# THE LAGRANGEAN SADDLE-FUNCTION AS A BASIS FOR AUTOMATED DESIGN

by

D.W. KELLY

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## ABSTRACT


The Lagrangean saddle-function is shown to form a useful basis for a unifying theory encompassing the different numerical procedures for constrained optimization. Dual formulations are then considered in detail in an attempt to define a procedure for assessing the merit of a given design and a simple active set strategy for optimization.

The stationary point of the Lagrangean function, which defines an optimum design, is first shown to be a saddle point and the primal and dual problems are identified. Several numerical procedures for automating the design process are then considered to demonstrate that a unifying theory can be proposed. A method is then presented for finding feasible solutions to both primal and dual problems and these solutions provide bounds on the optimum value of the cost function. Information about the active constraint set is available through Lagrange multipliers, and an active set strategy for design is proposed. It is also suggested that the saddle function and duality can be used to extend certain approximate redesign strategies. These procedures can then be drawn into the same framework which unifies the different mathematical programming procedures.

The examples considered are mainly concerned with the minimum mass design of aerospace structures. It is shown that a useful procedure for generating bounds to assess the merit of a given design has been proposed. The active set strategy is used to design the structures and the use of the dual to check the convergence of approximate redesign strategies is also discussed. Because these strategies can be identified with the unifying theory, it is possible to transfer to a more rigorous search strategy when the approximate procedures are not converging to the optimum design.

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# NOTATION

## Mathematical symbols

| | |
|---|---|
| $\underline{x}$ | a column vector of the design variables $x_i$ |
| $\underline{x}^*$ | a locally optimal design |
| $\underline{x}'$ | the current design, often the operating point for the formation of approximations |
| $f(\underline{x})$ | the cost or merit function |
| $g_i(\underline{x})$ | the ith constraint function |
| $\underline{\nabla f}$ | a column vector of the gradient of the cost function |
| $\underline{\nabla g_i}$ | a column vector of the gradient of the ith constraint function |
| $\underline{\nabla g}$ or $\underline{G}$ | the matrix whose ith column is $\underline{\nabla g_i}$ |
| $\underline{H}$ | the Hessian matrix with $H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ |
| $\underline{p}$ | a column vector giving a direction in design space |
| $L(\underline{x},\underline{\lambda})$ | the Lagrangean function |
| $\underline{\lambda}$ | a column vector of the Lagrange multipliers |
| $v(\underline{\delta},\underline{\Gamma})$ | the dual merit function of geometric programming |
| $\underline{\delta},\underline{\Gamma}$ | the column vectors of dual variables in the geometric programming formulation |
| $\underline{A}=\left[\underline{A}_1,\underline{A}_2\right]$ | the matrix of coefficients in the geometric programming dual, contains exponents of posynomials for the constraints |
| $\underline{D}$ | the matrix of exponents for terms in posynomial approximations for the cost function |

## Structural design symbols

| | |
|---|---|
| $E$ | Young's modulus of elasticity |
| $\rho$ | material density |
| $\sigma_{ij}$ | the stress in member $i$ under load case $j$ |

Structural design symbols (contd.)

| | |
|---|---|
| $u_{ij}$ | the deflection in direction i under load case j |
| $\underline{k}_e$ | the element stiffness matrix in the finite element method for structural analysis |
| $\underline{K}$ | the assembled structural stiffness matrix |
| $\underline{r}$ | a column vector of nodal displacements |
| $\underline{P}$, $\underline{R}$ | column vectors of applied loads |

Units used in examples

| | |
|---|---|
| kg | kilogram |
| m (cm) | metre (centimetre) |
| sec | second |
| N | Newton |

# CHAPTER 1

# INTRODUCTION AND CHAPTER SUMMARY

## 1.1 Introduction

The theory of mathematical programming for constrained optimization has grown historically out of initial work by Dantzig (1951) on the simplex method for linear problems [1], and a paper in the same year by Kuhn and Tucker on nonlinear programming [2]. Kuhn and Tucker in fact focused attention on the role of convexity and discovered the connection between Lagrange multipliers and saddle points. The concept of duality was also recognised at an early stage by Gale, Kuhn and Tucker [3] for linear programs, and Dorn [4] and Wolfe [5] devised dual problems for nonlinear programming problems. The fundamental ideas for duality can be obtained directly from Legendre's dual transformation [6] and duality theorems in convex analysis were derived by Fenchel [7]. All of this work of course followed the classical work of Fermat in the seventeenth century which led to the differential approach for finding interior optima of continuous functions of a single variable. This approach was extended by Lagrange in the eighteenth century by the development of a method for finding the extremum of a function subject to equality constraints. This method was based on the definition of undetermined parameters, now called Lagrange multipliers, and the Lagrangean function. With the advent of computers, however, the work on numerical methods after 1950 led to a sudden expansion in the use of optimization theory and a large number of numerical search procedures for optimization [8] have since been developed.

These procedures have been successfully applied to a wide range of problems but a number of considerations have restricted their use in fields where function evaluations can be computationally expensive. The most important consideration is that a very large number of function evaluations can be required by the search strategy so that the overall design process becomes prohibitively expensive. A second consideration is that it has not been possible to assess the merit of a given design without first conducting a search in either the primal or dual problem. Fiacco and McCormick [9] showed for example that a solution to the dual problem bounding the optimum could be determined, but only after the optimum point had been found for one of the sequence of problems which converges to the final solution. An adequate basis for assessing the improvement to the design which can be achieved in return for applying an optimization procedure has not been available. The designer has therefore been faced with the prospect of a potentially expensive design search with only an intuitive knowledge of the merit of his initial design.

One field of engineering in which the mathematical programming procedures are potentially very useful is structural design for minimum volume or weight. Basic theorems on the design of pin-jointed frameworks were published in 1869 by Maxwell [10], and in 1904 Michell [11] published his classical work on the limits of economy of material in frame structures. However the development of mathematical programming procedures and electronic computers after 1950 has been matched by the development of finite element methods for structural analysis [12, 13, 14] allowing the analysis process to be automated. The combination of mathematical programming procedures and finite element methods was first suggested by Schmit [15], and many large scale programs with practical operational capabilities have been developed [16, 17]. In addition several special techniques have been proposed for specific applications. These include the optimality criterion approach to the design of statically determinate structures subject to a single deflection constraint [18] and the Lagrange multiplier method [19] for design subject to frequency constraints.

The finite element analysis procedure can however be computationally expensive. The complexity of the designs which need to be optimized therefore often exceeds the limit of economic application of the mathematical programming procedures. The designs are also more general than the type of problems for which specialized techniques were defined. A number of approximate redesign procedures have therefore been applied to these problems. These include the stress ratioing technique to produce fully stressed designs [20], and the envelope procedure based on the optimality criterion approach but extending it to include multiple constraints [21]. This extension incorporates a degree of approximation into the redesign strategy to avoid having to find the values of Lagrange multipliers associated with the constraints. These approximate procedures then specify pre-conditions on the optimum design which lead to simple recursive expressions for redesign. They can therefore exhibit very rapid initial improvement in the design but are only guaranteed to converge to the optimum under certain special circumstances. The Kuhn-Tucker conditions [2] can be applied to determine whether the final design is optimal, but it has not in general been possible to assess the merit of a non-optimal design. There does however seem to be considerable scope in engineering design for strategies such as these which can accommodate the engineers' knowledge and experience to define the path to the optimum [45] .

The aim of the research carried out for this thesis has therefore been to isolate a basic theory common to the different mathematical programming procedures, and to search within this framework for a technique to enable the merit of a given design to be assessed. It is also suggested that correctly formulated approximate methods should lie within the same basic framework as the mathematical programming procedures and the approximations in these methods could then be identified. If this logical basis exists, then the degree of approximation could be reduced to recover a more rigorous algorithm capable of continuing the design process if the approximate method is not converging to the optimum design. This process would then represent an extension of the work of Razani [22] which suggested that a mathematical programming procedure should be used if the fully stressed design, produced by the approximate stress ratioing procedure, did not satisfy the Kuhn-Tucker conditions for optimality. The generation of bounds would here provide a more useful basis for deciding whether the extension of the design process is necessary.

It is perhaps not surprising that the search for the unifying theory led back to the classical Lagrangean saddle-function and the concepts of duality which appeared early in the history of mathematical programming. As well as the work by Fiacco and McCormick [9], Morris [23] recognised the usefulness of dual solution points in defining a lower bound on the minimum mass of a structure. These bounds were generated as a result of conducting the search for the optimum design in the dual provided by geometric programming. However it has been found that bounds can be obtained more directly using the geometric programming dual formulation and the results of this initial work were published by the author in [24]. This was closely followed by a paper by Bartholomew and Morris [25] based on the same strategies but making use of a different dual formulation. They also recognized that the mathematical programming procedure based on projected gradients could be recovered by removing approximations inherent in the fully stressing design procedure. Co-operation with these authors led to the work conducted as part of this thesis and published in [26] in which the unifying theory for mathematical programming procedures was detailed.

This unifying theory is again presented in this thesis and is based on the Lagrangean saddle-function and duality. In particular it is shown that the penalty function and projected gradient procedures, as well as certain approximate redesign strategies, can be derived from the saddle function and related theory. A dual problem can also be defined for any design problem which exhibits certain local convexity properties. If the cost function F for the design problem is to be minimized, feasible solutions to the primal and dual problems satisfy the inequality

$$F \geq F^* = V^* \geq V \quad ,$$

where V is the cost function for the dual problem and the asterisk indicates the optimum value. It is obvious therefore that feasible solutions to both primal and dual problems will bound the optimum value of the cost or merit function and define a range in which the optimum must lie. If these solutions converge simultaneously to the optimum then useful bounds will be obtained. It is also of particular significance that the dual variables are Lagrange multipliers which reflect constraint activity levels in the primal problem. Dual formulations should therefore be able to check the selection of the active set of constraints in the stress ratioing technique and evaluate the Lagrange multipliers appearing in the optimality criterion approach for multiple deflection constraints.

The path for developing these ideas is, however, not clear because the constraints imposed on the Lagrangean function to define the dual problem are closely related to the Kuhn-Tucker conditions for optimality. It has been stated [20] that these conditions can only indicate if a given design is optimal and give no further information as to how the design process should proceed. The definition of a dual point can be accomplished by the solution of the set of equations provided by these conditions, but there is no guarantee that a set of subsidiary conditions for feasibility, requiring that the dual variables be non-negative, will be satisfied. It was therefore necessary to develop a method enabling easy transition to a feasible solution to the dual problem satisfying the non-negativity conditions.

Finally a design procedure based on the saddle function and trying to make full use of the information available in both primal and dual problems is proposed. The extension of the design process when an approximate procedure has terminated at, or is converging to, a non-optimal design is also discussed.

## 1.2 Summary of chapter content

In Chapter 2 the design problem is defined together with the basic terminology which will be used throughout this thesis. The geometry of the optimum point in design space is also investigated and the Kuhn-Tucker necessary conditions for optimality are defined. These conditions are introduced here because in Chapter 3 it is shown that they can be recovered from the Lagrangean saddle-function for the design problem by imposing the zero derivative conditions which define the stationary point of this function. This stationary point is then shown to be a saddle point and duality concepts are introduced.

The strategies followed by the mathematical programming procedures for constrained problems are investigated in detail in Chapter 4, and the Lagrangean saddle-function and duality are proposed as the basis for relating the different automated design procedures. The classical approach of Lagrange, projected gradient and linear programming, and the penalty function methods are considered in detail. Approximate methods for structural design are also introduced and derived from the saddle function.

Chapter 5 is intended to clarify a number of points previously suggested without proof. Convex sets and functions are described and the simplifications arising in the design problem if the merit function is convex and the design space a convex region are discussed. The saddle point only exists if certain convexity and concavity conditions are satisfied, at least locally, by the merit function and constraints. In fact the theory of duality is derived in the branch of mathematics known as Convex Analysis [27] from certain conjugacy relationships between convex functions.

All linear programming problems are convex and the relationships used to define a solution to the dual problem corresponding to a particular feasible solution to the primal are identified. At the optimum a transition between feasible points in both primal and dual problems is possible. The generalization of these ideas, to allow the definition of corresponding pairs of primal and dual points which are feasible but not optimal, forms the basis of the following chapters. In preparation for this work the convex primal problem of geometric programming [28] is defined and the corresponding concave dual is derived from the Lagrangean saddle-function.

In Chapter 6 procedures for generating bounds are developed and an active set strategy for redesign is proposed. The general form of the primal and dual problems based on the Lagrangean saddle-function is considered first, but it is found that entry into the dual can be blocked by negative Lagrange multipliers. The same basic equations can however be recovered if a particular form of approximation is made to the primal problem and the geometric programming dual formulation used. The advantage of this approach is flexibility in that negative Lagrange multipliers can be removed and entry into the dual problem achieved. Considerable emphasis is placed on showing the similarity between this dual entry formulation and the corresponding procedures for the dual based on the original form of the primal problem and the saddle function and used in [25].

Pivotal relationships exist at the primal dual interface with sets of equations based on the same coefficient matrix defining either primal or dual solution points. The occurrence of singularity in this set of equations is investigated and a redesign strategy based on the same coefficient matrix is developed in this chapter. Since the dimension of the dual problem depends on the number of active constraints, it is proposed either to conduct a search in the dual problem if few constraints are active, or to work with active set strategies in the primal problem when many constraints are active. Bounds would be generated in both cases by the transfer between primal and dual problems with the dual used to select the active set in the second procedure. A basis of constraint vectors is also required to define the dual problem so that examination of the current primal design can assist the formation of the dual.

Practical considerations related to the application of the procedures based on the geometric programming dual formulation are discussed in Chapter 7. Three preliminary examples are then presented. These examples only have two or three design variables and the mathematical operations required can be performed with the aid of a desk calculator. However these examples illustrate the existence of the dual and the use of the procedures proposed to generate bounds. A simple active set strategy is used in one of the examples to produce a sequence of improving designs so that it can be shown that the bound generated will converge to the optimum.

Convergence of iterative design procedures is then discussed in Chapter 8. A convergence theorem is considered to isolate the minimal requirements which must be satisfied to rigorously guarantee that the redesign procedures will produce the optimum design. The use of the bound as a termination criterion for the design process is considered. It is also suggested that the dual procedures can be used to check the redesign strategies of approximate methods, and that simple transfer can be made to mathematical programming procedures based on the saddle function to extend the design process if necessary.

A set of detailed examples for application of the procedures based on the geometric programming formulation are then presented in Chapter 9. Although the methods are generally applicable, and the first preliminary example involves the minimum cost design of a chemical plant, these detailed examples all involve the minimum mass design of aerospace structures subject to stress, deflection and minimum size constraints. Examples are constructed to test the new procedures on a variety of problem types. Problems with linear and non-linear merit functions are considered together with problems in which either a few or many non-linear constraints are active at the optimum. A problem is also devised in which the matrix of coefficients in the pivotal operation between primal and dual problems is singular. It is acknowledged that the Michell structure [11] can be used as an absolute minimum mass design for certain constraints, against which the merit of a given design can be assessed. However it is shown that the formulation in this thesis for generating bounds is more general and allows the merit of a given design for a defined configuration and topology to be assessed. The application of the approximate redesign procedures, based on the fully stressed design concept and the optimality criterion approach to these examples, is also discussed.

The conclusions drawn in Chapter 10 relate firstly to the usefulness of the bounds in providing a method for assessing the merit of a given design. The identification of a unifying framework encompassing the mathematical programming procedures and the approximate redesign strategies is also considered significant. It is also concluded that the general strategy for design which is discussed could provide a useful extension to modern automated procedures. It involves attempting to identify, and use in an integrated fashion, those procedures which require a minimum of computational effort while guaranteeing that the process will converge to the optimum design.

Finally the appendices give details not contained in the text. A description of the computer programs developed in this research project can be found in [44].

# CHAPTER 2

# THE DESIGN PROBLEM

## 2.1 Introduction

In this chapter the design problem is defined together with the terminology which will be used throughout the thesis. The geometry of a constrained optimum is investigated and the Kuhn-Tucker necessary conditions for optimality are defined.

## 2.2 The mathematical model of the design problem

The optimum design problem can be stated mathematically as

$$\text{minimize (or maximize)} \quad f(x_1, x_2, \ldots, x_n) = f(\underline{X}) \tag{1}$$

subject to the satisfaction of a set of constraints which may consist of inequality constraints

$$g_i(\underline{X}) \leq 1 \quad , \quad i = 1, \ldots, \ell \quad , \tag{2}$$

and equality constraints

$$g_i(\underline{X}) = 1 \quad , \quad i = \ell+1, \ldots, m \quad . \tag{3}$$

The design variables are also usually restricted to be non-negative

$$x_i \geq 0 \quad , \quad , \quad i = 1, \ldots, n \quad , \quad \underline{X} \in E^n \quad . \tag{4}$$

Any design $\underline{X}$ which satisfies the constraints is said to be a feasible design, and a feasible design $\underline{X}^*$ which makes $f(\underline{X})$ an absolute minimum (maximum) is the optimum design.

The design variables $\underline{X}$ may be any quantifiable aspect of the design problem such that when numerical values have been chosen for them the design is specified. For the structural design applications they describe the configuration of the structure, the most common being member sizes and parameters describing the geometry of the structure. The merit or cost function $f(\underline{X})$ constitutes the basis for selection of one of several alternative feasible designs. It represents the most important single property of the design, such as cost or weight, or the weighted sum of a number of properties. The constraints $g_i(\underline{X})$, $i=1,\ldots,m$, consist of all restrictions which must be imposed for the design to be acceptable. For structural design these can include maximum stresses and deflections, minimum and maximum member sizes, limits on frequencies and others.

In this thesis only the problem of minimizing the merit function will be considered, but it should be noted that

$$\text{minimize} \quad ( -f(\underline{X}) )$$

will recover the maximization problem expressed in (1). Wherever possible, diagrams will be used to illustrate the geometry of design space and the axes for these diagrams will correspond to the design variables. The diagrams must necessarily be restricted to two, or at most three dimensions, but the concepts of planes and surfaces, representing contours of constant values of functions in three dimensions, generalise to hyperplanes and hypersurfaces in $n$-dimensions.

A simple design space in two dimensions is illustrated in Fig. 1 with three inequality constraints. At the point 0 on the

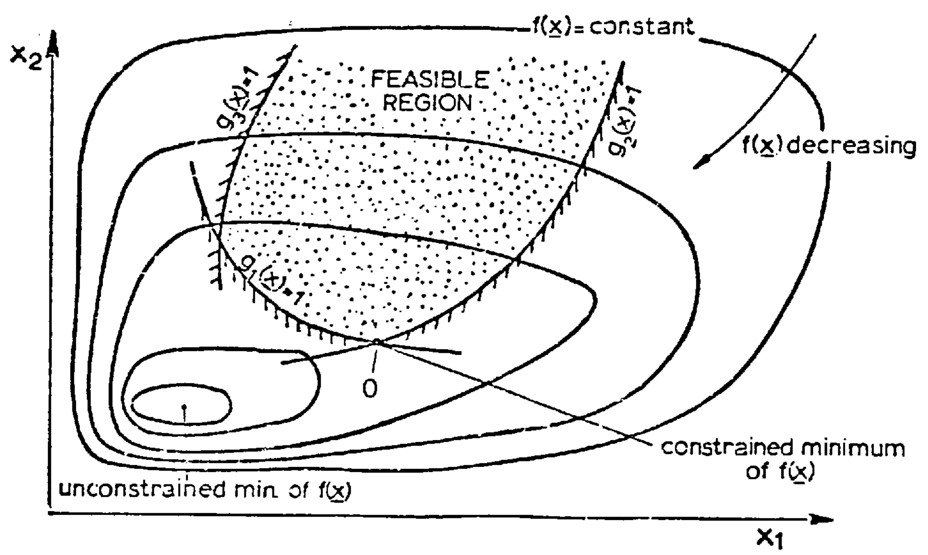



Figure 1 A simple design space with inequality constraints

boundary of the feasible region constraints $g_1(\underline{X})$ and $g_2(\underline{X})$ are satisfied as equalities and are said to be active. The inequality sign holds for $g_3(\underline{X})$ at the point 0 and this constraint is said to be not active.

Finally the gradient vector needs to be defined. For a general function $h(\underline{X})$ , where $\underline{X}$ is an $n$-dimensional vector, the gradient vector is given by

$$\underline{\nabla h}(\underline{X}) = \begin{bmatrix} \frac{\partial h(\underline{X})}{\partial x_1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial h(\underline{X})}{\partial x_n} \end{bmatrix}$$

Given an arbitrary direction $\underline{d}$ , $\underline{\nabla h}(\underline{X})^t \underline{d}$ gives the instantaneous rate of change of h along that direction; (here the norm of $\underline{d}$ is one).

## 2.3 The Kuhn-Tucker necessary conditions for optimality

Classically the minimum of a function $f(\underline{X})$ is found by setting the derivatives to zero, that is,

$$\left.\frac{\partial f}{\partial x_i}\right|_{\underline{X}=\underline{X}^*} = 0 \quad , \quad i = 1,\ldots\ldots,n \text{ or simply } \underline{\nabla f} = \underline{0} \ . \tag{5}$$

For the problem with inequality constraints illustrated in Fig. 1, it is clear that condition (5) must be modified as we are now seeking the constrained minimum of $f(\underline{X})$ rather than the free minimum. Geometrically, at the optimum point the negative of the gradient of the merit function, $-\underline{\nabla f}(\underline{X}^*)$ , lies in the cone formed by the gradients of the active inequality constraints as shown in Fig. 2. Hence $-\underline{\nabla f}(\underline{X}^*)$ is expressible as a non-negative linear combination of the gradients of the constraints active at the optimum.

This concept can be extended to equality constraints by noting that, if the constraint $g_2(\underline{X})$ had to be satisfied as an equality, neither the optimum nor the condition for optimality would change, although the feasible region would now only exist along the curve $g_2(\underline{X}) = 1$. However, if $g_3(\underline{X})$ was forced to be active, the optimum would be at point E and the gradient of $g_3(\underline{X})$ would have a negative coefficient to set up the constraint cone enclosing the negative gradient of the merit function. The sign of the coefficients of the gradients of the equality constraints is therefore unrestricted in this relationship.

Therefore the condition on the constrained derivative,

$$\frac{\partial f(\underline{X}^*)}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i(\underline{X}^*)}{\partial x_j} \geq 0 \text{ , with } \lambda_i \geq 0,\ i = 1,\ldots,\ell \text{ ,} \quad (6)$$
$$j = 1,\ldots,n \text{ ,}$$

must be satisfied at the optimum with equality holding for $x_j^* > 0$. The auxiliary conditions

$$\lambda_i \left[ g_i(\underline{X}^*) - 1 \right] = 0 \quad , \quad i = 1,\ldots,\ell \text{ ,} \quad (7)$$

for the inequality constraints (2) must also be satisfied to ensure that constraints not active at the optimum, such as $g_3(\underline{X})$ in Fig. 2, are eliminated (if $g_i(\underline{X}^*) < 1$ , $\lambda_i = 0$ ). Conditions (7) are automatically satisfied by the equality constraints (3).

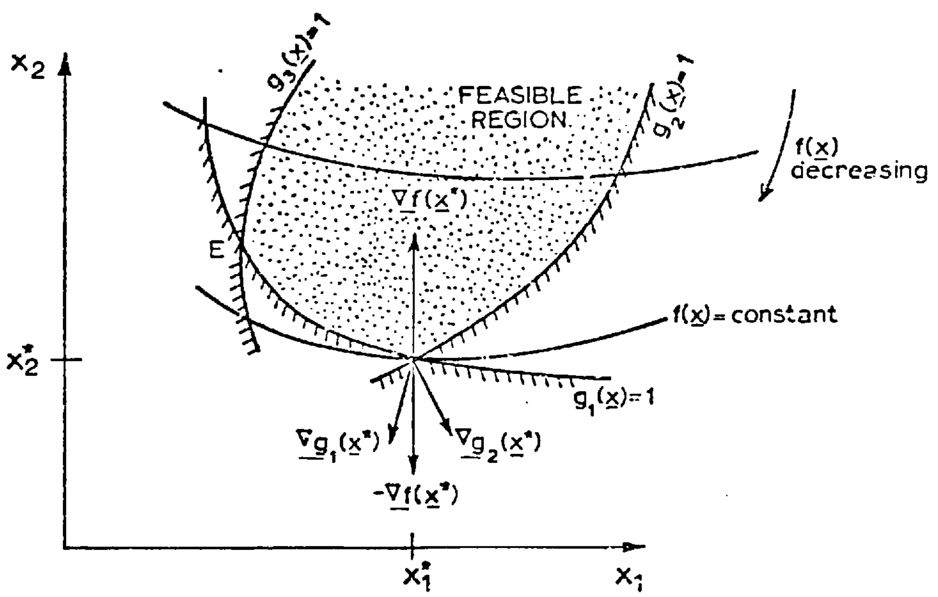



Figure 2 The geometry of a constrained optimum (all constraints inequalities)

Conditions (6) and (7) together with (4) are necessary conditions which must be satisfied for a design to be optimal and, together with a condition that $\underline{X}^*$ be feasible, they are called the Kuhn-Tucker conditions [2]. A certain constraint qualification must also be introduced to rule out irregular behaviour or singularities on the boundaries of the set of feasible solutions. This qualification requires that $\nabla \underline{g}_i(\underline{X})$ for all active constraints be positively linearly independent. The Kuhn-Tucker conditions can be shown to hold at the optimum providing only that this constraint qualification is met. Note also that the necessary condition for an extremum of an unconstrained problem is a special case of (6) when no constraints are active.

The Kuhn-Tucker conditions are qualified as necessary conditions because their satisfaction is not sufficient to define the design as the global optimum. Conditions (6) will be satisfied if $\underline{X}^*$ is a local minimum as well as the global minimum. At the local optimum $\underline{X}^*$ minimizes f over a feasible neighbourhood of $\underline{X}^*$, while at a global optimum $\underline{X}^*$ minimizes f over all feasible points in $E^n$. Throughout the thesis the asterisk will be used to denote both local and global optimality.

Certain convexity conditions must be satisfied for the necessary conditions to also be sufficient to ensure global optimality, and these will be discussed in Chapter 5. Considerable use will also be made in the following chapters of the geometric interpretation of the Kuhn-Tucker conditions discussed in this section.

## CHAPTER 3

## THE LAGRANGEAN SADDLE-FUNCTION AND DUALITY

### 3.1 Introduction

In this chapter the Lagrangean function is defined. The conditions for the stationary point of this function relate directly to the Kuhn-Tucker conditions set out in the previous chapter. This stationary point is a saddle point and the concept of duality is introduced. It is then suggested that the saddle function can form a basis relating the different techniques for automated design and that the existence of dual problems may enable bounds on the optimum value of the cost function to be found.

### 3.2 The Lagrangean saddle-function

Departing from geometric arguments, a general problem can be formulated from which the Kuhn-Tucker conditions can be derived. The design problem is first rewritten as

$$\text{minimize} \quad f(\underline{X}) \quad ,$$

$$\text{subject to} \qquad g_i(\underline{X}) \leq 1 \quad , \qquad i = 1,\ldots\ldots,p \ ,$$

$$\underline{X} \geq \underline{0} \quad \text{i.e.} \quad g_i(\underline{X}) = 1 - x_{i-p} \leq 1 \quad , \quad i = p+1,\ldots\ldots,\ell \ ,$$

$$\text{and} \qquad g_i(\underline{X}) = 1 \quad , \qquad i = \ell+1,\ldots\ldots,m \ ,$$

$$\ldots (8)$$

for $\underline{X} \in E^n$ . This problem can now be replaced by the problem of finding the stationary point of the Lagrangean function

$$L(\underline{x},\underline{\lambda}) = f(\underline{x}) + \sum_{i=1}^{m} \lambda_i \left[ g_i(\underline{x}) - 1 \right] \tag{9}$$

where the $\lambda_i$ are called Lagrange multipliers and

$$\lambda_i \geq 0 , \qquad i = 1,\ldots,\ell .$$

The non-negativity conditions on the design variables (4) have been incorporated into the inequality constraint set only to simplify the notation in the following derivations. Note that the Lagrange multipliers in (9) are unconstrained in sign for the equality constraints.

