} + \begin{pmatrix} f_{30} \cos 3az \\ g_{30} \cos 3az \\ h_{30} \sin 3az \end{pmatrix} \right] + O(\epsilon^2), \\ T = T_0 + \epsilon T_1 + O(\epsilon^2), \end{cases} \qquad (6.8)\ \text{a,b}$$

where

$$T_1 = (d_1)^{-1}. \qquad (6.9)$$

Furthermore we separate the "fast" time dependence from the "slow" growth of the disturbance by further expanding f_{10}, etc. in the form

$$\begin{cases} f_{np} = \sum_{m=-\infty}^{\infty} f_{np,2m}(\eta,\tau\,;\,a,T)\; e^{i2mt}, \\ (g_{np}, h_{np}) = \sum_{m=-\infty}^{\infty} \left(g_{np,2m-1}(\eta,\tau;a,T)\;,\; h_{np,2m-1}(\eta,\tau;a,T)\right) e^{i(2m-1)t}, \end{cases} \qquad (6.10)$$

and require $(f_{10,2m}\,;\,g_{10,2m-1})$ to behave like $k\,e^{\tau}\,(u_{2m}(\eta),\ v_{2m-1}(\eta))$ as $\tau \to -\infty$.

<u>The fundamental</u>

If we substitute from (6.8), (6.10) into (4.4), use (6.7) and equate terms of order $(\epsilon^{\frac{1}{2}})$, we obtain a partial differential system which describes the behaviour of the fundamental component of the disturbance. We obtain

$$\begin{cases} \left[\mathcal{L}_1 - 2i(2m)\right]\mathcal{L}_1\, f_{10,2m} - \dfrac{a^2 T_0}{2}\left[g_{10,2m-1}\, e^{-(1+i)\eta} + g_{10,2m+1}\, e^{-(1-i)\eta}\right] = 0, \\ \left[\mathcal{L}_1 - 2i(2m-1)\right] g_{10,2m-1} + (1+i)\, e^{-(1+i)\eta} f_{10,2m-2} + (1-i)\, e^{-(1-i)\eta} f_{10,2m} = 0, \\ f_{10,2m} = \partial f_{10}/\partial\eta = g_{10,2m-1} = 0 \qquad (\eta = 0), \\ f_{10,2m},\ \partial f_{10}/\partial\eta,\ g_{10,2m-1} \to 0 \qquad (\eta \to \infty), \end{cases} \qquad (6.11)\ \text{a,b,c,d}$$

$$(m = 0, \pm 1, \pm 2, \ldots.)$$

where

$$\mathcal{L}_j \equiv \frac{\partial^2}{\partial\eta^2} - (ja)^2 .$$

The similarity between the differential systems (6.11) and (5.5) suggests that we assume a separable solution for $(f_{10,2m}, g_{10,2m-1})$ of the form

$$(f_{10,2m}\, ,\, g_{10,2m-1}) = A(\tau)\ (u^{(0)}_{2m}(\eta)\, ,\, v^{(0)}_{2m-1}(\eta))\ , \qquad (6.12)$$

where $A(\tau)$ is an amplitude function (complex in general) which behaves like $A\, e^{\tau}$ when $\tau \to -\infty$, and $(u^{(0)}_{2m}, v^{(0)}_{2m-1})$ are as defined earlier.

The first harmonic

The differential system obtained by substituting from (6.8), (6.10) into (4.4) and equating terms of order ϵ describes the behaviour of the first harmonic of the disturbance $(f_{20,2m}, g_{20,2m-1})$. By taking (6.12) into account it can be shown that $(f_{20,2m}, g_{20,2m-1})$ are the solution of a linear non-homogeneous partial differential system parametrically dependent on τ through coefficients proportional to $A^2(\tau)$. Thus we assume

$$(f_{20,2m}, g_{20,2m-1}) = A^2(\tau)\ (F_{20,2m}(\eta), G_{20,2m-1}(\eta))\, , \qquad (6.13)$$

where $(F_{20,2m}(\eta), G_{20,2m-1}(\eta))$ are the solution of the following linear non-homogeneous ordinary differential system:

$$\begin{cases} \left[L_2 - 2i(2m)\right] L_2 F_{20,2m} - 2\alpha^2 T_0 \left[G_{20,2m-1}\, e^{-(1+i)\eta} + G_{20,2m+1}\, e^{-(1-i)\eta}\right] = \\ \qquad \alpha^2 T_0 \sum\limits_{j=-\infty}^{\infty} \left[v^{(0)}_{2j-1}\, v^{(0)}_{2m-(2j-1)}\right] + 2 \sum\limits_{j=-\infty}^{\infty} \left[u^{(0)}_{2j}\, u^{(0)III}_{2m-2j} - u^{(0)I}_{2j}\, u^{(0)II}_{2m-2j}\right], \\ \left[L_2 - 2i(2m-1)\right] G_{20,2m-1} + (1+i)\, e^{-(1+i)\eta} F_{20,2m-2} + (1-i)\, e^{-(1-i)\eta} F_{20,2m} = \\ \qquad \sum\limits_{j=-\infty}^{\infty} \left(u^{(0)}_{2j}\, v^{(0)I}_{(2m-1)-2j} - u^{(0)I}_{2j}\, v^{(0)}_{(2m-1)-2j}\right), \\ F_{20,2m} = F^{I}_{20,2m} = G_{20,2m-1} = 0 \qquad (\eta = 0)\, , \\ F_{20,2m}\ ;\ F^{I}_{20,2m}\ ;\ G_{20,2m-1} \to 0 \qquad (\eta \to \infty)\, , \end{cases} \qquad \begin{matrix} (6.14) \\ a,b,c,d \end{matrix}$$

$(m = 0, \pm 1, \pm 2, \ldots)$

where

$$L_n = \frac{d^2}{d\eta^2} - n^2 a^2 . \tag{6.15}$$

The distortion of the basic flow

The behaviour of the function $g_{01}(\eta,\tau\ ;\ a,T)$ which represents the distortion of the basic flow can also be obtained by substituting from (6.8), (6.10) into (4.4) and equating terms of order ϵ. If (6.12) and (6.13) are taken into account, we can assume

$$g_{01,2m-1} = A^2(\tau)\ \ G_{01,2m-1}(\eta) \quad , \tag{6.16}$$

where $G_{01,2m-1}(\eta)$ $(m = 0, \pm 1, \pm 2, \ldots)$ are the solutions of the following ordinary differential system

$$\begin{cases} \left[\dfrac{d^2}{d\eta^2} - 2i(2m-1)\right] G_{01,2m-1} = \displaystyle\sum_{j=-\infty}^{\infty} \left(U_{2j}^{(0)}\ Y_{(2m-1)-2j}^{(0)I} + U_{2j}^{(0)I}\ Y_{(2m-1)-2j}^{(0)} \right), \\ G_{01,2m-1} = 0 \ , \\ G_{01,2m-1} \to 0 \ . \end{cases} \tag{6.17 a,b,c}$$

$(m = 0, \pm 1, \pm 2, \ldots.)$

The distortion of the fundamental

If (6.8),(6.10) are again substituded into (4.4) and terms of order $(\epsilon^{3/2})$ are equated the system which describes the behaviour of the distortion of the fundamental component of the perturbation can be obtained. By using (6.12), (6.13) and (6.16) we find

$$\left\{\begin{array}{l}
\left[\mathcal{L}_1 - 2(2im)\right]\mathcal{L}_1 f_{11,2m} - \dfrac{\alpha^2 T_0}{2}\left[g_{11,2m-1}\, e^{-(1+i)\eta} + g_{11,2m+1}\, e^{-(1-i)\eta}\right] \\
\qquad = S_{2m}\dfrac{dA}{d\tau} + (\alpha^2 T_1 P_{2m})A + (\alpha^2 T_0 Q_{2m} + R_{2m})A^3, \\
\left[\mathcal{L}_1 - 2i(2m+1)\right] g_{11,2m-1} + (1+i)\, e^{-(1+i)\eta} f_{11,2m-2} + (1-i)\, e^{-(1-i)\eta} f_{11,2m} \\
\qquad = 2\, v^{(0)}_{2m-1}\dfrac{dA}{d\tau} + Z_{2m-1} A^3, \\
f_{11,2m} = \partial f_{11,2m}/\partial\eta = g_{11,2m-1} = 0 \qquad (\eta = 0), \\
f_{11,2m}\ ;\ \partial f_{11,2m}/\partial\eta\ ;\ g_{11,2m-1} \longrightarrow 0 \qquad (\eta \to \infty).
\end{array}\right. \tag{6.18 a,b,c,d}$$

$(m = 0, \pm 1, \pm 2, \ldots\ldots)$

where P_{2m}, Q_{2m}, R_{2m}, S_{2m}, Z_{2m-1} are functions of η given by

$$P_{2m} = \frac{1}{2}\left(v^{(0)}_{2m-1}\, e^{-(1+i)\eta} + v^{(0)}_{2m+1}\, e^{-(1-i)\eta}\right),$$

$$Q_{2m} = \frac{1}{2}\sum_{j=-\infty}^{\infty}\left(v^{(0)}_{2j-1}\, G_{20,2m-(2j-1)} + 2\, v^{(0)}_{2m-(2j-1)}\, G_{01,2j-1}\right),$$

$$R_{2m} = \sum_{j=-\infty}^{\infty}\Big[\frac{1}{2} F^{III}_{20,2j} F_{10,2m-2j} - \frac{1}{2} F^{I}_{20,2j} F^{II}_{10,2m-2j} - F^{III}_{10,2j} F_{20,2m-2j}$$
$$+ F^{I}_{10,2j} F^{II}_{20,2m-2j} - 3\alpha^2\Big(\frac{1}{2} F_{10,2m-2j} F^{I}_{20,2j} + F_{20,2m-2j} F^{I}_{10,2j}\Big)\Big],$$

$$S_{2m} = 2\left(U^{(0)II}_{2m} - \alpha^2 U^{(0)}_{2m}\right),$$

$$Z_{2m-1} = \sum_{j=-\infty}^{\infty}\Big(F_{20,2j}\, v^{(0)I}_{(2m-1)-2j} + U^{(0)}_{2j}\, G^{I}_{20,2m-1-2j} +$$
$$2\, G_{20,2m-1-2j}\, U^{(0)I}_{2j} + \frac{1}{2} v^{(0)}_{2m-1-2j} F^{I}_{20,2j} + 2 U^{(0)}_{2j} G^{I}_{01,2m-1-2j}\Big).$$

(6.19)
a,b,c,d,e

The infinite set of linear ordinary differential equations is non-homogeneous. An orthogonality condition is to be satisfied for the solvability of such differential system. An amplitude equation is thus obtained in the form

$$\frac{dA}{d\tau} + a_1 A + a_2 A^3 = 0 \quad , \tag{6.20}$$

where

$$a_1 = a^2 T_1 \sum_{m=-\infty}^{\infty} \frac{\int_0^\infty P_{2m} F^+_{2m} \, d\eta}{\int_0^\infty (S_{2m} F^+_{2m} + 2 v^{(o)}_{2m-1} G^+_{2m-1}) d\eta} \quad , \tag{6.21}$$

$$a_2 = \sum_{m=-\infty}^{\infty} \frac{\int_0^\infty (a^2 T_0 Q_{2m} + R_{2m}) F^+_{2m} + Z_{2m-1} G^+_{2m-1} \, d\eta}{\int_0^\infty S_{2m} F^+_{2m} + 2 v_{2m-1} G^+_{2m-1} \, d\eta} \quad . \tag{6.22}$$

If (6.4) and (6.9) are taken into account a_1 is found to be (-1). Furthermore on setting

$$A = \epsilon^{-\frac{1}{2}} A_1 \quad , \tag{6.23}$$

the amplitude equation can be rewritten in the form

$$\frac{dA_1}{dt} - \epsilon A_1 + a_2 A_1^3 = 0 \quad . \tag{6.24}$$

Equation (6.24) is a Bernoulli type equation of the kind discussed by Stuart (1960). After usual substitutions the solution is obtained in the form

$$A_1^2 = \frac{C \epsilon \exp (2\epsilon t)}{1 + C a_2 \exp (2\epsilon t)} \quad , \tag{6.25}$$

with C an arbitrary constant.

