DIGITAL FILTER STRUCMURE FROM CLASNICAL
ANALOGUE NET:ORKS

## By

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## ABGRict

Recent investigations have shom that a class of dicital filter structures ezists that posessses nuch lower attenuation distortion than the conventional direct or cascade forms. These structures can be derived from classical analogue doublyterminated lossless netrorks by using a one-port wave variable description for circuit elenents. The basic technique, due to Fettweis, consists of expressing the voltage-current relationship of an element in terms of incident and reflected waves and then applyins the bilinear transformation to sive the digital equivalent. These disital circuits are then interconncoted mint the aid of 'Adaptors'. in adaptor is simply the digital reulization of Kirchhof's t::o lavs for a parallel or series junction of $n$ ports. The use of waves in the derivation of the digital filter structures has led to the term 'Yave Digital , Filters' being applied to then. In this thezis it is sho:m that, by considarins each clement in the analogue netrork as a tro-port, a true simulation can be achieved for the corresponding dicital filter structure. In the new rethod, the adaptor, mhici is needed in the one-port description, is not required explicitly out is included as part of the equivalent wave-ilos diagran. The relationship betreen the sensitivity of the attenuation to first-order nultiplier variations and the analogue netrork element sensitivities is derived and it is sinom that the :ultiplier sensitivities are not generally zero at peints of maximum pseudopower transfer. A generalization of the Wave Digital filter concept of rettreis is also examined by considering the relationship between the wave variaoles and the voltarses and currents as a linear transiormation on the $: 13 C D$ matrix of the LS tro-port. A particular transformation is studied in detail and the associated signal-flo:r diagrans are derived. The sensitivity behaviour of structures derived usinc the generat linear transformation is also studied and it is shom that their behaviour is similar to that of Uave Digitel filters. We also consider in this thesis the computer-aided analysis of digital filter struciures and present a new algorithr for analysis which has many advantages over conventional methods.

To Chris

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## Conventions and Symbols

Most symbols are defined in the text as they appear and because a symbol may be used in several different contexts, it is not useful to define them here. However, the few symbols and abbreviations that are used throughout the text are introduced below:

| $\left\{x_{k}\right\}$ | Sequence of numbers,e.g. $x_{0}, x_{1}, \ldots \ldots \ldots \ldots, x_{n}$ |
| :---: | :---: |
| $\mathrm{Re}(\mathrm{z})$ | Real part of a complex variable Z |
| $\operatorname{Im}(Z)$ | Imaginary part of a complex variable $Z$ |
| $\|z\|$ | Absolute value of a complex variable 2 |
| Arg 2 | The phase of a complex variable 2 |
| Z* | The complex conjugate of 2 |
| $\Omega$ | Analogue angular frequency variable |
| $\omega$ or w | Digital angular frequency variable |
| $\mathrm{p}=\Sigma+j \Omega$ | Analogue complex frequency variable |
| $z=\exp (j \omega T)$ | Digital complex frequency variable |
| $\frac{\partial y}{\partial x}$ | Partial derivative of a function $y$ with respect to $x$ |
| $S_{x}^{y}=\frac{x}{y} \frac{\partial y}{\partial x}$ | Sensitivity function of y with respect to x |
| BP | Band-pass |
| BW | Bandwidth |
| DTLLN | Doubly-terminated lossless ladder network |
| ITA | Invariant Transfer Admittance |
| IVR | Invariant Voltage Ratio |
| IP | Low-pass |
| MAP | Maximum available power |
| SFD | Signal-flow diagram |
| SFG | Signal-flow graph |
| UE | Unit Element |
| VSWR | Voltage standing-wave ratio |
| WDF | Wave Digital Filter |
| WFD reflectance | Wave-flow diagram Reflection Coefficient |
| , the following symbol which is to be found in gures represents a delay element with transfer function $z^{-1}$ ay T seconds, unless otherwise stated. |  |

Finally, the following symbol which is to be found in many figures represents a delay element with transfer function $z^{-1}$ and delay T seconds, unless otherwise stated.

## Ghater 1

## Introduction

## Contents:

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Cnotor 1

## Introduction

1.1. Theory of Linear ne-Invariant Discrote-inc netrove

### 1.1.1 Dixith iman and Exocosors

In normal analogue sisnal processines, the input and output signals of a system are continuous in tine and the systen itself is conetructod froa analo.juc components. For example, ir the systen is an electrical network then the components may be resistors, inductors, capacitors, etc. It is the purpose of the wors contained in this thosis to consider signal processing where signals are expressed no lonfor in a continuous-time domain, but ratller in a discrote-tine doman. By discrete-tine domain, it is neant that the signals appear as sequences of resularly spaced nurbers. Nith reference to a systen that operatos on such signals, the input signal shall be denoted by $\left\{x_{k}\right\}$ and the output simal by $\left\{y_{k}\right\}$, k beine an integer indicatin: the discretc-time variable. The specific processiñ operations that rill be eramined are of the filtoring type with one input and one output. In Fir. 1.1 a blocl diacram of a digital signal processing systea is shown. The dicital siçal processors that will bo studied in this thosis shall be reforred to as dienital netorks or, less fenerally, as disital filters. The fundamental concepts of digital signal processing are sumarisod in this section. In addition, sone aspects of disitel filter realisation and allied topics are oxamined, analysed and aporaised.

### 1.1.2. Lincarity end rinc-Invariance [1]

A disital network is said to be linear if the response to an excitation $\alpha\left\{x_{z-1}\right\}+\beta\left\{x_{k 2}\right\}$ is $\alpha\left\{y_{y_{k-1}}\right\}+\beta\left\{y_{k 2}\right\}$ there $\left\{y_{k-1}\right\}$ is the response to an excitation $\left\{x_{k 1}\right\}$ and $\left\{y_{k 2}\right\}$ is the response to $\left\{x_{k 2}\right\}, \infty$ and $\beta$ are scalar constants.

A network is time-invariant if, givon the response $\left\{\mathrm{y}_{k}\right\}$ to an excitation $\left\{x_{k}\right\}$, the response to an excitation $\left\{x_{k-m}\right\}$ is $\left\{y_{1-m}\right\}$ for all values of $m$. Alternatively, it can be said that for tine-invariance, the output sequence, to a given input sequenco does not depend on the instant of time when the input was applied. A time-invariant network consists only of time-invariant elements.

It is important to nention that linear net:oris need not be time-inveriant.

### 1.1.3 - ansalittr [1].

A netror: is causal if, $\hat{\mathrm{O}}$ ( zero excitation, it gives zero response. That is, if

$$
\begin{aligned}
\left\{x_{k}\right\} & \rightarrow\left\{y_{k}\right\} \\
\text { and } \quad x_{k} & =0 \quad \forall k<k \\
\text { the: } \quad y_{k} & =0 \forall k<\mathbb{L}
\end{aligned}
$$

where ' $\rightarrow$ ' zeans 'gives rise to' and ' $\forall$ ' means 'for all'.

It is nomally ascuned, by convention, that $:=0$. That is, the first member oif $\left\{x_{i}\right\}$ is $x_{0}$.

### 1.1.4 Jinoar Disference Jaugtions [1]

Only linear time-invuriant di ital rilters will be considered and thesc filteramey be described by a set of lineur difference equations with constant coefincients [2]. An expression for the kth. output sample nay be written as follo:s,

$$
\begin{equation*}
y_{k}=\sum_{m=0}^{p} a_{m}=\bar{I}_{k-m}+\sum_{n=1}^{q} j_{n^{r} k-n} \tag{1.1}
\end{equation*}
$$

The iterntire nature of the ifference equation con be scen from eqn. (i.1). The kth. output depends on the $q$ previous values of $y$ and the $(p+1)$ nost recent values of $x$. The $\left\{a_{n}\right\}$ and $\left\{b_{n}\right\}$ are constants and are chosen accordins to the type of filterinc requireà.

In analogue network theory, a set of linear differential equations with constant coefficients corresponds to eqn. (1.1).

By observing the form of eqn. (1.1) it is obvious that it is necessary to be able to store certain previous values of $x$ and $y$. A particular value of $y$ will be required $q$ tines tinilst an $x$ will be required ( $p+i$ ) times. If $I$ is the sampling period in seconds
 the al;sorithm to evaluete $\left\{y_{1 s}\right\}$ usins eqn. (1.1) must take less thon I seconds per sample.

## 1.1 .5 z-mannsorm [3]

In analogue networl theory, the Fourier and Laplace transforms are used to transiorn the $\dot{\text { uifferential equations of the linear }}$ time-invariant systen into alGebraic equations, which are more convenient to manipulate. In addition, the variable $t$, time is replaced by p, the comple: frequency.variable. A similar concept exists in dicital filter theory. The one-sided z-transform of a sequase $\left\{Y_{n}\right\}, X(z)$ is lefined as follors,

$$
\begin{equation*}
X(x)=\sum_{0}^{\infty} x_{n} z^{-n} \tag{1.2}
\end{equation*}
$$

where $z$ is the complex variable.
'he correspondins inverse $z$-transform is

$$
\begin{equation*}
x_{n}=\frac{1}{2 \pi j} \oint X(z) z^{n-1} d z \tag{1.3}
\end{equation*}
$$

where $j=\sqrt{-1}$ and $\oint$ denotes integration around any closed curve in the z-plane nhich onclosos all the poles os a $(z)$ and the orieine.

The basic properties of the z-transfora may be sumanized as follows:

Let $z\left[\left\{x_{2}\right\}\right]$ denote the $z$-transtorm of the sequence $\left\{x_{1 i}\right\}$.
Linearity: If $\vec{X}_{1}(z)=Z\left[\left\{x_{1 z}\right\}\right]$
and $X_{2}(z)=\pi\left[\left\{x_{21 ;}\right\}\right]$
then $Z\left[\alpha\left\{x_{i k}\right\}+\beta\left\{x_{2 k}\right\}\right]=\alpha x_{1}(z)+\beta x_{2}(z)$.
Shiftinf: If $Z\left[\left\{x_{1}\right\}\right]=X(z)$
then $Z\left[\left\{x_{k-n}\right\}\right]=z^{-m_{X}(z)}$.
where $x_{-k}=0$ for $k>0$

$$
\begin{aligned}
& \text { Saving: } \quad \text { Is }\left[\left\{x_{1 z}\right\}\right]=X(z) \\
& \text { then } Z\left[\left\{a^{-k} x_{k c}\right\}\right]=X(a z) .
\end{aligned}
$$

### 1.1.6 Transfer functions and ancient Pernonses [3]

By applying the $z$-transform to eqn. (1.1) and letting

$$
\begin{aligned}
X(z) & =Z\left[\left\{x_{k}\right\}\right] \\
\text { and } \quad Y(z) & =Z\left[\left\{y_{k}\right\}\right]
\end{aligned}
$$

it is found that

$$
\begin{equation*}
Y(z)=H(z) X(z) \tag{1.4}
\end{equation*}
$$

where

$$
H(z)=\sum_{i=0}^{p} a_{i} z^{-i} /\left\{1+\sum_{j=1}^{q} b_{j} z^{-j}\right\}
$$

En. (1.4) relates the z-transform of the output to that of the input. The transfer function $\bar{I}(z)$ is a rational function in $z^{-i}$ and depends on the constant coefficients in the original difference equation.

Consider the input sequence

$$
x_{n}=\left\{\begin{array}{lll}
1 & & n=0  \tag{1.5}\\
& \text { for } & \\
0 & & n>0
\end{array}\right.
$$

The z-traneform of eqn. (1.5) is, on using eqn. (1.2), given by

$$
X(z)=1
$$

therefore,

$$
Y(z)=H(z)
$$

and

$$
\begin{equation*}
h_{n}=\frac{1}{2 \pi j} \oint H(z) z^{n-1} d z \tag{1.6}
\end{equation*}
$$

$\left\{h_{n}\right\}$ will be referred to as the impulse response of the digital filter.

### 1.1.7 Stability [i]

The stability condition may be expressed in terms of in $(\mathrm{z})$ or $\left\{\mathrm{h}_{\mathrm{k}}\right\}$.
$\therefore$ digital filter is stable if and only is the poles os If outside the unit-circle on the $z^{-i}$-plane or inside the unit-circle on the $z$-plane.

Equivalently, stability is assured if and only if

$$
\sum_{0}^{\infty}\left|\ln _{h_{2}}\right|<\infty
$$

Min condition implies that [ 4 ]

$$
\operatorname{Lim}_{k \rightarrow \infty} h_{k}=0
$$

### 1.1.8 Digital Filter Realisations [1]

It is now appropriate to consider the realization of transfer functions or difference equations. Three digital components are required (a) the adder, (b) the multiplier and (c) the delay. They are defined as follows (Tie. 1.2):

For two signals, $x$ and $y$, at its input, the adder produces an output $x+y$. It is assumed that the adder has tron inputs and one output. For an input simel $x$, the multiplier produces an output $\alpha x$ where $\alpha$ is the value of the multiplier and is real. Finally, for an input signal $x_{n}$, the delay produces the previous input signal $x_{n-1}$ and after the sampling period of $I$ seconds and a new input signal, $x_{n+1}$, produces $x_{n}$ at its output. Thus if $y_{n}$ is the output signal, then for $y_{0}=0$,

$$
\begin{equation*}
y_{n}=x_{n-1} \tag{1.7}
\end{equation*}
$$

On applying the z-transform to eqn. (1.7), it is found that

$$
\begin{equation*}
Y(z)=z^{-1} X(z) \tag{1.8}
\end{equation*}
$$

Thus the transfer function of the delay element is $z^{-1}$.

H( $z$ ), as defined in eqn. (1.4), or equivalently the algorithm defined in eqn. (1.1) may now be realised. The structure appears in Fig. 1.3.

### 1.1.2 Fronencr ?eoronse [3]

If it is assuned thet the input sicnal is a sampled complex exponential wave,

$$
\begin{equation*}
\text { i.e. } \quad x_{n}=e^{j n w T} \tag{1.9}
\end{equation*}
$$

then the solution $y_{n}$ to eqn. (1.1) is also an exponential wave which con be represented as

$$
\begin{equation*}
y_{n}=F\left(e^{j w T}\right) e^{j n w T} \tag{1.10}
\end{equation*}
$$

substituting for $y_{n}$ and $y_{n}$ fror eqns. (1.0) and (1.10) into eqn. (1.1) eives imediately that

$$
\begin{equation*}
F\left(e^{j w I}\right)=\sum_{r=0}^{p} a_{r} e^{-j r w T} /\left(1+\sum_{s=1}^{q} b_{s} e^{-j s w T}\right) \tag{1.11}
\end{equation*}
$$

Eqn. (1.11) is the same as $\bar{H}(z)$ in eqn. (i.4) iiith z replaced by $e^{j v T}$. The frequency response is therefore $H\left(e^{j w T}\right)$.

### 1.2 Desicn of Disital Pilters

### 1.2.1 Classification of Di-ital Pilters [1]

Returniñ to the difference equation of eqn. (1.1) it is clear that the order of the equation is $(q+1)$. Because of the recursive nature of this difference equation, that is $y_{k}$ depending on $y_{j}, j<k$, the digital filters that are realized using eqn. (1.1) or equivalentily eqn. (1.4) are called Recursive disital filters. If $b_{n}=0$ for all $n$, then eqn. (1.1) gives

$$
\begin{equation*}
y_{k}=\sum_{\mathrm{m}=\mathrm{d}}^{\mathrm{q}} \mathrm{a}_{\mathrm{m}} \mathrm{x}_{\mathrm{k}-\mathrm{m}} \tag{1.12}
\end{equation*}
$$

This equation defines the class of non-recursive diçital filters. The corresponding $z$-transform is

$$
\begin{aligned}
& Y(z)=I(z) X(z) \\
& H(z)=\sum_{m=0}^{q} a_{m} z^{-m} .
\end{aligned}
$$

The direct realisation of eqn. (1.12) appears in Fis. 1.4. A recursive disital filter has feedback whereas a non-recursive digital filter has not (c.i. Pigs. 1.3 and 1.4). Disital filters may be subdivided into another tro classes: (a) FIR, finite impulse response, or (b) IIR, infinite inpulse response. All non-recursive disital filters have IIR but the reverse is not necessarily true. All IIR.disital filters are recursive, asain the reverse is not senerally true.

### 1.2.2 Dinoxiation and Dowim [1]

In classical dirsital filter theory, once the coefficients of the desirod tranfor function had beon found by approxination, tho design was complete since the transfer function could be synthesised by inspection. The many different approximation methods have been adequately discussed in the literature and it is not proposed to say any more on this topic. It is therofore approprjate to mention the three classical disital filter structures that realize $\mathrm{H}(z)$ in eqn. (1.4). The first is the direct form, the first variant of which was illustrated in lis. 1.3. $H(z)$ may also be realized in the forr sho:m in Fig. 1.5. Jy expressins $H(z)$ as a product of 2nd order rational functions in $z^{-1}$, the cascade for $m$ may be derived (Fig. 1.6). Finally, by expressing $\mathrm{H}(z)$ in the followine way

$$
\mathrm{H}(z)=\gamma_{0}+\sum_{i} \frac{\gamma_{0 i}+\gamma_{1 i^{z}} z^{-1}}{1+\beta_{1 i^{z}} z^{-1}+\beta_{2 i^{z^{-2}}}}
$$

the parallel-forn realization may be derived (ize. 1.7). The structures in Fiss. $1.3,1.6$ and 1.7 are canonical in the sense that a minimun number of adders, multipliers and delays are used to realize eqn. (1.4) in the 'general case.

## 1.2 .3 Bilinear Iransformation Yethod of Degicn [5]

The bilinear transformation is defined as follows

$$
\begin{equation*}
p \rightarrow \frac{2}{T}\left\{\frac{1-z^{-1}}{1+z^{-1}}\right\} \tag{1.14}
\end{equation*}
$$

where $p=j \Omega$, is the anologue complex frequency variable and $z^{-1}=e^{-j i r i}$ is the digital complex frequency variable, it is the sampling rate. It has the property of uniquely nappins the ontire left-half plane of the p-plane into the outside of the unit circle in the $z^{-1}$-plane. Thus, stable analogue filters map into stable dicital filters. By this means, dicital filters can be designed froa a classical filter transfer function. It is inportant to note that the transformation in eqn. (1.14) distorts the $: \mathrm{F}$ frequency scale, but the desien procedure can take account of this. Once the analosue transfer function $H(p)$, which corresponds to the desired digital transfer characteristics, has been obtained, the bilinear trensformation noy then be anplien matheresultinc rational function in $z^{-1}$ realized in any of three forrs of section 1.2.2. The coefficients of $H(p)$ can be found by consulting standard filter desisn tables [6][7][8].

### 1.2.4 Prevuncy Erangomations [9]

For conventional analogue filter desich, the coefficients of the rational function $\mathrm{II}(\mathrm{p})$, or the element values of the possible realizations, are siven in taioles for normalized lon-pass filters only. By nornalized, it is meant that the cut-off frequency is 1 radian per second and the source resistiance (or load resistance) is 1 olm. Any other type of filter may then be found by using one of the four basic transformations (a) Low-pass to Low-pass, (b) Lowpass to Hifgh-pass, (c) Lov-pass to Jand-pass and (d) Low-pass to Bend-stop [10]. To achieve realizable element values, impedance scaliñ̈ may also be necessary [10]. In digital filter desicn, an analogous theory has been developed [9]. On referring bacl: to the previous section, it can be seon that there are t:ro alternatives to the desizn of dicital filters using the bilinear transformation. Havinc found the analogue nomalized lor-pass prototype from tables or otherrise, the frequency transformation may be performed either in the analocgue or in the digital frequency domain.

### 1.3 Finite Ford Lemeth iffects in Digital Wilters

### 1.3.1 Princios Problems

In analogue filters, a desired magnitude characteristic may change becouse of tomperature, ageing and inaccuracies in component values.

These effects are non-caistent or merli-ible in dirital filters. Hovever, there is a problen caused by the obvious limitation to a finite rord lensth. This restriction leads to three major problems [11]:

```
I: quantization of the input signal { }\mp@subsup{x}{n}{}}\mathrm{ into a
set of discrete levels.
II: Representation of the filter coerficients
```



``` coofficient quantization.
```

III: The accumiation of roundmofe errors comittod at arithmetic operations, i.e. round-off noice.

In addition there may be eriors in processing analogue si nals, that is eriors in samplins and in reconstruction using $\dot{A} / D$ and D/i convertors [ii].

### 1.3.2 mroes of irithmetic [11]

A binary representation vill be assumed for the data. There are two basic trpes of arithmetic one may use in a disital filter (aj fjxed-point and (b) iloatins-point. Floating-point arithmetic is to be preferred because of the greater range of values possible,
 Pixed-point arithmetic, on the other hand, is easier to inplement in hardvare form. sfter arithmetic operations, it is necessery to round or truncate and this will introduce errors. Mese errors will be nore severe in truncation. Burthermore, fized-point arithmetic introduces ecrors only in multiplication.

## 1.3 .3 Coefincient vantization

The quantization of the multiplier values in a digital filter will lead to distortion in the loss characteristic and in severe cases to instability as the poles move towards the unit-circle [i2]. i suitable measure of this effect is the function $\partial A / \partial \alpha \quad$. There $A$ is the attenuation and $\alpha$ is a particular multiplier [13]. Ideally, it would be desired to minimise the function $I$, given by

$$
I=\int_{0}^{\pi / T} \sum_{i}\left|\frac{\partial A(\omega)}{\partial \alpha_{i}}\right|^{2} d \omega \quad \text { or an }
$$

equivalent function thich takes account of the viriation of
$\partial: / \partial \alpha_{i}$ with frequency anu for each $\alpha_{i}$. It is lmom that the effect of coefificient quantization is more pronounced for a highorder filter when it is realized in direct form than when it is realized in parallel or cascade form [i1].

### 1.3.4 Round-0ff Error [11][20]

The round-ofic error, together with the coefficient quantization error, are the main sources of error. There are three factors which aetermine the style of the rouncori error for a ziven input signal, (i) the number of bits usei for the data, (ii) the type of arithmetic used and (iii) the disital structure realizins the desirea transfor function.

It has been shom that the output round-off noise is usually, but not always, himher for a filter with fixeci-point arithnetic than for a filter with floating-point arithnetic using the same number of dizits [14]. This is because of the automatic scaling provided by Sloatins-point arithmetic. Ho::ever, as was said in section 1.3.2, fixed-point hardware is more economical to implement than floating-point hardware.

### 1.3.5 Limit Cycles [11]

Limit cycles or overflow oscillations may appear at the output of the disitill filter cwon :hen thero is no cacitation. Where are tro types of lirit cycles for fixed-point realizations (i) overflows in the registers coused by limited dynamic rance and (ii) as a result of rounding after multiplication. It has been shown that (i) nay be eliminated completely [15][i6]. The second type of limit cycle, lmorn as the deadband effect, has jeen extensively studied for first and second-order filters [17][10]. The limit cycle effect is not significant in floatinemoint filters [19].
1.2.6 Connection betreen Round-Off Hoise and Coefficient uantization

In two recent papers, Fetweis has shom that a relationship exists between the round-ofe noise and coefficient quantization orror in a filter realised with fixed or floatjing point arithmetic [13][21]. There are, however, certain conditions that must be satisfied:
i) The input siemal must be sinusoidal.
2) The output noise function $\Phi(w)$ measures the round-off error due to a single multiplier.
3) The frequency of the output sional is the sane as that of the input.
and in adaition for fireci-point filters,
4). So as to be able to compare the noise in different filters, it is assurncd that the quantization step of the signal parameter is the semc.for all filters and that the available dynamic range is used to the same extent.

### 1.3.7 Smativity menctions

It has been observei that the difforentisl sensitivity plays an important part in the estimation of round-o noice and coefficient quantization error. his is because roundint, although a non-linear effect, is equivalent to small chanees in the coefficients and the signals. For filterinis, it is particularly useful to consider the loss function $L$ defined as follows,

$$
\begin{equation*}
L(i)=20 \log _{10}\left|\sim\left(i_{0}\right) / s(i)\right| \tag{1.15}
\end{equation*}
$$

where $G(w)$ is the transfer function of the digital filter at some frequency w and $\mathrm{w}_{0}$ is a reference frequency (e.g. d.c.).

If eqn. (1.15) is differentiated with respect to any multiplier then, with $|G|=\left|G\left(w_{0}\right) / G(w)\right|$,

$$
\frac{\partial L}{\partial \alpha}=-\frac{20}{L_{0} e_{e}^{10}} \quad \frac{1}{|G|} \cdot \frac{\partial|G|}{\partial \alpha}
$$

For convenience, we shall use the function ${ }_{S}|G|$, defined as follows

$$
\begin{equation*}
{ }_{S}^{|G|}=\frac{\alpha}{|G|} \frac{\partial|G|}{\partial \alpha} \tag{1.16}
\end{equation*}
$$

which differs fron $\frac{\partial L}{\partial \alpha}$ by a constant anount.

Ban. (1.16) is, of course, the classical definition of the relative sensitivity function [22]. Te shall be interested, in addition to eqn. (1.16), in the following sensitivity function

$$
\begin{equation*}
S_{D}^{|G|}=\frac{D}{|G|} \frac{\partial 1 G \mid}{\partial D} \tag{1.17}
\end{equation*}
$$

where $D$ is any delay element. Fie have used $D$, instead of $z^{-1}$, to avoid confusion with the complex frequency variable. The delay sensitivity has no practical significance since delay elements are not subject to chance. However, there are theoretical reasons and those will bo expanded in chapter 3.

Finally, in most computational alforithas, and in particular those to be described in Chapter 3 , it is $S_{\infty}^{G}$ and $S S_{D}^{\dot{G}}$ that are obtained. It is well know that

$$
\begin{equation*}
S_{\alpha}^{|G|}=\operatorname{Re}\left\{S_{\alpha}^{G}\right\} \tag{1.18}
\end{equation*}
$$

but let us consider the following,

$$
\begin{equation*}
G=|G| \exp (j \Phi) \tag{1.19}
\end{equation*}
$$

where $\Phi$ is the phase of $G$.

Then, on taking natural logarithms of both sides we have that

$$
\begin{equation*}
\log _{e} G=\log _{e}|G|+j \cdot \bar{\Phi} \tag{1.20}
\end{equation*}
$$

Differentiate both sides of eau. (1.20) w.r.t. (vT), then

$$
\begin{equation*}
S_{W T}^{G}=S_{V T}^{|G|}+j \Phi S \Phi_{w T} \tag{1.21}
\end{equation*}
$$

As all the functions in eqn. (1.21) are real, we have immediately that

$$
\begin{equation*}
{ }_{S}|G|=\operatorname{Re}\left\{S_{W T}^{G}\right\} \tag{1.22}
\end{equation*}
$$

But

$$
D=e^{-j v T}
$$

thereiore

$$
\begin{equation*}
S_{n T}^{D}=-j \div T \tag{1.23}
\end{equation*}
$$

Coabining eqn. (1.22) and (1.23) and notins that [22]

$$
\left.\begin{array}{l}
S_{y}^{z} S_{x}^{y}=S_{x}^{z} \text {, we have } \\
S \underset{D}{|\mathcal{N}|}=-j 2 c\left\{j S_{D}^{G}\right\}
\end{array}\right\}
$$

or, equivalently,

$$
\begin{equation*}
{ }_{S}^{|G|}=j \operatorname{In}\left\{S_{D}^{G}, ~\right\} \tag{1.24}
\end{equation*}
$$

1.4 Low Sensitivity Pronart: of Certain Clessical Analosie Filters

It is well known that ladder filters have low sensitivity to element variations in the stopband. Furthermore, doubly-teranated lossless filters can be desioned so that, at frequencies of minimum loss, the source delivers macimum power into the load. At these points of naxinum available pover ( $\because / \mathrm{L}$ ), the derivative of the loss with respect to any reactive component is zero. This fact, together witin the proof, have been collectively referred to as 'Crchard's arcument' [23]. It has led to the design of active filters by replacins the inductors in a doubly-terminated LC network by gyrators [24] and to Bruton's transfomation and the concept of frequency-dependent negative resistarice [25].

Let us examine the property of zero sensitivity more fully. The maximum power available from the resistive voltace generator (Fig. 1.8) is $\mathrm{V}_{\mathrm{O}}^{2} / 4 \mathrm{R}_{\mathrm{s}}$ and would be delivered to $\mathrm{R}_{\mathrm{L}}$, the load, if the reactance network II was replaced by an ideal transformer which exactly matched $R_{L}$ and $R_{s}$ (Fig. 1.9). The voltage, $V_{2}$, across $R_{L}$ would then be $\left(V_{o}^{2} R_{L} / 4 R_{S}\right)^{\frac{1}{2}}[10]$.

Therefore at MP points, the transier function $H(p)$ is given by the following expression,

$$
H(p)=V_{2} / V_{o}=\sqrt{R_{L} / 4 R_{s}}
$$

or

$$
\begin{equation*}
|\mathrm{H}|^{2}=R_{\mathrm{I}} / 4 R_{S} \tag{1.25}
\end{equation*}
$$

It is easy to see that, on using eqn. (1.25),
and $\quad{ }_{S^{|H|}}^{|H|}=-\frac{1}{2}$
ITo:, in Fig. 1.8 , let $Z_{11}=R_{11}+j x_{11}$ then the power $P_{\text {III }}$ into the filter is given by

$$
\begin{equation*}
P_{I I}=\frac{v_{0}^{2} R_{11}}{\left(R_{s}+R_{11}\right)^{2}+X_{11}{ }^{2}} \tag{1.27}
\end{equation*}
$$

As the network II is lossless the power at the output, $\mathrm{P}_{\mathrm{L}}$ is equal to $\mathrm{P}_{\mathrm{IIT}}$ 。

Let x be a reactive elenent in N then, on using the chain rule for differentiation,

$$
\begin{equation*}
\frac{\partial P_{L}}{\partial x}=\frac{\partial P_{L}}{\partial R_{11}} \frac{\partial R_{11}}{\partial x}+\frac{\partial P_{L}}{\partial X_{11}} \frac{\partial X_{11}}{\partial x} \tag{1.28}
\end{equation*}
$$

Using eqn. (1.27) we have that

$$
\begin{equation*}
\frac{\partial P_{L}}{\partial R_{11}}=\frac{\left(R_{S}^{2}-R_{11}^{2}+X_{11}^{2}\right) V_{0}^{2}}{\left\{\left(R_{S}+R_{11}\right)^{2}+X_{11}^{2}\right\}^{2}} \tag{1.29}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial P_{L}}{\partial X_{11}}=\frac{-2 R_{11} X_{11} V_{0}^{2}}{\left\{\left(R_{S}+R_{11}\right)^{2}+X_{11}^{2}\right\}^{2}} \tag{1.30}
\end{equation*}
$$

At points of maximur available power, $R_{11}=R_{s}$ and $X_{11}=0$ therefore eqns. (1.29) and (1.30) become

$$
\begin{equation*}
\frac{\partial P_{L}}{\partial R_{11}}=\frac{\partial P_{L}}{\partial X_{11}}=0 \tag{1.31}
\end{equation*}
$$

and hence eqn. (1.28) gives finally that

$$
\begin{equation*}
\frac{\partial P_{L}}{\partial x}=0 \tag{1.32}
\end{equation*}
$$

Since the power at the output, $P_{L}$ is given by the expression

$$
P_{L}=\frac{1}{2}\left|V_{2}\right|^{2} / n_{L}
$$

then eqn. (1.32) can be written as

$$
\begin{equation*}
\frac{\partial\left|v_{z}\right|}{\partial x}=0 \tag{1.33}
\end{equation*}
$$

i: have shom, therefore, that a doubly-teminated lossless networ: has the propert of zero attenuation sensitivity at points of naximun available power [20]. Although the number of itip points in the passband of a filter is finite, we may be fairly certain that at other passband points, the attenuation sensitivity will be snall, so lona as the maximu passband deviation is small enough. This can be achieved for the classical filter designs, e.c. Chebyshev, Butter:rorth and Elliptic [26].

Three well lenorn classes of doubly-terminated lossless netroork might be mentioned in conclusion. The first is the LC ladder which, in addition to zero MP sensitivity, has low sensitivity in the stopband. This is true also of a cascade of lossless transmission lines. Ho:rever, the third class, that is the Lattice is fnown to have poor stopband performence.

## 1. 5 Simulation of Doubly-Terminated Lossless Metwonks Usini Dicital Components

In previous sections, it has been seen how disital filters may be designed and the principal problems that exist in their realization. The question arises - can a dirital filtor structure be derived tiat initates the behaviour of a lom-sensitivity analogue network? A desired digital transfer function may be realized in nany different ways and therefore it is apposite to acl thether ono particular realization has the lowest coefficient quantization error and round-off noise. Consequently, if a doubly-teminated LC lader filter is taken and the voltage and current relationshins expressed in signal-flow graph form then it is certain that the unique
relstionships are meserved and thereere that the low sensitivity proporter is maintained [27]. It is necessary to find some means oi transformins the analosue irequency variable $p$ to the correswondinc dizital frezuency variable $z^{-1}$ in such a war as to preserve the transfor characteristics. A suitable transformation has already been introduced in section 1.2 , that is

$$
\begin{equation*}
p+\text { (1) } \frac{1-z^{-1}}{1+z^{-1}} \tag{1.34}
\end{equation*}
$$

The factor $2 / T$ has been dropyed Ior convonience but it is to be noted that there is a.constant multiplicr of unity in eqn. (1.34) which has the dirensions of sec. ${ }^{-i}$. The bilinear transformation of eqn. (1.34) has the property of mapping stable analosue transier functions onto stable disital transfer functions [28]. The entire jr-aris is mapped onto the unit-circle in the z-planc and thus the frequency characteristic euffers a contraction from $[0, \infty]$ in the $p$-plane to $[c, \pi / T]$ in the $z$-plane. This property has been reierred to as 'varping' [5]. It is always possible to adjust the cut-ofi frequencies in the analozue filter so that a specific cut-ofer may be obtained in the disital equivalent [5].

Before proceeding, it is inportant to mention the realizability condition for aisital filters. is the classical design methods autoraticelly satisfy this condition, there has not been a need to erplain it previously. To ensure that a digital filter is ralizable, it is necessamy that every loop in the structure contains at least one delay elenent. If this rule is violated then it would be necessary to use the output sisnal of some part of the digital filter structure to compute itself. is arithmetic operations tale a finite amount of time, this clearly would be impossible.

4s a simple example of the flow- craph method and why it leads to unrealisable structures, the doubly-teminated thiri-orier allpole netrori shom in Pig. i. 10 was chosen. If the structure is split into three t:ro-ports and two one-ports as shown in riz. i. 11 and if each tro-port is expressed in terms of its ABCD natrix then it is clear that

$$
\begin{gathered}
V_{0}-V_{i}=I_{i} R_{s} \\
{\left[\begin{array}{l}
V_{i} \\
I_{1}
\end{array}\right]=\left[\begin{array}{ll}
1 & 0 \\
p C_{1} & 1
\end{array}\right]\left[\begin{array}{l}
I_{2} \\
I_{2}
\end{array}\right]} \\
{\left[\begin{array}{l}
V_{2} \\
I_{2}
\end{array}\right]=\left[\begin{array}{ll}
1 & p I_{1} \\
0 & 1
\end{array}\right]\left[\begin{array}{l}
V_{3} \\
I_{3}
\end{array}\right]} \\
{\left[\begin{array}{l}
V_{3} \\
I_{3}
\end{array}\right]=\left[\begin{array}{ll}
1 & 0 \\
n C_{2} & 1
\end{array}\right]\left[\begin{array}{l}
V_{4} \\
I_{4}
\end{array}\right]}
\end{gathered}
$$

and $\quad V_{4}=I_{4} Z_{I}$

Let $V_{0}, V_{1}, V_{2}, V_{3}, V_{4}, I_{1}, I_{2}, I_{3}$ and $I_{4}$ be the node variables in the signal-iow graph then the realizetion of the above equations is as shown in ric. 1.12. It can imediately je observed that a delay-free loop exists through the variables $V_{1}, V_{4}, I_{4}, I_{i}$ and $V_{1}$. If the bilinear transiormation of eqn. (1.74) is applied to the signal-ilo:r graph, the branches $V_{2} I_{1}, V_{3} I_{3}$ and $V_{4} I_{3}$ are trangforned to branches of the forn shom in Fig. 1.13. Thus many more delay-free locps are formed. for this reason, the voltagecurrent approach to the simulation of analogue rilters with aisital components fails to vield realizable disital structures. If rlow-graph reduction techniques vere used [29]to eliminate the delay-frce loons then the ladder relationships would be destroyed and so possibly would the desirable low-sensitivity property.

A solution to the problem of realizability has been siven by Pettreis [ 30 ]. Instead of volteges and currents, he considors wave variables and the resulting filters he has called 'wave digital filters'. fhese filter structures will be eramined in the survey of the following section.

## 

### 1.6.1 Basic theory

:iave digital filters are derived fron classical analo gue filtors usins thrce fundamental concepts: (a) Scatterins paraneters [99], (b) The bilinear tronsformation [23] and (c) Sisnal-1lou Graphs [29]. The basic theory has been described in references [30], [31], [52], [53] and [ 34 ] and a short account of the salient points will be presented here.

Having observed that the direct approach using voltases and currents failea to sive realizable structures, Fettreis considered aves as the basic variables in the sicnal-flost graph. He derined the incident and reflected wavos $\dot{A}_{k}, F_{k}$ at a port $k$ in the follorins wav

$$
\begin{align*}
& A_{k}=V_{k}+R_{k} I_{k} \\
& B_{k}=V_{k}-R_{k} I_{k} \tag{1.35}
\end{align*}
$$

where $R_{k}$ is the port nomalieation resistance. It will be seem that, as a result of $R_{k}$ beins arbitrary, any delay-free loovs that nay arise in the signal-flow dissram can be eliminated. The transformation defined in eqn. (1.35) is a variation of the scatterinf matrix formulation in siccowave network theory [gj]. Hevinc definea the neans by which equivalent aicital filter structures nay be derived, the resul.t of applyiñ enn. (i.j5) to the various elements in a doubly-terminated lossiess analosue netirorl: will be exemined. Feitweis treats such a net:orls as a set of reactive and resistive one-ports ani t:o-jorts connected together by means of new elements called '́adupors'.

For an inductor, thereîore, the voltage-current relationship is (Fig. 1.14)

$$
\begin{equation*}
V=(\mathrm{pL}) I \tag{1.36}
\end{equation*}
$$

On using eqn. (1.35) and (1.36) to elirinate $V$ and $I$ we find thet

$$
\frac{B}{A}=\frac{n I-n}{p I+!}
$$

If we set $\mathrm{R}=\mathrm{L}$ then

$$
\frac{B}{A}=\frac{p-1}{p+i}
$$

and on applying the bilinear transtormation, which we have discussed in the previous section, we find (2iz. 1.15)

$$
\begin{equation*}
B=-z^{-i} i \tag{1.57}
\end{equation*}
$$

The signal-flow graw, or wave-flow diagraa (FD) as retweis has calleć it, of eqn. (1.37) is nothing more than a delay element. and sign inversion.
zor a canacitor, $C$, moy, in a similar way, find

$$
\begin{equation*}
B=z^{-1} \dot{A} \tag{1.32}
\end{equation*}
$$

where R has been set equal to $1 / \mathrm{C}$.

For a resis iive voltare source (15. 1.16) we have

$$
\begin{equation*}
V_{0}=V+R_{s} I \tag{1.39}
\end{equation*}
$$

On eliminatina $\bar{V}$ and $I$ between eans. (1.35) and (i.j9) we find that

$$
\begin{equation*}
A=V_{0} \tag{1.40}
\end{equation*}
$$

where we have ascumed $R=?_{s}$.
The :ify of eqn. (1.40) is a 'mave-source' (ris. 1.17).

For a teminatino resistance, $R_{L}$, (kig. 1.18) wo have

$$
\begin{equation*}
V=R_{L} I \tag{1.41}
\end{equation*}
$$

and on vsing eqn. (1.35) with $R=R_{D}$, we find that

$$
\begin{equation*}
B=0 \tag{1.42}
\end{equation*}
$$

The $W$ of cin. (1.12) is a 'wave-sinit (Fic. 1.19).

For a lossiess transissio:-line element or init nement (C:) [35] (Fis. : 20), the corresponding equatione are as follows [ 50 ],

$$
\left.\begin{array}{l}
B_{1}=z^{-\frac{1}{2} A_{2}} \\
B_{2}=z^{-\frac{1}{2} i_{1}}
\end{array}\right\}(1.4 \bar{y})
$$

where we have assumed $F_{4}=R_{2}=Z_{0}$, the characteristic inpedance. The Hisp of eqn. (1.43) appears in Tig. (1.21).

The port resistances are associated with particular ports and as we have set then equal to the corresponding element values, sone meens of connecting ports with different port resistances is required. This may be thoucht of as a problea on matching. Lat us consider $n$ forts $\because i t h$ port resistances $R_{1}, R_{2}, \ldots R_{n}$ respectively. If the forts are connected in parallel we have (Fis. 1.22), by ïjrchoíf's Law,

$$
\begin{equation*}
v_{1}=v_{2}=\ldots=v_{n} \tag{1.44}
\end{equation*}
$$

and

$$
I_{1}+I_{2}+\ldots+I_{n}=0
$$

On eliminating $V_{k}$ and $I_{k}$ betireen equations (1.35) and (1.44) for $k=1,2, \ldots n$, we find that $[30]$

$$
\begin{align*}
B_{k} & =\dot{A}_{0}-\dot{A}_{k}  \tag{1.45}\\
\text { where } \quad \dot{G}_{0} & =\sum_{1}^{n} \alpha_{k} A_{k} \\
\text { and } \quad \alpha_{k} & =2 G_{k} /\left(G_{1}+G_{2}+\ldots+\dot{G}_{n}\right), G_{k}=1 / r_{k}
\end{align*}
$$

Note that $\sum_{1}^{n} \alpha_{k}=2$, thus one multiplier may be eliminated.
These equations derine an 'n-port parallel adaptor'. It is represented scheatically in Fis. 1.23. If the $n$-ports are connected in series then we have (Fig. 1.24)

$$
\left.\begin{array}{ll} 
& I_{1}=I_{2}=I_{3}=\ldots=I_{n} \\
\text { and } & V_{1}+V_{2}+V_{3}+\ldots+V_{n}=0
\end{array}\right\}
$$

Therefore, on eliminating $T_{k}$ and $I_{k}$ as before, we find that [ 30 ]

$$
\begin{equation*}
B_{k}=A_{k}-\beta_{k} A_{0} \tag{1.47}
\end{equation*}
$$

where

$$
A_{0}=\sum_{1}^{n} A_{x}
$$

and

$$
\beta_{k}=2 R_{k} /\left(R_{1}+R_{2}+\ldots+R_{n}\right)
$$

Again, we have $\sum_{1}^{n} \beta_{:}=2$.
These equations deîine an 'n-port series adaptor', shown schenatically in Fi こ. 1.25.
fo be able to desicn a dizital filter from any classical doublyterainated reactive netiorl, it is necessary to have the $\because \mathrm{P}$ of tro more circuit elements: the series-tuncd circuit and the parallel-tuned circuit (inss. 1.26 and 1.27). It is woll lnom that a series-tunel circuit is equivalent to a cascade of two W's with one end oven-circuited (Fir. 1.28) [30]. The characteristic impedances $Z_{1}$ and $Z_{2}$ of these US's are siven by

$$
\begin{align*}
Z_{1} & =I+D  \tag{1.48}\\
Z_{2} & =(L+D) / L C \\
\text { where } \quad D & =1 / C .
\end{align*}
$$

A tro-nort cories adontor is roulured betwen the two W's since $z_{1} \neq z_{2}$. However it can be show that a t:ro-port series adaptor is the sa:ce as a tro-port parallel adaptor and so it shall be denoted schematically as in Fif. 1.29 [30]. The irp of an opencircuit is found sirply by seting $\mathrm{I}=0$ in éqn. (1.35), then $B=A$. For a short-circuit, $V=0$ and therefore $B=-\dot{A}$. The corresponding WFD of a series-tuned circuit can now be completed and is shom in Fig. 1.30. On realizing that a parallel-tuned circuit is a shortcircuited coscede of tro We's, the mid of Pis. 1.31 is apparent.

An inductence could have been treated as a short-circuited $U 3$ and it still would have been possiule to obtain the irp of fig. 1.15. Similarly, a capacitance is equivalent to an onen-circuited U. . This is the principle of Richard's transformations [ 36 ] and was considered by Crochiere as an alternative method or deriving the elements of the :Vave Digital Filter [32].

In a peper publishod in 1970 independently of Fettreis, Binchan [37] described a similar concent to that of the ave Digital pilter, but the work was never continued.

### 1.6.2 Desim or Wave Diritaj iliters

In the previous section, the of the various elenents, sources and conmections necessary to realize a wide variety of filters were exanined. To intcrconnect those 'buildin--blocks' the following three points must be observed [30]:

It is necessary to
I) Interconnect building-blocks port by port.
II) For every pair of terminals connected the two corresponding waves nust flow in the same direction.
III) The resulting must be realizable, that is, every loop must have at least one delay.

There are two classical analogue filters of interest, (i) the doublyterminated LC ladder and (ii) the doubly-terminsted cascade oí ve's. THo typical examples appear in Fis. 1.32 and 1.33 respectively. Then it is attermted to realise (i) in digital form, a problem exists due to cascade connctions of adaptors. Delay-iree loops are formed which, of course, leads to unrealizability. F'cttreis solved this problem [30] by introducing Unit : Dements at the source, or load, and applying Kuroda identities [3e]. These enable any number of U.3.'s to be shifted through the tro-port. The result of this procedure is to separate lumped elements by unit elements (7in. 1.34). It is therefore possible to realize (i) in digital fora since the WFD of a UR is a delay (Fig. 1.35). The lumped-distributed filters, of which lig. 1.34 is an ezample, have been called 'UnitBlement Filters' [38] and have importance in microwave theory [35]. The addition of unit elements adds to the compleaty of the dicital filter and methods have been dexived which male nore efficient use of them [39]. It may be aske? whether unit clements are required at all since they were introduced only to satisfy the realizability constraints? Indeed, it is possible by settine an appropriate adaptor nultiplier to unity to avoid delay-fres loons. This technique leads to digital filters whose elements bear a
one to one relationship rith the elcaents of the anclogue rilter［ $\bar{y} 4$ ］． The muver of rultipliers used is canonic in the sense that it is equal to the number of reactive components plus one，whilst the number of delays is equal to the number of reactive components．Thus for all－pole filters，the number of delays is equal to the filtor＇s degree．

Consider now the second type of filter，that is the cascade of w＇s． The oî fic． 1.33 appears in 21.5 ． 1.30 ［30］．Such filters alwas have realizajle ap although transfer functions are restricted to the all－pole fom．＇We nuber or milipliers is equal to one wore than the nurber of unit elements．

The design of a particular wave disital filter can be achieved either by usins standard analogue filter tables［6］［7］［0］［40］or by synthesisins the desized analosue netror：［11］［42］．：Why practical examles hove been siven in the literature，for lumed filtors one ray werer to refronces［39］［43］［4］［45］［10］［17］，for distributod filters to $[40]$ and for unit－slement filters to［j2］［30］［40］［50］［51］．

Recently，two papers ：iere publishel on the realization of wave disital filters from symetrical lattice netrorks，quite indapendently．The first［52］uses the equivalence bet：reen the Lattice and the Jaunann structure to derive the appopriate ave－flon diagran．The second［53］， Which is a simpler appronch，uses the fact the the transfer function of a y metricol latico ow be obtrined from the reflectences of the canonic impedances．In either case，a ne：four－port adaptor called the＇Lattice Adaptor＇is derived．The correspondine equations are as Collo：s（ゴラ．1．37）

$$
\begin{align*}
& B_{1}=\frac{1}{2}\left(\dot{A}_{3}+\dot{A}_{4}\right) \\
& B_{2}=\frac{1}{2}\left(-\dot{A}_{3}+\dot{A}_{4}\right) \\
& B_{3}=A_{1}-A_{2}  \tag{1.49}\\
& B_{4}=A_{1}+A_{2}
\end{align*}
$$

with the constraints that

$$
\begin{equation*}
A_{3}=S_{1} B_{3} \tag{1.50}
\end{equation*}
$$

and

$$
A_{4}=S_{2}{ }_{4}
$$

where $S_{k}=\frac{Z_{k}-R}{Z_{k}+R}$ ，is the reflectance of impedence

$$
\ddot{H}_{k}, k=i, 2 \text { and } a=r_{L}=a_{s}
$$

In particular, with $i_{2}=0$ the transfer function $3_{2} / \lambda_{1}$ is found to be $\frac{1}{2}\left(s_{2}-S_{1}\right)$, which is a simpler fom. The min advantage of usins the latice equivalent of a symetrical ladder netrorit is in the reduction of the number of components in the dicital structure. Ho:ever, as in the analogue equivalent, the dicital lattice shows high stopbnd sensitivity, :ithouth not as severe. In addition, only sfanetrical netroras which are equally torainated can be used.

Gay practical filter exmples nay be found in reforences [59] and [5] .
$\therefore$ method of desisning wave disitsl filters fron doubly-terinimed Lis laders $\because i$ th the advantare of a reduction in the number of delays has recently been describod [34][55]. The watiod relies or the fact that multiplication or division of ev my imosance in the anolorie netirorl dy the complez frecuency variable $\underline{x}$ leaves the transer function unchaneed. This technique was first put formard by Bruton [25], originally for eliminatirs inductances so that an active realication of the analogue transier fuction could be mede with the sane lor sensitivity as the passive prototype. If every impeance is aultiplied by $n$ then resistances are tronsformed into inductances, capsicitences into recistances, and inductances into now eloments callec 'superinductances'. If every impediance is divided by $p$ then resistances are transfor ind into copacitances, capacitinces into 'supercapacitances' and inductances into resistances. The .ris's of these ner analowe olements are derived in Rei. [55] and aramles may je found in Reĩs. [39] and [50].
 ainicomputer in real tine and for fired-point and floaing-point arithmetic. Buperimental results are fiven in References [j9][56]. Furthemore, descriptions of two iifforent haramure realisations may be found in Refs. [39] and [46]. In both cases, theoretical predictions have been confirmed.

### 1.6.3 Conficient ungtigetion Sror ani Zound-off ioise <br> in :Wve Dizital pilters

In the last two sections, the basic principles of desiming digital filters fron classical filter networks usine waves have been reviewer.

Whernin reason for using waves ass to preserve the lon ansitivity of the analogue prototype. Are the filters :re have jescriped potter than the conventional direct, cascade or parallel stinctures? The evidence from the literature is clearly affimative. The cocfficient quantisetion emor, which is the easiest to reasure, is smallest in Save Disital Pilters [32] [4]][50][5i] [36], at least for tho examples exmined.

The nain adventage of wave disitul filtors is shorter multiplier
 the wordenctil is usually fineu .nd therefore one is intcrested in the best alforithn under these conditions. lave airital filters rould therefore be more useful in hardrare siturtions, shere the arithmetic rould ésenerully be fixex-point, and because of their 10: quantization cror, could be roalized with multipliers equal to powers of two [46].

T:e round-oî noise has also been studied extensively for vave dicital filters, both for fized-point and floatins-point aritmetic.

In the floatinj-point case, nractical filter exanples have sinom a clear superiority for wave disital filters over conventional structures $[32][44]$. Howevor, in the fixed-noint situation the arcuments vary $[44][5 i]$, although, as has been said, the periomance of wave dirital filters is such that multipliers equal to porers of two may be used in certain circumstances [46].

Winally, the problem of limit cycles, or overflo: oscillations, in wave disital filters has been examined $[57]$. It was found that these parasitic eifects can be avoided by very simple means.

### 1.6.4 Soncitivity ne Gobility of Teve Difitul jijters

It has been shom that the sttenuation sensitivity is userul in estinating the error due to coefficient quantigation and, in particular circunstances, to measure tho round-off noise $[13][2 i]$. "o derive the sensitivity properties oi wave digital ililters, there are two approaches (i) direct, using the comespondence betreen the multiplier variables and the analogue elements, and (ii) using the concent of pseudopover [33][58].

Tho diroct oproush has been nade in dors. [s] and [50]. In these reioroncos, it was shom the the attonustion sonsitivity ras zero at points of naximun available pseudopower (ap). Wis property, of coursc, holds in the doubly-teminted lossless not:orle Erom wich the dicital filtor was derived. Ho:rever, the conclusions of Ronner and Gupta are quite misleading and rill be discussed in a moment. seîore this, horevor, the concept of pseucopower will be ownined.
por an malowe notront, the roal poner $=$ is doninal as follo:s [0]

$$
\begin{equation*}
P_{k}=\frac{1}{2} \operatorname{Re}\left\{T_{l k} I_{k} *\right\} \tag{1.51}
\end{equation*}
$$

where $k$ denotes the $:$ :th. port and $*$ donotes complex conjugate. Let us no:r express $F_{2}$ in term orin and usine en. (i.j5) then

$$
P_{k}=\frac{1}{2} \operatorname{Re}\left\{\frac{1}{2}\left(G_{k}+Y_{k}\right) \frac{1}{2}\left(G_{k}^{*}-3_{k}^{*}\right) G_{k}\right\}
$$

or,

$$
\begin{equation*}
P_{k}=\frac{1}{v}\left\{\left|\dot{L}_{1}\right|^{2}-\left|B_{1}\right|^{2}\right\} A_{k} \tag{1.52}
\end{equation*}
$$

since $\operatorname{Re}\left\{A_{k} P_{k}^{x}-i_{k}^{*} Z_{k}\right\}=0$ and $\quad G_{\mathrm{E}}=1 / \mu_{\mathrm{L}}$ is real.

For wave dicital filters, $P_{k}$ in eqn. (1.52) is called the 'steady-stute pseuioworri' at the kin. port. The constant, $\frac{t}{8}$ can be dronged fron ean. (1.52). Fettreis has shom that the disital filter equivalents to the passive analogue components (i.e. resistance, inductance, capacitance, U.a., transformer etc.) are 'pseudopassive'., that is $P \geqslant 0$ for $\operatorname{Re}(p) \geqslant 0$ where $\underline{p}=\sigma+j w$ is the complex frequency. In addition, inductance, capacitance, US have $P=0$ for 1 = jw and therefore are called 'pseudolossless' or 'pzoudoreactive'. In addition adaptors were show to be pseudoreactive. To derive the sensitivity properties, both Fettweis and Renner and cupta [3\%][48][50] used the following transfer function (Fic. 1.30)

$$
\begin{equation*}
G(z)=\sqrt{\frac{R_{S}}{R_{L}}} \frac{B_{2}}{V_{0}} \tag{1.53}
\end{equation*}
$$

Wheh comernom to the fomm trencnission coerficient in classical networls theore [10]. Howevor, the actual tronsfor function is $\mathrm{B}_{2} / \mathrm{V}_{0}$, that is the output signal then can be detected for an input $Y_{0}$ is $D_{2}$. Using the formule for $G(z)$ in ean. (1.53), it ws shown that at points of inP

$$
\frac{\partial A}{\partial \alpha}=0
$$

there $A=-\log _{\mathrm{e}}|G(\mathrm{wT})|$ and $\alpha$ is any multiplier.
$\therefore$ s has been said, the actual transier function $\hat{G}(z)$ is siven by

$$
\begin{equation*}
\hat{\vec{r}}(z)=\frac{3_{2}}{V_{0}} \tag{1.55}
\end{equation*}
$$

and thererore, combinins eqns. $(1.53),(1.54)$ and (1.55) we find that, at imp points, [59]

$$
\begin{align*}
\frac{\partial \hat{\Delta}}{\partial \alpha} & =\frac{1}{K} \frac{\partial:}{\partial \alpha}  \tag{1.56}\\
\mathrm{K} & =\sqrt{R_{\mathrm{S}} / R_{\mathrm{L}}}
\end{align*}
$$

Thus the attenuation sensitivity of interest is siven by eqn. (1.55) and is not zero in general. It was shom that the expression on the rishthand side of eqn. (1.55) depends only on the rultiplier values and therefore eives a flat loss to the attenuation, mich in practice can be ifonored [59]. Howover, the output yound-off noise is dependent on $\partial \hat{M} / \partial \alpha$ end therefore it is of interest to find ways of reducing it [59]. By observing the form of eqn. (1.56) it is clear that $\partial \hat{A} / \partial \alpha$ will be zero in $R_{S} / R_{L}$ is independent of $\alpha$. There are three cases for which this is truc [59], (a) by realizins a transfer function as a reflectance, in which case because the input and output sichale refer to the some port, the port resistancos are the same, (b) by realizins a transfer function in lattice form [52][53] and (c) by realizing as a particular type of unit-elenent structure. The disadvantages of (a) and (b. lie in the increased stopband sensitivity thilst for (c) we have an increase in the number of delay elements.

The non-zero nature of the attenuation sensitivity has also been examined by Long [4] using the concepts of pseudopower and
psendomassivity. In the mesent study, the attemation scasitivity :ill be derived without usins these concepts, by using the functional relationsing that exists betreen the analogue filtor and the correanondins rave disital filter.
rinally, it has been shom by Nettweis that wave disital filters are stable for wide raņes of their multiplior values [3j].

### 1.6.5 Some Other Properties of Tave Diritrl Iilters

Consider $\operatorname{ij} \mathrm{J}$. 1.30 in inheh for a sional input, $\dot{A}_{1}=V_{0}$, the required output sicmal is $3_{2}$ if $A_{2}=0$. It is mom that the output simal $\overline{3}_{1}$ is complementary to $\bar{B}_{2}$. in the sense int [J]

$$
\begin{equation*}
\left|s_{11}\right|^{2}+\left|s_{21}\right|^{2}=1 \tag{1.57}
\end{equation*}
$$

where $S_{11}=B_{1} / N_{0}$
and $\quad S_{21}=B_{2} / T_{0}$
Eqn. (1.57) is valid for equally terminated reactive networks only.