The necessary conditions for optimality, (6) and (7), are recovered by imposing the conditions which define the stationary point of this Lagrangean function,

$$\frac{\partial L}{\partial x_j} = 0 \text{ for } x_j \in \underline{X} \implies \frac{\partial f(\underline{x}^*)}{\partial x_j} + \sum_{i=1}^{m} \lambda_i^* \frac{\partial g_i(\underline{x}^*)}{\partial x_j} = 0 \qquad \ldots (10)$$

$$\text{and } \frac{\partial L}{\partial \lambda_j} = 0 \text{ for } \lambda_j \in \underline{\Lambda} \implies \lambda_j^* \left[ g_j(\underline{x}^*) - 1 \right] = 0 , \quad j = 1,\ldots,\ell ,$$

$$\text{and } \quad g_j(\underline{x}^*) = 1 , \quad j = \ell+1,\ldots,m . \qquad \ldots (11)$$

$$\text{where } \underline{\Lambda} = \left\{ \underline{\lambda} = (\lambda_1,\lambda_2,\ldots,\lambda_m)^t \;\middle|\; \lambda_i \geq 0, \quad i = 1,\ldots,\ell \right\}$$

$$\underline{X} = \left\{ \underline{x} \;\middle|\; x_j > 0 , \quad \underline{x} \in E^n \right\} .$$

The stationary point of this function $L(\underline{x}^*,\underline{\lambda}^*)$ therefore defines at least a locally optimum design if $\underline{x}^*$ is feasible.

To illustrate the nature of this stationary point, contours of the Lagrangean function (9) for the simple problem,

$$\text{minimize } f(x) = (x - 2)^2 \text{ subject to } x \leq 1 ,$$

are plotted on Fig. 3. The Lagrangean function for this single variable problem is

$$L(x,\lambda) = (x - 2)^2 + \lambda(x - 1) \quad ,$$

and the stationary point is defined by

$$\text{(i)} \quad \frac{\partial L}{\partial x} = 0 \implies 2(x - 2) + \lambda = 0 \quad ,$$

$$\text{(ii)} \quad \frac{\partial L}{\partial \lambda} = 0 \implies x = 1 \ .$$

Substitution for x from (ii) into (i) gives the stationary point A as the point (1,2) . It can be seen from the contour diagram that this stationary point is a saddle point with the point A being the focus of a minimization problem parallel to the x-axis and a maximization problem in the direction DD .

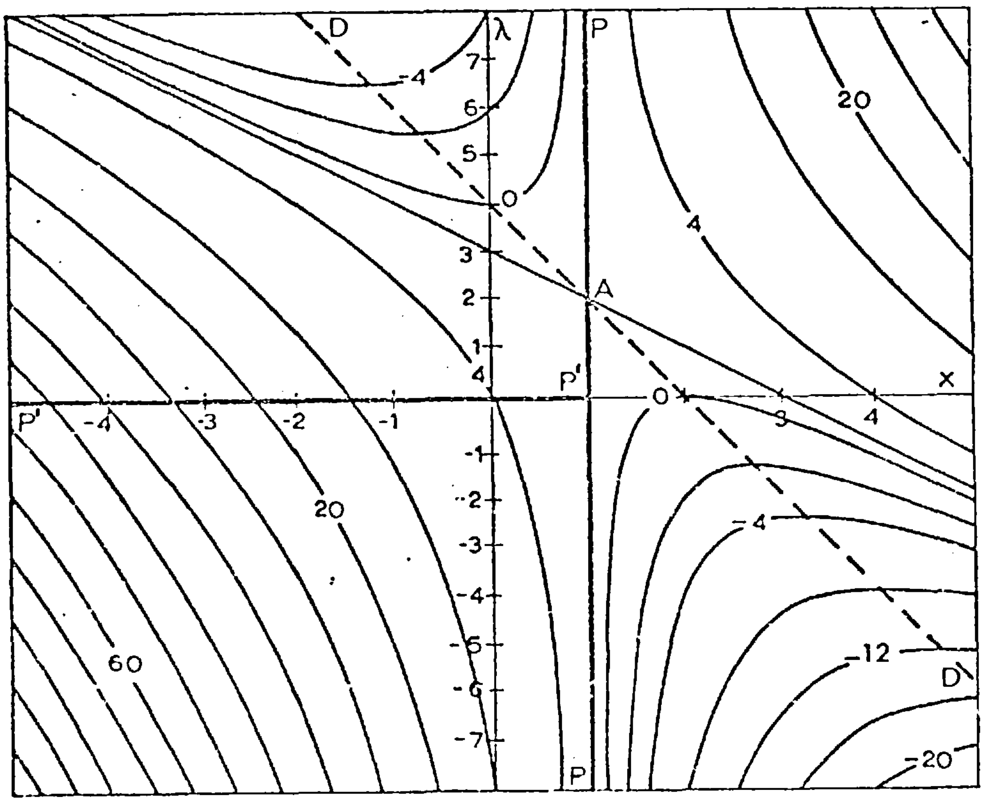



Figure 3 Contours of the Lagrangean saddle-function for the problem: minimize $f(x) = (x-2)^2$, subject to $x \leq 1$

The stationary point can be shown to be a saddle point for general problems if the merit function and constraints are convex near this point by identifying the primal and dual problems associated with the Lagrangean function.

## 3.3 Duality

The Lagrangean saddle-function gives rise to the duality concepts encountered in many mathematical programming procedures. If we invoke local convexity conditions near the saddle point, we can identify the following two problems.

1. The primal problem

Find the extremum of $L(\underline{X},\hat{\underline{\lambda}})$ for the variables $\underline{X}$ with $\hat{\underline{\lambda}}$ satisfying by maximization of $L(\underline{x},\lambda)$ with respect to $\underline{\lambda}$ either

$$\lambda_i \text{ or } \frac{\partial L(\underline{X},\underline{\lambda})}{\partial \lambda_i} = 0 \quad , \quad i = 1,\ldots,m \; .$$

$$\left(\text{with } \lambda_i = 0 \quad \text{for} \quad g_i(\underline{X}) < 1 \quad , \quad i = 1,\ldots,\ell\right) \quad \ldots (12)$$

It can be shown mathematically that this extremum is a minimum. However it is intuitively obvious since enforcing (12) ensures $L(\underline{X},\hat{\underline{\lambda}}) = f(\underline{X})$ , and a minimum of $f(\underline{X})$ in (9) is being sought.

This primal problem corresponds to selecting the minimum value of $L(x,\lambda)$ from points along the lines P'P' and PP for the problem illustrated in Fig. 3. The original design problem (8) can be recovered from this primal problem by noting that condition (12) requires either $\lambda_i = 0$ or $g_i(\underline{X}) = 1$ . In either case

$$\underset{\underline{X}\varepsilon\bar{\underline{X}}}{\text{minimize}} \; \underset{\underline{\lambda}\varepsilon\underline{\Lambda}}{\max} \; L(\underline{X},\underline{\lambda}) = \underset{\underline{X}\varepsilon\bar{\underline{X}}}{\text{minimize}} \; f(\underline{X})$$

$$\text{where } \bar{\underline{X}} = \left\{ \underline{X} \;\middle|\; g_i(\underline{X}) \le 1 \; , \quad i = 1,\ldots,\ell \; ; \quad g_i(\underline{X}) = 1 \; , \quad i = \ell+1,\ldots,m \; ; \; \underline{X} \ge \underline{0} \right\} \; .$$

2. The dual problem

Find the extremum of $L(\hat{\underline{x}},\underline{\lambda})$ for the non-negative variables $\underline{\lambda}$ , with $\hat{\underline{x}}$ defined to satisfy

$$\frac{\partial L}{\partial x_i} = 0 \quad , \qquad i = 1,\ldots,n \quad . \tag{13}$$

It can be shown mathematically, but is again intuitively obvious, that this extremum is a maximum since the $\lambda_j$ must be driven to zero for $g_j(\underline{x}) < 1$ to satisfy condition (7). This problem is called the dual problem and corresponds to the segment of the line DD for which $\lambda \geq 0$ .

The optimum for both primal and dual problems is the stationary point $(\hat{\underline{x}},\hat{\underline{\lambda}})$ . However, the direction of extremization in these problems indicates that the inequality

$$L(\hat{\underline{x}},\underline{\lambda}) \leq L(\hat{\underline{x}},\hat{\underline{\lambda}}) \leq L(\underline{x},\hat{\underline{\lambda}}) \tag{14}$$

is satisfied. The stationary point will therefore be a saddle point if the local convexity conditions are satisfied.

An interesting property of the primal and dual problems is that the directions P'P' of the primal problem and DD of the dual problem are conjugate with respect to the Hessian matrix of second partial derivatives. This can be demonstrated by determining the Hessian matrix $\underline{H}$ for the Lagrangean function illustrated in Fig. 3.

$$\underline{H} = \begin{bmatrix} \frac{\partial^2 L}{\partial x^2} & \frac{\partial^2 L}{\partial x \partial \lambda} \\ \frac{\partial^2 L}{\partial \lambda \partial x} & \frac{\partial^2 L}{\partial \lambda^2} \end{bmatrix}$$

and for $\qquad L(x,\lambda) = (x-2)^2 + \lambda(x-1)$

$$\underline{H} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad .$$

Now the primal direction P'P' is given by $\underline{p}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ .

The dual direction DD is defined by $\frac{\partial L}{\partial x} = 0 = 2(x - 2) + \lambda$ .

Therefore the dual direction is $\underline{p}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and

$$\underline{p}_1{}^t \underline{H} \underline{p}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 0 .$$

The directions $\underline{p}_1$ and $\underline{p}_2$ are therefore said to be $\underline{H}$-conjugate. This condition is sufficient to ensure that the primal and dual vectors, $\underline{p}_1$ and $\underline{p}_2$ , for the quadratic Lagrangean saddle-function are independent and therefore the vector set $\underline{p}$ is a basis in the two-dimensional vector space.

## 3.4 Summary

In this chapter the Kuhn-Tucker conditions for optimality have been shown to be equivalent to requiring the existence of a stationary point of the Lagrangean function for the design problem. The concepts of duality, commonly encountered in mathematical programming, are then recovered when it is recognised that the stationary point is a saddle point.

In addition, the inequality (14) indicates that feasible solutions to both primal and dual problems will define a range in which the optimum value of the merit function must lie. Therefore, if a feasible solution to the dual problem could be defined such that it would converge to the optimum as the design in the primal problem improves, then a method for assessing the merit of a given design would be available. It should also be noted that the dual variables would have to converge towards satisfying (7) and would reflect constraint activity levels. It should therefore be possible to use the dual to provide information to supplement search procedures in the primal problem, and these points will be pursued further in the following chapters. However, in order to demonstrate the unifying role played by the saddle function and duality in automated design, the search strategies of a number of mathematical programming procedures are studied in the next chapter.

# CHAPTER 4

# THE LAGRANGEAN SADDLE-FUNCTION AS A BASIS FOR AUTOMATED REDESIGN STRATEGIES

## 4.1 Introduction

In this chapter the Lagrangean saddle-function is shown to form a useful basis relating a number of mathematical programming procedures for automated design. The role this function plays in the sensitivity studies of post-optimal analysis is also discussed and the derivation of two approximate redesign strategies from the Lagrangean function is summarized. It is shown that all the design procedures considered attempt to either satisfy directly the conditions (10) and (11), which define the saddle point, or form approximations to the Lagrangean function itself.

## 4.2 A basis for mathematical programming procedures

### (i) The classical approach of Lagrange

In Section 2.3 it was established that the optimum design for problem (8) defines a stationary point of the Lagrangean function. If only equality constraints are considered the design problem becomes

$$\text{minimize} \quad f(x_1, \ldots, x_n) \quad = \quad f(\underline{x})$$

$$\text{subject to the constraints} \quad g_i(\underline{x}) = 1 \quad , \quad i = 1, \ldots, m \quad , \qquad \ldots (15)$$

with $m < n$. The definition of the stationary point of the corresponding Lagrangean function

$$L(\underline{x}, \underline{\lambda}) \quad = \quad f(\underline{x}) \;+\; \sum_{i=1}^{m} \lambda_i \left[ g_i(\underline{x}) - 1 \right] \qquad (16)$$

then requires only satisfaction of the conditions

$$\frac{\partial L}{\partial x_j} = 0 \ , \quad j = 1,\ldots\ldots,n \ , \tag{17}$$

and
$$\frac{\partial L}{\partial \lambda_j} = 0 \ , \quad j = 1,\ldots\ldots,m \ . \tag{18}$$

Since all the constraints are equalities there are no positivity conditions on the Lagrange multipliers to be satisfied. Note also that the restriction that $m$ be less than $n$ is imposed because if $m = n$ the solution of the constraint set in (15) will uniquely define the design variables, and if $m > n$ the design is overspecified.

Equations (17) and (18) are a system of $n + m$ equations in the $n + m$ unknowns $\underline{x}$ and $\underline{\lambda}$. Their solution will therefore lead to the set of variables $\underline{x}^*$ and $\underline{\lambda}^*$ which define the stationary point of the Lagrangean function and hence the optimum design. The satisfaction of the constraints is guaranteed because conditions (18) require the constraints to be active, that is,

$$\frac{\partial L}{\partial \lambda_j} = 0 \implies g_j(\underline{x}) = 1 \ .$$

This classical Lagrangean approach would appear to provide a very powerful technique for design when the active constraint set can be identified and the functions are continuous and differentiable within a given closed region. However the equation set defined by (17) and (18) will in general be non-linear and their solution can impose severe computational difficulties. The classical approach is also not practical when the active constraint set for the optimum design cannot be determined in advance. The numerical search procedures of mathematical programming have been developed to overcome these problems by conducting a search through design space for the constrained minimum of the merit function, including the inequality constraints in the formulation as they are encountered.

(ii) Penalty function methods

The simplest penalty function methods [9] are indirect numerical search procedures which try to set up an approximation to the Lagrangean function defined in Section 3.2. That is, they try to set up approximations to

$$L(\underline{X},\underline{\lambda}) = f(\underline{X}) + \sum_{i=1}^{m} \lambda_i \left[ g_i(\underline{X}) - 1 \right] \tag{19}$$

where $\underline{X} \in \underline{X}$ and $\underline{\lambda} \in \underline{\Lambda}$ .

The basis for these methods is then a numerical procedure to perform the unconstrained minimization of a multi-dimensional function. Numerous computationally efficient algorithms exist to perform this task [8], the most obvious (but seldom the most efficient) being the steepest descent procedure whose search strategy is summarized by the equation

$$\underline{X}^{\nu+1} = \underline{X}^{\nu} + \alpha \, \underline{\nabla\phi}(\underline{X}) \quad . \tag{20}$$

Here $\nu$ refers to the iteration number, $\alpha$ is a step length and $\underline{\nabla\phi}(\underline{X})$ is the column vector of the gradients of $\phi$ , the function being minimized. If we define $\phi(\underline{X})$ as

$$\phi(\underline{X}) = f(\underline{X}) + \sum_{i=1}^{m} \delta_i \, p_i \tag{21a}$$

where $\delta_i = 0$ if $g_i(\underline{X}) \le 1$ , $i = 1,\ldots,\ell$

or $g_i(\underline{X}) = 1$ , $i = \ell+1,\ldots,m$

i.e. constraints not violated . . . (21b)

and $\delta_i > 0$ if $g_i(\underline{X}) > 1$ , $i = 1,\ldots,m$

$\delta_i < 0$ if $g_i(\underline{X}) < 1$ , $i = \ell+1,\ldots,m$

i.e. constraints violated . . . (21c)

and $p_i = ( g_i(\underline{X}) - 1 )$ , $i = 1,\ldots,m$ , (21d)

then the analogy between (21a) and the Lagrangean function (19) is clear. This procedure would effectively form an approximation to the Lagrangean function setting the Lagrange multipliers in an artificial way.

Insight into how these pseudo-multipliers should be set, and a greater understanding of the analogy with the Lagrangean function, can be obtained by applying this procedure to the simple problem given in Fig. 3. If we start from a feasible design ($x<1$ and $\delta_1=0$) the unconstrained steepest descent search will generate a sequence of points along the line P'P' until the constraint is encountered at $x=1$. If $\delta_1$ remains equal to zero an unconstrained minimum of $\phi(\underline{X})$ is encountered at $x=2$, the intercept of the lines $\lambda=0$ and DD. However, if once the constraint was violated $\delta_1$ is set equal to one, in accordance with (21c), an unconstrained minimum of $\phi(\underline{X})$ would be found at $x=1.5$, the intercept of the lines $\lambda=1.0$ and DD. If we set $\delta_1>2.0$ the steepest descent search would drive x less than one, $\delta_1$ would have to be set to zero in accordance with (21b) until the constraint was again encountered, and the procedure would therefore oscillate about $x=1$ for $\delta_1>2.0$. The augmented function $\phi(\underline{X})$ is plotted in Fig. 4, and it can be seen that the solution to the unconstrained problem approaches the correct solution to the design problem at $x=1$ as $\delta_1$ is increased. Setting $\delta_i$ large early in the search for a problem with more than one constraint can lead to numerical difficulties which are indicated by the oscillatory behaviour just discussed when $\delta_1$ was set to a value greater than two.

A detailed description of the penalty function procedures is given in [9] where several more practical strategies for setting the parameters $\underline{\delta}$ and choosing the form of the penalties $p_i$ are discussed.

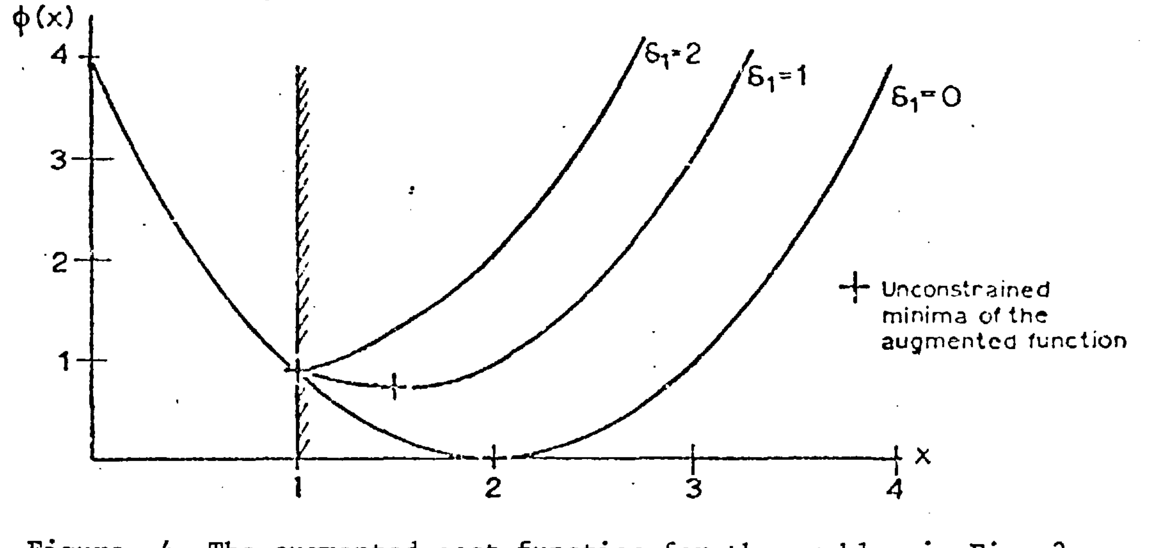



Figure 4 The augmented cost function for the problem in Fig. 3

Procedures similar to those described above are called exterior point algorithms. An alternative form is the barrier function or interior point algorithm which use an augmented function of the form

$$\phi(\underline{X}) = f(\underline{X}) - \sum_{i=1}^{m} \delta_i \frac{1}{(g_i(\underline{X}) - 1)} \quad . \tag{22}$$

The parameters $\underline{\delta}$ are given positive values at all times so that the surface of $\phi(\underline{X})$ represents the normal merit function with sharply rising barriers near the constraints. The design remains feasible during the search and a constrained optimum is approached as the values of $\delta_i$ are set small.

The direct analogy with the Lagrangean function (19) does not apply for the augmented function of (22). However, if $\phi(\underline{X})$ is minimized with set values for the parameters $\underline{\delta}$ then at the minimum

$$\frac{\partial\phi}{\partial x_i} = 0 \quad , \quad i = 1,\ldots,n \quad ,$$

and therefore

$$\frac{\partial f(\underline{X})}{\partial x_j} + \sum_{i=1}^{m} \frac{\delta_i}{(g_i(\underline{X}) - 1)^2} \frac{\partial g_i(\underline{X})}{\partial x_j} = 0 \quad , \quad i = 1,\ldots,n \quad . \qquad \ldots (23)$$

Equations (23) and (10) are the same, except that $\frac{+\delta_i}{(g_i(\underline{X}) - 1)^2}$ replaces $\lambda_i$. Since they both hold at the optimum $(\underline{X}^*,\underline{\lambda}^*)$

$$\lambda_i^* = \frac{+\delta_i}{(g_i(\underline{X}^*) - 1)^2} \quad , \quad i = 1,\ldots,m \quad .$$

Thus minimization of the augmented objective function (22) for given values of the parameters $\delta_i$ may be considered to be an evaluation of the corresponding dual problem with a $\underline{\lambda}$ other than $\underline{\lambda}^*$. The analogy only exists after each minimization because conditions (10) (and hence (23)) are the constraints of the dual problem and therefore must be satisfied.

These penalty function algorithms are possibly the most flexible of the mathematical programming procedures. The exterior point algorithms do not need feasible starting points and therefore can provide a powerful technique for generating feasible but not necessarily optimal designs, a task which is not always easy in engineering design. The exterior point algorithms can also consider equality and inequality constraints with equal ease and search procedures which do not require gradients can be defined if gradient evaluation is difficult. However, these procedures are limited to problems with only a relatively small number of variables unless the function evaluations are extremely simple because up to $40n(n+1)$ function evaluations could be required during the search for the optimum [30].

### (iii) Projected gradient and linear programming procedures

Projected gradient and linear programming procedures are direct search procedures which can be included in a rather broad class of feasible direction methods. In these methods a feasible direction is a direction in which a move can be made without violating the constraints, and a usable feasible direction is a feasible direction which also improves the merit function. The projected gradient method [31], for example, tries to move in the direction of the negative gradient of the merit function, projecting on to the surface of those constraints which would be violated.

An interesting derivation of the projected gradient method results if an attempt is made to define an algorithm which tries directly to satisfy (10). That is, it tries to satisfy

$$\underline{\nabla f}(\underline{X}^*) + \sum_{i=1}^{m} \lambda_i^* \, \underline{\nabla g}_i(\underline{X}^*) = \underline{0} \ , \quad \lambda_i \geq 0 \ , \quad i = 1, \ldots, \ell \qquad \ldots (24)$$

for $\quad x_i > 0 \ , \quad i = 1, \ldots, n \ ,$

and with $\lambda_i^* \left[ g_i(\underline{X}^*) - 1 \right] = 0 \ , \quad i = 1, \ldots, m \ .$

Condition (24) is a necessary condition for optimality which can be rewritten in matrix form as

$$\underline{\nabla f} + \underline{G}\,\underline{\lambda} = \underline{0} \quad \text{for} \quad \underline{\lambda} \in \underline{\Lambda} \quad . \tag{25}$$

For a given initial feasible design $\underline{\nabla f}(\underline{X})$ and $\underline{\nabla g}_i(\underline{X})$ for the currently active constraint set will be defined. It is therefore possible to check for optimality by solving (25) (with $\underline{G}$ containing only the active constraints) for the multipliers $\underline{\lambda}$ and checking that the positivity condition in (24) is satisfied. Since $\underline{G}$ will in general not be square a least squares fit for the multipliers must be obtained in the following way.

Rewriting (25) as

$$\underline{G}\,\underline{\lambda} = -\underline{\nabla f} \quad ,$$

premultiply by $\underline{G}^t$ to obtain

$$\underline{G}^t\underline{G}\,\underline{\lambda} = -\underline{G}^t\underline{\nabla f} \quad .$$

Hence

$$\underline{\lambda} = -(\underline{G}^t\underline{G})^{-1}\,\underline{G}^t\underline{\nabla f} \quad . \tag{26}$$

In general a least squares fit for $\underline{\lambda}$ will be obtained and (25) will not be satisfied identically. Therefore we can define

$$\underline{P} = \underline{\nabla f} + \underline{G}\,\underline{\lambda} \tag{27}$$

with $\underline{P} \neq \underline{0}$ implying that the current design is not optimal. Substituting for $\underline{\lambda}$ from (26) into (27) gives

$$\underline{P} = \underline{\nabla f} - \underline{G}(\underline{G}^t\underline{G})^{-1}\underline{G}^t\,\underline{\nabla f} \quad .$$

Therefore

$$\underline{P} = \left[\underline{I} - \underline{G}(\underline{G}^t\underline{G})^{-1}\underline{G}^t\right]\underline{\nabla f} \quad . \tag{28}$$

Now if $\underline{P} \neq \underline{0}$, (10) implies $\underline{\nabla L}(\underline{X},\underline{\lambda}) \neq \underline{0}$ . The Lagrange multipliers can be set by (26), and a steepest descent path followed to minimize $L(\underline{X},\underline{\lambda})$ with respect to $\underline{X}$ . That is, the design can be improved by defining

$$\underline{x}^{\nu+1} = \underline{x}^{\nu} - \alpha \underline{P} \tag{29}$$

where $\nu$ refers to the iteration number and $\alpha$ is a step length.

It can now be shown that the projected gradient algorithm has been recovered by deriving an expression for the projection $\underline{p}$ of the negative of the gradient of the merit function into the active constraint set, the direction used in the projected gradient search.

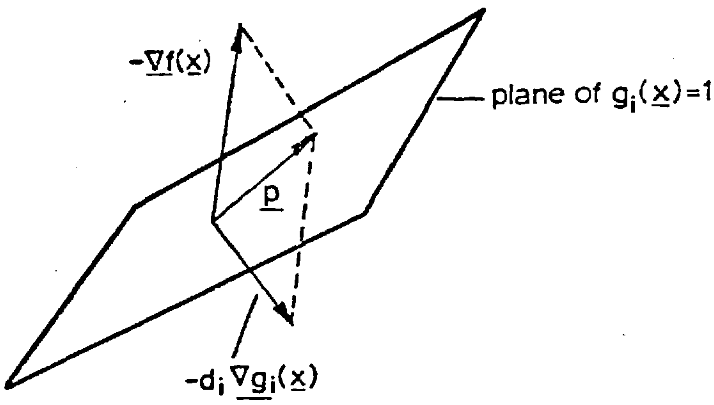



Figure 5 Projection of the gradient on the constraint hyperplane

Vectorial addition on Fig. 5 gives

$$\underline{p} = -\underline{\nabla f}(\underline{x}) - d_i \underline{\nabla g}_i(\underline{x})$$

and for projection into $m$ active constraints

$$\underline{p} = -\underline{\nabla f}(\underline{x}) - \sum_{i=1}^{m} d_i \underline{\nabla g}_i(\underline{x})$$

or
$$\underline{p} = -\underline{\nabla f}(\underline{x}) - \underline{G}\,\underline{D} \tag{30}$$

where
$$\underline{D} = \left[ d_1, d_2, \ldots, d_m \right]^t .$$

Now since $\underline{p}$ is perpendicular to each of the gradient vectors $\underline{\nabla g}_i(\underline{x})$

$$\underline{G}^t \underline{p} = \underline{0} = -\underline{G}^t \underline{\nabla f} - \underline{G}^t \underline{G}\,\underline{D} .$$

Hence $\underline{G}^t\underline{G}\underline{D} = -\underline{G}^t\underline{\nabla f}$

and $$\underline{D} = -(\underline{G}^t\underline{G})^{-1}\ \underline{G}^t\underline{\nabla f} \quad . \tag{31}$$

Substituting for $\underline{D}$ from (31) into (30) gives

$$\underline{p} = -\underline{\nabla f} + \underline{G}(\underline{G}^t\underline{G})^{-1}\ \underline{G}^t\underline{\nabla f}$$

and hence $$-\underline{p} = \left[\ \underline{I} - \underline{G}(\underline{G}^t\underline{G})^{-1}\ \underline{G}^t\ \right] \underline{\nabla f} \quad . \tag{32}$$

Comparison of (32) and (28) shows that the direction $-\underline{P}$ used in the steepest descent search of (29) is in fact the direction of search used in the projected gradient algorithm. Satisfaction of the non-negativity condition in (24) requires that the Lagrange multipliers $\lambda_i$, and hence the $d_i$, corresponding to inequality constraints be non-negative at the optimum. Inequality constraints with negative $\lambda_i$ or $d_i$ must be dropped from the active set and a new projection direction defined. Only one constraint can be dropped at a time, and a general rule is to eliminate the most negative Lagrange multiplier.