The condition that (6.25) allow the matching with the linear solution (5.4) gives

$$C = K \quad . \tag{6.26}$$

Furthermore, according as $a_2 < 0$ or $a_2 > 0$, (6.25) shows that sub-critical or supercritical disturbances decay from or amplify up to their equilibrium values respectively. To the present approximation the equilibrium amplitude A_e is given by

$$A_e^2 = \frac{\epsilon}{a_2} \quad . \tag{6.27}$$

6.4 - Results

The aim of the computation was the determination of the constants d_1 and a_2 which, to order $\epsilon^{3/2}$, characterize the behaviour of the amplitude function $A(\tau)$.

Equations (6.4) and (6.22) show that the knowledge of the pairs of functions $(u_{2m}^{(0)}(\eta),\ v_{2m-1}^{(0)}(\eta))$, $(F_{20,2m}(\eta),\ G_{20,2m-1}(\eta),\ G_{01,2m-1}(\eta),\ F_{2m}^{+}(\eta),\ G_{2m-1}^{+}(\eta))$ and some of their derivatives is needed in order to perform such computations. Thus the differential systems (6.14)(6.17) , (6.5) must be solved.

Each of such infinite systems was approximated by a finite set of ordinary differential equations by neglecting terms for $|m| > 4$ in the series expansions (6.10). Such approximation, which was chosen on the basis of the results of the linear theory, was "a posteriori" found to be satisfactory.

Each set of equations was solved numerically following a numerical procedure similar to that discussed in § 5.3. The numerical integration of each differential set was performed by means of the Runge-Kutta-Gill procedure of the IV order. The boundary conditions imposed at η equal η_∞ for each independent solution of the initial value problems were obtained by neglecting the centrifugal terms in (6.5) and the centrifugal and non-homogeneous terms in (6.14) and (6.17).

The values of some of the pairs of functions mentioned above are shown in figs. 21 - 32. They show that enough accuracy was achieved by retaining terms for $|m| < 4$ in (6.10). A check for the solution of the adjoint system (6.5) was obtained by comparing its eigenvalues with those associated with the linear system (5.5). The agreement was satisfactory (5 significant figures).

Only partial checks were available for the solutions of the systems

(6.14) and (6.17). For instance the structure of such systems shows that the functions ($G_{01,2m-1}$; $F_{20,2m}$; $G_{20,2m-1}$) are complex conjugates to ($G_{01,-2m+1}$; $F_{20,-2m}$; $G_{20,-2m+1}$) respectively and $F_{20,0}$ is real, except for an arbitrary constant. Such conditions were satisfied by the numerical solution with an accuracy of 3 significant figures. In fact the solution was affected by the error associated with two numerical integrations (the non-homogeneous terms of (6.14) and(6.17) were only approximately known).

If should be emphasized that the value chosen for η_∞ had to be large enough for the numerical quadratures (6.21),(6.22) to be performed with sufficient accuracy after replacing the limit of integration ∞ with η_∞. In the same time the opposite requirement was present that η_∞ should be small enough for the numerical integrations of the differential systems to lead to sufficiently modest errors. The value chosen was $\eta_\infty = 10$ with step length 0.1.

After obtaining the functions previously mentioned, the functions P_{2m}, Q_{2m}, R_{2m}, S_{2m}, Z_{2m-1} as given by (6.19) were determined. Finally the numerical quadratures present in (6.21),(6.22) could be performed numerically by means of Simpson rule. Thus the constants d_1 and a_2 were obtained. They are

$$d_1 = 0.00431 , \qquad (6.28)$$

$$a_2 = 1.51 . \qquad (6.29)$$

6.5 - Conclusions

The results given in § 6.4 show that an equilibrium amplitude solution exists in a supercritical neighbourhood of the marginal configuration. Such findings are in agreement with the experimental observations described in chapter 7. Also, as mentioned earlier, they are consistent with Joseph's (1972) results on the problem of bifurcation of quasi periodic solutions of the Navier Stokes equations from basic time periodic solutions of fixed frequency. Indeed, the eigenvalue $(-i\Omega)$ being zero at criticality the bifurcation leads to a stable supercritical branch of the same frequency as the basic flow.

The analysis shows that the type of non linear interaction discussed in this chapter cannot produce a steady component in the azimuthal direction as shown by the experimental observations in the second stage of instability. This suggests that such a component may be associated with an asimmetry of the disturbance. We will not discuss asymmetric disturbances here. However we notice that, by inspecting the differential system that governs their behaviour in the linear regime, one can show that a steady azimuthal component of the disturbance can exist coupled with a steady radial component both being periodic in the axial direction. Thus it is desirable to investigate the possibility that an interaction of such a mode with that discussed in the present work might be responsible for the second stage of instability.

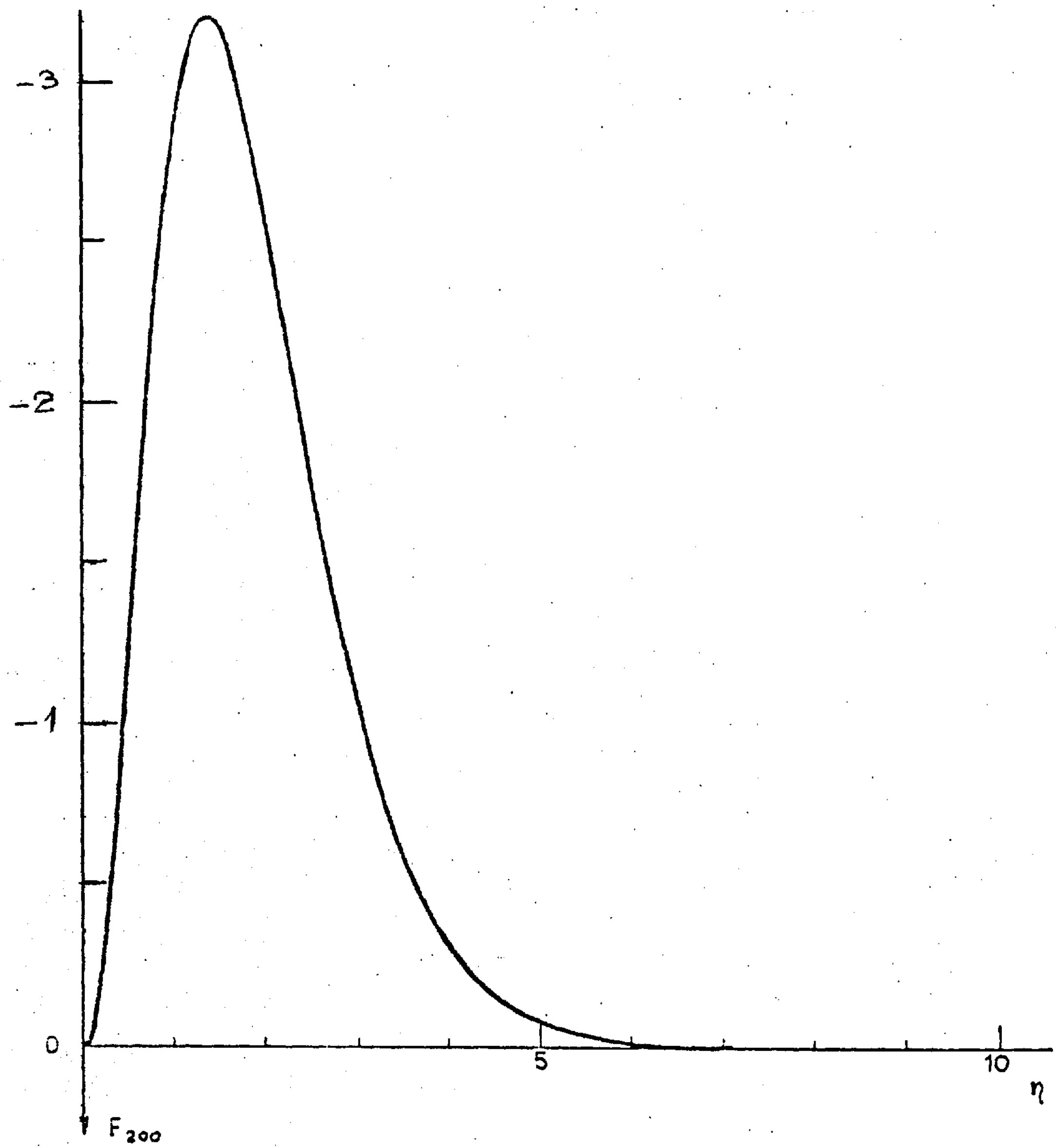


FIG. 21 - The function $F_{200}(\eta)$.

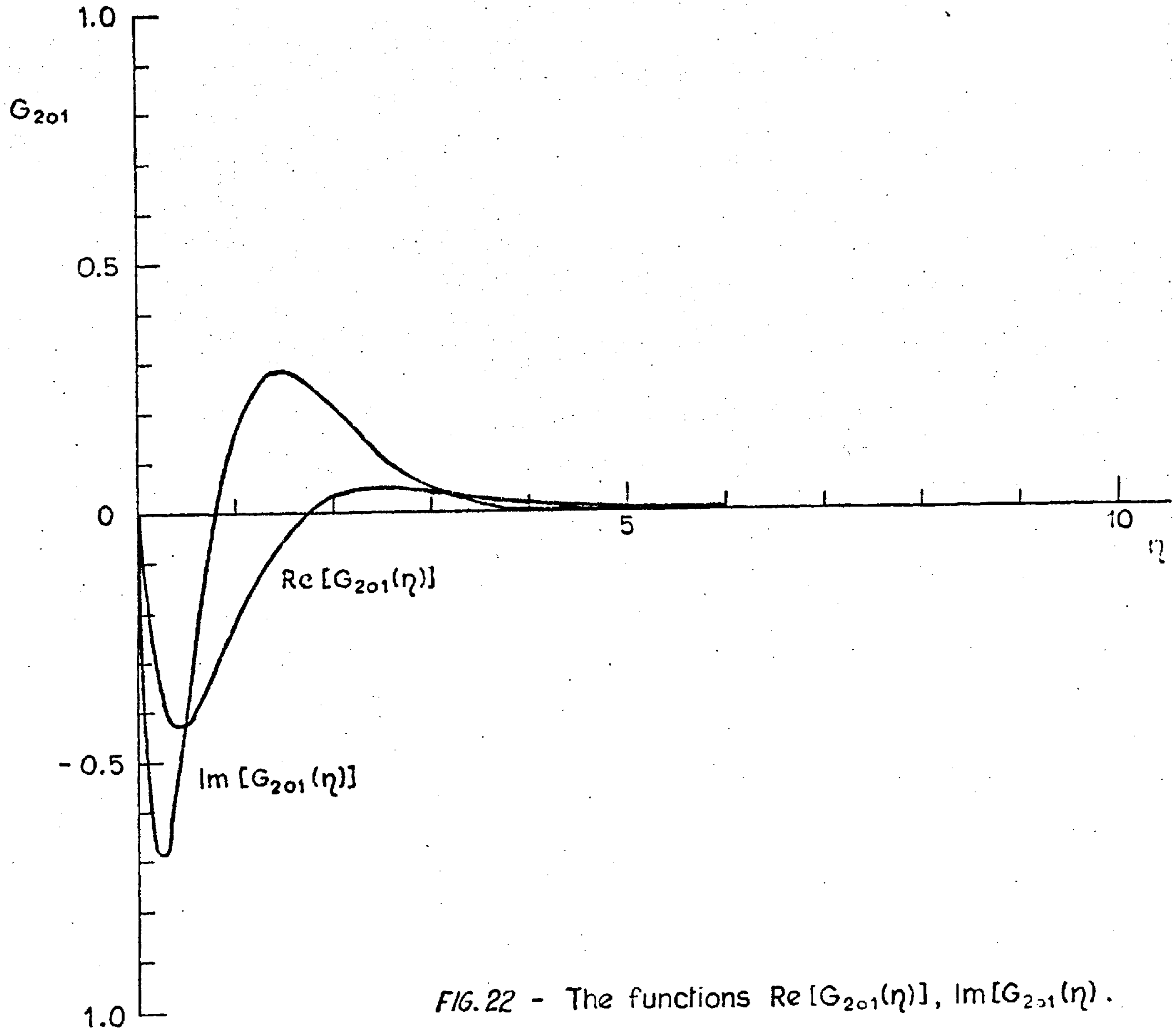


FIG. 22 - The functions $Re[G_{201}(\eta)]$, $Im[G_{201}(\eta)]$.