Thus if $S_{21}$ has a lo:pass characteristic, $S_{11}$ will have a high-pass characteristic. Similarly if $S_{21}$ has a band-pass characteristic, then $S_{11}$ has a band-stop. These arguments apply equally to the case when $A_{2}$ is the input, $\dot{A}_{1}=0$ and $B_{2}$ is the complementary outnut to $3_{1}$. As has beon snid previously, takinz the output from the sarme port as the input signal sives good passoand perfornance but the stopband sensitivity is high. Therefore a wave dicital filter may be used as a low and a hish-pass filtor simultaneously.

Fettreis has made a detailed study of the simificance of Bordowijk's concept of Inter-Reciprocity in wave disital filter theory [60]. iie has dorived theorams that are similar to pellegen's theorem for analogue networks. It was shom that, of the many types of transposition of a given network, two were of interest (a) 'intrinsic transnosition' and (b) 'elenentary transposition'. Intrinsic transposition corresponds to transposition in the classical sense, that is only networls consistins of non-reciprocal elements are affected. Since classical doubly-teminated lossless netyorls are reciprocal, intrinsic transposition has no eifect [60]. Dlementary tranoposition corresnonds to flow reversal, that is .
revereing the sifmel direction in each branch. Soth tymes of transposition yield structures with the sane number of multipliers and transfer function as the oricinal. Thus the concept of
 levels of round-off noise and parasitic oscillations [60].

### 1.7 Surver of titerature on Cther lon-Gonsitivitu Disital silter Structures

The innorthe of the atmeture of a dicitnl filter yith record to the problen of round-of $\hat{i}$ noise and coefficiont quontieation errors has been emphasised by Jackson [20]. In the previous section, one method for the reduction of noise was examined in thich the know lov-sensitivitü behaviour of certain analocue net:ows was preserved in the digitul realization. © shall biefly review some other deaisn techniques :inos air was to reduce the round-off noise.

Zitra and Chernood heve derived rocursive dicital filter structures using the continued fraction oxpansion well lenom in passive synthesis [01] [62][63]. Similer work has been done indepondently by Constantiniles [6c]. These structures do resemble ladders in apperrance but their sensitivity nerformance is poor [4<].

In another paper, constantinides [65] has exanined the syathesis of FIR transfer functions as ladders usiñ the continued fraction expansion. These structures, although recursive, have finite inpulse response. This vorl has been continued in lefs. [66] [67].

Gray and Mankel have presented a new type of disital structure synthesised using orthoconal nolynomial expansions [62][69]. It is claimed that, in the onse of namon bendiridths and clustered voles, the round-of noise properties are better than those of any of the convontional structures.
inother promising stiucture has been investicated by Lons [44]. its simal flow craph is of the sane moneral form as that of a passive ladder: Those structures, called nultiple-fcedback or leapiros, have beon used to desicn active aC filters. They have been shom to eahibit lower sonsitivity in the passband than the conventional mothod of casendins socom-ondor zections. However, as has been seon, ono cannot design a reali:jable dicital filtor froa an analogue
filtor uaine the sifnol-ifor sran bacea on volteyes and curronts. Lone's solution mas to metch the cocticients of the desired transier function to the cocificients of the trancfer function of the digital lean-fros siructure. This led to a set of non-lineer sinultansous equations, which were thon solved sivint the renuired desisn [4A]. The noice performance in the fixed-noint case mas then discussed. It was fowd that the romoroff noise, for a lo:-pans roalization, was similur to that of the conventional cascade ans that of the nuve dicital filter. Furthernore, even jetten neriommee was observed in the $c$ wse of nuron band filtors.

Oonevar, the bandpass lenpiroc realization sho:ed cansidorably poorar yorformonce then the comosponding caccade or vave filter reslisotion.

Buton has sugestei that the folloning tamsiormation could je used to derive disitel filter structures fron doubly-ternineted lossless ladders [27][\%],

$$
\begin{equation*}
p \rightarrow \frac{2}{2} \sinh \left(\frac{n T}{2}\right) \tag{1.58}
\end{equation*}
$$

He becins :ith the signal-flo:r graph or the laddar in the leapfrof forn, as lons has done, and then applies the tranctornation in eqn. (1.53) to every frefuency deponiont brancir. The resulting digital structure is then realizable, that is no loons uithout delar are fomed. the bilinear transiomation

$$
\begin{equation*}
p \rightarrow \frac{2}{T} \quad \tan \left(\frac{p T}{2}\right) \tag{1.59}
\end{equation*}
$$

if used in a similar way to eqn. (i.58) would, as has been shom, lead to unrealizable structures.
:.e may wite on. (1.53) as follow,

$$
\begin{equation*}
\Omega \rightarrow \frac{2}{T_{1}} \sin \frac{w T^{1}}{2} \tag{1.60}
\end{equation*}
$$

where $\Omega$ is the analogue frequency, $v$ is the aisital frequency and $T$ is the sampling rate. Thus the naxinum frequency $\Omega_{0}$ which maps onto the discrete-time domain is $2 / T$. The correspondins discrete frequency is $\pi / i$. Gerefore, only the tilter characteristic from 0 to $2 / I$ can be transferred into the disital range. The size of $I$
is linited by the hird:ane that is beins used. If 2 is uritten for $e^{n i}$ in ean. (1.50), then the trmocomation may be irition as

$$
\begin{align*}
& p \rightarrow \frac{1}{T}\left(z^{\frac{1}{2}}-z^{-\frac{1}{2}}\right)  \tag{1.61}\\
& p \rightarrow \frac{1}{1} \frac{1-z^{-1}}{z^{-\frac{1}{2}}} \tag{1.62}
\end{align*}
$$

Consider a pole $p_{i}$ of the analozue prototype, this is transformed by eqn. ( 1.62 ) into ti:0 poles which satisfy the follorine quadratic


$$
\begin{equation*}
z^{-1}+p_{-} T z^{-\frac{1}{2}}-1=0 \tag{1.63}
\end{equation*}
$$

If $\alpha$ and $\beta$ are the t:ro poles then

$$
\begin{equation*}
\alpha \beta=-1 \tag{1.64}
\end{equation*}
$$

Ine stability condition for disital trinsfer functions may no:i be applied to enn. (1.64). It is required that both $\alpha$ and $\beta$ should lie outsida the unit circle but this is inpossible since the product or their moduli is equal to unity. Thus filters desiened using the trensformation siven by ean. (1.58) are unstable.
1.8 inoroach Taien in this Thesis

Fhore is now un antensive litorature an wo Disitul filter theory and many papers have been publishea verifying the low noise properties which are a conseruence of simulatins doubly-teminated LC laiders or Uj cascades. The wori done un until now has concentrated on an approach in which each lumped elenent is treated as a one-port and strnctures are built up by connectins the sipnal-flo:t Eraphs of these elements with adantors. An adaptor is simply a neans of connecting $n$ ports with uneyual port normelization resistances in parallel or in series. the adaptor is necessary in the onc-port approach because the port rosistance has been set equal to the corresponding element value, thus constraining the one iree parameter. Similarly, the two-port UE has both of its port resistances set equal to the characteristic impedance. In this thesis, each arr of the ladder, excluding terminations, will be considered as a two-port with unequal port resistances and the correspondins wave-flow diagrams (mi) will be
derivad. It : :ill thon be possible to cascrie successive. without the use of the adaptor concent. This is a conse uence or allowing the port resistances to vary independently. Bimilarly, the nort resistances of the $W$ :ill also be allo:ed to very independontly and it rill be seen that digital filter structures cen be derivei froi distributed prototypes wint usins aneptors. It turns out, in fact, that the appropriate adaptor is included in the $\because$ of a particular element. whe a new apyonch to the design
 the concest tint is boing proposed asounes only the transionation from voltaces and currents to incident and reflected raves and the bilinser trensformetion. It is not necescamy to consider michard's transformations and unit elements is a lumed netrork equivalent is rezuired. yurther, the t:o-port anprow entibles more sencral transformations to be siudied. 价 the literature has concentrated on the following,

$$
\left.\begin{array}{l}
A_{k}=V_{k}+R_{k} I_{k} \\
B_{k}=V_{k}-R_{k} I_{k}
\end{array} \quad k=1,2 .\right\}
$$

but it may be of interest to consider, for a two-port, the followins tronsiomation [99],

$$
\begin{align*}
& A_{1}=p_{11} V_{1}+p_{12} I_{1}  \tag{1.66}\\
& B_{1}=p_{21}-V_{1} \div p_{22} I_{1}
\end{align*}
$$

and

$$
\begin{aligned}
& A_{2}=q_{11} V_{2}+q_{12} I_{2} \\
& B_{2}=q_{21} V_{2}+q_{22} I_{2}
\end{aligned}
$$

and to examinc the conditions on. $p_{i j}$ and $q_{i, j}$ to ensure that realizable netmorks are Porned.

The :ork in this thesis is divided as follows:

In Chaptor 2, the raions Fr , nececcary for the simulation of lumped ani distributed filters, are derived. Chapter 3 is concerned with the desjorn or dicital filters from doubly-terminated lossless ladder netrorks. In addition, the attenuation sensitivity characteristic is derived using the corresponience betreen the analogue and dicital elements and the low coefficient quantization error is
 initating the behaviour of doubly-tominated lossless tranmission line net:roris is described and the sensitivity properties are
 (1.67) is examined in Chapter 5. Conditions are aiven that are necesoniry to ensure theit the derived structures are realizable and, ialect, initate the classicel prototypo. In shapter 6, a varticular transiomation is studied in detail. Tne apmopriate sioncl-ilou sraphs are derived and several filter exariples aie given. Jour other transformations are studied in Chapter 7 and the correspondinc flo:j-srapis are derived.

Cownuter-aided analysis of digital filtors is the subject of Choytor 8. $\therefore$ nes alforitim for the fast anolysis of arbitrary diëital filter structures is presented and eramples are ívon sho:ing its superiority over tie conventional techniques.

In Chopter 9, other avenues of resesrch are sugcested in the field of lave Digital Filters that alay be fruitul.

Sone of the worl in this thesis has been presented as conference papers. The ideas of Chanter 2 tozather vith the desizn metiod of Chastor 3 appear in [71][96]. The sensitivity analyois of Chanter 3 appears in [72]. The concept discussed in Cnaptor 4 is the subject of [73]. ihe zenemal two-port transfomation of Chavter 5, and in particular the 'IVR' formulation of Chapter 6, appear in [74]. Finally, the ne:r analysis alforithm of Chapter 8 is to be found in [75].

At the same conference as [71] was presented, Stramy and Thyacarajen [76] gave a paper containing very similir results.


Fig. 1.1 Digital Signal Processor


Fig. 1.2 The three basic digital filter components.


Fig. 1.3 The first direct realisation of $H(z)$.


Fig. 1.4 Direct Realisation of Non-Recursive Transfer Function .


Fig. 1.5 The second direct realisation of $H(z)$. .


Fig. 1.6 Cascade of second-order sections (Each box represents a and. order non-recursive transfer function).


Fig. 1.7 Parallel form realisation ( Each box represents a
and. order non-recursive transfer function ).


Fig. 1.8 . Doubly-terminated lossless network .


Fig. 1.9 Doubly-terminated lossless network at a point of maximum available power .


Fig. 1.10 Doubly-terminated third-order all-pole network .


Fig. 1.11 The network of Fig. 1.10 partitioned into three 2-ports and two l-ports.


Fig. 1.12 . Signal-Flow Graph of Fig. 1.11


Fig. 1.13 Signal-Flow Graph of Typical Digital Branch .


Fig. 1.14 Inductance, L .


Fig. 1.16 Resistive Voltage Source . Fig.l. 17 WFD of Fig.1.16.


Fig. 1.18 Load Resistance,$R_{L}$


Fig.1.19 WFD of Fig.l.18.


Fig. 1. 20 Unit-Element (UE) .


Fig. 1. 21 WFD of Unit Element .


Fig. 1.23 n-port parallel adaptor $\cdot$

Fig. 1.22 Parallel Connection of $n$ ports .


Fig. 1.25 n-port series adaptor .

Fig. 1. 24 Series Connection of n ports .


Fig. 1.26 Series-Tuned Circuit .


Fig.1.27 Parallel-Tuned Circuit .


Fig. 1. 28 Unit-Element equivalent of Series-Tuned circuit .


Fig. 1.29 Two-port adaptor .


Fig.1. 30 Wave-Flow diagram of a series-tuned circuit .


Fig. 1.31 Waye-Flow diagram of a parallel-tuned circuit .


Fig.l. 32 Doubly-terminated LC ladder network .


Fig.l. 33 Doubly-terminated cascade of commensurate transmission-lines .


Fig.1. 34 Unit-element filter equivalent of Fig.l. 32 .


Fig. 1.35 Wave-Flow Diagram of Fig. 1.34


Fig.1. 36 Wave-Flow Diagram of Fig.1. 33 .


Fig. $1.37 \quad$ 4-Port Lattice Adaptor .


Fig. 1.38 Wave Digital Filter viewed from its ports .

## Thintom?

The T:o-Port Annroach

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## Chanter ?

The Tro-Port : Anpoach

### 2.1 Introduction: princingl Tdea of the Armroach.

In the amponch of Fettrois [ 30 ], each tro-terminal network olement is treated as a one-port and the corresponding wave-flow diagrans are interconnected using adaptors. In the procedure discussed in this section, the constitiont elements of a general ladder network are considered as two-norts. In addition, the port resistances are teken to be unequal. This enables us to interconnect such elements directly and thus avoid the necessity of introducins the concent of the 'adantor'. Concentually, the two-port aproach is more direct although the digital ifilter.structures derived using either method have similar pronerties.

In this chapter, we derive the Wave-Flow Diagrams (VFD) correspondirg to the circuit elements that are necessary to realise doubly-to-minated, loscless ladder or trancmission-line netrorts.

### 2.2 Scattorine Pomantors for a Passive Tyo-Port

We begin by stating some well-established results from classical analosue network theory [99]. Given a passive tro-port we may describe its behaviour at the t:10 ports by the APCD parameters in the form;

$$
\left[\begin{array}{l}
V_{1}  \tag{2.1}\\
I_{1}
\end{array}\right]=\left[\begin{array}{ll}
A & B \\
C & D
\end{array}\right]\left[\begin{array}{l}
V_{2} \\
I_{2}
\end{array}\right]
$$

We have assumed that both $I_{1}$ and $I_{2}$ flow into the network and therefore any negative signs arising when cascading. such networks are included in $B$ and $D$.

The relationship between the incident and reflected waves and the voltages and currents for the same network (Fig. 2.1) cen be written in matrix form as

$$
\left[\begin{array}{l}
A_{1} \\
B_{1}
\end{array}\right]=\left[\begin{array}{cc}
1 & R_{1} \\
1 & -R_{1}
\end{array}\right]\left[\begin{array}{l}
V_{1} \\
I_{1}
\end{array}\right]
$$

$$
\left[\begin{array}{l}
A_{2}  \tag{2.3}\\
B_{2}
\end{array}\right]=\left[\begin{array}{cc}
1 & R_{2} \\
1 & -R_{2}
\end{array}\right]\left[\begin{array}{c}
r_{2} \\
I_{2}
\end{array}\right]
$$

From the above equations we obtain the following directly,

$$
\left[\begin{array}{l}
A_{1}  \tag{2.4}\\
B_{1}
\end{array}\right]=\left[\begin{array}{ll}
\boldsymbol{\alpha} & \boldsymbol{\beta} \\
\boldsymbol{\gamma} & \boldsymbol{\delta}
\end{array}\right]\left[\begin{array}{l}
A_{2} \\
B_{2}
\end{array}\right]
$$

where

$$
\begin{aligned}
\boldsymbol{\alpha} & =\frac{1}{2}\left(A+C R_{1}+B G_{2}+D R_{1} G_{2}\right) \\
\boldsymbol{\beta} & =\frac{1}{2}\left(A+C R_{1}-B G_{2}-D R_{1} G_{2}\right) \\
\boldsymbol{\gamma} & =\frac{1}{2}\left(A-C R_{1}+B G_{2}-D R_{1} G_{2}\right) \\
\boldsymbol{\delta} & =\frac{1}{2}\left(A-C R_{1}-B G_{2}+D R_{1} G_{2}\right) \\
\text { and } G_{2} & =1 / R_{2}
\end{aligned}
$$

In Fig. 2.1, we have assumed that $A_{1}$ and $A_{2}$ are input variables and $B_{1}$ and $B_{2}$ aro outnut variables. This corresponds to the classical definition [99], althouch we are by no means tied to it. Thus, ve could consider the transformation

$$
\left[\begin{array}{c}
A_{k} \\
B_{k}
\end{array}\right]=\left[\begin{array}{cc}
1 & -R_{k} \\
1 & R_{k}
\end{array}\right]\left[\begin{array}{c}
V_{k} \\
I_{k}
\end{array}\right] ; k=1,2 .
$$

which, in effect, has reversed the roles of the variables $A_{k}$ and $B_{k}$ in eqns. (2.2) and (2.3).

In the present discussion, therefore, we are interested in expressing the reflected vaves $B_{1}, B_{2}$ in terms of the incident waves $A_{1}, A_{2}$. Let us define the relationship as follows,

$$
\left[\begin{array}{l}
B_{1}  \tag{2.5}\\
B_{2}
\end{array}\right]=\left[\begin{array}{ll}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{array}\right]\left[\begin{array}{l}
A_{1} \\
A_{2}
\end{array}\right]
$$

By rearrancint eqn. (2.4) into the form of eqn. (2.5) we find that

$$
\begin{aligned}
S_{11} & =\delta / \beta \\
S_{12} & =-\Delta / \beta \\
S_{21} & =1 / \beta \\
S_{22} & =-\alpha / \beta \\
\Delta & =\operatorname{det}\left[\begin{array}{cc}
\alpha & \beta \\
\gamma & \delta
\end{array}\right]=-R_{1} G_{2} .
\end{aligned}
$$

Thus far, we have introduced orly basic idens about seattering narametors [90]. In the next section, we shall use eqn. (2.5)


Before doins so, it is necessary to exanine different zeneral waveSlo: diamens representins eqn. (2.5) and the problems associntod wisth their interconmection.

The rave-flow diarmam (rpy) of eqn. (2.5) is illustrater in Fig. 2.2.
 for tro different netrorls. Interconnection between these jmplies that

$$
\begin{aligned}
& B_{2}^{(1)}=A_{1}^{(2)} \\
& A_{2}^{(1)}=B_{1}^{(2)}
\end{aligned}
$$

Interconnection is sho:m by broken lines. It is apparent from Fig. 2.3 that if we cascade several elements together, loops will be formed by the $S_{22}{ }^{(1)}$ transmittance and the $S_{11}$ transmittance of the second and subsequent olements. A similar arcument can be applied to each element in turn. Such feedback loops may contain no delay, a situation that is not acceptable due to the realisability condition discussed in Chapter 1. Therefore to avoid delay-free loons, it is necessary to ensure that either each $S_{11}$ or each $S_{22}$ tranmittance has no delay-free path. This is equivalent to sayine that the transmittance $S_{11}$ (or $S_{22}$ ) has a factor, $z^{-1}$ or that the constant term in its numerator is zero.

### 2.3 Derivation of Yave-FIow Diarrams for Series Elements

### 2.3.1 Introduction

A series impedonce $Z$ has ABCD matrix given by, in our notation,

$$
\left[\begin{array}{ll}
1 & -z \\
0 & -1
\end{array}\right]
$$

Substituting for $A, B, C$ and $D$ into eqn. (2.4) and (2.5) gives the following expressions,

$$
\begin{align*}
& S_{11}=\left(R_{2}-R_{1}+Z\right) /\left(R_{2}+R_{1}+Z\right) \\
& S_{12}=2 R_{1} /\left(R_{2}+R_{1}+Z\right)  \tag{2.6}\\
& S_{21}=2 R_{2} /\left(R_{2}+R_{1}+Z\right) \\
& S_{22}=\left(R_{1}-R_{2}+Z\right) /\left(R_{2}+R_{1}+Z\right)
\end{align*}
$$

It is important to note that the following constraints anply to eqn. (2.6),

$$
\begin{align*}
& s_{11}+s_{12}=1  \tag{2.7}\\
& s_{21}+s_{22}=1
\end{align*}
$$

These will be useful in simplifying the wave-flow diagrams, for if we combine eqn. (2.5) with eqn. (2.7) then

$$
\left.\begin{array}{l}
B_{1}=S_{11}\left(A_{1}-A_{2}\right)+A_{2}  \tag{2.8}\\
B_{2}=S_{22}\left(A_{2}-A_{1}\right)+A_{1}
\end{array}\right\}
$$

Thus we need only realise $S_{11}$ and $S_{22}$ to define the series impedance 2. The eqn. (2.8) could have been writter in three other ways because of the constraints in eqn. (2.7). However, as the realizability conditions, mentioned in the last section, involve only $S_{11}$ and $S_{22}$ it was preferable to eliminate $S_{12}$ and $S_{21}$.

We have established the relationship between the incident and reflected waves of a series impedance. If the bilinear transformation

$$
\begin{equation*}
p \rightarrow \frac{1-z^{-1}}{1+z^{-1}} \tag{2.9}
\end{equation*}
$$

is applied ( $\left.z^{-1}=e^{-j v T}\right)$ the wave relationshins are then taken into the z-domain of digital networks [?8].
2.3.2 Serien Renintnros, R.

Let $7=R$ in eqn. 2.6 then
and

$$
\begin{align*}
& S_{11}=\frac{R_{2}-R_{1}+R}{R_{2}+R_{1}+R}  \tag{2.10}\\
& S_{22}=\frac{R_{1}-R_{2}+R}{R_{2}+R_{1}+R}
\end{align*}
$$

To avoid delay-free loons, the constant term in the numerator of $S_{11}$ or $S_{22}$ must be zero and hence we have two cases

## Csise T:

He set $R_{2}-R_{4}+R=0$; the constant term in the numerator of $S_{11}$. This nroduces the follo:ring

$$
\left.\begin{array}{l}
R_{1}=R_{2}+R \\
S_{11}=0  \tag{2.1i}\\
S_{22}=\frac{R}{R-R}=\beta, \text { say }
\end{array}\right\}
$$

By combinins eqns. (2.11) and (2.8) the corresponding waveflow diagran (TFD) is derived as it annears in Fir. 2. 1.

## Case II:

The altemative case to the ons above is to set to zero the constant term in the numerator of $\mathrm{S}_{22}$

$$
\left.\begin{array}{rl}
\text { i.e. } R_{1}-R_{2}+R & =0 \\
\text { that is } R_{2} & =R_{1}+R \\
\text { and therefore } S_{11} & =\frac{R}{R_{1}+R}=\alpha_{1} \text {, say }  \tag{2.12}\\
\cdot S_{22} & =0
\end{array}\right\}
$$

The FrD for this case appears in Fis. 2.5.

### 2.3.3 Series Inductance, I

For an inductance in the series-arm we have $Z=p L$. Substituting.
this value into eqn. (2.6) and applying the bilinear transformation $p \rightarrow \frac{1-z^{-1}}{1-z^{-1}}$ we have the following relationships

$$
\begin{align*}
& S_{11}=\frac{\alpha_{1}+\alpha_{2} z^{-1}}{1+\alpha_{2} z^{-1}}  \tag{2.13}\\
& \text { and } \quad S_{22}=-\frac{\alpha_{3} \div \alpha_{1} z^{-1}}{1+\alpha_{2} z^{-1}} \\
&\text { where } \left.\quad \begin{array}{rl}
\alpha_{1} & =\left(R_{2}-R_{1}+L\right) /\left(R_{2}+R_{1}+I\right) \\
\alpha_{2} & =\left(R_{2}+R_{1}-I\right) /\left(R_{2}+R_{1}+I\right) \\
\alpha_{3} & =\left(R_{2}-R_{1}-I\right) /\left(R_{2}+R_{1}+I_{1}\right)
\end{array}\right\}, ~
\end{align*}
$$

We also note that these constants are not independent in that

$$
\alpha_{1}+\alpha_{2}=1+\alpha_{3}
$$

It is amarent that to avoid the possibility of delay-free loons ve must set either $\alpha_{1}=0$ or $\alpha_{3}=0$. Thus, again, two cases arise.

Case I: $\alpha_{1}=0$
This condition innlies that $R_{1}=R_{2}+L^{*}$ and therefore we have

$$
\begin{align*}
S_{11} & =\frac{\alpha_{3}^{z^{-1}}}{1+\alpha_{2}^{z^{-1}}}  \tag{2.14}\\
S_{22} & =\frac{-\alpha_{2}}{1+\alpha_{2^{z^{-1}}}} \\
\text { and } \alpha_{2} & =1+\alpha_{3}=R_{2} / \lambda_{1} .
\end{align*}
$$

Horeover it is observed that $S_{11}=-S_{22} z^{-1}$. The canonic realisation structure obtained by combiniñ eqns. (2.8) and (2.14) appears in Fir. 2.6. Note that the realisation structure is canonic in both mul.tipliers and delays.

## Footnote:

*The bilinear transformation we have used here, should be written

$$
p \rightarrow(1) \cdot\left(1-z^{-1}\right) /\left(1+z^{-1}\right)
$$

where the comdant, (1) has dimensions of angular frequency, that is sec. ${ }^{-1}$ (see Chapter 1). Thus the expression $R_{1}=R_{2}+L$ only anpears dimensionally incorrect. In fact, it could be written $\overline{R_{1}}=R_{2}+(1) . L$ where the constant (1) has dimensions of time. We may apply this argument to all frequency dependent elements (o.g. cnpacitors, tuned-circuits, ete.).

## Case IT: $\boldsymbol{\alpha}_{3}=0$

In this case we have

$$
R_{2}=R_{1}+L
$$

and the scattering parameters become

$$
\left.\begin{array}{rl}
s_{11} & =\alpha_{1} /\left(1+\alpha_{2} z^{-1}\right)  \tag{2.15}\\
s_{22} & =-\alpha_{1} z^{-1} /\left(1+\alpha_{2} z^{-1}\right)
\end{array}\right\}
$$

Purtinemore, ve ohsorve that

$$
S_{22}=-S_{11} z^{-1}
$$

The realisation structure obtained by using eqns. (2.15) in conjunction with eqn. (2.8) ampars in Fis. 2.7.

### 2.3.4 Series Gangoitance, C

For a capacitance in the series-arm, we have $Z=1 / n C=D / 0$ where $D=1 / C$. Cn anplying the bjilincar transformation to $Z$ we find that

$$
\begin{equation*}
z=\frac{1+z^{-1}}{1-z^{-1}} D \tag{2.16}
\end{equation*}
$$

Compare this with the expression for a serios inductance and we note immediately that $z^{-1}$ is replaced by $-z^{-1}$ and $L$ by $D$. Thus to obtain the URD of a sexies conncitance all we noed do is to alter the corresponding :TFD of a series inductance by adding to the delay branch a sicn inversion.* Alternatively, we mey nroceed as for a series inductance.

Substituting $D / p$ for $Z$ in eqn. (2.6) and applying the bilinear transformation produces the following results,

$$
\left.\begin{array}{l}
s_{11}=\left(\alpha_{1}+\alpha_{3} z^{-1}\right) /\left(1+\alpha_{2} z^{-1}\right) \\
s_{22}=\left(\alpha_{3}+\alpha_{1} z^{-1}\right) /\left(1+\alpha_{2} z^{-1}\right)
\end{array}\right\}
$$

## Footmote:

*This nrocess is compatible with the Low-Pass to Hish-Pess transformation as expected [9].

$$
\begin{aligned}
& \alpha_{1}=\left(n_{2}-R_{1}+D\right) /\left(n_{2}+P_{1}+D\right) \\
& \alpha_{2}=\left(D-n_{2}-p_{1}\right) /\left(R_{2}+R_{1}+D\right) \\
& \alpha_{3}=\left(D-n_{2}+R_{1}\right) /\left(n_{2}+R_{1}+D\right) \\
& \text { and in ardition } \alpha_{1}+\alpha_{3}=1+\alpha_{2}
\end{aligned}
$$

To nyoid delny-free loons there are tro vossihilities,
$\underline{\operatorname{rose} I: \alpha:=0}$
Mnis nontition mos $P_{4}=D_{n}+D$

$$
\begin{align*}
\text { sni } \quad s_{11} & =\alpha_{3} z^{-1} /\left(1+\alpha_{2} z^{-1}\right)  \tag{2.18}\\
s_{2 ?} & =\alpha_{3} /\left(1+\alpha_{2} z^{-1}\right)
\end{align*}
$$

to cether with $\alpha_{2}=\alpha_{3}-1=-?_{2} / n_{1}$.
Trote also that

$$
s_{11}=s_{2} z^{-1}
$$

The wip for this caso anears in Fis. 2.8.

Casc II: $\alpha_{3}=0$
This case zives $R_{2}=R_{1}+D$
for which

$$
\left.\begin{array}{l}
S_{11}=\alpha_{1} /\left(1+\alpha_{2} z^{-1}\right)  \tag{2.19}\\
S_{22}=\alpha_{1} z^{-1} /\left(1+\alpha_{2} z^{-1}\right)
\end{array}\right\}
$$

and

$$
\alpha_{2}=\alpha_{1}-1=-\mathrm{R}_{1} / \mathrm{R}_{2} .
$$

Furthermore $S_{22}=S_{11} z^{-1}$.

The :IFD corresponding to this case is show in Fi.s. 2.9.

### 2.3.5 Parallel-Tuned Circuit in Sories-Arm

For a parallel-tuned circuit in the series-aym (Fic. 2.10), $z=1 /\left(\mathrm{nC}+\frac{1}{\mathrm{pL}}\right)$. Let p be roplaced by $\left(1-z^{-1}\right) /\left(1+z^{-1}\right)$ then

$$
\begin{equation*}
z=\frac{L D\left(1-z^{-2)}\right.}{(L+D)+2(D-L) z^{-1}+(L+D) z^{-2}} \tag{2.20}
\end{equation*}
$$

Substituting for $Z$ in eqn. (2.6) gives

$$
\begin{align*}
& S_{11}=\frac{\alpha_{1}+\alpha_{2} z^{z^{-1}}+\alpha_{3} z^{z^{-2}}}{1+\alpha_{4}^{z^{-1}}+\alpha_{5}^{z^{-2}}}  \tag{2.2i}\\
& S_{22}=\frac{\alpha_{3}+\alpha_{2} z^{-1}+\alpha_{1} z^{-2}}{1+\alpha_{4}^{z^{-1}}+\alpha_{5} z^{-?}}
\end{align*}
$$

also ne note that $\alpha_{1}+\alpha_{5}=1+\alpha_{3}$
where

$$
\begin{aligned}
& \left.\alpha_{1}=\left\{\Omega_{2}-R_{1}\right)(I \div D)+I D\right\} / D E M O H \\
& \alpha_{2}=2(D-L)\left(P_{2}-R_{1}\right) \quad / \text { KC: } \\
& \alpha_{3}=\left\{\left(R_{2}-R_{1}\right)(L+D)-I D\right\} / \text { DEM: } \\
& \alpha_{4}=2(\Omega-L)\left(n_{2}+n_{1}\right) / D \operatorname{LiON} \\
& \alpha_{5}=\left\{\left(R_{2}+R_{1}\right)(I+D)-L D\right\} / \text { DINO } \\
& \operatorname{DNOM}=\left(R_{2}+R_{1}\right)(I+D)+L D \quad .
\end{aligned}
$$

If the realisability condition is applied, te must have either $\alpha_{1}=0$ or $\alpha_{3}=0$.

## Case I: $\alpha_{1}=0$

In this case we have

$$
R_{1}=R_{2}+\frac{I D}{L+D}
$$

and

$$
\begin{align*}
& S_{11}=\frac{\alpha_{2} z^{-1}+\alpha_{3} z^{-2}}{1+\alpha_{4} z^{-1}+\alpha_{5} z^{-2}} \\
& S_{22}=-\frac{\alpha_{3}+\alpha_{2} z^{-1}}{1+\alpha_{4} z^{-1}+\alpha_{5} z^{-2}} \tag{2.22}
\end{align*}
$$

$$
\begin{aligned}
& \text { 21:0 } \quad \alpha_{5}=-\quad \therefore \alpha_{z}=n_{2} / n_{1} \\
& \text { and } \quad \alpha_{2}=\left(\frac{D-I}{D} \frac{I}{I}\right)\left(\frac{?_{2}-B_{1}}{B_{1}}\right) \\
& \alpha_{3}=\frac{p_{2}-R_{1}}{R_{4}} \\
& \alpha_{4}=\left(\frac{D-L_{1}}{D+L}\left(\frac{R_{2}+R_{1}}{R_{1}}\right)\right.
\end{aligned}
$$




$$
\frac{s_{11}}{s_{22}}=\frac{-z^{-1}\left(\alpha_{2}+\alpha_{3} z^{-1}\right)}{\left(\alpha_{3}+\alpha_{2} z^{-1}\right)}
$$

Let $\quad \beta=\alpha_{2} i^{\prime} \alpha_{3}=\frac{D-I}{D+I} \quad$ then we have *

$$
\begin{equation*}
\frac{S_{11}}{S_{22}}=\frac{-z^{-1}\left(\beta \pm z^{-1}\right)}{1+\beta z^{-1}}=-T \tag{2.2:i}
\end{equation*}
$$

Let us choose $\alpha_{5}$ and $\beta$ as the independent set of multipliers in the above equations so that

$$
\begin{align*}
& \alpha_{2}=\beta\left(\alpha_{5}-1\right) \\
& \alpha_{3}=\alpha_{5}-1  \tag{2.25}\\
& \alpha_{4}=\beta\left(\alpha_{5}+1\right)
\end{align*}
$$

Using these values of $\alpha_{2}, \alpha_{3}, \alpha_{1}$, (en. (2.25) ) in association with en. (2.22) we obtain the following relationships

$$
\begin{align*}
S_{22} & =\frac{-\left(\alpha_{5}-1\right)\left(1+\beta z^{-1}\right)}{1+\beta\left(1+\alpha_{5}\right) z^{-1}+\alpha_{5^{2}}-2} \\
& =\frac{\left(1-\alpha_{5}\right)\left(1+\beta z^{-1}\right)}{\left(1+\beta z^{-1}\right)+\alpha_{5} z^{-1}\left(\beta+z^{-1}\right)} \\
& =\frac{\left(1-\alpha_{5}\right)}{\left(1+\alpha_{5} 1\right)} \tag{2.26}
\end{align*}
$$

Finally ens. (2.2f) and (2.26) together with (2.8) derive the WP D of Fig. (2.11). The HFD has been realised with two multipliers and tyro delays, ic. it is canonic in both types of elements.

## *Footnote:

Eqn. (2.24) represents the low-pass to band-pass transformation [9].

In this case re have

$$
\begin{align*}
& \mathrm{R}_{2}=\mathrm{R}_{1}+\frac{\mathrm{LD}}{\mathrm{~L}+\mathrm{D}} \\
\text { also } & \mathrm{S}_{11}=\left(\alpha_{1}+\alpha_{2} z^{-1}\right) /\left(1+\alpha_{4} z^{-1}+\alpha_{5} z^{-2}\right)  \tag{2.27}\\
\text { and } & S_{22}=-\left(\alpha_{2} z^{-1}+\alpha_{1} z^{-2}\right) /\left(1 \div \alpha_{4} z^{-1}+\alpha_{5} z^{-2}\right)
\end{align*}
$$

In addition,

$$
\begin{align*}
& \alpha_{1}=1-\alpha_{5} \\
& \alpha_{2}=\left(\frac{D-T_{1}}{D+L_{1}}\left(1-\frac{R_{1}}{D_{2}}\right)\right. \\
& \alpha_{4}=\left(\frac{D-I_{1}}{D+I_{1}}\left(1+\frac{R_{1}}{n_{2}}\right)\right.  \tag{3.28}\\
& \alpha_{5}=P_{1} / R_{2} .
\end{align*}
$$

Mo: consider

$$
\begin{align*}
& \frac{S_{22}}{S_{11}}=-\frac{\alpha_{2} z^{-1}+\alpha_{1} z^{-2}}{\alpha_{1}+\alpha_{2} z^{-1}} \\
& \beta=\alpha_{2} / \alpha_{1}=\frac{D-L}{D+L} \\
& \frac{S_{22}}{S_{11}}=-\frac{z^{-1}\left(\beta+z^{-1}\right)}{1+\beta z^{-1}}=-T
\end{align*}
$$

Again, we ernooss the multipliers in toms of $\alpha_{5}$ and $\beta$. Thus

$$
\begin{align*}
& \alpha_{1}=1-\alpha_{5} \\
& \alpha_{2}=\beta\left(1-\alpha_{5}\right)  \tag{2.30}\\
& \alpha_{4}=\beta\left(1+\alpha_{5}\right)
\end{align*}
$$

Using those values and en. (2.27) we derive the following

$$
\begin{align*}
S_{11} & =\frac{\left(1-\alpha_{5}\right)\left(1+\beta z^{-1}\right)}{\left(1+\beta z^{-1}\right)+\alpha_{5} z^{-1}\left(\beta+z^{-1}\right)} \\
& =\frac{\left(1-\alpha_{5}\right)}{1+\alpha_{5}^{5}} \tag{2.31}
\end{align*}
$$

Mon on ho chthined using ens. (2.8), (2.20) and (2.01). It ia if ?notrotod in Fin. 2.12.

### 2.3.5 Sories-rime? Circuit in Sories-Arm

 appropriate serins $\cap$ ont, in addition, wo mat adjust the port resistances ot, the intemoomection (see section 2.6.5). However,
 saving in the musher of inters ray he made. The impornoe 3 is


$$
\begin{equation*}
z=\frac{(I+D)+\frac{2\left(D-I^{\prime} z^{-1}+\left(L+D^{\prime} z^{-2}\right.\right.}{1-z^{-2}} .}{} \tag{2.22}
\end{equation*}
$$

Substituting for $a$ in nan. (2.6) rives

$$
\left.\begin{array}{l}
s_{11}=\left(\alpha_{1}+\alpha_{2} z^{-1}+\alpha_{3} z^{-2}\right) /\left(1+\alpha_{2} z^{-1}+\alpha_{4} z^{-2}\right)  \tag{3.33}\\
s_{22}=\left(\alpha_{3}+\alpha_{2} z^{-1}+\alpha_{1} z^{-2}\right) /\left(1+\alpha_{2} z^{-1}+\alpha_{4} z^{-2}\right)
\end{array}\right\}
$$

where

$$
\alpha_{1}+\alpha_{3}-\alpha_{4}=1
$$

and

$$
\begin{aligned}
& \alpha_{1}=\left(R_{2}-R_{1}+L+D\right) /\left(R_{2}+D_{1}+I+D\right), \\
& \alpha_{2}=2(D-L) \\
& \alpha_{2}=\left(R_{1}-p_{2}+I+D\right) / \\
& \alpha_{4}=\left(L+D-R_{1}-R_{2}\right) /
\end{aligned}
$$

If the molimpititar condition io mpried, wo have either $\alpha_{1}=0$ or $\alpha_{3}=0$.

Case I: $\alpha_{1}=0$
For this case we have

$$
R_{1}=R_{2}+L+D
$$

and hence,

$$
\left.\begin{array}{l}
s_{11}=\left(\alpha_{2} z^{-1}+\alpha_{3} z^{-2}\right) /\left(1+\alpha_{2} z^{-1}+\alpha_{4} z^{-2}\right) \\
S_{22}=\left(\alpha_{3}+\alpha_{2} z^{-1}\right) /\left(1+\alpha_{2} z^{-1}+\alpha_{4} z^{-2}\right)
\end{array}\right\}(2.34)
$$

Fuッ亡hロックロッロ，

$$
\begin{align*}
& \alpha_{3}-\alpha_{1}=1 \\
& \alpha_{2}=(D-I) / R_{1}  \tag{2.35}\\
& \alpha_{3}=(D+I) / R_{1} \\
& \alpha_{4}=-R_{2} / R_{1}
\end{align*}
$$

Nov define $\quad \beta=\alpha_{2} / \alpha_{3}=(D-L) /(D+I)$
then

$$
\alpha_{2}=\beta\left(1+\alpha_{4}\right)
$$

and

$$
\alpha_{3}=1+\alpha_{4}
$$

so that $i f^{*} m=\frac{z^{-1}\left(\beta+z^{-1}\right)}{\left(1+\beta z^{-1}\right)}$
then．
and

$$
\begin{align*}
& S_{11}=\frac{\left(1+\alpha_{4}\right) T}{1+\alpha_{4}^{\mathrm{T}}}  \tag{2.36}\\
& S_{22}=\frac{\left(1+\alpha_{4}\right)}{1+\alpha_{4}^{T}}
\end{align*}
$$

From those rolationchinm wo have the ronumod rip which guars in Pic．2．14．
$\operatorname{Cnse} I^{\top}: \alpha_{T}=0$
In this case，we find that

$$
R_{2}=R_{1}+I+D
$$

also
and

$$
\left.\begin{array}{l}
s_{11}=\left(\alpha_{1}+\alpha_{2} z^{-1}\right) /\left(1+\alpha_{2} z^{-1} \div \alpha_{4} z^{-2}\right) \\
s_{22}=\left(\alpha_{2} z^{-1}+\alpha_{1} z^{-2}\right) /\left(1+\alpha_{2} z^{-1}+\alpha_{4} z^{-2}\right)
\end{array}\right\}(2.37)
$$

2］．so

Furthermore，

$$
\begin{align*}
& \alpha_{1}=(I+D) / n_{2} \\
& \alpha_{2}=\left(D-I_{1}\right) / R_{2}  \tag{2.38}\\
& \alpha_{4}=-R_{1} / R_{2}
\end{align*}
$$

Te may define

$$
\beta=\alpha_{2} / \alpha_{1} \quad \text { and } \quad T=\frac{z^{-1}\left(\beta+z^{-1}\right)}{1+\beta z^{-1}}
$$

then

$$
\begin{align*}
& S_{11}=\left(1+\alpha_{4}\right) /\left(1+\alpha_{4} T\right) \\
& S_{22}=\left(1+\alpha_{4}\right) T /\left(1+\alpha_{4} T\right) \tag{2.39}
\end{align*}
$$

and This expression represents the low－pass to band－stop
transformation $[9]$ ．

The ronisation obtrined by combining eqns (2.30) and (2.8) is illustrated in Pic. 2.15.

### 2.4 Derivation of Vave-Flow Diarrans for Shiut Flements

### 2.1.1 Tntroduction

A shunt admittance $Y$ has ABCD ratrix given, in our notation, by

$$
\left[\begin{array}{rr}
1 & C \\
Y & -1
\end{array}\right]
$$

Substituting for A, B, C and D in eqns. (2.4) and (2.5) gives the following,

$$
\begin{align*}
& S_{11}=\left(G_{1}-G_{2}-Y\right) /\left(G_{1}+G_{2}+Y\right) \\
& S_{12}=2 G_{2} /\left(G_{1}+G_{2}+Y\right)  \tag{2.40}\\
& S_{21}=2 G_{1} /\left(G_{1}+G_{2}+Y\right) \\
& S_{22}=\left(G_{2}-G_{1}-Y\right) /\left(G_{1}+G_{2}+Y\right)
\end{align*}
$$

It is important to note that

$$
\begin{align*}
& S_{21}-s_{11}=1  \tag{2.41}\\
& s_{12}-S_{22}=1
\end{align*}
$$

These will be vseful in simnlifying the wave-flow diatrams for if we combine eqn. (2.5) with (?.41) then

$$
\begin{align*}
& B_{1}=\left(S_{11} A_{1}+S_{22} A_{2}\right)+A_{2}  \tag{2.42}\\
& B_{2}=\left(S_{11} A_{1}+S_{22} \dot{A}_{2}\right)+A_{1}
\end{align*}
$$

Again we have eliminated $S_{12}$ and $S_{21}$ from eqn. (2.5) as the realizability condition involves $S_{11}$ and $S_{22}$ only.

We shall proceed to derive the wave-flom diagrams of the shunt el.ements necessary to realise the digital equivalent of a ladder structure. The procedure adopted is similar to thet takon in sec. 2.3.

## 2.4.? Shunt Resistence, ?

Let $Y=1 / R=G$ in eqn. $(2.40)$ then

$$
\left.\begin{array}{l}
S_{11}=\left(G_{1}-G_{2}-G\right) /\left(G_{1}+G_{2}+G\right)  \tag{2.43}\\
S_{22}=\left(G_{2}-G_{1}-G\right) /\left(G_{1}+G_{2}+G\right)
\end{array}\right\}
$$

To avoid the delay-free loops, the numerator of $S_{11}$ or $S_{22}$ must be zero thus, as before, we have two cases.

## Cose I:

Let $G_{1}=G_{2}+G$
then $S_{11}=0$
and $S_{22}=\frac{-G}{G_{2}+G}=\beta$, say
The HFD may be obtained by comoining eqns. (2.42) and (2.44) and is sho:m in Fieg. 2.16.

Mase IT:
Let $G_{2}=G_{1}+G$
$\left.\begin{array}{l}\text { then } S_{11}=-\frac{G}{G_{1}+G}=\alpha \text {, say } \\ \text { and } S_{22}=0\end{array}\right\}$ (2.45)
The YRD may be obtained by combinins eqns. (2.42) and (2.45) and is shom in Fis. 2.17.

### 2.4.3 Shunt Cenacitance, $C$

For a capacitance in the shunt arm we set $Y=p C$. This
substitution in eqn. (2.40) and the use of the bilinear transformation produces the following

$$
\left.\begin{array}{ll} 
& \begin{array}{l}
s_{i 1}
\end{array}=\left(\alpha_{1}+\alpha_{3} z^{-1}\right) /\left(1+\alpha_{2} z^{-1}\right) \\
\text { and } & s_{22}=-\left(\alpha_{3}+\alpha_{1} z^{-1}\right) /\left(1+\alpha_{2} z^{-1}\right)
\end{array}\right\} \text { (2.46) }
$$

To avoid delay-free loops there are, as before, two possibilities, namply $\alpha_{1}=0$ or $\alpha_{3}=0$ which zives rise to two dirierent cases.

Case I.: $\alpha_{1}=0$
This condition gives the following

$$
\left.\begin{array}{l}
G_{1}=G_{2}+c \\
s_{11}=\alpha_{3} z^{-1} /\left(1+\alpha_{2} z^{-1}\right)  \tag{2.47}\\
s_{22}=-\alpha_{3} /\left(1+\alpha_{2} z^{-1}\right)
\end{array}\right\}
$$

and $\alpha_{2}=1-\alpha_{3}=a_{2} / a_{1}$.
It is further observed that

$$
s_{11}=-s_{22^{2}} z^{-1}
$$

The Th for this case apyears in Fig. 2.18.

Case II: $\alpha_{3}=0$
This condition yields the following relationships,

$$
\begin{align*}
G_{2} & =G_{1}+C \\
S_{11} & =\alpha_{1} /\left(1+\alpha_{2} z^{-1}\right)  \tag{2.48}\\
S_{? 2} & =-\alpha_{1} z^{-1} /\left(1+\alpha_{2} z^{-1}\right) \\
\alpha_{2} & =1+\alpha_{1}=G_{1} / G_{2} \\
\text { also } \quad S_{22} & =-S_{11} z^{-1}
\end{align*}
$$

The VID for this caso onments in Fis. 2.19.

### 2.1.4 Shunt Tnructerce, I

For an inductance in the shunt arm we have $Y=1 / p L=r / P$ On applying the bilinear transformation to $Y$ we find that

$$
\begin{equation*}
Y=\frac{1+z^{-1}}{1-z^{-1}} \Gamma \tag{2.49}
\end{equation*}
$$

On comparing this with the expression for a shunt conacitance, we immediately see that $z^{-1}$ is renlaced by $-z^{-1}$ and $C$ by $\Gamma$. Thus to obtain the of a shunt inductance we need only alter the corresnondine Vifd of a shunt canacitance by addinc to the delay branch a sicn inversion. Alternatively, we may proceed as follows.

Substitutins for $Y$ from eqn. (2.49) into eqn. (2.40) gives

$$
\left.\begin{array}{rl}
s_{11} & =\left(\alpha_{1}+\alpha_{3} z^{-1}\right) /\left(1+\alpha_{2} z^{-1}\right)  \tag{2.50}\\
\text { and } S_{22} & =\left(\alpha_{3}+\alpha_{1} z^{-1}\right) /\left(1+\alpha_{2} z^{-1}\right)
\end{array}\right\}
$$

rhere
and

$$
\begin{aligned}
& \alpha_{1}=\left(G_{1}-G_{2}-\Gamma\right) /\left(G_{1}+G_{2}+\Gamma\right) \\
& \alpha_{2}=\left(\Gamma-G_{1}-G_{2}\right) /\left(G_{1}+G_{2}+\Gamma\right) \\
& \alpha_{3}=\left(G_{2}-G_{1}-\Gamma\right) /\left(G_{1}+G_{2}+\Gamma\right) \\
& \alpha_{1}+\alpha_{2}+\alpha_{3}=-1
\end{aligned}
$$

To avoid delav-free loons there are tro possibilities.

## Case I: $\alpha_{1}=0$

This condition cives the folloring

$$
\left.\begin{array}{l}
G_{1}=G_{2}+\boldsymbol{r} \\
S_{11}=\alpha_{3} z^{-1} /\left(1+\alpha_{2} z^{-1}\right)  \tag{2.51}\\
S_{22}=\alpha_{3} /\left(1+\alpha_{2} z^{-1}\right) \\
\alpha_{2}=-\left(1+\alpha_{3}\right)=-G_{2} / S_{1}
\end{array}\right\}
$$

The roslication sononrs in Eig. ?.20.

Case II: $\alpha_{3}=0$
This condition implies that

$$
\begin{aligned}
& G_{2}=G_{1}+\Gamma \\
& S_{11}=\alpha_{1} /\left(1+\alpha_{2} z^{-1}\right) \\
& S_{22}=\alpha_{1} z^{-1} /\left(1+\alpha_{2} z^{-1}\right) \\
& \alpha_{2}=-\left(1+\alpha_{1}\right)=-G_{1} / G_{2}
\end{aligned}
$$

The TFD for this case is shorm in Fig. 2.21.

### 2.4.5 Series-Tuned Cirouit in Shunt-Arm

For a series-tuned circuit in the shunt-arm (Fig. 2.22) we have

$$
Y=1 /\left(p L+\frac{1}{p C}\right)
$$

Applying the bilinear transformation we have that

$$
\begin{equation*}
Y=\frac{C \Gamma\left(1-z^{-2}\right)}{(C+\Gamma)+2(\Gamma-c) z^{-1}+(C+\Gamma) z^{-2}} \tag{2.53}
\end{equation*}
$$

Substituting for $Y$ in en. (?.40) rives

$$
\left.\begin{array}{l}
s_{11}=\left(\alpha_{1}+\alpha_{2} z^{-1}+\alpha_{3} z^{-2}\right) /\left(1+\alpha_{4} z^{-1}+\alpha_{5} z^{-2}\right) \\
s_{22}=\left(\alpha_{3}+\alpha_{2} z^{-1}+\alpha_{1} z^{-2}\right) /\left(1+\alpha_{4} z^{-1}+\alpha_{5} z^{-2}\right)
\end{array}\right\}(2.54)
$$

where

$$
\begin{aligned}
& \alpha_{1}=\left\{\left(G_{1}-G_{2}\right)(C+\Gamma)-C \Gamma\right\} / D M O M, \\
& \alpha_{2}=\left\{2\left(G_{1}-G_{2}\right)(\Gamma-C)\right\} / D \operatorname{CND}, \\
& \alpha_{3}=\left\{\left(\omega_{1}-\hat{\sigma}_{2}\right)(C+\Gamma)+C \Gamma\right\} / D \operatorname{lom}, \\
& \alpha_{4}=\left\{2\left(a_{1}+G_{2}\right)(\Gamma-c)\right\} \text { /Draw, } \\
& \alpha_{5}=\left\{\left(G_{1}+G_{2}\right)(\sigma+\Gamma)-C \Gamma\right\} / \text { Damon, } \\
& \text { DiNO }=\left(G_{1}+G_{2}\right)(C+\Gamma)+C \Gamma
\end{aligned}
$$

and

$$
1+\alpha_{1}=\alpha_{3} \div \alpha_{5}
$$

If the realizaility condition is applied, we must have either $\alpha_{1}=0$ or $\alpha_{3}=0$.

Case I: $\alpha_{1}=0$
In this case we hove

$$
G_{1}=G_{2}+\frac{C \Gamma}{C+\Gamma}
$$

and

$$
\left.\begin{array}{l}
S_{11}=\left(\alpha_{2} z^{-1}+\alpha_{3} z^{-2}\right) /\left(1+\alpha_{4} z^{-1}+\alpha_{5} z^{-2}\right) \\
S_{22}=-\left(\alpha_{3}+\alpha_{2} z^{-1}\right) /\left(1+\alpha_{4} z^{-1}+\alpha_{5} z^{-2}\right)
\end{array}\right\}(2.55)
$$

Furthermore,
$\left.\begin{array}{l}\alpha_{2}=\left(\frac{\Gamma-C}{\Gamma+C}\right)\left(1-\frac{G_{2}}{G_{1}}\right) ; \quad \alpha_{3}=1-\frac{G_{2}}{G_{1}} \\ \alpha_{4}=\left(\frac{\Gamma-C}{\Gamma+C}\right)\left(1+\frac{G_{2}}{G_{1}}\right) ; \quad \alpha_{5}=1-\alpha_{3}=\frac{G_{2}}{G_{1}}\end{array}\right\}$
Let us consider dividing $S_{11}$ by $S_{22}$ using eqn. (2.55), then

$$
\begin{aligned}
\frac{S_{11}}{-S_{22}} & =\frac{\alpha_{2} z^{-1}+\alpha_{z^{z}} z^{-2}}{\alpha_{3}+\alpha_{2} z^{-1}} \\
\text { Int } \beta=\alpha_{2} / \alpha_{3} & =(\Gamma-\Gamma) /(\Gamma+0) \quad \text { then }
\end{aligned}
$$

$$
\begin{equation*}
\frac{S_{11}}{S_{22}}=\frac{-z^{-1}\left(\beta \div z^{-1}\right)}{1+\beta z^{-1}}=-T \tag{2.57}
\end{equation*}
$$

Expressing all multipliers in terms of $\alpha_{5}$ and $\beta$, we have

$$
\begin{align*}
\alpha_{2} & =\beta\left(1-\alpha_{5}\right) \\
\alpha_{3} & =1-\alpha_{5} \\
\alpha_{4} & =\beta\left(1+\alpha_{5}\right) \\
e_{22} & =\frac{\left(\alpha_{5}-1\right)\left(1+\beta z^{-1}\right)}{\left(1 \div \beta z^{-1}\right)+z^{-i} \alpha_{5}\left(\beta+z^{-1}\right)} \\
& =\frac{\left(\alpha_{5}-1\right)}{1+\alpha_{5}} \tag{2.58}
\end{align*}
$$

The THD in Fig. 2.23 may be derived by using en. (2.42) with ens. (2.57) and (2.58).

Case II: $\alpha_{3}=0$
In this case we have

$$
G_{2}=G_{1}+\frac{C}{C+\Gamma}
$$

together with

$$
\begin{align*}
& S_{11}=\frac{\alpha_{1}+\alpha_{2} z^{-1}}{1+\alpha_{4} z^{-1}+\alpha_{5} z^{-2}}  \tag{2.59}\\
& S_{22}=-\frac{\left(\alpha_{2} z^{-1}+\alpha_{1} z^{-2}\right)}{1+\alpha_{4}^{z^{-1}}+\alpha_{5}^{z^{-2}}}
\end{align*}
$$

and

$$
\begin{aligned}
& \alpha_{1}=\frac{G_{1}}{G_{2}}-1 ; \alpha_{2}=\left(-\frac{\Gamma-c}{\Gamma+C}\right)\left(\frac{G_{1}}{G_{2}}-1\right) \\
& \alpha_{4}=\left(\frac{\Gamma-C}{\Gamma+C}\right)\left(1+\frac{G_{1}}{G_{2}}\right) ; \alpha_{5}=1+\alpha_{1}=G_{1} / G_{2}
\end{aligned}
$$

In a similar way to that used in Cased we can show that if $\beta=\frac{\Gamma-C}{\Gamma+C}$, then

$$
\left.\begin{array}{l}
\frac{S_{22}}{S_{11}}=-\frac{z^{-1}\left(\beta+z^{-1}\right)}{1+\beta z^{-1}}=T \\
S_{11}=\frac{\alpha_{5}-1}{1+\alpha_{5} T}
\end{array}\right\}
$$

and

The wFD may be derived by using ean. (2.42) together with eqn. (2.60) and is shown in Fig. 2.24.

### 2.4.6 Parallel-Tuned Circuit in Shunt-Arm

In this case (Fig. 2.25) we may simply cascade a shunt $L$ with the appropriate shunt $C$. However, by considering the tuned-circuit as a whole, a saving in additions may be made.

The admittance $Y$ is $p C+\frac{1}{p L} \quad$ and after applying the bilinear transformation, we have

$$
\begin{equation*}
Y=\frac{(C+\Gamma)+2(\Gamma-C) z^{-1}+(C+\Gamma) z^{-2}}{1-z^{-2}} \tag{2.61}
\end{equation*}
$$

On substituting for $Y$ from eqn. (2.61) into eqn. (2.40) we find that

$$
\begin{align*}
s_{11} & =\frac{\alpha_{1}-\alpha_{2} z^{-1}+\alpha_{z} z^{-2}}{1+\alpha_{2} z^{-1}+\alpha_{4} z^{-2}} \\
s_{22} & =\frac{\alpha_{3}-\alpha_{2} z^{-1}+\alpha_{1} z^{-2}}{1+\alpha_{2} z^{-1}+\alpha_{4} z^{-2}} \tag{2.62}
\end{align*}
$$

where

$$
\begin{aligned}
& \alpha_{1}=\left(G_{1}-G_{2}-C-\Gamma\right) /\left(G_{1}+G_{2}+C+\Gamma\right) \\
& \alpha_{2}=2(\Gamma-C) /\left(G_{1}+G_{2}+C+\Gamma\right) \\
& \alpha_{3}=\left(G_{2}-G_{1}-C-\Gamma\right) /\left(G_{1}+G_{2}+C+\Gamma\right) \\
& \alpha_{4}=\left(\Gamma+C-G_{1}-G_{2}\right) /\left(G_{1}+G_{2}+C+\Gamma\right)
\end{aligned}
$$

$$
\alpha_{1}+\alpha_{3}+\alpha_{4}=-1
$$

If the realizability condition is applied we have either $\alpha_{1}=0$ or $\alpha_{3}=0$.

