MODELS OF EXTENDED HADRONS

by

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Abstract

The notion that a strongly interacting particle may be thought of as a spatially extended object has recently gained much in popularity. For example, dual models imply one-dimensional hadrons ('strings') whilst 'bag' models have maximally extended hadrons (i.e. three-dimensional).

Particular emphasis has been placed on inserting this idea of spatial extension into the familiar framework of relativistic local quantum field theory.

As a starting point it has been conjectured that hadron-like properties may be evident even at the classical level. Consequently much attention has been alotted to investigating field theories which admit potentially hadron-like solutions to the classical static field equations. The question then arises: what do the quantized versions of these classical solutions look like?

Up to now most investigations have been concerned with setting up a systematic perturbative approach to the quantization problem. Unfortunately this is only applicable to the (phenomenologically uninteresting) case of weak coupling e.g. quark confinement in bag models is only achieved in the strong-coupling régime.

The work contained in this thesis will be concerned with the development and application of techniques which are directly applicable to the problem of strong coupling.

The principal method adopted is a generalized Hartree-Fock procedure. This approximation scheme is phrased both in terms of trial state functions and functionals as well as in an unconven-

tional normal-ordering prescription. Moreover it is demonstrated that the latter two variational methods are in fact equivalent although superficially they are quite different. The applications we will be concerned with include a variety of bag models as well as an unconventional dual string model.

Whilst our Hartree-Fock scheme is unsystematic in application, the requirement of self-consistency acts as a powerful constraint on possible ansatzs. In fact, due to this latter point, the variational equations which result are generally quite intractable, although in certain circumstances useful qualitative results may be gleaned from them. We often find that classical intuition can be most misleading.

Preface

The work presented in this thesis was carried out between October 1972 and October 1975 under the supervision of Dr. R.J. Rivers. Except where stated, this work is original and has not been submitted for a degree of this or any other university.

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CHAPTER I

INTRODUCTION

AND

MOTIVATION

1.1 General Introduction

One way to look at the structure of a proton is to use the theory of local currents. Here one is led to think of a hadron as a collection of quasifree point particles endowed with quark quantum numbers. This theoretical prejudice has achieved considerable success, we need only quote SU(6)[1] and deep inelastic experiments.[2] However in order to make this scheme self-consistent it is necessary to provide a convincing explanation for the hitherto unobservability of these fundamental constituents of hadrons (which we shall name "quarks"). This quark confinement problem has motivated the introduction of a novel type of theory for describing hadrons. Here one imagines that all hadrons are, from the very outset, spatially extended objects. The dynamics of the theory then precludes the possibility that the fundamental hadronic constituents will ever escape from their confining region. This category of quark confinement schemes has two main contenders, the so-called MIT[3] and SLAC[4] bag models of hadrons.

Of course, whether or not such an intuitive starting point will be able to explain the complicated interactions of hadrons with themselves is questionable. Not surprisingly there exists an alternate view of hadrondynamics which has as its origin Veneziano's[5] inspired guess for a scattering amplitude. Here the strong interactions are taken to illuminate us about the fundamental structure of hadrons. Incredibly, one finds that hadrons are to be regarded as strings evolving in space-time i.e. hadrons have a one-dimensional spatial extent. Unfortunately, this introduction of a geometrical shape is intrinsically connected to the appearance of quantum fluctuations which mess up the relativistic invariance of the theory except in an

anomalous (and model-dependent) number of space-time dimensions. In this formalism abelian charges appear in the guise of singularities in the geometrical structure thus giving rise to no problem of quark confinement. Finally we would like to point out that this theory may be set up in the language of field theory, albeit multilocal field theory. We will now go on to give a brief review of both the bag and string models.

1.2 Dual Relatavistic Strings

This phenomenon of unidimensionality of hadrons, in the dual framework, seems at first sight rather peculiar. Because of this we will give a heuristic classical argument to motivate this viewpoint. Think of a hadron as a rotating d-dimensional object with uniform spatial extension R in all d-dimensions. The rest mass m_o of such an object will be determined by

$$m_0 \sim R^d \qquad (1)$$

whilst the angular momentum j will behave like

$$j \sim R^{d+1} \qquad (2)$$

Eliminating R from these two equations yields the result

$$j \sim m^{2\left[\frac{d}{1+d}\right]} \qquad (3)$$

The experimental evidence of linear Regge trajectories then informs us that d=1. We will now go on to briefly discuss the more technical

aspects of dual string theory.

Just as a point-particle traces out a world-line in space-time, so a string traces out a world sheet which may be described by four functions, each of two variables: $X_\mu(\tau,K)$. The K variable is taken to label the points on the string and hence has a finite range which we shall take to be $[0,\ell]$. By analogy with the point-particle case, we are led to minimize the area of the world-sheet in order to obtain the equations of motion for the string. To make sure the resulting dynamics is parametrization independent we write the action as[6]

$$S(\tau_i,\tau_f) = \frac{-1}{\pi\alpha'}\int_{\tau_i}^{\tau_f} d\tau \int_0^{\ell} dK \left[-\sigma_{\mu\nu}\sigma^{\mu\nu}\right]^{1/2} \qquad (4)$$

where $\sigma_{\mu\nu} = \dot{X}_\mu X'_\nu - \dot{X}_\nu X'_\mu$ and α' is a fundamental constant with the dimensions of (length)^2; the dot (prime) means differentiation with respect to τ (K). The equations of motion which result from minimizing the action (eqn. (4)) are non-linear but may fortunately be linearized by exploiting the two-dimensional topology of our τ-K manifold. Anyway, we end up with one linear differential equation and some constraint equations. They are

$$\ddot{X}_\mu(\tau,K) = X''_\mu(\tau,K) \qquad (5)$$

$$X'_\mu(\tau,0) = X'_\mu(\tau,\ell) \quad \text{(for open strings)} \qquad (6)$$

$$(\dot{X}_\mu \pm X'_\mu)^2 = 0 \qquad (7)$$

A simple consequence of eqns. (6) and (7) is that the end-points of a free string move with the speed of light. What about the zero-point energy of the string? By introducing a smooth cut-off it is possible to show that it goes like[7]

$$\frac{-1}{2p_V}\,\frac{d-2}{24\alpha'} + O(\Lambda) + O(\Lambda^{-1}) \qquad (8)$$

Here p_V is the momentum of the string along the (x_0+x_3) direction [we use null plane coordinates i.e. we work in a particular gauge] and d is the number of space-time dimensions. The cut-off independent term gives us a state with

$$M^2 = -\frac{(d-2)}{24\alpha'} \qquad (9)$$

In other words a tachyon has appeared as a result of the quantum fluctuations. It also turns out that, in this gauge, the first excited state is purely transverse, so that Lorentz covariance insists that its mass be zero. We therefore have

$$d = 26 \qquad (10)$$

This is the anomalous number of space-time dimensions referred to earlier.

How do strings interact with the electromagnetic field? Take the gauge invariant interaction term to be

$$S_{int} = e\int_{\tau_i}^{\tau_f} d\tau \int_0^{\ell} d\sigma \; \dot{x}_\mu x'_\nu F^{\mu\nu} \qquad (11)$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

After a simple integration by parts we find

$$S_{int} = e\int_{\tau_i}^{\tau_f} d\tau\ \dot{X}_\mu(\tau,1)\,A^\mu(X(\tau,1)) - e\int_{\tau_i}^{\tau_f} d\tau\ \dot{X}_\mu(\tau,0)\,A^\mu(X(\tau,0)) \qquad (12)$$

For a closed string

$$X_\mu(\tau,1) = X_\mu(\tau,0) \qquad (13)$$

We thus see that a closed string does not interact with a Maxwell field i.e. the closed string carries no charge. Physically, it is identified with the Pomeron (of Regge fame). For an open string, eqn. (12) informs us that the "electric" charges reside at the endpoints of the string and are opposite in sign. We are thus lead to an abelian quark picture of a meson. We may also continue to develop this approach to include the interactions of strings with themselves. The Born term in the string-string scattering amplitude is just the Veneziano ansatz. The model we have just reviewed is called (rather unsurprisingly) the Veneziano model. The salient features of this model are that its mass spectrum contains a scalar tachyon together with a massless vector meson and the whole theory operates in a world 26 space-time dimensions! This problem of the mass spectrum is very reminescent of a similar problem in quantum field theory which is there neatly solved by implementing the Higgs mechanism. Because of this is may be useful to set up an analogue model for the dual relatavistic string in the familiar framework of local field

theory. This is because some insight into the physical origin of the all too manifest deficiencies of the string model may result. In fact Nielsen and Olesen[8] have suggested that vortex-line solutions to Higgs-type field theories may be identified with the dual string.

Interestingly enough the Higgs model turns out to be the simplest theory which allows sensible static vortex solutions. This result is primarily due to Derrick's theorem which states that any finite energy solution to a wide class of scalar field theories is unstable under coordinate dilations in any number of space dimensions greater than one. The proof of this theorem is quite simple and revolves around a simple scaling argument. Consider a general scalar Lagrange density of the form

$$\mathcal{L}_\phi = \frac{1}{2}\left(\frac{\partial \phi^{(a)}(\underline{x},t)}{\partial t}\right)^2 - \frac{1}{2}\left(\vec{\nabla}\phi^{(a)}(\underline{x},t)\right)^2 - U\left(\phi^{(a)}(\underline{x},t)\right) \tag{14}$$

where 'a' labels the members of a finite set of scalar fields and $\underline{x}$ is a D-dimensional position vector. The classical energy of an (allowed) static field configuration is

$$E_C[\,\phi^{(a)}\,] = E_T + E_V \tag{15}$$

where $E_T = \int d^D x \, \frac{1}{2}\left(\vec{\nabla}\phi^{(a)}\right)^2$ and $E_V = \int d^D x \, U(\phi^{(a)})$

By definition, the set of static solutions to the field equations are

precisely those field configurations which make $E_c[\phi^{(a)}]$ stationary. Consequently if we dilate all D space coordinates by a scale factor K^{-1}, it then follows that $E_c[\phi^{(a)}]$ will be stationary with respect to variations in K at K=1. A simple change of integration variables shows that

$$E_c\left[\phi^{(a)}\left(\frac{x}{K}\right)\right] = K^{D-2}E_T + K^{D}E_V \qquad (16)$$

requiring that

$$\left.\frac{dE_c}{dK}\right|_{K=1} = 0 \quad \text{and} \quad \left.\frac{d^2E_c}{dK^2}\right|_{K=1} > 0 \qquad (17)$$

yields the following inequality

$$(D-2)E_T < 0 \qquad (18)$$

provided $U(\phi^{(a)})$ is monotonic increasing. Hence for $D \geq 2$ we have shown that $E_c[\phi^{(a)}]$ is sitting on top of a dilation maximum and is consequently unstable against small dilatory perturbations.

Derrick's theorem looses its power when we consider infinite energy solutions. Consider, for example, the following complex Goldstone model. Here the Lagrange density is

$$\mathcal{L}_D = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi^* + \frac{m^2}{2}|\phi|^2 - \frac{\lambda}{4}|\phi|^4 \qquad (19)$$

with

$$\phi = \phi_1 + i\,\phi_2 = |\phi|\exp[i\,\psi]$$

If $m^2 > 0$ then we have an infinitely degenerate ground state due to the $U(1)$ continuous symmetry. Associated with the spontaneous breakdown of this abelian symmetry is the presence of a massless Goldstone boson. Static vortex solutions are obtainable from eqn. (19) as can be seen by putting

$$|\phi| = f(r) \qquad \text{and} \qquad \chi = 2\pi n\,\theta \qquad (20)$$

and solving the corresponding field equations near to and far away from the origin. Unfortunately such solutions have infinite energy, due to the infinite range of the massless Goldstone particle. It should be noted that n in eqn. (20) must be nonzero in order to obtain vortex solutions. Obviously to obtain static vortex solutions of finite energy we must remove the massless agent of symmetry breakdown. This may be done by using the Higgs-Kibble mechanism. Here the massless vector field swallows the Goldstone boson and acquires mass, and with it an extra degree of freedom. In conclusion, we see that we are led to consider the Higgs model as the simplest prototype for the construction of static vortex solutions. Nielsen and Olesen picked the Higgs model for totally different reasons. Essentially they noticed that, in the static limit, the equations of motion obtained from the Higgs Lagrange density were formally the same as the Landau-Ginzburg equations of superconductivity.[9] This later pair of equations are well known to have vortex solutions-- the so-called Abrikosov vortex-lines. Hence they knew before they started that the Higgs model has the desired vortex solutions. We will now go on to present a brief résumé of the so-called bag models of hadrons.

1.3 Bag Models

We will motivate the introduction of bag models by considering a simple non-relativistic model, which although invented some time ago is just as successful phenomenologically as the more recent and sophisticated models. It is due to P. Bogoliubov.[10] The idea is to solve the Dirac equation for a quark whose mass is given by

$$\begin{aligned} m &= 0 \qquad r < R_0 \\ &= M \qquad r > R_0 \end{aligned} \tag{21}$$

and M will eventually be taken to infinity. The wave function has to be continuous at $r=R_0$. By demanding that the system be spherically symmetric, we endow the quark with a total angular momentum of $\frac{1}{2}$. This potential obviously has the merit of quark confinement. By filling the bag with the required number of quarks, one calculates (for example) the proton magnetic moment to be

$$\mu_p = 2.79 \tag{22}$$

This value is good to within 15%. The bag radius, R_0, is determined by the proton mass. However, even though this simple model for a hadron is successful phenomenologically, it suffers from two major setbacks. One is that the model is nonrelativistic, the other being that the potential is given in a purely ad-hoc manner.

The philosophy of the bag modellists is to correct these two deficencies. It turns out that the energy of a quark in our simple non-relativistic model is inversely proportional to the bag radius R_0. Consequently the system is really unstable. At this point one may make an apt analogy with cosmological models. Here we want the bag size to be constant in time, the exact opposite of what we

want the universe to be. The way to construct a static Einstein universe is to add the so-called cosmological term to the energy-momentum tensor. It is precisely this feature which is introduced in the MIT bag model[3], to which we now turn.

To expound the philosophy of the MIT approach, we will (for simplicity) consider a massless scalar field confined to a region of space R. The energy-momentum tensor is taken to be

$$T_{\mu\nu}(x) = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}(\partial_\alpha\phi\,\partial^\alpha\phi)\,g_{\mu\nu} - B\,g_{\mu\nu} \qquad (23)$$

Here B is the "bag constant" and has dimensions of (mass)^4. Inside the region R, the scalar field obeys the massless Klein-Gordon equation and we have

$$\partial_\mu T^{\mu\nu} = 0 \qquad (24)$$

By integrating eqn. (24) over the hypertube traced out by R we find

$$0 = \int d^4x\,\partial^\mu T_{\mu\nu} = \int_{t_1}^{t_2} dt \frac{d}{dt}\int_R d^3x\, n^\mu T_{\mu\nu} + \int_S dS\, n^\mu T_{\mu\nu} \qquad (25)$$

where n_μ is the spacelike normal four-vector of the bag surface, S being the spatial boundary of the bag.

Hence if we want well-defined momenta we must have

$$n_\mu T^{\mu\nu}(x) = 0 \quad \text{on } S \qquad (26)$$

This condition will be satisfied if

$$n_\mu \partial^\mu\phi = 0 \quad \text{on } S \qquad (27)$$

and $B + \frac{1}{2} \partial^\alpha \phi \partial_\alpha \phi = 0$ on S (28)

In this formalism, the bag surface $\chi_\mu(\tau, \eta_1, \eta_2)$ is not a dynamical variable and because of this the canonical quantization of the model is very complicated. In fact, in three spatial dimensions, no general solution to the quantized bag equations has been found, although one situation of phenomenological interest has been worked out. Here one has a static and spherically symmetric bag. One finds that the spherical bag of finite radius is a stable configuration. So we see that the bag term induces stability and explains the introduction of our potential, eqn. (21). In other words, a fully relativistic version of our earlier non-relativistic model demands energy exchange between the bag and field. The spherical proton bag yields the result $M_p = 2 \cdot b$ (29)

On this note we shall end our brief discussion of the MIT bag and we will now go on to describe its main competitor, the SLAC 'shell'.[4]

In the SLAC approach the potential of eqn. (21) is generated in a more conventional manner by making use of the increasingly popular idea of spontaneous symmetry breakdown. To illustrate their methods, consider a field theory of massless quarks bound together by a scalar 'glue' such that the quark-gluon interaction term is $\mathcal{L}_{int} = g \bar{\psi} \phi \psi$

Let the gluonic potential $V(\phi)$ have several minima situated at $\phi = \phi_i$. Then expanding around the ith minimum generates a fermion mass $m_i = g \phi_i$ (30)

Hence by different choices of the potential we may generate different masses for the quarks, depending on the expansion point of the potential. For example, for the case of a potential with two minima one can have massless and very massive quarks. So, if in different regions of space, one solution is preferred to another then one is led to a natural explanation of quark confinement. To be specific,

consider a potential with two minima, situated at $\pm \phi_0$. If both g and ϕ_0 are large, then the corresponding quark mass will be large. Let us now look at the case where the vacuum expectation value of ϕ changes continuously from $+\phi_0$ to $-\phi_0$ over a very small spatial domain. Continuity of $\langle \phi \rangle$ implies that the quark mass within this region be very small. Consequently the quarks will, for energy reasons, be contained within this (assumed spherically symmetric) shell.

In this model free quarks can exist, albeit with very high mass. The phenomenological result for the proton magnetic moment is

$$\mu_p = 3/2m \qquad (31)$$

where M is the average 56 mass. This completes our brief survey of models for extended hadrons. We will now give a short description of the layout of the rest of the thesis, which will be predominantly concerned with the qualitative changes that quantization induces in strongly coupled field theoretic systems, such as bags and strings.

In chapter two we give three examples of spatially extended systems (all described within the framework of local field theory). They are the Nielsen-Olesen dual vortex, the SLAC bag and the Creutz local field theory model for the MIT bag[11]. Chapter three contents itself with a presentation of two general methods for dealing with the strong coupling problem in quantized field theory. These are the Wentzel-Tomonaga-Pauli finite mode approximation[12] and the Rosen Gaussian functional variational procedure[13] (which is also shown to be equivalent to Coleman's normal-ordering variational calculation[14]). Chapter four contains various attempts at quantising the three systems encountered in chapter two. Chapter five contains the conclusion of this research programme.

CHAPTER II

CLASSICAL ANALYSIS OF SOME SIMPLE MODELS OF SPATIALLY EXTENDED HADRONS

Preamble

This chapter will be concerned with the classical discussion of three simple models of spatially extended hadrons. The first section deals with the Nielsen-Olesen dual vortex. Here we show how string-like solutions to Higgs-type field theories can come about. In the second section we present a quantitative and qualitative study of the SLAC 'bag'. In this case we show how shell-like field configurations naturally emerge from a detailed study of the equations of motion of the theory. The final section of this chapter deals with the Creutz local field theory model for the MIT 'bag', the aim being to construct a non-local bag state from a conventional local field theory (all manipulations being performed at the tree level). In this chapter we shall treat the words, classical and semiclassical, as synonymous.

2.1 Classical Analysis of the Nielsen-Olesen 'dual' Vortex

For reasons given earlier, Nielsen and Olesen decided to construct a string-like solution to the Higgs model and to then identify this object (the so-called 'vortex solution') with a dual string. This section will be devoted to a detailed exposition of their ideas. To start off with, the Higgs Lagrange density is

$$\mathcal{L}_0 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} |(\partial_\mu + ieA_\mu)\phi|^2 + C_2 |\phi|^2 - C_4 |\phi|^4 \qquad (32)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{and} \quad \phi = \phi_1 + i\phi_2$$

The equations of motion derived from eqn. (32) are

$$(\partial_\mu + ieA_\mu)^2\phi = -2C_2\phi + 4C_4\phi^2\phi^* \tag{33}$$

$$\partial^\nu F_{\mu\nu} \equiv j_\mu = \frac{ie}{2}\,\phi^* \overleftrightarrow{\partial_\mu}\phi + e^2 A_\mu |\phi|^2 \tag{34}$$

We now look for vortex solutions to the above equations. Now as we are going to identify a vortex line with a dual string, it seems sensible to require that the flux be quantized. This is because we can then attach a simple meaning to the tensor field $F_{\mu\nu}$: the field F_{12} measures the number of vortex lines going in the 3-direction per unit square in the (12)- plane. It is simple to show that this is indeed the case. The flux Φ is given by

$$\Phi = \int F_{\mu\nu}\, d\sigma^{\mu\nu} = \int A_\mu(x)\, dx^\mu \tag{35}$$

where $d\sigma_{\mu\nu}$ is a two-dimensional surface element in Minkowski space. Writing

$$\phi = |\phi| \exp[i\chi] \tag{36}$$

we obtain from eqn. (34) the result

$$A^\mu = \frac{1}{e^2}\frac{j^\mu}{|\phi|^2} + \frac{1}{e}\partial^\mu\chi \tag{37}$$

We now perform the integration in (35) around a closed loop which

does not contain any current. We find

$$\Phi = \frac{1}{e} \int \partial_\mu \Psi \, dx^\mu \qquad (38)$$

which has the general solution

$$\Phi = n \Phi_0$$

with $$\Phi_0 = \frac{2\pi}{e}$$

Hence, the flux of vortex lines is quantised, the quantum being $\frac{2\pi}{e}$. We will now show that the equations of motion (33) and (34) permit a string like solution. Consider the static case with the gauge choice $A_0=0$. We look for a cylindrically symmetric solution, the axis of symmetry being the z-axis. We write

$$\vec{A}(\underline{r}) = \frac{\underline{r} \wedge \underline{e}_z}{|\underline{r}|} A(r) \qquad (39)$$

where $\underline{e}_z$ is a unit vector along the z-direction. The flux is given by

$$\Phi(r) = 2\pi r A(r) \qquad (40)$$

so that

$$H(r) = \frac{1}{2\pi r} \frac{d}{dr} \Phi(r) = \frac{1}{r} \frac{d}{dr} (r A(r)) \qquad (41)$$

With cylindrical symmetry about the z-axis the equations of motion (33) and (34) give (with n = 1)

$$-\frac{1}{r}\frac{d}{dr}\left(r\frac{d|\phi|}{dr}\right)+\left[\left(\frac{1}{r}-eA\right)^2-2c_2+4c_4|\phi|^2\right]|\phi|=0 \qquad (42)$$

$$-\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(rA)\right)+|\phi|^2\left(Ae^2-\frac{e}{r}\right)=0 \qquad (43)$$

No one has yet succeeded in finding an exact analytical solution to the above pair of equations. Consequently we will be content with a solution of the form

$$|\phi| \to \text{constant} \quad \text{(for large r)} \qquad (44)$$

If we take $|\phi|$ to assume a constant value everywhere then equation (43) can be solved exactly. One obtains

$$A(r)=\frac{1}{\varepsilon r}+\frac{C}{\varepsilon}K_1(\varepsilon|\phi|r) \qquad (45)$$

where C is a constant of integration which will be evaluated later. Equation (45) has the asymptotic form

$$A(r)\xrightarrow{r\to\infty}\frac{1}{\varepsilon r}+\frac{C}{\varepsilon}\sqrt{\frac{\pi}{2\varepsilon|\phi|r}}\,e^{-\varepsilon|\phi|r} \qquad (46)$$

$$+ \text{ lower order terms}$$

The corresponding value of the H field may be now evaluated using equation ( ). One finds

$$H(r) = c|\phi| K_0(e|\phi|r) \qquad (46)$$

$$\xrightarrow[r\to\infty]{} \frac{c}{e}\sqrt{\frac{\pi|\phi|}{2er}}\, e^{-e|\phi|r} \qquad (47)$$

+ lower order terms

One sees that equation ( ) is then approximately satisfied if

$$|\phi| = \sqrt{\frac{C_2}{2C_4}} \qquad (48)$$

provided that C_2 and C_4 are large enough to take care of deviations of A(r) from 1/er.

