DIMENSIONAL CONTINUATION OF SPINOR FIELD THEORIES

by

Vinod Bihari Prasad

Thesis presented for the Degree of Doctor of Philosophy of the University of London and the Diploma of Membership of Imperial College.

Department of Theoretical Physics
Imperial College of Science & Technology
London S.W.7

June 1975

ABSTRACT

Recently a new and powerful method known as dimensional regularization[1] has been invented which has the merit to regularizing all the divergences and preserving all the symmetries[2] (like gauge invariance) of field theories at every stage of the perturbation calculations. This method of dimensional continuation for theories containing scalar and vector particles[1,3] has been applied by numerous people for studying ultraviolet and infrared divergences[4] in several contexts. However, they have evaded discussing theories containing spinor particles, chiefly because of the complicated Γ-matrix algebra involved for arbitrary dimensions. In this thesis we have studied the problems associated with spinor fields[5] in considerable detail and have applied our work to the following cases.

Firstly, we have examined the properties of bilinear currents $J_{(r)} \equiv (\bar{\psi}\,\Gamma_{(r)}\,\psi)$ and their associated anomalies with particular reference to the Thirring Model in the two dimensional limit[6]. By considering Lagrangians of the type $\sum_r J_{(r)}\,J^{(r)}$ we have extended weak interaction theory to arbitrary dimensions[7]; oddly enough we find two kinds of polar vector and also two kinds of axial vector among possible set of currents. One of these weak polar vector currents is not conserved except in four dimensions and undergoes a finite renormalization from quantum loops. Further on, we consider anomalous currents $J'_{(4)} = \psi\{\not{\partial},\ \Gamma_{(4)}\}\psi$. The interactions of the type $J'_{(4)}\ \phi^{(4)}$ which are naturally evanescent[8] because

they disappear in four dimensions, but lead to divergences on the basis of power counting, are proved to be non renormalizable. Finally, we have examined supersymmetry in the two and four dimensional limits. In each instance we have adopted a dimension independent definition of the supersymmetric Lagrangian and shown[9] the spinor Ward Identities to be anomaly free.

Preface

The work presented in this thesis was carried out in the Department of Theoretical Physics, Imperial College, between October 1972 and June 1975 under the supervision of Dr. R. Delbourgo. Except where otherwise stated, this work is original and has not been submitted for a degree of this or any other university.

The author wishes to thank Dr. R. Delbourgo for his guidance and encouragement, for valuable discussions and suggestions. He would also like to thank Prof. T.W.B. Kibble and all other members of the Theoretical Group at Imperial College.

The financial support of University of Patna and the British Council is gratefully acknowledged.

This thesis is dedicated to

my wife

SHOBHA

and to

SAIN BABA

CONTENTS

CHAPTER ONE

INTRODUCTION

The infinities appearing in higher order terms of the perturbation approach to quantum field theory have always presented a problem. The normal procedure has been first to regularize the divergent integrals by some means or another in order to isolate the infinite part and then to rename this infinity by the technique of renormalization. Unfortunately, if this regularization is performed in the most naive way one can violate the original symmetries of the Lagrangian, e.g. one can get an infinite mass in the photon self energy which conflicts with gauge invariance. Recently, a new technique called[1] dimensional regularization has been invented which has the great merit of regularizing all the divergences and preserving all the symmetries[2] (like gauge invariance) of the field theories. Besides the practical advantage of allowing us to manipulate our integrals with ease till the very end, it has the physical consequence of pinpointing the source of anomalous[5,10] quantum corrections as we shall see in Chapter III.

This method of dimensional regularization consists in setting up the theory in arbitrary dimensions (and if we wish to accommodate electromagnetism it is necessary to work in the even (2ℓ) dimensions from the very beginning) working out the Feynman integrals and then calculating them appropriately. The actual continuation to any dimension is a trivial affair in cases where only scalar or vector particles are involved[1,3]. This has been applied by numerous people for studying ultraviolet and infrared divergences[4] in several contexts.

However, they have purposely evaded discussing theories when spinors or abnormal parity objects arise because the generalization requires considerable care. In this thesis we have studied these problems which are all associated with spinor fields[6,7,8,9]. In Chapter II we shall give all properties of the generalized Γ-matrices, a brief resume of the properties of the Fierz reshuffling matrix for arbitrary ℓ, Majorana conditions etc., i.e. we give the "kinematical framework on which the rest of the thesis is based.

Chapter III is devoted first of all in studying the bilinear currents in 2ℓ dimensions, viz.

$$J_{(r)} \equiv \bar{\psi}(x)\,\Gamma_{(r)}\,\psi(x) \tag{1.1}$$

where $\Gamma_{(r)}$ is the antisymmetric product of Γ-matrices ($0 \leq r \leq 2\ell$); then we go on to discuss the Thirring model in arbitrary dimensions and evaluating its anomalies in two dimensions. It is the simplest example of an interaction Lagrangian formed out of these bilinear currents

$$\mathcal{L}_I \equiv \sum_r \left(\bar{\psi}(x)\,\Gamma_{(r)}\,\psi(x)\right)\left(\bar{\psi}(x)\,\Gamma^{(r)}\psi(x)\right) \tag{1.2}$$

In calculating higher order corrections to these four-Fermi interactions one encounters fermion loops. By Fierz transformation these can be recast in the form of meson self-energy parts $\Pi^{(s)}_{(r)}(x)$ which we evaluate in section (3.1) of this chapter for later reference. Then in section (3.2) we discuss in detail the Thirring model and show how to recover the Johnson-Hagen result[12] in the limit $\ell \to 1$ by using the

Fierz reshuffling matrix and the spinorial identities peculiar to two dimensions. In two dimensions we can define $\gamma_5 \equiv \Gamma_{(2)}$. The correct approach to extract the two dimensional axial anomaly[6,5] is to associate chirality with the set of dimension independent spinor transformations

$$\Psi \longrightarrow \exp\left(\tfrac{1}{2} i\, \theta^{[KL]}\, \Gamma_{[KL]}\right) \Psi \tag{1.3}$$

under which the kinetic energy term is <u>not</u> invariant. Hence the usual PCAC identity contains anomalous terms: for

$$\mathcal{L} = \bar{\Psi} i \not{\partial} \psi + \tfrac{1}{2} g (\bar{\Psi} \Gamma_M \Psi)(\bar{\Psi} \Gamma^M \Psi) \tag{1.4}$$

$$\delta \mathcal{L} = i\, \partial_{[K} \bar{\Psi} \Gamma_{L]} \Psi$$

$$= \tfrac{1}{2} \bar{\Psi} \{ \Gamma \cdot \partial , \Gamma_{[KL]} \} \Psi + g \bar{\Psi} \{ \Gamma_M , \Gamma_{[KL]} \} \Psi \, \bar{\Psi} \Gamma^M \Psi \tag{1.5}$$

The extra terms on the right hand side do not exist in two dimensions. But it would be wrong to delete them for $\ell = 1$. The rules of dimensional regularization (DR) say that we have first to evaluate all the matrix elements and then take the limit $\ell \to 1$; specifically one discovers a transition from vector current to axial divergence in a fermion loop. The scaling anomaly[13] is a similar affair which is also included in section (iii) of this chapter for completeness.

We also recapitulate the procedure for extracting axial anomalies in four dimensions at the end of this chapter. Again the procedure which relies on a proper identification of γ_5 as $\Gamma_{(4)}$, is a dimension independent[5] one.

In Chapter IV we apply DR to weak interactions[7]. One of the problems is to find the proper generalization of V-A theory to 2ℓ dimensions before the final descent to $\ell = 2$. Those computations which have appeared in the literature[14] all assume that the only weak currents are of vector-pseudovector type, viz.

$$\bar{\Psi}\Gamma_M(1 - i\Gamma_{-1})\Psi \quad \text{where } \Gamma_{-1} = \Gamma_0\Gamma_1 \cdots \Gamma_{2\ell-1} \tag{1.6}$$

the presumption being that there is a weak vector current, identical to the electromagnetic current, plus a $(2\ell-1)$ index antisymmetric pseudovector weak current brought in by the parity-violating lefthanded neutrinos. On the other hand, from the work on anomalous PCAC identities[5] mentioned before one has learnt that the hadronic axial current consists instead of three-index antisymmetric tensor (the pseudo scalar mesons being associated by four-index tensors), a conclusion which seems to be at variance with the form[1].

To reconcile these two view points we shall return to first principles and try to make a respectable guess at the structure of the four-Fermi weak Lagrangians for arbitrary integer ℓ, without categorically tying ourselves solely to the currents (1). There are three useful guides for the appropriate choice of leptonic Lagrangian:

(i) interactions should invalue ((lefthanded)) neutrinos;

(ii) they must reduce to V-A form when $\ell = 2$ and

(iii) the weak Lagrangian ought to be Fierz reshuffling invariant in 2ℓ dimensions, a property which we know to be true for $\ell = 2$.

(The last criterion is perhaps not totally compelling and we later discuss the consequences of relaxing it.)

Now on the basis of the classification of massless and massive particle states given in the 'kinematical' Chapter II one sees that neutrinos of the lefthanded variety have associated the projector $\frac{1}{2}(1-i\,\Gamma_{-1})$, the direct generalization of $\frac{1}{2}(1-i\,\gamma_5)$ in four dimensions. The next step is to exploit the properties of the Fierz transformation matrix[11] in order to find the crossing-invariant generalization of V-A theory. This we do in section and prove that

$$\mathcal{L}_W \propto K_-(1) - K_-(3) + K_-(5) - \cdots + (-1)^{l-1} K_-(2l-1) \tag{1.7}$$

$$K_-(r) \equiv \frac{1}{4}\,\bar{\psi}\,\Gamma_{[M_1\cdots M_r]}(1-i\Gamma_{-1})\psi\,\bar{\psi}\,\Gamma^{[M_1\cdots M_r]}(1-i\Gamma_{-1})\psi \tag{1.8}$$

The result is perhaps not so obvious and it leads to some unexpected consequences, the most important of which is the emergence of a new current $\bar{\psi}\,\Gamma_{(KLM)}\,\Gamma_{-1}\,\psi$ which is not conserved but which yet reduces to a vector current in four dimensions! Naturally at the classical true level this current behaves innocuously as a vector current but at the next level of computing quantum loops we meet some curious renormalization effects.

Having listed the leptonic effective Lagrangians for hadronic weak interactions in section (3.2) we elaborate on the renormalization of weak currents by evaluating the one-loop corrections of the axial-currents and the extraordinary new vector currents, in a particular model of strong interactions. We have the perplexing phenomenon of weak vector current undergoing renormalization and there is no natural way of eliminating this: one could cancel it off but then one is liable to destroy renormalizability of the initial Lagrangian as we show in the next chapter.

In Chapter V we examine anomalous currents such as

$$J'_{(4)} \equiv \bar{\Psi}\{\overleftrightarrow{\not\partial}, \Gamma_{(4)}\}\Psi \tag{1.9}$$

in axial current Ward identities, currents which disappear for $\ell = 2$ but whose matrix elements yield the Adler-Bell-Jackiw anomalies[10]. Interactions which fade away in four dimensions (or, stronger still, can not even be written down!) have been coined "evanescent" by Bollini and Giambiagi[8]. It is the interplay of their vanishing and the divergence of Feynman integrals for $\ell \to 2$ which is responsible for the interesting finite corrections to classical Ward identities.

Here we shall investigate a theory which has a primary evanescent interaction (unlike Bollini and Giambiagi[8] we shall adhere to purely local field couplings in the Lagrangian itself), viz,

$$\mathcal{L}_I = G\,\bar{\Psi}\{\overleftrightarrow{\not\partial}, \Gamma_{[KLMN]}\}\,\Phi^{KLMN} \tag{1.10}$$

where ϕ stands for the "pseudoscalar" field in 2ℓ dimensions. (Such an interaction could have been used as a counter term to cancel off the Adler anomalies.) On the basis of power counting $\mathcal{L}_I$ is singular as X^{-5} near $\ell = 2$, (signalled by

G having dimensions m^{-1}) and one would naturally argue that the divergences get worse in higher orders of perturbation theory. However, the interaction itself disappears at $\ell = 2$, so the question arises whether the theory is really infinite at all and if renormalizability is truly lost. We shall prove that the model is indeed non-renormalizable, but to arrive at this conclusion we need to go beyond the one loop level, i.e., the bad effects are at least of order $\hbar^2$. This means it is highly dangerous to cancel off anomalies in Lagrangian models without incurring difficulties with renormalizability. One has to accept the anomalies for what they are and not remove them by hand.

By this time it should be clear that the continuation of field theories away from four dimensions provides one of the clearest ways of regularization[1,3,5,6,7,8,9] and helps to clarify the role of anomalies in Ward identities. It is therefore quite natural to pursue the idea of dimensional continuation in connection with supersymmetries[15] and to compare the consequences with more traditional regularization schemes. This will be the subject of Chapter VI. If we want dimensional continuation of supersymmetry to resemble the four dimensional version as set out in (6.1), we must limit ourselves to 2,4,10,12, .. dimensions. Put differently, we should replace $n = 2\ell$ by $n = 4(2k+1)$ and do a continuation in k down to $k = 0$ in all kinematic quantities. This is discussed in section (6.2) of this chapter. We set down the transformation laws for superfields in multispinor form. Then comes the critical decision of having to assign the

Lagrangian to a supermultiplet. There are two viable alternatives and we adopt the dimension independent definition. As immediate consequence of this choice is that the action is no longer supersymmetric (and the associated spinor current $J_{\mu\varepsilon}$ is not conserved) except for $\ell = 2$. That is

$$\delta\mathcal{L} = \partial^{\mu} J_{\mu\epsilon} = \text{anomalous looking terms.} \qquad (1.11)$$

By contrast to the previous situations however, the anomalous terms do disappear in four dimensions to <u>all orders</u> of $\hbar$ showing the spinor Ward identity is anomaly free, and this agrees with other methods of regularization[16].

CHAPTER II

KINEMATICAL FRAMEWORK

2.1 The Spinor Representation of SO(n)

Generalisation of Γ-matrices to an arbitrary n-dimensional vector space can be achieved without many difficulties(5,6,11). We list some of the relevent properties beginning with the Clifford algebra,

$$\{\Gamma_M , \Gamma_N\} = 2\eta_{MN} \qquad (2.1)$$

Where the indices M,N run from 0, 1, 2,...to n-1. The metric of η is appropriate to the SO(n-1, 1) group, i.e. $\eta_{00} = 1$, $\eta_{0N} = 0$ and $\eta_{MN} = -\delta MN$.

When $M \geqslant 1$. This metric tensor η_{MN} can be used in the usual way of raising and lowering the indices, e.g. $\Gamma^M = \eta^{MN}\Gamma_N$. There are some differences between even and odd dimensional spaces, but on the whole these are not very significant to the later work:

(i) When $n = 2\ell$ is even, the Γ-matrices are of dimension $2^\ell \times 2^\ell$. There will be a total of $2^{2\ell}-1 = n^2-1$ matrices obtained by multiplication and these form a complete set. First there is the vector matrices Γ_M, and then we have the <<spin>> matrices

$$\Gamma_{(KL)} = \tfrac{1}{2} i \left(\Gamma_K, \Gamma_L\right) = i\,\Gamma_K \Gamma_L \quad , \quad K<L,$$

The <<axial>> matrices

$$\Gamma_{(KLM)} = i\,\Gamma_K \Gamma_L \Gamma_M \, , \quad K<L<M$$

the <<pseudoscalar>> matrices

$$\Gamma_{[JKLM]} \equiv \Gamma_J \Gamma_K \Gamma_L \Gamma_M \qquad (2.2)$$

and in general, we define the antisymmetric product

$$\Gamma_{[M_1 M_2 \cdots M_n]} \equiv \Gamma_{M_1} \Gamma_{M_2} \cdots \Gamma_{M_n} \tag{2.3}$$

up to a factor of i. The procedure terminates with

$$\Gamma_{[012\cdots 2l-1]} = \Gamma_0 \Gamma_1 \Gamma_2 \cdots\cdots \Gamma_{2l-1}^{\;*} \tag{2.4}$$

The analogue of γ_5 in four dimensions. This anti-commutes with all the vector matrices:

$$\{\Gamma_{-1}, \Gamma_M\} = 0 \quad \text{and} \quad (\Gamma_{-1})^2 = (-1)^{l+1} \tag{2.5}$$

For short we shall denote all these matrices by $\Gamma_{(r)}$, where r refers to the number of antisymmetric indices carried by Γ. Also there exists the generalization C of the charge conjugation matrix:

$$C \tilde{\Gamma}_M C^{-1} = -\Gamma_M \tag{2.6}$$

and the generalized charge conjugation

$$C \Gamma_{(r)} C^{-1} = \eta_r \tilde{\Gamma}_{(r)} \tag{2.7}$$

Where η_r = 1, -1, 1, -1, ... for r = 0, 1, 2, 3, 4, 5 ..

(ii) When n = $2\ell+1$ is odd the Γ's still have dimension $2^\ell \times 2^\ell$. The vector set $\Gamma_M^{(2\ell+1)}$ consist of the vector matrices $\Gamma_M^{(2\ell)}$ (M = 0, 1, ..., $2\ell-1$) appropriate to the even

---

*But we will see in next chapter that this choice of γ_5 is not appropriate for anomalies.

dimensional case as well as $\Gamma_{2\ell}^{(2\ell+1)} \equiv \Gamma_{-1}^{(2\ell)}$.

We can realize these matrices as the direct product of Pauli matrices. In the generalized Weyl representation, with a Euclidean metric, we can write down a recurrence relation between the matrices. Thus,

$$\Gamma_0^{(2l+2)} = 1 \times 1 \times \cdots \times 1 \times \sigma_1$$

$$\Gamma_1^{(2l+2)} = \underbrace{1 \times 1 \times \cdots \times \sigma_3}_{l} \times \sigma_2 \tag{2.8}$$

and $\Gamma_M^{(2l+2)} = \Gamma_{M-2}^{(2l)} \times \sigma_2$ where $2 \leq M \leq 2l+1$

Thus $\Gamma_{-1}^{(2l+2)} = \Gamma_{M-2}^{(2l)} \times \sigma_3 = \Gamma_{2l+2}^{(2l+3)}$

and $C^{(2l)} = \sigma_2 \times 1 \times \sigma_2 \times 1 \times \cdots \times \sigma_2$ when ℓ is odd

$= C^{(2l-2)} \times \sigma_3$ when ℓ is even. (2.9)

We can readily pass to the pseudo Euclidean metric by inserting the necessary factors of i.

From now on we shall suppose that the vector space is even-dimensional $n = 2\ell$, so that C exists. Dimensional regularization will correspond to continuation in ℓ. Clearly the product of an odd number of Γ's has vanishing trace, i.e.

$$2^{-l}\, \mathrm{Tr.}(\Gamma_1 \Gamma_2 \cdots \Gamma_r) = 0 \qquad \text{for} \quad = \text{odd}$$

and for the rest we have

$$2^{-l}\, \mathrm{Tr.}(\Gamma_M \Gamma_N) = \eta_{MN}$$

$$2^{-l}\ \mathrm{Tr.}\left(\Gamma_K\Gamma_L\Gamma_M\Gamma_N\right) = \eta_{KL}\eta_{MN} - \eta_{KM}\eta_{LN} + \eta_{KN}\eta_{LM}$$

etc., of the special significance is the fact

$$\mathrm{Tr.}\left(\Gamma_{-1}(\Gamma)^{n}\right) \equiv 0 \qquad \text{unless } n \geqslant 2l. \tag{2.10}$$

We define the normalization of other Γ-matrices by

$$2^{-l}\ \mathrm{Tr.}\left(\Gamma_{[KL]}\ \Gamma^{[MN]}\right) = \left(\delta_K^M\delta_L^N - \delta_K^N\delta_L^M\right) = \delta_{[KL]}^{[MN]}$$

$$2^{-l}\ \mathrm{Tr.}\left(\Gamma_{[IJK]}\ \Gamma^{[LMN]}\right) = \delta_{[IJK]}^{[LMN]}$$

and in general

$$2^{-l}\ \mathrm{Tr.}\left(\Gamma_{[M_1M_2\cdots M_r]}\ \Gamma^{[N_1N_2\cdots N_r]}\right) = (-1)^{[\frac{1}{2}r]}\cdot\delta_{[M_1\cdots M_r]}^{[N_1\cdots N_r]} \tag{2.11}$$

$$2^{-l}\ \mathrm{Tr.}\left(\Gamma_{[M_1M_2\cdots M_r]}(\Gamma)^{s}\right) = 0 \qquad \text{unless } s \geq r$$

$$2^{-l}\ \mathrm{Tr.}\left(\Gamma_{[M_1M_2\cdots M_r]}\Gamma_K\ \Gamma^{[N_1\cdots N_r]}\Gamma_L\right) = \tag{2.12}$$

$$= (-1)^{[3r/2]}\cdot$$

$$\cdot\left(\begin{array}{l} -\delta_K^{[N_1}\eta_{L[M_1}\delta_{M_2}^{N_2}\cdots\delta_{M_r]}^{N_r]} - \delta_L^{[N_1}\eta_{K[M_1}\delta_{M_2}^{N_2}\cdots\delta_{M_r]}^{N_r]} \\ +\ \eta_{KL}\,\delta_{[M_1}^{[N_1}\delta_{M_2}^{N_2}\cdots\delta_{M_r]}^{N_r]} \end{array}\right) \tag{2.13}$$

and so on.