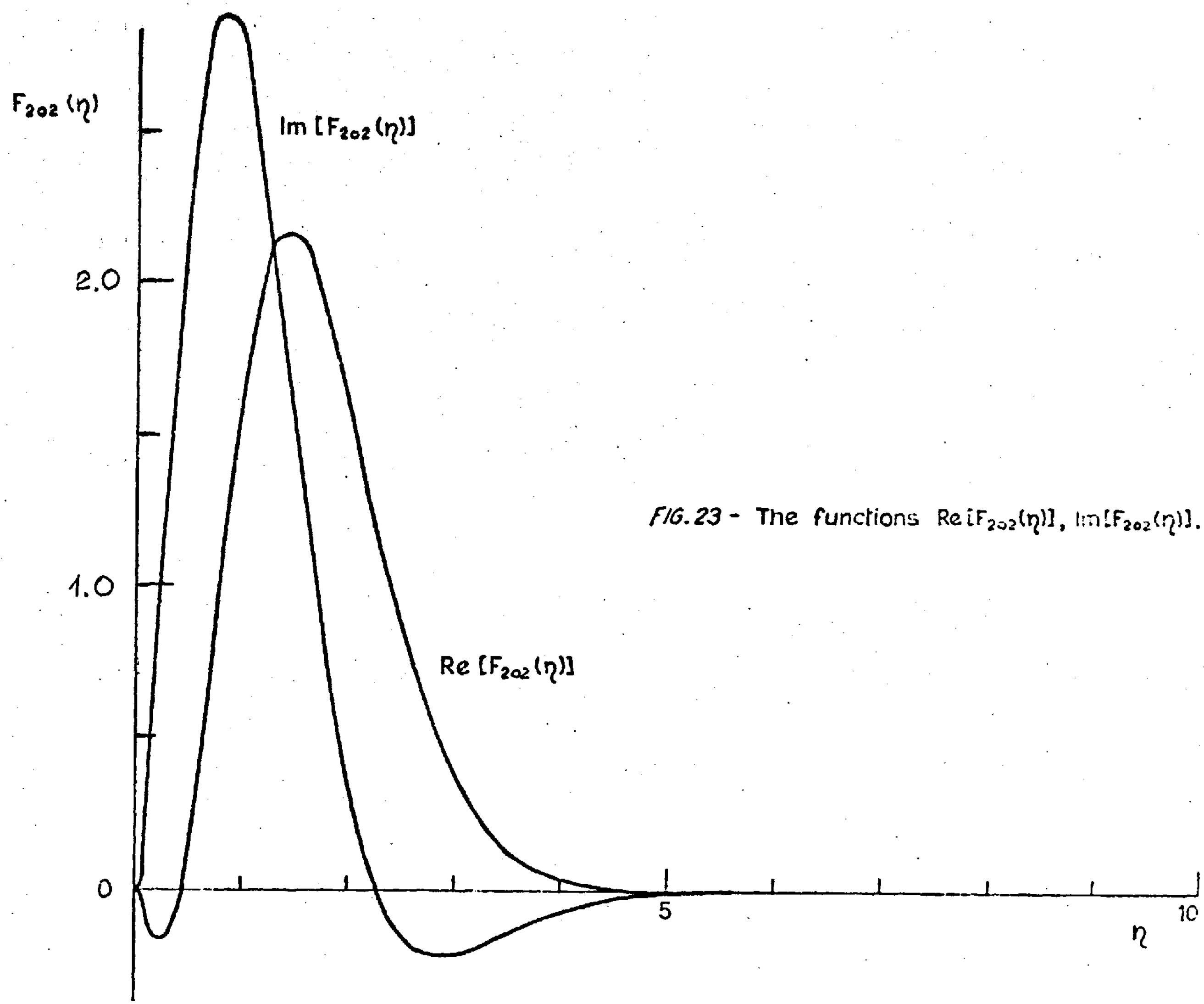


FIG. 23 - The functions $Re[F_{202}(\eta)]$, $Im[F_{202}(\eta)]$.

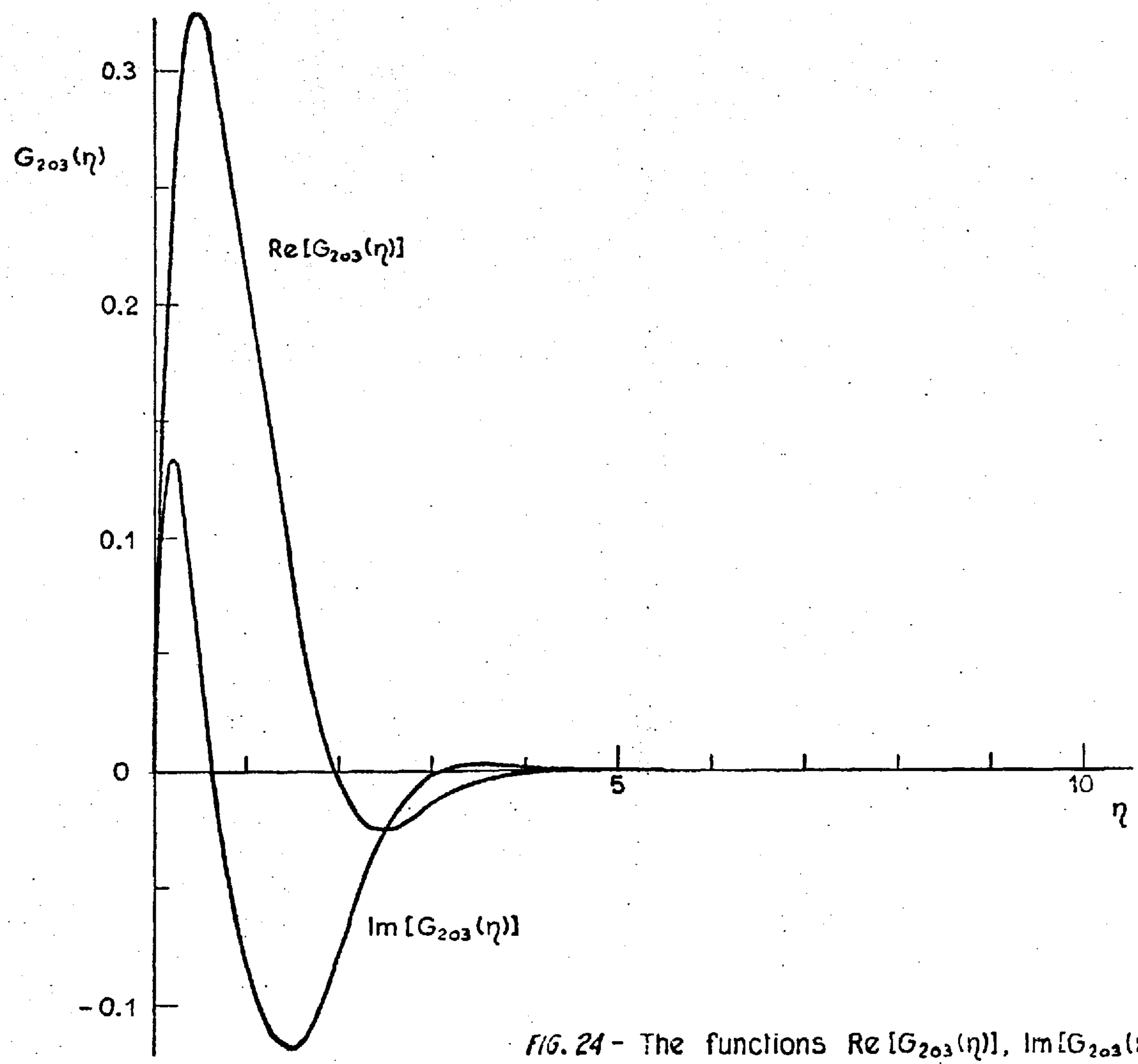


FIG. 24 - The functions $Re[G_{203}(\eta)]$, $Im[G_{203}(\eta)]$.

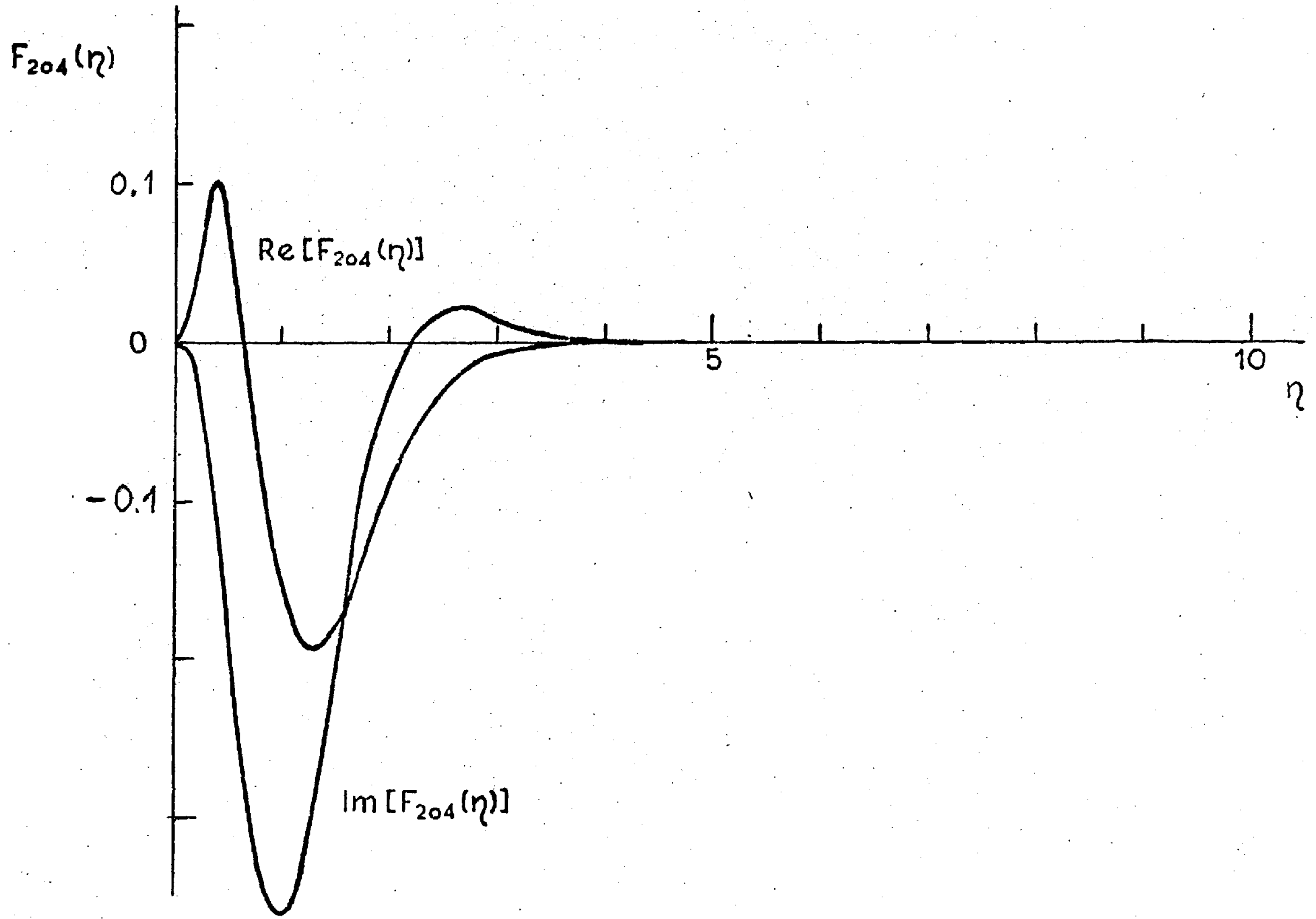


FIG. 25 - The functions $Re[F_{204}(\eta)]$, $Im[F_{204}(\eta)]$.

