POLE AND ZERO ASSIGNMENT AND OBSERVER DESIGN

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ABSTRACT

This thesis is concerned with pole and zero assignment by proportional feedback in linear time-invariant multivariable systems, and the associated problem of observer design.

The eigenvectors of a single-input system with feedback are given in a new form, and the associated re-diagonalized system is obtained directly. The problem of pole assignment in systems with restricted measurement access is assisted by a solution which yields the feedback gains for the assignment of a limited number of poles, together with the coefficients of a residual characteristic equation, yielding the unassigned poles.

Two solutions are given to the problem of the assignment of the poles and zeros of a scalar transfer function.

A simple step-by-step design procedure for state observers is given, and a general solution for the design of a linear functional observer, which removes the need for reduction to a canonical form. The procedure is extended to the design of low-order linear functional observers, yielding explicitly the constraints on the observer poles corresponding to any proposed observer

order. A design procedure for general degenerate observers is also given.

The properties of the dual observer are examined, and a new design procedure is presented, which yields a design of lower order than that available hitherto.

The problem of observers for systems with inaccessible inputs is considered, and new conditions for the existence of a type of observer suitable for such systems are obtained.

Finally, a simple general algorithm for pole assignment by output feedback is given, which exploits the high speed of operation of modern digital computers. This algorithm permits the inclusion of practical design constraints.

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CHAPTER 1.

INTRODUCTION TO POLE AND ZERO ASSIGNMENT AND OBSERVER DESIGN.

1.1 Introduction.

The pioneering work of Rosenbrock [R1] and Wonham [W1] on pole assignment, and of Luenberger [L1] on the theory of observers, has led to a large volume of published work on these and related topics in a little over ten years. Although much progress has been made, complete answers to all of the questions arising have not yet been obtained. For example, the necessary and sufficient conditions for the assignment of all poles of a linear system by output feedback are not known. The situation has now been reached where some workers in the field of control theory are expressing doubts as to the value of the state space approach, upon which the ideas of pole and zero assignment and observer theory are based. This probably is a natural reaction to the fact that the high hopes which were held at one stage have not been completely fulfilled. However, there can be little doubt that these techniques have value, and must take their place alongside other techniques, such as those based on frequency response, in the design of linear systems.

The systems considered in this thesis are linear and time-invariant, and the signals are deterministic. The matrices (A,B,C), which characterise the system, in the usual notation, are regarded as fixed, and not open to the choice of the designer. All external signals continue to be applied to the system after the feedback is applied, and no interchanging of external inputs, so as to apply them to other inputs, is permitted. The problems thus take a reasonably realistic form.

The broad aim of the research was to find methods for the design of feedback for such systems so as to obtain desired sets of closed-loop poles. The assignment of zeros is also considered, in so far as this can be achieved by simple feedback. Because of the importance of the position occupied by state vector feedback in pole and zero assignment, the question of the design of state observers, degenerate observers and dual observers has received attention. In such a wide subject, some measure of selection is necessary, and an important topic that has not been considered in this thesis is that of decoupling by output feedback.

Towards the end of the thesis, the question of closed-loop pole assignment by output feedback is re-examined, and an algorithm is presented which becomes practicable now that modern high-speed digital computers, such as the CDC7600, are available. This algorithm may be regarded almost as the opposite approach to that of modal control, which formed the starting point of this thesis. The modal control approach is based on an eigenvalue-eigenvector analysis of the system to be controlled, and depends upon the Jordan form into which the system matrix can be transformed. The algorithm presented here requires no preliminary transformation of the system equations, is completely general, and can deal with all system poles, whether they be real or complex, simple or multiple, before or after the feedback is applied. When this algorithm is used, the general theory of mode controllability and observability provides a background which helps in the understanding of the problems, but it does not form the basis of the technique.

1.2 Outline of the Thesis.

In Chapter 2, the problem of pole assignment where the system state vector is fully accessible for measurement is considered. After a brief review of the available methods, the explicit gain formula for the single-input case, which was reported in [M1] , [M2] , is developed, for completeness. New formulae are obtained for the eigenvectors of the system after the application of state vector feedback. These formulae are an advance on those given previously [M2] , in that they apply whether particular system eigenvalues are changed or not. These results will be useful in theoretical work, and a canonical form is obtained for the re-diagonalized system with feedback, which avoids the need for the re-calculation of eigenvectors by the usual methods.

Relationships are established between recent work of Fallside and Seraji and earlier work of Bass and Gura; also, between a method due to Willner, Ash and Roy and a technique proposed by Luenberger. The connection between this last method and the explicit gain formula of [M1] , [M2] , is demonstrated.

Other recently reported work on pole assignment by state vector feedback is reviewed, and the connections with existing methods are indicated.

The problem of pole assignment with restricted measurement access is discussed in Chapter 3, including the important results of Davison and Sridhar and Lindorff concerning the number of poles which can be assigned. A procedure for pole assignment for this case is given, which yields the feedback gain vector and, at the same time, the coefficients of the residual characteristic polynomial, from which the unassigned poles may be determined. The sensitivity of the unassigned poles to small changes in the assigned poles is also examined. The techniques available for multi-input systems are reviewed, and the limitations of some recently reported work in this area are pointed out.

The assignment of zeros as well as poles is considered in Chapter 4. The basic limitations arising from the limited number of variable parameters are discussed. Attention is then directed to the problem of designing the feedback gains to give a desired single scalar input-output transfer function. Two methods are presented, one of which is based on the concepts of modal control, and permits the results of modal control theory, such as the explicit gain formula for a single-input system, to be applied directly to the problem of zero assignment, as well as pole assignment. The second method presented is based on the transformation of the system to the companion form. Both techniques provide information on the possibilities available for zero assignment in individual cases.

In Chapter 5, the properties of state observers are reviewed, and the techniques available for the design of state observers of full dimension and of reduced dimension are discussed. Cumming's method, which is satisfactory for the design of reduced-order observers where full digital computer facilities are available, is discussed in some detail. A different approach is appropriate for design by pencil and paper, assisted by an electronic calculator, or a time-sharing computer terminal, using a limited programming language. A very simple step-by-step design method is presented, which is useful in such cases.

The linear functional observer is considered in Chapter 6. The established results are reviewed, and a design procedure is presented for an observer with arbitrary dynamics to provide any pre-specified linear functional of the state vector of a multi-output system.

This procedure differs from that originally proposed by Luenberger in that it does not require the transformation of the system to a special canonical form. The procedure, incidentally, can be used to provide an alternative proof of Luenberger's result concerning the existence of a linear functional observer of order (p-1), where p is the observability index. The procedure is extended to the case of a low-order linear functional observer, and provides sets of conditions which must be satisfied by the coefficients of the characteristic polynomial of the system matrix of the observer, if a linear functional observer of given order is to exist. This work parallels that of Fortmann and Williamson, which is based, like Luenberger's earlier work, on the use of canonical forms.

The more general problem of designing degenerate observers, to provide more than one pre-specified linear functional of the state vector, is considered in Chapter 7. The approach used in Chapter 6 is applied to this problem, to provide a routine design procedure for degenerate observers. It is not claimed, however, that this procedure will yield the degenerate observer of lowest possible order in any given case.

The dual observer is considered in Chapter 8. The properties of the dual observer are examined; in particular how these differ from those of the ordinary observer. A new technique for the design of a dual observer of reduced order is presented, and the existence of a design of lower order than that given by previous workers is established.

Chapter 9 deals with observers for systems with inaccessible inputs. A serious disadvantage of observers is their need to be provided with the inputs which are applied to the observed system. The work of Hostetter and Meditch on O-observers and k-observers, intended to overcome this difficulty, is reviewed, and extended by the provision of a simplified criterion for the existence of a k-observer for a single-input, single-output system, and a sufficient condition for the existence of a k-observer for a multi-input, multi-output system.

In Chapter 10, a return is made to the problem of pole assignment in multi-input, multi-output systems. Recent work is reviewed, and its limitations indicated. A general algorithm for closed-loop pole assignment is presented, which makes use of the availability of the modern high-speed digital computer. The algorithm is applied to some numerical examples.

In Chapter 11, the work covered by the thesis is reviewed, and some general conclusions are reached. Recommendations are made for future research, where this is considered to be promising. Some more general comments are also included, concerning the problems that arise in applying linear systems theory to practical numerical cases.

Concerning notation, this has been made as consistent as possible, within the limits imposed by the available symbols. According to convenience, the feedback gain matrix is sometimes defined with a negative sign, and sometimes with a positive sign. The convention used is stated in each case, and so this should not cause confusion.

1.3 Contributions of the Thesis.

In Chapter 2, the formulae for the eigenvectors for the system with feedback, and the canonical form for the re-diagonalized system equations, are new. In Chapter 3, the procedure for pole assignment for a system with restricted measurement access, which yields both the feedback gains corresponding to the assigned poles and the coefficients of the residual characteristic polynomial corresponding to the unassigned poles, is original. The treatment of the sensitivity of the unassigned poles to small changes in the assigned poles has, as far as is known, not been given before.

The two techniques for transfer function synthesis in the scalar case by state vector feedback are original, in Chapter 4.

In Chapter 5, the simple step-by-step design method for state observers is new. The design method for the linear functional observer in Chapter 6 is original, as are its extensions to the design of low-order linear functional observers and, in Chapter 7, to the general problem of degenerate observers.

In Chapter 8, the design method for dual observers is new, as is the result that a dual observer of order (q-1) exists, where q is the controllability index, permitting the arbitrary assignment of all poles of the overall system comprising the system and the dual observer. Some new conditions for the existence of k-observers and 0-observers are established in Chapter 9, particularly for the multi-input, multi-output case.

The algorithm for pole assignment for multi-input, multi-output systems by output feedback given in Chapter 10 is original. It is very simple, and becomes practicable only with the availability of the modern high-speed digital computer.

CHAPTER 2.

POLE ASSIGNMENT WITH FULLY ACCESSIBLE STATE VECTOR.

2.1 Introduction.

We consider a linear time invariant system described by the equations:

$$\dot{x} = Ax + Bu' + Bu \tag{2.1.1}$$

$$y = Cx \tag{2.1.2}$$

where x, u', u and y are vectors of dimensions n, r, r and m, representing the state, feedback input, external input, and output, respectively. The matrices A, B and C are constant, and the problem is to find a constant feedback matrix K such that, if:

$$u' = Ky \tag{2.1.3}$$

the closed-loop system has desired poles.

There are two broad types of approach to this problem, the first based on eigenvalue/eigenvector analysis, which we shall call 'spectral' methods, and the second based on directly changing the coefficients of the characteristic polynomial, which we shall call 'coefficient' methods.

2.2 Spectral Methods.

It is obvious that, if r = m = n, and B and C are non-singular, K can be found without difficulty. For, if

W is a matrix which transforms A into Jordan form, i.e., $W^{-1}AW = \Lambda$, where Λ is a Jordan matrix, then the change of state vector to z, where $z = W^{-1}x$, in (2.1.1) and (2.1.2) gives the equations as:

$$\dot{z} = \Lambda z + W^{-1}BKCWz + W^{-1}Bu \qquad (2.2.1)$$

If now Λ_1 is a diagonal matrix of desired eigenvalues of the closed-loop system, we have only to put:

$$K = B^{-1}W(\Lambda_1 - \Lambda)W^{-1}C^{-1} \qquad (2.2.2)$$

to achieve the required result.

In practice, it is rarely possible to use the result in (2.2.2), because the numbers of inputs and outputs available are usually very much less than the system order n. Porter [P1] has set out the conditions which must be satisfied for such a simple approach to be extended to systems in which m and r are less than n, and these are clearly very restrictive.

Rosenbrock [R1] , in his original paper on modal control, dealing with the case in which the eigenvalues of A are distinct, suggested a procedure for obtaining arbitrary assignment of m eigenvalues by choosing C as the first m row eigenvectors of A, and B as the first m column eigenvectors. This permits the assignment of m eigenvalues without affecting the remaining (n-m) eigenvalues. Rosenbrock suggested a procedure for approximating to this in practical cases. This procedure requires a separate feedback loop, between one input and one output, for each eigenvalue which is to be assigned. The method is obviously limited by the number of inputs and outputs available, and by the success one can achieve in approximating to eigenvectors in the input and output matrices.

Takahashi, Rabins and Auslander [T1] have described a similar approach, in which they distinguish between 'ideal' control, in which all the canonical states which are not having eigenvalues changed are both uncontrollable and unobservable with respect to the inputs and outputs used for the feedbacks, and 'non-ideal' control, in which either uncontrollability or unobservability is achieved, but not both. It is shown that, in the non-ideal case, although the system eigenvalues are changed in the desired manner, cross-couplings are introduced between the canonical states, which would not otherwise exist. The approach depends, in practice, upon a process of finding suitable measurement nodes, with respect to which certain canonical states are unobservable, or control nodes, with respect to which certain canonical states are uncontrollable. It would appear to be useful in giving some guidance in the choice of a control structure, although the amount of freedom of choice may be so limited in practice as to make this of no great value. This method, and that of Rosenbrock, are included in this chapter, because they represent partial applications of (2.2.2).

Simon and Mitter [S1] set out the theoretical foundations of modal control, based on the application of the fundamental work of Kalman [K1] , Gilbert [G1] , Wonham [W1] and Luenberger [L1] to the ideas suggested by Rosenbrock. They introduced the important concepts of mode controllability and mode observability, and showed that the conditions for these coincide with those for state controllability and state observability derived by Kalman and Gilbert. Simon and Mitter gave an important theorem covering the case in which the A matrix is derogatory. They showed that the minimum number of inputs necessary to permit full eigenvalue assignment in a controllable system in which the state vector is fully accessible, is equal to the greatest number of Jordan blocks having the same eigenvalue, in the Jordan canonical form representation. They gave an algorithm for changing a number of eigenvalues simultaneously, which requires the solution of a set of linear equations, and a recursive algorithm, in which one eigenvalue is changed at a time, and the system restored to canonical form at each step. Although they showed that a single-input system in which A has distinct eigenvalues can have all n eigenvalues assigned by state vector feedback, they did not obtain an explicit expression for the feedback gains necessary for this.

The problem of finding the feedback gains explicitly was solved by Mayne and Murdoch [M1] [M2] , and by Crossley and Porter [C1] , independently. These solutions are for the distinct eigenvalue case. The solution was extended to the multiple eigenvalue case by Retallack and Mac-Farlane [R2] , and by Gould, Murphy and Berkman [G2] .

The derivation of the result for the distinct eigenvalue case is given in the following section, as this will be required subsequently.

2.3 Explicit Gain Formula for Single-Input System.

A single-input system is described by the equations:

$$\dot{x} = Ax + bu' + bu \qquad (2.3.1)$$

$$y = Cx \qquad (2.3.2)$$

We require to find the feedback gain vector k^T, where:

$$u' = k^T y,$$

such that the closed-loop system matrix $(A + bk^TC)$ has a desired set of eigenvalues. x and y are n-vectors, u' and u are scalars, A has distinct eigenvalues, C is nonsingular, and the pair (A,b) is controllable.

Let W be a matrix of self-conjugate column eigenvectors of A, corresponding to the eigenvalues $(\lambda_1, .. \lambda_n)$.

Let $W^{-1}b = \alpha = [\alpha_1 \;..\; \alpha_n]^T$, and

$k^T CW = \beta^T = [\beta_1 \;..\; \beta_n]$.

If the eigenvalues of $(A + bk^T C)$ are $(\lambda_1^d, .. \lambda_n^d)$, then:

$$\underset{j=1,..n}{\alpha_j \beta_j} = \frac{\prod_{i=1}^{n}(\lambda_i^d - \lambda_j)}{\prod_{\substack{i=1\\ i\neq j}}^{n}(\lambda_i - \lambda_j)} \tag{2.3.3}$$

Proof.

We introduce a new state vector z, such that x = Wz. Then, (2.3.1) becomes, with feedback,

$$\dot{z} = (\Lambda + \alpha\, \beta^T)z + \alpha\, u \tag{2.3.4}$$

$$y = CWz \tag{2.3.5}$$

where $\Lambda = \mathrm{diag}(\lambda_1, .. \lambda_n)$.

The system poles are the eigenvalues of $(\Lambda + \alpha\,\beta^T)$. The characteristic polynomial of this matrix is:

$$\det(sI - \Lambda - \alpha\beta^T) \tag{2.3.6}$$

$$= |sI - \Lambda| \det(I - (sI - \Lambda)^{-1} \alpha\, \beta^T) \tag{2.3.7}$$

Applying the matrix identity det(I + EF) = det(I + FE), which is proved in [M1] , (2.3.7) becomes:

$$\prod_{i=1}^{n}(s - \lambda_i)\left(1 - \sum_{i=1}^{n}\frac{\alpha_i\,\beta_i}{s - \lambda_i}\right) \equiv \prod_{i=1}^{n}(s - \lambda_i^d) \tag{2.3.8}$$

Setting s = λ_j and rearranging gives (2.3.3). This proof is due to Mayne [M1] .

2.3.1 Comment.

As (A,b) is controllable, and A has distinct eigenvalues, $\alpha_j \neq 0$, j=1, .. n. It follows that the β_j can be determined from equation (2.3.3) for any set of λ_i^d, i=1, .. n. There is no restriction on the λ_i^d, which may be distinct or repeated. For physical realizability, the λ_i will be real, or in complex conjugate pairs, as will the λ_i^d. Some of the λ_i^d may be set equal to the λ_j, if desired, which simply corresponds to leaving some eigenvalues unchanged. k^T is found from:

$$k^T = \beta^T W^{-1} C^{-1} . \qquad (2.3.9)$$

If the pair (A,b) is not completely controllable, the formula (2.3.3) can still be used, as the derivation does not require that $\alpha_j \neq 0$. It is clear from the expression that, if $\alpha_j = 0$, at least one of the λ_i^d must be set equal to λ_j, in order to make the expression zero. β_j then becomes indeterminate, but the solution can otherwise be completed as for the controllable case.

The expression clearly applies equally to the case in which $k^T C$ is regarded as fixed, while b is varied. If the system is observable through $k^T C$, complete eigenvalue assignment is possible.

2.4 New Eigenvectors With Feedback.

It is useful to have the new row and column eigenvectors of the system with feedback, i.e., the row and column eigenvectors of $(\Lambda + \alpha \beta^T)$ in the canonical z co-ordinates. These are useful if it is desired to restore the system to diagonal form after feedback is applied, or if eigenvalue and eigenvector sensitivity studies are to be made.

Simon and Mitter [S1] obtained the new eigenvectors after changing a single eigenvalue, in connection with their recursive algorithm. Murdoch [M2] obtained the eigenvectors for the general case, when some or all of the eigenvalues are changed, where the eigenvalues of A are distinct. Using the results obtained in [M2], and the feedback gain expression (2.3.3), the following expressions are obtained for the new eigenvectors for a completely controllable system with distinct eigenvalues both before and after the application of feedback. Thus, $\alpha_i \neq 0$, i=1, .. n, $\lambda_i \neq \lambda_j$, $i \neq j$, and $\lambda_i^d \neq \lambda_j^d$, $i \neq j$.

2.4.1 Formulae for Eigenvectors.

(a) If V is a matrix of row eigenvectors of $(\Lambda + \alpha\beta^T)$, the element in the ith row and the jth column of V is given by:

$$v_{ij} = \frac{\prod_{\substack{q=1\\q\neq i}}^{n} (\lambda_q^d - \lambda_j)}{\alpha_j \prod_{\substack{q=1\\q\neq j}}^{n} (\lambda_q - \lambda_j)} \qquad (2.4.1)$$

(b) If W is a matrix of column eigenvectors of $(\Lambda + \alpha\beta^T)$, the element in the ith row and the jth column of W is given by:

$$w_{ij} = \frac{\alpha_i \prod_{\substack{q=1\\q\neq i}}^{n} (\lambda_j^d - \lambda_q)}{\prod_{\substack{q=1\\q\neq j}}^{n} (\lambda_j^d - \lambda_q^d)} \qquad (2.4.2)$$

The expressions (2.4.1) and (2.4.2) have been so chosen that:

$VW = I$,

where I is the identity matrix.

There is no requirement that the λ_j^d be not equal to the λ_i. If such equalities exist, these will result in the cancellation of some factors, and zero values for others, as appropriate.

Proof.

The proof is in three parts. (a) We first prove that a general row of V, v_r^T, is a row eigenvector of $(\Lambda + \alpha\beta^T)$ corresponding to the eigenvalue λ_r^d, where the jth element of v_r^T is given by (2.4.1), with i replaced by r. (b) We then prove that a general column of W, w_r, is a column eigenvector of $(\Lambda + \alpha\beta^T)$ corresponding to the eigenvalue λ_r^d, where the ith element of w_r is given by (2.4.2), with j replaced by r. (c) Finally, we show that $v_r^T w_r = 1$.

(a) Row Eigenvector.

v_r^T is a row eigenvector of $(\Lambda + \alpha\beta^T)$ corresponding to the eigenvalue λ_r^d if and only if the following equation is satisfied:

$$v_r^T(\Lambda + \alpha\beta^T) = \lambda_r^d v_r^T \qquad (2.4.3)$$

The jth element on the left hand side of this equation is:

$$v_{rj}\lambda_j + v_r^T\alpha\beta_j$$

where β_j is the jth element of β^T, and that on the right hand side is:

$$\lambda_r^d v_{rj}$$

Hence, we wish to show that:

$$v_r^T\alpha\beta_j = (\lambda_r^d - \lambda_j)v_{rj}$$

Substituting for v_{rj} from (2.4.1), with i = r, this equation is equivalent to:

$$v_r^T\alpha\beta_j = (\lambda_r^d - \lambda_j)\frac{\prod_{\substack{q=1\\q\neq r}}^{n}(\lambda_q^d - \lambda_j)}{\alpha_j\prod_{\substack{q=1\\q\neq j}}^{n}(\lambda_q - \lambda_j)}$$

Since $\alpha_j \neq 0$, this may be written as:

$$v_r^T \alpha \, \alpha_j \beta_j = \frac{\prod_{q=1}^{n} (\lambda_q^d - \lambda_j)}{\prod_{\substack{q=1 \\ q \neq j}}^{n} (\lambda_q - \lambda_j)} \qquad (2.4.4)$$

Using (2.3.3), with the appropriate change of symbols, the left hand side of (2.4.4) becomes:

$$v_r^T \alpha \frac{\prod_{q=1}^{n} (\lambda_q^d - \lambda_j)}{\prod_{\substack{q=1 \\ q \neq j}}^{n} (\lambda_q - \lambda_j)}$$

Clearly, the equation is satisfied if and only if

$$v_r^T \alpha = 1.$$

Substituting for the elements of v_r^T from (2.4.1), the left hand side of this equation becomes:

$$\sum_{j=1}^{n} \frac{\prod_{\substack{q=1 \\ q \neq r}}^{n} (\lambda_q^d - \lambda_j)}{\prod_{\substack{q=1 \\ q \neq j}}^{n} (\lambda_q - \lambda_j)} = \sum_{j=1}^{n} \frac{\alpha_j \beta_j}{\lambda_r^d - \lambda_j}$$

from (2.3.3).

(2.3.8) gives the identity:

$$\sum_{j=1}^{n} \frac{\alpha_j \beta_j}{s - \lambda_j} = 1 - \frac{\prod_{j=1}^{n} (s - \lambda_j^d)}{\prod_{j=1}^{n} (s - \lambda_j)} \qquad (2.4.5)$$

Setting $s = \lambda_r^d$ in this identity completes the proof for the row eigenvector.

(b) Column Eigenvector.

w_r is a column eigenvector of $(\Lambda + \alpha \beta^T)$ corresponding to the eigenvalue λ_r^d if and only if the following equation is satisfied:

$$(\Lambda + \alpha\beta^T)w_r = \lambda_r^d w_r \qquad (2.4.6)$$

The ith element of this equation on the left hand side is:

$\lambda_i w_{ir} + \alpha_i \beta^T w_r$, where α_i is the ith element of α,

and that on the right hand side is:

$$\lambda_r^d w_{ir}$$

Hence, we wish to show that:

$$\alpha_i \beta^T w_r = (\lambda_r^d - \lambda_i) w_{ir}$$

Substituting for w_{ir} from (2.4.2), with $j = r$, gives, on the right hand side:

$$\frac{\alpha_i \prod_{q=1}^{n} (\lambda_r^d - \lambda_q)}{\prod_{\substack{q=1\\q\neq r}}^{n} (\lambda_r^d - \lambda_q^d)}$$

The left hand side becomes:

$$\alpha_i \sum_{p=1}^{n} \frac{\alpha_p \beta_p \prod_{\substack{q=1\\q\neq p}}^{n} (\lambda_r^d - \lambda_q)}{\prod_{\substack{q=1\\q\neq r}}^{n} (\lambda_r^d - \lambda_q^d)}$$

which may be written:

$$\frac{\alpha_i \prod_{q=1}^{n} (\lambda_r^d - \lambda_q)}{\prod_{\substack{q=1\\q\neq r}}^{n} (\lambda_r^d - \lambda_q^d)} \sum_{p=1}^{n} \frac{\alpha_p \beta_p}{\lambda_r^d - \lambda_p}$$

Clearly this equation is satisfied if and only if:

$$\sum_{p=1}^{n} \frac{\alpha_p \beta_p}{\lambda_r^d - \lambda_p} = 1$$

but this was shown to be true in (a), and so the proof is complete for the column eigenvector.

(c) Product of Row Eigenvector and Column Eigenvector.

$$v_r^T w_r = \sum_{p=1}^{n} v_{rp} w_{pr}$$

Where v_{rp} is obtained by putting i = r and j = p in (2.4.1) and w_{pr} is obtained by putting i = p and j = r in (2.4.2). Then:

$$v_r^T w_r = \sum_{p=1}^{n} \frac{\prod_{\substack{q=1\\ q\neq r}}^{n} (\lambda_q^d - \lambda_p) \prod_{\substack{q=1\\ q\neq p}}^{n} (\lambda_r^d - \lambda_q)}{\prod_{\substack{q=1\\ q\neq p}}^{n} (\lambda_q - \lambda_p) \prod_{\substack{q=1\\ q\neq r}}^{n} (\lambda_r^d - \lambda_q^d)} \qquad (2.4.7)$$

Using (2.3.3), this may be written:

$$\frac{\prod_{q=1}^{n} (\lambda_r^d - \lambda_q)}{\prod_{\substack{q=1\\ q\neq r}}^{n} (\lambda_r^d - \lambda_q^d)} \sum_{p=1}^{n} \frac{\alpha_p \beta_p}{(\lambda_r^d - \lambda_p)^2} \qquad (2.4.8)$$

Differentiating the identity (2.4.5) with respect to s and setting s = λ_r^d, with j = p, gives:

$$\sum_{p=1}^{n} \frac{\alpha_p \beta_p}{(\lambda_r^d - \lambda_p)^2} = \frac{\prod_{q=1}^{n} (\lambda_r^d - \lambda_q) \prod_{\substack{q=1\\ q\neq r}}^{n} (\lambda_r^d - \lambda_q^d)}{\left(\prod_{q=1}^{n} (\lambda_r^d - \lambda_q)\right)^2} \qquad (2.4.9)$$

From (2.4.8) and (2.4.9), it follows that:

$$v_r^T w_r = 1.$$

This completes the proof.

2.4.2 Numerical Example.

To illustrate the application of the formulae of (2.4.1) and (2.4.2), consider the case in which:

$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = -3$$
$$\lambda_1^d = -1, \quad \lambda_2^d = -2, \quad \lambda_3^d = -3$$
$$\alpha_1 = 1, \quad \alpha_2 = -1, \quad \alpha_3 = 2.$$

This example includes one eigenvalue which is unchanged by the feedback.

Application of the formulae gives:

$$V = \begin{bmatrix} -2 & -3 & 0 \\ -1 & -2 & 0 \\ -\frac{2}{3} & -\frac{3}{2} & \frac{1}{12} \end{bmatrix}$$

$$W = \begin{bmatrix} -2 & 3 & 0 \\ 1 & -2 & 0 \\ 2 & -12 & 12 \end{bmatrix}$$

It is readily verified that VW = I.

The feedback vector required to give these new eigenvalues may be obtained from (2.3.3) as:

$$\beta^T = \begin{bmatrix} 2 & 6 & 0 \end{bmatrix} ,$$

whence the new system matrix with feedback is:

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 6 & 0 \\ -2 & -5 & 0 \\ 4 & 12 & -3 \end{bmatrix}$$

It then follows that:

$$VLW = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

as required.

This procedure is described in [M11].

2.5 Rediagonalization of Controllable System With Feedback.

If, in the system discussed in 2.4, a new state vector , p, is introduced, where z = Wp, and W is as given in (2.4.2), equation (2.3.4) becomes:

$$\dot{p} = \Lambda^d p + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ 1 \end{bmatrix} u \qquad (2.5.1)$$

where $\Lambda^d = \text{diag}(\lambda_1^d \ldots \lambda_n^d)$.

Derivation.

The new system matrix is diagonal, because the λ_j^d are distinct, and W is a matrix of column eigenvectors. The new input distribution vector is $V\alpha$, where V is given in (2.4.1), or, equivalently, as:

$$v_{ij} = \frac{\beta_j}{(\lambda_i^d - \lambda_j)}$$

Then the ith element of the new input distribution vector is:

$$\sum_{j=1}^{n} \frac{\alpha_j \beta_j}{(\lambda_i^d - \lambda_j)} = 1 \qquad (2.5.2)$$

Proof.

From (2.3.8),

$$\prod_{j=1}^{n}(s - \lambda_j)(1 - \sum_{j=1}^{n}\frac{\alpha_j\beta_j}{s - \lambda_j}) \equiv \prod_{j=1}^{n}(s - \lambda_j^d) \qquad (2.5.3)$$

First assume $\lambda_i^d \neq \lambda_j$, j=1, ... n, and set $s = \lambda_i^d$. Equation (2.5.2) follows.

Now let λ_i^d tend to one of the λ_j. The expression (2.5.2) is unaffected, because the term $(\lambda_i^d - \lambda_j)$ is always a factor of $\alpha_j\beta_j$, for each j, j=1, ... n. The expression (2.5.2) is therefore true generally.

2.6 Coefficient Methods.

An explicit solution to the problem of finding the feedback gains for arbitrary pole assignment in a controllable single-input system was given by Bass and Gura [B1]. For the system described by (2.3.1) and (2.3.2), in which C = I, the feedback gain vector k^T, where:

$$u' = k^T y \qquad (2.6.1)$$

is given by:

$$k = -\sum_{i=1}^{n}(\tilde{a}_{i-1} - a_{i-1})(A^T)^{i-1}d \qquad (2.6.2)$$

where: $$d = \left[D^{-1}\right]^T e_n \qquad (2.6.3)$$

in which e_n is the last column of the identity matrix, $\tilde{a}_{i-1}$ and a_{i-1} are coefficients of s^{i-1} in the characteristic polynomial with and without feedback, respectively. D is the controllability matrix:

$$D = \left[b \,\vdots\, Ab \,\vdots\, A^2 b \ \ldots \vdots\, A^{n-1} b \right] \tag{2.6.4}$$

This result can be obtained alternatively by transforming equation (2.3.1) to the phase-variable form. Using the method of Ramaswami and Ramar [R3] , the system may be placed in the phase-variable form:

$$\dot{p} = A_c p + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \end{bmatrix} u' + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \end{bmatrix} u \tag{2.6.5}$$

by the transformation $p = Tx$, where T is the observability matrix of (t_1^T, A), i.e.,

$$\begin{bmatrix} t_1^T \\ t_1^T A \\ \cdot \\ \cdot \\ t_1^T A^{n-1} \end{bmatrix} \tag{2.6.6}$$

and t_1^T is the last row of the inverse of the controllability matrix (2.6.4). A_c has the form:

$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & 0 \\ 0 & 0 & 1 & 0 & . & . & 0 \\ . & . & . & . & . & . & . \\ -a_o & -a_1 & . & & . & . & -a_{n-1} \end{bmatrix} \qquad (2.6.7)$$

The transformed state feedback gain vector is $k^T T^{-1}$, and it is clear from (2.6.5) that this gives the desired set of coefficients ($\tilde{a}_o$, ... $\tilde{a}_{n-1}$) to the closed-loop characteristic equation if:

$$k^T T^{-1} = -\left[\tilde{a}_o\ \tilde{a}_1\ .\ .\ \tilde{a}_{n-1}\right] + \left[a_o\ a_1\ .\ .\ a_{n-1}\right] \qquad (2.6.8)$$

Hence:

$$k^T = -\left[(\tilde{a}_o - a_o)\ (\tilde{a}_1 - a_1)\ .\ .\ (\tilde{a}_{n-1} - a_{n-1})\right] T \qquad (2.6.9)$$

Substituting for T from (2.6.6) gives:

$$k^T = -(\tilde{a}_o - a_o)t_1^T - (\tilde{a}_1 - a_1)t_1^T A\ ..\ -(\tilde{a}_{n-1} - a_{n-1})t_1^T A^{n-1} \qquad (2.6.10)$$

Transposing (2.6.10) gives (2.6.2).

2.7 The Method of Anderson and Luenberger.

The formula (2.6.2) cannot be used with a system which is not controllable through a single input.

Anderson and Luenberger [A1] considered the general case of a multi-input system, and derived a canonical form, based on generalised phase-variable forms obtained by Luenberger [L2] . In the single-input case, this form coincides with the phase-variable canonical form, and is unique. In the multi-input case, the canonical form is not unique, as it depends upon the scheme used in selecting sets of linearly independent vectors from the controllability matrix.

Power [P2] has described an extension to this method, in which advantage is taken of the possibility of replacing some zero terms by non-zero terms in the canonical forms, without affecting the characteristic polynomial, so as to obtain extra design freedom.

One difficulty that may arise in using the method of [A1] is that the initial non-unique formation of the canonical form decides the size of the real companion matrices of which it is composed. This places limitations on the choice of system poles, to achieve realizability. Power [P3] has suggested an extension to the method to overcome this difficulty, and provide increased design freedom.

2.8 The Method of Fallside and Seraji.

This method [F1] is based on the relationship:

$$k^T g(s) = F(s) - H(s) \qquad (2.8.1)$$

where F(s) and H(s) are the open-loop and closed-loop characteristic polynomials of the system, k^T is the feedback gain vector, and g(s) is an n-vector, the elements of which are the numerator polynomials of the transfer functions from the input concerned to each state variable. The relationship (2.8.1) was given by Bass and Gura [B1] . Multi-input systems are treated as single-input systems, by distributing scalar feedback amongst the inputs, as is done in other schemes. The method then involves equating coefficients to achieve a desired F(s).

Although it appears to be simple, this method probably involves, for large systems, about the same amount of work as the use of the phase variable canonical form, due to the need to find F(s) and g(s) initially. It is obviously closely related to the method of Bass and Gura.

2.9 The Method of Willner, Ash and Roy.

Willner, Ash and Roy [W2] have described a pole-placement algorithm which they claim to be new. This is neither a spectral nor a coefficient method. For a single-input system, this method is based on the solution of the equation:

$$TF - AT = bk^T \tag{2.9.1}$$

or:

$$TFT^{-1} = A + bk^T T^{-1} \tag{2.9.2}$$

The vector $k^T T^{-1}$ now represents the state feedback gain vector.

F is chosen to have the desired set of eigenvalues, and k^T is set to the sum vector $[1\ 1\ 1\ .\ .\ 1]$. Equation (2.9.1) is then solved for T.

This method is very similar to a method first described by Luenberger [L1].

In extending their method to multi-input systems, the authors use a technique of converting the multi-input system into a single-input system by distributing a scalar feedback amongst the inputs in such a way as to preserve controllability. It is claimed that this method can be used with any controllable system, but Murdoch [M3] has pointed out that this is a fallacy. As shown by Simon and Mitter [S1] , there is no single input through which the system could be controllable if the A matrix is derogatory. The significance of this statement is that, for a derogatory A matrix, there exists no vector d such that the pair (A,d) is controllable. This is easily proved because, since A is derogatory, there exists a polynomial in A of degree q, less than n, such that:

$$A^q + a_{q-1}A^{q-1} + \dots + a_1A + a_oI = 0. \qquad (2.9.3)$$

Post-multiplying (2.9.3) by the column vector d, and rearranging,

$$A^qd = -a_{q-1}A^{q-1}d - \dots - a_1Ad - a_od \qquad (2.9.4)$$

It follows that the controllability matrix $\left[d \; Ad \; A^2d \; . \; . \; A^{n-1}d\right]$ is singular, because the $(q+1)$th column is linearly dependent on the preceding columns.

2.9.1 Alternative Derivation of Explicit Feedback Gain Formula.

The method of Willner, Ash and Roy applied to a controllable single input system with distinct eigenvalues provides an interesting alternative derivation of the explicit feedback gain formula (2.3.3)

Consider the controllable single input system described by equations (2.3.1) and (2.3.2).

Let $A = W\Lambda W^{-1}$, where W is a matrix of self-conjugate column eigenvectors of A, and $\Lambda = \text{diag}(\lambda_1, \dots \lambda_n)$, where $(\lambda_1, \dots \lambda_n)$ are the eigenvalues of A, assumed distinct. Then (2.9.1) becomes:

$$TF - W\Lambda W^{-1}T = bk^T \qquad (2.9.5)$$

or:

$$W^{-1}TF - \Lambda W^{-1}T = W^{-1}bk^T \qquad (2.9.6)$$

Let $W^{-1}b = \alpha = [\alpha_1 \; . \; . \; \alpha_n]^T$, and let

$k^T = [1 \; 1 \; 1 \; . \; . \; 1]$.

Let $F = \Lambda^d = \text{diag}(\lambda_1^d \; . \; . \; \lambda_n^d)$, where $(\lambda_1^d, \; . \; . \; \lambda_n^d)$ are the desired closed-loop eigenvalues.