We also list some of the formulae for the multiplication rules,

$$\Gamma_{(1)}\ \Gamma_{(r)}\ \Gamma^{(1)} = (n-2r)(-1)^{r}\ \Gamma_{(r)} \tag{2.14}$$

(This is also calculated in section 2.2 of Fierz metrix)

$$\left[\Gamma_{(1)}\ ,\ \Gamma_{(r)}\right] = \begin{cases} \Gamma_{(r-1)} & \text{for } r \text{ even} \\ \Gamma_{(r+1)} & \text{for } r \text{ odd} \end{cases} \tag{2.15}$$

$$\left\{\Gamma_{(1)}\ ,\ \Gamma_{(r)}\right\} = \begin{cases} \Gamma_{(r+1)} & \text{for } r \text{ even} \\ \Gamma_{(r-1)} & \text{for } r \text{ odd} \end{cases} \tag{2.16}$$

and the contraction formulae

$$\delta^{N}_{[N}\delta^{N_1}_{M_1}\cdots\delta^{N_r}_{M_r]} = (n-r)\ \delta^{N_1}_{[M_1}\cdots\delta^{N_r}_{M_r]}$$

$$\Gamma^{N}\ \Gamma_{[NM_1\cdots M_r]} = (r+1)(2l-r)\ \Gamma_{[M_1M_2\cdots M_r]}$$

and

$$\Gamma_{[NM_1\cdots M_r]}\ \Gamma^{N} = (-1)^{r}\ (r+1)(2l-r)\ \Gamma_{[M_1M_2\cdots M_r]} \tag{2.17}$$

2.2 The Fierz Matrix

The completeness of the Γ-matrices in the spinor space ($\alpha = 1,2,\ldots 2^{\ell}$) means that we can write

$$\delta_\alpha^\beta \delta_\gamma^\delta = 2^{-l} \sum_{\text{distinct } r} \left(\Gamma_{(r)}\right)_\alpha^\delta \left(\Gamma^{(r)}\right)_\gamma^\beta \tag{2.18}$$

For general matrices F and G the reshuffling theorem follows:

$$F_\alpha^\beta G_\gamma^\delta = 2^{-l} \sum_r \left(F\,\Gamma_{(r)}\,G\right)_\alpha^\delta \left(\Gamma^{(r)}\right)_\gamma^\beta \tag{2.19}$$

Suppose now that we consider spinor-spinor interactions expanded in terms of the $2\ell+1$ kinematic covariants $K(r) = \Gamma_{(r)} \times \Gamma^{(r)}$ defined explicitly as

$$\begin{aligned} K(0)_{\alpha\gamma}^{\beta\delta} &= \delta_\alpha^\beta\ \delta_\gamma^\delta\ , \\ K(1)_{\alpha\gamma}^{\beta\delta} &= \left(\Gamma_M\right)_\alpha^\beta \left(\Gamma^M\right)_\gamma^\delta\ , \\ K(2)_{\alpha\gamma}^{\beta\delta} &= \left(\Gamma_{[MN]}\right)_\alpha^\beta \left(\Gamma^{[MN]}\right)_\gamma^\delta\ ,\ \ldots\ldots \end{aligned} \tag{2.20}$$

where the summation is taken over distinct Γ-matrices with no repetition.

By reshuffling rule (2.19) we can equally well use the crossed kinematics covariants

$$\tilde{K}(r)_{\alpha\gamma}^{\beta\delta} = \left(\Gamma_{(r)}\right)_\alpha^\delta \left(\Gamma^{(r)}\right)_\gamma^\beta \tag{2.21}$$

for the expansion. The linear relation between K and $\tilde{K}$ is

$$\begin{aligned} \tilde{K}(s)_{\alpha\gamma}^{\beta\delta} &= 2^{-l} \sum_r \left(\Gamma_{(s)}\,\Gamma_{(r)}\,\Gamma^{(s)}\right)_\alpha^\beta \left(\Gamma^{(r)}\right)_\gamma^\delta \\ &= \sum C^{2l}(s,r)\, K(r)_{\alpha\gamma}^{\beta\delta}\ , \end{aligned} \tag{2.22}$$

where $C^{2\ell}(s,r)$, the sr-element of the Fierz transformation in 2ℓ dimensions, is fixed through the equation

$$C(s,r)\,\Gamma_{[M_1 M_2 \cdots M_r]} =$$
$$= 2^{-\ell} \sum_{\text{distincts}} \Gamma_{[N_1 N_2 \cdots N_s]}\,\Gamma_{[M_1 M_2 \cdots M_r]}\,\Gamma^{[N_1 N_2 \cdots N_s]} \tag{2.23}$$

By double reshuffling we may be sure that $C^2 = 1$ and already know the boundary elements $C(0,r) = 2^{-\ell}$.

In order to determine the general element $C^{2\ell}(s,r)$ it is sufficient to pick a particular matrix $\Gamma_{(r)}$, e.g. $\Gamma_0\,\Gamma_1 \cdot\cdot\, \Gamma_{r-1}$ in (2.23). By summing over indices N which are and which are not included in $\Gamma_{(r)}$ one can arrive at the general formula

$$C(s,r) = 2^{-\ell}(-1)^{sr} \sum_q (-1)^q \binom{2\ell-r}{s-q}\binom{r}{q} =$$

$$= \sum_q \frac{2^{-\ell}(-1)^{sr+q}\,(2\ell-r)!\;r!}{(s-q)!\,(2\ell-r-s+q)!\,q!\,(r-q)!} \tag{2.24}$$

Hence the reflection rules

$$C(2\ell-s,r) = (-1)^r\,C(s,r)$$
$$C(s,2\ell-r) = (-1)^s\,C(s,r) \tag{2.25}$$

We can construct the matrices for 2,4,6,8,... from a knowledge of $C(o,r) = 2^{-\ell}$, $C(1,r) = 2^{-\ell}(-1)^r\ (2\ell-2r)$ and $C(2,r) =$ $2^{-\ell}(2(\ell-r)^2 - \ell)$ (2.26)

so we might as well note them.

$$C^2 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}, \quad C^4 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & -2 & 0 & 2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & 2 & 0 & -2 & -4 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$C^6 = \frac{1}{8}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & -4 & 2 & 0 & -2 & 4 & 6 \\ 15 & 5 & -1 & -3 & -1 & 5 & 15 \\ 20 & 0 & -4 & 0 & 4 & 0 & -20 \\ 15 & -5 & -1 & 3 & -1 & -5 & 15 \\ 6 & 4 & 2 & 0 & -2 & -4 & 6 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix}$$

$$C^8 = \frac{1}{16}\begin{pmatrix} 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 8 & 6 & 4 & -2 & 0 & 2 & -4 & 6 & -8 \\ 28 & 14 & 4 & -2 & -4 & -2 & 4 & 14 & 28 \\ 56 & -14 & -4 & 6 & 0 & -6 & 4 & 14 & -56 \\ 70 & 0 & -10 & 0 & 6 & 0 & -10 & 0 & 70 \\ 56 & 14 & -4 & -6 & 0 & 6 & 4 & -14 & -56 \\ 28 & -14 & 4 & 2 & -4 & 2 & 4 & -14 & 28 \\ 8 & 6 & 4 & 2 & 0 & -2 & -4 & -6 & -8 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix}$$

So far as the Fierz transformation is concerned, inspection of C^2 reveals that a vector is reshuffled into a scalar and a tensor (which equals the pseudoscalar in two dimensions) - in fact the Fierz invariants are K(0) - K(2), K(0) + K(1) - K(2) and K(0) - K(1) - K(2) with crossing eigenvalues 1, 1 and -1 respectively.

With no prior information about the origins of (2.24) the square property $C^2 = 1$ would be far from obvious. If one wants to prove this from scratch the best avenue is first to construct a generating for the Fierz matrix (which yields other dividends besides). One readily checks that

$$(1+z)^{2l}\cdot\left[1-\omega\left(\frac{1-z}{1+z}\right)\right]^{-1} \equiv$$

$$\equiv 2^{l}\sum_{r,s}(-1)^{sr}\,C^{2l}(s,r)\,\omega^{r}\,z^{s} \tag{2.27}$$

reproduces formula (2.24). Hence the representation

$$2^{l}(-1)^{sr}\,s!\;C(s,r) =$$

$$= \left(\frac{d}{dz}\right)^{s}\left[\{1-z\}^{r}\,\{1+z\}^{2l-r}\right]\Bigg|_{z=0} \tag{2.28}$$

It becomes straightforward to verify now that

$$\sum_{r} C(s,r)\,C(r,t) =$$

$$= 2^{-2l}\,\frac{1}{s!}\left(\frac{d}{dz}\right)^{s}\left[\left\{1+z-(-1)^{s+z}(1-z)\right\}^{t}\cdot\right.$$

$$\left.\cdot\left\{1+z+(-1)^{s+t}(1-z)\right\}^{2l-t}\right]\Bigg|_{z=0} = \delta(r,t) \tag{2.29}$$

The generating function (2.27) leads immediately to the reflection rules (2.25) and furthermore provides recurrence relations such as

$$2\,C^{(2l+2)}(s,r) = C^{2l}(s,r) + 2(-1)^r C^{2l}(s-1,r) + C^{2l}(s-2,r)\,, \tag{2.30}$$

$$2\,C^{(2l+2)}(s,r+2) = C^{2l}(s,r) - 2(-1)^r C^{2l}(s-1,r) + C^{2l}(s-2,r)\,, \tag{2.30'}$$

which, if desired, can be used to build up matrices of larger dimension from those of smaller dimension.

In the text (Chapter V) we will meet numerators of Feynman integrals of the type $\not{K}\,\Gamma_{(4)}\,\not{p}\,\Gamma'_{(4)}\,\not{K}\,\Gamma^{(4)}$ where K is an internal and p is an external momentum which are simplified as follows after symmetrical integration

$$\begin{aligned}
\not{K}\,\Gamma_{(4)}\,\Gamma'_{(4)}\,\not{K}\,\Gamma^{(4)} &\longrightarrow (2l)^{-1} K^2\, \Gamma_{(1)}\Gamma_{(4)}\Gamma'_{(4)}\Gamma^{(1)}\Gamma^{(4)} \\
&= (4l)^{-1} K^2\, \Gamma_{(1)}\Gamma_{(4)} \left(\left[\Gamma'_{(4)}, \Gamma^{(1)}\right] + \left\{\Gamma'_{(4)}, \Gamma^{(1)}\right\} \right) \Gamma^{(4)} \\
&= 2^l (4l)^{-1} K^2\, \Gamma_{(1)} \left(C(4,3) \left[\Gamma'_{(4)}, \Gamma^{(1)}\right] + C(4,5) \left\{\Gamma'_{(4)}, \Gamma^{(1)}\right\} \right) \\
&= \\
&= 2^l (4l)^{-1} K^2\, \Gamma'_{(4)} \begin{bmatrix} C(4,3)\left\{C(1,4) - C(1,0)\right\} + \\ + C(4,5)\left\{C(1,4) + C(1,0)\right\} \end{bmatrix}
\end{aligned} \tag{2.31}$$

$$\Gamma_{(1)}\Gamma_{(4)}\not{k}\,\Gamma'_{(4)}\,\Gamma^{(1)}\Gamma^{(4)} = 2^{l-2}\,\Gamma_{(1)}\Gamma_{(4)}\,\cdot$$

$$\cdot\left[\begin{array}{l}\left\{\{\not{k},\Gamma'_{(4)}\},\Gamma^{(1)}\right\}+\left[\{\not{k},\Gamma'_{(4)}\},\Gamma^{(1)}\right]+ \\ +\left\{[\not{k},\Gamma_{(4)}],\Gamma^{(1)}\right\}+\left[[\not{k},\Gamma'_{(4)}],\Gamma^{(1)}\right]\end{array}\right]$$

$$\rightarrow 2^{l-2}\Gamma_{(1)}\left(C(4,2)\left\{[\not{k},\Gamma_{(4)}],\Gamma^{(1)}\right\}+C(4,4)\left[[\not{k},\Gamma'_{(4)}],\Gamma^{(1)}\right]\right)$$

$$= 2^{l-2}[\not{k},\Gamma_{(4)}]\left(\begin{array}{l}C(4,2)\{C(1,3)+C(1,0)\}+ \\ +C(4,4)\{C(1,3)-C(1,0)\}\end{array}\right) \qquad (2.31')$$

2.3 Majorana Conditions

On charge conjugation

In the Weyl representation for instance the charge conjugation matrix is given by formula (2.9), i.e.,

$$C^{2l} = \sigma_2\times 1\times\sigma_2\times 1\times\cdots\times\sigma_2 \qquad \text{when } \ell \text{ is odd}$$

and $$C^{2l} = \sigma_2\times 1\times\sigma_2\times 1\times\cdots\times\sigma_3 \qquad \text{when } \ell \text{ is even}$$

$$= C^{(2l-2)}\times\sigma_3 \qquad (2.9)$$

For example when

$l=1$, $C=\sigma_2$

$l=2$, $C=\sigma_2\times\sigma_3$

$l=3$, $C=\sigma_2\times 1\times\sigma_2$

$l=4$, $C=\sigma_2\times 1\times\sigma_2\times\sigma_3$ etc.

we observe the following property

$$\tilde{C} = (-1)^{[\frac{1}{2}l]}\, C \tag{2.32'}$$

where (ν) denotes the nearest integer equal to or above ν.

Beginning with

$$C^{-1}\Gamma_M C = -\tilde{\Gamma}_M$$

$$C^{-1}\Gamma_{[M}\Gamma_{N]}C = -\widetilde{(\Gamma_{[M}\Gamma_{N]})}$$

etc.

we establish the general formula

$$C^{-1}\Gamma_{(n)}C = (-1)^{\frac{1}{2}n(n+1)}\,\tilde{\Gamma}_{(n)} \tag{2.33}$$

which can be recast in the index form

$$\left(\Gamma_{(n)}C\right)_{\beta\delta} = (-1)^{[\frac{1}{2}l]+\frac{1}{2}n(n+1)}\left(C\,\Gamma_{(n)}\right)_{\delta\beta} \tag{2.34}$$

If ψ and θ are Majorana Spinors, they are defined by the self conjugacy relations

$$\Theta_\beta = C_{\beta\gamma}\,\bar{\Theta}^\gamma ,$$

$$\bar{\Psi}^\alpha = (C^{-1})^{\alpha\delta}\,\Psi_\delta \tag{2.35}$$

(In the Majorana representation, this becomes a reality property.)

$$\bar{\psi}\,\Gamma_{(r)}\,\theta = (-1)^{\frac{1}{2}(r-1)(r+2)+\left[\frac{1}{2}l\right]}\,\bar{\theta}\,\Gamma_{(r)}\,\psi$$

$$\equiv \epsilon_r\,\bar{\theta}\,\Gamma_{(r)}\,\psi \tag{2.36}$$

A little table will help to provide a picture of these symmetry properties. Taking $\psi = \theta$ it follows that some of the bilinears vanish identically.

$l=1$ 2. dims.		$l=2$ 4 dims		$l=3$ 6 dims		$l=4$ 8 dims.		$l=5$ 10 dims.		$l=6$ 12 dims.	
r	ϵ_r	r	ϵ_r	r	ϵ_r	r	ϵ_r	r	ϵ_r	r	ϵ_r
0	1	0	1	0	-1	0	-1	0	1	0	1
1	-1	1	-1	1	1	1	1	1	-1	1	-1
2	-1	2	-1	2	1	2	1	2	-1	2	-1
		3	1	3	-1	3	-1	3	1	3	1
		4	1	4	-1	4	-1	4	1	4	1
				5	1	5	1	5	-1	5	-1
				6	1	6	1	6	-1	6	-1
						7	-1	7	1	7	1
						8	-1	8	1	8	1
								9	-1	9	-1
								10	-1	10	-1
										11	1
										12	1

It is clear that to have $\bar{\theta}\theta \neq 0$ we must work in 2,4,10,12,18,20 ... dimensions. Put differently, we should replace $n = 2\ell$ by $n = 4(2K+1)$ and do a continuation in K down to $K = 0$ in all our kinematic quantities in order to deal with Majorana Spinors. Having decided on these dimensions, the surviving antisymmetric bilinears involve the scalars $C_{\alpha\beta}$, the axials

$$\left(\Gamma_{(3)} C\right)_{\alpha\beta} = \left(\Gamma_{[KLM]} C\right)_{\alpha\beta} ,$$

the pseudovectors

$$\left(\Gamma_{(4)} C\right)_{\alpha\beta} \equiv \left(\Gamma_{[KLMN]} C\right)_{\alpha\beta} , \tag{2.37}$$

and so on, similarly to four dimensions.

As far as quadrilinears are concerned, note that not all of these are independent because of the Majorana reshuffling relation

$$\bar{\Theta}\,\Gamma_{(r)}\,\Theta\,\bar{\Theta}\,\Gamma_{(s)}\,\Theta = -2^{-\ell} \sum_{\substack{t \\ \text{antisymmetric}}} \bar{\Theta}\,\Gamma_{(r)}\,\Gamma_{(t)}\,\Gamma_{(s)}\,\Theta\,\bar{\Theta}\,\Gamma^{t}\,\Theta \tag{2.38}$$

Obviously one could amplify much more on these spinorial expansions but that will not be necessary in our work.

2.4 Current Algebra

Let us now consider the current algebras following from the generalization to 2ℓ dimensions of the equal time commutator

$$\left\{ \psi(t,x)\, ,\psi^{\dagger}(t,\underline{0}) \right\} = i\,\delta^{2l-1}(\underline{x}) \tag{2.39}$$

Firstly, we define the currents for $\ell \neq 2$:

$$J_{(1)} = J_N = \bar{\psi}\,\Gamma_N\,\psi \qquad \text{a true vector,}$$

$$J_{(2)} = J_{[MN]} = \bar{\psi}\,\Gamma_{[MN]}\,\psi$$

$$J_{(3)} = J_{[MNP]} = \bar{\psi}\,\Gamma_{[MNP]}\psi \qquad \text{an <<axial>> vector} \tag{2.40}$$

$$J_{(4)} = J_{[KLMN]} = \bar{\psi}\,\Gamma_{[KLMN]}\,\psi \qquad \text{pseudoscalar, ...}$$

$$J_{(2l-3)} = i\,\bar{\psi}\,\Gamma_{[KLM]}\Gamma_{-1}\psi \qquad \text{the <<pseudoaxial>> vector say}$$

$$J_{(2l-1)} = i\,\bar{\psi}\,\Gamma_N\Gamma_{-1}\psi \qquad \text{the true pseudovector}$$

and finally

$$J_{(2l)} = \bar{\psi}\,\Gamma_{-1}\,\psi$$

are all distinct from one another.

We shall now work out the equal time commutation relations for the currents. We know that the time component of the vector current, $\bar{\psi}\,\gamma_0\,\gamma_5\,\psi$, is equal to $\psi^{+}\,\gamma_5\,\psi$. Hence it will be sufficient to consider the currents

$$j_{(r)}(x) = \psi^{\dagger}(x)\,\Gamma_{(r)}\,\psi(x) \tag{2.41}$$

in general, and the equal time commutation relations

$$\left[j_{(r)}(t,\underline{x})\, ,\, j_{(s)}(t,\underline{0}) \right] = \psi^{\dagger}(x)\left[\Gamma_{(r)}\, ,\, \Gamma_{(s)} \right]\psi(0)\,\delta^{2l-1}(\underline{x}) \tag{2.42}$$

To simplify this we need the commutation relations of Γ-matrices which are obtained from

$$\left[AB, CD \right] = A\{C,B\}D - AC\{D,B\} + \\ + \{C,A\}DB - C\{D,A\}B \tag{2.43}$$

Starting with $r = s = 4$ (the time components of axial vectors), we obtain therefore

$$\left[\dot{j}_{[KLMN]}(t, \underline{x}) , \dot{j}_{[PQRS]}(t, \underline{0}) \right] \\ = i\, \delta^{2\ell-1}(\underline{x}) \left(\eta_{[KP}\eta_{LQ}\eta_{MR}\, \dot{j}_{NS]} + \eta_{[KP}\, \dot{j}_{LMNQRS]} \right) \tag{2.44}$$

leading us to new currents $\dot{j}_{(AB)}$, $\dot{j}_{(LMNPQR)}$ beyond the vector and the axial vector set. In fact, commuting these six-index currents produces new currents with eight and ten indices, and so on until the procedure ends when the number of indices exceeds 2ℓ. Therefore we have to take $r = s =$ even in (2-42).

To generalize the classification of algebra in arbitrary 2ℓ dimensions, we first work out the algebra in 2,4 and 6 dimensions.

<u>2 dimensions</u>

The only even Γ-matrices are $\Gamma_{(0)} = 1$, $\Gamma_{(2)} = \Gamma_{(-1)} = \Gamma_0 \Gamma_1$ the spin matrix. Hence the trivial commutators

$$\left(\Gamma_{(0)}, \Gamma_{(0)}\right) = \left(\Gamma_{(0)}, \Gamma_{(2)}\right) = \left(\Gamma_{(2)}, \Gamma_{(2)}\right) = 0$$

The projectors are $(1 \pm \Gamma_{-1})$ which are in the Weyl representation given by the (2×2) matrix of the form

$$(1 + \Gamma_{-1}) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right) \quad \text{and}$$

$$(1 - \Gamma_{-1}) = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 1 \end{array}\right) \tag{2.45}$$

Since the spinor in 2-dimensions has two components, each projection operates on only one component and the complete algebra is given by $U(1) \times U(1)$.

<u>4-dimensions</u>

The even Γ-matrices are

$$\Gamma_{(0)} = 1 \;, \quad \Gamma_{(2)} = \sigma_{\mu\nu} = \Big(\Gamma_0\Gamma_1 \,,\, \Gamma_0\Gamma_2 \,,\, \Gamma_0\Gamma_3 \,,\, \underbrace{\Gamma_1\Gamma_2 \,,\, \Gamma_2\Gamma_3 \,,\, \Gamma_3\Gamma_1}_{\underline{\sigma}}\Big) \tag{2.46}$$

and $$\Gamma_{(4)} = \Gamma_{-1} = \Gamma_0\Gamma_1\Gamma_2\Gamma_3$$

The commutators are

$$[\Gamma_{(0)}, \Gamma_{(0)}] = [\Gamma_{(0)}, \Gamma_{(2)}] = [\Gamma_{(0)}, \Gamma_{(4)}] = [\Gamma_{(2)}, \Gamma_{(4)}] = [\Gamma_{(4)}, \Gamma_{(4)}] = 0$$

and $$[\Gamma_{(2)}, \Gamma_{(2)}] \equiv [\Gamma_\mu\Gamma_\nu \,,\, \Gamma_\sigma\Gamma_\rho]$$

$$= \eta_{\sigma\nu}\Gamma_\mu\Gamma_\rho - \eta_{\nu\rho}\Gamma_\mu\Gamma_\sigma + \eta_{\sigma\mu}\Gamma_\rho\Gamma_\nu - \eta_{\rho\mu}\Gamma_\sigma\Gamma_\nu \equiv \Gamma_{(2)}$$

Thus commuting two spin currents we get a spin current.