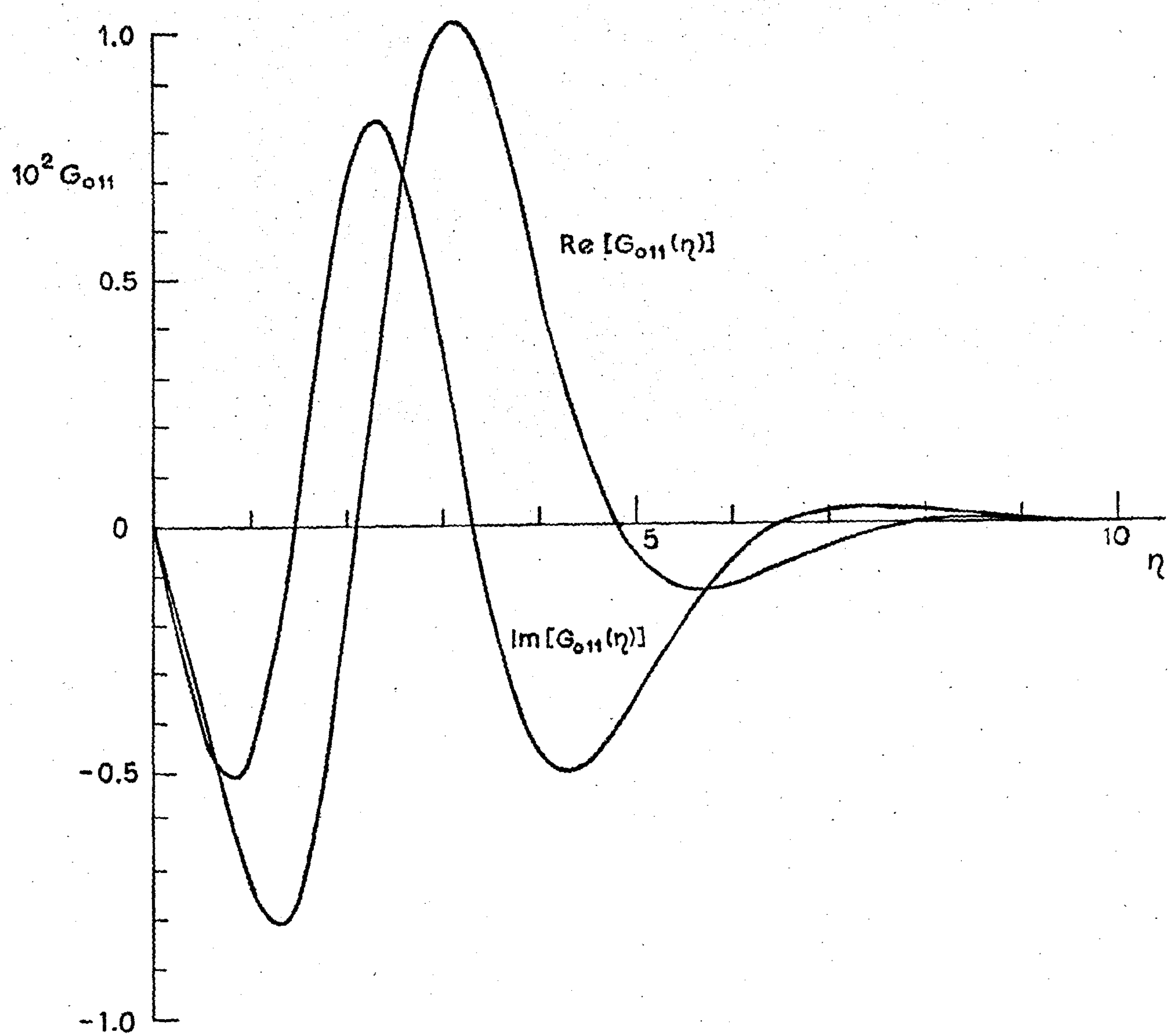


FIG. 26 - The functions $Re\,[G_{o11}(\eta)]$, $Im\,[G_{o11}(\eta)$.

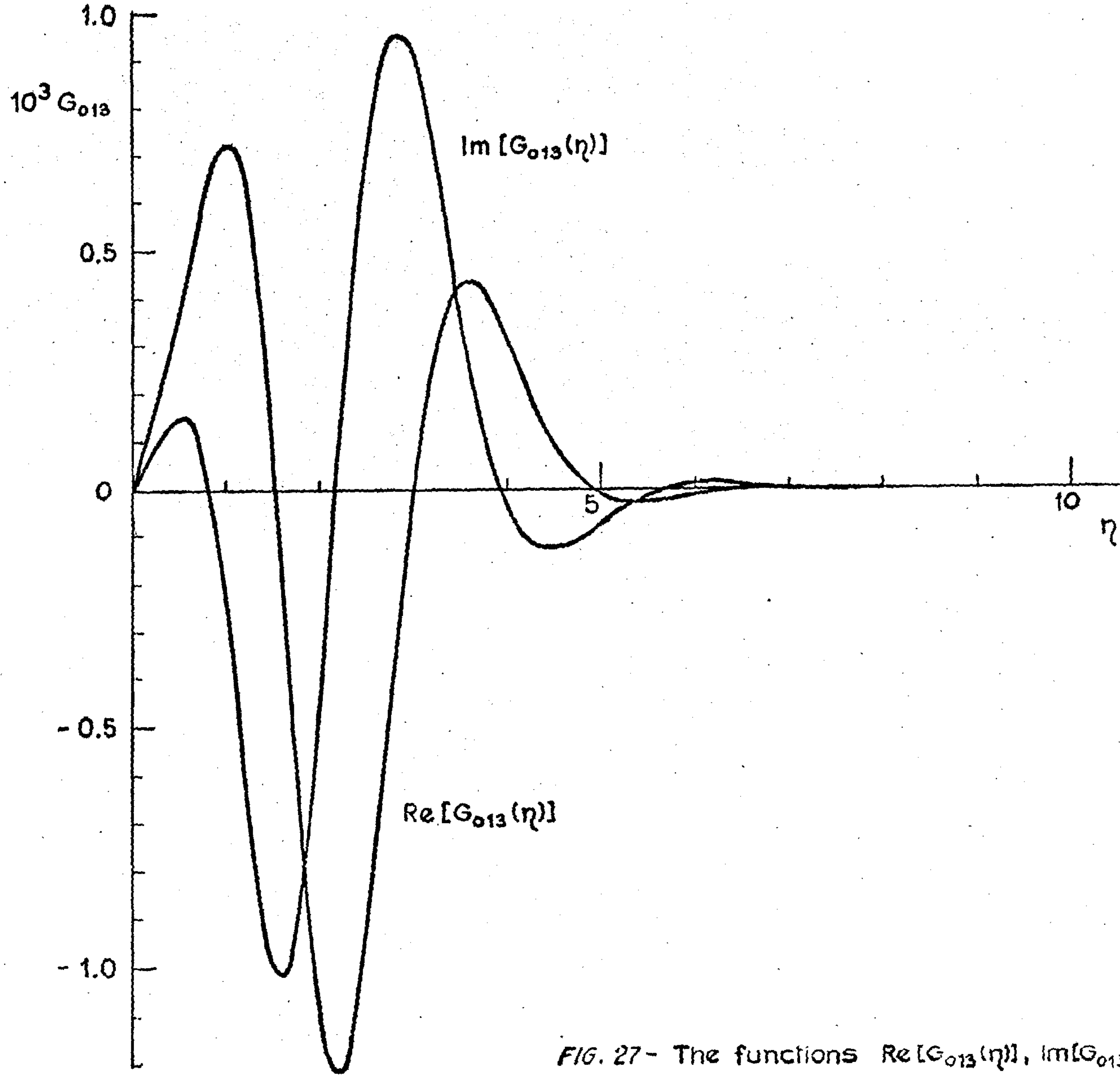


FIG. 27 - The functions Re[$G_{o13}(\eta)$], Im[$G_{o13}(\eta)$].

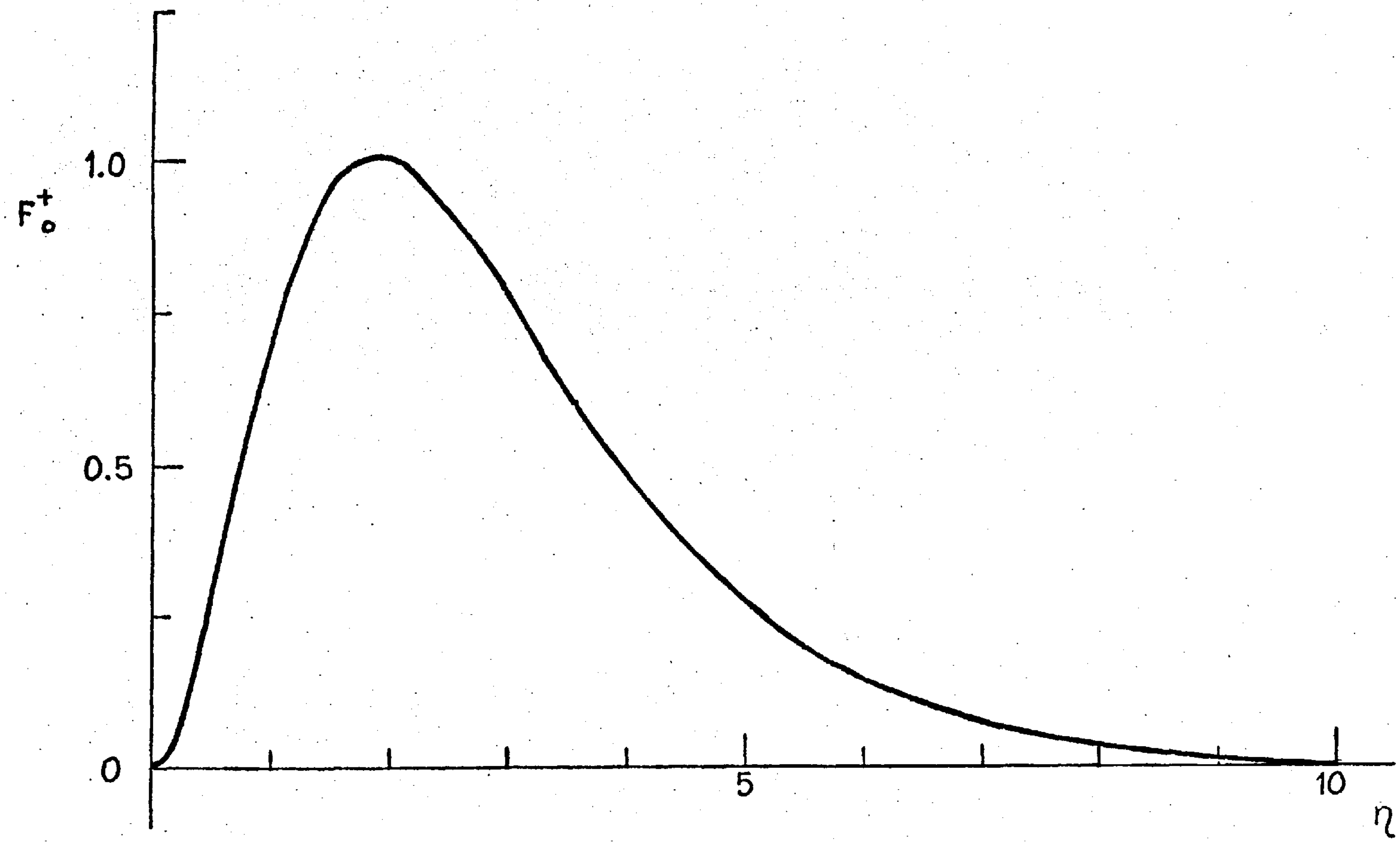


FIG. 28 - The function $F_o^+(\eta)$.