We now define a characteristic length λ by

$$\lambda = \frac{1}{e|\phi|} = \sqrt{\frac{2C_4}{e^2 C_2}} \qquad (49)$$

From eqn. ( ) we see that λ measures the region over which the H-field is appreciably different from zero. To compute the spatial variation of $|\phi|$ we note that eqn. ( ) gives the minimum value of the potential. In other words

$$|\phi| = \phi_0 = \sqrt{\frac{C_2}{2C_4}} \qquad (50)$$

is the vacuum-value of the field $|\phi|$. Consequently we decompose $|\phi|$ as follows

$$|\phi| = \phi_0 + \rho(r) \qquad (51)$$

where $\rho(r)$ measures the fluctuations of $|\phi|$ around its vacuum value. Substituting eqn. (51) into eqn. (42) gives us a Yukawa-type solution for large r

$$\rho(r) \sim \exp\left[-\sqrt{2C_2}\, r\right] \qquad (52)$$

whilst for small r we have

$$|\phi| \sim r\, e^{i\theta} \qquad (53)$$

We now define another characteristic length K

$$K = \frac{1}{\sqrt{2C_2}} \qquad (54)$$

Obviously K measures the distance that it takes before the $|\phi|$ - field assumes its vacuum value. It is easy to see that λ and K must be of the same order of magnitude if we are to have a well-defined vortex line. The vacuum state of our theory is defined by H=0 and $|\phi| = \phi_0$. When the interactions are turned on, the string manifests itself with a width given by $\lambda \sim K$. In conclusion, we have demonstrated that the Higgs Lagrangian allows a string-like solution. The constant of integration C introduced in eqn. (45) can be evaluated by requiring that the flux $\Phi(r) = 2\pi r A(r)$ tends to zero for $K \ll r \ll \lambda$. Hence using the relation

$$K_1(e|\phi|r) \simeq \frac{1}{e|\phi|r} \quad \text{for } e|\phi|r \ll 1 \qquad (55)$$

we find that

$$C = -e|\phi| \qquad (56)$$

Immediately, we are lead to ask: how does our 'string' move, as a consequence of the time-dependent field equations? The answer to this question is as yet unknown, although in the case of a sufficiently thin string of not too large curvature it is very plausible that the string will move roughly according to the equations of motion resulting from the Nambu action (eqn. ( )). This is because the Nambu action is the only parametrization invariant action one can construct without using higher derivatives.

The Lagrangian we have been discussing is the one used by Higgs to illustrate the so-called Higgs mechanism . He was able to show that, by picking an appropriate gauge, the theory could be revealed for what it really was- namely a theory of massive scalar and vector mesons interacting with one another. The gauge choice he made was one where the phase of the charged scalar field became everywhere zero. Making this gauge choice and writing

$$\phi(x) = \phi_0 + \rho(x) \qquad (57)$$

where $\phi_0 = \sqrt{\frac{c_2}{N\,2c_4}}$ we immediately find that

$$\mathcal{L}_1 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2}(\partial_\mu \rho)^2 + \frac{e^2}{2} A_\mu^2 (\phi_0 + \rho)^2 + \frac{c_2}{2c_4} - 2c_2\rho^2 - 2\sqrt{2c_2 c_4}\,\rho^3 - c_4\rho^4 \qquad (58)$$

From the above we may read off the masses of the scalar (M_s) and

vector (M_v) mesons

$$M_V = g\dot{\phi}_0 = g\sqrt{\frac{C_2}{2C_4}} \qquad (59)$$

$$M_S = \sqrt{2C_2} \qquad (60)$$

The interesting point to be made now is that the Compton wavelengths of the above two mesons give the two characteristic lengths of the string i.e.

$$\lambda = M_V^{-1} \quad \text{and} \quad K = M_S^{-1} \qquad (61)$$

We now need to tie this analysis up with the formal theory of the dual relatavistic string. According to the 'classic' paper of Goldstone, Goddard, Rebbi and Thorn[15] the energy density along the string is given by

$$\text{energy density} = \frac{\gamma_\perp}{2\pi\alpha'} = \frac{1}{2\pi\alpha'(1-V_\perp^2)^{1/2}} \qquad (62)$$

where $V_\perp$ is the transverse velocity of the string. We are now in a position to make a classical evaluation of the functional dependence of the universal Regge slope α' on the three parameters C_2, C_4 and g which appear in the Higgs Lagrangian. This may be achieved by simply calculating the energy density at rest for the vortex solution given earlier and comparing it to the energy density of the dual string given above (evaluated when the string is also at rest, of course). The magnetic energy-density along the vortex is given by

$$\frac{1}{2}\int_0^\infty \mathcal{H}^2(r)\, 2\pi r\, dr = \frac{C_2}{C_4}\frac{\pi}{2}\int_0^\infty K_1^2(y)\, y\, dy \qquad (63)$$

and the later integral is of order unity. A crude estimate of the energy per unit length of the string due to the fluctuating scalar field $|\phi|$ is given by a term whose order of magnitude is

$$K^2 C_2 \phi_0^2 \sim \phi_0^2 \qquad (64)$$

since $C_2 \frac{1}{r_0^2}$ is the relevant energy density. Hence the comparison of the energy density of the stated vortex line with that of a static dual string yields the relation

$$\frac{1}{2\pi\alpha'} \sim \phi_0^2 \sim \frac{C_2}{C_4} \qquad (65)$$

The characteristic length of a hadron is given in the dual formalism by $\sqrt{\alpha'}$ which in our vortex-line language becomes

$$\sqrt{\alpha'} \sim \sqrt{\frac{C_4}{C_2}} \qquad (66)$$

But in order to have strings at all we need to have this characteristic length much greater than the width of our vortex state i.e.

$$\sqrt{\alpha'} \gg K, \lambda \qquad (67)$$

This implies that both $\frac{C_4}{C_2}$ and $\sqrt{\frac{C_4}{C_2}}$ are both much greater than unity i.e. we have a strongly-coupled system. These last two requirements may also be compactly written as

$$M_V, M_S \gg \frac{1}{\sqrt{\alpha'}} \qquad (68)$$

which means that the particles corresponding directly to the local

fields have masses M_V and M_S much larger than typical hadron masses. Here the trouble starts. This is because we have arrived at a situation where both coupling constants in our theory are very big, which tends to make us suspicious of our considerations, as they were based entirely on classical field theory. Another way of saying the same thing is by stating that the limit we have arrived at is superquantum-mechanical. This means that the typical action associated with our limit is much smaller than Planck's constant $\hbar$. This is at present unclear because we set $\hbar = 1$ in our analysis.

To make a crude estimate of the typical action corresponding to our vortex state we assume that the width λ of the vortex is of the same order of magnitude as the vortex length $\frac{1}{\sqrt{\alpha'}}$. We now have

Typical action= vortex line energy X characteristic time

$$\sim \frac{\lambda}{\alpha'} \frac{1}{M_V} \tag{69}$$

$$\sim \frac{1}{e^2} \ll 1 \tag{70}$$

in the strong coupling limit.

So we see that from the view of the strong coupling limit, our typical quantum of action is very much smaller than Planck's constant. Hence our theory is, in fact, very far removed from its classical limit and is extremely quantum mechanical in nature. Following on from this, we see that the quantization of a vortex state such as we have been considering will be, in general, a very difficult problem.

2.2 Classical Analysis of the SLAC 'bag'

Before proceeding directly to a purely mathematical analysis of the SLAC 'bag', we shall present the basic idea of their approach by considering a simple intuitive picture of a quark bound state. The basic idea is to show how the strong interaction of an elementary quark (fermion)-field with a self-coupled scalar field can lead to a low mass bound state. Consider a quark described by a wave function ψ interacting with a neutral scalar field σ with the Hamiltonian

$$H = \int d^3x \, \mathcal{H}_0(x)$$

$$\mathcal{H}_0(x) = \psi^\dagger \left(\frac{\alpha}{i} \cdot \nabla \right) \psi + g \bar{\psi} \psi \sigma + \frac{1}{2} |\nabla \sigma|^2 + \frac{1}{2} |\nabla \sigma|^2 + H (\sigma^2 - f^2)^2 \qquad (71)$$

where $g, H \gg 1$ are large dimensionless coupling constants and f has the dimensions of a mass. It is seen that the quartic self-interaction term of the σ-field permits invariance of the theory under the discrete transformation $\sigma \to -\sigma$. In the language of quantum field theory, eqn. (71) describes a theory with spontaneous symmetry breaking, the scalar field having a non-vanishing vacuum expectation value. When in the vacuum state, σ takes on one of two values, $\pm f$. Small vibrations about one or other of these two vacuum states are studied by making the field translation $\sigma \to \sigma' + f$. It is seen that the small σ-vibrations have a mass $m_\sigma^2 = 8 H f^2$

while the small ψ-vibrations have a mass $M_\psi = 4f$. By assumption, the bare quark mass is

$$M_0 = 8f >> 1 \text{ GeV} \qquad (72)$$

We consider the specific choice of Hamiltonian, eqn. (71), as one within a wide class of renormalizable field theories exhibiting spontaneous breakdown.

In order to develop an intuitive picture of nonperturbative solutions to the field equations obtained from eqn. (71) we will tackle this problem from the purely classical viewpoint.

In the one-fermion sector, the charge

$$Q = \int d^3x \, \psi^\dagger \psi \qquad (73)$$

has unit eigenvalue and we are solving a Dirac equation for the quark in the presence of a scalar potential σ. As is usual, the problem of negative energy states rears its head. To counter this, we simply specify that all the negative energy states (in the presence of the scalar potential) are filled. We can then focus our attention on the lowest positive energy eigenvalue.

We now proceed classically with Q=1. By looking at the energy of our system, eqn. (71), we are led to expect that the quark wave-function and the field amplitude will tend to avoid one another as shown in fig.(1).

The significance of this high mass energy effect, eqn. (72), increases as the magnitude of the quark mass increases. Simultaneously, of course, there are other effects which work against the

formation of such a 'hole' into which the quark traps itself. These are the energies associated with the curvatures of the localized quark wave-function and the σ-field and the energy associated with the potential term $H(\sigma^2-f^2)$ extending over the volume where $\sigma \simeq \pm f$. We will illustrate these qualitative remarks by considering the simple potential of fig. 1 . We denote by D the thickness of the shell in which the σ-amplitude falls from +f to 0. The σ-field effectively vanishes inside a region of radius R. The contributions to the total energy, eqn. (71), are

$$\int d^3x \; \psi^\dagger \frac{\underline{\alpha}\cdot\underline{\nabla}}{i} \psi \sim \frac{1}{R} \tag{74}$$

$$\int d^3x \; \frac{1}{2} |\underline{\nabla}\sigma|^2 \sim \frac{1}{2}\left(\frac{f}{D}\right)^2 4\pi R^2 D \tag{75}$$

$$\int d^3x \; H(\sigma^2-f^2) \sim Hf^4\left[\frac{4\pi}{3}R^3 + 4\pi R^2 D\right] \tag{76}$$

Estimate (74) follows from the uncertainty principle and $k \sim 1$ is a shape-dependent number. Consequently the total energy of this configuration is given by

$$E(R,D) \sim \frac{1}{R} + 2\pi R^2 \frac{f^2}{D} + Hf^4\left[\frac{4\pi}{3}R^3 + 4\pi R^2 D\right] \tag{77}$$

We now minimize this expression with respect to R and D to obtain the dependence of the lowest energy state on these parameters. Now

$$\frac{\partial E}{\partial D} = 0 \Rightarrow D \sim \frac{1}{\sqrt{H f}} \qquad (78)$$

whilst

$$\frac{\partial E}{\partial R} = 0 \Rightarrow R \sim \frac{1}{\sqrt{H f}} \quad \text{if } \sqrt{H f} \gg R^{-1} \qquad (79)$$

i.e. if the volume energy dominates over the surface energy. Consequently the lowest energy state is given by

$$E = \min E(D, R) = \frac{4}{3R} \sim f^{\frac{1}{2}} H^{\frac{1}{2}} \qquad (80)$$

In this particular case we note that $\frac{D}{R} \sim H^{-1/4} \ll 1$ (81) which is consistent with a thin transition shell region in the strong coupling limit. Comparing this with eqn. (72) we note that a localized bound state is formed if $\frac{D}{R} \gg H^{1/4}$ and by eqn. (74) we see that we are in the strong coupling regime. A more careful and systematic treatment of Hamiltonian (71) shows us

that (classically) we have a thin shell model of a hadron where the σ-field changes rapidly from +f to -f in a region of thickness $D \sim H^{-1/2} f^{-1} \ll R$ and with the quark confined to a thin shell centered at r=R with a spread of $g^{-1} f^{-1}$. In this case we find that the energy goes like $H^{1/2} f$ rather than the behaviour $H^{1/4} f$ found in eqn. ( ). The more accurate solution is illustrated in fig. ( ).

We have just seen how a heuristic semiclassical discussion of the Hamiltonian ( ) suggests the possible existence of bound states with masses much lower than the bare masses of the constituents. We will now show how such a semiclassical picture can emerge from a canonical quantum field theory. As our theory is a strong-coupling theory, a non-perturbative approach is essential.

The equations of motion which result from eqn. ( ) are (in the static limit)

$$\nabla^2 \sigma - \tfrac{1}{4} H \sigma (\sigma^2 - f^2) = g \bar{\psi} \psi \qquad (82)$$

$$\left(\frac{\alpha \cdot \nabla}{i} + \beta g \sigma \right) \psi = \varepsilon \psi \qquad (83)$$

Here ε appears as a Lagrange multiplier because we have normalized our ψ states to unity

$$\int d^3x \; \psi^\dagger \psi = 1 \qquad (84)$$

We now set about solving this coupled set of differential equations.

An exact solution has, up to the present time, not been found. However it is relatively easy to obtain the leading order solution, together with the order of magnitude of the small corrections, in the strong coupling limit. As we are looking for the lowest energy state, we expect the classical field $\sigma(x)$ and its source $\bar{\psi}(x)\psi(x)$ to be spherically symmetric. The equations we are now lead to consider are

$$\left(\frac{\underline{\alpha}\cdot\underline{\nabla}}{i} + \beta g\sigma\right)\psi = \varepsilon\psi \qquad (85)$$

$$\frac{d^2\sigma}{dr^2} + \frac{2}{r}\frac{d\sigma}{dr} - \frac{1}{4}\mu^2\sigma(\sigma^2 - f^2) = g\bar{\psi}\psi \qquad (86)$$

Following a procedure developed by Lee and Wick[15], we solve eqn. (86) first by neglecting the $\bar{\psi}\psi$ source term and $\frac{2}{r}\frac{d\sigma}{dr}$.

So we are left with

$$\frac{d^2\sigma}{dr^2} - \frac{1}{4}\mu^2\sigma(\sigma^2 - f^2) = 0 \qquad (87)$$

The solution to this equation is

$$\sigma(r) = f\tanh\left[\frac{1}{2\sqrt{2}}\mu f(r-R)\right] \qquad (88)$$

Here one of the integration constants has been chosen so that $\sigma(x)$ assumes the value +f at large distances whilst the other constant,

R, will be chosen later so as to minimize the total energy. It will be shown later that the two neglected terms cancel "on the average".

The precise form of the Dirac wave function ψ in the transition region depends on the relative magnitudes of q and H. However the total energy of our state (together with the optimum choice of R) is governed by H alone. To clarify these statements we shall now consider the following two extreme cases:

$$\text{(i) } \sqrt{H} \gg q \gg 1 \quad \text{and} \quad \text{(ii) } q \gg \sqrt{H} \gg 1 \tag{89}$$

(i) $\sqrt{H} \gg q \gg 1$

In this case we may validly replace $\sigma(x)$ by a square well potential. Hence we are lead to solve the Dirac equation in the following potential

$$\begin{aligned} \sigma(x) &= +f \qquad r > R \\ &= -f \qquad r < R \end{aligned} \tag{90}$$

The standard method for solving this problem starts by making the following decomposition

$$\psi = \begin{pmatrix} \dfrac{i\, g_j(r)}{r}\, \phi_{jm} \\ \dfrac{F_j(r)}{r}\, \dfrac{\sigma \cdot r}{r}\, \phi_{jm} \end{pmatrix} \tag{91}$$

Consistency of this form with eqn. (91) can only be achieved by taking

$j = \frac{1}{2}$ (or $l = 0$). From now on we will restrict ourselves to the case $l = 0$. The radial wave-functions g_0 and F_0 satisfy the following equations

$$\frac{dg_0}{dr} = \frac{1}{r} g_0 + (q_0 \sigma + \varepsilon) F_0 \qquad (92)$$

$$\frac{dF_0}{dr} = -\frac{1}{r} F_0 + (q_0 \sigma - \varepsilon) g_0 \qquad (93)$$

The solutions to these equations in the limit $q_f \gg \varepsilon$ are

$$g_0 = A \sinh(\lambda r)$$

$$F_0 = -\frac{A\lambda}{\varepsilon - q_f}\left[\cosh(\lambda r) - \frac{\sinh(\lambda r)}{\lambda r}\right] \qquad (94)$$

both for $r < R$

And

$$g_0 = B e^{-\lambda r}$$

$$F_0 = -\frac{B\lambda}{\varepsilon + q_f}\left[1 + \frac{1}{\lambda r}\right] e^{-\lambda r} \qquad (95)$$

for $r > R$

In the above $\lambda = (q_f^2 - \varepsilon^2)^{1/2}$ and the energy eigenvalue ε

$$\varepsilon = \frac{1}{R} \qquad (96)$$

is determined by the requirement of the continuity of g_0/f_0 at r = R. The normalization condition determines

$$A = \frac{\beta e^{-\lambda R}}{\sinh \lambda R} = \sqrt{\frac{\lambda}{2\pi}}\, e^{-\lambda R} \qquad (97)$$

Note that the quark wave function ψ is concentrated in the region r $\lesssim$ R. By direct evaluation we find that

$$\psi^\dagger \psi = \frac{\lambda}{4\pi R^2}\, e^{-\lambda |r-R|} \qquad (98)$$

$$\bar{\psi}\psi = \frac{1}{\lambda R}\left[\left|1 - \frac{r}{R}\right| + \frac{1}{2\lambda R}\right]\psi^\dagger\psi \qquad (99)$$

These two quantities are only appreciably different from zero when they are near the transition region, see fig. (3) .

We now have to consider the effect of including the source term. By requiring that the two neglected terms vanish on the average we obtain a condition for R. This condition comes about by multiplying eqn. (96) by $\frac{dr}{dr}$ and integrating with respect to r over the transition region. We find

$$\int dr\, \frac{2}{r}\left(\frac{dr}{dr}\right)^2 = \int dr\, \frac{dr}{dr}\, g\bar{\psi}\psi \qquad (100)$$

By making use of eqns. (88), (93) and (99) we find that eqn. (100)

gives

$$\frac{1}{R_1^2} = \frac{8}{3}\pi\sqrt{\kappa H}\,\rho^3 \qquad (101)$$

which is independent of ρ_1 (as stated earlier). The value of R so obtained has the merit of minimizing the total energy. To see how this comes about we take the derivative of eqn. (71) with respect to R. We find

$$\frac{\partial E}{\partial R} = 8\pi R\int dr\left(\frac{d\sigma}{dr}\right)^2 + \frac{\partial \mathcal{E}}{\partial \rho_1} \qquad (102)$$

Using eqn. (93) we obtain

$$\frac{\partial \mathcal{E}}{\partial R} = -4\pi R^2 q\int dr \frac{d\sigma}{dr}\,\bar{\psi}\psi \qquad (103)$$

Hence $\frac{\partial E}{\partial R} = 0$ implies that

$$\frac{2}{R_1}\int dr\left(\frac{d\sigma}{dr}\right)^2 = q\int dr\,\frac{d\sigma}{dr}\,\bar{\psi}\psi \qquad (104)$$

which is the same as eqn. (100)

(ii) $q \gg \sqrt{\kappa H} \gg 1$

It turns out that in this case the Dirac wave-function ψ is

still given by eqns. (98) and (99) when $|r-R| \gg (2H)^{-1/2} f^{-1}$. However in the transition region changes take place. In terms of the variables $u_\pm$ where

$$u_\pm = g_0 \pm F_0 \tag{105}$$

$$u_+ = \rho u_- \tag{106}$$

we see that eqns. (92) and (93) become

$$\frac{du_-}{dr} = -g\sigma u_- + \left(\frac{1}{r} + \varepsilon\right)\rho u_- \tag{107}$$

$$\frac{d\rho}{dr} = 2g\sigma\rho + \left(\frac{1}{r} - \varepsilon\right) - \rho^2\left(\frac{1}{r} + \varepsilon\right) \tag{108}$$

For a solution which is +f outside the well ($r > R$) and -f inside the well ($r < R$), we see that ρ = +1 at r = 0 and rapidly decreases away from the origin. Simultaneously $u_-(r)$ increases in an exponential fashion as r approaches the bag radius R. Because of these facts we only have to solve our set of equations away from the origin where they reduce to

$$\frac{du_-}{dr} = -g\sigma u_- \tag{109}$$

$$\frac{d\rho}{dr} = 2g\rho + \left(\frac{1}{r} - \varepsilon\right) \tag{110}$$

The first equation can be solved immediately to give

$$\underline{U} = C \left\{ \cosh\left[\sqrt{2H}\, f(r-R)\right] \right\}^{-\frac{q}{\sqrt{2H}}} \qquad (111)$$

For the second equation to have a stable solution when $\sigma(x)$ changes sign we must have $\frac{1}{r} \approx \mathcal{E}$. Hence the quark energy eigenvalue is given by

$$\mathcal{E} = \frac{1}{R} \qquad (112)$$

Note that the two functions $U_\pm$ vary much more rapidly than $\sigma(x)$. The half-width of $U_\pm$ is given by $q^{-1} f^{-1} H^{-1/2}$ which is to be compared to the half-width $q^{-1} f^{-1}$ found in the previously discussed case. Due to the smallness of the half-width it is valid to make the following approximation

$$q\sigma(x) \approx q\sqrt{2H}\, f(r-R) \quad \text{for } r \approx R \qquad (113)$$

The ρ equation can now be solved to give

$$\rho = \frac{1}{2q\sqrt{2H}\, f^2 R^2} \ll 1 \qquad (114)$$

The normalization condition for the quark wave-function is

$$\int dr\, \underline{u}^2(r) = \frac{1}{2\pi} \qquad (115)$$

Also

$$\bar{\Psi}\Psi = \frac{1}{R^2}\rho\, \underline{u}^2 = \frac{1}{2q\sqrt{2H}\, f^2 R^4}\, \underline{u}^2 \qquad (116)$$

The solutions just found are illustrated in fig. (4) and interestingly enough give the same value for R as that found previously.

In conclusion, we have proven the important result that the size and energy of our bound states are determined by H alone. This result is independent of whether $1 \ll q \ll \sqrt{H}$ or $1 \ll \sqrt{H} \ll q$. In fact this conclusion is valid in the intermediate range of parameters $q \sim \sqrt{H} \gg 1$ as well, although now the detailed shape of the wave-function is sensitive to the value of q in the transition region i.e. as $\sigma(x)$ changes from +f to -f.

The extension of these ideas to multiquark states is straight forward. The energy functional for a system of N quarks or anti-quarks is

$$E = \sum_{i=1}^{N} \varepsilon_i + \int d^3x \left[\frac{1}{2} |\vec{\nabla}\sigma|^2 + H(\sigma^2 - f^2)^2 \right] \qquad (117)$$

where the quark energies ε_i are given by the solutions of the following Dirac equations

$$\left(\frac{1}{i} \underline{\alpha} \cdot \underline{\nabla} + \beta q \sigma \right) \psi_i = \varepsilon_i \psi_i \qquad (118)$$

Similarly, $\sigma(x)$ is determined by

$$\nabla^2 \sigma - 4H\sigma(\sigma^2 - f^2) = q \sum_{i=1}^{N} \bar{\psi}_i \psi_i \qquad (119)$$

Just by looking at these two equations, we see that the solutions we will obtain from them will be identical in structure to those obtained for the single quark system. By a generalization of our discussion of the single quark system, it is easy to show that the energy of a state filled with N quarks or antiquarks (all in the ground state) is given by

$$E_N = \frac{2}{3}\frac{N}{R_N} = \frac{2}{3}\frac{N^{3/3}}{R_0} \qquad (120)$$

where R_0 is the radius of a single quark state.