It is obvious that (2.46) can be generated by the elements of the group

$$\{1 \,,\, \underline{\sigma} \,,\, \Gamma_{-1} \,,\, \underline{\sigma}\,\Gamma_{-1}\} \tag{2.47}$$

The projectors are $(1 + i\,\Gamma_{-1})$ which in the Weyl representations are given by the (4×4) matrix

$$(1 + i\,\Gamma_{-1})(1, \underline{\sigma}) = \left(\begin{array}{c|c} 2\times 2 & 0 \\ \hline 0 & 0 \end{array}\right)$$

$$(1 - i\,\Gamma_{-1})(1, \underline{\sigma}) = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & 2\times 2 \end{array}\right)$$

where both 2×2 matrices are unitary, Hence (2.47) is summarised by the generators $(1 + i\,\Gamma_{-1})$, $(1 + i\,\Gamma_{-1})\underline{\sigma}$ and $(1 - i\,\Gamma_{-1})$, $(1 - i\,\Gamma_{-1})\underline{\sigma}$ each forming a U(2) group separately; hence the complete algebra is U(2) x U(2).

In this particular example the U(1) x U(1) subalbegra of 1, Γ_{-1} is closed.

<u>6-dimensions</u>

Here the even Γ-matrices are

$$\Gamma_{(0)} = 1, \quad \Gamma_{(2)} = \sigma_{MN} = (\Gamma_0\Gamma_i\,,\ \Gamma_i\Gamma_j)$$

$$\Gamma_{(4)} = \Gamma_{LMPQ} \qquad \text{and} \quad \Gamma_{(6)} = \Gamma_{-1} = \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5$$

The commutation relations for them are

$$[\Gamma_{(0)}, \Gamma_{(0)}] = [\Gamma_{(0)}, \Gamma_{(2)}] = [\Gamma_{(0)}, \Gamma_{(4)}] = [\Gamma_{(0)}, \Gamma_{(0)}] = 0$$

$$[\Gamma_{(2)}, \Gamma_{-1}] = [\Gamma_4, \Gamma_{-1}] = [\Gamma_{-1}, \Gamma_{-1}] = 0$$

$$[\Gamma_{(2)}, \Gamma_{(2)}] = \Gamma_{(2)}$$

$$[\Gamma_{(2)}, \Gamma_{(4)}] = [\Gamma_{PQ}, \Gamma_{KLMN}] =$$

$$= (\eta_{PK}\Gamma_{QLMN} - \eta_{QK}\Gamma_{PLMN} + \text{KLMN perm.}) + \Gamma_{(2)}$$

$$= \Gamma_{(4)} + \Gamma_{(2)}$$

Finally $\left(\Gamma_{(4)}, \Gamma_{(4)}\right) \equiv \Gamma_{(2)} + \Gamma_{(6)}$ as we've already seen. Hence the set (2.48) can be generated by the elements of the group

$$\left(1, \sigma_{MN}, \Gamma_{-1}, \sigma_{MN}\Gamma_{-1} \right)$$

Similarly to 2 & 4 dimensions, the projectors are $(1 \pm \Gamma_{-1})$ in 6-dimensions which can be written in the Weyl representation as (8 x 8) matrices. It follows that the complete algebra in this case is U(4) x U(4).

Thus the complicated current algebra for large ℓ certainly closes. The precise classification of this algebra in 2ℓ dimensions is $U(2^{\ell-1}) \times U(2^{\ell-1})$. In any case it contains the physically relevant current subalgebra U(N) x U(N) generated by the space integrals

$$F^{V}(t) = \int \bar{\psi}(t,\underline{x})\, \Gamma_0\, \lambda\, \psi(t,\underline{x})\, d^{2l-1}\underline{x} \quad ,$$

with

$$F^{A}(t) = i \int \bar{\psi}(t,\underline{x})\, \Gamma_1 \Gamma_2 \Gamma_3\, \lambda\, \psi(t,\underline{x})\, d^{2l-1}\underline{x} \quad ,$$

$$\left[F^{V}, F^{V}\right] = \left[F^{V}, F^{A}\right] = \left[F^{A}, F^{A}\right] = 0$$

The crucial point is that <u>even for massless fermions</u>; although $F^A = 0$, $\dot{F}^A \neq 0$, because the fermion kinetic energy is not invariant under 'chiral transformation as we shall see.

2.5 Classification of Particle States

We present here a resume of the classification of particle states in 2ℓ dimensions with particular attention paid to the massless limit. We begin with the generalization of the Poincare group $SO(2\ell-1, 1) \wedge T(2\ell)$ generated by the linear momentum operators P_L and the angular-momentum operators J_{MN} which satisfy the usual commutation rules,

$$[P_M, P_N] = 0,$$

$$[J_{MN}, P_L] = i(\eta_{NL} P_M - \eta_{ML} P_N)$$

$$[J_{MN}, J_{PQ}] = i(\eta_{NP} J_{MQ} + \eta_{MQ} J_{NP} - \eta_{MP} J_{NQ} - \eta_{NQ} J_{MP})$$

Because of the changed dimensionality we can define $(\ell-1)$ sets of Pauli-Lubanski <<spin>> operators $W_{(r)}$

$$W_{(3)} \text{ or } W_{[KMN]} = P_{[J} J_{MN]},$$

$$W_{(5)} \text{ or } W_{[JKLMN]} = P_{[J} J_{KL} J_{MN]}, \ldots,$$

$$W_{2\ell-1} \text{ or } W_{[M_1 M_2 \ldots M_{2\ell-1}]} = P_{[M_1} J_{M_2 M_3} \cdots J_{M_{2\ell-2} M_{2\ell-1}]} \tag{2.57}$$

all of which are translationally invariant

$$(P_N, W_{(r)}) = 0, \qquad r = 3, 5, \ldots, 2\ell-1,$$

and whose squares $W_{(r)} W^{(r)}$ are casimir operators of the inhomogeneous group like P^2.

To understand better the significance of the W, suppose first that we are dealing with $P^2 > 0$ vectors. In that case we can induce all momenta from a frame where P is at rest: $\hat{P} = (m,\underline{o})$. The little group is SO(2ℓ-1) and is generated by all the spatial J_{KL}; the $W_{(r)}$ have one index equal to zero for rest states and are given by direct products of the spatial J. In fact since an SO(2ℓ-1) representation is described by ℓ-1 casimirs, these invariants are precisely related to the $W^2_{(r)}$. The remaining $\frac{1}{2}\ell(\ell-1)$ labels needed to specify the state vector fully correspond to picking out particular W components and subcasimirs.

Next we take $P^2 = 0$. Here we only permitted to induce our vectors from a frame in which the momentum has 0 and 3 components: $\hat{p} = (E,0,0,E,0,\ldots,0)$ say. Now the little group is the Euclidean group in 2ℓ-2 dimensions, $SO(2\ell-2) \wedge T(2\ell-2)$; the rotation generators are $J_{k\ell}$, $k,\ell = 1,2,4,\ldots, 2\ell-1$, and the translation generators are $J_{k0} - J_{k3}$. Again these Euclidean operators are just what the $W_{(KLM)}$ reduce to on such a momentum eigenvector, the higher $W_{(r)}$ being direct products of P and J. In practice one is only interested in the finite-dimensional basis wherein the translations are trivially represented; in these circumstances the little group is effectively SO(2ℓ-2) with its ℓ-1 casimirs and further $\frac{1}{2}(\ell-2)(\ell+1)$ labels for designating the weights. In particular one has

$$W[M_1 M_2 \ldots M_{2\ell-1}] = \lambda\, \epsilon_{M_1 M_2 \ldots M_{2\ell}} P^{M_{2\ell}} \tag{2.58}$$

where λ is the $SO(2\ell-2)$ casimir of degree $\ell-1$. In four dimensions, of course, λ has the significance of helicity. The finite-dimensional spinor representation of $O(2\ell-1,1)$ holds special interest because it is the direct generalization of the four-dimensional Dirac spinor. For massive particles the <<Dirac equation>> $(\Gamma.p-m)u = 0$ serves to cut down the number of degrees of freedom from 2^{ℓ} to $2^{\ell-1}$ and these correspond to the spinor representation of the little group $O(2\ell-1)$. However for massless particles therefore is the Γ_{-1} invariance of the equation which breaks up u instead into left - and right-handed $2^{\ell-1}$ compnent spinors:

$$U_- = \tfrac{1}{2}(1 - i\ \Gamma_{-1})u$$

and

$$U_+ = \tfrac{1}{2}(1 + i\ \Gamma_{-1})u$$

corresponding here to the little group $SO(2\ell-2)$. In fact when $m = o$ the Hamiltonian $u^+\ \Gamma_o\ \underline{\Gamma}.p\ u$ can be re-expressed as $|p|\ \lambda\ u^+\ \Gamma_{-1}\ u$, where λ is the $O(2\ell-2)$ casimir, represented spinorially now by

$$\underbrace{1 \times 1 \times \cdots \times \underline{\sigma}.\hat{\underline{p}}}_{\ell} \times 1$$

with eigen values ± 1. The other commuting $W_{(r)}$ operators which serve to remove the degeneracy in the classification of neutrino states have a similar structure, viz.

$$\underline{\sigma}.\hat{\underline{p}} \times 1 \times \cdots \times 1$$

$$1 \times \underline{\sigma}.\hat{\underline{p}} \times \cdots \times 1 \qquad \text{etc.}$$

For $\hat{p}$ along the z-axis the $W_{(z)}$ eigenvalues are given by diagonal matrices like

$$\sigma_3 \times 1 \times \cdots \quad , \qquad 1 \times \sigma_3 \times \quad \cdots \quad ,$$

e.g. in 4 dimensions by

$$W_{(1)} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

$$W_{(2)} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

CHAPTER III

ONE LOOP CALCULATIONS IN SPINOR THEORIES

3.1 Generalized Self-Energies

In calculating higher-order corrections to four-Fermi interactions one encounters fermion loops. By Fierz transformation these can be recast in the form of meson self-energy parts

$$\Pi^{(s)}_{(r)}(x) = i\left\langle T\left[\bar{\psi}(x)\Gamma_{(r)}\psi(x)\bar{\psi}(0)\Gamma^{(s)}\psi(0)\right]\right\rangle$$

which we now proceed to evaluate for later reference.- For the moment let us be as general as possible by supposing the fermion is massive. Fixing $r \geq s$ for definiteness we have to evaluate the momentum-space integral

$$\Pi^{[N_1\cdots N_s]}_{[M_1\cdots M_r]}(k) =$$

$$= ie^2\int\frac{d^{2\ell}p}{(2\pi)^{2\ell}}\;\frac{\mathrm{Tr}\left(\Gamma_{[M_1\cdots M_r]}(\not{p}+m)\Gamma^{[N_1\cdots N_s]}(\not{p}+\not{k}+m)\right)}{(p^2-m^2)\left[(p+k)^2-m^2\right]}$$

Introducing a Feynman parameter α in the usual way and utilizing (2.23) one arrives at

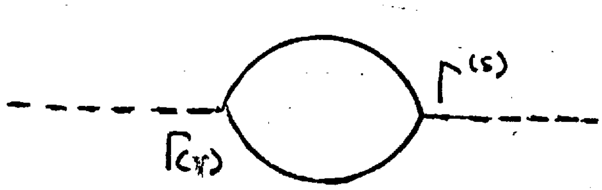

Fig. 0

$$\Pi^{[N_1\cdots N_s]}_{[M_1\cdots M_\gamma]}(K) = \frac{\Gamma(1-\ell)\,e^2}{(2\pi)^\ell}\cdot$$

$$\cdot\int_0^1 \frac{d\alpha(-1)^r(2\ell-r-1)}{[m^2-K^2\alpha(1-\alpha)]^{1-\ell}}\,\delta^{[N_1\cdots N_s]}_{[M_1\cdots M_r]} - \frac{\Gamma(2-\ell)}{(2\pi)^\ell}\int_0^1 \frac{d\alpha}{[m^2-K^2\alpha(1-\alpha)]^{2-\ell}}\cdot$$

$$\cdot\left(m^2\{-(-1)^r\}\,\delta^{[N_1\cdots N_s]}_{[M_1\cdots M_r]} - mK^{[N_1}\delta^{N_2\cdots N_s]}_{[N_1\cdots M_r]} - 2\alpha(1-\alpha)K_{[M_1}K^{[N_1}\delta^{\cdots N_s]}_{\cdots M_r]}\right) \tag{3.1}$$

The results for $\gamma = 1$ and $\gamma = 3$ have been given previously[1,5]. In particular, we observe that when the fermion mass is zero Π is diagonal in the number of indices:

$$\Pi^{[N_1\cdots N_r]}_{[M_1\cdots M_r]}(K) = \frac{\Gamma(1-\ell)e^2}{(2\pi)^\ell}\int_0^1 \frac{d\alpha}{[-K^2\alpha(1-\alpha)]^{1-\ell}}\cdot$$

$$\cdot(-1)^r(2\ell-r-1)\,\delta^{[N_1\cdots N_r]}_{[M_1\cdots M_r]} + 2(\ell-1)\,K_{[M_1}K^{[N_1}\delta^{N_2\cdots N_r]}_{M_2\cdots M_r]}\Big/K^2 . \tag{3.2}$$

The vector self-energy complies with the gauge invariance

$$\Pi^{MN}(K) = \frac{2\Gamma(2-\ell)e^2}{(2\pi)^\ell}\int_0^1 \frac{d\alpha}{[-K^2\alpha(1-\alpha)]^{1-\ell}}\cdot\left(\eta^{MN} - \frac{K^M K^N}{K^2}\right)$$

$$= \left(\eta^{MN} - \frac{K^M K^N}{K^2}\right)\pi(K^2) \tag{3.3}$$

and its x-space transform can be usefully expressed in the form

$$i\,\Pi^{MN}(x) = 2^{\ell}(\ell-1)(2\ell-1)^{-1}\left(\partial^M\partial^N - \eta^{MN}\partial^2\right) D^2(x) \tag{3.3'}$$

corresponding to the Fourier transform

$$i \int \exp[-iK\cdot x]\,\Pi(K^2)\, d^{2\ell}K/K^2(2\pi)^{2\ell}$$
$$= 2^{\ell}(\ell-1)(2\ell-1)^{-1} D^2(x) \quad , \tag{3.3''}$$

where the massless causal propagator is (see Appendix)

$$i\,D(x) = \Gamma(\ell-1)(-x^2+i\epsilon)^{1-\ell}/4\pi^{\ell} \tag{3.4}$$

In the two-dimensional limit $\Pi_{MN} \to (\eta_{MN} - K_M K_N/K^2)/\pi$ is finite! For future reference we also remark that near $\ell = 1$

$$(\ell-1)\, D^2(x) \to \ln(-x^2+i\epsilon)/8\pi^2 = -i\,D(x)/2\pi \tag{3.4'}$$

Two dimensional accidents

From the Fierz transformation in two dimensions it is clear that a Feynman diagram like Figure 2b will vanish in two dimensions; this intuitive guess is substantiated by a careful passage to $\ell = 1$ only because the vector self-energy happens to be finite in this limit and multiplies a fading

Fierz matrix factor $C(1,1) = 2^{1-\ell}(1-\ell)$. We can anticipate many happy simplications for the Thirring model therefore. In any other dimension but two the argument could not be pushed through. There exist a number of purely spinorial identities peculiar to two dimensions which are vital for summing the $\ell \to 1$ perturbation series. The general identity

$$\Gamma_L \Gamma_M \Gamma_N - \Gamma_N \Gamma_M \Gamma_L = -2i\,\Gamma_{[LMN]} \tag{3.5}$$

reduces in two dimensions to

$$\gamma_\lambda \gamma_\mu \gamma_\nu = \gamma_\nu \gamma_\mu \gamma_\lambda \tag{3.5'}$$

providing we do not close the fermion lines between which this matrix acts (or if we do so, providing the fermion loop integral converges). Letting $S(p) = (p.\Gamma)^{-1}$ stand for the massless fermion propagator, another general identity is

$$S(p)\Gamma_M S(p-k) = \left(k_M + i k^N \Gamma_{[MN]}\right)\left(S(p-k) - S(p)\right)/k^2 - \\ - 2i\, p^L k^N \Gamma_{[LMN]} S(k) S(p-k)/p^2 , \tag{3.6}$$

and it collapses into the two dimensional form

$$S(p)\gamma_\mu S(p-k) = k^\nu\left(\eta_{\mu\nu} - \epsilon_{\mu\nu}\gamma_5\right)\left(S(p-k) - S(p)\right)/k^2 \tag{3.6'}$$

because $\Gamma_{(\mu\nu)} \to i\,\epsilon_{\mu\nu}\,\gamma_5 \equiv i\,\epsilon_{\mu\nu}\,\gamma_o\,\gamma_1$. In view of 1(3.5'), an equivalent way to express (3.6') is

$$S(p)\,\gamma_\mu\, S(p-k) = S(k)\,\gamma_\mu\, S(p-k) - S(p)\,\gamma_\mu\, S(k)$$

(3.6")

Rules (3.5'), (3.6') and (3.6") are absolutely crucial in summing the perturbation graphs[12] for the Thirring model and in proving that $\dot{j}=\bar{\psi}\gamma\psi$ behaves like a free field[17].

3.2 <u>Perturbation Theory for the Thirring Model</u>

We shall now focus on the lowest-order quantum loop corrections to the Thirring model in 2ℓ dimensions:

$$\mathcal{L} = \bar{\psi}(i\not{\partial}-m)\psi + \frac{1}{2}g(\bar{\psi}\Gamma_M\psi)(\bar{\psi}\Gamma^M\psi)$$

(3.7)

in an effort to understand the miracles which make the perturbation graphs summable in the limit $\ell \to 1$. We will essentially be repeating the calculations of Mueller and Trueman[12] except that instead of introducing regulator fields and nonlocal interactions we shall be relying on ℓ-continuation. The simplest calculation, that of $\Pi_{MN}(x) = i\langle T(\dot{j}_M(x)\ \dot{j}_N(o))\rangle$, has already been done in lowest order and is stated in (3.3). Higher-order corrections to $\Pi_{\mu\nu}$ amount to summing the bubble graphs because other diagrams give zero at $\ell = 1$ after Fierz reshuffling; it is not hard to see that the complete sum is

$$\Pi_{\mu\nu}(k) = \left(\eta_{\mu\nu} - k_\mu k_\nu/k^2\right)\Big/(\pi+g)$$

For our second task let us evaluate the second-order fermion self-energy: the two graphs depicted in Fig. 1(a) and (b) (the latter being zero at $\ell=1$). For general ℓ they add up to

$$\Sigma(p) = i g^2 (2\pi)^{-2\ell} \int d^{2\ell}K\, \Gamma_{(1)} S(p+K) \left[\Pi^{(1)}_{(1)} \Gamma^{(1)} + \sum_{r} C(1,r) \Pi^{(r)}_{(1)} \Gamma_{(r)} \right].$$

since Π is diagonal and vectorial and since $C(1,1) = 2^{\ell-1}(1-\ell)$ we get

$$\Sigma(p) = i g^2 \left[1 + 2^{1-\ell}(1-\ell) \right] \int \frac{d^{2\ell}K}{(2\pi)^{2\ell}} \frac{\Gamma_M \left(\eta^{MN} - K^M K^N / K^2 \right) \Gamma_N}{(\not{p} + \not{K})} . \Pi(K^2)$$

(3.8)

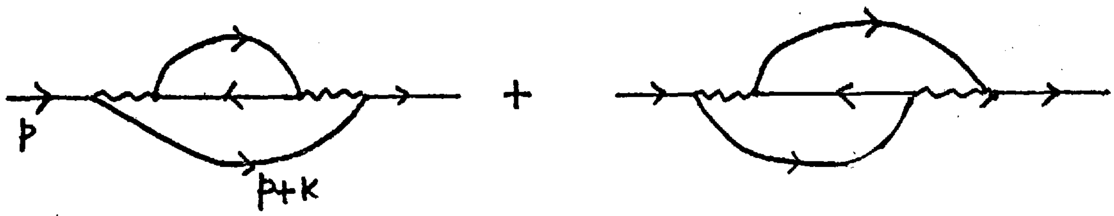

Fig. 1 - Fermion self-energy graphs in order g^2

Now one of the important consequence of DR is that $\int d^{2\ell} K$, and $\int \frac{d^{2\ell} K}{K^2}$ (Appendix A) can be consistently set equal to zero (which incidentally explains why Σ vanishes to order g).