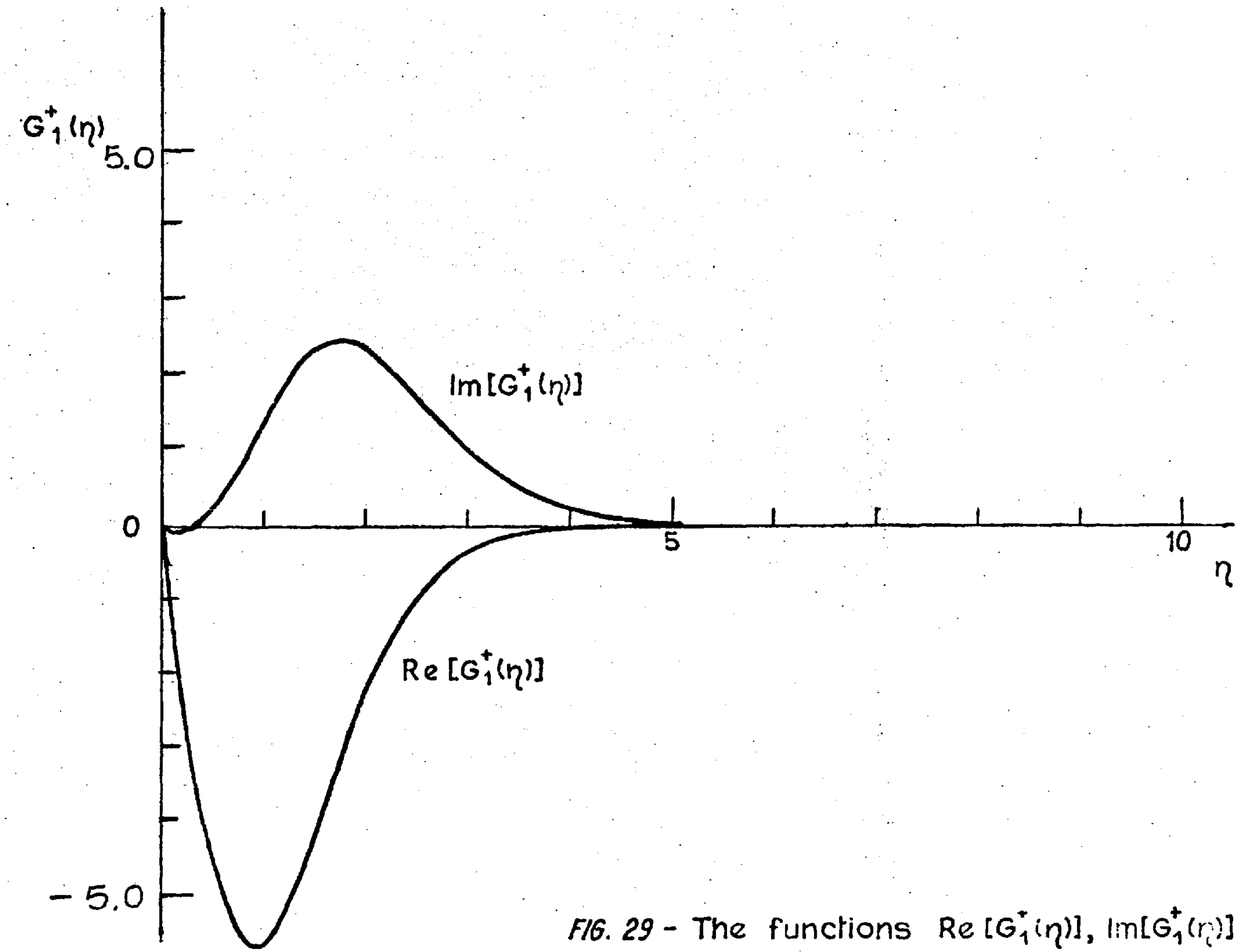


FIG. 29 - The functions $\mathrm{Re}[G_1^+(\eta)]$, $\mathrm{Im}[G_1^+(\eta)]$.

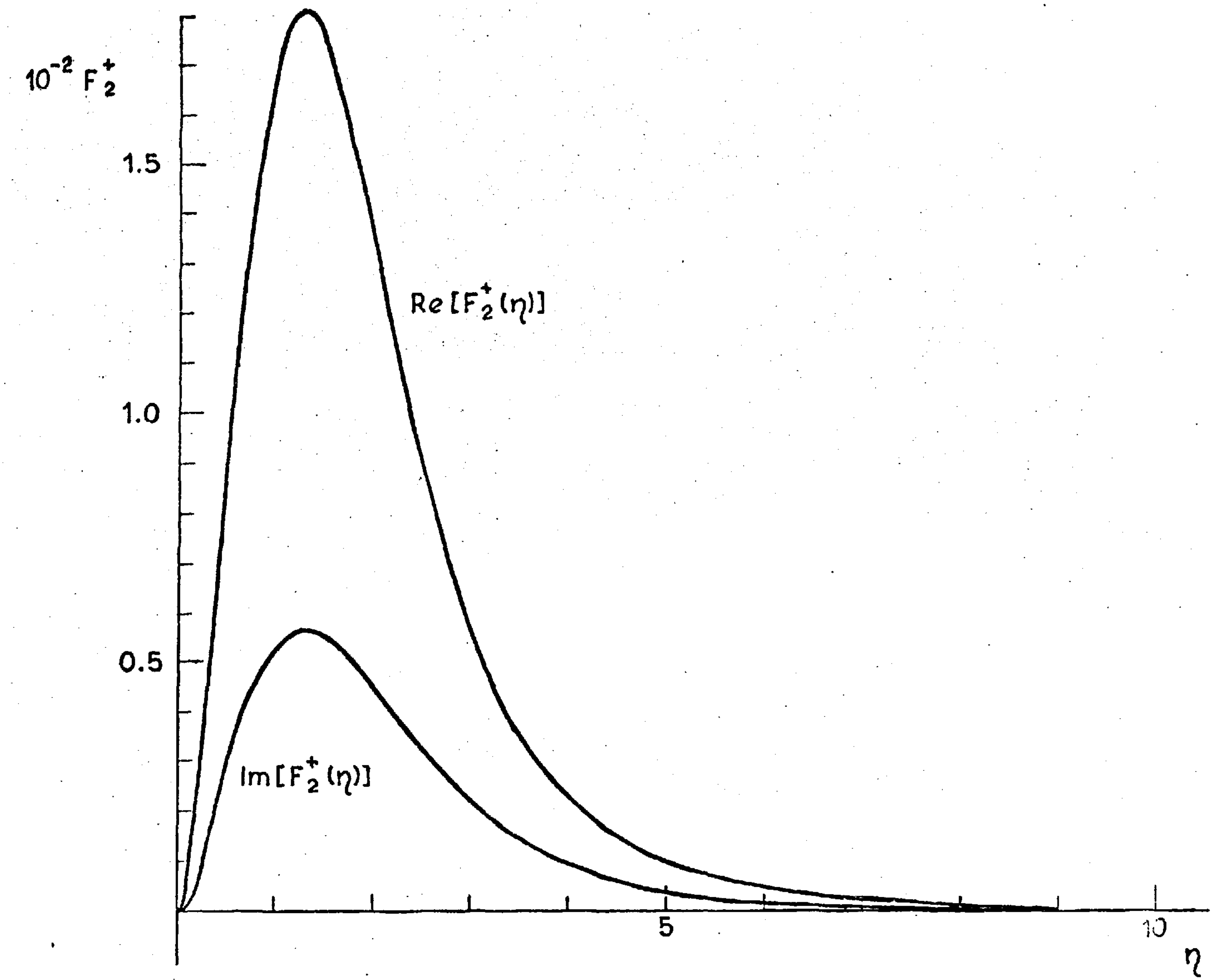


FIG. 30 - The functions $Re[F_2^+(\eta)]$, $Im[F_2^+(\eta)]$.

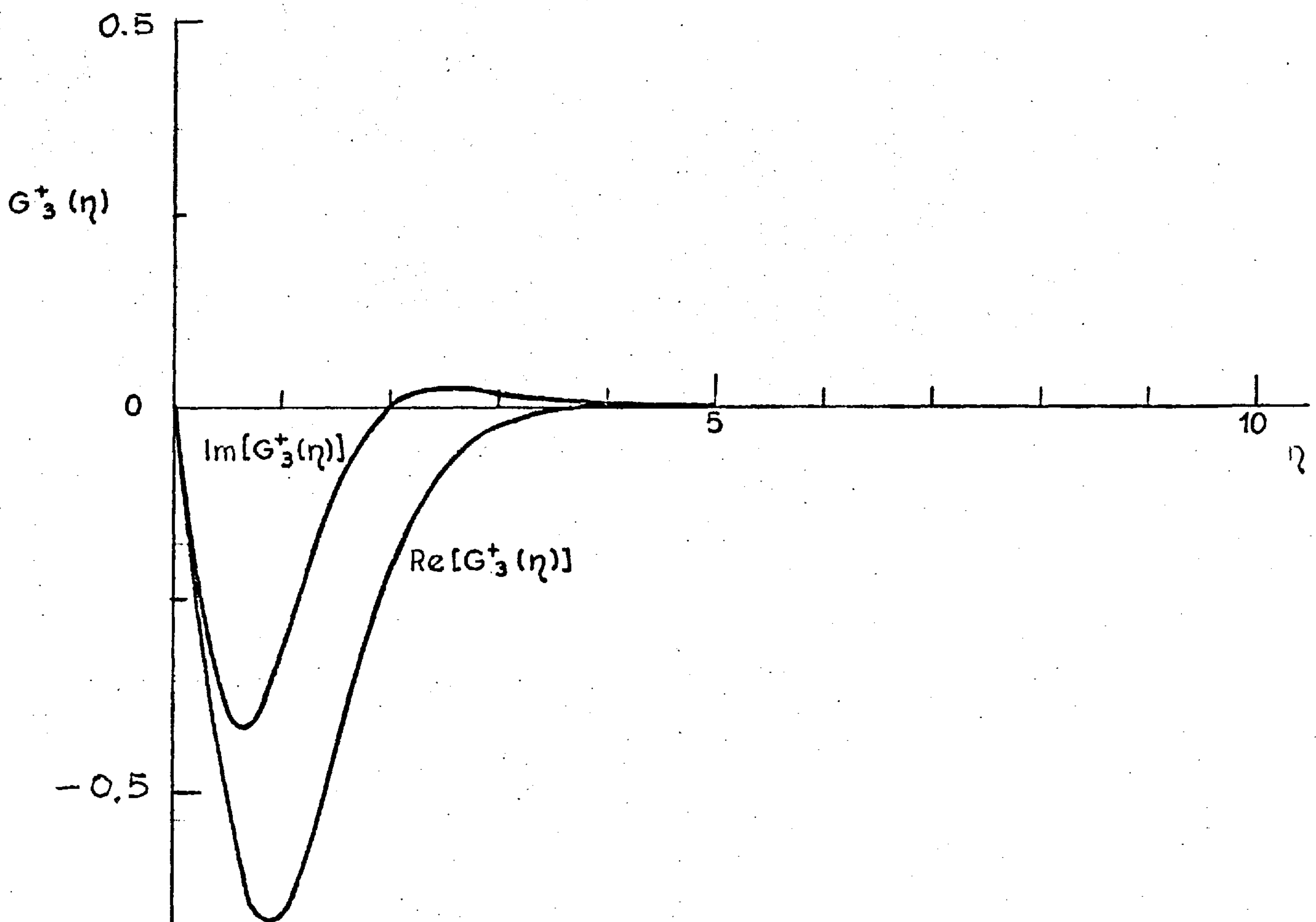


FIG. 31 – The functions $\mathrm{Re}[G_3^+(\eta)]$, $\mathrm{Im}[G_3^+(\eta)]$.