Before ending this section we would like to point out that our treatment of fermion states in the SLAC bag has been somewhat lax. This is because we have treated the quarks as bosons due to the fact that fermion states have no classical counterpart. It turns out that this situation may easily be rectified by using coherent boson states and quasi-fermions. Through the use of these objects one may reduce the true quantum field theory problem to one of solving the classical equations of motion (82) and (83). As this is essentially a technical point, we demote a discussion of these considerations to an appendix.

2.3 Classical Analysis of the Creutz Model for the MIT 'bag'

Starting from an explicitly local theory we will construct states of finite spatial extent. These states will become the states of the MIT 'bag' model when the parameters of the theory

take on a specific limiting form. The vacuum state of the local theory corresponds to the case where the field vanishes everywhere. However the theory also permits another state (metastable with respect to the vacuum) which is realised for some non-zero constant value of the field. In order to form the bag state we form the metastable state in a finite region of space.[16] Well outside of this region the field will assume its stable value zero. Of course, there will be a transition region (or "skin") which smoothly connects up these two different regions. The field affects this connection in such a way that the energy per unit area of skin is minimized.

Such a state is, of course, unstable by simple energetics. So, in order to stabilize our bag we fill it with other fields (the quarks) which we shall take to carry conserved quantum numbers. The coupling between the quark field and the bag is chosen in such a manner that the effective quark mass is large outside the bag but zero inside. From this it follows that it is energetically preferable to create quarks inside the bag and not outside of it. Hence we realise quark confinement. The quark kinetic energy (which increases as the bag size decreases) balances those forces tending to make the bag collapse. We shall show that there exists a limit of the parameters of the theory such that the energy and thickness of the bag skin vanish whilst, simultaneously, all masses (except the interior quark mass) go to infinity. It is this limit which manufactures the MIT bag model. In the discussion we will now present, a single charged scalar field will be taken to mimic the quarks, whilst a neutral scalar field will represent the (vector) gluons. It can be shown that this rather alarming approximation is valid a poster-

iori. The discussion we now present involves only classical unquantized systems, the extension to the quantized case being left to a later chapter.

Our system is defined by the following Lagrange density

$$\mathcal{L}_\phi = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \partial_\mu\psi^*\partial^\mu\psi - V(\phi) - \lambda\psi^*\psi(\phi-\beta)^2 \qquad (121)$$

where $$V(\phi) = \alpha\left[\frac{\phi^4}{4} - (\beta+\gamma)\frac{\phi^3}{3} + \beta\gamma\frac{\phi^2}{2}\right] \qquad (122)$$

The parameters α, β, γ and λ are all positive. In the above $\phi(x)$ is the real scalar field which will produce the bag whilst $\psi(x)$ is the complex quark field. By implementing the following constraint

$$\frac{1}{2} < \frac{\gamma}{\beta} < 1 \qquad (123)$$

we see that $V(\phi)$ has two minima. The lowest one is situated at the origin whilst the other sits at $\phi=\beta$. The potential $V(\phi)$, under constraint (123), is shown in fig. 5.

The energy density $\mathcal{H}(x)$ corresponding to the Lagrange density of eqn. (121) is

$$\mathcal{H}(x) = \frac{1}{2}\left[\dot\phi^2 + |\nabla\phi|^2\right] + |\dot\psi|^2 + |\nabla\psi|^2 + V(\phi) + \lambda|\psi|^2(\phi-\beta)^2 \qquad (124)$$

When the quark field $\psi(x)$ is absent, the vacuum state (i.e. the state of lowest energy) is realised when $\phi(x) = 0$ everywhere but

there also exists a metastable state characterised by $\phi(x) = \beta$. The equations of motion obtained from the Lagrange density of eqn. (121) are

$$\Box\phi(x) = -\alpha\phi(x)[\phi(x)-\gamma][\phi(x)-\beta] - 2\lambda\psi^\dagger\psi[\phi(x)-\beta] \tag{125}$$

and

$$\Box\psi(x) = -\lambda\psi(x)[\phi(x)-\beta]^2 \tag{126}$$

The quark field $\psi(x)$ has a conserved current j_μ associated with it

$$j_\mu(x) = \psi^\dagger\partial_\mu\psi - [\partial_\mu\psi^\dagger]\psi \tag{127}$$

$$\partial_\mu j^\mu = 0 \tag{128}$$

and its associated charge

$$Q = \int d^3x\, j_0(x) \tag{129}$$

$$\frac{dQ}{dt} = \dot{Q} = 0 \tag{130}$$

By varying the parameters α, β, γ and λ we shall obtain the MIT bag model. The dynamics of the MIT bag is governed by the following Lagrangian[3]

$$\mathcal{L}_{eff}(t) = \int_{R(t)} d^3x\,[\partial_\mu\psi^\dagger(x)\,\partial^\mu\psi(x) - B]$$

$$+ \sum_{R(t)} \int d^3x\,[\partial_\mu\psi^\dagger(x)\,\partial^\mu\psi(x) - \mu^2\psi^\dagger(x)\psi(x)]$$

when $\mu \to \infty$ (131)

Here $R(t)$ is the region of space referred to as the "bag" and $\bar{R}(t)$ is its complement. The field $\psi(x)$ represents the quarks and corresponds to our field $\psi(x)$. The bag Hamiltonian resulting from L_{bag} is

$$H_{bag} = \int_{R(t)} d^3x \left[|\dot{\psi}(x)|^2 + |\vec{\nabla}\psi(x)|^2 + B \right] \qquad (132)$$

$$+ \int_{\bar{R}(t)} d^3x \left[|\dot{\phi}(x)|^2 + |\vec{\nabla}\phi(x)|^2 + \mu^2 |\phi(x)|^2 \right]$$

In order to obtain the theory given above in eqns. (131) and (132) from our local theory we construct bag states which are characterised by the fact that $\phi(x)$ assumes the value v inside the bag. The same field $\phi(x)$ adjusts itself outside the bag in a manner which minimizes the total energy of the system. Of course, it will be necessary to show that the spatial variation of $\phi(x)$ takes place only over a vanishingly small region of space (centred around the surface of the bag) by the parameters of our theory taking on a special limiting form. The possibility of particle creation due to the fluctuations of $\phi(x)$ about its average value inside or outside the bag is made negligible by making the effective mass of such excitations (particles) very large. The mass associated with the excitations of the quark field $\psi(x)$ is taken to be large outside of the bag although inside the bag the quarks will be massless and effectively free. We will now go on to make these qualitative considerations quantitative.

The effective mass of quark excitations outside the bag is

$$m^2_{\psi,E} = \lambda v^2 \qquad (133)$$

Hence we require

$$\Lambda_\beta^2 \to \infty \qquad (134)$$

The exterior ϕ mass gives us

$$m_{\phi,E}^2 = \alpha\beta\gamma \to \infty \qquad (135)$$

whilst the interior ϕ mass tells us that

$$m_{\phi,I}^2 = \left.\frac{d^2V(\phi)}{d\phi^2}\right|_{\phi=\beta} = \alpha\beta(\beta-\gamma) \to \infty \qquad (136)$$

Because of the constraint we imposed earlier, eqn. (133), eqns. (135) and (136) are equivalent to the one condition

$$\alpha\beta^2 \to \infty \qquad (137)$$

As we will need a term in our Hamiltonian equal to B times the bag volume we will take

$$V(\beta) = \frac{\alpha\beta^4}{6}\left(\frac{\gamma}{\beta} - \frac{1}{2}\right) \equiv B > 0 \qquad (138)$$

We have also to adjust the parameters of our theory so that the thickness and energy of the bag skin tend to zero. We proceed to make a crude estimate of these quantities by using a simple variational procedure. To do this we approximate the skin shape by a simple linear form, knowing all the while that the "true skin" will have a lower energy content. Consequently, at a particular time, we consider a cross-section of skin with $\phi(x)$ shown as in fig. 6.

The quantity Λ parametrizes the thickness of the skin. In the following we shall take $\dot{\phi}(x) = \dot{\psi}(x) = \psi'(x) = 0$ although later on we shall argue that the presence of ψ' doesn't in fact

alter things. The energy per unit area of our skin is

$$E_S(\Lambda) = \int_0^{\Lambda} dx \left[\frac{1}{2} \left(\frac{1}{1+x} \right)^2 + V(x) \right]$$

$$= \frac{1}{2} \frac{\Lambda^2}{\Lambda} + \frac{1}{12} \alpha_\beta^4 \Lambda \left(\frac{\gamma}{6} - \frac{3}{5} \right) \qquad (139)$$

Minimizing this expression with respect to Λ yields

$$\Lambda^2_{min} = \frac{b}{\alpha_\beta^2} \left(\frac{\gamma}{6} - \frac{3}{5} \right)^{-1} \qquad (140)$$

$$E_{S,min} = \left[\frac{\alpha_\beta^4 b}{6} \left(\frac{\gamma}{6} - \frac{3}{5} \right) \right]^{1/2} \qquad (141)$$

So in order to have Λ_{min} and $E_{s,min}$ both vanishingly small we require that

$$\alpha_\beta^2 \to \infty \qquad (142)$$

$$\alpha_\beta^4 \to 0 \qquad (143)$$

Note that conditions (137) and (142) are equivalent.

Inside the bag we want the quarks (the ψ-particles) to become effectively free. Now even though the ϕ excitations in the bag have a large mass they can still produce an effective coupling between the quarks in much the same way as massive intermediate vector bosons produce effective four-fermion couplings at low energy in theories of weak interactions. To avoid this taking place we require the term $\Lambda |\psi(x)|^2 \left(\frac{1}{2} \phi^2 - \epsilon \right)$ to become insignificant inside the bag and as the variations in $\phi(x)$ are parametrized by $m_{\phi,I}$ we are led to impose the following constraint

$$\frac{\Lambda}{m^2_{\phi,I}} \sim \frac{\Lambda}{\alpha_\beta^2} \to 0 \qquad (144)$$

Finally we come back to the behaviour of $\psi(x)$ in the vicinity of the skin. Note that by combining eqns. (110) and (123) we see that

$$m^2_{\gamma,E}\ \Delta^2_{min} \sim \frac{\lambda}{\alpha} \qquad (145)$$

Now as conditions (113), (143) and (144) imply that the right-hand side of eqn. (145) goes to zero in the limit we are considering we see that unless the ψ field has important contributions coming from momenta large compared to the external gluon mass $m_{\gamma,E}$ then the quarks will effectively be blind to the thickness of the gluon transition region. We still have to make sure that the quark-gluon interaction term does not alter the properties of the skin. This forces us to demand that

$$\Delta_{min} \langle \lambda \psi^*(x)\psi(x)[\phi(x)-B]^2\rangle_{skin} \ll E_S \qquad (146)$$

This condition will be satisfied if

$$\langle \psi^*(x)\psi(x)\rangle_{skin} \ll \frac{\alpha B^2}{\lambda} \qquad (147)$$

Now the right-hand side of the above inequality goes to infinity in our limit by condition (114) and since $\psi(x)$ vanishes on the skin in this same limit, condition (147) will be automatically satisfied. In conclusion, we are led to the following set of conditions

necessary for bag formation in our model

$$\lambda_\rho^2 \to \infty \qquad (147)$$

$$\alpha_\rho^2 \to \infty \qquad (149)$$

$$\frac{\gamma}{\beta} = \frac{1}{2} + \frac{dB}{\alpha_\rho^4} \qquad (150)$$

$$\alpha_\rho^4 \to 0 \qquad (151)$$

$$\frac{\lambda}{\alpha_\rho^2} \to 0 \qquad (152)$$

Conditions (H_1) and (15_1) tell us that $\beta \to 0$ which means that only a small numerical shift in the gluon field is needed in order to produce the bag state. By parametrizing α and λ in the following manner

$$\alpha = R_1 \beta^{-(p_1+1)} \qquad (153)$$

$$\lambda = R_2 \beta^{-(p_2+2)} \qquad (154)$$

(here R_1 and R_2 are two positive but otherwise arbitrary constants) the above conditions, eqns. (H_2) - (15_2), may be compactly written as

$$\beta \to 0$$

$$0 < p_1 < p_2 < 2 \qquad (155)$$

$$\gamma = \frac{B}{2}\left(1 + \frac{12B}{n_1}\beta^{n_1}\right)$$

We see that as the limit $\rho \to 0$ is taken the two minima of the gluon potential $V(\rho)$ rapidly approach one another whilst all the time they are separated by a barrier of ever increasing height. The difference in height between these two minima is given by the bag constant B .

CHAPTER III

STRONG COUPLING METHODS

IN

QUANTUM FIELD THEORY

Preamble

In this chapter we will give a general discussion of two superficially distinct variational approximation schemes which share the merit of taking into partial account the nonlinearity of the theories we are dealing with. Because of this latter fact, it is hoped that the methods we give here are superior to the naive semiclassical tree approximations used in the first half of this thesis. For example, our techniques can accommodate an infinite mass renormalization. It will become apparent that the two methods of approximation are equivalent when we consider the ground states of systems. Both techniques are phrased in the Schrodinger picture version of quantum field theory. However we shall first discuss a more primitive strong coupling scheme (due originally to Pauli, Tomonaga and Wentzel) which happens to lend itself naturally to a finite mode approximation.

3.1 The Wentzel-Tomonaga-Pauli Approximation Scheme

To illustrate the WTP approximation procedure[12] we will take a simple model field theory defined by the following Lagrange density

$$\mathcal{L}_\phi = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \frac{m^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \qquad (156)$$

It is plainly seen that the theory exhibits spontaneous breakdown of the discrete symmetry $\phi \to -\phi$. We now choose a set of (orthonormal) functions $f_n(x)$ which will be taken to describe the various modes of vibration of the field $\phi(x)$. As the whole purpose of this approximation scheme is to describe possible bound states of our system we expect that some of the $f_n(x)$ already represent bound state wave functions. Decomposing the field up into its constituent modes of vibration yields

$$\phi(x,t) = \sigma + \sum_n q_n(t)\, f_n(x) \qquad (157)$$

where the amplitudes $q_n(t)$ describe the time dependence of the system and the c-number σ is just the vacuum expectation value of the field

$$\sigma = \frac{m}{\sqrt{\lambda}} \qquad (158)$$

To second-quantize our system we require that

$$[q_n, \dot{q}_m] = i\delta_{nm} \qquad (159)$$

where

$$\dot{q}_m = \frac{dq_m}{dt} \qquad (160)$$

Obviously eqn. (159) may be realised by the following substitutions

$$q_m \rightarrow \frac{1}{i}\frac{\partial}{\partial q_m} \qquad (162)$$

The Hamiltonian corresponding to the Lagrange density of eqn. (156) is

$$H = \int d^3x \left[\frac{\dot{\phi}^2}{2} + \frac{|\nabla\phi|^2}{2} - \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 \right] \qquad (163)$$

We now substitute eqn. (157) into eqn. (163). We construct the "free" Hamiltonian

$$H_0^{(n)} = \frac{1}{2}\left[-\frac{\partial^2}{\partial q_n^2} + \sum_i \left(\nabla^{(n)}\right)^2 - m_1^2 \sum_i (q_n)^2 - m_1^2 H_0^{(n)} \sigma^2 \right.$$
$$- 2m_1^2 H_1^{(n)} \sigma q_n + 3\lambda H_0^{(n)} \sigma^4 + 6\lambda H_1^{(n)} \sigma^3 q_n$$
$$\left. + 3\lambda H_2^{(n)} \sigma^2 (q_n)^2 + 6\lambda H_3^{(n)} \sigma (q_n)^3 + \frac{\lambda}{2} H_4^{(n)} (q_n)^4 \right] \qquad (164)$$

where

$$\left[\nabla^{(n)}\right]^2 = \int d^3x \, |\vec{\nabla}\mathcal{F}|^2 \qquad (165)$$

$$H_N^{(n)} = \int d^3x \, (\mathcal{F}_n)^N \qquad (166)$$

$$H_0^{(n)} = \text{infinite spatial volume} \qquad (167)$$

$$H_2^{(n)} = 1 \qquad (168)$$

By subtracting out the vacuum energy

$$\frac{1}{2}\int d^3x\left[-\pi_1^2+\frac{\Lambda}{2}\phi^4\right]=-\frac{1}{4}\frac{\pi_1^4}{\lambda}+H_0^{(2)} \tag{169}$$

We can define the following zeroth order Hamiltonian

$$H_0=\sum_n H_0^{(n)} \tag{170}$$

and

$$H_{10}^{(n)}=\frac{1}{2}\left[-\frac{\partial^2}{\partial a_n^2}+\frac{1}{2}\left[\chi^{(n)}\right]^2+m_1^2\right](a_n)^2 + 2\lambda A_3^{(n)}(a_n)^3+\frac{\lambda}{4}H_4^{(n)}(a_n)^4\Big] \tag{171}$$

The interaction Hamiltonian H_{int} is given by the difference

$$H_{int}=H-H_0 \tag{172}$$

and it originates from the kinetic energy part

$$\frac{1}{2}\int d^3x\left[\dot{\phi}^2+\right] \tag{173}$$

and the self-coupling term

$$\frac{\lambda}{4}\int d^3x\,\phi^4 \tag{174}$$

Clearly H_{int} contains cross terms of the a_n's up to the fourth order whilst all "diagonal" products have been included in $H_o^{(n)}$.

The WTP strong coupling approximation states that if the coupling constant λ is sufficiently large, we may treat H_{int} as a (presumably small) perturbation on the (hopefully more manageable) H_o. The eigenstates and eigenvalues of H_o correspond to (approximate) physical meson states and their rest masses. The lowest energy state is the vacuum and the first excited state corresponds to one physical meson state. All the other excited states, which are stable under the approximation $H_{int} = 0$, can decay into physical mesons by the perturbation H_{int} provided such decay processes are allowed by total energy-momentum and total angular momentum conservation laws.

We now have to decide how to choose our set of orthonormal functions. Here we shall use a variational principle. In other words, we shall introduce variational parameters λ_j into the $f_n : f_n(\underline{x}; \lambda_j)$. The parameters $A_N^{(n)}$ and $\lambda^{(n)}$ in the zeroth order Hamiltonian $H_o^{(n)}$ now depend on the λ_j. Consequently, the eigenvalues $E_N(\lambda_j)$ and eigenvectors (in the Fock space) of $H_o^{(n)}$ also depend on λ_j. Hence by minimizing $E_N(\lambda_j)$ with respect to the λ_j's, approximate upper bounds on the total energy E_N (with corresponding values of the λ_j's) can be computed. In practice, of course, the inclusion of only one or two modes is sufficient to cause computational hardship.

In the next chapter we shall use this approach to quantise the MIT bag in a single mode approximation. It is seen that, rather than extending the approximation piecemeal (i.e. mode by mode), it is necessary to adopt more realistic approximation schemes that take the infinity of field modes directly into account, and it is to two such (equivalent) approximation schemes that we now turn.

3.2 Rosen's Functional Variational Scheme (13)

Before presenting Rosen's functional variational scheme, we shall first make some general comments on the functional formulation of Schrodinger picture quantum field theory. Obviously, this formulation is equivalent to the field mode representation that we have just considered.

An abstract quantum field-theoretic state $|\Psi\rangle$ may be realized by a wave-functional. For a field theory involving the field operator $\phi(x)$, the wave-functional (in the Schrodinger picture) is a functional of a c-number $\phi(\underline{x})$ and of the time t:

$$|\Psi\rangle \longrightarrow \Psi[\phi;t] \qquad (175)$$

The action of the operator $\phi(x)$ on $|\Psi\rangle$ is realized (in the so-called field representation) by multiplying $\Psi[\phi;t]$ by $\phi(\underline{x})$:

$$\phi(x)|\Psi\rangle \longrightarrow \phi(\underline{x})\Psi[\phi;t] \qquad (176)$$

The only other independent operator in the theory is the canonical momentum $\pi(x)$. The action of this operator on $|\Psi\rangle$ is realized ...

(continued overleaf)

by functional differentiation

$$\pi(x)\,|\Psi\rangle \rightarrow \frac{1}{i}\frac{\delta}{\delta\phi(x)}\,\Psi[\phi;t] \qquad (177)$$

Finally, the inner product is defined by functional integration

$$\langle\Psi_1|\Psi_2\rangle = \int \mathcal{D}[\phi]\,\Psi_1^*[\phi;t]\,\Psi_2[\phi;t] \qquad (178)$$

More precisely, we are introducing eigenstates of $\phi(x)$ at some fixed time. When these states are denoted by $|\phi\rangle$, the wave function $\Psi[\phi]$ is simply (as the time is fixed, we shall suppress it)

$$\langle\phi|\Psi\rangle = \Psi[\phi] \qquad (179)$$

Energy eigenstates satisfy the Schrodinger equation

$$\int d^3x\, \mathcal{H}\left[\frac{1}{i}\frac{\delta}{\delta\phi(x)},\,\phi(x)\right]\Psi[\phi] = E\,\Psi[\phi] \qquad (180)$$

where $\mathcal{H}[\pi,\phi]$ is the Hamiltonian density. The development in time is given by

$$\Psi[\phi;t] = e^{-iEt}\,\Psi[\phi] \qquad (181)$$

A direct solution of the functional integro-differential equation is of course impossible. Let us consider the variational principle which can be used to derive eqn. (180). We demand that

$$\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \frac{\int \mathcal{D}[\phi]\int d^3x\,\Psi^*[\phi]\,\mathcal{H}\,\Psi[\phi]}{\int \mathcal{D}[\phi]\,\Psi^*[\phi]\,\Psi[\phi]} \qquad (182)$$

be stationary with respect to arbitrary variations of the wave

functional $\Psi[\phi]$. This procedure yields the exact, but intractable, eqn. (180). To make some progress we shall now make a Rayleigh-Ritz-type of ansatz for the ground-state functional, of the form

$$\Psi_0[\phi] = \exp\left[-\int d^3x\, d^3y\, [\phi(x) - \bar{\phi}(x)]\, f(x,y)\, [\phi(y) - \bar{\phi}(y)]\right] \tag{183}$$

where the function f ($\underline{x}$, $\underline{y}$) is a symmetric distribution

$$f(x,y) = f(y,x) \tag{184}$$

We now compute the normalized expectation value of H in the trial state, eqn. (183), and we require this quantity to be stationary against arbitrary variations in $\bar{\phi}$ ($\underline{x}$) and f ($\underline{x}$, $\underline{y}$).