Case I: $\alpha_{1}=0$
We have, in this case,

$$
G_{1}=G_{2}+C+\Gamma
$$

and thus

$$
\begin{equation*}
S_{11}=\frac{-\alpha_{2} z^{-1}+\dot{\alpha}_{3} z^{-2}}{1+\alpha_{2} z^{-1}+\alpha_{4} z^{-2}} \tag{2.63}
\end{equation*}
$$

$$
\begin{equation*}
s_{22}=\frac{\alpha_{3}-\alpha_{2} z^{-1}}{1+\alpha_{2} z^{-1}+\alpha_{4} z^{-2}} \tag{2.63}
\end{equation*}
$$

Where

$$
\begin{aligned}
& \alpha_{2}=(\Gamma-c) / a_{i}, \\
& \alpha_{3}=-(\Gamma+c) / G_{1}, \\
& \alpha_{4}=-G_{2} / G_{1}
\end{aligned}
$$

and

$$
\alpha_{3}+\alpha_{4}=-1
$$

Letting

$$
\beta=-\alpha_{2} / \alpha_{3}=(\Gamma-c) /(\Gamma+c)
$$

and $T=\frac{z^{-1}\left(\beta+z^{-1}\right)}{1 \div \beta z^{-1}}$
, it is not too difficult
to show that

$$
\begin{equation*}
S_{11}=\frac{\alpha_{3}^{\dot{T}}}{1+\alpha_{4^{T}}} \tag{2.6i}
\end{equation*}
$$

and

$$
S_{22}=\frac{\alpha_{3}}{1+\alpha_{4}^{T}}
$$

The VFD is then obtained using the values of $S_{11}$ and $S_{22}$
 Fir. ?.25.

Case II: $\alpha \dot{3}=0$
In this case, we find that

$$
G_{2}=G_{1}+C+\Gamma
$$

together with

$$
\left.\begin{array}{l}
s_{11}=\left(\alpha_{1}-\alpha_{2} z^{-1}\right) /\left(1+\alpha_{2} z^{-1}+\alpha_{4} z^{-2}\right) \\
s_{22}=\left(-\alpha_{2} z^{-1}+\alpha_{1} z^{-2}\right) /\left(1+\alpha_{2} z^{-1}+\alpha_{4} z^{-2}\right)
\end{array}\right\}(2.65)
$$

where
and

$$
\begin{aligned}
& \alpha_{1}=-(c+\Gamma) / G_{2} \\
& \alpha_{2}=(\Gamma-C) / G_{2} \\
& \alpha_{4}=-G_{1} / G_{2} \\
& \alpha_{1}+\alpha_{4}=-1
\end{aligned}
$$

Again, by letting $\beta=-\alpha_{2} / \alpha_{1}=\frac{\Gamma-C}{\Gamma+C}$

$$
\text { and } T=\frac{z^{-1}\left(\beta+z^{-1}\right)}{1+\beta z^{-1}} \quad \text { we can }
$$

easily show that

$$
\begin{align*}
S_{11} & =\frac{-\left(1+\alpha_{4}\right)}{1+\alpha_{4}}  \tag{2.66}\\
\text { and } \quad S_{22} & =\frac{-\left(1+\alpha_{1}\right) T}{1+\alpha_{4} T}
\end{align*}
$$

The Tn obtained by combining ens. (2.42) and (2.65) is jJ. lustrated in Fir. 2.27.
2.5 Derivation of Yrve-prow Diampams Cor a Tosaloss Irmentionortino lamont

Consider the general lossless-line element of Pis. 2.28. The voltage and current equations can be written as follows [10]

$$
\left[\begin{array}{l}
V_{1}  \tag{2.67}\\
I_{1}
\end{array}\right]=\vdots\left[\begin{array}{ll}
A & B \\
C & D
\end{array}\right]\left[\begin{array}{l}
V_{2} \\
I_{2}
\end{array}\right]
$$

where

$$
\begin{aligned}
& A=\cos \hat{\beta} 1 \\
& B=-j \sin (\hat{\beta} 1) z_{0} \\
& C=j \sin (\hat{\beta} 1)_{0} \\
& D=-\cos \hat{\beta} 1
\end{aligned}
$$

and where

$$
\begin{aligned}
& \hat{\boldsymbol{\beta}}=\hat{\forall} \sqrt{L C}, \\
& Z_{o}=\sqrt{L / C} \quad \text { is the characteristic impedance, } \\
& Y_{0}=1 / Z_{o}, \\
& I \text { is the line length and } \hat{w} \text { is the lumped frequency } \\
& \text { variable. }
\end{aligned}
$$

On substituting the above values for A, B, C, D into ear. (2.4) we find that

$$
\left.\begin{array}{l}
\alpha=\frac{1}{2}\left\{\left(1-R_{1} G_{2}\right) \cos \theta+j\left(R_{1} Y_{0}-G_{2} Z_{0}\right) \sin \theta\right\} \\
\beta=\frac{1}{2}\left\{\left(1+R_{1} G_{2}\right) \cos \theta+j\left(R_{1} Y_{0}+G_{2} J_{0}\right) \sin \theta\right\} \\
\gamma=\frac{1}{2}\left\{\left(1+R_{1} G_{2}\right) \cos \theta-j\left(R_{1} Y_{0}+G_{2} Z_{0}\right) \sin \theta\right\} \\
\delta=\frac{1}{2}\left\{\left(1-R_{1} G_{2}\right) \cos \theta-j\left(R_{1} Y_{0}-G_{2} Z_{0}\right) \sin \theta\right\}
\end{array}\right\}(2.68)
$$

W have imodiotaly from one. (2.ro) that

were * denotes complex conjugate.

Now consider the substitution

$$
\begin{align*}
& z^{\frac{1}{2}}=\cos \theta+j \sin \theta  \tag{2.70}\\
& \text { where } z=\exp (j w T)
\end{align*}
$$

On equating real and imaginary parts we see that

$$
\begin{equation*}
\theta=\frac{1}{2} \mathrm{wT} \tag{2.71}
\end{equation*}
$$

By virtu n of on. (2.70) we have

$$
\begin{align*}
\cos \theta & =z^{\frac{1}{2}}+z^{-\frac{i}{2}}  \tag{2.72}\\
\text { and } 2 j \sin \theta & =z^{\frac{1}{2}}-z^{-\frac{1}{2}}
\end{align*}
$$

and therefore, on replacing $\cos \theta$ and $\sin \theta$ in eqn. (2.68) ye find that

$$
\left.\begin{array}{l}
\alpha=\frac{1}{4}\left\{\left(1-R_{1} G_{2}+R_{1} Y_{0}-G_{2} Z_{0}\right) z^{\frac{1}{2}}\right. \\
\left.\quad+\left(1-R_{1} G_{2}-R_{1} Y_{0}+G_{2} Z_{0}\right) z^{-\frac{1}{2}}\right\} \\
\left.\begin{array}{rl}
\beta=\frac{1}{4}\left\{\left(1+R_{1} G_{2}\right.\right. & \left.+R_{1} Y_{0}+G_{2} Z_{0}\right) z^{\frac{1}{2}} \\
& \left.\quad\left(1+R_{1} G_{2}-R_{1} Y_{0}-G_{2} Z_{0}\right) z^{-\frac{1}{2}}\right\}
\end{array}\right\} \tag{2.73}
\end{array}\right\}
$$

where $\gamma$ and $\delta$ are related to $\alpha$ and $\beta$ according to en. (2.60).

The scattering parameters defined in eqn. (2.5) may now be derived as

$$
\begin{aligned}
& S_{11}=\delta / \beta=\alpha^{*} / \beta \\
& S_{12}=R_{1} S_{2} / \beta \\
& S_{21}=1 / \beta \\
& S_{22}=-\alpha / \beta
\end{aligned}
$$

+hntis
where
$D$ DOI $=\left(1+R_{1} C_{2}+R_{1} Y_{0}+G_{2} Z_{0}\right)+\left(1+R_{1} G_{2}-R_{1} Y_{0}-G_{2} Z_{0}\right) z^{-1}$.
 case, the realizobility condition still holds on $S_{11}$ ard $S_{22}$ and imnlies ofther that $1-R_{1} G_{2}-R_{1} Y_{0}+G_{2} Z_{c}=0$ or that $1-R_{1} G_{2}+R_{1} Y_{0}-G_{2} Z_{0}=0$.

## Cose I: $1-D_{1} G_{2}-Q_{1} Y_{0}+G_{2} Z_{0}=0$

This cordition implies thot

$$
R_{1}=\frac{:+Q_{0}}{G_{2}+Y_{0}}=Z_{0}
$$

Lettinc $R_{1}=7_{0}$ in eqn. (2.74) gives the followjnc,

$$
\begin{align*}
S_{11} & =\frac{R_{2}-Z_{0}}{R_{2}+Z_{0}} z^{-1} \\
S_{12} & =\frac{2 Z_{0}}{R_{2}+Z_{0}} z^{-\frac{1}{2}}  \tag{2.75}\\
S_{21} & =\frac{2 R_{2}}{R_{2}+Z_{0}} z^{-\frac{1}{2}} \\
S_{22} & =-\frac{R_{2}-Z_{0}}{R_{2}+Z_{0}} \\
\text { Iet } \quad K_{1} & =\frac{R_{2}-Z_{0}}{R_{2}+Z_{0}} \quad \text { in eqn: (2.75) then } \\
S_{11} & =K_{1} z^{-1} \\
S_{12} & =\left(1-K_{1}\right) z^{-\frac{1}{2}} \\
S_{21} & =\left(1+K_{1}\right) z^{-\frac{1}{2}}  \tag{2.76}\\
S_{22} & =-K_{1}
\end{align*}
$$

We can ranlise these eqns. using one multiplier. and two delors. The VFD arnears in Tig. 2.?2.


This condition innlies thet

$$
\begin{aligned}
r_{2} & =\frac{1+I_{1} Y_{0}}{R_{1}+Z_{0}}=Y_{0} \\
\text { or } R_{2} & =Z_{0}
\end{aligned}
$$

Ietting $?_{?}=Z_{0}$ in ean. (?.74) rives the follorine,
$S_{11}=-\frac{R_{1}-Z_{0}}{R_{1}+Z_{0}}$
$S_{12}=\frac{2 R_{1}}{R_{1}+Z_{0}} z^{-\frac{1}{2}}$
$S_{21}=\frac{2 Z_{0}}{i_{q}+7_{0}} z^{-\frac{1}{2}}$.
$S_{22}=\frac{R_{1}-Z_{0}}{R_{1}+Z_{0}} z^{-1}$
on letting $K_{2}=\frac{R_{1}-Z_{0}}{R_{1}+Z_{0}}$ in enn. (2.77) ye have finally that

$$
\begin{align*}
& S_{11}=-V_{2} \\
& S_{12}=\left(1+K_{2}\right) z^{-\frac{1}{2}} \\
& S_{21}=\left(1-K_{2}\right) z^{-\frac{1}{2}}  \tag{2.78}\\
& S_{22}=K_{2} z^{-1}
\end{align*}
$$

The realisation of these equations usins one multiplier and two delays annears in $\operatorname{Fiz}$. 2.30.

Note that if $K_{1}=0$ in eqn. (2.76) or if $K_{2}=0$ in eqn. (2.78) the corresponding WrD simplifies to that of the unit elcnent [30].

### 2.6 Sourcos, Terminations ard Interconnections

### 2.6.1 Introinotion

In the previous sections we have discussed the two-port approach with resard to lossless Iumped cr distributed elements. We heve allowed the port resistances $R_{1}$ and $R_{2}$ to vary indenendently where the obvious choice would have been to set $R_{1}=R_{2}$. However, When we consider sources and terminations, which are one-port elements, there is only one port resistance, $R$, to vary. Again, the obvious choice for $R$ is the value of the corresponding element. In this section we shall allow $R$ to remain variable and consider special cases.
2.6.2 nnatativa Ynltnmesmon

The resistive voltare source is shom in Fig. (2.31) and its equations are as follows,

$$
\begin{align*}
& A=V \perp R I \\
& B=V-B I  \tag{2.79}\\
& V_{O}=V+R_{S} I
\end{align*}
$$

Te wish to climinate $V$ and $I$ and thus obtain a relationshin between $B$ and $A$. In matrix form we may write eqn. (?.79) as

$$
\left[\begin{array}{l}
A  \tag{2.80}\\
B
\end{array}\right]=\left[\begin{array}{cc}
1 & R \\
1 & -R
\end{array}\right]\left[\begin{array}{l}
V \\
I
\end{array}\right]
$$

together with

$$
\begin{align*}
V_{0} & =\left[\begin{array}{ll}
1 & R_{S}
\end{array}\right]\left[\begin{array}{l}
V \\
I
\end{array}\right]  \tag{2.81}\\
\therefore \quad V_{0} & =\left[\begin{array}{ll}
1 & R_{S}
\end{array}\right]\left[\begin{array}{rr}
1 & R \\
1 & -R
\end{array}\right]^{-1}\left[\begin{array}{l}
A \\
B
\end{array}\right] \tag{2.82}
\end{align*}
$$

On simplifying eqn. (2.82) we find that

$$
\text { Let } \quad \begin{align*}
A & =\frac{2 R}{R+R_{S}} V_{0}+\frac{R_{S}-R}{R_{S}+R} B  \tag{2.83}\\
\beta & =\frac{R_{S}-R}{R_{S}+R} \quad \text { then } \\
A & =(1-\beta) V_{0}+\boldsymbol{\beta} \tag{2.84}
\end{align*}
$$

The wave-flov dierrom is shown in Fir. 2.32. The eqn. (?.84) is the reneral exnression but it is useful to consider the following cases, as the delay-free path from $B$ to $A$ may cause problems when elements are interconnected tomether.
(i) If $R=R_{S}$ then $\beta=0$ and $A=V_{0}$ which is the 'wave-source' ceiven by Fettreis in [30] , (Fis. 2.33).
(ii) For an ideal voltage source, $R_{S}=0$ and hence $\beta=-1$ so that $A=2 V_{O}-B$ (Fig. 2.34), a condition given in [30].

### 2.6.3 Resistive Current Source

The resistive current source, which is illustrated in Fig. 2.35, can be defined as follows,

$$
\begin{align*}
& A=V+R I \\
& B=V-R I  \tag{2.85}\\
& I_{0}=G_{S} V+I
\end{align*}
$$

On eliminating $V$ and $I$ between the three equations, we find that

$$
\begin{equation*}
A=\frac{2}{\left(G+G_{S}\right)} I_{0}+\frac{G-G_{S}}{G+G_{S}} B \tag{2.86}
\end{equation*}
$$

Now let $\mathrm{E}=\mathrm{I}_{0} / \mathrm{I}_{\mathrm{S}}$ then

$$
A=\frac{2 G_{S}}{G+G_{S}} \quad E+\frac{G-G_{S}}{G+G_{S}} \quad B
$$

Let

$$
\begin{align*}
\alpha & =\frac{G-G_{S}}{G+G_{S}}=\frac{R_{S}-R}{R_{S}+R} \text { then } \\
A & =(1-\alpha) E+\alpha B \tag{2.87}
\end{align*}
$$

The form of eqn. (2.87) is similar to that of eqn. (2.84). Again scveral important cases arise,
(i) If $R=R_{S}$ then $\alpha=0$ and $A=E$, which is a wave-source.
(ii) For an ideal current source, $G_{S}=0$ then, on using eqn. (2.86), we find that [30], $A=2 R I_{0}+B$ for which the WFD appears in Fig. 2.36.

### 2.6.4 Terminating Imnedance

Let us consider an impedance 2 , shown in Fig. 2.37, whose properties car be written as follows,

$$
\begin{align*}
& \mathrm{A}=\mathrm{V}+\mathrm{RI} \\
& \mathrm{~B}=\mathrm{V}-\mathrm{RI} .  \tag{2.88}\\
& \mathrm{V}=\mathrm{Z} \mathrm{I}
\end{align*}
$$

Thus on eliminating $V$ and $I$ between the three equations we find that

$$
\begin{equation*}
\frac{B}{A}=\frac{Z-R}{Z+R} \tag{2.89}
\end{equation*}
$$

Again several important cases arise, however we shall concern ourselves at present with the terminating resistance, $R_{\mathrm{I}}$.
$\operatorname{Tnn} 2=n_{\mathrm{T}}$ nnd

$$
\begin{align*}
B & =\gamma A  \tag{2.00}\\
\text { where } \quad \gamma & =\left(R_{I}-R\right) /\left(R_{L}+R\right)
\end{align*}
$$

If $R=R_{L}$ then $\gamma=0$ and $B=0$, which renresents the ' wavesink' referred to by Fettweis in [30].

### 2.6.5 Intercomentions

Consider the interconnection of two ports with port resistanees $R_{1}$ and $R_{2}$ (fig. 2.38). The equations cefining the interconnoction are as follows

$$
\begin{align*}
{\left[\begin{array}{l}
A_{2} \\
B_{k}
\end{array}\right] } & =\left[\begin{array}{cc}
1 & R_{1} \\
1 & -R_{k}
\end{array}\right]\left[\begin{array}{l}
V_{1} \\
I_{k}
\end{array}\right] \quad K=1,2  \tag{2.91}\\
\text { and } \quad\left[\begin{array}{l}
V_{1} \\
I_{1}
\end{array}\right] & =\left[\begin{array}{ll}
1 & 0 \\
0 & -1
\end{array}\right]\left[\begin{array}{l}
V_{2} \\
I_{2}
\end{array}\right] . \tag{2.92}
\end{align*}
$$

We may eliminate $V_{k}$ and $I_{k}$, $K=1,2$ between eqnṣ. (2.91) and (2.92) to obtain [30].

$$
\left.\begin{array}{l}
B_{1}=A_{2}+\alpha\left(A_{2}-A_{1}\right)  \tag{2.93}\\
B_{2}=A_{1}+\alpha\left(A_{2}-A_{1}\right)
\end{array}\right\}
$$

where

$$
\alpha=\left(r_{1}-n_{2}\right) /\left(n_{1}+n_{2}\right)
$$

Fettreis calls this structure a 'two-port adaptor'. In the NaveFlow Diasrams derived using the two-port approach, the port resistances $R_{a}, R_{b}$ of a narticular elenent are related by a linear expression of the form $R_{a}=R_{b}+\theta$ or $R_{b}=R_{a}+\theta$ where $\theta$ denends on the element value ( $s$ ). Ye may thus say thet either $R_{a}$ or $R_{b}$ is arbitrary, but once one of them is defined, so also is the other. Thus at any intorconnoction we my bo cortain that either $R_{1}$ or $R_{2}$ is arbitrary and therfore we may set $R_{1}=R_{2}$. This substitution in eqn. (2.93) gives $B_{1}=A_{2}$ and $B_{2}=A_{1}$, that is a direct connection. We may sumarise this by saying that tro ports may be directly connected so long as their appropriate port resistances are equal.

Let us look at the number of discrete components necessary for the realisations riven in this chapter. The $W P D$ of a series $L$ (or C) and a shunt $C$ (or L) each require 1 delay and 1 multiplier which is the minimum. The number of adders used is 5 and it is knorm that an equivalent structure using adantors can be realised with 4 adders [34]. It is not difficult to show that the WWD of the reactive elements derived in this chanter can also be realised with only 4 adders. For ezample, let us consider a series inductor $I$ with $\alpha_{1}=0$. The corresponding equations are as follo:rs (see section 2.3 .3 ),
$\left.\begin{array}{l} \\ \text { where } \quad \begin{array}{l}B_{1}=S_{11}\left(A_{1}-A_{2}\right)+A_{2} \\ B_{2}=S_{22}\left(A_{2}-A_{1}\right)+A_{1} \\ \text { and }\end{array} \quad S_{22}=\left(1-\alpha_{2}\right) /\left(1+\alpha_{2^{z}}-1\right) \\ S_{11}=-S_{22^{z-1}}\end{array}\right\} \quad$ (2.94)

Now from egn. (2.94) we see that

$$
\begin{align*}
B_{1} & =S_{22^{z-1}}\left(A_{2}-A_{1}\right)+A_{2} \\
& =z^{-1}\left(B_{2}-A_{1}\right)+A_{2} \tag{2.95}
\end{align*}
$$

As $S_{11}+S_{12}=1$ and $S_{21}+S_{22}=1$ we can write the scattering equations as follows

$$
\left.\begin{array}{l}
B_{1}=\left(1-S_{12}\right) A_{1}+S_{12} A_{2} \\
B_{2}=S_{21} A_{1}+\left(1-S_{21}\right) A_{2}
\end{array}\right\}
$$

where $\quad S_{21}=\alpha_{2} S_{12}$

Therefore $B_{2}=\alpha_{2} S_{12}\left(A_{1}-A_{2}\right)+A_{2}$

$$
\begin{equation*}
=\quad \alpha_{2}\left(A_{1}-B_{1}\right)+A_{2} \tag{2.97}
\end{equation*}
$$

We may now realise eans. (2.95) and (2.97) with only 4 adders (Fig. 2.39).

In the realisation of tuned-circuits we have used 2 delays and 2 . multipliers which again is the minimum. The number of adders used is 7 which is a saving on the method of Fettreis [30]. Ho:jever, Crochiere [77] has realised a tuned-circuit usinc only 6 adders and this seems a minimum. Finally, the transmjssion-
line element hes been realised usine 1 multinlier and 2 delays. We shall see in Chanter 4 that we can combine the 2 delays together to form a delay of twice the duration rithout affecting the amplitude response of a network. The number of adders used was 3 and this number seems a minimum [30] [^8]

In this chanter we have been concerned only with the elements of a filter and their corresponding inve-Flow Dingrams. In the noxt two chanters we shall see how we may conrect NWD togother to nroduce structures that are anable of simal-fitterinz.


Fig. 2.1 General representation of a two-port network N .


Fig. 2.2 Wave-Flow Diagram (WFD) of general two-port .


Fig. 2.3 WFD of interconnection of two general two-ports .


Fig. 2. 5 WFD of Series Resistance with $\mathrm{S}_{22}=0$.


Fig. 2. 4 WFD of Series Resistance with $S_{11}=0$.


Fig. 2.6 WFD of Series Inductance with $\alpha_{1}=0$.


Fig. 2.7 WFD of Series Inductance with $\alpha_{3}=0$.


Fig. 2.8 WFD of Series Capacitance with $\alpha_{1}=0$.


Fig. 2.9 WFD of Series Capacitance with $\alpha_{3}=0$.


Fig. 2.10 Parallel-Tuned Circuit in Series Arm .


Fig. 2.11 WFD of Fig. 2.10 for Case I .


Fig. 2.12 WFD of Fig. 2.10 for Case II .


Fig. 2.13 Series-Tuned Circuit in Series Arm .


Fig. 2.14 WFD of Fig. 2.13 for Case I .


Fig. 2.15 WFD of Fig. 2.13 for Case II .


Fig. 2.17 WFD of Shunt Resistance with $S_{22}=0$.


Fig. 2.16 WFD of Shunt Resistance with $S_{11}=0$.


ほiç. 2.18 IFD of Shunt Capacitance with $\alpha_{1}=0$.


Fig. 2.19 NTD of Shunt Capacitance :iith $\alpha_{3}=0$.


Fis. 2.20 of Shunt Inductance with $\alpha_{1}=0$.

iig. 2.21 Wip shunt Inductance with $\alpha_{3}=0$.


Fig. 2.22 Series-Tuned Circuit in Shunt Arm .


Fig. 2.23 WFD of Fig. 2.22 for Case I .
${ }_{a}^{A^{-}}$


Fig. 2.24 WFD of Fig. 2.22 for Case II .


Fig. 2.25 Parallel-Tuned Circuit in Shunt Arm .


Fig. 2.26 WFD of Fig. 2.25 for Case I .


Fig. 2.27 WFD of Fig. 2.25 for Case II .


Fig. 2.28 Representation of a transmission-line element .


Fig. 2.29 WFD of Fig. 2.28 for Case I .


Fig. 2.30 WFD of Fig. 2.28 for Case II.


Fig. 2. 31 Resistive Voltage Source .


Fig. 2.32 WFD of Fig. 2.31.


Fig. $2.33 \begin{aligned} & \text { WFD of Fig. } 2.31 \\ & \text { with } R=R_{s}\end{aligned}$,


Fig. 2.34 WFD of Fig. 2.31 with $\mathrm{R}_{\mathrm{s}}=0$.


Fig. 2.35 Resistive Current Source .


Fig. 2.37 Terminating Impedance Z .


Fig. 2.38 Interconnection of two ports .


Fig. 2.39 WFD of Series Inductance with $\alpha_{1}=0$ realized with 4 adders only .

## 「nantor z

Desim and Sensitirioy Analysis oi UDF Imiteting DTIMT.

Contents:
3. 1 Desisn Procedure.
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## Chanter 3

Desion and Sonsitivitr Aralugis of Gve Digital Filters Imitatine Doublr-merminated Iossioss Iadder Met:orks.

### 3.1 Desim Procetume

Having establishod, in the last chapter, the wave-flo: diagrams of the pessive components and sources, we are ro\% in a position to consjder thoir intorconnection to form n complete ladder notrork.
 opproriate port nometization recistances aro equal. Ton anot
 daluy-fren nath flomirs only frow botom to ton of the wion ond (b) with a doly-free peth only from tom to bottom. Pespizable netrores may be corstructed trereforo so long as all tio constituent elements belon; to (3) or (b) but rot botr. There are thus tro Altornotive dixitnl structuras airulatins the nnalonan nowork. These are chomen in Fis. 3.1. It should te noted that it is necessary to choose the appropriate meve-non diagroms for the
 frec loons. These may occur ns inter-olemont loops or acrose the overall strusture throusin the cuccossive $S_{12}$ and $S_{21}$ patis. If: we begin our design procedure at the source-ond, the port resistance at the input of the first $L C$ element is constrained to he the souron nostronce. All suheonuent nort resistances in the LC filter are then also dcfined by virtue of the linear constraints on the port resistances and the element values. Since the valua of tho port rocistance at. the outrut of the last LC elcment is not generally the same as that of the termination, we must use the general UFD for the load resistance. Similar arguments annly to the design from the load-end but in this case ve must use the general $W F D$ for the voltage source.

Before considering an example of the design procedure, let us look at the transfer functions obtained rith the digital stmeturos of TiEG 3.1.

## (a)

The transfer function of a doubly-tominated two-nort is given by $[10]$,

$$
\begin{equation*}
H(n)=\frac{1}{\left(\Lambda-3 G_{I_{1}} \div ?_{\mathrm{S}}\left(5-\lambda \lambda_{\mathrm{L}}\right)\right.} \tag{3.1}
\end{equation*}
$$

where $G_{L}=1 / R_{L}$ and $A, B, C$ and $D$ are the network parameters determined by assuming the port currents to be entoring the networ?. In Fig. 3.1 ( $a$ ), the equivalent ratio $\hat{H}(n)$ is defined in terms of wave variables as $\mathrm{B}_{2} / \mathrm{V}_{0}$. The overall fave equations describing the two-port are as follows

$$
\begin{aligned}
& B_{1}=S_{11 A_{1}}+S_{12} A_{2} \\
& B_{2}=S_{21 A_{1}}+S_{22 A_{2}}
\end{aligned}
$$

with the terminal conditions $\mathrm{H}_{2}=0$

$$
\begin{array}{rlrl}
\text { and } & & A_{1} & =(1-\hat{\alpha}) V_{0}+\hat{\alpha} B_{1} \\
\text { mere } & \hat{\alpha} & =\left(R_{s}-R_{1}\right) /\left(I_{s}+n_{1}\right) *
\end{array}
$$

fron :hinc. :re find that

$$
\begin{equation*}
H(p)=\frac{(1-\hat{\alpha}) S_{21}}{\left(1-\hat{\alpha} S_{11}\right)} \tag{3.2}
\end{equation*}
$$

It has been establiched that in reneral (see section 2.2)

$$
\begin{align*}
& s_{11}=\delta / \beta  \tag{3.3}\\
& s_{21}=1 / \beta
\end{align*}
$$

Fhere

$$
\begin{aligned}
& \beta=\frac{1}{2}\left(\therefore+C R_{1}-B G_{L}+D R_{1} A_{L}\right) \\
& \delta=\frac{1}{2}\left(A-C R_{1}-B R_{L}+D R_{1} G_{L}\right)
\end{aligned}
$$

on combinine equs. (3.2) and (3.3) : have

$$
\begin{align*}
\hat{H}(\eta) & =\frac{1-\hat{\alpha}}{\beta-\hat{\alpha} \delta} \\
& =\frac{2 R_{1}}{R_{S}\left(C R_{1}-D R_{1} G_{L}\right)+R_{1}\left(\hat{A}-B A_{I}\right)} \\
& =\frac{2}{R_{S}\left(C-D R_{I}\right)+\left(A-D G_{I}\right)} \tag{3.4}
\end{align*}
$$

Comparing (3.1) and (3.4) we finally have

$$
\begin{equation*}
\hat{H}(p)=2 H(p) \tag{3.5}
\end{equation*}
$$

## Fontnote

* The condition $\Lambda_{2}=0$ renresents the sotterinc equation for a torminating resistance where the value of the anpropriate port rosirtanos has boon sot equal to that of the element (see section 2.6.4). The cther condition renresents the scatterinc equation for a resistive voltage source in its most ceneral fom (see section 2.6.2).

As the bilinear transformation, $n \rightarrow \frac{1-z-1}{1+z-1}$ leaves the level of the transfer function unalterea [5] we may conclude therefore that the transfor function of the diçital filter is twice that of the bilinearly transformed analogue filter. The multipli.cative constant, 2 , being the result of the transformation from voltages and currents to mue variables.

## (b) Desim from the source-ond

In Fig. $3.1(\mathrm{~b})$, the transfer function $\hat{H}(\mathrm{p})$ is defined as $\mathrm{B}_{2} / \mathrm{v}_{0}$.

The rave equations describing the two-port are as follows,

$$
\begin{aligned}
& B_{1}=S_{11} A_{1}+S_{12} \dot{A_{2}} \\
& B_{2}=S_{21} A_{1}+S_{22} A_{2}
\end{aligned}
$$

with the terminal conditions $\Lambda_{2}=\hat{\beta} B_{2}$ and $A_{1}=V_{0}$ where $\hat{\beta}=\left(R_{L}-R_{2}\right) /\left(R_{L}+R_{2}\right)$ from which we find that

$$
\begin{equation*}
H(p)=\frac{S_{21}}{1-\hat{\beta}^{S_{22}}} \tag{3.6}
\end{equation*}
$$

Recalling eqns. (2.4) and (2.5) we have from eqn. (3.6) the following

$$
\begin{align*}
\hat{H}(p) & =\frac{1}{\beta+\hat{\beta} \alpha} \\
& =\frac{R_{I}+R_{2}}{R_{L}(\beta+\alpha)+R_{2}(\beta-\alpha)} \\
& =\frac{1+\frac{R_{2}}{R_{L}}}{\left(A-B G_{I}\right)+R_{S}\left(C-D G_{L}\right)} \tag{3.7}
\end{align*}
$$

Comparine eqns. (3.7) and (3.1) cives finally that

$$
\begin{equation*}
\hat{H}(\mathrm{~F})=\left(1+\frac{R_{2}}{P_{L}}\right) \quad H(p) \tag{3.8}
\end{equation*}
$$

Thus we have shown that the transfer function of a wave digital filter will differ from that of the original analogue filter bilinearly transformed by a constant multiplicative factor of $\left(1+R_{2} / R_{L}\right)$.

Finally, let us look at the transfer function of a 'Fettweis-type' wave digital filter shown in Fig. 3.1(c). We have immediotely that

$$
\begin{aligned}
& \hat{H}(p)=\frac{B_{2}}{V_{0}}=S_{21} \\
& S_{21}=\frac{1}{\beta}=\frac{2}{\left(A+C R_{S}-B G_{L}+D R_{S} G_{L}\right)}
\end{aligned}
$$

and on comparing with eqn. (3.1) we have always

$$
\hat{H}(p)=2 H(p) .
$$

## 3.? A Filter Branle

For a simple nontrivial example a third-order Chebyshev normalized filter with $0.1 d B$ ness band ripple and equal terminations was chosen, the component values of which are show in Fig. 3.2 [6].

Let us design a wave digital filter from the source-end, that is we need to use elements that have been realized with a domvard delay-free path (or, equivalently, S2? divisible by z-1).

The design equations are as follows (Firs. 3.3),

$$
\begin{aligned}
& R_{1}=R_{S}=1 \\
& G_{1}=1 / R_{1}=1 \\
& G_{2}=G_{1}+C_{1}=2.0316 \\
& \alpha_{1}=G_{1} / G_{2}=0.4922 \\
& R_{2}=1 / G_{2}=0.4922 \\
& R_{3}=R_{2}+L=1.6396 \\
& \alpha_{2}=R_{2} / R_{3}=0.3002 \\
& G_{3}=1 / R_{3}=0.6099 \\
& G_{4}=G_{3}+C_{2}=1.6415 \\
& \alpha_{3}=G_{3} / G_{4}=0.3716 \\
& R_{4}=1 / G_{4}=0.6093 \\
& \alpha_{4}=\left(R_{L}-R_{4}\right) /\left(R_{L}+R_{4}\right)=0.2429
\end{aligned}
$$

The complete WFD is illustrated in Fig. 3.4. As a check on the validity of the method, the transfer function, $G(z)$, of the third-order digital filter structure of Fig. 3.4 was derived, into which was substituted the multiplier values found from the design procedure. It was found that

$$
\begin{equation*}
G(z)=K \frac{\left(1+z^{-1}\right)^{3}}{1+b_{1} z^{-1}+b_{2^{2}} z^{-2}+b_{3^{2}}} \tag{3.9}
\end{equation*}
$$

where
and

$$
\begin{aligned}
& b_{1}=\beta_{1}+\beta_{2}+\beta_{3} \beta_{4} \\
& b_{2}=\beta_{1} \beta_{2}+\alpha_{1} \alpha_{2}+\beta_{5} \alpha_{4}-\beta_{3} \beta_{4} \\
& b_{3}=\alpha_{1} \alpha_{2} \beta_{1}-\alpha_{4} \beta_{3}
\end{aligned}
$$

$$
\begin{aligned}
\beta_{1} & =\alpha_{3}-\alpha_{4}+\alpha_{3} \alpha_{4} \\
\beta_{2} & =2 \alpha_{1}+2 \alpha_{2}-\alpha_{1} \alpha_{2}-1 \\
\beta_{3} & =1-2 \alpha_{1}+\alpha_{1} \alpha_{2} \\
\beta_{4} & =\alpha_{3}+\alpha_{3} \alpha_{4}-1 \\
\beta_{5} & =1-2 \alpha_{2}+\alpha_{1} \alpha_{2} \\
K & =\alpha_{1} \alpha_{3}
\end{aligned}
$$

On substituting for $\alpha_{1}, \alpha_{2}, \alpha_{3}$ and $\alpha_{4}$ we have

$$
\begin{aligned}
\mathrm{b}_{1} & =0.3614 \\
\mathrm{~b}_{2} & =0.4644 \\
\mathrm{~b}_{3} & =-0.0073 \\
\text { and } \mathrm{K} & =0.1829
\end{aligned}
$$

The transfer function was then derived for the original LC filter as given in Fig. 3.2 and the bilinear transformation applied to it. It was found that

$$
H(p)=\frac{R_{T}}{a_{3} p^{3}+a_{2} p^{2}+a_{1} p+a_{0}}
$$

where

$$
\mathrm{a}_{3}=\mathrm{R}_{\mathrm{S}} \mathrm{R}_{\mathrm{I}}, \mathrm{C}_{1} \mathrm{C}_{2} \mathrm{I}
$$

$$
a_{2}=\left(C_{1} R_{3}+C_{2} R_{I}\right) L
$$

$$
a_{1}=L+R_{S} R_{I}\left(C_{1}+C_{2}\right)
$$

and.

$$
a_{0}=R_{s}+R_{L}
$$

and after applying the transformation $p \rightarrow \frac{1-z^{-1}}{1+z^{-1}}$ we obtain the expression
where

$$
\begin{equation*}
G^{\prime}(z)=K^{\prime} \frac{\left(1+z^{-1}\right)^{3}}{1+b_{1}^{1} z^{-1}+b_{2}^{1} z^{-2}+b_{3}^{1} z^{-3}} \tag{3.10}
\end{equation*}
$$

where

$$
\begin{aligned}
b_{3}^{\prime} & =\left(a_{2}+a_{0}-\left(a_{3}+a_{1}\right)\right) / b_{0}^{\prime} \\
b_{2}^{\prime} & =\left(3\left(a_{3}+a_{0}\right)-\left(a_{2}+a_{1}\right)\right) / b_{0}^{\prime} \\
b_{1}^{\prime} & =\left(3\left(a_{0}-a_{3}\right)+\left(a_{1}-a_{2}\right)\right) / b_{0}^{\prime}
\end{aligned}
$$

$$
\begin{gathered}
b_{0}^{\prime}=a_{0}+a_{1}+a_{2}+a_{3} \\
\text { and } K^{\prime}=R_{L} / b_{0}^{\prime} \\
\text { On substitutine for } R_{S}, R_{L}, L, C_{1} \text { and } C_{2} \text { we have } \\
b_{1}^{\prime}=0.3614 \\
b_{2}^{\prime}=0.4643 \\
b_{3}^{\prime}=-0.0073 \\
\text { and } K^{\prime}=0.1136
\end{gathered}
$$

Thus $G(z)$ and $G^{\prime}(z)$ differ, as expected, only in the constant multiplicative factor.

The desien procedure we have illustrated in this section is guite systematic so that ladder filters of any size may be transformed to equivalent disital filters. With the aid of filter design tables [6][8] it is an easy matter to calculate the digital multiplier values and hence the digital circuit required.

### 3.3 The Derivation of MAP Sensitivity Characteristics

### 3.3.1 Introduction

It has been shom that [33][50], as a result of preserving the relationships between the incident and reflected waves, the proverty of low attenuation distortion in the passband of a doubly-terminated lossless ladder is transmitted to the digital filter. In this section, we continue the discussion by examining the implications of Orchard's arsument [23] when applied to Wave Digital Filters and apply it to the problem of finding an expression for the first-order attenuation sensjitivity to multiplier variations in tems of the analogue component sensitivities. The significence of this expression is then discussed for Fettweis Vave Digital Filters and for those desirned with the procedure in section 3.1. It is further shom that the multiplier sensitivities are not at their expected zero value at the noints of maximum nseudo-power transfer (MAP). This fact becomes significant when we consider the accumulation of round-off noise [13][21].
3.3.2 Sensitivity of the Attenuation to Multinlier Variations.

In this section, we shall derive an expression for the sensitivity wi thout the use of concepts such as pseudo-power and pseudolosslessness described by Fettweis [33] and explained in Chapter 1. Further, the transfer function we shall use is not the sane as that used by Fettweis in [37] It was felt that the ratio of
outnut to innut, namely $\mathrm{B}_{2} / \mathrm{T}_{0}$, ws the mat locical choice whilct Fettweis used the function

$$
\frac{B_{2}}{V_{O}} \sqrt{\frac{R_{S}}{R_{L}}}
$$

Consider a doubly-terminated lossless ladder aith elements $R_{S}, R_{L}, L_{1}, I_{2}, \ldots . I_{m}, C_{1}, C_{2}, \ldots . C_{n}$, that is a total of $N=m+n+2$ elements. The transfer function $V$ out $/ V$ in ve shall denote by $H(p)$.

Suppose now that the wave digital filler is derived from the analoeve filter ahove and let its multinlior values be denoted by $\alpha_{1}, \alpha_{2}, \ldots \alpha_{n-1}$. It is clear thet, as a result of the desim nrocess, any multinlier value $\alpha=\alpha_{i}$ is a function of the original passive varisbles. The transfor funstion of this digeital filter will be denoted by $G(z)$. It has been shom, in section 3.1, that if we apnly the bilinear transfomation
 rational function in $z^{-1}$ as $\hat{G}(z)$ then

$$
\begin{equation*}
G(z) \equiv K \hat{T}(z) \tag{3.11}
\end{equation*}
$$

where K is indenendent of z but mey depend on the nascive components.

Therefore for any multiplier $\alpha_{\kappa}$, we have from oqn. (3.11)

where $S_{x}^{y}=\frac{x}{y} \frac{\partial y}{\partial z}$ is the relative sensi.tivi.ty function [?2].

Hence we may state imediately the following

$$
\left.\hat{G}(z) \equiv H(p)\right|_{p}=\frac{1-z^{-1}}{1+z^{-1}}
$$

and therefore it follows that *

## Footnote:

* $H(p)$ is a function of the pessive component variables in addition to beins a function of the complex frequency variable n. To chall refon to this finction, for convenjonce, as it. Similarly, ve shall drop the variablezfrom the functions $G(z)$ and $\hat{G}(z)$.

$$
\begin{aligned}
& S_{x}^{\hat{G}}=\left.S_{x}^{H}\right|_{\underline{D}}=\frac{1-z^{-1}}{1+z^{-1}}
\end{aligned}
$$

where $x$ is a passive component.

We kno: that each multiplior is a function of the passive comonent valuns. Je shall assume for the moment that we may express each nessive variable as a function of the multiplier variables (in the next section, we shall see that this assumption cannot be made in the case of delay elemonts). Ne can therefore write, by virtue of the chain rule of differentiation of function of several variables [78],

$$
\begin{align*}
\frac{\partial H}{\partial \alpha_{k}}=\frac{\partial H}{\partial R_{L}} \frac{\partial R_{L}}{\partial \alpha_{k}}+\sum_{\rho} \frac{\partial H}{\partial L_{p}} \frac{\partial L_{p}}{\partial \alpha_{k}} & +\sum_{q_{1}} \frac{\partial H}{\partial C_{q}} \frac{\partial C_{q}}{\partial \alpha_{k}}  \tag{3.13}\\
& +{\frac{\partial H}{\partial R_{s}} \frac{\partial R_{s}}{\partial \alpha_{k}}}
\end{align*}
$$

On multinlying both sides of eqn. (3.13) by $\frac{\alpha_{k}}{H}$ we obtain,

The sensitivity invariant for a transfer function $H(\underline{p})$ is giver, hur [70],

$$
\begin{equation*}
S{ }_{R_{S}}^{H}+S \frac{H}{R_{L}}-\sum_{q} S_{C_{q}}^{H}+\sum_{p} S{\underset{L}{p}}_{H}^{L_{p}}=0 \tag{3.15}
\end{equation*}
$$

Thus we may eliminate one term from enn. (3.14). Let us choose $S{ }_{R_{S}}^{H}$ in which case ean. (3.14) becomes
$S_{\alpha_{k}}^{H}=S{ }_{R_{L}}^{H}\left(S_{\alpha_{k}}^{R_{L}}-S_{\alpha_{k}}^{R_{S}}\right)+\sum_{p} S_{L_{p}}\left(S_{\alpha_{k}}^{L_{p}}-S_{\alpha_{k}}^{R_{s}}\right)$

$$
+\sum_{q_{v}} S_{C_{q}}^{H}\left(S_{\alpha_{k}}^{C_{q}}+S_{\alpha_{b}}^{b_{s}}\right)
$$

Noting that $S_{x}^{z y}=S_{x}^{z}+S S_{x}^{q}$ and $S_{x}^{z / y}=S_{x}^{z}-S_{x}^{y}$ we find that eqn. $(3.16)$ can be written as follows,

We are particularly interested in the attenuation function - Log $|\mathrm{II}|$ and the sensitivity function associated with it, namely
$S_{\alpha_{k}}^{|r|}=\operatorname{Re} S_{\alpha_{k}}^{H}$. On taking the real part of eqn. (3.17) and noting that expressions like $S_{\alpha_{k}} / R_{s}$ are real functions we obtain the result,

Ve are now in a nosition to ammy Orohard's argment (soe Chapter 1) to eqn. (3.18) at points of maximum power transfer. At these points,

$$
S{ }_{L_{p}}^{|H|}=S_{C_{q}}^{|H|}=0 \text { and }
$$

thus
and further for a well-desiened filter we mieht also expect eqn. (3.19) to hold approximately at other points in the passband.

Taking the real part of eqn. (3.12) we have

$$
\begin{align*}
{ }_{S \alpha_{k}}^{|\sigma|} & =S_{\alpha_{k}}^{K}+{ }_{S \alpha_{k}}^{|\hat{f}|} \\
& =S_{\alpha_{k}}^{K}+S_{\alpha_{k}}^{|H|} \tag{3.20}
\end{align*}
$$

and on substituting for $S_{\alpha_{k}}^{|y|} \mid$ from eqn. (3.19) into eqn. (3.20) ve obtain the expression

$$
\begin{equation*}
S_{\alpha_{k}}^{|\sigma|}=\frac{K}{S_{\alpha_{k}}}+S_{R_{L}}^{|\mathrm{K}|} S_{\alpha_{k}}^{R_{\mathrm{L}} / R_{S}} \tag{3.21}
\end{equation*}
$$

It was shom in Chapter 1 that $S_{\mathrm{S}_{\mathrm{L}}}^{|\mathrm{H}|}=\frac{1}{2}$, thus

$$
\underset{S_{\alpha_{k}}^{|G|}}{\mid G} \quad \begin{array}{r}
K  \tag{3.22}\\
S_{\alpha_{k}}
\end{array}+\frac{1}{2} S_{\alpha_{k}}^{R_{L} / R_{S}}
$$

We shall now discuss the terms $S_{\alpha_{k}}^{K}$ and $S_{\alpha}{ }_{R_{j}} / D_{S}$ for different wave realizations. We shall consider Fettweis type realizations and refer to them as having been designed usinç Method A. In addition, we shall consider the Wave Digital Filters discussed in section 3.1. These will be referred to as having
been designed using liethod $B$.

For the sake of argument, we shall use a general odd-order lov-pass filter with zeros at infinity (Fig. 3.5).

## I: Method A desirned from source-end

In this case, $K=2$ (see section 3.1) and therefore eau. (3.22) becomes

$$
\begin{equation*}
S_{\alpha_{k}}^{\left|T_{\tau}\right|}=\frac{1}{2} S_{\alpha_{k}}^{R_{L} / R_{S}} \tag{3.23}
\end{equation*}
$$

Consider the $: T D$ of a digital filter (Fie. 3.6) designed from the analogue network of Fir. 3.5. The necessary dosicn equations are as follows [34] (see Chanter 1)

$$
\begin{aligned}
\alpha_{1} & =\varepsilon_{1} / s_{3} \\
\alpha_{2} & =r_{3} / r_{5} \\
\alpha_{3} & =\varepsilon_{5} / g_{7} \\
\alpha_{4} & =r_{7} / r_{9} \\
& \vdots \\
\alpha_{2 n-1} & =2 E_{4 n-3} / g \\
\alpha_{2 n} & =2 \varepsilon_{4 n-2} / E \\
\text { where } \quad E & =E_{4 n-3}+E_{4 n-2}+g_{4 n-1} \text { and } R_{s}=r_{1}, R_{L}=r_{4 n-1}
\end{aligned}
$$

Elimination of all internal port resistances from these equations produces the following expression for the ratio of termination resistances

$$
\frac{R_{I_{1}}}{R_{S}}=\frac{\alpha_{1} \alpha_{3} \cdots \cdots \cdots \alpha_{2 n-3} \alpha_{2 n-1}}{\alpha_{2} \alpha_{4} \cdots \alpha_{2 n-2}\left(2-\alpha_{2 n-1}-\alpha_{2 n}\right)} \text { (3.24) }
$$

Hence we have

$$
S_{S} / \mathrm{G} \left\lvert\,=\frac{1}{2} S_{\alpha_{p}}^{R_{L} / R_{S}} \alpha_{p}=\left\{\begin{array}{cc}
\frac{1}{2} & \text { for } p \text { odd }(\neq 2 n-1) \\
-\frac{1}{2} & \text { for } p \text { even }(\neq 2 n) \\
\frac{1}{2}\left(2-\alpha_{2 n}\right) /\left(2-\alpha_{2 n-1}-\alpha_{2 n}\right) \text { for } n=2 n-1 \\
\frac{1}{2} \alpha_{2 n} /\left(2-\alpha_{2 n-1}-\alpha_{2 n}\right) & \text { for } n=2 n
\end{array}\right\}\right. \text { (3.25) }
$$

## II: Wethod A desimed from load-end

For this case $K=2$ also, and therefore eqn. (3.23) applies.

The VIFD of the dicital filter designed using this method, from the networls of Fig. 3.5, appears in Fic. 3.7.

The necessary design equations are as follows,

$$
\begin{aligned}
\alpha_{1} & =E_{1} / \varepsilon_{3} \\
\alpha_{2} & =r_{3} / r_{5} \\
\alpha_{2 n-2} & =r_{4 n-5} / r_{4 n-3} \\
\alpha_{2 n-1} & =2 g_{4 n-3} / g \\
\alpha_{2 n} & =25_{4 n-2} / g \\
\text { where } \quad E & =E_{4 n-3}+E_{4 n-2}+E_{4 n-1} \\
\text { and } \quad R_{I} & =r_{1}, R_{s}=r_{4 n-1}
\end{aligned}
$$

Elimination of all intermal port resistances from these expressions yields the followins rosult

$$
\begin{equation*}
\frac{R_{L}}{R_{S}}=\frac{\alpha_{2} \alpha_{4} \cdots \cdots \alpha_{2 n-2}\left(2-\alpha_{2 n-1}-\alpha_{2 n}\right)}{\alpha_{1} \alpha_{3} \cdots \cdots \cdots \alpha_{2 n-3} \alpha_{2 n-1}} \tag{3.26}
\end{equation*}
$$

which is seen to be the reciprocal of eqn. (3.24). It follows therefore that

$$
S_{S_{p}|G|}^{\alpha_{p}}=\left\{\begin{array}{ccc}
-\frac{1}{2} & \text { for } p \text { odd } & (\neq 2 n-1) \\
\frac{1}{2} & \text { for } p \text { cven } & (\neq 2 n) \\
\frac{1}{2}\left(\alpha_{2 n}-2\right) /\left(2-\alpha_{2 n-1}-\alpha_{2 n}\right) & \text { for } p=2 n-1 \\
-\frac{1}{2} \alpha_{2 n} /\left(2 \cdot-\alpha_{2 n-1}-\alpha_{2 n}\right) & \text { for } p=2 n
\end{array}\right\} \text { (3.2.7) }
$$

## III: Method B desioned from source-end

For this realisation (Fir. 3.8) it has already been show that

$$
\begin{equation*}
\mathrm{K}=1+\frac{\mathrm{r} 2 \mathrm{n}}{\mathrm{R}_{\mathrm{L}}} \tag{3.28}
\end{equation*}
$$

There $r_{2 n}$ is the port resistance of the load. Moreover, we hove the desim expressions,

$$
\begin{aligned}
\alpha_{1} & =E_{1} / g_{2} \\
\alpha_{2} & =r_{2} / r_{3} \\
\alpha_{3} & =g_{3} / \xi_{4} \\
& \vdots \\
\alpha_{2 n-1} & =g_{2 n-1} / \xi_{2 n}
\end{aligned}
$$

and also $\quad R_{s}=r_{1}$

On eliminating all internal port resistances we find that

$$
\begin{equation*}
\frac{\mathrm{R}_{\mathrm{I}}}{\mathrm{R}_{\mathrm{S}}}=\frac{\alpha_{1} \alpha_{3} \ldots \ldots \alpha_{2 n-1}\left(1+\alpha_{2 n}\right)}{\alpha_{2} \alpha_{4} \ldots \alpha_{2 n-2}} \frac{\left(1-\alpha_{2 n}\right)}{(1)} \tag{3.29}
\end{equation*}
$$

As a consequence of this me obtain

$$
S \alpha_{\alpha_{p}}^{R_{L} / R_{S}}=\left\{\begin{array}{cl}
1 & \text { for p odd } \\
-1 & \text { for } p \text { even }(\dot{t} 2 n) \\
2 \alpha_{2 n} /\left(1-\alpha_{2 n}^{2}\right) & \text { for } p=2 n
\end{array}\right\}(3.30)
$$

Now consider the following expression,

$$
\alpha_{2 n}=\frac{R_{L_{1}}-r_{2 n}}{R_{I}+r_{2 n}}
$$

from which wo may deduce that

$$
\begin{equation*}
\frac{r_{2 n}}{R_{L}}=\frac{1-\alpha_{2 n}}{1+\alpha_{2 n}} \tag{3.31}
\end{equation*}
$$

Therefore on combining eqns. (3.28) and (3.31), we find that

$$
K=\frac{2}{1+\alpha_{2 n}}
$$

Hence

$$
\left.S_{\alpha_{p}}^{K} \quad \begin{array}{ll}
0 & p \neq 2 n  \tag{3.32}\\
\frac{-\alpha_{2 n}}{1+\alpha_{2 n}} & p=2 n
\end{array}\right\}
$$

Combinine cans. (3.22), (3.30) and (3.32) gives. the following,

$$
{ }_{S}{ }_{\alpha_{p}}|G|=\left\{\begin{aligned}
\frac{1}{2} & \text { for } p \text { odd } \\
-\frac{1}{2} & \text { for } p \text { even }(\neq 2 n) \\
\alpha_{2 n}^{2} /\left(1-\alpha_{2 n}^{2}\right) & \text { for } \underline{p}=2 n
\end{aligned}\right\} \text { (3.33) }
$$

## IV: hethod $P$ desimed from load-end

For this realization (Fin. 3.0) it has beon shom that $K=2$ and therefore eqn. (3.23) applies. The desi.gn equations are as follows,

$$
\begin{aligned}
\alpha_{1} & =g_{1} / \xi_{2} \\
\alpha_{2} & =r_{2} / r_{3} \\
\alpha_{2 n-1} & =\xi_{2 n-1} / \xi_{2 n} \\
\text { and } \alpha_{2 n} & =\left(R_{s}-r_{2 n}\right) /\left(\Omega_{s}+r_{2 n}\right) \quad \text { also } R_{L}=r_{1}
\end{aligned}
$$

On eliminating all internal port resistances we find that

$$
\frac{\mathrm{R}_{\mathrm{L}}}{\mathrm{R}_{\mathrm{S}}}=\frac{\alpha_{2}}{\alpha_{1}} \frac{\alpha_{4} \cdots \cdots \alpha_{2 n-2}}{\alpha_{3} \cdots \cdots \alpha_{2 n-1}} \frac{\left(1-\alpha_{2 n}\right)}{\left(1+\alpha_{2 n}\right)}(3.31)
$$

It is noted that the right-hand-side of eqn. (3.34) is the reciprocal of eqn. (3.29) and therefore we have immodiatoly that

$$
s_{\alpha_{p}}^{|G|}\left\{\begin{aligned}
-\frac{1}{2} & \text { for } p \text { odd } \\
\frac{1}{2} & \text { for } p \text { even }(\neq 2 n) \\
-\alpha_{2 n} /\left(1-\alpha_{2 n}^{2}\right) & \text { for } p=2 n
\end{aligned}\right\} \text { (3.35) }
$$

Finally, in this section let us examine those sensitivities that are not equal to $\pm \frac{1}{2}$.

We may consider I and II together as the correspondinc absolute values of sensitivity are equal. We have that

$$
{ }_{S_{\alpha}}^{|G|}=\frac{\frac{1}{2}\left(2-\alpha_{2 n-1}\right)}{2-\alpha_{2 n-1}^{-\alpha_{2 n}}}
$$

and

$$
S{ }_{\alpha}^{|G|}=\frac{\frac{1}{2} \alpha_{2 n}}{2-\alpha_{2 n-1}-\alpha_{2 n}}
$$

We vould like these sensitivities to be as small as possible. Consider, for example, the values of $\alpha_{2 n-1}$ and $\alpha_{2 n}$ that minimise the sum of squares of the moduli sensitivities. That is we wish to find positive $\alpha_{2 n-1}$ and $\alpha_{2 n}$ such that
 By observinf the form of in, namely

$$
\frac{1}{2} \frac{\left(\alpha_{2 n}^{2}-2 \alpha_{2 n}+2\right)}{\left(2-\alpha_{2 n-1}-\alpha_{2 n}\right)^{2}}
$$

ve moy easily deduce that $\alpha_{2 n}$ and $\alpha_{2 n-1}$ shorle be as close to zero as is possible.

In III the sensitivity of interest is $\begin{aligned} & |G| \\ & \rho_{2 n}\end{aligned}=\alpha_{2 n}^{2} /\left(1-\alpha_{2 n}^{2}\right)$ and this may be made small by ensurine that $\alpha_{2 n}$ is small. Finally, in IV where $\frac{|G|}{2 n}=-\alpha_{2 n} /\left(1-\alpha_{2 n}^{2}\right)$ we may again be certain of lo: sensitivity if $\alpha_{2 n}$ is small, ideally, of course, we should lite $\alpha_{2 n}$ to be as close to zero as is possible.

The general conclusion of this section is that the sensitivity of attenuation characteristic to multiplier variations is not generally zero at points of mazimum power transfer (HAP). Although the results have been derived for an all-nole low-nass filter jit is an easy mattor to extend tho armment to cover all LC ladder filtors. For a tuncd-circuit (see Chepter 2), one of the multipliers is a ratio of nort resistaness and therefore the method of this section may be anolied. The second multiplier is a function only of the $L$ and $C$ in the tuned-circuit and therefore, by virtue of eqn. (3.22), the attenuction sensitivity is zero at mp points.