Let $W^{-1}T = \Gamma$. Then (2.9.5) becomes:

$$\Gamma \Lambda^{d} - \Lambda \Gamma = \alpha [1 \; 1 \; 1 \; . \; . \; 1] \tag{2.9.7}$$

Assuming that the λ_i and the λ_j^d have no terms with common values, we may solve for the elements of Γ, obtaining, for the term in the ith row and jth column:

$$\gamma_{ij} = \frac{\alpha_i}{\lambda_j^d - \lambda_i} \tag{2.9.8}$$

Writing (2.9.7) in the form:

$$\Gamma \Lambda^{d} \Gamma^{-1} = \Lambda + \alpha [1 \; 1 \; 1 \; . \; . \; 1] \Gamma^{-1} \tag{2.9.10}$$

and comparing with equation (2.3.4), we find:

$$\beta^{T} = [1 \; 1 \; 1 \; .. \; 1] \Gamma^{-1} \tag{2.9.11}$$

For clarity, we shall write this in full for a third order system:

$$\begin{bmatrix}\beta_1 & \beta_2 & \beta_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix} \frac{\alpha_1}{\lambda_1^d - \lambda_1} & \frac{\alpha_1}{\lambda_2^d - \lambda_1} & \frac{\alpha_1}{\lambda_3^d - \lambda_1} \\ \frac{\alpha_2}{\lambda_1^d - \lambda_2} & \frac{\alpha_2}{\lambda_2^d - \lambda_2} & \frac{\alpha_2}{\lambda_3^d - \lambda_2} \\ \frac{\alpha_3}{\lambda_1^d - \lambda_3} & \frac{\alpha_3}{\lambda_2^d - \lambda_3} & \frac{\alpha_3}{\lambda_3^d - \lambda_3} \end{bmatrix}^{-1} \qquad (2.9.12)$$

Suppose we wish to find β_1. This is the sum of the terms in the first column of the inverse of the 3×3 matrix in (2.9.12). Hence, we may write:

$$\beta_1 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ \frac{\alpha_2}{\lambda_1^d - \lambda_2} & \frac{\alpha_2}{\lambda_2^d - \lambda_2} & \frac{\alpha_2}{\lambda_3^d - \lambda_2} \\ \frac{\alpha_3}{\lambda_1^d - \lambda_3} & \frac{\alpha_3}{\lambda_2^d - \lambda_3} & \frac{\alpha_3}{\lambda_3^d - \lambda_3} \end{vmatrix}}{\begin{vmatrix} \frac{\alpha_1}{\lambda_1^d - \lambda_1} & \frac{\alpha_1}{\lambda_2^d - \lambda_1} & \frac{\alpha_1}{\lambda_3^d - \lambda_1} \\ \frac{\alpha_2}{\lambda_1^d - \lambda_2} & \frac{\alpha_2}{\lambda_2^d - \lambda_2} & \frac{\alpha_2}{\lambda_3^d - \lambda_2} \\ \frac{\alpha_3}{\lambda_1^d - \lambda_3} & \frac{\alpha_3}{\lambda_2^d - \lambda_3} & \frac{\alpha_3}{\lambda_3^d - \lambda_3} \end{vmatrix}} \qquad (2.9.13)$$

By removing the factors α_2 and α_3, and considering the formation of identical columns and identical rows, it is clear that the numerator determinant of (2.9.13) has the following form, disregarding signs:

$$\frac{\alpha_2\alpha_3(\lambda_2-\lambda_3)(\lambda_1^d-\lambda_2^d)(\lambda_2^d-\lambda_3^d)(\lambda_3^d-\lambda_1^d)}{(\lambda_1^d-\lambda_2)(\lambda_1^d-\lambda_3)(\lambda_2^d-\lambda_2)(\lambda_2^d-\lambda_3)(\lambda_3^d-\lambda_2)(\lambda_3^d-\lambda_3)} \tag{2.9.14}$$

Similarly, the denominator determinant has the form of (2.9.14), multiplied by:

$$\frac{\alpha_1(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}{(\lambda_1^d-\lambda_1)(\lambda_2^d-\lambda_1)(\lambda_3^d-\lambda_1)} \tag{2.9.15}$$

The ratio is thus:

$$\beta_1 = \frac{(\lambda_1^d-\lambda_1)(\lambda_2^d-\lambda_1)(\lambda_3^d-\lambda_1)}{\alpha_1(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} \tag{2.9.16}$$

The sign is determined by considering the effect of λ_1 becoming large in magnitude.

(2.9.16) is the formula (2.3.3) for n=3, j=1. The corresponding expressions for β_2 and β_3 can be found in a similar way, and the generalisation to larger systems can be understood from the form of (2.9.14) and (2.9.15). We shall not deal with the case where some of the λ_i^d are specified as equal to some λ_j. In such cases, it is only necessary to replace the appropriate 1's in k^T by 0's.

In this procedure, we were solving for the new column eigenvectors of the system with feedback, and finding the feedback gains in the process.

2.10. Other Recent Work on Pole Assignment by State Vector Feedback.

Paraskevopoulos and Tzafestas [P5] have recently described a procedure for closed-loop pole assignment based on finding the transformation matrix T such that, for given A, B and L, the matrix equation:

$$A + BF = TLT^{-1} \qquad (2.10.1)$$

is satisfied. L is chosen to be in the diagonal or the Jordan form, with the desired set of eigenvalues. A and B have their usual significance, and F is the unknown feedback gain matrix. Although claimed to be new, this is similar to the method described by Willner, Ash and Roy [W2] , and by Luenberger [L1] . The method of [P5] differs from earlier procedures in that the system is first transformed to the phase-variable form, or to the generalised phase-variable form of Anderson and Luenberger [A1] . However, when this is done, there seems to be no advantage in going to the trouble of finding the transformation matrix T, since the last rows of the companion blocks of the A matrix can then be changed directly to give any desired set of coefficients to the closed-loop characteristic polynomial.

It is claimed that the method described yields the full degrees of freedom in the choice of F, but this claim does not seem to be justified in the numerical examples which have been given. For example, in Ex. 3 of [P5] , the system given is described by the equation:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 11 & 30 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 7 & 12 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u \qquad (2.10.2)$$

in which the desired closed-loop poles are to be at 0, 1, -1 and 2. The authors reach the conclusion that:

$$F = \begin{bmatrix} -11 & -29 & 0 & 0 \\ 0 & 0 & -5 & -11 \end{bmatrix} \qquad (2.10.3)$$

However, it is clear that this is not the most general form of F possible, because the last two elements of the first row could be given any values or, alternatively, the first two elements of the second row could be given any values, without affecting the closed-loop poles of the system. This is clear from the quasi-upper-triangular or quasi-lower-triangular structure of the system matrix, respectively, in the two cases. No indication is given in [P5] as to how the degrees of freedom in F may be used.

The problem of output feedback is also treated in [P5] , and this will be discussed in Chapter 3.

Flower [F7] has described a procedure for pole assignment for a single-input system, based on the fact that the coefficients of the characteristic polynomial of a matrix are given by the generalised traces, i. e., the sums of all possible determinants which can be formed from the original matrix by omitting columns and corresponding rows. As is well known, the coefficient of $(-1)^n s^0$ is obtained by omitting no column or row, giving the determinant of the matrix itself, and the coefficient of $-s^{n-1}$ is obtained by omitting (n-1) rows and columns, giving the usual trace.

When this technique is applied to the matrix:

$$(A + bk^T),$$

a set of linear equations in the elements of k^T is obtained, so as to give a desired set of closed-loop poles.

The method becomes rather cumbersome with high-order systems, due to the large number of determinants, of different orders, which have to be computed. Apart from the small advantage of avoiding complex arithmetic, this method seems to have no advantage over the naïve technique of inserting each desired closed-loop eigenvalue in turn, to form the matrix:

$$D_j = (A - \lambda_j^d I) = \left[d_{j1} \vdots d_{j2} \vdots \cdots \right] \qquad (2.10.4)$$

and then forming the set of linear equations in the elements of k^T, k_1, k_2, etc., in which each equation is of the form:

$$|D_j| + k_1 \left| b \vdots d_{j2} \vdots d_{j3} \vdots \cdot\cdot \right| + k_2 \left| d_{j1} \vdots b \vdots d_{j3} \vdots \cdot\cdot \right| + \cdot\cdot = 0 \qquad (2.10.5)$$

In practical applications, it is unlikely that the closed-loop system will be required to have multiple poles. If this is required, however, this simple method can be adapted by considering the derivatives of $D_j + bk^T$ with respect to λ_j^d, the corresponding number of which must also be set to zero for a multiple eigenvalue. These derivatives correspond to the generalised traces considered in [F7] , but, of course, in this alternative approach, these need be considered only when multiple eigenvalues are specified.

2.11 Conclusion.

The coefficient methods have the advantages that they involve only real numbers, and that they admit of pole assignment in systems in which the A matrix has multiple eigenvalues, provided that it is nonderogatory, without any change in the method.

The spectral methods provide greater insight and, although they generally involve complex arithmetic, it would, in any case, be necessary to compute the open-loop poles when contemplating feedback, and so the disadvantage of this is not great.

On the question of accuracy, some earlier workers in this field tended to avoid the use of eigenvalues and eigenvectors, and to favour the coefficient methods. Direct methods of computing the transformations to the phase variable form [S2,T2,C2,J1,R4] involve repeated multiplications by the A matrix, and this can cause numerical difficulties. The problem of the powers of the A matrix tending to become more nearly singular, or the converse of their having very large determinants, can be overcome by the use of time scaling, which simply involves multiplying the A matrix by a constant. This procedure has been suggested by Davison and Chow [D1] .

The reason for the avoidance of eigenvalue methods probably was due to the lack of a satisfactory method for computing the eigenvalues of a non-symmetric matrix. However, the introduction of the QR transform method of Francis [F2] has resulted in the availability of reliable, accurate programs for determining eigenvalues. The situation now is that the determination of eigenvalues by the direct matrix method is preferred to procedures which first find the coefficients of the characteristic polynomial, and then find the roots of this to obtain the eigenvalues.

It should be remembered, however, that, if it is desired to use methods based on the coefficients of the characteristic polynomial, these coefficients and the transforming matrices needed to give the phase variable form can be found by taking advantage of the accurately determined eigenvalues which are now available. Procedures based on the use of this information have been described by Johnson and Wonham [J2] for the distinct eigenvalue case, and an extension to the multiple eigenvalue case has been given by Mufti [M16] . These methods require the computation of the determinant and of the inverse of the Vandermonde matrix, and explicit solutions to both of these problems are available [W3,A2,T3].

In the problem of determining the feedback gain vector for a single-input system, one may choose to transform to the phase variable form, or some variant of this, and then determine the elements of the feedback gain vector directly. Alternatively, one may choose to use the spectral method, with accurately computed eigenvalues and eigenvectors. At the final stage of this process, however, one has to compute the feedback gains from an expression of the type (2.3.3), which involves the products and quotients of differences between eigenvalues, which could introduce undesirable magnification of errors. Basically, the feedback gains which are to be found, consist of such products, and are, therefore, closely related computationally to the coefficients of the characteristic polynomial.

A result obtained recently by Davison and Wang [D2] reveals that almost all feedback laws make a controllable observable system controllable through a single input, and cause it to have distinct eigenvalues. The effect of this is that, if closed-loop pole assignment is the only consideration, any controllable observable system can be treated as a single-input system with distinct eigenvalues. The preliminary step of applying almost arbitrary feedback to achieve this condition can always be assumed to be possible, provided that the system is controllable and observable. Any change of system poles caused by this initial feedback can be allowed for in the final pole assignment.

CHAPTER 3.

POLE ASSIGNMENT WITH RESTRICTED MEASUREMENT ACCESS.

3.1 Introduction.

This chapter is concerned with pole assignment in the system described by equations (2.1.1),(2.1.2) and (2.1.3), in which $m<n$.

Davison [D3] has shown that a controllable observable system with m independent outputs (i.e., rankC = m), can always have m closed-loop eigenvalues made arbitrarily close, but not necessarily equal to any desired set, by suitable choice of K, subject, of course, to complex conjugate pairing to ensure realizability. An algorithm for finding K is given, but nothing can be said about the remaining (n-m) unassigned eigenvalues, which will, in general, have been changed by the feedback in an unpredictable way. Jameson [J3] has reached a similar conclusion, and has given an algorithm for finding K based on a least-squares fit of the actual eigenvalues achieved to the desired set.

The application of the method of Fallside and Seraji to this case is discussed in 3.4.

3.2 The Result of Sridhar and Lindorff.

Using the approach of Retallack and MacFarlane [R2], Sridhar and Lindorff [S3] have given a proof that, in a controllable, observable linear time-invariant system, max(m,r) closed-loop poles can be assigned almost arbitrarily.

This result follows at once from Davison's result [D3], that m closed-loop poles can be assigned almost arbitrarily. This is so because the eigenvalues of (A+BKC) and of $(A^T+C^TK^TB^T)$ are the same. Hence the conditions of [D3] that C has rank m and (A,B) is controllable, when applied to the transposed matrix, coincide with the conditions that B has rank r and (A,C) is observable. Thus, if $r > m$, it follows that r eigenvalues can be assigned almost arbitrarily.

A further comment in [S3] is that Davison's conclusion that the poles which cannot be assigned correspond to the zeros of the various transfer functions existing in the multivariable system applies only to single-input, single-output systems, and is not general.

3.3 Pole Assignment and Determination of the Residual Characteristic Equation.

The method based on obtaining a least-squares fit of closed-loop poles, suggested by Jameson, is mathematically attractive, and would be expected to give satisfactory results if the desired poles were suitably weighted. This type of approach is considered more generally in Chapter 10.

Davison's method is not entirely satisfactory as, although it permits the assignment, or near assignment, of m poles, it gives no information about the remaining (n-m) poles. There is, therefore, the necessity to compute all the system poles after the feedback has been determined, a process that will probably reveal some unsatisfactory poles, calling for reconsideration of the m assigned poles.

A reasonable approach seems to be to use a method which assigns the m poles, while facilitating the finding of the remainder. In the following procedure, the feedback vector is found so that m poles are assigned to be arbitrarily close to a desired set, and, at the same time, the coefficients of the residual characteristic equation are found, the roots of which are the remaining (n-m) unassigned poles.

3.3.1 System Description.

A linear time invariant system is described by the equations:

$$\dot{x} = Ax + bu' + bu \qquad (3.3.1)$$

$$y = Cx \qquad (3.3.2)$$

where x is a $n \times 1$ state vector, u' and u are scalar feedback and external inputs, respectively, and y is an $m \times 1$ output vector. A has distinct eigenvalues (λ_1, .. λ_n), C has rank m, and the triple (A,b,C) is controllable and observable.

3.3.2 Problem Statement.

The problem is to find for the system (3.3.1), (3.3.2) the $1 \times m$ feedback gain vector $k^T = [k_1 \ldots k_m]$, such that the feedback input:

$$u' = -k^T y \tag{3.3.3}$$

gives m preassigned eigenvalues ($\lambda_1^d, \ldots \lambda_m^d$) to the closed-loop system matrix:

$$(A - bk^T C) \tag{3.3.4}$$

We also require to find the (n-m) coefficients ($a_o, \ldots a_{n-m-1}$) of the residual characteristic equation:

$$s^{n-m} + a_{n-m-1}s^{n-m-1} + \ldots + a_1 s + a_o = 0 \tag{3.3.5}$$

The roots of (3.3.5) are the remaining (n-m) unassigned eigenvalues of (3.3.4).

3.3.3 Solution.

Let W be a matrix of self-conjugate column eigenvectors of A. Let $\Lambda = \text{diag}(\lambda_1 \ldots \lambda_n)$, and let:

$$\alpha = [\alpha_1 \ldots \alpha_n]^T = W^{-1} b \tag{3.3.6}$$

Let $P = [p_1 \ldots p_n]^T$ (3.3.7)

where:

$$p_j = - \frac{\prod_{i=1}^{m} (\lambda_j - \lambda_i^d)}{\alpha_j \prod_{\substack{i=1 \\ i \neq j}}^{n} (\lambda_j - \lambda_i)} \quad j=1,\ldots n \tag{3.3.8}$$

Then the elements of k^T and the coefficients $a_o, \dots a_{n-m-1}$ are given as the solution of the set of linear equations:

$$\left[W^TC^T \vdots P \vdots \Lambda P \vdots \Lambda^2 P \vdots \dots \vdots \Lambda^{n-m-1}P\right] \begin{bmatrix} k_1 \\ \cdot \\ \cdot \\ k_m \\ a_o \\ \cdot \\ \cdot \\ a_{n-m-1} \end{bmatrix} = -\left[\Lambda^{n-m}P\right] \qquad (3.3.9)$$

<u>3.3.4 Proof.</u>

Let the eigenvalues of (3.3.4) be $(\lambda_1^d, \dots \lambda_n^d)$, of which $(\lambda_1^d, \dots \lambda_m^d)$ are specified, whilst $(\lambda_{m+1}^d, \dots \lambda_n^d)$ are unspecified. Let:

$$\beta^T = [\beta_1 \dots \beta_n] = k^TCW \qquad (3.3.10)$$

Introducing the transformation $x = Wz$, equations (3.3.1),(3.3.2) and (3.3.3) give:

$$\dot{z} = (\Lambda - W^{-1}bk^TCW)z + W^{-1}bu \qquad (3.3.11)$$

From (3.3.6) and (3.3.10), this becomes:

$$\dot{z} = (\Lambda - \alpha\beta^T)z + \alpha u \qquad (3.3.12)$$

From (2.3.3), we have:

$$\alpha_j \beta_j = \frac{\prod_{i=1}^{n} (\lambda_j - \lambda_i^d)}{\prod_{\substack{i=1 \\ i \neq j}}^{n} (\lambda_j - \lambda_i)} \qquad j=1, \ldots n \tag{3.3.13}$$

As the system is controllable, $\alpha_j \neq 0$, j=1, ..n, ans so (3.3.13) determines the β_j uniquely, from the λ_i^d.

Transposing (3.3.10) and substituting from (3.3.13),

$$W^T C^T k = \begin{bmatrix} \dfrac{\prod_{i=1}^{n} (\lambda_1 - \lambda_i^d)}{\alpha_1 \prod_{i=2}^{n} (\lambda_1 - \lambda_i)} \\ \cdot \\ \cdot \\ \dfrac{\prod_{i=1}^{n} (\lambda_n - \lambda_i^d)}{\alpha_n \prod_{i=1}^{n-1} (\lambda_n - \lambda_i)} \end{bmatrix} \tag{3.3.14}$$

The jth element in the vector on the right hand side of (3.3.14) may be written as:

$$\frac{\prod_{i=1}^{m}(\lambda_j - \lambda_i^d)}{\alpha_j \prod_{\substack{i=1 \\ i \neq j}}^{n}(\lambda_j - \lambda_i)} \cdot \prod_{i=m+1}^{n}(\lambda_j - \lambda_i^d) \qquad (3.3.15)$$

Consider the polynomial:

$$s^{n-m} + a_{n-m-1}s^{n-m-1} + \ldots + a_1 s + a_o \triangleq \prod_{i=m+1}^{n}(s - \lambda_i^d) \qquad (3.3.16)$$

Comparison of (3.3.16) with the factors of the form

$$\prod_{i=m+1}^{n}(\lambda_j - \lambda_i^d)$$

in (3.3.15) shows that, in each case, the coefficients of the powers of λ_j are the coefficients of the corresponding powers of s in (3.3.16).

Let P be defined as in (3.3.7) and (3.3.8). Then rearrangement of (3.3.15) and the use of (3.3.16) give (3.3.9). This completes the proof.

3.3.5 Comment.

If equations (3.3.9) are inconsistent, the desired set of m specified eigenvalues cannot be obtained. However, Davison's result [D3] shows that consistency can always be achieved by making small adjustments to the specified λ_i^d, i=1, .. m.

If the choice of the λ_i^d is such that all the elements of P are non-zero, it follows, since the λ_i are distinct, that the last (n-m) columns of the coefficient matrix in (3.3.9) are linearly independent, and the vector on the right hand side is independent of these columns. Also, since W is non-singular and C has rank m, the first m columns are a linearly independent set.

The condition for consistency may be stated in the form that, if the augmented coefficient matrix is written, echelon reduction of the columns of this should annihilate the last column. This may be used as the basis of a procedure for finding relationships amongst the λ_i^d which represent inadmissible choices. Avoidance of choices of λ_i^d satisfying these relationships will then guarantee the admissibility of the set of λ_i^d chosen.

The procedure is illustrated by application to the simple third order system with two outputs, used as an example in 3.2. In this example,

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

The augmented coefficient matrix for this system, assuming that λ_1^d and λ_2^d are to be specified, is:

$$\left[\begin{array}{c|c|c|c} 1 & 1 & -\frac{1}{2}(1+\lambda_1^d)(1+\lambda_2^d) & \frac{1}{2}(1+\lambda_1^d)(1+\lambda_2^d) \\ 1 & -1 & (2+\lambda_1^d)(2+\lambda_2^d) & -2(2+\lambda_1^d)(2+\lambda_2^d) \\ 1 & 1 & -\frac{1}{2}(3+\lambda_1^d)(3+\lambda_2^d) & \frac{3}{2}(3+\lambda_1^d)(3+\lambda_2^d) \end{array}\right] \quad (3.3.17)$$

which, on echelon reduction of the first row, becomes (3.3.18), overleaf.

$$
\left[\begin{array}{c:c:c:c}
0 & 0 & -\frac{1}{2}(1+\lambda_1^d)(1+\lambda_2^d) & 0 \\[2em]
1+\dfrac{2(2+\lambda_1^d)(2+\lambda_2^d)}{(1+\lambda_1^d)(1+\lambda_2^d)} & -1+\dfrac{2(2+\lambda_1^d)(2+\lambda_2^d)}{(1+\lambda_1^d)(1+\lambda_2^d)} & (2+\lambda_1^d)(2+\lambda_2^d) & -(2+\lambda_1^d)(2+\lambda_2^d) \\[2em]
1-\dfrac{(3+\lambda_1^d)(3+\lambda_2^d)}{(1+\lambda_1^d)(1+\lambda_2^d)} & 1-\dfrac{(3+\lambda_1^d)(3+\lambda_2^d)}{(1+\lambda_1^d)(1+\lambda_2^d)} & -\frac{1}{2}(3+\lambda_1^d)(3+\lambda_2^d) & (3+\lambda_1^d)(3+\lambda_2^d)
\end{array}\right]
$$

(3.3.18)

It is clear that echelon reduction of the last column by the first two columns will not be possible, in this case, if the 2×2 matrix in the bottom left hand corner of the augmented coefficient matrix is singular. This condition yields:

$$\frac{1 + \lambda_1^d}{3 + \lambda_1^d} = \frac{3 + \lambda_2^d}{1 + \lambda_2^d} \qquad (3.3.18a)$$

Thus, any choice of λ_1^d and λ_2^d satisfying (3.3.18a) is inadmissible. We note that this condition is satisfied by the choice ($\lambda_1^d=0$, $\lambda_2^d=-4$) which was used in the example.

It would be possible to use this method of checking for inadmissible choices, or a variation of it, in which some of the λ_i^d, i=1, .. m, are given numerical values, for large systems. However, this would be cumbersome, and a simple trial and error search procedure probably would be satisfactory. The problem of avoiding inadmissible choices of λ_i^d is a general one, and has been discussed by Davison [D3] .

3.3.6 Numerical Example

The occurrence of inadmissible choices of eigenvalues is exceptional, and we now demonstrate the use of this method in the normal case, by application to a simple example.

Given the system:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The matrix A has the eigenvalues:

$\lambda_1 = -1, \quad \lambda_2 = -2, \quad \lambda_3 = -3.$

It is required that $\lambda_1^d = -4$, $\lambda_2^d = -5$. We wish to find the feedback gain vector $k^T = [k_1 \ k_2]$, where $u' = -k^T y$, to give these eigenvalues, and to find the coefficient of the residual characteristic equation which, in this case, is of degree 1.

A matrix of column eigenvectors of A is:

$$W = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}, \text{ and } W^{-1} = \begin{bmatrix} 3 & \frac{5}{2} & \frac{1}{2} \\ -3 & -4 & -1 \\ 1 & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

$$\alpha = W^{-1}b = \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}$$

which shows that the system is controllable, and

$$CW = \begin{bmatrix} 0 & -1 & -2 \\ 2 & 5 & 10 \end{bmatrix}$$

which shows that the system is observable. Whence:

$$W^T C^T = \begin{bmatrix} 0 & 2 \\ -1 & 5 \\ -2 & 10 \end{bmatrix}$$

From (3.3.8),

$$p_1 = -\frac{(-1 + 4)(-1 + 5)}{3(-1 + 2)(-1 + 3)} = -2$$

and $\lambda_1 p_1 = -1 \times -2 = 2$

$$p_2 = -\frac{(-2 + 4)(-2 + 5)}{-5(-2 + 1)(-2 + 3)} = -\frac{6}{5}$$

and $\lambda_2 p_2 = -2 \times -\frac{6}{5} = \frac{12}{5}$

$$p_3 = -\frac{(-3 + 4)(-3 + 5)}{2(-3 + 1)(-3 + 2)} = -\frac{1}{2}$$

and $\lambda_3 p_3 = -3 \times -\frac{1}{2} = \frac{3}{2}$

Equation (3.3.9) then becomes:

$$\begin{bmatrix} 0 & 2 & -2 \\ -1 & 5 & -\frac{6}{5} \\ -2 & 10 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ a_o \end{bmatrix} = \begin{bmatrix} -2 \\ -\frac{12}{5} \\ -\frac{3}{2} \end{bmatrix}$$

which has the solution:

$k_1 = 4$, $k_2 = \frac{14}{19}$, $a_o = \frac{33}{19}$, so that the residual characteristic equation is:

$$s + \frac{33}{19} = 0,$$

and the third eigenvalue is $-\frac{33}{19}$.

3.3.7 Sensitivity of Unassigned Poles to Changes in Assigned Poles.

In the solution given in 3.3.3, the (n - m) unassigned poles will often be found to be unsatisfactory, because they represent an unstable, slow, or inadequately damped response.

The next step is to adjust the values of the assigned poles, in order to improve the locations of the unassigned poles, with a view to arriving at an acceptable compromise. In this process, it is useful to have some idea of the likely effect of changing each of the assigned poles. In particular, knowledge of the direction of the change, and the relative effects of changing different assigned poles would be helpful. This information is given, for small changes, by the following relationships.

The partial derivative of each element in the vector of unknowns in (3.3.9) with respect to variation of one of the assigned poles, λ_q^d, $1 \leq q \leq m$, is given as the solution of the set of linear equations (3.3.19).

$$\left[\lambda_q^d I - \Lambda\right]\left[W^T C^T \,\vdots\, P \,\vdots\, \Lambda P \,\vdots\, \cdot\cdot \,\vdots\, \Lambda^{n-m-1} P\right]\begin{bmatrix} \frac{\partial k_1}{\partial \lambda_q^d} \\ \cdot \\ \cdot \\ \frac{\partial k_m}{\partial \lambda_q^d} \\ \frac{\partial a_o}{\partial \lambda_q^d} \\ \cdot \\ \cdot \\ \frac{\partial a_{n-m-1}}{\partial \lambda_q^d} \end{bmatrix} = W^T C^T \begin{bmatrix} k_1 \\ \cdot \\ \cdot \\ k_m \end{bmatrix} \qquad (3.3.19)$$

The small changes in the unassigned eigenvalues, $\Delta\lambda_l^d$, $l = (m+1), \dots n$, corresponding to given small changes in the coefficients of the residual characteristic equation, $\Delta a_o, \dots \Delta a_{n-m-1}$, are given as the solution of the following set of linear equations:

$$\begin{bmatrix} -1 & -1 & \cdot & -1 \\ \sum_{l=m+2}^{n} \lambda_l^d & \sum_{\substack{l=m+1 \\ l \neq m+2}}^{n} \lambda_l^d & & \sum_{l=m+1}^{n-1} \lambda_l^d \\ \cdot & \cdot & \cdot & \cdot \\ (-1)^{n-m}\lambda_{m+2}^d \cdot\cdot \lambda_n^d & & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \Delta\lambda_{m+1}^d \\ \cdot \\ \cdot \\ \cdot \\ \Delta\lambda_n^d \end{bmatrix} = \begin{bmatrix} \Delta a_{n-m-1} \\ \cdot \\ \cdot \\ \cdot \\ \Delta a_o \end{bmatrix} \qquad (3.3.20)$$

The element in the ith row and the jth column of the $(n - m) \times (n - m)$ coefficient matrix is the sum of all products of $\lambda^d_{m+1}, \ldots \lambda^d_n$, taken $(i - 1)$ at a time, excluding λ^d_j, multiplied by $(-1)^i$.

Proof.

Differentiating (3.3.9) partially with respect to one of the assigned eigenvalues λ^d_q, $1 \leq q \leq m$, gives:

$$\left[0 \;\vdots\; \frac{\partial P}{\partial \lambda^d_q} \;\vdots\; \Lambda \frac{\partial P}{\partial \lambda^d_q} \;\vdots\; \cdots \;\vdots\; \Lambda^{n-m-1} \frac{\partial P}{\partial \lambda^d_q} \right] \begin{bmatrix} k_1 \\ \cdot \\ \cdot \\ k_m \\ a_o \\ \cdot \\ \cdot \\ a_{n-m-1} \end{bmatrix} +$$

$$\left[W^T C^T \;\vdots\; P \;\vdots\; \Lambda P \;\vdots\; \cdots \;\vdots\; \Lambda^{n-m-1} P \right] \begin{bmatrix} \frac{\partial k_1}{\partial \lambda^d_q} \\ \cdot \\ \cdot \\ \frac{\partial k_m}{\partial \lambda^d_q} \\ \frac{\partial a_o}{\partial \lambda^d_q} \\ \cdot \\ \frac{\partial a_{n-m-1}}{\partial \lambda^d_q} \end{bmatrix} = - \left[\Lambda^{n-m} \frac{\partial P}{\partial \lambda^d_q} \right] \quad (3.3.21)$$

Differentiating (3.3.8), we obtain:

$$\frac{\partial p_j}{\partial \lambda_q^d} = \frac{\prod_{\substack{i=1 \\ i \neq q}}^{m} (\lambda_j - \lambda_i^d)}{\alpha_j \prod_{\substack{i=1 \\ i \neq j}}^{n} (\lambda_j - \lambda_i)} \qquad (3.3.22)$$

$j=1, \ldots n$

$q=1, \ldots m$

from which:

$$(\lambda_q^d - \lambda_j) \frac{\partial p_j}{\partial \lambda_q^d} = p_j \qquad (3.3.23)$$

$j=1, \ldots n$

$q=1, \ldots m$

Equations (3.3.23), (3.3.21) and (3.3.9) then give (3.3.19).

Equation (3.3.20) is obtained by writing the coefficients of the residual characteristic equation as the sums of the products of the unassigned eigenvalues, with appropriate signs, and finding the total differential of each of these expressions.

3.3.8 Comment.

The expression (3.3.19) for the partial derivatives of the feedback gains and of the coefficients of the residual characteristic equation is convenient to use, as, apart from the diagonal matrix ($\lambda_q^d I - \Lambda$), the two coefficient matrices are already available from (3.3.9). The k_j, j=1, .. m, were obtained in the solution of (3.3.9), together with the a_l, l=0, .. (n-m-1), which yield the λ_p^d, p=(m+1), .. n, required in (3.3.20).

The only part of (3.3.19) which has to be altered to yield the sensitivities with respect to a different assigned eigenvalue is the diagonal matrix ($\lambda_q^d I - \Lambda$) on the left hand side.

The expressions provide guidance in the directions in which to change the assigned eigenvalues, and in the choice as to which assigned eigenvalues have the greatest effect on the unassigned eigenvalues it is desired to influence.

It is important that the steps chosen in the changes introduced in the assigned eigenvalues are not so large that the derivatives fail to provide a reasonably valid prediction. The step size can be reduced if necessary, after initial trials.

The sensitivity expression also gives guidance in the adjustment of the assigned eigenvalues to effect a reduction in some of the feedback gains, where it was found that these were undesirably high. The effect of any proposed change on the unassigned eigenvalues can be seen at the same time.

3.3.9 Numerical Example.

We now apply the sensitivity expressions (3.3.19) and (3.3.20) to the example considered in 3.3.6. Here we have: $\lambda_1^d = -4$.

For changes in λ_1^d, (3.3.19) becomes:

$$\begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 2 & -2 \\ -1 & 5 & -\frac{6}{5} \\ -2 & 10 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{\partial k_1}{\partial \lambda_1^d} \\ \frac{\partial k_2}{\partial \lambda_1^d} \\ \frac{\partial a_o}{\partial \lambda_1^d} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -1 & 5 \\ -2 & 10 \end{bmatrix} \begin{bmatrix} 4 \\ \frac{14}{19} \end{bmatrix}$$

from which: $\frac{\partial a_o}{\partial \lambda_1^d} = \frac{60}{19^2}$

For changes in λ_2^d, we have: $\lambda_2^d = -5$, and (3.3.19) is similar to the foregoing expression, except for the diagonal matrix on the left hand side, which is:

$$\begin{bmatrix} -4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

from which: $$\frac{\partial a_o}{\partial \lambda_2^d} = \frac{20}{19^2}$$

In this case, equations (3.3.20) give:

$$\Delta\lambda_3^d = - \Delta a_o$$

We conclude that the unassigned eigenvalue is three times more sensitive to changes in λ_1^d than to changes in λ_2^d. Also, the signs are such that, if we wish to move the unassigned eigenvalue to the left, along the real axis, this can only be achieved by moving either or both of the assigned eigenvalues to the right. These conclusions are, of course, only valid over a limited range.

If we now reassign the eigenvalues as:
$\lambda_1^d = -3$, and $\lambda_2^d = -4$, we obtain:

$$p_1 = -1$$
$$p_2 = \frac{2}{5}$$
$$p_3 = 0$$

and (3.3.9) becomes:

$$\begin{bmatrix} 0 & 2 & -1 \\ -1 & 5 & \frac{2}{5} \\ -2 & 10 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ a_o \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{4}{5} \\ 0 \end{bmatrix}$$

from which:

$$k_1 = \frac{5}{2}, \quad k_2 = \frac{1}{2}, \quad a_o = 2.$$

Hence, $\lambda_3^d = -2$.

It is interesting to see how the prediction from the sensitivity analysis compares with the exact result found above. We have:

$$\Delta\lambda_3^d = \frac{\partial\lambda_3^d}{\partial\lambda_1^d}\Delta\lambda_1^d + \frac{\partial\lambda_3^d}{\partial\lambda_2^d}\Delta\lambda_2^d$$

$$= -\frac{60}{19^2}\times 1 - \frac{20}{19^2}\times 1 = -\frac{80}{19^2}$$

and the new λ_3^d is given by:

$$\lambda_3^d = -\frac{33}{19} - \frac{80}{19^2} = -1.96.$$

It is seen that, in this case, the agreement is very good.

3.4 The Method of Fallside and Seraji Applied to the Restricted Measurement Access Case.

This method will be discussed by application to the numerical example 3.3.6.

In this procedure, it is first necessary to transform the state vector so that the two outputs are the first two elements of the new state vector. This is achieved by making the transformation:

$$z = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} = Tx, \quad \text{where:}$$

$$T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and:

$$T^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad (3.4.1)$$

so that the A, b and C matrices in the new co-ordinates become:

$$A_o = \begin{bmatrix} 1 & -1 & 2 \\ -10 & 4 & -10 \\ -11 & 5 & -11 \end{bmatrix}, \qquad b_o = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$C_o = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{3.4.2}$$

We next find the vector g(s) of the numerators of the transfer functions from the scalar input to the states. This is given by:

$$\frac{g(s)}{F(s)} = \frac{1}{F(s)} \begin{bmatrix} g_1(s) & g_2(s) & g_3(s) \end{bmatrix}^T$$

$$= (sI - A_o)^{-1} b_o \tag{3.4.3}$$

where F(s) is the open-loop characteristic polynomial. Hence,

$$\frac{g(s)}{F(s)} = \frac{1}{s^3+6s^2+11s+6} \begin{bmatrix} (s^2+8s+7) \\ (s^2-10s+1) \\ (s^2-11s+6) \end{bmatrix} \tag{3.4.4}$$

The third element of the vector in (3.4.4) will not be used, as feedback is taken only from z_1 and z_2. Then, with feedback $-k^Ty = -k^TCx = -k_1z_1-k_2z_2$, we use the relationship:

$$H(s) = F(s) + k_1g_1(s) + k_2g_2(s),$$

where H(s) and F(s) are the closed-loop and open-loop characteristic polynomial, respectively. Let:

$H(s) = s^3 + a_2's^2 + a_1's + a_0'$, where the a_j' are fixed by the choice of closed-loop poles. Then,

$$s^3+a_2's^2+a_1's+a_0' = s^3+6s^2+11s+6 + k_1(s^2+8s+7) + k_2(s^2-10s+1).$$

Equating coefficients,

$$\begin{aligned} a_2' &= 6 + k_1 + k_2 \\ a_1' &= 11 + 8k_1 - 10k_2 \\ a_0' &= 6 + 7k_1 + k_2 \end{aligned} \qquad (3.4.5)$$

from which we obtain:

$$3a_0' - a_1' - 13a_2' + 71 = 0 \qquad (3.4.6)$$

Equation (3.4.6) represents a constraint on the coefficients of the new characteristic polynomial. If this polynomial has as roots the assigned poles -4 and -5, and the unassigned pole $-\rho$, it may be written:

$$s^3 + (9+\rho)s^2 + (20+9\rho)s + 20\rho \qquad (3.4.7)$$

Substituting from (3.4.7) into (3.4.6) gives:

$\rho = \frac{33}{19}$, so that the unassigned pole is at $-\frac{33}{19}$.