Define

$$E[\bar{\phi}, f] = \frac{\langle \Psi_0 | H | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} \tag{185}$$

We require

$$\frac{\delta E}{\delta \bar{\phi}} = \frac{\delta E}{\delta f} = 0 \tag{186}$$

This variational procedure gives us a set of coupled equations for $\bar{\phi}$ ($\underline{x}$) and f ($\underline{x}$, $\underline{y}$). Note that $\bar{\phi}$ ($\underline{x}$) is just the expectation value of the field ϕ ($\underline{x}$) in the trial state $\Psi_0[\phi]$ i.e. the approximate local vacuum expectation value of ϕ ($\underline{x}$) is $\bar{\phi}$ ($\underline{x}$). The energy functionality $E[\bar{\phi}, f]$ can be easily

evaluated. Firstly, we define the symmetric function g ($\underline{x}$, $\underline{y}$) by the following orthogonality relation

$$\int d^3z\, g(\underline{x},\underline{z})\, f(\underline{z},\underline{y}) = \delta^{(3)}(\underline{x}-\underline{y}) \tag{187}$$

It then follows that if

$$H = \int d^3x \left[\frac{\Pi^2}{2} + \frac{|\vec{\nabla}\Phi|^2}{2} + V(\Phi) \right] \tag{188}$$

then

$$E[\Phi, f] = \int d^3x \left[\frac{1}{4} f(\underline{x},\underline{x}) + \frac{1}{4} \vec{\nabla}_x \cdot \vec{\nabla}_y\, g(\underline{x},\underline{y})\Big|_{\underline{x}=\underline{y}} + \frac{1}{2}|\vec{\nabla}\Phi|^2 + U(\Phi,g) \right] \tag{189}$$

where

$$\langle \Psi_0 | V(\Phi) | \Psi_0 \rangle \equiv U(\Phi, g) \tag{190}$$

behaves like an effective potential. For clarity we shall now restrict our attention to a specific form for $V(\Phi)$, we shall take

$$V(\Phi) = \frac{m_1^2}{8\Phi_0^2} \left(\Phi^2 - \Phi_0^2\right)^2 \tag{191}$$

A direct functional computation then yields

$$U(\Phi, g) = V(\Phi) + \frac{m_1^2}{8\Phi_0^2} \left[3\Phi^2(\underline{x})\, g(\underline{x},\underline{x}) - \Phi_0^2\, g(\underline{x},\underline{x}) + \frac{3}{4} g^2(\underline{x},\underline{x}) \right] \tag{192}$$

Upon differentiation we obtain

$$\frac{\partial U}{\partial \Phi} = \frac{m^2}{2\phi_0^2} \bar{\Phi}(\underline{x}) \left[\bar{\Phi}^2(\underline{x}) - \left(\phi_0^2 - \frac{3}{2} g(\underline{x}, \underline{x}) \right) \right] \tag{193}$$

and

$$\frac{\partial^2 U}{\partial \bar{\Phi}^2} = 4 \frac{\partial U}{\partial g} = \frac{m^2}{2\phi_0^2} \left[3\bar{\Phi}^2(\underline{x}) - \left(\phi_0^2 - \frac{3}{2} g(\underline{x}, \underline{x}) \right) \right] \tag{194}$$

Varying E $[\bar{\Phi}, f]$ with respect to $\bar{\Phi}(\underline{x})$ and f $(\underline{x}, y)$ gives the following 'equations of motion'

$$\nabla^2 \bar{\Phi}(\underline{x}) = \frac{\partial U}{\partial \Phi}$$

$$= \frac{m^2}{2\phi_0^2} \bar{\Phi}(\underline{x}) \left[\bar{\Phi}^2(\underline{x}) - \left(\phi_0^2 - \frac{3}{2} g(\underline{x}, \underline{x}) \right) \right] \tag{195}$$

and

$$g^{(3)}(\underline{x} - \underline{z}) = \int d^3 z \, g(\underline{x}, \underline{z}) \left[-\nabla_z^2 + \frac{\partial^2 U}{\partial \bar{\Phi}^2} \right] g(\underline{z}, \underline{z})$$

$$= \int d^3 z \, g(\underline{x}, \underline{z}) \left[-\nabla_z^2 + \frac{m^2}{2\phi_0^2} \left\{ 3\bar{\Phi}^2(\underline{z}) - \left(\phi_0^2 - \frac{3}{2} g(\underline{z}, \underline{z}) \right) \right\} \right] g(\underline{z}, \underline{z}) \tag{196}$$

It goes without saying that eqns. (195) and (196) are essentially

insoluble (remembering that we are here interested in the case of strong coupling so that the usual linearization methods become inapplicable). However, one solution which is immediately obtainable is the one corresponding to the physical vacuum (which is, as usual, assumed to be spatially homogeneous and isotropic). In this case g (x , y) only depends on $|x-y|$ so that the unique solution of eqn. (196) can be found by using a Fourier convolution. We find

$$g(\underline{x},\underline{y}) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\underline{k}\cdot(\underline{x}-\underline{y})}}{[\underline{k}^2+m_1^2]^k} \qquad (197)$$

where $$m_1^2 \equiv V(\underline{x},\underline{x}) \qquad (198)$$

3.3 Coleman's Fock Space Variational Scheme

This section introduces a rather clever variational scheme, due originally to Coleman[14]. In a world of only one spatial dimension, the application of the principle is straightforward and causes no trouble. Unfortunately in three spatial dimensions this is not the case, due to the fact that a certain amount of technical 'fiddling' is required (a fact that tends to obscure the main points of the argument). Consequently we shall here present Coleman's variational scheme in a world of only one spatial dimension, leaving the extension of the scheme to the case of three dimensions till afterwards.

Operators in the Schrodinger picture are given as functions of the field $\phi(x)$ and the canonical momentum density $\pi(x)$, where x represents the spatial coordinates only. If we define the operators a (k , m) by the following mode decompositions

$$\phi(x) = \int \frac{dk}{\sqrt{4\pi}} \frac{1}{\sqrt{\omega(k,m)}} \left[a(k,m)\, e^{-ikx} + a^{\dagger}(k,m)\, e^{ikx} \right] \qquad (199)$$

and

$$\pi(x) = i \int \frac{dk}{\sqrt{4\pi}} \sqrt{\omega(k,m)} \left[a^{\dagger}(k,m)\, e^{ikx} - a(k,m)\, e^{-ikx} \right] \qquad (200)$$

where

$$\omega(k,m) = (k^2 + m^2)^{1/2}$$

then the normal-ordered Schrodinger operator corresponds to the operator rearranged with all the a's on the right and all the

a^+'s on the left.

Because this prescription does not inform us what value m is to assume we will not specify it and we will denote by N_m the normal-ordering operation defined by the mass m.

We now turn our attention to the following Hamiltonian density

$$\mathcal{H}_0 = \frac{1}{2}\pi^2 + \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + V(\phi) \qquad (201)$$

with

$$V(\phi) = \frac{m_1^2}{2\phi_0^2}\left(\phi^2 - \phi_0^2\right) \qquad (202)$$

Now Wick's theorem tells us that, for a free field of mass m, and for any space-time function J(x)

$$\exp\left[i\int dx\, J(x)\,\phi(x)\right] =$$

$$N_m\left\{\exp\left[i\int dx\, J(x)\,\phi(x)\right]\exp\left[-\frac{1}{2}\int dx\,dy\, J(x)\,\Delta(x-y;m)\,J(y)\right]\right\} \qquad (203)$$

where Δ is the free-field two-point Wightman function. For small spacelike separation

$$\Delta(x;m) = \frac{-1}{4\pi}\ln cm^2x^2 + O(x^2) \qquad (204)$$

where c is a numerical constant and

$$x^2 = -x_\mu x^\mu \qquad (205)$$

We regularise the theory by replacing $\Delta(x, m)$ by

$$\Delta(x;m_1) - \Delta(x;\Lambda) \equiv \Delta^\Lambda(x;m) \qquad (206)$$

where Λ is a large mass, the cut off. This form has the merit of being nonsingular at the origin,

$$\Delta^{\Lambda}(0;m) = -\frac{1}{4\pi} \ln\left(\frac{m^2}{\Lambda^2}\right) \qquad (207)$$

Using eqn. (203) with J a δ-function yields

$$e^{i\beta\phi} = N_m \left\{ e^{-\frac{1}{2}\beta^2 \Delta^{\Lambda}(x;m^2)} \; e^{i\beta\phi} \right\} \qquad (208)$$

Although this result has been derived in the interaction picture, it is also true in the Schrodinger picture because of the fact that it only involves fields at the same time.

We will now prove the following useful result. Given an arbitrary scalar polynomial potential

$$V(\phi) = \sum_{n=0}^{N} a_n \phi^n(x) \qquad (209)$$

then

$$V(\phi) = N_m \left[e^{\frac{\Delta^{\Lambda}}{2} \frac{\delta^2}{\delta\phi^2}} \; V(\phi) \right] \qquad (210)$$

Now

$$V(\phi) = \sum_{n=0}^{N} a_n \phi^n(x)$$

$$= V\left(\frac{1}{i}\frac{d}{d\beta}\right) e^{i\beta\phi(x)} \Big|_{\beta=0} \qquad (211)$$

But Wick's theorem informs us that

$$e^{i\beta\phi} = N_m \left\{ e^{-\frac{1}{2}\beta^2 \Delta^{\Lambda}(x;m^2)} \; e^{i\beta\phi} \right\} \qquad (212)$$

Consequently

$$V\left(\frac{1}{i}\frac{d}{d\beta}\right) e^{i\beta\phi} = N_m\left\{ V\left(\frac{1}{i}\frac{d}{d\beta}\right) e^{-\frac{1}{2}\beta^2 \Delta^\wedge(x;m^2) + i\beta\phi}\right\}$$

$$= N_m\left\{ e^{\frac{\phi^2}{2\Delta^\wedge}} V\left(\frac{1}{i}\frac{d}{d\beta}\right) e^{-\frac{\Delta^\wedge}{2}\left[\beta - \frac{i\phi}{\Delta^\wedge}\right]^2}\right\} \qquad (213)$$

Using the following identity (well-known from the functional formalism of quantum field theory)

$$V\left(\frac{1}{i}\frac{d}{d\beta}\right) e^{-\frac{\Delta^\wedge}{2}\beta^2} = e^{-\frac{\Delta^\wedge}{2}\beta^2} e^{-\frac{\Delta^\wedge}{2}\frac{\delta^2}{\delta\phi^2}} V(\phi) \qquad (214)$$

we may rearrange the r.h.s. of eqn. (213) to give

$$N_m\left[e^{-\frac{\beta^2}{2}\Delta^\wedge + i\beta\phi} e^{\frac{\Delta^\wedge}{2}\frac{\delta^2}{\delta\phi^2}} V(\phi)\right] \qquad (215)$$

Hence, finally we may write, using eqn. (211)

$$V(\phi) = N_m\left[e^{\frac{\Delta^\wedge}{2}\frac{\delta^2}{\delta\phi^2}} V(\phi)\right] \qquad (216)$$

Hence, given a Hamiltonian density

$$\mathcal{H}_b = \frac{1}{2}\pi^2 + \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + V(\phi) \equiv \mathcal{H}_{b_0} + V \qquad (217)$$

we know how to normal order $V(\phi)$. The remainder of $\mathcal{H}_b$ is straightforward to normal order since it is just a quadratic form in the fundamental fields, $\phi(x)$ and $\pi(x)$. From eqns. (199) and (200) it is easy to see that

$$\mathcal{H}_{b_0} = N_m \mathcal{H}_{b_0} + \int \frac{dk}{8\pi} \frac{2k^2+m^2}{\omega(k,m)} \qquad (218)$$

Hence

$$N_m \mathcal{H}_{cc} = N_\mu \mathcal{H}_0 + \frac{1}{8\pi}(\mu^2 - m_1^2) \qquad (219)$$

We will now proceed to apply these results to the specific potential

$$V(\phi) = \frac{m_1^2}{8\phi_0^2}(\phi^2 - \phi_0^2)^2 \qquad (220)$$

Eqn. (216) allows us to write

$$V(\phi) = N_m\left[\frac{m_1^2}{8\phi_0^2}\left\{\phi_0^4 - 2\phi_0^2(\phi^2 + \Delta^\wedge) + \phi^4 + \{\Delta^\wedge\phi^2 + 3(\Delta^\wedge)^2\}\right\}\right] \qquad (221)$$

Consequently

$$\mathcal{H}_0 = N_\mu\left[\frac{\pi^2}{2} + \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{m_1^2}{8\phi_0^2}(\phi^2 - \phi_0^2)^2\right]$$

$$= N_\mu\left[\frac{\pi^2}{2} + \frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{1}{8\pi}(\sigma^2 - \mu^2) + \frac{m_1^2}{8\phi_0^2}\left\{\phi^4 - 2\phi_0^2(\phi^2 + \Delta) + \phi^4 + \{\Delta\phi^2 + 3\Delta^2\}\right\}\right] \qquad (222)$$

where

$$\Delta = \frac{1}{4\pi}\ln\left(\frac{\mu^2}{\sigma^2}\right) \qquad (223)$$

To gain some insight into our variational scheme let us first consider the following Hamiltonian density

$$\mathcal{H}_\beta = N_m\left[\mathcal{H}_0 + \frac{1}{2}\beta^2\phi^2\right] \qquad (224)$$

This is just a free- field Hamiltonian density, normal-ordered with respect to a mass which is not necessarily equal to β .

We now attempt to find the ground state of this theory by using the Rayleigh-Ritz variational method. We will use as trial states the vacuum states appropriate to a free field of mass μ . These states are defined by

$$a(k,\mu)\,|0,\mu\rangle = 0 \qquad (225)$$

The computation is made trivial by renormal-ordering. Using eqn. (216) we have

$$N_m\left(\frac{\beta^2}{2}\phi^2\right) = N_\mu\left(\frac{\beta^2}{2}\phi^2\right) - \frac{\beta^2}{8\pi}\ln\left(\frac{\mu^2}{m^2}\right) \qquad (226)$$

From this, and eqn. (219)

$$\mathcal{H}_\beta = N_\mu\left(\mathcal{H}_0 + \frac{\beta^2}{2}\phi^2\right) + \frac{1}{8\pi}\left(\mu^2 - m^2 - \beta^2\ln\left(\frac{\mu^2}{m^2}\right)\right) \qquad (227)$$

whence

$$\langle 0,\mu|\,\mathcal{H}_0\,|0,\mu\rangle = \frac{1}{8\pi}\left(\mu^2 - m^2 - \beta^2\ln\left(\frac{\mu^2}{m^2}\right)\right) \qquad (228)$$

The right hand side of this equation, considered as a function of μ , assumes its minimum value when μ is β . This is, of course, the correct result.

We will now perform the same computation for the theory of eqn. (220). Here the previously discussed variational scheme

has to be generalised slightly in order to allow for the introduction of spontaneous symmetry breaking.[17] We now write

$$\phi(x) = \bar{\phi} + \eta(x) \qquad (229)$$

where $\bar{\phi}$ is to be determined and η (x) has the usual decomposition into a and a+ as in eqn. (199).

The trial ground state is denoted by $|\bar{\phi}, p\rangle$ and we assume that

$$a(k,p)|\bar{\phi},p\rangle = 0 \qquad (230)$$

Evaluating

$$\langle \bar{\phi},p|\mathcal{H}|\bar{\phi},p\rangle = U(\bar{\phi},p) \qquad (231)$$

and imposing the two stationarity requirements

$$\frac{\partial U}{\partial \bar{\phi}} = \frac{\partial U}{\partial p} = 0 \qquad (232)$$

leads, respectively, to the following two equations

$$\bar{\phi}\left[\bar{\phi}^2 - \phi_0^2 + 3\Delta\right] = 0 \qquad (233)$$

and

$$\rho^2 = \frac{m^2}{2g_0^2}\left[3\bar{\phi}^2 - \phi_0^2 + 3\Delta\right] \qquad (234)$$

It is interesting to note the following identity

$$\frac{\partial U}{\partial \rho} = \frac{1}{4\pi\rho}\left[\rho^2 - \frac{\partial^2 U}{\partial \bar{\phi}^2}\right] \qquad (235)$$

which follows directly from eqns. (216) and (231). How are these equations related to eqns. (195) and (196)?

To answer this question, we will extend Coleman's approximation scheme to n spatial dimensions. The main ideas of the scheme are already implicit in the previously discussed 1 + 1 dimensional model.

Assume we are given a Lagrange density of the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi) \qquad (236)$$

where $V(\phi)$ is some polynomial in the scalar field ϕ. We may write (compare with eqn. (222))

$$\mathcal{E}(\bar{\phi}, m) = \Omega_1(m) + U(\bar{\phi}, m) \qquad (237)$$

where

$$\Omega_1(m) = \frac{1}{4(2\pi)^n} \int \frac{d^n k \, (2k^2 + m^2)}{(k^2 + m^2)^{1/2}} \qquad (238)$$

and

$$U(\bar{\phi}, m) = \exp\left[\frac{1}{2} A \frac{\delta^2}{\delta \phi^2}\right] V(\phi)\Big|_{\phi = \bar{\phi}} \qquad (239)$$

where

$$A = \frac{1}{2(2\pi)^n} \int \frac{d^n k}{(k^2 + m^2)^{1/2}} \qquad (240)$$

The variational equations are

A)

$$\frac{\delta \mathcal{E}}{\delta \bar{\phi}} = 0 \Rightarrow \frac{\partial U[\bar{\phi}, m]}{\partial \bar{\phi}} = 0 \qquad (241)$$

B) $$\frac{\delta \mathcal{E}}{\delta m} = 0 \Rightarrow \frac{\partial \Omega}{\partial m} + \frac{\partial U}{\partial m} = 0 \qquad (242)$$

But

$$\frac{\partial \Omega}{\partial m} = \frac{1}{4(2\pi)^n} m^2 \int \frac{d^n k}{(k^2+m^2)^{3/2}} \qquad (243)$$

Whilst

$$\frac{\partial U}{\partial m} = \frac{1}{2}\frac{\partial A}{\partial m}\frac{\delta^2}{\delta \tilde{f}^2}\left[e^{\frac{1}{2}A\frac{\delta}{\delta \tilde{f}^2}} V(\tilde{f}) \right]$$

$$= \frac{1}{2}\frac{\partial A}{\partial m}\frac{\delta^2 U[\tilde{f}, m]}{\delta \tilde{f}^2}$$

$$= -\frac{1}{4(2\pi)^n} m \int \frac{d^n k}{(k^2+m^2)^{3/2}} \frac{\delta^2 U}{\delta \tilde{f}^2} \qquad (244)$$

Consequently, eqn. (242) informs us that

$$\frac{m}{4(2\pi)^n}\left(\int \frac{d^n k}{(k^2+m^2)^{3/2}} \right)\left(m^2 - \frac{\delta^2 U}{\delta \tilde{f}^2} \right) = 0 \qquad (245)$$

i.e.

$$m^2 = \frac{\delta^2 U}{\delta \tilde{f}^2} \qquad (246)$$

Eqn. (241) corresponds to eqn. (195), whilst eqn. (246) is Rosen's definition of m^2. The main point is that Rosen uses A and $\tilde{f}$ as variational parameters and m^2 is defined in terms of A. Coleman, on the other hand, has m and $\tilde{f}$ as variational parameters and A is defined in terms of m^2. Hence we see that (for

the ground state only!) the two variational schemes are computationally equivalent. This is a rather remarkable result (in my opinion).

Hence we have found two simple and equivalent ways for deriving the Hartree-Fock vacuum equations. It turns out that if we have a theory which involves fermions, Coleman's method is preferable (as it circumvents canonical quantisation difficulties).

Consider now the theory of eqn. (71), where we have a scalar field $\phi(x)$ which exhibits spontaneous symmetry breakdown and a fermion field $\psi(x)$. We expand the Schrodinger picture fields $\phi(x)$ and $\psi(x)$ as

$$\phi(x) = \bar{\phi} + \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega_k}} \left[a(\underline{k}, \mu, \bar{\rho}) e^{-i\underline{k}\cdot\underline{x}} + a^\dagger(\underline{k}, \mu, \bar{\rho}) e^{i\underline{k}\cdot\underline{x}} \right] \tag{247}$$

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k \left[\frac{M}{E_k} \right]^{1/2} \left[b(\underline{k}, M) u(\underline{k}) e^{-i\underline{k}\cdot\underline{x}} + d^\dagger(\underline{k}, M) v(\underline{k}) e^{i\underline{k}\cdot\underline{x}} \right] \tag{248}$$

with equivalent expressions for the canonically conjugate fields $\pi(x)$ and $\bar{\psi}(x)$. The parameters $\bar{\rho}$, μ and M are not specified, and can be chosen freely.

We define the normal ordering operation $N_{\mu,M,\bar{\rho}}$ by inserting these expressions in the Schrodinger picture Hamiltonian density (71) and rearranging with all a's, b's on the right, and all a^+'s, d^+'s on the left, etc. It follows that

$$\mathcal{H}_c(x) = N_{\mu,M,\bar{\Phi}}\Big[\bar{\psi}^i\Big(-i\frac{\lambda}{2}\cdot\nabla + M(\bar{\Phi})\Big)\psi^i + \frac{1}{2}\pi^2 + \frac{1}{2}|\nabla\bar{\Phi}|^2$$
$$+ B\sum_{r<s}(\bar{\Phi}_r-\bar{\Phi}_s)^2 + \tfrac{3}{4}\bar{\Phi}^2 g(M) - \bar{\Phi}^2 J(M) + \tfrac{3}{4}\bar{\Phi}^2 M^2$$
$$+ \frac{1}{4(2\pi)^3}\int d^3k\left[\frac{2k^2+M^2}{[k^2+M^2]^{1/2}} - \frac{4(k^2+M(\bar{\Phi}))}{(k^2+M^2)^{1/2}}\right]\Big] \qquad (249)$$

where

$$g(M) = 2A(0) = \frac{1}{(2\pi)^3}\int\frac{d^3k}{(k^2+M^2)^{1/2}}$$

The Hartree-Fock trial vacuum $|\mu, M, \bar{\Phi}\rangle$ is defined by

$$a(\underline{k},\mu,\bar{\Phi})\,|\mu,M,\bar{\Phi}\rangle = 0 \qquad (250)$$

$$b(\underline{k},M)\,|\mu,M,\bar{\Phi}\rangle = 0 \qquad (251)$$

The energy density

$$\mathcal{E}(\mu,M,\bar{\Phi}) = \langle\mu,M,\bar{\Phi}|\,\mathcal{H}_c(x)\,|\mu,M,\bar{\Phi}\rangle \qquad (252)$$

of the Hartree-Fock vacuum may now be worked out and the set of variational equations deduced. In the next Chapter we shall do this for the SLAC bag in an attempt to understand its quantization.

CHAPTER IV

QUANTUM ANALYSIS OF THE MODELS OF CHAPTER II

Preamble

In this chapter we shall examine the application of the strong coupling quantisation approaches of Chapter III to the hadron models of Chapter II.

We first consider heuristic arguments that indicate the probable instability of the Goldstone vortex (in the strong coupling limit). These arguments, whilst suggestive, are not quantitative in any useful manner. To be more concrete, we then consider a single mode approximation to the quantisation of the Creutz model for the MIT bag. Again, our prejudices about strong coupling are confirmed. As stated earlier, the intractability of the finite mode quantisation scheme becomes apparent. We reserve the most realistic quantisation approach (the relativistic Hartree-Fock scheme due to Coleman and Rosen) for the most realistic classical model, the SLAC bag (which has been discussed in detail previously). This approximation scheme, which has the merit of accommodating the infinite number of degrees of freedom of the system, still yields the same qualitative conclusions that the more naive approaches led us to.

4.1 The Quantum Description of a Zero-Width 'Dual' Vortex

In this section we will examine the effects that quantization induces in a simple model for a 'dual' vortex.[18]

Two arguments will be given for the probable dynamical instability of the zero-width (Goldstone) vortex. The first approach is based on a somewhat heuristic path-integral argument whilst the second is comparatively more rigorous (it involves the canonical quantization of a scalar field in the Schrodinger picture via a mode decomposition of the field). Fortunately these two very different methods yield corroborative results. It is important to note that we are unable to extend our basic arguments to include the case considered by Nielsen and Olesen, namely the Higgs vortex. This is due to the limitations inherent in the techniques we use. The basic difficulty we encounter is in trying to canonically quantize a local gauge theory with spontaneous symmetry breaking. We end up with a Hamiltonian which is nonpolynomial in the fields. This makes it difficult to even use a finite mode decomposition of the fields for any useful purpose.

Consequently to make any progress at all we have to abolish the gauge field from the Higgs model i.e. we are lead back to the Goldstone model. As stated earlier the total energy corresponding to a static vortex solution is not well-defined due to the presence of the Goldstone boson. For this reason we only consider the vortex proper to be liable to quantitative treatment. Outside of the vortex region our techniques yield no information. This approximation is fairly reasonable since near the origin $A_i(r) \simeq 0$ so that as a first step we may ignore the presence of the electro-

magnetic field. Consequently the solutions we obtained for the scalar field near the origin in the Higgs model are essentially unaffected upon removal of the electromagnetic field.

To be specific we now consider the theory of a complex scalar field ϕ with global $U(1)$ symmetry described by the following density

$$\mathcal{D}_{\phi} = \partial_\mu \phi^* \partial^\mu \phi - \frac{m^2}{2\phi_0^2}\left(\phi^*\phi - \frac{1}{2}\phi_0^2\right)^2 \qquad (253)$$

Defining 'radial' and 'angular' fields ρ, θ by

$$\sqrt{2}\,\phi = \rho \exp[i\theta] \qquad (254)$$

$\mathcal{D}_{\phi}$ becomes

$$\mathcal{D}_{\phi} = \frac{1}{2}\left[(\partial_\mu \rho)(\partial^\mu \rho) + \rho^2 (\partial_\mu \theta)(\partial^\mu \theta)\right] - \frac{m^2}{2\phi_0^2}\left(\rho^2 - \phi_0^2\right)^2 \qquad (255)$$

Furthermore, defining η by $\rho = \phi_0 + \eta$, $\mathcal{D}_{\phi}$ can be written

$$\mathcal{D}_{\eta} = \frac{1}{2}\left[(\partial_\mu \eta)(\partial^\mu \eta) + (\phi_0 + \eta)^2 (\partial_\mu \theta)(\partial^\mu \theta)\right] - \frac{m^2}{2}\eta^2\left(1 + \frac{\eta}{2\phi_0}\right)^2 \qquad (256)$$

That is, we have the well-known fact that the radial field η acquires mass m by virtue of the spontaneous symmetry breakdown of the vacuum. Moreover, m sets the scale of the coupling constant for the self-interacting η field.