Movement in the projection direction will only continue to satisfy the active constraints if they are linear, and iterative procedures may have to be defined to enable the algorithm to return to the active constraint surface after each step. However, for linear constraints the procedure can move in the projection direction until either $f(\underline{X})$ attains a local minimum or a new constraint set becomes active. If the merit function is also linear then a local minimum will not exist and the optimum will lie at an intercept of the constraints.

When the merit function and constraints are all linear the design problem is called a linear programming problem. To demonstrate the existence of a dual problem, the linear programming dual will be derived here. The linear programming formulation will also be used in the next chapter to introduce the new procedures for generating feasible solutions to both the primal and dual problems.

Consider the problem

$$\text{minimize } \underline{q}^t\underline{X}$$

subject to the constraints $\underline{A}\underline{x} \geq \underline{b}$

and $\underline{x} \geq \underline{0}$ .

$$ \quad \ldots (33)$$

The dual problem as defined in Section 3.3 is

$$\text{maximize} \quad L(\underline{x},\underline{\lambda}) = \underline{q}^t\underline{x} + \underline{\lambda}^t(\underline{b} - \underline{A}\underline{x}) \quad ,$$

subject to the constraints $\frac{\partial L}{\partial x_j} \geq 0$ , $j = 1,\ldots,n$ with equality holding for $x_j > 0$. That is,

$$\underline{q}^t - \underline{\lambda}^t\underline{A} \geq \underline{0}$$

and $\underline{\lambda} \geq \underline{0}$ .

Now $\underline{q}^t - \underline{\lambda}^t\underline{A} \geq \underline{0}$ , with equality holding for those $x_j > 0$ , implies

$$\underline{q}^t\underline{x} - \underline{\lambda}^t\underline{A}\underline{x} = \underline{0} \quad .$$

Therefore the dual problem becomes

$$\text{maximize} \quad \underline{\lambda}^t\underline{b} \quad ,$$

subject to the constraints $\underline{\lambda}^t\underline{A} \leq \underline{q}^t \implies \underline{A}^t\underline{\lambda} \leq \underline{q}$ ,

and $\underline{\lambda} \geq \underline{0}$ .

$$ \quad \ldots (34)$$

The simplex algorithm [32] takes full advantage of the simplifications made possible by the linearity of these problems. However the selection of the active constraint set is still based on the Lagrange multipliers. The sensitivity coefficients for slack variables, which reflect the activity levels of the inequality constraints, are given by these multipliers [29]. The Lagrange multipliers are also the dual variables in (34). This indicates why the optimal values of the dual variables are given by the sensitivity coefficients for the slack variables in the optimum simplex tableau generated by the simplex algorithm.

(iv) Sensitivity studies

Sensitivity studies are a form of post-optimal analysis which do not involve a definite search strategy. However they form an important branch of mathematical programming which is again based on the Lagrangean saddle-function. These studies are aimed at determining the sensitivity of the optimum design to changes in the constraint limits and cost coefficients, and can also be used to assess the sensitivity of the solution to inaccuracies in the mathematical model of the design problem.

For the design problem

$$\text{minimize } f(\underline{x}) \text{ subject to } g_i(\underline{x}) \le b_i \quad , \quad i = 1,\ldots,m \quad ,$$

the derivative of $f(\underline{x}^*)$ with respect to $b_i$ is given by the chain rule as

$$\left.\frac{\partial f}{\partial b_i}\right|_{\underline{x}^*,\underline{\lambda}^*} = \sum_{j=1}^{n} \left.\frac{\partial f}{\partial x_j}\right|_{\underline{x}^*,\underline{\lambda}^*} \left.\frac{\partial x_j}{\partial b_i}\right|_{\underline{x}^*,\underline{\lambda}^*} \quad . \tag{35a}$$

Considering only the $m'$ active constraints, for which $g_k(\underline{x}) = b_k$, we can write

$$\sum_{j=1}^{n} \left.\frac{\partial g_k}{\partial x_j}\right|_{\underline{x}^*,\underline{\lambda}^*} \left.\frac{\partial x_j}{\partial b_i}\right|_{\underline{x}^*,\underline{\lambda}^*} = \bar{\delta}_{ik} \quad , \quad k = 1,\ldots,m' \quad ,$$

where $\bar{\delta}_{ik}$ is the Kronecker delta ($\bar{\delta}_{ik} = 1,\ i = k;\quad \bar{\delta}_{ik} = 0,\ i \neq k$). Multiplying this last equation by $\lambda_k^*$, summing over all k's and adding the sum to (35a) gives

$$\left.\frac{\partial f}{\partial b_i}\right|_{\underline{x}^*,\underline{\lambda}^*} \doteq -\sum_{k=1}^{m'} \lambda_k^* \bar{\delta}_{ik} + \sum_{j=1}^{n} \left\{ \left.\frac{\partial f}{\partial x_j}\right|_{\underline{x}^*,\underline{\lambda}^*} + \sum_{k=1}^{m'} \lambda_k^* \left.\frac{\partial g_k}{\partial x_j}\right|_{\underline{x}^*,\underline{\lambda}^*} \right\} \frac{\partial x_j}{\partial b_i} \quad .$$

The term in the brackets has been shown to be zero. Therefore

$$\left.\frac{\partial f}{\partial b_i}\right|_{\underline{x}^*,\underline{\lambda}^*} = -\lambda_i^* \quad . \tag{35b}$$

The Lagrange multipliers therefore give the perturbation to the optimum value of the merit function caused by altering the constraint limiting value. This relationship forms the basis for sensitivity studies. The sensitivity to changes in the coefficients of the cost function follows from direct differentiation of $f(\underline{x}^*)$ with respect to the coefficients. The relation (35b) is a further property of the saddle function which will be studied in Chapter 6.

## 4.3 The development and extension of approximate redesign procedures

In addition to the mathematical programming procedures described in the preceding section, a number of special strategies for structural design can be derived from the Lagrangean function. The examples which will be considered in Chapters 7 and 9 involve the minimum mass design of aerospace structures and two approximate redesign strategies will be discussed. These are the stress ratioing procedure for stress limited design for a given topology and geometry, and the envelope procedure based on the optimality criterion approach to design for deflection constraints under the same configuration restrictions. The derivation of these two procedures will be summarized in this section. The approximations arise because the redesign formulae are applied to conditions not defined in the Lagrangean formulation.

### (i) The fully stressed design concept and the stress ratioing procedure

The merit function considered is the mass of the structure given by

$$m = \sum_{i=1}^{n} \rho_i c_i x_i \quad ,$$

where $\rho_i$ is the density of the material and $c_i$ is a positive constant which depends on the type of structural member being considered. The design is subject only to a general stiffness requirement

$$\tfrac{1}{2}\underline{R}^t\underline{r} = k \quad . \tag{36}$$

The Lagrangean formulation for this problem requires that the stationary point of the function

$$L(\underline{x},\lambda) = \sum_{i=1}^{n} \rho_i c_i x_i + \lambda(\tfrac{1}{2}\underline{R}^t\underline{r} - k)$$

be found. The condition $\frac{\partial L}{\partial \lambda} = 0$ is automatically satisfied if the constraint (36) is imposed. The second condition $\frac{\partial L}{\partial \underline{x}} = 0$ requires

$$\rho_i c_i + \lambda \frac{\partial(\tfrac{1}{2}\underline{R}^t\underline{r})}{\partial x_i} = 0 \quad , \qquad i = 1,\ldots,n$$

or $$\frac{\frac{\partial(\frac{1}{2}\underline{R}^t\underline{r})}{\partial x_i}}{\rho_i c_i} = -\frac{1}{\lambda} = \text{constant} , \quad i = 1,\ldots,n \quad .$$

Following the derivation in [33] leads to the criterion

$$\frac{u_i}{x_i c_i \rho_i} = \text{constant} \tag{37}$$

where $u_i$ is the total strain energy in the $i^{th}$ element. For elements in which the stress remains constant

$$u_i = \frac{\sigma_i^2 x_i c_i}{2 E_i} \quad .$$

Substitution into (37) now gives

$$\frac{\sigma_i^2}{E_i \rho_i} = \text{constant} , \quad i = 1,\ldots,n \quad . \tag{38}$$

This is a condition which must be satisfied at the optimum and gives a criterion for optimality. If the density and Young's modulus $E$ is constant throughout the structure (38) requires that the stress in all the members be the same. If the same limiting stress is applied to all members then the fully stressed design concept is recovered.

For statically determinate pin-ended bar structures the stress in each member is given by

$$\sigma_i = \frac{k_i}{x_i} , \quad i = 1,\ldots,n \quad .$$

For these structures the optimality criterion (38) can be satisfied in a single redesign step, after analysing the structure for stresses, using the formula

$$x_i^{\nu+1} = x_i^{\nu} \frac{\sigma_i^{\nu}}{\sigma_{lim}} , \quad i = 1,\ldots,n \quad . \tag{39}$$

Here $\nu$=1 and $x_i^{\nu+1}$ is the optimal value for a statically determinate structure.

The stress ratioing formula (39) is only approximate for redundant structures and iterative procedures result with $\nu$ being the iteration number. When multiple stiffness requirements, such as multiple load cases, are applied to the design problem more than one Lagrange multiplier appears in the Lagrangean formulation and prohibits the recovery of a simple redesign formula such as (39). However, this formula has also been applied to these problems giving an approximate iterative redesign procedure. The fully stressed criterion is modified to require that each member be stressed at the limiting value in at least one of the load cases. Difficulties have however been encountered when the members are subjected to widely differing stress limits and material properties.

Because of the existence of multiple Lagrange multipliers which are not evaluated there is no guarantee that the fully stressed design, if it can be obtained, will be optimal when multiple stiffness requirements are considered. However, since the stress ratioing procedure is iterating towards the intercept of a set of constraints in design space, the Kuhn-Tucker conditions could be checked for optimality. If the constraint intercept does not define the optimum negative Lagrange multipliers will be found and the projected gradient scheme described in Section 4.2 could be used to extend the design process and find the optimum design. Alternatively a method which adjusts the constraint limiting values has to be derived so that the new intercept will define the optimum design.

(ii) The optimality criterion approach for deflection constraints

Consider again the design of a pin-ended bar truss but now apply only a single deflection constraint,

$$\bar{r} = \bar{r}_{lim} ,$$

and make the design variables the cross-sectional areas of the members. Then if $F_i$ is the force in the member $i$ caused by the application of the load system, and if $U_i$ is the force in member $i$ caused by the application of a virtual unit load to the joint and in the direction corresponding to $\bar{r}_{lim}$, the displacement constraint can be written as

$$\sum_{i=1}^{n} \frac{F_i U_i c_i}{E_i x_i} = \bar{r}_{lim} \quad .$$

Again the condition $\frac{\partial L}{\partial \lambda} = 0$ for the stationary point of the Lagrangean function for this problem is satisfied if this constraint is forced to be active. The condition $\frac{\partial L}{\partial x} = 0$ gives

$$\rho_j c_j + \lambda \sum_{i=1}^{n} \frac{\partial}{\partial x_j} \left[ \frac{F_i U_i c_i}{E_i \bar{r}_{lim}} \cdot \frac{1}{x_i} \right] = 0 \quad . \tag{40}$$

Now terms of the form

$$\sum_{i=1}^{n} \frac{U_i c_i}{E_i \bar{r}_{lim}} \cdot \frac{1}{x_i} \frac{\partial F_i}{\partial x_j} \quad ,$$

and similar terms involving derivatives of $U_i$, can be shown to form self-equilibrating load systems for redundant structures and sum to zero. For statically determinate structures they are automatically zero because the forces $F_i$ and $U_i$ are constant. Equation (40) therefore becomes

$$\rho_i c_i - \frac{\lambda F_i U_i c_i}{E_i \bar{r}_{lim}} \cdot \frac{1}{x_i^2} = 0 \quad .$$

The Lagrange multiplier can now be eliminated [21] to give the redesign formula

$$x_i = \sqrt{\frac{F_i U_i}{E_i \bar{r}_{lim}}} \sum_{j=1}^{n} c_j \sqrt{\frac{F_j U_j}{E_j \bar{r}_{lim}}} \quad , \qquad i = 1,\ldots\ldots,n \quad . \qquad \ldots (41)$$

This redesign formula must again be applied iteratively for redundant structures because the forces $F_i$ and $U_i$ vary as the design changes. When multiple deflection constraints are considered the Lagrangean function takes the form

$$L(\underline{x},\underline{\lambda}) = \rho_i c_i x_i + \sum_{j=1}^{m} \lambda_j \left( \sum_{i=1}^{n} \frac{F_i c_i U_i}{E_i x_i} - \bar{r}_{lim_j} \right)$$

and the condition $\frac{\partial L}{\partial x_i} = 0$ gives

$$\rho_i c_i = \sum_{j=1}^{m} \lambda_j \frac{F_j c_j U_j}{E_i x_i^2} \quad , \qquad i = 1,\ldots\ldots,m \quad . \tag{42}$$

Again multiple Lagrange multipliers have appeared in the formulation and cannot be eliminated to recover a redesign formula similar to (41). However an envelope procedure similar to the method used to apply the stress ratioing procedure to problems with multiple load cases has been proposed in [21]. In this procedure the structure is designed for each deflection constraint (and stress and minimum size constraint if they are applied) independently. The largest value of the design variable from all the designs is then taken as the new design value. However this procedure is approximate and the projected gradient method would again have to be applied if the design produced by this scheme does not satisfy the Kuhn-Tucker conditions. Alternatively, a dual formulation could be proposed in which the conditions (42) would become the dual constraints and the Lagrange multipliers would then be evaluated as the dual variables in the search for the optimum.

## 4.4 Summary

In this chapter a number of redesign strategies have been shown to be based on the Lagrangean saddle-function. The reason for isolating this unifying theory is to define a framework within which to search for a method for generating bounds on the optimum value of the cost function. If the method is to be generally applicable it should be defined within this unifying framework.

The identification of a unifying theory will also assist the investigation of approximate redesign strategies. It should be possible to draw on the mathematical programming procedures, which share a common basis in the saddle function with the approximate methods, to tighten the redesign strategy and continue the design process when the approximate methods do not produce the optimum design.

# CHAPTER 5

# CONVEXITY, AND CONVEX-PRIMAL CONCAVE-DUAL FORMULATIONS

## 5.1 Introduction

Properties of convex and concave functions and convex sets will now be investigated. It has already been necessary to resort to assumptions that the merit function and constraints are locally convex in the proof of the existence of the saddle point and duality. The property of convexity also ensures the uniqueness of the local minimum to which the numerical search procedures of the previous chapter converge, so that the global minimum is found. It is of particular importance that convex regions contain no re-entrant corners and that the gradients of convex and concave functions vary in a unimodal fashion.

The dual and the conjugacy relationships in the saddle function can be derived through convex analysis [27], indicating that the unifying theory proposed has its foundations in this branch of mathematics. Certain useful properties are also defined in convex analysis, including uniqueness of the dual problem and conditions under which equivalent points can be defined in both the primal and the dual. These properties are illustrated for linear programming in this chapter and the generalization of these relationships will form the basis of the work which is to follow. In preparation a convex-primal concave-dual formulation is defined.

## 5.2 Convex functions and convex sets

A function $f(\underline{X})$ is said to be a convex function if

$$f\left[ \alpha x_1 + (1 - \alpha)x_2 \right] \leq \alpha f(x_1) + (1 - \alpha)f(x_2) \quad , \quad 0 < \alpha < 1 \quad , \qquad \ldots (43)$$

for any two points $x_1$ and $x_2$. The function is said to be strictly convex if the inequality holds always in (43). That is, a strictly convex function is never under-estimated by a linear interpolation between any two points. For a concave function the direction of the inequality in (43) is reversed so that a strictly concave function is one whose negative is strictly convex, and is never over-estimated by a linear interpolation. This definition also applies to multi-dimensional functions if $x_1$ and $x_2$ are replaced by vectors $\underline{X}_1$ and $\underline{X}_2$ in (43).

Convex and concave functions must be continuous but they need not be differentiable. However for those that are twice differentiable it can be shown that if $f(\underline{X})$ is convex then the related function

$$\phi(\alpha) = f\left[\alpha x_1 + (1-\alpha)x_2\right] \quad , \quad 0 \leq \alpha \leq 1 \quad ,$$

satisfies the inequality

$$\frac{\partial^2\phi}{\partial\alpha^2} \geq 0 \quad , \quad 0 \leq \alpha \leq 1 \quad . \tag{44}$$

This result can be defined in terms of the Hessian matrix $\underline{H}$, where

$$H_{ij} = \frac{\partial^2 f(\underline{X})}{\partial x_i \partial x_j} \quad .$$

The function $f$ is convex if the quadratic form

$$Q = \sum_{i,j=1}^{n} \left.\frac{\partial^2 f}{\partial x_i \partial x_j}\right|_{\underline{X}} q_i q_j \tag{45}$$

is positive semi-definite for all points $\underline{X}$. Furthermore, it is strictly convex if this quadratic form is positive definite. Conversely a function is concave if the Hessian is negative semi-definite.

These functions therefore have only one minimum or maximum and are said to be unimodal. Given a point $\underline{X}'$ it is possible to tell from the gradients $\left.\frac{\partial f}{\partial x_i}\right|_{\underline{X}'}$ on which side of $\underline{X}'$ the relative extremum lies, and this gradient varies monotonically as the point $\underline{X}'$ moves in either direction.

Convex sets are now defined in terms of convex functions using the following two theorems.

1. If $g(\underset{\sim}{X})$ is convex, the set $\underset{\sim}{R} = \left\{ \underset{\sim}{X} \mid g(\underset{\sim}{X}) \leq \underset{\sim}{k} \right\}$, which means $\underset{\sim}{R}$ is the set of vectors $\underset{\sim}{X}$ which satisfies the inequality $g(\underset{\sim}{X}) \leq \underset{\sim}{k}$, is convex for all positive $\underset{\sim}{k}$.

2. The intersection of a number of convex sets is convex. That is, the region defined by the constraints

$$g_i(\underset{\sim}{X}) \leq 1 \quad , \quad i = 1,\ldots,m$$

is convex if each $g_i(\underset{\sim}{X})$ is convex.

If the feasible region of a design problem is convex a straight line between any two points $\underset{\sim}{X}_1$ and $\underset{\sim}{X}_2$ in the feasible region lies entirely within that region. The important result is that if $f(\underset{\sim}{X})$ is strictly convex in a convex feasible region, then $f$ is unimodal. That is, there is only one minimum in the feasible region which is therefore the global minimum. Contours of a strictly convex function in a convex region are shown in Fig. 6. Fig. 7 shows a region which is not convex.

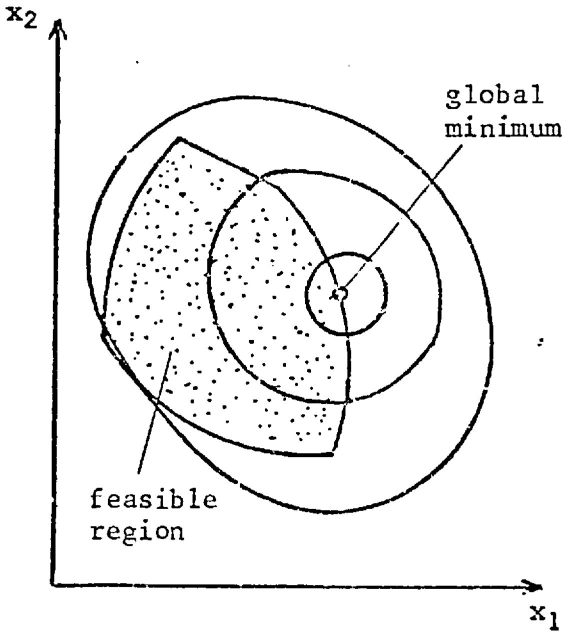



Figure 6 A convex function defined on a convex region

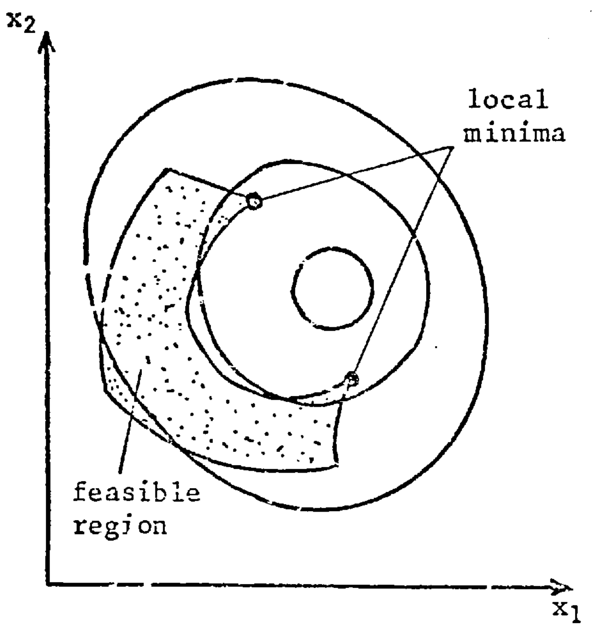



Figure 7 A convex function defined on a non-convex region

It is obvious from Fig. 6 that problems involving the minimization of a convex function subject to a set of convex constraints have associated with them a well-behaved design space with no re-entrant corners and a merit function whose variation is unimodal. In the next chapter procedures will be proposed to predict information about the constraint set active at the optimum. It is obvious from this brief discussion and Fig. 6 that it would be highly desirable, if not essential, for the merit function and the feasible region to both be convex in the space of the variables in which these predictions are to be made.

In the following section it is shown that the properties of convex functions and regions make the Kuhn-Tucker necessary conditions, derived in Section 2.3, also sufficient for optimality. In a later section a certain form of convex function, which will allow a convex approximation to a real design problem to be formed, is defined.

It should be noted that only inequality constraints have been considered in this discussion. The set

$$\underline{R} = \left\{ \underline{X} \mid g_i(\underline{X}) = k \right\}$$

is convex only if $g_i(\underline{X})$ is a linear function of $\underline{X}$ .

## 5.3 Sufficient conditions for optimality

One simplifying feature of a convex design problem, in which the merit function and constraints are all convex, is that the Kuhn-Tucker necessary conditions derived in Section 2.3 are also sufficient for optimality. If $f(\underline{X})$ is strictly convex or strictly concave, the unimodal and second derivative properties ensure that there is only one point for which $\frac{\partial f}{\partial \underline{X}} = \underline{0}$ , and that point must also be the global optimum. Therefore this condition is both necessary and sufficient to define the unconstrained extremum of convex and concave functions.

The simplest way to extend this argument to strictly convex design problems with constraints is to follow the geometric argument given in [29]. Fig. 8 shows contours of a strictly convex function in a convex region. The zero constrained derivative condition (6) for only inequality constraints defines a local minimum which can be shown to be unique in this case. Consider any point $\underline{X}$ in the feasible region and the straight line connecting it with the global minimum $\underline{X}^*$. Since the feasible region is convex, all points $\alpha \underline{X}^* + (1-\alpha)\underline{X}$ on this line are also in the feasible region. However, strict convexity of $f(\underline{X})$ implies

$$f\left[\alpha \underline{X}^* + (1-\alpha)\underline{X}\right] < \alpha f(\underline{X}^*) + (1-\alpha) f(\underline{X})$$

$$= f(\underline{X}) + \alpha\left[f(\underline{X}^*) - f(\underline{X})\right] < f(\underline{X}) .$$

The last inequality holds because $\alpha\left[f(\underline{X}^*) - f(\underline{X})\right]$ can only be negative, since $\alpha>0$ and $\underline{X}^*$ is the global minimum. It follows that all points on the line are better than the arbitrary point $\underline{X}$, so that $\underline{X}$ cannot be a local minimum. Therefore the minimum for a strictly convex problem is unique and the Kuhn-Tucker conditions are both necessary and sufficient to define the global optimum.

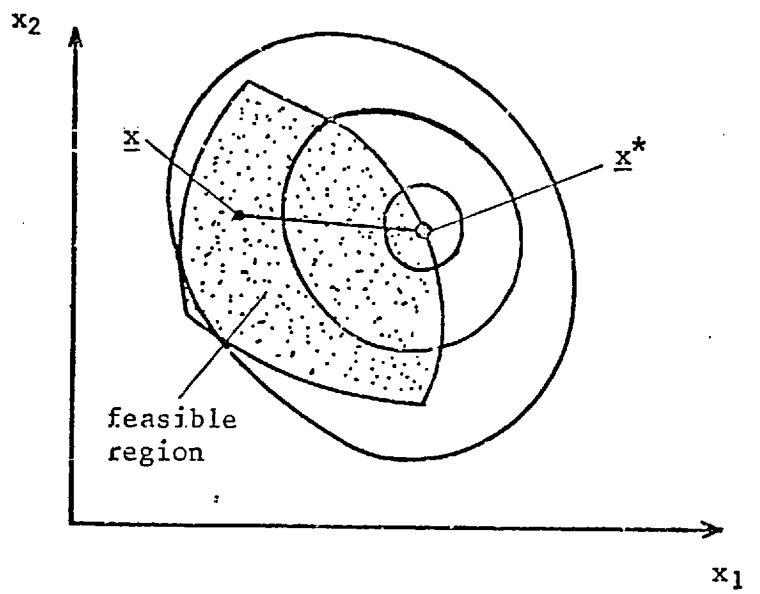



Figure 8. Uniqueness of the minimum of a convex problem.

The existence of the saddle point can also be guaranteed for a convex problem. Consider the following Lagrangean function in which

$f(\underline{X})$ and $g_i(\underline{X})$ are all strictly convex,

$$L(\underline{X},\underline{\lambda}) = f(\underline{X}) + \sum_{i=1}^{\ell} \lambda_i \left[ g_i(\underline{X}) - 1 \right] \quad , \quad \lambda_i \geq 0 \ . \qquad \ldots (46)$$

It was shown in Section 3.2 that satisfaction of the Kuhn-Tucker conditions implied

$$\frac{\partial L}{\partial x_i} = 0 \quad \text{for} \quad x_i > 0 \qquad (47a)$$

and

$$\frac{\partial L}{\partial \lambda_j} = 0 \quad \text{for} \quad \lambda_j > 0 \ . \qquad (47b)$$

These conditions are sufficient to define a stationary point of the Lagrangean function. However, if the inequality

$$L(\underline{X}^*,\underline{\lambda}) \leq L(\underline{X}^*,\underline{\lambda}^*) \leq L(\underline{X},\underline{\lambda}^*) \quad , \qquad (48)$$

where the asterisk denotes optimal values, is also satisfied then the stationary point is a saddle point.

For this inequality relationship to hold $L(\underline{X},\underline{\lambda}^*)$ must first be convex in $\underline{X}$ . This function is a positive linear combination of strictly convex functions in the design variables $\underline{X}$ and this combination is also convex [28]. This convexity property together with (47a) is then sufficient to show that

$$L(\underline{X}^*,\underline{\lambda}^*) \leq L(\underline{X},\underline{\lambda}^*) \ .$$

The second condition (47b) requires that

$$\lambda_j^*(g_j(\underline{X}^*) - 1) = 0$$

so that $L(\underline{X}^*,\underline{\lambda}) \leq L(\underline{X}^*,\underline{\lambda}^*)$ .