We then find:

$$a_2' = (9+\rho) = \frac{204}{19}$$

$$a_1' = (20+9\rho) = \frac{677}{19}$$

$$a_0' = 20\rho = \frac{660}{19}$$

and, from (3.4.5), we obtain:

$$k_1 = 4, \quad k_2 = \frac{14}{19}.$$

These results agree with those obtained in 3.3.6.

3.4.1 Comment.

Although the method of Fallside and Seraji is fairly simple to apply in the case of systems of low order, as in this example, the following points should be noted.

(a) There is the initial transformation of the state vector so that the first m elements of the new state vector are the m system outputs.

(b) The formation of the vector g(s) would, in larger systems, involve the use of the Leverrier algorithm, or some similar process suitable for use with a computer.

(c) There are, in general, (n-m) constraint equations corresponding to (3.4.6), in the coefficients of the new characteristic polynomial. These equations involve sums of products of the assigned and unassigned poles, and are not easy to interpret. In the example considered, there was a single unassigned pole, and this fact simplified the solution.

3.5 Multi-input Systems.

For a controllable, observable system with restricted measurement access, and more than one input, the result given in 3.3.3 for the single-input case with distinct eigenvalues can be used as follows.

If the system has multiple eigenvalues, almost arbitrary feedback can be applied initially to give distinct eigenvalues, as shown by Davison and Wang [D2] .

If the system is controllable through a single input, this input can be used for the feedbacks. Otherwise, a vector α can be found such that $(A, B\alpha)$ is controllable. Wonham [W1] has shown that such a vector always exists for a controllable system in which the A matrix is non-derogatory. In fact, there is considerable freedom in choosing α, which can be chosen in such a way as to provide desired relative magnitudes of control signals applied to the various inputs. This can be done, of course, even if the system is controllable through a single input. α can be chosen initially from this point of view, and modified as necessary to achieve controllability.

This corresponds to the 'relative tightness' of control referred to by Fallside and Seraji [F1] .

A procedure has been described recently by Paraskevopoulos and Tzafestas [P5] . This is as follows:

Suppose, in the usual notation, that $y = Cx$, where y is an m-vector, $m < n$, and rank $C = m$. We seek a non-singular $m \times m$ transformation S such that:

$$SC = [\hat{C} \vdots\ 0]$$

where $\hat{C}$ is non-singular, and the remaining elements of SC are zero. It is claimed that such a transformation always exists, but this claim is unjustified, since $\hat{C}$ will be singular if the first m columns of C form a singular matrix. It would be possible to obtain a non-singular $\hat{C}$ by re-ordering the elements of the state vector, but it is not, in general, possible to obtain the required zero elements elsewhere in SC. The method proceeds as follows:

(1) Solve the eigenvalue problem using state vector feedback, i.e., the input $u = Fx$.

(2) Partition F as $F = [\hat{F} \vdots \hat{\hat{F}}]$, where $\hat{F}$ has m columns. Then the gain matrix P of the output feedback Py is obtained from:

$$P = \hat{F}[\hat{C}]^{-1}$$

(3) The arbitrary elements of $\hat{F}$ are then restricted

by the equations involved in $\hat{\hat{F}} = 0$.

Even if the above steps can be carried out in special cases, it is claimed only to permit m closed-loop poles to be pre-specified. Nothing can be said about the remaining (n-m) closed-loop poles. It is seen that the method is based on invalid assumptions and, in any case, it is of little use to assign some of the closed-loop poles, if the remainder may move to unacceptable locations.

In a numerical example, Ex. 5 of [P5], a system is considered which has two inputs and two outputs, and which is of order three. It is found that two of the eigenvalues can be changed to desired values, whilst the third eigenvalue remains unchanged. The reason given is that two eigenvalues only can be controlled because the rank of the C matrix is two. This reason is incorrect. In the particular example chosen, the third eigenvalue is unobservable, and so could not be affected by output feedback.

It is interesting to note, as has been pointed out by E. J. Davison (in a verbal communication), that a third order system with two independent inputs and two independent outputs, which is controllable and observable, can always have its eigenvalues made equal to, or arbitrarily close to, any prescribed set of values, by the use of constant feedback between output and input. In the usual case, therefore, one would expect to be able to assign all three eigenvalues in the example given in [P5] .

CHAPTER 4.

POLE AND ZERO ASSIGNMENT BY STATE VECTOR FEEDBACK.

4.1 Introduction.

Although the problem of designing constant state vector feedback to achieve desired closed-loop pole assignment is now well understood, the question of designing such feedback to achieve desired transfer function zeros, as well as poles, is currently the subject of investigation. The problem of zero assignment is clearly very much more difficult than pole assignment.

Suppose it is desired to assign all poles and the zeros in q scalar transfer functions by state vector feedback applied to r inputs. To obtain a rough idea of the problem, it may be assumed that each transfer function has the maximum of (n-1) zeros, so that the total number of (zeros + poles) to be assigned is q(n-1) + n. If state vector feedback is applied to the r inputs, the number of available feedback gain parameters is nr, so that, to provide sufficient parameters for the zero-pole assignment, it is required that:

$$nr \geq q(n-1) + n$$

or $$q \leq \frac{n}{n-1}(r-1) \qquad (4.1.1)$$

For a system of appreciable order, $\frac{n}{n-1} \simeq 1$, so that the number of scalar transfer functions is limited to about (r-1). Whilst there will be exceptional cases in which it is possible to achieve assignment of all zeros and poles in a greater number of scalar transfer functions than this argument suggests, it is clear that, in general, there are enough feedback parameters to permit assignment of zeros and poles in only a very limited number of transfer functions. This fact was pointed out, in a slightly different way, by Chen [C3].

It is apparent from the form of (4.1.1) that systems of low order will give more favourable results than high order systems. Thus, the illustration of proposed techniques by application to, say, second or third order systems, which is quite common in the literature, does not give a true picture of what can be expected in general.

Simon and Mitter [S4] have considered the synthesis of transfer function matrices with invariant zeros, by making use of the property of invariance of transfer function zeros in the presence of feedback to the input concerned. This approach, however, is limited to systems in which each input controls only a subset of the system eigenvalues, and the subsets are disjoint. The field of application thus is not very great.

Rosenbrock [R5] has investigated the allocation of poles and zeros in the McMillan form of the transfer function matrix, but his method is based on the choice of the system output matrix C, a choice which is not usually available to the designer. This approach, also, seems to have limited application.

Loscutoff, Schenz and Beyer [L3] have studied the effects of invariant zeros in connection with closed-loop pole assignment. They have provided means for identifying invariant zeros, and have proposed to overcome their undesirable effects, where these arise, by cancellation with poles, when they are in the left half-plane.

Power [P4] has investigated the effect of state variable feedback on the numerators of transfer functions, but his approach is purely analytical, and makes no contribution to the problem of zero assignment.

An interesting approach was used by Chen [C3], yielding a sequential zero-pole placement technique. This provides a means for checking zero assignability, and permits more than one input-output transfer function to have complete pole-zero assignment in some cases, but gives little guidance as to how to proceed in those cases in which there is no solution.

Furthermore, although the method is general in principle, the solution is found as a system of linear algebraic equations only in the case of a 2-input system. When the number of inputs is greater than 2, the equations become non-linear, and hence difficult to solve.

Fallside and Seraji [F3] have studied the problem of pole and zero assignment using unity-rank feedback. By this it is meant that a linear functional of the state vector or of the output vector is formed, giving a single scalar variable. This scalar variable is then applied to the system inputs through different constant gains. It has been shown that unity-rank feedback is very restrictive and that, for an nth-order system, with full state-vector feedback applied to r inputs, there are only (r-1) degrees of freedom available to meet specifications other than pole assignment. Thus, for a system with two inputs, only one zero could be assigned, together with all the poles. If more zeros are required to be assigned, this can only be achieved at the expense of pole assignment. Fallside and Patel [F4] have described a procedure for achieving an approximation to a desired pole-zero pattern. However, there is no guarantee that a satisfactory approximation can be achieved in the general case.

Wang and Desoer [W4] have given a complete solution to the problem of 'exact model matching', by which is meant finding a state feedback law for a given system which makes the overall system transfer function exactly equal to a given transfer function. In this context, 'transfer function' means transfer function matrix. Their procedure is in two stages. First, by using an $r \times r$ gain matrix between the external inputs and the system inputs, an $r \times n$ state feedback gain matrix, and a co-ordinate transformation, the system is put in a special canonical form, in which the A matrix is quasi-diagonal, with each diagonal block of companion form, but with all eigenvalues zero. In the second step, a further $r \times r$ input gain matrix and $r \times n$ state feedback gain matrix are found, so as to give the desired transfer function matching, where this is possible. The required laws are given as the solution of a matrix equation, the conditions for solvability of which give the conditions for the existence of a solution of the problem.

The method is of considerable interest, but it is subject to limitations similar to those given in (4.1.1). The introduction of the additional design freedom represented by the $r \times r$ input gain matrix changes the condition on q to:

$$q \leq \frac{n}{n-1}(r - 1 + \frac{r^2}{n}) \qquad (4.1.2)$$

For a system of low order, the introduction of the input gain matrix may permit an appreciable increase in q. For a high order system with a relatively small number of inputs, the increase is not likely to be very great.

The method of Wang and Desoer does not give any clear guidance as to how to proceed when the conditions for solvability are not satisfied, and the introduction of the various gain matrices and the co-ordinate transformations tend to make the problem rather obscure, in such cases.

Moore and Silverman [M15] have approached the exact model matching problem in a different way, without using initial co-ordinate transformations. They have also considered 'dynamic state feedback', by which is meant feedback obtained from the original system augmented by the addition of a number of integrators, and have given a set of necessary and sufficient conditions for one system to be transfer function equivalent via such dynamic state feedback to a specified model system.

In this thesis, the pole-zero assignment problem considered is that of finding state vector feedback to provide completely specified poles and zeros in a single scalar input-output transfer function. It is clear from the foregoing discussion that, in general, the possibilities of zero assignment in high order systems with a relatively small number of inputs are rather limited. There are many practical cases in which the number of inputs available for manipulation is small, and where one input-output transfer function is of major importance, whilst other input-output transfer functions are of secondary importance. The most unfavourable case of this sort arises where there are just two inputs, the desired transfer function between one of these inputs and some output being specified. The second input is available for the purpose of applying feedback so as to permit the numerator of the transfer function from the first input to be changed. It is this case which is considered.

Two methods are described, the first of which permits the results of 'modal control' theory to be applied directly to the problem of assigning transfer function zeros. The second method is based on the transformation of the system to the companion form and the formation of a set of linear equations from which the coefficients of the characteristic polynomial corresponding to the desired transfer function numerator coefficients can be found. State vector feedback can then be calculated to provide this set of characteristic polynomial coefficients when applied to the second input. In both methods, the second stage is to find the state vector feedback which, when applied to the first input, will give the desired closed-loop poles. This feedback does not affect the numerator of the transfer function, which was established in the first stage.

It is well known that the numerator of a transfer function is unaffected by state vector feedback applied to the input to which the transfer function applies. However, a simple proof of this is given in the next section, as the result is used in the pole-zero assignment procedures which follow.

4.2 Proof of Invariance of Transfer Function Numerator.

Given a linear system described by the equation:

$\dot{x} = Ax + Bu$, where $B = [b_1 \vdots \ldots \vdots b_r]$,

let state vector feedback be applied to the first input, so that the equation becomes:

$$\dot{x} = (A + b_1 k^T)x + Bu \qquad (4.2.1)$$

The input-state transfer functions are obtained by taking the Laplace Transform of (4.2.1), with zero initial conditions, giving:

$$(sI - A - b_1k^T)\bar{x} = B\bar{u} \qquad (4.2.2)$$

The transfer function from the first input is obtained from:

$$(sI - A - b_1k^T)\bar{x} = b_1\bar{u}_1 \qquad (4.2.3)$$

Each element of $\bar{x}$ may be found by using Cramer's rule, as the ratio of two determinants. The denominator determinant is $\det(sI - A - b_1k^T)$, and the numerator determinant corresponding to $\bar{x}_i$ is the same determinant, but with the ith column replaced by b_1. It is obvious that, in this numerator determinant, all the remaining elements of k^T can be removed by subtracting suitable multiples of the ith column from all the other columns, without changing the value of the determinant. Hence, the transfer function numerator is invariant with respect to k^T.

It is clear that the removal of the remaining elements of k^T would not have been possible if the transfer function considered had related to an input other than that to which the feedback was applied.

Use of both of these results is made in the pole-zero assignment techniques to be described.

4.3 Pole and Zero Assignment by State Vector Feedback.

4.3.1 Introduction.

A procedure is now described which enables specified poles and zeros of a scalar transfer function of a controllable, observable, linear system to be obtained by using state vector feedback to two inputs. The number of zeros is equal to the number of zeros in the transfer function, before feedback is applied, from one input or the other to the output concerned, whichever is the greater. Those zeros which can be changed, and those which cannot, are identified. The former can be made equal to, or arbitrarily close to, any assigned values, and the poles can be assigned arbitrarily.

This procedure is described in [M6] and [M12].

4.3.2 System Description.

A linear time-invariant system is described by the equations:

$$\dot{x} = Ax + Bu \qquad (4.3.1)$$

$$y = c^T x \qquad (4.3.2)$$

where x is an $n \times 1$ state vector, $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$ is a 2×1 input vector, and y is a scalar output. $B = \begin{bmatrix} b_1 \vdots b_2 \end{bmatrix}$, where the n-vectors b_1 and b_2 are linearly independent, and the system is completely controllable through b_2 alone. This latter condition can always be met by the use, if necessary, of suitable feedback [D4] , since (A,B) is controllable. c^T is a constant measurement vector.

4.3.3 Problem Statement.

The problem is to find the feedback vectors k_1^T and k_2^T such that the system:

$$\dot{x} = (A + \left[b_1 \vdots b_2\right] \begin{bmatrix} k_1^T \\ k_2^T \end{bmatrix})x + Bu \; ; \quad y = c^T x, \qquad (4.3.3)$$

has a transfer function between y and u_1 with, as far as possible, specified poles and zeros.

4.3.4 Procedure.

Two sequences of scalars, S_1 and S_2, are formed:

$$S_1 = c^T b_1, \; c^T A b_1, \; c^T A^2 b_1, \; \ldots \; , \; c^T A^{p-1} b_1 \quad \text{and}$$

$$S_2 = c^T b_2, \; c^T A b_2, \; c^T A^2 b_2, \; \ldots \; , \; c^T A^{q-1} b_2$$

where, in each case, the sequence terminates at the first non-zero term. The method to be described requires that $q \geq p$. If this condition is not satisfied, a proportion h of the input u_1 is added to u_2, so that b_1 becomes $(b_1 + hb_2)$. This will make $q = p$. It will, from now on, be assumed that this has been done, if necessary, and that b_1 has been changed accordingly. The procedure is in two stages.

Stage 1.

k_2^T is first determined so as to locate the zeros. Let k_1^T be a zero vector at this stage. Using a result obtained by Brockett [B2], the zeros of the transfer function relating y to u_1 are eigenvalues of the matrix:

$$\left\{ I - \frac{b_1 c^T (A + b_2 k_2^T)^{p-1}}{c^T (A + b_2 k_2^T)^{p-1} b_1} \right\} (A + b_2 k_2^T) \qquad (4.3.4)$$

It is proved in 4.3.8 that, since $q \geq p$,

$$c^T (A + b_2 k_2^T)^{p-1} = c^T A^{p-1} \qquad (4.3.5)$$

The matrix (4.3.4) may thus be written:

$$A_o + b_o k_2^T \qquad (4.3.6)$$

where:

$$A_o = \left\{ I - \frac{b_1 c^T A^{p-1}}{c^T A^{p-1} b_1} \right\} A$$

$$b_o = \left\{ I - \frac{b_1 c^T A^{p-1}}{c^T A^{p-1} b_1} \right\} b_2 \qquad (4.3.7)$$

The pair (A_o, b_o) is checked for controllability, using any method that permits the identification of the uncontrollable eigenvalues. Kalman's canonical decomposition method is suitable, or Gilbert's method, extended, if necessary, to the case of multiple eigenvalues. It is proved in 4.3.9 that A_o has the eigenvalue 0 of multiplicity at least p, and it is further proved in 4.3.10 that the eigenvalue 0 of multiplicity p is uncontrollable through b_o. The remaining (n-p) eigenvalues of (4.3.6) are the zeros of the transfer function. Of these, any that are uncontrollable through b_o cannot be changed, whilst all the rest may be assigned arbitrarily by using modal control theory [M1] ,[R2] , to determine k_2^T. The eigenvalue 0 of multiplicity p has no physical significance, and arises only because the degree of the numerator of the transfer function is (n-p).

Stage 2.

The system poles will have been changed by the application of feedback k_2^T, and k_1^T is now determined so as to locate the poles as required. It is first necessary to check for controllability the pair:

$$((A + b_2k_2^T), b_1) \qquad (4.3.8)$$

If this test is satisfied, k_1^T may be found, by again using modal control theory [M1] , [R2] , to move the poles to any desired locations. As was shown in 4.2, the application of the feedback k_1^T will have no effect on the zeros which were established in Stage 1, because this feedback is applied to the input from which the transfer function is taken.

If the test for controllability of (4.3.8) fails, controllability can be achieved by making small adjustments to k_2^T, which means that the zeros in Stage 1 can now be made only arbitrarily close to assigned values, and not necessarily equal to them. This statement is justified by the following theorem.

4.3.5 Theorem.

If the pair (A,b_2) is controllable, and $b_1 \neq 0$, a vector k^T can be chosen, with elements arbitrarily close to those of a given vector k_2^T, such that $((A + b_2k^T),b_1)$ is controllable.

Proof.

Applying to this case a lemma of Heymann [H1], there exists a vector k^T such that $((A + b_2k^T),b_1)$ is controllable. Let k^T be so chosen. Now change the first element of k^T, noting that there is a finite number of values of the change which give uncontrollability. We may. therefore, choose a value which makes this element either equal to or arbitrarily close to the first element of k_2^T, whilst preserving controllability. Repetition of this process for each element of k^T in turn, retaining the changed value at each step, completes the proof.

<u>4.3.6 Coefficient of s^{n-p} in the Numerator Polynomial.</u>

The coefficient of the highest power of s, s^{n-p}, in the transfer function numerator polynomial, is $c^TA^{p-1}b_1$, and this is independent of k_1^T and k_2^T, so that this coefficient cannot be assigned by the state vector feedback.

<u>Proof.</u>

Let the characteristic polynomial of $(A + b_1k_1^T + b_2k_2^T)$ be:

$$s^n + d_1s^{n-1} + \dots + d_{n-1}s + d_n \qquad (4.3.9)$$

Let the adjoint of this matrix be:

$$D(s) = s^{n-1}D_o + s_{n-2}D_1 + \dots + sD_{n-2} + D_{n-1} \qquad (4.3.10)$$

where the D_j are $n \times n$ constant matrices.

The matrix coefficient of s^{n-p} in the transfer function numerator polynomial is then:

$$c^TD_{p-1}b_1 \qquad (4.3.11)$$

From the Faddeev-Leverrier algorithm [F5][Z1] ,

$$D_o = I$$

$$D_1 = D_o(A + b_1k_1^T + b_2k_2^T) + d_1I$$

$$D_2 = D_1(A + b_1k_1^T + b_2k_2^T) + d_2I$$

$$= (A + b_1k_1^T + b_2k_2^T)^2 + d_1(A + b_1k_1^T + b_2k_2^T) + d_2I$$

and so on.

Continuing in this way gives:

$$D_{p-1} = (A + b_1k_1^T + b_2k_2^T)^{p-1} + d_1(A + b_1k_1^T + b_2k_2^T)^{p-2} + \dots$$
$$\dots + d_{p-1}I \qquad (4.3.12)$$

It then follows from (4.3.12), and the rules of formation of the sequences S_1 and S_2 that the coefficient of s^{n-p}, $c^TD_{p-1}b_1$, is $c^TA^{p-1}b_1$. This completes the proof.

4.3.7 Numerical Example.

The procedure is illustrated by a simple example, in which:

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 5 & 0 & -3 \end{bmatrix}; \quad b_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \end{bmatrix}; \quad b_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

$$c^T = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}$$

The transfer function between y and u_1 is to have zeros at -2 and -4, and poles at -1.5, -2.5, -3.5 and -4.5.

Forming the sequences S_1 and S_2 reveals that $p = q = 2$, so that there will be two zeros.

Applying (4.3.7) to this case gives:

$$A_o = \begin{bmatrix} -2 & -10 & -3 & 27 \\ -1 & -2 & -1 & 9 \\ 0 & -3 & -1 & 0 \\ -2 & -3 & -2 & 15 \end{bmatrix} ; \quad b_o = \begin{bmatrix} -11 \\ -4 \\ 0 \\ -7 \end{bmatrix}$$

A_o has the eigenvalues 0, 0, 0.101 and 9.899, of which only the last two are controllable through b_o. These are the zeros of the transfer function without feedback. Using modal control theory to move these eigenvalues to -2 and -4 gives:

$$k_2^T = \begin{bmatrix} -1.5 & 8.125 & 0 & 0 \end{bmatrix}$$, and $(A + b_2k_2^T)$ becomes:

$$\begin{bmatrix} -5 & 34.5 & 0 & 0 \\ -1.5 & 10.125 & 0 & 0 \\ 0 & -3 & -1 & 0 \\ -4.5 & 29.375 & 0 & -3 \end{bmatrix}$$

This matrix has the eigenvalues 4.8952, 0.2298, -1 and -3, all of which are controllable through b_1. Again using modal control theory to move these eigenvalues to -1.5, -2.5, -3.5 and -4.5 gives:

$$k_1^T = \begin{bmatrix} -6.146 & 6.433 & -2.188 & -0.560 \end{bmatrix}$$

This completes the procedure.

4.3.8 Proof of (4.3.5).

By considering vectors of the form:

$$c^T(A + b_2k_2^T) = c^TA + c^Tb_2k_2^T = c^TA, \text{ if } c^Tb_2 = 0,$$

$$c^T(A + b_2k_2^T)^2 = c^TA(A + b_2k_2^T) = c^TA^2 + c^TAb_2k_2^T = c^TA^2,$$

if $c^T b_2 = 0$ and $c^T A b_2 = 0$, and so on, it is easily seen that, since:

$$c^T A^i b_2 = 0, \text{ for } i = 0, \ldots (q-2),$$

$$c^T (A + b_2 k_2^T)^i = c^T A^i, \text{ for } i = 0, \ldots (q-1). \quad (4.3.13)$$

Equation (4.3.5) follows, since $q \geq p$.

4.3.9 Proof that A_o has the Eigenvalue 0 of Multiplicity At Least p.

The eigenvalues of A_o are the roots of the characteristic equation:

$$\left| sI - A + \frac{b_1 c^T A^p}{c^T A^{p-1} b_1} \right| = 0 \quad (4.3.14)$$

which may be written:

$$\left| sI - A \right| \left| I + \frac{(sI - A)^{-1} b_1 c^T A^p}{c^T A^{p-1} b_1} \right| = 0.$$

Using the result, proved in [M1], that $\left| I + fg^T \right| = (1 + g^T f)$, where f and g^T are column and row vectors, respectively, the equation becomes:

$$\left| sI - A \right| \left\{ 1 + \frac{c^T A^p (sI - A)^{-1} b_1}{c^T A^{p-1} b_1} \right\} = 0.$$

Setting $s = 0$ makes the expression in brackets zero, so that 0 is a root. Differentiating the expression in brackets with respect to s, and setting $s = 0$, makes this expression zero, because $c^T A^{p-2} b_1 = 0$. Repeating this operation (p-1) times, and remembering that $c^T A^i b_1 = 0$, for $i = 0, \ldots (p-2)$, shows that the expression in brackets has the root 0 of multiplicity p, so that A_o has the eigenvalue 0 of multiplicity at least p.

4.3.10 Proof that, in (4.3.6), the Eigenvalue 0 of Multiplicity p is Uncontrollable Through b_o.

The matrix (4.3.4) is of the same form as A_o, except that A is replaced by $(A + b_2 k_2^T)$. It follows from (4.3.13), and from the rules of formation of the sequences S_1 and S_2, that the values of p and q are unchanged if, in these sequences, A is replaced by $(A + b_2 k_2^T)$. Thus, by the same argument as in 4.3.9, the matrix (4.3.4), and hence (4.3.6), has the eigenvalue 0 of multiplicity at least p, for all k_2^T. This statement implies that the eigenvalue 0 of multiplicity p is uncontrollable through b_o.

4.3.11 Conclusion.

The procedure described permits the identification of those zeros which can be changed, and those which cannot. All the zeros that can be changed can be made equal to, or arbitrarily close to, any assigned values, and the poles can be assigned arbitrarily, by using the established techniques of modal control.

It follows from these results, and from 4.3.6, that the transfer function is completely determined on completion of the procedure described.

4.4 Transfer Function Synthesis by State Vector Feedback.

4.4.1 Introduction.

A procedure is described for the design of state vector feedback for a time-invariant linear system to give a desired scalar input-output transfer function. Any constraints on the design are revealed, and means are provided for checking all other transfer functions. The problem considered is the determination of the state vector feedbacks needed to give a desired scalar transfer function between one input and one output of a system which has two inputs. The number of inputs available often is limited in practical cases, and so the system considered may be regarded as representing the most unfavourable multi-input case. The procedure gives guidance at each stage on any constraints on the design. Means are provided for monitoring all other input/output and input/state transfer functions during the design process.

4.4.2 System Description.

A linear system is described by the equations:

$$\dot{\underline{x}} = A\underline{x} + B\underline{u} \qquad (4.4.1)$$

$$\underline{y} = C\underline{x} \qquad (4.4.2)$$

where $\underline{x}$ is an $n \times 1$ state vector,

$$\underline{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \text{ is a } 2 \times 1 \text{ input vector,}$$

$$\underline{y} = \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ y_m \end{bmatrix} \text{ is an } m \times 1 \text{ output vector,}$$

$$B = \left[\underline{b}_1 \vdots \underline{b}_2\right] \qquad (4.4.3)$$

and

$$C = \begin{bmatrix} \underline{c}_1^T \\ \cdot \\ \cdot \\ \underline{c}_m^T \end{bmatrix} \qquad (4.4.4)$$

The system is observable, and is completely controllable through $\underline{b}_2$ alone. The n-vectors $\underline{b}_1$ and $\underline{b}_2$ are linearly independent.

Note.

If (A,B) is controllable, $(A+BK,\underline{b}_2)$ can be made to be controllable by the use of suitable initial feedback K [D4] .

4.4.3 Problem Statement.

The problem is to find the feedback vectors $\underline{k}_1^T$ and $\underline{k}_2^T$ such that the system:

$$\dot{\underline{x}} = (A + \left[\underline{b}_1 \vdots \underline{b}_2\right]\begin{bmatrix} \underline{k}_1^T \\ \underline{k}_2^T \end{bmatrix})\underline{x} + B\underline{u}, \quad \underline{y} = C\underline{x}, \qquad (4.4.5)$$

has a transfer function relating $y_1(s)$ to $u_1(s)$, given by:

$$\frac{p_{n-1}s^{n-1} + p_{n-2}s^{n-2} + \ldots + p_1 s + p_o}{s^n + a'_{n-1}s^{n-1} + \quad \ldots \quad + a'_1 s + a'_o} \qquad (4.4.6)$$

in which the p_j, $j = 0, \ldots, (n-2)$, and the a'_i, $i = 0, \ldots, (n-1)$, are to be given preassigned values, as far as possible. The value of p_{n-1} is $c_1^T b_1$, and cannot be assigned.

4.4.4 Preliminary Results.

A system of the type represented in (4.4.1), (4.4.2) has a transfer function relating an input corresponding to a general column $\underline{b}$ of B to an output corresponding to a general row $\underline{c}^T$ of C given by:

$$\underline{c}^T(sI - A)^{-1}\underline{b} \qquad (4.4.7)$$

where $$(sI - A)^{-1} = \frac{\text{adj}(sI - A)}{\det(sI - A)} \qquad (4.4.8)$$

Now, $$\text{adj}(sI - A) = Is^{n-1} + G_{n-2}s^{n-2} + \ldots + G_1 s + G_o \qquad (4.4.9)$$

where the G_j can always be computed in a routine manner, e.g., by the Faddeev-Leverrier algorithm, but, in the case in which the A matrix is in the companion form, A_c, the G_j have a particularly simple

form, so that they can be written down. The formulae for writing these matrices are given in 4.4.6.
The matrix A_c for a fifth order system is of the form:

$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -a_o & -a_1 & -a_2 & -a_3 & -a_4 \end{bmatrix} \qquad (4.4.10)$$

and the G_j matrices are given in full for this case in equations (4.4.11).

$$G_4 = I$$

$$G_3 = \begin{bmatrix} a_4 & 1 & 0 & 0 & 0 \\ 0 & a_4 & 1 & 0 & 0 \\ 0 & 0 & a_4 & 1 & 0 \\ 0 & 0 & 0 & a_4 & 1 \\ -a_o & -a_1 & -a_2 & -a_3 & 0 \end{bmatrix} ; \quad G_o = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & 1 \\ -a_o & 0 & 0 & 0 & 0 \\ 0 & -a_o & 0 & 0 & 0 \\ 0 & 0 & -a_o & 0 & 0 \\ 0 & 0 & 0 & -a_o & 0 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} a_3 & a_4 & 1 & 0 & 0 \\ 0 & a_3 & a_4 & 1 & 0 \\ 0 & 0 & a_3 & a_4 & 1 \\ -a_o & -a_1 & -a_2 & 0 & 0 \\ 0 & -a_o & -a_1 & -a_2 & 0 \end{bmatrix} \qquad (4.4.11)$$

$$G_1 = \begin{bmatrix} a_2 & a_3 & a_4 & 1 & 0 \\ 0 & a_2 & a_3 & a_4 & 1 \\ -a_o & -a_1 & 0 & 0 & 0 \\ 0 & -a_o & -a_1 & 0 & 0 \\ 0 & 0 & -a_o & -a_1 & 0 \end{bmatrix}$$

From (4.4.7) and (4.4.9), the coefficient p_q of s^q in the transfer function numerator polynomial is given by:

$$p_q = \underline{c}^T G_q \underline{b} \qquad (4.4.12)$$

This relationship enables all the coefficients to be found for all the transfer functions, by the appropriate choice of $\underline{c}^T$, $\underline{b}$ and G_q. Where the A matrix is in the companion form, it is clear that the G_j are functions of the a_i, the coefficients of the characteristic polynomial of the A matrix. It is then possible to formulate a set of equations from which the a_i can be found so as to give desired coefficients p_q in a given transfer function numerator polynomial. These equations are set out in full in (4.4.13), for a system of fifth order, where:

$$\underline{c}^T = \begin{bmatrix} c_1 & c_2 & \cdots & c_5 \end{bmatrix}$$

and $$\underline{b} = \begin{bmatrix} b_1 & b_2 & \cdots & b_5 \end{bmatrix}^T.$$

In 4.4.7, formulae are given which enable the equations to be written down for a system of any order.

$$\begin{bmatrix} (c_1b_1+c_2b_2+c_3b_3+c_4b_4) & -c_5b_4 & -c_5b_3 & -c_5b_2 & -c_5b_1 \\ (c_1b_2+c_2b_3+c_3b_4) & (c_1b_1+c_2b_2+c_3b_3) & -(c_4b_3+c_5b_4) & -(c_4b_2+c_5b_3) & -(c_4b_1+c_5b_2) \\ (c_1b_3+c_2b_4) & (c_1b_2+c_2b_3) & (c_1b_1+c_2b_2) & -(c_3b_2+c_4b_3+c_5b_4) & -(c_3b_1+c_4b_2+c_5b_3) \\ c_1b_4 & c_1b_3 & c_1b_2 & c_1b_1 & -(c_2b_1+c_3b_2+c_4b_3+c_5b_4) \end{bmatrix} \begin{bmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \\ a_o \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_3 \\ p_2 \\ p_1 \\ p_o \end{bmatrix} - \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ 0 & c_1 & c_2 & c_3 \\ 0 & 0 & c_1 & c_2 \\ 0 & 0 & 0 & c_1 \end{bmatrix} \begin{bmatrix} b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} \qquad (4.4.13)$$

4.4.5 Procedure.

The design proceeds in two stages. In the first stage, the feedback vector $\underline{k}_2^T$ is found, so as to give the desired numerator polynomial to the transfer function:

$$\underline{c}_1^T(sI - A - \underline{b}_2\underline{k}_2^T)^{-1}\underline{b}_1 \qquad (4.4.14)$$

This feedback changes the system eigenvalues to some new set. In the second stage, the feedback vector $\underline{k}_1^T$ is found so as to change the eigenvalues to the desired set corresponding to the required system poles.

Since the system (4.4.1), (4.4.2) is controllable through $\underline{b}_2$, it can be transformed into the companion form by the state vector transformation $\underline{z} = T\underline{x}$, so that the $\underline{b}_2$ vector becomes, in the $\underline{z}$-space, $T\underline{b}_2 = \underline{e}_n$, where $\underline{e}_n$ is a unit column n-vector with a 1 in the last row, and zeros elsewhere. In this form, the feedback vector $\underline{k}_2^T T^{-1}$ can be written down to change the a_i in the last row of the A_c matrix to any desired values.

Let $\underline{c}^T = \underline{c}_1^T T^{-1}$, and $\underline{b} = T\underline{b}_1$, in equations (4.4.13). If these equations are consi-tent, they may be solved for the a_i to give the desired p_q. Otherwise, row reduction of these equations will provide a set of linear constraints on the p_q which must be satisfied to give a solution.

The right hand side of equations (4.4.13) has been arranged in such a way that, if the row reduction operations on the left hand side coefficient matrix are carried across to the right hand side also, each zero row on the left hand side on completion of the row reduction yields a linear condition on the p_q.

$\underline{k}_1^T$ is now found to give the desired closed-loop poles. As has been shown in 4.2, the application of this feedback will not disturb the numerator polynomials relating to this input, which were obtained in the first stage. It is first necessary to check for controllability the pair $((A + \underline{b}_2\underline{k}_2^T),\underline{b}_1)$. If this test is satisfied, $\underline{k}_1^T$ may be found so as to give any preassigned set of closed-loop poles. Otherwise, small changes in $\underline{k}_2^T$, and hence in the p_q, must be introduced, so as to achieve controllability. The validity of this procedure has been established in 4.3.5.

The determination of $\underline{k}_1^T$ may be achieved by transforming the matrix $(A + \underline{b}_2\underline{k}_2^T)$ to the companion form, by a suitable co-ordinate transformation, so that the vector $\underline{b}_1$ becomes $\underline{e}_n$. Thus, the denominator coefficients $a_i^!$ are assigned as desired.

4.4.6 Rules for Writing the G_j Matrices.

$G_{n-1} = I_n$, and $a_n = 1$.

For $j < (n-1)$,

The first $(j + 1)$ elements on the main diagonal are a_{j+1}, and the rest are zeros.

The first $(j + 1)$ elements on the i'th diagonal above the main diagonal are a_{j+i+1}, and the rest are zeros.

For $(j + i + 1) > n$, all diagonal entries are zero.

The last $(n - j - 1)$ elements on the i'th diagonal below the main diagonal are $-a_{j-i+1}$, and the rest are zeros.

For $(j - i + 1) < 0$, all the diagonal entries are zero.

4.4.7 Rules for Writing the Linear Equation Set.

The general term in the i'th row and the j'th column in the coefficient matrix on the left hand side of the equations generalised from (4.4.13) is:

$$\sum_{p=1}^{n-i} c_p b_{p+i-j} \quad \text{for } j \leq i \text{, and:}$$

$$-\sum_{p=n-i+1}^{n} c_p b_{p+i-j} \quad \text{for } j > i.$$

The formation of the generalised right hand side will be clear from equations (4.4.13).

4.4.8 Derivation of the Formulae for the G_j and the Linear Equation Set.

The formulae for the G_j may be derived by applying the Faddeev-Leverrier algorithm [Z1] to the case in which the A matrix is of the companion form, A_c. According to this algorithm,

$$G_j = G_{j+1}A_c + a_{j+1}I$$

$$\text{and} \quad G_{n-1} = I \qquad (4.4.15)$$

Now,

$$A_c = H - \underline{e}_n \left[a_o \; a_1 \; \ldots \; a_{n-1} \right]$$

where H is a matrix with 1's in the first diagonal above the main diagonal, and zeros elsewhere. $\underline{e}_n$ is a unit column n-vector, with a 1 in the last row, and zeros elsewhere.

Hence, in (4.4.15),

$$G_j = G_{j+1}H - G_{j+1}\underline{e}_n \left[a_o \; a_1 \; \ldots \; a_{n-1} \right] + a_{j+1}I \qquad (4.4.16)$$

The G_j are obtained by starting with $j = (n-2)$, and working downwards in j. The effect of post-multiplying G_{j+1} by H is to shift all columns one place to the right, and to make the first column zero. Let the resulting matrix be $\overline{\overline{G}}_j$. $G_{j+1}\underline{e}_n$ is the last column of G_{j+1}, which is found to be $\underline{e}_{j+2}$, a unit column n-vector with a 1 in the (j+1)th row, and zeros elsewhere. Hence, the vector $[a_0\ a_1\ \ldots\ a_{n-1}]$ is subtracted from the (j+2)th row of $\bar{\bar{G}}_j$, to form $\bar{G}_j$, say. The addition of a_{j+1} to each diagonal element of $\bar{G}_j$ then gives G_j. The formation of the G_j matrices will be understood from this, and from a consideration of the matrices in (4.4.11). The formulae in 4.4.6 are derived from this.