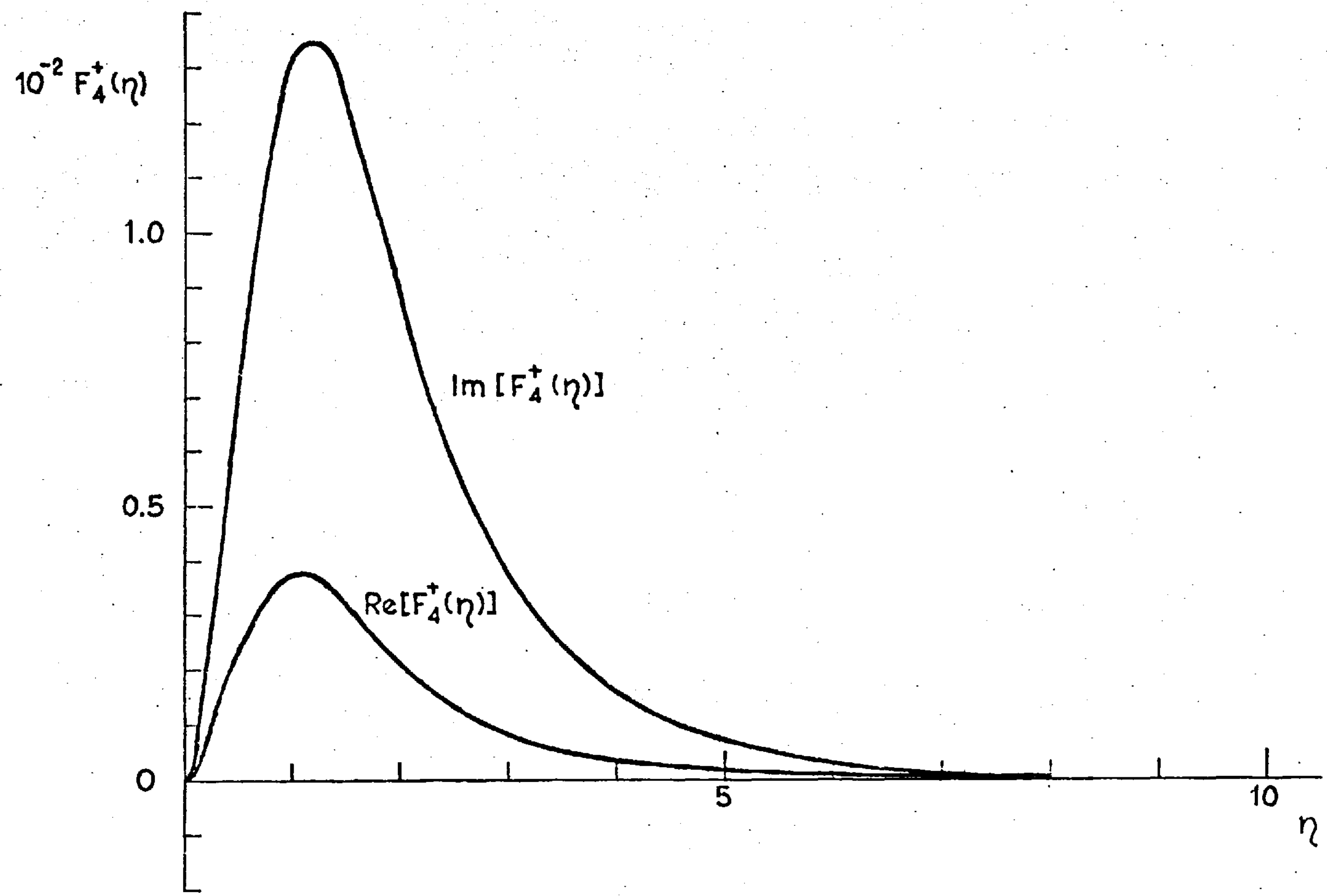


FIG. 32 - The functions $\mathrm{Re}[F_4^+(\eta)]$, $\mathrm{Im}[F_4^+(\eta)]$.

CHAPTER 7

EXPERIMENTAL OBSERVATIONS

7.1 - Apparatus

The apparatus consisted of two concentric cylinders. The gap was filled with water. The outer cylinder, of relatively "large" diameter, was transparent and stationary. The inner cylinder, of much small diameter, was driven in simple harmonic angular motion by a scotch-yoke (an eccentric mounted on the shaft of an electric motor). A variac connected with the motor provided a continuous variation of the frequency of the oscillation. The amplitude of the oscillation could be adjusted by changing the position of the eccentric. The apparatus is shown by Fig. 34.

The dimensions of the apparatus were as in Fig. 33.

The gap between the cylinders was wide enough to ensure that the basic flow and the disturbance would not be affected by the presence of the outer cylinder. Furthermore the cylinders were carefully set up in the vertical position so that no induced secondary motion occurred.

The indicator dye used in the visualization technique was normal ink whose density was adjusted to be the same as that of water by adding a sufficient quantity of alcohol. By means of a pipette lifted mechanically a uniform streak of dye was deposited on the well polished surface of the inner cylinder with the apparatus at rest.

7.2 - Observations

The apparatus was set in motion with the amplitude Δ of the oscillation held fixed and the movement of the streak was observed. The frequency of the oscillation varied in the range 0.5 Hz to 2.0 Hz whilst the Stokes layer thickness then varied in the range 0.4÷0.8 mm.

At low oscillation frequencies the motion was purely in the azimuthal direction as the dye remained within the layer where it had been injected (Fig.35a).When the frequency, and hence T, was increased beyond a critical value of about 1.18 Hz, axial and radial components of velocity were detected. A Taylor vortex type flow then developed with vortices evenly spaced in the axial direction(Fig.35b-g).The vortex motion appeared to be dominantly steady. Very small periodic oscillations could be observed on careful examination. The approximate value of T_c associated with the above critical frequency was 210. The critical value of the wavenumber was found to be $a_c \simeq 0.88$.

It should be emphasized that the purpose of the experiment was to observe the phenomenon more than to perform careful measurements. The above values of T_c and a_c are then to be taken as qualitative estimates.

The frequency was then increased further. The flow pattern was not significantly altered till a frequency of about 1.8 Hz was reached ($T \simeq 260$). At that stage some vortices appeared to interact with each other forming bigger vortices with $a \simeq 0.17$ (Fig. 35h-1). The present visualization technique was not good enough to let us draw definite conclusions about the new flow configuration. However, during the short interval when the dye had not completely diffused, this second "mode" of instability appeared to contain steady tangential as well as radial components of velocity. However, unlike the steady Taylor vortex problem the second stage of instability does not appear to lead to a wavy vortex regime.

The above observations could not be compared with those performed by Taneda (1971) whose results were not available to the present author. However Kuwabarà and Takaki (1975) plot some of Taneda's (1971) results for the critical value of a Reynolds number above which secondary flow was observed, as a function of the wavenumber of the perturbation and of the frequency parameter $\sqrt{\frac{\nu}{\omega}}/R$. The number of data reported is quite small and difficult to interpret. Thus one can only infer an order of magnitude for the critical configuration. After expressing this results in terms of T and a one finds that the critical values $T_c^{(T)}$ and $a_c^{(T)}$ observed by Taneda (1971) lie within the range

$$T_c^{(T)} \simeq 100 \div 400$$

$$a_c^{(T)} \simeq 0.85 \div 1.0$$

It is seen that both the theoretical and the experimental results for T_c and a_c obtained in the present work lie within Taneda's range.

The flow pattern observed by Taneda (1971) as reported by Kuwabara & Takaki (1975) also seems to agree with that observed in the present work in the first stage of instability. No mention is made in Kuwabara & Takaki (1975) of a second stage of instability.

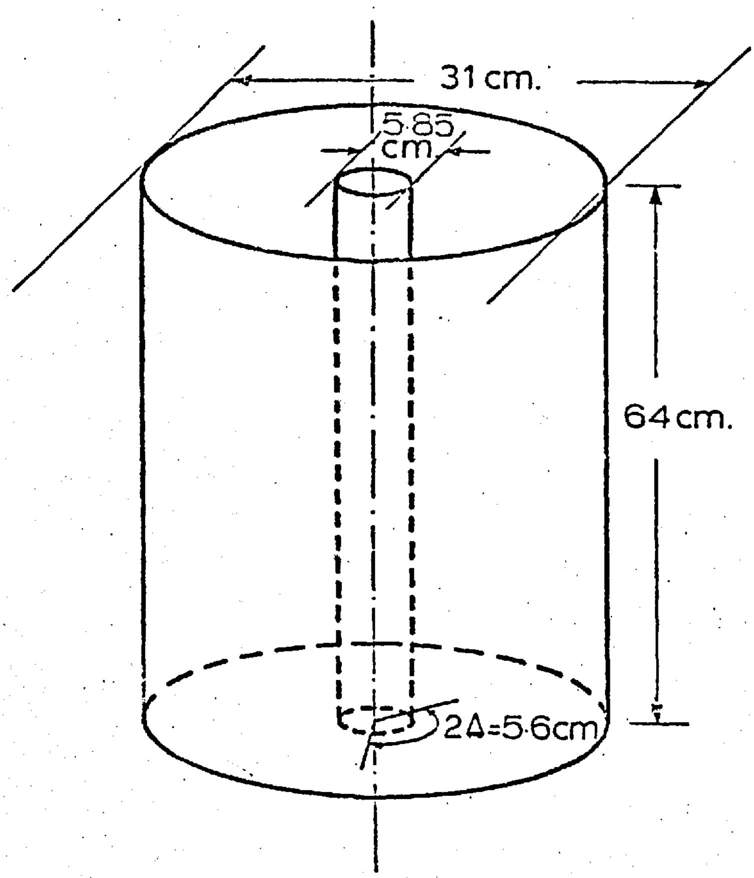


Fig.33: Sketch of the apparatus.

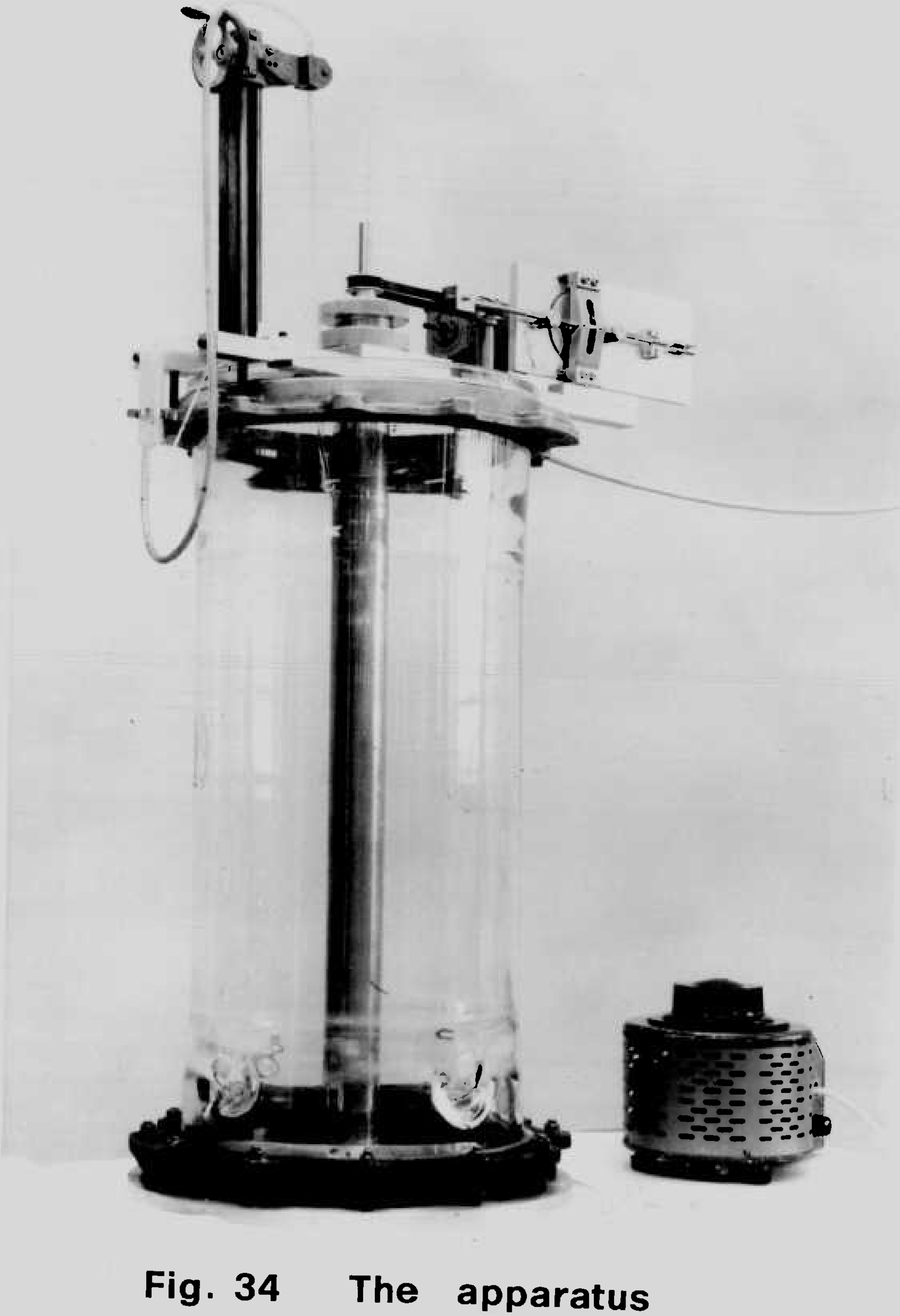


Fig. 34 The apparatus

Fig.35 The development of the instability.