This follows because $g_j(\underline{X}^*) < 1$ for those constraints $j$ which are not active and $\lambda_j$ must remain non-negative. The inequality (48) is therefore satisfied if the primal problem is convex and the existence of the saddle function is guaranteed.

## 5.4 The primal and dual problems of linear programming

In [27] it is shown that the concave dual for a convex primal problem can be derived from concepts of conjugates of convex functions and mappings. This confirms the conclusion drawn in Chapter 3 that the directions defined by the primal and dual problems on the surface of the Lagrangean function are conjugate. Certain uniqueness theorems are also proved. In particular, if certain continuity conditions are satisfied and $F$ is the mapping function from the primal to the dual problem, then

$$(F^*)^* = F$$

where $F^*$ is the conjugate of $F$. In other words, the dual of the dual is the primal. In addition it is shown that the Lagrange multipliers will satisfy the Kuhn-Tucker conditions for the primal problem, that is,

$$\lambda_i(g_i(\underline{X}) - 1) = 0 \quad , \quad i = 1, \ldots, m \quad ,$$

and $$\underline{\nabla f} + \underline{\lambda} \underline{\nabla g} = \underline{0} \quad ,$$

if $\underline{\lambda}$ is the optimum solution to the dual problem. When the primal problem is solved the Kuhn-Tucker conditions can therefore be used to define the Lagrange multipliers at the optimum enabling a dual solution point to be defined which is feasible and optimal.

The simplest primal-dual formulation which can be used to investigate these relationships further is provided by linear programming. The convex primal problem for linear programming was given by (33) as

$$\text{minimize} \quad f = \underline{q}^t \underline{x}$$

subject to the constraints $\underline{A}\underline{x} \geq \underline{b}$ and $\underline{x} \geq \underline{0}$ . . . . (49)

The dual derived in the same section was

$$\text{maximize} \quad h = \underline{\lambda}^t \underline{b}$$

subject to the constraints $\underline{A}^t \underline{\lambda} \leq \underline{q}$ and $\underline{\lambda} \geq \underline{0}$. . . . (50)

The optimum point 0 in the two-dimensional primal problem depicted in Fig. 9 corresponds to the intercept of two of the constraints giving equations of the form

$$\underline{A}_1\underline{x} = \underline{b} \quad . \tag{51}$$

A dual solution point corresponding to the inclusion of the same constraints in the dual problem will be defined by an equation of the form

$$\underline{A}_1{}^t\underline{\lambda} = \underline{q} \quad . \tag{52}$$

Equation (52) can however be written as

$$\underline{\nabla f} - \underline{\lambda \nabla g} = \underline{0} \tag{53}$$

which is the Kuhn-Tucker condition (6) with the sign change caused by the direction of the inequality in the constraints of (49). A geometric interpretation has already been applied to this equation in Chapter 2 and will be used again here. At the point 0 in Fig. 9 conditions (53) can be satisfied by vector addition with a non-negative set of dual variables $\underline{\lambda}$ and solution of (52) therefore gives a feasible solution to the dual problem. However the solution of a similar set of equations at B now including constraints $g_2(\underline{X})$ and $g_3(\underline{X})$ will lead to one negative dual variable because the gradient of the merit function lies outside the cone formed by the gradients of these constraints.

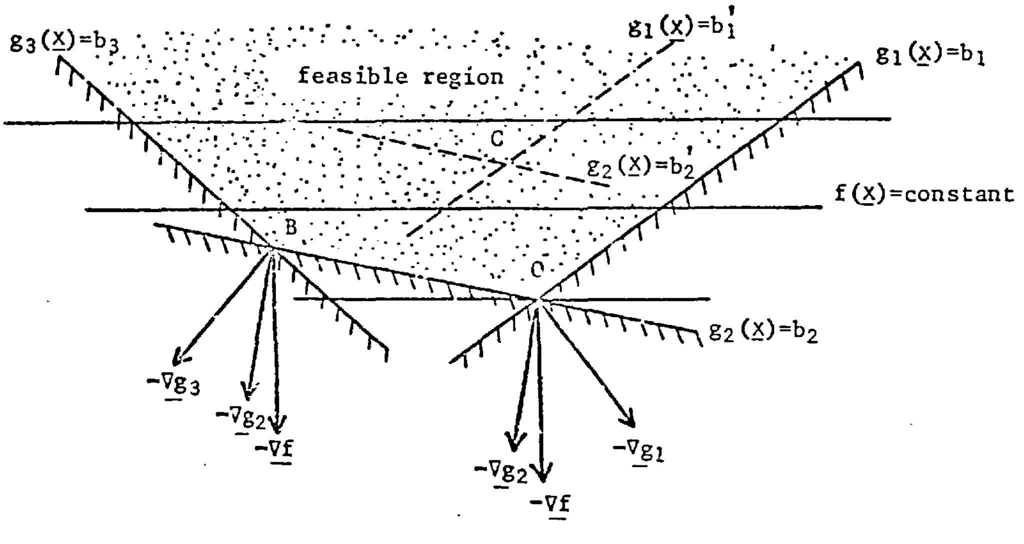



Figure 9 Design space for a linear programming problem

The non-negativity conditions on the Lagrange multipliers will not be satisfied and a feasible dual solution point will not be found.

Therefore corresponding feasible solutions to both the primal and dual problems can be defined at least at the optimum with a unique dual point being found by solving (52). The original primal point can be recovered from (51) if necessary. However, if the primal design under consideration moves away from the optimum on the constraint surface, (53) cannot be satisfied with a non-negative set of dual variables if $\underline{\nabla g}$ contains only gradients for active constraints. It will therefore not in general be possible to define a unique feasible dual point corresponding to each feasible primal point if the additional Kuhn-Tucker condition

$$\lambda_i \left[ g_i(\underline{x}) - 1 \right] = 0 \quad ,$$

is satisfied. However, if the requirement to satisfy this condition is removed, points such as C in the feasible region in Fig. 9 can be considered. At this point (53) can be satisfied by the same set of dual variables as those defined at O and a feasible solution to the dual problem will be found.

The generation of corresponding feasible solutions to both primal and dual problems will be considered further in Chapter 6. The success of transition from primal to dual problems will rely on the development of schemes to remove negative Lagrange multipliers without having to resort to considering all possible constraint combinations until a positive set of dual variables is found. This operation would effectively solve the linear programming problem.

In the following sections a special non-linear convex problem will be defined and its concave dual derived. For the non-linear problem gradients in (53) will not be the same at points O and C . Therefore solutions to the dual corresponding to these points will be different. Analogous formulations however arise, in particular the paired equations like (51) and (52), so that the new developments represent a generalization of the ideas presented here.

## 5.5 Posynomials and a convex primal problem

In this section a particular form of polynomial, called a posynomial, will be defined. A primal problem in which both the merit function and a set of inequality constraints are all posynomials is called the geometric programming primal problem [28]. While posynomials are not convex, these non-linear functions can be made convex by a simple variable transformation. The concave dual for the resulting convex programming problem will be derived in the next section, and the non-linear convex-primal concave-dual formulation will then provide a more flexible framework within which to study the interface between the primal and dual problems than the linear programming problem discussed in the previous section.

The exponential function $e^t$ is a strictly convex function on its domain $E_1$. This can be verified by evaluating the second derivative of (44) and showing that the strict inequality holds. All linear functions on $E_n$ are also convex, but not strictly convex because their second partial derivatives are identically zero. Therefore the function

$$f(\underline{t}) = \sum_j a_{ij} t_j$$

is convex for constant $a_{ij}$. To combine these functional forms the following theorem [27] is used.

> If $h$ is a monotone non-decreasing function on $E_1$ (i.e. $h(s) \leq h(t)$ when $s \leq t$), and $h$ is also convex on $E_1$ ; and if $f$ is a convex function on its convex domain $D \subset E_n$, then the composite function $h(f)$ is a convex function on $D$.

Hence it follows that

$$f(\underline{t}) = e^{\sum_j a_{ij} t_j}$$

is convex because $e^{t_i}$ is clearly monotone increasing in $t_i$. Also a positively weighted sum of convex functions is convex [28] so that

$$f(\underline{t}) = \sum_i c_i \, e^{\sum_j a_{ij} t_j} \qquad (54)$$

is convex if all $c_i > 0$ .

Introducing a change of variables

$$x_j = e^{t_j} \quad , \quad j = 1,\ldots,n \quad , \qquad (55)$$

recovers the posynomial form

$$f(\underline{x}) = \sum_i c_i \prod_j x_j^{a_{ij}} \quad . \qquad (56)$$

The fact that posynomials are not necessarily convex can be observed by considering the simple function $x^{\frac{1}{2}}$ . However, they can obviously be converted into convex functions by reversing the variable transformation to recover (54).

The following posynomial programming problem can now be defined.

$$\text{Minimize } f(\underline{x}) = \sum_{i=1}^{n_0} K_i \prod_{j=1}^{n} x_j^{a_{ij}} \quad , \quad K_i > 0 \; , \; x_i > 0 \; , \qquad \ldots (57)$$

subject to the constraints

$$\sum_{i \varepsilon J[k]} K_i \prod_{j=1}^{n} x_j^{a_{ij}} \leq 1 \quad , \quad K_i > 0 \quad , \quad k = 1,\ldots,m, \qquad (58)$$

where $J[k] = \left\{ m_k, m_k+1, m_k+2, \ldots, p_k \right\}$ , $k = 1, \ldots, m$ and

$m_1 = n_0+1, \; m_2 = p_1+1, \ldots, \; m_m = p_{m-1}+1, \; p_m = p$ .

Here, $a_{ij}$ are real constants. This is the geometric programming primal problem and $p$ is the total number of terms in the problem. On making the change of variable defined by (55), the following convex programming problem is obtained.

$$\text{Minimize } h_0(\underline{t})$$

subject to $h_k(\underline{t}) \leq 1$ , $k = 1,\ldots,m$ , where from (57)

$$h_0(\underline{t}) = \sum_{i=1}^{n_0} K_i \, e^{\sum_{j=1}^{n} a_{ij} t_j} \quad , \quad K_i > 0 \quad ,$$

and from (58)

$$h_k(\underline{t}) = \sum_{i \varepsilon J[k]} K_i \, e^{\sum_{j=1}^{n} a_{ij} t_j} \quad , \quad K_i > 0 \quad , \quad k = 1,\ldots,m \quad .$$

. . . (59)

## 5.6 Derivation of a dual for the convex programming problem and proof that the dual is concave

In this section the dual for the convex programming problem (59) is defined and it is shown that this dual problem is concave. The derivation follows that given in [34] and is only summarized here.

The first step in this derivation is to collect the coefficients of the exponents into the matrices $\underline{B}$ and $\underline{b}$ such that

$$h_0(\underline{y}) = \sum_{i=1}^{n_0} e^{y_i}$$

and

$$h_k(\underline{y}) = \sum_{i \varepsilon J[k]} e^{y_i} \quad , \quad k = 1,\ldots,m \quad ,$$

with

$$\underline{y} = \underline{Bt} + \underline{b} \quad .$$

After taking natural logarithms the primal problem (59) is reformulated as

$$\text{Minimize} \quad \ln h_0(\underline{y})$$

$$\text{subject to} \quad \ln h_k(\underline{y}) \leq 0 \quad , \quad k = 1,\ldots,m \quad ,$$

$$\text{and} \quad \underline{y} - \underline{Bt} = \underline{b} \quad ,$$

. . . (60)

where both $\underline{y}$ and $\underline{t}$ are variables.

It is shown in [34] that (60) is a convex programming problem with convex merit function and constraints. The dual problem defined in Section 3.3 now becomes

$$\text{maximize} \quad \ln h_0(\underline{y}) + \sum_{k=1}^{m} \lambda_k \ln h_k(\underline{y}) + \underline{\delta}^t ( \underline{b} + \underline{\underline{B}}\underline{t} - \underline{y} )$$

$$\text{subject to} \quad \underline{\nabla} \ln h_0(\underline{y}) + \sum_{k=1}^{m} \lambda_k \underline{\nabla} \ln h_k(\underline{y}) + \underline{\delta}^t \underline{\nabla} ( \underline{b} + \underline{\underline{B}}\underline{t} - \underline{y} ) = \underline{0}$$

$$\text{and} \quad \lambda_k \geq 0 \quad , \quad k = 1,\ldots,m \;.$$

From the definition of $\underline{y}$ at the beginning of this section this becomes

$$\text{maximize} \quad \ln h_0(\underline{y}) + \sum_{k=1}^{m} \lambda_k \ln h_k(\underline{y}) + \underline{\delta}^t ( \underline{b} + \underline{\underline{B}}\underline{t} - \underline{y} ) \qquad (61a)$$

$$\text{subject to} \quad \frac{e^{y_i}}{h_0(\underline{y})} - \delta_i = 0 \quad , \quad i = 1,\ldots,n_0 \;, \qquad (61b)$$

$$\frac{\lambda_k e^{y_i}}{h_k(\underline{y})} - \delta_i = 0 \quad , \quad i \in J[k] \;,\; k = 1,\ldots,m \;, \qquad (61c)$$

$$\underline{\underline{B}}^t \underline{\delta} = \underline{0} \;, \qquad (61d)$$

$$\text{and} \quad \underline{\lambda} \geq \underline{0} \;. \qquad (61e)$$

It can be shown [34] that constraints (61c) are redundant and need not be considered. It is also shown that if $(\underline{\lambda}, \underline{\delta}, \underline{y})$ is feasible for this dual problem, then

$$\underline{\delta} \geq \underline{0}$$

$$\text{and} \quad \sum_{i\in J[k]} \delta_i = \lambda_k \quad , \quad k = 1,\ldots,m \;. \qquad (62)$$

Recognising that (61d) gives $\underline{\delta}^t \underline{\underline{B}}\underline{t} = \underline{0}$ leads, after some algebraic manipulation to (61) being rewritten in the form

$$\text{maximize} \quad \ln v(\underline{\delta}) \quad ,$$

subject to $\underline{B}^t \underline{\delta} = \underline{0}$ ,

$\sum_{i=1}^{n_o} \delta_i = 1$ ,

and $\underline{\delta} \geq \underline{0}$ .

Here $$v(\underline{\delta}) = \prod_{i=1}^{p} \left(\frac{c_i}{\delta_i}\right)^{\delta_i} \prod_{i=1}^{m} \left(\lambda_i\right)^{\lambda_i} \quad ,$$

with $$\lambda_k = \sum_{i \varepsilon J[k]} \delta_i \quad , \quad k = 1,\ldots,m \quad ,$$

and $$c_i = e^{b_i} \quad , \quad i = 1,\ldots,p \quad ,$$
$$= K_i \quad , \quad i = 1,\ldots,p \qquad \ldots (63)$$

To ensure $\delta_i \ln \delta_i$ is continuous for $\delta_i=0$ , $0 \ln 0$ is defined equal to zero.

This is the geometric programming dual problem corresponding to the posynomial primal problem defined by (57) and (58). It is markedly different from the normal dual program defined in Chapter 3 in which the Lagrangean function appears as the merit function. The special property which will be exploited in the next chapter is that the dual variable set $\underline{\delta}$ has been expanded larger than the Lagrange multiplier set $\underline{\lambda}$ corresponding to the inequality constraints in the primal problem. The Lagrange multipliers can still however be recovered through Equation (62).

To prove $\ln v(\underline{\delta})$ concave requires that it be shown that the Hessian matrix, $\underline{H}$ , is negative semi-definite where

$$H_{ij} = \frac{\partial^2 \ln v}{\partial \delta_i \partial \delta_j} \quad , \quad i, j = 1,\ldots,p \quad .$$

The quadratic form arising from $\underline{H}$ is

$$Q(\underline{q}) = \sum_i \sum_j q_i H_{ij} q_j \quad .$$

Now $$H_{ii} = -\frac{1}{\delta_i} \quad , \quad H_{ij}\Big|_{i \neq j} = 0 \quad , \quad j = 1,\ldots,p \; , \quad i = 1,\ldots n_o$$

and

$$H_{ii} = \frac{1}{\lambda_k} - \frac{1}{\delta_i} \; ; \; H_{ij}\Big|_{i \neq j} = \frac{1}{\lambda_k} \text{ if } j \in J[k] \; ; \; H_{ij} = 0 \text{ if } j \notin J[k]$$

$$j = 1,..,p \quad : \quad i \in J[k], \; k = 1,..,m.$$

Therefore $Q(\underline{q})$ reduces to

$$Q(\underline{q}) = -\sum_{i=1}^{n_0} \frac{q_i^2}{\delta_i} + \sum_{k=1}^{m} \left[ \frac{(\lambda(\underline{q}))^2}{\lambda_k(\underline{\delta})} - \sum_{i\varepsilon J[k]} \frac{q_i^2}{\delta_i} \right] .$$

The Cauchy inequality [28] states that

$$\left( \sum_{r=1}^{n} x_r y_r \right)^2 \leq \sum_{r=1}^{n} (x_r)^2 . \sum_{r=1}^{n} (y_r)^2 .$$

Therefore

$$\left[ \sum_{i\varepsilon J[k]} q_i \right]^2 = \left[ \sum_{i\varepsilon J[k]} \frac{q_i}{\delta_i^{\frac{1}{2}}} \delta_i^{\frac{1}{2}} \right]^2 \leq \left( \sum_{i\varepsilon J[k]} \delta_i \right) \left( \sum_{i\varepsilon J[k]} \frac{q_i^2}{\delta_i} \right)$$

which means that

$$\frac{(\lambda(\underline{q}))^2}{\lambda_k(\underline{\delta})} - \sum_{i\varepsilon J[k]} \frac{q_i^2}{\delta_i} \leq 0 .$$

Therefore $Q(\underline{q}) \leq 0$ .

The Hessian matrix $\underline{H}$ is therefore negative semi-definite proving that $\ln v(\underline{\delta})$ is concave.

## 5.7 Summary

When the merit function and constraints are all convex the design space for the given design problem will be similar to that illustrated in Fig. 6 , and the merit function will only have one local minimum in the feasible region. This is essential for the subsequent procedures based on geometric programming as (47) is then sufficient to define the optimum and the existence of the dual problem and saddle point is guaranteed.

The difficulty of generating feasible solutions to the dual problem has already been discussed in connection with the linear programming primal and dual problems. This investigation will be continued in the next chapter. It should perhaps be noted here that duals can be defined for unconstrained problems through the Legendre Transformation, and a dual will exist in the geometric programming formulation for an unconstrained problem [35].

CHAPTER 6

# THE DEVELOPMENT OF BOUND GENERATION PROCEDURES AND REDESIGN STRATEGIES

## 6.1 Introduction

The generation of corresponding feasible solutions to both primal and dual problems will now be considered in detail. A particular geometric programming dual formulation which allows negative Lagrange multipliers to be removed from an infeasible set of dual variables will be defined. Certain pivotal relationships in the transfer between primal and dual problems will also be identified. These pivotal relationships present the opportunity to define a redesign procedure based on an active set strategy.

## 6.2 The primal and dual problems

Consider the following design problem including only inequality constraints. Find values for the $n$ variables $\underline{X}$ which

$$\text{minimize} \quad f(\underline{X}) \tag{64a}$$

$$\text{subject to} \quad g_i(\underline{X}) \leq 1 \quad , \quad i = 1,\ldots,\ell \quad . \tag{64b}$$

The corresponding dual problem, defined in terms of the Lagrangean function, is to find the variables $\underline{\lambda}$ which

$$\text{maximize} \quad L(\underline{X},\underline{\lambda}) = f(\underline{X}) + \sum_{i=1}^{\ell} \lambda_i \,(g_i(\underline{X}) - 1) \tag{65a}$$

$$\text{subject to} \quad \underline{\nabla f}(\underline{X}) + \sum_{i=1}^{\ell} \lambda_i \,\underline{\nabla g_i}(\underline{X}) = \underline{0} \tag{65b}$$

$$\text{and} \quad \underline{\lambda} \geq \underline{0} \quad . \tag{65c}$$

If a feasible design $\underline{x}^*$ is optimal then the Kuhn-Tucker conditions require that a set of multipliers $\underline{\lambda}$ should exist such that

$$\underline{\nabla f}(\underline{x}^*) + \sum_{i=1}^{\ell} \lambda_i \, \underline{\nabla g_i}(\underline{x}^*) = \underline{0} \tag{66}$$

with $\underline{\lambda} \geq \underline{0}$ , (67)

and $\lambda_i \left[ g_i(\underline{x}^*) - 1 \right] = 0$ . (68)

The analogy between the Kuhn-Tucker conditions and the constraints for the dual problem has already been pointed out. Effectively Equation (68) is relaxed for feasible dual points and only finally satisfied by the search for the maximum of the dual problem.

If $\ell = n$ then Equation (65b) can be rewritten

$$\underline{G}\,\underline{\lambda} = -\underline{\nabla f} \quad . \tag{69}$$

with $\underline{G}$ a square matrix. This set of equations can now be solved for those $\underline{\lambda}$ corresponding to values of $\underline{G}$ and $\underline{\nabla f}$ for a given design $\underline{x}'$ . If a sequence of improving designs is defined for the primal problem then the solution to (69) must converge to satisfying the Kuhn-Tucker conditions, and in particular (68). It would appear therefore that the sequence of Lagrange multipliers $\underline{\lambda}$ thus generated could be used to gain some indication of whether constraints will be active at the optimum as this design is approached, with those multipliers corresponding to non-active constraints converging to zero.

It should also be noted that if the set of multipliers $\underline{\lambda}$ satisfying (69) also satisfy the non-negativity condition (67) then a feasible solution to the dual problem will be obtained. The Lagrangean saddle-point (14) which defines the optimum design will then have been spanned from a feasible design in the primal problem to a feasible solution point for the corresponding dual problem. Equation (14) can be rewritten as

$$L(\underline{\hat{x}},\underline{\lambda}) \leq L(\underline{x}^*,\underline{\lambda}^*) = f(\underline{x}^*) \leq f(\underline{x}')$$

where $(\hat{\underline{X}},\underline{\lambda})$ are feasible solutions to the dual problem defined by (65), and $\underline{X}'$ satisfies (64b). Entry into the dual problem will therefore not only offer information about the constraints but will also provide a lower bound on the reduction to the merit function which can be achieved by further redesign. The appearance of negative Lagrange multipliers associated with the inequality constraints will however lead to an infeasible dual point and prevent definition of the bound.

Equality constraints were not included in the primal problem considered in this section. However these constraints can be included directly to enable bounds to be formed. It should be noted that no restriction is placed on the sign of the Lagrange multipliers for equality constraints and the constraints must be automatically considered active.

## 6.3 Generating feasible dual solution points

The basic ideas suggested in the previous section will now be investigated in more detail using the geometric programming primal and dual problems derived in Chapter 5. The following convex primal problem, in which the merit function is a linear posynomial and the constraints are single term posynomial functions, will be used.

$$\text{Minimize} \quad f(\underline{x}) = \sum_{i=1}^{n} c_i x_i \quad , \quad c_i > 0 \quad , \quad x_i > 0 \qquad \ldots (70)$$

subject to the constraints

$$g_j(\underline{x}) = K_j \prod_{k=1}^{n} x_k^{a_{jk}} \leq 1 \quad , \quad j = 1,\ldots\ldots,n \qquad (71)$$

in which the number of constraints is again equal to the number of design variables. The dual geometric programming problem (63) for this problem can be written

$$\text{maximize} \quad \ln v(\underline{\delta},\underline{\Gamma}) \qquad (72)$$

$$\text{subject to} \quad \underline{A}^t\underline{\Gamma} = -\underline{\delta} \quad , \qquad (73)$$

$$\sum_{i=1}^{n} \delta_i = 1 \quad , \tag{74}$$

$$\underline{\delta}, \underline{\Gamma} \geq \underline{0} \quad , \tag{75}$$

where $$v(\underline{\delta}, \underline{\Gamma}) = \prod_{i=1}^{n} \left(\frac{c_i}{\delta_i}\right)^{\delta_i} \prod_{j=1}^{n} (K_j)^{\Gamma_i} \tag{76}$$

and where $\underline{A}$ is the matrix of coefficients $a_{jk}$ in the constraints (71). The dual variables have here been redefined from (63) as $(\underline{\delta}, \underline{\Gamma})$ with

$$\delta_i = \delta_i \quad , \quad i = 1, \ldots, n$$

and $$\Gamma_j = \delta_{j+n} \quad , \quad j = 1, \ldots, n \; .$$

An attempt can be made to define a feasible solution to the dual problem for a given set of design variables $\underline{X}'$ by defining from (61.b)

$$\delta_i = \frac{c_i x_i'}{f(\underline{x}')} \quad , \quad i = 1, \ldots, n \tag{77}$$

and solving (73) for the dual variables $\underline{\Gamma}$. The $\delta_i$ defined by (77) will automatically satisfy the normality condition (74) so that if the dual variables $\underline{\Gamma}$ are positive, a feasible solution to the dual problem will have been found. If some of the $\Gamma_i$ are negative they can be arbitrarily set to zero and (73) used to redefine $\underline{\delta}$. Scaling of the complete dual variable set will again satisfy the normality condition (74) and if the $\underline{\delta}$ set is now positive a feasible dual solution point will have been found.

Obviously this procedure can fail with some of either the variables $\underline{\delta}$ or $\underline{\Gamma}$ remaining negative. However, for the optimum design the dual constraints (61b) must be satisfied. Therefore

$$c_i x_i^* = \delta_i^* \, v(\underline{\delta}^*, \underline{\Gamma}^*) \quad \text{with} \quad v(\underline{\delta}^*, \underline{\Gamma}^*) = f(\underline{x}^*) \tag{78}$$

and $\underline{\delta}$ defined by (77) will be optimal. In this solution to the dual problem the positivity conditions (75) will automatically be satisfied so that failure to remove the negative dual variables by the procedures proposed above would indicate that the design $\underline{X}'$ is far from optimal.

An interesting analogy can now be drawn between this primal-dual formulation and the standard Lagrangean dual problem given in Section 6.2. Consider the case when the single term posynomials (71) are approximations to the real constraints. A suitable method for generating these approximations is given in [28] and described here in Appendix A. The approximations are given by

$$g_i(\underline{X},\underline{X}') = g_i(\underline{X}') \prod_{j=1}^{n} \left(\frac{x_j}{x_j'}\right)^{a_{ij}} \quad , \quad a_{ij} = \left[\frac{x_j}{g_i}\frac{\partial g_i}{\partial x_j}\right]_{\underline{X}=\underline{X}'} \quad ,$$

$$j = 1,\ldots,n \; , \qquad \ldots (79)$$

where the operating point $\underline{X}'$ is the point about which the approximation is required. This approximation can be written

$$g_i(\underline{X}) = c_i \prod_{j=1}^{n} x_j^{a_{ij}} \quad , \quad c_i > 0 \quad ,$$

and takes the form required by the primal problem in (71). The matrix $\underline{A}$ in (73) now contains terms

$$a_{ij} = \left[\frac{x_j}{g_i}\frac{\partial g_i}{\partial x_j}\right]_{\underline{X}=\underline{X}'} \; .$$

With the dual variables $\underline{\delta}$ defined by (77), the Equation (73) can be written

$$\sum_{i=1}^{n} \left[\frac{x_j}{g_i}\frac{\partial g_i}{\partial x_j}\right]_{\underline{X}=\underline{X}'} \Gamma_i = -\frac{c_j x_j'}{f(\underline{X}')} \quad , \quad j = 1,\ldots,n \; .$$

Therefore $$\sum_{i=1}^{n} \left[\frac{\partial g_i}{\partial x_j}\right]_{\underline{X}=\underline{X}'} \frac{\Gamma_i}{g_i(\underline{X}')} = -\frac{c_j}{f(\underline{X}')} \quad , \quad j = 1,\ldots,n \quad ,$$

or $$\sum_{i=1}^{n} \left[\frac{\partial g_i}{\partial x_j}\right]_{\underline{X}=\underline{X}'} \frac{f(\underline{X}')}{g_i(\underline{X}')} \Gamma_i = -c_j \quad , \quad j = 1,\ldots,n \; .$$

$$\ldots (80)$$

Comparing (80) with (69) indicates that

$$\lambda_i = \frac{f(\underline{X}')}{g_i(\underline{X}')} \Gamma_i \qquad (81)$$

where $\lambda_i$ are the Lagrange multipliers for the classical Lagrangean saddle-function corresponding to the primal problem. Effectively the same equation set is therefore being solved when (73) is solved for $\underline{\Gamma}$ or (69) is solved for the Lagrange multipliers $\underline{\lambda}$.