The linear equation set is derived by forming the coefficient of each power of s in the transfer function numerator polynomial, by using $\underline{c}^T$, $\underline{b}$ and the G_j, equating this to the desired coefficient, and performing simple algebraic manipulation. The formulae in 4.4.7 are obtained by considering the generalisation of the process of formation of the equations (4.4.13).

4.4.9 Numerical Example.

The procedure is illustrated by application to the same numerical example as was used in 4.3.7.

Here,

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 5 & 0 & -3 \end{bmatrix}; \quad \underline{b}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \end{bmatrix}; \quad \underline{b}_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\underline{c}_1^T = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}$$

The transfer function between y and u_1 is to have zeros at -2 and -4, and poles at -1.5, -2.5, -3.5 and -4.5.

We first note that $p_3 = \underline{c}_1^T \underline{b}_1 = 0$, so that there are at most two zeros. The transformation matrix T to give the companion form representation of A, $\underline{b}_2$ is:

$$T = \frac{1}{240}\begin{bmatrix} -15 & 69 & 20 & -3 \\ -15 & 33 & -20 & 9 \\ -15 & 141 & 20 & -27 \\ -15 & 57 & -20 & 81 \end{bmatrix}$$

and:

$$T^{-1} = \begin{bmatrix} -18 & -12 & 10 & 4 \\ -3 & -1 & 3 & 1 \\ 9 & -6 & -3 & 0 \\ 1 & -3 & -1 & 3 \end{bmatrix}$$

whence:

$$T\underline{b}_1 = \begin{bmatrix} 0.075 \\ 0.025 \\ 0.175 \\ 0.725 \end{bmatrix}$$

and $\underline{c}_1^T T^{-1} = \begin{bmatrix} -7 & -14 & 5 & 0 \end{bmatrix}$

The linear equation set corresponding to (4.4.13) is now formed as:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -2.625 & -0.875 & -0.125 & -0.375 \\ -1.225 & -0.175 & -0.525 & 0.925 \end{bmatrix} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_o \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_2 \\ p_1 \\ p_o \end{bmatrix} - \begin{bmatrix} 1.0 \\ -11.375 \\ -5.075 \end{bmatrix}$$

The first row gives the constraint $p_2 = 1$. We require $p_1 = 6$, $p_o = 8$.
It is unnecessary to perform the row reduction in this

simple case, as the two remaining equations obviously are consistent. The solution is non-unique, and so additional specifications can be included. In this case, however, for the purpose of permitting direct comparison with the method of 4.3, two of the eigenvalues of $(A + \underline{b}_2\underline{k}_2^T)$ will be specified as -1 and -3. This gives the following set of equations:

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 27 & -9 & 3 & -1 \\ 2.625 & 0.875 & 0.125 & 0.375 \\ 1.225 & 0.175 & 0.525 & -0.925 \end{bmatrix} \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_o \end{bmatrix} = \begin{bmatrix} 1 \\ 81 \\ -17.375 \\ -13.075 \end{bmatrix}$$

The solution is:

$$\begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_o \end{bmatrix} = \begin{bmatrix} -1.125 \\ -16.375 \\ -10.875 \\ 3.375 \end{bmatrix}$$

Without feedback, the coefficients of the characteristic polynomial of A are: 1, -7, -1 and 6. Hence,

$$\underline{k}_2^T T^{-1} = \left[(-3.375+6)\ (10.875-1)\ (16.375-7)\ (1.125+1)\right]$$

whence:

$$\underline{k}_2^T = \begin{bmatrix} -1.5 & 8.125 & 0 & 0 \end{bmatrix}$$

This result agrees with that obtained in 4.3.7.

The second stage of the procedure, to locate the poles, is the same as in 4.3.7.

4.4.10 Conclusion.

The procedure described gives a general method of approach for determining the state vector feedbacks required for the synthesis of the scalar transfer function. It is clearly closely related to the method of [W4] , but gives more information at each stage of the design process. Constraints on the design are revealed, and can be allowed for at the appropriate stage. By the use of the relations of the form (4.4.7) and (4.4.9), the coefficients of the numerator polynomials of any transfer functions can be examined. Where these transfer functions relate to the input u_1, the numerators will not change when the feedback $\underline{k}_1^T$ is applied. The numerators of transfer functions relating to other inputs in general will change when $\underline{k}_1^T$ is applied. Hence, (4.4.7) and (4.4.9) must be used accordingly.

Although the solution of the linear equation set is generally non-unique, as in the numerical example considered, the scope for including additional specified requirements is rather limited. It is possible to obtain another linear equation set corresponding to another row of C, to permit specification of the transfer function to another output, from the same input, and to seek a solution of both sets of equations. In this case, linearity is preserved. With low-order systems, this may sometimes permit the specification of two or possibly more scalar transfer functions from the same input, depending upon the orders of the numerators of the transfer functions.

The procedure has been described in [M13].

4.5 General Comments.

Either of the procedures 4.3 or 4.4 gives a complete solution to the problem of designing state vector feedback for a system with two inputs, to provide, as far as possible, a specified scalar input-output transfer function. The first method achieves this by making direct use of the technique of modal control. The problem of locating zeros is transformed into a problem of locating the eigenvalues of a related matrix.

The procedure of 4.4 reduces the zero assignment problem to the solution of a set of linear equations. The consistency of these equations provides full information regarding any constraints on the choice of coefficients of the transfer function numerator polynomial.

The non-uniqueness of the solution appears in 4.3 where only the controllable eigenvalues of the matrix used to determine the zeros are assigned by means of an n-vector of state feedback gains.

In 4.4, the non-uniqueness appears more directly in the solution of the linear equation set, and permits the consideration of other linear equation sets for simultaneous solution, if desired, to enable other transfer functions to be specified also. The method of 4.4 also provides convenient means for checking all transfer function numerators, using information available in the procedure.

CHAPTER 5.

STATE OBSERVERS.

5.1 Introduction.

The pole assignment and pole-zero assignment procedures described in the preceding chapters nearly all require feedback of specified linear functionals of the system state vector. In most practical cases, the number of available outputs is less than the system order, so that it is not possible to form any arbitrarily chosen linear functional of the state vector as a linear functional of the system outputs. One solution to this problem for deterministic systems is provided by the state observer, which is a linear dynamic system driven from the inputs and outputs of the system under consideration, so as to permit the construction of an estimate of the system state vector continuously in time.

5.1.1 Observer Properties.

The arrangement of an observer is shown in Fig. 5.1. The system is described by the equations:

$$\dot{x} = Ax + Bu$$

$$y = Cx \qquad (5.1.1)$$

where x, y and u are vectors of dimension n, m and r, respectively, representing the system states, outputs and inputs.

The state observer is described by the equation:

$$\dot{z} = Dz + Ky + Gu \qquad (5.1.2)$$

where z is the observer state vector, of dimension l. The matrices A, B, C, D, K and G are constant, and of appropriate dimensions.

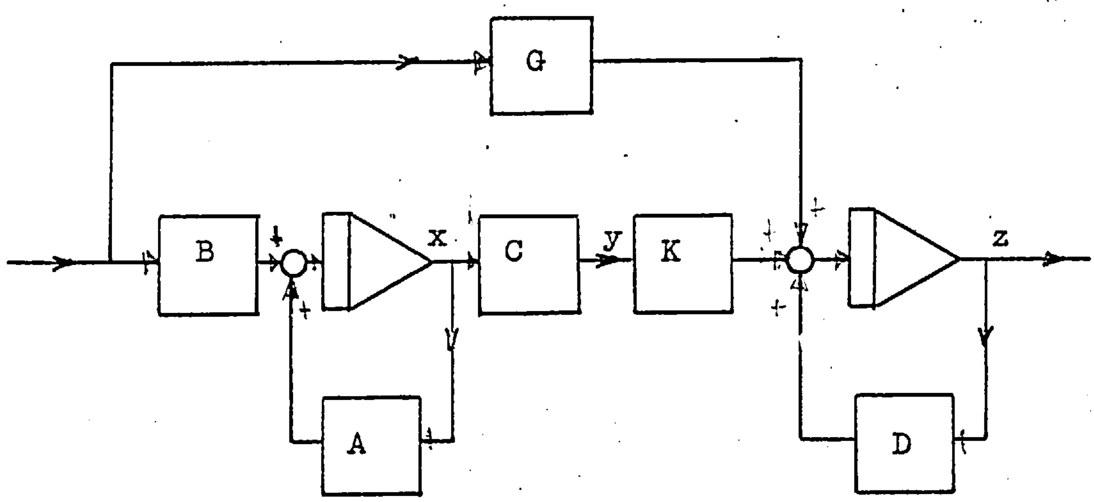


Fig. 5.1.

The theory of state observers was given by Luenberger [L1][L4][L5] and the following brief treatment is included for completeness.

Combining (5.1.1) and (5.1.2) gives the overall system differential equation as:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & 0 \\ KC & D \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ G \end{bmatrix} u \qquad (5.1.3)$$

Taking the Laplace transform of (5.1.3), and denoting the Laplace transform of a time variable by a bar, gives the solution as:

$$\begin{bmatrix} \bar{x} \\ \bar{z} \end{bmatrix} = \begin{bmatrix} (sI_n-A) & 0 \\ -KC & (sI_l-D) \end{bmatrix}^{-1} \begin{bmatrix} x(0) \\ z(0) \end{bmatrix} + \begin{bmatrix} (sI_n-A) & 0 \\ -KC & (sI_l-D) \end{bmatrix}^{-1} \begin{bmatrix} B \\ G \end{bmatrix} \bar{u} \qquad (5.1.4)$$

whence:

$$\bar{x} = (sI_n-A)^{-1}x(0) + (sI_n-A)^{-1}B\bar{u} \qquad (5.1.5)$$

and:

$$\bar{z} = (sI_l-D)^{-1}KC(sI_n-A)^{-1}x(0) + (sI_l-D)^{-1}z(0) + (sI_l-D)^{-1}KC(sI_n-A)^{-1}B\bar{u} + (sI_l-D)^{-1}G\bar{u} \qquad (5.1.6)$$

Now let $G = TB$, where T satisfies the matrix equation:

$$TA - DT = KC \qquad (5.1.7)$$

and let $z(0) = Tx(0) + w(0)$ (5.1.8)

G can be chosen by the designer, and (5.1.8) introduces no loss of generality.

Substituting for G, KC and z(0) in (5.1.6) gives:

$$\bar{z} = (sI_l-D)^{-1}(TA-DT)(sI_n-A)^{-1}x(0) + (sI_l-D)^{-1}(Tx(0)+w(0)) + (sI_l-D)^{-1}(TA-DT)(sI_n-A)^{-1}B\bar{u} + (sI_l-D)^{-1}TB\bar{u} \qquad (5.1.9)$$

$$= (sI_l-D)^{-1}w(0) + (sI_l-D)^{-1}(TA-DT)(sI_n-A)^{-1}x(0) + (sI_l-D)^{-1}T(sI_n-A)(sI_n-A)^{-1}x(0) + (sI_l-D)^{-1}(TA-DT)(sI_n-A)^{-1}B\bar{u} + (sI_l-D)^{-1}T(sI_n-A)(sI_n-A)^{-1}B\bar{u} \qquad (5.1.10)$$

$$= (sI_l-D)^{-1}w(0) + (sI_l-D)^{-1}(TA-DT+Ts-TA)(sI_n-A)^{-1}x(0) + (sI_l-D)^{-1}(TA-DT+Ts-TA)(sI_n-A)^{-1}B\bar{u} \qquad (5.1.11)$$

$$= (sI_l-D)^{-1}w(0) + T(sI_n-A)^{-1}x(0) + T(sI_n-A)^{-1}B\bar{u} \qquad (5.1.12)$$

$$= (sI_l-D)^{-1}w(0) + T\bar{x} \qquad (5.1.13)$$

from (5.1.5).

If the observer is stable, i.e., D represents a stable system, the time response corresponding to the first term of (5.1.13) will represent a decaying response to the initial 'mismatch' w(0). When this response has decayed, there remains:

$$\bar{z} = T\bar{x}, \text{ and so } z = Tx \qquad (5.1.14)$$

due to the linearity and uniqueness properties of the Laplace transform. In the time domain, equation (5.1.13) becomes:

$$z = e^{Dt}w(0) + Tx \qquad (5.1.15)$$

The equation (5.1.15) expresses a fundamental property of the observer, which is that, after any initial mismatch has decayed, the state vector of the observer becomes and remains a fixed linear transformation of the system state vector, for all system inputs.

A question arises as to whether the matrix T can be found as a solution of the equation (5.1.7). Gantmacher [G3] has shown that an equation of the form of (5.1.7) always has a unique solution if the matrices A and D have no common eigenvalue. Now, D is at the disposal of the designer of the observer, so that the conditions (i) D should have all eigenvalues with negative real parts and (ii) D should have no eigenvalue in common with A are easily met. Apart from these two conditions, D can be chosen arbitrarily. However, these conditions only ensure that there will be some solution matrix T. The general problem of observer design is concerned with the dynamics of D, and with the properties of T.

5.1.2 Observer Used with Feedback.

Since the observer is to be used for applying feedback, it is important to consider the properties of a linear system with observer and feedback. The general arrangement is shown in Fig. 5.2. Feedback is obtained both from the system outputs and the observer state vector.

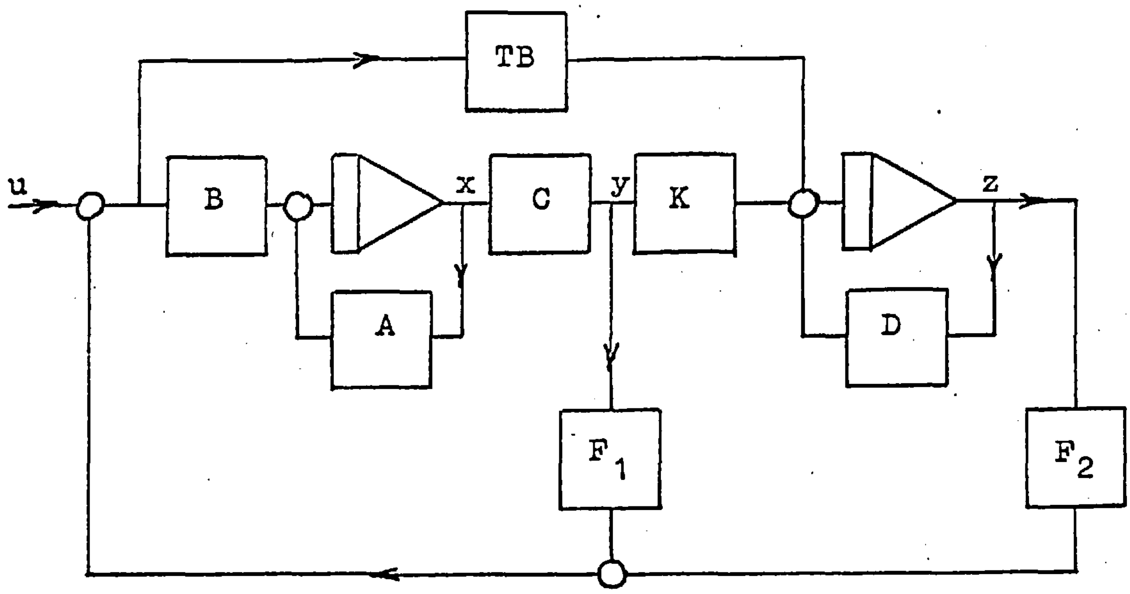


Fig. 5.2.

This system is represented by the following equations:

$$\dot{x} = Ax + Bu + BF_1y + BF_2z \qquad (5.1.16)$$

$$\dot{z} = Dz + Ky + TBu + TBF_1y + TBF_2z \qquad (5.1.17)$$

Setting y = Cx, and KC = TA - DT, gives:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} (A+BF_1C) & BF_2 \\ (TA-DT+TBF_1C) & (D+TBF_2) \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ TB \end{bmatrix} u \qquad (5.1.18)$$

Introducing a change of state vector to $\begin{bmatrix} x \\ w \end{bmatrix}$, where:

$$\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ T & I_1 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$

gives the differential equation in the form:

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} (A+BF_1C+BF_2T) & BF_2 \\ 0 & D \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u \qquad (5.1.19)$$

The form of (5.1.19) reveals that the overall system with feedback has the eigenvalues of $(A+BF_1C+BF_2T)$ and the eigenvalues of D. This is the 'separation' property, and permits the feedback to be designed as for the original system, but with the measurement matrix C augmented to $\begin{bmatrix} C \\ \cdots \\ T \end{bmatrix}$.

To obtain the transfer functions, the Laplace transform of (5.1.19), with zero initial conditions, is taken. This gives:

$$\begin{bmatrix} \bar{x} \\ \bar{w} \end{bmatrix} = \begin{bmatrix} (sI_n-A-BF_1C-BF_2T) & -BF_2 \\ 0 & (sI_1-D) \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} \bar{u} \qquad (5.1.20)$$

$$\begin{bmatrix} (sI_n-A-BF_1C-BF_2T)^{-1} & (sI_n-A-BF_1C-BF_2T)^{-1}BF_2(sI_1-D)^{-1} \\ 0 & (sI_1-D)^{-1} \end{bmatrix} \begin{bmatrix} B \\ 0 \end{bmatrix} \bar{u}$$

(5.1.21)

whence:

$$\bar{x} = (sI_n-A-BF_1C-BF_2T)^{-1}B\bar{u} \qquad (5.1.22)$$

Equation (5.1.22) shows that, as far as input-state, and hence input-output transfer functions are concerned, the system with observer behaves as a system of order n, with the measurement matrix augmented as described. It is this property which, together with the separation property, makes the observer particularly useful in connection with pole assignment and pole-zero assignment by feedback.

5.1.3 State Observer Design.

So far, nothing has been said about the value of l, the order of the observer. The objective is to reconstruct the state vector, and the simplest solution is to let $T = I$. This gives $z = x$, and the observer equation (5.1.7) takes the form:

$$A - D = KC,$$

or

$$D = (A - KC) \qquad (5.1.23)$$

From modal control theory, if the pair (A,C) is observable, the matrix D in (5.1.23) can be given any arbitrary set of eigenvalues, by suitable choice of K.

Recognising that some linear combinations of the state variables are already available in the outputs y, Luenberger showed that the entire state vector could always be reconstructed for an observable system by means of an observer of order (n-m), where m is the number of linearly independent system outputs, and that the observer dynamics could be chosen almost arbitrarily, subject only to the conditions (i) and (ii) in 5.1.1. Luenberger gave a design procedure based on the transformation of the system to a canonical form in which the A matrix assumes one of the companion forms, or has diagonal blocks of this form.

The design problem will now be considered based on the equation (5.1.7) directly. The approach is somewhat similar to that used by Newmann [N1] .

First suppose, in (5.1.7), that D is of dimension (n-m), and that T is chosen with rows linearly independent of the rows of C. This will enable the state vector to be recovered, if the observer functions correctly, from: $x = \begin{bmatrix} C \\ T \end{bmatrix}^{-1} \begin{bmatrix} y \\ z \end{bmatrix}$.

Then, (5.1.7) gives:

$$\left[K \vdots D\right] = TA \begin{bmatrix} C \\ T \end{bmatrix}^{-1} \qquad (5.1.24)$$

Hence, K and D are uniquely determined. If the solution for D obtained in this way represents an unstable or otherwise undesirable observer, another T must be tried. This approach is, therefore, unsatisfactory.

Now suppose that, for an observer of the same dimension, in (5.1.7), D and K are chosen. The equation may then be solved for T. For example, if both A and D have distinct eigenvalues, A may be written $A = W\Lambda^a W^{-1}$, and D may be written $U\Lambda^d U^{-1}$, where Λ^a and Λ^d are diagonal matrices of eigenvalues, and W and U are matrices of column eigenvectors. The equation (5.1.7) then becomes:

$$TW\Lambda^a W^{-1} - U\Lambda^d U^{-1}T = KC, \text{ or:}$$

$$U^{-1}TW\Lambda^a - \Lambda^d U^{-1}TW = U^{-1}KCW \qquad (5.1.25)$$

which may be solved, element by element for the matrix $U^{-1}TW$, and hence for T. The method may be adapted to cater for multiple eigenvalues, but is, in any case, unsatisfactory, because, although D can be chosen to have satisfactory dynamics, there is no control over the solution T. If this is found to have rows which are linearly dependent on the rows of C, the choice of dynamics of D will have to be changed, and the process repeated.

The general solution to this problem was obtained by Cumming, and is described in the next section. The approach used in establishing this method of solution is different from that used by Cumming.

5.1.4 Cumming's Method.

In Cumming's method [C4][C5] , a vector z_o is formed as:

$$z_o = z + Ry \tag{5.1.26}$$

where R is a constant matrix to be determined, and so

$$z = z_o - RCx$$

The $(n-m)\times n$ matrix T_o is chosen arbitrarily, with rows linearly independent of the rows of C. The action of the observer is to make z_o tend asymptotically to $T_o x$, and z tends to Tx, so that:

$$T = T_o - RC \tag{5.1.27}$$

Inserting (5.1.27) in (5.1.7) gives:

$$(T_o - RC)A - D(T_o - RC) = KC$$

$$(T_o - RC)A = \left[(K - DR) \vdots D\right]\begin{bmatrix} C \\ \hdashline T_o \end{bmatrix}$$

and so:

$$\left[(K - DR) \vdots D\right] = T_o A\begin{bmatrix} C \\ \hdashline T_o \end{bmatrix}^{-1} - RCA\begin{bmatrix} C \\ \hdashline T_o \end{bmatrix}^{-1} \tag{5.1.28}$$

$$= \left[K_1 \vdots K_2\right] - R\left[K_3 \vdots K_4\right] \tag{5.1.29}$$

where the partitioning on the right hand side coincides with that on the left. Then, from (5.1.29),

$$D = K_2 - RK_4 \tag{5.1.30}$$

and $$K = DR + K_1 - RK_3 \tag{5.1.31}$$

In (5.1.30), according to modal control theory, D can be assigned any arbitrary set of eigenvalues by suitable choice of R, if the pair (K_2,K_4) is observable. It will now be proved that (K_2,K_4) is observable if (A,C) is observable.

Proof.

Let (A,C) be observable, and let C have m rows, which are linearly independent. This introduces no loss of generality, since linearly dependent rows can be ignored.

Let:

$$\begin{bmatrix} C \\ \hdashline T_o \end{bmatrix}^{-1} = \left[P \,\vdots\, Q \right] \tag{5.1.32}$$

where the partitioning is such that P is $n \times m$, and Q is $n \times (n-m)$.

Since:

$$\begin{bmatrix} C \\ \hdashline T_o \end{bmatrix} \left[P \,\vdots\, Q \right] = I_n \tag{5.1.33}$$

$$\begin{aligned} CP &= I_m \; ; \qquad CQ = 0 \\ T_oP &= 0 \; ; \qquad T_oQ = I_{n-m} \end{aligned} \tag{5.1.34}$$

From (5.1.29),

$$K_2 = T_oAQ \; ; \qquad K_4 = CAQ \tag{5.1.35}$$

The pair (K_2,K_4) is observable if the matrix H has rank (n-m), where:

$$H = \begin{bmatrix} K_4 \\ K_4K_2 \\ K_4K_2^2 \\ \cdot \\ \cdot \\ K_4K_2^{n-m-1} \end{bmatrix} \qquad (5.1.36)$$

Substituting from (5.1.35),

$$H = \begin{bmatrix} CAQ \\ CAQT_oAQ \\ CAQT_oAQT_oAQ \\ \cdot \\ \cdot \end{bmatrix} \qquad (5.1.37)$$

Since:

$$\left[P \vdots Q\right] \begin{bmatrix} C \\ \hdashline T_o \end{bmatrix} = I_n \qquad (5.1.38)$$

$$QT_o = I_n - PC \qquad (5.1.39)$$

Substituting from (5.1.39) in (5.1.37),

$$H = \begin{bmatrix} CA \\ CA^2 - CAPCA \\ CA^3 - CA^2PCA - CAPCA^2 + CAPCAPCA \\ \cdot \\ \cdot \end{bmatrix} Q \qquad (5.1.40)$$

By elementary row operations, it can be shown that H has the same rank as:

$$G = \begin{bmatrix} CA \\ CA^2 \\ \cdot \\ \cdot \\ CA^{n-m} \end{bmatrix} Q \qquad (5.1.41)$$

Now consider the matrix product:

$$G_1 = \begin{bmatrix} C \\ CA \\ CA^2 \\ \cdot \\ \cdot \\ CA^{n-m} \end{bmatrix} Q \qquad (5.1.42)$$

The rank of Q is (n-m), because this comprises (n-m) linearly independent columns of $\begin{bmatrix} C \\ \hdashline T_o \end{bmatrix}^{-1}$, and the rank of the first factor in (5.1.42) is n, because (A,C) is observable and C has rank m. Hence the rank of the product matrix G_1 is at least n - n + (n-m) = (n-m). The rank also is at most (n-m), because Q has this rank. Hence G_1 has rank (n-m). But, from (5.1.34), CQ = 0, and so G has rank (n-m). This completes the proof.

5.2 Simple Design of State Observer.

Although Cumming's method provides a complete solution to the problem of designing a state observer having arbitrarily chosen eigenvalues, which is suitable for programming on a digital computer so as to provide an automatic solution, it is not an easy procedure for pencil and paper calculations. The matrix operations and the embedded modal control problem may present considerable difficulties when dealing with high-order systems without the aid of a full computer program. The availability of electronic calculating machines of the hand and desk types, and of time-sharing computer terminals which provide packages for standard matrix manipulations, but not specialised programs for control engineering work, makes it useful to consider methods of design which permit these facilities to be used to aid pencil and paper design.

In the method which is now to be described, a state observer is designed as a succession of scalar observer designs. The eigenvalue of each scalar observer must be real, but this is no disadvantage in most practical cases. Otherwise, the eigenvalues can be chosen almost arbitrarily, and the computations involved are very simple.

5.2.1 Properties of Scalar Observer.

For a linear system described by the equations:

$$\dot{x} = Ax + Bu$$
$$y = Cx \qquad (5.2.1)$$

where x, y and u are vectors of dimension n, m and r respectively, a scalar observer is described by the equation:

$$\dot{z} = dz + k^T y + g^T u \qquad (5.2.2)$$

where z and d are scalars, and k^T and g^T are row vectors. The system (5.2.1) is controllable and observable.

In this case, equation (5.1.7) takes the form:

$$t^T A - dt^T = k^T C \qquad (5.2.3)$$

where t^T is a row vector of dimension n, such that z tends to $t^T x$.

d is the eigenvalue of the observer, and equation (5.2.3) may be solved for t^T, if d is not an eigenvalue of A, as:

$$t^T = k^T C(A - Id)^{-1} \qquad (5.2.4)$$

The scalar observer design requires that d is chosen to be negative and real, and that the pair $(A, k^T C)$ is observable. Since (A,C) is observable, a k^T can always be found to satisfy this condition iff A is non-derogatory. However, since, in the case under consideration, the state observer is being designed for the purpose of providing feedback for pole/zero assignment, there is no reason why arbitrary feedback from the system outputs should not be applied initially so as to separate the eigenvalues of a derogatory A matrix, and so render it non-derogatory [D2] . The final design of feedback can allow for this initial feedback.

The design procedure rests on the following theorem:

<u>Theorem.</u>

The observer eigenvalue d in (5.2.4) can be chosen so that the vector t^T is linearly independent of any arbitrary set of (n-1) linearly independent vectors T_a.

<u>Proof.</u>

If t^T is linearly dependent on T_a, the determinant:

$$\begin{vmatrix} t^T \\ T_a \end{vmatrix} = 0 \qquad (5.2.5)$$

Substituting in (5.2.5) from (5.2.4),

$$\begin{vmatrix} k^T C(A - Id)^{-1} \\ T_a \end{vmatrix} = 0 \qquad (5.2.6)$$

Assume initially that A has distinct eigenvalues. d is not an eigenvalue of A, so that the inverse exists. Let W be a matrix of column eigenvectors of A. Then (5.2.6) may be written:

$$\left| \begin{array}{c} k^T CW \begin{bmatrix} \frac{1}{\lambda_1 - d} & 0 & 0 & . \\ 0 & \frac{1}{\lambda_2 - d} & . & . \\ 0 & 0 & . & . \\ . & . & . & . \end{bmatrix} \\ \hline T_a W \end{array} \right| \left| W^{-1} \right| = 0 \qquad (5.2.7)$$

Let $k^T CW = \gamma^T = \begin{bmatrix} \gamma_1 & \cdots & \gamma_n \end{bmatrix}$ (5.2.8)

and let $T_a W = T_w$ (5.2.9)

T_w has (n-1) linearly independent rows, because W is non-singular. It then follows from (5.2.7) that:

$$\left| \begin{array}{cccc} \frac{\gamma_1}{\lambda_1 - d} & \frac{\gamma_2}{\lambda_2 - d} & & \frac{\gamma_n}{\lambda_n - d} \\ \hline & T_w & & \end{array} \right| = 0 \qquad (5.2.10)$$

In (5.2.10), the γ_j, j=1,...n, are all non-zero, because $(A, k^T C)$ is observable. Expanding the determinant by the first row, and multiplying by $\prod_{j=1}^{n} (\lambda_j - d)$, yields a polynomial equation in d of degree (n-1), which is not identically zero because, since T_w has (n-1) linearly independent rows, at least one of the co-factors of the terms in the first row is non-zero. It follows that there are at most (n-1) values of the observer eigenvalue d giving a t^T which is linearly dependent upon T_a. This completes the proof for the case in which A has distinct eigenvalues.

Where A has multiple eigenvalues, but is non-derogatory, the Jordan form will have a Jordan block corresponding to each different multiple eigenvalue. There will be no two blocks with the same eigenvalue. The proof will be extended to this case by considering one such block, of dimension 3, for definiteness. W then includes generalised eigenvectors.

Corresponding to a third-order Jordan block, with eigenvalue λ_1, the inverse in (5.2.7) will have a diagonal block of the form:

$$\begin{bmatrix} \frac{1}{\lambda_1-d} & -\frac{1}{(\lambda_1-d)^2} & \frac{1}{(\lambda_1-d)^3} \\ 0 & \frac{1}{\lambda_1-d} & -\frac{1}{(\lambda_1-d)^2} \\ 0 & 0 & \frac{1}{\lambda_1-d} \end{bmatrix} \qquad (5.2.11)$$

whence the corresponding part of the first row of (5.2.10) will assume the form:

$$\left\{\frac{\gamma_1}{\lambda_1-d}\right\}\left\{-\frac{\gamma_1}{(\lambda_1-d)^2}+\frac{\gamma_2}{\lambda_1-d}\right\}\left\{\frac{\gamma_1}{(\lambda_1-d)^3}-\frac{\gamma_2}{(\lambda_1-d)^2}+\frac{\gamma_3}{(\lambda_1-d)}\right\}$$

(5.2.12)

The condition of observability implies that $\gamma_1 \neq 0$, whilst the values of γ_2 and γ_3 are immaterial. It again follows that expansion of the determinant by the first row, and multiplication by $(\lambda_1 - d)^3$, and by factors corresponding to the other eigenvalues yields a polynomial equation of degree (n-1), which is not identically zero. Hence the theorem is true generally.

5.2.2 Design Procedure.

k^T is first chosen so as to preserve observability. This value is retained throughout the procedure.

The first observer eigenvalue d_1 is chosen, and the corresponding t^T vector, t_1^T is found from (5.2.4). t_1^T is then checked for linear independence of C, by echelon reduction, or otherwise. If it is found to be linearly dependent, a different value of d_1 is chosen, and the process repeated. When linear independence is established, the next observer eigenvalue d_2 is chosen, and t_2^T is found from (5.2.4). t_2^T is then checked for linear independence of $\begin{bmatrix} C \\ t_1^T \end{bmatrix}$, and so on.

The process is continued until (n-m) eigenvalues have been chosen. The state vector can then be estimated by inverting the non-singular matrix:

$$\begin{bmatrix} C \\ t_1^T \\ \cdot \\ t_{n-m}^T \end{bmatrix} .$$

Each scalar observer has the external inputs applied to it as:

$$g^T = t^T B \qquad (5.2.13)$$

Whore means arc available for finding eigenvectors; and A has distinct eigenvalues, the process can be further simplified so as to avoid the matrix inversion for each value of d chosen. Equation (5.2.4) may then be written in the form:

$$t_j^T W = k^T C W \begin{bmatrix} \frac{1}{\lambda_1 - d_j} & 0 & 0 \\ 0 & \frac{1}{\lambda_2 - d_j} & 0 \\ & & \vdots \\ 0 & 0 & \cdot \end{bmatrix} \qquad (5.2.14)$$

This method will involve complex arithmetic if A has complex eigenvalues.

<u>5.3 Numerical Example.</u>

A simple numerical example, which is a modified form of an example given by Luenberger [L5] , will now be used to illustrate the design of state observers by both Cumming's method and the simple method of 5.2.

In this example, the state vector is to be estimated by a minimum-order observer having the eigenvalues -1 and -2, if possible.

Here,

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

5.3.1 Cumming's Method.

Choose $T_o = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Then: $$\begin{bmatrix} C \\ T_o \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and: $$\begin{bmatrix} C \\ T_o \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_oA\begin{bmatrix} C \\ T_o \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -2 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$CA\begin{bmatrix} C \\ T_o \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Thus:

$$K_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad ; \qquad K_2 = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \quad ;$$

$$K_3 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \quad ; \qquad K_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This gives:

$$D = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} - R\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

K_2 has the eigenvalues 0 and -2, and D is to have the eigenvalues -1 and -2. This is achieved with:

$$R = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and then: $D = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$

and: $K = DR + K_1 - RK_3$

$$= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and $G = TB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

5.3.2 Simple Scalar Method.

We first choose any k^T such that (A, k^TC) is observable. $k^T = \begin{bmatrix} 1 & 1 \end{bmatrix}$ satisfies this requirement.

Then: $k^TC = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$

First scalar observer: $d_1 = -1$

$$t_1^T = k^TC(A - Id_1)^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 2 & -2 \end{bmatrix}$$

This is linearly independent of the rows of C, and so is acceptable.

$$g_1 = t_1^TB = \begin{bmatrix} 1 & 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = -2$$

Second scalar observer: $d_2 = -2$

$$t_2^T = k^TC(A - Id_2)^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 0 & -1 & 2 & -1 \end{bmatrix}$$

This is linearly independent of the rows of $\begin{bmatrix} C \\ t_1^T \end{bmatrix}$, and so is acceptable.

$$g_2 = t_2^T B = \begin{bmatrix} 0 & -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = -1$$

The overall design obtained by this method is thus:

$$D = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} ; \quad K = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} ;$$

$$G = \begin{bmatrix} -2 \\ -1 \end{bmatrix} .$$

The corresponding design obtained by Cumming's method was:

$$D = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} ; \quad K = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} ;$$

$$G = \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

5.4 Conclusions.

Cumming's method has been derived in a uniform treatment based on Luenberger's observer equation, and a proof of observer eigenvalue assignability has been given. Where the observer is used for providing feedback, the matrix R introduced in Cumming's method need have no physical existence. For example, if the overall feedback is to be $F_1y + F_2z_o$, this is equal to $F_1y + F_2(Ry + z) = (F_1 + F_2R)y + F_2z = F_1'y + F_2z$, so that the effect of R can be included in a new system output feedback gain matrix F_1'. R may then be regarded as an artifice used to facilitate the design of the observer.

Cumming [C4] has shown that the condition that the observer should have no eigenvalue in common with the original system is not necessary. Although this point is of some theoretical interest, it is not important where the observer is used in connection with closed-loop pole assignment. Since all the closed-loop poles are to be assigned, there seems to be no merit in choosing observer poles which coincide with those of the original system.

Whilst Cumming's method undoubtedly provides a complete and satisfactory solution to the problem of state observer design in general, there are advantages in considering a computationally simpler method. The simple scalar design method presented in 5.2 permits a minimum-order observer to be designed in every case, with almost arbitrary real eigenvalues, whilst using only routine matrix operations which are available in most time-sharing and similar computer libraries. Whilst the method involves an element of trial and error, this is not a serious disadvantage, because, as is clear from the theorem given, the occurrence of a t_j^T which is linearly dependent upon the rows of C and the t_i^T, i=1, ... (j-1), is exceptional.

Linearly dependent t_j^T could be avoided altogether at each stage by forming a T_a matrix, in the theorem, from C, and the t_i^T, i=1, ... (j-1), and augmenting this with (n-m-j) other rows, chosen arbitrarily, but linearly independent of these, and then solving the polynomial equation in d_j, which yields all the values of d_j giving linear dependence. If d_j is then chosen so as not to have any of these values, nor any of the eigenvalues of A, it may otherwise be chosen freely. However, the extra trouble involved in this process does not seem to be justified.

CHAPTER 6.

LINEAR FUNCTIONAL OBSERVER.

6.1 Introduction.

A controllable single-input linear system can have all closed-loop eigenvalues assigned by state vector feedback. It follows that, if an observer is to be used with such a system, all that is required is to develop an estimate of a single pre-specified linear functional of the state vector. Furthermore, any controllable multi-input system in which the A matrix is non-derogatory can have a single input distributed amongst the system inputs in such a way that the system is completely controllable with respect to this input. This statement follows from the fact that, if (A,B) is controllable, and A is non-derogatory, there exists a vector g such that (A,Bg) is controllable [W1] .

If the A matrix is derogatory, but (A,B,C) is controllable and observable, arbitrary output/input feedback K can be applied initially to separate the system eigenvalues, and make the system matrix non-derogatory [D2] . This feedback will not affect the controllability, as ((A+BKC),B) will be controllable.

It follows from these remarks that any controllable, observable linear time-invariant system can be treated as a single-input system, as far as pole assignment is concerned, so that pole assignment can be achieved in any such system by feeding back a specified linear functional of the state vector.