Therefore,

$$-i S(p) \Sigma(p) S(p) =$$
$$= g^2 \left[1 + 2^{1-\ell}(1-\ell) \right] (2\pi)^{-2\ell} \int \frac{d^{2\ell}K\; \Pi(K^2)}{K^2 (\not{p} + \not{K})} .$$

If we put $\ell = 1$ at once, $\Pi(K^2) \to \frac{1}{\pi}$ and we recognize the Fourier transform of g^2 D(x) S(x)/π, the free massless scalar and spin propagators being

$$i\,D(x) \to \ln(-x^2+i\epsilon)/4\pi \quad ,$$

$$S(x) = i\ \gamma\cdot\partial\, D(x) = \gamma\cdot x/2\pi(x^2-i\epsilon) \quad ,$$

However if we reserve $\ell \to 1$ till the very end and make use of (3.3") we deduce that

$$i\,S'(x) = \langle T(\psi(x)\bar{\psi}(0))\rangle$$

$$= i\,S(x)\left[1+g^2\{2^\ell+2(1-\ell)\}\frac{(\ell-1)\,D^2(x)}{(2\ell-1)} + \cdots\right].$$

(3.9)

The Johnson Hagen answer is recovered[12] by recalling the limit formula (3.4') so that

$$i\,S'(x) \to i\,S(x)\left[1-\frac{i g^2 D(x)}{\pi}+\cdots\right].$$

Summing over higher-order graphs in the manner of Mueller and Trueman (which simply corresponds to dressing internal vector lines with bubbles) one ends up with

$$i\,S'(x) = i\,S(x)\exp\left[-g^2 D(x)/(\pi+g)\right].$$

(3.10)

Consider next the vector Green's function

$$\langle \psi(x)\, j^\mu(z)\, \bar{\psi}(y)\rangle$$

which also receives two first order contributions (Fig. 2(a) and (b). Following the same line of reasoning which led to eq. (3.8) we obtain the momentum-space amplitude

$$J^M(p, p-k) = g\left[2^l + 2(1-l)\right] S(p)\left(\eta^{MN} - k^M k^N/k^2\right)\Gamma_N S(p-k)$$

$$= ig\left[1 + 2^{1-l}(1-l)\right]\Pi(k^2)\,\Gamma_{MN}\, k^N \left[S(p-k) - S(p)\right]\Big/k^2$$

(3.11)

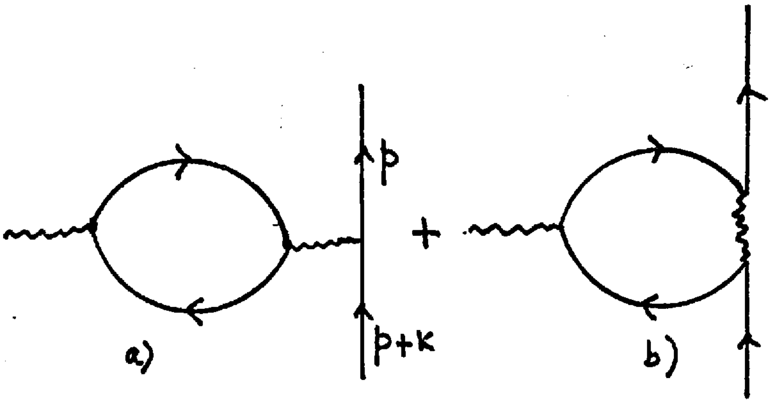

Fig. 2 - Vertex corrections in order of g^2

Taking its Fourier transform we get the x-space Green's function

$$g(2l-1)^{-1}\left[2^l + 2(1-l)\right](l-1)\,\Gamma_{[MN]}\,\partial^N\left[D^2(z-x) - D^2(z-y)\right]S(x-y),$$

and as $l \to 1$ we can recognize this as the first-order term in the complete answer

$$\left[\eta_{\mu\nu} + \epsilon_{\mu\nu}\gamma_5\left(1+\frac{g}{\pi}\right)^{-1}\partial^\nu\left[D(z-x) - D(z-y)\right].\, i\, S'(x-y)\right.$$

(3.12)

which comes by summing the bubble graphs and dressing the fermions. To close this section let us sketch how one may work out the four-point Green's function

$$\langle T\left[\psi(x)\bar{\psi}(y)\psi(x')\bar{\psi}(y')\right]\rangle.$$

Up to order g^2 we have to contend with six diagrams as well as their crossed versions (Fig. 3(a) to (f)). The main thing to expose is how vector exchange can be replaced by scalar and pseudoscalar

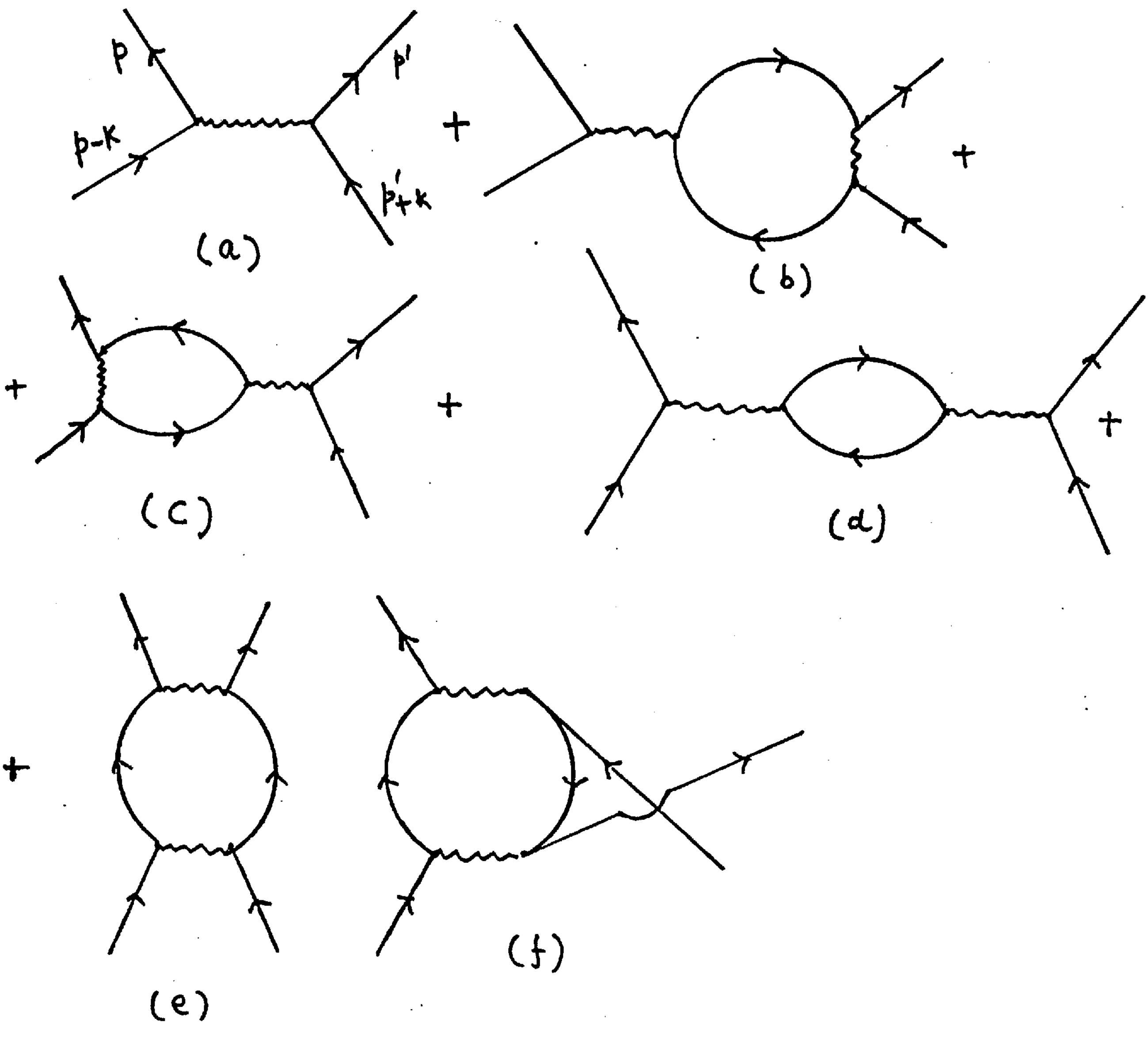

Fig. 3 - Fermion-fermion scattering in order g^2

exchange through a judicious use of (3.6'). For instance, in lowest order

$$g\ S(p)\gamma_\mu S(p-k) \otimes S(p')\gamma^\mu S(p'+k)$$
$$= g\, k^\nu (\eta_{\mu\nu} + \epsilon_{\mu\nu}\gamma_5)[S(p-k) - S(p)] \otimes$$
$$\otimes\ k_\lambda (\eta^{\mu\lambda} + \epsilon^{\mu\lambda}\gamma_5)[S(p') - S(p'+k)] / k^4$$
$$= g\,[S(p-k) - S(p)](1\otimes 1 + \gamma_5 \otimes \gamma_5)[S(p') - S(p'+k)] / k^2 .$$

(3.13)

Together with its crossed version amplitude (3.13) has the x-transform

$$g(1\otimes 1 - \gamma_5 \otimes \gamma_5)\cdot$$
$$\cdot [D(x-x') - D(x-y') - D(x'-y) + D(y-y')]\, S(x-y)\, S(x'-y')$$

In order g^2 diagrams 3(b), (c) and (d) have already been discussed and it is easy to see that 3(c) and (f) have cancelling ultraviolet characterics as $\ell \to 1$; both lead to the amputed amplitude.

$$(2\pi)^{2\ell}\, T(p, p', k)$$
$$= \int d^{2\ell}q\, \Gamma_M S(q) \Gamma_N \otimes \left(\Gamma^M S(p+p'-q)\Gamma^N + \right.$$
$$\left. + \Gamma^N S(p'-p+k+q)\Gamma^M \right)$$
$$= i\int d^{2\ell}q\, \Gamma_M S(q) \Gamma_N \otimes \Gamma^M \left(S(p+p'-q) + S(p'-p+k+q)\right)\Gamma^N,$$

if we drop all $\Gamma_{(MLN)}$ terms. By persistently applying (3.6), the Green's function can be expressed as linear combinations of integrals like

$$\left(1 \otimes 1 - \gamma_5 \otimes \gamma_5\right) \int \frac{d^{2l}q}{(2\pi)^{2l}} \; \frac{1}{q^2 \, (k-q)^2} \cdot$$
$$\cdot \left[S(p-k) - S(p-q) \right] \otimes \left[S(p'+k) - S(p'+q) \right] ,$$

which are in turn recognizable as Fourier transforms of mixed terms

$$\left(1 \otimes 1 - \gamma_5 \otimes \gamma_5\right) \; D(x-x') \, D(x-y') \, S(x'-y')$$

All this is just to indicate how the general sum of eikonal graphs(4) can be performed. With dressed fermions, the complete Green's function is

$$\exp\left[i g \left\{ 1 \otimes 1 - \gamma_5 \otimes \gamma_5 \left(1 + g/\pi\right)^{-1} \right\} \cdot \right.$$
$$\left. \cdot \left\{ D(x-x') - D(x-y') - D(x'-y) + D(y-y') \right\} \right] \cdot$$
$$\cdot \; i\, S'(x-y) \cdot i\, S'(x'-y') , \tag{3.14}$$

a result which otherwise appears rather mysterious.

3.3 Anomalies

In two dimensions one identifies γ_5 with the spin tensor $\Gamma_0 \, \Gamma_1$ (although its charge parity is negative). The correct approach to extract the two dimensional axial anomaly(17) is to associate chirality with the set of spin transformations $\psi \to \exp\left(\frac{1}{2}\, i\theta^{KL}\, \Gamma_{(KL)}\right)\psi$ and to take the $\ell \to 1$ limit of the axial Ward Identity. Thus, under an infinitesimal <<chiral>> transformation, the charge in $\mathcal{L}$, expressed in the two ways

$$\begin{aligned}\delta\mathcal{L} &= \frac{1}{2}\, i\, \delta\theta^{KL}\left(i\bar{\Psi}\left\{\tfrac{1}{2}\overleftrightarrow{\partial\!\!\!/},\, \Gamma_{[KL]}\right\}\right)\Psi \;+\\ &\quad + g\bar{\Psi}\left\{\Gamma_M, \Gamma_{[KL]}\right\}\Psi\bar{\Psi}\Gamma^M\Psi \;,\\ &= -\frac{1}{4}\,\delta\theta^{KL}\partial^M\left(\bar{\Psi}\left[\Gamma_M, \Gamma_{[KL]}\right]\Psi\right),\end{aligned}$$

yields the PCAC relation

$$i\,\partial_{[K}\bar{\Psi}\Gamma_{L]}\Psi = \frac{1}{2}\bar{\Psi}\left\{\Gamma\cdot\overleftrightarrow{\partial},\, \Gamma_{[KL]}\right\}\Psi + g\bar{\Psi}\left\{\Gamma_M, \Gamma_{KL}\right\}\Psi\bar{\Psi}\Gamma^M\Psi. \tag{3.15}$$

The anomalous terms on the right-hand side of (3.15) are non existent in two dimensions. But it would be wrong to delete them[5] for $\ell = 1$ owing to fermion loop corrections. Specifically, there is a transition from vector current to axial divergence

$$\left\langle i\,\partial_{[K}\bar{\Psi}\Gamma_{L]}\Psi,\, \bar{\Psi}\Gamma_M\Psi\right\rangle = \frac{1}{2}\left\langle\bar{\Psi}\left\{\overleftrightarrow{\partial\!\!\!/},\, \Gamma_{[KL]}\right\},\, \bar{\Psi}\Gamma_M\Psi\right\rangle$$

where the $(\bar{\psi}\,\Gamma\psi)^2$ anomalous term can be disregarded because tadpole graphs are necessarily involved. In momentum space the calculation devolves to the product of an integral which diverges as $\ell \to 1$ and a trace which disappears as $\ell \to 1$.

$$K_{[K\Pi_L]M} = \int \frac{d^{2\ell}p}{(2\pi)^{2\ell}} \frac{\mathrm{Tr.}\left[\{2\not p+\not K, \Gamma_{[KL]}\}\not p\,\Gamma_M(\not p+\not K)\right]}{(p+K)^2\,p^2}$$

$$= \frac{2(1-\ell)}{\ell(2\pi)^{\ell}}\left(\eta_{KM}K_L - \eta_{KL}K_M\right)\int_0^1 \frac{d\alpha\;\Gamma(1-\ell)}{\left[K^2\alpha(\alpha-1)\right]^{1-\ell}}$$

$$\xrightarrow[\ell\to 1]{} \left(\eta_{KM}K_L - \eta_{KL}K_M\right)/\pi \quad (3.17)$$

If we were to introduce electromagnetism $e\bar{\psi}\Gamma.A\psi$ into the Thirring model, one would interpret (3.17) as an anomalous PCAC equation

$$\partial^{\mu} j_{\mu 5} = -e\,\epsilon^{\mu\nu}\partial_{\mu}A_{\nu}/\pi$$

in lowest order. Summing the higher-order bubble graphs, the effect, as usual, is to replace π by $\pi + g$. It is perhaps important to stress that no other anomalous amplitudes such as $\langle i\,\partial_{(K}\bar{\Psi}\,\Gamma_{L)}\Psi,\ \bar{\psi}\psi\bar{\psi}\,\Gamma_M\,\psi\rangle$ survive the limit $\ell \to 1$ because the fermion loop integrals are finite and multiply zero kinematic traces. As we are discussing the Thirring model in the context of dimensional continuation we might as well show how the scaling anomaly too can be consistently determined from the trace of the stress tensor

$$\Theta^{M}_{M} = (\ell-1)\,g\,(\bar{\Psi}\Gamma_M\Psi)(\bar{\Psi}\Gamma^M\Psi)$$

$$= 2(\ell-1)\,\mathcal{L}_{\mathrm{INT.}} \quad (3.18)$$

which does not vanish except when $\ell = 1$. The anomalous scale dimension of ψ can be obtained from pair of relation

$$i \langle \psi(x) \int \theta^M_M(z)\, d^{2\ell}z\, \bar{\psi}(0) \rangle =$$

$$= 2(\ell-1)\, i \langle \psi(x) \int \mathcal{L}_{int}\, d^{2\ell}z\, \bar{\psi}(0) \rangle \qquad (3.18')$$

which is written in perturbation theory as

$$2(\ell-1)\, i \langle \psi(x) \int L_I\, d^{2\ell}z \,.\, e^{i\int L_I\, d^{2\ell}z}\, \bar{\psi}(0) \rangle$$

$$= 2(\ell-1)\, i \langle \psi(x)\, g \int Z\, d^{2\ell}z\, e^{ig\int Z\, d^{2\ell}z}\, \bar{\psi}(0) \rangle$$

$$= 2(\ell-1) \langle \psi(x)\, g \frac{\partial}{\partial g}\, e^{i\int L_I\, d^{2\ell}z}\, \bar{\psi}(0) \rangle$$

$$= 2(\ell-1)\, g \frac{\partial}{\partial g}\, S'(x)$$

Now we define the dilation current

$$D_M = \theta^N_M\, X_N$$

and the divergence of the dilation current is

$$\partial^M D_M = \theta^M_M \equiv D$$

where

$$i[D, \psi] = (x.\partial - 2d)\psi$$

With the help of this relation we can write (3.18') as

$$2(\ell-1)\, g \frac{\partial}{\partial g} S'(x) = \left(x \cdot \frac{\partial}{\partial x} - 2d \right) S'(x)$$

(3.19)

Having already seen that near $\ell = 1$

$$S'(x) = S(x) \left[1 + 2g^2 (\ell-1)\, D^2(x) + \cdots \right]$$

we can proceed to the two-dimensional limit

$$\lim_{\ell \to 1} 2(\ell-1)\, g \frac{\partial}{\partial g} S'(x) = \lim_{\ell \to 1} 8 g^2 (\ell-1)^2 D^2(x)\, S(x)$$

$$= - \frac{g^2}{2\pi^2} S(x)$$

(3.20)

giving the anomalous scale corrections of $g^2/4\pi^2$, to this order. If we sum the higher-order bubble graphs, the total scale dimension is amended to

$$d = \frac{1}{2} + \frac{g^2/4\pi^2}{1 + g/\pi}$$

(3.21)

which could in fact have been immediately deduced from the nonperturbative answer (3.10) as did Wilson[18]. Since a change of gauge alters the scale dimension without affecting S-matrix elements, Mueller and Trueman have questioned the physical significance of d.

Now we sketch briefly the method of extracting anomalies in four dimensions. There are in fact many anomalous matrix elements which can be obtained by the same principle, but we discuss in detail the cases of the <AVV> and <AAA> anomalies like the previous case of anomaly in two dimensions. We start with the dimension independent definition[5] of $\gamma_5 = \Gamma_{(4)}$. This has been tested by several authors as the correct way to extracting anomaly. Once the question of γ_5 has been settled we can write down (see Section 2.4), the axial current as

$$\bar{\psi}\,\Gamma_{(KLM)}\,\psi$$

and pseudoscalar as

$$\bar{\psi}\,\Gamma_{(KLMN)}\,\psi$$

As our previous two dimensional anomaly, we again associate chirality with the set of transformations

$$\psi \to \exp(i\ \theta^{(KLMN)}\ \Gamma_{(KLMN)})\psi .$$

Thus under this infinitesimal <<chiral>> transformation, the change in the Lagrangian can be expressed in two ways which gives the PCAC relation

$$\partial_{[K}\bar{\Psi}\,\Gamma_{LMN]}\,\Psi = 2mi\,\bar{\Psi}\,\Gamma_{[KLMN]}\,\Psi + i\,\bar{\Psi}\left(i\overleftrightarrow{\partial}^{J} + 2eA^{J}\right)\Gamma_{[JKLMN]}\,\Psi$$

where the extra term on righthand does not exist in 4 dimensions. It should be noted that even for m = 0 the kinetic energy is not invariant under these chiral transformations. In fact because of this we meet a new overall-pseudoscalar current which is precisely the axial vector anomaly and can be obtained by calculating the closed fermion loops coupling to other two vector currents. That is if we calculate a triangle Feynman graph with two vertex coupling to two vectors and the third one with $(2P+K)^J \; \Gamma_{(JKLMN)}$ (<AVV> anomaly) we obtain the same Adler anomaly in the limit $\ell \to 2$. The contribution from the new current is

$$T'^{[IJKL]}_{MN}(K, K') =$$

$$= \frac{2ie^2}{4!(2\pi)^\ell} \int \frac{d^{2\ell}p}{(2\pi)^{2\ell}} \left\{[(p-k')^2 - m^2][p^2 - m^2][(p+k)^2 - m^2]\right\}^{-1} .$$

$$. \,\mathrm{Tr.}\left(\left\{2\not{p}+\not{k}-\not{k}', \Gamma^{[IJKL]}\right\}\left[(\not{p}-\not{k}')+m\right]\Gamma_M(\not{p}+m)\,\Gamma_N\left[(\not{p}+\not{k})+m\right]\right)$$

This integral can be very much simplified in this case because we are only interested in anomalies. We drop all $\{\not{K}-\not{K}', \Gamma_{(4)}\}$ terms (we will see why in Chapter V), and consider the fermion to be massless. In fact, we can also drop all K's in the denominator because we are only interested in the divergent part of the integrand. We find

$$T^{[IJKL]}_{MN} \longrightarrow \frac{2ie^2}{4!} \int \frac{d^{2\ell}p}{(2\pi)^{2\ell}} \frac{\mathrm{Tr.}\left(\{\not{p}, \Gamma^{IJKL}\}\not{k}'\,\Gamma_M \not{p}\,\Gamma_N \not{k}\right)}{(p^2 - m^2)^3} \tag{2.23}$$

Using the symmetrical integration and Fierz transformation formula, the above trace can be further reduced to

$$\rightarrow \quad \frac{ie^2 C(1,2)}{4!} \int \frac{d^{2\ell} p}{(2\pi)^{2\ell}} \frac{p^2 \mathrm{Tr}\left(\Gamma^{IJKL} [\not{k}, \Gamma_M][\Gamma_N, \not{k}]\right)}{(p^2 - m^2)^3}$$

because the anticommutator of K's and Γ's does not survive the trace. After integration (see appendix) and taking the limit $\ell \to 2$ with $C(1,2) \to \frac{1}{2}(\ell-2)$, then

$$T_{MN}^{[IJKL]}(K,K') \longrightarrow \frac{2e^2}{4!\,(2\pi)^{\ell}} \, \delta^{[IJKL]}_{[MNRS]} \, K^R K'^S$$

In 4-dimensions this corresponds to $T^5_{\mu\nu}(K,K') = \frac{e^2}{4\pi} \epsilon_{\mu\nu\sigma\rho} K^\sigma K'^\rho$ which is the famous axial anomaly. This calculation amply confirms the interpretation we have given to axial vectors and pseudoscalars in the context of DR. It means that one can now study interactions involving 0^- and 1^+ currents with some degree of confidence and compute overall abnormal amplitudes. Now we shall examine the triple axial anomaly[5]

$$\langle T(J_{IJK}(-K-K')\, J_{LMN}(K)\, J_{L'M'N'}(K'))\rangle$$

due to the spinor loop which is the only other anomaly peculiar to this simple model, (no internal SU(n) indices involved in the current).