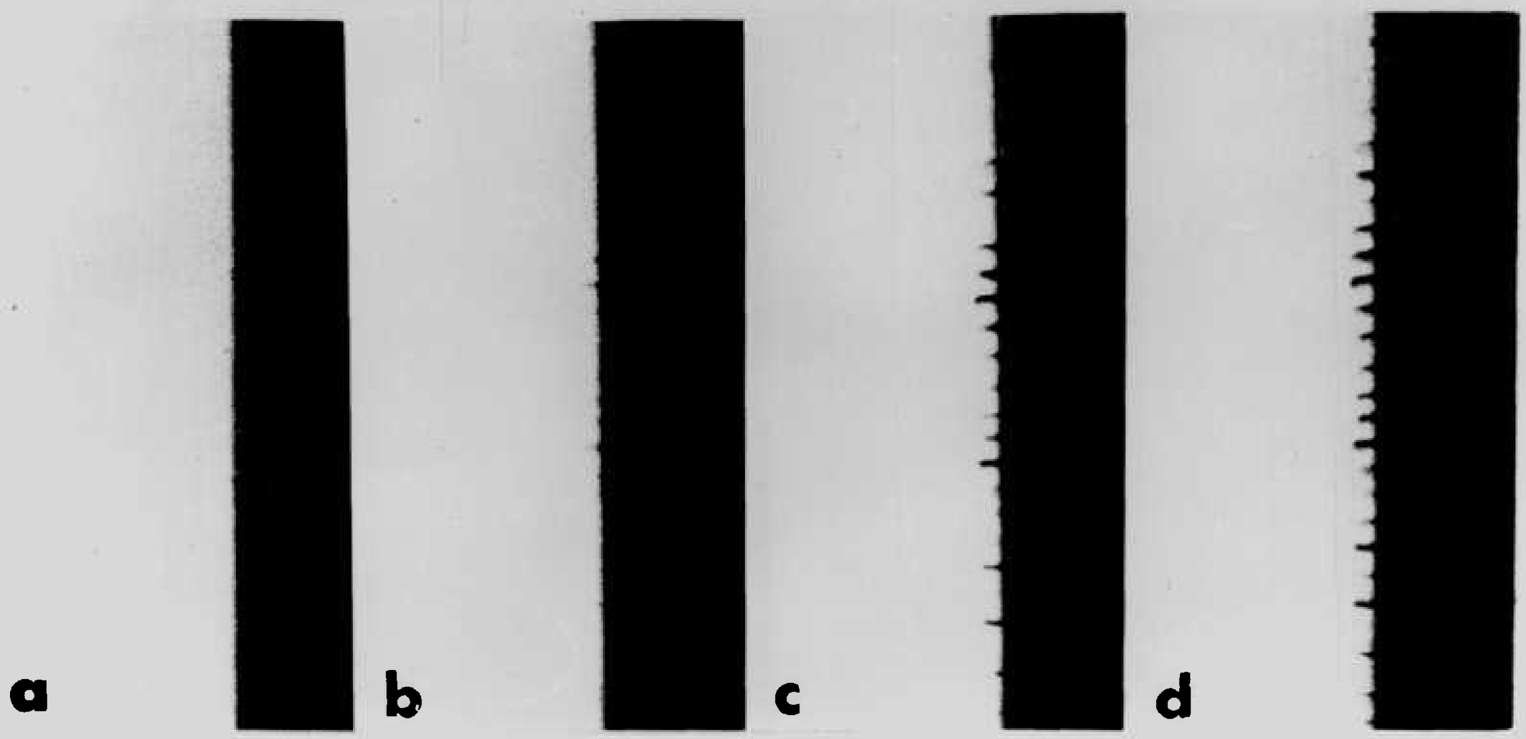


Fig. 35 cont.

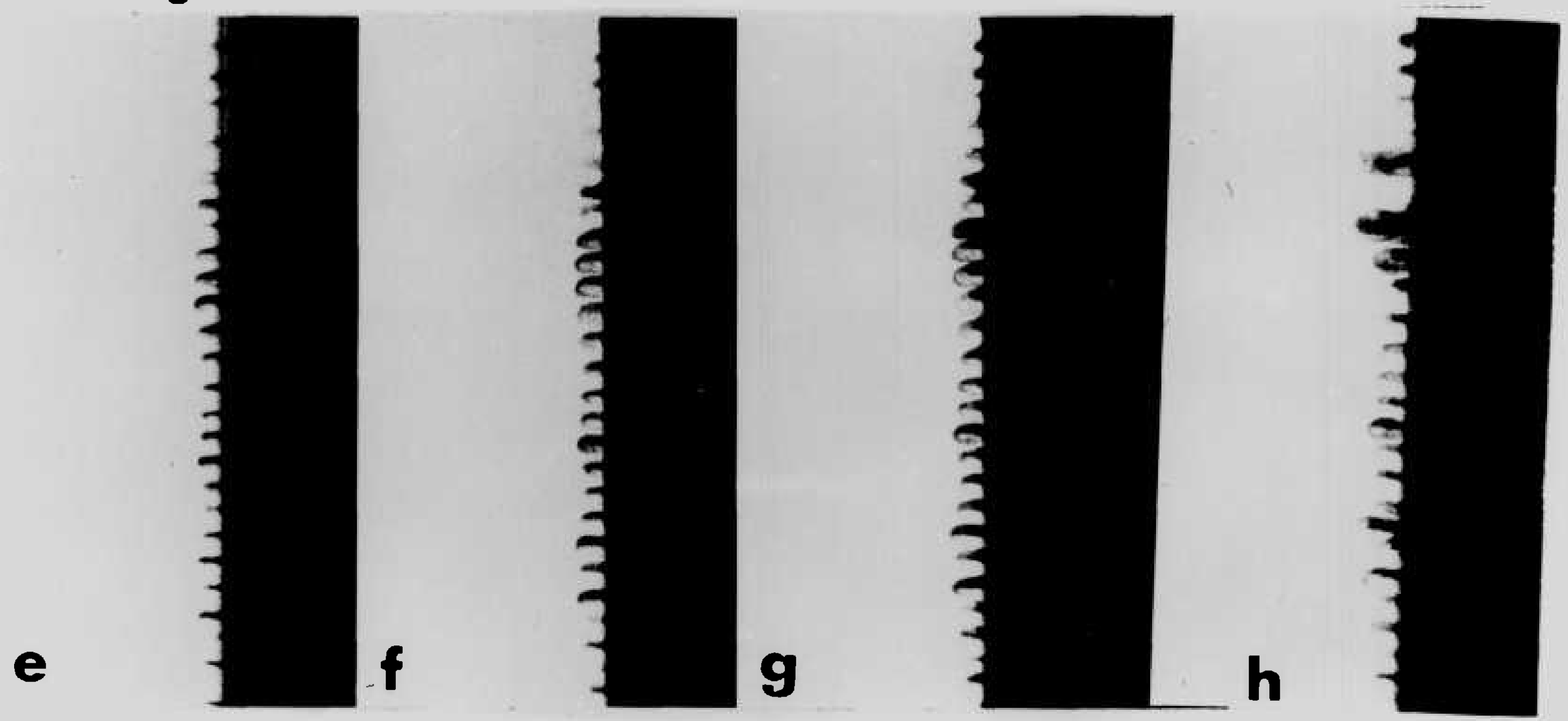
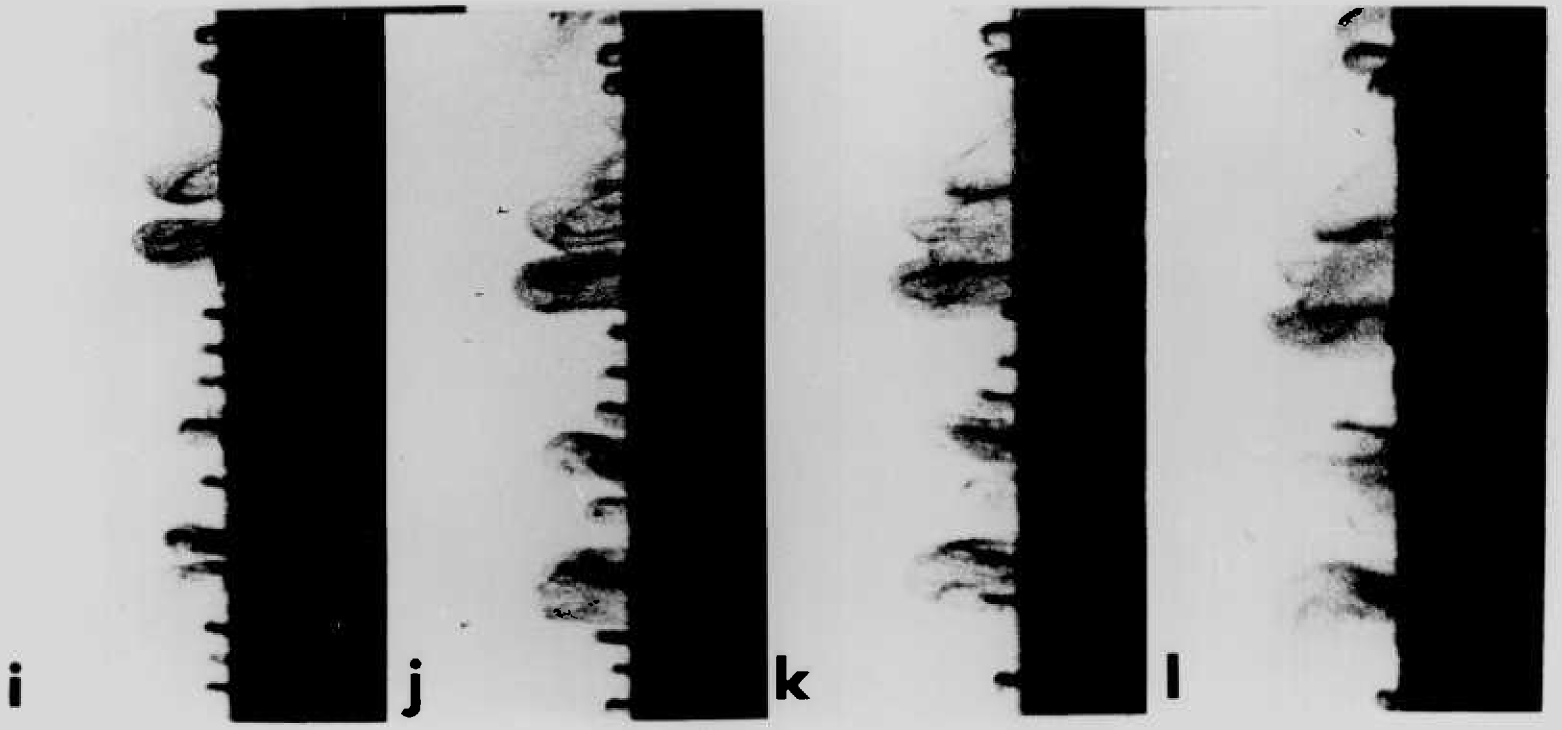


Fig. 35 cont.