### 3.3.3 Sensitivity of the Attomuntion to Dolew Pement Varintions

Aithoush, as we have shown, the first-order sensitivities to multiplier variations are not cenerally zero at MP, we shall see that the sencitivities to delay variations are zero, excent in the case of tuned-circuits. In practice, however, we can be certain.
that the delay olement vill not contribute to deviations in the attenuation characteristic and therefore sensitivities to such elemonts are only of theoratical intorest.

Consider first a series inductor $I$ whose impedence $Z=p I$. Let the tronefer function of the analoge notnore be $\underline{I}(n)$ and that of the derived dicitol filter be $G(z)$ then

$$
\left.G(z) \equiv \operatorname{IH}(p)\right|_{n=\frac{1-z^{-1}}{1+z^{-1}}}
$$

and if is indeorndent of frequency.



 dolay brersh nar tho dixital complex frequonor vamiable. If :r not anyly tine bilinear trenaformation to on impodance $Z=p i$ then

$$
\begin{equation*}
\mathrm{pL} \rightarrow \frac{1-z^{-1}}{1+z^{-1}} \quad \mathrm{~L}=\frac{1-D_{0}}{1+D_{0}} \quad \mathrm{~L}, \mathrm{sey} \tag{3.37}
\end{equation*}
$$

Motice that $D_{0}$ is not ronosenrily the sare as $D$ in then sense thet thore mer be a furctionel relationshin betreen $I$ and $D_{0}$ such that, elvoneh $D=\eta_{c}, \frac{\partial D}{\partial D_{0}} \neq 1$. This is imnortant $s=$ we shall soc prosently, For exenvle consider

$$
\begin{aligned}
& D=D_{0}+\left(z^{-1}-D_{0}\right) D_{0} \\
& D_{0}=z^{-1}, \text { honern } D=D_{0} \text { rot } \\
& \frac{\partial D}{\partial D_{0}}=1+z^{-1}-2 D_{0} \\
& \text { and when } D_{0}=z^{-1}, \frac{\partial D}{\partial D_{0}}=1-z-1
\end{aligned}
$$

「o find the relationsiter botwon $D$ and $D_{0}$, we need to derive the THD of the seriss induntor $I$ in terms of the trensformetion $p \rightarrow \frac{1-D_{0}}{1 \div D_{0}}$ and find the relationship betreen the actual realised delay imanch and $D_{C}$. In this ase, it is easily shom that $D=D_{0}$ but we shall see in the case of tuned-circuits that this is not senerally so.

As pl is the independent variable and $D_{O}$ and $D$ are dependent we can urite

$$
\begin{equation*}
S_{p L}^{G}=S_{D}^{G} S_{D_{0}}^{D} S_{p L}^{D_{0}} \tag{3.38}
\end{equation*}
$$

However, as $G=k H$ when $D=D_{0}$ eqn. (3.38) becomes
as

$$
\begin{align*}
& S_{p L}^{H}=S_{D}^{G} S_{p}^{D_{O}}  \tag{3.39}\\
& p=\frac{1-D_{O}}{1+D_{O}} \quad \text { it follows that } \\
& S_{p}^{D_{O}}=\frac{-2 n}{1-n^{2}}=\frac{-2 i \Omega}{1+\Omega^{2}}=-j \operatorname{sinvT}
\end{align*}
$$

where $\Omega=\tan \frac{W T}{2}$ is the analogue frequency variable and hence

$$
\begin{equation*}
S_{D}^{G}=j S_{L}^{H}\left\{\frac{1+\Omega^{2}}{2 \Omega}\right\}=j \operatorname{cosec} v^{T} S_{L}^{H} \tag{3.40}
\end{equation*}
$$

We are interested in $S_{D}^{|r|}$ wich may be written as $\operatorname{jim} S{ }_{D}^{G}$ (see Chapter 1).

Therefore eqn. (3.n0) yields finally the result

$$
\begin{equation*}
S{ }_{D}^{|G|}=j S \underset{L}{|F|} \frac{1+\Omega^{2}}{2 \Omega}=j \operatorname{cosec} \pi T S{ }_{L}^{|H|} \tag{3.41}
\end{equation*}
$$

 rind thet

$$
S \underset{D}{|\sim|}=0
$$

In a similar vay, we can derive equivalent expressions for a series canacitor, a shunt inductor and a shunt copacitor.

Jet us now consider a series-tuned circuit in the shunt-arm. Then the impedance $Z$ is given by the expression

$$
z=p L+\frac{1}{p C}
$$

Here we have two delay elements $D_{1}$ and $D_{2}$ which are related individually to ni and to $1 / \mathrm{pc}$. Wh may write genemaly that
and




$$
\begin{aligned}
& {\left[\begin{array}{l}
\mathrm{S}_{\mathrm{F}} \\
\mathrm{D}_{1} \\
\mathrm{~S}_{\mathrm{G}} \\
\mathrm{D}_{2}
\end{array}\right]=\left[\begin{array}{ll}
\alpha & \beta \\
\gamma & \delta
\end{array}\right]^{-1}\left[\begin{array}{l}
\mathrm{S} \\
\mathrm{~S} \\
\mathrm{I} \\
\mathrm{~S} \\
\mathrm{~S} \\
\mathrm{C}
\end{array}\right]}
\end{aligned}
$$

Thus wa haven that a mationchin aorists hetwon the LS sensitivities and the delay sensitivities．To find the reletionshins between $D_{1}, D_{2}$ and $D_{L}$ and $D_{C}$ ne need to derive the rad of the series－tuned circuit in the shunt arm using the transformations

$$
\underline{O L} \rightarrow \frac{1-D_{L}}{1+D_{I}} \quad \mathrm{~L} \quad \text { and } \quad \underline{D C} \rightarrow \frac{1-D_{C}}{1+D_{C}} \quad C
$$

It is found that，（see Appendix I）

$$
\left.\begin{array}{rl}
D_{1} & =\frac{\frac{1}{2}\left(D_{\mathrm{I}}+D_{C}\right)}{1+\frac{1}{2}\left(D_{I}-D_{C}\right)} \\
D_{1} D_{2} & =\frac{-\frac{1}{2}\left(D_{I_{1}}-D_{C}\right)+D_{I_{1}} D_{C}}{1+\frac{1}{2}\left(D_{L}-D_{C}\right)} \tag{3.41}
\end{array}\right\}
$$

It is easily checked that，when $D_{\mathrm{I}}=D_{C}=z_{-1}$ ，then $D_{1}=z^{-1}=D_{2}$ ．

If we obtain $D_{2}$ by the ratio of these last two equations then

$$
\begin{equation*}
D_{2}=\frac{-\frac{1}{2}\left(D_{T_{1}}-D_{C}\right)+D_{I} D_{C}}{\frac{1}{2}\left(D_{L}+D_{C}\right)} \tag{3.45}
\end{equation*}
$$

It is easily show that, on using first eqn. in (3.41) and eqn. (3.4.5),

$$
\begin{aligned}
& S \begin{array}{l}
D_{1} \\
D_{\mathrm{L}}
\end{array}=\frac{1}{2}(1-z-1) \\
& S \begin{array}{l}
D_{1} \\
D_{c}
\end{array}=\frac{1}{2}(1+z-1) \\
& S \begin{array}{l}
D_{2} \\
D_{\mathrm{L}}
\end{array}=\frac{1}{2}(1-z)
\end{aligned}
$$

and $\quad S \frac{D_{2}}{D_{C}}=\frac{1}{2}(1+z)$
Now $\quad S{ }_{p}^{D_{\mathrm{L}}}=S{\underset{p}{\mathrm{D}}}_{\mathrm{D}_{\mathrm{C}}}=-j \sin \mathrm{~T}$
therefor s on substituting from eqns. (3.45) and (3.47) into en. (3.43) we have

$$
\begin{aligned}
& {\left[\begin{array}{c}
S \\
S_{1}^{G} \\
D_{1} \\
S \\
D_{2}
\end{array}\right]=k\left[\begin{array}{cc}
(1+z) & -(1-z) \\
-\left(1+z^{-1}\right) & \left(1-z^{-1}\right)
\end{array}\right]\left[\begin{array}{c}
S \\
I \\
\\
S \\
S \\
c
\end{array}\right]} \\
& \text { where } k=\frac{1}{2} \operatorname{cosec}^{2} v T .
\end{aligned}
$$

We may therefore write the following, since $S{ }_{S}^{|c|}=j \operatorname{Im} S{ }_{D_{i}}^{G}$,

$$
\begin{aligned}
&{ }_{S}^{|G|} \frac{j k}{D_{1}}=\left\{\operatorname { s i n } T \left(S \underset{L}{|H|}+\underset{c}{|H|}+(1+\cos T) \Phi S \frac{\Phi}{I}\right.\right. \\
&\left.+(\cos w T-1) \Phi S \frac{\Phi}{c}\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
& \left.+(1-\cos r T) \Phi \mathrm{S} \frac{\Phi}{\mathrm{c}}\right\}
\end{aligned}
$$

Where $\quad \Phi=\arg H$.
and therefore at MAP pts. we have

Thus we have seen that for a series-tuned circuit the attenuation sensitivity to delay variation is not generally zero at points of maximm nover trangfer. Tn a similar manner one can derive the corresponding formulae for the other tuned-circuits where the basic problom is one of finding the relationshin betmeen $D_{1}, D_{2}$ and $D_{L}$ and $D_{C}$ and solving ean. (3.12). It is noted that. eqns. (3.42) and (3.43) hold for nny tuned-circuit element. Furthemore, these equations cannot be used at zero frequency since when $w=0, k=\frac{1}{2} \operatorname{cosec}{ }^{2} w^{m}=\infty$. However, at zero frequency all sensitivity functions are real and therefore ${ }_{S}{ }_{D_{i}}^{|G|}=\operatorname{Im} S{ }_{D_{i}}^{G}=0$, for any delay element.

If we had used the equivalence between $a$ tuned-circuit and an appronriately terminated cascade of two unit elements, then it is easy to see that there is a one to one correspondence between $D_{L}$ and $D_{1}$ and between $D_{C}$ and $D_{2}$. Thus if we use the wave-flow diafrar for a tmed-circuit derived by Fett:feis [30], we can be certain that the attenuation sensitivity to delay variation is zero at points of maximum pseudopower transfer. However, such wavo-ilow diagrens use more adders then those discussed in soctions 2.3 and 2.4 [77] and therefore, as delay sensitivity is of theoretical interest only, the latter structure is to be preferred.

### 3.4 Sensitivity Eyamnle

In this section, a third-order nornalizcd Chebyshev filter with 0.1 dB passband ripnle and equal terminations is analysed, first in its analogue form then in its direct digital. form and finally in each of three different wave realisations.

## (a) Analopue filter

The netrork is shom in Fig. 3.10 and the element values were taken from reforence [5]. The filter wes onolyeed
at 19 froquency points and the amplitude characteristic (Fig. 3.11) ws plotted agrinst $2 \tan ^{-1} \Omega$ so that it may be compared with the amplitude response of the digital equivalent. In addition the attenuation sensitivity was plotted for each component against $2 \tan ^{-1} \Omega$ (Mis. 3.12).

## (b) Direct Smthesis Dicital Pilter

On applyins the bilinear transformation $p \rightarrow \frac{1-z^{-1}}{1+z^{-1}}$ to the analogue transfer function $H(n)$ we obtain

$$
G(z)=K \frac{\left(1+z^{-1}\right)^{3}}{\left(1+b_{1} z^{-1}+b_{2} z^{-2}+b_{3} z^{-3}\right)}
$$

where

$$
\begin{aligned}
\mathrm{K} & =0.1135 \\
\mathrm{~b}_{1} & =0.3514 \\
\mathrm{~b}_{2} & =0.4644 \\
\text { and } \mathrm{b}_{3} & =-0.0073
\end{aligned}
$$

The signal-floy sramh of $G(z)$ appears in Fis. (3.13).
The attenuation sensitivity was plotted for each multiplier ( $b_{1}, b_{2}$ and $b_{3}$ ) over the entire frequency spectrum (Fis. 3.14).

## (c) Wothod A desimed from the source-ond

The Wave-plow Diagmin is shom in Pis. 3.15 ond the desion equations are as follows:

$$
\begin{aligned}
r_{1}=R_{S}, g_{2} & =C_{1}, r_{4}=L, E_{6}=C_{2}, r_{7}=R_{L} \\
g_{3} & =g_{1}+g_{2}=2.0316 \\
\alpha_{1} & =g_{1} / \varepsilon_{3}=0.4922 \\
r_{5} & =r_{3}+r_{4}=1.6396 \\
\alpha_{2} & =r_{3} / r_{5}=0.3002 \\
\xi & =g_{5}+\delta_{6}+\varepsilon_{7}=2.6415 \\
\alpha_{3} & =2 \varepsilon_{5} / g=0.4618 \\
\alpha_{4} & =2 \varepsilon_{6} / \varepsilon=0.7811
\end{aligned}
$$

The wave digital filter with the multiplier values above
was then analysed and its attenuation sensitivity was plotted over the frequency spoctrum to each of the multipliers (Fig. 3.16).

## (d) Method B desicned from the source-end

The wave-flow diagram is show in Fis. 3.17 and the design equations are as follows:

$$
\begin{aligned}
& r_{1}=r_{s} \\
& g_{2}=s_{1}+C_{1}=2.0316 \\
& \alpha_{1}=E_{1} / r_{2}=0.4972 \\
& r_{3}=r_{2}+L=1.6396 \\
& \alpha_{2}=r_{2} / r_{3}=0.3002 \\
& s_{4}=\varepsilon_{3}+c_{2}=1.6415 \\
& \alpha_{3}=\varepsilon_{3} / \widetilde{c}_{4}=0.3716 \\
& \alpha_{4}=\left(R_{L}-r_{4}\right) /\left(R_{L}+r_{4}\right)=0.2429
\end{aligned}
$$

Similar analysis to (c) was performed and results are to be found in Pig. 3.18.

## (e) Mothor B Aerimed from the load-ond.

The WFD is shom in Fir. 3.19 and the design ecuations are as follons:

$$
\begin{aligned}
& r_{1}=R_{I} \\
& \delta_{2}=\varepsilon_{1}+C_{2}=2.0316 \\
& \alpha_{1}=\varepsilon_{1} / \varepsilon_{2}=0.4922 \\
& r_{3}=r_{2}+L=1.6396 \\
& \alpha_{2}=r_{2} / r_{3}=0.3002 \\
& \xi_{4}=E_{3}+C_{1}=1.6415 \\
& \alpha_{3}=\varepsilon_{3} / \varepsilon_{4}=0.3716 \\
& \alpha_{4}=\left(R_{s}-r_{4}\right) /\left(R_{s}+r_{4}\right)=0.2429
\end{aligned}
$$

Similar analysis to (c) was performed and results are to be found in Fig. 3.20.

The most important observation to be made is. that for wave dirital filters, the attenuation sensitivities are approximately constant in the passband. This confirms the arsument made by Orchard [23]. The consequence of almost constant sensitivities is low distortion [59], that is for small chances in the multiplier values we have a constant shift of the attenuation curve. In the examoles ot the end of this chonter, we shall see this more clearly. However, non-zero sensitivities are sienificant rith recard to round-off noise [13][21] and we shall look at this in a later chapter.

As a concluding note to this section, let us use the expressions derived in section 3.3 to calculate the attenuation sensitivities and compare the results to those found by analysis. The comparison may be made by examining Fig. 3.21 where the required arreement is clearly show.

### 3.5 Example I: 3rd order Ellintio Jow-Pars Tittor showing attenuation curve and sensitivity characteristics.

Let us consider a network example that may verify the various formulae derived in this chantor. A third-order normalized lowposs olliptic filtor with 0.177 ds pasoband ripple vas choncn [7] (Fis. 3.22). It was recided to design a wave dieital filtor using the two-port approach from the source-ond. The wave-flow block diagram appears in Fis. 3.23 and the design equations are as follows:

$$
\begin{aligned}
R_{s}=R_{J}=1, L_{1} & =L_{3}=1.1672, I_{2}=0.029, C_{2}=1.1231 \\
R_{1} & =R_{s}+L_{1}=2.1672 \\
\alpha_{1} & =R_{s} / R_{1}=0.461425 \\
G_{2} & =G_{1}+\frac{C_{2} \Gamma_{2}}{C_{2}+\Gamma_{2}}=1.5491 \\
\alpha_{2} & =G_{1} / G_{2}=0.207867 \\
\alpha_{3} & =\left(\Gamma_{2}-C_{2}\right) \cdot /\left(\Gamma_{2}+C_{2}\right)=0.936915 \\
R_{3} & =R_{2}+L_{3}=1.812736
\end{aligned}
$$

$$
\begin{aligned}
& \alpha_{4}=R_{2} / R_{3}=0.355112 \\
& \alpha_{5}=\left(R_{L}-R_{3}\right) /\left(R_{L}+R_{3}\right)=-0.288949
\end{aligned}
$$

The complete :Wh anners in Tir. 3.24. The dimitel network was then analysed and Fig. 3.25 shows the nominal attenuation curve, Fie. 3.25 shors the multinljer sensitivities and Fis. 3.27 shors the celay sensitivities. Iet us now compre the WP sensitivities found by analysis with those using the formlae of this chapter. These sensitivities have beon tabulnted in Fis. 3.28 and the required ajrement may be seen. Several points arise and they are as follow:

1. The sensitivity $S_{\alpha}^{|N|}$ is seen to tend to infinity at the zero of transmission (Fig. 3.26). This is not serions since $\alpha_{3}$ is derived from the tunedcircuit and at resnnance, the tured-circuit becones a short-sircuit.
 first lap noint (i.e. w $=0$ ) but not at the second point. This is a consequence of enn. (3.49) to eether With the fact, that at zero frojuency, all sensitivitu functions ne real and as $S_{d_{i}}^{|c|}=j \operatorname{InS} \frac{d_{i}}{d_{i}}$, we have $S_{d i}^{|c|}=0$.
3.6 Jramp TT: 5 th-omor R11intic Tow-pnas Iilter shovinu offects of coerciciont amntization

Let us now consider a Nave Digital filter using the tro-port approach for the purnose of examining the effects of reducing the multiplier wordlength. In practical realisations, it is desirable to have as small a wordlength as possible and we shall see that for wave diontal filter etructures we may reduce the number of bits representing the multiplier value to a's little as three without a serious decay in the attenuation curve.

Considor a firth-order nomalized elliptic low-pass filter mith 0.090 d 3 passband rinnle. The element valucs wore taken from Reference [ 8 ] and the network is shown in Fig. 3.20. The waveflou blook dinerom appoars in ric. 3.30 and the desien equations
are as follows:

$$
\begin{aligned}
& R_{S}=R_{L}=1, C_{1}=1.00877, C_{2}=0.06800, I_{2}=1.29869 \\
& C_{3}=1.0028 R, C_{4}=0.18583, L_{4}=1.15805, C_{5}=0.08555 . \\
& G_{1}=G_{S}+C_{1}=2.08877 \\
& \alpha_{1}=G_{S} / G_{1}=0.478751 \\
& R_{2}=R_{1}+\frac{L_{2} D_{2}}{L_{2}+D_{2}}=1.67193 \\
& \alpha_{2}=R_{1} / R_{2}=0.286346 \\
& \alpha_{3}=\left(D_{2}-I_{2}\right) /\left(D_{2}+L_{2}\right)=0.837513 \\
& G_{3}=G_{2}+C_{3}=2.400991 \\
& \alpha_{4}=G_{2} / G_{3}=0.249110 \\
& R_{4}=R_{3}+\frac{L_{4} D_{4}}{L_{4}+D_{4}}=1.369464 \\
& \alpha_{5}=R_{3} / R_{4}=0.304129 \\
& \alpha_{6}=\left(D_{4}-L_{4}\right) /\left(D_{4}+L_{4}\right)=0.645819 \\
& G_{5}=G_{4}+C_{5}=1.715772 \\
& \alpha_{7}=G_{4} / G_{5}=0.425588 \\
& \alpha_{8}=\left(G_{5}-G_{L}\right) /\left(G_{5}+G_{L}\right)=0.263561
\end{aligned}
$$

The complete wiphears in Fic. 3.3i. The difitul notronk was then analysed at 50 frequency points, first for the nomins? multiplier values above, then for the multiplier values rounded to 3 decimal places and finally to 1 decimal place. The analysis is shom in eramical form in Pig. 3.32 and confirrs the prediction of a constant shift in the roseonse with little distortion.

### 3.7 Eyample III: 6th order Blintio Band-Pass Tilter shomine effects of coofficient ruantization.

As a further example of the effects of reducinc the multiplier wordlength, let us take the third-order Plliptic Lou-Pass Filter of Example I (fic. 3.22) and apply to it the Low-Pass to BandPass transformation [10]. The resultinc filter, which is of the 6 th order, anpears in Fig. 3.33. The bandwidth was chosen to be 0.1 HZ . and the centre frequency to be 1 HZ . The wave-flow
block diagrom appears in Fig. 3.34 and the design equations are as follous:

$$
\begin{aligned}
R_{S}=R_{L}=1, L_{1} & =L_{4}=11.672, C_{1}=C_{4}=0.085675 \\
L_{2}=0.79126, C_{2} & =2.1845, L_{3}=0.45778, C_{3}=1.26380 . \\
R_{1} & =R_{S}+L_{1}+D_{1}=24.344016 \\
\alpha_{1} & =-R_{S} / R_{1}=-0.0410779 \\
\alpha_{2} & =\left(D_{1}-L_{4}\right) /\left(D_{1}+L_{4}\right)=0 \\
G_{2} & =G_{1}+\frac{C_{2} \Gamma_{2}}{C_{2}+\Gamma_{2}}=0.841699 \\
\alpha_{3} & =G_{1} / G_{2}=0.0488035 \\
\alpha_{4} & =\left(\Gamma_{2}-C_{2}\right) /\left(\Gamma_{2}+C_{2}\right)=-0.266999 \\
G_{3} & =G_{2}+\frac{C_{3} \Gamma_{3}}{C_{3}+\Gamma_{3}} \\
\alpha_{5} & =G_{2} / G_{3}=0.512509 \\
\alpha_{6} & =\left(\Gamma_{3}-C_{3}\right) /\left(\Gamma_{3}+C_{3}\right)=0.266992 \\
R_{4} & =R_{3}+L_{4}+D_{4}=23.9529145 \\
\alpha_{7} & =-R_{3} / R_{4}=-0.0254206 \\
\alpha_{8} & =\left(D_{4}-L_{4}\right) /\left(D_{4}+L_{4}\right)=0 \\
\alpha_{9} & =\left(R_{L}-R_{4}\right) /\left(R_{L}+R_{4}\right)=-0.919849
\end{aligned}
$$

We have used the rip's for a series-tuned circuit in the series-arm and for a parallel-tuned circuit in the shunt-arm. Note that, as a result of the fact that $I_{1} C_{1}=I_{4} C_{4}=1$, the number of multipliers is reduced from nine to seven. The complete : FFD appoars in Fig. 3.35. The digital network was then analysed at 50 frojuency moints over the whole spectrum $(0, \pi / \pi)$ and then at 21 points in the passband only. The analysis is shom in graphical form in Firs. 3.36 for the following cases (i) nominal multiplier values \& (ij) multipliers rounded to 3 decimal places.
3.8 Conclusions

In this chapter we have seen hov, with the aid of analocue filter design tables, to design wave disital filters using the two-port approach. The examples eiven were of normalized filters but this does not imply a restriction, since any classical LC ladder filter has a wave digital equivalent.

A theoretical approach has been presented in which it was shown that wave dirital filters do not possess the property of zero first-order attenuation sensitivity to multinlier variations at noints of maximum nover transfer. The theory was extended to cover the exact behnviour at these points. It was sound that the multinlier sonsitivities were approximately constant in the passband which implies a constant shift in the attenuation curve. This has significance rith rogard to round-off noise.


Fig. 3.1 WFD of DMLUN designed (a) from the load, (b) from the source and (c) using Fettweis theory .


Fig. 3.2 3rd. Order Chebyshev filter .


Fig. 3.3 Wave-Flow Block Diagram of Fig. 3.2


Fig. 3.4 Complete Wave-Flow Diagram of Fig. 3.3


Fig. 3.5 General Odd-order Low-Pass Fillter with zeros at infinity .


Fig. 3.6 WFD of Fig. 3.5 designed using method A from source -


Fig. 3.7 WFD of Fig. 3.5 designed using method A from load .


Fig. 3.8 WFD of Fig. 3.5 designed using method B from source .


Fig. 3.9 WFD of Fig. 3.6 designed using method $B$ from load.


Fig. 3.10 3rd. Order Chebyshev Filter .


Fig. 3.13 Signal-Flow graph of Direct Synthesis Digital Filter .


Fig. 3.15 WFD of Fig. 3.10 designed using method A from source .


Fig. 3.17 WFD of Fig. 3.10 designed using method B from source .


Fig. 3.11 Amplitude characteristic of third-order Chebyshev analogue filter plotted against a warped frequency axis .

Fij. 3.12 Attenuation sensitivity with respect to first-order element variations for a 3rá.order Cnebyshev analogue filter .



Fig. 3.16 Attenuation sensitivity in Wave Digital Pilter of Method A (from source).
 of Hethod B(from source).



Fig. 3.19 WFD of Fig. 3.10 designed using method $B$ from load .


Fig. 3.22. Example I:Third-Order Elliptic Filter .


Fig. 3.23 Wave-flow olock diagram of Fig. 3.22 .

Fig. 3.20 Attenuation sensitivity in Wave Digital Filter of Method B (from load).


| Method |  | $S_{\alpha_{i}}$ |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | $\alpha_{2}$ | $\alpha_{3}$ | $\alpha_{4}$ |
| A (from source) | Observed | 0.5 | -0.5 | 0.805 | 0.516 |
|  | Predicted | 0.5 | -0.5 | 0.8050 | 0.5153 |
| $B$ (from source) | Observed | 0.5 | -0.5 | 0.5 | 0.05 |
|  | Predicted | 0.5 | -0.5 | 0.5 | 0.65270 |
| B (from load) | Observed | -0.5 | 0.5 | -0.5 | -0.253 |
|  | Predicteà | -0.5 | 0.5 | -0.5 | -0.2581 |

Fiog. 3.21 Table comparing attenuation sensitivities found by. analysis and those found using the formulae of section 3.3.2.




Fig.3.27 First-order attenuation sensitivities to delay element variation for Example I


|  | $\delta_{\alpha}^{161}$ |  |
| :--- | :---: | :---: |
|  | Observed | Predicted |
|  | -0.5 | -0.5 |
| $\alpha_{2}$ | 0.5 | 0.5 |
| $\alpha_{3}$ | $2.55 \times 10^{-17}$ | 0 |
| $\alpha_{4}$ | -0.5 | -0.5 |
| $\alpha_{5}$ | 0.091097 | 0.091097 |
| $d_{1}$ | $-2.12 \times 10^{-5}$ | 0 |
| $d_{2}$ | 0.4426 | 0.444 |
| $d_{3}^{5}$ | -0.4453 | -0.444 |
| $d_{4}$ | $-2.59 \times 10^{-5}$ | 0 |

The multiplier sensitivities are valid at both MAP points . The delay sensitivities are valid at the second MAP point only, i.e. $\omega T=1.432566$. At the rirst point i.e. $\omega T=0$ all delay sensitivities are zero.

Fig. 3.28 Table of comparison for attenuation sensitivities in thirdorder elliptic filter .


Fig. 3.29 Example II: 5th. Order Elliptic Low-Pass Filter .


Fig. 3.30. Wave-Flow Block Diagram of Fiģ. 3.29.


Fig. 3.33 Example III: 6th.Order Elliptic Band-Pass Filter .


Fig. 3.34 Wave-Flow Block Diagram of Fig. 3.33 .

$\alpha_{1}=0.478751$
$\alpha_{2}=0.286346$
$\alpha_{3}=0.837513$
$\alpha_{4}=0.249110$
$\alpha_{5}=0.304129$
$\alpha_{6}=0.645819$
$\alpha_{7}=0.425588$
$\alpha_{8}=0.263561$

$\underset{1}{\underset{\sim}{t}}$

Fig. 3.31 Complete Wave-Flow Diagram of Fig. 3.30 .


$\alpha_{1}=-0.0410779$
$\alpha_{3}=0.0488035$
$\alpha_{4}=-0.266999$
$\alpha_{5}=0.512509$
$\alpha_{6}=0.266992$
$\alpha_{7}=-0.0254206$
$\alpha_{9}=-0.919849$


Fig. 3.35 Complete Wave-Flow Diagram of Fig. 3.34 .


## Chanter 4

Desim and Sansitivjty Analysis of Wave Dirital Filters Imitatine Doubly-Terminater Lossless Transmissjon-Jine Netrorls.

## Contents:

4. 1 Desimm Procedure.
4.2 The Derivation of MP Benajivitv Characteristics.
4.3 Example I: 3rd.Order IP Filter.
4.4 Examle II: 7th. Order JP Filter.
4.5 Discussion.
4.6 Conclusions.

## Chantor 4

Desim and Sensitivity Analarsis of Tave Digital Tiltera Tmitatinm Doubly-merminated Jossless Transmission-Line Metrorks.
4.1 Desien Procedure

The distributed nrototyne we shall use in this chanter is made un from a cascade of commensurate stenned-imnedance transmission lines [35] terminatod at hoth ends by resistances. The basic networle element necessam, the losslons tranmission line or unit element, was described in Chanter 2. The netrork made un from n soctions of this terne annears in Fig. 4.1 and the trancfer function will he of the form [80].

$$
\begin{equation*}
\hat{G}(z)=\frac{z^{-n / 2}}{E_{n}\left(z^{-1}\right)} \tag{4.1}
\end{equation*}
$$

where $H_{n}\left(E^{-1}\right)$ is an nth-degren nolymonial in $z^{-1}=e^{-2 \tau n}, \tau$ is onemay delay of lines and $D$ is the lumned-olement commer frequency variable. Thus we are rastricted to all-nole filters and tables for such filters can be found in reforences [40],[81] and [82].

Jet us now consider the dirital equivalent of $\sin$. 4.1. In the tro-nort annroach to the renresentation of transmission lines adonted in thonter ?, te have for a line three nararnetors $\left(R_{1}, R_{2}, Z_{0}\right)$, ton of which $\left(R_{1}, Z_{0}\right)$ or $\left(R_{2}, Z_{0}\right)$ are constraned and the other free. Thus the free noremeter rey be chosen to facilitate interconnection. There are, therefore, two different diefital confimeretions for a riven distributed nrototume as was the case for tic filters. The wave-flow diacram on desionine from the source annears in Fir. A.?, thilst that on derignine from the load annears in Fig. 4.3. It is to be noted that the tro delavs of $z^{-\frac{1}{2}}$ in each section may be combined to form a delav $z^{-1}[32][50]$. The effect of this combination is to produce in the transfer function a factor of $z^{-\frac{1}{2}}$ in the odi-order case. This factor has no effect on the mamitude resmonse and causes . a linear shift in the nhase resnonse.

The values of the multinliers in the Wave Dirotal Filter of Fig. 4.? are

$$
\left.\begin{array}{rl}
\alpha_{1} & =\left(R_{s}-z_{1}\right) /\left(R_{s}+z_{1}\right) \\
\alpha_{k} & =\left(z_{k-1}-z_{k}\right) /\left(z_{k-1}+z_{k}\right) \quad k=2,3, \ldots n \\
n+1 & =\left(R_{T}-z_{n}\right) /\left(R_{L}+z_{n}\right)
\end{array}\right\}(1.2)
$$

and in the Wave Diopital Pilter of Rie. 4.3

$$
\left.\begin{array}{l}
\alpha_{0}=\left(R_{s}-z_{1}\right) /\left(R_{s}+z_{1}\right) \\
\alpha_{k}=\left(r_{k+1}-z_{k}\right) /\left(z_{k+1}+z_{k}\right) \quad K=1,2, \ldots, r_{n-1} \\
\alpha_{n}=\left(R_{L}-z_{n}\right) /\left(R_{L}+z_{n}\right)
\end{array}\right\}(1, z) .
$$

where $Z_{k}$ is the kth Characteristic Imedance.

There are tro tynes of distributed filter of interest (a). MarterWave transformors and (b) Yalf-Wave Filters [8?]. The anventare of (b) over (a) lies in the fact that the jmedraces of sucoessive stens alternately increase and decrease, usually oscillatint about, from unity ofton to extremely larme values, nossibly $10^{20}$ or more [ 10 ]. Te shall concentrate on the Falfflave Filter and, in narticular, the tables of J,evr [40] for equi-rimple mamiture resmonses. The amnlitude function for a Chobyshov equi-rinnle resnonse is given by the exnression

$$
\begin{equation*}
|\hat{r}|^{2}=\frac{1}{1+h^{2} T_{n}^{2}\left(\frac{\sin }{\sin } \theta_{0}\right)} \tag{4.4}
\end{equation*}
$$

where $n_{n}$ denoter the Chebyhev function of the first kind [a0]of derpee $n, \theta=\beta l \quad$ and $\theta_{0}$ is the cut-off maramoter. A trical resnonse jis illustrated in Tis. 4.4. Ievy has tabulated the characteristic imedances for values of $n$ from 2 to 21 , for values of Bandridth ( $B H$ ) $=\Lambda \Theta_{0} / \pi$ and for values of Voltame Standine vave Ratio (VGIR) $=1+2 h^{2}+2 \sqrt{h^{2}+h^{4}}$.

Thus having chosen the desired filter characteristics i.e. the values of $n, B \in$ and VS:R, the tables will vield the $n$ characteristic imnedances. For odd-order filters ve shatl have [40],

$$
\begin{aligned}
& R_{s}=R_{L}=1 \\
& Z_{k}=Z_{n-k+1}
\end{aligned}
$$

whereas for even-order filters,

$$
R_{s}=1, R_{L}=V S U R
$$

and $\quad z_{k}{ }^{7}{ }_{n-k+1}=V S^{\prime} \cdot R$.
4.? The Dorivation of Mep Sensitivite Characteristics
4.2.1 Introdiction

We sholl inse Orchard's armment [23] to show that, for Mave Difital. Finters imitatine distributed filters, the first-order attrnugtion sensitivitor to multinlier variation is non-zero at noints of mavimum porer transfor. The method used here is similer to thot used in section 3.3. Teet us first stato Orchard's armument amain [3]. For any lossloss network onorntinm betwoon resistive terminations, the first-ordor attennation gensitivity to variations in each of the reactive comnonents is zern at noints of maximum nower transfer. Clearly, a netrorle consistine of cascaded commensurate lossless transmission lines terminated by resistances must satisfy orchard's rearirements. As each line element is lossless (reactive) we may state at once that $\mathrm{S}_{\mathrm{Z}_{i}}^{\mathrm{HI} \mid}=0$ at noints of maximum nower transfer where $|\mathrm{H}|$ is the mamitude of the transfer function and $Z_{i}$ is the ith characteristic imnedance. Te are now in a nosition to derive an exnession for the sensitivity of the wave digital filter transfer function to motinlier vamiations.

### 1.2.2 Derivation of Formpae

If $G(z)$ is the transfer function of the wave dimital filter, and if $\hat{T}(z)$ is the transfer function of the transmission line filter where $z=e^{3 y T}$ then

$$
\begin{equation*}
G(z) \equiv K \hat{G}(z) \tag{4.5}
\end{equation*}
$$

Where $K$ is a constant whose value denents on the elements of the distributed filter. For the wave dirital filter desirned from the source end, we may use the discussion in section 3.1 and therefore

$$
\begin{equation*}
\mathrm{K}=1+\frac{Z_{n}}{\eta_{L}} \tag{4.6}
\end{equation*}
$$

For the wave direital filter desimod from the load-end ve similarly deduce that

$$
\begin{equation*}
K=2 \tag{4.7}
\end{equation*}
$$

We may write, from eqn. (4.5), that *

$$
\begin{equation*}
S_{\alpha_{\underline{p}}}^{G}=S_{\alpha_{\underline{\eta}}}^{K}+S_{\alpha_{\eta}}^{\hat{G}} \tag{4.8}
\end{equation*}
$$

where $\alpha_{n}$ is a multinlier.
Let us consider the term $S \hat{\Lambda}^{\hat{r}} \alpha_{0}$ in ean. (4.8), which by using chain differentiation we find that

$$
\begin{equation*}
S \alpha_{p}^{\hat{G}}=\sum_{i} S_{Z}^{\hat{G}} S_{\alpha}^{\hat{Z}}{ }_{\alpha}^{\dot{j}}+S_{R_{J}}^{\hat{G}} S_{\alpha}^{R_{D}}+S_{R_{s}}^{\hat{G}} S_{\alpha}^{R_{p}} \tag{4.9}
\end{equation*}
$$

The only assmotion we have made in eqn. (4.9) is.that we my exnress $R_{L}, R_{s}, Z_{i}$ each in terms of the $\alpha_{n}$. This is valid because of that which follows. We shall use the Sensitivity Invariant for tranmission lines mhich may be expressed as follows [83],

$$
\begin{equation*}
\sum_{i=1}^{n} S{ }_{Z_{i}}^{\hat{G}}+S_{R_{I_{1}}}^{\hat{G}}+S_{R_{S}}^{\hat{G}}=0 \tag{4.10}
\end{equation*}
$$

We may eliminate one sensitivity fanction from eqn. (4.9) by virtue of eqn. (4.10) and for convenience, let us choose $S \hat{G}_{R_{S}}^{\hat{G}}$ then we have

$$
\begin{equation*}
S_{\alpha}^{\hat{G}}=\sum_{i} S_{Z_{i}}^{\hat{G}} S_{\alpha_{p}}^{Z_{i} / R_{s}}+S_{R_{L}}^{\hat{G}} S_{\alpha_{p}}^{R_{L} / R_{s}} \tag{4.11}
\end{equation*}
$$

Footnote:

* For convenience, we have dromped the ( $z)$ from $G(z)$ and $\hat{r}(z)$, but their functional denendence on $z$ is still to be nresumed.

Taring, the real part of eq. (4.11) and noting that $S_{\alpha_{p}} Z_{i} / R_{S_{a n d}}$ ${ }_{S} R_{\alpha_{\mathrm{D}}} / R_{S}$ are real, we have

At points of maximum power transfer we have $S^{|\hat{G}|}=0$ and therefore

$$
{ }_{S}^{|\hat{G}|} \begin{gather*}
\alpha_{p}
\end{gathered}={ }_{S}^{|\hat{G}|} \quad{ }_{S} \quad \begin{gathered}
R_{L} / R_{S}  \tag{4.13}\\
R_{D}
\end{gather*}
$$

Taring the real nat of eau. (4.8) and combining with eqn. (4.13) gives

$$
\begin{equation*}
S_{\alpha_{p}}^{|G|}=S \alpha_{p}^{K}+S_{\alpha_{D}}^{|\hat{G}|} S{ }_{R_{L}}^{R_{L} / R_{S}} \tag{4.14}
\end{equation*}
$$



$$
\begin{equation*}
S_{\alpha_{D}}^{|G|}=S \alpha_{\underline{p}}^{K}+\frac{1}{2} S_{\alpha_{D}}^{R_{I} / R_{S}} \tag{4.15}
\end{equation*}
$$

Let us now consider the two methods of desjom:

## I: Desire from the source-end.

For this case $K=1+\frac{Z_{n}}{R_{L}}$ and noting that, from en. (4.?),

$$
\alpha_{n+1}=\frac{R_{I}-Z_{n}}{R_{I_{\mu}}+Z_{n}}
$$

we can easily show that

$$
K=\frac{2}{1+\alpha_{n+1}}
$$

thus

$$
{ }_{S}^{K} \alpha_{p}^{K}=\left\{\begin{array}{ll}
0 & \text { for } p=1,2, \ldots n \\
\frac{-\alpha_{n+1}}{1+\alpha_{n+1}} & \text { for } p=n+1
\end{array}\right\}
$$

On rearranging the relationshins of ear. (4.2) we have.

$$
\begin{aligned}
\frac{Z_{1}}{R_{S}} & =\frac{1-\alpha_{1}}{1+\alpha_{1}} \\
\frac{Z_{k}}{Z_{k-1}} & =\frac{1-\alpha_{k}}{1+\alpha_{k}} \quad \text { for } k=2, \ldots, n \\
\frac{Z_{n}}{R_{L}} & =\frac{1-\alpha_{n+1}}{1+\alpha_{n+1}}
\end{aligned}
$$

from which we observe that

$$
\frac{R_{L}}{{\underset{R}{S}}}=\frac{Z_{1}}{R_{S}} \frac{Z_{2}}{Z_{1}} \cdots \cdots \cdot \frac{Z_{n}}{Z_{n-1}} \frac{R_{T,}}{Z_{n}}
$$

and hence

$$
\begin{equation*}
\frac{R_{\mathrm{L}}}{R_{\mathrm{S}}}=\frac{\left(1-\alpha_{1}\right)\left(1-\alpha_{2}\right) \ldots \ldots \ldots\left(1-\alpha_{n}\right)\left(1+\alpha_{n+1}\right)}{\left(1+\alpha_{1}\right)\left(1+\alpha_{2}\right) \ldots \ldots\left(1+\alpha_{n}\right)\left(1-\alpha_{n+1}\right)} \tag{4.17}
\end{equation*}
$$

It cen be seen from sean. (4.17) that

$$
{ }_{S}^{R_{L} / R_{S}} \alpha_{p}=\left\{\begin{array}{ll}
\frac{-2 \alpha_{n}}{\left(1-\alpha^{2} n_{n}\right)} & \text { for } n=1,2, \ldots n \\
\frac{2 \alpha_{n}}{\left(1-\alpha^{2}\right)} & \text { for } n=n+1
\end{array}\right\} \text { (4.18) }
$$

On combining ans. (4.15), (4.16) and (4.18) we see finally that

$$
\begin{array}{ll}
|G| \\
\alpha_{n}
\end{array} \quad=\left\{\begin{array}{ll}
\frac{-\alpha_{n}}{\left(1-\alpha_{n}^{2}\right)} & \text { for } p=1,2, \ldots n \\
\frac{\alpha^{2} n}{\left(1-\alpha_{n}^{2}\right)} & \text { for } \underline{p}=n+1
\end{array}\right\} \text { (4.10) }
$$

II: Desim from the Ioad-3nd
For this case $K=2$ and therefore eqn. (1.15) becomes

$$
\begin{equation*}
\underset{\alpha_{p}}{|G|}=\frac{1}{2} S_{\alpha_{p}}^{R_{T} / R_{S}} \tag{4.20}
\end{equation*}
$$

On rearrancing the desim exnression of eqn. (4.3), we heve that

$$
\begin{aligned}
\frac{z_{1}}{R_{s}} & =\frac{1-\alpha_{0}}{1+\alpha_{0}} \\
\frac{z_{k}}{z_{k+1}} & =\frac{1-\alpha_{k}}{1+\alpha_{k}} ; k=1, \ldots, n-1 \\
\frac{z_{n}}{R_{L}} & =\frac{1-\alpha_{n}}{1+\alpha_{n}}
\end{aligned}
$$

Therefore

$$
\frac{R_{L}}{R_{S}}=\frac{Z_{1}}{R_{S}} \frac{Z_{2}}{Z_{1}} \cdots \cdots \frac{Z_{n}}{Z_{n-1}} \frac{R_{L}}{Z_{n}}
$$

$$
\text { or } \frac{R_{L}}{R_{S}}=\frac{\left(1-\alpha_{0}\right)\left(1+\alpha_{1}\right) \ldots \ldots\left(1+\alpha_{n-1}\right)\left(1+\alpha_{n}\right)}{\left(1+\alpha_{0}\right)\left(1-\alpha_{1}\right) \ldots \ldots\left(1-\alpha_{n-1}\right)\left(1-\alpha_{n}\right)}
$$

from wich it is easy to see that

$$
\underset{\alpha_{\mathrm{p}}}{R_{\mathrm{J}} / R_{\mathrm{S}}}=\left\{\begin{array}{ll}
-\frac{2 \alpha_{n}}{\left(1-\alpha_{\underline{n}}^{2}\right)} & ; p=0 \\
\frac{2 \alpha_{0}}{\left(1-\alpha_{n}^{2}\right)} & ; p=1,2, \ldots, n
\end{array}\right\}(4.21)
$$

On substitutino for $\quad S_{\mathrm{S}_{\mathrm{D}} / \mathrm{R}_{\mathrm{S}}}$ from eqn. (4.21) into eqn. (4.20) we have

$$
S_{\alpha_{p}}^{|G|}=\left\{\begin{array}{cl}
-\frac{\alpha_{n}}{\left(1-\alpha_{n}^{2}\right)} & p=0 \\
\frac{\alpha_{0}}{\left(1-\alpha_{n}^{2}\right)} & ; n=1,2, \ldots, n
\end{array}\right\}
$$

The formulae in ean. (4.19) and (4.2?) are valid at maxjmm psendonower transfer noints [50], hovevor for a 'vell-desirned' filter it, has been nostulated that at other noints in the Dassband, the attenuation sensitivito should be close to its value at MAP [33]. This fact is imnortant with repard to attenuation distortion [59].

It shomid be noted that the VID of the filter desioned from the source differs from that of the filter desimed from the load onjy in the connections of the last multiplier (Figs. 4.2 and 4.3). Murthemore, it should be noted that the IFD obtained using the tro-nort annroach are not dissimilar to thone obtained br Fettreis [30].

Finallv, let us summarise the results of this section br sayin ; that we have shorm that the first-order attenuation sensitivities to multinlier variations are not reneralle zero at mp points. This result clearly contradicts that of Renner and Gunta [ $\because 0]$. Their proof rel.ies on the assumption that ${ }_{S}^{|\hat{G}|} R_{L}={ }_{S}^{|\hat{G}|}{ }_{R_{S}}=0$ at VAP , which is clearly false. $(\hat{G}(z)$ is the transfer function of the transmission-line filter).

### 4.3 Example I: 3rd Order Jow-Pass Fijter.

Consider a third-order equally terninated Chobvshev trensmission line filter havinz a bandmidth, FH , of 0.2 and $\mathrm{VS} / \mathrm{R}=1.2$. The tebles of Jevy [ 10 ] yield the followins characteristic impedances,

$$
\begin{aligned}
z_{1} & =5.256 \\
Z_{2} & =0.1475 \\
z_{3} & =Z_{1} \\
\text { also } R_{s} & =R_{L}=1
\end{aligned}
$$

If we desion from the load-end then, on applyinc the design formulae of eqn. ( 1,3 ) we have

$$
\left.\begin{array}{l}
\alpha_{0}=\frac{R_{S}-Z_{1}}{R_{S}+Z_{1}}=-0.6803 \\
\alpha_{1}=\frac{Z_{2}-Z_{1}}{Z_{2}+Z_{1}}=-0.9454
\end{array}\right\}
$$

$$
\begin{align*}
& \alpha_{2}=\frac{Z_{3}-Z_{2}}{Z_{3}+Z_{2}}=0.9454  \tag{4.23}\\
& \alpha_{3}=\frac{R_{L}-Z_{3}}{R_{I}+Z_{3}}=-0.6803
\end{align*}
$$

The wave-flo: diagram appears in Fir. 4.5. The dirital structure was analysed and the results in graphical form appear in pies. 4.6 and 4.7. The former contains the amplitude response and the latter the attenuation sensitivity characteristics for each multiplier.

At points of maximum nseudooower transfer we have from the rank of Firs. 4.7, the following values for the attenuation sensitivities,

$$
\begin{align*}
& S_{\alpha_{0}}^{|G|}=1.2664 \\
& S_{\alpha_{1}}^{|G|}=-8.9005  \tag{4.24}\\
& S_{\alpha_{1}}^{|G|}=0.9005 \\
& { }_{\alpha_{2}}^{|G|}=-1.2664
\end{align*}
$$

Using the theory of section 4.2 and, in particular, eqn. (4.22) we have

$$
\begin{aligned}
& \mathrm{S}_{\alpha_{0}}^{|G|}=-\frac{\alpha_{0}}{1-\alpha_{0}^{2}}=1.2664 \\
& \mathrm{~S}_{\alpha_{1}}^{|G|}=\frac{\alpha_{1}}{1-\alpha^{2} 1_{1}}=-8.9005 \\
& \mathrm{~S}_{\alpha_{2}}^{|G|}=\frac{\alpha_{2}}{1-\alpha_{2}^{2}}=8.0005
\end{aligned}
$$

$$
\text { and } \quad{ }_{S^{\prime}}^{|G|}=\frac{\alpha_{3}}{1-\alpha_{3}^{2}}=-1.2664
$$

Thus the theoretical sensitivities are seen to be the same as those found by analysis.
4.4 Eramnle TT: 7th Order Iov-Pass Pinter.

Consider a 7th order eaually terminated Chebvihev transmissionline filter havine a bandridth of 0.8 and $\mathrm{VS}: \mathrm{R}=1.5$. The tables of Levy [40] yie]d the folloring

$$
\begin{aligned}
z_{1}=z_{7} & =2.270 \\
z_{2}=z_{6} & =0.4956 \\
z_{3}=z_{5} & =3.342 \\
z_{4} & =0.4391 \\
R_{L}=R_{s} & =1
\end{aligned}
$$

Tf we desim from the source-end then, on annlying the design formulae of eqn. (4.2) we have

$$
\begin{aligned}
& \alpha_{1}=\left(R_{s}-z_{1}\right) /\left(R_{s}+z_{1}\right)=-0.38838 \\
& \alpha_{2}=\left(z_{1}-z_{2}\right) /\left(z_{1}+z_{2}\right)=0.64160 \\
& \alpha_{3}=\left(z_{2}-z_{3}\right) /\left(z_{2}+z_{3}\right)=-0.74171 \\
& \alpha_{4}=\left(z_{3}-z_{4}\right) /\left(z_{3}+z_{4}\right)=0.76774 \\
& \alpha_{5}=\left(z_{4}-z_{5}\right) /\left(z_{4}+z_{5}\right)=-\alpha_{4} \\
& \alpha_{6}=\left(z_{5}-z_{6}\right) /\left(z_{5}+z_{6}\right)=-\alpha_{3} \\
& \alpha_{7}=\left(z_{6}-z_{7}\right) /\left(z_{6}+z_{7}\right)=-\alpha_{2} \\
& \alpha_{8}=\left(z_{5}-z_{7}\right) /\left(R_{L}+z_{7}\right)=\alpha_{1}
\end{aligned}
$$

The complete (rym annears in Fig. 4.8. The network was analysed at 61 frequency noints and Firs. 4.9 shows the nominal attenuation curve together with the curres corresponding to multinliers rounded to 3, ? and 1 decimal place respectively. It is observed that the effect of reducing the wordlength is to rive a constant shift in the attemation curve. This confirms Orchard's armument for 'vell-desirned' filters at points, other than MAP, in the passband. Pinally, the transfer function of the wave digital
filter in Fis. 4.8 is given by the emression

$$
\begin{equation*}
G(z)=\left(1+\frac{Z_{n}}{R_{L}}\right) \hat{G}(z) \tag{4.25}
\end{equation*}
$$

but

$$
\mathrm{R}_{\mathrm{L}}=1 \text { and } \mathrm{Z}_{\mathrm{n}}=2.27
$$

therefore $\hat{r}(z)=3.27 \times \hat{G}(z)$.
At MiP points, $|\hat{G}|=\frac{1}{2}$
therefore $|G|=1.635$.

The theoretical MP sensitivities are given below:

$$
\begin{aligned}
& { }_{S}^{|G|}=-\frac{\alpha_{D}}{\left(1-\alpha_{D}^{2}\right)} \quad ; \quad v=1 \text { (1) } 7 \\
& \text { and } S_{\alpha_{D}}^{|G|}=\frac{\alpha^{2} \underline{p}}{\left(1-\alpha_{p}^{2}\right)} \quad ; p=8 \\
& { }_{S}^{|G|}=0.1574 \\
& S_{\alpha_{2}}^{|G|}=-S_{\alpha_{7}}^{|G|}=-1.0905 \\
& S_{\alpha_{3}}^{|\mathrm{F}|}=-\mathrm{S}_{\alpha_{6}}^{|G|}=1.6487 \\
& s_{\alpha_{4}}^{|G|}=-S_{\alpha_{5}}^{|G|}=-1.8609 \\
& s_{\alpha_{8}}^{|g|}=0.1776
\end{aligned}
$$

4.5_Dj._scussion

1. It is to he noted that the frequency resonse of the distributed filter is contained in $\left[0, \frac{\pi}{2}\right]$ (Ric. 4.4) whilst that of the digital filter is contained in $[0, \pi]$ (Firs. 4.6 and 4.9). This is a direct result of the transformation (see section 2.5)

$$
z^{\frac{1}{2}}=\cos \theta+j \sin \theta
$$

where $z=e^{j w T}, \theta=\hat{v} \tau \quad$ which may be written as
$\mathrm{WT}=20$
2. The distributed filter prototyoe we have used, enables us to realize the equivalent wave dirital filter with fewer additions than the corresvondins structure realised fron a lumned nrototyne satisfvino the same snecification [ 18 ]. However, the orototyne we have used in this chaoter dons restrict us to all-nole transfer functions. By allorine stubs [35] into tho distributed filter we can, of course, extend the ranme to cover the more oeneral. 'Unit-Therent' (UR) Rilters [38] . The derivation of wave dixitel fillers from $T T_{\text {f }}$ filters has been extensively covered in the literature and references were oiven in Chenter 1. The adrentore of usine the simbler soscade of transmission lines, lies in the fact. that numerous filter tables are available for both Chebyshev and Hammally-Flat aporoximation thus mal-ino desim an easy task.
3. Althoull the dolay sensitivities :neronot derived roy transmission-line filters, the methods of section 3.3 mav be applied to obtain similar results.
4. In section 4.1, the Voltace Standine Wave Ratio, S, was intronuced and a formula was riven relating $S$ to the rinnle factor $h$, namely

$$
\begin{equation*}
S=1+2 h^{2}+2 \sqrt{h^{2}+h^{4}} \tag{4.26}
\end{equation*}
$$

We would be interested to lnow the nassband rinnle in $d B$, fiven the value of $S$. We mav arrance eqn. (4.26) so that $h$ apnears as a function of $S$. It is found that

$$
\begin{equation*}
h^{2}=\frac{(S-1)^{2}}{4 S} \tag{4.27}
\end{equation*}
$$

By obsorving fir. 4.4 we note that the maximum nassband rinnle, $\eta$, is riven by

$$
\begin{equation*}
\eta=10 \operatorname{Iog}_{10}\left(1+h^{2}\right) \tag{4.28}
\end{equation*}
$$

Thus, on combinine eqns. (4.27) and (4.28) we find that

$$
\begin{equation*}
\eta=10 \operatorname{Tos}_{10} \frac{(S+1)^{2}}{45} d B \tag{4.29}
\end{equation*}
$$

A table relatinr $\eta$ to $S$ using eqn. (4.29) is iven in Fis. 4.10.

### 4.6 Conclusions

He have seen that the procerlures pronosed by Fettreis [30] for the desien of wave dirital filters can be viewed from an unconstrained tro-nort description of the individual lines. The characteristics of the attenuation sensitivity to multiplier variations were examined and it was found that the behaviour of the filter in the nassbend could be predicted. Horeover; it vas found thrt the attenuation sensitivity to first-order variations of the multinliers is not zero at noints of maximum nseudonower transfer but exhibj.ts a constant shift from zero. This result is of narticular imnortance since the noise due to roundoff at the outnit of the filter is denencent on these sensitivities $[13][21]$.


Fig. 4.1 Distributed Filter made up from a cascade of $n$ commensurate lines.


Fig. 4.2 Wave-Flow Diagram of Fig. 4.1 obtained by design from source .


Fig. 4.3 Wave-Flow Diagram of Fig. 4.1 obtained by design from load.


Fig. 4.4 Typical response of distributed Chebyshev filter .


Fig. 4.5 Wave-Flow Diagram of Example I:3rd.order Chebyshev filter •

Fig.4.8 Wave-FIow Diagram of Example II:7th.order Chebyshev LP Filter .

Fig. 4.6 Amplitude Response of 3rd.order Chebyshev Low-Pass Wave Digital Filter •
$\qquad$



Fig. 4.7 Attenuation sensitivity with respect to lst.order multiplier variations for a 3rd.order Chebyshev filter .


| $s$ | $\eta \quad(\mathrm{~dB})$ |
| :---: | :---: |
| 1.01 | $1.075 \times 10^{-4}$ |
| 1.02 | $4.258 \times 10^{-4}$ |
| 1.05 | $2.584 \times 10^{-3}$ |
| 1.10 | $9.859 \times 10^{-3}$ |
| 1.20 | $3.604 \times 10^{-2}$ |
| 1.50 | $1.773 \times 10^{-1}$ |
| 2.00 | $5.115 \times 10^{-1}$ |

Fig̀ 4.10 Table of Pass-Band Ripples ( $\eta$ ) aşainst VSVR's ( $S$ ).

## Chapter

The Ceneral Tro-Port Trenaformation Contents:
5.1 Introduction: Princinel Tdea.
5.2 Derivation of $\sigma$-Parameters.
5.3 Rasic Gauations for Series Blements.
5.4 Basic Fquations for Shunt Jlements.
5.5 Rasic Equations for Sources, Terminations and Interconnections.
5.6 Study of Realisability Conditions.
5.7 Derivation or MiP Sensitivity Characteristics.
5.8 Study of Some Soccial Cases.
5.9 Discussion of a Condjtion for Canonic SPD.
5.10 General Discussion (includins table of vave fomulations lonom to vield realisable disital filter stmuctures).