One way to obtain the required linear functional feedback is to use a state observer, and derive an estimate of the linear functional from this. Luenberger [L4] , however, has shown that an estimate of any specified linear functional of the state vector of a linear system can be provided by means of an observer of order (p-1), with arbitrary dynamics, where p is the 'observability index' of the system, defined as the least integer p for which the $mp \times n$ matrix Q has rank n, where:

$$Q = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ CA^{p-1} \end{bmatrix} \qquad (6.1.1)$$

Such an observer, which is known as a 'linear functional observer', may be of considerably lower order than the corresponding state observer, for a multioutput system.

Luenberger [L4] gave a method for designing a linear functional observer, based on the reduction of the system to a special canonical form, one of the companion forms.

A method will now be described, which does not require any change in the representation of the system, and which can be applied to any observable linear time-invariant system. An extension of the method leads to a procedure for designing linear functional observers of order lower than (p-1). Again, the method forms the basis of an approach to the design of degenerate observers, to provide estimates of a number of specified linear functionals of the state vector.

6.2 Design Procedure for Linear Functional Observer.

6.2.1 System Description.

We consider a linear time-invariant system described by the equations:

$$\dot{x} = Ax + Bu \tag{6.2.1}$$

$$y = Cx \tag{6.2.2}$$

where x, u and y are vectors of dimension n, r and m, representing the states, inputs and outputs, respectively, and A, B and C are constant matrices of appropriate dimensions. The system is observable, with observability index p, where p is defined as in (6.1.1).

6.2.2 Problem Statement.

The problem is to design for the system (6.2.1), (6.2.2) an observer with arbitrary dynamics, such that a suitable combination of y and the (p-1)-dimensional observer state vector z will give a specified linear functional h^T of x, i.e., such that, asymptotically,

$$f^Ty + g^Tz = h^Tx \tag{6.2.3}$$

where f^T, g^T and h^T are row vectors, of which h^T is specified.

The observer is described by the equation:

$$\dot{z} = Dz + Ky + Gu \tag{6.2.4}$$

where D, K and G are constant real matrices.

The observer dynamics are determined by D, which may be chosen arbitrarily, with the mild restriction that its eigenvalues, $(d_1, \dots d_{p-1})$, are distinct and different from the eigenvalues of A. Real eigenvalues must, of course, be negative, to provide a stable observer, and complex eigenvalues must occur in conjugate pairs with negative real parts, for physical realizability and stability. The matrix K, which couples the system to the observer, and the matrix G, which couples the external inputs to the observer, are to be found, as are the vectors f^T and g^T.

6.2.3 Solution.

Let U be a matrix of self-conjugate column eigenvectors of D, so that:

$$D = U \Delta U^{-1} \qquad (6.2.5)$$

where $\Delta = \text{diag}(d_1 \dots d_{p-1})$.

Let:

$$g^T = e^T U^{-1} \qquad (6.2.6)$$

where e^T is the (p-1)-dimensional sum vector $[1\ 1 \dots 1]$. This introduces no loss of generality.

Then:

$$K = UM \qquad (6.2.7)$$

where f^T and the rows of M, $(m_1^T, \dots m_{p-1}^T)$, are obtained as the solution of the set of linear equations (6.2.8).

$$\left[f^T \,\vdots\, m_1^T \,\vdots\, \cdots \,\vdots\, m_{p-1}^T\right] \begin{bmatrix} C\prod_{j=1}^{p-1}(A-d_jI) \\ C\prod_{j=2}^{p-1}(A-d_jI) \\ C\prod_{\substack{j=1\\j\neq 2}}^{p-1}(A-d_jI) \\ \vdots \\ C\prod_{j=1}^{p-2}(A-d_jI) \end{bmatrix} = h^T\prod_{j=1}^{p-1}(A-d_jI) \qquad (6.2.8)$$

I is the identity matrix of order n.

The ith m-rowed block of the coefficient matrix on the left hand side of (6.2.8), for i = 4, .. , (p-1), is:

$$C\prod_{\substack{j=1\\j\neq i-1}}^{p-1}(A-d_jI)$$

G is obtained from:

$$G = URB \qquad (6.2.9)$$

where R is a $(p-1)\times n$ matrix with rows $(r_1^T, \ldots r_{p-1}^T)$, where:

$$r_j^T = m_j^T C(A-d_jI)^{-1} \qquad (6.2.10)$$

$j=1, \ldots (p-1)$

Equations (6.2.8) are consistent, and the solution is unique if mp = n, and non-unique if mp > n.

This completes the solution.

Proof.

The matrix T relating the state vectors of the observer and the original system is the solution of the observer equation [L4] :

$$TA - DT = KC \qquad (6.2.11)$$

This equation has a unique solution if A and D have no common eigenvalue. This condition is satisfied. Setting $U^{-1}T = R$, and $U^{-1}K = M$, (6.2.5) and (6.2.11) give:

$$RA - \Delta R = MC \qquad (6.2.12)$$

From the rows of (6.2.12), since Δ is diagonal, we obtain (6.2.10). Equations (6.2.3) and (6.2.6) give the vector equation:

$$f^T C + e^T R = h^T \qquad (6.2.13)$$

Substituting (6.2.10) into (6.2.13) gives:

$$f^T C + \sum_{j=1}^{p-1} m_j^T C(A - d_j I)^{-1} = h^T \qquad (6.2.14)$$

Postmultiplication of (6.2.14) by:

$$\prod_{j=1}^{p-1} (A - d_j I)$$

and rearranging give (6.2.8). Equation (6.2.7) follows from the definition of M. From [L4] , G = TB. Hence, (6.2.9) follows from the definition of R given above.

The only unknowns in (6.2.8) are the mp components of the vector $\left[f^T \vdots m_1^T \vdots \ldots \vdots m_{p-1}^T\right]$. It can be verified by elementary row operations on the coefficient matrix on the left hand side of (6.2.8) that, provided the d_j are distinct, this matrix has the same rank as Q in (6.1.1), namely rank n, by definition. This statement will now be justified.

The coefficient matrix in (6.2.8) can be reduced to the form of Q if there exists a set of p^2 real scalars $\theta_{oo}, \ldots \theta_{o(p-1)}, \theta_{1o}, \ldots \theta_{1(p-1)}, \theta_{(p-1)o}, \ldots, \theta_{(p-1)(p-1)}$, such that, for q = 0, ... (p-1),

$$\theta_{qo}\prod_{j=1}^{p-1}(A - d_jI) + \sum_{i=1}^{p-1}\theta_{qi}\prod_{\substack{j=1\\j\neq i}}^{p-1}(A - d_jI) = A^{p-q-1} \qquad (6.2.15)$$

Equating coefficients of like powers of A on both sides of (6.2.15) gives a set of linear equations corresponding to each value of q, in the form:

$$\begin{bmatrix}\theta_{qo} & \cdots & \theta_{q(p-1)}\end{bmatrix}\begin{bmatrix}1 & -\sum d_j & \sum\limits_{s\neq t} d_sd_t & -\sum\limits_{r\neq s\neq t} d_rd_sd_t & \cdots & (-1)^{p-1}d_1\cdots d_{p-1}\\ 0 & 1 & -\sum\limits_{j\neq 1} d_j & \sum\limits_{s\neq t\neq 1} d_sd_t & \cdot & (-1)^{p-1}d_2\cdots d_{p-1}\\ 0 & 1 & -\sum\limits_{j\neq 2} d_j & \cdot & \cdot & (-1)^{p-1}d_1d_3\cdot d_{p-1}\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 1 & -\sum\limits_{j\neq p-1} d_j & \cdot & \cdot & (-1)^{p-1}d_1\cdot d_{p-2}\end{bmatrix}$$

$$= \begin{bmatrix}0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0\end{bmatrix} \qquad (6.2.16)$$

↑qth position

Equations (6.2.16) will have a solution for each value of q if the $p \times p$ coefficient matrix on the left hand side of (6.2.16) has full rank, i.e., if the determinant of this matrix does not vanish. The determinant is of degree $(p-1)(p-2)/2$ in the d_j. It is zero if $d_i = d_j$, for any $i \neq j$, as this gives identical rows, so that the determinant has the factors:

$$(d_{p-1} - d_{p-2})(d_{p-1} - d_{p-3})\ldots(d_{p-2} - d_{p-3})\ldots(d_2 - d_1)$$

There are $(p-1)(p-2)/2$ such factors, which is equal to the degree of the determinant, so that these are the only factors, apart from, possibly, a non-zero numerical multiplier. It can be shown, by comparing the coefficient of any term, that this multiplier is unity, so that the determinant may be written as:

$$\prod(d_i - d_j),\ i > j\ ;\ i=2,\ \ldots\ (p-1)\ ;\ j=1,\ \ldots\ (p-2) \qquad (6.2.17)$$

It follows that the determinant is non-zero under the given condition of distinct observer eigenvalues. This completes the justification of the statement concerning the coefficient matrix in (6.2.8).

Equations (6.2.8), hence, are consistent. From (6.1.1), $mp \geq n$. It is clear that the solution is unique if $mp = n$. If $mp > n$, the solution is non-unique, and comprises the sum of a particular solution and the solutions of the homogeneous form of equation (6.2.8) multiplied by arbitrary scalar constants.

6.2.4 Solution for Real Observer Eigenvalues.

The solution can be simplified slightly if it is desired that the observer should have only real eigenvalues. In this case, D may be chosen to be diagonal, so that $D = \Delta$, and $U = I$, giving $K = M$, $G = RB$ and $g^T = e^T$.

6.2.5 Numerical Example.

The method of solution is illustrated by application to a simple numerical example given by Luenberger [L5], although the advantages of the procedure become more apparent as larger systems are considered.

In this example:

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad h^T = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}$$

p = 2, and the single observer eigenvalue is to be d = -3, so that the observer is trivially diagonal and, according to (6.2.6), $g^T = 1$. M has a single row, m^T, and (6.2.8) becomes:

$$\begin{bmatrix} f^T \vdots m^T \end{bmatrix} \begin{bmatrix} C(A - Id) \\ C \end{bmatrix} = h^T(A - Id) \quad .$$

$$(A - Id) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ -1 & 0 & 0 & 3 \end{bmatrix}$$

$$C(A - Id) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$h^T(A - Id) = \begin{bmatrix} -1 & 1 & 1 & 3 \end{bmatrix}$$

Hence, if $f^T = \begin{bmatrix} f_1 & f_2 \end{bmatrix}$, and $m^T = \begin{bmatrix} m_1 & m_2 \end{bmatrix}$, (6.2.8) becomes:

$$\begin{bmatrix} f_1 & f_2 & m_1 & m_2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 & 3 \end{bmatrix} ,$$

which yields $f_1 = 1$, $f_2 = 3$, $m_1 = -2$, $m_2 = -5$; whence:

$$K = M = \begin{bmatrix} -2 & -5 \end{bmatrix}$$

In this case, the solution is unique, because mp = n.

The row vector r^T is obtained from (6.2.10) as:

$$r^T = m^T C(A - Id)^{-1}$$

$$= \begin{bmatrix} -2 & -5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ -1 & 0 & 0 & 3 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -1 & 1 & -3 & 1 \end{bmatrix}.$$

Hence, $G = r^T B, = 1$.

These results agree with those obtained in [L5].

6.2.6 Conclusion.

The procedure described enables an observer to be designed to provide any specified linear functional of the system state vector. There is almost complete freedom of choice of the observer matrix D, which may have real or complex eigenvalues, provided only that these are distinct, and different from those of A.

The procedure is particularly suitable for use with a digital computer, in dealing with large systems.

This procedure has been described in [M4].

6.3 Low Order Linear Functional Observer.

Although the method of 6.2 permits the design of a linear functional observer of order (p-1), with arbitrary dynamics, where p is the observability index, it has been pointed out by Fortmann and Williamson [F6] that a reduction in observer order can be achieved by permitting the observer poles to be determined during the design process. The method described by Fortmann and Williamson is based on the companion canonical form approach of Luenberger, and requires the transformation of the system to a number of single-output sub-systems in the general case.

A procedure will now be described for the design of an observer of low order to provide a specified linear functional of the state vector of a linear system. The method is based on the procedure of 6.2, and is suitable for direct application to single-output and multi-output systems. The procedure yields information on the existence of an observer of given order, and on any constraints on the choice of observer poles. The method permits the investigation of observers of increasing order, until an acceptable solution is found.

6.3.1 System Description.

We consider a linear time-invariant system described by the equations:

$$\dot{x} = Ax + Bu$$
$$y = Cx \qquad (6.3.1)$$

where x, u and y are vectors of state, input and output, of dimension n, r and m, respectively, and A, B and C are constant matrices. The pair (A,C) is observable.

6.3.2 Problem Statement.

The problem is to design an observer described by the equation:

$$\dot{z} = Dz + Ky + Gu \qquad (6.3.2)$$

where z is the q-dimensional observer state vector, and D, K and G are constant matrices, such that, for a specified n-vector h^T, $(f^Ty + g^Tz)$ tends asymptotically to h^Tx. The matrices D, K and G, and the row vectors f^T and g^T are to be found such that D has acceptable, but not necessarily arbitrary, eigenvalues, and the dimension, q, of the observer, is to be as small as possible.

6.3.3 Procedure.

We postulate the existence of an observer of order q, to provide a linear functional specified by the vector h^T.

Let the characteristic polynomial of D be:

$$s^q + \beta_{q-1}s^{q-1} + \dots + \beta_1 s + \beta_o \quad . \qquad (6.3.3)$$

Then constraints on the coefficients β_i correspond to constraints on the choice of observer poles.

The following array is formed:

$$\begin{matrix} C \\ CA \\ \cdot \\ \cdot \\ CA^q \\ h^T \\ h^T A \\ \cdot \\ \cdot \\ h^T A^q \end{matrix} \qquad (6.3.4)$$

We now perform a row reduction of the array (6.3.4). The last (q+1) rows are reduced in the process, but are not used in reducing other rows.

If the last (q+1) rows are reduced to zero, there are no conditions on the β_i, and the observer poles can be chosen arbitrarily. Otherwise, each non-zero column in the last (q+1) rows, say, $\left[\gamma_1 \dots \gamma_{q+1}\right]^T$, provides a linear relationship among the β_i given by:

$$\sum_{i=0}^{q-1} \beta_i \gamma_{i+1} + \gamma_{q+1} = 0 \qquad (6.3.5)$$

The set of all equations of the type of (6.3.5) provides a set of constraints which must be satisfied by the β_i. If this set is inconsistent, there is no solution for the chosen value of q.

When a set of distinct observer poles has been chosen to satisfy the constraints, the design may be completed by following the procedure described in 6.2, substituting (q+1) for p therein, wherever it occurs. The matrix, U, of column eigenvectors of D, may be chosen arbitrarily, provided that the requirements of complex pairing are satisfied.

Proof.

Postulating an observer of order q leads to the equations of 6.2, with (q+1) replacing p. The necessary and sufficient condition for the consistency of equations (6.2.8), viz., that the vector on the right hand side lies in the space spanned by the rows of the coefficient matrix on the left hand side, is used to form conditions on the β_i. The coefficient matrix may be reduced by elementary row operations to the matrix:

$$\begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ CA^q \end{bmatrix} \qquad (6.3.6)$$

and the vector on the right hand side may be written as:

$$h^T A^q + \beta_{q-1} h^T A^{q-1} + \ldots + \beta_1 h^T A + \beta_o h^T \qquad (6.3.7)$$

where the β_i are the coefficients of the characteristic polynomial of D, as in (6.3.3).

Let $m(q+1) = q'$

Let the first q' rows of the array (6.3.4) be:
$v_1^T, \ldots v_{q'}^T$.

When the row reduction has been completed, the last (q+1) rows of (6.3.4) will have the form:

$$h^T - \alpha_{11} v_1^T - \alpha_{12} v_2^T - \ldots - \alpha_{1q'} v_{q'}^T$$

$$h^T A - \alpha_{21} v_1^T - \alpha_{22} v_2^T - \ldots - \alpha_{2q'} v_{q'}^T$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$h^T A^q - \alpha_{q+1,1} v_1^T - \ldots - \alpha_{q+1,q'} v_{q'}^T$$

where the α_{ij} are scalars resulting from the reduction process.

Application of the conditions (6.3.5) then gives:

$$(h^TA^q - \alpha_{q+1,1}v_1^T - \dots - \alpha_{q+1,q'}v_{q'}^T) +$$

$$\beta_{q-1}(h^TA^{q-1} - \alpha_{q,1}v_1^T - \dots - \alpha_{q,q'}v_{q'}^T) +$$

$$\cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot$$

$$+ \beta_1(h^TA - \alpha_{21}v_1^T - \dots - \alpha_{2q'}v_{q'}^T) +$$

$$\beta_o(h^T - \alpha_{11}v_1^T - \dots - \alpha_{1q'}v_{q'}^T) = 0^T$$

where 0^T is an n-dimensional vector of zeros.

Hence,

$$h^TA^q + \beta_{q-1}h^TA^{q-1} + \dots + \beta_1h^TA + \beta_oh^T =$$

$$(\alpha_{q+1,1}v_1^T + \dots + \alpha_{q+1,q'}v_{q'}^T)$$

$$+ \beta_{q-1}(\alpha_{q,1}v_1^T + \dots + \alpha_{q,q'}v_{q'}^T)$$

$$\dots + \beta_1(\alpha_{21}v_1^T + \dots + \alpha_{2,q'}v_{q'}^T) + \beta_o(\alpha_{11}v_1^T + \dots +)$$

which shows that the vector (6.3.7) lies in the space spanned by v_1^T, ... $v_{q'}^T$; hence, in the space spanned by the first m(q+1) rows of (6.3.4). This completes the proof.

The complete row reduction clearly results in the creation of the least number of conditions of the form of (6.3.5), because as many as possible of the columns of the last (q+1) rows of (6.3.4) are reduced to zero. If all the columns of the last (p+1) rows are reduced to zero, there are no conditions on the β_i.

6.3.4 Numerical Example.

The procedure is illustrated by application to an example from [F6] . Here,

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -3.5 \\ 0 & 0 & 0 & 1 & -1.5 \end{bmatrix} \qquad B = b = \begin{bmatrix} -3 \\ 5 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

$$C = c^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad ; \; h^T = \begin{bmatrix} 0.83 & -0.08 & -0.31 & 0.19 & 0.32 \end{bmatrix}$$

The array (6.3.4) is formed as:

$$\begin{array}{llllll} c^T & = 0 & 0 & 0 & 0 & 1 \\ c^TA & = 0 & 0 & 0 & 1 & -1.5 \\ c^TA^2 & = 0 & 0 & 1 & -1.5 & -1.25 \\ h^T & = 0.83 & -0.08 & -0.31 & 0.19 & 0.32 \\ h^TA & = -0.08 & -0.31 & 0.19 & 0.32 & -0.365 \\ h^TA^2 & = -0.31 & 0.19 & 0.32 & -0.365 & -0.3325 \end{array} \qquad (6.3.8)$$

Inspection reveals that no solution is possible if q = 0 or 1. With q = 2, row reduction in this case obviously leaves only the first two columns non-zero in the last three rows. This gives the condition:

$$[-0.31\ 0.19] + \beta_1[-0.08\ -0.31] + \beta_o[0.83\ -0.08] = [0\ \ 0]$$

from which $\beta_o = 0.423$, $\beta_1 = 0.505$, and the observer poles are at $(-0.25 \pm j0.6)$, which agrees with the result in [F6] .

If we are not satisfied with these poles, we may consider a third-order observer, and this simply involves including the following extra rows in the appropriate places in the array (6.3.8):

$c^TA^3 =$	0	1	-1.5	-1.25	5.13
$h^TA^3 =$	0.19	0.32	-0.365	-0.3325	0.75625

Row reduction in this case leaves only the first column of the last four rows non-zero, and gives the condition:

$$0.19 - 0.31\beta_2 - 0.08\beta_1 + 0.83\beta_o = 0 \qquad (6.3.9)$$

We may then specify observer poles at, say, -1 and -2, and an unknown λ. Inserting these in (6.3.9) gives $\lambda = -0.81$.

6.3.5 Conclusion.

The method, which has been described in [M9] , enables a linear functional observer of low order to be designed in a routine manner. The existence of a design of any given order is established, and the constraints imposed on the observer poles are obtained as a set of linear equations in the coefficients of the characteristic polynomial of the D matrix of the observer.

The numerical example demonstrates how the procedure deals with all possible cases. If it had been found that the row reduction of (6.3.8) eliminated the last row of the array with q = 0, this would have meant that the required h^T happened to lie in the space spanned by the rows of C, so that no observer was needed. There was, in fact, no solution for q = 0 or 1, and q = 2 gave conditions which required the use of a certain pair of observer eigenvalues, and no others. With q = 3, the choice was widened, so that two observer eigenvalues could be chosen, and the third was then determined. If q = 4 had been tried, this would have resulted in elimination of the last 5 rows of the array, revealing that there were no conditions on the observer eigenvalues. This is consistent with normal observer theory since, in this case, q = (n-m).

6.4 Linear Functional Observer with Repeated Eigenvalues.

6.4.1 Introduction.

The condition that the observer eigenvalues are distinct leads to the simple solution which has been described. However, this condition is not necessary, and the method of solution in the case in which any number of the observer eigenvalues are repeated will now be described.

6.4.2 System Description and Problem Statement.

The system description is as given in 6.2.1, and the problem statement is as in 6.2.2, except that the eigenvalues of D are not now required to be distinct, although they are to be different from the eigenvalues of A.

6.4.3 Solution.

In the solution, we now assume that Δ has a Jordan canonical form, in which each eigenvalue is found in only one Jordan block. U is then a matrix of self-conjugate column eigenvectors and generalised eigenvectors of D, so that D is real. Then:

$$D = U \Delta U^{-1} \qquad (6.4.1)$$

where, for example, if d_1 has multiplicity 3, Δ may have the form shown in (6.4.2).

$$\Delta = \begin{bmatrix} d_1 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & d_1 & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & d_1 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & d_4 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \qquad (6.4.2)$$

Let $g^T = e^T U^{-1}$ (6.4.3)

where e^T is the (p-1)-dimensional sum vector $[1 \; 1 \; 1 \; .. \; 1]$.

Then:

$$K = UM \qquad (6.4.4)$$

where f^T and the rows of M, (m_1^T, ... m_{p-1}^T), are obtained as the solution of the set of linear equations (6.4.5) where, for example, $d_{q+2} = d_{q+1} = d_q$.

In (6.4.5), I is the identity matrix of order n.

$$\left[f^T \vdots m_1^T \vdots \cdots \vdots m_{p-1}^T\right] \begin{bmatrix} C\prod_{j=1}^{p-1}(A-d_jI) \\ \vdots \\ C\left\{\prod_{\substack{j=1\\ j\neq q}}^{p-1}(A-d_jI) + \prod_{\substack{j=1\\ j\neq q\\ j\neq q+1}}^{p-1}(A-d_jI) + \prod_{\substack{j=1\\ j\neq q\\ j\neq q+1\\ j\neq q+2}}^{p-1}(A-d_jI)\right\} \\ C\left\{\prod_{\substack{j=1\\ j\neq q}}^{p-1}(A-d_jI) + \prod_{\substack{j=1\\ j\neq q\\ j\neq q+1}}^{p-1}(A-d_jI)\right\} \\ C\prod_{\substack{j=1\\ j\neq q}}^{p-1}(A-d_jI) \\ \vdots \\ C\prod_{j=1}^{p-2}(A-d_jI) \end{bmatrix}$$

$$= h^T\prod_{j=1}^{p-1}(A-d_jI). \qquad (6.4.5)$$

G is obtained from:

$$G = URB \qquad (6.4.6)$$

In the coefficient matrix of (6.4.5), each eigenvalue of multiplicity v gives rise to v blocks, each of m rows, where, in each block, the product term is replaced by the sum of product terms, each of which contains one less factor $(A-d_jI)$ than its predecessor, as indicated for the particular case v = 3.

In (6.4.6), R is a $(p-1)\times n$ matrix with rows $(r_1^T, \ldots r_{p-1}^T)$, where, for each simple eigenvalue of D,

$$r_j^T = m_j^T C(A-d_jI)^{-1} \tag{6.4.7}$$

whilst for each eigenvalue of multiplicity v, there is a block of v rows, given by:

$$r_j^T = m_j^T C(A-d_jI)^{-1} + m_{j+1}^T C(A-d_jI)^{-2} + \ldots$$

$$\ldots + m_{j+v-1}^T C(A-d_jI)^{-v}$$

$$r_{j+1}^T = m_{j+1}^T C(A-d_jI)^{-1} + m_{j+2}^T C(A-d_jI)^{-2} + \ldots$$

$$\ldots + m_{j+v-1}^T C(A-d_jI)^{-v+1}$$

$$\vdots$$

$$r_{j+v-1}^T = m_{j+v-1}^T C(A-d_jI)^{-1} \tag{6.4.8}$$

Equations (6.4.5) are consistent, and the solution is unique if mp = n, and non-unique if mp > n. This completes the solution.

Proof.

The matrix T relating the state vectors of the observer and the original system is the solution of the observer equation:

$$TA - DT = KC \qquad (6.4.9)$$

This equation has a unique solution if A and D have no common eigenvalue. This condition is satisfied in this case. Setting $U^{-1}T = R$, and $U^{-1}K = M$, equation (6.4.9) becomes:

$$RA - \Delta R = MC \qquad (6.4.10)$$

Δ is now in the Jordan form, and equation (6.4.10) may be solved row-by-row, to give (6.4.7) and (6.4.8).

The vector equation (6.2.13) applies in this case:

$$f^T C + e^T R = h^T \qquad (6.4.11)$$

Substituting (6.4.7) and (6.4.8) in (6.4.11) gives, for example, when $d_3 = d_2 = d_1$, and d_4 is simple,

$$f^T C + m_1^T C\,(A-d_1 I)^{-1} + m_2^T C\left\{(A-d_1 I)^{-1} + (A-d_1 I)^{-2}\right\}$$
$$+ m_3^T C\left\{(A-d_1 I)^{-1} + (A-d_1 I)^{-2} + (A-d_1 I)^{-3}\right\} + m_4^T (A-d_4 I)^{-1}$$
$$= h^T \qquad (6.4.12)$$

Post-multiplication of (6.4.12) by:

$$\prod_{j=1}^{p-1} (A-d_j I)$$

and rearranging give (6.4.5). Equation (6.4.4) follows from the definition of M. From [L4] , G = TB. Hence, (6.4.6) follows from the definition of R given above.

The only unknowns in (6.4.5) are the mp elements of the vector $\left[f^T \vdots m_1^T \vdots \cdots \vdots m_{p-1}^T\right]$. It will now be shown that the coefficient matrix on the left hand side of (6.4.5) has the same rank as Q in (6.1.1), namely rank n, by definition. This is done by showing that the coefficient matrix can be reduced to the form of Q by elementary row operations. We first note that this matrix can be reduced to the form:

$$\begin{bmatrix} C\prod_{j=1}^{p-1}(A-d_jI) \\ \vdots \\ C\prod_{\substack{j=1\\ j\neq q\\ j\neq q+1\\ j\neq q+2}}^{p-1}(A-d_jI) \\ C\prod_{\substack{j=1\\ j\neq q\\ j\neq q+1}}^{p-1}(A-d_jI) \\ C\prod_{\substack{j=1\\ j\neq q}}^{p-1}(A-d_jI) \\ \vdots \\ C\prod_{j=1}^{p-2}(A-d_jI) \end{bmatrix} \qquad (6.4.13)$$

The proof will be facilitated by introducing a more general notation. Suppose the linear functional observer eigenvalues are to be:

d_1 with multiplicity v_1,
d_2 with multiplicity v_2,
. . . .
. . . .
d_w with multiplicity v_w,

so that:

$$v_1 + v_2 + \ldots + v_w = p - 1$$

With this notation, and a rearrangement of the m-rowed blocks, the matrix (6.4.13) may be written as in (6.4.14).

$$\begin{bmatrix} C(A-d_1I)^{v_1}(A-d_2I)^{v_2} \;.\;.\; \;..\; \;..\; (A-d_wI)^{v_w} \\ C(A-d_1I)^{v_1-1}(A-d_2I)^{v_2} \;.\; \;..\; \;..\; (A-d_wI)^{v_w} \\ C(A-d_1I)^{v_1-2}(A-d_2I)^{v_2} \;.\; \;..\; \;..\; (A-d_wI)^{v_w} \\ \cdot \qquad \cdot \qquad \qquad \cdot \\ C(A-d_1I)(A-d_2I)^{v_2}(A-d_3I)^{v_3} \;..\; (A-d_wI)^{v_w} \\ C(A-d_2I)^{v_2}(A-d_3I)^{v_3} \;.\; \;..\; \;..\; (A-d_wI)^{v_w} \\ C(A-d_1I)^{v_1}(A-d_2I)^{v_2-1} \;.\; \;..\; \;..\; (A-d_wI)^{v_w} \\ \cdot \qquad \cdot \qquad \qquad \cdot \\ C(A-d_1I)^{v_1}(A-d_2I)(A-d_3I)^{v_3} \;..\; (A-d_wI)^{v_w} \\ C(A-d_1I)^{v_1}(A-d_3I)^{v_3} \;.\; \;..\; \;..\; (A-d_wI)^{v_w} \\ \cdot \qquad \cdot \qquad \qquad \cdot \\ C(A-d_1I)^{v_1}(A-d_2I)^{v_2} \;.\; \;..\; \;..\; (A-d_{w-1}I)^{v_{w-1}} \end{bmatrix} \qquad (6.4.14)$$

The first m-rowed block of (6.4.14) contains all the factors of the type $(A-d_jI)$ raised to powers equal to their multiplicities, v_j. In successive m-rowed blocks, one such factor has its power reduced by one at each step until zero power is reached, whilst the other factors remain unchanged. This process is repeated for each factor in turn, with all other factors raised to powers equal to their multiplicities.

We wish to show that the $mp \times n$ matrix (6.4.14) can be reduced to that of (6.1.1) by elementary row operations. The following lemma is required in the proof.

Lemma.

Given the vector R in (6.4.15):

$$R = \begin{bmatrix} (s-d_1)^{v_1}(s-d_2)^{v_2} \;\; .. \;\; .. \;\; .. \;\; .. \;\; (s-d_w)^{v_w} \\ (s-d_1)^{v_1-1}(s-d_2)^{v_2} \;\; .. \;\; .. \;\; .. \;\; (s-d_w)^{v_w} \\ (s-d_1)^{v_1-2}(s-d_2)^{v_2} \;\; .. \;\; .. \;\; .. \;\; (s-d_w)^{v_w} \\ \cdot \qquad \cdot \qquad \cdot \\ (s-d_1)(s-d_2)^{v_2}(s-d_3)^{v_3} \;\; .. \;\; .. \;\; (s-d_w)^{v_w} \\ (s-d_2)^{v_2}(s-d_3)^{v_3} \;\; .. \;\; .. \;\; .. \;\; (s-d_w)^{v_w} \\ (s-d_1)^{v_1}(s-d_2)^{v_2-1} \;\; .. \;\; .. \;\; .. \;\; (s-d_w)^{v_w} \\ \cdot \qquad \cdot \qquad \cdot \\ (s-d_1)^{v_1}(s-d_2)(s-d_3)^{v_3} \;\; .. \;\; .. \;\; (s-d_w)^{v_w} \\ (s-d_1)^{v_1}(s-d_3)^{v_3} \;\; .. \;\; .. \;\; .. \;\; (s-d_w)^{v_w} \\ \cdot \qquad \cdot \qquad \cdot \\ (s-d_1)^{v_1}(s-d_2)^{v_2} \;\; .. \;\; .. \;\; .. \;\; (s-d_{w-1})^{v_{w-1}} \end{bmatrix} \qquad (6.4.15)$$

in which s is a scalar variable, $d_j \neq 0$, j=1, ... w, and $d_i \neq d_j$, $i \neq j$, then there exists a constant $p \times p$

matrix $\hat{T}$, such that:

$$\hat{T}R = \begin{bmatrix} 1 \\ s \\ s^2 \\ \cdot \\ \cdot \\ s^{v_1+v_2+\ldots+v_w} \end{bmatrix} \qquad (6.4.16)$$

where:

$$p = (1 + \sum_{j=1}^{w} v_j)$$

The formation of the scalar factors in (6.4.15) coincides with that described for the m-rowed blocks of (6.4.14).

Proof.

We first note that each row of (6.4.16) represents a polynomial equation of degree $(p-1)$ in s, so that we may seek to determine the p elements of each row of $\hat{T}$ by substituting different values of s in this equation, or its derivatives with respect to s, so as to provide p conditions. When these substitutions are made in (6.4.16), the conditions are applied to all p rows of $\hat{T}$ simultaneously.

The method of proof is to substitute different values of s in equation (6.4.16), and its derivatives with respect to s, according to a particular scheme which will be described, until the required number of conditions are applied. The p vector equations obtained are written as a single matrix equation, and this is examined for the existence of a solution for $\hat{T}$. The matrix equation has the form:

$$\hat{T}\hat{R} = \hat{S} \qquad (6.4.17)$$

where $\hat{R}$ and $\hat{S}$ are constant $p \times p$ matrices.

The columns of $\hat{R}$ and $\hat{S}$ are formed as follows, where, for brevity, S represents the right hand side of (6.4.16):

The first columns of $\hat{R}$ and $\hat{S}$ are obtained by setting s=0 in R and S respectively.

The second columns of $\hat{R}$ and $\hat{S}$ are obtained by differentiating R and S (v_1-1) times with respect to s, and setting $s=d_1$.

The third columns of $\hat{R}$ and $\hat{S}$ are obtained by differentiating R and S (v_1-2) times with respect to s, and setting $s=d_1$.

We continue in this way, until the (v_1+1)th columns of $\hat{R}$ and $\hat{S}$ are obtained by setting $s=d_1$ in R and S respectively.

The (v_1+2)th columns of $\hat{R}$ and $\hat{S}$ are obtained by differentiating R and S (v_2-1) times with respect to s, and setting $s=d_2$.

This process is continued to completion, giving p columns in (6.4.17).

It is clear that $\hat{R}$ is a lower triangular matrix, and this matrix will be non-singular if each diagonal element is non-zero. The diagonal elements, in order, are:

$$\begin{array}{l}
(-d_1)^{v_1}(-d_2)^{v_2} \;\cdots\cdots\cdots\; (-d_w)^{v_w} \\
(v_1-1)!(d_1-d_2)^{v_2} \;\cdots\cdots\; (d_1-d_w)^{v_w} \\
(v_1-2)!(d_1-d_2)^{v_2} \;\cdots\cdots\; (d_1-d_w)^{v_w} \\
\quad\cdot \qquad \cdot \qquad\qquad \cdot \\
(d_1-d_2)^{v_2} \;\cdots\cdots\; (d_1-d_w)^{v_w} \\
(v_2-1)!(d_2-d_1)^{v_1}(d_2-d_3)^{v_3} \;\cdots\; (d_2-d_w)^{v_w} \\
(v_2-2)!(d_2-d_1)^{v_1}(d_2-d_3)^{v_3} \;\cdots\; (d_2-d_w)^{v_w} \\
\quad\cdot \qquad \cdot \qquad \cdot \qquad \cdot \\
(d_2-d_1)^{v_1}(d_2-d_3)^{v_3} \;\cdots\; (d_2-d_w)^{v_w} \\
\quad\cdot \qquad \cdot \qquad \cdot \qquad \cdot \\
(d_w-d_1)^{v_1}(d_w-d_2)^{v_2} \;\cdots\; (d_w-d_{w-1})^{v_{w-1}}
\end{array} \tag{6.4.18}$$

These diagonal elements are all non-zero, under the given conditions:

$d_j \neq 0, \quad j=1, \ldots w$, and

$d_i \neq d_j, \quad i \neq j$,

so that $\hat{R}$ is non-singular. Hence, (6.4.17) can be solved for $\hat{T}$. This completes the proof of the lemma.

Proof of the Main Result.

The proof of the main result follows at once from the lemma, since the row operations implied by the constant transformation matrix $\hat{T}$ of the lemma, when applied to the m-rowed blocks of (6.4.14), will reduce (6.4.14) to the form of (6.1.1).

It follows that equations (6.4.5) are consistent. The solution is unique if mp = n, and non-unique if mp > n.

Note.

This general proof also provides an alternative to the proof given in 6.2.3 for the particular case of a linear functional observer with distinct eigenvalues.

6.4.4 An Observer Theorem.

From the results of this section and 6.2, the following theorem can be stated.

Theorem.

For any controllable, observable, linear time-invariant system, there exists an observer of order (p-1), where p is the observability index, such that, by the use of constant feedback to the system inputs from the system outputs and the observer state vector, the eigenvalues of the composite system can be assigned arbitrarily.

The proof is given on the following page.

Proof.

The system (A,B,C) may be assumed to be controllable through a single input. If this condition is not satisfied at the outset, it may be achieved by applying almost arbitrary feedback between the system outputs and inputs [D2] . This feedback will not disturb the observability of the system. If the application of this feedback makes any of the system eigenvalues equal to those required for the observer, the feedback is changed, so as to make the system eigenvalues distinct from the required observer eigenvalues, whilst retaining the single-input controllability [D2] .

The system is now represented by ((A+BKC),b,C), where K is the feedback matrix, and b is the input vector through which the system is controllable.
A linear functional observer of order (p-1), with arbitrary simple or multiple stable eigenvalues will then permit the arbitrary assignment of the closed-loop system eigenvalues, using the results of 6.2 and 6.4, because a single linear functional of the state vector is sufficient to achieve the required eigenvalue assignment.

6.4.5 Conclusion.

Although the use of an observer with repeated eigenvalues is unlikely to be a requirement, the results of this section permit the generalisation of the technique presented in 6.2, and lead to the theorem of 6.4.4, from which it may be concluded that, if pole assignment is the only consideration, this can always be achieved for a controllable observable system with an observer of order (p-1), where p is the observability index. In other words, the results obtained by Brasch and Pearson [B3] , using general dynamic compensation, can be achieved with an observer, with its attendant advantage of not imposing its poles on the input-output transfer functions of the overall system.