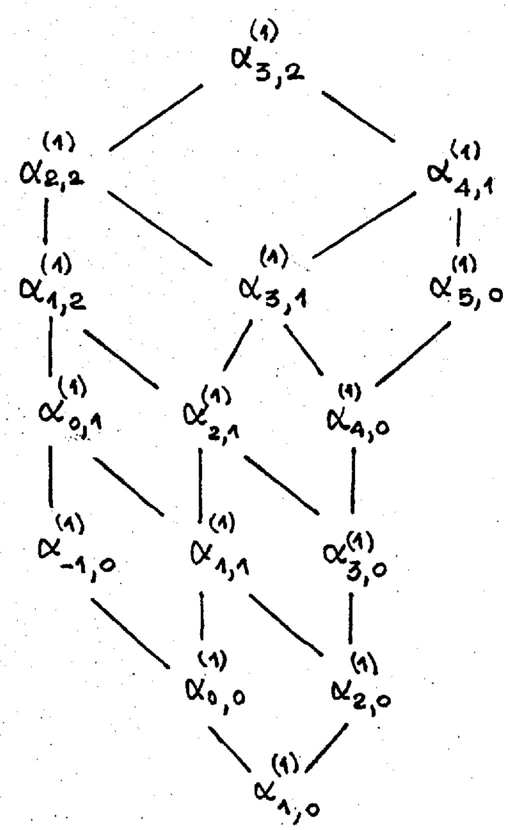
APPENDIX A

A PROOF OF CONVERGENCE

For the sake of clarity it is convenient to write a few terms of the infinite determinant associated with the linear algebraic system (5.14),(5.15) explicitly.

Let us consider the general element of the matrix, namely the series $\left\{ \sum_{n=0}^{\infty} \alpha_{p,n}^{(M)} \left[\alpha_{M,0}^{(M)} \right]^{-1} \right\}$ and let us prove that such a series converges.

The recurrence relationships (5.10)(5.11)(5.12)(5.13) show that each coefficient $\alpha_{p,n}^{(M)}$ can be expressed in terms of $\alpha_{M,0}^{(M)}$. It may be worth showing schematically the series of steps needed for a particular term, say with M = 1, p = 3, n = 2.



$$\left| \begin{array}{cccccccc}
\cdots & \dfrac{\sum_{n=0}^{\infty} \beta^{(-2)}_{-2n}}{\beta^{(-2)}_{-20}} & \dfrac{\sum_{n=0}^{\infty} \alpha^{(-1)}_{-2n}}{\alpha^{(-1)}_{-10}} & & & & & \cdots \\[2ex]
\cdots & - & \dfrac{\sum_{n=0}^{\infty} \alpha^{(-1)}_{-1n}}{\alpha^{(-1)}_{-10}} & \dfrac{\sum_{n=0}^{\infty} \alpha^{(0)}_{-1n}}{\alpha^{(0)}_{00}} & - & & & \cdots \\[2ex]
\cdots & \dfrac{\sum_{n=0}^{\infty} \beta^{(-2)}_{0n}}{\beta^{-2}_{-20}} & \dfrac{\sum_{n=0}^{\infty} \alpha^{(-1)}_{0n}}{\alpha^{(-1)}_{-10}} & \dfrac{\sum_{n=0}^{\infty} \alpha^{(0)}_{0n}}{\alpha^{(0)}_{00}} & \dfrac{\sum_{n=0}^{\infty} \beta^{(0)}_{0n}}{\beta^{(0)}_{00}} & \dfrac{\sum_{n=0}^{\infty} \alpha^{(1)}_{0n}}{\alpha^{(1)}_{10}} & & \cdots \\[2ex]
\cdots & & & \dfrac{\sum_{n=0}^{\infty} (\sigma^{(0)}+2n)\alpha^{(0)}_{0n}}{\alpha^{(0)}_{00}} & \dfrac{\sum_{n=0}^{\infty} (a+2n)\,\beta^{(0)}_{0n}}{\beta^{(0)}_{00}} & \dfrac{\sum_{n=0}^{\infty} (\sigma^{(1)}+(1-i)+2n)\alpha^{(1)}_{0n}}{\alpha^{(1)}_{10}} & & \cdots \\[2ex]
\cdots & & & & - & \dfrac{\sum_{n=0}^{\infty} \alpha^{(1)}_{1n}}{\alpha^{(1)}_{10}} & \dfrac{\sum_{n=0}^{\infty} \alpha^{(2)}_{1n}}{\alpha^{(2)}_{20}} & \cdots \\[2ex]
\cdots & & & & & \dfrac{\sum_{n=0}^{\infty} \alpha^{(1)}_{2n}}{\alpha^{(1)}_{10}} & \dfrac{\sum_{n=0}^{\infty} \alpha^{(2)}_{2n}}{\alpha^{(2)}_{20}} & \cdots
\end{array} \right| = 0 \qquad \text{(A-1)}$$

The indices (n_i, p_i) associated with each of the coefficients generated at step i, are related to those associated with each of the coefficients of the previous step (i-1) by the following relationship

$$2n_i + |M - p_i| = \{2n_{(i-1)} + |M - p_{(i-1)}|\} + 1 \qquad \text{(A.2)}$$

Thus the integral quantity $\{2n_i + |M - p_i|\}$ with M fixed increases when i increases.

Let us now consider the recurrence relationship (5.10) and denote by $F^{(M)}_{p,n}$ (p even) the expression on the right hand side. Thus we write

$$F^{(M)}_{p,n} = \frac{a^2 T}{2} \left\{ \left[\sigma^{(M)} + 2n + |M-p| + (p-M)i \right]^2 - \left[\sigma^{(p)} \right]^2 \right\}^{-1} \times$$
$$\left\{ \left[\sigma^{(M)} + 2n + |M-p| + (p-M)i \right]^2 - a^2 \right\}^{-1}.$$
$$\text{(A.3)}$$

It is easy to prove that

$$\left| F^{(M)}_{p,n} \right| \leq \frac{a^2 T}{2\{2n + |M-p|\}^2} \qquad \text{(A.4)}$$

if

$$2n + |M - p| > N_f \qquad \text{(A.5)}$$

where N_f is a positive integer which satisfies the inequalities

$$N_f + 2\,\mathrm{Re}(\sigma^{(M)}) > 1 \,, \qquad \text{(A.6) a,b}$$

$$(N_f + 2\mathrm{Re}(\sigma^{(M)}) - 1)\, N_f > -2\,\mathrm{Re}\,(-i\Omega) \,,$$

for fixed values of a, M, $\mathrm{Re}(-i\Omega)$, $\mathrm{Im}(-i\Omega)$, T.

Let us consider the recurrence relationship (5.12) and denote by $G^{(M)}_{p,n}$ the factor present on the right hand side. We can write

$$G^{(M)}_{p,n} = - \left\{ \left[\sigma^{(M)} + 2n + |M - p| + (p - M)i \right]^2 - \left[\sigma^{(p)} \right]^2 \right\}^{-1} \qquad \text{(A.7)}$$

It can be proved that

$$\left| G_{p,n}^{(M)} \right| \leq \frac{1}{\{2n + |M - p|\}} \qquad \text{(A.8)}$$

if

$$2n + |M - p| > N_g \qquad \text{(A.9)}$$

where N_g is a positive integer which, for fixed values of a, M, Re(-iΩ), Im(-iΩ), satisfies the inequality

$$N_g + 2\mathrm{Re}(\sigma^{(M)}) > 1 \qquad \text{(A.10)}$$

Thus, if n is sufficiently large (depending on a, T, m , M, Re(iΩ), Im(iΩ)) we have

$$\left| \alpha_{p,n+1}^{(M)} \right| \leq \frac{a^2 T}{\{2n + |M-p|\}^3} \left| \alpha_{p,n}^{(M)} \right| \qquad \text{(A.11)}$$

So by the ratio test the series in question converges absolutely.

REFERENCES

Benney, D.J. & Rosenblat, S. 1964. Phys. Fluids 7, 1385.

Blennerhasset, P. 1976. Ph. D. Thesis, University of London.

Bouthier, M. 1972. J. de Mecanique 11, 599.

Bouthier, M. 1973. J. de Mecanique 12, 75.

Brewster, D.B., Grosberg, P. & Nissan, A.H. 1959. Proc. R.Soc. Lond. A 251, 76.

Caro, C.G. 1973. Ciba Foundation Symposium on Atherogenesis: Initiating factors. Elsevier, Amsterdam.

Chandrasekhar, S. 1961. Hydrodynamic and hydromagnetic stability. Oxford University Press.

Chen, C.F. & Kirchner, R.P. 1971. J.Fluid Mech. 48, 365.

Courant, R. & Hilbert, D. 1953. Methods of Mathematical Physics, Interscience.

Davey, A., Di Prima, R.C. & Stuart, J.T. 1968. J. Fluid Mech. 31, 17.

Dean, W.R. 1928. Proc. R.Soc. Lond. A 121, 402.

Di Prima, R.C. & Stuart, J.T. 1972. J. Fluid Mech. 54, 393.

Di Prima, R.C. & Stuart, J.T. 1975. J. Fluid Mech. 67, 85.

Donnelly, R.J. 1964. Proc. R.Soc. Lond. A 281, 130.

Drazin, P. 1974. Q. Jl. Mech. Appl. Math. 1974.

Eagles, P.M. & Weissman, M.A. 1975. J.Fluid Mech. 69, 241.

Eagles, P.M. 1971. J.Fluid Mech. 49, 529.

Fox, L. & Mayers, D. 1968. Computing Methods for Scientists and Engineers, Clarendon.

Gibson, D.R. & Cook, A.E. 1974. Q.Jl. Mech. appl. Math. 27, 149.

Hämmerlin, G. 1958. Arch. rat. Mech. Anal. 1, 212.

Hall, P. 1973. Ph. D. Thesis, University of London.

Hall, P. 1975 (a). J. Fluid Mech. 67, 29.

Hall, P. 1975 (b). Proc. Roy Soc. 344, 453.

Iooss, G. 1972. Arch. Rational Mech. Anal., 40, 166.

Joseph, D.D. 1972. Remarks about bifurcation and stability of quasi periodic solutions which bifurcate from periodic solutions of the Navier Stokes equations, in "Non Linear Problems in the Physical Sciences and Biology" Lecture Notes in Mathematics # 322, Springer-Verlag.

Kelly, E.R. 1976. J. Fluid Mech. 27, 657.

Kerczek, C. & Davis, S.H. 1974. J.Fluid Mech. 62, 753.

Krueger, E.R., Gross, A. & Di Prima, R.C. 1966. J.Fluid Mech. 24, 521.

Kuwabara, S. & Takaki, R. 1975. J.Phys. Soc. Japan 38, 1180.

Lyne, W.H. 1971. J. Fluid Mech. 45, 13.

Longuet-Higgins, M.S. 1953. Phil. Trans. A 245, 535.

Mchlachlan, N.W. 1947. Theory and Application of Mathieu functions, Clarendon.

Meksyn, D. 1950. Proc Roy Soc 203, 253.

Nerem, R.M., Rumberger, J.A., Gross, D.R., Hamlin, R.L. & Geiger, G.L. 1974. Circ. Res. 34, 193.

Reid, W.H. 1958. Proc. R. Soc. Lond. A 244, 186.

Riley, D. & Lawrence, R. 1976. J. Fluid Mech. To appear.

Rosenblat, S. 1968. J. Fluid Mech. 33, 321.

Rosenblat, S. & Herbert, D.M. 1970. J. Fluid Mech. 43, 385.

Sattinger, D. 1971. Arch. Rational Mech. Anal. 41, 66.

Seed, W.A. & Wood, N.B. 1971. Cardiovasc. Res. 5, 319.

Sexl, T. 1930. Z.Phys 61, 349.

Shen, S.F. 1961. J. Aero. Sci. 28, 397.

Smith, F. 1975. J. Fluid Mech. 71, 15.

Stewartson, K. & Stuart, J.T. 1971. J. Fluid Mech. 48, 529.

Stuart, J.T. 1960. J. Fluid Mech. 9, 353.

Stuart, J.T. 1963. Laminar Boundary Layers, ed. Rosenhead.

Stuart, J.T. 1966. J. Fluid Mech. 24, 527.

Taneda, S. 1971. Proc. IUTAM Symposium 2, 1165.

Thompson, R. 1968. Ph. D. Thesis, Dept. Meteorology, M.I.T.

Venezian, G. 1969. J. Fluid Mech. 35, 243.

Yudovitch, V. 1970. Soviet Math. Dokl. 11, 1473.