The bounds defined by the general Lagrangean dual and the geometric programming formulation can also be compared if a positive set of Lagrange multipliers is obtained. The merit function for the Lagrangean dual is given by

$$L(\underline{X}',\underline{\lambda}) = f(\underline{X}') + \sum_{i=1}^{n} \lambda_i \,(g_i(\underline{X}') - 1) \,. \qquad (82)$$

For the geometric programming dual the merit function (76) can be written as

$$v(\underline{\delta},\underline{\Gamma}) = \prod_{i=1}^{n} \left(\frac{c_i x_i'}{\delta_i}\right)^{\delta_i} \prod_{i=1}^{n} \left[ K_i \prod_{j=1}^{n} x_j'^{\,a_{ij}} \right]^{\Gamma_i} . \qquad \ldots (83)$$

If the $\underline{\delta}$ set is defined by (77) then the first product of $n$ terms in (83) will be equal to the current value of the merit function so that this equation can be rewritten

$$v(\underline{\delta},\underline{\Gamma}) = f(\underline{X}') \,.\, \prod_{i=1}^{n} \left[ g_i(\underline{X}') \right]^{\Gamma_i} .$$

For regions near the optimum design $g_i(\underline{X}') \doteqdot 1$ for those $\Gamma_i$ not equal to zero. By the binomial expansion

$$(1 + \varepsilon)^a \doteqdot 1 + a\varepsilon \quad \text{for } \varepsilon \text{ small.}$$

Therefore we can write $\left[ g_i(X') \right]^{\Gamma_i} \doteqdot 1 + \Gamma_i \,(g_i(\underline{X}') - 1)$ .

Hence

$$v(\underline{\delta},\underline{\Gamma}) \doteqdot f(\underline{X}') \,\circ\, \prod_{i=1}^{n} \left[ 1 + \Gamma_i \,(g_i(\underline{X}') - 1) \right]$$

$$\doteqdot f(\underline{X}') + \sum_{i=1}^{n} f(\underline{X}') \,\Gamma_i \left[ g_i(\underline{X}') - 1 \right] .$$

Substituting for $\underline{\Gamma}$ from (81) now gives

$$v(\underline{\delta},\underline{\Gamma}) \doteqdot f(\underline{X}') + \sum_{i=1}^{n} \lambda_i \, g_i(\underline{X}') \left[ g_i(\underline{X}') - 1 \right] \quad . \quad (84)$$

The dual functions (82) and (84) can now be compared and it can be seen that near the optimum the dual function $v(\underline{\delta},\underline{\Gamma})$ corresponding to feasible designs with $g_i(\underline{X}) \leq 1$ will in general be greater than $L(\underline{X},\underline{\lambda})$. In fact, for the examples considered in Chapters 7 and 9, values of the bounds given by (76) are near optimal and this dual function is particularly flat near the optimum.

It therefore follows that if a convex approximation to the design problem in the form of (70) and (71) can be formed then a corresponding feasible solution to the dual geometric programming problem can be found even when some of the Lagrange multipliers are negative. The generation of linear posynomials for the merit function is straight forward since a linear function in the required form can be found by matching the current value and first derivatives. Any constant can be absorbed into $f(\underline{X})$ and positive coefficients ensured by using the variable transformation

$$x_i' = k - x_i$$

with $k$ a suitably defined constant, to remove any negative signs.

The effect on the bound of this approximation and the formation of the approximations (79) to the constraints must also be discussed. A solution to the dual problem defined by (65) can obviously be obtained if the current value and first derivatives of the merit function and constraints are known. If a positive set of Lagrange multipliers is defined by the first solution of (73) for the dual variables $\underline{\Gamma}$ in the geometric programming formulation, the dual solution point will therefore provide a bound on the optimum value of the merit function for the true design problem. This follows because the current value and first derivatives are correctly matched by the approximations in the form of (70) and (71). In other words the dual defined by (61) would be the same irrespective of whether the primal merit function and constraint functions in (60) are exact or only match the current values and first derivatives. However, if negative $\Gamma_j$ appear in the initial solution and are set to zero, the dual variables are updated to a different solution point in the dual corresponding to the approximating primal problem. Since the minimum of this approximating problem may be higher than the true optimum, there is no longer any guarantee that the new feasible solution to the dual will provide a lower bound on the true optimum value of the merit function. A bound on the true optimum is therefore only obtained when the initial solution of (73) leads to a positive set of Lagrange multipliers.

The main advantage of using the geometric formulation is the fact that it is possible to generate a dual feasible point even when negative Lagrange multipliers have been encountered. A similar flexibility could have been built into the dual defined by (65) if the merit function was non-linear. For example, if

$$f(\underline{X}) = \sum_{i=1}^{n} \frac{c_i}{x_i} ,$$

then the vector $\underline{\nabla f}$ consists of terms of the form $-\frac{c_i}{x_i^2}$ . An adjustment to the right-hand side of (69), corresponding to setting a negative Lagrange multiplier to zero, could now be considered to cause an adjustment to the design variables $\underline{X}$ . If a non-negative set of variables $\underline{X}$ and Lagrange multipliers $\underline{\lambda}$ were thus obtained a feasible solution to a dual problem based on linear approximations to the constraints (since $\underline{G}$ in (69) remained constant) would have been found. The fact that the Lagrangean dual is inflexible if the merit function has the linear form of (70) agrees with the linear programming results found in Section 5.4.

## 6.4 The development of redesign strategies

The bound on the minimum value of the cost function will provide a valuable supplement to a redesign procedure because it will allow the merit of a given design to be assessed and therefore provide a criterion for terminating the redesign process. However, gradients had to be found to generate the bound and an attempt should be made to propose redesign strategies which take advantage of these gradients and the information about constraint activity levels provided by the Lagrange multipliers.

To make the dual procedures more generally applicable consider the case when the primal problem contains a large number of constraints. The convex approximation to the primal problem given by (70) and (71) now contains $m$ single term posynomial constraints, with $m \geq n$, and each constraint generated using (79). The $\underline{A}^t$ matrix can now be partitioned into $\underline{A}_1^t$ and $\underline{A}_2^t$ giving (73) as

$$\left[ \underline{A}_1^t \; \vdots \; \underline{A}_2^t \right] \begin{bmatrix} \underline{\Gamma}_1 \\ \cdots \\ \underline{\Gamma}_2 \end{bmatrix} = - \left[ \underline{\delta} \right] , \qquad (85)$$

where $\underline{A}_1^t$ is an $n \times n$ matrix. It should be noted that $n$ of the $m$ constraints are sufficient though not always necessary, to define the optimum point in the $n$-dimensional design space. Since each column of $\underline{A}^t$ contains the coefficients $a_{ij}$ for a particular constraint, the partitioning of $\underline{A}^t$ can be done by including the $n$ active, or nearest to active constraints in $\underline{A}_1^t$. This choice will ensure that if the current design is optimal then the correct active constraint set will be included in $\underline{A}_1^t$.

Given a design $\underline{x}'$ for the primal problem the procedures for defining a dual feasible point can now follow those given by (77) and the solution of (73), but now setting $\underline{\Gamma}_2 = \underline{0}$ and solving the subset of equations

$$\underline{A}_1^t \, \underline{\Gamma}_1 = -\underline{\delta} \qquad (86)$$

for the dual variables $\underline{\Gamma}_1$. The procedures for removing negative $\Gamma_j$ follow those defined in Section 6.3.

The selection of the active constraint set included in $\underline{A}_1^t$ can be checked by recognising that $\ln v(\underline{\delta},\underline{\Gamma})$ is concave and the dual constraints are linear equality constraints. Gradients of $\ln v(\underline{\delta},\underline{\Gamma})$ w.r.t. $\Gamma_k$ should therefore vary monotonically from their current value to zero if $\Gamma_k$ remains $>0$ along a line joining the current dual solution point to the optimum point. Therefore if those variables $\Gamma_j$ which have been set to zero in (85) are included in the independent set and gradients of the form $\frac{\partial \ln v}{\partial \Gamma_j}$ evaluated, then for any gradient which is positive the corresponding

$\Gamma_j$ would take positive values in a search for the optimum of the dual problem. The corresponding constraint would therefore be active at the optimum for the approximating problem defined by (70) and (71) and should have been included in the active set in $\underline{A}_1^t$ .

It should be noted that the dual problem contains $n+m$ variables $\underline{\delta}$ and $\underline{\Gamma}$ and $n+1$ linear equality constraints. There are therefore a total of $m-1$ independent variables which can best be selected from the $m$ variables $\underline{\Gamma}$ . The variable $\Gamma_k$ not included in this set could be arbitrarily selected to be the one with the largest positive value. Gradients can then be obtained in the simplest way by a finite difference scheme incrementing the appropriate $\Gamma_j$ , adjusting $\Gamma_k$ to satisfy (74) and redefining $\underline{\delta}$ by matrix multiplication in (85). The dual function $\ln v(\underline{\delta},\underline{\Gamma})$ can then be re-evaluated.

When the given design in the original primal problem defined by (64) does not lie at the intercept of $n$ constraints, the nearest to active set is used to construct the matrix $\underline{A}_1^t$ . If a positive set of Lagrange multipliers is now obtained from the solution of (73), then the optimum for the problem would be correctly defined by the intercept of these constraints so long as the gradients of the merit function and constraints do not change significantly between the current design and the optimum. The gradients of the merit function and constraints contribute to the equation sets (69) or (73) used to find the Lagrange multipliers. Therefore when significant changes can occur to the gradients, pseudo-limits for the constraints should be set to some suitable value between the true limit and the current value if the intercept of the constraint set is to define an improved design. This intercept can now be found by noting that the active single term posynomial constraints in (71) have the form

$$g'_i(\underline{X}) = K'_i \prod_{k=1}^{n} x_k^{a_{jk}} = 1$$

where the constants $K'_i$ include the effect of defining the pseudo-limits. Taking logarithms of both sides of these equations leads to the set of linear equations

$$\underline{A}_1 \ln \underline{X} = -\ln \underline{K}' \quad , \tag{87}$$

which can be solved for the design variables $\underline{X}$ at the intercept of these constraints in design space.

The point to be emphasised is that any interpretation placed on the Lagrange multipliers with respect to constraint activity levels is only locally accurate. However, returning to the concave dual problem defined by (72) to (76) the following considerations also apply. If $\Gamma_j$ is zero or positive and the gradient $\frac{\partial \ln v}{\partial \Gamma_j}$ is positive with $v(\underline{\delta},\underline{\Gamma})$ including the real constraint limiting values, then the constraint will also be active at the optimum of the approximating dual problem defined by (70) and (71). If the approximations were reasonably accurate this should apply with respect to constraint B in Fig. 10(a). Alternatively if $\Gamma_j$ is zero in the feasible dual solution and $\frac{\partial \ln v}{\partial \Gamma_j}$ is negative, then the constraint will not be active at the optimum of the approximating dual problem - a situation in which this would occur being depicted in Fig. 10(b). The third alternative is for $\Gamma_j$ to be positive but for $\frac{\partial \ln v}{\partial \Gamma_j}$ to be negative. It is now not conclusive whether the corresponding constraint would be active at the optimum of the approximate problem. This would depend on whether the gradient would go to zero before $\Gamma_j$ had been driven to zero in the search for the maximum of the dual problem. A situation in which this might occur is depicted on Fig. 10(c) but it should be noted that restricting the range over which the information is considered correct should lead to the definition of a pseudo-constraint similar to B'.

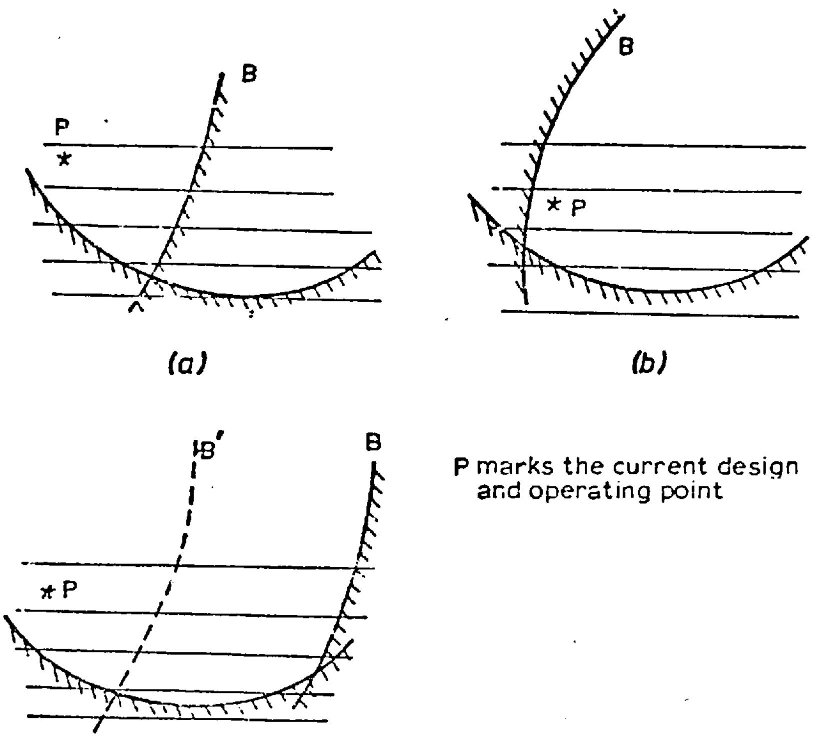



Figure 10 Constraint intercept configurations in design space

It is obvious from these considerations that the greatest difficulty in these procedures is going to arise when the Lagrange multipliers are small, since large positive values for these multipliers indicate that a constraint will be definitely binding, if the limiting value lies in the near region of design space where the gradients $\underline{\nabla f}$ and $\underline{\nabla g}$ cannot vary significantly. However, the cone formed by the multipliers in the Kuhn-Tucker conditions derived in Section 2.3 indicates that the absolute values of the Lagrange multipliers reflect the degree of influence of the corresponding constraints in constraining the design. For the simple two-dimensional situation shown in Fig. 11(a) the Lagrange multipliers must have nearly equal values to satisfy the condition

$$- \underline{\nabla f} = \underline{\nabla G} \; \underline{\lambda}$$

which is represented by the vector addition shown in the diagram. In Fig. 11(b) however constraint 2 will be instrumental in blocking the progress of a search in the direction $- \underline{\nabla f}$, while the influence of constraint 1 will be relatively small. It is obvious from the figure that the values of $\lambda_1$ and $\lambda_2$ will be correspondingly large and small respectively. A design process utilizing information generated in the

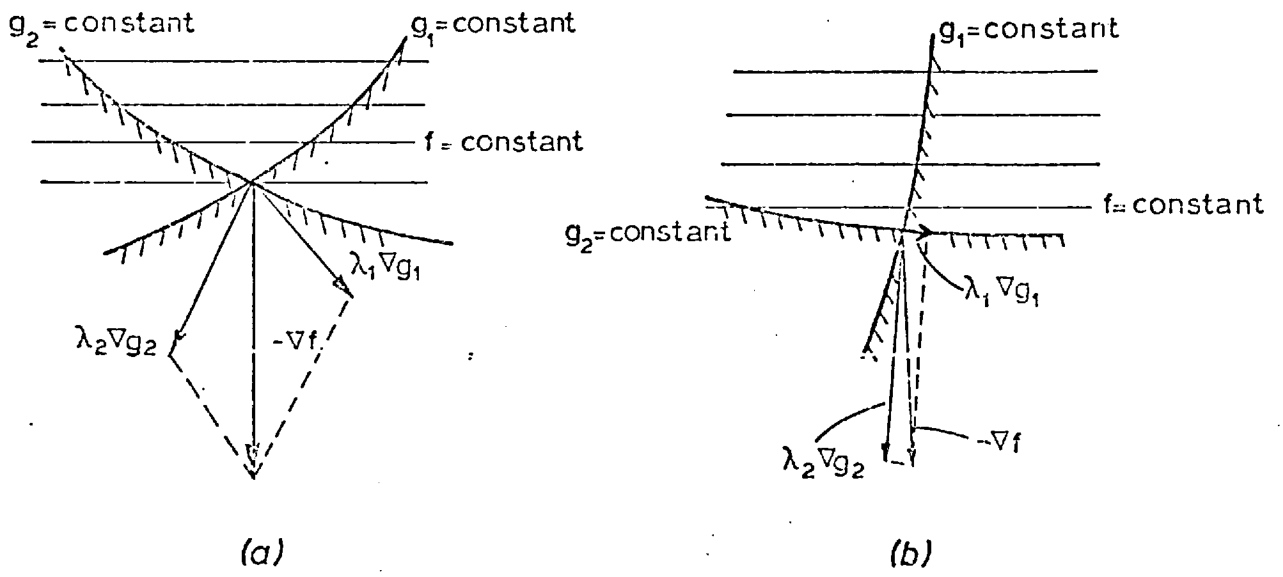



Figure 11 Vector diagrams for the Kuhn-Tucker conditions at constraint intercepts in design space

dual about active constraint sets should therefore be relatively insensitive to inaccuracy in the prediction of whether constraint 1 in Fig. 11(b) is active or not. This concept is reinforced by (35b), that is,

$$\left.\frac{\partial f}{\partial b_i}\right|_{\underline{X}^*,\underline{\lambda}^*} = -\lambda_i^* \quad \text{if} \quad g_i(\underline{X}) \le b_i \quad .$$

This relation was originally derived for the sensitivity studies but serves here to show that, if the Lagrange multiplier is small, the optimum will be insensitive to errors in the limiting value of the corresponding constraint.

If a positive set of Lagrange multipliers is obtained by solution of (86) and any negative gradients of the dual function are not large enough to drive the corresponding dual variable to zero in a limited search, then (87) can be solved for the new design with relatively large changes allowed in design space. However, if negative Lagrange multipliers were found in the initial solution of (86) and were subsequently set to zero, this simple active set redesign procedure could not be used directly. If the gradient of the dual function for the variable $\Gamma_k$ corresponding to this Lagrange multiplier were positive then the constraint could be considered active and a limited step taken. However, if this gradient is negative the constraint is not active.

A situation in which this might occur is shown in Fig. 12 if the current operating point were the point C . At the optimum however

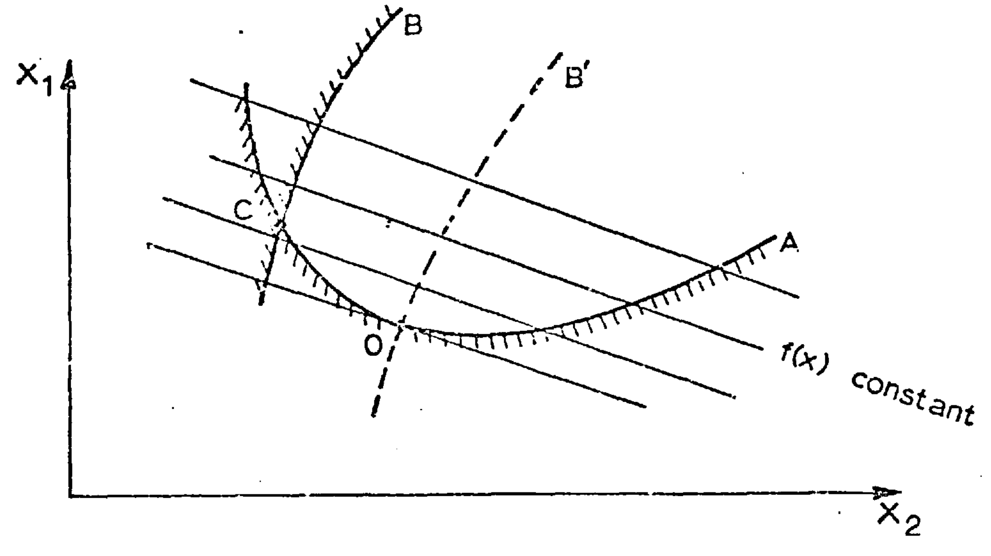



Figure 12 A pseudo-constraint which defines the optimum

$$\frac{\partial \ln v(\underline{\delta},\underline{\Gamma})}{\partial \Gamma_j} = 0 \tag{88}$$

for those independent $\Gamma_j$ not set to zero. It is therefore possible to update the limiting values for the constraints predicted to be non-active to define the pseudo-constraint B' in Fig. 12. Updating the limiting value till (88) is satisfied will provide a prediction of the value of the constraint function $g_j(\underline{X})$ at the optimum, and hence the limiting value required for this constraint to be active at that point. It should be noted that setting $\Gamma_j$ equal to zero removes the constraint $g_j(\underline{X})$ from the dual function (76). A small positive value brings the constraint into the formulation so that satisfaction of (88) with $\Gamma_j$ equal to zero indicates that the optimum value of the merit function does not change if the limiting value of the constraint $g_j(\underline{X})$ is modified slightly from the value which makes it active at the optimum. This interpretation follows geometrically for constraint B' in Fig. 12 but is dependent on the design space being convex.

The pseudo-limits defined using (88) will only be accurate when the maximum of the dual problem has been found. Some search in the dual will therefore be necessary if pseudo-constraints have to be defined and some of the gradients are large. The search however can be limited to removing the large gradients because the optimum has already been shown to be insensitive to errors in the constraint limiting value if the corresponding Lagrange multiplier is small. The range of accuracy of the active set prediction will also invariably restrict the pseudo-limit update. The most effective dual search would then probably be a steepest ascent strategy with

$$\Gamma_j^{\nu+1} = \Gamma_j^{\nu} + \alpha \underline{\nabla v}$$

for $\Gamma_j$ in the independent set, $\nu$ defining the iteration number, $\alpha$ a step length and $\underline{\nabla v}$ the vector of gradients $\frac{\partial \ln v}{\partial \Gamma_j}$.

The active set strategy for redesign can now be extended to the case when a full set of $n$ constraints is not active at the optimum. The limiting values of the active constraints included in $\underline{A}_1^t$ are set either to the true limiting values or to reflect the range over which the predictions of activity levels are considered correct. This active set can then be augmented by the pseudo-constraints derived above in order to produce a square $n$x$n$ set of coefficients in $\underline{A}_1^t$ . Solution of (87) will then define the new design variables.

It is interesting to note the pivotal nature of the $\underline{A}_1$ matrix with entry to the dual problem being based on the solution of (85), that is,

$$\underline{A}_1^t \underline{\Gamma}_1 = -\underline{\delta} \quad . \qquad \text{(dual solution)} \qquad (89)$$

The definition of points in the primal design space now however is based on the solution of the equations (87), or

$$\underline{A}_1 \ln \underline{X} = - \ln \underline{K}. \quad \text{(primal solution)} \qquad (90)$$

Indeed if a Cholesky decomposition were used in the initial solution of (89) the operation would not have to be repeated to carry out the redesign with (90). Similar "pivotal relationships" have already been identified in linear programming in Section 5.4. They will be discussed further in the next section before an attempt is made in the following chapter to demonstrate the existence of the dual and the use of the design procedures for a number of problems. Iterative redesign procedures naturally result because the posynomial approximations to the constraints based on (79) need to be updated as the design improves.

## 6.5 Singular matrices in the pivotal relationships at the primal-dual interface

The pivotal relationships (89) and (90) can be rewritten

$$\underline{\Gamma}_1 = -\left[ \underline{A}_1{}^t \right]^{-1} \underline{\delta} \quad \text{and} \quad \ln \underline{X} = -\left[ \underline{A}_1 \right]^{-1} \ln \underline{K} \ .$$

Any point in either primal or dual design space can be defined by adjusting the constants $\underline{\delta}$ or $\underline{K}$ respectively. The columns of $\underline{A}_1{}^t$ therefore form a set of vectors which is a basis for the dual problem, while the columns of $\underline{A}_1$ form a basis for the primal design space of the variables $\underline{X}$ . A basis set of vectors must be independent and the recognition of the transposed relationships in (89) and (90) is useful if the $\underline{A}_1$ matrix is singular. The occurrence of a singular matrix would prevent the solution of these equations and stop entry into the dual problem from the primal, or entry into the primal from the dual.

The cause of the singularity can be investigated by considering the following set of equations,

$$x_1 + x_2 + 2x_3 = 9$$
$$4x_1 - 2x_2 + x_3 = 4$$
$$5x_1 - x_2 + 3x_3 = 1 \quad .$$

Forming an augmented matrix and performing a Gauss reduction gives

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 4 & -2 & 1 & 4 \\ 5 & -1 & 3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & \frac{7}{6} & \frac{16}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad .$$

The third row of the final matrix indicates that

$$0x_1 + 0x_2 + 0x_3 = 1 \quad ,$$

so that the equation set is clearly inconsistent. A dependence must exist between the columns of the coefficient matrix of the form

$$c_3 = bc_1 + dc_2 \quad ,$$

where $c_i$ denotes the ith column of the matrix. From the final matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{7}{6} \end{bmatrix}$$

giving $d = \frac{7}{6}$ , $b = \frac{5}{6}$ ,

and $$c_3 = \frac{5}{6}c_1 + \frac{7}{6}c_2 \quad . \qquad (91)$$

A different form of dependence is exhibited by the equation set

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 6 \\ 2x_1 + x_2 + 2x_3 &= 3 \\ 8x_1 + 9x_2 + 4x_3 &= 21 \end{aligned} \quad .$$

Again forming an augmented matrix and performing a Gauss reduction gives

$$\begin{bmatrix} 1 & 3 & -1 & 6 \\ 2 & 1 & 2 & 3 \\ 8 & 9 & 4 & 21 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & -1 & 6 \\ 0 & 1 & -\frac{4}{5} & \frac{9}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad .$$

The third row of the final matrix now correctly indicates that

$$0x_1 + 0x_2 + 0x_3 = 0$$

and the equation set is consistent. However from the final matrix

$$x_1 + 3x_2 - x_3 = 6 \quad \text{and} \quad x_2 - \frac{4}{5}x_3 = \frac{9}{5} \quad .$$

Substituting for $x_2$ from the second equation into the first gives

$$x_1 = \frac{3}{5} - \frac{7}{5}x_3$$

and

$$x_2 = \frac{9}{5} + \frac{4}{5}x_3 \quad . \qquad \ldots (92)$$

Clearly $x_3$ can be arbitrarily set. The choice of the variable to be set is not unique and the existence of the dependence between the equations indicates only that there is a degree of freedom in the equation set.

The coefficient matrices in (89) and (90) are the same except that a transpose operation has been performed. When attempting to define the dual problem the main difficulty arises in selecting the constraint set to form the basis. If the matrix $\underline{A_1}^t$ is singular it will therefore invariably be caused by a column dependence. This could be removed by replacing one of the constraints with another, effectively changing one of the columns of $\underline{A}_1^t$. The constraint interchange may, however, not be easy to select and it may be simpler to recognise that a row dependence will exist in (90) if the $K_i$ correspond to the current operating point. One of the design variables $\underline{X}$ can then be arbitrarily set for the given design cycle, reducing the dimension of $\underline{A}_1$ by one and modifying the coefficients $K_i$ in the dual cost function (76). The singularity will be removed and a dual point will then be obtained corresponding to the primal design problem with the additional constraint that the selected design variable does not change or takes a particular value.

Finally since each column of $\underline{A}_1^t$ corresponds to the exponents in the single term posynomial for a given constraint, the physical interpretation of a dependence between the columns of this matrix is not difficult. There would however appear to be no similar reasoning which could anticipate a linear dependence between the equations of (89) or the columns of (90) and arbitrarily setting a Lagrange multiplier would not alter the equation set in (90).