CHAPTER 7.

DEGENERATE OBSERVER.

7.1 Introduction.

The generalisation of a linear functional observer, in which estimates of more than one linear functional of the state vector of a linear system are required, is termed a 'degenerate' observer. Such an observer may be needed in connection with pole and zero assignment. For example, in the cases considered in 4.3 and 4.4, of providing a desired scalar transfer function between one input and one output, two linear functionals of the state vector are required.

The problem of designing a degenerate observer in the general case is considerably more difficult than that of designing either a linear functional observer or a state observer, and the question of achieving a minimum order design has not yet been solved. Cumming [C5] obtained a sufficient condition for the existence of a degenerate observer to provide estimates of specified linear functionals, although this condition does not ensure that the observer will be stable. Fortmann and Williamson [F6] obtained necessary and sufficient conditions for the existence of an observer with specified poles, in the case of a single-output system.

By an extension of the procedures described in 6.2 and 6.3, methods will now be described for the design of stable degenerate observers to provide estimates of any required number of linear functionals of the state vector. The procedures may result in designs of quite low order, but there is no reason to suppose that they provide minimum-order designs in general. Two procedures are given, one for the design of a degenerate observer with arbitrary poles, and another for the case in which some constraint on the choice of observer poles is accepted in order to achieve reduction in observer order.

7.2 Degenerate Observer with Arbitrary Poles.

The problem considered is the design of an observer of reduced order, with arbitrary dynamics, to provide estimates of a number of specified linear functionals, $h_1^T x$, $h_2^T x$, etc., of the state vector, x, of a time-invariant linear system described by the matrix triple (A,B,C). One possible solution is to design a state observer of dimension (n-m), where n and m are the dimensions of the system state vector and output vector, respectively, and to obtain the required linear functionals from this. Another approach is to design a separate linear functional observer for each linear functional required. Either of these methods may, however, result in an observer of unnecessarily large dimensions.

In the present method, a solution of lower order is sought by designing a succession of observers, one for each linear functional, in which each observer is driven by all the preceding observers, as well as by the system outputs and inputs.

7.2.1 Design Procedure.

At the first stage, a linear functional observer is designed for the system (A,B,C), using the procedure of 6.2, to provide an estimate of the first linear functional, $h_1^T x$. The dimension of this observer is (p_1-1), where p_1 is the observability index of (A,C).

At each subsequent stage, say the jth, a linear functional observer is designed for the system:

$$\left\{ A, B, \begin{bmatrix} C \\ T_1 \\ \cdot \\ \cdot \\ T_{j-1} \end{bmatrix} \right\},$$

using the procedure of 6.2 to provide an estimate of $h_j^T x$. The dimension of this observer is (p_j-1), where p_j

is the observability index of:

$$\left\{ A, \begin{bmatrix} C \\ T_1 \\ \cdot \\ \cdot \\ T_{j-1} \end{bmatrix} \right\}.$$

In the procedure, at each stage, the matrix T_j is found, corresponding to the matrix T in 6.2.

The validity of the procedure is established by the following theorem.

Theorem.

An s-stage observer designed according to the above procedure yields asymptotic estimates of the linear functionals $h_1^T x$, $h_2^T x$, ... $h_s^T x$. The complete observer has as its poles the poles of all the individual linear functional observers.

Proof.

The state vector, z_j, of the jth linear functional observer is governed by the differential equation:

$$\dot{z}_j = D_j z_j + K_j y + \sum_{i=1}^{j-1} K_{ji} z_i + L_j u \qquad (7.2.1)$$

where K_j, the K_{ji}, and L_j are matrices coupling this observer to the system outputs y, the state vectors of the other observers z_i, and the external inputs u.

$L_j = T_jB$. The summation is zero for $j = 1$.

The estimate of h_j^Tx is obtained as:

$$f_j^Ty + \sum_{i=1}^{j} g_{ji}^T z_i \tag{7.2.2}$$

Defining $v_j = z_j - T_jx$, it is easily shown from (7.2.1), the original system equations,and the relationships obtained in 6.2, that:

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \cdot \\ \cdot \\ \dot{v}_s \end{bmatrix} = \begin{bmatrix} D_1 & 0 & 0 & 0 & \cdots & 0 \\ K_{21} & D_2 & 0 & 0 & \cdots & 0 \\ K_{31} & K_{32} & D_3 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ K_{s1} & K_{s2} & \cdot & \cdot & \cdot & D_s \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \cdot \\ \cdot \\ v_s \end{bmatrix} \tag{7.2.3}$$

It is clear from (7.2.3) that the complete observer has as its poles the eigenvalues of $(D_1, \ldots, D_s)$. Provided that these eigenvalues are chosen to be stable, $v_j \to 0$, as $t \to \infty$, so that $z_j \to T_jx$. This completes the proof.

Remark.

The observability index at each stage, p_j, is less than or equal to that for the preceding stage, p_{j-1}, because T_{j-1} must contain at least one row which is linearly independent of the rows of C, and of the T_i, $i=1, \ldots, (j-2)$. Otherwise, the $(j-1)$th linear functional observer would have been unnecessary, since $h_{j-1}^T x$ could have been obtained from y and the $z_1, \dots z_{j-2}$.

7.2.2 Conclusion.

As an example of the saving in observer dimensions which may be achieved by using this procedure, a system of dimension 12, with 3 outputs, having an observability index 4, for which 2 linear functionals were required, would need a state observer of dimension 9, or two separate linear functional observers of dimension 3, giving a total observer dimension of 6. Using the procedure described, if the observability index at the second stage were 3, the total dimension of the observer would be 5. If three linear functionals were required with this system, the corresponding dimensions would be 9, 9 and 7.

This procedure has been described in [M8] .

7.3 Low Order Degenerate Observer.

The method of 6.3 may be applied at each stage of the procedure described in 7.2, so that, instead of designing a linear functional observer of dimension (p_j-1) at the jth stage, an attempt is made to achieve a lower order design, by investigating the resulting constraints on the choice of observer poles.

The procedure may or may not give an overall degenerate observer design of lower order, depending upon the particular problem. Reduction in the order of the jth stage linear functional observer will give a smaller number of rows to T_j, and this will make it less probable that a reduction in order of the (j+1)th stage can be achieved.

The procedure has been reported in [M10].

7.3.1 Numerical Example.

The procedure described for the design of a degenerate observer of low order may be illustrated by extending the example of a low-order linear functional observer given in 6.3.4.

The design in 6.3.4 may be completed by choosing the eigenvector matrix of D as:

$$U = \begin{bmatrix} -0.8350j & 0.8350j \\ 0.5 - 0.2104j & 0.5 + 0.2104j \end{bmatrix}$$

giving:

$$D = \begin{bmatrix} -0.5040 & 1 \\ -0.4221 & 0 \end{bmatrix}$$

$$f^T = \begin{bmatrix} 0.2849 \end{bmatrix}$$

$$g^T = \begin{bmatrix} -0.5040 & 2 \end{bmatrix}$$

$$K = \begin{bmatrix} -0.3763 \\ -0.01471 \end{bmatrix}$$

$$G = \begin{bmatrix} 0.3195 \\ -0.08 \end{bmatrix}$$

and:

$$T = \begin{bmatrix} -0.1801 & 0.4604 & -0.1560 & -0.1157 & 0.1242 \\ 0.3696 & 0.0760 & -0.1943 & 0.6585 & 0.0488 \end{bmatrix}$$

The observability index of:

$$(A, \begin{bmatrix} c^T \\ T \end{bmatrix}) \qquad (7.3.1)$$

is 2, so that a linear functional observer of order 1 may be designed for the system (7.3.1), with arbitrary dynamics, to provide an estimate of any specified linear functional of the system state vector.

Thus, the two linear functionals may be provided by a degenerate observer of total order 2 + 1 = 3, for a system of order 5, with one output. This represents a saving, in this case, of 1 order, compared with a state observer, from which the two linear functionals could have been obtained, with arbitrary observer dynamics.

CHAPTER 8.

DUAL OBSERVER.

8.1 Introduction.

The concept of the dual observer, which is attributed to Brasch, has been reported by Luenberger [L5] . The dual observer is a special kind of controller which, when used with a time-invariant linear system, permits the assignment of all the closed-loop poles of the composite system. The important feature of the dual observer is that, whereas, in the case of an observer, the dimension of the observer is determined by the number of system outputs, or by the observability index, the dimension of the dual observer is determined by the number of system inputs, or by the controllability index. The dual observer, thus, may offer an advantage in cases in which the system has more inputs than outputs, or where the controllability index is lower than the observability index.

8.2 Linear System with Dual Observer.

We consider a linear time-invariant plant described by the equations:

$$\dot{x} = Ax + Bu$$

$$y = Cx \qquad (8.2.1)$$

where x, y and u are vectors of dimension n, m and r, representing the system states, outputs and inputs, respectively. The triple (A,B,C) is controllable and observable.

The arrangement of the dual observer, together with the plant, is shown in Fig. 8.1.

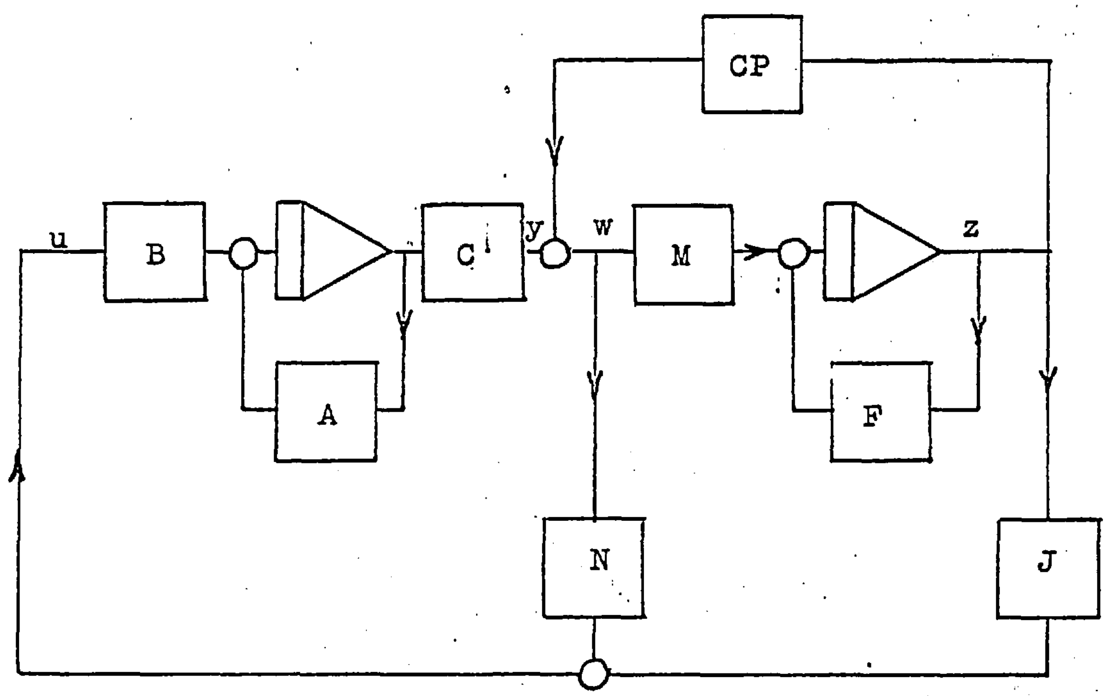


Fig. 8.1.

The dual observer is of dimension ρ, and is governed by the following equations:

$$\dot{z} = Fz + Mw \qquad (8.2.2)$$

$$w = y + CPz \qquad (8.2.3)$$

$$u = Jz + Nw \qquad (8.2.4)$$

where z is the ρ-dimensional dual observer state vector, and P is the solution of the matrix equation:

$$AP - PF = BJ \qquad (8.2.5)$$

The dimensions of the constant matrices F, J, M, N and P, and of the variable vector w, will be clear from the equations. The matrix L is defined as:

$$L = PM + BN \qquad (8.2.6)$$

Equations (8.2.1) to (8.2.6) give:

$$\dot{x} = Ax + BJz + BNCx + BNCPz \qquad (8.2.7)$$

$$\dot{z} = Fz + MCx + MCPz \qquad (8.2.8)$$

Pre-multiplying (8.2.8) by P, and adding to (8.2.7), gives:

$$(\dot{x} + P\dot{z}) = (A + LC)(x + Pz) \qquad (8.2.9)$$

Setting x = v - Pz, the overall differential equation of the system with dual observer is:

$$\begin{bmatrix} \dot{v} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} (A + LC) & 0 \\ MC & F \end{bmatrix} \begin{bmatrix} v \\ z \end{bmatrix} \qquad (8.2.10)$$

It is clear from (8.2.10) that the eigenvalues of the composite system are those of F, and those of (A + LC). The dual observer thus exhibits the 'separation' property possessed by the observer. Note that no assumption has, so far, been made concerning the value of ρ.

It is now necessary to show that the eigenvalues of the composite system can be assigned arbitrarily. Luenberger [L5] has given the following theorem, as Theorem 5: 'Corresponding to an nth-order completely controllable and completely observable system having r linearly independent inputs, a dynamic feedback system of order (n-r) can be constructed such that the (2n-r) eigenvalues of the composite system take any preassigned values.' A proof of this theorem will now be given, by using a construction which is a dual of Cumming's method for designing state observers. The construction also constitutes a procedure for designing a dual observer of order (n-r).

8.3 Construction of Dual Observer of Order (n-r), and Proof of its Properties.

It has been shown that the eigenvalues of the composite system are the eigenvalues of (A + LC) and the eigenvalues of F. Let ρ = (n-r). It then follows that, since (A,C) is observable, the (2n-r) eigenvalues of the composite system can all be given preassigned values if the matrix L and the eigenvalues of F can be chosen arbitrarily. Let L be determined, by using modal control theory, so that (A + LC) has any desired set of eigenvalues. The following problem then remains:

Given A, B, C and L, in equations (8.2.1) to (8.2.6), to find J, M, N, P and F, such that F has arbitrarily chosen eigenvalues.

Choose any $n \times (n-r)$ constant matrix Q, such that the $n \times n$ matrix $\left[B \vdots Q\right]$ is non-singular. It may be assumed, without loss of generality, that the columns of B are linearly independent.

Form the matrix:

$$\left[B \vdots Q\right]^{-1} AQ = \begin{bmatrix} S \\ \hdashline T \end{bmatrix} \qquad (8.3.1)$$

where the partitioning is such that S is $r \times (n-r)$, and T is $(n-r) \times (n-r)$.

and the matrix:

$$\left[B \vdots Q\right]^{-1} AB = \begin{bmatrix} U \\ \hdashline V \end{bmatrix} \qquad (8.3.2)$$

where the partitioning is such that U is $r \times r$, and V is $(n-r) \times r$. Then:

$$F = T + VR \qquad (8.3.3)$$

and $$J = S + UR - RF \qquad (8.3.4)$$

where R is an $r \times (n-r)$ matrix to be determined.

It will be shown that the pair (T,V) is controllable, and so, from equation (8.3.3), the eigenvalues of F may be assigned arbitrarily by using modal control theory. When R and F have been determined in this way, J may be found from (8.3.4). M and N are found from (8.3.5):

$$\begin{bmatrix} N \\ M \end{bmatrix} = \left[B \,\vdots\, (Q + BR) \right]^{-1} L \qquad (8.3.5)$$

This completes the construction.

<u>Proof.</u>

Let Q be chosen as described, and let:

$$P = Q + BR \qquad (8.3.6)$$

Substituting in (8.2.5) gives:

$$AQ + ABR = \left[B \,\vdots\, Q \right] \begin{bmatrix} (J + RF) \\ F \end{bmatrix} \qquad (8.3.7)$$

Pre-multiplication by the inverse of $\left[B \,\vdots\, Q \right]$ and partitioning as in (8.3.1) and (8.3.2) gives (8.3.3) and (8.3.4). Equation (8.3.5) follows from (8.2.6) and (8.3.6). The inverse exists in (8.3.5) because the rank of the matrix to be inverted is the same as that of $\left[B \,\vdots\, Q \right]$, namely n, due to the way in which Q was chosen.

<u>Proof of Controllability of (T,V).</u>

Let:

$$\left[B \,\vdots\, Q \right]^{-1} = \begin{bmatrix} D \\ E \end{bmatrix} \qquad (8.3.8)$$

where the partitioning is such that D is $r \times n$, and E is

$(n-r) \times n$. It follows that:

$$DB = I_r \quad ; \quad DQ = 0$$
$$EB = 0 \quad ; \quad EQ = I_{n-r} \tag{8.3.9}$$

From (8.3.1) and (8.3.2),

$$T = EAQ \quad ; \quad V = EAB \tag{8.3.10}$$

The pair (T,V) is controllable if the $(n-r) \times (n-r)r$ matrix H has rank $(n-r)$, where:

$$H = \left[V \vdots TV \vdots T^2V \ \dots \ \vdots T^{n-r-1}V\right] \tag{8.3.11}$$

Substituting from (8.3.10) in (8.3.11),

$$H = \left[EAB \vdots EAQEAB \vdots EAQEAQEAB \vdots \ \dots \ \right] \tag{8.3.12}$$

Since:

$$\left[B \vdots Q\right] \begin{bmatrix} D \\ E \end{bmatrix} = I_n,$$

$$QE = I_n - BD \tag{8.3.13}$$

Substituting from (8.3.13) in (8.3.12),

$$H = E\left[AB \vdots A^2B - ABDAB \vdots A^3B - ABDA^2B - A^2BDAB + ABDABDAB \vdots \dots\right]$$

By elementary column operations, it may be shown that H has the same rank as:

$$G = E\left[AB \vdots A^2B \vdots A^3B \vdots \ \dots \ \vdots A^{n-r}B\right] \tag{8.3.14}$$

Now consider the matrix product:

$$E\left[B \vdots AB \vdots A^2B \vdots \ \dots \ \vdots A^{n-r}B\right] \tag{8.3.15}$$

The rank of E is $(n-r)$, and the rank of the second matrix in (8.3.15) is n, because (A,B) is controllable and the columns of B are linearly independent. Hence, the product matrix has rank (n-r). But EB = 0, from (8.3.9), and so G has rank (n-r). This completes the proof.

8.4 Design Procedure for Reduced-Order Dual Observer.

Since the dual observer is a particular form of dynamic compensator, it is to be expected, from the results obtained by Brasch and Pearson, [B3], that a dual observer of order (q-1) could be designed to permit arbitrary assignment of the (n+q-1) poles of the composite system, where q is the controllability index. A procedure is now described that establishes the existence of such a design, and provides a convenient computational procedure, for the case in which the system A matrix is non-derogatory.

8.4.1 System Description.

A linear time-invariant plant is described by the equations:

$$\dot{x} = Ax + Bu \qquad (8.4.1)$$

$$y = Cx \qquad (8.4.2)$$

where x is an n-dimensional plant state vector, u is an r-dimensional input vector, and y is an m-dimensional output vector. The triple (A,B,C) is observable and controllable, with a controllability index q, where q is the least integer for which the matrix Q has rank n, where:

$$Q = \left[B \,\vdots\, AB \,\vdots\, A^2B \,\vdots\, \ldots \,\vdots\, A^{q-1}B\right] \qquad (8.4.3)$$

The matrix A is non-derogatory.

The dual observer and its coupling with the plant are described by the equations:

$$\dot{z} = Fz + Mw \qquad (8.4.4)$$

$$w = y + CPz \qquad (8.4.5)$$

$$u = Jz + Nw \qquad (8.4.6)$$

where z is the (q-1)-dimensional state vector of the dual observer. P is a $n \times (q-1)$ constant matrix that satisfies the equation:

$$AP - PF = BJ \qquad (8.4.7)$$

and the $n \times m$ constant matrix L is given by:

$$L = PM + BN \qquad (8.4.8)$$

The dimensions of the constant matrices F, M, J and N, and of the variable vector w, will be clear from the equations.

8.4.2 Problem Statement.

Given A, B and C, the problem is to find L, M, J, P, N and F, such that the (n+q-1) eigenvalues of the composite system have preassigned values.

8.4.3 Solution.

It is shown in 8.2 that the eigenvalues of the composite system are the n eigenvalues of (A + LC) and the eigenvalues of F. We now assume that $\rho = (q-1)$.

Since the pair (A,C) is observable and A is non-derogatory, there exists an m-dimensional row vector l^T, such that the pair (A,l^TC) is observable [W1] . Let l^T be chosen, otherwise arbitrarily, to satisfy this condition, and let $L = hl^T$, where h is an n-dimensional column vector to be determined. Then the eigenvalues of (A + LC) can be given preassigned values by using known results in modal control theory to determine h. Let:

$$\left.\begin{aligned} M &= gl^T \\ \text{and} \qquad N &= fl^T \end{aligned}\right\} \tag{8.4.9}$$

so that equation (8.4.8) becomes:

$$Pg + Bf = h \tag{8.4.10}$$

Choose the matrix F to give the desired dual-observer dynamics. The eigenvalues of F may be real or complex, with the mild restriction that they should be distinct and different from the eigenvalues of A. They should, of course, be real and negative, or occur in complex-conjugate pairs with negative real parts.

Let U be a matrix of self-conjugate column eigenvectors of F, so that:

$$F = U\Lambda U^{-1} \tag{8.4.11}$$

where $\Lambda = \text{diag}(f_1, \ldots f_{q-1})$, in which $(f_1, \ldots f_{q-1})$

are the eigenvalues of F.

Let g = Ue, where e is the (q-1)-dimensional column sum-vector $[111..11]^T$. Then:

$$J = TU^{-1} \tag{8.4.12}$$

where f and the columns of T, $(t_1, \ldots t_{q-1})$, are obtained as the solution of the set of linear equations (8.4.13):

$$\left[\prod_{j=1}^{q-1}(A-f_jI)B \,\vdots\, \prod_{j=2}^{q-1}(A-f_jI)B \,\vdots\, \ldots \,\vdots\, \prod_{j=1}^{q-2}(A-f_jI)B\right]\begin{bmatrix} f \\ t_1 \\ \cdot \\ \cdot \\ t_{q-1}\end{bmatrix} = \prod_{j=1}^{q-1}(A-f_jI)h \tag{8.4.13}$$

where I is the identity matrix of order n, and the ith term in the coefficient matrix on the left-hand side is:

$$\prod_{\substack{j=1 \\ j\neq(i-1)}}^{q-1} (A-f_jI)B \;, \quad i=2, \ldots, q.$$

P is obtained from:

$$P = RU^{-1} \tag{8.4.14}$$

where R is a $n\times(q-1)$ matrix with columns $(r_1, .., r_{q-1})$, given by:

$$r_j = (A-f_jI)^{-1}Bt_j \quad , \; j=1, .., (q-1) \tag{8.4.15}$$

Equations (8.4.13) are consistent, and the solution is unique if rq = n, and nonunique if rq > n. This completes the solution.

Proof.

Equation (8.4.7) has a unique solution if A and F have no common eigenvalue. Setting PU = R, and JU = T, equations (8.4.7) and (8.4.11) give:

$$AR - R\Lambda = BT \qquad (8.4.16)$$

From the columns of equation (8.4.16), since Λ is diagonal, we obtain equation (8.4.15). Substituting (8.4.15) into (8.4.10) gives:

$$Bf + \sum_{j=1}^{q-1} (A - f_j I)^{-1} Bt_j = h \qquad (8.4.17)$$

Premultiplication of (8.4.17) by:

$$\prod_{j=1}^{q-1} (A - f_j I)$$

and rearranging gives (8.4.13). The relations of (8.4.12) and (8.4.14) follow from the definitions of R and T.

The only unknowns in equations (8.4.13) are the rq elements of f, and the t_j, j=1, ... , (q-1). It can be verified by elementary column operations on the coefficient matrix on the left-hand side of (8.4.13), that, privided that the f_j are distinct, this matrix has the same rank as Q in (8.4.3), namely n, by definition. Hence equations (8.4.13) are consistent. From (8.4.3), $rq \geq n$. It is clear that the solution is unique if rq = n.

If $rq > n$, there are $(rq-n)$ degrees of freedom in the nonunique solution, which may be obtained in the usual way as the sum of a particular solution and the solutions of the homogeneous equation multiplied by arbitrary constants.

8.4.4 Solution for Real Dual Observer Eigenvalues.

The solution can be simplified slightly if it is desired that the dual observer should have only real eigenvalues. In this case, F may be chosen to be diagonal, so that $F = \Lambda$, and $U = I$, giving $J = T$, $P = R$, and $g = e$.

8.4.5 Conclusion.

The existence of a dual observer of order $(q-1)$ has been established for a system of controllability index q. The $(n+q-1)$ eigenvalues of the composite system may be given any preassigned values, with the mild restriction that the $(q-1)$ eigenvalues of the dual observer should be distinct, and different from the eigenvalues of the original system. A procedure for the design of such a dual observer and all the associated coupling matrices has been described. The dual observer is a particular form of dynamic feedback compensator, and the existence of such a compensator of order $(q-1)$, permitting arbitrary eigenvalue assignment, is consistent with the conclusions reached by Brasch and Pearson [B3] . This procedure has been described in [M5] .

CHAPTER 9.

OBSERVERS FOR SYSTEMS WITH INACCESSIBLE INPUTS.

9.1 Introduction.

In the theory of observers, as discussed in Chapters 5-7, it is a requirement that the external inputs applied to the observed system should also be applied to the observer. The inputs are applied in such a way as to correspond to the transformation of the system state vector to which the observer state vector tends asymptotically. There are some situations in which it is physically impossible to obtain signals representing inputs for application to the observer. Examples are wind gusts in the case of aircraft control, and internally generated noise or disturbances in the system itself.

A closely related problem arising in the theory of optimal control was studied by Johnson [J4] [J5] [J6], who used the idea of representing the unknown inputs by finite power series in time, t. Hostetter and Meditch [H3] [H4] [M17] have placed this technique in the context of observer theory.

The treatment of this chapter follows the approach of Hostetter and Meditch, but from a more general viewpoint, and some new results are presented concerning the existence of observers for systems with inaccessible inputs.

9.2 O-Observers and k-Observers.

9.2.1 Definition of O-Observer.

A linear time-invariant system is described by the equations:

$$\dot{x} = Ax + Bu \tag{9.2.1}$$

$$y = Cx + Du \tag{9.2.2}$$

in which x, y and u are vectors of dimensions n, m and r, respectively, and A, B, C and D are constant matrices. The inputs u are unknown and not accessible for direct measurement. Then the system:

$$\dot{z} = Fz + Gy + Hu \tag{9.2.3}$$

$$w = Ly + Mz \tag{9.2.4}$$

is a O-observer for the system (9.2.1), (9.2.2), if, for constant u, and for some initial z, depending upon the initial conditions and inputs of the observed system, there exists a linear transformation T, such that:

$$w = T \begin{bmatrix} x \\ u \end{bmatrix} \tag{9.2.5}$$

9.2.2 Definition of k-Observer.

A kth-degree observer, or k-observer, is defined similarly to the O-observer, except that now the system (9.2.3), (9.2.4) is defined as a k-observer for the system (9.2.1), (9.2.2) if, for inputs u in the form of any linear combination of powers of time t, up to and including t^k, e.g., the jth element of u may be:

$$u_j = \alpha_0 + \alpha_1 t + \dots + \alpha_k t^k \qquad (9.2.6)$$

there exists a linear transformation T such that (9.2.5) is satisfied, for some initial z, depending upon the initial conditions and inputs of the observed system.

9.2.3 Note.

For the purpose of this section, it is assumed that all the system inputs are inaccessible. Any inputs which are accessible may be treated in the usual way, and applied to the observer, so that such inputs need not be considered further.

9.2.4 System Augmentation.

An input of the form (9.2.6) may be represented as a differential equation of the form:

$$\begin{bmatrix} \dot{u}_{j1} \\ \cdot \\ \cdot \\ \cdot \\ \dot{u}_{j(k+1)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & 0 \end{bmatrix} \begin{bmatrix} u_{j1} \\ \cdot \\ \cdot \\ \cdot \\ u_{j(k+1)} \end{bmatrix} \qquad (9.2.7)$$

where $u_{j1} = u_j$, and the other variables are defined as the derivatives of this with respect to time.

The system (9.2.1), (9.2.2) may then be augmented so as to include the equations (9.2.7). The resulting system is shown in (9.2.8), (9.2.9), for a system with two inputs, and with k = 1, for clarity, although the generalisation is obvious. Here, $B = \begin{bmatrix} b_1 & b_2 \end{bmatrix}$, and $D = \begin{bmatrix} d_1 & d_2 \end{bmatrix}$, where the b's and d's are vectors of dimension n and m, respectively. The u_{ij} are scalars.

$$\begin{bmatrix} \dot{x} \\ \dot{u}_{11} \\ \dot{u}_{12} \\ \dot{u}_{21} \\ \dot{u}_{22} \end{bmatrix} = \begin{bmatrix} A & b_1 & 0 & b_2 & 0 \\ & 0 & 1 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{bmatrix} \qquad (9.2.8)$$

$$y = \begin{bmatrix} C & d_1 & 0 & d_2 & 0 \end{bmatrix} \begin{bmatrix} x \\ u_{11} \\ u_{12} \\ u_{21} \\ u_{22} \end{bmatrix} \qquad (9.2.9)$$

9.2.5 Principles of the 0-Observer and k-Observer.

A stable observer designed for the free system (9.2.8), (9.2.9) will provide an asymptotic estimate of the state vector of this augmented system. The rate of convergence of the estimate will depend upon the observer dynamics. The signals used to drive this observer are all obtained from the available outputs of the original system, without augmentation, and so the observer will provide an asymptotic estimate of the state vector of the original system, and of the system inputs and their derivatives with respect to time, provided that these inputs are representable by power series in t of degree not greater than k.

9.3 Conditions for Existence of 0-Observer and k-Observer.

9.3.1 Single-Input, Single-Output System.

Hostetter and Meditch [H4] have investigated the conditions forrthe existence of a 0-observer and a k-observer, by using the Luenberger canonical form. They have obtained a necessary and sufficient condition in two forms, according to whether or not the A matrix is singular. In the following treatment, the canonical forms are not used, and a single necessary and sufficient condition is obtained, which is valid in all cases. This same single condition is also expressed in an alternative form, for use when the system equations are in the Luenberger canonical form.

In this case, in equations (9.2.1), and (9.2.2), B becomes a vector b, C becomes a vector c^T, and D is a scalar, d.

Theorem.

The necessary and sufficient condition for the existence of a 0-observer or a k-observer for an observable single-input, single-output system is that:

$$c^T(\mathrm{adj}A)b \neq d.\det A \qquad (9.3.1)$$

If the system is in the Luenberger canonical form in which:

$$A = \begin{bmatrix} -a_{n-1} & 1 & 0 & . & . & . \\ . & 0 & 1 & 0 & . & . \\ . & . & . & . & . & . \\ -a_o & 0 & 0 & 0 & . & 0 \end{bmatrix} \qquad (9.3.2)$$

$$c^T = \begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 \end{bmatrix}$$

the condition reduces to:

$$b_n \neq -d.a_o \qquad (9.3.3)$$

where b_n is the element in the nth row of b in this canonical representation, and a_o is the constant term in the monic characteristic polynomial of A.

<u>Proof.</u>

For clarity, the proof will be given for a 2-observer, the augmented equations of which will have the form:

$$\begin{bmatrix} \dot{x} \\ \dot{u}_{11} \\ \dot{u}_{12} \\ \dot{u}_{13} \end{bmatrix} = \begin{bmatrix} A & b & 0 & 0 \\ & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ u_{11} \\ u_{12} \\ u_{13} \end{bmatrix} \qquad (9.3.4)$$

$$y = \begin{bmatrix} c^T & d & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ u_{11} \\ u_{12} \\ u_{13} \end{bmatrix} \tag{9.3.5}$$

The 2-observer exists for the system represented by (9.3.4), (9.3.5) if and only if this augmented system is observable. Writing the observability matrix for this system gives:

$$\begin{bmatrix} c^T & d & 0 & 0 \\ c^TA & c^Tb & d & 0 \\ c^TA^2 & c^TAb & c^Tb & d \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ c^TA^{n+2} & c^TA^{n+1}b & c^TA^nb & c^TA^{n-1}b \end{bmatrix} \tag{9.3.6}$$

The condition for observability is that this $(n+3)\times(n+3)$ matrix should have full rank.

If the characteristic polynomial of A is:

$$s^n + a_{n-1}s^{n-1} + \ldots + a_1s + a_o \tag{9.3.7}$$

then, from the Cayley-Hamilton theorem,

$$A^n = -a_{n-1}A^{n-1} - \ldots - a_1A - a_oI \tag{9.3.8}$$

Applying (9.3.8) to the row-reduction of (9.3.6) gives:

$$\begin{bmatrix} c^T & d & 0 & 0 \\ c^TA & c^Tb & d & 0 \\ c^TA^2 & c^TAb & c^Tb & d \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ c^TA^{n-1} & c^TA^{n-2}b & c^TA^{n-3}b & c^TA^{n-4}b \\ 0 & h & x & x \\ 0 & 0 & h & x \\ 0 & 0 & 0 & h \end{bmatrix} \qquad (9.3.9)$$

where:

$$h = c^TA^{n-1}b + a_{n-1}c^TA^{n-2}b + \ldots + a_1c^Tb + a_od \qquad (9.3.10)$$

and the x's denote numbers which are not of interest.

Since the original system is observable, the matrix:

$$\begin{bmatrix} c^T \\ c^TA \\ \cdot \\ \cdot \\ c^TA^{n-1} \end{bmatrix}$$

has rank n. It follows that (9.3.9) has rank (n+3) if and only if $h \neq 0$. The condition, thus, is:

$$c^T A^{n-1} b + a_{n-1} c^T A^{n-2} b + \dots + a_1 c^T b + a_o d \neq 0 \quad (9.3.11)$$

Now, $adj(-A) = A^{n-1} + a_{n-1} A^{n-2} + \dots a_1 I$ (9.3.12)

Combining (9.3.11) and (9.3.12) gives:

$$c^T adj(-A)b + a_o d \neq 0 \quad (9.3.13)$$

Since $adj(-A) = (-1)^{n-1} adjA$, and $a_o = (-1)^n detA$, the condition (9.3.13) becomes:

$$c^T (adjA)b \neq d.detA \quad (9.3.14)$$

as required.

If A is in the Luenberger canonical form of (9.3.2), the condition (9.3.14) reduces to:

$$(-1)^{n-1} b_n \neq d.(-1)^n a_o,$$

or

$$b_n \neq -d.a_o \quad (9.3.15)$$

as required.

It is clear from the structure of (9.3.9) that the existence condition is the same for a 0-observer or for a k-observer, since, for each addition of 1 to k, a new column is formed in the observability matrix (9.3.9), which has a non-zero element in the appropriate place if the condition (9.3.14) is satisfied.

9.3.2 Multi-Input, Multi-Output System.

In this case, the system representation is as shown in (9.2.8), (9.2.9), where, for clarity, only two inputs are shown, and it is assumed that $k = 1$. However, the generalisation is obvious. The following theorem gives a sufficient condition for the existence of a O-observer or a k-observer, in this case.

Theorem.

Let the observability index of the pair (A,C) be p, so that the matrix:

$$\begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ CA^{p-1} \end{bmatrix} \qquad (9.3.16)$$

has rank n.

Then there exists a $m \times pm$ matrix Q such that:

$$CA^{p} = Q \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ CA^{p-1} \end{bmatrix} \qquad (9.3.17)$$

A sufficient condition for the existence of a O-observer or a k-observer for the system (9.2.1), (9.2.2), having r inputs u_1, ... u_r, is that the following r m-vectors be linearly independent:

$$CA^{p-1}b_j - Q\begin{bmatrix} Cd_j \\ Cb_j \\ CAb_j \\ \cdot \\ \cdot \\ CA^{p-2}b_j \end{bmatrix} \qquad (9.3.18)$$

$j=1,\ldots r$

where $B = \begin{bmatrix} b_1 & \ldots & b_r \end{bmatrix}$, and $D = \begin{bmatrix} d_1 & \ldots & d_r \end{bmatrix}$

<u>Proof.</u>

The observability matrix for the augmented system (9.2.8), (9.2.9) is:

$$\begin{bmatrix} C & d_1 & 0 & d_2 & 0 \\ CA & Cb_1 & d_1 & Cb_2 & d_2 \\ CA^2 & CAb_1 & Cb_1 & CAb_2 & Cb_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ CA^{p-1} & CA^{p-2}b_1 & CA^{p-3}b_1 & CA^{p-2}b_2 & CA^{p-3}b_2 \\ CA^p & CA^{p-1}b_1 & CA^{p-2}b_1 & CA^{p-1}b_2 & CA^{p-2}b_2 \\ CA^{p+1} & CA^pb_1 & CA^{p-1}b_1 & CA^pb_2 & CA^{p-1}b_2 \end{bmatrix}$$

(9.3.19)

Application of the relationship of (9.3.17) and row-reduction of (9.3.19) gives:

$$\begin{bmatrix} C & d_1 & 0 & d_2 & 0 \\ CA & Cb_1 & d_1 & Cb_2 & d_2 \\ CA^2 & CAb_1 & Cb_1 & CAb_2 & Cb_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ CA^{p-1} & CA^{p-2}b_1 & CA^{p-3}b_1 & CA^{p-2}b_2 & CA^{p-3}b_2 \\ 0 & h_1 & x & h_2 & x \\ 0 & 0 & h_1 & 0 & h_2 \end{bmatrix}. \tag{9.3.20}$$

where h_1 is the vector of (9.3.18) with j=1, and h_2 is the corresponding vector with j=2. It is clear from (9.3.20) that the matrix has full rank if h_1 and h_2 are linearly independent. It is also clear that this condition cannot be satisfied unless $m \geq r$, and that the condition for a k-observer is the same as that for a 0-observer, by the same reasoning as was used in the single-input, single-output case.