When we take the divergence at one of the axial vertices, the Ward identity,

$$[(\not p+\not k)-m]^{-1}\, K_{[H}\Gamma_{IJK]}\, [\not p-m]^{-1}$$

$$= [(\not p+\not k)-m]\{2m\, \Gamma_{[HIJK]} - (2p+k)^{G} \cdot$$

$$\cdot\, \Gamma_{[GHIJK]}\}[\not p-m]^{-1} \; +$$

$$+ [(\not p+\not k)-m]^{-1}\, \Gamma_{[HIJK]} + \Gamma_{[HIJK]}\, (\not p-m)^{-1}$$

means that we shall have to cater for (i) the usual pseudo-scalar-axial-axial (PAA) vertex; (ii) two self-energy like graphs due to contraction of propagators* plus (iii) the anomalous term involving the antisymmetric product of five matrices. It is not too difficult to carry out the perturbation calculation in spite of the profusion of indices if we recognize that the four dimensional limit $\ell \to 2$ is to be taken at the end and that the anomaly emerges as the product of an integral which diverges in this limit multiplied by a kinematic term carrying a tracing factor $(\ell-2)$ which vanishes in this limit. Kinematic factors like $\delta^{(PQRST)}_{(JKLMN)}$ which have no place in four dimensions can be disregarded. These important technical aspects of DR are well substantiated by detail computation. For sake of simplicity we shall set $K^2 = K'^2 = 0$ to demonstrate the workings:

(i) The <PAA> vertex is described by the tensor element,

---

*These graphs also occur in the <VVA> anomaly but happen to give a zero answer for arbitrary ℓ. This is not true in the triple axial case.

$$T^{[HIJK]}_{LMN}(K,K') =$$

$$= \frac{4i}{(2\pi)^{2\ell}} \int \frac{d^{2\ell}p \; dx\,dy\; \Theta(1-x-y)}{(p^2 + 2K\cdot K'\,xy - m^2)^3} \cdot$$

$$\cdot \mathrm{Tr}\left[\{\not p - \not K x - \not K'(1-y) + m\}\,\Gamma_{[L'M'N']}\{\not p - \not K x + \not K' y + m\}\cdot\right.$$

$$\left.\cdot\; \Gamma_{[LMN]}\{\not p + \not K(1-x) + \not K' y + m\}\,\Gamma^{[HIJK]}\right]$$

and the trace gives rise to the typical kinematic terms $\eta_{[LL'}\delta^{[HIJK]}_{MN\,M'N']}$ and $K_{[L}K_{L'}\delta^{[HIJK]}_{[MN\,M'N']}$. The former factor does not exist in four dimensions and multiplies a finite integral

$$(\ell-2)\int d^{2\ell}p\; p^2 (p^2 + 2K\cdot K'\,xy - m^2)^{-3}\, dx\,dy\,\Theta(1-x-y).$$

while the latter factor multiplies the integral

$$m(3-\ell)\,(2\pi)^{-\ell}\int dx\,dy\,(2x+2y-1)(2K\cdot K'\,xy - m^2)^{\ell-3} \cdot$$

$$\cdot\; \Theta(1-x-y)$$

which supplies the traditional answer when $\ell \to 2$.

(ii) The self-energy integrals boils down to

$$T''^{[HIJK]}_{[LMN][L'M'N']} =$$

$$= -4i \, \mathrm{Tr}\left[\Gamma_{[LMN]} \Gamma_{[L'M'N']} \Gamma^{[HIJK]} \right] .$$

$$\cdot \int \frac{d^{2\ell}p}{(2\pi)^{2\ell}} \frac{(3-\ell)\, p^2/\ell + m^2}{(p^2 - m^2)^2}$$

and must be added to the

(iii) Anomalous integral associated with the new vertex

$$\{ i\overleftrightarrow{\not\partial}, \Gamma_{[HIJK]} \}$$

In fact the sum of (ii) and (iii) yields the <AAA> anomaly

$$K_{[L} K'_{L'} \delta^{[HIJK]}_{MN M'N']} \Gamma(3-\ell) \int dx\, dy\, (2x+2y-1)\, \Theta(1-x-y) ,$$

which is one third of the <VVA> anomaly.

Following work of Akyeampong and Delbourgo[19], Kee has worked out all the anomalies and these include the overall normal anomalies, where normal objects are defined to be (letting P stand for parity).

$$(-1)^J . P = 1$$

and the abnormal objects are defined to be

$$(-1)^J . P = -1$$

The definition of normal amplitudes is

$$(-1)^{\Sigma J_i} . \Pi \, P_i = 1$$

and that of abnormal amplitudes is

$$(-1)^{\Sigma J_i} . \Pi \, P_i = -1$$

These extra normal anomalies contribute a term in the divergence of axial vector current, in addition to the abnormal Bardeen[19] term, i.e.,

$$\partial_\mu j_5^\mu = 2m\, j_5 \quad + \text{Abnormal} + \text{Normal}.$$

This means that even for normal amplitudes like $\langle A_\mu \, A_\nu \rangle$

$$\langle \partial^\mu A_\mu, A_\nu \rangle - \langle 2mP, A_\nu \rangle \neq 0$$ but is given by

$$\langle P', A_\mu \rangle = i K_\mu \left(\frac{K^2}{6} - m^2 \right) / 2\pi^2$$

Other anomalous normal terms are

$$\langle P'P \rangle = m \left(K^2/6 - m^2 \right) / \pi^2$$

$$\langle P'PS \rangle = \left(3K^2 + 3KK' + K'^2 - 18m^2 \right) / 6\pi^2$$

$$\langle P'A_\mu S \rangle = i m K'_\mu / \pi^2$$

$$\langle P'PV_\mu \rangle = m (K - K')_\mu / 3\pi^2 ,$$

$$\langle P' A_\mu V_\nu \rangle = i\Big[\left(K'_\mu K_\nu + 2K_\mu K_\nu + 3K_\mu K'_\nu\right) - \\ - \eta_{\mu\nu}\left(3K^2 + K'^2 + 3KK' - 6m^2\right)\Big]\Big/ 6\pi^2 .$$

where K refers to the momentum carried by the pseudoscalar field P' having the anomalous kinetic interaction and K' refers to the other unnatural leg of the three-point vertex.

CHAPTER IV

WEAK INTERACTIONS

4.1 The Weak Leptonic Lagrangian

The two basic properties enjoyed by the usual four-Fermi current-current interaction

$$\mathcal{L}_W \propto \bar{\Psi}_1 \gamma_\mu (1-i\gamma_5)\Psi_2 \bar{\Psi}_3 \gamma^\mu (1-i\gamma_5)\Psi_4 \tag{4.1}$$

are

(i) left-handedness of the 2-component lepton spinors;

(ii) Fierz-transformation invariance of the Lagrangian, i.e.

$$\mathcal{L}_W \propto \bar{\Psi}_3 \gamma_\mu (1-i\gamma_5)\Psi_2 \bar{\Psi}_1 \gamma^\mu (1-i\gamma_5)\Psi_4 . \tag{4.2}$$

Since we have no prior knowledge about the structure of weak interactions in arbitrary dimensions (2ℓ) let us for the present assume that characteristics (i) and (ii) are retained for all ℓ. Property (i) means (see section 2.5) that the lepton fields have $2^{\ell-1}$ components and are represented by the spinors $\frac{1}{2}(1-i\,\Gamma_{-1})\psi$, where, according to eq. (1.6), Γ_{-1} is the obvious generalization of γ_5 to 2ℓ dimensions. To investigate requirement (ii) we have to know something about Fierz reshuffling[6] which we have shown in section (2.2). From eq. (2.21) $K(r) \equiv \Gamma_{(M_1 \ldots M_r)} \times \Gamma^{(M_1 \ldots M_r)}$ stand for the $2\ell+1$ parity-conserving kinematic covariants pertaining to one channel, the covariants in the crossed channel $\tilde{K}(r)$ are from (2.22) is

$$\tilde{K}(s) = \sum_{S} C(s,r)\, K(r) \tag{2.22}$$

where C(s,r) is defined by eq. (2.23) and (2.24).

In the case of weak interactions there is the minor modifications that we should be dealing with the parity-violating left-handed covariants $K_-(r)$ as defined by eq. (1.8). By crossing, for s odd, we obtain

$$\tilde{K}_-(s) = 2^{-\ell}\sum_{r}\frac{1}{4}\Gamma_{(s)}(1-i\Gamma_{-1})\Gamma_{(r)}\Gamma^{(s)}(1-i\Gamma_{-1})\otimes\Gamma^{(r)}$$

$$= \sum_{r\ \mathrm{odd}}\frac{1}{2}C(s,r)\,\Gamma_{(r)}(1-i\Gamma_{-1})\otimes\Gamma^{(r)}$$

$$= \sum_{r\ \mathrm{odd}} C(s,r)\,K_-(r)\,. \qquad (4.3)$$

Thus Fierz reshuffling for these particular covariants involves just the odd-odd entries in C (up to r, s = $\ell-1$ by reflection symmetry). We now are in a position to prove that the sum (1.7) is crossing invariant. We make use of the representation (2.27), then, from (4.3) and (2.27).

$$\sum_{s\ \mathrm{odd}}(-1)^{\frac{1}{2}(s-1)}\tilde{K}_-(s)$$

$$= i\,2^{-\ell}\sum_{s,r\ \mathrm{odd}}\frac{1}{s!}\left(i\frac{d}{dz}\right)^{s}\cdot$$

$$\cdot\left\{(1-z)^{r}(1+z)^{2\ell-r}\right\}\Big|_{z=0}K_-(r)$$

$$\sum_{s\ \mathrm{odd}} (-1)^{\frac{1}{2}(s-1)} \tilde{K}_-(s) = i\, 2^{-\ell-1} \sum_{n\ \mathrm{odd}} \left(\exp\left[i \frac{d}{dz}\right] - \exp\left[-i\frac{d}{dz}\right] \right) .$$

$$\cdot \left\{ (1-z)^n (1+z)^{2\ell-n} \right\} \Big|_{z=0} K_-(n)$$

$$= (-1)^{\frac{1}{2}\ell} \sum_{n\ \mathrm{odd}} (-1)^{\frac{1}{2}(n-1)} K_-(n)$$

Showing that the sum (1.7) is a crossing eigenvector with eigenvalue $(-1)^{\frac{1}{2}\ell}$.

Finally taking note of the anticommutativity of spinor fields we conclude that the appropriate weak four-Fermi interaction in 2ℓ dimensions is

$$\mathcal{L}_W \propto \sum_{n\ \mathrm{odd}}^{\ell-1} \bar{\Psi}_1 \Gamma_{(n)} (1 - i\Gamma_{-1}) \Psi_2\, \bar{\Psi}_3\, \Gamma^{(n)} (1 - i\Gamma_{-1}) \Psi_4$$

$$= (-1)^{\frac{1}{2}\ell+1} \sum_{n\ \mathrm{odd}}^{\ell-1} \bar{\Psi}_1\, \Gamma_{(n)} (1 - i\Gamma_{-1}) \Psi_4\, \bar{\Psi}_3\, \Gamma^{(n)} (1 - i\Gamma_{-1}) \Psi_2 . \tag{4.5}$$

In particular the leptonic Lagrangian, with correct normalization, written in current-current form, is

$$\mathcal{L}_W = \sqrt{2}\, G_\mu \sum_{n\ \mathrm{odd}}^{2\ell-1} J^{\dagger}_{(n)}\, J^{(n)} \tag{4.6}$$

with

$$J_{(n)} \equiv \frac{1}{2} \bar{\Psi}\, \Gamma_{[M_1 \cdots M_n]} (1 - i\Gamma_{-1}) \Psi . \tag{4.7}$$

At the tree graph level, which may be the only sensible way to look upon the four-Fermi $\mathcal{L}_W$, we observe that in the limit $\ell = 2$ there occur two currents which are replicas of one another, namely

$$J_\nu = \frac{1}{2}\bar{\psi}\gamma_\nu(1-i\gamma_5)\psi$$

and

$$J_{\kappa\lambda\mu} = \frac{1}{2} i \bar{\psi}\gamma_{[\kappa}\gamma_\lambda\gamma_{\mu]}(1-i\gamma_5)\psi$$

(The same currents arise in

$$\mathcal{L}_W = e\sum_{n\,\mathrm{odd}}\left(J^\dagger_{(n)}W^{(n)} + h.c.\right), \tag{4.8}$$

the weak-intermediate-boson variant of weak interactions.) Such repetitions of V-A in $J_{(1)}$ and $J_{(3)}$ for $\ell = 2$ are uninteresting in themselves if we stick to the Born diagrams. However as soon as we calculate higher-order quantum loops differences begin to show up. Indeed for $\ell \neq 2$ the currents

$$J_N = \bar{\psi}\Gamma_N\psi, \qquad \text{a true vector,}$$

$$i\bar{\psi}\Gamma_N\Gamma_{-1}\psi, \qquad \text{a true pseudovector,}$$

$$\bar{\psi}\Gamma_{[KLM]}\psi \qquad \text{an <<axial>> vector and}$$

$$i\bar{\psi}\Gamma_{[KLM]}\Gamma_{-1}\psi \qquad \text{The <<pseudoaxial >> vector, say,} \tag{4.9}$$

are all distinct from one another. In fact J_N is the only truly conserved current; the pseudovector current is conserved in the zero-mass fermion limit when chiral transformations

$\psi \to \exp(\theta\ \Gamma_{-1})\psi$ become an exact symmetry of the theory; but the axial and pseudoaxial currents are not at all conserved as one knows from our work on anomalies[5,6] in Chapter III. We shall retrun to the consequences of this extraordinary fact shortly.

4.2 Effective Hadronic Weak Lagrangian

It is a good idea to list of all the 2ℓ-dimensional forms of the Lagrangians responsible for the weak semileptonic decays before we come to the question of quantum loops. The baryons are represented by 2^{ℓ} component spinors and we have to bear in mind that pseudoscalar mesons are 4-index tensors. Beginning with the semi-leptonic baryon decays $B \to B'\ \ell\bar{\nu}$, the straight generalization of (4.6) and (4.8) is indicated, and we should include in (4.7) the hadronic-current contribution

$$J_{(n)}^{\text{baryon}} = \frac{1}{2}\bar{B}\,\Gamma_{(n)}(1 - i\,\Gamma_{-1})\,B \tag{4.10}$$

before renormalization. Naturally one expects radiative corrections due to current nonconservation to be significant, so one should modify the effective hadronic currents to

$$J_{(1)} \to \frac{1}{2}\bar{B}\ \Gamma_M(g_1 - i\,g_{2\ell-1}\Gamma_{-1})\,B,$$

$$J_{(3)} \to \frac{1}{2}\bar{B}\ \Gamma_{[KLM]}(g_3 - i\,g_{2\ell-3}\Gamma_{-1})\,B,$$

$$J_{(5)} \to \frac{1}{2}\bar{B}\ \Gamma_{[IJKLM]}(g_5 - i\,g_{2\ell-5}\Gamma_{-1})\,B,$$

etc.,

where g_r are coupling constant renormalizations. The vector current is of course conserved so $g_1 = 1$; one would also be inclined to suppose that $g_{2\ell-3}$ is unrenormalized at unity since the associated pseudoaxial current looks vectorial in four dimensions and is conserved in that limit. That however is contrary to the rules of DR which stipulate that all perturbation calculations have to be performed before going to the four dimensional limit. In fact we shall prove later that $g_{2\ell-3}$ differs from 1 by a finite amount, calculable in any given model of the strong interactions. Thus we have the bizarre fact that the weak vector current is renormalized and

$$\frac{g_A}{g_V} = \lim_{\ell\to 2} \frac{g_3 + g_{2\ell-1}}{1 + g_{2\ell-3}} . \tag{4.12}$$

We shall enlarge upon this curiosity in the next section, but for the present let us carry on writing effective Lagrangian for purely leptonic and semileptonic decays, $P \to \ell\bar{\nu}$ and $P \to P'\ell\,\nu$ respectively. The effective weak currents $J_{(r)}$ which couple the mesons to the lepton currents turn out to be

$$\begin{aligned} & i f_+ P^\dagger_{JKLM} \overleftrightarrow{\partial}_N P^{JKLM}, && \text{vector,} \\ & i f_- \partial_N \left(P^\dagger_{JKLM} P^{JKLM}\right), && \text{pseudovector,} \\ & f\ \partial^J P_{JKLM}/M && \text{axial vector,} \end{aligned} \tag{4.13}$$

4.3 One Loop Renormalizations

Let us suppose that strong interactions are renormalizable in four dimensions. The Yukawa meson-baryon coupling

$$\mathcal{L}_{strong} = g \bar{B} \Gamma_{[M_1 \cdots M_s]} B \phi^{[M_1 \cdots M_s]} \qquad (4.17)$$

can serve as a suitable model for the purpose of the following discussions. Using the propagators

$$\langle \varphi_{[M_1 \cdots M_n]}(k) \varphi^{[N_1 \cdots N_s]}(-k) \rangle = \delta^{[N_1 \cdots N_s]}_{[M_1 \cdots M_n]} / (k^2 - \mu^2)$$

$$\langle B(p) \quad \bar{B}(p) \rangle = i / (\not{p} - m)$$

we can enquire about the nature of one-loop renormalizations of the weak currents. Before we plunge into an analysis of the vertex parts let us treat the wave function renormalization. The fermion loop contribution to the meson self-energy parts (see Fig. o) is given in e.g. (3.2) for massless fermions. To determine the associated wave function renormalization constants $Z_\phi(r)$ we note that in the vector sector (r = 1) according to (3.3),

$$\Pi^{MN}(k) = \frac{2e^2 \Gamma(2-\ell)}{(2\pi)^{\ell}} \int_0^1 \frac{d\alpha \, (\eta^{MN} - k^M k^N / k^2)}{(-k^2 \alpha(1-\alpha))^{1-\ell}} \qquad (3.3)$$

whereas in the pseudovector sector ($r = 2\ell-1$) we meet the dual

$$\Pi^{MN}_{-1\,-1}(K) \equiv \epsilon^{MM_1\cdots M_n}\epsilon_{NN_1\cdots N_n}\,\Pi^{[N_1\cdots N_n]}_{[M_1\cdots M_n]}(K)\Big/(2\ell-1)!$$

$$= \Pi^{MN}(K)$$

because $\underline{\Gamma_{-1}\ \text{chirality}}$ is a good symmetry for m = o. On the other hand the axial vector sector is obtained by putting r = 3 in eq. (3.2), that is

$$\Pi^{[LMN]}_{[IJK]}(K) = 2e^2\,\frac{\Gamma(1-\ell)}{(2\pi)\ell}\,\cdot \tag{4.18}$$

$$\cdot\int_0^1 \frac{dx}{[-K^2\alpha(1-\alpha)]^{1-\ell}}\Big\{(\ell-2)\,\delta^{[LMN]}_{[IJK]} + $$

$$+(\ell-1)\,K_{[I}K^{[L}\delta^{MN]}_{JK]}\Big/K^2\Big\},$$

and dual pseudoaxial self-energy (r = 2ℓ-3) is its image:

$$\Pi^{-1\,[IJK]}_{-1\,[IJK]}(K) \equiv \epsilon_{IJKR_4\cdots R_{2\ell}}\,\epsilon^{LMNS_4\cdots S_{2\ell}}\,.\,\Pi^{R_4\cdots R_{2\ell}}_{S_4\cdots S_{2\ell}}(K)$$

$$= \Pi^{[LMN]}_{[IJK]}(K)\,.$$

Thus for m = o, $Z_\phi(1) = Z_\phi(2\ell-1)$

i.e. vector = pseudovector renormalization

and $Z_\phi(3) = Z_\phi(2\ell-3)$

axial = pseudoaxial renormalization

However the difference between vector and axial renormalizations

$$Z_\varphi(2\ell-3) - Z_\varphi(1) =$$

$$= \lim_{\ell \to 2} \frac{2e^2(\ell-2)\Gamma(1-\ell)}{(2\pi)^\ell K^2} \int_0^1 \frac{d\alpha}{[-K^2\alpha(1-\alpha)]^{1-\ell}}$$

$$= -\frac{e^2}{12\pi^2} \tag{4.19}$$

is finite and non vanishing! The explanation for the difference between vector and pseudoaxial renormalizations is the same one that is offered when one meets anomalous Ward identities, viz. because the pseudoaxial current is only conserved in four dimensions, we have the product of a kinematic factor which vanishes as $\ell \to 2$ and a singular factor $(\ell-2)^{-1}$ due to the divergent quantum loop. The same discrepancy[5] between axial and pseudovector renormalizations was observed some time ago, but is not so striking because the axial current will not in general be conserved for $m \neq o$.

Consider next the fermion self-energy due to the interaction (4.17), as depicted in Fig. 4. Again, the fermion wave function renormalization constant Z_ψ can for simplicity be determined by setting the fermion mass equal to zero.