Classical vortex-line solutions to the Lagrangian can easily be found. In cylindrical coordinates r, θ and z, the equations of motion derived from permit the following static solution

$$\chi = \chi(\theta) = n\theta \quad \text{for integral } n \tag{257}$$

$$\eta = \eta(r) \quad \text{with} \quad \eta(r) \to 0 \quad \text{for } r \to r_n \tag{258}$$

and r_n is defined to be the first zero of $J_n'(mr)$. One can easily show that for small r (i.e. $r \ll r_n$) η may be approximated by

$$\eta(r) \approx \eta_0\left[1 - \frac{J_n(mr)}{J_n(mr_n)}\right] \tag{259}$$

The Hamiltonian for the classical fields η and χ satisfying (257) and (258) is

$$H = \frac{m^2}{2\eta_0^2}\int d^3x\,(\eta_0^2+\eta^2)(\eta_0^2-\eta^2) \tag{260}$$

which gives the vortex-line an energy per unit length

$$E \approx \frac{m^2\pi}{2} r_n^2 \eta_0^2 \sim \eta_0^2 \tag{261}$$

(which is independent of m for large values of η_0). Anyway, in the infinite mass limit (which corresponds to infinite coupling or zero vortex width) the Hamiltonian and Lagrangian densities of the static vortex-line of equations (257) and (258) both behave like two-dimensional delta-functions. These infinitely thin classical vortices are now taken to be candidates for relatavistic dual strings.

We now wish to quantize our theory. In other words we are

interested in computing the relative probability amplitude $\langle \phi(\underline{\Delta}), t \mid \phi_0(\underline{\Delta}), t_0 \rangle$ for the configuration of the complex ϕ field to change from $\phi_0(\underline{\Delta})$ at time t_0 to $\phi(\underline{\Delta})$ at time t. The noncovariance of this amplitude presents no difficulties of interpretation since we shall have in mind a situation in which $\phi_0(\underline{\Delta})$ is a classical static vortex-line solution. That is, we shall watch the vortex from the frame in which it is initially at rest.

Now we can express $\langle \phi(\underline{\Delta}), t \mid \phi_0(\underline{\Delta}), t_0 \rangle$ as a path integral

$$\langle \phi(\underline{\Delta}), t \mid \phi_0(\underline{\Delta}), t_0 \rangle = \int \mathcal{D}[\phi] \mathcal{D}[\phi^*] \exp\left[i \int_{V, t_0}^{t} \mathcal{L}_\phi \, d^4x \right] \tag{262}$$

By changing variables from ϕ and ϕ^* to η and η^* this becomes

$$\langle \eta, \eta^*, t \mid \eta_0, \eta_0^*, t_0 \rangle = \int \mathcal{D}[\eta] \mathcal{D}[\eta^*] \exp\left[i \int_{V, t_0}^{t} d^4x \left(\mathcal{L}_\eta + \mathcal{L}_{\eta^*}' \right) \right] \tag{263}$$

where

$$\mathcal{L}_\eta' = -i \delta^4(0) \, \mathcal{L}_\eta \left(1 + \frac{\eta}{\eta_0} \right) \tag{264}$$

and the whole system is contained in a volume V. We emphasize that $\mathcal{L}_\eta'$ is a pure quantum effect arising from the Jacobian of the variable change.

The trick is to compare the amplitude $\langle \eta, \eta^*, t \mid \eta_0, \eta_0^*, t_0 \rangle$ with the amplitude $\langle \eta, \eta^*, t \mid \eta_0, \eta_0^*, t_0 \rangle_0$ for the free fields η and η^* which are in turn governed by the Lagrangian density

$$\mathcal{L}_\eta^0 = \frac{1}{2} \left[\partial_\mu \eta \, \partial^\mu \eta + \mu^2 \eta \eta^* \right] - \frac{\lambda}{4} \eta^2 \tag{265}$$

which has the same values of ρ_0 and m as does eqn. (25). In order to examine the large m limits of $\mathcal{L}_\eta$ and $\mathcal{L}^0_\eta$ we work with the scaled field ψ', defined by $m\eta = \psi'$

in terms of which

$$\mathcal{L}_\eta = \frac{1}{2m^2}\partial_\mu\psi'\partial^\mu\psi' + \frac{\rho_0^2}{2}\left(1+\frac{\psi'}{m\rho_0}\right)\partial_\mu\theta\,\partial^\mu\theta - \frac{1}{2}\psi'^2\left(1+\frac{\psi'}{2m\rho_0}\right)^2 \qquad (26b)$$

and

$$\mathcal{L}^0_\eta = \frac{1}{2m^2}\partial_\mu\psi'\partial^\mu\psi' + \frac{\rho_0^2}{2}\partial_\mu\theta\,\partial^\mu\theta - \frac{\psi'^2}{2} \qquad (26c)$$

Now if $\partial_\mu\psi'$ is bounded in some sense as $m\to\infty$ then both $\mathcal{L}_\eta$ and $\mathcal{L}^0_\eta$ have the same trivial limiting form as $m\to\infty$

$$\mathcal{L}_{\rho_0} = \frac{\rho_0^2}{2}\partial_\mu\theta\,\partial^\mu\theta - \frac{\psi'^2}{2} \qquad (26d)$$

It can be seen that $\mathcal{L}_{\rho_0}$ gives the equation of motion $\psi'=0$ and it is in this sense that infinitely massive classical fields do not occur. Unfortunately the quantum correction term $\mathcal{L}'$ has a somewhat more ambiguous limiting form, namely

$$\mathcal{L}' \sim -\frac{i\delta^4(0)\,\psi'}{m\rho_0} \quad \text{as } m\to\infty \qquad (26e)$$

Since the Lagrangian density $\mathcal{L}^0_\eta$ is not derivable from a single complex field ψ in the same way that $\mathcal{L}_\eta$ is, the path integral for $\langle \eta_2, N_2, t_2 | \eta_1, N_1, t_1\rangle$ will not contain a term $\mathcal{L}'$

in the action. However the effect of $\frac{1}{\tau}$ in the infinite mass limit is only to displace the τ integration and by doing so introduce a multiplicative phase

$$\ell(T) = \exp\left[-\frac{i}{2}\left(\frac{\beta^{+}(0)}{m_{\beta}}\right)^{2} T\right] \quad \text{where } T = t - t_0 \tag{270}$$

which will have no effect on probabilities. Thus we are lead to expect that in the infinite mass limit the modulus of the ratio of the two relative probability amplitudes tends to unity i.e.

$$\left| \frac{\langle \eta, \alpha, t \,|\, \eta_0, \alpha_0, t_0 \rangle}{\langle \eta, \alpha, t \,|\, \eta_0, \alpha_0, t_0 \rangle_0} \right| \longrightarrow 1 \quad \text{as } m \to \infty \tag{271}$$

It is apparent that the argument just given is extremely crude. Because of this we will now look at the quantisation problem from another point of view-- that of the canonical quantization of our theory in the Schrodinger picture. It is reassurring that the physical results obtained by these two different methods are in fact in concord.

For orientation, consider a cylinder of length L and radius R. We wish to find a complete orthonormal basis for describing the possible excitations of our field which is to be confined within the boundaries of the cylinder. It is easy to demonstrate that the following set of functions fulfils our needs

$$f_{pkm} = \frac{1}{\sqrt{V}\, J'_m(kh)}\, e^{ipz}\, J_m(kr)\, e^{im\theta} \qquad (272)$$

where V is the volume of the enclosing cylinder and we have set

$$e^{ipL} = 1 \quad \text{and} \quad J_m(kh) = 0 \qquad (273)$$

The orthonormality and completeness relations are respectively

$$\int dV\, f^*_{pkm}(z,r,\theta)\, f_{p'k'm'}(z,r,\theta) = \delta_{pp'}\,\delta_{kk'}\,\delta_{mm'} \qquad (274)$$

$$\sum_{p,k,m} f^*_{pkm}(z,r,\theta)\, f_{pkm}(z',r',\theta') = \delta(z-z')\,\delta(r-r')\,\delta(\theta-\theta') \qquad (275)$$

whilst each f_{pkm} satisfies

$$\nabla^2 f_{pkm} = -(p^2+k^2)\, f_{pkm} \qquad (276)$$

This basis is in fact a little too general because we will now restrict our analysis to 2+1 dimensions by ignoring the z-dependence. This is a small change to make considering the violent truncations of our theory that our approximation scheme will require. We are now in a position to quantize our theory using canonical methods, all the time working in the Schrodinger picture. We write the Schrodinger field $\psi(r,\theta)$ as

$$\psi(r,\theta) = a_0 f_{00} + \eta(r,\theta) \qquad (277)$$

where η has an expansion into its radial (k) and angular (m) modes as follows

$$\eta(r,\theta) = \sum_{k,m} q_{km} f_{km}(r,\theta) \qquad (278)$$

The functions $f_{k,m}$ constitute an orthonormal set and can be taken to be

$$f_{km}(r,\theta) = \frac{1}{\sqrt{A}} \frac{J_m(kr)}{J_{m+1}(kR)} e^{im\theta} \qquad (279)$$

and A is now the crossectional area i.e. $A = \pi R^2$ whilst the q_{km} form a set of self adjoint operator coefficients. If the initial field configuration is that of a classical vortex $\phi_0(r,\theta)$ labelled by n (by which we mean that the state of the system $|\phi_0(x), t_0\rangle$ is an eigenstate of $\hat{J}$ at time t_0 with eigenvalue J_0) then as a first approximation we may restrict ourselves to considering only coherent excitations i.e. we will only quantize those field modes with the same value of n. For this case

$$\eta(r,\theta) = \sum_k \exp[in\theta]\, q_{0k} f_k(r)$$

$$\equiv \exp[in\theta]\, \eta(r) \qquad (280)$$

$$\text{and } \sqrt{A}\,\phi(r,\theta) = e^{in\theta}[\phi_0 + \eta(r)] \qquad (281)$$

In terms of $\hat{\eta}$ the Lagrangian (256) reads

$$\mathcal{L}_\eta = \partial_\mu \eta^* \partial^\mu \eta - m^2 \eta^2 \left(1 + \frac{\eta}{2\phi_0}\right)^2 - \frac{n^2}{r^2}(\phi_0 + \eta)^2 \qquad (282)$$

The momentum conjugate to η is

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \eta)} = 2\,\partial_0 \eta \qquad (283)$$

As usual, the Hamiltonian may now be constructed and is given by

$$H = \int_0^\infty 2\pi r\,dr\left[\pi(r)\,\partial_0\eta(r) - \mathcal{L}\right] \qquad (284)$$

Or explicitly

$$H = \int_0^\infty 2\pi r\,dr\left[\frac{\pi^2(r)}{4} + |\nabla\eta|^2 + m^2\eta^2\left(1+\frac{\eta}{2\phi_0}\right)^2 + \frac{\eta^2}{r^2}(\phi_0+\eta)^2\right] \qquad (285)$$

The canonical commutation relations read

$$[\pi(r), \eta(r')] = \frac{-i\,\delta(r-r')}{2\pi r} \qquad (286)$$

These commutation relations are satisfied by the following realisations

$$\hat{\eta}(r) \longrightarrow \eta(r) \qquad (287)$$

$$\hat{\pi}(r) \longrightarrow \frac{-i}{2\pi r}\frac{\delta}{\delta\eta(r)}$$

Discarding the centrifugal term derived from $\phi_0^2\,\partial_\mu\chi\,\partial^\mu\chi$ (as it makes no contribution in the limit we are considering) we end up with the following Hamiltonian

$$\mathcal{H} = \int_0^\infty 2\pi r\,dr \left[\frac{-1}{4(2\pi)^2} \frac{\delta^2}{\delta\eta^2} - \eta \nabla^2 \eta + m^2 \eta^2 \left(1 + \frac{\eta}{2f_0}\right)^2 + \frac{n^2}{r^2} \eta \left(1 + \frac{\eta}{2f_0}\right)^2 \right] \quad (288)$$

As our approximation scheme involves stripping the field down into its proper modes it is now necessary to rewrite the above expression in terms of the q's and f's of eqn. (). This has the effect of changing functional differential operators into partial differential operators.

As before we decompose the $\eta(r)$ field into the field mode operators

$$\hat{\eta}(r) = \sum_k \hat{q}_k f_k(r) \quad (289)$$

where the set of functions $f_k(r)$ is complete and orthonormal. Correspondingly our state functional becomes a function of the

$$\Psi[\eta] = \Psi[\{q_k\}] \quad (290)$$

One can easily show that

$$\frac{\delta \Psi}{\delta \eta(r)} = \sum_k \frac{\partial \Psi}{\partial q_k} f_k(r) \quad (291)$$

and

$$\frac{\delta^2 \Psi}{\delta \eta(r)^2} = \sum_{k,m} \frac{\partial^2 \Psi}{\partial q_k \partial q_m} f_k(r) f_m(r) \quad (292)$$

Consequently we may rewrite our Hamiltonian functional operator $\mathcal{H}[\eta, \frac{\delta}{\delta \eta}]$ which acts on $\Psi[\eta]$ as a partial differential operator $\mathcal{H}[\{q_k\}, \{\frac{\partial}{\partial q_k}\}]$ which now acts on $\Psi[\{q_k\}]$.

We find (upon neglecting the centrifugal term as before)

$$\mathcal{H} = -a_{nm}\frac{\partial^2}{\partial q_n \partial q_m} + b_{nm} q_n q_m + \ldots \tag{293}$$

where

$$a_{nm} = \frac{1}{4}\int_0^\infty \frac{dr}{2\pi r} f_n(r) f_m(r)$$

and $$b_{nm} = (k_n^2 + m^2)\delta_{nm}$$

We immediately see from the above that the a_{nm} term is not in general diagonal. Hence even when the interaction is turned off the modes are still coupled. This difficulty is easily overcome by constructing a new set of operator coefficient functions which do diagonalize the a_{nm}. As an example we may take a plane wave basis which definitely diagonalizes the a_{nm}. The relationship between the q_{km} and the plane-wave basis $q_{k_1 k_2}$ is found to be

$$q_{nk} = J_n(kR)\sum_\theta q_{k\sin\theta, k\cos\theta}\, e^{in\theta} \tag{294}$$

where $q_k \equiv q_{kn}$ and $k_1 = k\sin\theta$ whilst $k_2 = k\cos\theta$.
In terms of the new coefficients (i.e. after diagonalisation) the Hamiltonian takes the form

$$\mathcal{H} = \sum_i \left(-\frac{\partial^2}{\partial Q_i^2} + \omega_i^2 Q_i^2\right) + \frac{m^2}{r_0}\sum_{n,m,r} C_{nmr} q_n q_m f_r + \left(\frac{m^2}{r_0}\right)^2 \sum_{n,m,r,s} D_{nmrs} q_n q_m q_r q_s \tag{295}$$

and the Q_i are suitable linear combinations of the $\{f_k\}$

To investigate the dynamics of this system in the large mass limit we will first of all consider the extreme situation where we quantize just one mode (Q, say). The extension of this argument to a finite number of degrees of freedom is transparent and the extension to an infinite number of degrees of freedom plausible.

For one mode the effective Hamiltonian is taken to be

$$H = -\frac{1}{2}\frac{\partial^2}{\partial q^2} + \frac{\omega^2}{2} q^2 (1+\sigma q)^2 \qquad (296)$$

where, without loss of generality (since cubic terms in Q are relatively suppressed in the large ω limit) we have taken the interaction term to be a perfect square and σ is proportional to ρ_0^{-1}.

Rather than compute $\langle q,t | q_0, t_0 \rangle_{\omega,\sigma}$ directly from eqn. (296) as a perturbation series in

$$V = \frac{\omega^2}{2} q^2 \left[(1+\sigma q)^2 - 1 \right] \qquad (297)$$

(and therefore as a series in the singular distribution of eqn. (310)) it is more convenient to work in a less singular coherent state basis.

That is, if $|\alpha, t\rangle$ is a coherent state with frequency ω labelled by α at time t we compute $\langle \alpha, t | \alpha, t_0 \rangle_{\omega,\sigma}$

Expanding in powers of V gives

$$\langle \alpha, t | \alpha, t_0 \rangle_{\omega,\sigma} = \langle \alpha, t | \alpha, t_0 \rangle_{\omega,0} - i \langle \alpha, t | \alpha, t_0 \rangle^{(1)}_{\omega,\sigma} + i^2 \langle \alpha, t | \alpha, t_0 \rangle^{(2)}_{\omega,\sigma} \qquad (298)$$

where

$$\langle \alpha, t | \alpha_0, t_0 \rangle_{\omega,\sigma} = \exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\alpha|^2 + \alpha\beta e^{-i\omega(t-t_0)}\right]$$

and

$$\langle \alpha, t | \alpha_0, t_0 \rangle^{(1)}_{\omega,\sigma} = \int \frac{d^2\beta}{\pi} \frac{d^2\gamma}{\pi} \int_{t_0}^{t} d\tau \, \langle \alpha, t | \beta, \tau \rangle_{\omega,\sigma} \langle \beta | V | \gamma \rangle \langle \gamma, \tau | \alpha_0, t_0 \rangle_{\omega,\sigma}$$

and so on.

To compute $\langle \alpha, t | \alpha_0, t_0 \rangle^{(1)}$ we need to evaluate $\langle \beta | V | \gamma \rangle$ i.e. the matrix element of the potential between the two coherent states $|\beta\rangle$ and $|\gamma\rangle$.

A correctly normalized coherent state has the representation

$$\psi(x,\alpha) = e^{\frac{1}{2}(\alpha^2 - |\alpha|^2)} \left(\frac{\omega}{\pi}\right)^{1/4} \exp\left[-\left(\frac{\omega}{2}x - \alpha\right)^2\right] \tag{299}$$

so that

$$\int |\psi(x,\alpha)|^2 dx = 1 \quad \text{and}$$

$$\int \psi^*(x,\beta)\psi(x,\alpha)\,dx = \exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \bar{\beta}\alpha\right] \tag{300}$$

We need to evaluate $\langle \beta | V | \alpha \rangle$ where

$$\langle \beta | V | \alpha \rangle = \int \psi(x,\beta)\, V(x)\, \psi(x,\alpha)\, dx \tag{301}$$

and

$$V(x) = \sigma\omega^2 x^3\left[1 + \frac{x}{2}\right] \qquad (302)$$

One may show by direct integration that

$$\langle\rho|V|\alpha\rangle = \langle\rho|\alpha\rangle\left[\mu_0 + \mu_1(\rho+\gamma) + \mu_2(\rho+\gamma)^2 + \mu_3(\rho+\gamma)^3 + \mu_4(\rho+\gamma)^4\right] \qquad (303)$$

where

$$\mu_0 = \frac{3}{8}\sigma^2, \quad \mu = \frac{3}{4}\sigma^2$$

and

$$\mu_4 = \frac{1}{8}\cdot\sigma^2$$

whilst

$$\mu_1 = \frac{3}{2}\sigma\sqrt{\frac{\omega}{2}}$$

and

$$\mu_3 = \frac{\sigma}{2}\sqrt{\frac{\omega}{2}}$$

It is now straightforward to show that (via a long and tedious calculation)

$$\frac{\langle\alpha,t|\alpha,t_0\rangle^{(1)}_{\omega,\sigma}}{\langle\alpha,t|\alpha,t_0\rangle_{\omega,\sigma}} = \frac{3}{8}\sigma^2 T\left[1 + O(e^{-i\omega T}) + O\left(\frac{\sigma}{\sqrt{\omega}}\right)\right] \qquad (304)$$

where $T = t - t_0$.

Whilst for the sake of interest

$$O(e^{-i\omega T}) = 4|\alpha|^2 e^{-i\omega T} + 2|\alpha|^4 e^{-i2\omega T} \qquad (305)$$

By explicit calculation of further terms it becomes apparent that eqn. (304) generalises to

$$\frac{\langle \alpha', t | \alpha, t_0 \rangle_{\omega,\sigma}}{\langle \alpha', t | \alpha, t_0 \rangle_{\omega,0}} = \exp\left[-i\frac{3}{8}\sigma^2 T\right] \left(1 + O\left(e^{-i\omega T}\right) + O\left(\frac{\sigma}{\omega}\right)\right) \tag{306}$$

If, by the $n \to \infty$ limit we mean that rapidly oscillating terms $O(e^{-i\omega T})$ are neglected in eqn. (306) then the effect of the interaction in the single mode approximation is just to change the phase of the free field amplitude. If we generalise the above to a finite number of interacting modes we get the same result i.e. in the sense that rapidly oscillating terms are neglected.

$$\frac{\langle \{\alpha\xi\}, t | \{\alpha\xi\}, t_0 \rangle_{\omega,\sigma}}{\langle \{\alpha\xi\}, t | \{\alpha\xi\}, t_0 \rangle_{\omega,0}} \sim \exp[ikT] \tag{307}$$

for some real k.

It is plausible that the effect of extending this quantisation scheme to an infinite number of modes will do no more than introduce an (infinite) phase. Although this is not proven the argument of the main text indicates that we should not expect anything serious to go amiss. The probability interpretation of eqn. (307) would thus be unaltered.

Before concluding this section we will now briefly review the properties of a free field η of mass m given by the η-dependent

part of ϕ_η^0 eqn. (2.5) . The extension of our analysis to include the massless field V is straightforward and will be omitted here. Let us expand η in terms of real orthonormal field modes $f_n(x)$, satisfying $\nabla^2 f_n(x) = -k_n^2 f_n(x)$, as

$$\eta(x) = \sum_n q_n f_n(x) \qquad (308)$$

The functional probability amplitude $\langle \eta(x), t | \eta_0(x), t_0 \rangle$ then becomes a function of the $\{q_n\}$.

$$\langle \eta, t | \eta_0, t_0 \rangle \equiv \langle \{q_n\}, t | \{q_{0n}\}, t \rangle = \prod_n \langle q_n, t | q_{0n}, t \rangle \qquad (309)$$

where, if $T = t - t_0$ and $\omega_n^2 = k_n^2 + m^2$

$$\langle q_n, t | q_{0n}, t \rangle = F(T) \exp\left[\frac{i\omega_n}{2 \sin \omega_n T} \{ (q_n^2 + q_{0n}^2) \cos \omega_n T - 2 q_n q_{0n} \} \right] \qquad (310)$$

with

$$F(T)^{-2} = 2\pi i \, \omega_n^{-1} \sin \omega_n T$$

We see from eqn. (310) that if the harmonic oscillator is at q_{on} at time t_o, then for almost all subsequent times, the oscillator can be anywhere with equal probability. In particular at times $T = \frac{\pi}{\omega_n} m$, $\frac{(2m+1)\pi}{\omega_n} m$ (for all integer m)

the oscillator is at q_{on+1} - q_{on} respectively. It follows that, for a free field with a specified configuration η_c at time t_o, for almost all t thereafter any field configuration η is equally likely. This is true for arbitrarily large mass, with the proviso that the probability distribution is not well-defined in the limit $m \to \infty$.

In conclusion it seems that we have a situation where, whatever the initial configuration of the interacting fields η and $\hat{\eta}$, eqn. (25b), in the large mass limit a vortex-line configuration is no more likely than any other configuration (at almost all subsequent times). In particular, if the initial configuration is a classical vortex-line it is instantaneously annihilated by quantum fluctuations.