In the particular case of a system with a single input and multiple outputs, the condition requires that the vector (9.3.18), with j=1, should have at least one non-zero element.

9.4 Steady-State Frequency Response of k-Observer.

In [H4] , a numerical example is given of a second-order system having scalar input and output, and a 1-observer of dimension 4. The results of a simulation study of this system indicate, as might be expected, that the estimates of the system state variables and of the scalar input given by the observer are very good, after the decay of initial transients, when the input is in the form of a ramp, a square wave, or a triangular wave. These inputs are all of the type for which the 1-observer is intended. It is surprising, however, that the results for a sinusoidal input also appear to be very satisfactory. In view of this result, the system has been examined a little more closely from the point of view of steady-state frequency response, principally to discover whether the results reported in [H4] were merely the outcome of a fortunate choice of input frequency. The system considered has:

$$A = \begin{bmatrix} -2 & 1 \\ -3 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad (9.4.1)$$

$$c^T = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad d = 1$$

Full details of the 1-observer are not given, but it is reported to have four eigenvalues at -3. Based on this information, the observer equation is:

$\dot{z} = Fz + ky$, where

$$F = \begin{bmatrix} -3 & 16 & 45 & 0 \\ 27 & 0 & 91 & 0 \\ -189 & 0 & -189 & 16 \\ -324 & 0 & -324 & 0 \end{bmatrix} \frac{1}{16} \qquad k = \begin{bmatrix} -29 \\ -75 \\ 189 \\ 324 \end{bmatrix} \frac{1}{16} \tag{9.4.2}$$

It is easily shown that the Laplace transform of the system state, $\bar{x}$, with zero initial conditions, is:

$$\bar{x} = (sI - A)^{-1} b\bar{u} \tag{9.4.3}$$

and the Laplace transform of the observer state vector, $\bar{z}$, with zero initial conditions, is:

$$\bar{z} = (sI - F)^{-1} k(c^T(sI - A)^{-1} + d)\bar{u} \tag{9.4.4}$$

$\bar{z}_3$ is the estimate of $\bar{u}$, and, inserting $s = j\omega$ in (9.4.4) yields the frequency response shown in Table 9.4.1.

Table 9.4.1.

Freq. ω in rad sec^{-1}	0	0.5	1.0	2.0	2.5	3.0
$\frac{\bar{z}_3(j\omega)}{\bar{u}(j\omega)}$	1.0	0.9957	0.9946	1.1047	1.2056	1.2970

Freq. in rad sec^{-1}	3.5	4.5	5.0	5.5	7.0	10.0
$\frac{\bar{z}_3(j\omega)}{\bar{u}(j\omega)}$	1.3614	1.4046	1.3938	1.3696	1.2566	1.0142

The poles of the system are located at $(-1 \pm j1.414)$, and those of the observer at -3. The input frequency reported in [H4] was about 1 rad sec^{-1}, and it is seen from Table 9.4.1 that, at this frequency, the amplitude of the estimated input is very nearly correct. The phase error also is small. At frequencies just above this, however, there is a sharp increase in the amplitude ratio, which reaches a peak of about 1.4 at a frequency of 4.5 rad sec^{-1}. The phase error then also becomes considerable.

Comparison of $\bar{z}_1$, the estimate of $\bar{x}_1$, by finding the amplitude ratio:

$$\frac{\bar{z}_1(j\omega)}{\bar{x}_1(j\omega)}$$

as a function of frequency, yields Table 9.4.2.

Table 9.4.2.

Freq. ω in rad sec^{-1}	0.5	1.0	2.0	3.0	4.0	5.0
$\frac{\bar{z}_1(j\omega)}{\bar{x}_1(j\omega)}$	1.0389	1.0899	1.2553	1.5237	1.7227	1.8294

The estimate of $\bar{x}_1$ is quite reasonable up to a frequency of 1 rad sec^{-1}, although not as good as the estimate of $\bar{u}$. Above this frequency, however, the amplitude ratio rises sharply.

It is not intended to investigate this matter in detail here, but enough has been included to indicate that there is an interesting field of study in the investigation of what order of k-observer is needed, and how the observer eigenvalues should be chosen, so as to provide a good frequency response over a wide frequency band. This would ensure a satisfactory response to general input signals.

9.5 Conclusion.

Hostetter and Meditch obtained the condition for the existence of a 0-observer or a k-observer in the case of a single-input, single-output system in the form of two criteria, the choice of which is determined by the singularity, or otherwise, of the A matrix. In applying their method, the system must first be transformed into the Luenberger canonical form. The single criterion obtained herein is more compact, and is given in a general form, requiring no preliminary transformation of the system state vector, and in a special version suitable for use when the system is represented in the Luenberger canonical form.

The treatment of multi-input, multi-output systems by Hostetter and Meditch requires that the system has the same number of outputs as inputs. If this condition is not satisfied at the outset, some adjustment is needed. The problem is then treated in the manner suggested by Luenberger, by obtaining a representation of the system as a number of single-output sub-systems, coupled only at their inputs. The method presented here is more general, and does not impose any upper limit on the number of system outputs.

The question of the dimensions of a 0-observer and of a k-observer has been discussed by Hostetter and Meditch [H3] . The conclusions reached are equivalent to the statement that such an observer, with arbitrary dynamics, can be designed to observe the state vector, or a linear functional of the state vector, of the augmented system, and that the observer dimension corresponds to that which one would expect from applying Luenberger observer theory to this case.

The results reported [H4] from a simulation of a system with a 1-observer to which various inputs are applied are surprisingly good. With a maintained sine wave input to the system, an input for which the 1-observer is not specifically designed, the observer rapidly adjusts, and subsequently reproduces the state vector and the system input with little error. It is argued that the higher the value of k, the more faithfully will the state vector of the augmented system be reproduced. This seems, intuitively, to be likely, and this subject presents an interesting field for further study.

CHAPTER 10.

GENERAL POLE ASSIGNMENT BY OUTPUT FEEDBACK.

10.1 Introduction.

In Chapter 3, the problem of closed-loop pole assignment with restricted measurement access was discussed, for the case of a single-input system. Multi-input systems were included only as an extension of the single-input case, giving unity-rank feedback. This restriction results in a reduction in the number of variable parameters in the $r \times m$ feedback gain matrix K from rm to (r+m). This chapter deals with the general case, in which the rank of K is unrestricted, so that full advantage can be taken of all available variable parameters.

10.2 Output Feedback Derived From State Feedback.

10.2.1 The Methods of Munro and Vardulakis.

Munro and Vardulakis [M18] have considered the following problem. Given the system described by the equations:

$$\dot{x} = Ax + Bu \qquad (10.2.1)$$

$$y = Cx \qquad (10.2.2)$$

in which (A,B) is controllable, find the necessary and sufficient conditions for arbitrary assignment of all the system poles, using only constant output feedback, and find a formula for the feedback gain matrix.

The approach used is first to find any state feedback gain matrix K_x such that the matrix $(A - BK_x)$ has the desired eigenvalue spectrum, and then to seek a solution to the matrix equation:

$$K_y C = K_x \tag{10.2.3}$$

where K_y is the output feedback gain matrix. The matrix C has dimensions $m \times n$, where $m < n$, and may be assumed, without loss of generality, to have rank m. The so-called g_1-inverse, C^{g_1}, of C, is used, defined by the property:

$$CC^{g_1}C = C \tag{10.2.4}$$

The condition for the consistency of (10.2.3) is expressed as the condition that:

$$K_x C^{g_1} C = K_x \tag{10.2.5}$$

The solution for K_y is given as:

$$K_y = K_x C^{g_1} \tag{10.2.6}$$

and it is stated that other solutions for K_y can be obtained from the equation:

$$K_y = K_x C^{g_1} + Z(I_m - CC^{g_1}) \tag{10.2.7}$$

where Z is an arbitrary $r \times m$ matrix.

It will now be shown that the degrees of freedom in the solution for K_y represented by the matrix Z in (10.2.7) are not available if C has rank m.
In equation (10.2.4), let $CC^{g_1} = R$, where R is an unknown $m \times m$ matrix. Then:

$$RC = C \qquad (10.2.8)$$

Since C has rank m, the columns on both sides of (10.2.8) may be rearranged, if necessary, to give:

$$R \left[C_1 \vdots C_2 \right] = \left[C_1 \vdots C_2 \right] \qquad (10.2.9)$$

where C_1 is non-singular.
Hence, $RC_1 = C_1$, and $R = I_m$, so that, in (10.2.7), $CC^{g_1} = I_m$, and the bracketed term is a null matrix. Thus, (10.2.7) always reduces to (10.2.6), and there is no arbitrary Z.

The condition for the consistency of equation (10.2.3) expressed as in (10.2.5), in terms of the g_1-inverse, seems to be unnecessary, because, as the authors have noted, this condition is equivalent to the condition that, in (10.2.3), the rows of K_x should lie in the row space of C. This latter condition is clear from (10.2.3), and is easy to apply directly.

Patel [P6] has pointed out that the condition for the consistency of (10.2.3) may be stated as the condition that there exists a state vector transformation $x = Tz$, such that, in the z-co-ordinates, the output feedback becomes incomplete state feedback, obtained from m state variables. This fact is self-evident from a consideration of (10.2.3), when it is remembered that a state-vector transformation of the type considered will change (10.2.3) to:

$$K_y CT = K_x T \qquad (10.2.10)$$

Reverting to (10.2.3), since C has rank m, we may perform a column reduction on both sides of (10.2.3) until there are precisely m non-zero columns in the reduced form of C. For consistency, all the zero columns of the reduced C must have corresponding zero columns in the reduced T. Noting that the column reduction can be expressed as the product of elementary matrices, which will form T, the result follows.

Unfortunately, the conditions obtained in [M18] are necessary and sufficient conditions for the existence of a solution to (10.2.3), and not necessary and sufficient conditions for arbitrary pole assignment. It is first necessary to find K_x to satisfy the conditions. No guidance is given in this most important matter, although it is suggested that, since C^{g_1} is non-unique, other

g_1-inverses could be tried, and a method for generating other g_1-inverses is given. However, Seraji [S5] has pointed out that, if any one g_1-inverse fails to satisfy the condition of (10.2.5), then no g_1-inverse exists which satisfies this, so that there is no point in seeking other g_1-inverses. This conclusion also follows from the fact that the necessary and sufficient condition for satisfaction of (10.2.5) coincides with the condition that K_x lies in the row space of C.

Munro [M19] has extended the approach of [M18] and has reached the conclusion that the output feedback must be chosen in such a way that the pair (A_c,B) has the same controllability indices as the pair (A,B), and that the pair (A_c,C) must have the same observability indices as the pair (A,C), where A_c is the system matrix with feedback, $(A-BK_yC)$. These conditions are sufficient to permit the design of the feedback using the method of Anderson and Luenberger, so that the canonical structure of the system is unchanged. However, the necessity of the condition does not follow.

10.2.2 The Method of Bengtsson and Lindahl.

Bengtsson and Lindahl [B4] have described a different method which is also based on the initial determination of state feedback. As far as possible, the notation of [B4] will be used in discussing this method. The authors distinguish between a 'single constrained feedback structure', and a 'multiple constrained feedback structure'. The latter refers to the use of local feedback in systems which comprise subsystems geographically far apart, such as electrical power systems. The approach to the single constrained feedback structure forms the basis of that used in the multiple case, and the present discussion is limited to the former, as this permits comparison with the other methods considered.

For the system represented by (10.2.1), (10.2.2), the first step is to determine the $r \times n$ matrix L such that the state feedback $u = Lx$ gives the desired eigenvalue spectrum to the matrix $(A + BL)$. The problem then centres on consideration of the equation corresponding to (10.2.3), which, in the present notation, becomes:

$$KC = L \qquad (10.2.11)$$

where the output feedback:

$$u = Ky \qquad (10.2.12)$$

is applied, and K is $r \times m$, and C $m \times n$, with rank m.

Let Q be a $n \times p$ matrix of vectors forming a real basis for the eigenspace corresponding to a set of p symmetric eigenvalues of (A + BL). Then if K is a solution of the equation:

$$KCQ = LQ \qquad (10.2.13)$$

the set of p eigenvalues will also be eigenvalues of (A + BKC). In this way, if a solution of (10.2.13) exists, a matrix K is found which preserves p of the closed-loop eigenvalues assigned by the application of the state feedback L.

If there is more than one solution to (10.2.13), that solution which minimises the feedback gains is given by the generalised inverse [N2] as:

$$K = LQ(R^{-1}CQ)^{+}R^{-1} \qquad (10.2.14)$$

where R is a non-singular $m \times m$ matrix used to scale the output variables.

Where a solution to (10.2.13) does not exist, an approximate solution is obtained by use of the generalised inverse [N2] as:

$$K = LQW(CQW)^{+} \qquad (10.2.15)$$

This solution minimises the norm:

$$\|(KCQ - LQ)W\| \qquad (10.2.16)$$

Here, W is a diagonal matrix to weight the columns of Q, so as to influence the importance of the different eigenvalues. The norm is defined by:

$$\|M\| = (tr(MM^T))^{\frac{1}{2}} \qquad (10.2.17)$$

In both of these cases, the effect of neglecting (n-p) eigenvalues is quite unpredictable. It is claimed that these remaining eigenvalues can be restricted to the less dominant modes, but it is not stated how this can be assured.

In the second case, it does not follow that the minimisation of (10.2.16) yields a matrix K giving the best possible fit of p eigenvalues of (A + BKC) to the desired set. This statement will be justified by considering the case in which, for simplicity, all eigenvalues are real and distinct, and it is desired to preserve all the eigenvalues of (A + BL) in (A + BKC). We may write:

$$(A + BL)Q = Q\Lambda^d \qquad (10.2.18)$$

where Λ^d is a diagonal matrix of the eigenvalues of (A + BL). This follows from the definition of Q. Now suppose the application of output feedback K gives an eigenvalue spectrum represented by the diagonal matrix Λ. Then:

$$(A + BKC)Q_o = Q_o\Lambda \quad (10.2.19)$$

where Q_o is a matrix of eigenvectors of $(A + BKC)$. Introducing a diagonal weighting matrix W, and noting that Q and Q_o are non-singular, (10.2.18) and (10.2.19) give:

$$(\Lambda - \Lambda^d)W = Q_o^{-1}(A+BKC)Q_oW - Q^{-1}(A+BL)QW \quad (10.2.20)$$

The norm of (10.2.17) applied to (10.2.20) clearly gives the square root of the weighted sum of squares of the differences between the actual and desired eigenvalues, and so is a measure of the 'fit' of the eigenvalue spectrum provided by the feedback K. Minimisation of this norm would yield the 'least squares' solution. This solution would involve determining both Q_o and K. The introduction of the assumption $Q_o = Q$ in (10.2.20) reduces this expression to:

$$Q^{-1}B(KCQ - LQ)W \quad (10.2.21)$$

The variable factor of this expression, (KCQ-LQ)W, coincides with the matrix of which the norm is minimised in the method of Bengtsson and Lindahl. Clearly, this could not be expected to yield the matrix K which gives the best possible fit of the eigenvalue spectrum. This point is confirmed in 10.7.3 , where the eigenvalue spectrum corresponding to a solution for K obtained by a direct algorithm is compared with the results given in [B4] .

10.3 Patel's Method.

Patel [P7] has described an interesting recursive pole-assignment procedure for multi-input, multi-output systems, which places no restriction on the rank of the feedback gain matrix. For a system described by the equations (10.2.1) and (10.2.2), the $r \times m$ output feedback gain matrix K, such that:

$$u = -Ky \qquad (10.3.1)$$

is sought, for arbitrary pole assignment. The matrix K is partitioned into rows as:

$$K = \begin{bmatrix} k_1^T \\ \cdot \\ \cdot \\ k_r^T \end{bmatrix} \qquad (10.3.2)$$

It is shown that the characteristic polynomial with feedback, $D_c(s)$, and the characteristic polynomial without feedback, $D_o(s)$, are related by:

$$D_c(s) = D_o(s) + k_1^T\emptyset_1(s)b_1 + k_2^T\emptyset_2(s)b_2 + \,..\, + k_r^T\emptyset_r(s)b_r \qquad (10.3.3)$$

where the $m \times n$ polynomial matrix $\emptyset_q(s)$ is given by:

$$\emptyset_q(s) = C\mathrm{adj}(sI - A + \sum_{p=q+1}^{r} b_p k_p^T C) \qquad (10.3.4)$$

for $q = 1, 2, \ldots (r-1)$,

and:

$$\emptyset_r(s) = C\mathrm{adj}(sI - A) \qquad (10.3.5)$$

b_i, i=1, ... r, is the ith column of the input distribution matrix B in (10.2.1).

It follows that the change in the closed-loop characteristic polynomial, $\Delta D_c^i(s)$ resulting from a change Δk_i^T in the ith row of K is given by:

$$\Delta D_c^i(s) = \Delta k_i^T \emptyset_1^i(s) b_i \qquad (10.3.6)$$

where:

$$\emptyset_1^i(s) = C\mathrm{adj}(sI - A + \sum_{p=1}^{i-1} b_p k_p^T C + \sum_{p=i+1}^{r} b_p k_p^T C) \qquad (10.3.7)$$

By equating coefficients of like powers of s on each side of (10.3.6), a numerical equation is obtained as:

$$J^i \Delta k_i^T = d^i \qquad (10.3.8)$$

where J^i is an $n \times m$ matrix of coefficients obtained from the m-dimensional column polynomial vector $\emptyset_1^i(s) b_i$, and d^i is an n-dimensional column vector of coefficients of $\Delta D_c^i(s)$, excluding the coefficient of s^n. The least-squares solution of (10.3.8) is obtained as:

$$\Delta \hat{k}_i^T = \left[J^i\right]^+ d^i \qquad (10.3.9)$$

where $\left[J^i\right]^+$ is the pseudoinverse of J^i.

The ith row of K then becomes $k_i^T + \Delta\hat{k}_i^T$, and:

$$\Delta D_c^{i+1}(s) = \Delta D_c^i(s) - \Delta\hat{k}_i^T \phi_1^i(s) b_i \text{ , or}$$

$$d^{i+1} = d^i - J^i \Delta\hat{k}_i^T \qquad (10.3.10)$$

Starting from some arbitrary initial K, this process is repeated until, to complete one cycle of operations, each row of K has been included. The cycles are continued until the norm of the d^i, $\|d\|$, at any step becomes sufficiently small, or fails to decrease over a cycle.

It is shown that the recursive process is convergent, and a modification is included which permits the specification of individual poles, instead of the coefficients of the characteristic polynomial. The reason for considering this alternative approach is stated as the desire to avoid computational inaccuracy which sometimes accompanies the use of characteristic polynomial coefficients as a means of specifying pole positions.

10.4 Comment.

In the absence of any direct way of ensuring that the state feedback gain matrix lies in the row space of the C matrix, the approach used by Munro and Vardulakis does not seem to be of great assistance in the general pole-assignment problem.

A disadvantage of an approach based on the initial determination of state feedback, as used by Munro and Vardulakis, and by Bengtsson and Lindahl, is that some of the freedom in designing the output feedback is lost when the state feedback is determined. The state feedback for a given closed-loop pole configuration often is non-unique, so that the choice of a different state feedback could yield a better solution to the output feedback problem.

Patel's method does not suffer from this disadvantage, and it represents a useful approach. This is a recursive method, however, and, if such a method is to be used, it is worth while ensuring, as far as possible, that the recursive algorithm includes all the design constraints it is desired to impose. The use of an algorithm in which one row of the feedback matrix K is considered at a time does not lend itself readily to this.

A further consideration is that, even if a direct general solution of the problem of pole assignment in multi-input, multi-output systems were found, it would be of limited use in practice. The reason for this is that the rigid specification of all closed-loop poles requires the designer to specify his problem more completely than he is, with knowledge, able to do. The practical situation usually is that the locations of some closed-loop poles, normally those near the origin of the complex plane, are critical. The locations of the remaining poles, within broad limits, are not critical. If the designer is forced to specify these poles rigidly, he may be unconsciously imposing severe constraints on the system, tending to make the critical pole locations difficult to achieve, and demanding high feedback gains.

The foregoing considerations suggest an approach in which the desired closed-loop poles are specified with weighting factors, so that their relative importance may be taken into account, and in which the elements of the feedback matrix K are considered one at a time. It then becomes possible to apply limits to the values of the elements of K, and to allow for the inclusion of only certain elements of K, if desired. An example of a case in which this facility would be useful was given by Bengtsson and Lindahl, in the problem of the control of three interconnected power stations, geographically far apart. Another situation in which it is useful to have freedom to consider the elements of K individually is in the problem of determining the feedback gains needed to maintain stability of the system, in the event of loss of some of the feedback loops, due to malfunction.

In the next section, a general recursive pole-assignment procedure is described, in which one element of K is considered at a time. This procedure is intended to meet the requirements set out in the preceding paragraph. This may be regarded as a direct, practical approach to the problem of closed-loop pole assignment by output feedback, in which advantage is taken of the availability of the high-speed digital computer.

10.5 A General Procedure for Pole Assignment in Multi-Input, Multi-Output Systems.

10.5.1 System Description and Problem Statement.

A linear time-invariant system is described by the equations:

$$\dot{x} = Ax + Bu + Bu' \qquad (10.5.1)$$

$$y = Cx \qquad (10.5.2)$$

where x, y, u and u' are vectors of dimension n, m, r and r, respectively.

$$B = \begin{bmatrix} b_1 & \ldots & b_r \end{bmatrix} \qquad (10.5.3)$$

and

$$C = \begin{bmatrix} c_1^T \\ \cdot \\ \cdot \\ c_m^T \end{bmatrix} \qquad (10.5.4)$$

The problem is to find the feedback gain matrix K, where:

$$u' = -Ky \qquad (10.5.5)$$

such that the poles of the closed-loop system approach as closely as possible a given set of desired poles, $\lambda_1^d, \ldots \lambda_n^d$. A negative sign for the feedback gain in (10.5.5) is introduced for convenience.

The relative importance attached to the deviation of each pole from the desired value is to be capable of adjustment. Arrangements are to be provided to permit only certain chosen elements of K to be included, if desired, and provision is to be made to limit the values of the elements of K, so that these do not exceed some arbitrarily chosen values.

10.5.2 Development of the Algorithm.

The problem is equivalent to that of finding K so that, subject to the constraints, the eigenvalues of the matrix (A - BKC) approach as closely as possible the desired set $\lambda_1^d, \dots \lambda_n^d$. Considering one element of K alone, say k_{ij}, and setting all other elements of K to zero, reduces this matrix to:

$$(A - b_i c_j^T k_{ij}) \qquad (10.5.6)$$

The characteristic polynomial of (10.5.6) is:

$$\det(sI - A + k_{ij} b_i c_j^T) \qquad (10.5.7)$$

and this may be written as:

$$\det(sI - A) \left| I + (sI - A)^{-1} k_{ij} b_i c_j^T \right|$$

$$= \det(sI - A)(1 + k_{ij} c_j^T (sI - A)^{-1} b_i) \qquad (10.5.8)$$

using a matrix identity proved in [M1] .

The expression (10.5.8) may be written as:

$$\det(sI - A) + k_{ij} c_j^T \operatorname{adj}(sI - A) b_i \qquad (10.5.9)$$

It is desired that the closed-loop characteristic polynomial (10.5.9) be zero when s takes each of the values $\lambda_1^d, \dots \lambda_n^d$. Departure from the zero value is a measure of how imperfectly the problem has been solved. A criterion of success is given as the weighted sum of the squares of the moduli of the expression (10.5.9), evaluated for each of the desired poles, $\lambda_1^d, \ldots \lambda_n^d$.

Let:

$$g(s) = \det(sI - A) \tag{10.5.10}$$

and

$$f(s) = c_j^T \operatorname{adj}(sI - A)b_i \tag{10.5.11}$$

The sum of the squares of the moduli for all poles λ_q^d, real or complex, is given by:

$$\sum w_q(g(\lambda_q^d) + k_{ij}f(\lambda_q^d))(\bar{g}(\lambda_q^d) + k_{ij}\bar{f}(\lambda_q^d)) \tag{10.5.12}$$

where w_q is a positive weighting factor.

Differentiating the expression (10.5.12) with respect to k_{ij} gives:

$$\sum \left\{ w_q(g(\lambda_q^d) + k_{ij}f(\lambda_q^d))\bar{f}(\lambda_q^d) + w_q(\bar{g}(\lambda_q^d) + k_{ij}\bar{f}(\lambda_q^d))f(\lambda_q^d) \right\} \tag{10.5.13}$$

The expression (10.5.13) may be re-written as:

$$\sum \left\{ w_q(g(\lambda_q^d) + k_{ij}f(\lambda_q^d))\bar{f}(\lambda_q^d) + w_q(g(\bar{\lambda}_q^d) + k_{ij}f(\bar{\lambda}_q^d))\bar{f}(\bar{\lambda}_q^d) \right\} \tag{10.5.14}$$

and, since each complex λ_q^d will have its conjugate, $\bar{\lambda}_q^d$, included in the summation, with the same weighting factor, the summation may be written, in general, as:

$$2\sum_{q=1}^{n} w_q(g(\lambda_q^d) + k_{ij}f(\lambda_q^d))\overline{f}(\lambda_q^d) \qquad (10.5.15)$$

The weighted sum of the squares of the moduli of the characteristic polynomial has a stationary value where:

$$k_{ij} = -\frac{\sum w_q g(\lambda_q^d)\overline{f}(\lambda_q^d)}{\sum w_q f(\lambda_q^d)\overline{f}(\lambda_q^d)} \qquad (10.5.16)$$

There is clearly only one stationary value, and the weighted sum of squares of the moduli tends to + infinity as k_{ij} tends to + or - infinity, so that the stationary value given by (10.5.16) is a minimum. It follows that the value of k_{ij} given by (10.5.16) is the 'best' value, according to the chosen criterion, where only the element k_{ij} is permitted to change.

The specification of repeated poles calls for a modification of the expressions. A pole of multiplicity p will satisfy the characteristic polynomial (10.5.9) and its first (p-1) derivatives with respect to s. Hence the squares of these derivatives are included in the summation for the criterion. The summations of (10.5.16) then will include the terms:

$$w_{q1}g(\lambda_q^d)\overline{f}(\lambda_q^d) + w_{q2}g'(\lambda_q^d)\overline{f}'(\lambda_q^d) + \ldots + w_{qp}g^{(p-1)}(\lambda_q^d)\overline{f}^{(p-1)}(\lambda_q^d) \quad (10.5.17)$$

in the numerator, and:

$$w_{q1}f(\lambda_q^d)\overline{f}(\lambda_q^d) + w_{q2}f'(\lambda_q^d)\overline{f}'(\lambda_q^d) + \ldots + w_{qp}f^{(p-1)}(\lambda_q^d)\overline{f}^{(p-1)}(\lambda_q^d) \quad (10.5.18)$$

in the denominator.

It should be noted that the existence of repeated eigenvalues of A is immaterial.

The weighting factors applied to the various derivatives need not be equal, but the weighting factors for a complex pole must be equal to those for the conjugate pole in the original expressions and the derivatives.

The algorithm for pole assignment can now be stated.

10.5.3 Algorithm for Pole Assignment.

The algorithm is described as follows.

1. Starting with a particular element of K, say k_{11}, find the corresponding f(s) and g(s), using the Leverrier method.

2. Calculate k_{11} from (10.5.16).

3. Replace A by $(A - k_{11}b_1c_1^T)$, and return to 1, but, this time, using, say, k_{12}.

4. Continue in this way, dealing with each desired element of K in turn, according to some regular scheme. When all the elements have been computed once, the cycle is repeated.

5. The process continues for a definite number of cycles, or until the absolute values of all the computed k_{ij} increments are less than some preassigned number, over a cycle. At each step, the total value of the element k_{ij} calculated may be compared with a corresponding assigned limit value. If the limit is exceeded, the calculated value is replaced by the limit value, and the process continued.

6. In order to avoid the accumulation of errors, at each step, the matrix (A - BKC) is calculated, using the latest matrix K.

10.5.4 Refinements of the Algorithm.

When applied to simple test problems, the algorithm was found to give rather rapid variations in the elements of K initially, and then to settle to a slow asymptotic type of approach to the final values.

Once the steady approach stage had been reached, it was found that a considerable acceleration of convergence was achieved by the use of Aitken's extrapolation formula [A3] . This formula takes three values of each element of K, equally spaced in number of iterations, and uses these to predict the final values of these elements. K is then adjusted so that each element has this calculated final value, and the iterations are continued. The extrapolation formula is applied as often as necessary until the matrix K is stationary.

The extrapolation formula takes the following form. If a variable y has three values, y_1, y_2 and y_3, where the interval of the independent variable between y_1 and y_2 is equal to that between y_2 and y_3, the predicted final value of y, y_f, assuming that the approach to the final value is exponential, is obtained as:

$$y_f = y_1 - \frac{(y_2 - y_1)^2}{y_3 - 2y_2 + y_1}$$

In an algorithm in which each element of K is computed separately, there is a tendency for the elements of K which happen to be computed early in the process to be given large values. This tendency was reduced by the use of a 'slow turn on' feature. Here, during the early part of the procedure, the computed values of the elements of K were multiplied by positive constants of magnitude less than 1. These multiplying constants were gradually increased to 1 as the process continued, so that the full calculated values were then applied.

10.6 Comment.

The algorithm is conceptually very simple as, at each step, the variation of one scalar feedback gain only is involved, and this corresponds to the familiar scalar root locus approach. The algorithm finds the value of this scalar gain which brings the set of closed-loop poles as close as possible to the desired set, taking account of the weighting factors. At the next step, the action is similar, except that a different scalar gain is involved, and a different set of root loci would apply.

Variation of the specified weighting factors, or of the desired poles, provides the designer with a means whereby he may acquire a feel for the system. It is not necessary, at the outset, to have a knowledge of the quantitative effects of varying the weighting factors. This will be acquired after a few trials.

It is not a necessary part of the procedure to compute the system poles, but a program to do this is included, so that the progress of the algorithm can be monitored at intervals.

As a by-product of the program, the numerator and denominator polynomial of every input-output transfer function of the system with feedback can be obtained, as this information is already available.

10.7 Applications of the Pole Assignment Algorithm.

10.7.1 Example 1, 3rd Order System.

The algorithm was applied to an example considered by Patel [P7] . In this example,

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 0 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The specified closed-loop poles are -6 and $(-12 \pm j5)$.

Patel obtained the solution:

$$K = \begin{bmatrix} 32.0 & 11.8681 \\ -108.407 & -49.7246 \end{bmatrix}$$

The solution to this problem is non-unique.

A solution was obtained using the algorithm with equal pole weightings, and no gain constraint. This gave the result:

$$K = \begin{bmatrix} 32.0000 & 167.748 \\ 3.50537 & -5.75702 \end{bmatrix}$$

with the poles:

-6.0000 and $-12.0000 \pm j5.0000$

This solution was obtained after 151 calculations of the matrix K.

In order to investigate the operation of the feedback gain limit feature, the feedback gain limit matrix was set as:

$$\begin{bmatrix} 100 & 20 \\ 1000 & 1000 \end{bmatrix}$$

The same problem was run again, but with the gain limit in operation, and the algorithm set off as before, but encountered the feedback gain limit on k_{12} at -20. This limit was applied 323 times, and the algorithm then left this limit and found a solution in which the gain matrix was:

$$K = \begin{bmatrix} 32.0000 & 12.6294 \\ -105.961 & -51.4096 \end{bmatrix}$$

giving the closed-loop poles:

-5.9999 and $-12.0000 \pm j5.0000$.

This program required 1,536 calculations of K, and the execution time on the CDC7600 computer was 2.827 seconds.

10.7.2 Example 2, 3rd Order System - Repeated Poles.

To check the operation of the algorithm for a problem in which repeated poles were specified, it was applied to the same system as was considered in 10.7.1, but with the specified closed-loop poles: -6, -12 and -12.

A solution was obtained at a count of 512 calculations of the K matrix as:

$$K = \begin{bmatrix} 32.0000 & 203.142 \\ 9.68123 & 35.9605 \end{bmatrix}$$

giving the closed-loop poles: -6.0000 and $-12.0000 \pm j0.0000$.

The execution time for this program, on the CDC7600 computer was 0.913 second.

It appears from this example that the gain matrix is slower to converge for repeated poles than for simple poles.

10.7.3 Boiler Control Problem.

The algorithm was applied to a steam boiler control problem described by Bengtsson and Lindahl [B4] , in which the elements of the state vector, input vector and output vector have the following physical significance:

x_1 = drum pressure (bar)
x_2 = drum liquid level (m)
x_3 = drum liquid temperature (deg. C)
x_4 = riser wall temperature (deg. C)
x_5 = steam quality (per cent)
u_1 = heat flow to the risers (kJ/sec.)
u_2 = feedwater flow (kg/sec.)
$y_1 = x_1$
$y_2 = x_2$

For a power station boiler with a maximum steam flow of 350 t/h, drum pressure 140 bar, operating at 90% full load, the matrices A, B and C are as follows:

$$A = \begin{bmatrix} -0.129 & 0.000 & 0.396\times10^{-1} & 0.250\times10^{-1} & 0.191\times10^{-1} \\ 0.329\times10^{-2} & 0.000 & -0.779\times10^{-4} & 0.122\times10^{-3} & -0.621 \\ 0.718\times10^{-1} & 0.000 & -0.100 & 0.887\times10^{-3} & -3.851 \\ 0.411\times10^{-1} & 0.000 & 0.000 & -0.822\times10^{-1} & 0.000 \\ 0.361\times10^{-3} & 0.000 & 0.350\times10^{-4} & 0.426\times10^{-4} & -0.743\times10^{-1} \end{bmatrix}$$

$$B = \begin{bmatrix} 0.000 & 0.139\times10^{-2} \\ 0.000 & 0.359\times10^{-4} \\ 0.000 & -0.989\times10^{-2} \\ 0.249\times10^{-4} & 0.000 \\ 0.000 & -0.543\times10^{-5} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The desired closed-loop eigenvalues are:

$$-0.490\times10^{-1}$$

$$-0.755\times10^{-1} \pm j0.511\times10^{-1}$$

$$-0.141 \pm j0.170\times10^{-1}$$

In Figs. 10.1 to 10.6, the closed-loop poles are shown on the complex plane. In each figure, the desired poles, which are the same in every case, are shown as crosses, whilst the poles which were obtained are shown as circled points.

Bengtsson and Lindahl [B4] first attempted to assign only three poles, leaving the other two to assume any values. This was done by applying the first of their methods, and yielded the closed-loop poles:

$$-0.493075 \times 10^{-1}$$
$$-0.594323 \times 10^{-1} \pm j0.544265 \times 10^{-1}$$
$$-0.955001 \times 10^{-1} \pm j0.113494$$

These are shown in Fig. 10.1, and it is seen that, whilst the real pole is correctly located, the complex poles are not near the desired locations.

The next two results in [B4] were obtained by using the second method described in this reference, which comprised a least squares technique with weighting. These yielded the closed-loop poles:

$$-0.326633 \times 10^{-1}$$
$$-0.592370 \times 10^{-1} \pm j0.553355 \times 10^{-1}$$
$$-0.121923 \pm j0.576148 \times 10^{-1}$$

which are shown in Fig. 10.2, and:

$$-0.251520 \times 10^{-1}$$
$$-0.589330 \times 10^{-1} \pm j0.549513 \times 10^{-1}$$
$$-0.131879 \pm j0.288136 \times 10^{-1}$$

which are shown in Fig. 10.3.

The algorithm developed in this chapter was applied to the problem initially with equal weightings, and yielded the gain matrix:

$$K = \begin{bmatrix} 0.481513 \times 10^5 & 0.112829 \times 10^6 \\ 0.528289 \times 10^2 & 0.322502 \times 10^3 \end{bmatrix}$$

which gave the closed-loop poles:

$$-0.360368 \times 10^{-1}$$
$$-0.746516 \times 10^{-1} \pm j0.490695 \times 10^{-1}$$
$$-0.142585 \pm j0.169954 \times 10^{-1}$$

These are shown in Fig. 10.4, and are seen to coincide with the specified values, except for the real pole, the value of which is specified as -0.49×10^{-1}.

A second run was taken with pole weightings 5,2,2,1,1, so as to bring the real pole closer to the specified value. This gave the gain matrix:

$$K = \begin{bmatrix} 0.550159 \times 10^5 & 0.940841 \times 10^5 \\ 0.610184 \times 10^2 & 0.307994 \times 10^3 \end{bmatrix}$$

with corresponding closed-loop poles:

$$-0.392154 \times 10^{-1}$$
$$-0.736004 \times 10^{-1} \pm j0.474900 \times 10^{-1}$$
$$-0.147478 \pm j0.159734 \times 10^{-1}$$

These are shown in Fig. 10.5.

A third run was taken with weightings 10,2,2,1,1, so as to bring the real pole nearer still to the specified value. This gave the gain matrix:

$$K = \begin{bmatrix} 0.596941 \times 10^5 & 0.757852 \times 10^5 \\ 0.665364 \times 10^2 & 0.290672 \times 10^3 \end{bmatrix}$$

with corresponding closed-loop poles:

$$-0.416929 \times 10^{-1}$$
$$-0.724652 \times 10^{-1} \pm j0.460415 \times 10^{-1}$$
$$-0.150899 \pm j0.141252 \times 10^{-1}$$

These are shown in Fig. 10.6.

In each application of the algorithm, the K matrix was calculated approximately 1,000 times, with a run of 256 calculations of K before switching in the fast convergence algorithm. The execution time on the CDC7600 computer was limited to 6 seconds.