## 6.6 Least-squares solutions to define dual points

In general less than $n$ constraints (where $n$ is the number of design variables) are active at the optimum and pseudo-constraints have to be defined in the redesign process. Difficulties can however arise in the definition of the initial dual solution point and bound. The number of equations in (65b) or (73) is equal to the number of design variables. Therefore these equations cannot be solved directly for the dual variables unless it is possible to select a full set of $n$ constraints which are either active at the optimum, or are near-active and can be updated to form pseudo-constraints.

For the initial dual solution point the matrix $\underline{A}_1$ could be constructed only from those constraints which are currently active or expected to become active at the optimum. A least squares fit to (86) could then be found in the same way as for the solution of (25) in Chapter 4. That is, taking

$$\underline{A_1}^t \, \underline{\Gamma} = - \underline{\delta} \tag{93}$$

premultiply by $\underline{A_1}$ :

$$\underline{A_1} \, \underline{A_1}^t \, \underline{\Gamma} = - \underline{A_1} \, \underline{\delta} \quad ,$$

giving

$$\underline{\Gamma} = - \left[ \underline{A_1} \, \underline{A_1}^t \right]^{-1} \underline{A_1} \, \underline{\delta} \quad . \tag{94}$$

The dual variables $\underline{\Gamma}$ given by (94) may not satisfy (93) identically if $\underline{A_1}$ is not a square matrix, but any small errors together with any negative $\Gamma_j$ could be removed by multiplying in (93) to reset the variables $\underline{\delta}$ . Scaling the complete dual variable set would then lead to satisfaction of the normality condition (74) and hence a feasible dual solution point.

It has however already been recognised that the removal of negative $\Gamma_j$ could lead to negative values for some of the set $\underline{\delta}$ . Negative values could also result here from the removal of the least squares error in the solution (94). This difficulty did not arise in any of the design examples considered in this thesis and procedures for countering it were not developed. However, the dual formulation proposed in [25] is based on the Lagrangean dual (65) which has no flexibility for removing negative Lagrange multipliers if the cost function is linear. A Newton updating scheme is suggested for driving out any negative multipliers and this would appear to be the most appropriate method when difficulties are encountered in the solution of (93). The similarity between the least squares solution of (94) and the projection direction in the projected gradient method has also been pointed out in Section 4.2. This method could be used to improve the design, at least until a better selection of the active constraints can be made and a dual solution point found. At the optimum the Kuhn-Tucker conditions (6) must be satisfied and the least squares solution (94) must then be exact since essentially the same set of equations is being solved. Note however that the direct correspondence between the Lagrange multipliers $\underline{\lambda}$ and the variables $\underline{\Gamma}$ expressed in (81) is lost in the least squared solution of (94).

## 6.7 Summary

It has been shown that feasible dual solution points corresponding to a given design in the primal space can be defined and used to generate bounds. The correspondence between the equations solved to find the solution to the dual and the Kuhn-Tucker conditions for optimality indicate that the bound will equal the current value of the cost function if the design is optimal. The relative rates of convergence to the optimum of the bound and an improving sequence of primal designs will be investigated in the examples. It has been suggested in Section 6.3 that the bound should remain near-optimal for non-optimal primal designs.

An active set strategy for redesign has also been suggested in an attempt to utilise fully the information available in the saddle function with the Lagrange multipliers providing information about the active constraint set. Alternative redesign schemes could be based on the projected gradient or penalty function procedures, and acceleration procedures exist for the second method if predictions of the optimum value of the cost function (here provided by the bound) are known. If only a few constraints are active however a search for the optimum of the dual problem would form a computationally efficient scheme if the single term posynomial approximations to the constraints are accurate. The number of independent dual variables is one less than the number of constraints in this case and would therefore be small. Once the maximum of the dual has been found (78) can be used to define the new design variables. The redesign procedure would then be similar to the sequence of geometric programming problems suggested in [36] with new approximations to the cost function and constraints being obtained at each new design point.

The difficulties which can be encountered in defining a bound when few constraints are active at the optimum has also been considered in Section 6.6. If the search for the optimum were to be conducted in the dual plane then standard procedures for defining the initial dual solution point [8], which can incorporate a linear programming step, could be used.

In the following chapters several examples will be presented to demonstrate the use of the dual procedures to generate bounds and the active set strategy for redesign. The use of the bound as a termination criterion in the design process will also be discussed.

CHAPTER 7

# CONSIDERATIONS FOR PRACTICAL APPLICATION AND PRELIMINARY EXAMPLES

## 7.1 Introduction

In this chapter a number of preliminary examples are presented to illustrate the existence of the dual problem, and the use of the convex-primal concave-dual geometric programming formulation in the design process. A number of points related to the practical use of this formulation are first discussed in the next section.

## 7.2 Considerations for practical application

All the examples presented in this chapter and Chapter 9 make use of the geometric programming formulation proposed in Section 6.3. This formulation is used because feasible dual solution points can be defined even when negative Lagrange multipliers are encountered. However, it has already been recognised that only $m-1$ of the $m$ dual variables $\underline{\Gamma}$ can be considered independent and this can make the dual procedures cumbersome. In the following work on stress constrained problems the dual was considered a problem in $m$ variables $\underline{\Gamma}$ with the search directions projecting into the single constraint posed by the normality condition (74) at each step. Constrained derivatives evaluated by taking a finite difference step and projecting onto the normality condition were also used in the definition of the pseudo-constraints using (88).

It is interesting to note that this projection scheme can be recovered by introducing the variable transformation

$$\delta_i = \frac{\delta_i'}{\Delta} \quad , \quad \Gamma_j = \frac{\Gamma_j'}{\Delta} \quad , \quad i = 1,\ldots,n \quad ; \quad j = 1,\ldots,m \quad . \qquad \ldots (95)$$

into the dual problem. Substitution into Equations (72) to (76) gives the dual problem as

maximize $\ln v(\underline{\delta}',\underline{\Gamma}',\Delta)$

where
$$v(\underline{\delta}',\underline{\Gamma}',\Delta) = \prod_{i=1}^{n} \left(\frac{c_i}{\delta_i'}\right)^{\frac{\delta_i'}{\Delta}} \Delta \prod_{i=1}^{m} (K_i)^{\frac{\Gamma_i'}{\Delta}} , \quad (96)$$

$$\underline{\delta}' = - \underline{A}^t\underline{\Gamma}' ,$$

$$\Delta = \sum_{i=1}^{n} \delta_i' ,$$

and
$$\underline{\delta}', \underline{\Gamma}' \geq \underline{0} .$$

Now
$$\ln v(\underline{\delta}',\underline{\Gamma}',\Delta) = \sum_{i=1}^{n} \frac{1}{\Delta} (\delta_i' \ln c_i - \delta_i' \ln \delta_i') + \ln \Delta + \sum_{i=1}^{m} \frac{\Gamma_i'}{\Delta} \ln K_i .$$

Therefore
$$\frac{\partial \ln v}{\partial \Gamma_j'} = f(\underline{\delta}', \frac{\partial \underline{\delta}'}{\partial \Gamma_j'}, \Delta, \frac{\partial \Delta}{\partial \Gamma_j'}) + \frac{1}{\Delta} \ln K_j - \frac{\Gamma_j'}{\Delta^2} \ln K_j \frac{\partial \Delta}{\partial \Gamma_j'} .$$

Pseudo-limits are only defined for those $\Gamma_j'$ equal to zero so that the last term drops out when these limits are being defined. Recognition that constraints of the form

$\sigma \leq \sigma_{lim}$ have been rewritten $\frac{\sigma}{\sigma_{lim}} \leq 1$

to recover the posynomial form required in (71), then shows that updating the limiting value leads to modification to $K_j$ . In the following examples the computer implementation of this procedure involved scaling $K_j$ a small amount, re-evaluating the derivative and extrapolating the scaling factor linearly (which is exact) to the pseudo-limit giving zero derivative.

The final practical consideration for the application of the new design procedures is that posynomial approximations will in general be required for both the merit function and the constraints. An iterative procedure was therefore required in the redesign process and the operating point for the approximations (79) was taken to be the current design at each step. An attempt can also be made to improve the accuracy of these approximations by generalising the convex primal problem to the form originally considered in (57) and (58). Use is made of this generalised form in two problems considered with non-linear merit functions. In Appendix A approximations to these functions in the form

$$f'(\underline{x}) = \sum_{i=1}^{n} c_i x_i^{d_i} + K \tag{97}$$

can be found matching first and second derivatives at the operating point. Two-term posynomial approximations defined in Appendix A were also formed for deflection constraints by matching second derivatives of the form $\frac{\partial^2 g}{\partial x^2}$ as well as first derivatives at the operating point. These constraints may define more than one design variable so the increased accuracy is an advantage. The active constraint conditions (87) are now replaced by the conditions at the optimum that

$$K_i \prod_{j=1}^{n} x_j^{a_{ij}} = \frac{\Gamma_i}{\Delta_k} \quad \text{where } i \in J[k] \text{ and } J[k] \text{ was defined for (58)}, \tag{98}$$

for the multi-term constraints.

In summary the steps in the dual procedures can be detailed as follows:

1 SETTING UP THE DUAL PROBLEM

1.1 Determine a posynomial approximation for the merit function

1.2 Select the active constraint set to form a basis for the dual

1.3 Evaluate gradients and form single term posynomials for the constraints selected in 1.2 using (79)

2 BOUND GENERATION

2.1 Set the dual variables $\underline{\delta}$ using (77)

2.2 Solve (73) for $\underline{\Gamma}$

2.3 Check for negative $\Gamma_j$ , if any are present set them to zero and use (73) to redefine $\underline{\delta}$ . Scale all the dual variables to satisfy (74). If negative $\delta_j$ now appear set them to zero and return to 2.2

2.4 Evaluate the bound

3 THE ACTIVE SET STRATEGY FOR REDESIGN

3.1 Reduce the feasible region about the operating point to an area in which the posynomials are accurate by tightening the constraints

3.2 Scan the Lagrange multipliers for zero or small values and associated negative gradients

3.3 If pseudo-limits are to be defined conduct a limited dual search to remove large gradients and define the new constraints using (88)

3.4 Solve (87) for the new design variables

The only difficulty which can arise in the application of these procedures and which has not been explored is the appearance of negative $\delta_j$ in the dual search of 3.3. These negative values should however not occur because each of the dual variables $\underline{\delta}$ should be related through (77) to a minimum size constraint on each of the variables $\underline{X}$ .

## 7.3 Preliminary examples

Three examples are presented in this section to illustrate the existence of the dual problem and the easy transfer between primal and dual feasible points which is possible using the procedures proposed in Chapter 6. Active set strategies are used to improve the designs and it is shown that the bounds converge to the optimum as the design in primal space improves.

(i) Example 1

The minimum cost design of a chemical plant

a. The design problem

The minimum value is sought for the annual operating cost of the chemical plant depicted in Fig. 13 and described in detail in [29]. The annual operating cost of the plant is given by

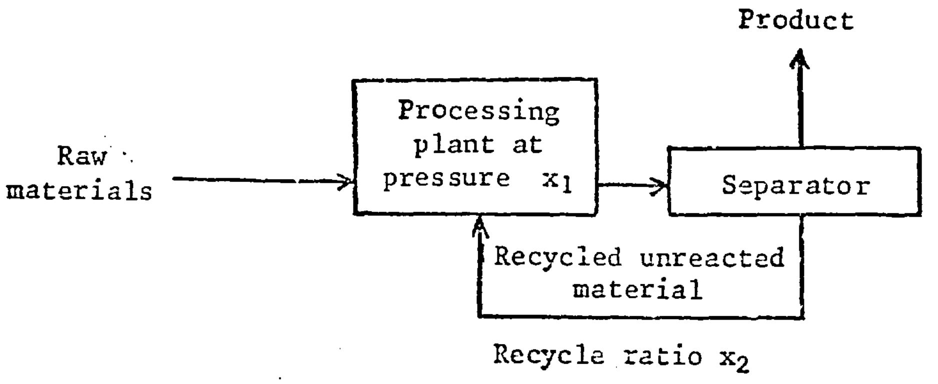



Figure 13  A hypothetical chemical plant

$$y(\underline{X}) = 1000x_1 + 4 \times 10^9 x_1^{-1} x_2^{-1} + 2.5 \times 10^5 x_2 , \quad (99)$$

and the design is constrained by safety and design codes which require that the non-negative variables $x_1$ and $x_2$ satisfy

$$\frac{x_2}{8} \le 1 \quad , \quad \frac{x_1 x_2}{9000} \ge 1 \quad , \quad \frac{x_1}{2200} \le 1 \quad . \quad (100)$$

Design space for this problem is plotted in Fig. 14.

The merit function (99) does not have the correct form required to apply the procedures proposed in Section 6.3. An approximation to the merit function based on the generalised form (97) was therefore found. A feasible design $\underline{X}'$ for the plant, which was used as the operating point for this approximation, is given by

$$x_1' = 1700 \quad , \quad x_2' = 6.5 \quad , \quad (101)$$

for which $y(\underline{X}') = 3.687 \times 10^6$. The approximation matching the current value of $y(\underline{X})$, first partial derivatives and second derivatives of the form $\frac{\partial^2 y}{\partial x_i^2}$ at this design point is given by

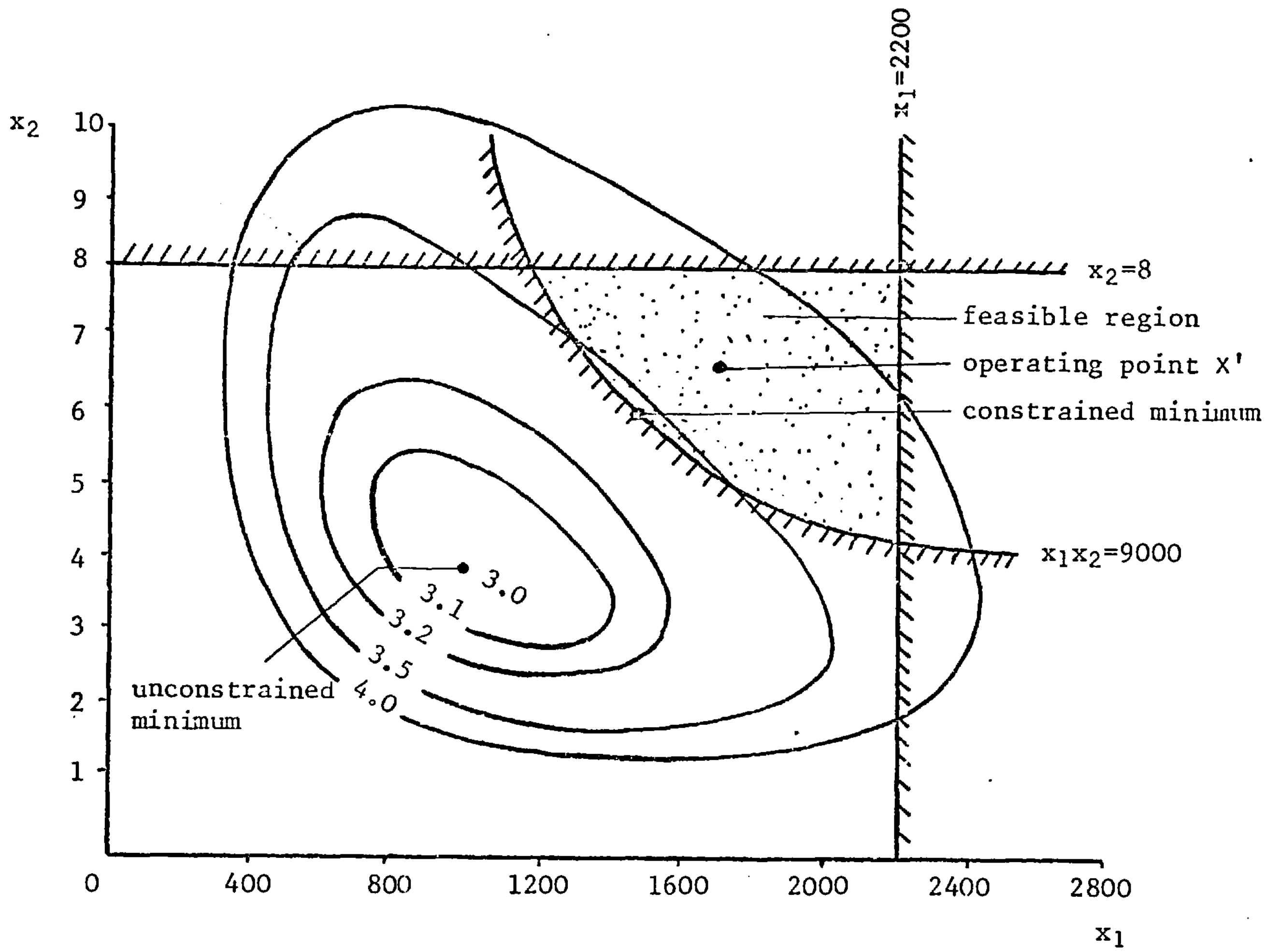



Figure 14 Design space for the chemical plant problem

$$Y(\underline{x}) - 2.0159 \times 10^6 = 9.1247\, x_1^{1.5411} + 4.2240 \times 10^4\, x_2^{1.5732} .$$

The primal merit function is now set as

$$f(\underline{x}) = Y(\underline{x}) - 2.0159 \times 10^6 \quad \text{and} \quad f(\underline{x}') = 1.6711 \times 10^6 .$$

b. Dual entry procedures

The saddle inequality (14) now gives

$$f(\underline{x}) \geq f(\underline{x}^*) = v(\underline{\delta}^*, \underline{\Gamma}^*) \geq v(\underline{\delta}, \underline{\Gamma})$$

where $v(\underline{\delta},\underline{\Gamma}) = \left(\frac{9.1247}{\delta_1}\right)^{\delta_1} \left(\frac{4.2240 \times 10^4}{\delta_2}\right)^{\delta_2} \left(\frac{1}{8}\right)^{\Gamma_1} (9000)^{\Gamma_2} \left(\frac{1}{2200}\right)^{\Gamma_3}$

. . . (102)

and the dual variables $\underline{\delta}$ and $\underline{\Gamma}$ must satisfy the constraints

$$\begin{bmatrix} \underline{A}_1 & & \vdots & \underline{A}_2 \\ 0 & -1 & \vdots & 1 \\ 1 & -1 & \vdots & 0 \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \end{bmatrix} = - \begin{bmatrix} 1.5411\ \delta_1 \\ 1.5732\ \delta_2 \end{bmatrix} \qquad (103)$$

$$\delta_1 + \delta_2 = 1 \quad , \qquad (104)$$

and $\underline{\delta}, \underline{\Gamma} \geq \underline{0}$ . (105)

The nearest to active of the constraints (100) at the operating point (101) have been included in $\underline{A}_1$ .

Applying (77) to define the dual variables $\underline{\delta}$ gives for the operating point (101),

$$\delta_1 = 0.5196 \quad , \quad \delta_2 = 0.4804 \quad .$$

Setting $\Gamma_3 = 0$ and solving (103) then gives

$$\Gamma_1 = 0.0449 \quad \text{and} \quad \Gamma_2 = 0.8007 \quad .$$

Since these dual variables satisfy the non-negativity conditions (105) a feasible dual solution point has been obtained and substitution of these values into (102) gives

$$v(\underline{\delta},\underline{\Gamma}) = 1.404 \times 10^6 \quad .$$

It is of interest to note that setting $\Gamma_1$ to zero and solving (103) for $\Gamma_2$ and $\Gamma_3$ would have given

$$\Gamma_2 = 0.75576 \quad , \quad \Gamma_3 = -0.0449 \quad .$$

The new dual solution point does not satisfy the non-negativity condition

(105) and therefore is not feasible. However, the procedures proposed for removing negative variables in the $\underline{\Gamma}$ set can be applied by setting $\Gamma_3$ to zero and performing a matrix multiplication in (103) to redefine $\delta_1$ and $\delta_2$. Scaling both $\underline{\delta}$ and $\underline{\Gamma}$ to give $\delta_1 + \delta_2 = 1$ gives

$$\delta_1 = 0.5051 \quad , \quad \delta_2 = 0.4949 \quad , \quad \Gamma_2 = 0.7785$$

and $\Gamma_1 = \Gamma_3 = 0$ .

. . . (106)

A feasible dual solution point has been recovered and substitution into (102) gives

$$v(\underline{\delta},\underline{\Gamma}) = 1.428 \times 10^6 \quad .$$

The third variation in the solution of (103) would be to set $\Gamma_2 = 0$ and solve for $\Gamma_1$ and $\Gamma_3$ . This obviously gives negative values for both $\Gamma_1$ and $\Gamma_3$ and there is no way to define a feasible dual solution point in which the second constraint and $\Gamma_2$ do not participate.

The optimum design [29] has an annual operating cost of $3.444 \times 10^6$ with $x_1^* = 1500$ and $x_2^* = 6$ . This solution corresponds to $f(\underline{X}^*) = 1.428 \times 10^6$ so that both of the dual solution points have given excellent bounds on the reduction in the annual operating cost which can be achieved by optimizing the design. The second dual solution point is optimal because with $\Gamma_3$=0 the three conditions defined by (103) and the normality condition on the variables $\underline{\delta}$ uniquely define $\delta_1$, $\delta_2$ and $\Gamma_2$ and hence define the optimum point.

## c. Recovery of Lagrange multipliers

The recovery of the Lagrange multipliers for the general Lagrangean form (65) from the dual variables $\underline{\Gamma}$ can be demonstrated for this simple problem. Equation (81) defines

$$\lambda_i = \frac{y(\underline{X}')}{g_i(\underline{X}')}\,\Gamma_i \quad \text{giving} \quad \lambda_1 = 1.6711 \times 10^6 \times \frac{8}{6.5} \times 0.0499$$

$$= 9.2 \times 10^4$$

$$\text{and} \quad \lambda_2 = 1.6711 \times 10^6 \times \frac{11050}{9000} \times 0.8007$$

$$= 1.6 \times 10^6$$

The value of the Lagrange multipliers can also be found from the dual constraints

$$\underline{\nabla f}(\underline{x}') + \underline{\nabla g}(\underline{x}')\,\underline{\lambda} = \underline{0} \tag{107}$$

for the Lagrangean function for this problem. These constraints give the equation set

$$\begin{bmatrix} 0 & -0.0004791 \\ 0.125 & -0.1253 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = - \begin{bmatrix} 787.12 \\ 194298.65 \end{bmatrix} . \tag{108}$$

Solution of (108) gives

$$\lambda_1 = 9.2 \times 10^4 \quad , \quad \lambda_2 = 1.6 \times 10^6 .$$

d. Definition of pseudo-constraints

The updating of limiting values for non-active constraints so that pseudo-constraints active at the optimum are defined can also be demonstrated. Consider the solution to the dual problem given by (106). Since $\Gamma_3$ was originally negative when (103) was solved for this constraint set, $\frac{\partial \ln v}{\partial \Gamma_3}$ gives a prediction of whether the third of the constraints (100) will be active at the optimum.

$$\ln v = \delta_1 \ln 9.1247 - \delta_1 \ln \delta_1 + \delta_2 \ln 42240 - \delta_2 \ln \delta_2$$
$$- \Gamma_1 \ln 8 + \Gamma_2 \ln 9000 - \Gamma_3 \ln x_{1_{lim}} . \qquad \ldots (109)$$

But from (103)

$$-\Gamma_2 + \Gamma_3 = -1.5411\,\delta_1$$
$$-\Gamma_2 = -1.5732\,\delta_2 \qquad \ldots (110)$$

and the normality condition $\delta_1 + \delta_2 = 1$ must also be satisfied. These three equations can now be used to eliminate $\delta_1$, $\delta_2$ and $\Gamma_2$ from (109). Differentiation and setting $x_{1_{lim}} = 2200$ and substituting for the dual variables from (106) gives

$$\frac{\partial \ln v}{\partial \Gamma_3} = -0.3801 \quad .$$

The constraint is therefore not active at the optimum and in order to define a pseudo-constraint which is active $x_{1_{lim}}$ must be decreased until $\frac{\partial \ln v}{\partial \Gamma_3} = 0$ . Setting the derivative obtained from (109) to zero gives

$$\ln x_{1_{lim}} = 7.316$$

or $$x_{1_{lim}} = 1504 \quad .$$

The active set strategy now requires simultaneous satisfaction of the constraints

$$\frac{x_1}{1504} = 1 \quad \text{and} \quad \frac{9000}{x_1 x_2} = 1$$

giving (to the accuracy of the calculations)

$$x_1 = 1504 \quad , \quad x_2 = 5.98 \quad .$$

Recall once again that the optimum solution is

$$x_1 = 1500 \quad , \quad x_2 = 6.0 \quad .$$

e. Sensitivity studies

The final feature of the saddle function which can be illustrated using this example is the well-known use of the Lagrange multipliers in sensitivity studies. For the dual solution given by (106) which is optimal, the optimal value of the Lagrange multiplier for the active constraint is given from (81) as

$$\lambda_2 = 1.428 \times 10^6 \times 1 \times 0.7785 = 1.1117 \times 10^6 \quad .$$

The constraint corresponding to this Lagrange multiplier can be written

$$(x_1 \; x_2)^{-1} \leq \frac{1}{9000}$$

A linear approximation to the change in the merit function if the limiting value of the original constraint is raised from 9000 to 9500 is given from (35b) as

$$\Delta y = -\lambda \Delta b = -1.1117 \times 10^6 \left( \frac{1}{9500} - \frac{1}{9000} \right) = 5.8508 \times 10^4 .$$

It has already been stated that the dual solution (106) is optimal so that the optimum value of the merit function for the updated constraint is given as

$$y(\underline{X}^*) = 1.428 \times 10^6 \times \left( \frac{9500}{9000} \right)^{.7785} = 1.489 \times 10^6 .$$

Therefore

$$\Delta y = 1.489 \times 10^6 - 1.428 \times 10^6 = 6.1 \times 10^4 .$$

Application of the procedure described by (35b) has therefore led to an accurate prediction of the sensitivity of the optimum design to the given change to the active constraint.

(ii) Example 2

The minimum mass design of a 3-bar truss subject to stress constraints

a. The design problem

The 3-bar truss shown in Fig. 15 was designed for minimum mass with geometry fixed. The design variables $x_i$, $i=1,.,3$ were the cross-sectional areas of the bars. Two load cases were applied and the design was subject to stress constraints on each member.

Design space for this example has been drawn on Fig. 16. Symmetry of the design and the applied loads ensures that $x_1$ and $x_3$ can be set equal so that a two-dimensional drawing is sufficient to illustrate the design space. The constraints shown in the figure by the hatched lines ( ⁄⁄⁄⁄ ) include the stress constraints for members 1 and 3 for the first and second load cases respectively, and the constraint for member 2 which is the same for both load cases. It can be shown analytically that these constraints are given by setting $\sigma_{11} = \sigma_{21} = 13790$ in

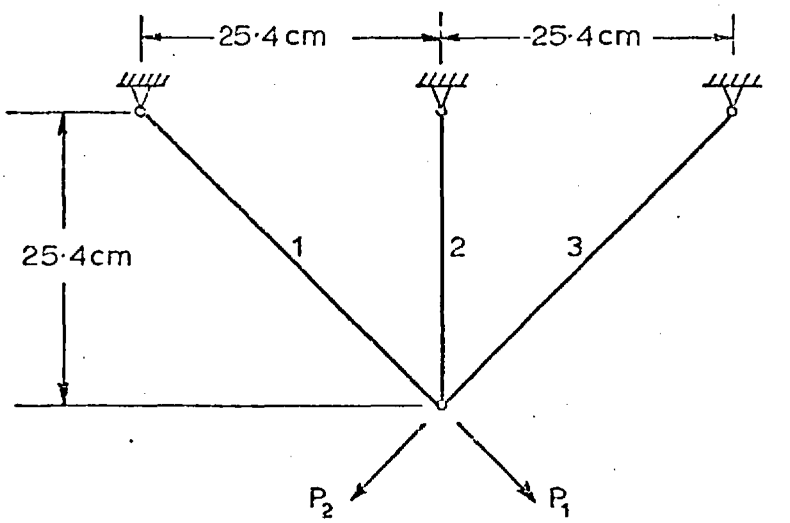
$$\sigma_{11} = \frac{88960\ (x_2 + \sqrt{2}\, x_1)}{2x_1x_2 + \sqrt{2}\, x_1^2} \quad \text{and} \quad \sigma_{21} = \frac{\sqrt{2}\ \cdot\ 88960\ x_1}{(2x_1x_2 + \sqrt{2}\, x_1^2)} \tag{111}$$

where $\sigma_{ij}$ is the stress in member $i$ for load case $j$.