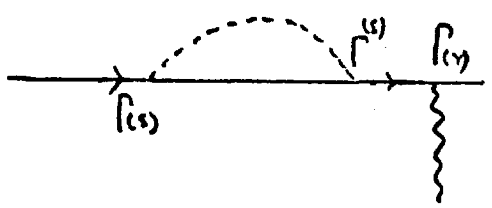

Fig. 4

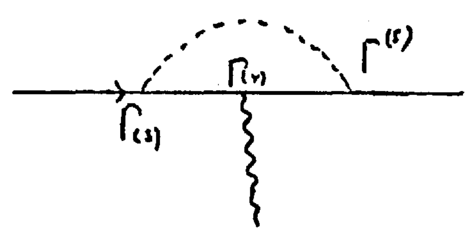

Fig. 5

Thus

$$\Sigma(p) = -i\, g^3 \int \frac{d^{2\ell}k}{(2\pi)^{2\ell}} \frac{\Gamma_{(s)}\, \Gamma_{\cdot}\,(p-k)\,\Gamma^{(s)}}{(p-k)^2\,(k^2-\mu^2)}$$

By standard procedures outlined in Appendix A this can be reduced to the parametric integral

$$\Sigma(p) = \not{p}\; g^2 \frac{C(s,1)}{(4\pi)^{\ell}}\, \Gamma(2-\ell) \int_0^1 \frac{d\alpha}{\{(\mu^2-p^2\alpha)(1-\alpha)\}^{2-\ell}} ,$$

(4.20)

Near four dimensions,

$$Z_\psi - 1 \approx g^2 \frac{C(s,1)}{16\pi^2(\ell-2)}$$

Finally we can turn to the vertex part of Fig. 5 which we shall evaluate at zero momentum transfer

$$\Lambda_{(v)}(p) = i\, g^2 \int \frac{d^{2\ell}k}{(2\pi)^{2\ell}} \frac{\Gamma_{(s)}(\not{p}-\not{k})\,\Gamma_{(v)}(\not{p}-\not{k})\,\Gamma^{(s)}}{(p-k)^4\,(k^2-\mu^2)}$$

(4.21)

The $\Lambda_{(r)}$ are logarithmically infinite at $\ell = 2$ and have each to be renormalized by a vertex factor, say $Z_g(r)$. Clearly, since the vector vertex satisfies Ward identity

$$\Lambda_M(p) = \frac{\partial \Sigma(p)}{\partial p^M} ,$$

the vector renormalization constant equals the fermion wave function renormalization

$$Z_g(1) = Z_\psi ,$$

but there is no reason in general to expect similar equalities for the remaining constants; in our zero-fermion mass model however it happens that

$$Z_g(2l-1) = (-1)^{S+1} Z_\psi .$$

Of all the other vertex parts the pseudoaxial is the most intriguing and it can be deduced from the parametric integral representation of eq. (4.21):

$$\Lambda_{(n)}(p) = \frac{g^2}{(4\pi)^l} \int_0^1 \alpha\, d\alpha \left[\frac{\Gamma(3-l)\,\Gamma_{(s)}\,\not p\,\Gamma_{(v)}\,\not p\,\Gamma^{(s)}}{[(\mu^2 - p^2\alpha)(1-\alpha)]^{3-l}} - \frac{C(1,n)\,C(s,n)\,\Gamma_{(n)}\,(2-l)}{[(\mu^2 - p^2\alpha)(1-\alpha)]^{2-l}} \right]$$

Thus

$$Z_g(2l-3) - 1 = -\lim_{l \to 2} C(1,2l-3)\,C(3,2l-3)\,\cdot\, \frac{g^2\,\Gamma(2-l)}{(4\pi)^l} \int_0^1 \frac{\alpha\, d\alpha}{[(\mu^2 - p^2\alpha)(1-\alpha)]^{2-l}} \qquad (4.22)$$

The anomalous finite difference between vector and pseudo-axial vertex renormalizations is

$$Z_g(1) - Z_g(2\lambda-3)$$
$$= \lim_{\lambda \to 2} \frac{g^2\left(C(1,1)C(s,1) - C(1,2\lambda-3)C(s,2\lambda-3)\right)}{32\pi^2(\lambda-2)}$$

(4.23)

and for values up to s = 4 equal to

$$\frac{g^2}{32\pi^2}\left[-\frac{1}{4}\delta_{s0} + \delta_{s1} - \delta_{s2} - \frac{1}{3}\delta_{s3} + \frac{11}{12}\delta_{s4}\right]$$

(4.23')

A similar difference arises between pseudovector and axial vertex renormalizations:

$$Z_g(3) - Z_g(2\lambda-1)$$
$$= \lim_{\lambda \to 2} \frac{g^2\left(C(1,3)\,C(s,3) - C(1,2\lambda-1)C(s,2\lambda-1)\right)}{32\pi^2(\lambda-2)}$$

(4.23")

Altogether the one-loop current coupling renormalizations in (4.11) follow from the formula

$$g_r = Z_\varphi^{\frac{1}{2}}(r)\; Z_g^{-1}(r)\; Z_\psi$$

since all bare couplings are normalized to unity. If the renormalized couplings g_r truly refer to weak interactions we can disregard differences between $Z_\phi(r)$ as being of order e^2 and therefore small (remember however that Z_ϕ is not r-independent and we need only concern ourselves with the strong corrections to Z_g and Z_ψ. Besides the expected result $g_1 = 1$ connoting absence of vector current renormalization, we have the unexpected result

$$g_{2l-3} = 1 + \frac{g^2}{32\pi^2}\left[-\frac{1}{4}\delta_{s0} + \delta_{s1} - \delta_{s2} - \frac{1}{3}\delta_{s3} + \frac{11}{12}\delta_{s4}\right]$$

as the one-loop renormalization of the other weak vector current. The axial and pseudovector renormalizations are infinite if s is even because (4.17) violates Γ_{-1} chirality, but finite if s is odd. For instance a vector model (s = 1) gives

$$g_{2l-1} = g_1 = 1 \quad , \quad g_3 = g_{2l-3} = 1 + \frac{g^2}{32\pi^2} \quad ,$$

while an axial vector strong interaction (s = 3) gives

$$g_{2l-1} = g_1 = 1 \quad , \quad g_3 = g_{2l-3} = 1 - \frac{g^2}{96\pi^2} \quad ,$$

In each of these cases

$$g_A/g_V = \frac{g_1 + g_{2l-3}}{g_{2l-1} + g_3} = 1$$

because of Γ_{-1} chiral symmetry. In more general circumstances when every current (with the exception of the pure vector) is nonconserved we anticipate that $g_{2\ell-3} \neq 1$ and $g_{2\ell-1} \neq g_3$. However the previous discussion shows how and why $g_{2\ell-3}-1$ and $g_3-g_{2\ell-1}$ are <u>finite</u> and <u>calculable</u> to any order in perturbation theory - when the strong interactions are renormalizable the differences depend only on the magnitude g and character of the strong coupling.

Apart from certain conformal <u>anomalies</u>[20] in quantum gravity, the renormalization of this weak vector current is the first unconventional result to come out of DR and one may wonder how it can be avoided. One possibility is to contrive a cancellation of $g_{2\ell-3}-1$ with a counter-Lagrangian, but this can surely be dismissed as being too artificial, rather ugly, and possibly dangerous in view of the conclusions to come in Chapter V. A second possibility is to reject the Fierz invariance requirement and to return to the vector and pseudovector currents, $\bar{\psi}\, \Gamma_N\, \psi$ and $i\, \psi\, \Gamma_N\, \Gamma_{-1}\, \psi$; this too seems fraught with difficulties for the simple reason that hadronic contributions to axial currents must satisfy anomalous PCAC identities (even if the anomaly cancels in toto); the pseudovector $i\, \bar{\psi}\, \Gamma_N\, \Gamma_{-1}\, \psi$ is potentially incapable of yielding any anomaly (since Γ_{-1} - invariance is an exact symmetry of zero mass fermions) is contrast to the axial $i\, \bar{\psi}\, \Gamma_{(KLM)}\, \psi$ and its partner $i\, \bar{\psi}\, \Gamma_{(KLM)}\, \Gamma_{-1}\, \psi$; therefore even if we abandon the notion of Fierz symmetry of $\mathcal{L}$ we are obliged to consider at least these four kinds of vector current and their attendant renormalizations. One final possibility is to cancel off the anomalous renormalizations altogether (by increasing the number of fermion fields, including V+A interactions, etc.) in the accustomed manner which ensures the renormalizability of unified gauge theories.

CHAPTER V

THE NONRENORMALIZABILITY OF EVANESCENT COUPLINGS

5.

In this chapter we shall investigate a theory which has a primary evanescent interaction which is given in (1.10), and we shall prove that the model is non-renormalizable. To arrive at this conclusion we will need to go beyond one-loop level. The basic reason is as follows: at one-loop level the divergent Feynman integrals contain a pole term $(\ell-2)^{-1}$ and these multiply a factor $(\ell-2)$ which must be present for all form factors associated with kinematic terms that survive the 4-dimensional limit (because $\mathcal{L}_I \to 0$ as $\ell \to 2$). The product of these yields a polynomial in external momenta and masses at four dimensions. At the next, two loop level we may encounter double integrals which contain second order poles $(\ell-2)^{-2}$ but only a single factor $(\ell-2)$ in the numerator, signifying a divergence. (Another way of stating this is to note that the 1-loop polynomial suffers a further divergent integration with no further compensating zero from $\mathcal{L}_I$). As these divergences get progressively worse in higher orders of G there is no hope of renormalizing the theory. The final result, that evanescent couplings with bad powercounting characteristics, are non-renormalizable after all, is useful in restricting the class of Lagrangian models that are viable in the context of DR. Let us substantiate these statements by giving a few details of our investigation. For our free Lagrangian we shall take a massive ψ and a massless boson ϕ:

$$\mathcal{L}_0 = -\phi^{+KLMN}\partial^2\phi_{KLMN} + \bar{\psi}(i\not{\partial}-m)\psi \tag{5.1}$$

in order to simplify some of the Feynman integrals without affecting the ultraviolet behaviour in question. We also once again write down our interaction Lagrangian $\mathcal{L}_I$ from (1.10)

$$\mathcal{L}_I = G\bar{\Psi}\{\overleftrightarrow{\not\partial}, \Gamma_{KLMN}\}\phi^{KLMN} \qquad (1.10)$$

The fact that $\mathcal{L}_0$ can lead to ghost mesons in some of the ϕ components will not concern us unduly, since none arise when $\ell \to 2$. The classical tree graphs evidently give zero identically in the 4-dimensional limit, so the first interesting results occur at the one-loop level. In momentum space the vertex factor arising from $\mathcal{L}_I$ in a perturbation development is $\{2\not p + \not k, \Gamma_{KLMN}\}$ where p and p and p+k stand for the incoming and outgoing fermion momenta. These have to be combined with the propagators $s(p) = i(\not p - m)^{-1}$ and $D_{M_1 \ldots M_4}^{N_1 \ldots N_4} = i\ \delta_{M_1 \ldots M_4}^{N_1 \ldots N_4} / k^2$ by the standard Feynman rules. As we shall be interested in kinematic terms produced from Feynman graphs which survive the passage to 4-dimensions one can set <u>external</u> momenta equal to <u>zero</u> at each vertex.

We may now determine some simple one-loop diagrams.

<u>Boson self energy</u>

To order G^2 retaining the part which survives four dimensions

$$\Pi^{(N_1\cdots N_4)}_{(M_1\cdots M_4)}(K) =$$
$$= 4iG^2\int\frac{d^{2l}p}{(2\pi)^{2l}}\;\frac{\mathrm{Tr.}\left[\{\not p, \Gamma_{[M_1\cdots M_4]}\}(\not p+m)\{\not p, \Gamma^{[N_1\cdots N_4]}\}(\not p+\not K+m)\right]}{(p^2-m^2)\left[(p+K)^2-m^2\right]}$$

Introducing a Feynman parameter α, shifting the integral, and dropping all $\{\not K, \Gamma_{(4)}\}$ terms, the usual manipulations lead us to

$$\Pi^{(N_1\cdots N_4)}_{(M_1\cdots M_4)}(K) = \frac{G^2}{(2\pi)^l}\,\Gamma(3-l)\,\delta^{[N_1\cdots N_4]}_{[M_1\cdots M_4]}\,\frac{16(2l+1)}{l(l-1)}\cdot$$
$$\cdot\int\left\{m^2-K^2\alpha(1-\alpha)\right\}^l d\alpha$$

$$\underset{l\to 2}{\longrightarrow}\; G^2\left(K^4-10m^2K^2+30m^4\right)\delta^{[N_1\cdots N_4]}_{[M_1\cdots M_4]}\Big/3\pi^2 \qquad (5.2)$$

As promised the quartic divergence has disappeared owing to the vanishing trace.

Fermion self energy

Since we shall presently take this graph to be part of a larger graph the integral to be evaluated is

$$\Sigma(p) = -ig^2\int\frac{d^{2l}k}{(2\pi)^{2l}}\;\frac{\{2\not p+\not k, \Gamma_{[M_1\cdots M_4]}\}(\not p+\not k+m)\{2\not p+\not k, \Gamma^{[M_1\cdots]}\}}{\left[(2p+k)^2-m^2\right]\,k^2}$$

The calculation of the numerator here (as well as that of the vertex part to follow) is greatly facilitated by the methods set out in section (2.2). Using (2.31) and (2.31') the final answer is a polynomial in $\not p$:

$$\Sigma(p) \sim G^2\,\Gamma(3-\ell)(2\ell-1)(2\ell-3)\int_0^1 d\alpha\,\cdot$$

$$\cdot\Big(\big[\not p\{3(\ell-1)-\alpha(1+\ell)\}+m\ell\big]\{p^2\alpha(1-\alpha)-m^2\alpha\}^{\ell-1}-$$

$$-(\ell-1)p^2(2-\alpha)^2\{(1-\alpha)\not p+m\}\{p^2\alpha(1-\alpha)-\alpha m^2\}^{\ell-2}\Big)$$

$$\xrightarrow[\ell\to 2]{} G^2\left[(\not p+2m)(p^2+\tfrac{1}{2}m^2)+\frac{1}{6}\not p\,p^2\right]$$

Vertex part

$$V_{M_1\cdots M_4}(p',p) = -iG^3\int\frac{d^{2\ell}K}{(2\pi)^{2\ell}}\cdot$$

$$\cdot\frac{\{\not K,\Gamma_{[N_1\cdots N_4]}\}(\not p+\not K+m)\{2\not K,\Gamma_{M_1\cdots M_4}\}(\not p+\not K+m)\{\not K,\Gamma^{[N_1\cdots N_4]}\}}{\left[(p'+K)^2-m^2\right]K^2\left[(p+K)^2-m^2\right]}$$

If we are only interested in kinematic terms which survive $\ell\to 2$. Combining denominators with Feynman parameters and using simplification methods outlined in the eq. (2.31) and (2.31') we end up with

$$V_{M_1\cdots M_4}(p',p) = X\,\Gamma_{[M_1\cdots M_4]}+Y\left[\not p'-\not p,\Gamma_{[M_1\cdots M_4]}\right] \tag{5.4}$$

where the form factors X and Y tend to

$$X \rightarrow Y/4\,05\,m \propto G^3 \left(p^2 + p'^2 - 4m^2\right) \tag{5.4'}$$

for four dimensions.

<u>Other one-loop graphs</u>

The systematics should by now be obvious. Every one-loop graph is divergent due to the momentum factor in the vertex, but this is cancelled by a zero caused by the disappearance of $\mathcal{L}_I$. Always we are left with a polynomial in external momenta whose degree increases with G - as it must from simple dimensional analysis.

We may now wonder if this phenomenon carries over to higher loops and if all Feynman diagrams are finite. The answer is "no" and is most simply illustrated by examination of the vacuum graphs. The simplest two loop graph (conveniently treated in x-space) does indeed happen to be finite, but this is only an accident due to the masslessness of our boson. Thus

$$Z = G^2 \int d^{2\ell}x \, \mathrm{Tr}\left[\left\{\overleftrightarrow{\not{\partial}}, \Gamma_{[M_1 \cdots M_4]}\right\} S(x) \left\{\overleftrightarrow{\not{\partial}}, \Gamma^{[M_1 \cdots M_4]}\right\} S(x)\right] \cdot D(x)$$

Retaining the most singular terms in the integrand and dropping all ∂^2 D terms upon rotation to Euclidean space,

$$\begin{aligned} Z &\sim G^2 \int d^{2\ell}x \, 2^\ell D \partial^M \partial^N D \, \partial_M \partial_N D \, (\ell-2) \\ &= G^2 \int d^{2\ell}x \, 2^\ell \, 8\ell(2\ell-1)(\ell-1)^2(\ell-2) \, D^3/x^4 \end{aligned}$$

$$\rightarrow 192\,(\ell-2)\,G^2 \int d^{2\ell}x \; D^3/x^4$$

$$\sim (\ell-2)\,G^2 \int_0^\infty (r^2)^{\ell-6}\, dr^2$$

which is ultraviolet finite as $\ell \to 2$. Another way of seeing this is to work in momentum space, carry out the fermion loop integral, obtaining π (a polynomial at $\ell = 2$), and then working out the integral

$$\int d^{2\ell}K \; \Pi^{M_1 \cdots M_4}_{M_1 \cdots M_4}(K) \Big/ K^2 \quad .$$

In DR this is zero for polynomial Π. However had we chosen to give the mesons a mass μ we would instead have obtained the divergent answer $G^2(\mu^2)^{2\ell-1}/(\ell-2)$.

The problem is much clearer at the three loop level. Consider first the vacuum diagram of figure 6. If we carry out the meson loop integrations first we are left with

$$Z \sim G^2 \int d^{2\ell}p \; \mathrm{Tr}\left(\Sigma(p)(\not{p}-m)^{-1}\,\Sigma(p)(\not{p}-m)^{-1}\right)$$

and since $\Sigma(p)$ is finite near $\ell = 2$ (see eq. (5.3)) this final integral is bound to diverge near four dimensions. A more relevant example is the meson-meson scattering diagram of figure 7.

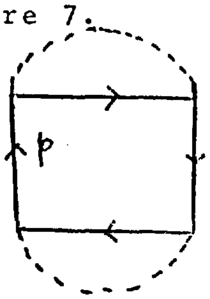

Fig. 6

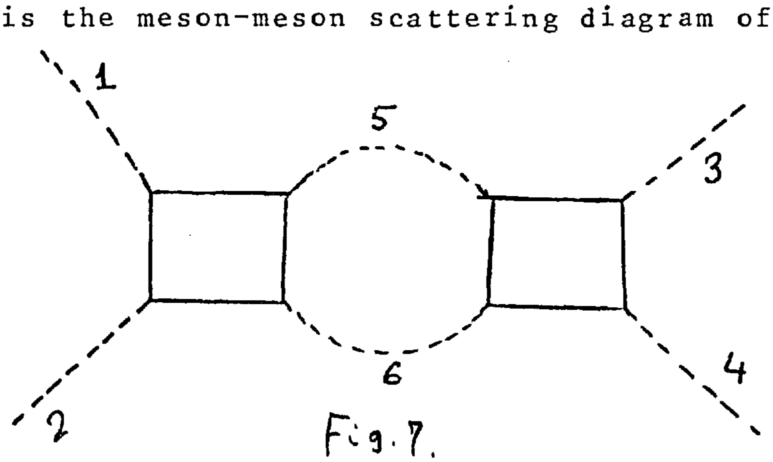

Fig. 7.

Carrying out the fermion loop integration first

$$M = \int d^{2\ell}K_5\, d^{2\ell}K_6\, \delta(K_5+K_6-K_1-K_2)\cdot$$
$$\cdot M_4(K_1K_2, K_5K_6)\, M_4(K_5K_6, K_3K_4) \Big/ K_5^2 K_6^2$$

But M_4 is a finite polynomial in K near $\ell = 2$. Hence the final integration produces a pole term $(\ell-2)^{-1}$.

The inescapable conclusion then is that the higher graphs diverge in general. Because these diagrams are associated with higher powers of G they require ever increasing numbers of subtractions and the theory is therefore non-renormalizable. We can thus class all theories with evanescent interactions and coupling constants having dimensions of inverse mass powers as undesirable in spite of appearances. More importantly, this means that <u>if we start with a renormalizable theory and happen to meet anomalous currents in the context of Ward identities, we should never attempt to cancel them off with evanescent counter Lagrangians</u>.

CHAPTER VI

SUPERSYMMETRY

6.1 The Four Dimensional Theory

The concept of the combination of fermions and bosons, known as supersymmetry (possibly supergauged) has come into existence very recently. The origin of this symmetry can be traced back to dual model; the theory works in 2 dimensions and takes the form of a local symmetry where it plays a vital role in the elimination of ghosts. More recently, Wess and Zumino took a decisive role in formulating a global Fermi-Bose symmetry in 4-dimensional space time. Afterwards, Salam and Strathdee systematized this 4-dimensional supersymmetry. They were led to consider a superfield $\phi(x,\theta)$ defined on an 8-dimensional space which is the extension of ordinary space-time $X_\mu(\mu = 0,1,2,3)$, to include a 4-dimensional space whose points are labelled by the anticommuting Majorana spinor $\theta_\alpha(\alpha = 1,2,3,4)$. The action of the Poincare' group on the space X and θ is given by

$$X_\mu \rightarrow \Lambda_{\mu\nu} X_\nu + b_\mu$$

$$\theta_\alpha \rightarrow a_\alpha^\beta(\Lambda)\ \theta_\beta$$

where $a(\Lambda)$ denotes the Dirac spinor representation of the homogeneous Lorentz transformation Λ. In particular, space reflections are associated with the mapping

$$\theta_\alpha \rightarrow i(\gamma_0\theta)_\alpha$$

The action of a supertranslation on the space X and θ is defined by

$$X_\mu \rightarrow X_\mu + \frac{i}{2}\bar{\epsilon}\gamma_\mu\theta$$

$$\theta_\alpha \longrightarrow \theta_\alpha + \epsilon_\alpha$$

where the parameter ϵ_α must of course be an anticommuting Majorana spinor. This larger group comprising the Poincare group and supertranslations leaves invariant the interval $(x' - x - i\bar{\theta}'\gamma\theta)^2$. The expansion of the scalar superfield terminates at θ^4 and is given by

$$\Phi(x,\theta) = A(x) + \bar{\theta}\psi(x) + \frac{1}{4}\bar{\theta}\theta F(x) + \frac{1}{4}\bar{\theta}\gamma_5\theta G(x) +$$
$$+ \frac{1}{4}\bar{\theta}\, i\gamma_\nu\gamma_5\theta A_\nu(x) + \frac{1}{4}\bar{\theta}\theta\bar{\theta}\chi(x) + \frac{1}{32}(\bar{\theta}\theta)^2 D(x)$$

where the coefficients A,F, G , A_ν and D are ordinary Bose fields, and ψ and χ are Fermi fields. The behaviour of these components under the action of the Poincare' group and parts is clear: A,F and D are scalars, G is a pseudoscalar, A_μ is an axial vector, ψ and χ are Dirac spinors, all up to an overall intrinsic parity factor. These components are complex in general. However, it is possible to impose a reality condition on the superfield,

$$\bar{\Phi}(x,\theta)^* = \Phi(x,\theta)$$

where the complex conjugation is understood to reverse the order of anticommuting vectors. The important thing to notice in the expansion of the superfield that the expansion stops at θ^4 which is due to anticommutivity of θ. This superfield is reducible. In particular one can project out chiral irreducible fields for which

$$D_\pm = -\partial^2 A_\pm \quad , \quad \chi_\pm = -i \not{\partial} \psi_\pm$$

$$G_\pm = \pm i F_\pm \quad , \quad A_{\nu\pm} = \pm i \partial_\nu A_\pm$$

and the resulting superfields can be compactly expressed as

$$\phi_\pm(x,\theta) = \exp\left[\mp \frac{1}{4} \bar{\theta} \gamma_\mu \gamma_5 \theta \frac{\partial}{\partial x_\mu}\right] \cdot$$
$$\cdot \left[A_\pm(x) + \bar{\theta} \psi_\pm(x) + \frac{1}{4} \bar{\theta}(1 \pm i\gamma_5)\theta F_\pm(x)\right]$$

with the expansion now terminating at θ^2 because θ are two component spinors.