4.2 A Simple Quantum Treatment of the Creutz Model for the M.I.T. Bag

In this section we attempt to consider the quantized version of the Creutz model for the M.I.T. bag, previously discussed in chapter two (section 3). For our present purposes we shall make two modifications to the previously discussed model. They are:

a) to replace the dynamical charged quarks by a fixed (in space and time) neutral source (the reason for this will become apparent later) and

b) instead of performing our manipulations in three spatial dimensions, we shall instead work in only one. This is because static solutions to such field theories as we have been led to consider (namely, scalar field theories) are unstable against perturbations which dilate the coordinates in any number of spatial dimensions greater than one (Derrick's theorem; see chapter one for details). Now with respect to quantization we immediately see that we are going to have problems due to the fact that our system is (infinitely) strongly coupled. Consequently some sort of variational principle for dealing with the quantized system will have to be employed. We shall, in fact, use a one-mode version of the Wentzel-Tomonaga-Pauli strong coupling approximation scheme.

Our model, which exists in a space-time of 1 + 1 dimensions, is defined by the following Lagrange density

$$\mathcal{L}_\varphi = \frac{1}{2}(\partial_\mu \varphi)^2 - V(\varphi) - \lambda\pi(\varphi - \varphi_0)^2 \qquad (311)$$

where $V(\varphi) = \lambda\left[\frac{1}{4}\varphi^4 - (\varphi_0)\frac{\varphi^3}{3} + \varphi_0^2\frac{\varphi^2}{2}\right]$ as in eqn. (122). The quark dynamics has been replaced by the source term J(x).

Taking the spirit of the mode decomposition to its extreme, we assume that the gluon field $\phi(x)$ may be well (qualitatively) represented by a simple-minded one-mode-of-vibration approximation

$$\phi(x) = q(t) f(x) \qquad (312)$$

The function f(x) is arbitrary and for static field configurations (which will concern us here) the function q(t) simply becomes a constant parameter q.

Let us define the quantity Q in the following manner

$$Q = \int_{-\infty}^{\infty} dx\, J(x) \qquad (313)$$

We shall now choose f(x) and J(x) to have box-like shapes.

We have

$$f(x) = 1 \quad \text{for } x \in [0, L] \qquad (314)$$
$$= 0 \quad \text{otherwise}$$

and

$$J(x) = J \quad \text{for } x \in [0, L_0] \qquad (315)$$
$$= 0 \quad \text{otherwise}$$

The edges of the gluon-box have to be 'smooth' in order that the term in the energy involving $\int f'^2 dx$ can be made vanishingly small. We now define η by

$$\eta Q = \int_0^{L_0} f(x) J(x) \qquad (316)$$

Considering the case of $\eta \leq 1$, we find that the total energy H' of our system can be written as

$$H' = H + \lambda \rho^2 Q (1 - \eta) + \lambda \eta Q (q - \beta)^2 \qquad (317)$$

with

$$H = \frac{L}{2} \dot{q}^2 + \frac{L}{2} V(q) \qquad (318)$$

when the gluon field vanishes everywhere, the total energy H' assumes the value E_o

$$H'(\phi=0) = \lambda_\beta^2 q \equiv E_0 \qquad (319)$$

We will now show that (classically, at least) it is possible for the energy of our system to take on a value less than E_o for a nonvanishing value of ϕ. The 'potential' U of our system is

$$U = \frac{L}{2} V(\phi) + \lambda_\beta^2 q(1-\eta) + \lambda\eta\phi(\phi-\beta)^2 \equiv V(\phi) + \lambda_\beta^2 q(1-\eta) \qquad (320)$$

where all the q-dependence has been inserted into V. Now for $q = \beta$ to become a global minimum of V requires that

$$E_0 - U(\beta) > 0 \qquad (321)$$

which will be satisfied if

$$\frac{\Lambda^2}{4L}\left[\Lambda^2 + \frac{4}{3}(2\beta-\gamma)\Lambda + \frac{2}{L}\beta(\beta-\gamma) + 2\lambda\eta\beta\right] > 0 \qquad (322)$$

where $\Lambda = \phi - \beta$. Condition (322) will be satisfied if

$$\frac{(2\beta-\gamma)^2}{9} < \frac{\beta(\beta-\gamma)}{2} + \frac{\lambda\eta\beta}{2L} \qquad (323)$$

as can be seen by completing the square. This inequality is easily seen to be satisfied in the 'bag formation' limit, previously discussed in chapter two (section three).

A similar analysis can be performed for the case $\gamma > 1$.

The general dependence of $U(q)$ on L is shown in fig.(7). The size of the source L_o plays an important dynamical role as it tells us the preferred size of the gluon distribution. The gluons fill out exactly that region of space occupied by the quarks. Hence, classically, the bag exists and is energetically stable, in the sense that the bag 'radius' R is happiest when it is nonzero and finite and equal to L_o. We see that the source term acts as a stabilizer and prevents our gluon field system from shrinking to a point.

We now turn to a rough quantum theoretic version of our classical system. Emphasis will be placed on the form the energy takes on as a function of the 'running variable' β. To quantize our system, we implement the following substitutions

$$\varphi_0 \to \varphi_0 \quad \text{and} \quad \dot{\varphi}_0 \to \frac{-i}{L}\frac{d}{d\varphi_0} \qquad (324)$$

We now compute the expectation value of H' (eqn. (317)) in a variational trial state $\Psi(\varphi_0; \bar{\varphi}_0, a^2)$ of the form

$$\Psi(\varphi_0; \bar{\varphi}_0, a^2) = N \exp\left[\frac{-a^2}{2}(\varphi_0 - \bar{\varphi}_0)^2\right] \qquad (325)$$

The constant N is the normalization factor, so that

$$\int d\varphi_0 \, \Psi(\varphi_0)^2 = 1 \qquad (326)$$

The energy of this state will be minimized with respect to the two variational parameters $\bar{\varphi}_0$ and a^2 (we only insist on stationarity

for variations in L) in order to obtain a rough upper bound on the true ground state energy of our system.

From eqns. (317) and (324), the total energy operator is

$$\hat{H}' = \frac{-1}{2L}\frac{d^2}{dq_0^2} + LV(q_0) + \lambda\rho^2 y(1-\eta) + \lambda\eta y(q_0-\rho)^2 \qquad (327)$$

And

$$E(q_0, q_0^2, L) \equiv \langle\Psi'|\hat{H}'|\Psi'\rangle$$
$$= LV(q_0) + \lambda\eta y(q_0-\rho)^2 + \lambda\rho^2 y(1-\eta)$$
$$+ \frac{\lambda}{2}\eta\rho x + \frac{\alpha\beta\gamma}{4}Lx + \frac{3\alpha}{16}Lx^2$$
$$+ \frac{3\alpha}{4}Lxq_0^2 - \frac{\alpha}{2}(\rho+\delta)Lxq_0 + \frac{1}{2Lx} \qquad (328)$$

where

$$x = q^{-2} \quad \text{and} \quad \langle q_0|\Psi'\rangle = \Psi'(q_0)$$

Stationarity requires that

$$\frac{\partial E}{\partial q_0} = \frac{\partial E}{\partial x} = \frac{\partial E}{\partial L} = 0 \qquad (329)$$

Minimization, with respect to q and x, is achieved if

$$\frac{\partial^2 E}{\partial q_0^2}, \frac{\partial^2 E}{\partial x^2} > 0 \quad \text{and} \quad \left(\frac{\partial^2 E}{\partial x\,\partial q_0}\right)^2 < \frac{\partial^2 E}{\partial x^2}\frac{\partial^2 E}{\partial q_0^2} \qquad (330)$$

Written out in full, the first three equations of eqn. (329) are (in consecutive order from the left):

$$\bar{q}_0^3 - (\gamma+\beta)\bar{q}_0^2 + \left(\beta\gamma + 2\frac{\lambda}{\alpha}\frac{\eta}{L} + \frac{3}{2}\chi\right)\bar{q}_0 - 2\frac{\lambda\beta}{\alpha}\frac{\eta}{L} - (\beta+\gamma)\frac{\lambda}{2} = 0 \qquad (331)$$

$$\chi^3 + \frac{4}{3}\left[\frac{\lambda}{\alpha}\frac{\eta}{L} + \frac{\beta\gamma}{12} + \frac{3}{2}\bar{q}_0^2 - \bar{q}_0(\beta+\gamma)\right]\chi^2 - \frac{4}{3}\frac{1}{\alpha L^2} = 0 \qquad (332)$$

$$\bar{q}_0^4 - \frac{4}{3}(\beta+\gamma)\bar{q}_0^3 + 2\beta\gamma\bar{q}_0^2 + \bar{q}_0^2 - 2\beta\chi\bar{q}_0 + \beta\gamma\chi + \frac{3}{4}\chi^2 - \frac{2}{\alpha L^2\chi} = 0 \qquad (333)$$

The inequalities $\frac{\partial^2 E}{\partial \bar{q}_0^2} > 0$, $\frac{\partial^2 E}{\partial \chi^2} > 0$ will both be satisfied provided that

$$3\bar{q}_0^2 - 2(\gamma+\beta)\bar{q}_0 + \beta\gamma + 2\frac{\lambda}{\alpha}\frac{\eta}{L} + \frac{3}{2}\chi > 0 \qquad (334)$$

The final inequality of eqn. (330) is rather complicated to write out in full and anyway turns out to be easily satisfied a posteriori.

Before proceeding to attempt to solve eqns. (331) - (333) in the Creutz limit (eqn. (155)) we take time off to make a few general comments. If, instead of using our modified Creutz Lagrange density (eqn. (311)), we had stuck to Creutz's original choice

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \partial_\mu\psi\,\partial^\mu\psi^* - V(\phi) - \lambda|\psi|^2(\phi-\beta)^2 \qquad (335)$$

then our trial state (corresponding to eqn. (325)) would have

looked something like

$$\Psi(\underline{r}_0;\underline{r}_{01},\underline{r}_{02};\underline{r}_{11},\underline{r}_{12};\underline{r}_{21},\underline{r}_{22};\underline{r}_{31},\underline{r}_{32};\underline{r}_{41},\underline{r}_{42};\underline{r}_{51},\underline{r}_{52};\underline{r}_{61},\underline{r}_{62};\underline{r}_{71},\underline{r}_{72};\underline{r}_{81},\underline{r}_{82};\underline{r}_{91},\underline{r}_{92};\underline{r}_{A1},\underline{r}_{A2};\underline{r}_{B1},\underline{r}_{B2};\underline{r}_{C1},\underline{r}_{C2}) =$$
$$N\cdot\exp\left[-\frac{1}{2}\underline{r}_{A1}\cdot(\underline{r}_0-\underline{r}_{A1})^2-\frac{1}{2}\underline{r}_{B1}\cdot(\underline{r}_0-\underline{r}_{B1})^2-\frac{1}{2}\underline{r}_{C1}\cdot(\underline{r}_0-\underline{r}_{C1})(\underline{r}_0-\underline{r}_{B1})\right] \qquad (336)$$

The energy of our system in this trial state would have been parametrized by six (potentially independent) variables. We would than have six simultaneous (and highly 'mixed') stationarity equations to solve as well as a host of minimisation conditions to satisfy as well. So, in order to reduce the number of variational variables to a more manageable level it is necessary to disregard the quark degree of freedom by replacing it by a source term. It is important to stress that in the strong coupling regime there is no reason whatsoever for assuming that the ground state of the whole system is simply the product of the ground states of the constituent systems (i.e. the notion that the vacuum state is nonseparable). This is why the term associated with the parameter c^2 (which causes alot of computational trouble) cannot be dismissed. Another point is that because of the fact that eqns. (331) - (333) are highly 'mixed', a straightforward solution is ruled out-- instead we must seek a self-consistent solution.

We now restrict our attention to the specific case $n_1 = 1$. Using this fact we may compound equations (332) and (335) to give us an L-independent equation which turns out to be

$$\frac{3}{4}X^2+\left[-\frac{A}{2}\frac{9}{\underline{r}_0}+3\underline{r}_0^2+2\underline{r}_0X\right]X-\underline{r}_0^2\left[\underline{r}_0^2-\frac{4}{3}(2A)\frac{\underline{r}_0}{\underline{r}_0}+\frac{4}{9}(2A)^2\right]=0 \qquad (337)$$

After some tedium it is possible to convince oneself that the leading order self-consistent solution to eqns. (331) and (337) is

$$\frac{q}{\tau} = \frac{\beta}{2} \quad \text{and} \quad \chi = \frac{\beta}{\sqrt{3}} \qquad (338)$$

Here we have used the fact that we wish to keep relation (155) intact. These solutions imply that

$$L^2 = \frac{1}{R_1}\sqrt{\frac{3}{\beta}}\,\rho^{1+f_1} \qquad (339)$$

What do these results mean? Equation (338) tells us that the width and displacement from the origin of our Gaussian trial state become vanishingly small in the bag formation limit of $\beta \to 0$. Also the vacuum expectation value of the τ-field has undergone a finite renormalization.

Finally a note on the $\rho \to 0$ limit. By inserting eqns. (338) and (339) into eqn. (328), we are led to the result that the minimum value of E is realised at some non-zero value of β, say β_0. The exact value of β_0 is 'model-dependent', in that it depends on the particular values assigned to the 'arbitrary' parameters of eqn. (155). The fact that the limit $\beta \to 0$ cannot be exactly taken is only due to the presence of the intrinsic quantum fluctuations of our system. The important point is that the classical role of L_o has been completely usurped! If it were not for the uncertainty relations, the energetically preferred configuration would be the one with L = 0 (as opposed to the classical situation where $L = L_o$ was the ideal state of affairs). Hence, at least within the context of our crude approximation scheme, the quantized version of our classical bag is energetically unstable and essentially prefers to contract down to the minimum

possible size. Hence we have the result that the source term is not sufficient to bring stability to our quantized bag.

4.3 Quark Trapping in the Quantised SLAC Bag

In this section we attempt to determine whether or not the semiclassical results of section 2.2 are maintained in a more realistic quantization scheme[19] than that described in the appendix .

Before going into any detail, let us first explain why we might expect strong coupling effects to be large. Consider a quark field ψ interacting with a scalar field ϕ via the Hamiltonian

$$H = \int d^3x \left[\psi^\dagger \left(\frac{\alpha}{i} \cdot \nabla + \beta\phi \right) \psi + \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} |\vec{\nabla}\phi|^2 + \frac{1}{4}(\phi^2 - f^2)^2 \right] \tag{340}$$

This is the prototype quark trapping Hamiltonian of section 2.2. The stable semi-classical vacuum corresponds to $\langle\phi\rangle = \pm f$, the normal vacuum being unstable.

From the detailed analysis of section 2.2, we know that semiclassical bags can exist. At large distances the semiclassical $\langle\phi\rangle$ field must attain one of the stable vacuum values. Because of the existence of two inequivalent stable vacua it is possible for $\langle\phi\rangle$ to take values + f outside the bag and - f in the interior, giving rise to a potential well which traps the ψ field.

By definition, in the semiclassical approximation the stable vacua are always $\langle\phi\rangle = \pm f$. In more realistic approximation schemes this is not necessarily so. For example, in a self-consistent Hartree-Fock approximation for the scalar field alone it happens that, in the strong coupling limit, it is the normal vacuum $\langle\phi\rangle = 0$ that becomes stable.
If this were true for the Hamiltonian (340) we would expect the quark trapping to fail because the ϕ field will now merely give a penetrable barrier.

We shall first briefly consider the vacuum stability for the Hamiltonian (340) and then examine its consequences for quark trapping. We shall find that there are domains of coupling constant strengths for which the semi-classical bags are destroyed by quantum effects. Equally, there are domains of coupling strengths for which quark trapping plausibly exists.

Consider the energy density E of the vacuum in the theory given by the Hamiltonian (340). We assume that with respect

to the vacuum the mass of the ψ field is M, that of the ϕ field μ, and $\langle\phi\rangle = \bar{\phi}$. In the Hartree-Fock approximation, obtained most easily by normal ordering (as described in chapter three), it is straightforward to see that

$$\mathcal{E}[\mu, M, \bar{\phi}] = \left[\frac{1}{4(2\pi)^3}\right] 4\pi R \left\{ \frac{2k^2+\mu^2}{(k^2+\mu^2)^{1/2}} - \frac{2(k^2+M(4\bar{\phi}))}{(k^2+M^2)^{1/2}} \right\} \Bigg]$$
$$+ H\left[(\bar{\phi}^2 - f^2)^2 + 3\bar{\phi}^2 g(\mu) - f^2 g(\mu) + \frac{3}{4} g^2(\mu)\right] \Bigg] \tag{341}$$

where

$$g(\mu) = \frac{1}{(2\pi)^3} \int d^3k \, \frac{1}{(k^2+\mu^2)^{1/2}} \tag{342}$$

Varying with respect to M, μ and $\bar{\phi}$ shows that the extrema of E occur when

a) $$M = 4\bar{\phi} \tag{343}$$

b) $$\mu^2 = 4H\left[3\bar{\phi}^2 - \left(f^2 - \frac{3}{2} g(\mu)\right)\right] \tag{344}$$

c) $$\bar{\phi}\left[\bar{\phi}^2 - \left(f^2 - \frac{3}{2} g(\mu)\right)\right] = \frac{M}{8\pi} g(M) \tag{345}$$

Inserting eqn. (343) into eqn. (345) gives the following two solutions

A. Normal Vacuum

$$\bar{\phi} = 0 = M \tag{346}$$

$$\mu_0^2 = -4H\left(f^2 - \frac{3}{2} g(\mu_0)\right) \tag{347}$$

$$E[\mu_0,0,0] = \left[\frac{1}{4(2\pi)^3}\int d^3k\left\{\frac{2k^2+\mu_0^2}{(k^2+\mu_0^2)^{1/2}} - 2\sqrt{k^2}\right\}\right] + H\left[f^4 - f^2 g(\mu_0) + \frac{3}{4} g^2(\mu_0)\right] \quad (348)$$

B. Abnormal Vacuum

$$\bar{\Phi}^2 = \left(f^2 - \frac{3}{2} g(\mu)\right) + \frac{g^2}{8H} g\left(\frac{4}{g}\bar{\Phi}\right) \quad (349)$$

$$\mu^2 = 4H\left[3\bar{\Phi}^2 - \left(f^2 - \frac{3}{2} g(\mu)\right)\right] \quad (350)$$

$$E[\mu,M,\bar{\Phi}] = \left[\frac{1}{4(2\pi)^3}\int d^3k\left\{\frac{2k^2+\mu^2}{(k^2+\mu^2)^{1/2}} - (k^2+g^2\bar{\Phi}^2)^{1/2}\right\}\right] + H\left[(\bar{\Phi}^2-f^2)^2 + 3\bar{\Phi}^2 g(\mu) - f^2 g(\mu) + \frac{3}{4} g^2(\mu)\right] \quad (351)$$

In solving these two sets of equations we will consider the following domains of coupling constant strengths.

a. Solutions for $H \gg g^4 \gg 1$

The coupling strength g^2 does not occur in the equations for the normal vacuum. Observing that $g\left(\frac{4}{g}\bar{\Phi}\right) \sim \frac{4}{g}\bar{\Phi}\ln\frac{4}{g}\bar{\Phi}$ we see that when $\frac{g^2\bar{\Phi}^2}{8H} \ll 1$ we can neglect the last term in eqn. (349) to give

$$\mu^2 = 3H\left(f^2 - \frac{3}{2} g(\mu)\right) \quad (352)$$

$$M^2 = g^2\bar{\Phi}^2 \sim \frac{g^2\mu^2}{8H} \ll \mu^2 \quad (353)$$

From eqn. (353) we see that E loses its dependence on g and the quark field drops out of the problem, which now becomes the much

simpler one of the stability of the ϕ^4 vacuum. The equations to be solved are now (347) and (352). Eqn. (352) has been discussed in detail in ref. 20 . The solutions to these equations are then inserted into the energy difference

$$\Delta \mathcal{E} = E[\mu_0, 0, 0] - E[\mu, M, \bar{\Phi}]$$
$$= \Delta\omega + H\left[\bar{\Phi}^4 + \left(g(\mu) - g(M)\right)\left[\bar{\Phi}^2 + \frac{3}{4}\left(g(\mu) - g(M)\right)\right]\right] \qquad (354)$$

where $\Delta\omega$ is the difference of the vacuum fluctuation integrals (bracketed first terms in eqns. (348) and (351)).

For three spatial dimensions (although not for less) there is some ambiguity in the definition of the infinite integrals since their infinite parts cannot be simultaneously eliminated in the two phases represented by eqns. (347) and (352). The two obvious tactics are to replace integrals by their finite parts(21) or to regularise.(22) Whichever is adopted, we reach the same result that for $H \gg 1$, $\Delta\mathcal{E} < 0$ and the normal vacuum is stable.

b. Solutions for $g^4 \gg H \gg 1$

Again there is ambiguity over the definition of infinite integrals. We shall replace them everywhere by their finite parts with a single subtraction point $\bar{\Phi}^2 = \lambda^2$. Denoting the finite part of g (written $F.P.\, g$) by $\bar{g}$, the equations become

$$16H\bar{\Phi}^2 = 2\mu^2 - g^2\bar{g}(g\bar{\Phi})$$
$$= 2\mu^2 - \frac{g^4\bar{\Phi}^2}{8\pi^2}\ln\left(\frac{g^2\bar{\Phi}^2 e}{4\lambda^2}\right) \qquad (355)$$

$$\mu^2 = +H\left[3\bar{\phi}^2 - f^2 + \frac{3}{16\pi^2}\mu^2 \ln\left(\frac{\mu^2 e}{4\Lambda^2}\right)\right] \qquad (356)$$

and $$\mu_0^2 = -H\left[\ldots\right]^2 - \frac{3}{16\pi^2}\mu_0^2 \ln\left(\frac{\mu_0^2 e}{4\Lambda^2}\right)\Big] \qquad (357)$$

(implying infinite renormalisation of f^2). We see that the main hope for the abnormal vacuum to remain stable is in the second integrand of eqn. (356) becoming large. This would automatically happen if $\bar{\phi}$ were independent of q, implying a dependence of the subtraction point Λ on q as $4\Lambda^2 = q^2 e$, for finite α. Eqn. (355) then has the unique solution

$$\bar{\phi} = \alpha \qquad (358)$$

In turn eqn. (357) has the unique solution

$$\mu_0 = q\bar{\phi} = M \qquad (359)$$

and eqn. (356) the solution

$$\mu = q\bar{\phi} = M \qquad (360)$$

(with the additional possibility that $\mu^2 = O(-q^2)$ if $3\alpha^2 > f^2$)

The energy difference is now easily seen to be

$$\Delta\mathcal{E} \simeq \text{F.P.}\left[\frac{1}{2(2\pi)^3}\int d^3k\left(\sqrt{k^2+q^2}-\sqrt{k^2}\right)\right] > 0$$

if $\mu^2 \simeq M^2$, and (361)

$$\Delta\mathcal{E} \simeq \text{F.P.}\left[\frac{1}{2(2\pi)^3}\int d^3k\left(\frac{4k^2+3q^2}{\sqrt{k^2+q^2}}-4\sqrt{k^2}\right)\right] + \frac{q^4\lambda^4}{48\pi^2} > 0$$

if $\mu^2 \simeq O(\lambda q^2)$ (362)

Thus, in the limit $q^4 \gg \mu^4 \gg 1$ the abnormal vacuum remains stable.

We are only stating the obvious when we stress that the above manipulations are very crude. However, taking them seriously we are led to the conclusion, for the reasons mentioned earlier, that we would expect quantum bags to exist for $q^4 \gg \mu^4 \gg 1$, but not for $\mu^4 \gg q^4 \gg 1$.

It is, of course, more compelling if we can replace such plausibility arguments by detailed calculations. In the following, we shall consider the latter case in detail, to show explicitly how quantised bags fail to occur when $\mu^4 \gg q^4 \gg 1$, even though classical bags exist.

The first case of interest is $\mu^4 \gg q^4 \gg 1$ for which, as we have just seen, the normal vacuum $\langle\phi\rangle = 0$ becomes stable. For this domain, the quantum effects of the quark field become negligible. Taking this cue (which will be justified later),

we shall also neglect quark quantum effects in the Hartree-Fock approximation for quark trapping.

A straightforward way to do this is to replace the Hamiltonian density $\mathcal{H}_D(x)$ of eqn. (340) by the simpler

$$\mathcal{H}_0(x) = \psi^\dagger\left[\frac{\sigma\cdot\nabla}{i} + q\phi(1+\sigma\cdot\hat{r})\right]\psi + \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}|\vec{\nabla}\phi|^2 + B(\phi^2-1)^2 \tag{363}$$

where ψ is now a two-component non-relatavistic spinor and $\underline{L} = -i\underline{x}\wedge\underline{\nabla}$ is the orbital angular momentum.