## Chapter 5

The feneral Trooport Transformation

## 5.1 - Introduction: Princinal Idea

In previous chanters, we have show how a dipital filter may be desioned in such a wav that it imitates the behaviour of a doubly-terminated lossless analogue network. The methor relied on wave variable thenry by means of which it was honed to preserve the low-sensitivity pronerties of the analorue prototyme. The procedure consisted of transforming the port voltage and current variables for each element in the networls to nes port variables $A_{k}$ and $R_{k}$ such that

$$
\begin{align*}
& A_{k}=V_{k}+R_{k} I_{k}  \tag{5.1}\\
& B_{k}=V_{k}-R_{k} I_{k}
\end{align*}
$$

Bach reactive ejement in the network was treated as a two-port and the relationship between its new port variables was established usine ean. (5.1) tofether with eqn. (5.2) which relates the voltares and currents.

$$
\left[\begin{array}{l}
V_{1}  \tag{5.2}\\
I_{1}
\end{array}\right]=\left[\begin{array}{ll}
A & B \\
C & D
\end{array}\right]\left[\begin{array}{l}
V_{2} \\
I_{2}
\end{array}\right]
$$

The transformation in eqn. (5.1) will be referred to as the 'Voltage-Tave formulation' or more succinctlv as the 'Voltage formulation'. This formulation has been studied extensively, almost to the exclusion of other forms (see Chanter 1). Hovever, tin other formulations are known
(i) the Current-ifave formulation which is defined in matrix form as follows,

$$
\left[\begin{array}{c}
A_{k}  \tag{5.3}\\
B_{k}
\end{array}\right]=\left[\begin{array}{cc}
G_{k} & 1 \\
G_{k} & -1
\end{array}\right]\left[\begin{array}{c}
V_{k} \\
I_{k}
\end{array}\right]
$$

and (ii) the Power-lave formulation which is defined in matrix
form as follovs,

$$
\left[\begin{array}{c}
A_{k}  \tag{5.4}\\
B_{k}
\end{array}\right]=\frac{1}{2} \sqrt{R_{k}}\left[\begin{array}{cc}
G_{k} & 1 \\
G_{k} & -1
\end{array}\right]\left[\begin{array}{c}
V_{k} \\
I_{k}
\end{array}\right]=\frac{1}{2 \sqrt{R_{k}}}\left[\begin{array}{cc}
1 & R_{k} \\
1 & -R_{k}
\end{array}\right]\left[\begin{array}{c}
V_{k} \\
I_{I_{E}}
\end{array}\right]
$$

The transformations defined in eqns. (5.1), (5.3) and (5.4) are to be found in the theory of scattering parameters, were $R_{k}$ is known as the port resistance and $G_{k}=1 / R_{k}$ as the port conductance [31]. The properties of the current-wave fomulation are similar to those of the voltare-wave formulation in that the derived dicital filter structures are canonic in delays and multinliers [60]. The nover-wave formulation would have heen the more obvious choise but the disital filter structures so formed are not canonic in multipliers [30].

The main purnose of this chanter is to show that there are other transformations which enable us to sunthesise a dicital filter from a classical analome netrork. Furthermore, it is shom that there exists a subset of these transformations that admit canonic dipital structures.

Jet us concider, therefore, instead of voltares and currentr, four nev variables and let them be related to oort voltares and currents by the followin expressions (Tir. 5.1) *

$$
\begin{align*}
& {\left[\begin{array}{l}
X_{1} \\
Y_{1}
\end{array}\right]=P\left[\begin{array}{l}
V_{1} \\
I_{1}
\end{array}\right]}  \tag{5.5}\\
& {\left[\begin{array}{l}
x_{2} \\
Y_{2}
\end{array}\right]=Q\left[\begin{array}{l}
V_{2} \\
I_{2}
\end{array}\right]} \tag{5.6}
\end{align*}
$$

Where $X_{1}$ and $X_{2}$ are input variables and $Y_{1}$ and $Y_{2}$ are outnut variables. Also $P$ and $?$ are $2 \times 2$ non-sincular transformation matrices. The oroblen therefore is to detemine the elements of $P$ and ? that rill enble the desime of a dirital filter imjtating the behaviour of a doubly-terminated lossless netrork.

## Footnote:

* The general transformation defined by eqns. (5.5) and (5.6) was first investigated by Carlin and Giordano in their excellent book 'INetwork 'Theory' [99]. However, they used it only for analogue networls.

For the first part of this chapter, we shall abandon the port resistance concept and shall assume $P$ and ? each to have four indenendent coefficients. The only necessary constraint is that of realizability, that is no delay-free loons must be formed on interconnectins the siznal-flov diarrams of correswonding analopue elements.

### 5.2 Derivation of $\sigma$-Parameters

Let us set

$$
T=\left[\begin{array}{ll}
A & B  \tag{5.7}\\
C & D
\end{array}\right]
$$

Then we mey combine eqns. (5.2), (5.5), (5.6) and (5.7) to eliminate the voltares and currentis. Ve find that

$$
\left[\begin{array}{l}
X_{1}  \tag{5.8}\\
Y_{1}
\end{array}\right]=P Z^{-1}\left[\begin{array}{l}
X_{2} \\
Y_{2}
\end{array}\right]
$$

Let $R=\mathrm{PT}^{-1}$, then we have more concisely,

$$
\left[\begin{array}{l}
X_{1}  \tag{5.9}\\
Y_{1}
\end{array}\right]=R\left[\begin{array}{l}
X_{2} \\
Y_{2}
\end{array}\right]
$$

He should like to express $Y_{1}$ and $Y_{2}$ in terms of $X_{1}$ and $X_{2}$ for realisation ourvoses. Let the relationship be defincd in the following way

$$
\left[\begin{array}{l}
Y_{1}  \tag{5.10}\\
Y_{2}
\end{array}\right]=\sigma\left[\begin{array}{l}
x_{1} \\
X_{2}
\end{array}\right]
$$

The relationship between the elements of $R$ and those of $\sigma$ is therefore as follows,

$$
\begin{align*}
\sigma_{11} & =R_{22} / R_{12} \\
\sigma_{12} & =-\Delta R / R_{12} \\
\sigma_{21} & =1 / R_{12} .  \tag{5.11}\\
\sigma_{22} & =-R_{11} / R_{12}
\end{align*}
$$

where $\Delta R$ is the determinant of $R$. is $R=P P^{-1}$ we have

$$
\begin{align*}
\Delta R & =\Delta\left(P Q^{-1}\right) \\
& =(\Delta P)(\Delta T)\left(\Delta Q^{-1}\right) \\
& =-(\Delta P) / \Delta Q \tag{5.12}
\end{align*}
$$

where $\Delta T=-1$ if notwork is reciprocal (the currents are assumed to flow into the network).

Finally, let

$$
P=\left[\begin{array}{ll}
p_{11} & p_{12} \\
p_{22} & p_{22}
\end{array}\right] \quad \text { and } \quad Q^{-1}=\left[\begin{array}{ll}
q_{11} & q_{12} \\
q_{21} & q_{22}
\end{array}\right]
$$

then eqn. (5.11) becomes

$$
\left.\begin{array}{l}
\sigma_{11}=\left\{\begin{array}{l}
\left.p_{21} q_{12} A+p_{21} q_{22} B+p_{22} q_{12} C+p_{22} q_{22} D\right\} / R_{12} \\
\sigma_{12}= \\
-\Delta R / R_{12} \\
\sigma_{21}= \\
\sigma_{22}=\left\{p_{11} q_{11} A+p_{11} q_{21} B+p_{12} q_{11} C+p_{12} q_{21} D\right\} / R_{12} \\
R_{12}= \\
p_{11} q_{12} A+p_{11} q_{22} B+p_{12} q_{12} C+p_{12} q_{22} D
\end{array}\right\}(5.13) .
\end{array}\right\}
$$

The general sisnal-flow diagram (sip) of eqn. (5.10) appears in Fig. 5.2.

## 5.3_._Basic Equations for Series Blements

5.3.1 Series Tmpadance

Consider 11 in Pig . (5.1) to be a series impedance 2 then

$$
T=\left[\begin{array}{cc}
1 & -2 \\
0 & -i
\end{array}\right]
$$

On using eqn. (5.13) we find that

$$
\left.\begin{array}{l}
\sigma_{11}=\left(p_{21} q_{12}-p_{22} q_{22}-p_{21} q_{22} z\right) / \text { denom }  \tag{5.14}\\
\sigma_{12}= \\
\sigma_{21}= \\
\sigma_{22} \cdot \\
\sigma_{22}= \\
\left(p_{12} q_{21}-p_{11} q_{11}+p_{11} q_{21} z\right) / \text { denom } \\
\hline
\end{array}\right\}
$$

where denom $=p_{11}{ }_{12}-p_{12} q_{22}-p_{11} q_{22}{ }^{2}$
The next stage is to replace $z$ by $\mathrm{pL}, \frac{1}{\mathrm{pC}}, \mathrm{pL}+\frac{1}{\mathrm{pC}}$
and $1 /\left(\mathrm{pc}+\frac{1}{\mathrm{pI}}\right)$ for the series inductance, capseitance,
series-tmed circuit and parallel-tuned circuit respectively.

### 5.3.2 Series Inductance

For a series inductance, $Z=p L$ and if :e then apply the bilinear transiormation, that is $D \rightarrow\left(1-z^{-1}\right) /\left(1+z^{-1}\right)$, eqn. (5.14) becones

$$
\left.\begin{array}{ll}
\sigma_{11}=\left[\left(p_{21} q_{12}-p_{22} q_{22}-p_{21} q_{22} L\right)\right. & \left.+\left(p_{21} q_{12}-p_{22} q_{22} \div p_{21} q_{22} L\right) z^{-1}\right] \\
\sigma_{i 2}= & / \text { denom } \\
\sigma_{21}= & / \Delta R\left(1+z^{-1}\right) \\
& / \text { denom } \\
\sigma_{22}=\left[\left(p_{12} q_{21}-p_{11} q_{11}+p_{11} q_{21} L\right)+\left(p_{12} q_{21}-p_{11} q_{11}-p_{11} q_{21} L\right) z^{-1}\right]
\end{array}\right\}
$$

where

$$
\text { denom }=\left(p_{11} q_{12}-p_{12} q_{22}-p_{11} q_{22} L\right)+\left(p_{11} q_{12}-p_{12} q_{22}+p_{11} q_{22}\right)_{2}^{-1}
$$

To avoid delay-free loops on interconnection of t:ro-ports we have, as in Chanter 2, that either $\sigma_{11}$ or $\sigma_{22}$ must have a factor of $z^{-1}$. This condition imblies that either

$$
\begin{align*}
& p_{21} a_{12}-p_{22} a_{22}-p_{21} q_{22} I=0  \tag{5.16}\\
& p_{12} q_{21}-p_{11} a_{11}+p_{11} q_{21} I=0 \tag{5.17}
\end{align*}
$$

Substitutiñ first condition (eqn.(5.16)) into eqn. (5.15) yi.elds the following,

$$
\begin{align*}
& \sigma_{11}=\alpha_{1} z^{-1} /\left(1+\alpha_{2} z^{-1}\right) \\
& \sigma_{12}=\alpha_{3}\left(1+z^{-1}\right) /\left(1+\alpha_{2} z^{-1}\right) \\
& \sigma_{21}=\alpha_{4}\left(1+z^{-1}\right) /\left(1+\alpha_{2} z^{-1}\right)  \tag{5.18}\\
& \sigma_{22}=\left(\alpha_{5}+\alpha_{6} z^{-1}\right) /\left(1+\alpha_{2} z^{-1}\right)
\end{align*}
$$

where

$$
\begin{aligned}
& \alpha_{1}=2 p_{21}^{2} \mathrm{~L} / \Delta \mathrm{p} \\
& \alpha_{2}=1+\left\{2 p_{11} p_{21} \mathrm{~L} / \Delta \mathrm{p}\right\} \\
& \alpha_{3}=\underline{p}_{21} \Delta a_{1} / q_{22} \\
& \alpha_{4}=p_{21} /\left[{ }_{22} \Delta p\right] \\
& \alpha_{5}=p_{21}\left(p_{12} \underline{q}_{21}-p_{11} \underline{q}_{11}+p_{11} q_{21} \mathrm{~L}\right) /\left[q_{22} \Delta p\right] \\
& \alpha_{6}=p_{21}\left(p_{12} q_{21}-p_{11} q_{11}-p_{11} q_{21} L\right) /\left[q_{22} \Delta p\right]
\end{aligned}
$$

Substituting second condition, eqn. (5.17) into en. (5.15) gives the following,

$$
\begin{align*}
& \sigma_{11}=\left(\alpha_{1}+\alpha_{2} z^{-1}\right) /\left(1+\alpha_{3} z^{-1}\right) \\
& \sigma_{12}=\alpha_{4}\left(1+z^{-1}\right) /\left(1+\alpha_{3^{z}}\right)  \tag{5.19}\\
& \sigma_{21}=\alpha_{5}\left(1+z^{-1}\right) /\left(1+\alpha_{3} z^{-1}\right) \\
& \sigma_{22}=\alpha_{6} z^{-1} /\left(1+\alpha_{3} z^{-1}\right)
\end{align*}
$$

where

$$
\begin{aligned}
& \alpha_{1}=-\underline{-n}_{21}\left(\underline{p}_{21}{ }^{a_{12}}-\underline{n}_{22} \underline{a}_{22}-\underline{p}_{21}{ }^{\text {a }} 22^{L}\right) /\left[\underline{n}_{11} \Delta_{a}\right]
\end{aligned}
$$

$$
\begin{aligned}
& \alpha_{3}=1-\left\{2 q_{21} q_{22} L / \Delta q\right\} \\
& \alpha_{4}=-\underline{q}_{21} \Delta \mathrm{p} / \underline{p}_{11} \\
& \alpha_{5}=-q_{21} /\left[\begin{array}{ll}
p_{11} & \Delta q
\end{array}\right] \\
& \alpha_{6}=2 \underline{q}_{21}^{2} L / \Delta q
\end{aligned}
$$

### 5.3.3 Series Capacitance

For a series capacitance, $Z=1 / \mathrm{pC}$ and, as we have seen in Chanter 2, we may replace $z^{-1}$ by $-z^{-1}$ and $L$ by $1 / C$ in equations (5.13) and (5.19) to obtain the appropriate equations.

### 5.3.4 Parallel-Tuned Circuit in the Series-Arn

In this case, $Z=1 /\left\{\frac{1}{\underline{D} I}+\underline{D C}\right\}$ and if we apply the bilinear transformation then

$$
\begin{equation*}
z=(\Gamma+c)+\frac{\left(1-z^{-2}\right)}{(\Gamma-c) z^{-1}+(\Gamma+c) z^{-2}} \tag{5.20}
\end{equation*}
$$

Let $\beta=(\Gamma-c) /(\Gamma+c)$ then en. (5.20) may be written thus

$$
z=\frac{1-z^{-2}+\beta z^{-1}-\beta z^{-1}}{(\Gamma+c)\left\{1+2 \beta z^{-1}+z^{-2}\right\}}
$$

or, alternatively, as

$$
\begin{equation*}
z=\frac{1}{(\Gamma+c)} \frac{1-z^{-1} \frac{\left(\beta+z^{-1}\right)}{\left(1+\beta z^{-1}\right)}}{1+z^{-1} \frac{\left(\beta+z^{-1}\right)}{\left(1+\beta z^{-1}\right)}} \tag{5.21}
\end{equation*}
$$

On letting $T=\frac{z^{-1}\left(\beta+z^{-1}\right)}{\left(1+\beta z^{-1}\right)}$ in eqn. (5.21) we see that

$$
\begin{equation*}
z=\frac{1}{(\Gamma+C)} \cdot(1-\Gamma) \tag{5.22}
\end{equation*}
$$

A+ $T$ is an a.ll-pass function $N$ event. In fact, when $C=0$, for example, then $\beta=1$ and $T=z^{-1}$.

Substituting for $Z$ from eq. (5.22) into eqn. (5.14) gives the following,

$$
\begin{align*}
& \sigma_{11}=\left[\left\{\left(p_{21} a_{12}-p_{22^{q} 22}\right)(\Gamma+c)-p_{21} a_{22}\right\}\right. \\
& \left.+\left\{\left(\underline{p}_{21} q_{12}-p_{22}{ }^{n} 22\right)(\Gamma+c)+p_{21} \underline{q}_{22}\right\} \mathrm{T}\right] / \text { denom } \\
& \sigma_{12}=-\Delta R(\Gamma+c)(1+T) / \text { debora }  \tag{5.23}\\
& \sigma_{21}=(\Gamma+c)(1+T) \quad / \text { denom } \\
& \sigma_{22}=\left[\left\{\left(p_{12}{ }^{0} 21-p_{11} \sigma_{11}\right)(\Gamma+c)+p_{11} \sigma_{21}\right\}+\right. \\
& \left.\left\{\left(\underline{p}_{12} \underline{o}_{21}-\underline{p}_{11} \tilde{o}_{11}\right)(r+c)-\underline{p}_{11} \underline{q}_{21}\right\} T\right] / \text { denom }
\end{align*}
$$

where

$$
\begin{aligned}
\text { denom }= & \left\{\left(p_{11} a_{12}-p_{12} a_{22}\right)(\Gamma+c)-p_{11} a_{22}\right\}^{-} \\
& +\left\{\left(p_{11} a_{12}-\underline{p}_{12}{ }^{n} 22\right)(\Gamma+c)+p_{11} \underline{a}_{22}\right\} T
\end{aligned}
$$

To avoid delay-free loops on interconnection, actin we have two alternative conditions, either
or

$$
\begin{align*}
& \left(p_{21} a_{12}-p_{22} a_{22}\right)(\Gamma+c)-p_{21} a_{22}=0  \tag{5.24}\\
& \left(p_{12} a_{21}-p_{11} a_{11}\right)(\Gamma+c)+p_{11} a_{21}=0 \tag{5.25}
\end{align*}
$$

Substituting first condition, eqn. (5.24) into eqn. (5.23) gives the folloring,

$$
\begin{align*}
& \sigma_{11}=\alpha_{1} \mathrm{~T} / \cdot\left(1+\alpha_{2} \mathrm{~T}\right) \\
& \sigma_{12}=\alpha_{3}(1+\mathrm{T}) /\left(1+\alpha_{2} \mathrm{~T}\right)  \tag{5.26}\\
& \sigma_{21}=\alpha_{4}(1+\mathrm{T}) /\left(1+\alpha_{2} \mathrm{~T}\right) \\
& \sigma_{22}=\left(\alpha_{5}+\alpha_{6} \mathrm{~T}\right) /\left(1+\alpha_{2} \mathrm{~T}\right)
\end{align*}
$$

where

$$
\begin{aligned}
& \alpha_{1}=2 p_{21}^{2} /[(\Gamma+c) \Delta p] \\
& \alpha_{2}=1+2 p_{11} p_{21} /[(\Gamma+c) \Delta v] \\
& \alpha_{3}=p_{21} \Delta a / a_{22} \\
& \alpha_{4}=p_{21} /\left[\begin{array}{ll}
n \\
-22 & \Delta p
\end{array}\right] \\
& \left.\alpha_{5}=p_{21}\left(p_{12}^{n} 21-p_{11} q_{11}+p_{11} r_{21} /(\Gamma+c)\right) / q_{2.2} \Delta n\right] \\
& \alpha_{6}=\underline{p}_{21}\left(p_{12}{ }^{n} 21-\underline{p}_{11} n_{1,1}-\underline{p}_{11} n_{21} / /(\Gamma+c)\right) /\left[q_{22} \Delta p\right]
\end{aligned}
$$

Substitutine second condition, eqn. (5.25) into eqn. (5.23) gives the followint,

$$
\begin{aligned}
& \sigma_{11}=\left(\alpha_{1}+\alpha_{2} \mathrm{~T}\right) /\left(1+\alpha_{3} \mathrm{~T}\right) \\
& \sigma_{12}=\alpha_{4}(1+\mathrm{T}) /\left(1+\alpha_{3} \mathrm{~T}\right) \\
& \sigma_{21}=\alpha_{5}(1+\mathrm{T}) /\left(1+\alpha_{3} \mathrm{~T}\right) \\
& \sigma_{22}=\alpha_{6} \mathrm{~T} /\left(1+\alpha_{3} \mathrm{~T}\right)
\end{aligned}
$$


where

$$
\begin{aligned}
& \left.\alpha_{1}=-q_{21}\left(p_{21}{ }^{n} \cdot 12-p_{22}{ }^{a_{22}}-p_{21} a_{22} /(\Gamma+c)\right) \cdot p_{11} \Delta a\right] \\
& \alpha_{2}=-q_{21}\left(\underline{p}_{21} \tilde{q}_{12}-\underline{n}_{22} q_{22}+\underline{p}_{21}{ }_{2}{ }_{22} /(\Gamma+c)\right) /\left[n_{11} \Delta a\right] \\
& \alpha_{3}=1-2 n_{21} n_{22} /[\Delta n(\Gamma+C)] \\
& \alpha_{4}=-q_{21} \Delta p / p_{11} \\
& \alpha_{5}=-\overbrace{21}\left[{ }_{10} \Delta r\right] \\
& \alpha_{6}=2 \underline{q}_{21}^{2} /[\Delta \underline{n}(\Gamma+c)]
\end{aligned}
$$

If we compare eqns. (5.18) and eqn. (5.26) we see that we may obtain the latter by the follorin? substitution in the former equation,

$$
z^{-1} \text { replaced by } T
$$

and

$$
\text { L replaced by } 1 /(\Gamma+C)
$$

This is obvious when we consider ean. (5.22) and compere it 1 ith the followins for a series inductance,

$$
\begin{equation*}
\mathrm{Z}=\mathrm{pL}=\frac{1-z^{-1}}{1+z^{-1}} \mathrm{~L} \tag{5.28}
\end{equation*}
$$

Thus we do not need to derive the equations for the tuned-circuits. In fact only one set of equations for the series elements is necessary. The same is true for shunt elements.

### 5.3.5 Series-Tuned Circuit in the Series-1m

In this case $Z=p I+\frac{1}{\underline{D C}}$ and on apnlying the bilinear transformation, we find that

$$
\begin{equation*}
Z=(L+D)\left\{\frac{1+T}{1-T}\right\} \tag{5.29}
\end{equation*}
$$

where $T=z^{-1}\left(\beta+z^{-1}\right) /\left(1+\beta z^{-1}\right)$
and

$$
\dot{\beta}=(D-L) /(D+L)
$$

If we conbaro eqns. (5.28) and (5.29), then the following substitutions are evident,

$$
\begin{array}{ll} 
& z^{-1} \text { replaced by }-T \\
\text { and } & L \quad \text { replaced by } L+D \\
\text { in eqns. } & (5.18) \text { and }(5.19) .
\end{array}
$$

5.4 $\qquad$
5.4.1 Shunt Acrinttance

Consider ir in Pig. 5.1 to be a shunt admittance $Y$ then

$$
T=\left[\begin{array}{cc}
1 & 0 \\
Y & -1
\end{array}\right]
$$

On using eqn. (5.13) we find that
and

$$
\text { denom }=p_{11} q_{12}-p_{12} q_{22}+p_{12} q_{12} Y
$$

5.4.2 Shunt Capacitance

For a shunt capacitance $C$, we have $Y=D C$ and towe ther with the bilinear transformation, eqn. (5.30) becomes
where

$$
\text { denom }=\left(p_{11} q_{12}-p_{12} q_{22}+p_{12} q_{12} c\right)+\left(p_{11} q_{12}-p_{12} q_{22}-p_{12} q_{12}\right)_{z}^{-1}
$$

To avoid delay-free loops on interconnection we have either

$$
\begin{array}{ll}
\text { or } & p_{21} q_{12}-p_{22} q_{22}+p_{22} q_{12} c=0 \\
& p_{12} q_{21}-p_{11} q_{11}-p_{12} q_{11} c=0 \tag{5.33}
\end{array}
$$

Applying the first condition to eqn. (5.31) gives the folloring,

$$
\left.\begin{array}{l}
\sigma_{11}=\alpha_{1} z^{-1} /\left(1+\alpha_{2} z^{-1}\right)  \tag{5.34}\\
\sigma_{12}=\alpha_{3}\left(1+z^{-1}\right) /\left(1+\alpha_{2} z^{-1}\right) \\
\sigma_{21}=\alpha_{4}\left(1+z^{-1}\right) /\left(1+\alpha_{2} z^{-1}\right) \\
\sigma_{22}=\left(\alpha_{5}+\alpha_{6} z^{-1}\right) /\left(1+\alpha_{2} z^{-1}\right)
\end{array}\right\}
$$

where

$$
\begin{aligned}
& \alpha_{1}=-2 p_{22}^{2} c / \Delta p \\
& \alpha_{2}=1-2 p_{11} p_{22} c / \Delta p \\
& \alpha_{3}=\Delta q p_{22} / q_{12} \\
& \left.\alpha_{4}=p_{22} \dot{q}_{12} \Delta p\right] \\
& \alpha_{5}=p_{22}\left(p_{12} q_{21}-p_{11} q_{11}-p_{12} q_{11} c\right) /\left[q_{12} \Delta p\right] \\
& \alpha_{6}=p_{22}\left(p_{12} q_{21}-p_{11} q_{11}+p_{12} q_{11} c\right) /\left[{ }_{-12} \Delta p\right]
\end{aligned}
$$

Applying the second condition to eqn. (5.31) gives the following equations,

$$
\left.\begin{array}{rl}
\sigma_{11} & =\left(\alpha_{1}+\alpha_{2} z^{-1}\right) /\left(1+\alpha_{3} z^{-1}\right) \\
\sigma_{12} & =\alpha_{4}\left(1+z^{-1}\right) /\left(1+\alpha_{3} z^{-1}\right) \\
\sigma_{21} & =\alpha_{5}\left(1+z^{-1}\right) /\left(1+\alpha_{3} z^{-1}\right)  \tag{5.35}\\
\sigma_{22} & =\alpha_{6} z^{-1} /\left(1+\alpha_{3} z^{-1}\right) \\
\text { where } \alpha_{1} & =-q_{11}\left(p_{21} q_{12}-p_{22} q_{22}+p_{22^{q}}{ }_{12} c\right) /\left[p_{12} \Delta q\right] \\
\alpha_{2} & =-q_{11}\left(p_{21} q_{12}-p_{22} q_{22}-p_{22^{-}} \underline{q}_{12} c\right) /\left[p_{12} \Delta q\right] \\
\alpha_{3} & =1+2 q_{11} q_{12} c / \Delta q \\
\alpha_{4} & =-q_{11} \Delta p / p_{12} \\
\alpha_{5} & \left.=-q_{11} \dot{p}_{12} \Delta q\right] \\
\alpha_{6} & =-2 q_{11}^{2} c / \Delta q
\end{array}\right\}
$$

### 5.4.3. Shunt Inductance

In this case, $Y=1 / 0 \mathrm{~L}$ and as we have seen in Chapter 2 we may reolace $z^{-1}$ by $-z^{-1}$ and $C$ by $\Gamma(=1 / L)$ in equations (5.34) and (5.35) to obtain the appropriate equations.

### 5.4.4 Series-tuned circuit in shunt-arm

In this case, $Y=1 /\left(p L+\frac{1}{p C}\right)$ and if we apply the bilinear transformation, we find that

$$
\begin{equation*}
Y=\frac{\left(1-z^{-2}\right)}{(L+D)+2(D-L) z^{-1}+(I+D) z^{-2}} \tag{5.36}
\end{equation*}
$$

On letting $\beta=\left(\frac{D-\frac{L}{D}+\frac{L}{L}}{}\right)$ and $T=\frac{z^{-1}\left(\beta+z^{-1}\right)}{1+\beta z^{-1}}$
we can write eqn. (5.36) as follows,

$$
\begin{equation*}
Y=\frac{1}{(L+D)} \frac{1-T}{1+T} \tag{5.37}
\end{equation*}
$$

For a shunt, capacitance,

$$
\begin{equation*}
Y=p C=\frac{1-z^{-1}}{1+z^{-1}} c \tag{5.38}
\end{equation*}
$$

Therefore, we may obtain the appropriate $\sigma$-paraneters for a series-tuned circuit by renlacinc $z^{-1}$ by $T$ and $C$ by $1 /(L+D)$ in eans. (5.34) and (5.35).

### 5.4.5 Parallel-Tuned Gircuit in Ghunt-im

In this case, $Y=p C+\frac{1}{p J}$ and on using the bilinear transformation we find that $Y$ may be mitten as follows,

$$
\begin{equation*}
Y=(C+\Gamma) \quad \frac{1}{1}+\frac{T}{T} \tag{5.39}
\end{equation*}
$$

where $T$ is as beiore.

Therefore, we mav obtain the $\sigma$-parameters for a parallel-tuned circuit by replacins, $z^{-1}$ by $-T$ and $C$ by $C+\Gamma$ in eqns. (5.34) and (5.35).

### 5.5 Basic Zquations for sourcos, Tominations and Intorcomections

### 5.5.1 Introduction

In the previous sections, as we dealt with 2-ports, it was clear that the matrix $P$ defined the transformation on the input voltages and currents, whilst 2 defined the transformation on the output voltares and currents. In the case of i-ports, however, ve can use either $P$ or $Q$. Thus, for each 1-nort, there are two alternative equations.

### 5.5.2 Resistive Voltare Source

rhe equations of the voltage source (ing. 5.3) are, in terms of P , as follous,

$$
\begin{align*}
& V_{0}=V+R_{s} I \\
& X=p_{11} V+p_{12} I  \tag{5.40}\\
& Y=p_{21} V+p_{22} I
\end{align*}
$$

In matrix form, eqn. (5.40) can be written as

$$
V_{0}=\left(\begin{array}{ll}
1 & R_{S} \tag{5.4i}
\end{array}\right)\binom{V}{I}
$$

and $\quad\binom{X}{Y}=P\binom{V}{I}$
On eliminating $\binom{V}{I}$ between eqns. (5.41) and (5.42) we find that

$$
v_{0}=\left(\begin{array}{ll}
1 & R_{s}
\end{array}\right) P^{-1}\binom{X}{Y}
$$

from which we find that

$$
\begin{aligned}
X & =\frac{p_{12}-p_{11} R_{S}}{p_{22}-p_{21} R_{S}}+\frac{\Delta_{p}}{p_{22}-p_{21} R_{s}} V_{0} \\
\text { or } \quad X & =\beta_{2} Y+\beta_{1} V_{o}
\end{aligned}
$$

The sional-flow diagran appears in Fif. (5.4). The delay-free path from $Y$ to $X$ may be eliminated by setting $p_{12}=p_{11} R_{s}$ and then eqn. (5.43) becomes

$$
\begin{equation*}
X=p_{i!} V_{0} \tag{5.44}
\end{equation*}
$$

that is, a wave-source [30].

Let us nov consider using \& to uerive a relationship between $Y$ and X . Clecurly, the defining equation is similur to eqn. (5.4j) with the elements of a renlacing the corresponding elerents of F. Eowever, in teras of the elements of $z^{-1}$ we have

$$
\begin{equation*}
x=-\frac{\left(q_{i 2}+q_{22} R_{s}\right)}{\left(q_{11}+q_{21} R_{s}\right)} Y+\frac{1}{\left(q_{1 i}+q_{2 i} R_{s}\right)} V_{0} \tag{5.45}
\end{equation*}
$$

The delay-free path frorn $Y$ to $X$ may be eliminated by setting $q_{12}=-q_{22} R_{s}$ and then eqn. (5.45) becomes

$$
\begin{equation*}
x=\frac{q_{22}}{\Delta q} v_{0} \tag{5.46}
\end{equation*}
$$

Egain, this represents a wave-source [30].

### 5.5.3 Load Impedance

The equations of a load impedance $Z$ (Fig. 5.5) are, in terms of $P$, as follows

$$
\begin{align*}
& V=Z I \\
& X=p_{11} V+p_{12} I  \tag{5.47}\\
& Y=p_{21} V+p_{22} I
\end{align*}
$$

On eliminating $V$ and $I$ we find that .

$$
\begin{equation*}
Y=\frac{p_{2 i}^{2}+p_{22}}{p_{1 i^{2}}+p_{12}} \quad X \tag{5.48}
\end{equation*}
$$

We are particularly interested in the case when $:=R_{L}$, load resistance and eqn. (5.48) therefore becones

$$
\begin{equation*}
Y=\frac{p_{21} R_{L}+p_{22}}{p_{11} R_{L}+p_{12}} \quad X \tag{5.49}
\end{equation*}
$$

The delay-free path from $X$ to $Y$ may be eliminated by setting

$$
p_{21} R_{L}+p_{22}=0
$$

in which case

$$
\mathrm{Y}=0 \text {, a wave-sink }[乡 0] .
$$

If we had used 2 to define the lond resistance then eqn. (5.49) would have been

$$
\begin{equation*}
Y=\frac{-q_{21} R_{L}+q_{11}}{q_{22^{2}}{ }^{2}}-q_{12} \tag{5.50}
\end{equation*}
$$

where $2^{-1}=\left[q_{i j}\right]$.
The delay-free path from $X$ to $Y$ may be eliminated by seting

$$
q_{i 1}-q_{2 i} R_{L}=0
$$

in which case, we have as before a mave-sint [ 30 ].

### 2.5.4...Intoromocitions

To connect two elenents together, we must apply Kirchhoff's Laws at the common junction (Fig. 5.6), that is

$$
\begin{aligned}
& V_{1}=V_{2} \\
& I_{1}=-I_{2}
\end{aligned}
$$

or equivalently,

$$
T=\left[\begin{array}{rr}
1 & 0  \tag{5.51}\\
0 & -1
\end{array}\right]
$$

Combining eqns. (5.13) and (5.51) we find that

$$
\left.\begin{array}{l}
\sigma_{i 1}=\left(p_{21} q_{12}-p_{22} q_{22}\right) / \text { denom. }  \tag{5.52}\\
\sigma_{12}= \\
\sigma_{21}= \\
\sigma_{22} \\
\sigma_{22}=\left(p_{12} q_{21}-p_{11} q_{11}\right) / \text { denom. } \\
\text { denom. }
\end{array}\right\}
$$

```
and denory = p p11 q12 - pi2q}\mp@subsup{q}{22}{
```

To avoid delay-free loops, we have the same condition as for reactive two-port elements, namely that the constant term in the numerator of $\sigma_{11}$ or $\sigma_{22}$ must be zero. Thererore, we have two alternative conditions,

$$
\begin{array}{ll} 
& p_{21} q_{12}-p_{22} q_{22}=0 \\
\text { or } \quad & p_{12} q_{2 i}-p_{11} q_{11}=0 \tag{5.54}
\end{array}
$$

Applying the first condition to eqn. (5.52) gives

$$
\left.\begin{array}{l}
\sigma_{11}=0 \\
\sigma_{12}=p_{21} \Delta q / q_{22} \\
\sigma_{21}=p_{21} / q_{22} \Delta p \\
\sigma_{22}=p_{21}\left(p_{12} q_{21}-p_{11} q_{1 i}\right) / q_{22} \Delta p
\end{array}\right\} \text { (5.55) }
$$

and applying the second condition to eqn. (5.52) gives

$$
\begin{align*}
& \sigma_{1 i}=-p_{i 2} \Delta q\left(p_{21} q_{12}-p_{22} q_{22}\right) / q_{11} \\
& \sigma_{12}=-q_{11} \Delta p / p_{12} \\
& \sigma_{2 i}=-q_{1 i} / p_{i 2} \Delta q  \tag{5.56}\\
& \sigma_{22}=0
\end{align*}
$$

In the discussion on interconnections in Chapter 2, it was desired to have a dircct connection betwcen successive two-ports. In this chapter, however, we are dealing more senerally with interconnections. Let us, nevertheless, consider the case when $\sigma_{11}=\sigma_{22}=0$ and then eqn. (5.52) sives for $\sigma_{12}$ and $\sigma_{21}$ the folloring,

$$
\begin{align*}
& \sigma_{12}=p_{21} \Delta q / q_{22}  \tag{5.57}\\
& \sigma_{21}=p_{21} / q_{22} \Delta p
\end{align*}
$$

Hany equivalent expressions for $\sigma_{12}$ and $\sigma_{21}$ may be found because eqns. (5.53) and (5.54) must be satisfied simultaneously.

### 5.6 Study of Additional Realigability. Conditions

### 5.6.1 Introduction

We have seen, in previous chanters, that it is necessary for realizability, to have no delay-free loops in the disital structure. There are however, additional constraints which must be satisfied. The first of these shall be referred to as the 'cascade' condition and may be expressed thus: The transmission matrix of a disital t::o-port netrork talen as a whole must be equal to that formed by the product of the transmission matrices of the constituent two-ports. The significance or this condition will be discussed in section 5.6.2. The second condition whim shail be referred to as the 'Transfer-function' condition states that if $H(0)$ is the transfer function of the analorue prototype and $G(z)$ that of the derived dirital filter then

$$
\begin{equation*}
G(z) \equiv K \hat{G}(z) \tag{5.58}
\end{equation*}
$$

where $\quad K$ is independent of frequency
and $\quad \hat{G}(z)$ is the dicital transfer function obtained on applying the bilinear transformation to $H(p)$. Fe shall look at this more closely in sections 5.6 .3 and 5.6.4.

Finally, the tinird condition concerns the individual circuit elements and is discussed further in section 5.6.5.

### 5.6.2 Cascade Condition

Let us consider arain a two-port (nir. 5.1) described by its modified transinission matrix, $T=\left[\begin{array}{ll}A & B \\ C & D\end{array}\right]$, and subpose ve decompose the network into a cascade of two-ports (e.g. series impedances, shunt admittances or unit elements). Let $P$ and $Z$ be, as before, $2 \times 2$ non-sinrular matrices such that

$$
\left[\begin{array}{l}
X_{1} \\
Y_{1}
\end{array}\right]=P\left[\begin{array}{l}
V_{1} \\
I_{1}
\end{array}\right]
$$

and

$$
\left[\begin{array}{l}
x_{2} \\
y_{2}
\end{array}\right]=2\left[\begin{array}{l}
v_{2} \\
I_{2}
\end{array}\right]
$$

We may write therciore, by eliminating $V_{1}, V_{2}, I_{1}$ and $I_{2}$, that

$$
\left[\begin{array}{l}
X_{1}  \tag{5.59}\\
Y_{1}
\end{array}\right]=P T ?^{-1}\left[\begin{array}{l}
X_{2} \\
Y_{2}
\end{array}\right]
$$

The matrix $R=P T Q^{-1}$ is then the transmission matrix for the nev variables $X_{1}, X_{2}, Y_{1}$ and $Y_{2}$. Let us now transform, in a similar way, each constituent two-port. For the kth-section, we have

$$
\left[\begin{array}{l}
X_{1}^{(k)}  \tag{5.60}\\
Y_{1}^{(k)}
\end{array}\right]=P_{k} T_{k} Q_{k}^{-1}\left[\begin{array}{l}
X_{2}^{(k)} \\
Y_{2}^{(k)}
\end{array}\right]
$$

Hotice that even an interconnection between, say a series inductance and a shunt capacitance can be treated as a constituent two-port. As a consequence of this, we have at a junction (Fir. 5.7)

$$
\begin{aligned}
& x_{1}^{(k+1)}=Y_{2}^{(k)} \\
& x_{2}^{(k)}=Y_{1}^{(k+1)}
\end{aligned}
$$

or, equivalently in matrix form

$$
\left[\begin{array}{l}
X_{2}^{(k)}  \tag{5.61}\\
Y_{2}^{(k)}
\end{array}\right]=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\left[\begin{array}{l}
X_{1}^{(k+1)} \\
Y_{1}^{(k+1)}
\end{array}\right]
$$

If we now cascade $n$ sections tofether (Fir. 5.8) we find that

$$
\begin{align*}
& {\left[\begin{array}{l}
X_{1}^{(1)} \\
Y_{1}^{(1)}
\end{array}\right]=P_{1} T_{1} Q_{1}^{-1} C P_{2} T_{2} 0_{2}^{-1} \ldots \ldots P_{n}^{T} n^{-1}\left[\begin{array}{c}
X_{2}^{(n)} \\
Y_{2}^{(n)}
\end{array}\right]} \\
& \text { where } C=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] . \tag{5.62}
\end{align*}
$$

The equivalent expression for the whole network is

$$
\left[\begin{array}{l}
X_{1}^{(1)}  \tag{5.63}\\
Y_{1}(1)
\end{array}\right]=P T Q^{-1}\left[\begin{array}{l}
X_{2}^{(n)} \\
Y_{2}(n)
\end{array}\right]
$$

Let $\hat{T}$ be the conventional transmission natrix of a tro-port and $\hat{T}_{k}$ the matrix for the kth. constituent two-port element. Fe have inmediately that

As

$$
\begin{equation*}
\hat{T}=\hat{T}_{1} \hat{T}_{2} \ldots \hat{T}_{n} \tag{5.64}
\end{equation*}
$$

$$
\left[\begin{array}{l}
V_{1} \\
I_{1}
\end{array}\right]=T\left[\begin{array}{l}
V_{2} \\
I_{2}
\end{array}\right] \text { and }\left[\begin{array}{c}
V_{1} \\
I_{1}
\end{array}\right]=\hat{T}\left[\begin{array}{c}
V_{2} \\
-I_{2}
\end{array}\right] \text { we have }
$$

that

$$
\begin{equation*}
\hat{T}=T D \tag{5.65}
\end{equation*}
$$

where

$$
D=\left[\begin{array}{rr}
1 & 0 \\
0 & -1
\end{array}\right]
$$

and therefore eqn. (5.64) becomes

$$
\begin{equation*}
T=T_{1} D T_{2} D \ldots T_{n} \tag{5.66}
\end{equation*}
$$

He are now in a position to compare eqns. (5.62) and (5.63). It is clear that

$$
\left.\begin{array}{l}
P_{1}=P \\
Q_{k}^{-1} C P_{k+1}=D \quad ; \quad k=1,2, \ldots,(n-1) \\
Q_{n}^{-1}=Q^{-1}
\end{array}\right\} \text { (5.67) }
$$

Thus eqn. (5.67) represents the cascade cordition. For fenerality, we have allowed each constituent two-port to be transformed by different matrices but henceforth shall consj.der the case in which $P_{k}=P,\left\{_{k}=Q\right.$ for all $k$. Thus ean. (5.67) reduces to

$$
\begin{equation*}
Q^{-1} C P=D \tag{5.68}
\end{equation*}
$$

As an example, let us consider the current-wave formulation for wich

$$
P=\left[\begin{array}{cc}
G_{1} & 1 \\
G_{1} & -1
\end{array}\right], \quad 0=\left[\begin{array}{cc}
G_{2} & 1 \\
G_{2} & -1
\end{array}\right]
$$

On using eqn. (5.68) in the fom $C P=2 D$, we find that

$$
C P=\left[\begin{array}{cc}
G_{1} & -1 \\
G_{1} & 1
\end{array}\right]
$$

$$
\text { and } \quad D=\left[\begin{array}{cc}
G_{2} & -1 \\
G_{2} & 1
\end{array}\right]
$$

Therefore, $C D=Q D$ if $G_{1}=G_{2}$. As the cascade condition is concerned with connections of tro-ports and as the theory described in Chapter 2 allo: us to interconnect directly two-ports if their appropriate port resistances (or conaductances) are equal, the current-vave formulation clearly satisfies the cascade condition.

### 2.6.3 The Transfer Function Condition

In this section, we shall investigate the conditions necessary to realise a disital filter whose transfer function $G(z)$ is related to that of the analorue prototype by the folloning,

$$
\begin{equation*}
G(z) \equiv K \hat{G}(z) \tag{5.69}
\end{equation*}
$$

where $K$ is independent of $z$ and $\hat{G}(z)$ is the function obtained by applying the bilinear transformation to $H(p)$.

Consider the networl of Pis. 5.9. The transfer function $H(p)$ is given by the expression,

$$
\begin{equation*}
H(p)=\frac{1}{\left(A-B / R_{L}\right)+R_{S}}\left(C-D / R_{I}\right) \tag{5.70}
\end{equation*}
$$

where

$$
\left[\begin{array}{l}
V_{1} \\
I_{1}
\end{array}\right]=\left[\begin{array}{ll}
A & B \\
C & D
\end{array}\right] \cdot\left[\begin{array}{l}
V_{2} \\
I_{2}
\end{array}\right]
$$

There are three altermative digital equivalents of Fig. 5.9,
(i) Peitreis-type design (Pis. 5.10), (ii) Design from the source (Fie. 5.11) and (iii) Design from the load (Fig. 5.12). Each will be discussed in turn.

## (i) Fettreis-Type Design

In this case, the delay-iree paths of both the resistive source and the load resistance have been eliminated. :le have therefore imposed the following constraints (see sections 5.5.2 and 5.5.3)
and

$$
\left.\begin{array}{l}
p_{12}=p_{11} R_{s} \quad \text { or } \quad q_{12}=-q_{22} R_{s} \\
p_{22}=-p_{21} R_{L} \text { or } \quad \dot{q}_{11}=q_{21} R_{L}
\end{array}\right\}
$$

These conditions lead to the following, since the relationships between the elements of $P$ and $\gamma$ defined in earn. (5.71) can not chance,

$$
\frac{p_{12}}{p_{11}}=R_{1} \quad \text { or } \quad \frac{\underline{q}_{12}}{q_{22}}=-R_{2}
$$

and

$$
\frac{\underline{p}_{22}}{\mathrm{p}_{21}}=-\mathrm{R}_{1} \quad \text { or } \quad \frac{\underline{q}_{11}}{\mathrm{q}_{21}}=\mathrm{R}_{2}
$$

The variables $R_{1}$ and $R_{2}$ were chosen so as to connect the know transformations to the present discussion.

Jet us consider the transfer function of Firs. 5.10 and recall that

$$
\left[\begin{array}{l}
Y_{1}  \tag{5.72}\\
Y_{2}
\end{array}\right]=\left[\begin{array}{ll}
\sigma_{11} & \sigma_{12} \\
\sigma_{21} & \sigma_{22}
\end{array}\right]\left[\begin{array}{l}
X_{1} \\
X_{2}
\end{array}\right]
$$

The terminal conditions are as follows,

$$
\begin{array}{ll}
x_{2} & =0 \\
\text { and } \quad X_{1}=\beta_{1} V_{0} \text { where } \beta_{1}=p_{11} \text { or } \frac{q_{22}}{\Delta q} \text { (see } \\
\text { section 5.5.2). Thus the transfer function } G(z)=Y_{2} / V_{0}
\end{array}
$$

can be written as

$$
\begin{equation*}
G(z)=\beta_{1} \sigma_{21} \tag{5.73}
\end{equation*}
$$

Recalling eqn. (5.13) we have from eqn. (5.73) that

$$
G(z)=\frac{\beta_{1}}{p_{11}\left(q_{12} A+q_{22} B\right)+p_{12}\left(q_{12} C+q_{22} \bar{D}\right)}
$$

or, equivalently as

$$
\begin{equation*}
G(z)=\frac{\beta_{1}}{\underline{p}_{11} q_{12}} \frac{1}{\left(A+\frac{q_{22}}{q_{12}} B\right)+\frac{p_{12}}{p_{11}} \cdot\left(C+\frac{q_{22}}{q_{12}}\right.} \tag{5.74}
\end{equation*}
$$

On equating coefficients of $A, B, C$ and $D$ in eqn. (5.74) and eqn. (5.70) and noting eqn. (5.69), we find that

$$
\begin{aligned}
& K=\frac{\beta_{1}}{p_{11} q_{12}} \\
& \frac{q_{22}}{q_{12}}=-\frac{1}{R_{L}}
\end{aligned}
$$

and

$$
\frac{p_{12}}{p_{11}}=R_{s}
$$

The last two conditions lead, for reasons given before, to the following

$$
\frac{p_{12}}{p_{11}}=R_{1}, \quad \frac{q_{12}}{q_{22}}=-R_{2}
$$

Combining conditions for Pettweis-type structures, we have

$$
\frac{p_{12}}{p_{11}}=R_{1}, \frac{q_{12}}{q_{22}}=-R_{2}
$$

and

$$
\frac{p_{22}}{p_{21}}=-q_{1} \quad \text { or } \frac{q_{11}}{q_{21}}=R_{2}
$$

also

$$
K=\frac{1}{q_{12}} \text { or } \frac{1}{2 p_{1} i_{12} q_{11}}
$$

All these results are tabulated at the end of this section.

## (ii) Design from the source

In this case only the delay -free path of the resistive source has been eliminated. : We have therefore imposed the constraint

$$
p_{12}=p_{11} R_{s} \text { or } q_{12}=-q_{22} R_{s}
$$

These conditions lead to the following.

$$
\frac{p_{12}}{p_{11}}=R_{1} \quad \text { or } \cdot \frac{q_{12}}{q_{22}}=-R_{2}
$$

Let us consider the transfer function of Fig. 5.11, the terminal conditions are

$$
\text { and } \begin{aligned}
X_{1} & =\beta_{1} V_{0} \\
X_{2} & =\alpha Y_{2}
\end{aligned}
$$

These, to ce the wi. th eqn. (5.72) give

$$
\begin{equation*}
G(z)=\frac{Y_{2}}{V_{0}}=\frac{\beta_{1} \sigma_{21}}{\left(1-\alpha \sigma_{22}\right)} \tag{5.75}
\end{equation*}
$$

On combining eqns. (5.75) and (5.13), we find that

$$
\begin{gathered}
\left.G(z)=\frac{\beta_{1}}{p_{11}\left(q_{12}+\alpha q_{11}\right)} \frac{1}{A+\frac{q_{22}+\alpha q_{21}}{q_{12}+\alpha q_{11}}+\frac{p_{12}}{p_{11}}\left(C+\frac{q_{22}+\alpha q_{21}}{q_{12}+\alpha q_{11}}\right.}\right) \\
\end{gathered}
$$

Comparing eqns. (5.76), (5.69) and (5.70) and equating coefficients of $A, B, C$ and $D$ torether with $K$, we find that

$$
\begin{align*}
& K=\frac{\beta_{1}}{p_{11}\left(q_{12}+\alpha q_{11}\right)} \\
& \frac{p_{12}}{p_{11}}=R_{s} \tag{5.77}
\end{align*}
$$

and

$$
\frac{q_{22}+\alpha q_{21}}{q_{12}+\alpha q_{11}}=-\frac{1}{R_{L}} \quad \int
$$

There are two possible values for $\alpha$, depending on whether $P$ or $\ell$ was used for the derivation of the sismal-inlow diagram for $R_{L}$ (see section 5.5.3).

If $\quad \alpha=\frac{p_{21} R_{L}+p_{22}}{p_{11} R_{L}+p_{12}}$ then the last expression in eqn. (5.77) can be written as follo:rs,

$$
\begin{equation*}
\frac{q_{22}}{q_{12}} \frac{\left(p_{12}+p_{11} R_{L}\right)+q_{21}\left(p_{22}+p_{21} R_{L}\right)}{\left(p_{11} R_{L}\right)+q_{11}\left(p_{22}+p_{21} R_{L}\right)} \equiv-\frac{1}{R_{L}} \tag{5.78}
\end{equation*}
$$

Clearly eqn. (5.75) must be satisîied for all values of $R_{L}$. This leads to the following set of constraints,

$$
\begin{align*}
& \frac{q_{22}}{q_{21}}=\frac{-p_{21}}{p_{11}} \\
& \frac{q_{12}}{q_{11}}=\frac{p_{22}}{p_{12}}  \tag{5.79}\\
& \frac{q_{21}}{q_{11}}=\frac{p_{11}}{p_{12}}
\end{align*}
$$

and

If $\alpha=\frac{q_{11}-q_{21} R_{L}}{-q_{12}+q_{22} R_{L}}$ then we have, from eqn. (5.77), that

$$
\frac{q_{22}\left(-q_{12}+q_{22} R_{L}\right)-q_{21}\left(q_{i 1}-q_{21} R_{L}\right)}{q_{12}\left(-q_{12} q_{22}^{i} q_{22_{L}}\right)+q_{11}\left(q_{11}-q_{2 i} R_{L}\right)} \equiv-\frac{1}{R_{L}}
$$

which leads to the following

$$
\begin{aligned}
q_{22}^{2} & =q_{21}^{2} \\
\text { and } \quad q_{11}^{2} & =q_{12}^{2}
\end{aligned}
$$

It is easy to show that when $q_{22}=q_{21}$ then $q_{11}=-q_{12}$
and when $q_{22}=-q_{21}$ then $q_{11}=+q_{12}$.

Note also that the expression for $K$ in eqn. (5.77) depends on $\alpha$ and $\beta$, and since there are two possible values for each, $K$ has four alternative values. These values appear in the summary at the end of this section.

## (iii) Desim from the load

In this case only the delay-free path of the load has been eliminated and therefore the following constraint has been applied (see section 5.5.5),

$$
p_{22}=-p_{21} R_{L} \quad \text { or } \quad q_{11}=q_{21} R_{L}
$$

These conditions lead to the following,

$$
\frac{p_{22}}{p_{21}}=-R_{1} \quad \text { or } \frac{q_{11}}{q_{21}}=R_{2}
$$

Let us consider the transfer function of Fig . 5.12, the terminal conditions of which are

$$
\begin{aligned}
X_{1} & =\beta_{1} V_{0}+\beta_{2} Y_{i} \\
\text { and } \quad X_{2} & =0
\end{aligned}
$$

These, together with eqn. (5.72) give

$$
\begin{equation*}
G(z)=\frac{Y_{2}}{\bar{V}_{0}}=\frac{\beta_{1} \sigma_{21}}{\left(1-\beta_{2}^{\sigma_{1 i}}\right)} \tag{5.80}
\end{equation*}
$$

On combining eqns. (5.80) and (5.13), we find that

$$
\begin{gathered}
\left.G(z)=\frac{\beta_{1}}{q_{12}\left(p_{11}-\beta_{2} p_{21}\right)} \frac{1}{\pi+\frac{q_{22}}{q_{12}} B+\frac{1}{\left(p_{12}-\beta_{2} p_{22}\right)}(1)+\frac{q_{22}}{p_{11}-\beta_{2} p_{21}}}\right) \\
\text { (5.81). }
\end{gathered}
$$

Comparing eqn. (5.81) with eqns. (5.69) and (5.70) and equating coefficients of $\dot{A}, B, C, D$ and $K$ we find that

$$
\begin{align*}
& \left.K=\frac{\beta_{i}}{q_{12}\left(p_{11}-\beta_{2} p_{21}\right.}\right) \\
& \frac{q_{22}}{q_{12}}=-\frac{1}{R_{L}}  \tag{5.82}\\
& \frac{p_{12}-\beta_{2} p_{22}}{p_{11}-\beta_{2} p_{21}}=R_{s}
\end{align*}
$$

We have to consider the alternative values for $\beta_{1}, \beta_{2}$ depending on whether $P$ or $Q$ is used for the voltane source.