We shall not bother to spell out the transformation properties and the rules for combining supermultiplets but will immediately proceed to the generalization to 2ℓ-dimensions which is obvious. Then we shall determine the transformation rules for all dimensions.

In its simplest formulation, superspace can be characterised by 2ℓ space time parameters $X_M (M = 0,1,\ldots, 2\ell-1)$ and 2^ℓ spinor parameters $\theta_\alpha (\alpha = 1,2,\ldots, 2^\ell)$. A Majorana constraint $\theta_\alpha = C_{\alpha\beta} \bar{\theta}^\beta$ is normally invoked to make the spinors essentially real, and if we wish to do this for all ℓ it is sufficient to work in even dimensions (ℓ integral) where in the charge conjugation matrix C exists. The generalised superfield expansion for $\ell > 2$ is then given by

$$\Phi(x,\theta) = A(x) + \bar{\theta}^{\alpha}\psi_{\alpha}(x) + \frac{1}{2}\bar{\theta}^{\alpha}\bar{\theta}^{\beta}F_{[\alpha\beta]} + \\ + \bar{\theta}^{\alpha}\bar{\theta}^{\beta}\bar{\theta}^{\gamma}\chi_{[\alpha\beta\gamma]}/3! \\ + \frac{1}{4!}\bar{\theta}^{\alpha}\bar{\theta}^{\beta}\bar{\theta}^{\gamma}\bar{\theta}^{\delta}D_{[\alpha\beta\gamma\delta]} + \\ + \bar{\theta}^{\alpha}\bar{\theta}^{\beta}\bar{\theta}^{\gamma}\bar{\theta}^{\delta}\bar{\theta}^{\epsilon}\eta_{[\alpha\beta\gamma\delta\epsilon]}/5! + \cdots \tag{6.1}$$

and terminates at $\theta^{2^{\ell}}$. This superfield is reducible and by projecting out the left or right handed components

$$\theta_{\pm} \equiv \frac{1}{2}\left(1 \pm i\Gamma_{-1}\right)\theta \equiv \frac{1}{2}\left(1 \pm i\Gamma_0\Gamma_1\cdots\Gamma_{2\ell-1}\right)\theta ,$$

we can pick out the irreducible chiral fields,

$$\Phi^{\pm}(x,\theta) = \exp\left(\mp\frac{1}{4}\bar{\theta}\not{\partial}\Gamma_{-1}\theta\right) . \\ \cdot\begin{bmatrix} A^{\pm} + \bar{\theta}_{\pm}^{\alpha}\psi_{\alpha}^{\pm}\psi_{\alpha}^{\pm} + \frac{1}{2}\bar{\theta}_{\pm}^{\alpha}\bar{\theta}_{\pm}^{\beta}F^{\pm}_{[\alpha\beta]} + \\ + \frac{1}{6}\bar{\theta}_{\pm}^{\alpha}\bar{\theta}_{\pm}^{\beta}\bar{\theta}_{\pm}^{\gamma}\chi_{[\alpha\beta\gamma]} + \cdots \end{bmatrix} \tag{6.2}$$

with the expansion now terminating at $\theta^{2^{\ell-1}}$. An infinitesimal supertransformation is given by

$$\delta\Phi = \bar{\epsilon}\left[\partial/\partial\bar{\theta} + \frac{1}{2}(i\not{\partial}\theta)\right]\Phi \equiv \bar{\epsilon}D\Phi \tag{6.3}$$

where

$$D_\alpha = \frac{\partial}{\partial \bar{\theta}^\alpha} + \frac{1}{2}(i\not{\partial}\theta)_\alpha$$

and

$$\bar{D}^\alpha = \frac{\partial}{\partial \theta_\alpha} - \frac{1}{2}(i\theta\not{\partial})^\alpha$$

so

$$\delta\bar{\Phi} = \bar{\epsilon}^\alpha\Big[\Psi_\alpha + \bar{\theta}^\beta F_{[\alpha\beta]} + \bar{\theta}^\beta\bar{\theta}^\gamma \chi_{[\alpha\beta\gamma]}/2! + $$

$$+ \bar{\theta}^\beta\bar{\theta}^\gamma\bar{\theta}^\delta D_{[\alpha\beta\gamma\delta]}/3! + \frac{1}{4!}\bar{\theta}^\beta\bar{\theta}^\gamma\bar{\theta}^\delta\bar{\theta}^\epsilon \eta_{[\alpha\beta\gamma\delta\epsilon]} + \cdots \tag{6.3'}$$

$$-\frac{1}{2}(\bar{\theta}\, i\not{\partial} C\bar{\epsilon})\Big[A + \bar{\theta}^\alpha\Psi_\alpha + \bar{\theta}^\alpha\bar{\theta}^\beta F_{[\alpha\beta]}/2! + $$

$$+ \bar{\theta}^\alpha\bar{\theta}^\beta\bar{\theta}^\gamma \chi_{[\alpha\beta\gamma]}/3! + \cdots\Big]$$

The component field changes are then as follows .

$$\delta A = \bar{\epsilon}^\alpha \Psi_\alpha$$

$$\delta\Psi_\alpha = \bar{\epsilon}^\beta\Big(F_{[\alpha\beta]} - \frac{1}{2} i(\not{\partial} C)_{\alpha\beta} A\Big), \ldots, \tag{6.4}$$

$$\delta D_{[\alpha\beta\gamma\delta]} = \bar{\epsilon}^\epsilon\Big(\eta_{[\alpha\beta\gamma\delta\epsilon]} - \frac{1}{2} i(\not{\partial} C)_{[\epsilon\delta}\chi_{\alpha\beta\gamma]}\Big),$$

for the reducible field components, and

$$\delta A^\pm = \bar{\epsilon}^\alpha_+ \Psi^\pm_\alpha$$

$$\delta\psi_\alpha^{\pm} = \bar{\epsilon}_{\pm}^{\beta}\left(F^{\pm}_{[\alpha\beta]} - i(\not{\partial}C)_{\alpha\beta}A^{\pm}\right)$$

$$\delta F^{\pm}_{[\alpha\beta]} = \bar{\epsilon}_{\pm}^{\gamma}\left(\chi^{\pm}_{[\alpha\beta\gamma]} - i\left(\not{\partial}C\right)_{[\alpha\beta}\psi^{\pm}_{\gamma]}\right), \cdots \tag{6.5}$$

for the chiral field components. It is clear that η and $\chi^{\pm}$ vanish only in four dimensions.

6.2 The Theory for Arbitrary Dimensions

The two dimensional model of Supersymmetry is rather curious in the sense that we have to take the spinor superfield to construct the Lagrangian because the scalar superfield does not give us right type of Lagrangian. Let us examine the two dimensional case of the supersymmetry: similarly to (6.1) the expansion of the scalar superfield is given by

$$\phi(x,\theta) = A + \bar{\theta}\psi + \frac{1}{2}(\bar{\theta}\theta)F \tag{6.6}$$

where the coefficients A and F are Bose fields, and ψ is the Fermi field. The behaviour of the component fields under the action of an infinitesimal supergauge transformation is easily deduced from (6.3').

$$\delta\phi(x,\theta) = \bar{\epsilon}\left[\frac{\partial}{\partial\bar{\theta}} - \frac{1}{2}i\not{\partial}\theta\right]\phi \tag{6.7}$$

by substituting the expansion (6.6), we get

$$\delta A = \bar{\epsilon}\psi$$

$$\delta\psi = (F + \tfrac{1}{2} i \not{\partial} A)\epsilon$$

$$\delta F = \tfrac{1}{2}\bar{\epsilon} i \not{\partial}\psi \tag{6.8}$$

The Lagrangian of the scalar superfield is given by -

$$\mathcal{L}(A\psi F) = \frac{1}{4}\frac{\partial^2}{\partial\bar{\theta}\partial\theta}[\phi\phi]$$

$$= \frac{1}{4}\frac{\partial^2}{\partial\bar{\theta}\partial\theta}\left(A^2 + 2\bar{\theta}\psi A + (\bar{\theta}\theta)FA - \tfrac{1}{2}(\bar{\theta}\theta)(\bar{\psi}\psi)^2\right)$$

$$= FA - \tfrac{1}{2}\bar{\psi}\psi \tag{6.9}$$

which is quite unacceptable since it contains no kinetic energy terms.

To construct a sensible Lagrangian we have to take the spinor superfield

$$\phi_\alpha = A_\alpha + \bar{\theta}^\beta \psi_{\alpha\beta} + \tfrac{1}{2}(\bar{\theta}\theta)F_\alpha \tag{6.10}$$

and impose the covariant condition (see Appendix C for the algebra).

$$(\bar{D}D)\phi_\alpha = (i\not{\partial}\phi)_\alpha$$

(6.11)

which yields after comparison

$$F_\alpha = \frac{1}{2}(i\not{\partial}A)_\alpha$$

$$i\not{\partial}_\beta^{\,\gamma}\,\psi_{\alpha\gamma} = i\not{\partial}_\alpha^{\,\gamma}\,\psi_{\gamma\beta}$$

(6.12)

(6.12')

We substitute (6.12) and (6.12') into (6.10) we obtain

$$\phi_\alpha = A_\alpha + \bar{\theta}^\beta\left((\gamma_\mu C)_{\alpha\beta}\,\epsilon^{\mu\nu}\,\partial_\nu\,\psi(x) + (\gamma_5 C)_{\alpha\beta}\,\psi_5(x)\right) + \frac{1}{4}(\bar{\theta}\theta)(i\not{\partial}A)_\alpha$$

(6.13)

Thus the Lagrangian (6.9) is given by

$$\mathcal{L} = \bar{A}^\alpha F_\alpha - \frac{1}{2}\bar{\psi}^{\alpha\beta}\psi_{\alpha\beta}$$

$$= \frac{1}{2}\bar{A}^\alpha\, i\not{\partial}A_\alpha + \left((\partial\psi)^2 - \psi_5^{\,2}\right)$$

which has the proper form of a free Lagrangian.

In the four dimensional case however, it is well known that the scalar superfield will do for constructing acceptable free Lagrangians and these read

$$\mathcal{L} = \frac{1}{8}(\bar{D}D)^2(\phi_+\phi_-) + \frac{1}{4}\bar{D}D\, m(\phi_+^2 + \phi_-^2)$$

$$= \frac{1}{2}\left[(\partial A)^2 + (\partial B)^2 + \bar{\Psi} i \not{\partial} \Psi + F^2 + G^2\right] -$$

$$- \frac{1}{2}\left[2FA - 2GB + m\bar{\Psi}\Psi\right]$$

The question is how one is to proceed in higher dimensions using the scalar superfield (scalar because we shall be interested in the 4-dimensional limit). Let us return to (6.1) - (6.5). From section (2.3) we observe that in 6,8, 14,16,... dimensions the Majorana bilinear $\bar{\theta}\theta$ (among others) vanishes identically. Therefore if we want our theory to resemble to four dimensional versional as far as possible we must limit ourselves to 2,4,10,12,... dimensions. For such dimensions one could re-expand the multispinors appearing in (6.1) to (6.5) in terms of Γ-matrices but this is an unimportant detail.

The discussion so far has been perfectly straightforward and required little ingenuity. But we now come to the first (and only) critical decision: to what member of the supermultiplet must one assign the Lagrangian? There are two viable alternatives: (A) The Lagrangian is taken as the last term in the relevant θ expansion i.e. the term of order $\theta^{2^{\ell}}$ in (6.1) or the term of order $\theta^{2^{\ell-1}}$ in (6.2). (B) The Lagrangian is contained in the θ^4 term of expression (6.1) and the θ^2 term of expansion (6.2), exactly as in four dimensions. If we adopt alternative (A) the action is given by

$$S = \int \left[(\bar{D}D)^{2^{\ell-1}} K(\Phi) + (\bar{D}D)^{2^{\ell-2}} V(\Phi_{\pm}) \right] d^{2\ell}x \tag{6.14}$$

with the number of θ derivatives varying with ℓ. Because of this the Lagrangian alters its degree of polynomiality in the component fields derivatives with ℓ, so that in fact the theory changes its character with the dimension. (In fact the kinetic energy includes $\phi(\partial^2)^{2^{\ell-2}}\phi$.) This change of theory is contrary to the spirit of DR, and in any case conflicts with the dimension-independent procedure necessary to extract the <u>axial anomaly</u>[5]. Thus we advocate alternative (B) whereby the action is

$$S = \int (\bar{D}D) \left[\bar{D}D\, K(\Phi) + V(\Phi_{\pm}) \right] d^{2\ell}x \tag{6.15}$$

and the kinetic energy properly contains 2 derivatives of the scalar fields.

6.3 Ward Identities

An immediate consequence of this dimension-independent choice is that the action is no longer supersymmetric. In particular if we extract the Lagrangian as the Lorentz invariant part of superfield products,

$$\mathcal{L} = (C^{-1})^{\alpha\beta}\left((C^{-1})^{\gamma\delta} D'_{[\alpha\beta\gamma\delta]} + F'^{+}_{[\alpha\beta]} + F^{-}_{[\alpha\beta]}\right)$$

(6.16)

where D' and F' (see appendix for detail) are composed of the original superfield component products in the usual way, then our finds from (6.3) and (6.4) that

$$\int \delta\mathcal{L}\, d^{2\ell}x = (C^{-1})^{\alpha\beta}\int d^{2\ell}x\left((C^{-1})^{\gamma\delta}\bar{\epsilon}^{\epsilon}\eta'^{+}_{[\alpha\beta\gamma\delta\epsilon]} + \bar{\epsilon}^{\gamma}\chi'_{[\alpha\beta\gamma]} + \bar{\epsilon}^{\gamma}\chi'^{-}_{[\alpha\beta\gamma]}\right)$$

(6.17)

(Again η' and χ' refer to those particular component field products which are generated for the super transformation.) Therefore if $J_{\mu\alpha}$ is a spinor current whose time-component space-integral defines the spinor generator, we obtain the spinor Ward identity,

$$\partial^{\mu} J_{\mu\epsilon} = (C^{-1})^{\alpha\beta}\left((C^{-1})^{\gamma\delta}\eta'^{+}_{[\alpha\beta\gamma\delta\epsilon]} + \chi'^{+}_{[\alpha\beta\epsilon]} + \chi'^{-}_{[\alpha\beta\epsilon]}\right)$$

(6.18)

It looks anomalous in so far the r.h.s. of (6.18) contains terms which are not present in 4 dimensions. For example in the ϕ^3 supersymmetric model,

$$\mathcal{L} = 2^{-\ell+1}(\bar{D}D)^2(\Phi_+\Phi_-) - 2^{-\ell}\bar{D}D\left[m(\Phi_+^2+\Phi_-^2)+\right.$$
$$\left. + g(\Phi_+^3+\Phi_-^3)\right],$$

the offending terms on the right of (6.17') are

$$\chi'^{\pm}_{[\alpha\beta\gamma]} = 2m\left[A^{\pm}\chi^{\pm}_{[\alpha\beta\gamma]} + F^{\pm}_{[\alpha\beta]}\psi^{\pm}_{\gamma} + F^{\pm}_{[\beta\gamma]}\psi^{\pm}_{\alpha} + F^{\pm}_{[\gamma\alpha]}\psi^{\pm}_{\beta}\right]$$

$$+ g\left[3A^{\pm 2}\chi^{\pm}_{[\alpha\beta\gamma]} + 6A^{\pm}(F^{\pm}_{\alpha\beta}\psi^{\pm}_{\gamma} + F^{\pm}_{[\beta\gamma]}\psi^{\pm}_{\alpha} + \right.$$
$$\left. + F^{\pm}_{[\gamma\alpha]}\psi^{\pm}_{\beta}) - 6\psi^{\pm}_{\alpha}\psi^{\pm}_{\beta}\psi^{\pm}_{\gamma}\right]$$

with an analogous but more complicated expression for $\eta'_{(\alpha\beta\gamma\delta\varepsilon)}$.

In the event, ones immediate reaction is to conclude that Ward identity will lead to anomalous corrections. However, the effect does not stand up to more careful scrutiny. It is indeed true that $\partial^{\mu} J_{\mu}$ differs from zero, but the difference (6.18) gives matrix elements with kinematic factors which vanish in 4-dimensions irrespective of the loop order to which we work - in marked contrast to the axial and scaling anomalies where the anomalous terms in the Ward identities survive the four-dimension limit at the quantum level. The final result is that the spinor current Ward identity is anomaly-free, in spite of first appearances. Reassuringly, agrees with the Pauli-Villars approach (Wess and Zumino[16]) and the kinetic regularization method (Iliopolous and Zumino[16])

Conclusion

In this thesis we have seeked to provide an account of the use of DR techniques to various problems in the spinor field theories and gain some insight into the obstacles that must be surmounted in the dimensional continuation. One of the important lessons we have learnt during the course of our work is that the limit to integer four dimensions must be left to last. This remark is at its most powerful when we take the divergence of the axial vector current in arbitrary dimensions: this modifies the PCAC law by giving an anomalous term on the right which does not exist in four dimensions, but is precisely the axial vector anomaly when we descend to four dimensions. We might be tempted to get rid of such anomalies by adding local evanescent interactions (which disappear in four dimensions). However the work of chapter V has shown that to remove them in this way is liable to make the theory non renormalizable. Another crucial point about the whole analysis is that we have always stuck to dimension free definitions of our Lagrangians and currents. This point has been exemplified in chapter VI dealing with supersymmetry where we have checked the absence of the spinor anomaly by the dimensional continuation method. In all these aspects we have verified the utility of DR by its success in providing a beautifully elegant regularization technique and in bringing to light any possible anomalies in Ward identities. However, in weak interactions we are not totally satisfied with the method, for we have seen that in generalizing the

current-current form of the weak Lagrangian to arbitrary dimensions we get <u>two kinds</u> of polar vector and <u>two kinds</u> of axial vector among the possible set of currents. One of these weak polar vector currents is not conserved except in four dimensions and undergoes a finite renormalization from quantum loops. The renormalization of this weak vector current is the first unconventional result to come out of DR and one may wonder how it can be avoided. One possibility is to cancel it off with a counter Lagrangian but this seems very artificial and beset with renormalization difficulties. A second possibility is to reject the Fierz invariance requirement and to return to the vector and pseudovector currents, $\bar{\psi}\,\Gamma_N\,\psi$ and $i\bar{\psi}\,\Gamma_N\,\Gamma_{-1}\,\psi$;
this too confronts us with the same difficulty, for the simple reason that hadronic contributions to axial currents must satisfy anomalous PCAC identities so one cannot avoid introducing $\bar{\psi}\,\Gamma_{KLM}\,\psi$ and its curious companion $\bar{\psi}\,\Gamma_{KLM}\,\Gamma_{-1}\,\psi$. Perhaps the best way to restore polar vector current conservation is to double the fermions and reverse the sign of their abnormal current interactions with mesons.

Finally let us see what new problems suggest themselves in spinor field theories. (i) First, one might repeat the work done on Dynamical Rearrangement of symmetry as applied say to the Nambu-Jona-Lasinio[25] model. In this model of massless four fermi interaction, despite the invariance of the Lagrangian under the γ_5 transformation $\psi \to \psi \exp\left(i\alpha\ \gamma_5\right)\psi$ the chiral symmetry is dynamically broken and the physical fermion mass is non zero through the existence of a bound nucleon antinucleon pair with pseudoscalar properties. Now the same problem could be tackled by DR. Here the Lagrangian is not in any case invariant under the generalized chiral transformation $\psi \to \exp\left(i\,\theta_{(4)}\Gamma^{(4)}\right)\psi$, once we adopt the dimensional independent definition of $\gamma_5 \equiv \Gamma_{(4)}$. Also the invariant cut off Λ^2 characteristic of the conventional approach is replaced by pole terms $^1/_{(n-4)}$ in the amplitudes. (ii) A second interesting problem might be the nature of gauge theories in arbitrary dimensions. Pure vector gauge theories like quantum electrodynamics, invariant under the gauge transformations

$$\psi \to e^{i\alpha}\psi$$
$$A_M \to A_M - \partial_M \alpha \,,$$

generalize readily to arbitrary dimensions. But as soon as we extend the gauges to include chiral transformations we should expect a set of bosons

$$W_{(3)} = W_{LMN}, \quad W_{(5)} \equiv W_{LMNOP}, \quad \ldots, W_{(2\ell-1)}$$

associated with the gauge transformations

$$\psi \to e^{i\alpha_{(4)}\Gamma^{(4)}}\psi$$
$$W_{LMN} \to W_{LMN} - \partial^K W_{KLMN} \quad \text{etc.}$$

The Lagrangians for these models have to be constructed. (iii) A third problem of interest might be an investigation infinite dimensional unitary representations of the O(n-1, 1) group in the limit $n \to 4$. Here we have to concentrate on those representations which are characterised by two Casimir labels to obtain a correspondence with O(3,1) limit.