In order to apply the procedures proposed in Section 6.3 to this design problem single term posynomial approximations had to be found for the stresses because the expressions given by (111) do not have the required form given by (71). These approximations were found using (79) with the operating point taken to be $x_1 = x_3 = 6.45$ and $x_2 = 1.61$, and are given by

$$\sigma_{11}' = 6.624 \times 10^4\ x_1^{-.889}\ x_2^{-.111} \quad \text{and}$$

$$\sigma_{21}' = 4.570 \times 10^4\ x_1^{-.739}\ x_2^{-.261}.$$




| | | | |
|---|---|---|---|
| Applied loads: | load case 1 | $P_1 = 88960$ N | $P_2 = 0$ |
| | load case 2 | $P_1 = 0$ | $P_2 = 88960$ N |
| Material properties: | Young's modulus = 6895000 N/cm$^2$ | | |
| | Density = 2.768 gm/cm$^3$ | | |
| Constraints: | Stress $\lvert\sigma_i\rvert \le 13790$ N/cm$^2$, $i = 1,..,3$ | | |

Figure 15 Design details for the 3-bar truss

The curves obtained by setting $\sigma_{11}' = \sigma_{21}' = 13790$ have been plotted as the dashed line (- - - -) on Fig. 16. Notice that the powers in each posynomial sum to -1 and the approximations are tangent to the true constraint curves at points corresponding to scaling the structure from the operating point A . No approximation was required for the merit function since the mass of the structure is given by $m = \sum c_i x_i$ in which $c_i > 0$ so that the expression has the correct posynomial form.

b. Dual entry procedures

The vector diagrams for the gradients of the constraints at points (1) and (2) on Fig. 16 indicate that a positive set of dual variables should be obtained at point (1) by the solution of (69), but a negative Lagrange multiplier will be associated with one of the

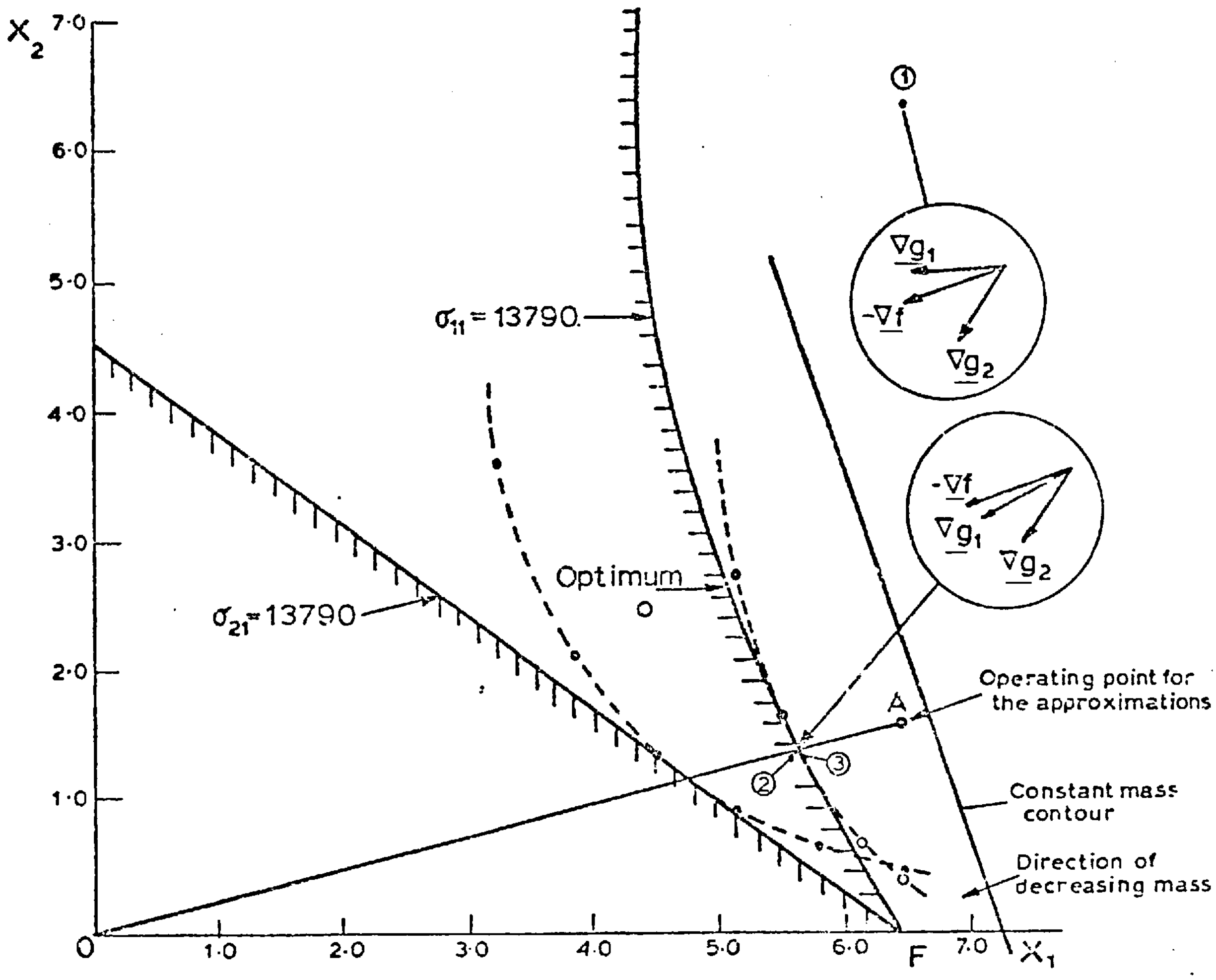



Figure 16 Design space for the 3-bar truss problem

constraints at point 2 . In fact the dual entry procedures proposed in Section 6.3 which are effectively solving (69) gave

for point (1) $\Gamma_1 = 0.576$ , $\Gamma_2 = 0.216$ , $\Gamma_3 = 0.207$ . . . (112)

and for point (2) $\Gamma_1 = 0.482$ , $\Gamma_2 = -0.201$ , $\Gamma_3 = 0.719$ . . . . (113)

In deriving these values three constraints were included in (73) and all three design variables were included in the problem. The dual variables $\Gamma_1$ and $\Gamma_3$ are not equal because the stress constraint for member 1 was from the first load case, while those for members 2 and 3 were from the second load case and the matrix A in (73) is not symmetric.

The point (1) is given by $(x_1, x_2, x_3) = (6.45, 6.45, 6.45)$ with $m = 1.736$ . The lower bound on the minimum mass obtained by substituting the dual solution point corresponding to (112) into (76) was 1.094.

The point (2) on the other hand is given by (5.60, 1.38, 5.60) with $m = 1.211$ . This design was deliberately chosen to be infeasible to show that the dual procedures are not dependent on the operating point being feasible. The lowest weight feasible design found by scaling this design to point (3) has a mass of 1.214. The dual variables including $\underline{\Gamma}$ in (113) do not however define a feasible dual solution point. Setting $\Gamma_2$ to zero, re-evaluating the $\underline{\delta}$ set and normalising gave

$$\Gamma_1 = 0.402 \quad , \quad \Gamma_2 = 0 \quad , \quad \Gamma_3 = 0.598 \quad .$$

A feasible dual solution point had now been obtained giving a lower bound on the minimum mass of 1.189.

The mass of the two designs and the bounds obtained have been entered in Table 1. The optimum design for this truss given in [22] is (5.058, 2.723, 5.058) with a mass of 1.197 kgm. The feasible primal designs and the dual solution points therefore correctly define a range in which the optimum must lie with the dual solution point being near optimal.

| | Design 1 | Design 2 | Optimum |
|---|---|---|---|
| mass | 1.736 | 1.214 | 1.197 |
| bound | 1.094 | 1.189 | |

Table 1 Feasible design masses and corresponding bounds of the 3-bar truss

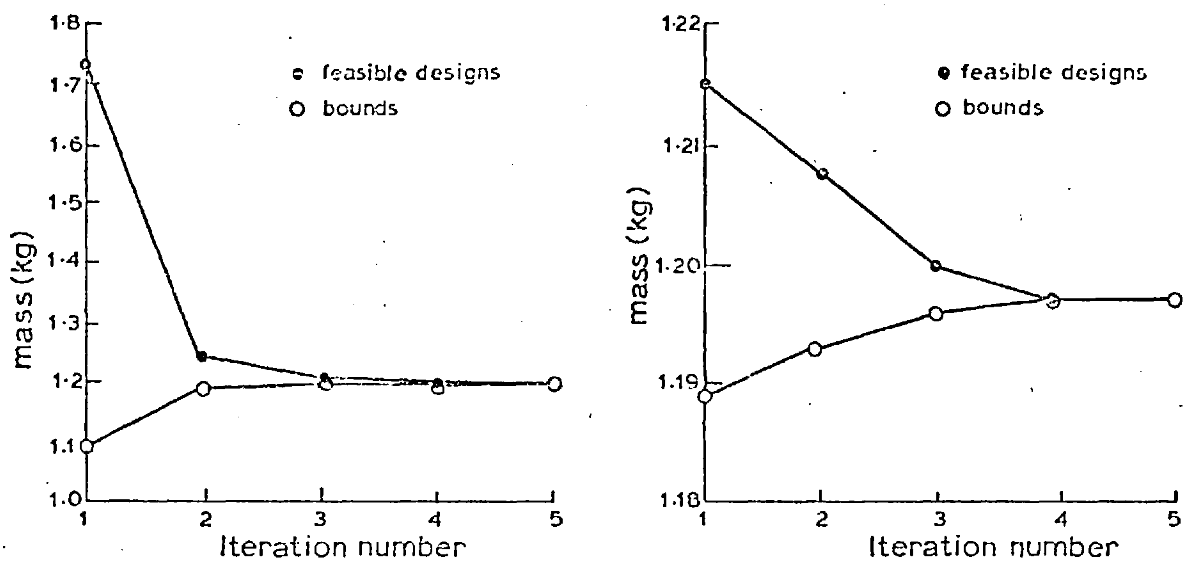
c. Convergence of the bound to the optimum as the primal design improves

In order to show that the bound will converge to the optimum as the design in primal space improves the active set strategy was applied to design the truss. The design given as point (1) on Fig. 16 was taken as the initial design. When negative Lagrange multipliers were encountered and pseudo-constraints had to be defined, a maximum of two line searches in steepest ascent directions were conducted in the dual at each step. Design space was not otherwise artificially constrained.

The sequence of designs produced is detailed in Table 2 together with the pseudo-limits defined for the constraint on member 2 to be used in the design update. Convergence of the bounds and the mass of the designs to the optimum is plotted in Fig. 17. The stresses in members 1 and 3 are of course 13790 at the optimum. The values of the dual variables $\underline{\Gamma}$ from the initial solution of (73) at each step are also given in Table 2 to show that $\Gamma_2$ converges to zero verifying the conclusions drawn from the vector diagrams in Fig. 11 .

d. Alternative design procedures applied to the truss

The stress ratioing procedure described in Section 4.3 attempts, for this design problem, to force a stress constraint active for each




a. Design point 1 in Fig. 16 as the initial design

b. Design point 2 in Fig. 16 as the initial design

Figure 17 Iteration histories for the application of the active set strategy to the design of the 3-bar truss

member of the truss. The iterative redesign process therefore converges to the design F on Fig. 16. This design is given by (6.45, 0.0, 6.45) with a mass of 1.283 kgm. Because it is iterating towards an intercept in design space this procedure is essentially similar to the active set strategy proposed in Section 6.4. Indeed truncation of the posynomial in (71), while maintaining the sum of the exponents equal to −1 to ensure that scaling the design variables will scale the stresses by an equal amount, recovers the stress ratioing redesign formula. However the stress ratioing procedure does not update the intercept to which it is converging and therefore does not achieve the optimum design. The truncation of the posynomial form also leads to inaccuracy in the simulation of structural behaviour as the design changes, resulting in slow convergence to the optimum design. Designs produced after 20 and 200 iterations of the stress ratioing method are given in Table 3.

| Iteration | Design ($cm^2$) $x_1$ | $x_2$ | $x_3$ | Dual variables from initial solution of (73) $\Gamma_1$ | $\Gamma_2$ | $\Gamma_3$ | Feasible dual set $\Gamma_1$ | $\Gamma_2$ | $\Gamma_3$ | Design mass (kg) | Bound | Pseudo-limit for $\sigma_2$ ($N/cm^2$) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 6.45 | 6.45 | 6.45 | 0.576 | 0.216 | 0.207 | | | | 1.736 | 1.094 | |
| 2 | 5.693 | 1.565 | 5.693 | 0.483 | -0.176 | 0.693 | 0.411 | 0.0 | 0.589 | 1.242 | 1.190 | 10954. |
| 3 | 5.343 | 2.025 | 5.343 | 0.604 | -0.092 | 0.488 | 0.553 | 0.0 | 0.446 | 1.2049 | 1.194 | 10462. |
| 4 | 5.187 | 2.381 | 5.187 | 0.494 | -0.036 | 0.542 | 0.477 | 0.0 | 0.523 | 1.1989 | 1.196 | 10199. |
| 5 | 5.107 | 2.580 | 5.107 | 0.509 | -0.007 | 0.499 | 0.505 | 0.0 | 0.495 | 1.197 | 1.197 | 10116. |
| Optimum | 5.058 | 2.723 | 5.058 | | | | | | | 1.197 | | 9986. |

Table 2. Iteration history for the design of the 3-bar truss

| Design | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| Initial | 5.600 | 1.380 | 5.600 |
| After 20 iterations | 6.200 | 0.354 | 6.200 |
| After 200 iterations | 6.419 | 0.044 | 6.419 |
| Point F on Fig. 16 | 6.452 | 0. | 6.452 |

Table 3. Designs produced by the stress ratio method

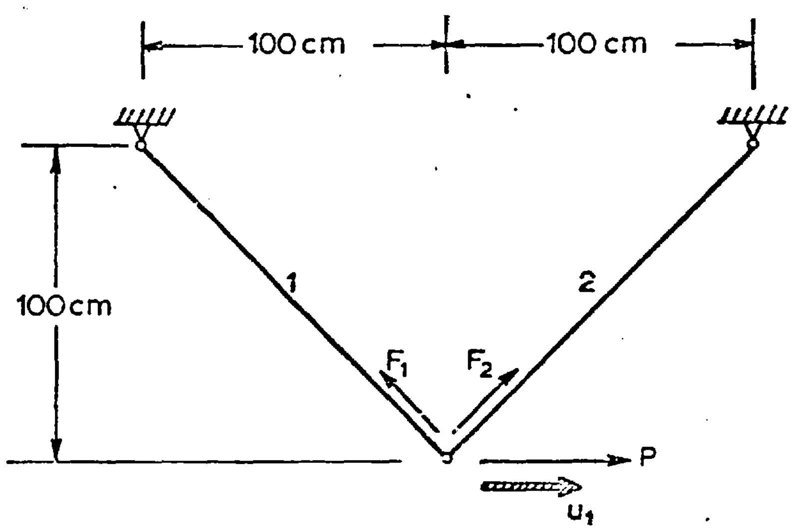
(iii) Example 3

The minimum mass design of a 2-bar truss subject to a single deflection constraint.

In this problem only a single deflection constraint is imposed on a 2-bar truss being designed for minimum mass. Since the values of two design variables are controlled by a single constraint, it provides an opportunity to test the least squares scheme for entering the dual problem proposed in Section 6.6.

a. The design problem

Design details for the truss are given in Fig. 18. The design variables are the cross-sectional areas of the members and the structure is designed for minimum mass with geometry fixed.




| | |
|---|---|
| Applied load: | P = 100000 N |
| Material properties: | Young's modulus E = 6.895 $MN/cm^2$<br>Density $\rho$ = 2.8 $gm/cm^3$ |
| Design constraint: | Deflection $u_1 \leq 1.0$ cm |

Figure 18 Design details for the 2-bar truss

The truss is statically determinate and simple statics gives the internal loads $F_i$ in the members for an applied load P as

$$F_1 = \frac{P}{\sqrt{2}} \quad , \quad F_2 = -\frac{P}{\sqrt{2}} \quad . \tag{114}$$

where the sign convention for $F_1$ and $F_2$ is defined in Fig. 18. The deflection $u_1$ shown on this figure is given by the Unit Load Theorem as

$$u_i = \sum_{i=1}^{2} \frac{F_i U_i L_i}{E x_i} \tag{115}$$

where $L_i$ is the length of member i ,
and $U_i$ is the force in member i caused by the application of a virtual unit load corresponding to $u_1$ in direction and point of application.

Substitution of the design details and (114) into (115) gives

$$u_1 = 1.0255\, x_1^{-1} + 1.0255\, x_2^{-1} \quad . \tag{116}$$

The mass of the truss is to be minimized and is given by

$$w = \sum_{i=1}^{2} \rho_i L_i x_i \quad .$$

Substitution of the design details into this equation gives

$$w = 0.39598\, x_1 + 0.39598\, x_2 \quad . \tag{117}$$

Design space for this problem can now be drawn and is given in Fig. 19.

b. The dual entry procedure

The operating point for the initial design was taken to be $x_1 = 0.8$, $x_2 = 1.2$ . The lowest mass feasible design having this material distribution can be found by scaling and has a mass of 1.6920.

A single term posynomial for the deflection constraint can now be generated using (79) which gives

$$u_1 = 2.0101\, x_1^{-.6}\, x_2^{-.4} \quad .$$

The dual for the design problem is now given by

$$v(\underline{\delta},\underline{\Gamma}) = \left(\frac{0.39598}{\delta_1}\right)^{\delta_1} \left(\frac{0.39598}{\delta_2}\right)^{\delta_2} (2.0101)^{\Gamma_1} \quad (118)$$

subject to
$$\begin{bmatrix} -0.6 \\ -0.4 \end{bmatrix} \Gamma_1 = - \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} , \quad (119a)$$

with $\delta_1 + \delta_2 = 1$ and $\underline{\delta},\underline{\Gamma} \geq \underline{0}$ . (119b)

Applying (77) gives $\delta_1 = 0.4$ , $\delta_2 = 0.6$ so that (119a) becomes

$$\begin{bmatrix} -0.6 \\ -0.4 \end{bmatrix} \Gamma_1 = - \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} .$$

Applying the least squares solution procedure

$$\begin{bmatrix} -0.6 & -0.4 \end{bmatrix} \begin{bmatrix} -0.6 \\ -0.4 \end{bmatrix} \Gamma_1 = - \begin{bmatrix} -0.6 & -0.4 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} .$$

Hence $0.52\ \Gamma_1 = 0.48$

and $\Gamma_1 = 0.9231$ .

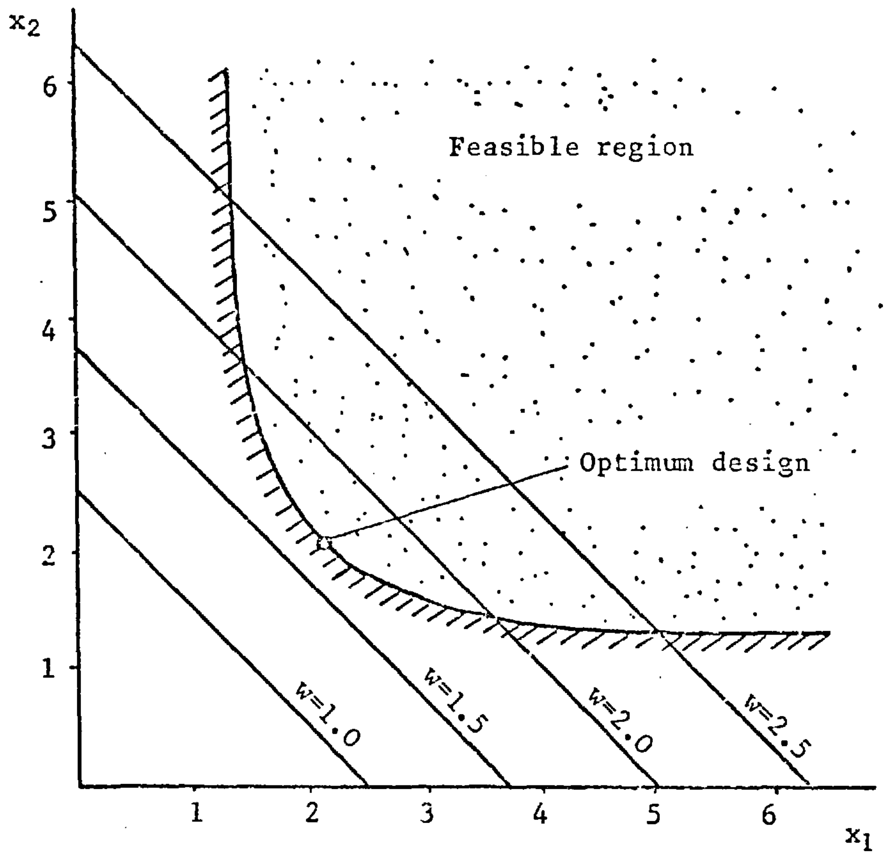



Figure 19 Design space for the 2-bar truss problem

Using (119a) to redefine $\underline{\delta}$ and scaling to satisfy the normality condition in (119b) gives the feasible dual solution point

$$\delta_1 = 0.6 \quad , \quad \delta_2 = 0.4 \quad \text{and} \quad \Gamma_1 = 1 \quad ,$$

for which $v(\underline{\delta},\underline{\Gamma}) = 1.5602$ .

The optimum in Fig. 19 has $x_1 = x_2 = 2.051$ and $w = 1.5602$ . The lowest feasible mass found by scaling the design given as the operating point onto the constraint surface was 1.6920 . This mass and the bound given by the dual solution point therefore correctly define a range in which the optimum lies.

This example can also be used to demonstrate the use of the projected gradient scheme to redesign the structure. The feasible design found by scaling the design $x_1 = 0.8$ , $x_2 = 1.2$ onto the constraint surface is $x_1 = 1.7092$ , $x_2 = 2.5638$ and of course $w = 1.6920$ . Equation (26) gives

$$\lambda = -(\underline{G}^t \underline{G})^{-1} \underline{G}^t \underline{\nabla f} \quad .$$

For this design

$$\underline{G} = \begin{bmatrix} -0.35103 \\ -0.15602 \end{bmatrix} \quad \text{and} \quad \underline{\nabla f} = \begin{bmatrix} .39598 \\ .39598 \end{bmatrix} \quad .$$

Hence applying (26) gives $\lambda = 1.3607$ .
Now the projection direction is given by

$$\underline{P} = \underline{\nabla f} + \lambda \underline{G} = \begin{bmatrix} -0.08167 \\ 0.18368 \end{bmatrix} \quad .$$

Equation (29) for the new design now gives

$$\underline{x}^{\nu+1} = \underline{x}^{\nu} - \alpha \underline{P} \quad .$$

With $\alpha = 1.0$ , $\underline{x}^{\nu+1} = \begin{bmatrix} 1.7909 \\ 2.3801 \end{bmatrix}$ .

Comparing this new design with

$$\underline{x}^{\nu} = \begin{bmatrix} 1.7092 \\ 2.5638 \end{bmatrix}$$

and the optimum $\underline{x}^* = \begin{bmatrix} 2.051 \\ 2.051 \end{bmatrix}$ shows that the new design has moved towards the optimum.

A similar projection scheme can be based on the least squares solution to the dual variables $\underline{\Gamma}$ in (94). The projection directions will not be the same however because the scheme based on $\underline{\Gamma}$ is projecting into the constraints of (60) in order to finally satisfy the Kuhn-Tucker conditions for the problem defined therein.

## 7.4 Summary

A number of practical points related to the application of the primal-dual procedures proposed in Chapter 6 have been discussed, and various features of the primal-dual formulations demonstrated for three simple examples. The linearity in the dual constraints enables the search to maximize the dual function with respect to the variables $\underline{\Gamma}$ to be subject only to the normality constraint. Here (73) would be used only to define $\underline{\delta}$. Projection into this normality condition involves scaling the dual variable set. The dual search procedures are therefore very simple and the linearity has also been shown to extend to the definition of the pseudo-constraints.

A number of examples requiring the use of a digital computer are presented in Chapter 9. However, convergence of the new design procedures and the use of the bound as a termination criterion in the design process will first be considered in the next chapter.

# CHAPTER 8

# CONVERGENCE OF ITERATIVE REDESIGN PROCEDURES AND USE OF THE BOUND TO TERMINATE THE DESIGN PROCESS

## 8.1 Introduction

The Kuhn-Tucker conditions can be used to check whether a redesign procedure has achieved at least a locally optimal design. However, the search procedures can be computationally expensive and the bound would be a valuable supplement if it could be used to terminate the redesign process before the optimum is found. There must however be some form of guarantee that the design is converging to the optimum before a termination criterion based on the bound can be used to stop the redesign process.

A convergence theorem is introduced in this chapter and the use of the bound as a termination criterion is considered. In particular, the difficulties arising when negative Lagrange multipliers are found are discussed. The bound is then only a bound on the optimum defined by the posynomial approximation to the primal problem.

The use of the dual formulations to monitor the strategy of approximate redesign procedures is also considered. Here the Lagrange multipliers can be used to check the selection of the active constraint set in the stress ratioing and envelope procedures based on the optimality criterion approach to structural design. The ease with which certain mathematical programming procedures can be brought into the design process when the approximate procedures fail to improve a non-optimal design is again pointed out.

## 8.2 A convergence theorem for iterative redesign algorithms

In this section the first of a number of convergence theorems given in [34] is presented. This theorem relies on the notion of a point-to-set map which must be defined, together with closedness and compactness of sets.

Redesign algorithms are iterative procedures which calculate a sequence of points $\{\underline{x}^k\}_{k=1}^{\infty}$. At the heart of the algorithm is a recursive process that given a point $\underline{x}^k$ calculates a successor point $\underline{x}^{k+1}$. This recursive process can in general be defined in terms of a point-to-set map $A_k:\underline{V} \rightarrow \underline{V}$. That is, for any point $\underline{x} \varepsilon \underline{V}$, $A_k(\underline{x})$ is a set in $\underline{V}$ and in terms of the algorithm

$$\underline{x}^{k+1} \ \varepsilon \ A_k(\underline{x}^k) \quad .$$

Furthermore, any point in the set $A_k(\underline{x}^k)$ is a possible successor point $\underline{x}^{k+1}$.

Before considering the convergence theorem, we must consider closed maps and compact sets. Closedness is an extension of the function continuity concept to maps. A point-to-set map is closed at $\underline{x}^{\infty}$ if

| | | | |
|---|---|---|---|
| | (a) | $\underline{x}^k \longrightarrow \underline{x}^{\infty}$ | $k \ \varepsilon \ \underline{K}$ |
| | (b) | $\underline{y}^k \ \varepsilon \ A(\underline{x}^k)$ | $k \ \varepsilon \ \underline{K}$ |
| and | (c) | $\underline{y}^k \longrightarrow \underline{y}^{\infty}$ | $k \ \varepsilon \ \underline{K}$ |
| imply | (d) | $\underline{y}^{\infty} \ \varepsilon \ A(\underline{x}^{\infty})$ | |

where $\underline{K}$ defines the infinite sequence generated by the map. The map is said to be closed if it is closed at each point where it is defined.