If $P$ was used then, from section 5.5 .2

$$
\beta_{1}=\frac{\Delta \frac{p}{p_{22}-p_{21} R_{s}}}{\text { a }}
$$

and

$$
\beta_{2}=\frac{p_{12}-p_{11} R_{S}}{p_{22}-p_{21} R_{s}}
$$

and together with e.un. (5.82) give

$$
K=1 / q_{12}
$$

and

$$
\begin{aligned}
\frac{p_{12}-\beta_{2} p_{22}}{p_{11}-\beta_{2} p_{21}} & =\frac{p_{12}\left(p_{22}-p_{2 i} R_{g}\right)-p_{22}\left(p_{i 2}-p_{1 i} R_{s}\right)}{p_{11}\left(p_{22}-p_{2 i} R_{s}\right)-p_{21}\left(p_{12}-p_{11} R_{s}\right)} \\
& =R_{s}
\end{aligned}
$$

If $Q$ was used then, from section 5.5 .2

$$
\begin{aligned}
& \beta_{1}=\frac{1}{\left(q_{11}+q_{21} R_{s}\right)} \\
& \beta_{2}=-\frac{\left(q_{12}+q_{22} R_{s}\right)}{\left(q_{11}+q_{21} R_{s}\right)}
\end{aligned}
$$

and together with eqn. (5.82) we find that
$\frac{p_{12}\left(q_{11}+q_{21} R_{s}\right)+\left(q_{12}+q_{22} R_{s}\right) p_{22}}{p_{11}\left(q_{11}+q_{2 i} R_{s}\right)+\left(q_{12}+q_{22} q_{s}\right) p_{21}} \equiv R_{s}$
Mors eau. ( 5.85 ) must be satisfied for all $R_{s}$. This leads to the following set of equations,

$$
\begin{align*}
& \frac{p_{12}}{p_{22}}=-\frac{q_{12}}{q_{11}} \\
& \frac{p_{11}}{p_{21}}=-\frac{q_{22}}{q_{21}} \tag{5.84}
\end{align*}
$$

and

$$
\frac{p_{21}}{p_{22}}=-\frac{q_{21}}{q_{1 i}}
$$

### 5.6.4 Summary of the Transfer Function Condition

(i) Fettueis-mype Design
(a) Matrix $P$ for source and load.

$$
\begin{aligned}
& \frac{p_{12}}{p_{11}}=-\frac{p_{22}}{p_{21}}=R_{i} \\
& \frac{q_{12}}{q_{22}}=-R_{2}, \quad \mathrm{~F}=\frac{1}{q_{12}}
\end{aligned}
$$

(b) Matrix $P$ for source, i for load.

$$
\begin{aligned}
\frac{p_{12}}{p_{11}} & =R_{1}, \frac{q_{11}}{q_{21}}=-\frac{q_{12}}{q_{22}}=R_{2} \\
K & =\frac{1}{q_{12}}
\end{aligned}
$$

(c) Matrix \& for source, $P$ for load.

$$
\begin{aligned}
& \frac{p_{12}}{p_{11}}=-\frac{p_{22}}{p_{21}}=R_{1} \\
& \frac{q_{12}}{q_{22}}=-R_{2}, \quad K=\left\{\frac{q_{22}}{p_{11} \Delta q}\right\} \quad \frac{1}{q_{12}}
\end{aligned}
$$

(d) Natrix 2 for source and load.

$$
\begin{aligned}
\frac{p_{12}}{p_{1 i}} & =R_{1}, \frac{q_{11}}{q_{21}}=-\frac{q_{12}}{q_{22}}=R_{2} \\
K & =\left\{\frac{1}{2 p_{1 i} q_{11}}\right\} \frac{1}{q_{12}}
\end{aligned}
$$

(ii) Design from the source-end
(a) Hatrix $P$ for source and load.

$$
\begin{aligned}
& \frac{\underline{p}_{12}}{\underline{p}_{11}}=R_{1}, \quad K=\frac{-p_{12}}{q_{11} \Delta p}\left(p_{11}+\frac{p_{12}}{R_{L}}\right) \\
& \frac{\underline{q}_{22}}{q_{21}}=-\frac{p_{21}}{p_{11}}, \quad \frac{q_{12}}{q_{11}}=-\frac{p_{22}}{p_{12}} \\
& \frac{q_{21}}{q_{11}}=\frac{p_{11}}{p_{12}}
\end{aligned}
$$

(b) Latrix $P$ for source, $Z$ for load.

$$
\begin{aligned}
\frac{p_{12}}{\underline{p}_{11}} & =R_{1}, q_{22}= \pm q_{21}, q_{11}=\mp q_{12} \\
K & =\frac{1}{2}\left\{\frac{1}{q_{12}}-\frac{1}{q_{22} R_{L}}\right\}
\end{aligned}
$$

(c) Watrix 2 for source, $P$ for load.

$$
\begin{aligned}
& \frac{p_{12}}{p_{11}}=R_{1}, \frac{q_{12}}{q_{22}}=-R_{2} \\
& \frac{q_{22}}{q_{21}}=-\frac{p_{21}}{p_{11}}, \frac{q_{12}}{q_{11}}=-\frac{p_{22}}{p_{12}} \\
& \frac{q_{21}}{q_{22}}=\frac{p_{11}}{p_{12}}, K=\left\{\frac{q_{22}}{p_{11}} \cdot\right\}\left\{\frac{-p_{12}}{q_{11} \Delta p}\left(p_{11}+\frac{p_{12}}{R_{L}}\right)\right\}
\end{aligned}
$$

(d) Matrix \& for source and load.

$$
\begin{aligned}
& \frac{p_{12}}{p_{11}}=R_{1}, \frac{q_{12}}{q_{22}}=-R_{2} \\
& q_{22}= \pm q_{21}, \dot{q}_{11}=\mp q_{12}, K=\frac{q_{22}}{2 p_{11} \Delta q}\left\{\frac{1}{q_{12}}-\frac{1}{q_{22} R_{L}}\right\}
\end{aligned}
$$

## (iii) Design from the load-end

(a) Matrix P for source and load.

$$
\frac{p_{22}}{p_{21}}=-R_{1}, \frac{q_{12}}{q_{22}}=-R_{2}, r=\frac{1}{q_{12}}
$$

(b) Matrix $P$ for source, $२$ for load.

$$
\frac{q_{11}}{q_{21}}=-\frac{q_{12}}{\underline{q}_{22}}=R_{2}, \quad K=\frac{1}{q_{12}}
$$

(c) Matrix Q for source, $P$ for load.

$$
\begin{aligned}
& \frac{p_{22}}{p_{21}}=-R_{1}, \frac{q_{12}}{q_{22}}=-R_{2} \\
& \frac{p_{12}}{p_{22}}=-\frac{q_{12}}{q_{11}}, \frac{p_{11}}{p_{21}}=-\frac{q_{22}}{q_{21}} \\
& \frac{p_{21}}{p_{22}}=-\frac{q_{21}}{q_{11}}, \quad K=\frac{1}{q_{12}}\left\{\frac{p_{22}}{q_{11} \Delta p}\right\}
\end{aligned}
$$

(d) Matrix $\geq$ for source and load.

$$
\begin{aligned}
& \frac{q_{11}}{q_{21}}=-\frac{q_{12}}{q_{22}}=\dot{R}_{2} \\
& \frac{p_{12}}{p_{22}}=-\frac{q_{12}}{q_{11}}, \quad \frac{p_{11}}{p_{21}}=-\frac{q_{22}}{q_{21}} \\
& \cdot \\
& \frac{p_{21}}{p_{22}}=-\frac{q_{21}}{q_{11}}, \quad K=\frac{1}{q_{12}}\left\{\frac{1}{2 p_{22}^{n}-22}\right\}
\end{aligned}
$$

The transfer function condition reintroduces the port resistance concept which was abandoned at the beginning of this chapter. Several important points arise and we shall discuss them here.

It is clear that, disregardine for the moment the terminations, $P$ is associated with the input port and its resistance $R_{1}$ and 2 with the output port and $R_{2}$. However, we may use either $P$ or $Q$ to derive the SFD's of the terminations. Let us consider, for example, the value of K in eqn. (5.74), namely, $\beta_{1} / p_{11} q_{12}$. The quantity $p_{11}$ is definitely associated with the input and $\underline{a}_{12}$ with the output. Horeover $\beta_{1}$ is associated with input irrespective of whether $\beta_{1}$ is derived from $P$ or 2 . As another example, let us consider eqn. (5.77) where $K=\beta_{1} /\left\{\underline{p}_{11}\left(q_{12}+\alpha q_{11}\right)\right\}$. Again we have that $p_{11}$ and $\beta_{1}$ are associated with the input and $q_{11}$ and $q_{12}$ with the output. The multiplier variable $\alpha$ may be derived using $P$ or 2 , but it is still associated with the outnut port and its correspondins resistance $R_{2}$. This discussion immediately implies that the constraints of eqn. (5.79) need only hold then $R_{1}=R_{2}$ since $P$ and 2 refer to the same port. A similar armument may be applied to eqn. (5.84).

In the expression for K , the separation of variables into those at the input and those at the output will have important consequences in the nert section when :e discuse sensitivity.

Finally, the simificance of expressions such as $p_{12} / p_{11}=R_{1}$ lies in the fact that $R_{1}$ must be finite and non-zero and therefore both $p_{11}$ and $p_{12}$ must be non-zero.

We have nowhere assumed that $P$ and $Q$ are the same, althouch there are cases in which the know transfomations exhibit this similarity. For example, the voltage formulation, which is defined as follows

$$
P=\left[\begin{array}{rr}
1 & R_{1} \\
1 & -R_{1}
\end{array}\right], \quad Q=\left[\begin{array}{rr}
1 & R_{2} \\
1 & -R_{2}
\end{array}\right]
$$

satisfies the condition $P=Q$ when $R_{1}=R_{2}$. In this case, and also for the current and power-wave formulations, using $P$ or $Q$
to derive the simal-flow diamans leads to the same structure and thus the argument used in this and the precedine section could be simplificd. However, :re shall consider, in a later chapter, transformations for wich the condition $P=$ ? when $R_{1}=R_{2}$ never holds and this justifies the more seneral approach taken here.

We could discuss the three cases above as special cases of the feneral tro-port configuration shom in Pig. 5.13. The transfer function may be derived as follows,

We require

$$
\begin{equation*}
G(z)=\frac{B_{2}}{V_{0}} \tag{5.85}
\end{equation*}
$$

and we kno: that

$$
\left[\begin{array}{l}
B_{1}  \tag{5.86}\\
B_{2}
\end{array}\right]=\left[\begin{array}{ll}
\sigma_{11} & \sigma_{12} \\
\sigma_{21} & \sigma_{22}
\end{array}\right]\left[\begin{array}{l}
A_{1} \\
A_{2}
\end{array}\right]
$$

The terminal conditions may be written in matrix form as

$$
\left[\begin{array}{l}
A_{1} \\
A_{2}
\end{array}\right]=\left[\begin{array}{ll}
\beta_{2} & 0 \\
0 & \alpha
\end{array}\right]\left[\begin{array}{l}
3_{1} \\
3_{2}
\end{array}\right]+\beta_{1} V_{0}\left[\begin{array}{l}
1 \\
0
\end{array}\right] \text { (5.87) }
$$

Eliminatine $\left[\begin{array}{l}A_{1} \\ A_{2}\end{array}\right]$ between (5.86) and (5.87) gives

$$
\left[\begin{array}{c}
B_{1}  \tag{5.83}\\
B_{2}
\end{array}\right]=\left[\begin{array}{cc}
1-\beta_{2} \sigma_{11} & -\alpha \sigma_{12} \\
-\beta_{2} \sigma_{21} & 1-\alpha \sigma_{22}
\end{array}\right]^{-1}\left[\begin{array}{c}
\sigma_{11} \\
\sigma_{21}
\end{array}\right]\left(\beta_{1} V_{0}\right)
$$

It is easy to show that, using ean. (5.88) and (5.85), we may write the transier function $G(z)$ in the followins way,

$$
\begin{equation*}
G(z)=\frac{\beta_{1} \sigma_{21}}{\left(1-\alpha \sigma_{22}-\beta_{2} \sigma_{11}+\alpha \beta_{2} \Delta \sigma\right)} \tag{5.89}
\end{equation*}
$$

where $\Delta \sigma$ is the determinant of $\sigma$, that is $\sigma_{11} \cdot \sigma_{22}-\sigma_{12} \sigma_{21}$.

It can be seen from Pis. 5.13 that a delay-free loop can be avoided if $\sigma_{12}$ or $\sigma_{21}$ has a factor of $z^{-1}$. Such a property
is a characteristic of filters made up from a cascade of commensurate lossless transmission-line elements. ie shall not take this feneral approach any further, since all practical filters fall into one or more of the simpler cases.

### 5.6.5 Individual Circuit Element Condition

The last condition concerms the individual two-port circuit elements. By observing the form of the coefficients $\left\{\alpha_{k}\right\}$ in the $\sigma$-parameters (see sections 5.3 and 5.4 ), it is clear that a condition is necessary in order to ensure the non-degeneracy of the digital structure. In other words, we must examine the restrictions necessary so that $\left|\alpha_{k}\right|<\infty \quad$ for some $k$.

In this section we shall consider only design from the source and design from the load, since Fettreis-type two-port elements using adaptors are similar to those discussed in Chapter 2.

If we design our network from the source-end, then it is necessary that $\sigma_{22}$, for each two-port element, is divisible by $z^{-1}$.

For a series inductance, let us recall eqn. (5.17), that is

$$
\begin{equation*}
p_{12} q_{21}-p_{11} q_{11}+p_{11} q_{21} L=0 \tag{5.90}
\end{equation*}
$$

and eqn. (5.19),

$$
\begin{align*}
& \alpha_{1}=-q_{21}\left(p_{21} q_{12}-p_{22} q_{22}-p_{21} q_{22} L\right) /\left[p_{11} \Delta q\right] \\
& \alpha_{2}=-q_{21}\left(p_{21} q_{12}-p_{22} q_{22}+p_{21} q_{22} L\right) /\left[p_{11} \Delta q\right] \\
& \alpha_{3}=1-2 q_{21} q_{22} L / \Delta q  \tag{5.91}\\
& \alpha_{4}=-q_{21} \Delta p / p_{11} \\
& \alpha_{5}=-q_{21} /\left[p_{11} \Delta q\right] \\
& \alpha_{6}=2 q_{21}^{2} L / \Delta q
\end{align*}
$$

For a shunt capacitance, let us recall eqns. (5.33) and (5.35), that is

$$
\begin{equation*}
p_{12} q_{21}-p_{11} q_{11}-p_{12} q_{11} c=0 \tag{5.92}
\end{equation*}
$$

and

$$
\left.\begin{array}{l}
\left.\alpha_{1}=-q_{11}\left(p_{21} q_{12}-p_{22} q_{22}+p_{22} q_{12} c\right) / p_{12} \Delta q\right]  \tag{5.93}\\
\alpha_{2}=-q_{11}\left(p_{21} q_{12}-p_{22} q_{22}-p_{22} q_{12} c\right) /\left[_{12} \Delta q\right] \\
\alpha_{3}=1+2 q_{11} q_{12} c / \Delta q \\
\alpha_{4}=-q_{11} \Delta p / p_{12} \\
\alpha_{5}=-q_{11} /\left[p_{12} \Delta q\right] \\
\alpha_{6}=-2 q_{11}^{2} c / \Delta q
\end{array}\right\}
$$

We do not need to consider any other circuit cloments since their equations can be obtained from either eqn. (5.91) or eqn. (5.93) by transformations which are independent of $P$ and $\bar{q}$.

It is clear that $p_{1 i}$ and $p_{12}$ can never be zero otherrise we heve infinite valued multipliers. Also $q_{11}$ must be non-zero, since if $q_{1 i}=0$ then ean. (5.92) sives $q_{2 i}=0$ and then $\Delta q=q_{1 i} q_{22}-q_{12} q_{21}=0$. which leads to infinite valued multipliers. Tinally $q_{2}$; must be non-zero, since if $q_{21}=0$ then en. (5.50) sives $q_{11}=0$ and then $\Delta_{q}=0$.

Thus for net:rori:s designed from the source-end, the following conditions are necessary:

$$
\begin{equation*}
p_{11} \neq 0, p_{12} \neq 0, q_{1 i} \neq 0, q_{21} \neq 0 \tag{5.94}
\end{equation*}
$$

Usins a similar argument, rith networks designed from the loadend, and eqns. (5.16), (5.18), (5.32) and (5.34) we find the following conditions are necessary:

$$
\begin{equation*}
\mathrm{p}_{21} \neq 0, \mathrm{p}_{22} \neq 0, \mathrm{q}_{12} \neq 0, \mathrm{q}_{22} \neq 0 \tag{5.95}
\end{equation*}
$$

### 5.6.6 Conclusions

We are no: in a position to discuss possible values for $P$ and $i$ and their associate $\dot{d}$ structures. It is to be noted that the last condition was derivea only for ladder network elements and hencciorth we shall consider only doubly-teminated lossiess ladder networks. The arguments may, however, de applied to unitelement filters. Although we abandoned the port resistance concept at the beginning of the chapter, we have found it necessary to return. We may indeed restate the first condition with this in mind, since
at a junction, the port resistances are equal and therefore the condition is

$$
\begin{equation*}
\left.Q^{-i} C P\right|_{R_{1}=R_{2}}=D \tag{5.96}
\end{equation*}
$$

and if ve express eqn. (5.96) in terms of the elements of $P$ and Q we have

$$
\left.\begin{array}{l}
p_{11}=Q_{21}, p_{12}=-Q_{22}  \tag{5.97}\\
p_{21}=Q_{11}, p_{22}=-Q_{12}
\end{array}\right\}
$$

again when $R_{1}=R_{2}$.
In the special case when $\left.P \equiv Q\right|_{R_{i}=R_{2}}$ we have from . $\quad$ ( 5.97 ),

$$
\left.\begin{array}{l}
p_{11}=p_{21}, p_{12}=-p_{22} \\
Q_{11}=Q_{21}, Q_{12}=-\hat{q}_{22}
\end{array}\right\}
$$

The three known transformations all satisiy $P \equiv E$ when $R_{1}=R_{2}$ and it is easy to verify that they also satisiy eqn. (5.98).

A table of other transformations that satisfy the three conditions will be given at the end oi this chapter.

### 5.7 Derivation of MP Sensitivity Characteristics

We are ultinately interested in digital filters that have no more than one multiplier for each reactive element. Thus for any userul configuration, linear relationships must exist between the multiplier variables $\alpha_{i}, \ldots . \alpha_{6}$ in eqn. (5.18), for example. It is not unreasonable to assume, therefore, that the one indenendent multiplier will be a function of the passive variable $R$, L or $C$. This assumption is precisely the one made for the derivation of HiP sensitivity formulae in Chapter 3. Consequently at points of maximum available pseudopover or MAP We shall have

$$
\begin{equation*}
S_{\alpha}^{|G|}=S S_{\alpha}^{K}+\frac{1}{2} S{ }_{\alpha}^{R_{L} / R_{S}} \tag{5.99}
\end{equation*}
$$

for a multiplier $\alpha$, where $G(z) \equiv \operatorname{KI}\left(\left.p\right|_{p=\frac{1-z^{-1}}{1+z^{-1}}, ~} ^{p}\right.$
and $K$ is independent of $p$ and $z^{-1}$.

The importance of the sensitivity function in estimating roundoff noise has already been established [13][21] and so it is expedient to minimise the absolute value of $S_{\alpha}^{|Q|}$.

There are two conditions for which

$$
S_{\alpha}^{|G|}=0 \quad \gamma \alpha
$$

Either

or
(b) $K=k \sqrt{\frac{\mathrm{R}_{S}}{R_{L}}} \quad, k$ constant.

Condition (a) has been discussed in detail by Petweis for :ave Digital Filters based on voltage waves [59]. Iie found three classes of filters that satisfy (a). They are as follows:
(1) Transfer functions realised as Reflectances. The main disadvantage is that stopband sensitivity is high.
(2) Lattice Filter realisations. Disadvantage as (1).
(3) Certain types of Unit-Rlement Filters. The main disadvantages are increased number of delays and design can only be achieved by optimisation. Ve vill not discuss the implication of (a) on the general tro-port transformation.

Condition (b) is easily derived by writing eqn. (5.99) as follors

$$
\begin{equation*}
S_{\alpha}^{|G|}=S_{\alpha}^{K \sqrt{R_{L} / R_{s}}} \tag{5.100}
\end{equation*}
$$

Therefore, if $K=1=\sqrt{R_{S} / Z_{I}}$ where $I$ is a constant, then we have
the desired result , Altematively, we may consider the transfer function $\hat{H}(p)$ which may be derined thus

$$
\begin{equation*}
\hat{H}(p)=k \sqrt{\frac{R_{s}}{R_{L}}} H(p) \tag{5.101}
\end{equation*}
$$

At points of maximum power transfer,

$$
|H(p)|=\frac{1}{2} \sqrt{\frac{R_{L}}{R_{s}}} \text { end }
$$

therefore

$$
|\hat{H}(p)|=\frac{1}{2} \mathrm{k}
$$

and thus

$$
S_{x}^{|\hat{H}|}=0 \text { for all passive variables } x \text {. }
$$

Again, we have the cesired result using the arguments of Chapter 3.

Let us now apply this constraint to the three cases aiscussed in the last section. It is important to recall the comment made in the last section, that is to separate clearly the variables associated with the input and those with the output.
(i) Fettreis-type design (see 5.6.4)

For (a) and (b), $\vec{k}=\frac{i}{q_{12}}$. No solution is possible in this case since ${\underset{i}{12}}$ is associated only with the output port and its associated resistance $R_{2}=R_{L} \cdot \operatorname{For}(c), K=\left\{\frac{q_{22}}{p_{i 1} \Delta q}\right\} \frac{1}{q_{i 2}}$ where the bracketed expression is associated witin the input port. Therefore as $K=k \sqrt{\frac{\hat{R}_{S}}{\hat{R}_{L}}}$
we have

$$
\begin{equation*}
\left\{\frac{q_{22}}{p_{11} \Delta q}\right\} \frac{1}{q_{12}}=k \sqrt{\frac{R_{s}}{R_{L}}} \tag{5.103}
\end{equation*}
$$

Fie have immediately that

$$
\begin{align*}
q_{12} & =k_{1} \sqrt{R_{2}}  \tag{5.104}\\
\text { and } \quad q_{22} & =-k_{1} / \sqrt{R_{2}}
\end{align*}
$$

Ban. (5.103) then becomes

$$
\frac{q_{22}}{p_{11} \Delta q}=k k_{i} \sqrt{R_{s}}
$$

Applying the cascade condition $Q_{11}=p_{2 i}$ we have

$$
\frac{p_{21}}{p_{11}}=k k_{1} \sqrt{R_{s}}
$$

Let $p_{11}=k_{2} / \sqrt{R_{s}}$, without loss of generality, then

$$
p_{21}=k k_{i} k_{2}
$$

Therefore

$$
\begin{aligned}
p_{12} & =k_{2} \sqrt{R_{1}} \\
\text { and } \quad p_{22} & =-k k_{1} k_{2} R_{1}
\end{aligned}
$$

We thus have $P$ and 2 in the followings form,

$$
\begin{aligned}
& P=\left[\begin{array}{ll}
k_{2} & k_{2} \sqrt{R_{1}} \\
\sqrt{R_{1}} & \\
k k_{1} k_{2} & -k k_{1} k_{2} R_{1}
\end{array}\right] . \\
& Q=\left[\begin{array}{ll}
k k_{1} k_{2} & k k_{1} k_{2} R_{2} \\
k_{2} & -k_{2} \sqrt{R_{2}}
\end{array}\right]
\end{aligned}
$$

However, we must check $Q$ to see that eqn. (5.104) is satisfied.

$$
\begin{equation*}
\text { How } q_{12}=-\frac{Q_{12}}{\Delta Q}=\frac{1}{2} \frac{1}{k_{2}} \sqrt{R_{2}} \tag{5.105}
\end{equation*}
$$

On combining eqns. (5.104) and (5.105) we have

$$
k_{1} k_{2}=\frac{1}{2}
$$

and therefore the final form for $P$ and 2 is as follows,

$$
\begin{aligned}
P & =\left[\begin{array}{cc}
k_{2} & k_{2} \sqrt{R_{1}} \\
\sqrt{R_{1}} & -\frac{i}{2} \leq R_{1}
\end{array}\right] \\
\frac{1}{2} x & \frac{i}{2} k R_{2} \\
\text { and } \quad Q & =\left[\begin{array}{cc}
\frac{i}{2} k & -k_{2} \sqrt{R_{2}}
\end{array}\right]
\end{aligned}
$$

Finally, for (d)

$$
\begin{equation*}
K=\left\{\frac{1}{2 p_{11} q_{11}}\right\} \frac{1}{q_{12}}=k \sqrt{\frac{R_{S}}{R_{L}}} \tag{5.106}
\end{equation*}
$$

therefore, we have immediately that

$$
\begin{aligned}
q_{12} & =k_{1} \sqrt{R_{2}} \\
\text { and } \quad q_{22} & =-k_{1} / \sqrt{R_{2}}
\end{aligned}
$$

Ban. (5.106) becomes

$$
2 p_{11} q_{11}=1 /\left(k k_{1} \sqrt{R_{s}}\right)
$$

Let $p_{11}=k_{2} / \sqrt{R_{s}}$ as before then

$$
\mathrm{q}_{11}=\frac{1}{2 \mathrm{kk} \mathrm{k}_{2}}
$$

and hence

$$
q_{21}=\frac{1}{2 k k_{1} k_{2} R_{2}}
$$

Thus we may write, by invertin5 $Q^{-1}$, that

$$
Q=\left[\begin{array}{ll}
\mathrm{kk}_{1} k_{2} & k k_{1} k_{2} R_{2} \\
\frac{1}{2 k_{1} \sqrt{R_{2}}} & \frac{-\sqrt{R_{2}}}{2 k_{1}}
\end{array}\right]
$$

and using the cascade condition,

$$
P=\left[\begin{array}{cc}
\frac{1}{2 k_{1} \sqrt{k_{1}}} & \frac{\sqrt{R_{1}}}{2 k_{1}} \\
k k_{1} k_{2} & -k k_{1} k_{2} R_{1}
\end{array}\right]
$$

but $p_{11}=k_{2} / \sqrt{R_{1}}$ therefore

$$
k_{1} k_{2}=\frac{1}{2}
$$

and the final solution is the same as for (c).

For the design from the source-end and design from the load-and, no solutions are possible. In the former case, 2 is associated with $R_{2}$ but $R_{2} \neq R_{L}$ and in the latter case, $P$ is associated with $R_{1}$ but $R_{1} \neq R_{s}$.

To conclude this section let us sumnarise the main results. We have shom that the sensitivity properties of digital filters derived using the general linear transiormation are similar to the sensitivity properties of Wave Digital Filters. Purthermore, there exists a transformation with the property of zero attenuation sensitivity at NAP points, althoumh the presence of irrational terms such as $\sqrt{R_{1}}$ and $\sqrt{R_{2}}$ means that the dirital filter will not be canonic [30]. Se shall not proceed thereiore with this transformation but will concentrate on those that are canonic.
5.8 Study of some special cases.

### 5.8.1 Introduction

The current-wave and voltare-wave formulations have the property of yieldinf canonic disital filter structures in the sense that the number of multipliers is equal to the numbor of reactive components plus one and the number of delays is equal to the number of reactive components. Therefore any nev transformation should be at least as sparing in the number of discrete components. In this section we are roing to examine the effects of applying certain constraints to the general two-port transformation that are know to apply to the current and voltare formulations.

### 2.8.2 Port Resjistance Concept

We shall now consider the case when the port resistance concept is retained. In mathenatical terms wo may write

$$
\begin{equation*}
\frac{p_{12}}{p_{11}}=-\frac{p_{22}}{p_{21}}=R_{1} \tag{5.107}
\end{equation*}
$$

Also

$$
\begin{equation*}
\frac{Q_{12}}{Q_{11}}=-\frac{Q_{22}}{Q_{21}}=R_{2} \tag{5.108}
\end{equation*}
$$

where $Q_{i j}$ is an element of $Q$.
If $q_{i j}$ is an element of $\Omega^{-1}$ then eqn. (5.108) can be written as follows

$$
\begin{equation*}
\frac{q_{11}}{q_{21}}=-\frac{q_{12}}{q_{22}}=R_{2} . \tag{5.109}
\end{equation*}
$$

Now, as $\Delta p=p_{11} p_{22}-p_{12}{ }^{p} 21$
and from eqn. (5.107)

$$
0=p_{11} p_{22}+p_{12} p_{21}
$$

then

$$
\begin{equation*}
\Delta \mathrm{p}=2 p_{11} p_{22}=-2 p_{12} p_{21} \tag{5.110}
\end{equation*}
$$

similarly

$$
\begin{equation*}
\Delta q=2 q_{11} q_{22}=-2 q_{12} q_{21} \tag{5.111}
\end{equation*}
$$

Using eqns. (5.107), (5.109), (5.110) and (5.111) together with the appropriate eqns. for the various elements in section 5.3, 5.4 and 5.5 , we have the following results, which have been summarised. Note that we need only consider the series inductance and the shunt capacitance to ether with the source, termination and interconnection.

For the series inductance we have, therefore,
either

$$
\begin{aligned}
& R_{1}=R_{2}+L \\
& \alpha_{1}=-\frac{p_{21}}{p_{12}} L \\
& \alpha_{2}=1-\frac{L}{R_{1}}=\frac{R_{2}}{R_{1}} \\
& \alpha_{3}=2 p_{21} q_{11} \\
& \alpha_{4}=-1 / 2 p_{12} q_{22} \\
& \alpha_{5}=\frac{q_{21}}{q_{22}} \frac{L}{R_{1}} \\
& \alpha_{6}=0 \\
& R_{2}=R_{1}+L \\
& \alpha_{1}=\frac{p_{21}}{p_{11}} \frac{L}{R_{2}} \\
& \alpha_{2}=0 \\
& \alpha_{3}=1-\frac{L}{R_{2}}=\frac{R_{1}}{R_{2}} \\
& \alpha_{6}=-\frac{q_{21}}{q_{12}} I \\
& \alpha_{5}=1 / 2 p_{11} a_{12} \\
& \alpha_{21} \\
& \alpha_{21} \\
& \alpha_{1}
\end{aligned}
$$

or

For the shunt capacitor we have either

$$
\begin{aligned}
& G_{1}=G_{2}+C \\
& \alpha_{1}=-\frac{p_{22}}{p_{11}} c \\
& \alpha_{2}=1-\frac{c}{G_{1}}=\frac{G_{2}}{G_{1}} \\
& \alpha_{3}=-2 q_{21} p_{22} \\
& \alpha_{4}=1 / 2 p_{11} q_{12} \\
& \alpha_{5}=-\frac{q_{11}}{q_{12}} \frac{c}{G_{1}} \\
& \alpha_{6}=0 \\
& G_{2}=G_{1}+c \\
& \alpha_{1}=\frac{p_{22}}{p_{12}} \frac{c}{G_{2}} \\
& \alpha_{2}=0 \\
& \alpha_{3}=1-\frac{c}{G_{2}}=\frac{G_{1}}{G_{2}} \\
& \alpha_{4}=2 p_{21} q_{11} \\
& \alpha_{5}=-1 / 2 p_{12} q_{22} \\
& \alpha_{6} \\
& q_{22} \\
& q_{11}
\end{aligned}
$$

or

For the resistive voltage-source we have, for example, using $P$ that

$$
X=\left(\frac{R_{1}-R_{s}}{R_{1}+R_{s}}\right) \quad\left(\frac{-p_{11}}{p_{21}}\right) Y+\frac{2 p_{11} R_{1}}{R_{1}+R_{s}} V_{0}
$$

and to avoid the delay-free path from $Y$ to $X$, we can set $R_{1}=R_{S}$
and then $X=p_{11} V_{0}$.

For the load resistance we have using $P$,

$$
Y=\left(\frac{R_{L}-R_{2}}{R_{L}+R_{2}}\right) \quad\left(\frac{p_{21}}{p_{11}}\right) X
$$

and to avoid delay-free path we can set $R_{2}=R_{L}$ and then $Y=0$. Finally for interconnections we find

$$
\text { and } \quad \begin{aligned}
R_{1} & =R_{2} \\
\sigma_{11} & =0 \\
\sigma_{12} & =2 p_{21} q_{11} \\
\sigma_{21} & =-1 / 2 p_{12} \underline{a}_{22} \\
\sigma_{22} & =0
\end{aligned}
$$

that is, a direct connection.

Clearly the port resistance concept is not sufficient to yield canonic structures. The number of multipliers has only been reduced by one. Notice that equations live $R_{1}=R_{2}+L$ are the same as those found in the theory of Chapter 2.
$5.8 .3 \quad \sigma$-parameter constraints
The voltage formulation also satisfies constraints of the form

$$
\left.\begin{array}{l}
\sigma_{11}+\sigma_{12}=1  \tag{5.112}\\
\sigma_{21}+\sigma_{22}=1
\end{array}\right\} \text { for series impedances }
$$

and

$$
\left.\begin{array}{l}
\sigma_{21}-\sigma_{11}=1  \tag{5.113}\\
\sigma_{12}-\sigma_{22}=1
\end{array}\right\} \text { for shunt admittances }
$$

Thus the equation

$$
\left[\begin{array}{l}
\mathrm{Y}_{1} \\
\mathrm{Y}_{2}
\end{array}\right]=\left[\begin{array}{ll}
\sigma_{11} & \sigma_{12} \\
\sigma_{21} & \sigma_{22}
\end{array}\right]\left[\begin{array}{l}
\mathrm{X}_{1} \\
\mathrm{x}_{2}
\end{array}\right]
$$

becones for series impedances,

$$
\left.\begin{array}{l}
y_{1}=\sigma_{11}\left(x_{1}-x_{2}\right)+x_{2}  \tag{5.114}\\
y_{2}=\sigma_{22}\left(x_{2}-x_{1}\right)+x_{1}
\end{array}\right\}
$$

and for shunt admittances,

$$
\left.\begin{array}{l}
Y_{1}=\left(\sigma_{11} X_{1}+\sigma_{22} x_{2}\right)+X_{2} \\
Y_{2}=\left(\sigma_{11} x_{1}+\sigma_{22} X_{2}\right)+X_{1}
\end{array}\right\} \text { (5.115) }
$$

Let us now apply eqns. (5.112) and (5.113) to the two-nort elenents. We find that for the series inductance either

$$
\left.\begin{array}{l}
\alpha_{3}=1, \alpha_{1}+\alpha_{3}=\alpha_{2}, \alpha_{4}+\alpha_{5}=1, \\
\alpha_{4}+\alpha_{6}=\alpha_{2}
\end{array}\right\}
$$

or

$$
\left.\begin{array}{l}
\alpha_{1}+\alpha_{4}=1, \alpha_{2} \div \alpha_{4}=\alpha_{3}, \alpha_{5}=1, \\
\alpha_{5}+\alpha_{6}=\alpha_{3}
\end{array}\right\}
$$

and for the shunt capacitance, either

$$
\left.\begin{array}{l}
\alpha_{4}=1, \alpha_{4}-\alpha_{1}=\alpha_{2}, \alpha_{3}-\alpha_{5}=1, \\
\alpha_{3}-\alpha_{6}=\alpha_{2}
\end{array}\right\}(5.118)
$$

or

$$
\left.\begin{array}{l}
\alpha_{5}-\alpha_{1}=1, \alpha_{5}-\alpha_{2}=\alpha_{3}, \alpha_{4}=1, \\
\alpha_{4}-\alpha_{6}=\alpha_{3}
\end{array}\right\}
$$

In each case, the constraints have reduced the number of multipliers from six to two.

Let us consider, as an example, the equations for a series inductance with no delay-free patin in $\sigma_{11}$. We have therefore from eqns. (5.13) and (5.116)

$$
\begin{aligned}
& \sigma_{11}=\alpha_{1} z^{-1} /\left(1+\alpha_{2}^{z^{-1}}\right) \\
& \sigma_{12}=\left(1+z^{-1}\right) /\left(1+\alpha_{2} z^{-1}\right) \\
& \sigma_{21}=\alpha_{4}\left(1+z^{-1}\right) /\left(1+\alpha_{2} z^{-1}\right) \\
& \sigma_{22}=\left(\alpha_{5}+\alpha_{6} z^{-1}\right) /\left(1+\alpha_{2} z^{-1}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_{1} & =\alpha_{2}-1 \\
\alpha_{6} & =\alpha_{5}+\alpha_{1} \\
\text { and } \alpha_{4} & =1-\alpha_{5}
\end{aligned}
$$

The Signal-קlow Diagram (SFD) of eqn. (5.114) appears in Fis. 5.14. For a canonic $S T$ we need to eliminate one nore multiplier, or equivalenily a relationship must exist between $\sigma_{11}$ and $\sigma_{22}$ such that the Sid of onc includes the other. This does not exclude the case there $\sigma_{11}$ or $\sigma_{22}$ is zero. is an exomple, if $\alpha_{5}=0$ then $\alpha_{6}=\alpha_{i}$ and $\sigma_{1 i}=\sigma_{22}$ and if $\alpha_{6}=0$ then $\alpha_{5}=-\alpha_{1}$ and $\sigma_{11}=-\sigma_{22^{2}}{ }^{-1}$. The latter is the case for the voltage formulation.

It can therefore be seen that equations of the form of eqns. (5.112) and (5.113) co reduce the numer of multipliers and enable canonic structures to be built up.

In the next two chanters se shall discuss transformations that do yield canonic structures.

## $5.2 \quad$ i condition for Cenonic 3 3

In this section wo propose a hypothesis conceming canonic signalflou diagrams. First let us derine what we mean by Canonic:

A canonic $S i d$ is one in which there is a mininum number of delays and rultipliers with the exception that multiplications by powers
of 2 are not counted.

Of course, one could express any multiplier as a sum of povers of two but clearly this would imply a larce number of additions. ilthoush adders are not specisically mentioned in the definition, it is accopted that we should wish to reduce their numbers too.

Let us now state the hypothesis:

It is sufficient for canonic $S P D$ that the transformation matrices be of the same type, i.e. voltage or current.

For example, in tine transiormation

$$
\begin{array}{ll}
X_{1}=V_{1}+R_{1} I_{1} & X_{2}=R_{2} I_{2} \\
Y_{1}=-R_{1} I_{1} & \text { and }
\end{array}
$$

the variailes $X_{i}, X_{2}, Y_{1}$ and $Y_{2}$ are voltage variables and therefore the transformation satisfies the condition.

Hovever, the transiormation

$$
\begin{array}{ll}
X_{1}=G_{i} V_{1}+I_{1} & X_{2}=V_{2}+R_{2} I_{2} \\
Y_{1}=V_{1}-R_{1} I_{1} & \text { and }
\end{array}
$$

does not satisfy the condition since
$X_{1}, I_{2}$ are current and $Y_{1}, X_{2}$ are voltase variables.

We shall consider these transformations more thoroughly in subsequent chapters.
5.10 General Iiscussion

In this section we shall sumarise the main ideas of the chanter. :Te have stuaied a general two-pori transformation on the classical doubly-teminatod lossless netrork. It was found that certain conditions had to be satisfied to ensure that the resulting digital filter was realisable and incieed imitated the classical prototype. the sensitivity behaviour was found to be similar to
that of "ave Digital ijlters and Iurthemore, a transformation exists that yielis disital filters :ith zero $\operatorname{di} P$ sensitivity but, uniontunately, such filters are not canonic. \& surficient condition was given for canonic OP which seems to fit all formulations round so far. Finally, in this chanter, we give a table of transiornations that satisíy the three conditions, that is the cascade condition, the trinsier function condition and the individual circuit elenent condition. This table which appears in $\operatorname{lig} .5 .15$ is by no means exhaustive. Of the tinirteen trinsiormations listed, ten are canonic. the first colum zives the elements of $P$, the second pives the elements of 2 . The third colum fives a key to hov a realisable dicital filter may be designed. Jor example, (i)(a) means that a disital 今ilter of the Pettweis-type, :iith matrix P used for both source and load, can je desisned (see section 5.6.4). If the number appears on its o:m, then we mav use $P$ and $a$ in any combination for the source and load. Hotice that only the three lnorm transformations, the voltage, current and poner formulations are unrestricted in this way. .hen we introduce zero elements into $P$ and $\because$, the number of realisations reduces to one. It is to be noted also that only the voltage, current and power formulations satisfy the condition $P \equiv Q$ when $P_{y}=R_{2}$. Finally, the fiftr colum gives the name of the transformation where applicable.

In the followinc two chapters we shall examine the properties of the transfonnations, listed in rig. 5.15, more closely.


Fig. 5.1 General Representation of a two-port network N.


Fig. 5.2 Signal-Flow Diagram of general 2-port.


Fig. 5.3 Resistive Voltage Source .


Fig. 5.4 SFD of Fig. 5.3.


Fig. 5.5 Load Impedance Z . Fig. 5.6 Analogue junction of two ports . .


[^0]

Fig. 5.8 Cascade of $n$ two-port sections.


Fig. 5.9 Doubly-terminated two-port .


Fig. 5.10 Fettweis-type design .
Fig. 5.11 Design from Source.


Fig. 5.12 Design from Load .


Fig. 5.13 General 2-port.


Fig. 5.14 Signal-Flow Diagram of Series Inductance .

| P | Q |  | Class(es) | Ref.No. | Name(is any) |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $1 \quad \mathrm{R}_{1}$ |  |  | I;II;III | FI | Voltage Haves |
| $1-\quad-{ }_{1}$ |  |  |  |  |  |
| $\mathrm{G}_{1} \quad 1$ |  | 1 | I;II;III | F2 | Current Waves |
| $\mathrm{G}_{1}-1$ |  | $-1$ |  |  |  |
| $\frac{11}{\sqrt{R_{1}}} \frac{1}{2} \sqrt{R_{1}}$ | $\frac{11}{1 R_{2}}$ | $\frac{1}{2} \sqrt{R_{2}}$ | I;II;III | F3 | Power Haves |
| $\frac{11}{\frac{1}{R_{1}}}-\frac{1}{2} \sqrt{R_{1}}$ |  | $-\frac{1}{2} \sqrt{R_{2}}$ |  |  |  |
| $\mathrm{G}_{1} \quad 1$ | 1 | $\mathrm{R}_{2}$ | $I ; I I(a)(c i ;$ | F4 |  |
| $1-\mathrm{R}$ | $\mathrm{G}_{2}$ | -1 | $\operatorname{III}(a)(0)$ |  |  |
| $1 . \quad \mathrm{R}_{1}$ | $\mathrm{G}_{2}$ | 1 | $I ; I I(a)(c) ;$ | F5 | . |
| $\mathrm{G}_{1} \quad-1$ |  |  | $\operatorname{III}(\mathrm{a})(\mathrm{b})$ |  |  |
| 10 |  |  | III( ${ }^{\text {a }}$ | F6 | IVR |
| $1 \quad-\mathrm{R}_{1}$ |  | 0 |  |  |  |
| 01 |  | 1 | III (a) | F7 | ITA |
| $\mathrm{G}_{1}-1$ |  | $-1$ |  |  |  |
| $0 \quad \mathrm{R}_{1}$ |  |  | III (a) | F8 |  |
| $1 \mathrm{l}_{1}$ |  |  |  |  |  |
| $\mathrm{G}_{1} \quad 0$ | $\mathrm{G}_{2}$ | 1 | III (a) | F9 |  |
| $\mathrm{G}_{1}-1$ |  | 0 |  |  | . |
| $1 \quad \mathrm{R}_{1}$ | 1 | 0 | II (a) | F10 |  |
| 10 |  | $-\mathrm{R}_{2}$ |  |  |  |
| $\mathrm{G}_{1} \quad 1$ |  | 1 | II (a) | Fl]. |  |
| $0-1$ | $\mathrm{G}_{2}$ | -1 |  |  |  |
| $1 \mathrm{R}_{1}$ |  |  | II( ${ }^{\text {a }}$ | F12 |  |
| $0 \quad-\mathrm{R}_{1}$ |  | $-\mathrm{R}_{2}$ |  |  |  |
| $\mathrm{G}_{1} \quad 1$ |  |  | II( a ) | F13 |  |
| $\mathrm{G}_{1} \quad 0$ |  |  |  |  |  |

Fig. 5.15 Table of transformations known to satisfy the three conditions discussed in section 5.6 .

## Chapter 6

## The 'Invariant Voltase Ratio' Transformation

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## Chantor 6

The 'Invariant Voltaze Ratio' Iransformation

### 6.1 Introduction

In this chopter we exanine a transiomation given in the table at the end of the last chapter. Ie consider first the derivation of the sirnal-flow diacrams for each analogue element and for the teminations and interconnections. The design and sensitivity analysis of digital filters usine this transformation rill then be examined and examples will be presented illustrating the characteristics of this formulation. The discussion will then tum to the latice filter realisation first examined by iouta [52]. Although such realications have poor sensitivity characteristics in the stopband, they exhibit some advantages which are to be examined. Tinally, a comparison will be made between seven methods of synthesising a digital filter from a third-order elliptic filter with particular reference to the number of components used and the effects of coefficient quantisation.
6.2 Derivation of SED for the basic elenents, sources, terminations and interconnections.

### 6.2.1 Introduction

The transformation that we shall consider will be referred to as the 'Invariant Voltage Ratio' transformation (IVR) for reasons which will become apparent. Ye may define it as follows,

$$
\begin{align*}
& P=\left[\begin{array}{cc}
1 & 0 \\
1 & -R_{1}
\end{array}\right]  \tag{6.1}\\
& Q=\left[\begin{array}{ll}
1 & R_{2} \\
1 & 0
\end{array}\right] \tag{6.2}
\end{align*}
$$

The IVR transformation satisfies the three conditions of Chapter 5 but we may design only frou the load-end. We shall see that each element has only one SFD and not tro, as in the
voltage-vave formulation.

Let us derive the $\sigma$-parameters directly from eqns. (6.1) and (6.2) so that we may observe any important relationships between them.

We have immediately

$$
\left[\begin{array}{l}
X_{1}  \tag{6:3}\\
Y_{1}
\end{array}\right]=\left[\begin{array}{ll}
1 & 0 \\
1 & -R_{1}
\end{array}\right]\left[\begin{array}{ll}
A & B \\
C & D
\end{array}\right]\left[\begin{array}{ll}
1 & X_{2} \\
1 & 0
\end{array}\right]^{-1}\left[\begin{array}{l}
X_{2} \\
Y_{2}
\end{array}\right]
$$

and further

$$
\left[\begin{array}{l}
X_{1}  \tag{6.4}\\
Y_{1}
\end{array}\right]=R\left[\begin{array}{l}
X_{2} \\
Y_{2}
\end{array}\right]
$$

where

$$
\begin{aligned}
& R_{11}=B G_{2} \\
& R_{12}=A-B G_{2} \\
& R_{21}=B G_{2}-D R_{1} G_{2} \\
& R_{22}=A-C R_{1}-B G_{2}+D R_{1} G_{2}
\end{aligned}
$$

Therefore

$$
\left.\begin{array}{l}
\sigma_{11}=\left(A-C R_{1}-B G_{2}+D R_{1} G_{2}\right) /\left(A-B G_{2}\right)  \tag{6.5}\\
\sigma_{12}=A_{1} G_{2} /\left(A-B G_{2}\right) \\
\sigma_{21}=1 /\left(A-B G_{2}\right) \\
\sigma_{22}=-B G_{2} /\left(A-B G_{2}\right)
\end{array}\right\}
$$

For a series impedance, $A=1, B=-Z, C=0$ and $D=-1$ and therefore eqn. (6.5) becomes

$$
\begin{aligned}
& \sigma_{11}=\left(R_{2}-R_{1}+z\right) /\left(R_{2}+Z\right) \\
& \sigma_{12}=R_{1} /\left(R_{2}+Z\right) \\
& \sigma_{21}=R_{2} /\left(R_{2}+Z\right) \\
& \sigma_{22}=z /\left(R_{2}+Z\right)
\end{aligned}
$$

It is observed that

$$
\begin{aligned}
\sigma_{11}+\sigma_{12} & =1 \\
\text { and } \quad \sigma_{21}+\sigma_{22} & =1
\end{aligned}
$$

For a shunt admittance, $A=1, B=0, C=Y, D=-1$ and thus eqn. (6.5) becomes

$$
\begin{align*}
& \sigma_{11}=\left(G_{1}-G_{2}-Y\right) / G_{1} \\
& \sigma_{12}=G_{2} / G_{1}  \tag{6.8}\\
& \sigma_{21}=1 \\
& \sigma_{22}=0
\end{align*}
$$

It is clear from eqn. (6.6) that the delay-free path cannot be removed from $\quad \sigma_{22}$ and thus only configurations with the delayfree path eliminated from $\sigma_{11}$ need be considered.

We shall continue with the method of this section although we could, of course, use the general results of Chapter 5 with the elements of $P$ and 2 substituted for from ens. (6.1) and (6.2).

### 6.2.2 Series Elements

For a series inductance we have $Z=p L$ in eqn. (6.6) and together with the bilinear transformation gives the following,

$$
\left.\begin{array}{l}
\sigma_{11}=\frac{\left(R_{2}-R_{1}+L\right)+z^{-1}\left(R_{2}-R_{1}-L\right)}{\left(R_{2}+L\right)+z^{-1}\left(R_{2}-L\right)} \\
\sigma_{22}=\frac{\left(1-z^{-1}\right) L}{\left(R_{2}+L\right)+z^{-1}\left(R_{2}-L\right)} .
\end{array}\right\} \text { (6.9) }
$$

The parameters $\sigma_{12}$ and $\sigma_{21}$ due to eqn. (6.7) need not be considered.

To avoid delay-free loops, we must set $R_{1}=R_{2}+L$ and then eq n. . (6.9) becomes

$$
\begin{aligned}
& \sigma_{11}=\frac{\alpha_{1} z^{-1}}{1+\alpha_{2} z^{-1}} \\
& \sigma_{22}=\frac{\alpha_{3}\left(1-z^{-1}\right)}{1+\alpha_{2} z^{-1}}
\end{aligned}
$$

where

$$
\begin{aligned}
& \alpha_{1}=-2 L / R_{1} \\
& \alpha_{2}=1+\alpha_{1} \\
& \alpha_{3}=L / R_{4}
\end{aligned}
$$

For realisation purposes, $\alpha_{1}$ and $\alpha_{3}$ must be expressed in terms of $\alpha_{2}$. Thus

$$
\begin{align*}
& \alpha_{1}=\alpha_{2}-1  \tag{6.11}\\
& \alpha_{3}=\frac{1}{2}\left(1-\alpha_{2}\right)
\end{align*}
$$

The signal-flow equations are

$$
\begin{align*}
& Y_{1}=\sigma_{11}\left(X_{1}-X_{2}\right)+X_{2}  \tag{6.12}\\
& Y_{2}=\sigma_{22}\left(X_{2}-X_{1}\right)+X_{1}
\end{align*}
$$

The SID of the series inductance can be derived using eq. (6.12) together with eqns. (6.10) and (6.11), and is illustrated in Fig. 6.1.

For other series elements we may. refer back to Chapter 5 from which we find that for a series capacitance, $C$, the $S T P$ is simply Fig. 6.1 with $z^{-1}$ replaced by $-z^{-1}$. and $L$ replaced by $D=1 / C$.

The parallel-tuned circuit has a $S$ ID which can be derived from that of the series inductance by the following replacements, (see section 5.3.4)

$$
\text { and } \quad \begin{array}{lll}
z^{-1} & \text { by } & T \\
& \text { by } & 1 /(\Gamma+C)
\end{array}
$$

inhere

$$
T=z^{-1}\left(\beta+z^{-1}\right) /\left(1+\beta z^{-1}\right)
$$

and

$$
\beta=(\Gamma-C) /(\Gamma+C)=(D-L) /(D+L)
$$

and

$$
\Gamma=1 / L, D=1 / C .
$$

The series-tuned circuit has a Sri which can be derived from that of the series inductance by the following substitutions, (see section 5.3.5)

|  | $\mathbf{z}^{-1}$ | by | $-T$ |
| :--- | :--- | :--- | :--- |
| and | $L$ | by | $L+D$ |
| where | $D$ | $=$ | $1 / C$. |

The SFD of these series elements can be found in Figs. 6.2, 6.3 and 6.4.

### 6.2.3 Shunt Elements

For a shunt capacitance, $C$, we have $Y=p C$ and together with the bilinear transformation, eqn. (6.8) becomes,

$$
\begin{equation*}
\sigma_{11}=\frac{\left(G_{1}-G_{2}-c\right)+z^{-1}\left(G_{1}-G_{2}+c\right)}{G_{1}\left(1+z^{-1}\right)} \tag{6.13}
\end{equation*}
$$

Note that other $\sigma$-parameters do not depend on frequency and therefore we need not consider them at present.

To avoid delay-free loops we must have

$$
G_{1}=G_{2}+C
$$

and thus eqn. (6.13) becomes

$$
\begin{equation*}
\sigma_{11}=\frac{\alpha_{1} z^{-1}}{1+z^{-1}} \tag{6.14}
\end{equation*}
$$

together with

$$
\sigma_{12}=\alpha_{2}
$$

where
and

$$
\begin{align*}
& \alpha_{1}=2 C / G_{1}  \tag{6.15}\\
& \alpha_{2}=G_{2} / G_{1}
\end{align*}
$$

For realisation purposes, let us express $\alpha_{1}$ and $\alpha_{2}$ in terms of $\quad \alpha=c / G_{1}$, thus

$$
\begin{equation*}
\text { and } \alpha_{2}=1-\alpha \tag{6.16}
\end{equation*}
$$

The signal-flow equations are


The SiD of the shunt capacitance can be derived using eqns. (6.17), (6.14) and (6.16) and is showm in Fi.5. 6.5.

For the other shunt elements we have, from the theory of Chapter 5, the following substitutions:

The SFD of a shunt inductance can be found by replacing $z^{-1}$ by $-z^{-1}$ and $C$ by $\Gamma=1 / \mathrm{L}$ in Fis. 6.5 (see section 5.4.3). The SFD of a series-tuned circuit can be found (see section 5.4.4) by replacing $z^{-1}$ by $T$ and $C$ by $1 /(\mathrm{L}+\mathrm{D})$ in Fig. 6.5 where $T=z^{-1}\left(\beta+z^{-1}\right) /\left(1+\beta z^{-1}\right)$ and $\quad \beta=(D-L) /(D+L)$. Finally, the SFP of a parallel-tuned circuit is the same as Fig. 6.5 with $z^{-1}$ replaced by $-T$ and $C$ by $C+\Gamma$ (see section 5.4.5). The Sid of these shunt elements can be found in Figs. 6.6, 6.7 and 6.8.

### 6.2.4 Resistive Voltare Source

Let us recall the transfer function condition of Chapter 5. For the IVR transformation defined by eqns. (6.1) and (6.2), we see that realisable structures can be built up only if $P$ is used. for the source and load.

The signal-flow equation for the resistive voltage source is, from section 5.5.2,

$$
\begin{equation*}
X=\left\{\frac{p_{12}-p_{11} R_{s}}{p_{22}-p_{21} R_{s}}\right\} Y+\left\{\frac{\Delta p}{p_{22}-p_{21} R_{s}}\right\} \nabla_{0} \tag{6.18}
\end{equation*}
$$

On using the values for $P$ from eqn. (6.1) in eqn. (6.18) we have that

$$
\begin{equation*}
X=\beta\left(Y-V_{0}\right)+V_{0} \tag{6.19}
\end{equation*}
$$

where

$$
\beta=R_{s} /\left(R+R_{s}\right)
$$

The SFP appears in Figs. 6.9.

### 6.2.5 Load Resistance

Recalling eqn. (5.49) from section 5.5.3, that is.

$$
\begin{equation*}
Y=\left\{\frac{p_{21} R_{L}+p_{22}}{p_{11} R_{L}+p_{12}}\right\} X \tag{6.20}
\end{equation*}
$$

and using the values for $P$ from eqn. (6.1) we find that

$$
\begin{equation*}
Y=\left\{\frac{R_{L}-R}{R_{L}}\right\} \quad X \tag{6.21}
\end{equation*}
$$

On setting $R=R_{I}$, we have $Y=0$, a wave-sink.

### 6.2.6 Interconnections

As we have seen, the ABCD matrix of an interconnection is $\left[\begin{array}{rr}1 & 0 \\ 0 & -1\end{array}\right]$ and therefore, on using eqn. (6.5) we find that

$$
\begin{aligned}
\sigma_{11} & =\frac{R_{2}-R_{i}}{R_{2}} \\
\sigma_{12} & =\frac{R_{1}}{R_{2}} \\
\sigma_{21} & =1 \\
\text { and } \quad & \sigma_{22}
\end{aligned}
$$

To avoid delay-iree loops on interconnection, we must have $\sigma_{i 1}=0$, therefore $R_{2}=R_{1}$ and hence

$$
\sigma_{12}=\sigma_{21}=1
$$

These equations are those of a direct interconection betreen t:ro ports.

We have therefore derived the SID of all necessary lumped elenents and in the next section shall consider complete dirital netroris.

### 6.3 Design and Sensitivity inalysis of Digital Bilters Initatine Doubly-Terninated Lossless Ledder ietworls.

### 6.3.1 Desim Procedure

For the IVR formulation we may only design from the load-end. zor the analozue prototype or sig. 6.10 there is thereiore one digital equivalent (iig. 6.11). The port resistance at the output of the Lis net:orl: is then constrained to save the value of the lond resistance. ill subsequent port resistances are then also derined by virtue of the linear relationships bet:een the port resistances and the element values. Jince the value of the port resistance at the input of the LC netirork is not generally the same as that of the source resistance, we must use the general SFD for the resistive voltage source.

There is no problen about interconnecting ind of successive twoports since at every junction we can ensure that the corresponding port resistances are equal.

The transfer ifunction of the resulting digital filter structure, denoted by $G\left(z^{\prime}\right)$, differs from the function obtained by oilinear transformin; the analocue transfer function $H(p)$ by a multiplicative factor K. From the theory of Chapter 5 we know that $i=1 / q_{12}$ where $q_{12}$ is an element of $Q^{-1}$. It is easy to see, from eqn. (6.2) that $q_{12}=1$ and therefore $K=1$. Thus the transfer function $G(z)$ is identical to the function $H(p)=V_{2} / V_{0}$ after the application of the bilinear transformation.

### 6.3.2 Sensitivity Characteristics

We may state immediately that at MiP points,

$$
\begin{equation*}
S_{\alpha}^{|G|}=S_{\alpha}^{K}+\frac{1}{2} S_{\alpha}^{R_{N} / R_{S}} \tag{6.22}
\end{equation*}
$$

where $\alpha$ is any multiplier.

Ban. ( 6.22 ) is an immediate consequence of the arguments of section 5.7. For the IVR formulation, we have $\mathrm{K}=1$ and therefore eqn. (6.22) becomes

$$
\begin{equation*}
S_{\alpha}^{|G|}=\frac{1}{2} S_{\alpha}^{R_{L} / R_{S}} \tag{6.23}
\end{equation*}
$$

For all distal two-ports using the IVR formulation there is one multiplier which denends on the port resistances. Furthermore, for tuned-circuits the additional multiplier depends only on the comesnondinE element values and therefore the corresponding sensitivity is zero. Thus we have

```
    for series-elements, \(\quad \alpha=2 \frac{R_{2}}{R_{1}}-1\)
for shunt-elemonts, \(\quad \alpha=1-\frac{G_{2}}{G_{1}}\)
```

and for the voltage source, $\alpha=\frac{R_{s}}{\pi_{1}}+r_{s}$

Let us consider a two-port network consisting of series and shunt elements terminated resistively at both ends. In Chapter 3, we expressed $R_{L} / R_{s}$ as a product of port-resistance ratios, that is

$$
\begin{equation*}
\frac{R_{L}}{R_{s}}=\frac{R_{L}}{R_{n}} \cdot \frac{R_{n}}{R_{n-1}} \ldots \cdots \cdot \frac{R_{2}}{R_{1}} \cdot \frac{R_{1}}{R_{s}} \tag{6.24}
\end{equation*}
$$

and then we expressed each multiplier as a ratio of port resistances.

For series elements

$$
\frac{R_{k}}{R_{K-1}}=\frac{1}{2}(1+\alpha)
$$

whilst for shunt elements

$$
\frac{R_{k}}{R_{k-1}}=\frac{1}{(1-\alpha)}
$$

and for the voltage source

$$
\frac{R_{1}}{R_{s}}=\frac{(1-\alpha)}{\alpha}
$$

Mee sensitivities cen no:: be derived and suistituted in eqn. (6.23). fe find that for series elements

$$
\begin{equation*}
s_{\alpha}^{|G|}=\frac{\alpha}{2(1+\alpha)} \tag{6.25}
\end{equation*}
$$

and for shunt elements

$$
\begin{equation*}
S_{\alpha}^{|G|}=\frac{\alpha}{2(1-\alpha)} \tag{6.26}
\end{equation*}
$$

and finally for the voltage-source

$$
\begin{equation*}
s_{\alpha}^{|G|}=\frac{-1}{2(1-\alpha)} \tag{6.27}
\end{equation*}
$$

### 6.4 Jxample: 3 Od Crder Chebyshev Piltor

As a simple example of the ideas of the last three sections, let us taise the third-order example used in section 3.2 , that is a Chebyshev normalised filter with 0.1 d3 passbend ripple and equal terminations. The circuit and values are shown in Fig. 6.12. The design equations are as follows (inc. ú.15),

$$
\begin{aligned}
& G_{3}=G_{I}+C=2.0316 \\
& \beta_{4}=C / G_{3}=0.50778 \\
& R_{2}=R_{3}+I=1.63962 \\
& \beta_{3}=\left(R_{3}-L\right) / R_{2}=-0.39959 \\
& G_{1}=G_{2}+C=1.64150 \\
& \beta_{2}=C / G_{1}=0.62845 \\
& \beta_{1}=R_{s} /\left(R_{s}+R_{1}\right)=0.62143
\end{aligned}
$$

The complete SFD appears in Fis. 6.14 whilst the amplitude response and the attenuation sensitivity characteristics are shom in rigs. 6.15 and 6.16 respectively.