(iv) Finally it would be interesting to generalize our work from the Dirac representation

$$\{\gamma_M , \gamma_N\} = 2\eta_{MN}$$

to the Kemmer representation:

$$\beta_M \beta_N \beta_L + \beta_L \beta_N \beta_M = \beta_M \delta_{NL} + \beta_L \delta_{NM} .$$

Appendix - A

On 2ℓ-Dimensional Integrals

$$\int d^{2\ell}x\, f(x) = i \int f(x)\, r^{2\ell-1} dr\, \sin\theta_{2\ell-1}\, d\theta_{2\ell-1} \cdot$$
$$\cdot \sin^{2l-3}\theta_{2l-2}\, d\theta_{2l-2} \cdots d\theta_1 \tag{A.1}$$

with $0 \leq \theta_i \leq \pi$, except $0 \leq \theta_1 \leq 2\pi$.
If f(x) depends only on $r = \sqrt{x_1^2 + \cdots + x_n^2}$ one may perform the integration over angles using

$$\int_0^\pi \sin^m\theta\, d\theta = \sqrt{\pi}\, \frac{\Gamma(m+1/2)}{\Gamma(m+2/2)} , \tag{A.2}$$

leading to

$$\int d^{2\ell}x\, f(x) = \frac{i\pi^\ell}{\Gamma(\ell)} \int f(x)\, (r^2)^{\ell-1} dr^2 , \tag{A.3}$$

similarly

$$\int d^{2\ell}k\, f(k^2) = \frac{i\,\pi^\ell}{\Gamma(\ell)} \int f(k)\, (k^2)^{\ell-1} dk^2 , \tag{A.4}$$

In particular the volume element can also be written as

$$\int d^{2\ell}p = \frac{\pi^{\ell-1}}{\Gamma(\ell-1)} \int (k^2)^{\ell-1} dk^2 (\sinh\chi)^{2l-2} d\chi\, (\sin\theta)^{2l-3} d\theta \tag{A.4'}$$

where all other angles are integrated over

$$-i\int\frac{d^{2\ell}K}{(2\pi)^{2\ell}}\,\frac{(K^2)^{\beta}}{(K^2-m^2)^{\alpha}} = \frac{(-1)^{\beta-\alpha}}{(4\pi)^{\ell}}\,\frac{\Gamma(\ell+\beta)\,\Gamma(\alpha-\beta-\ell)}{\Gamma(\ell)\,\Gamma(\alpha)\,(m^2)^{\alpha-\beta-\ell}} \tag{A.5}$$

This gives us a very important result

$$\int\frac{d^{2\ell}K}{(K^2)^{\alpha}} = 0 \tag{A.6}$$

for all $\alpha < \ell$. That means the total volume element and the massless tadpole graphs are to be interpreted as zero in DR for $\ell = 2$.

We also list some the symmetric integration relations:

$$\int\frac{d^{2\ell}K}{(2\pi)^{2\ell}}\,f(K^2)\,K_M K_N = \frac{\eta_{MN}}{2\ell}\int\frac{d^{2\ell}K}{(2\pi)^{2\ell}}\,K^2 f(K^2) \tag{A.7}$$

and

$$\int\frac{d^{2\ell}K}{(2\pi)^{2\ell}}\,f(K^2)\,K_L K_M K_N K_Q \tag{A.8}$$

$$= \frac{\eta_{LM}\eta_{PQ}+\eta_{LP}\eta_{MQ}+\eta_{LQ}\eta_{MP}}{2\ell(2\ell+2)}\,.$$

$$\cdot\int\frac{d^{2\ell}K}{(2\pi)^{2\ell}}\,f(K^2)\,K^4$$

These integrals can be easily evaluated with the help of (A.5).

The general Feynman parametrization of integrals is given by

$$\frac{\Gamma(n_1)\Gamma(n_2)\cdots\Gamma(n_N)}{A_1^{n_1}\,A_2^{n_2}\cdots A_N^{n_N}} =$$

$$= \Gamma(n_1+n_2+\cdots+n_N)\int_0^1 \frac{\prod dx_i\,\delta(1-x_1-x_2-\cdots-x_N)\,x_1^{n_1-1}\,x_2^{n_2-1}\cdots x_N^{n_N-1}}{\left[A_1x_1+A_2x_2+\cdots+A_nx_N\right]^{n_1+n_2+\cdots+n_N}} \quad \text{(A.9)}$$

The Fourier transform of a function is defined by

$$F(x) = \int\frac{d^{2l}k}{(2\pi)^{2l}}\,e^{-ik.x}\,\hat{F}(k)$$

$$\hat{F}(k) = \int d^{2l}x\;e^{ik.x}\,F(x)$$

(A.10)

In particular

$$\int\frac{d^{2l}k}{(2\pi)^{2l}}\,(-k^2)^a\,e^{-ik.x} = \frac{i\,2^{l+2a}}{(2\pi)^l(-x^2)^{l+a}}\,\frac{\Gamma(l+a)}{\Gamma(-a)}$$

(A.11)

while the boson self-energy part in the Thirring model is given by the invariant function

$$\Pi(K^2) = 2\,\frac{\Gamma(2-l)}{(2\pi)^{2l}}\int_0^1 \frac{d\alpha}{\left[-K^2\alpha(1-\alpha)\right]^{1-l}}$$

$$\rightarrow (-K^2)^{1-l}\, 2\,\Gamma(2-l)/(2\pi)^{2l}\cdot\frac{\Gamma(l)\cdot\Gamma(l)}{\Gamma(2l)} \qquad \text{(A.12)}$$

and thus its Fourier transform is

$$\int \frac{d^{2l}K}{(2\pi)^{2l}}\,\frac{\Pi(K^2)}{K^2}\,e^{-iK.x} = -i\,\frac{2^l\,(l-1)\,D^2(x)}{(2l-1)} \qquad \text{(A.12')}$$

Appendix - B

The Propagators

The two point Green's can be expressed in terms of one homogeneous and one inhomogeneous function which are given by

$$\langle 0|[\varphi(x), \varphi(0)]|0\rangle = i\,\Delta(x)$$
$$\langle 0|T[\varphi(x), \varphi(0)]|0\rangle = i\,\Delta_c(x) \tag{B.1}$$

A few other propagators are given by

$$-i\langle 0|\varphi(x)\varphi(0)|0\rangle = \Delta_+(x) = \theta(x)\Delta_c(x) + \theta(-x)\Delta_c^*(x)$$
$$+i\langle 0|\varphi(0)\varphi(x)|0\rangle = \Delta_-(x) = \theta(x)\Delta_c^*(x) + \theta(-x)\Delta_c(x)$$
$$-i\langle 0|R[\varphi(x)\varphi(0)|0\rangle = \Delta_R(x) = \theta(x)\Delta(x)$$
$$-i\langle 0|A[\varphi(x)\varphi(0)]|0\rangle = \Delta_A(x) = \theta(-x)\Delta(x) \tag{B.2}$$

The Green's functions $\varepsilon(x)\ \Delta(x)$ and $\Delta_c(x)$ can be easily continued to arbitrary ℓ, but those involving with $\theta(x)$ lose their invariant significance in the continuation.

One can show that the propagators are biven by

$$i\epsilon(x)\Delta(x) = -\frac{i\pi\,\theta(x^2)}{(2\pi)^{\ell}}\left(\frac{m}{\sqrt{x^2}}\right)^{\ell-1} J_{1-\ell}\left(m\sqrt{x^2}\right)$$

and

$$i\,\Delta_c(x) = \frac{1}{(2\pi)^\ell}\left(\frac{m}{\sqrt{(-x^2+i\epsilon)}}\right)^{\ell-1} K_{1-\ell}\left(m\sqrt{(-x^2+i\epsilon)}\right)$$

$$= \frac{\Theta(-x^2)}{(2\pi)^\ell}\left(\frac{m}{\sqrt{-x^2}}\right)^{\ell-1} K_{1-\ell}\left(m\sqrt{-x^2}\right) - \frac{\Theta(x^2)}{(2\pi)^\ell}\cdot$$

$$\left(\frac{m}{\sqrt{x^2}}\right)^{\ell-1}\frac{1}{2}i\pi\left[J_{1-\ell}(m\sqrt{x^2}) - iY_{1-\ell}(m\sqrt{x^2})\right] \qquad \text{(b.3)}$$

In the zero-mass limit (propagators are denoted by D)we have

$$i\,\epsilon(x)\,D(x) = -\frac{i\Theta(x^2)}{2\pi^{\ell-1}}\quad\frac{(x^2)^{1-\ell}}{\Gamma(2-\ell)}$$

and

$$i\,D_c(x) = \Gamma(\ell-1)(-x^2+i\epsilon)^{1-\ell}\Big/4\pi^{\ell} \qquad \text{(B.4)}$$

which carry all the light cone singularities since as generalized function[21] with ℓ integer $\geqslant 2$

$$\Theta(x^2)\;(x^2)^{1-\ell}\Big/\Gamma(2-\ell) \;\rightarrow\; \delta^{(\ell-2)}(x^2)$$

and

$$(-x^2+i\epsilon)^{1-\ell}\;\Gamma(\ell-1)$$

$$\rightarrow (-x^2)^{1-\ell}\;\Gamma(\ell-1) - i\pi\delta^{(\ell-2)}(x^2) \qquad \text{(B.5)}$$

The four dimensional case[22] $\ell = 2$ is familiar to everybody. The generalized functions (B.3) and (B.4) can be continued to other ℓ-values giving us the Sonine-Gegenbauer dispersion integral

$$\Delta_c(x) = \frac{1}{2\pi}\int_0^\infty \frac{i\,\epsilon(x')\,\Delta(x')\,dx'^2}{x'^2 - x^2 + i\epsilon} \quad ,$$

which is initially defined in the region

$$\mathrm{Re}\sqrt{-x^2} > 0 \qquad \text{and} \quad \tfrac{1}{2} < \mathrm{Re}\,\ell < 2 \quad .$$

Let us then note the propagators in two dimensions (ℓ = 1)

$$i\epsilon(x)\,\Delta(x) = -\tfrac{1}{2}\,i\,\theta(x^2)\,J_0(m\sqrt{x^2}) \rightarrow -\tfrac{1}{2}\,i\,\theta(x^2)$$

$$\begin{aligned} i\,\Delta_c(x) &= K_0\left(m\sqrt{(-x^2+i\epsilon)}\right)\Big/2\pi \\ &\rightarrow \frac{\ln(-x^2+i\epsilon)}{4\pi} + i\,D_c(0) \end{aligned} \qquad \text{(B.6)}$$

and the propagators in the vicinity of four dimensions (ℓ = 2),

$$\begin{aligned} i\epsilon(x)\,\Delta(x) = \frac{i\,\theta(x^2)\,m}{4\pi\sqrt{x^2}}\Big[& J_1(m\sqrt{x^2})\Big\{1+(\ell-2)\ln\Big(\frac{m}{2\pi\sqrt{x^2}}\Big)\Big\} + \\ & + \tfrac{1}{2}\pi(\ell-2)\Big\{\frac{2J_0(m\sqrt{x^2})}{m\sqrt{x^2}} - Y_1(m\sqrt{x^2})\Big\} + O(\ell-2)^2\Big] \end{aligned}$$

$$\xrightarrow[m\to 0]{} -i(2\pi)^{-1}\Big[\delta(x^2) + (\ell-2)\Big\{\frac{\theta(x^2)}{x^2} - (\gamma+\ln\pi)\,\delta(x^2)\Big\} + \\ + \ldots\Big]$$

$$i\Delta_c(x) = \frac{m}{4\pi^2\sqrt{(-x^2+i\epsilon)}}\Big[K_1\left(m\sqrt{(-x^2+i\epsilon)}\right)\cdot$$

$$\cdot\left\{1+(l-2)\ln\left(\frac{m}{2\pi\sqrt{(-x^2+i\epsilon)}}\right)\right\}+$$

$$+\frac{(l-2)\,K_0\left(m\sqrt{(-x^2+i\epsilon)}\right)}{m\sqrt{(-x^2+i\epsilon)}} + O(l-2)^2\Big]$$

$$\xrightarrow[m\to 0]{} -\frac{1}{4}\pi^{-2}(x^2-i\epsilon)^{-1}\Big[1+(l-2)\cdot$$

$$\cdot\left\{(x^2-i\epsilon)\ln(-x^2+i\epsilon)\right\}-(\gamma+\ln\pi)+\dots\Big] \tag{B.7}$$

It is these next to leading behaviours of propagators which are partly responsible for the anomalies in quantum loop corrections.

Equal Time Commutators

The continuation of time-ordered products to arbitrary dimensions is simple because we encouter just the invariant functions of $(x^2 - i\varepsilon)$. We adopt the BJL definition[(23)] of the equal time commutator between two fields, viz.

$$\left[A(\underline{x},0),\, B(0)\right] = \left(\lim_{t\to 0_+} - \lim_{t\to 0_-}\right) T\left[A(x)\, B(0)\right]$$

This may be used to calculate the matrix elements to any order in perturbation theory from the products of causal

Green's functions. For free scalar fields it gives the same result as obtained by the definition of Gelfand-Shilov, i.e.,

$$\langle 0|[\varphi(\underline{x},0),\varphi(0)]|0\rangle = \langle 0|[\varphi(\underline{x},0),\partial_r\varphi(0)]|0\rangle = 0$$

but

$$\langle 0|[\varphi(\underline{x},0),\dot{\varphi}(0)]|0\rangle = \lim_{t\to 0}\frac{\partial}{\partial t}\left[\frac{\Gamma(\ell-1)}{2\pi^{\ell}(-x^2+i\epsilon)}\right]$$

$$= \lim_{t\to 0}\frac{\partial}{\partial t}\left[\frac{1}{2}i(-\pi)^{1-\ell}\,\delta^{(\ell-2)}(x^2)\right]$$

$$= i\,\delta^{2\ell-1}(x) \tag{B.8}$$

These results rather clear in momentum space. In ℓ-continuation method we do not separate the points in field products provided 2ℓ is away from an integer point. Hence the Schwinteger terms can be calculated in the context of DR. For example, the commutator of two e.m. currents to order e^2 is

$$\langle 0|T[\dot{J}_M(x)J_N(0)]|0\rangle =$$

$$= e^2\,2^{\ell}\left[2\partial_M\Delta_c\,\partial_N\Delta_c-\eta_{MN}\partial\Delta_c\cdot\partial\Delta_c+m^2\Delta_c^2\right]$$

$$\longrightarrow \frac{(\Gamma(\ell))^2}{\pi^{2\ell}}\,2^{\ell-2}\,\frac{[2x^Mx^N-(x^2-i\epsilon)\,\eta^{MN}]}{(-x^2+i\epsilon)^{2\ell}} \tag{B.9}$$

and by B.J.L. recipe it follows that

$$\langle 0|[j_0(x), j_0(0)]|0\rangle = 0$$

and $$\langle 0|[j_r(x), j_s(0)]|0\rangle = 0 \qquad (B.9)$$

at equal times because the time ordered products are even in t.

On the other hand,

$$\langle 0|[j_0(\underline{x},0), j_r(0)]|0\rangle$$

$$= \lim_{t\to 0} e^2[\Gamma(l)]^2 2^l \frac{t.\, x_r}{\pi^{2l}(-x^2+i\epsilon)^{2l}}$$

$$= \frac{ie^2}{\pi^l\,\Gamma(2l)}\left(-2\cdot\frac{\partial}{\partial \underline{x}^2}\right)^{l-1}\partial_r\,\delta^{2l-1}(\underline{x})$$

(B.10)

by using the Gelfand-Shilov representation[24] for spatial δ-function.

Thus the Schwinger term is interpreted as $(\underline{\partial}^2)^{\ell-1}$ multiplying the spatial derivative, rather than an infinite constant multiplying it.

In momentum space, it is proportional to $k_r(\underline{k}^2)^{\ell-1}$ rather than $k_r(\Lambda^2)^{\ell-1}$, where Λ is a cut-off in the conventional point separation method. In a way this conclusion is rather obvious. DR does not tolerate the occurrence of cut-off terms Λ^2 which must therefore be replaced by m^2 or external momentum factors.

Appendix - C

The Supersymmetry Algebra in 2-dimensions

The Γ-matrices in two dimensions are defined to be

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1$$

$$\gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = i\sigma_2$$

and

The projectors are given by

$$\left(\frac{1 \pm \gamma_5}{2}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The charge conjugation matrix C is defined to be $C = i\sigma_2 = -\gamma_1$ which has the properties

$$C^{-1}\gamma_\mu C = -\tilde{\gamma}_\mu$$

$$C^{-1}\gamma_5 C = -\tilde{\gamma}_5$$

The differential operators D_α are defined in (5.3') satisfy the anticommutation relation

$$\{D_\alpha , D_\beta\} = -(\not{p}C)_{\alpha\beta} \qquad \text{(C.1)}$$

Hence from (C.1)

$$(C^{-1}\gamma_5)^{\alpha\beta}\{D_\alpha , D_\beta\} = \mathrm{Tr}(\not{p}\gamma_5) = 0$$

or $$\bar{D}\gamma_5 D = 0 \qquad (C.2)$$

and

$$(C^{-1}\gamma_\mu)^{\alpha\beta}\{D_\alpha, D_\beta\} = -2p_\mu$$

or $$\bar{D}\gamma_\mu D = -p_\mu \qquad (C.2')$$

With the help of (C.2) and (C.2'), the product of two D's can be expressed as

$$D_\alpha D_\beta = \frac{1}{2}(\not{p}C)_{\alpha\beta} + \frac{1}{2}C_{\alpha\beta}\bar{D}D \qquad (C.3)$$

Products of three D's are comprised in the formula

$$D_\alpha(\bar{D}D) = -\frac{1}{2}(\bar{D}D)D_\alpha - \frac{1}{2}(\not{p})_\alpha^\beta D_\beta \qquad (C.4)$$

Finally the product of four D's is

$$|\bar{D}D|^2 = -p^2 \qquad (C.5)$$

Projectors can also be formed out of D's. These are nonlocal,

$$\left(1 \pm i\frac{\bar{D}D}{\sqrt{p^2}}\right) = E_\pm \qquad (C.6)$$

And the mass term in supersymmetric Lagrangian is given by

$$\frac{m}{4}(\bar{D}D)(\Phi_+^2 + \Phi_-^2) = \frac{m}{4}\Big[A_+(C^{-1})^{\beta\gamma}F_{(\beta\gamma)} - 2\bar{\Psi}_+^\beta\Psi_{+\beta} + A_-(C^{-1})^{\beta\gamma}F_{(\beta\gamma)} - 2\bar{\Psi}_-^\beta\Psi_{-\beta}\Big] \qquad (C.7)$$

References

1. G. 't Hooft and M. Veltman, Nucl. Phys., 44B, 189 (1972).
   C.G. Bollini and J.J. Giambiagi, Phys. Letters 40B, 566 (1972).
   J. Ashmore, Lett. Nuovo Cimento, 4, 289 (1972).
2. In theories of gravitation this was used by -
   G. 't Hooft and M. Veltman, Nucl. Phys. 50B, 318 (1972).
   D.M. Capper and M.R. Medrano, Phys. Rev. 9, 1641 (1974).
   M. Brown, Nucl. Phys. B56, 194 (1973).
3. K.G. Wilson, Phys. Rev. 2D, 1478 (1970).
4. R. Gastmans and R. Menldermans, Nucl. Phys. B63, 277 (1973).
   W.J. Marciano and A. Sirlin, B88, 86 (1975).
5. D.A. Akyeampong and R. Delbourgo, Nuovo Cimento, 17A, 578 (1973);
   18A, 94 (1973).
6. R. Delbourgo and V.B. Prasad, Nuovo Cimento, 21A, 32 (1974).
7. R. Delbourgo and V.B. Prasad, Nuovo Cimento, 23A, 257 (1974).
8. C. Bollini and J. Giambiagi, Acta Phys. Austriaca, 38, 211 (1973).
   R. Delbourgo and V.B. Prasad, J. Phys. A, 8, 782 (1975).
9. R. Delbourgo and V.B. Prasad, J. Phys. G, 1, 377 (1975).
10. J. Schwinger, Phys. Rev. 82, 664 (1951).
    J. Bell and R. Jackiw, Nuovo Cimento, 60, 47 (1969).
    S. Adler, Phys. Rev., 177, 2426 (1969).
11. R. Brauer and H. Weyl, Am. J. Math., 57, 425 (1935).
    K.M. Case, Phys. Rev., 97, 810 (1955).
    M. Fierz, Z. Physik, 104, 533 (1937).

12. W. Thirring, Ann. of Phys. 3, 91 (1958).

K. Johnson, Nuovo Cimento, 20, 773 (1961).

C.R. Hagen, Nuovo Cimento, 51B, 169 (1967).

A.H. Mueller and T.L. Trueman, Phys. Rev., 4D, 1635 (1971).

13. D.A. Akyeampong and R. Delbourgo, 19A, 219 (1974).

J.C. Collins and A.J. Macfarlane, D10, 1201 (1974).

J.C. Collins, D10, 1213 (1974).

14. W.A. Bardeen, R. Gastmans and B. Lautrup, Nucl. Phys., B46, 319 (1972).

T. Applelquist, J. Carrazone, T. Goldmanaud, H. Quinn, Phys. Rev., 8, 1747 (1973).

W.J. Marciano and A. Sirlin, Phys. Rev., 8, 3612 (1973).

15. P. Ramond, Phys. Rev. D3, 2415 (1971).

J.L. Gervais and B. Sakita, Nucl. Phys. B34, 6321 (1971).

J. Wess and B. Jumino, Nucl. Phys., B70, 39 (1974); Phys. Letters 49B, 52 (1974).

A. Salam and J. Strathdee, Nucl. Phys., B76, 477 (1974).

16. J. Wess and B. Jumino, Phys. Letters, B49, 52 (1974).

J. Illiopoulos and J. Zumino, B76, 301 (1974).

17. H. Georgi and J.M. Rawls, Phys. Rev., D3, 874 (1971).

18. K.G. Wilson, Phys. Rev. 2D, 1473 (1970).

19. W.A. Bardeen, Phys. Rev. 184, 1848 (1969).

R. Kee, Axial Vector Anomalies in an SU(n) Model via Dimensional Regularization, Imperial College preprint ICTP/73/8.

20. D. Capper and M. Duff, Nuovo Cimento, 23A, 173 (1974).

21. I.M. Gel'fand and G.E. Shilov, Generalized Functions, Vol. 1, (Academic Press, London 1964).

22. J. Schwinger, Phys. Rev., 75, 651 (1949) and all subsequent standard textbooks on field theory.

23. J.D. Bjorken, Phys. Rev., 148, 1467 (1966).
K. Johnson and F.E. Low, Suppl. Prog. Theo. Phys. 37 and 38, 74 (1968).

24. C.J. Isham, Abdus Salam and J. Strathdee, Phys. Rev. D5, 2548 (1972).
Abdus Salam, "Computation of Renormalization Constants", Coral Gables Conference, Gordon & Breach (1971).

25. Y. Nambu and G. Jona-Lasinio, Phys. Rev., 122, 345 (1961).
D. Lurie, A.J. Macfarlane, Phys. Rev. 136, 816 (1964).
Y. Freundlich and D. Lurie, Nucl. Phys., B18, 141 (1970).