It can be seen that $\mathcal{H}_0$ gives the identical static classical field distribution for the one-quark state to that of the SLAC Hamiltonian (71). We observe that the classical equations for the static one-quark state arising from $\mathcal{H}_0$ are

$$\nabla^2\phi - 4B\phi(\phi^2-1) = q\psi^\dagger[1+\sigma\cdot\hat{r}]\psi \tag{364}$$

$$\left[\frac{\sigma\cdot\nabla}{i} + q\phi(1+\sigma\cdot\hat{r})\right]\psi = \varepsilon\psi \tag{365}$$

where ε is the Lagrange multiplier associated with the constraint

$$\int d^3x\,\psi^\dagger(x)\,\psi(x) = 1 \tag{366}$$

Requiring ϕ to be purely radial $\phi = \phi(r)$ forces ψ to have the form

$$\psi(x) = \frac{1}{r}\left[iG(r) + \frac{\sigma\cdot x}{r}F(r)\right]u \tag{367}$$

where u is a constant spinor satisfying $u^+ u = 1$. Inserting this form for ψ in eqn. (365) gives

$$\frac{dg_0}{dr} = \frac{f_0}{r} + F_0(\varepsilon + q\phi) \qquad (368)$$

$$\frac{df_0}{dr} = -\frac{F_0}{r} - g_0(\varepsilon - q\phi)$$

The normalisation condition becomes

$$4\pi \int dr\,(F_0^2 + g_0^2) = 1 \qquad (369)$$

with

$$\psi^+(1 + \alpha\cdot\hat{r})\psi = \frac{1}{r^2}(g_0^2 - F_0^2) \qquad (370)$$

for real g_0, F_0. Equations (368), (369) and (364), using eqn. (370), are identical to the equations for the static one-quark state obtained from $\psi_0(\Delta)$ of eqn. (71). In chapter two (section one), a detailed solution to these equations was found when $\#\Delta \gg 1$.

In displaying the Hartree-Fock approximation it is most convenient to work in the Schrodinger picture field representation ($\psi \to \psi$, $\dot{\psi} \to -i\psi$). Following Rosen[23], we take the one-quark state functional to have the form

$$|\Phi_0\rangle \equiv |[\bar{\psi}(\underline{x}), f(\underline{x},\underline{y}), \psi(\underline{x})]\rangle$$

$$= N^{-1}\left[\exp\left(-\frac{1}{2}\right)\int d^3x\, d^3y \left(\bar{\psi}(\underline{x}) - \bar{\psi}(\underline{x})\right) f(\underline{x},\underline{y}) \left(\phi(\underline{y}) - \bar{\phi}(\underline{y})\right)\right]$$

$$\int \psi^\dagger(\underline{x})\, \psi(\underline{x})\, d^3x\, |\Omega_0\rangle \qquad (371)$$

In eqn. (371), $|\Omega_0\rangle$ is the non-relatavistic quark vacuum $\psi(\underline{x})|\Omega_0\rangle = 0$ and $\bar{\phi}(\underline{x})$, $\psi(\underline{x})$ and $f(\underline{x},\underline{y}) = f(\underline{y},\underline{x})$ are c-number distributions to be determined by variational method. The normalisation constant N is chosen so that $\langle \Phi_0 | \Phi_0 \rangle = 1$.

In this single quark state we have

$$\bar{\psi}(\underline{x}) = \langle \Phi_0 | \phi(\underline{x}) | \Phi_0 \rangle \qquad (372)$$

$$\psi(\underline{x}) = \langle \Phi_0 | \psi(\underline{x}) | \Phi_0 \rangle \qquad (373)$$

$$\frac{1}{2} g(\underline{x},\underline{y}) = \langle \Phi_0 | \phi(\underline{x})\, \phi(\underline{y}) | \Phi_0 \rangle - \bar{\phi}(\underline{x})\, \bar{\phi}(\underline{y}) \qquad (374)$$

where $g(\underline{x},\underline{y}) = g(\underline{y},\underline{x})$ is the distribution inverse to $f(\underline{x},\underline{y})$

$$\int d^3x\, f(\underline{x}',\underline{x})\, g(\underline{x},\underline{x}'') = \delta^{(3)}(\underline{x}' - \underline{x}'') \qquad (375)$$

It immediately follows that the energy of the one-quark state is

$$E_0[\bar{\psi}, \psi, N] = \int d^3x\, \langle \Phi_0 | \mathcal{H}_0(\underline{x}) | \Phi_0 \rangle$$

$$= \int d^3x \left[\frac{1}{4} f(\underline{x},\underline{x}) + \frac{i}{4} \vec{\nabla}_x \cdot \vec{\nabla}_y\, g(\underline{x},\underline{y})\Big|_{\underline{x}=\underline{y}}\right.$$

$$+ \psi^\dagger\left(\frac{\underline{\sigma}\cdot\underline{\Sigma}}{i} + q\phi(1 + \underline{\sigma}\cdot\underline{\tau})\right)\psi +$$

$$\left.\frac{1}{2}\left[\left(\bar{\psi}(\underline{x})^2 - \tau^2\right)^2 + 3\bar{\psi}^2(\underline{x})\, g(\underline{x},\underline{x}) - f^2 g(\underline{x},\underline{x}) + \frac{3}{4} f^2(\underline{x},\underline{x})\right]\right. \qquad (376)$$

Varying E_q with respect to $\bar{\Phi}, f$ and $\psi^\dagger$ gives the coupled equations

$$\nabla^2\bar{\Phi}(\underline{x}) - 4H\bar{\Phi}(\underline{x})\left[\bar{\Phi}^2(\underline{x}) - \left(f^2 - \tfrac{3}{2}g(\underline{x},\underline{x})\right)\right] = g\psi^\dagger(\underline{x})\left[1 + \underline{\tau}\cdot\underline{I}\right]\psi(\underline{x}) \qquad (377)$$

$$\delta^{(3)}(\underline{x}'-\underline{x}'') = \int d^3x\, g(\underline{x}',\underline{x})\left[-\nabla_x^2 + 4H\left\{3\bar{\Phi}^2(\underline{x}) - \left(f^2 - \tfrac{3}{2}g(\underline{x},\underline{x})\right)\right\}\right]g(\underline{x},\underline{x}'') \qquad (378)$$

$$\left[\frac{\underline{\sigma}\cdot\underline{Y}}{i} + g\bar{\Phi}(\underline{x})\left(1 + \underline{\tau}\cdot\underline{I}\right)\right]\psi(\underline{x}) = \varepsilon\psi(\underline{x}) \qquad (379)$$

where ε is the Lagrange multiplier associated with the constraint

$$\int d^3x\, \psi^\dagger(\underline{x})\psi(\underline{x}) = 1 \qquad (380)$$

We now go on to consider approximate solutions to these equations.

We first observe that the quark field equation (379) has the same form as the classical equation (93), and can be solved in a similar fashion. We look for bag-like solutions to these equations centred on the origin $\underline{x} = 0$. That is, we are only interested in those solutions for which ψ falls off fast for large $r = |\underline{x}|$. Now, as $r \to \infty$, we must have $\bar{\Phi}(\underline{x}) \to \bar{\Phi}(r)$, the 'true' vacuum value. Thus, for large distances the ψ equation (written in the form of eqn. (363)) becomes

$$\frac{d^2 G_0}{dr^2} = (g^2 \bar{\phi}^2 - \mathcal{E}^2) G_0 \qquad (381)$$

$$\frac{d^2 F_0}{dr^2} - \frac{2}{r}\frac{dF_0}{dr} = (g^2 \bar{\phi}^2 - \mathcal{E}^2) F_0$$

We see that eqns. (381) can only have bag-like solutions if

$$g^2 \bar{\phi}^2 > \mathcal{E}^2 \geq 0 \qquad (382)$$

That is, we can only have quark trapping for the abnormal stable vacuum.

Of course, in the $r \to \infty$ limit, equations (377) and (378) must be equivalent to the vacuum stability equations discussed earlier. That they are so is trivial to see. For example, consider the abnormal vacuum solution $\bar{\phi}(r) \to \bar{\phi} \neq 0$ as $r \to \infty$, in which limit $\mathcal{G}(\underline{x},\underline{y}) \to \bar{\mathcal{G}}(\underline{x},\underline{y})$. Eqns. (377) and (378) are solved as

$$\bar{\phi}^2 = f^2 - \frac{3}{2}\mathcal{G}(\underline{x},\underline{x}) \qquad (383)$$

$$\bar{\mathcal{G}}(\underline{x},\underline{y}) = \frac{1}{(2\pi)^3}\int \frac{d^3k\; e^{i\underline{k}\cdot(\underline{x}-\underline{y})}}{\left[\underline{k}^2 + 4H\left\{3\bar{\phi}^2 - \left(f^2 - \frac{3}{2}\mathcal{G}(\underline{x},\underline{x})\right)\right\}\right]^{1/2}} \qquad (384)$$

Observing that $\bar{\mathcal{G}}(\underline{x},\underline{x})$ is independent of $\underline{x}$, eqn. (384) becomes more familiar if we write it as

$$\mu^2 = 4H\left[3\bar{\phi}^2 - \left(f^2 - \frac{3}{2}\mathcal{G}(\mu)\right)\right] \qquad (385)$$

$$\mathcal{G}(\mu) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{\left[\underline{k}^2 + \mu^2\right]^{1/2}} \qquad (386)$$

with $\mathcal{G}(\mu) = \bar{\mathcal{G}}(\underline{x},\underline{x})$. Eqns. (384) and (386) are just

the Hartree-Fock mass equation of (352). That is, we have reproduced the abnormal vacuum equations (342), (349) and (350) in the $f/\mu < 1$ limit.

We have already said that in this limit, for $H \gg 1$, the normal vacuum becomes stable, and so from the previous argument quark trapping will not occur. We will not reproduce the details of the proof of normal vacuum stability, since it is computationally tedious. We now show the absence of quark trapping directly and simply by approximately solving eqns. (371), (378) and (371) in the limit $H \gg 1$.

We adopt the WKB approximation to eqn. (378), that[24]

$$g(\underline{x},\underline{y}) = \frac{1}{(2\pi)^3}\int \frac{d^3k\; e^{i\underline{k}\cdot(\underline{x}-\underline{y})}}{\left(\underline{k}^2 + Z_1(\underline{x},\underline{y})\right)^{1/2}} \qquad (387)$$

where

$$Z_1(\underline{x},\underline{x}) = 4H\left[3f^2(\underline{x}) - \left(f^2 - \tfrac{3}{2}g(\underline{x},\underline{x})\right)\right] \qquad (388)$$

Defining $f^2(\underline{x})$ by $f^2(\underline{x}) = f^2 - \frac{3}{2}g(\underline{x},\underline{x})$

we can rewrite eqn. (387) as

$$f^2(\underline{x}) = f^2 - \frac{3}{2(2\pi)^3}\int \frac{d^3k}{\left[\underline{k}^2 + 4H\left(3f^2(\underline{x}) - f^2(\underline{x})\right)\right]^{1/2}} \qquad (389)$$

Eqn. (377) now becomes

$$\nabla^2\bar{\phi}(x) - 4H\bar{\phi}(x)\left[\bar{\phi}^2(x) - f^2(x)\right] = \phi\,\mathcal{M}^4\left(1 + \Sigma\cdot\frac{\bar{\phi}}{\mathcal{M}}\right)^{\mathcal{N}} \qquad (390)$$

To solve eqn. (390) we first absorb the infinite part of the integral into the definition of f^2 (infinite mass renormalisation) as

$$f^2(x) = f^2 - \frac{3H}{4\pi^2}\left(3\bar{\phi}^2(x) - f^2(x)\right)\ln\left[\frac{3\bar{\phi}^2(x) - f^2(x)}{4\cdot\Lambda^2}\right] \qquad (391)$$

where Λ^2 is related to some arbitrary subtraction point.

Suppose that a bag solution exists. Then $\bar{\phi}(r) \to \bar{\phi} \neq 0$ as $r \to \infty$. Similarly, from eqn. (390) we will have $f(x) \to \bar{f}$ also as $r \to \infty$. Thus $\bar{f}$ and Λ are related by

$$\bar{f}^2 = f^2 - \frac{3H}{4\pi^2}\bar{\phi}^2\ln\left(\frac{\bar{\phi}^2}{\Lambda^2}\right) \qquad (392)$$

This enables us to eliminate Λ^2, to give

$$\frac{f^2(x)}{2}\left(3 - \frac{f^2}{\bar{\phi}^2}\right) = f^2 + \frac{3}{2}\bar{\phi}^2(x)\left(1 - \frac{\bar{f}^2}{\bar{\phi}^2}\right) - \frac{3H}{4\pi^2}\left(3\bar{\phi}^2(x) - f^2(x)\right)\ln\left[\frac{3\bar{\phi}^2(x) - f^2(x)}{2\bar{\phi}^2}\right] \qquad (393)$$

For large H, eqn. (393) can be solved to give

$$f^2(x) = 3\bar{\phi}^2(x) - 2\bar{f}^2 + \frac{4\pi^2}{H}\left(\bar{\phi}^2(x) - \bar{\phi}^2\right) + o(H^{-2}) \qquad (394)$$

whence eqn. (390) becomes

$$\nabla^2\bar{\phi}(x) + g^2\left[H + 3\lambda\bar{\phi}^2 + O(H^{-1})\right]\bar{\phi}(x)\left[\bar{\phi}^2(x) - \bar{f}^2\right] = g\,\bar{\psi}^+(x)\left(1 + \gamma\cdot\partial\right)\psi(x) \qquad (395)$$

Comparing eqn. (395) to eqn. (364) we see that for large H the potential has effectively changed from

$$V(\phi) = H\left(\phi^2 - f^2\right)^2 \qquad (396)$$

to

$$V(\phi) = -2\left[H + 3\lambda\bar{\phi}^2 + O(H^{-1})\right]\left(\phi^2 - \bar{\phi}^2\right)^2 \qquad (397)$$

reversing sign. (Note that for small H, $\bar{f}^2(x) = f^2 + O(H)$, whence eqn. (390) and eqn. (364) become identical). This suggests very strongly that the stable vacuum is now $\bar{\phi} = 0$. (The value of H at which the effective potential changes sign depends on both f and Λ i.e. on the renormalised ϕ mass and the subtraction point). In particular, the prototype equation

$$\frac{d^2\phi}{dr^2} = \frac{\partial V}{\partial\phi} \qquad (398)$$

with solution

$$\phi(r) = f\tanh\left(\sqrt{\lambda H}\,f\,(r - R)\right) \qquad (399)$$

for the quark trapping potential would be replaced by

$$\frac{d^2\bar{\phi}}{dr^2} = \frac{\partial\bar{V}}{\partial\bar{\phi}} \qquad (400)$$

The solution to eqn. (400) is[17]

$$\Phi(r) = \sqrt{2}\,\Phi \operatorname{sech}\left(\sqrt{2}\,\Phi\,(r-R)\right) \qquad (401)$$

which tends to zero as $r \to \infty$ (i.e. instead of a potential well we have merely a potential barrier). This, however, contradicts our initial assumption that $\Phi(r) \to 0$ for large r.

Thus, for $H \gg 1$ quark trapping does not occur when $H \gg G^4$. Only for small H, when the potential has not changed sign, can trapping occur, but this would not correspond to realistic hadrons.

We will now make some comments on the Hartree-Fock-like approximation scheme when self-consistent quantisation of the quark field is also taken into account. From our earlier investigations on the self-consistent vacuum we see that such quantization must be included in the strong quark coupling limit $G^4 \gg H \gg 1$.

We don't know which approximation scheme for one-quark states reduces to the equations (355), (356) and (357) in the large distance limit. However, assuming separability on the lines of eqn. (371) we would expect the following equations for ψ, Φ and g (defined as expectation values in the one-quark state as in eqn. (372)).

a) The ψ field equation will retain the simple form

$$\left[\frac{\alpha \cdot \nabla}{i} + g\beta\Phi\right]\psi = \varepsilon\psi \qquad (402)$$

This is a consequence of the fact that the Hartree-Fock approximation does not alter the degree of polynomial field equations.

b) Since $\Gamma^{(2)}(x,x)$ (eqn. (388)) is the second derivative (with respect to ϕ) of the Hartree-Fock effective potential, and the field couples linearly to the quark field, eqn. (378) relating ϕ to j will be unaltered. That is, assuming $\Gamma \to 0$ as $r \to \infty$, the effective self-interaction potential $V(\phi)$ (eqn. (396)) will be replaced by $V(\phi)$ (eqn. (397)). In consequence, having eliminated $S(x,y)$, the ϕ equation (previously eqn. (395)) will become

$$\nabla^2 \phi(x) - V'(\phi(x)) = g\left[\phi(x) N(x) + S(x,x)\right] \qquad (403)$$

where $S(\underline{x}, \underline{y})$ will be determined by $\mathrm{Tr}\langle q | \psi(x) \bar{\psi}(y) | q \rangle$. It follows that S will depend on ϕ, on the lines of

$$S(x,x) \sim g\,\phi(x) \int \frac{d^3k}{\left[k^2 + g^2 \phi^2(x)\right]^{1/2}} \qquad (404)$$

where the expression for S has to match with the large distance identification

$$\mathrm{Tr}\langle q | \psi \bar{\psi} | q \rangle \sim M \int \frac{d^3k}{\left[k^2 + M^2\right]^{1/2}} \qquad (405)$$

with $M = g\phi$.

In the case $H \gg 1$ the term in S on the right hand side of eqn. (403) becomes negligible and we have the relatavistic variant of eqn. (316), justifying the approximation of the previous analysis in this coupling strength domain. For $H \gg 1$ this term cannot

be ignored. In fact, $\psi^{(1)}(x,x)$ will dominate $\psi^{(0)}(x)$. In consequence, we have no contradiction with the initial assumption that $\psi^{(0)} \to 0$ at large distances. We are, however, totally unable to suggest the nature of the solution to the new $\psi^{(0)}$ eqn. (403).

So far we have only considered quark trapping for single-quark states. In this context the classical quark is 'trapped' in the region where the classical field has flipped from its vacuum value. Realistic hadrons must be multiquark states. In chapter two we briefly reviewed the main semiclassical results for multiquark states. We shall now briefly consider the multiquark Hartree-Fock states for the particular case $\lambda^4 \ll H$. As in the previous analysis, we assume that, in this domain of coupling strengths, we can use the non-relatavistic quark Hamiltonian of eqn. (363). Restricting ourselves to N quark states we see that the N-quark Hartree-Fock state $|N_q\rangle$ has the form

$$|N_q\rangle \equiv |[\psi^{(0)}, f, \phi_i]\rangle$$
$$= \lambda^{-1} \exp\left[-\frac{1}{2}\int d^3x\, d^3y \left(\psi(x) - \psi^{(0)}(x)\right) f(x,y) \left(\psi(y) - \psi^{(0)}(y)\right)\right]$$
$$\prod_{i=1}^{N} \left(\int d^3x\, \psi^\dagger(x)\, \phi_i(x)\right) \Omega_0 \qquad (406)$$

It immediately follows that the Hartree-Fock energy functional becomes

$$E_{N_q}[\psi^{(0)}, \phi_i] \equiv \int d^3x\, \langle N_q | \mathcal{H}_0(x) | N_q \rangle$$
$$= \sum_i \varepsilon_i + \int d^3x \left[\frac{1}{4} f(x,x) + \frac{1}{4} \nabla_x \cdot \nabla_y\, g(x,y)\Big|_{x=y} + \mathcal{H}\left(\psi^{(0)2} - \psi^{(0)3} + 3\psi^{(0)2} g - \psi^{(0)} g + \frac{3}{4} g^2\right)\right] \qquad (407)$$

where the quark energies ϵ_i are determined by

$$\left[\frac{\underline{\alpha}\cdot\underline{\nabla}}{i} + q\bar{\tau}(\underline{x})\left(1+\tau_3\right)\right]\psi_i(\underline{x}) = \epsilon_i\psi_i(\underline{x}) \qquad (408)$$

and

$$\nabla^2\bar{\Phi} - \mu^2\bar{\tau}\left[\bar{\tau}^2 - \left(G - \tfrac{3}{2}\vartheta\right)\right] = q\sum_i^N \psi_i^\dagger\left(1+\tau_3\right)\psi_i \qquad (409)$$

(denoting $g(\underline{x}, \underline{x})$ by g). The equation relating $\vartheta(\underline{x}, \underline{y})$ to $\bar{\tau}$ is eqn. (378), as before.

The analysis of the preceding paragraphs goes straight through as before, giving the result that for $H \gg 1$ the effective potential changes signs to give (to order H^{-1})

$$\nabla^2\bar{\Phi}(\underline{x}) + g\left[\pi + \alpha\pi^2 - \bar{\tau}^2\right]\bar{\tau}(\underline{x})\left(\bar{\tau}(\underline{x}) - \bar{\tau}\right) = q\sum_i^N \psi_i^\dagger\left(1+\tau_3\right)\psi_i \qquad (410)$$

(assuming $\bar{\tau}(\underline{x}) \to \bar{\tau}$ as $r \to \infty$). Again we would have no bag-like solutions.

The relatavistic Hartree-Fock like approximation would again generalise in a similar way, justifying the above comments for $\mu^4 \ll H$. For $\mu^4 \gg H$ multiquark trapping is again plausible, although detailed solutions are unattainable.

We will end this section with a conclusion and summary of the results obtained. We have considered quark trapping in the basic Hamiltonian (340) of the SLAC bag. We argued that quark trapping requires the stability of the abnormal vacuum, and examined vacuum stability in the Hartree-Fock approximation. We reached the anticipated conclusions that, for relatively weak quark coupling ($H \gg \mu^4 \gg 1$) the abnormal vacuum is unstable, stability being

restored in the strong quark coupling domain ($g^4 \gg H \gg 1$).

In consequence, we do not expect quark trapping in the relatively weak quark coupling situation. To domonstrate explicitly how quark trapping fails in this case we have examined single quark states (in an equivalent classical theory with non-relatavistic quarks), also in the Hartree-Fock approximation. Quark trapping fails in this case because, in some sense, the Hartree-Fock effective potential reverses sign when $H \gg 1$. Plausible inferences for a relatavistic approximation only confirm this simpler model.

For the strong quark coupling situation ($g^4 \gg H \gg 1$) we are unable to provide a detailed solution. Nonetheless, our expectations from the simple vacuum stability arguments are reinforced.

CHAPTER V

CONCLUSION

In this thesis we have examined the effects that quantization induces on well-defined classical field configurations. We often found that the corresponding quantum fluctuations completely annihilated the classical configuration.

The approximation schemes we adopted all used a variational principle of one kind or another. Consequently we were not able to judge how far we should believe our results. Hence, it is important to test our techniques on (nontrivial) exactly soluble field theory models. Some progress in this direction has already been made.[25]

The extension of our methods to encompass gauge fields is also a major problem for the future. But, in my opinion, the major hurdle to jump is that of the nonseparability of the vacuum for strongly coupled field systems. Nonetheless, I feel that this particular area of research appears most promising.

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APPENDIX

Derivation of the Classical Field Equations from a Variational Principle

In this appendix we will show how we may translate the theory of the SLAC bag (which is phrased in the language of quantum field theory) into the mathematically more amenable language of classical field theory.

Consider the model Lagrange density

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \sigma)^2 - H(\sigma^2 - f^2)^2 + \bar{\psi}(i\gamma^\mu \partial_\mu - g\beta\sigma)\psi \qquad (411)$$

Here $\sigma(x)$ and $\psi(x)$ are scalar and spinor fields respectively whilst f is a constant parameter endowed with the dimensions of mass and $g, H > 0$ are dimensionless coupling constants.

In the variational scheme we will employ we will only need to consider the system eqn. (4.1) at a fixed time t, which we shall take to be t = 0.

We now go on to expand out the two field operators, σ and ψ, in terms of a normal mode Fock space basis.