Using the appropriate formulae for the theoretical NP sensitivity in section 6.3 we have

$$
\begin{aligned}
S_{S}^{|G|} & =-1.321 \\
\beta_{1} & = \\
{ }_{S}{ }^{|G|} & =0.8459 \\
\beta_{2} & \\
{ }_{S} \begin{array}{l}
|G| \\
\beta_{3}
\end{array} & =-0.3329 \\
\text { and } \quad{ }_{S} \begin{array}{l}
|G| \\
\beta_{4}
\end{array} & =0.5157
\end{aligned}
$$

The computer analysis shown in Fiz. 6.16 gives the followinc for the same sensitivities $-1.3203,0.34572,-0.33277$ and 0.51531 resnectively thus showing the desired agreenent.

### 6.5 3naple II: 5th Order 3lintic Lompass iliter

Let us now considor an example for the purnose of examining the effects of reducing the multiplier wordlenfth. The example we shall use is the same as that in Chapter 3 (section 3.6). The element values are displayed in Fig. 6.17 and the sional-ilow block diagram appears in Fig. 6.18. The design equations are as follows,

$$
\begin{aligned}
& C_{1}=1.08077, C_{2}=0.06809, I_{2}=1.29869, C_{3}=1.00288 \\
& C_{4}=0.18583, L_{4}=1.15805, C_{5}=0.98556, R_{S}=R_{L}=1 \\
& G_{1}=G_{L}+C_{5}=1.98556 \\
& \alpha_{1}=C_{5} / G_{1}=0.496354 \\
& R_{1}=1 / G_{1}=0.503536 \\
& R_{2}=R_{1}+L_{4} C_{4} /\left(L_{4}+C_{4}\right)=1.456607 \\
& \alpha_{2}=\left(2 R_{1} / R_{2}\right)-1=-0.308480 \\
& \alpha_{3}=\left(D_{4}-L_{4}\right) /\left(D_{4}+L_{4}\right)=0.645819 \\
& G_{2}=1 / R_{2}=0.686527 \\
& G_{3}=G_{2}+C_{3}=2.489407 \\
& \alpha_{4}=C_{3} / G_{3}=0.724221 \\
& R_{3}=1 / G_{3}=0.401702 .
\end{aligned}
$$

$$
\begin{aligned}
& R_{4}=R_{3}+L_{2} D_{2} /\left(L_{2}+D_{2}\right)=1.594882 \\
& \alpha_{5}=\left(2 R_{3} / R_{4}\right)-1=-0.496261 \\
& \alpha_{6}=\left(D_{2}-L_{2}\right) /\left(D_{2}+L_{2}\right)=0.837513 \\
& G_{4}=1 / R_{4}=0.627006 \\
& G_{5}=G_{4}+C_{1}=1.715776 \\
& \alpha_{7}=C_{1} / G_{5}=0.634564 \\
& \alpha_{8}=G_{5} /\left(G_{S}+G_{5}\right)=0.631781
\end{aligned}
$$

The complete SFD appears in Fig. 6.19. The digital network was then analysed at 50 frequency points first with the nominal multiplier values above, then vith the multipliers rounded to 3 decimal places and finally with the nultipliers rounded to 1 deciral place. The analysis is shom in eraphical form in Fig. 6.20 and confirus the hypothesis about low sensitivity.

## 6.6.- Brample III: 6th Order Plintic Band-Pass Pilter

As a further example of the IVR transiormation, let us consider the 6th order elliptic band-pass filter first introduced in section 3.7. The circuit and element values anpear in Fig. 6.21 and the signal-flom block diagram appears in Fig. 6.22. The design equations are as follows,

$$
\begin{aligned}
& R_{s}=R_{L}=1, L_{1}=L_{4}=11.672, C_{1}=C_{4}=0.085675 \\
& L_{2}=0.79126, C_{2}=2.1845, L_{3}=0.45778, C_{3}=1.26380 . \\
& R_{1}=R_{L}+L_{4}+D_{4}=24.344016 \\
& \alpha_{1}=1-\left(2 R_{L} / R_{1}\right)=0.917844 \\
& \alpha_{2}=\left(D_{4}-L_{4}\right) /\left(D_{4}+L_{4}\right)=0 \\
& G_{1}=1 / R_{1}=0.0410779 \\
& G_{2}=G_{1}+\Gamma_{3} C_{3} /\left(\Gamma_{3}+C_{3}\right)=0.841690 \\
& \alpha_{3}=G_{1} / G_{2}=0.0488040 \\
& \alpha_{4}=\left(\Gamma_{3}-C_{3}\right) /\left(\Gamma_{3}+C_{3}\right)=0.266992 \\
& G_{3}=G_{2}+\Gamma_{2} C_{2} /\left(\Gamma_{2}+C_{2}\right)=1.642311 \\
& \alpha_{5}=G_{2} / G_{3}=0.512503 \\
& \alpha_{6}=\left(\Gamma_{2}-C_{2}\right) /\left(\Gamma_{2}+C_{2}\right)=-0.266999
\end{aligned}
$$

$$
\begin{aligned}
& R_{3}=1 / G_{3}=0.6!8398 \\
& R_{4}=R_{3}+L_{1}+D_{1}=23.952915 \\
& \alpha_{7}=1-\left(2 R_{3} / R_{4}\right)=0.949159 \\
& \alpha_{8}=\left(D_{1}-L_{1}\right) /\left(D_{1}+L_{1}\right)=0 \\
& \alpha_{9}=R_{s} /\left(R_{4}+R_{s}\right)=0.0400755
\end{aligned}
$$

As a result of the fact that $L_{1} C_{1}=L_{4} C_{4}=1$ the number of multipliers is reduced from nine to seven. The complete SFD is shom in Pis. 6.23. The digital networls was then anilysed at 50 frequency points over the entire spectrum ( $0, \pi / T$ ) and then at 21 points in the passband only. The attenuation curves for the net:rorls with (a) nominal multiplier values riven above, (b) multiplier values rounded to 3 decimal places and (c) nultiplier values rounded to 1 decinal place are shom in Fis. 6.24.
6.7 The 'IVR' Lattice Pilter, its Derivation and Properties.
6.7.1 Introduction

In this section we shall consider the disital equivalent of a doubly-terminated lossless symmetrical lattice network using the IVR transformation. It is well knom that there is a lattice equivalent for every synnetric ladder and fur thernore the number of distinct elenents in the lattice is, in general, less than that of the ladder [53]. The lo: sonsitivity propert:r in the passband of doubly-terminated LC networks obviously applies to the lattice. However, the stopband sensitivity is high and therefore, as analogue networks require tuning and are subject to ageing, this disadvantage is serious. jith digital filters, thourh, no such problem exists althourh we may need slichtly longer wordienrths to accomplish the same accuracy as the ladder filter. This fact must be weished against the reduction in the number of discrete components.

We follow slichhtly different lines to references [52][53] in the derivation of the appropriate signal-flow diagrans.

### 6.7.2 Perivation of

The $A B C D$ parameters of a symetrical lattice (ins. 6.25) are, as is well know,

$$
\begin{align*}
& A=\left(z_{2}+z_{1}\right) /\left(z_{2}-z_{1}\right) \\
& B=2 z_{1} z_{2} /\left(z_{2}-z_{1}\right)  \tag{6.28}\\
& C=2 /\left(z_{2}-z_{1}\right) \\
& D=\left(z_{2}+z_{1}\right) /\left(z_{2}-z_{1}\right)
\end{align*}
$$

The transfer function, $H(p)$, of a doubly-terminated networl may be written as follows (ri.n. 6.26)

$$
\begin{equation*}
H(p)=\frac{1}{\left(A+\frac{D_{1}}{R_{L}}\right)+R_{s}\left(C+\frac{D}{R_{L}}\right)} \tag{6.29}
\end{equation*}
$$

On substituting for the lattice parameters, eqn. (6.29) becomes,

$$
\begin{equation*}
H(p)=\frac{R_{L}\left(z_{2}-Z_{1}\right)}{\left(Z_{1}+R_{S}\right)\left(Z_{2}+Z_{L}\right)+\left(Z_{1}+R_{L}\right)\left(Z_{2}+R_{S}\right)} \tag{6.30}
\end{equation*}
$$

We shall consider only the case where $R_{L}=R_{S}=R$ and therefore eqn. (6.30) becomes

$$
\begin{equation*}
H(p)=\frac{R\left(z_{2}-z_{1}\right)}{2\left(Z_{1}+R\right)\left(z_{2}+R\right)} \tag{6.31}
\end{equation*}
$$

Let us now consider the IVR transformation which can be written as follo::s,

$$
\begin{align*}
& {\left[\begin{array}{l}
A_{1} \\
B_{1}
\end{array}\right] }=P\left[\begin{array}{l}
V_{1} \\
I_{1}
\end{array}\right], \quad P=\left[\begin{array}{cc}
1 & 0 \\
1 & -R_{1}
\end{array}\right]  \tag{6.32}\\
& \text { and } \quad\left[\begin{array}{l}
A_{2} \\
B_{2}
\end{array}\right]=Q\left[\begin{array}{l}
V_{2} \\
I_{2}
\end{array}\right], \quad Q=\left[\begin{array}{ll}
1 & R_{2} \\
1 & 0
\end{array}\right]
\end{align*}
$$

As in the Lattice realisations of Fettreis [53] we wish to find a simple relationship between $H(p)$ and the reflectences of the canonical impedances $z_{1}$ and $z_{2}$.

For an inpedance $z$ we hove two nossible reflectances depending on thether we use $P$ or 2. Using $?$ we have

$$
\begin{equation*}
S_{p}=\frac{B}{A}=\frac{Z}{Z}-R \tag{6.34}
\end{equation*}
$$

and using 2,

$$
\begin{equation*}
S_{\underline{q}}=\frac{B}{A}=\frac{Z}{Z+R} \tag{6.35}
\end{equation*}
$$

For the voltace formulation as $P \equiv 2$ when $R_{1}=R_{2}$, we have only one possible resloctance nanely $(z-R) /(Z+R)[j 0][5 j]$.

Let the reflectonces of $z_{1}$ and $Z_{2}$ in eqn. (6.31) be $S_{1}$ and $S_{2}$ respectively. Usins $P$ to deine $B_{1}$ and $S_{2}$ we have therefore that

$$
\begin{equation*}
s_{k}=\frac{z_{k}-\boldsymbol{R}}{Z_{k}} \quad, k=1,2 \tag{6.36}
\end{equation*}
$$

and on elirinatine $z_{1}$ and $Z_{2}$ from eqn. (6.31) usinc eqn. (6.36) we find that

$$
H(p)=\frac{S_{2}-S_{1}}{2\left(2-S_{1}\right)\left(2-S_{2}\right)}
$$

which can be written as

$$
\begin{equation*}
H(p)=\frac{1}{4}\left(T_{2}-T_{1}\right) \tag{6.37}
\end{equation*}
$$

where

$$
\begin{equation*}
T_{k}=\left(1-\frac{1}{2} S_{k}\right)^{-1}, \quad k=1,2 \tag{6.38}
\end{equation*}
$$

Combining eqn. (6.38) and eqn. (6.36) gives

$$
T_{k}=\frac{2 Z_{k}}{Z_{k}+R}
$$

which is the reflectance obtained using Q. Thus we have found a simple relationship between $H(p)$ and the reflectances, nanaely

$$
\begin{equation*}
H(p)=\frac{1}{2}\left(S_{2}-S_{1}\right) \tag{6.39}
\end{equation*}
$$

where $S_{k}=\frac{Z_{k}}{Z_{k}+R}, k=1,2$.

If we had choson a to deîine the reilectonces and eliminated $z_{1}$ and $z_{2}$ from ean. $(6.31)$, :e :rould have found that

$$
H(p)=\frac{1}{2}\left(S_{2}-S_{1}\right) \text { as before. }
$$

Thus if the lattice impedances $Z_{1}$ and $Z_{2}$ are realised as reflectances in digital filter form, then the arithmetic difierence between these reflectances gives the desired transier Aunction. The STD of the lattice realisation or砛) apeurs in フig. 6.27.

It is not too difficult to show that, if we had used. $P$ for $Z_{i}$ and $i$ for $\ddot{Z}_{2}$, the transfer function rould have been the same as in eqn. (6.39). Similarly, if te hed used $a$ for $z_{1}$ and $P$ for $Z_{2}$.

## 6.7 .3 2xample

Consider as an example a third-order elliptic filter realised in lattice form (ig. 6.28). The lattice impedances $Z_{1}$ and $\mathrm{Z}_{2}$ are as follows,

$$
\begin{align*}
& z_{1}=p L_{1} \\
& z_{2}=p L_{2}+\frac{1}{\mathrm{pC}_{2}} \tag{6.40}
\end{align*}
$$

The reflectances $S_{1}$ and $S_{2}$ are found using eqn. (6.35). Thus

$$
S_{1}=\frac{p L_{1}}{p L_{1}+R}
$$

and on applyins the bilinear transformation, $p \rightarrow\left(1-z^{-1}\right) /\left(1+z^{-1}\right)$ we have

$$
\begin{aligned}
s_{1} & =\frac{\alpha_{1}\left(1-z^{-1}\right)}{1+\alpha_{2} z^{-1}} \\
\text { where } \alpha_{i} & =L_{1} /\left(R+L_{1}\right) \\
\text { and } \alpha_{2} & =\left(R-L_{1}\right) /\left(R+L_{1}\right) \\
\text { also } 2 \alpha_{1} & +\alpha_{2}=1
\end{aligned}
$$

The $S_{2}$ of $S_{1}$ appears in Fis. (6.29).
Aiso

$$
S_{2}=\frac{\mathrm{pL}_{2}+\frac{1}{\mathrm{pC}_{2}}}{\mathrm{pL}_{2}+\frac{1}{\mathrm{pC}_{2}}+\mathrm{R}}
$$

and on applying the bilinear transformation we find that

$$
\begin{equation*}
S_{2}=\frac{\alpha_{1}+2 \alpha_{2} z^{-1}+\alpha_{1} z^{-2}}{1+2 \alpha_{2} z^{-1}+\alpha_{3} z^{-2}} \tag{6.42}
\end{equation*}
$$

$$
\begin{aligned}
& \text { where } \alpha_{1}=\left(D_{2}+L_{2}\right) /\left(D_{2}+L_{2}+R\right) \\
& \text { and } \quad \alpha_{2}=\left(D_{2}-L_{2}\right) /\left(D_{2}+\dot{L}_{2}+R\right) \\
& \quad \alpha_{3}=\left(D_{2}+L_{2}-R\right) /\left(D_{2}+L_{2}+R\right) \\
& \text { also } \quad 2 \alpha_{1}-\alpha_{3}=1
\end{aligned}
$$

If we lei $\beta=\alpha_{2} / \alpha_{1}=\left(D_{2}-I_{2}\right) /\left(D_{2}+I_{2}\right)$
then we can write eqn. (6.42) as

$$
\begin{align*}
S_{2} & =\frac{\alpha_{1}(1+T)}{1+\alpha_{3}^{T}}  \tag{6.43}\\
\text { where } T & =\frac{z^{-1}\left(\beta+z^{-1}\right)}{1+\beta z^{-1}}
\end{align*}
$$

The SFD of $\mathrm{S}_{2}$ is shom in Fig. 6.j0. File could have derived eqn. (6.43) by noting that (see section 5.3.5)

$$
\begin{equation*}
z_{2}=\left(I_{2}+D_{2}\right)\left\{\frac{1+T}{1-T}\right\} \tag{6.44}
\end{equation*}
$$

Then on combining eqns. (6.44) and (6.55) we have the desired expression for $S_{2}$ in eqn. ( 6.43 ). In both the 3 Pip of $S_{1}$ in. Fig. 6.29 and the $\sin$ of $S_{2}$ in lis. 6.30 we have a multiplier of value $\frac{1}{2}$. The only eifect of dropping these multipliers is to give a scaling to the transfer function. In fact, if $G(z)$ is the transfer function of the digital lattice and $H(p)$ is that of the analogue lattice then, on using eqn. (6.39), we have

$$
G(z) \equiv 4 H(p) \left\lvert\, p=\frac{1-z^{-1}}{1+z^{-1}}\right.
$$

## 6.7 .4 Jiscussion

:e have shom thit a lattice digital filter, based on the IF? formulation, can be derived wich is similar to that derived by rettreis [5j]. The transfer function is sirmly the difference bet:een the reilectunces of the canonical impedances $\vec{Z}_{1}$ and $Z_{2}$. For the given thirà-order elliptic filter example, the lattice realisation has used three delays and three multipliers which is less than the correspondine lader realisation. Furthermore, the number of additions used was nine which again is less than the ladaer.

##  Wilter frona ahird-Crder aliptic Filter

### 6.8.1 Introduction

In this section we examine the disital filter structures of seven methoas of reulising a thira-orajer eiliptic low-pass filter. :ie compare not only the coefficient quantisation properties but also the number of discrete components needed. The analocue lader filter is shom in Fig. 6.31 and the lattice equivalent in Fig. 6.32. This example :as used in Chanter 3.

### 6.8.2 Direct Synthesis

The transfer function of the analogue filter example is given by the following

$$
\begin{equation*}
H(p)=\frac{z_{2}}{\left(1+z_{1}\right)\left(i+z_{1}+2 z_{2}\right)} \tag{6.45}
\end{equation*}
$$

where

$$
z_{1}=p L_{1}=\mathrm{pL}_{3}
$$

and

$$
\mathrm{z}_{2}=\mathrm{pL}_{2}+1 / \mathrm{pC} \mathrm{C}_{2}
$$

We have assumed $R_{s}=R_{L}=1$.
If we apply the bilinear transformation directly to eqn. (6.45) we ootain the following,

$$
\begin{equation*}
G(z)=\frac{\alpha_{0}+\alpha_{1} z^{-1}+\alpha_{1} z^{-2}+\alpha_{0} z^{-3}}{1+\beta_{i} z^{-i}+\beta_{2} z^{-2}+\beta_{3} z^{-3}} \tag{6.46}
\end{equation*}
$$

$$
\text { where } \quad \begin{aligned}
\alpha_{0} & =(1+a) / \mathfrak{f} \\
\alpha_{1} & =(3-a) / f \\
\beta_{i} & =(3 b+c-d-3 e) / f \\
\beta_{2} & =(j b-c-d+3 e) / f \\
\beta_{j} & =(b-c+d-e) / f \\
a & =L_{2} C_{2} \\
\text { and } \quad & =2 \\
c & =2 L_{1}+C_{2} \\
d & =2 C_{2}\left(L_{1}+L_{2}\right) \\
e & =L_{i} C_{2}\left(L_{i} \div 2 L_{2}\right) \\
f & =b+c+d+e
\end{aligned}
$$

The values of the coaificients are as follows,

$$
\begin{aligned}
\alpha_{0} & =0.105899 \\
\alpha_{1} & =0.304356 \\
\beta_{1} & =0.200227 \\
\beta_{2} & =0.479347 \\
\text { and } \beta_{3} & =-0.0386328
\end{aligned}
$$

 and the attenuation characteristics, for the cases ri?ere the multiplier values have been rounded to 3 and 2 decimal places together with the nominal case above, appears in Fig. 6.34.

### 6.8.3 Cascode Jynthesis

Tie may decompose eqn. (6.45) as follows,

$$
\begin{align*}
& \text { Let } H_{1}(p)=\frac{1}{1+Z_{1}}  \tag{6.47}\\
& \text { and } H_{2}(p)=\frac{z_{2}}{1+z_{1}+2 z_{2}}  \tag{6.48}\\
& \text { then } H(p)=H_{i}(p) H_{2}(p) \tag{6.49}
\end{align*}
$$

$\therefore$ Dplyine the bilinear transforation to $\mathrm{E}_{\mathrm{i}}$ and $\mathrm{H}_{2}$ we find tinat

$$
\begin{align*}
G_{1}(z) & =\frac{\alpha_{1}\left(1+z^{-1}\right)}{1+\alpha_{2} z^{-1}}  \tag{6.50}\\
\text { where } \alpha_{2} & =\left(1-L_{1}\right) /\left(1+L_{1}\right) \\
\text { and } 2 \alpha_{1} & =1+\alpha_{2} \\
G_{2}(z) & =\frac{\hat{\alpha}_{0}+\hat{\alpha}_{1} z^{-1}+\hat{\alpha}_{0} z^{-2}}{1+\hat{\beta}_{1} z^{-1} \div \hat{\beta}_{2} z^{-2}} \tag{6.5i}
\end{align*}
$$

ilso
where

$$
\begin{aligned}
\hat{\alpha}_{0} & =\left(1+L_{2} C_{2}\right) / \eta \\
\hat{\alpha}_{1} & =2\left(i-L_{2} C_{2}\right) / \eta \\
\hat{\beta}_{1} & =2\left(2-C_{2} L_{1}-2 C_{2} L_{2}\right) / \eta \\
\text { and } \quad \hat{\beta}_{2} & =\left\{2+C_{2}\left(L_{1}+2 L_{2}-1\right)\right\} / \eta \\
\text { anc } \quad \eta & =2+C_{2}\left(1+L_{1}+2 L_{2}\right) .
\end{aligned}
$$

The values of the multipliers are as follows,

$$
\begin{aligned}
\alpha_{2} & =-0.0771502 \\
\hat{\alpha}_{0} & =0.229505 \\
\hat{\alpha}_{1} & =0.450053 \\
\hat{\beta}_{1} & =0.277576 \\
\text { and } \quad \hat{\beta}_{2} & =0.500747
\end{aligned}
$$

The Sirl appears in Fieg. 6.35 and the attenuation characteristics in Fis. 6.36 for the cases there the multiplier values have been rounded to 3 and 2 decimal places togethor with the nominal case as given above.
6.S.4 Voltare-iave indder Synthesis ،

The design of a digital filter from the third-order elliptic prototype of Fig. 6.31 using the voltage formulation has been done in Chapter 3. For convenience, we give here the multiplier values and the SFD.

Thus

$$
\begin{aligned}
& \alpha_{1}=0.461425 \\
& \alpha_{2}=0.297867 \\
& \alpha_{3}=0.936915 \\
& \alpha_{4}=0.356112 \\
& \alpha_{5}=-0.288949
\end{aligned}
$$

The SFD appears in Fig. 6.37 and the attenuation characteristics may be found in Fic. 6.38 (multiplier values rounded to $6,3,2$ and 1 decinal places).

### 6.8.5 'IV?' Ladder Synthesis

The signal-flo: block diagran appears in Fis. 6.39 and the design equations, using the princinles of this chapter, are as follons,

$$
\begin{aligned}
& R_{1}=R_{L}+L_{3}=2.1672 \\
& \alpha_{1}=\left(R_{L}-L_{3}\right) /\left(R_{L}+L_{3}\right)=-0.0771502 \\
& G_{1}=1 / R_{1}=0.461425 \\
& G_{2}=G_{1}+r_{2} C_{2} /\left(\Gamma_{2}+C_{2}\right)=1.549099 \\
& \alpha_{2}=G_{1} / G_{2}=0.297867 \\
& \alpha_{3}=\left(r_{2}-C_{2}\right) /\left(r_{2}+C_{2}\right)=0.936915 . \\
& R_{2}=1 / G_{2}=0.645536 \\
& R_{3}=R_{2}+L_{1}=1.812736 \\
& \alpha_{4}=\left(R_{2}-L_{1}\right) /\left(R_{2}+L_{1}\right)=-0.287777 \\
& \alpha_{5}=R_{s} /\left(R_{s}+R_{3}\right)=0.355526
\end{aligned}
$$

The SFD of the complete network is shom in Fis: 6.40 and the attenuation characteristics appear in Pig. 6.41.

### 6.8.6 Sedineyer-Fetweis Ladder Synthesis

In the case of ladder netrorls that can be bisected so that one half is the mirror-image of the other, we may use the theory of aave difital filtors to eliminate one multiplier from the resulting network [39][59]. For the third-order example of Fig. 6.31, the corresponding wave digital filter appears in Fig. 6.42. Hotice that the central adaptor has two port resistances equal and therefore the correspondins multipliers are equal. The adaptor equations may therefore be written so as to require only one multiplier for realisation. In the ladder realisations of sections 6.8 .4 and 6.8 .5 we required 5 multipliers in all whilst we need only 4 rultipliors in the realisation or Fiç. 6. i2. The complete SFD appears in lig. 6.43 and the desisn equations are as follows,

$$
\begin{aligned}
R_{s}=R_{L} & =1, L_{1}=L_{3}=1.1672, C_{2}=1.1231, L_{2}=0.029 . \\
R_{1} & =R_{s}+L_{1}=2.1672 \\
R_{2} & =R_{L}+I_{3}=2.1672 \\
\alpha_{1} & =\alpha_{3}=R_{s} / R_{1}=0.461425 \\
R_{3} & =L_{2}+D_{2}=0.919393 \\
\beta & =\left(I_{2}-D_{2}\right) /\left(I_{2}+D_{2}\right)=-0.936915 \\
\alpha_{2} & =2 G_{2} /\left(G_{1}+G_{2}+G_{3}\right)=0.459010
\end{aligned} .
$$

The attenuation characteristics appear in Fig. 6.44.

### 6.8.7 'IVR' Lattice Synthesis

He have already derived the SFD of the reflectances $S_{1}$ and $S_{2}$ for the third-order elliptic lattice filter. The complete SFD appears in rig. 6.45 and the design equations, using the component values of Fig. 6.32, are as follows,

$$
\begin{aligned}
& \alpha_{1}=\frac{R-I_{1}}{R+L_{1}}=-0.0771502 \\
& \alpha_{2}=\frac{D_{2}+I_{2}-R}{D_{2}+I_{2}+R}=0.500747 \\
& \alpha_{3}=\frac{D_{2}-I_{2}}{D_{2}+L_{2}}=0.184826
\end{aligned}
$$

$$
\text { were } R=R_{L}=R_{S}=1
$$

The attenuation characteristics are illustrated in Fig. 6.46.

### 6.8.8_Tetweis Lattice jynthesis

Finclly, we consider the synthesis of a vave disital lattice filter fron the circuit of Fig. 6.32 using the method described in reierence [j5]. The signal-flow block diacrom is siom in Fig. 6.47 and the desim equations are as follors,

$$
\begin{aligned}
& L_{1}=1.1672, L_{2}=1.2252 \\
& C_{2}=0.56155, R=R_{s}=R_{L}=1 \\
& \alpha_{1}=\left(R-L_{1}\right) /\left(R+L_{1}\right)=-0.0771502 \\
& \alpha_{2}=2 D_{2} /\left(D_{2}+I_{2}+R\right)=0.839062 \\
& \alpha_{3}=2 I_{2} /\left(D_{2}+I_{2}+R\right)=0.611685
\end{aligned}
$$

Tine complete $S: D$ is shom in Fig. 6.48 and the attenuation characteristics in Fis: 6.49.

### 6.8.2 Discussion

The number of discrete components used in the seven realisations have been sumarised in Fig. 6.50. The lattice realisations are clearly the most economical, however their high stopband sensitivity could be a disadvantafe. In the passband, lattice filter structures behave excellently and this is the result of their zero mip sensitivity [59]. In the stopband, however, ve see the expected shift in the attenuation pole which also occurs in the classical prototype. :ie can, of course, predict accurately the number of bits required in the disital realisation for a siven accuracy in the filter response. It may vell be feasible to use a higher wordlensth in the lattice realisation, as compared with the ladder, to have the advantage of fewer multiplications. Although the example chosen in this section is mainly for illustrating the ideas of the present chapter, the digital filter structures derived using linear transformations do seem to behave better than those derived using conventional synthesis teclniques. This fact has been borne out by the large amount of simulation studies reported in the literature (see.Chapter 1).
6.9 3ng le: fth Order Tnobysiev In Filter

In this section we shall consider the 'IVZ' lattice realisation of the 5 tin order Chebyshev Loir-Pass Filter used by Fettreis in Reference [53]. Thus we shall be able to compare the digital structure derived using the voltage formulation and that derived using the IVR formulation.

In Fig. 6.51 may be found the lattice impedance $Z_{1}$ and $Z_{2}$ and the corresponding element values.

Let us consider first the reflectance $s_{1}$ for $Z_{1}$.

$$
\text { Brow } \quad z_{1}=1 /\left\{p C_{1}+1 / p \bar{L}_{1}\right\}
$$

and on applying the bilinear transformation, we know from Chapter 5 that

$$
\begin{aligned}
& z_{1}=\frac{1}{\left(\Gamma_{1}+C_{1}\right)} \frac{1-T}{1+T} \\
& \text { where } \Gamma_{1}=1 / L_{1} \\
& \text { and } \quad T=z^{-1}\left(\beta_{1}+z^{-1}\right) /\left(1+\beta_{1} z^{-1}\right), \\
& \beta_{1}=\left(\Gamma_{1}-C_{1}\right) /\left(\Gamma_{1}+C_{1}\right) . \\
& \text { As } \quad S_{1}=\frac{z_{1}}{Z_{1}+R}, R=Z_{L}=R_{S}=1 \text { we have }
\end{aligned}
$$

on using eqn. (6.52) that

$$
\begin{align*}
S_{1} & =\frac{\frac{1}{2}\left(1-\alpha_{1}\right)(1-T)}{1+\alpha_{1} T}  \tag{6.53}\\
\text { where } \quad \alpha_{1} & =\left(\Gamma_{1}+c_{1}-1\right) /\left(\Gamma_{1}+c_{1}+1\right)
\end{align*}
$$

For the lattice impedance $Z_{2}$, we have a shunt capacitor $C_{2}$ terminated by an impedance $Z_{3}=\mathrm{pL}_{3}+\frac{1}{\mathrm{pC}} 3$. The SFD of the shunt capacitor $C_{2}$ is already lnorm as it was discussed in section 6.2. The SFD of $z_{3}$ was derived in section 6.7. It was found that

$$
\begin{equation*}
S_{3}=\frac{z_{3}}{Z_{3}+i_{1}}=\frac{\frac{i}{2}\left(1+\alpha_{3}\right)(1 \div 7)}{1+\alpha_{3} T} \tag{6.54}
\end{equation*}
$$

there

$$
\begin{aligned}
\alpha_{3} & =\left(D_{3}+L_{3}-R_{1}\right) /\left(D_{3}+L_{3}+R_{1}\right) \\
T & =z^{-i}\left(\beta_{2}+z^{-i}\right) /\left(1+\beta_{2} z^{-1}\right)
\end{aligned}
$$

and

$$
\beta_{2}=\left(D_{3}-L_{3}\right) /\left(D_{3}+L_{3}\right)
$$

Hotice that $R_{1}$ is the port resistance betreen the shunt cepacitance $\mathrm{C}_{2}$ and $Z_{3}$. irom the theory of section 6.2 it is linom that


$$
\begin{align*}
& G_{1}=G+C_{2}  \tag{6.55}\\
& G_{1}=1 / R_{1} \text { and } G=1 / R .
\end{align*}
$$

The correspondins nultiplier $\alpha_{2}$, for $C_{2}$ is defined as follons,

$$
\begin{equation*}
\alpha_{2}=C_{2}^{\prime \prime} G_{1} \tag{6.56}
\end{equation*}
$$

The complete $S T$ of the disital latice filter is show in Fig. 6.52. It is to be noted that the factors of $\frac{1}{2}$ in eqn. (6.53) and (6.54) have been absorbed. The former into the output and the latter also into the output anc, in adaition, into the SFD of the shunt cepacitor $C_{2}$. The overall transfer function is thuas four tines that of the analogue prototype. The $D C$ value :ill therefore be 2 , as opyosed to $1 / 2$. The multiplier values for Fis. 6.52 are as follows,

$$
\begin{aligned}
& \alpha_{1}=0.392670 \\
& \beta_{i}=-0.514106 \\
& \alpha_{2}=0.634503 \\
& \alpha_{3}=0.74485 i \\
& \beta_{2}=-0.43630 j
\end{aligned}
$$

The cigital structure was analysed first with the values above, then, as in [53], with the multiplier values truncated to 5 bits in an equivalent decimal representation. The attenuation characteristics are shown in $\mathrm{Fig} \cdot 6.53$.
$\therefore$ 'e huve used 5 multipliers, 5 delays and 14 auders in this realisation. In the asuivilent retweis structure usins the voltare formulation, the sane number of components mere used [33].

### 6.10 Conclusions

In this chanter we have emlored the fnvariant Voltare mino t:ans ormation in sone detail. Fe have derived the signnl-flow diactrans of the constituent elenents of a doubly-terminated IC Fíltoi anu exaninei the sensitivity properties of conolete filters. Mrouch the examples, we have shom that the properties of digital networins derived using the $I V A$ transformation are sinilar to those of the :iave jirital rillters discussed in a previous chapter. In the next chapter we shall examine other transiormations and the correspondins ciisital structures, althoush in not as much detail as we have discussed the IV:? transfornation.


Fig. 6.1 Signal-Flow Diagram of series inductance .


Fig. 6.2 Signal-Flow Diagram of series capacitance .


Fig. 6.3 Signal-Flow Diagram of parallel-tuned circuit in series-arm .


Fig. 6.4 Signal-Flow Diagram of Series-'luned Circuit in Series-Arm .


Fig. 6.5 Signal-Flow Diagram of Shunt Capacitance .


Fig. 6.6 Signal-Flow Diagram of Shunt Inductance -


Fig. 6.7 Signal-Flow Diagram of Series-Tuned Circuit in Shunt-Arm .


Fig. 6.3 Signal-Flow Diagram of Parallel-Tuned Circuit in Series-Arm .


Fig. 6.9 Signal-Flow Diagram of Resistive Voltage Source .


Fig. 6.10 Doubly-Terminated Classical Analogue Network .


Fig. 6.1l IVR Digital equivalent of Fig. 6.10.


Fig. 6.12 3rd.order Chebyshev filter .


Fig. 6.13 Signal-flow block diagram of Fig. 6.12.


Fig. 6.14 Complete SFD of 3 rd. order Chebyshev filter showing multiplier values .

Fig. 6.15 Amplitude response of 3 rd.order Chebyshev IVR digital filter .


Fig. 6.16 Attenuation sensitivity in 3rd.order Chebyshev IVR digital filter.


Fig. 6.17 Example II: 5th. order Elliptic Low-Pass filter .


Fig. 6.18 Signal-Flow Block Diagram of Fig. 6.17.


Fig. 6.21 Example III: 6th. order Elliptic Band-Pass filter .


Fig. 6.22 Signal-Flow Block Diagram of Fig. 6.21.


Fig. 6.19 Complete SFD of 5th. order Elliptic Low-Pass Digital Filter .



$$
\begin{aligned}
& \alpha_{1}=0.917844 \\
& \alpha_{3}=0.0488040 \\
& \alpha_{4}=0.266992 \\
& \alpha_{5}=0.512503 \\
& \alpha_{6}=-0.266999 \\
& \alpha_{7}=0.949159 \\
& \alpha_{9}=0.0400755
\end{aligned}
$$



Fig. 6.23 Complete SFD of 6th.order Elliptic Band-Pass Digital Filter .



Fig. 6.25 Symmetrical Lattice Structure .


Fig. 6.26 Doubly-terminated network .


Fig. 6.27 SFD of general digital lattice filter .


Fig. 6.28 3rd.order Elliptic Low-Pass Lattice Filter .


Fig. 6.29 SFD of Reflectance $S_{1}$.


Fig. 6.30 SFD of Reflectance $S_{2}$.


Fig. 6.31 Example I: 3rd.order Elliptic Low-pass Filter .


Fig. 6.32 Lattice equivalent of Fig. 6.31


Fig. 6.33 SFD of Direct Synthesis .


Fig. 6.35 SFD of Cascade Synthesis .


Fig. 6.39 Signal -flow block diagram of IVR synthesis .




Fig. 6.37 Complete SFD of 3rd.order Elliptic Low-Pass Digital Filter using Voltage-Wave Transformation .



$$
\begin{aligned}
& \alpha_{1}=-0.0771502 \\
& \alpha_{2}=0.297867 \\
& \alpha_{3}=0.936915 \\
& \alpha_{4}=-0.287777 \\
& \alpha_{5}=0.355526
\end{aligned}
$$



Fig. 6.40. Complete SFD of 3rd.order Elliptic Low-Pass Digital Filter using IVR transformation.



Fig. 6.43 Complete SFD of 3rd.order Elliptic Low-Pass Digital Filter using Sedlmeyer-Fettweis synthesis .




Fig. 6.45 Complete SFD of 3rd.order Elliptic Low-Pass Lattice Digital filter using IVR transformation .


Fig. 6.47 Signal-Flow Block Diagram of 3rd.order Elliptic Low-Pass Lattice Filter using Fettweis synthesis .



Fig. 6.48 Complete SFD of 3rd.order Elliptic Low-Pass Lattice Digital Filter using Fettweis Synthesis •

| Synthesis | - Number of Components. |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  | D | $\square$ | ${ }_{(+1}$ | $\nabla_{\frac{1}{2}}$ |
| Direct | 5 | 3 | 6 | 0 |
| Cascaded 2nd.order | 5 | 3 | 7 | 0 |
| Voltage-Wave Ladder | 5 | 4 | 16 | 0 |
| IVR Ladder | 5 | 4 | 18 | 2 |
| Sedlmeyer-Fettweis Lad. | 4 | 4 | 15 | 0 |
| IVR Lattice | 3 | 3 | 9 | 0 |
| Fettweis Lattice | 3 | 3 | 10 | 0 |

Fig. 6.50 Table showing number of discrete components used in various realisations .



Fig. 6.51 Lattice Impedances of 5th.order Chebyshev Low-Pass Filter •


Fig. 6.52 Complete SFD of 5th.order Chebyshev Low-Pass Lattice Digital Filter using the IVR transformation.


## Chanter 7

## A Study of Four Pransformations.

Contents:
7.1 Introduction.
7.2 The 'Invariant Pransfer Admittance' Transformation.
7.3 Study of Sinilar Canonic Transformations.
7.4 Study of One ITon-Canonic Transformation.
7.5 Discussion and Conclusion.

## Chapter 7

## A Study of Pour transformations

### 7.1. Introduction

In the previous chapter the 'IVR' transformation was studied in some detail and the properties of the corresponding digital filter structures were considered. In this chapter, four other linear transformations are examined, the appropriate signalflow diagrams are derived and the corresponding digital filter transfer functions synthesized. The transformations given in this section are found in the table of section 5.10 and therefore they sutisiny the three existence conditions of section $5 . \sigma$. Canonic transíorrations will be chiefly studied, al though one transformation will be examined which is not. The examples provided help to confirm by construction the validity of the sufficient condition of section 5.9.
7.2 The 'Invariant Transfer Admittance' (ITA) Transformation The transformation nay be written thus (see section 5.10),

$$
\begin{align*}
P & =\left[\begin{array}{cc}
0 & 1 \\
G_{1} & -1
\end{array}\right]  \tag{7.1}\\
\text { and } Q & =\left[\begin{array}{cc}
G_{2} & 1 \\
0 & -1
\end{array}\right] \tag{7.2}
\end{align*}
$$

From Chapter 5, we have in addition

$$
\left[\begin{array}{l}
Y_{1}  \tag{7.3}\\
Y_{2}
\end{array}\right]=\left[\begin{array}{ll}
\sigma_{11} & \sigma_{12} \\
\sigma_{21} & \sigma_{22}
\end{array}\right]\left[\begin{array}{l}
X_{1} \\
X_{2}
\end{array}\right]
$$

where

$$
\begin{aligned}
& \sigma_{11}=R_{22} / R_{12} \\
& \sigma_{12}=-\Delta R / R_{12} \\
& \sigma_{21}=1 / R_{12}
\end{aligned}
$$

$$
\text { and } \quad \sigma_{22}=-\mathrm{R}_{11} / \mathrm{R}_{12}
$$

and

$$
\begin{equation*}
\mathrm{R}=P T Q^{-1} \tag{7.4}
\end{equation*}
$$

with

$$
T=\left[\begin{array}{ll}
A & B \\
C & D
\end{array}\right]
$$

and $\Delta R=-\Delta P / \Delta Q$ if network is reciprocal.

Combining eqns. (7.1), (7.2) and (7.4) together with (7.3) we deduce the following expressions,

$$
\begin{align*}
\sigma_{11} & =\left\{A G_{1} R_{2}-B G_{1}-C R_{2}+D\right\} / R_{12}  \tag{7.5}\\
\sigma_{12} & =G_{1} R_{2} / R_{12} \\
\sigma_{21} & =1 / R_{12} \\
\text { and } \sigma_{22} & =-C R_{2} / R_{12} \\
\text { where } R_{12} & =C R_{2}-D
\end{align*}
$$

For the series impedance $Z, A=1, B=-2, C=0$ and $D=-1$ therefore eqn. (7.5) becomes

$$
\begin{align*}
& \sigma_{11}=\left(R_{2}-R_{4}+2\right) / R_{1} \\
& \sigma_{12}=R_{2} / R_{1}  \tag{7.6}\\
& \sigma_{21}=1 \\
& \sigma_{22}=0
\end{align*}
$$

For the shunt admittance $Y, A=1, B=0, C=Y, D=-1$ and therefore eqn. (7.5) becomes

$$
\left.\begin{array}{l}
\sigma_{11}=\left(G_{1}-G_{2}-Y\right) /\left(G_{2}+Y\right) \\
\sigma_{12}=G_{1} /\left(G_{2}+Y\right) \\
\sigma_{21}=G_{2} /\left(G_{2}+Y\right) \\
\sigma_{22}=-Y /\left(G_{2}+Y\right)
\end{array}\right\}(7.7)
$$

Moreover we have from eq. (7.7)

$$
\text { and } \begin{align*}
\sigma_{12}-\sigma_{11} & =1  \tag{7.8}\\
& \sigma_{21}-\sigma_{22}
\end{align*}=1
$$

and therefore eqn. (7.3) becomes

$$
\left.\begin{array}{l}
Y_{1}=\sigma_{11}\left(x_{1}+X_{2}\right)+X_{2} \\
Y_{2}=\sigma_{22}\left(X_{1} \div X_{2}\right)+X_{1}
\end{array}\right\}(7 \cdot 9)
$$

He need only consider the SFD of the series inductance and the shunt capacitance as all other $S \because D$ can easily be derived from them as it will be illustrated.

It has been established in Chapter 5, and section 5.10 in particular, that the ITA formulation satisfies the transfer function coniition in only one instance. That is, realizable structures can be obtained only if design is begun at the loadend. In addition, the transformation matrix $P$ must be used to derive the SFD of both teminations. is desim must begin from the load-end, it is necessary to ensure that the SiP for each two-port has no delay-iree path in its $\sigma_{11}$ transuittance. For the series inductance, $z=p L$ and $p=\frac{1-z^{-1}}{1+z^{-1}}$ then we have from eqn. (7.6),

$$
\sigma_{11}=\frac{\left(R_{2}-R_{1}+L\right)+\left(R_{2}-R_{1}-L\right) z^{-1}}{R_{1}\left(1+z^{-1}\right)}
$$

and $\sigma_{12}=\frac{R_{2}}{R_{1}}$

To avoid delay-free loops we must have $R_{2}-R_{1}+L=0$; that is we must set $R_{4}=R_{2}+L$ from which condition we obtain

$$
\sigma_{11}=\frac{-2 L}{R_{1}} \frac{z^{-1}}{\left(1+z^{-1}\right)}
$$

Let $\quad \alpha=R_{2} / R_{1}$
then $\sigma_{11}=\frac{2(1-\alpha) z^{-1}}{1+z^{-1}}$
and $\sigma_{12}=\alpha$

One paraneter only appears in eqn. (7.10) i.e. $\alpha$ in addition to one delay. Thus a canonic structure can be realised as the SFD shom in Pig. 7.1.

For the other series elements we need only modify fig. 7.1 in the following way (see section 5.3):

For a series capacitor, we can replace $z^{-1}$ by $-z^{-1}$ and $L$ by $D=(1 / C)$, for a parallel-tuned circuit, replace $z^{-1}$ by $T$ and $L$ by $1 /(\Gamma+C)$ and for a series-tuned circuit, replace $z^{-1}$ by $-T$ and $L$ by $L+D$, where $T=z^{-1}\left(\beta+z^{-1}\right) /\left(1+\beta z^{-1}\right)$ and $\quad \beta=(D-L) /(D+L)$ or $(P-C) /(\Gamma+C)$.
For the shunt capacitance, $Y=p C$ and $p=\frac{1-z^{-1}}{1+z^{-1}}$ then we have, on using eqn. (7.7), that

$$
\sigma_{11}=\frac{\left(G_{1}-G_{2}-C\right)+\left(G_{1}-G_{2}+C\right) z^{-1}}{\left(G_{2}+C\right)+\left(G_{2}-C\right) z^{-1}}
$$

and

$$
\sigma_{22}=\frac{-C\left(1-z^{-1}\right)}{\left(G_{2}+C\right)+\left(G_{2}-C\right) z^{-1}}
$$

To avoid delay-free loops we rust have

$$
G_{1}-G_{2}-C=0
$$

that is we must set $G_{1}=G_{2}+C$ and then

$$
\begin{aligned}
\sigma_{11} & =\frac{2 C}{G_{1}} \frac{z^{-1}}{1+\left(1-\frac{2 C}{G_{1}}\right) z^{-1}} \\
\text { and } \quad \sigma_{22} & =\frac{-C}{G_{1}} \cdot \frac{\left(1-z^{-1}\right)}{1+\left(1-\frac{2 C}{G_{1}}\right)} z^{-1}
\end{aligned}
$$

Now let $\beta=1-\frac{2 C}{G_{1}}$ then

$$
\sigma_{11}=\frac{(1-\beta) z^{-1}}{1+\beta z^{-1}}
$$

and

$$
\sigma_{22}=\frac{-1}{\left.-\frac{1}{2}-\beta\right)\left(1-z^{-1}\right)} \frac{1+\beta z^{-1}}{}
$$

The SFD appears in Fig. (7.2) and was derived using eqns. (7.11) and (7.9).

For the other shunt elements we need only modify Fig. 7.2. For a shunt inductor replace $z^{-1}$ by $-z^{-1}$ and $C$ by $\Gamma(=1 / L)$, for $a$ series-tuned circuit replace $z^{-1}$ by $T$ and $C$ by $1 /(L+D)$ and for $a$ parallel-tuned circuit replace $z^{-1}$ by $-T$ and $C$ by $C+\Gamma$, where $T$ is as for series elements.

Henceforth we shall examine only the SFD of the series $L$ and the shunt $C$. : Fe nay obtain fron them the SFD of the series $C$, shunt $L$ and tined-circuita usin: the appropriate triansormations given above. If the SAD of the series $L$ is canonic, then it is easily checked that any SFD derived fron it will also be canonic. Te may arcue in a similar way for shunt elenents.

Let us now consider the terminations and interconnections for the 'ITA' transformation. The transfer function condition implies that only $P$ must be used to derive the SSD of the terminations. The equations for the resistive voltase source are therefore as follows,

$$
\begin{align*}
X & =I \\
Y & =G_{1} V-I \\
\text { and } \quad V_{0} & =V+R_{\dot{s}} I \tag{7.12}
\end{align*}
$$

On eliminatine $V$ and I fron eqn. (7.12) we find that

$$
\begin{aligned}
X & =\alpha Y+\beta V_{0} \\
\text { where } \quad \alpha & =-R_{1} /\left(R_{1}+R_{s}\right) \text { and } \quad \beta=1 /\left(R_{1}+R_{s}\right) .
\end{aligned}
$$

Note that we do not need a multiplier for $\beta$ since we can excite the filter with a source $\hat{V}_{0}=\beta V_{0}$. The resulting transfer function will then differ by a constant from the desired one.

$$
\begin{align*}
X & =I \\
Y & =G_{2} V-I  \tag{7.13}\\
V & =R_{L} I
\end{align*}
$$

from which we see that,

$$
Y=\left(G_{2}-G_{L}\right) X
$$

As we have to design from the load, we must set $G_{2}=G_{L}$ and then $Y=0$; a signal sink.

Finally, let us examine the interconnection of two-ports with port resistances $R_{1}$ and $R_{2}$ respectively. üsing eqns. (7.5) with $A=1, B=C=0$ and $D=-1$, we have

$$
\begin{aligned}
& \sigma_{11}=G_{1} R_{2}-1 \\
& \sigma_{12}=G_{1} R_{2} \\
& \sigma_{21}=1 \\
& \sigma_{22}=0
\end{aligned}
$$



To avoid delay-free loops on interconnection, we must have $\sigma_{11}=0$, that is $\mathrm{R}_{1}=\lambda_{2}$ and therefore eqns. (7.14) correspond to a direct connection.

The external connections of a two-port using the ITA formulation are shown in is. 7.3. The actual transfer function, $\hat{G}(z)$, is given. by $Y_{2} / \hat{V}_{0}$ but the desired, $G(z)$ is given by $Y_{2} / V_{0}$. We know, from Chapter 5, that

$$
\begin{equation*}
\left.G(z) \equiv K H(p)\right|_{p=} \frac{1-z^{-1}}{1+z^{-1}} \tag{7.15}
\end{equation*}
$$

where $K=1 / R_{L}$ for the 'ITA' transformation and $I(D)$ is the transfer function of the analogue filter. Thus

$$
\left.\hat{G}(z) \equiv \dot{K}_{1} H(p)\right|_{p=} \frac{1-z^{-1}}{1+z^{-1}}
$$

where $K_{1}=\left(R_{1}+R_{s}\right) / R_{L}$.

The reason for calling the transformation defined by eqns. (7.1) and (7.2) the Invariant Iransfer Adrattance may be explained as follows:

We have seen, from eqn. (7.15), that $K=1 / R_{L}$ and therefore

$$
\begin{aligned}
\mathrm{KH}(\mathrm{p}) & =\mathrm{V}_{2} / \mathrm{V}_{0} R_{\mathrm{L}} \\
\text { but } \quad V_{2} & =I_{2} R_{\mathrm{L}} \quad \text { therefore } \\
\mathrm{KH}(\mathrm{p}) & =I_{2} / V_{0}
\end{aligned}
$$

The SFD usiñ the ITA formulation are similar to those using the IVR formulation. However, it is to be observed that, whereas in the former the SFD corresponding to the series elements have a simpler form, in the latter it is those that correspond to the shunt elements.

As a result of the argunents of section 5.7 , it is apparent that the sensitivity properies of filters derived usinc the IMA transiormation are similar to those already described. Thus the effect of multiplier roundins would be a constant shift in the attenuation curve.

## 7. 3 Study of Similar Canonic Iransfomations

### 7.3.1 Introduction

In this section, we consider two other transformations from the table of section 5.10. Te shall derive the SJ of the series inductance, shunt capacitance, terminations and interconnections. All other required SFD can be derived using principles outlined in the previous section. As the IVR and ITA formulations admit only design from the load, we shall exemine here tiro formulations that allow only design from the source-end, namely P10 and I12.

### 7.3.2 The Transformation 110

The transformation may be defined as follows,

$$
P=\left[\begin{array}{ll}
1 & R_{i}  \tag{7.16}\\
1 & 0
\end{array}\right]
$$

$$
Z=\left[\begin{array}{cc}
1 & 0  \tag{7.i7}\\
1 & -R_{2}
\end{array}\right]
$$

Combining eqns. (7.4), (7.16), (7.17) and (7.3) we find that,

$$
\left.\begin{array}{l}
\sigma_{11}=B G_{2} /\left(B G_{2}+D R_{1} G_{2}\right) \\
\sigma_{12}=-R_{1} G_{2} /\left(D G_{2}+D Z_{1} G_{2}\right) \\
\sigma_{21}=-1 /\left(B G_{2}+D R_{1} G_{2}\right) \\
\sigma_{22}=\left(A+B G_{2}+C R_{1}+D R_{1} G_{2}\right) /\left(B G_{2}+D R_{1} G_{2}\right)
\end{array}\right\} \text { (7.18) }
$$

For series impedances, $A=1, B=-Z, C=0, D=-i$ and there ore eqn. (7.18) becomes

$$
\left.\begin{array}{l}
\sigma_{11}=z /\left(R_{1}+z\right) \\
\sigma_{12}=R_{1} /\left(R_{1}+z\right) \\
\sigma_{21}=R_{2} /\left(R_{1}+z\right) \\
\sigma_{22}=\left(R_{1}-R_{2}+z\right) /\left(R_{1}+z\right)
\end{array}\right\} \text { (7.19) }
$$

Note also that

$$
\sigma_{11}+\sigma_{12}=i
$$

and

$$
\begin{equation*}
\sigma_{21}+\sigma_{22}=1 \tag{7.20}
\end{equation*}
$$

For a series inductance, $z=p L$ and $p=\frac{1-z^{-1}}{1+z^{-1}}$ therefore we have from eqn. (7.19),

$$
\left.\begin{array}{c}
\sigma_{11}=L\left(1-z^{-1}\right) /\left\{\left(R_{1}+L\right)+\left(R_{1}-L\right) z^{-1}\right\} \\
\sigma_{22}=\left\{\left(R_{1}-R_{2}+L\right)+\left(R_{1}-R_{2}-L\right) z^{-1}\right\} / \\
\left\{\left(R_{1}+L\right)+\left(R_{1}-L\right) z^{-1}\right\}
\end{array}\right\}(7.21)
$$

As we have to design from the source-end, we must ensure that for each element the corresponding $\sigma_{22}$ has no delay-free path. Thus we must set

$$
\begin{aligned}
& R_{1}-R_{2}+L=0 \\
\text { that is } & R_{2}=R_{i}+L
\end{aligned}
$$

and therefore an. (7.21) becomes

$$
\left.\begin{array}{rl}
\sigma_{11} & =\frac{\frac{1}{2}(1-\alpha)}{\left(1+\alpha z^{-1}\right)} \frac{\left(1-z^{-1}\right)}{\left(1+\alpha z^{-1}\right)} \quad \text { where } \quad \alpha=\frac{2^{R_{1}}}{R_{2}}-1
\end{array}\right\} \text { (7.22) }
$$

The realization appears in fig. 7.4 and was obtained using eqns. (7.3), (7.20) and (7.22).

For shunt admittances, $\mathrm{A}=1, \mathrm{~B}=0, \mathrm{C}=\mathrm{Y}, \mathrm{D}=-1$ and therefore eqn. (7.18) becomes

$$
\begin{align*}
& \sigma_{11}=0 \\
& \sigma_{12}=1 \\
& \sigma_{21}=G_{1} / G_{2}  \tag{7.23}\\
& \sigma_{22}=\left(G_{2}-G_{1}-Y\right) / G_{2}
\end{align*}
$$

For a shunt capacitance, $Y=p C$ and $p=\frac{1-z^{-1}}{1+z^{-i}}$ and we find that

$$
\begin{equation*}
\sigma_{22}=\frac{\left(G_{2}-G_{i}-C\right)+\left(G_{2}-G_{i}+C\right) z^{-1}}{G_{2}\left(1+z^{-1}\right)} \tag{7.24}
\end{equation*}
$$

To avoid delay-free loops, $\sigma_{22}$ must have no constant in the numerator therefore,

$$
G_{2}=G_{1}+C
$$

and

$$
\begin{equation*}
\sigma_{22}=\frac{2(1-\beta) z^{-1}}{\left(1+z^{-1}\right)} \tag{7.25}
\end{equation*}
$$

where

$$
\beta=G_{1} / G_{2} .
$$

The realization appears in Fis. 7.5 and was obtained using eqns. (7.3), (7.23) and (7.25).

For interconnections, $\dot{A}=1, B=0, C=0$ and $D=-1$ and on using eqn. (7.18) together in th the realizability condition, we find that $R_{1}=R_{2}$ and

$$
\begin{aligned}
& \sigma_{11}=\sigma_{22}=0 \\
& \sigma_{12}=\sigma_{21}=.1
\end{aligned}
$$

that is, a direct connecion.

Let us now consider the terminations, noting that we must use $P$ for both source and load.

For the resistive voltage source,

$$
\begin{aligned}
& X=V+R_{1} I \\
& Y=V
\end{aligned}
$$

$$
\text { and } \quad V_{0}=V+R_{s} I
$$

As we are forced to design from the source-end, by the transfer function condition, we must set $R_{f}=R_{s}$ and then

$$
X=V_{0} .
$$

For the load-resistance,

$$
\begin{aligned}
& X=\nabla+R_{2} I \\
& \mathbf{Y}=\nabla \\
& V=R_{L} I
\end{aligned}
$$

therefore,

$$
Y=\left\{R_{L} /\left(R_{L}+R_{2}\right)\right\} X
$$

The resulting digital filter structure for a general doublyterminated LC two-port is showh in Fig. 7.6.