Examination of Figs. 1 to 6 reveals that the pole locations obtained by using the algorithm are considerably closer to the desired locations than those achieved by the methods described in [B4] . The effects of varying the weighting factors have been demonstrated. In this example, feedback gain limits were not imposed, as it was desired to make a direct comparison with the results obtained in [B4] .

Fig. 10.1

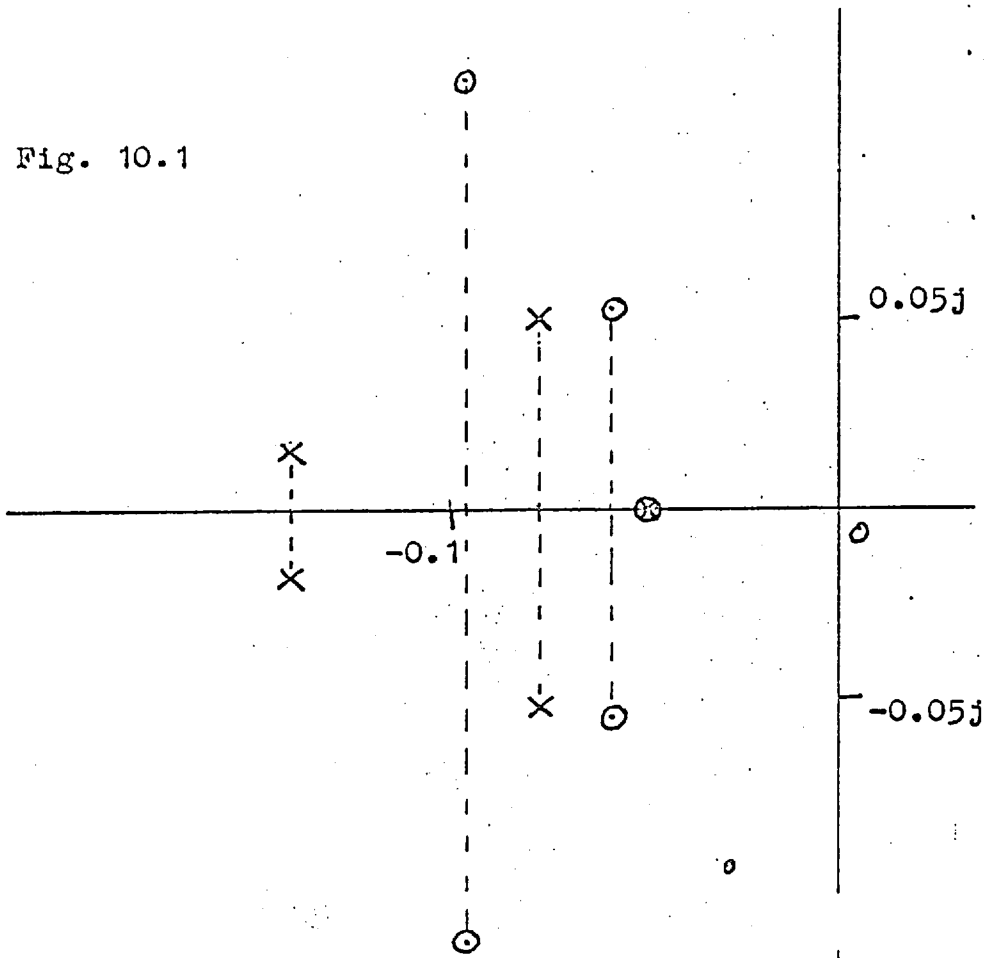


Fig. 10.2

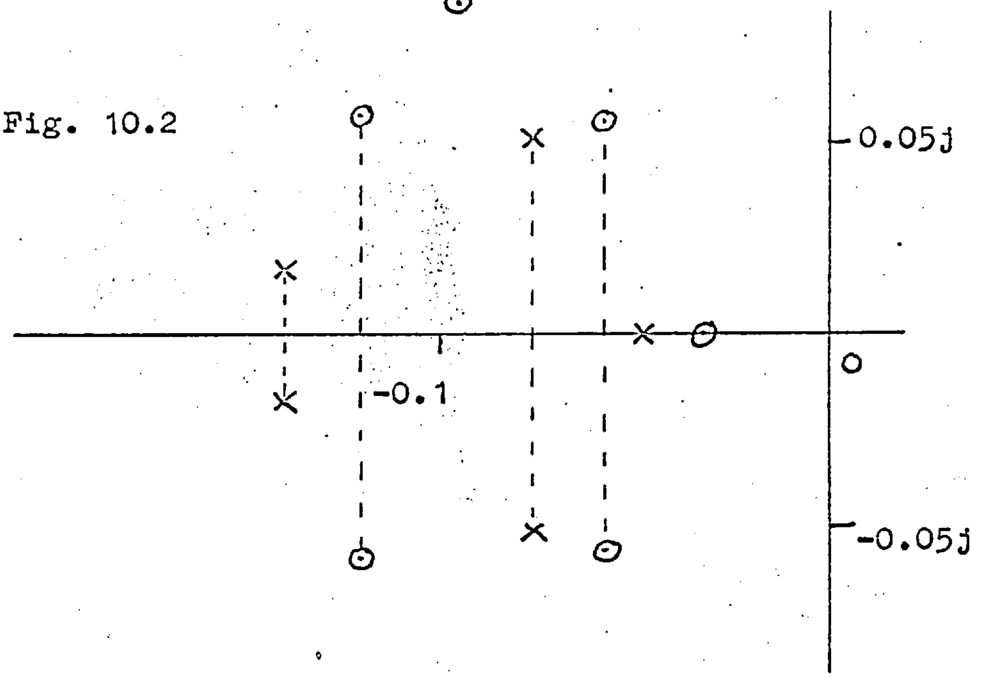


Fig. 10.3

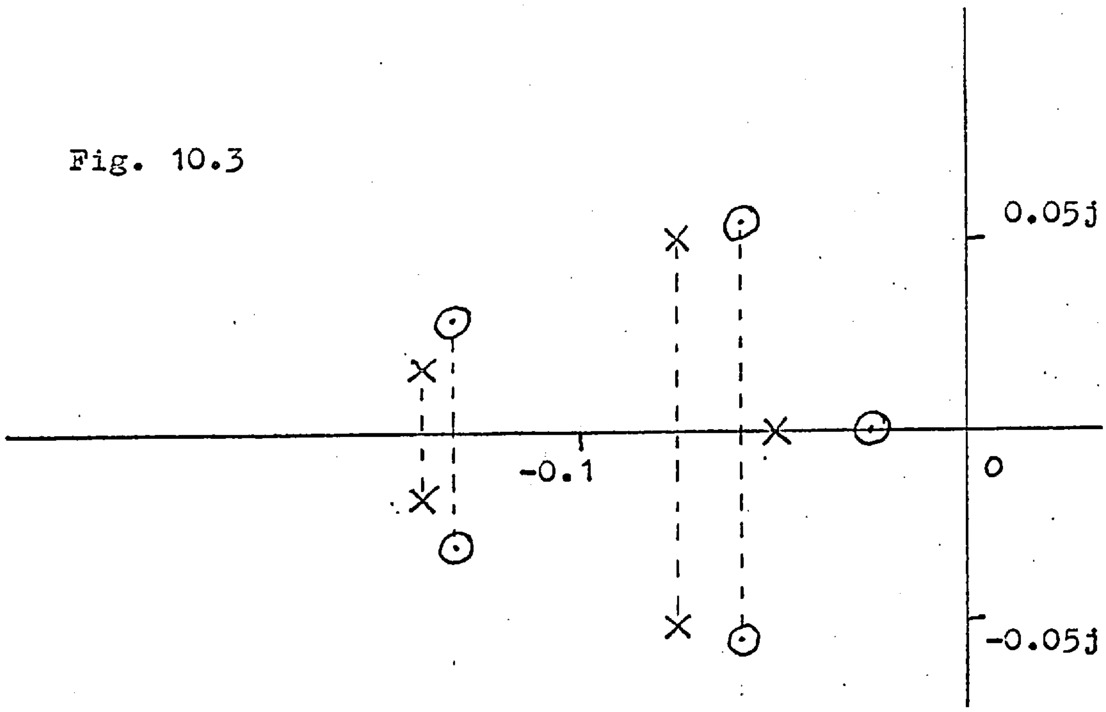


Fig. 10.4

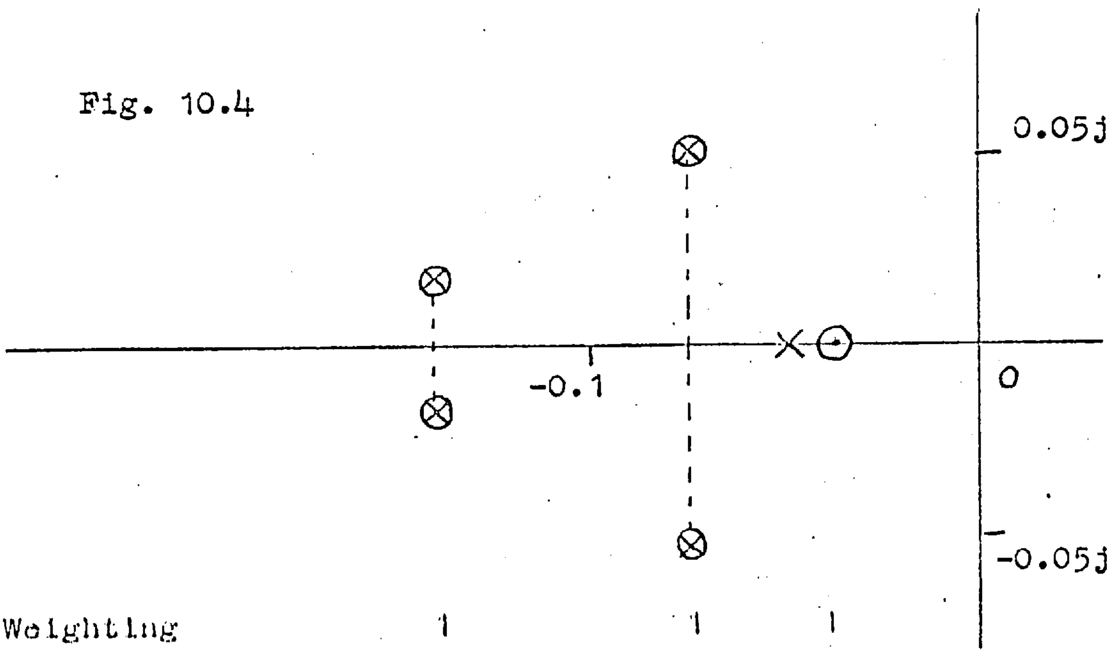
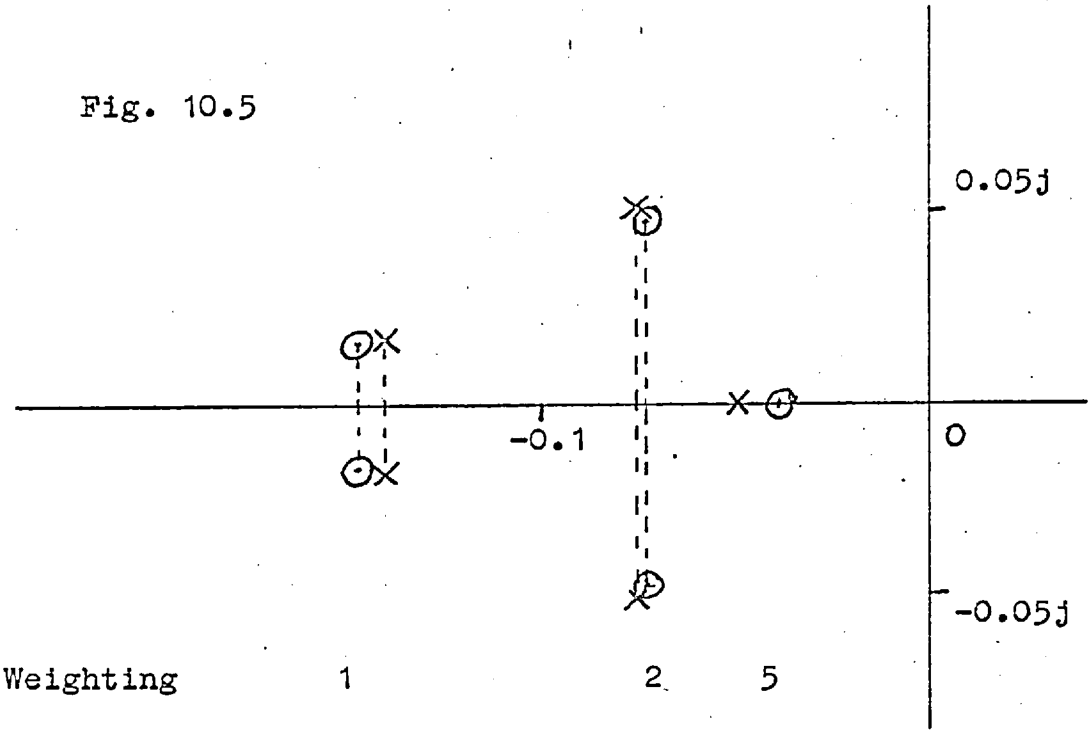
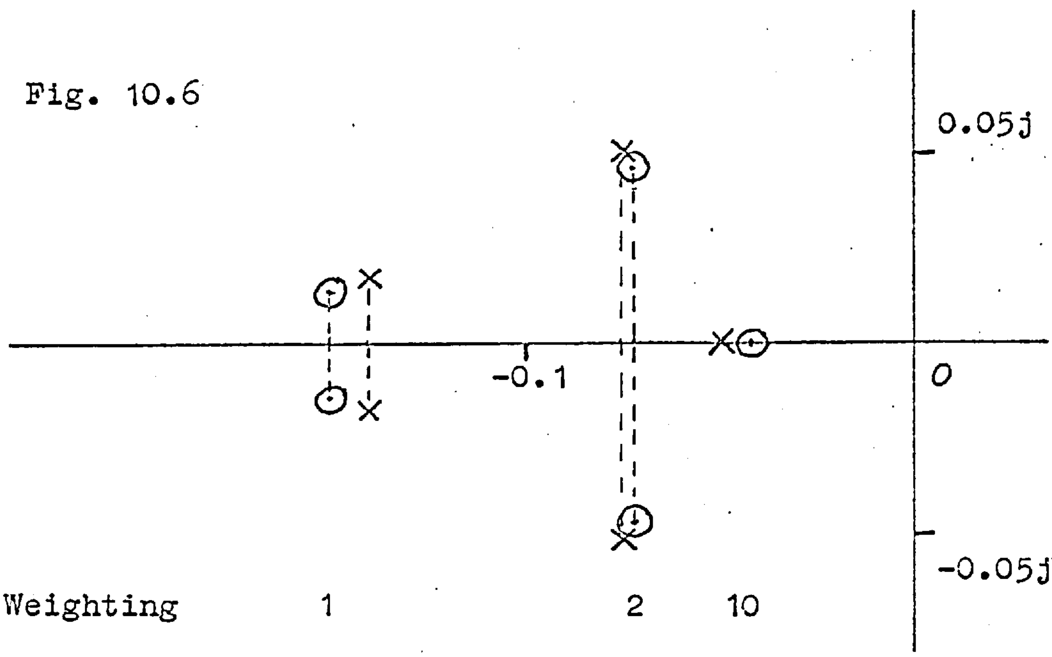


Fig. 10.5



Weighting 1 2 5

Fig. 10.6



Weighting 1 2 10

10.7.4 Power System Control Problem.

A power system control problem was also considered by Bengtsson and Lindahl [B4] . In a reduced model of the Scandinavian network, there are three generators, one on North Sweden (GNOSVE), one in South Sweden (GSYSVE), and one in Norway (GNGE). The generators in North Sweden and Norway have hydro turbines, and the generator in South Sweden has a steam turbine. The linearized equations for the power system are:

$$\dot{x} = Ax + Bu \qquad (10.7.1)$$

$$y = Cx \qquad (10.7.2)$$

The vectors x, u and y are of dimension 15, 7 and 10, respectively, and the variables have the following significance:

x_1	rotor angle, GNOSVE
x_2	rotor angular velocity, GNOSVE
x_3	flux linkage of field winding, GNOSVE
x_4	excitation voltage, GNOSVE
x_5	velocity of water, GNOSVE
x_6	rotor angle, GSYSVE
x_7	rotor angular velocity, GSYSVE
x_8	flux linkage of field winding, GSYSVE
x_9	excitation voltage, GSYSVE

x_{10}	steam pressure, GSYSVE
x_{11}	rotor angle, GNGE
x_{12}	rotor angular velocity, GNGE
x_{13}	flux linkage of field winding, GNGE
x_{14}	excitation voltage, GNGE
x_{15}	velocity of water, GNGE

u_1	excitation input, GNOSVE
u_2	gate opening, GNOSVE
u_3	excitation input, GSYSVE
u_4	steam valve setting, GSYSVE
u_5	fuel flow, GSYSVE
u_6	excitation input, GNGE
u_7	gate opening, GNGE

y_1	rotor angular velocity, GNOSVE
y_2	terminal voltage, GNOSVE
y_3	excitation voltage, GNOSVE
y_4	rotor angular velocity, GSYSVE
y_5	terminal voltage, GSYSVE
y_6	excitation voltage, GSYSVE
y_7	steam pressure, GSYSVE

y_8 rotor angular velocity, GNGE

y_9 terminal voltage, GNGE

y_{10} excitation voltage, GNGE

The operating condition considered corresponds to the expected peak load in 1975, with high transmission from North Sweden to South Sweden. The numerical values of the elements of A, B and C, for this condition, are given in Appendix I.

It was established that satisfactory operation was obtained with state feedback giving the system poles:

$$-7.33\times10^{-3}$$
$$-2.09\times10^{-1}$$
$$-2.77\times10^{-1}\pm j3.55\times10^{-1}$$
$$-3.17\times10^{-1}$$
$$-3.83\times10^{-1}\pm j2.53\times10^{-1}$$
$$-5.14\times10^{-1}$$
$$-1.36\pm j3.12$$
$$-1.37\pm j4.18$$
$$-1.49\pm j3.79\times10^{-2}$$
$$-2.4613$$

Considering the wide geographical separation of the three generating plants, there are obvious advantages in using only local feedback at each generating plant, instead of feedback of the full state vector. The problem, then, is to find the feedback matrix K, where

$$u = - Ky \qquad (10.7.3)$$

such that the system has approximately the desired set of poles, but where the form of the matrix K is restricted to have non-zero elements in only the following positions:

$$\begin{array}{cccccccccc}
k_{11} & k_{12} & k_{13} & & & & & & & \\
k_{21} & k_{22} & k_{23} & & & & & & & \\
 & & & k_{34} & k_{35} & k_{36} & k_{37} & & & \\
 & & & k_{44} & k_{45} & k_{46} & k_{47} & & & \\
 & & & k_{54} & k_{55} & k_{56} & k_{57} & & & \\
 & & & & & & & k_{68} & k_{68} & k_{6,10} \\
 & & & & & & & k_{78} & k_{79} & k_{7,10}
\end{array}$$

Using their method, the authors of [B4] found a K matrix which gave the following system poles:

$$-5.56 \times 10^{-8}$$

$$-3.33 \times 10^{-1} \pm j2.70 \times 10^{-1}$$

$$-3.44 \times 10^{-1} \pm j2.52 \times 10^{-1}$$

$$-3.84 \times 10^{-1} \pm j2.10 \times 10^{-1}$$

$$-4.65 \times 10^{-1}$$

$$-7.95 \times 10^{-1} \pm j3.00$$

$-1.19 \pm j4.05$

-1.47

-1.66

-2.44

The algorithm developed in this chapter was applied to this problem. When the desired poles as given were inserted in the program, it was found that, even after a long run, there were some poles which were not approaching the desired locations. The reason for this was that the particular pole pattern specified had some poles which were almost indistinguishable from multiple poles. An example is the complex pair at

$$-1.49 \pm j3.79 \times 10^{-2}$$

which is very close to being a double pole at -1.49. There is also a cluster of 8 poles close to the origin of the complex plane.

The situation was improved by specifying the following set of desired closed-loop poles:

-7.33×10^{-3}

-2.63×10^{-1} double real pole

$-3.33 \times 10^{-1} \pm j3.04 \times 10^{-1}$ double complex pair

-5.14×10^{-1}

$-1.36 \pm j3.12$

$-1.37 \pm j4.18$

-1.49 double real pole

-2.4613

using various weighting factors.

It became reasonably clear that the specified poles were not attainable, with the given constraints on the form of K. It followed that there was no point in specifying the desired closed-loop poles in the pole-assignment algorithm as those which were actually required. False desired pole locations could be used with advantage to 'draw' the closed-loop poles towards desired values. Using this technique, and after trying various weighting factors, the following result was obtained:

The closed-loop poles specified as desired in the algorithm were:

$-0.60 + j0.00$, with multiplicity 8

$-1.49 + j0.00$, with multiplicity 2

$-1.36 \pm j3.12$

$-1.37 \pm j4.18$

$-2.46 + j0.00$

The corresponding weighting factors were:
10^0, 10^1, 10^2, 10^3, 10^4, 10^5, 10^6, 10^7, 10^{-2}, 10^{-1}, 10^{-4}, 10^{-4}, 10^{-4}, 10^{-4}, 10^{-4}.

The choice of weighting factors was based initially on consideration of the magnitudes which would be assumed by the moduli of the specified poles when raised to the relevant powers.

There is a possibility, when using an algorithm of this type, in which one element of the K matrix is considered at a time, that the desired pole locations could be achieved by the first few values of K computed. This would have the effect of imposing the full burden of control on the first generating plant, an undesirable condition. The use of the slow turn-on feature would make this eventuality unlikely, but, as an additional attempt to spread the control burden as evenly as possible, the sequence in which the elements of K were computed was arranged in a way which covered the three generating plants in every four steps, as shown in Fig. 10.7. This shows the elements of the K matrix as numbers, indicating their order in a single cycle of computation.

1	5	9							
13	17	21							
			2	14	4	16			
			6	18	8	20			
			10	22	12	24			
							3	7	11
							15	19	23

Fig. 10.7.

The following closed-loop poles were obtained:

$-1.320188 \pm j4.150066$

$-0.685477 \pm j2.760519$

$-1.803196 \pm j0.748796$

$0.030663 + j0.00$

$-0.125816 \pm j0.388958$

$-0.638441 \pm j0.541223$

$-0.697708 \pm j0.186223$

$-2.094821 \pm j0.033927$

The corresponding K matrix is:

K(1, 1) = -0.336957+03	K(1, 2) = 0.123511+02
K(1, 3) = 0.774286+01	K(2, 1) = 0.239849+00
K(2, 2) = 0.101597-01	K(2, 3) = -0.276290-01
K(3, 4) = -0.110809+04	K(3, 5) = 0.287821+02
K(3, 6) = 0.286085+01	K(3, 7) = -0.788294+02
K(4, 4) = -0.187359+01	K(4, 5) = -0.208864+00
K(4, 6) = 0.112062+00	K(4, 7) = 0.244979+01
K(5, 4) = 0.319288+03	K(5, 5) = -0.525062+01
K(5, 6) = 0.201785+02	K(5, 7) = 0.394995+03
K(6, 8) = 0.110463+03	K(6, 9) = 0.142131+02
K(6,10) = 0.190445+01	K(7, 8) = -0.403856+00
K(7, 9) = -0.185914+00	K(7,10) = -0.807061-01

The above solution was obtained in a single run of the program, in which the K matrix was computed 300 times. The fast convergence algorithm was activated after 150 computations of K. The total execution time on the CDC7600 computer was less than 400 seconds. This time included the monitoring operation of finding the roots of the fifteenth order characteristic polynomial three times.

The solution obtained is seen to give a closed-loop pole pattern reasonably close to the desired pattern, but unacceptable without modification, due to the presence of an unstable pole. Other solutions were obtained which were completely stable, but which had lightly damped poles. It should be noted that the solution obtained in [B4] includes a pole which is very close to the origin of the complex plane, and so represents a dominant time constant many times greater than that said to be required.

The present design was completed by computing additional mild feedback between the output and input of generator GSYSVE only, so as to meet the requirement for local feedback. The additional feedback was computed in the following way:

The input matrix for GSYSVE, say B_2, consists of the three middle columns of B. It was decided to calculate the feedback on the basis of a single-input system, and so a distribution vector, g, was chosen arbitrarily, so that the equivalent single-input vector $b = B_2 g$, where g was chosen to be:

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

The negative sign in the last element was chosen so as to avoid having rather small elements in b. Using the method of [M1] , the state feedback vector was found which would maintain all the closed-loop poles unchanged, except the unstable real pole, which would be moved to the desired position -0.733×10^{-2}. The positive feedback vector, h^T, required for this was found to be:

-0.224632-02

-0.577750+00

-0.129410+00

-0.485069-01

-0.136880+00

-0.778877-03

-0.231656+00

-0.929728-02

-0.209143-02

0.133485-01

-0.496277-02

-0.688096+00

-0.724758-01

-0.101034+00

-0.162647+00

The state vector is not available for feedback, and the use of observers is not permitted, so that an approximation to this feedback, using only the available outputs from GSYSVE, was obtained, using the generalised inverse. If the output matrix of GSYSVE, the middle four rows of C, is represented by C_2, and the output feedback vector is k^T, then k was found as:

$$k = (C_2C_2^T)^{-1}C_2h.$$

This gave the solution for k as:

0.573343-01

-0.104032+01

-0.209143-02

0.133485-01

Applying this feedback, the eigenvalues of the overall system matrix $(A - BKC + B_2gk^TC_2)$ were found to be:

$-1.172409 \pm j4.914084$

$-0.909607 \pm j2.824345$

$-2.157151 \pm j0.074121$

$-1.310579 \pm j0.606576$

$-0.877863 \pm j0.755744$

$-0.285257 \pm j0.536374$

$-0.157726 + j0.00$

$-0.577581 \pm j0.335272$

This solution appears to be more satisfactory than that obtained in [B4] , as the closed-loop pole pattern is reasonably close to the desired pattern, and the dominant time-constant is considerably smaller.

CHAPTER 11.

CONCLUDING REMARKS AND RECOMMENDATIONS.

11.1 Computer Programs.

Computer programs were written, where appropriate, for the procedures described in this thesis. The language used was FORTRAN 4, and several subroutines for the standard operations were adapted from those published by Melsa and Jones [M14]. The programs were run on an ICL1903A computer and on a CDC7600 machine.

11.2 General Conclusions and Recommendations for Further Research.

The problem of closed-loop pole assignment by state vector feedback may now be regarded as solved. There is a choice of techniques available, and the combination of initial arbitrary feedback to remove multiple poles, and to render the system 'normal', in the sense that it is controllable from every input, followed by the use of the explicit gain formula for single-input systems, provides a solution in every case, provided the system is controllable.

The case in which there is restricted measurement access may be approached through the results of Davison [D3], for the assignment, or near-assignment, of m poles, where the system has m independent outputs, permitting the use of the single-input explicit gain formula in a restricted form. The remaining (n-m) closed-loop poles must be determined, as they move in an unpredictable way. A solution has been presented for this problem, in which the feedback gains are obtained, whilst, at the same time, the coefficients of the residual characteristic polynomial, from which the remaining (n-m) poles may be found, are given.

Very recently, Kimura [K2] has shown that a controllable, observable, system with r independent inputs and m independent outputs, can have all n poles made equal to, or arbitrarily close to, any assigned values by proportional feedback, if $m+r \geq n+1$. A similar conclusion has been reached by Davison and Wang [D5] , who have also shown that, for almost all (B,C) pairs, min(n, m+r-1) poles can be assigned arbitrarily. The approach, used by Davison, of considering the assignment of poles arbitrarily close to desired values, so avoiding the difficulties associated with the hypersurfaces of unattainable poles, has contributed greatly to the understanding of the theory in this area. Although the precise assignment of poles has no meaning in engineering applications, due to limitations in the accuracy of the parameters concerned, changes in their values with time, and so on, the theoretical existence of a solution which approaches arbitrarily close to an unattainable value may be accompanied by practical difficulties, such as the use of unduly high loop gains. Conclusions about the properties of 'almost all' systems could be regarded as having doubtful value, since the engineering designer will draw little comfort from the knowledge that most other systems would have a desirable property if the system with which he is confronted lacks this property. He would, presumably, be encouraged to look for other outputs or inputs, in the hope that the changed system would have the property required. The results obtained by Kimura and by Davison and Wang are important, but they do not represent a final answer to the general problem of the conditions which must be satisfied by the (A,B,C) matrices to permit arbitrary closed-loop pole assignment. There is scope for further interesting research in this direction.

The assignment of zeros as well as poles is an obvious development of state vector feedback, and two procedures are presented [M7] for the design of state vector feedback to provide, as far as possible, a specified scalar input-output transfer function. These procedures are useful in the commonly occurring case where, as in scalar system design, one input-output transfer function is of major importance. The problem has been approached more generally by Wang and Desoer, and there are some cases in which more than one transfer function can be specified. However, the limited number of variable parameters available in state vector feedback makes it unlikely that very much further progress will be made in this direction.

The complete freedom of pole assignment provided by state vector feedback makes the use of observers very attractive, especially since, provided that an observer is fed with the system inputs, its presence does not increase the order of the input-output transfer functions. The design method of Cumming for a state observer is satisfactory where full computer facilities are available, but it is cumbersome otherwise. A simple step-by-step procedure for the design of reduced-order state observers suitable for pencil-and-paper design, assisted by an electronic calculator, or a time-sharing computer terminal using a simple programming language, has been presented.

A design procedure has been given for the linear functional observer with arbitrary dynamics, which avoids the need to transform the system into a special canonical form. This procedure has been extended to permit the design of low-order linear functional observers in which the reduction of observer order is achieved by accepting restrictions on the choice of the observer poles. The problem of the design of the linear functional observer may be regarded as solved. The techniques have been used to provide a step-by-step procedure for the design of degenerate observers in general, to provide more than one specified linear functional of the state vector. This procedure may, in most cases, be expected to result in a worthwhile reduction in the order of the degenerate observer, but it is not claimed to yield a design of the lowest possible order. There is scope for further research in the problem of designing a degenerate observer of minimum order to provide a set of specified linear functionals of the state vector.

The properties of the dual observer are different from those of the ordinary observer and, whilst its field of application is less extensive than that of the ordinary observer, it could be useful in regulator systems having more inputs than outputs. A design procedure for a dual observer is presented, and it has been shown that a design of order (q-1) can be obtained, where q is the controllability index of the system, such that the poles of the overall system are assigned arbitrarily.

The need for an observer to be fed with the external inputs which are applied to the observed system is a disadvantage in practical cases. The design of O-observers and k-observers is an attempt to overcome this by designing the observer for a suitably augmented system. The work of Hostetter and Meditch in this field has been examined, and some new conditions for the existence of these observers have been presented. This is a promising field for further research.

Approaches to the general pole assignment problem for a multi-input, multi-output system by output feedback through the initial determination of state feedback have been examined, and do not seem very promising. On the other hand, the availability of modern digital computers operating at high speeds makes it possible to employ a simple algorithm which is based on a direct approach to the solution. Such an algorithm is presented. It is iterative, but not incremental in operation, and is shown to be acceptably fast, to yield solutions which are superior to those obtainable by other methods, and to accommodate design constraints without difficulty. Further research based on the application of this type of approach to large systems and to non-linear systems is likely to be rewarding.

From a more general point of view, there is a need for further research to improve the relationships between the general theoretical work and practical numerical problems. Many of the theoretical results, for example, in Kalman's controllability and observability criteria, and in Luenberger's canonical forms, rest upon the idea of the linear independence of vectors. In practical numerical work, this question does not always have a clear-cut answer, as has been demonstrated recently by LaPorte and Vignes [L6] , who have considered the related problem of determining whether a numerical matrix is singular. They have given examples in which a singular matrix would, if treated in the usual way on a finite word length digital computer, be regarded as non-singular, and of the converse case of a non-singular matrix which would be regarded as singular. A quantitative measure of the linear independence of a vector relative to a given set of linearly independent vectors is clearly very useful, and such a measure is provided by the ratio of the Gram determinant including the candidate vector to that without it. This ratio [G3] gives the square of the length of the component of the candidate vector normal to the space spanned by the given vectors. By the repeated application of this procedure, 'best' sets of linearly independent vectors can be built up for use, for example, in connection with Kalman's tests, or Luenberger's canonical forms. Such ideas as these could be applied to the general theory of control systems so that they would no longer be regarded as simply controllable or uncontrollable, etc., but as having these properties in varying degrees, subject to quantitative measurement.

APPENDIX I.

The numerical values of the elements of the A, B and C matrices of the system considered in 10.7.4 are given in this Appendix. The signed integer at the right hand side of each number indicates the power of ten by which it is to be multiplied.

A-MATRIX.

A(1, 2) = 0.314159+03	A(2, 1) = -0.242249-01
A(2, 2) = -0.322929+01	A(2, 3) = 0.162980+00
A(2, 5) = 0.340985+00	A(2, 6) = 0.113810-01
A(2, 7) = -0.864848-02	A(2, 8) = -0.684552-02
A(2,11) = 0.128439-01	A(2,12) = -0.998426-02
A(2,13) = -0.712887-02	A(3, 1) = -0.213677-01
A(3, 2) = -0.676581-01	A(3, 3) = -0.304433+00
A(3, 4) = 0.250453+00	A(3, 6) = 0.147243-01
A(3, 7) = -0.884903-02	A(3, 8) = 0.554234-03
A(3,11) = 0.664346-02	A(3,12) = 0.672120-02
A(3,13) = 0.827741-02	A(4, 4) = -0.769231-01
A(5, 5) = -0.140858+01	A(6, 7) = 0.314159+03

A(7, 1) = 0.310481-01	A(7, 2) = -0.207229-01
A(7, 3) = -0.138751-01	A(7, 6) = -0.499094-01
A(7, 7) = -0.242749+01	A(7, 8) = 0.465573-01
A(7,10) = 0.159024+00	A(7,11) = 0.188614-01
A(7,12) = -0.141278-01	A(7,13) = -0.993110-02
A(8, 1) = 0.180088-02	A(8, 2) = 0.602946-02
A(8, 3) = 0.695069-02	A(8, 6) = -0.227243-02
A(8, 7) = -0.282597-01	A(8, 8) = -0.360985+00
A(8, 9) = 0.336492+00	A(8,11) = 0.471545-03
A(8,12) = 0.390225-02	A(8,13) = 0.403555-02
A(9, 9) = -0.100000+00	A(10,10) = -0.732244-02
A(11,12) = 0.314159+03	A(12, 1) = 0.130472-01
A(12, 2) = -0.106559-01	A(12, 3) = -0.791938-02
A(12, 6) = 0.649000-02	A(12, 7) = -0.511231-02
A(12, 8) = -0.462964-02	A(12,11) = -0.195372-01
A(12,12) = -0.887218+00	A(12,13) = 0.165403+00
A(12,15) = 0.441134+00	A(13, 1) = 0.728335-02
A(13, 2) = 0.410596-02	A(13, 3) = 0.636214-02
A(13, 6) = 0.723718-02	A(13, 7) = -0.444029-02
A(13, 8) = -0.930530-04	A(13,11) = -0.145205-01
A(13,12) = -0.536106-01	A(13,13) = -0.288581+00
A(13,14) = 0.247081+00	A(14,14) = -0.769231-01
A(15,15) = -0.183560+01	

The 156 elements of the A-matrix which are not listed are zero.

B-MATRIX.

B(2, 2) = -0.227323+00	B(4, 1) = 0.769231-01
B(5, 2) = 0.140858+01	B(7, 4) = 0.162229+00
B(9, 3) = 0.100000+00	B(10, 4) = -0.783733-02
B(10, 5) = 0.730000-02	B(12, 7) = -0.294089+00
B(14, 6) = 0.769231-01	B(15, 7) = 0.183560+01

The 95 elements of the B-matrix which are not listed are zero.

C-MATRIX.

C(1, 2) = 0.100000+01	C(2, 1) = -0.212729-01
C(2, 2) = 0.932214+00	C(2, 3) = 0.889955+00
C(2, 6) = 0.194903-01	C(2, 7) = -0.100683-01
C(2, 8) = 0.734920-02	C(2,11) = 0.178260-02
C(2,12) = 0.227022-01	C(2,13) = 0.232592-01
C(3, 4) = 0.100000+01	C(4, 7) = 0.100000+01
C(5, 1) = -0.913649-01	C(5, 2) = 0.240934+00
C(5, 3) = 0.233823+00	C(5, 6) = 0.162358+00
C(5, 7) = 0.277790+00	C(5, 8) = 0.286660+00
C(5,11) = -0.709931-01	C(5,12) = 0.159072+00

C(5,13) = 0.143982+00	C(6, 9) = 0.100000+01
C(7,10) = 0.100000+01	C(8,12) = 0.100000+01
C(9, 1) = 0.108115-01	C(9, 2) = 0.114114-01
C(9, 3) = 0.151457-01	C(9, 6) = 0.126540-01
C(9, 7) = -0.743007-02	C(9, 8) = 0.117922-02
C(9,11) = -0.234655-01	C(9,12) = 0.102604+01
C(9,13) = 0.912437+00	C(10,14) = 0.100000+01

The 116 elements of C which are not listed are zero.

EIGENVALUES OF A.

$-1.240614 \pm j4.132591$

$-0.754666 \pm j2.970554$

$-2.383217 + j0.00$

$-1.408580 + j0.00$

$-1.835600 + j0.00$

$0.518948 \;\; 10^{-5} + j0.00$

$-0.732187 \;\; 10^{-2} + j0.00$

$-0.332077 + j0.00$

$-0.100000 + j0.00$

$-0.369861 + j0.00$

$-0.076923 + j0.00$

$-0.422286 + j0.00$

$-0.076923 + j0.00$

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