For the scalar field we choose an expansion in terms of plane waves

$$\sigma(\underline{x}) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left[a_k e^{i\underline{k}\cdot\underline{x}} + a_k^\dagger e^{-i\underline{k}\cdot\underline{x}} \right] \tag{4.12}$$

where $\omega_k = \sqrt{\underline{k}^2 + m_\sigma^2}$ and $m_\sigma^2 = 8Hf^2$

The above system may now be quantized by imposing the usual equal-time commutation relations appropriate to a scalar field

$$[\Phi(\underline{x}), \pi(\underline{x}')] = i\delta^{(3)}(\underline{x} - \underline{x}') \tag{4.13}$$

where $\pi(\underline{x}) = \dot{\Phi}(\underline{x})$.

For the fermi field $\psi(\underline{x})$ we do something a little different. Here we expand out $\psi(\underline{x})$ in terms of the eigenfunctions of the Dirac equations in an external potential (which will be specified later when we come to do the actual variation calculation)

$$\psi(\underline{x}) = \sum_n B_n U_n(\underline{x}) + D_n^\dagger V_n(\underline{x}) \tag{4.14}$$

The positive and negative energy eigenfunctions $U_n(\underline{x})$ and $V_n(\underline{x})$ satisfy the following orthonormality conditions

$$\int d^3x\, u_n^*(\underline{x})\, u_m(\underline{x}) = \int d^3x\, v_n^*(\underline{x})\, v_m(\underline{x}) = \delta_{nm}$$

$$\int d^3x\, u_n^*(\underline{x})\, v_m(\underline{x}) = 0 \qquad (4.5)$$

The non-vanishing equal-time anti-commutation relations are

$$\{\psi(\underline{x}),\, \psi^\dagger(\underline{x}')\} = \delta^{(3)}(\underline{x} - \underline{x}')$$

$$\{B_n,\, B_m^\dagger\} = \{D_n,\, D_m^\dagger\} = \delta_{nm} \qquad (4.6)$$

We are now in a position to construct the Hilbert space of states for our system (at time t = 0) by applying the three types of "particle" creation operators $a_k^\dagger$, $D_m^\dagger$, $B_n^\dagger$ to the translationally <u>noninvariant</u> no-particle state $|0_L\rangle$ which is characterized by

$$a_k |0_L\rangle = B_n |0_L\rangle = D_m |0_L\rangle = 0 \qquad (4.7)$$

Of course it is possible to relate this expansion to the one in terms of plane waves and a translationally invariant no-particle state $|0_P\rangle$. Such a change of basis states is well-known and has been dubbed the Bogoliubov-Valatin transformation. For example, the relation between the two no-particle states $|0_L\rangle$ and $|0_P\rangle$ is given by

$$|0_L\rangle = \frac{1}{N} \prod_n B_n D_n |0_P\rangle \qquad (4.8)$$

where N is a normalization factor. We now come to the problem of

deciding how to properly define the Hamiltonian of eqn. (4.1), this is because the meaning of a product of field operators at the same space-time point is ambiguous and has to be properly defined. In other words we need a renormalization procedure to purge our theory of its inherent ultraviolet divergences. As the theory, eqn. (4.1), is a strongly coupled theory , the invention of a sensible renormalisation programme for it is a formidable problem. Here we will content ourselves by defining our Hamiltonian by a naive normal-ordering prescription. This normal-ordering prescription has the drawback that it depends on the particular expansion chosen for the field operators. In the following we shall ignore these differences in normal-ordering prescriptions. Hence the Hamiltonian we are working with is accurate only in the so-called 'tree' approximation. To be specific, we shall ignore the difference in energy between a theory with H normal-ordered in the basis (4.4) as constructed for the one-fermion sector and a theory normal-ordered relative to a translationally invariant (trial) vacuum state.

After these preliminaries we shall now go on to construct a trial state for our system. What motivating principle can we use to make a good ansatz for the trial state? Intuitively, we would expect the σ-field to develop a position-dependent expectation value which is peaked in the environment of the fermi source. How do we describe such a situation mathematically? Fortunately boson coherent states give a mathematical realization of the intuitive picture discussed above. The boson coherent states

we shall employ are defined by

$$|\vartheta\rangle = \exp\left[-i\int d^3x\, \vartheta(\underline{x})\,\dot{\sigma}(\underline{x})\right]|0_L\rangle$$
$$\equiv U(\vartheta)\,|0_L\rangle \qquad (4\text{-}19)$$

It is easy to see that the unitary operator $U(\vartheta)$ acts as a translation operator in σ-space, i.e.

$$U^{-1}(\vartheta)\, f(\sigma)\, U(\vartheta) = f(\sigma+\vartheta) \qquad (4\text{-}20)$$

It follows directly from eqn. (4-20) that if $f(\sigma)$ is any polynomial function of $\sigma(\underline{x})$ which is normal-ordered term by term relative to the localized no-particle state $|0_L\rangle$ then

$$\langle\vartheta|f(\sigma)|\vartheta\rangle = \langle 0_L|f(\sigma+\vartheta)|0_L\rangle$$
$$= f(\vartheta) \qquad (4\text{-}21)$$

We see from eqn. (4-21) that the tree approximation rule for taking the expectation value of a function of σ in a boson coherent state is to replace $\sigma(\underline{x})$ by the c-number amplitude $\vartheta(\underline{x})$. In a similar way we shall also want to replace the fermion field operator $\psi(\underline{x})$ by an arbitrary c-number Dirac spinor wavefunction when we evaluate the expactation value of the energy operator H in our trial state. In the case of a trial state of fermion number one, one can do this by using

$$|S\rangle = B_n^{+}\,|0_L\rangle \qquad (4\text{-}22)$$

where B_n^{+} is the operator which creates a fermion in the nth excited state and $|0_L\rangle$ is the localized no-particle state referred to earlier eqn. (4-17). It is straightforward to

show that the expectation value of an operator bilinear in the fermion field and normal-ordered relative to $|0_L\rangle$ is given by

$$\langle s| : \psi^\dagger(\underline{x}) \Gamma \psi(\underline{x}) : |s\rangle = u_n^\dagger(\underline{x}) \Gamma u_n(\underline{x}) \qquad (423)$$

The arbitrary wave-function $u_n(\underline{x})$ is to be self-consistently determined by a variational calculation.

We are now in a position to implement our energy variational principle. Firstly, we compute the energy of our system, eqn. ( ), in a trial state which we guess to be

$$|\text{ansatz}\rangle = U(g) B_0^\dagger |0_L\rangle \qquad (424)$$
$$= \exp\left[-i \int d^3x\, g(\underline{x})\, \dot{\sigma}(\underline{x})\right] B_0^\dagger |0_L\rangle$$

where $U(g)$ creates the coherent boson state whilst B_0^+ creates a ground-state fermion. The important point, that will be made explicit later, is that this procedure reduces the quantum field theory problem to a classical form. Anyway, assuming that we normal-order the Hamiltonian of our system term by term relative to $|0_L\rangle$, we can evaluate the energy in our trial state, eqn. (424). The result is

$$E = \langle \text{ansatz} | H | \text{ansatz} \rangle$$
$$= \int d^3x \left[\psi^\dagger \left(\frac{\underline{\alpha} \cdot \underline{\nabla}}{i} + \beta g \right) \psi + \frac{1}{2} |\underline{\nabla} g|^2 + H(g^2 - f^2)^2 \right]$$

(425)

Here all zero-point energies have been dropped. Up to now we have said nothing about the expansion basis $\{u_n, v_m\}$. The idea behind the variational principle is that the best choice of the trial state is realised when the corresponding forms for $g(\underline{x})$ and $\psi(\underline{x})$ minimize the total energy. Obviously this procedure will only make sense if the energy operator is positive-definite for all g and ψ. As is well known, the Dirac part of eqn. (425) does not satisfy this requirement. To circumvent this difficulty we shall proceed as follows: given any form for $g(\underline{x})$, we solve the Dirac equation exactly

$$\left[\frac{\underline{\alpha}\cdot\underline{\nabla}}{i} + \beta g\right]\psi = \varepsilon(g)\,\psi \qquad (426)$$

We then take the lowest positive eigenvalue $\varepsilon_{min}(g)$. This only makes sense because the solutions for the positive and negative energy portions of the energy spectrum are clearly separated (there is no Klein paradox for our system).

The energy E of our system is given by

$$E = \int d^3x \left[\left|\frac{1}{i}\underline{\nabla} g\right|^2 + H(g^2-f^2)^2\right] + \varepsilon(g) \qquad (427)$$

provided that we normalise the Dirac spinor ψ to unity, i.e.

$$\int d^3x\, \psi^\dagger(\underline{x})\,\psi(\underline{x}) = 1 \qquad (428)$$

We now require that E be a minimum with respect to an arbitrary variation of $g(\underline{x})$. It is straightforward to show that by

imposing the condition

$$\frac{\delta E}{\delta g(\underline{x})} = 0 \qquad (429)$$

on eqn. (427), we end up with the following equation of motion for $g(\underline{x})$,

$$\nabla^2 g(\underline{x}) - 4Hg(g^2-f^2) = G\,\Psi^{\dagger}\Psi \qquad (430)$$

Here we have used the fact that

$$\frac{\delta \mathcal{E}(g)}{\delta g(\underline{x})} = G\,\Psi^{\dagger}(\underline{x})\,\Psi(\underline{x}) \qquad (431)$$

Note that E(g) appears as a Lagrange multiplier enforcing the normalization condition, eqn. (428). Evidently eqns. (430) and (426) are just the classical equations of motion of our system (compare with eqns. ( ) and ( )), which is the desired result.

FIGURES

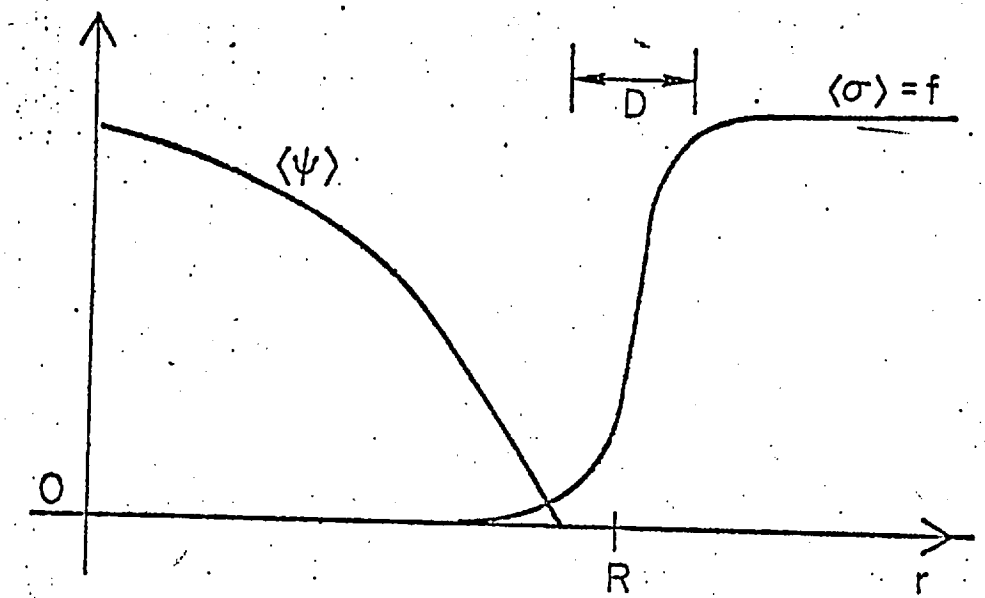

Fig. 1

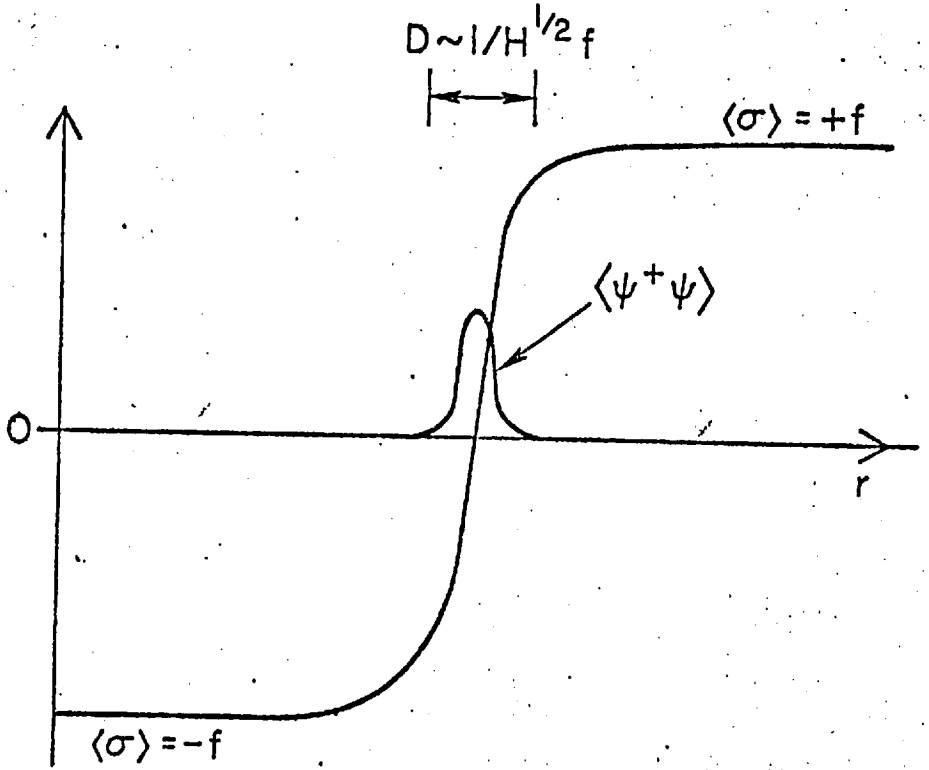

Fig. 2

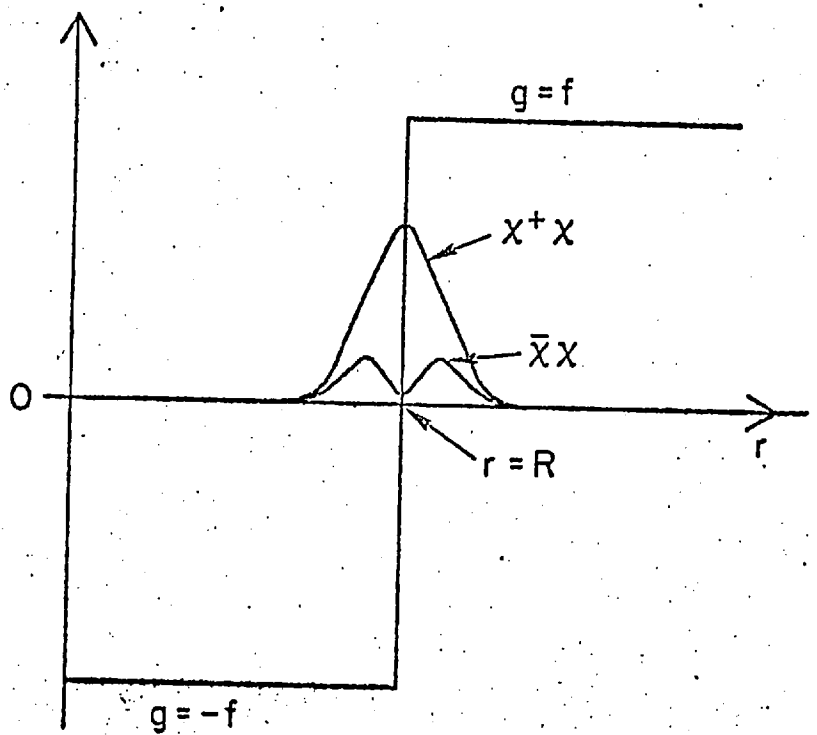

Fig. 3

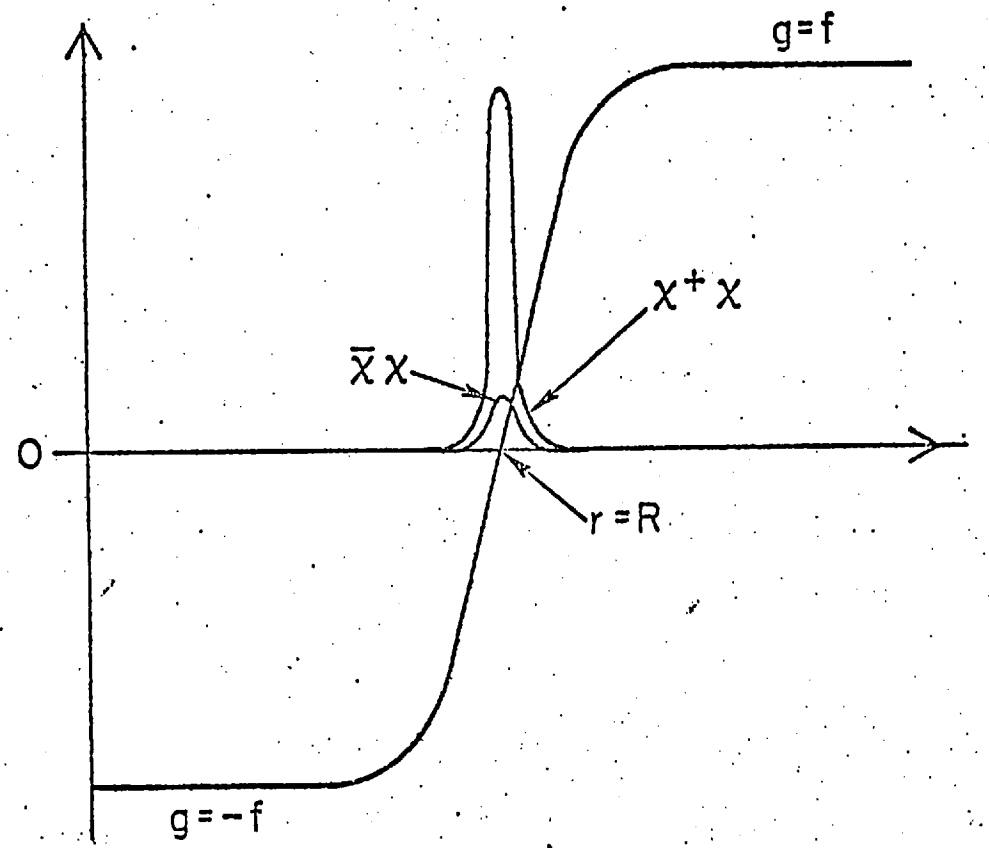

Fig. 4

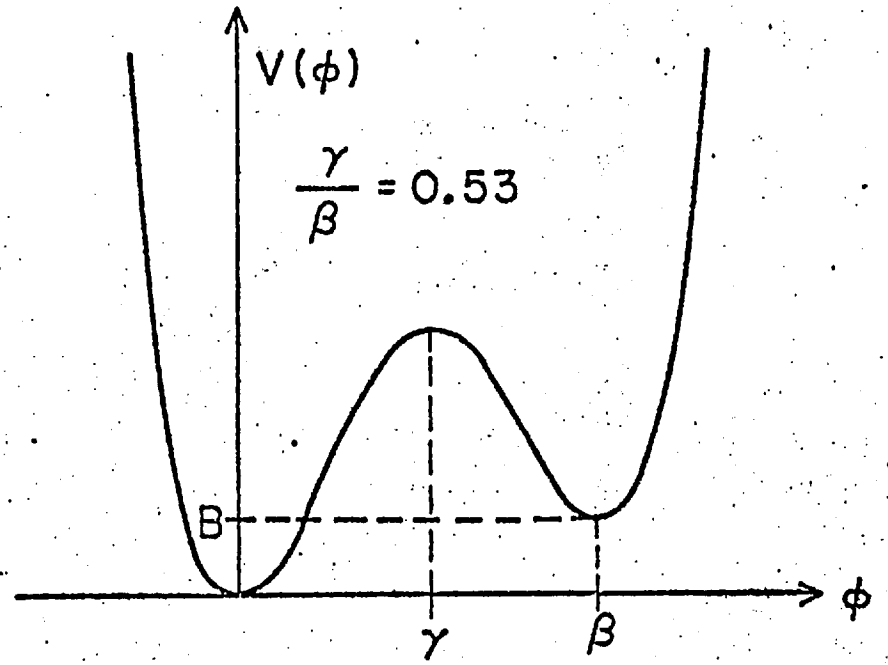

Fig. 5

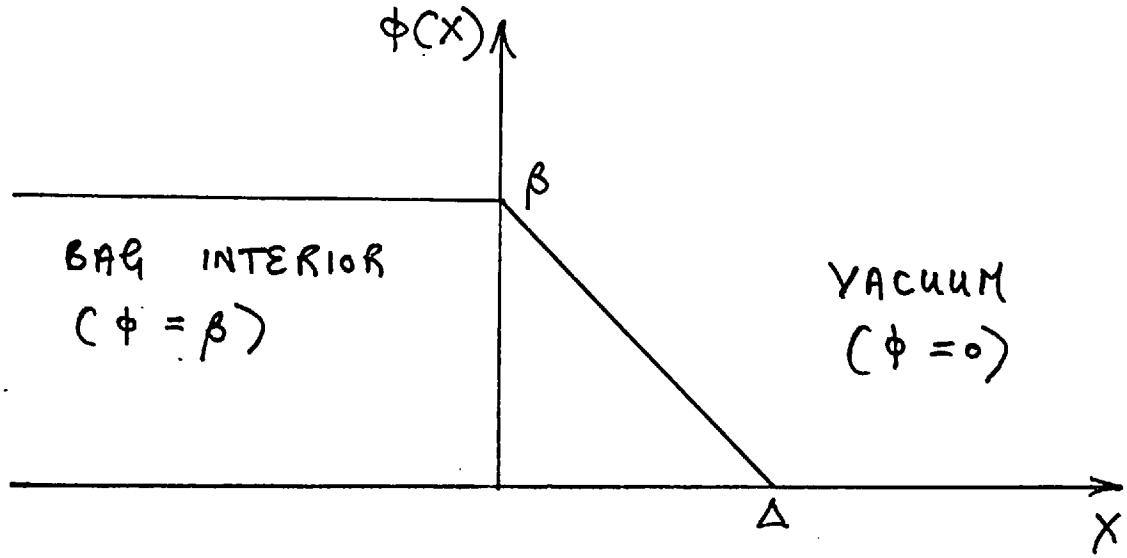

Fig. 6

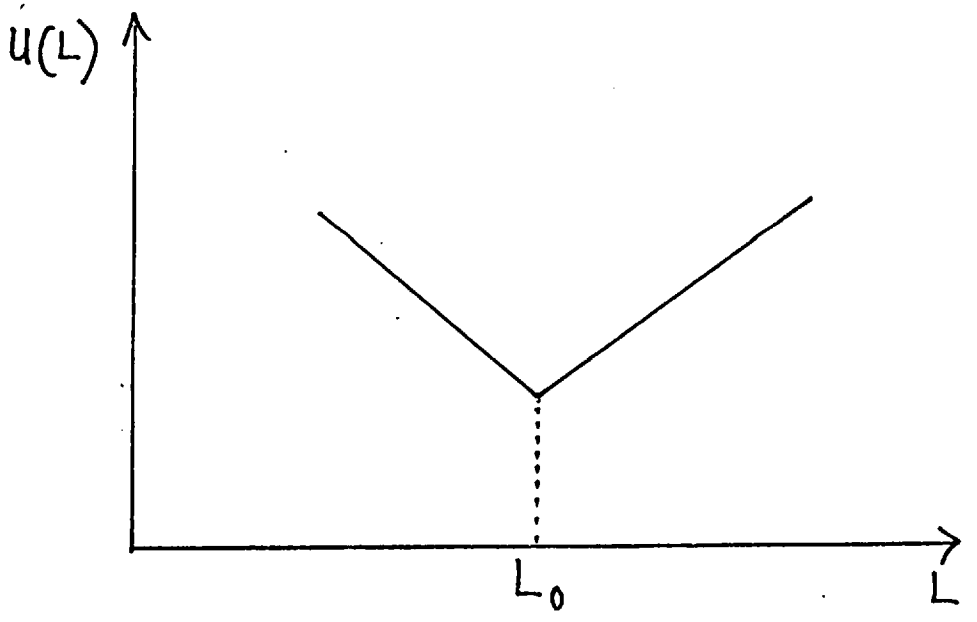

Fig. 7