## 7. 2.4 The Transformation P12

This transformation may be deîined as follows,

$$
\begin{align*}
& P=\left[\begin{array}{ll}
1 & R_{i} \\
0 & -R_{i}
\end{array}\right]  \tag{7.26}\\
& Q=\left[\begin{array}{ll}
0 & R_{2} \\
1 & -R_{2}
\end{array}\right] \tag{7.27}
\end{align*}
$$

Combining eqns. (7.4), (7.26) and (7.27) and (7.3) we see that

$$
\begin{align*}
& \sigma_{11}=-C R_{1} / R_{12} \\
& \sigma_{12}=R_{1} G_{2} / R_{12} \\
& \sigma_{21}=1 / R_{12}  \tag{7.28}\\
& \sigma_{22}=-\left(A+B G_{2}+C R_{4}+D R_{1} G_{2}\right) / R_{12} \\
& R_{12}=A+C R_{4}
\end{align*}
$$

For the series impedance 2, eqn. (7.28) becomes

$$
\begin{align*}
& \sigma_{11}=0 \\
& \sigma_{12}=R_{1} / R_{2}  \tag{7.29}\\
& \sigma_{21}=1 \\
& \sigma_{22}=\left(R_{1}-R_{2}+2\right) / R_{2}
\end{align*}
$$

For a series inductance $L$ we find that on using eqn. (7.29)

$$
\begin{equation*}
\sigma_{22}=\left\{\left(R_{1}-R_{2}+L\right)+\left(R_{1}-R_{2}-L\right) z^{-1}\right\} / R_{2}\left(1+z^{-1}\right) \tag{7.30}
\end{equation*}
$$

To avoid delay-free loons, we must set $R_{2}=R_{1}+L$ and therefore.

$$
\begin{equation*}
\sigma_{22}=\frac{2(\alpha-1) z^{-1}}{1+z^{-1}} \tag{7.31}
\end{equation*}
$$

where $\alpha=R_{1} / R_{2}$.

The realization is show in Fig. 7.7 and was obtained using eqns. (7.3), (7.29) and (7.31).

For the shunt admittance, $Y$, eqn. (7.28) becomes

$$
\left.\begin{array}{l}
\sigma_{11}=-Y /\left(G_{1}+Y\right) \\
\sigma_{12}=G_{2} /\left(G_{1}+Y\right) \\
\sigma_{21}=G_{1} /\left(G_{1}+Y\right) \\
\sigma_{22}=\left(G_{2}-G_{1}-Y\right) /\left(G_{1}+Y\right)
\end{array}\right\} \text { (7.32) }
$$

It is noted that

$$
\begin{align*}
& \sigma_{21}-\sigma_{11} \tag{7.33}
\end{align*}=1 .
$$

For the shunt capacitance $C$, we find that, on using eqn. (7.j2) and applying the realizability condition,

$$
\left.\begin{array}{rl}
G_{2} & =G_{1}+C \\
\sigma_{11} & =\frac{1}{2}(\beta-1)\left(1-z^{-1}\right) /\left(1+\beta z^{-1}\right) \\
\text { and } \quad \sigma_{22} & =(1-\beta) z^{-1} /\left(1+\beta z^{-1}\right) \\
\text { where } \quad \beta & =2 \frac{G_{1}}{G_{2}}-1
\end{array}\right\} \text { (7.34) }
$$

The SFD appears in Fig. $7.8^{\circ}$ and was obtained on using eqns. (7.3), (7.33) and (7.34).

On using en. (7.28) it is easy to show that we may directly connect two ports so long as their appropriate port resistances are equal.

Finally, let us consider the terminations, which must be derived using $P$ as a result of the transfer function condition.

For the resistive voltage source,

$$
\begin{aligned}
& X=V+R_{1} I \\
& Y=-R_{1} I \\
& V_{0}=V+R_{s} I
\end{aligned}
$$

As we are forced to design from the source-end, we must set $R_{1}=R_{s}$ and therefore $X=V_{0}$.

For the load resistonce,

$$
\begin{aligned}
& X=V+R_{2} I \\
& Y=-R_{2} I \\
& V=R_{I} I
\end{aligned}
$$

Tnerefore, on eliminating $V$ and $I$, we find that

$$
Y=\left\{\frac{-R_{2}}{R_{2}+R_{L}}\right\} X
$$

The representation of a general doubly-terminated tro-port using the F12 transformation, appears in Fis̃. 7.9.

### 7.3.4 Discussion

Certain important observations need to be mentioned with regord to the signal-flow diagrams derived in this chapter. The SPD of the series inductance and shunt capacitance using the $F 10$ transformation (ifigs. 7.4 and 7.5 ) can be derived frouthe correaponding SFD using the IVR transformation (ijgs. 6.1 and 6.5) by intercionsing input and output. That is, by interchonging $X_{1}$ and $X_{2}, Y_{1}$ and $Y_{2}, R_{4}$ and $R_{2}$. This fact becomes apparent when we realise that the FlO transformation is just the IVR transformation with $P$ and $Q$ interchanged. By observing the entries in Fig. 5.15, we may verify that this property holds between $F 4$ and $F 5, F 7$ and $F 11, F 8$ and $F 12$, and $F 9$ and $F 13$. We. would therefore expect to obtain the SID of one from the corresponding SMD of the other.

Let us now compare the SFD of F12 with those of the ITA transformation. The SFD of the series inductance and shunt capacitance using the F12 transformation can be derived from the corresponding SED using the ITA transformation (Figs. 7.7, 7.8, 7.1 and 7.2) by reversing the flow and interchanging $A_{1}$ and $B_{2}$, $A_{2}$ and $B_{1}$, $R_{4}$ and $R_{2}$. If we ezarine the corresponding $P$ and ? matrices, we find that to obtain the appropriate pair for P 12 all we need to do is interchanse the $P$ and 2 matrices for the ITA transformation and multiply then by their respective port resistances. This property also holds between P 6 and $\mathrm{F} 13, \mathrm{FB}$ and F11, and F9 and F10 (see Fir. 5.15). Furthermore, in the special case when $P=Q$ when $R_{1}=R_{2}$, interchansins $P$ and $Q$ has no effect. Thereifore, the voltace and current formulations may be derived from one another by respectively multiplyina by the port conductance and the port resistance. The relationship bet:reen the SFD obtained using the voltage and current formulations has been examined in Reference [60].

Finally, if we reverse the flow in Fig. 7.3 together with the interchenging of $R_{L}$ and $R_{s}, A_{1}$ and $B_{2}$, and $A_{2}$ and $B_{1}$ we obtain Fig. 7.9. This last point implies that a networle designed using the ITA formulation may be transformed by flow reversal into one designed using the F12 formulation. Sinilar comments may be made about $F 1$ and $F 2, F 6$ and $F 13, F S$ and $F 11$, and $F 9$ and $F 1 C$. For convenience, we have illustrated the ceneral tro-nort for $F 6$ to F13 inclusive in Fig. 7.10.

## 7. 4 - Study of One Hon-Ganonic Trensformotion

Let us examine the transformation, F4, defined as follows

$$
\begin{align*}
& P=\left[\begin{array}{ll}
G_{1} & 1 \\
1 & -R_{1}
\end{array}\right] \\
& Q=\left[\begin{array}{ll}
1 & R_{2} \\
G_{2} & -1
\end{array}\right] \tag{7.36}
\end{align*}
$$

So far we have studied, almost exclusively, transformations that yield canonic digital filter structures. By the term 'canonic' we mean that the individual sid use the minimum number of multipliers and delays althourh the overall digital structure may not. All the transformations that have been examined, satisfy the sufficient condition of section 5.9. In fact, of the thirteen transformations listed in. Fig. 5.15 ten are known to be canonic and the other three are not canonic. All the canonic transformations of $\operatorname{Fig} .5 .15$ satisfy the sufficient condition. le nay indeed propose that the condition is not only sufficient but necessary. Mo transformation has yet been found which contradicts this.

Returning to the present discussion, namely the discussion of the F4 transformation, we have on combining eqns. (7.35), (7.36), (7.3) and (7.4) the following expressions,

$$
\begin{align*}
& \sigma_{11}=\left(A R_{2}-B-C R_{1} R_{s}+D R_{1}\right) / R_{12} \\
& \sigma_{12}=2 / R_{12} \\
& \sigma_{21}=2 / R_{12}  \tag{7.37}\\
& \sigma_{22}=-\left(A G_{1}+B G_{1} G_{2}+C+D G_{2}\right) / R_{12} \\
& R_{12}=\left(A G_{1} R_{2}-B G_{1}+C R_{2}-D\right)
\end{align*}
$$

For the series impedance, $Z$, eqn. (7.37) becomes

$$
\left.\begin{array}{l}
\sigma_{11}=R_{1}\left(R_{2}-R_{1}+z\right) /\left(R_{2}+R_{1}+z\right)  \tag{7.38}\\
\sigma_{12}=2 R_{1} /\left(R_{2}+R_{1}+z\right) \\
\sigma_{21}=\sigma_{12} \\
\sigma_{22}=G_{2}\left(R_{1}-R_{2}+z\right) /\left(R_{2}+R_{1}+z\right)
\end{array}\right\}
$$

For the shunt admittance, $Y$, eq. (7.37) can be written thus,

$$
\begin{align*}
& \sigma_{11}=R_{1}\left(G_{1}-G_{2}-Y\right) /\left(G_{1}+G_{2}+Y\right) \\
& \sigma_{12}=2 G_{2} /\left(G_{1}+G_{2}+Y\right) \\
& \sigma_{21}=\sigma_{12}  \tag{7.39}\\
& \sigma_{22}=G_{2}\left(G_{2}-G_{1}-Y\right) /\left(G_{1}+G_{2}+Y\right)
\end{align*}
$$

Let us consider the series inductance, $L$, as an example, then eqn. (7.38) becomes

$$
\left.\begin{array}{l}
\sigma_{11}=R_{1}\left\{\left(R_{2}-R_{1}+L\right)+\left(R_{2}^{*}-R_{1}-L\right) z^{-1}\right\} / \text { denom. }  \tag{7.40}\\
\sigma_{12}=2 R_{4}\left(1+z^{-1}\right) / \text { denom } \\
\sigma_{21}=\sigma_{12} \\
\sigma_{22}=G_{2}\left\{\left(R_{1}-R_{2}+L\right)+\left(R_{1}-R_{2}-L\right) z^{-1}\right\} / \text { denom }
\end{array}\right\}
$$

To avoid delay-free loops on interconnection either $R_{y}=R_{2}+L$ or $R_{2}=R_{1}+L$. If we let $R_{1}=R_{2}+L$ in eqn. (7.40) then

$$
\begin{aligned}
& \sigma_{11}=-L z^{-1} /\left(1+\alpha z^{-1}\right) \\
& \sigma_{12}=\left(1+z^{-1}\right) /\left(1+\alpha z^{-1}\right) \\
& \sigma_{22}=\left(G_{2}-G_{1}\right) /\left(1+\alpha z^{-1}\right) \\
& \text { where } \alpha=R_{2} / R_{1} .
\end{aligned}
$$

The SFD for the series inductance is formed using en. (7.41) with eqn. (7.3). However, it is clear that no structure can be built up using only one multiplier, and one delay. To realise these equations, we would require at least three multipliers and two delays, since there are no useful relationships bet:reen the $\sigma$-parameters and between $\mathrm{I},\left(G_{2}-G_{1}\right)$ and $\alpha$. Thus the $\sin$ of a series inductance is not canonic. We may deduce, in a similar way, that the $\operatorname{SiP}$ of a shunt. capacitance is also noncanonic. We need not, therefore, concern ourselves with the sources, terminations and interconnections of the F4 transformation.

Finclly, it is ooserved that the $i 5$ transformation can be obtained from $P 4$ by interchonging $P$ and 2 . Thus, we may equally apply the arguments of this section to F 5 .

## 1. 5 Discussion and Conclusion

In this chapter we have examined $\hat{\jmath}$ our transformations and shown that the SND of an element using one particular transformation can be obtained by elementary operations on the $S P D$ of the same element using a different transformation. We have found that $F 4$ can be obtained from F5 by interchangins $P$ and 2 which is equivalent to interchansins $X_{1}$ and $X_{2}, Y_{1}$ and $Y_{2}$ and $R_{1}$ and $R_{2}$. Similarly, $\bar{F} 6$ and 710 , 77 and $F 11, ~ F S$ and $F 12, ~ F 9$ and $F 13$ are so related.

We have also established that a disital structure derived using F6 may be obtained from that derived using 113 by flow reversal. A similar relationshin holds beticen 77 and $\mathcal{F 1 2}$, $\bar{m}$ and $F 11$, $F 9$ and $F 10$, and $F 1$ and 72 .

We may thus classify the thirteen transformations given in section 5.10 into 5 divisions. The first consists of Pl and I 2 , the second of $\overline{-3}$ only, the third consists of $P 4$ and 55 , the fourth of F6, F9, Fio, F13 and the inith of F7, F0, F11, F12. The transiormations in each class are related by one or both of the pronerties mentioned ebove.

The transformations given in Fig. 5.15 and discussed here are, by no neans, the entire set of transformations that satisfy the three existence conditions. Clearly, there is more work needed to find nev transcomations and examine their properties, in particular the attenuation sensitivity to multiplier variation and round-off noise effects.


Fig. 7.1 SFD of Series Inductance (ITA).


Fig. 7.2 SFD of Shunt Capacitance (ITA) .


Fig. 7.3 General digital two-port (ITA).


Fig. 7.5 SFD of shunt capacitance (F1O).


Fig. 7.6 General Digital Two-Port
(F10) .


Fig. 7.8 SFD of Shunt Capacitance (F12) •


Fig. 7.9 General Digital Two-Port (F12). .


Fig. 7.10 General Digital Two-Port for F6 to F13 inclusive .

## Chapter 8

Computer-Aided inalysis oî Dirsital Filter Structures.

Contents:
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8.3 Mew Analysis Alsorithra.
8.4 Experimental Results.
8.5 Discussion.

## Chapter 8

## Computer-Aided Analysis of Disital Filter Structures

8.1 Introduction

The many filter examples given in this thesis vere analysed using procrans writton in APL [84]. This lanfuage was chosen for the following reasons, (a) it is fully interactive, (b) it is very concise, and (c) it has an extensive built-in netrix handling facility. Thus programs can be written and tested very quicilly.

As a difital filter structure can be thought of as a linear signal-flow graph [85] [86], all major projrans use a cozmon input procedure in mich the signal-îlow Eraph (SFG) of the dirital filter is specified. The following four rules describe the properties of a time-invariant SFG [29]:
I. Signals flow along branches in the direction of the arrors.
II. A signal flowing along any branch is multiplied by the transmittance of that branch.
III. The value of the variable represented by any node is the sum of all signals entering the node.
IV. The value of the variable represented by any node is transmitted to all branches leaving that node.

A node can be any noint on a branch in the SFG, however we are restricted here because we admit only two types of branch transmittance (a) a real constant or multiplier and (b) a fired complez delay, $z^{-1}$ where $z=\exp (j w T)$. Thus every branch transmittance must be uniquely defined by two nodes, the order of which defines the direction of the signal. The nodes of an SFG are numbered from 1 to $n$ where $n$ is the total number of nodes. Then every branch is defined by three numbers, the first two are the nodes to mich the branch is connected and the third is the
value of the transmittance if a multiplier and zoro otherrise, i.e. if a delay. As an example, consider the disital circuit of Fig. 8.1 and its associated SFG in Fi . 3.2 . The appropriate nodes have been numbered and in Fir. 8.3 we see the SFG information in list form.

It is the purpose of this chapter to describe how sensitivity derivatives are evaluated and to discuss a nev alcorithm which is helpful in calculating the chnnge in a netrorl function given simultaneous chanses in value of one or more multipliers.
8.2 Tinite Chnmensitivitu

Consider a linear SFG with $n$ nodes. Te may describe it by a set of linear algebraic equations of the form [ 07 ],

$$
\begin{equation*}
\underline{x}=\Delta \underline{x}+\underline{e} \tag{8.1}
\end{equation*}
$$

where $\underline{\underline{I}}$ represents the signal values at the nodes.
e is the vector of nodal inputs or signal sources.
and A is the $n \geq n$ connection matrix such that $A_{j i}$ is the transmittance from node $i$ to node $j$. A can be vritten dow by inspection of the $S T G$.

Eqn. (8.1) may be solved for $x$ to give

$$
\begin{equation*}
\underline{x}=T e \tag{8.2}
\end{equation*}
$$

where $T=(I-A)^{-1}$ and $I$ is the unit matrix.

It must be assumed, of course, that ( $I-A$ ) is non-singular. If. $V$ is the output node and $u$ is the input node then the transier function $G(z)$ is defined as $x_{V} / e_{u}$ or TVu The partial derivative of $G$ with respect to a transmittance $\alpha$ connected from node $i$ to $\mathbf{j}$ (see Fig. 8.4; is given by [88][89][90][91],

$$
\begin{equation*}
\frac{\partial G}{\partial \alpha}=T_{V j} T_{i u} \tag{8.3}
\end{equation*}
$$

Thus the sensitivity is simply the product of two elements from the matrix $工$. To ovaluate all the sensitivities for the transfer function $G(z)$, we need only the uth. column and vth. row of r.

To find the entire matrix $T$ requires $n^{3}$ operations (multiplÿ-add) whilst to find one roir and colurn require $\frac{1}{3} n^{3}+2 n^{2}$ at each frequency of interest [92][97][98]. The progran SOLVE finds the former and as we shall see this is necessary if are are to require a finite chance algorithm. A listing of SOLVE is given in Appendix II.

The principal problem of this section is to derive an expression which is numerically efficient that enables us to calculate the change in $G(z)$ given simultaneous changes in multiplier values. This expression is required because for Disital Filters we are interested in the eifocts of reiucins. the multiplier iordlenstin overall.

Let us first consider the folloring; if we perturb $T$ by an amount
$\Delta T$ and $=I-A$, Anl then wotill have the corresponding
change in $A^{\prime}, \Delta A^{\prime}$, will be such that

$$
\begin{equation*}
\left(A^{\prime}+\Delta A^{\prime}\right)(T+\Delta T)=I \tag{8.4}
\end{equation*}
$$

where $I$ is the unit matrix.

Eqn. (8.4) can be mritten as follows,

$$
\begin{equation*}
\left(A^{\prime}+\Delta A^{\prime}\right) \Delta T=-\Delta A^{\prime} T \tag{8.5}
\end{equation*}
$$

or as

$$
\begin{equation*}
\Delta T=-\left(A^{\prime}+\Delta A^{\prime}\right)^{-1} \Delta i^{\prime} T \tag{8.6}
\end{equation*}
$$

We may write eqn. (8.6), since A'T = $I$, as

$$
\begin{equation*}
\Delta T=-\left(I+T \Delta^{\prime}\right)^{-1} T \Delta A^{\prime} T \tag{8.7}
\end{equation*}
$$

But

$$
\Delta A^{1}=-\Delta \mathrm{A}
$$

therefore eqn. (8.7) becomes

$$
\begin{equation*}
\Delta T=(I-T \Delta A)^{-1} T \Delta A T \tag{8.8}
\end{equation*}
$$

We know that $A$ is linear in each transmittance, therefore

$$
\begin{equation*}
\Delta A=\sum_{j} \frac{\partial A}{\partial \alpha_{j}} \Delta \alpha_{j} \tag{3.9}
\end{equation*}
$$

where $\alpha_{j}$ is the $j$ th branch transuittance and the sumnation is taken over, say, $m$ of then. The zatrix $\frac{\partial A}{\partial \alpha_{j}}$ consists only of ones or zeros. Thus, if we use eqn. (8.8) in conjunction with eqn. (0.9) we may certainly find the chance in $n(z)$ aiven a change, $\Delta \alpha_{j}$, in the $j$ th. transmittance for $j$ from 1 to m . However, we need to invert not only $\Lambda^{\prime}$ but also ( $I-T \Delta A$ ).

We shall next consider an alternative and more efficient approach. Expressions have been found for the chance in $G$ due to only one multiplier chaning. Crochiere derived his formula using the Taylor Series [85][80] rhilst Spence [37] derived a similar expression using tine substitution Theorea [93]. Ue shall use the latter apnroach here in a more seneral way.

Consider changing $a$ of the branch transnittances by finite amounts. Let a typical branch have nominal transmittance $\boldsymbol{\beta}_{\mathrm{k}}$ and associated chanise $\Delta \beta_{k}$ (IVis. 0.5). The tranciittanco $\Delta \beta_{k}$ can be replaced by a source iith value $\Delta \beta_{k} X_{I_{k}}$ at node $J_{k}$ without affecting the overall structure of the eraph. This replacement (show in Fig. 8.6) can be performed for all transmittances that chanse thus reducing the structure to its original form with additional sources at the relevant nodes.

Let us denote $T_{v u}=x_{v} / e_{u}$ as the nominal transfer function of the sysien.

The chance in the output due to changes of the transmittances will be given by

$$
\begin{equation*}
\Delta x_{v}=\sum_{k=1}^{m} T_{\nabla J_{k}} e_{J_{k}} \tag{8.10}
\end{equation*}
$$

where $T_{V J k}$ represents the transmittance from node $J_{k}$ to the output node v. Equivalently, we can write eqn. (8.10) in the form

$$
\begin{equation*}
\Delta x_{V}=\underline{I}^{\prime} v J e_{J} \tag{8.11}
\end{equation*}
$$

Where
$T_{\sigma J}$ and $\underline{e}_{J}$ are column vectors, with the prime denoting transpose.

From superposition, we can write, therefore, that

$$
x_{I_{k}}=T_{I_{k} u} e_{u}+\sum_{I=1}^{m} T_{I_{k} J_{I_{1}}}{ }^{{ }^{\prime} J_{I}}
$$

or equivalently

$$
\begin{equation*}
\underline{\underline{x}}_{I}=\underline{T}_{I u} e_{u}+T_{I J} \underline{e}_{J} \tag{8.12}
\end{equation*}
$$

where $X_{I}$ and $T_{I u}$ are column vectors and $T_{I J}$ is an $n ~ m$ matrix.

Furthermore, we can write the expression

$$
\begin{align*}
{ }^{e_{J}} & =\Delta \beta_{k} \bar{I}_{I_{k}} \\
\text { or } \quad \underline{e}_{J} & =\Delta \beta x_{I} \tag{8.13}
\end{align*}
$$

where $\Delta \beta$ is an n m diagonal matrix.

On combining eqns. (8.11), (8.12) and (8.13) we obtain

$$
\begin{equation*}
\Delta x_{v}=\frac{T^{\prime}}{V^{\prime}} \quad\left[(\Delta \beta)^{-1}-T_{I J}\right]^{-1} \stackrel{T}{I}^{e} u \tag{8.14}
\end{equation*}
$$

where $(\Delta \beta)^{-1}$ is a diagonal matrix whose elements are $\left\{1 / \Delta \beta_{k}\right\}$.

On comparing eqn. (8.14) with (8.8) we see that knowledge of $T$ is required in both, however, whereas in the latter we have to invert another $n \times n$ matrix, in the former it is an $m \times m$ matrix. Although the total number of branches is generally greater than $n$, the number of multipliers of interest will always be much less $n$. Therefore eqn. (8.14) is more efficient to use than eqn. (8.8).

At any frequency of interest, we may apply eqn. (8.14) to find the chance in the transfer function given a simultaneous change in m branch transmittances. The only term that changes in eqn. (8.14),
for different sets of changes in the branch transmitiance, is $(\Delta \beta)^{-1}$.

A program, called FIIIITM, which embodies the finite chance algorithm just described is listed in ippendix II and is to be used in conjunction vith SOLV.

### 8.3 Ferinalysis :1rorithr

We have considered, thus far, $\beta_{15}$ to be a general transmittance. For digital filter structures, $\beta_{k}$ could be a multiplier or a delay. By considerins a finite change in each of the delays, we may devolop on analysis alsorithr :Thich has many advantases over conventional nethods.

Let the change in each delay be equal to $\delta z^{-1}$. This has the effect of changing the frequency from its nominal value. We can write, therefore, that

$$
\begin{equation*}
\Delta \beta=\quad \delta z^{-1} I \tag{8.15}
\end{equation*}
$$

where

$$
\delta z^{-1}=\left.z^{-1}\right|_{w=w_{1}}-\left.z^{-1}\right|_{w=w_{0}} \quad \text {, a scalar }
$$

and $I$ is the unit matrix.

Thus eqn. (8.14) becomes,

$$
\begin{equation*}
\Delta x_{v}=I_{v J}^{\prime} \quad\left[\delta z I-T_{I J}\right]^{-1} \cdot I_{I u} e_{u} \tag{8.16}
\end{equation*}
$$

where $\quad \delta z=1 / \delta z^{-1}$.

The algorithn will be as follows:

1. At some frequency $w=W_{0}$, solve eqn. (8.1), that is invert $A^{\prime}$.
2. Use eqn. (8.16) to find $\Delta x_{\nabla}$ for some frequency $w=w_{1}$ by considering delay elements as the changing transmittances.
3. Apply step 2 for other frequency values of interest.

It should be noted that, since eqn. (8.16) is exact, we may choose any value for $\delta z$. Thus the complete analysis of the systen is necessary at one frequency point only, which may be chosen at convenience. In fact, the complete analysis nay be done at zero frequency when all transmittances are real and therefore matrix calculations are simpler. The subsequent analysis using step 2 above, requires the inversion of an $m$ x m matrix where $m$ is the number of delay branches. For the digital filter structures described in previous chapters, in is very much less than $n$. The ideas of this section have been incorporated into a progran called HYGOLV, a listing of which is given in sippendix II.

### 8.4 Zoperimental Results

### 8.4.1..Theoretical Pradictions

He have developed the alsorithris in the previous two sections, so as to be able to compute efficiently the change in the loss characteristic of a digital filter structure due to chances in the multiplier values. Let us therefore compare these algorithms with the conventional method of triangular decomposition, which is the most efficient way of solving a general system of linear equations.

> Let $n$ be the number of nodes, $n$ be the number of multipliers of interest, $d$ be the number of delavs, $f$ be the number of sets of chanpes and $f$ be the number of frequency points.

## then for

ifethod $A$ : Trianeular Decomposition of $A^{\prime}=I-A$ at every frequency point.
The number of arithmetic operations, $T$, is given by [92]

$$
\begin{equation*}
T_{A}=\left(\frac{1}{3} n^{3}+n^{2}\right)(c+1) f \tag{8.17}
\end{equation*}
$$

Nethod B: :ultiple finite chance algorithn together with inversion of $\mathrm{A}^{\mathbf{2}}$ at every frequency point.

The number oi operations, $T_{D}=\left(n^{3}+m^{3} c\right) f$
and Hethod C: INew Analysis Alsorithm.

$$
\begin{equation*}
T_{c}=\left\{n^{3}+d^{3}(f-1)\right\}(c+1) \tag{8.19}
\end{equation*}
$$

### 8.4.2 Exaple

Let us consider a Jave Digital Pilter with

$$
\begin{aligned}
& \mathrm{n}=14, \mathrm{n}=4, \cdot \mathrm{~d}=4, \mathrm{c}=1,4 \text { and } 6, \text { and } \\
& \mathrm{f}=10 \text { and } 20 .
\end{aligned}
$$

The filter was analysed usinc iethods $A, B$ and $C$ with the aid oî an APL terminal. The CPU tines were noted anc tabulated in Fig. 8.7 where they may be compared with the theoretical predictions. The CPU times have enabled us to derive the following expressions for the three methods which can estinate the corresponding CPU tine for any $c$ and $f$ value.

$$
\left.\begin{array}{l}
\tilde{T}_{A}=424(c+1) f \\
\tilde{T}_{B}=(978+135 c) f \\
\tilde{T}_{c}=\{1008+91(\mathbf{f}-1)\}(c+1)
\end{array}\right\}(8.20)
$$

Fig. 8.7 shows, at least for one filter, that method $C$ and method $B$ are superior to method $A$ and that method $C$ is superior to method B. In fact, method B becomes more efficient as $c$ increases since $m \ll n$ and method $C$ becomes more efficient as $f$ increases since $d \ll n$.

Finally, Fig. 8.8 tabulates the theoretical estimates using methods $A, B$ and $C$ for five different networks and it can be seen that the conclusions dram above are not unreasonable:

### 8.5 Discussion

In this chapter two new ways of examininf finite wordlength effects in digital filter structures have been discussed. Both methods avail themselves of the formula for multiple parameter finite change.* The first uses the basic inversion of the connection matrix tosether with the finite change algorithn at each frequency of interest. This method is particularly useful when the number of sets of different changes is large and $m$ is less than $n$.

The second method uses the basic inversion only at the first frequency point. At subsequent points, the multiple paraneter finite change algorithm is used to modify the frequency response. It is of greatest advantage when the number of delay branches, $d$ is small compared with the number of nodes, $n$, and when the number of frequency points is large.

Both methods show improved results when compared with the conventional trianfular decomposition method.

Although the advantages of these two methods have been arcued for Wave Digital Filters, it is expected that the sane advantages will apply to the analysis of Digital Ladder Filters [62][65], Leapfrog Digital Filters [44][45] and the Digital Filters of Gray and Zarkel [68][59].

Footnote:

* An efficient method for the computation of the chanse in an analogue network function due to simultaneous finite chances in two or more netrork parameters has recently been published [94]. The formulae were derived using the Adjoint Hetwork Concopt [95].


Fig. 8.1 A Digital Circuit Example .


Fig. 8.2 Signal-Flow Graph of Fig. 8.1 -

| 1 | 2 | 0 |
| :--- | :--- | :--- |
| 2 | 3 | 0 |
| 3 | 4 | 0 |
| 4 | 5 | $a_{3}$ |
| 3 | 5 | $a_{2}$ |
| 2 | 5 | $a_{1}$ |
| 1 | 5 | $a_{0}$ |
| 2 | 1 | $b_{1}$ |
| 3 | 1 | $b_{2}$ |
| 4 | 1 | $b_{3}$ |

Fig. 8.3 Tabular Form of Fig. 8.2


Fig. 8.4 Transmittance $\alpha$ connected from node $i$ to node $j$ -


Fig. 8.5 Transmittance $\beta_{k}+\Delta \beta_{k}$ connected from node $I_{k}$ to $J_{k}$.


Fig. 8.6 Equivalent representation of Fig. 8.5 .

| Method | OBSERVED (in units of time) |  |  | THEORETICAL(no. of operations) |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\mathrm{c}=1$ | $c=4$ | $\mathrm{c}=6$ | $\mathrm{c}=1$ | $\mathrm{c}=4$ | $\mathrm{c}=6$ |
|  | $\mathrm{f}=10$ | $\mathrm{f}=10$ | $\mathrm{f}=20$ | $\mathrm{f}=10$ | $f=10$ | $\mathrm{f}=20$ |
| A | 8480 | 21200 | 59360 | 22220 | 55550 | 155540 |
| B | 11130 | 15180 | 35760 | 28080 | 30000 | 62560 |
| C | 3654 | 9135 | 19159 | 6640 | 16600 | 27720 |

Fig. 8.7 Comparison between methods $A, B \& C$ for one : example .

| $\underbrace{\text { Miethod }}_{\text {Network }}$ | A | B | C |
| :---: | :---: | :---: | :---: |
|  | 37 | $\begin{aligned} & 64(c=0) \\ & 115:(c=5) \\ & 125(c=\infty) \end{aligned}$ | $\begin{aligned} 64 & (f=1) \\ 15 & (f=10) \\ 8 & (f=\infty) \end{aligned}$ |
| $\begin{aligned} & N 2 \\ & n=10, m=4, d=3 \end{aligned}$ | 433 | $\begin{gathered} 1000(c=0) \\ 220(c=5) \\ 64(c=\infty) \end{gathered}$ | $\begin{aligned} 1000 & (f=1) \\ 138 & (f=10) \\ 27 & (f=\infty) \end{aligned}$ |
| $\underset{n}{N 3} \mathbf{N} 8, m=4, d=3$ | 2268 | $\begin{gathered} 5832(c=0) \\ 1025(c=5) \\ 64(c=\infty) \end{gathered}$ | $\begin{aligned} -5832 & (f=1) \\ 675 & (f=10) \\ 27 & (f=\infty) \end{aligned}$ |
| $\begin{gathered} N 4 \\ n=14, m=4, d=4 \end{gathered}$ | 1111 | $\begin{aligned} 2744 & (c=0) \\ 511 & (c=5) \\ 64 & (c=\infty) \end{aligned}$ | $\begin{aligned} 2744 & (f=1) \\ 369 & (f=10) \\ 64 & (f=\infty) \end{aligned}$ |
|  | 37 | $\begin{aligned} & 64(c=0) \\ & 33(c=5) \\ & 27 .(c=\infty) \end{aligned}$ | $\begin{aligned} 64 & (f=1) \\ 15 & (f=10) \\ 8 & (f=\infty) \end{aligned}$ |

Values tabulated are no. of operations per frequency change .

Fig. 8.8 Theoretical Estimates for methods $A, B$ \& C applied to five different networks .

## Further Research.

## Chapter 9

Further Research.

Given below are suggestions as to what research could usefully be done on some of the topics discussed in this thesis.

In Chapter 5, the general two-port linear transformation was introduced. It still remains to find a provable necessary and sufficient condition for canonic signal-flow diafrans. the one given in this thesis seems not to be provable, although it is satisfied by the thirteen transformations listed in Chapter 5. Furthermore, are there other canonic transformations apart from the ten listed in section 5.10 ?

With resard to the Cascade condition, is there any interesting and useful consequence of allowing $P$ and $Z$ to be different for each constituent series and shunt arm? Of course, at junctions the condition $\imath_{k}^{-1} C P_{k+1}=D$ still holds.

In the same chapter, the LuP sensitivity characteristics were investigated and two conditions were found for zero sensitivity. The second condition was studied and it was found that only non-canonic SED satisfied it. The first condition was not investigated and it would be interesting to know :rhether canonic transformations exist for which $\eta_{I} / R_{s}$ is independent of every multiplier variable.

Recently, Fettreis examined a way of reducing the number of delay elements in a wave digital filter by applying to the analogue network a Bruton transformation [54] [55]. This had the effect of producing new types of elements, the so-called supercapacitances and super-inductances. Towever, the sood sensitivityr characteristics were preserved. It would therefore be interesting to know whether such techniques could be applied to the 'IVR', 'ITA' and other transformations discussed in Chapters 6 and 7.

The noise properties of disital filters derived using 'IVR', 'I'TA', etc. need to be studied. Comparisons has been made between the roundooff noise generated in "ave Digital jilters and conventional
digital filters [ 44 ]. It was found that, in certain circumstances, the former yielded the lowest noise. It is expected that the disital filter structures discussed in Chapters 6 and 7 will behave in a similar way. As a step towards a rimorous analysis of the round-off noise, a comparison between seven different realisations of the same third-order elliptic transfer function was made. This may be found in ippendix III.

No mention has been made of the transient response and clearly work is needed in this area. In addition, the parasitic effects, i.e. limit cycles need to be studied for the ne:r transformations.

Finally, $x i$ th regard to computer-aided analysis a start has been made on writinc software for a PDP-15 computer with an interactive graphics unit. In this way, a disital filter structure may be entered into the computer by drawing it with a light pen on the eraphics terminal screen. At the present, the structure can be analysed in the frequency domain and appropriate response curves plotted. In addition, the differential sensitivity is calculated and displayed. Firther worl needs to be done in the follo:ring areas, (i) Large-change sensitivity, (ii) Transient analysis, (iii) Facilities for enterins very large structures in sections and (iv) Hoise analysis.

## Conclusions

1. A method has been described in which digital filter structures were derived from classical analogue doubly-terminated lossless ladder net:rorks using the two-port approach. The method consists of treating each series and shunt ladder arm as a tro-port and derivinf the correspondinz :rave-flow diasram. These wave-flow diacrams may then be directly connected tosether thus obviating the need for adaptors. The principles of design were then illustrated by means of several examples. These examples also verified that the coeificient quantization error in these wave digital filter structures was lower than that in conventional direct and cascade structures.

The method was then exterded to cover filters made up from a cascade of̂ comnensurate stepped-inpedance transaission lines.
2. The attenuation sensitivity to first-order aultiplier variations was then exarined. A formula was derived relating this sensitivity to that of the analocue networs fron which the disital filter was derived. It vas found that the first-order attenuation sensitivities were approximately constant in the pessiond. Lons has derived similar formulae but in a completely different way using the concept of pseudopower [44].

The consequence of approzimately constant sensitivities is a constant shift in the attenuation curve, after multiplicr rounding.
3. The tro-port approach enables digital filters to be desicned simply with the aid of standard filter tables. The analozue prototypes may be laddor netrorks or networks made up from a cascade of comensurate transmission lines.
4. A generalization of the Wave Digital rilter concept was also presented. This was made possible by considering, instead or voltages and currents, new variables which were related to the former by linear transformations as follows,

$$
\left[\begin{array}{l}
X_{1} \\
Y_{1}
\end{array}\right]=P\left[\begin{array}{l}
V_{i} \\
I_{i}
\end{array}\right] \text { and }\left[\begin{array}{l}
X_{2} \\
X_{2}
\end{array}\right]=\left[\begin{array}{l}
V_{2} \\
I_{2}
\end{array}\right]
$$

The matrices $P$ and 2 being of dimension $2 \times 2$ and non-sincular. Conditions heve been found on the elenents of $P$ and 2 such that the resultins aisital filter structure was realizable, canonic and, indeed, imitated the analogue filter from which it was derived. The general linear transformation on a two-port was first examined by Carlin and Giordano [99], although they considered only analogue networks. Using the same method as above, the sensitivity bchaviour was investigated and was found to be of a similar character to that of the Nave Dirital $\mathfrak{i l}$ iler. Furthermore, a trunsiomation was found to exist that yielded structures with zero attenuation sensitivity at points of mazimun pseudopower transfer but, unfortunately, such structures vere found not to be canonic. i table was given, listinf thirteen transiormations, fron which the Invariant Voltage Ratio transformation was taken and studied in detail. Several filter eamples were given that verified the low attenuation distoriion property.
5. Other transformations, from the table, were also examined. It was found, for example, that structures derived using one transformation could be obtained from structures derived using another, by simple operations on the corresponding signal-flow diastam.
6. Finally, the computer-aided analysis o $\hat{i}$ digital filter structures was investigated and various methods of andysis were discussed, from the point of viev of the number of arithmetic operations needed. A new algorithn was presented in which the formula for calculating the chance in the frequency response, given changes in any number of branch transmittances, was used. The branch transmittances changed were the delays and in this way, the frequency response at different frecuencies could be calculated for only one complete analysis of the structure. "It was further shom thit the new algorition was more eificient to use than conventional techniques for the analysis of most practical filter structures.

## Appendices

## Contents:

I: Delay 3lement Sensitivities.
II: APL Analysis Procrams.
III: Round-Off lloise Analysis.

## Appendices

Appendix I: Delay element Sensitivities.
The aim of this appendix is to derive the functional relationships between $D_{L}, D_{c}, D_{1}$ and $D_{2}$ for the case of a series-tuned circuit in the shunt arm. For this circuit, the admittance $Y$ is given by

$$
\begin{equation*}
Y=1 /\left\{\mathrm{pL}+\frac{1}{\mathrm{pC}}\right\} \tag{A.1}
\end{equation*}
$$

On applying the bilinear transformation to eq. (A.1) in the form

$$
\begin{aligned}
\mathrm{pL} & \rightarrow \frac{1-\mathrm{D}_{\mathrm{L}}}{1+\mathrm{D}_{\mathrm{L}}} \mathrm{~L} \\
\text { and } \quad \mathrm{pC} & \rightarrow \frac{1-\mathrm{D}_{\mathrm{c}}}{1+\mathrm{D}_{\mathrm{c}}} \mathrm{C}
\end{aligned}
$$

we find that

$$
\mathbf{I}=\frac{\left(1+D_{L}\right)\left(1-D_{c}\right)}{\left(1-D_{L}\right)\left(1-D_{C}\right) L+\left(1+D_{L}\right)\left(1+D_{c}\right) D}
$$

which can be written as

$$
\begin{equation*}
I=\frac{1+\left(D_{L}-D_{C}\right)-D_{L} D_{c}}{(D+L)+(D-L)\left(D_{L}+D_{C}\right)+(D+L) D_{L} D_{C}} \tag{A.2}
\end{equation*}
$$

Let $\alpha=\frac{1}{(D+L)}$ and $\beta=\left(\frac{D-L}{D+L}\right)$ then eqn. (A.2) becomes

$$
\begin{equation*}
Y=\frac{\alpha\left\{1+\left(D_{L}-D_{C}\right)-D_{L} D_{C}\right\}}{1+\beta\left(D_{L}+D_{C}\right)+D_{L} D_{C}} \tag{A.3}
\end{equation*}
$$

Now let us consider the realization of the series-tuned circuit. On applying the bilinear transformation to eqn. (A.1) we have that

$$
Y=\frac{1-z^{-2}}{(D+L)+2(D-L) z^{-1}+(D+L) z^{-2}}
$$

or

$$
Y=\frac{\alpha\left(1-z^{-2}\right)}{1+2 \beta z^{-1}+z^{-2}}
$$

Eqn. (A.4) nay be written as follows

$$
Y=\frac{\alpha\left\{1+\beta z^{-1}-z^{-1}\left(\beta+z^{-1}\right)\right\}}{1+\beta z^{-1}+z^{-1}\left(\beta+z^{-1}\right)}
$$

or, finally, as

$$
\begin{equation*}
Y=\frac{\alpha(1-T)}{(1+T)} \tag{A.5}
\end{equation*}
$$

where $T=$ $\frac{z^{-1}\left(\beta+z^{-1}\right)}{1+\beta z^{-1}}$

In the actual realization (see rigs. 2.23 and 2.24) we have

$$
\begin{equation*}
T=\frac{D_{1}\left(\beta+D_{2}\right)}{1+\beta D_{1}} \tag{A.6}
\end{equation*}
$$

On substituting for from eqn. (A.6) into eqn. (A.5) we have

$$
\begin{equation*}
Y=\frac{\alpha\left(1-D_{1} D_{2}\right)}{1+2 \beta D_{1}+D_{1} D_{2}} \tag{A.7}
\end{equation*}
$$

We may put eqn. ( 4.3 ) into the following form,

$$
Y=\frac{\alpha\left\{1+\frac{1}{2}\left(D_{L}-D_{c}\right)+\frac{1}{2}\left(D_{L}-D_{c}\right)-D_{L} D_{c}\right\}}{1+\frac{1}{2}\left(D_{L}-D_{c}\right)+\beta\left(D_{L}+D_{c}\right)+D_{L} D_{c}-\frac{1}{2}\left(D_{L}-D_{c}\right)}
$$

On dividing both numerator and denominator of eqn. (A.S) by
$1+\frac{1}{2}\left(D_{L}-D_{C}\right)$ and comparing with eqn. (A.7), it is not too difficult to see that

$$
\begin{equation*}
D_{1}=\frac{\frac{1}{2}\left(D_{\mathrm{L}}+D_{C}\right)}{1+\frac{1}{2}\left(D_{\mathrm{L}}-D_{C}\right)} \tag{A.9}
\end{equation*}
$$

and

$$
\begin{equation*}
D_{1} D_{2}=\frac{D_{L} D_{c}-\frac{1}{2}\left(D_{L}-D_{c}\right)}{1+\frac{1}{2}\left(D_{L}-D_{c}\right)} \tag{A.10}
\end{equation*}
$$

which is the desired result.

We may obtain $D_{2}$ explicitly by dividing eqn. (A.10) by eqn. (A.9). He find that

$$
\begin{equation*}
D_{2}=\frac{D_{L} D_{c}-\frac{1}{2}\left(D_{L}-D_{c}\right)}{\frac{1}{2}\left(D_{L}+D_{c}\right)} \tag{A.11}
\end{equation*}
$$

It is easily checked that, when $D_{L}=D_{C}=z^{-1}$ then $D_{1}$ and $D_{2}$ are also botin equal to $z^{-1}$.

## ADpendix II: APL Analysis Prorrans

Listings of the programs described in Chapter 8 are given here, that is SOLVA, FIitite Mid neisolv. In addition, four functions are civen, namely $\mathrm{HOD}, \angle R G$, GUUL and CDIY. . These four functions are called from within the analysis prograns. The input requirements of these programs are summarised below:
(i) salve

(ii) FITITE

This program is to be used in conjunction rith SOLV. Therefore a call to PHIPI must be made in SOLVS. It is suggested that a suitable place would be between lines 25 and 26. Furthemore lines 26 to 45 should be removed. The fourth quantity in the branch information, which is stored in DA'PA, should then contain the desired chañe for that branch. A vector called BarCu acts as an indicator to the branches that are required to change in value.

```
-315-
```

(iii) margow


```
        \(\nabla ら O L!R[\square] \nabla\)
    \(\square\) SOIVF
    [1.] BRMT+(MR,4) คDATA
    [2] \(\quad I+(H 1-M O) \div \| F R O-1+H F R Q=1\)
    [3.] \(e^{T+1}\)
    [4] \(\quad \Gamma, 4: T \nless 1\)
```



```
    [6] A1+A2ヶ(Mn, ill \() \rho 0\)
    [7] \(\quad T 3: I 1+B R 14 T[T ; 1]\)
    [8] M2+RRMTTLT:2.]
```



```
    \([10] \rightarrow(\pi 3=0) / K 1\)
```



```
    \([12] \rightarrow L 2\)
    [13] \(I 1: A 1[\| 2 ; 1 J 1]+A 1[\% 2 ; H 1]-20 W ?\)
    [14] A2[172;!17 \(7-A 2[112 ; 111]+101 / ?\)
    [15] \(I, 2: I<T+1\)
    [16] \(\rightarrow(T \leq I \Pi) / I, 3\)
    [17] \(I<1\)
    [18.] \(I 5: A 1[T ; T]<1\)
[19] \(I+T+1\)
    [20] \(\rightarrow(T=4) / 15\)
    [21] A3* pal
```




```
    [2.4] \(\quad G+T 1[V ; U], \operatorname{T2}[V ; U]\)
```



```
    [2.6] \(I+1\)
    [2.7]
    [28] \(I, 6: \rightarrow(R 2 i+2[T ; 4]=0) / T, 9\)
    [29] \(M 1+R R M[T ; 1]\)
    [30] \(/ 12+R P: I T[J ; 2]\)
    [31] \(133+B R: \%[I ; 3]\)
```




```
    [.34]
    [35] DTPF+Z1 Citlit, \(\quad 2\)
```



```
    [37] \(\rightarrow(1 / 3=0) / I, 7\)
    [38] DTPF+GRWに×133
    [39] \(\rightarrow\) [ 8
    [40] \(I, 7: 73+(20 N T),-(10 \% m)\)
    [41] DTMTかERMS GIUT Z3
    [42] \(I 8 ;{ }^{\prime} \quad\) '; T; ' \({ }^{\prime}\); DJTP
    [43] \(I, 9: T \leftrightarrow T+1\)
    \([44] \rightarrow(J \leq 1 / B) / I, 6\)
    [45]
    [46] \(e^{T}+\mathrm{c}^{T}+.1\)
    \([47] \rightarrow\left(e^{T} \leq M F R Q\right) / T, 4\)
    \([48] \rightarrow 0\)
        \(\nabla\)
```

```
        \nablaTMTMR[.%]
    \nabla FTHTTF
[1] I!<مBRAMCll
[2] nRIB\leftarrow(N,M)\rho0
[3] eTeT<IT+N\rho0
[4] K<1
[5] T1:IT[K] RRMTT[.RRAINCH[K];1]
[6] eTeT[K]+RR::T[RRAMCH[K];2]
```



```
[8] K<K+1.
[@] ->(K\leqi4)/T,1
[10] S.1+DRILB-T1[TJ; [TJ ]
[11] S2*-?2[IT; TeT.]
[12] S3+i1[V; 的T]
[13] S4<㕣2[V;eTeT]
[14] s5<T1[TT;|]
[15] S6<-J2[TI;㣙]
[16] ת7ヶEG1
```



```
[18] S2<-51+. xS2+. xS7
[19] S7+(53+. xS1)-54+. xS2
[20] S.1+(54+.x51)+53+.x52
[21] S2+(\Omega7+.x55)-51+.x55
[22] ת77+(31+. x55)+57+.x56
[23] r1<-(r+52,,57
[24] ' ';MOD G.1;' ';ARG G1
\nabla
```

$\operatorname{AARG[1]D}$
$\nabla R+A R G$ 分;
$[1] S+(-30!2[2] \div 6[1]) \times 180 \div 01$
$[2] \rightarrow((7[1]>0) \wedge \pi[2] \geq 0) / T, 1$
$[3] \rightarrow((\prime[.1]>0) \wedge$ 名 [. 2.$]<0) / I, 2$
$[4] \rightarrow((7[1]<0) \wedge Z[2]>0) / L, 3$
[5] $R+S+180$
[6] $\rightarrow 0$
[7] $\quad 1,1: \cap \leftarrow \pi$
[8.] $\rightarrow 0$
[9] $I T 2: M+360-5$
$[10.] \rightarrow 0$
[11] $T, 3: R \leftarrow 180-\Omega$
[12] $\rightarrow 0$
$\nabla$
จchtvedrop
$\nabla R+A \quad C D J V B ; D$
[1] $R \leftarrow 11$
$[2] \quad D+(n[1] * 2)+(n[2] * 2)$
$[3] \quad R[1]+((A[1] \times R[1])+(A[2] \times R[2])) \div n$
$[4] \quad i[2]+((A[2.] \times B[1 . \mathrm{j})-(A[1] \times n[2])) \div 1)$
$\nabla$
$\nabla$ callic[]] $\nabla$
$\nabla R+A$ rimut $B$
[1] $R+11$
[2. $\quad R[1]+.(A[1] \times R[1])-(A[2.] \times R[2]$.
[3] $\quad P[2]+(A[1] \times B[2])+(A[2] \times P[1])$
$\nabla$
$\nabla M O D[C] \nabla$
$\nabla R+\because \cap D \quad 7$
[1] $R \leftarrow((2[1] * 2)+.(7[2] * 2)) \div 0.5$


```
\(\nabla\) In, MOT:
```



```
[2] \(\quad!+(\because 1-\because 0) \div \because \Gamma 0-1+!\Gamma ? 0=1\)
[3] \(e^{T<-1}\)
```



```
\([5] \rightarrow\left(e^{T}>1\right) / T, 12\).
[6] \(\quad T \nleftarrow 1\)
```




```
\([0] \quad 112 \leftarrow T R \| 5[T ; 2]\)
[10] MЗ《-isinit? [T;3]
\([11] \rightarrow(\pi 3=0) / T, 1\)
```



```
[1.3] \(\rightarrow\)-2.
[14] \(T 1: A 1[72 ; i J 1] \leftarrow A 1[12 ; 271]-20: 10\)
```



```
[16] \(T, 2: T \leftarrow T+1\)
[17] \(\rightarrow(T\) siv \() / T, 3\)
\([18] \quad T \leftarrow 1\)
[19] \(\quad 5: A 1[T ; T.] \leftarrow 1\)
[20] \(\quad T<T+1\)
```



```
[2.2] A3<5月1
[23] \(\quad 314 A 1+A 2+. \times A 3+. \times A 2\).
[24] \(22<-21+. \times A 2+\ldots \times 3\)
\([25] \quad G<71[V ;(1], T 2[V ; 0]\).
```



```
[27] it~nBRAllCll
[.2.8] ereT \(T T \nleftarrow \because 0\)
[29] K+1
```



```
[31] eTeT[K]*BRUT[RRAMCH[K];2.]
[32] \(\mathrm{K}^{2}-\boldsymbol{r}+1\)
[.33] \(\rightarrow(K \leq 1) /[13\)
[34] \(\quad 11 \leftarrow(1 H) 0\) = 14
[35] S3*T1[V; \(\left.\mathrm{T}_{\mathrm{e}} T\right]\)
[36] \(34+T 2\left[V ; \mathrm{Ce}_{\mathrm{e}} \mathrm{T}\right]\)
[37] ふ5-T1[TT;U]
[38] \(\because \because 642[T T ; U]\)
[39] \(\rightarrow T, 14\)
```





```
[43] S24(DRIS[. 2\(] \times I 1)-T 2\left[J T ; T_{\mathrm{e} T}\right]\)
```



```
[45] S2世-6 \(52+61+. \times 57+. \times 51\)
[46] \(51+-52+{ }_{0} \times 51+. \times 57\)
\([47] \quad \Omega 7+(53+. \times 171)-54+. \times 52\)
\([48] \quad 51+(54+. \times 51)+53+. \times 52\)
[49] \(52+(57+. \times 55)-31+. \times 56\)
\([50] \quad 57+\left(51+{ }_{0} \times 55\right)+S 7+. \times 56\)
[51] G14G+S2,S7
```



```
[53] \(\quad 1,14: e^{T *} T+1\).
[54] \(\rightarrow\left(. e^{T} \leqslant M P l \Omega Q\right) / 1,4\)
\([55] \rightarrow 0\)
\(\nabla\)
```


## 

It has been show that $[100]$ the total round-oin noise in the output of a fixed-point digital filter has a power density spectrum in (r) seven by the following,

$$
N(w)=\sigma_{0}^{2} \sum_{m=1}^{M_{s}} K_{m}\left|T_{v j_{m}}\right| 2
$$

where $\sigma_{0}^{2}$ is the variance of the round-off noise. If it is assumed that the rounding errors are uniformly distributed with zero mean then [100]

$$
\sigma_{0}^{2}=\Delta^{2} / 12
$$

where $\Delta$ is the spacing of the quantization steps.
$\mathrm{II}_{\mathrm{S}}$ is the number of nodes (or branches) with error sources.
$\mathrm{K}_{\mathrm{m}}$ is the integer number of error inputs to the $j_{\mathrm{m}}$ th node.
$T_{\nabla j_{m}}$ is the transfer response from the $j_{m}$ th node to the output node of the filter (Fig. A.1).

Bach error source (rounding operation) is assured to inject white noise of uniform porer-spectral density $H_{0}$.

A program has been written in APL to calculate the function $\mathrm{H}(\mathrm{v})$. It is called RONSA and a listing appears at the end of this appendix. It has been assumed that $K_{m}=1$ for all values of $m$, which is not unreasonable. Furthermore $\sigma_{0}^{2}$ has been dropped from the formula in eqn. (A.12), since it is a constant. The summation of eqn. (A.12) is taken over all non-unity multipliers.

The variance corresponding to $\mathbb{N}(\mathbb{W})$ is given by the following [44],

$$
\begin{equation*}
\sigma^{2}=\frac{T}{\pi} \int_{0}^{\pi / T} \pi(v) d v \tag{A.13}
\end{equation*}
$$

To calculate $\sigma^{2}$, the Trapezoidal Rule [44] has been used and thus,

$$
\sigma^{2} \simeq \frac{1}{N_{p}}\left\{\frac{1}{2} I(0)+\frac{1}{2}(\pi / \mathbb{I})+\sum_{i=1}^{N_{p}-1} N\left(w_{i}\right)\right\}
$$

where

$$
w_{i}=\frac{\pi}{\Pi_{p} T} \quad i, \quad i=1,2, \ldots, \quad N_{p}-1
$$

and $\quad\left(N_{p}+1\right)$ is the number of frequency points.
For the seven methods of realizing the third-order elliptic filter discussed in Chapter 6, ROMSA has been used to calculate the corresponding if (w) and $\sigma^{2}$. Liz. (A.2) shows the power density spectrum expressed in dB's for each of the seven realizations. The table of Fig. (A.3) summarizes the variance estimates. Although the figure for the voltage ladder is high, this value does not seen unreasonable [44].

## DROHSA[.]D

## $\checkmark$ ROMSA

[1] $B M M T+(M B, 4) \rho D A T A$
[2] $H+(M 1-M O) \div M F R Q-1+M P R Q=1$
[3] $e^{T+1}$
[4] $V A R+0$
[5] $\quad L 4: J \leftarrow .1$
[6] $\quad i T T+W O+\| \times T-1$
[7] $A 1+A 2+(I m, I I H) \cap 0$
[8] $I 3: I 1+R R: 4 T[J ; 1]$
[9] $12+B R \pm T[T ; 2]$
[10] $B 3 * R R i T T[I ; 3]$
[11] $\rightarrow(173=0) / L, 1$

[13] $\rightarrow L 2$.
[14] $I 1: A 1[N 2 ; M 1]+A 1[H 2 ; H 1]-20 H / T$
[15] A2[172;:1]+A2[H2;M11+10:7?
[16] $L 2: T-T+1$
[17] $\rightarrow(T \leq m b) / I, 3$
[18] $T<1$
[1.9] $L 5: A 1[T][I]+1$
[20] $I+T+1$
[21]. $\rightarrow(T \leq M I) / I, 5$
[22] A3世点A1
[23] $T^{2} 1+\operatorname{tin} 1+A 2+. \times A 3+. \times A^{2}$
[24] T2+-? $1+. \times A 2+. \times A 3$
[25] G+T1[V;U],T2[V;V]
[26] MOTSR+0
[27] $I+1$
$[28] L 6: \rightarrow(B 124 T[T ; 4]=0) / L 9$
[29] M2+BRMTTT:2]
[30] \#1-T1[V;12],T2[V;12]
[31] MOISK*MOISF+(Z1[1]*2)+(21[2]*2)
[32] $L, 9: I+I+1$
[33] $\rightarrow(T \leqslant I D P) / I, 6$
[34] $e^{T+e T+1}$
[35] $\rightarrow\left(\left(e^{T}=0\right) \vee\left(. e^{T}=\| P R Q\right)\right) / I T$
[36] VAR+VAR+IOTSR
[37] $\rightarrow L$ 8
[38] $L 7:$ VAli $+V A R+H O T S E: 2$

[40] $\rightarrow(e T \leq \operatorname{MPRO}) / L, 4$
[41] VAR\&-VAR:HFRQ-1
[42] 'VARTABGE RSTIAATE = ';VAR
[43] $\rightarrow 0$
$\nabla$


Fig. A.l Signal-Flow Graph of Digital Filter with one multiplier extracted .

| Synthesis | $\hat{\sigma}_{0}^{2}$ |
| :--- | :---: |
| Direct Form | 2.43 |
| Cascaded 2nd.order | 2.29 |
| Voltage-Wave Ladder | 8.68 |
| Voltage-Wave Lattice | 5.23 |
| IVR Ladder | 1.78 |
| IVR Lattice | 4.19 |
| Sedlmeyer Ladder | 3.11 |

Fig. A. 3 Noise Variance Sumary Table .


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