Zangwill [34] offers the following intuitive rule for determining whether a map is closed at a point $\underline{x}$. Let $\underline{y} \ \varepsilon \ A(\underline{x})$ so that $\underline{y}$ is a possible successor to $\underline{x}$. Now perturb $\underline{x}$ slightly to $\underline{x}'$ and let $\underline{y}' \ \varepsilon \ A(\underline{x}')$. If $\underline{y}'$ is close to $\underline{y}$, then $A$ probably has the continuity property (i.e. closedness). Basically, a slight change to $\underline{x}$ should produce a slight change in $\underline{x}$'s successor. If this statement seems correct at an intuitive level, closedness of the map can usually be mathematically verified.

The remaining notion which has to be introduced is that of compactness. In Euclidean spaces, compact sets correspond to closed and bounded sets. Thus a compact set must contain all of its edges and not extend to infinity in any direction. The points generated by most algorithms are contained in such sets.

The convergence theorem can now be defined in terms related to an algorithm which tries to determine a solution point such that the merit function $Z(\underline{X})$ is within a certain tolerance of a minimum. Let the algorithm define a point-to-set map $A:\underline{V} \rightarrow \underline{V}$ that given a point $\underline{X}' \in \underline{V}$ generates the sequence $\{\underline{X}^k\}_1^\infty$ .

Suppose

(1) All points $\underline{X}^k$ are in the compact set $\underline{X} \subset \underline{V}$ .

(2) There is a continuous function $Z:\underline{V} \rightarrow E_1$ such that:

(a) if $\underline{X}$ is not a solution, then for any $\underline{y} \in A(\underline{X})$

$$Z(\underline{y}) \leq Z(\underline{X})$$

(b) if $\underline{X}$ is a solution, then either the algorithm terminates or for any $\underline{y} \in A(\underline{X})$

$$Z(\underline{y}) \leq Z(\underline{X})$$

and (3) The map A is closed at $\underline{X}$ if $\underline{X}$ is not a solution.

Then either the algorithm stops at a solution, or the limit of any convergent subsequence is a solution.

This convergence theorem is the same as that given in [34] but for the definition of the solution set which here consists of designs with values of the cost function within a certain tolerance of the optimum value. Condition (1) guarantees that the sequence of points generated does not diverge to infinity while Condition (3) is required to prohibit the discontinuities that may cause nonconvergence.

This theorem now specifies a precise procedure for proving convergence. Once the map A has been determined the compactness property can generally be assumed to hold, as it usually does in practice. The merit function Z must be identified and Condition (2) proved. Finally A must be shown to be closed at any point that is not a solution.

## 8.3 The bound as a termination criterion in the design process

It was implied in the design problem and convergence theorem in the previous section that it is seldom necessary to find the exact optimum. The design process could be terminated if it is known that the design is within a certain tolerance of the optimum. This would also allow the possible improvement to the design to be balanced against the computational expense of the redesign process. This termination criterion does however require the generation of bounds because there is no other way to assess the merit of a given design without allowing the redesign process to finally terminate and then checking the Kuhn-Tucker conditions for optimality.

If a positive set of Lagrange multipliers is obtained from the initial solution of (73) then bounds on the optimum to which the algorithm is converging will be found. It has already been shown that these bounds will converge to the optimum as the design improves when the nearest to active constraint set is included in the dual. The new termination criterion can therefore be applied.

The bound, however, is not guaranteed to envelop the optimum when negative Lagrange multipliers are encountered in the initial dual solution point. To obtain the bound the values of the dual variables $\underline{\delta}$ have to be modified from those given by (77) so that the negative $\Gamma_j$ can be set to zero. However the single term posynomial approximations to the constraints based on (79) match the current value and first derivatives at the operating point. Therefore, if the current design at each step is taken as the new operating point, the value of a new bound generated will only be equal to the current value of the cost function at the optimum. A gap should therefore always exist between the value of the cost function and the bound when the design is not at least a local optimum, even though the convergence need not follow the pattern obtained in Fig. 17.

Negative Lagrange multipliers were encountered in the second example in Chapter 7 and bounds on the optimum were still obtained when these multipliers were set to zero. A sequence of bounds must however be generated in this situation to check that the design and bounds are converging monotonically to the final design. On the other hand, if a positive set of Lagrange multipliers is obtained in the initial solution of (73), the final design can be checked against any of the bounds generated in the preceding design sequence. The appearance of negative Lagrange multipliers need not preclude the use of the bound as a termination criterion but a more careful control needs to be applied and a tighter criterion imposed.

## 8.4 Convergence of the redesign procedures

A number of redesign strategies have been considered in the preceding chapters and their ability to produce optimal designs can now be discussed with reference to the convergence theorem. Of particular interest are the approximate procedures for structural design. and the active set strategy proposed in Chapter 6, because each of the mathematical programming procedures described in Chapter 4 are shown in [34] to produce a sequence of designs satisfying the convergence theorem.

With the approximate redesign strategies it will still in general be acceptable to follow Zangwill's suggestion and assume the compactness property holds because divergence to infinity will at least be recognised if the application of the procedure is monitored. The closedness or continuity property can also be checked using the simple differencing scheme proposed. The difficulty however arises in satisfying Condition (2) because it is not, in general, possible to guarantee that a sequence of improving designs will be produced.

The dual procedures for generating the bounds and the Lagrange multipliers do however provide an opportunity to monitor the redesign process. For the type of constraints considered in the structural design problems in this thesis, feasible designs can always be produced by scaling the structure. The improvement to the least weight design at each step can therefore be checked. The stress ratioing and envelope procedures also operate with an active constraint set and Lagrange multipliers can be found by solution of (65b) to check whether the correct constraint set is being considered. If a positive set of multipliers is obtained the approximate strategy would appear to be iterating towards an intercept in design space which correctly defines the optimum. However, the appearance of negative values would indicate that the converse is true and the design process should be stopped, especially if the improvement to a sequence of feasible designs cannot be monitored.

The use of (65b) to monitor the convergence immediately suggests the strategy which should be followed when the approximate redesign process is stopped before the termination criterion based on the bound is satisfied. The projected gradient scheme described in Chapter 4 is based on a least squares solution to this equation. The transfer to this search strategy, which is guaranteed to satisfy the conditions of the convergence theorem and therefore will lead to at least a locally optimal design, should therefore be simple.

The alternative strategy which could be used, if the dual procedures to monitor convergence are based on the geometric programming formulation, is the active set strategy described in Chapter 6. Convergence of this procedure to the optimum cannot however be proved, because of the form of the posynomial approximations. If it could be shown that the feasible region for each approximating primal problem would always be a subset of the real feasible region, then a proof of convergence could follow those given in [36] and [37]. However this condition is not generally satisfied. Convergence should in general occur, especially if the constraint tightening procedures depicted in Fig. 20 are further utilized to ensure that at each step the feasible

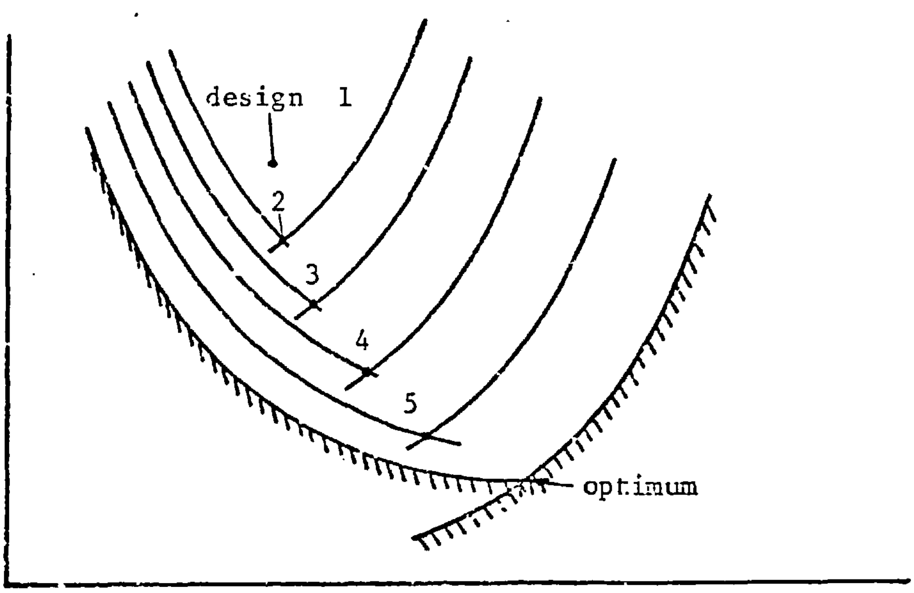



Figure 20 Pseudo-constraints limiting the design change at each step

design space is constrained to a region in which the posynomial approximations are accurate. However, this convergence cannot be guaranteed.

## 8.5 Summary

The use of the bound as a termination criterion in the design process has been considered in this chapter. If no negative Lagrange multipliers appear in the dual formulation, a bound on the optimum is found which will converge to the optimum as the primal design improves. There is then no difficulty in using the bound to terminate the redesign process. If negative Lagrange multipliers do appear in the dual, the accuracy of the posynomial approximations to the primal problem must be considered if the dual formulation based on geometric programming is used. Procedures for generation two-term posynomials, which match second as well as first derivatives at the operating point, are described in Appendix B. However the analogy between these dual procedures and the Lagrangean dual of (65) would be lost if these approximations are used. A better procedure would therefore be to generate a sequence of bounds as the primal design improves and check if the convergence pattern in Fig. 17 is being followed. Alternatively a Newton update procedure could be used in the geometric programming formulation to remove negative Lagrange multipliers. This procedure would update the operating point, and hence the posynomial approximations, in an attempt to remove negative Lagrange multipliers while improving the primal design. This procedure is followed in [25] for the dual based on the Lagrangean function (65). A bound on the optimum is obtained when an operating point giving a positive set of Lagrange multipliers in the solution of (65b) or (73) is found. This bound could then be used in a criterion to terminate the design process.

The use of dual formulations to monitor the constraint selection in approximate design strategies has also been discussed. The main reason for these procedures not producing an optimal design is recognised as the inability to guarantee that the design will improve at each step. The Lagrange multipliers can, however, be used to sense convergence to a non-optimal design. A more rigorous mathematical programming procedure could then be used in the design process.

# CHAPTER 9

# DETAILED EXAMPLES FROM STRUCTURAL DESIGN

## 9.1 Introduction

A number of examples demonstrating the implementation of the new procedures for generating bounds and redesign are presented in this chapter. All these examples are concerned with minimum mass structural design. The finite element method, which will be used to analyse the structures, can be computationally expensive on all but the simplest problems. These examples will therefore reflect a field of design in which the direct application of mathematical programming procedures would be impractical if the problem contains more than a few design variables.

The emphasis on minimum mass structural design is in some respects unfortunate because the procedures and ideas are generally applicable to automated design and optimal policy selection problems. However, application to structural design does enable a variety of special examples to be constructed to test the procedures proposed.

These examples supplement the preliminary examples in Chapter 7 and therefore start with Example 4. This first example illustrates the application of the new dual procedures to a problem in which no pseudo-constraints have to be defined and a constraint is active at the optimum for each design variable. The number of design variables is moderately large but convergence to the optimum is obtained in two iterations when the active set strategy is used for redesign.

A singular dual entry matrix $\underline{A}_1$ was encountered in Example 5 and the methods proposed in Section 6.5 were used to determine the cause of the singularity. The stress ratioing procedure fails to produce the optimum design for this structure and pseudo-constraints have to be defined in the active set strategy proposed in Chapter 6.

Example 6 is a multiple deflection constrained problem for which the envelope procedures based on the optimality criterion approach do not produce satisfactory convergence to the optimum design. The use of dual procedures to augment this approach is discussed and it is shown that monotonic convergence towards the optimum can be achieved.

In Example 7 the shape of a 15-bar cantilever was allowed to vary so that the merit function as well as the constraints were non-linear. In structural design the Michell structure for transmitting a given set of loads to support points can provide a measure of the absolute minimum mass design which can be achieved for a restricted range of constraints. The usefulness of the Michell structural mass should therefore not be ignored and comparison is made in this example. However it is shown that the bounds based on the dual formulation are more useful as they allow a given design to be assessed within a defined geometry and topology.

The final example, Example 8, provides a design space in which the range of accuracy of the single-term posynomial approximations to the constraints is limited compared to the design changes which could occur. For one application of the active set strategy the constraint tightening procedures proposed in the previous chapterwere required to ensure convergence.

Various computational aspects for the implementation of the new procedures in these examples are discussed in Appendix B. A flow chart for the computer program used in these examples is also given. The dimensions, loads and material properties for all but the first and last example were chosen to allow the results to be compared with those which have already appeared in the literature.

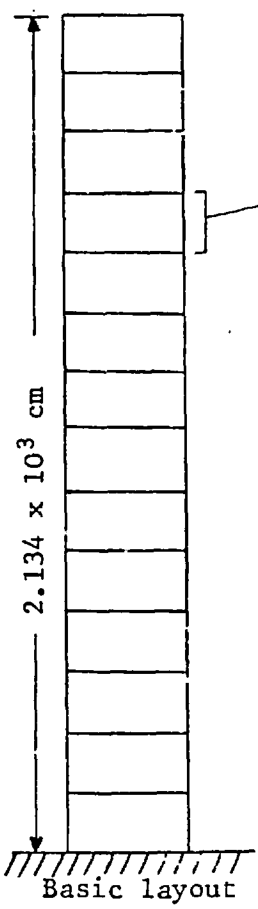
## 9.2 Example 4 - A tower with 252 design variables

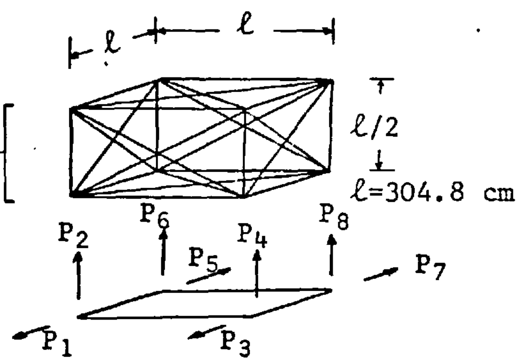
a. The design problem

The minimum mass design was sought for the pin-jointed tower shown in Fig. 21 by varying the cross-sectional area of each member. The design was subject to constraints on the maximum stress levels and minimum cross-sectional areas. Two load conditions were applied to the structure and the redesign process involved 252 variables. The pin-jointed bar is, however, possibly the simplest of all structural members since it carries only an axial load. Each stress constraint is therefore related to a particular member and hence a particular design variable. An active set strategy based on a set of active constraints, equal in number to the design variables, can therefore be applied directly to this problem.

b. Design by stress ratioing

The iteration history for the application of the stress ratioing procedure (39) to the design of the tower is given in Fig. 22. After each iteration the structure was scaled to remove violation of the constraints and the mass of the resulting feasible design is given. A lower bound on the minimum mass was also found to assess the merit of each design. It can be seen that the design produced after six iterations of the stress ratioing procedure was near optimal.




Loads (N) applied to the top level

| Load | Load Case 1 | Load Case 2 |
|---|---|---|
| $P_1$ | 44480 | 0 |
| $P_2$ | 0 | 44480 |
| $P_3$ | 44480 | 0 |
| $P_4$ | 0 | 44480 |
| $P_5$ | 0 | 44480 |
| $P_6$ | 44480 | 0 |
| $P_7$ | 0 | 44480 |
| $P_8$ | 44480 | 0 |

Material properties:
Young's modulus = 6.895 $MN/cm^2$
Density = 2.768 $g/cm^3$

Design constraints:
$\underline{X} \geq 0.645\ cm^2$
$|\underline{\sigma}| \leq 1.379 \times 10^4\ N/cm^2$

Figure 21 Design details for the skeletal tower

The stress ratioing procedure does not always converge to the optimum design. Any confidence that it has produced the optimum design in this example is not based on the low weight reduction over the last three iterations (which could at best indicate that a fully stressed design had been achieved). Rather it is based on convergence with the bound and the fact that a positive set of Lagrange multipliers was obtained in the solution of (73) indicating that constraint intercept to which the redesign procedure was converging would satisfy the Kuhn-Tucker conditions.

c. Procedures used for generation of the bounds

The problem had been posed in the following way,

$$\text{minimize} \quad w = \sum_{i=1}^{252} c_i x_i \quad , \quad c_i > 0 \quad , \quad x_i > 0 \quad ,$$

subject to the constraints

$$0.645 x_i^{-1} \le 1 \quad , \quad \frac{|\sigma_{ij}|}{13790} \le 1 \quad , \quad i = 1,\ldots,252 \quad , \quad j = 1,2 \ ,$$

where $\sigma_{ij}$ was the stress in member $i$ under load conditions $j$ , and the variables $x_i$ were the cross-sectional areas of the members. To find each bound single term posynomial approximations had to be derived for those stress constraints which were nearer to being active than

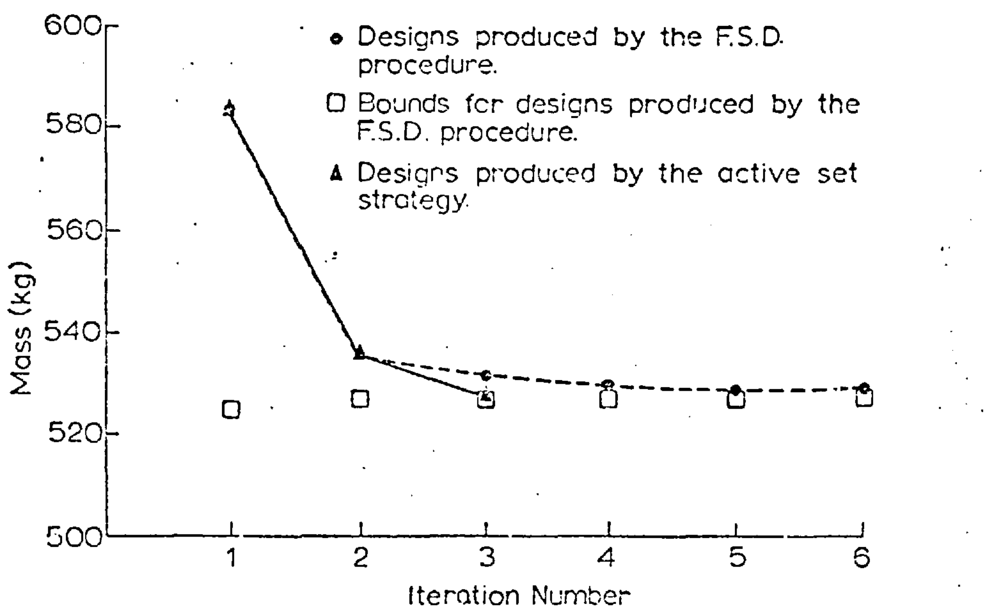



Figure 22 Design histories and bounds for the skeletal tower

the minimum size constraint for each member. These posynomials were derived using (79) with the current design at each iteration as the operating point. Only the local element groups depicted in Fig. 23 were considered in these approximations since the stress redistribution caused by the change in area of any one element was localized. Therefore the posynomial for the stress constraint identified with the member marked with an asterisk in Fig. 23 involved non-zero exponents on the design variables for those members shown in the same group. The computational expenditure to generate the stress gradients was then reduced by evaluating several gradients simultaneously. If a finite difference scheme had been used to evaluate the gradients, rather than the matrix equations given in Appendix B, this procedure could be easily seen to involve the simultaneous variation of members from different groups selected such that no overlap occurred.

The nearest to active constraint for each member was included in the matrix $\underline{A}_1$ in (85). The methods described in Section 6.3 corresponding to the application of (77) and the solution of (73) were applied and the bounds obtained are shown in Fig. 22. Normally only a single bound would have been required to check the merit of the final design when the stress ratioing procedure was terminated. However, additional bounds were found to demonstrate the accuracy of the bound as a prediction of the optimum mass when the operating point was far from optimal. The bounds were obtained from the initial entry to the dual problem and therefore enveloped the true optimum design as the dual variable set defined from the solution of (73) was non-negative. No updating or maximization procedures were required in the dual and the

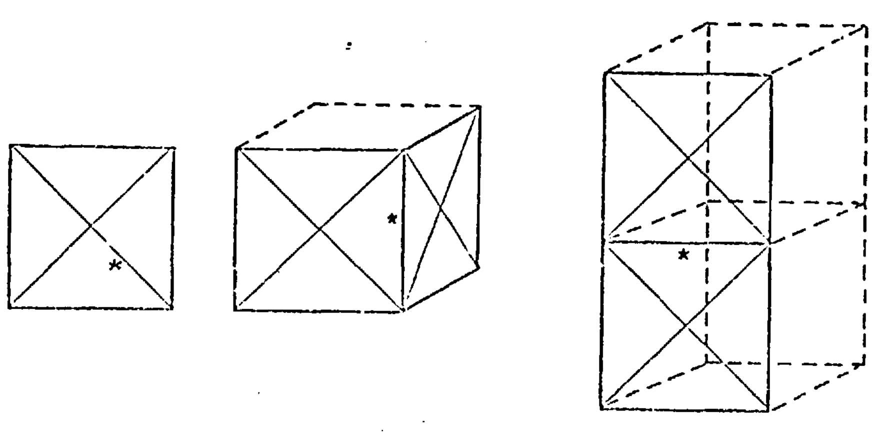


Figure 23 Local element groups considered in the stress posynomials

the bounds were recognized as being near-optimal from the low values of the dual gradient set.

d. Improving the redesign step

Since the active constraint set can be easily identified in this problem, convergence to the optimum design could be expected in a single step if the redesign formulae were accurate. It should be noted that the stress ratioing procedure assumes

$$\sigma_i \propto x_i^{-1} \quad , \tag{120}$$

whereas the single term posynomials used to generate the bounds take the form

$$\sigma_i \propto \prod_{j\in J} x_j^{b_j} \quad ,$$

where the set J defines the element groups given in Fig. 23. It is of course not acceptable to use approximations matching those for the stress ratioing procedure in the posynomials for the bounds because this would lead directly to the prediction that the fully stressed design was optimal. The only situation in which this is undoubtedly correct is if the structure is statically determinate and then the exact posynomials do take the form of (120) with other exponents zero. The degree of truncation of the posynomials which will be acceptable in any given problem will depend on the problem being considered.

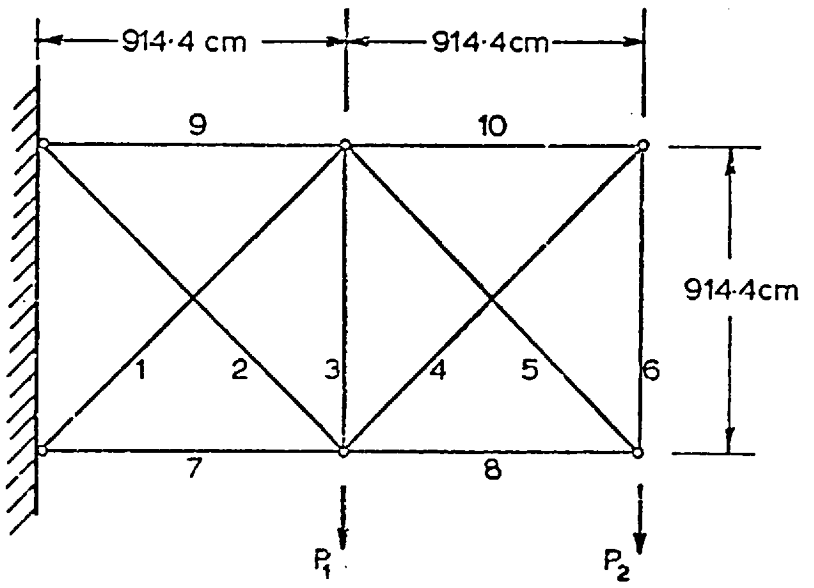
To demonstrate the effect of improving the approximations to the active stress constraints, an active set strategy, based on the solution for the intercept of the single term posynomials used to define the bounds, was applied to design the tower. The initial design was the same as the initial design to which the stress ratioing procedure was applied. The redesign process was however terminated after just two iterations because convergence had occurred with the bound. This bound was generated automatically on each step when entering the dual to evaluate the Lagrange multipliers and check the selection of the active constraint set. The masses of feasible designs found by scaling the structure onto the constraint surface have been superimposed on the results for the stress ratioing procedure in Fig. 22.

This first example has demonstrated the application of the procedures proposed to a problem well suited to the active set type of redesign strategy. It was not necessary to define pseudo-constraints and the single term posynomial approximations to the stress constraints were shown to be accurate even in the truncated form selected. The ease with which the dual solutions were generated is encouraging and the bounds were near optimal.

## 9.3 Example 5 - A more difficult problem with a singular dual entry matrix

### a. The design problem

The 10-bar truss shown in Fig. 24 was designed for minimum mass by varying the cross-sectional areas of the members. The single load case and design constraints are also given in the figure. In this initial form the design problem is very similar to the tower




Material properties: $E = 68950\ MN/m^2$
$\rho = 2.768\ gm/cm^2$

Single load case: $P_1 = P_2 = 4.448 \times 10^5\ N$

Initial constraint set: $|x_i| \geq 0.645\ cm^2$, $|\sigma_i| \leq 172.36\ MN/m^2$ (25000 psi) — Case A

Figure 24 Design details for the 10-bar truss

considered in the previous example and the stress ratioing procedure was applied to design the truss. The optimal design, with the area distribution given as Case A in Table 4 and a mass of 722.7 kg, was obtained.

The constraint set was then modified by raising the limiting value for the stress in member 5. If the cross-sectional area of this member is reduced to keep the stress equal to the limiting value, member 5 becomes more flexible and the load is redistributed to members 4, 6 and 10. The design problem within the augmented constraint set, given as Case B below, then served to illustrate a number of points related to the primal-dual formulation and the application of active set strategies for design.

Constraint set Case B $\quad |x_i| \geq 0.645 \text{ cm}^2$

$$|\sigma_i| \leq 172.36 \text{ MN/m}^2 \quad , \quad i \neq 5 \quad . \quad (25000 \text{ psi})$$

$$|\sigma_5| \leq 482.65 \text{ MN/m}^2 \qquad (70000 \text{ psi})$$

a. Application of the stress ratioing procedure to design for constraint Case B

The derivation of the stress ratioing procedure given in Section 4.3 indicates that this procedure would lead to the optimum design for this truss when the limiting value was the same for all the stress constraints. It was however recognised that this redesign procedure may fail to produce the optimum design when the structure contains materials with markedly different allowable stresses. It was therefore anticipated that application of the stress ratioing procedure to design the truss for the new constraint set might lead to a non-optimal design.

The fully stressed design for constraint Case A given in Table 4 was used as the starting design. The new application of the stress ratioing procedure produced a design with a mass of 782.6 kg. The iteration history for designs found by scaling the structure to remove stress violation are given in Fig. 25, and the final design areas are detailed in Table 4. The fully stressed designs produced by the stress ratioing procedure for the constraint sets Case A and Case B are also compared diagramatically in Fig. 26 to show the alternative load path utilized in the second design.

| Bar No. | Case A FSD & OPT | Case B Design produced by stress ratio | Case B Design from ASS | Case B Optimum [41] |
|---|---|---|---|---|
| 1 | 35.929 | 0.968 | 35.604 | 35.583 |
| 2 | 37.063 | 72.024 | 37.388 | 37.408 |
| 3 | 0.645 | 0.645 | 0.645 | 0.645 |
| 4 | 0.645 | 35.528 | 0.895 | 0.912 |
| 5 | 35.929 | 0.645 | 24.063 | 23.721 |
| 6 | 0.645 | 25.122 | 0.645 | 0.645 |
| 7 | 52.013 | 76.735 | 52.243 | 52.258 |
| 8 | 25.406 | 0.685 | 25.176 | 25.161 |
| 9 | 51.212 | 26.491 | 50.982 | 50.968 |
| 10 | 0.645 | 25.122 | 0.645 | 0.645 |
| Mass | 722.7 | 782.6 | 680.5 | 679.3 |

Table 4 Design areas ($cm^2$) and mass (kg) for the 10-bar truss

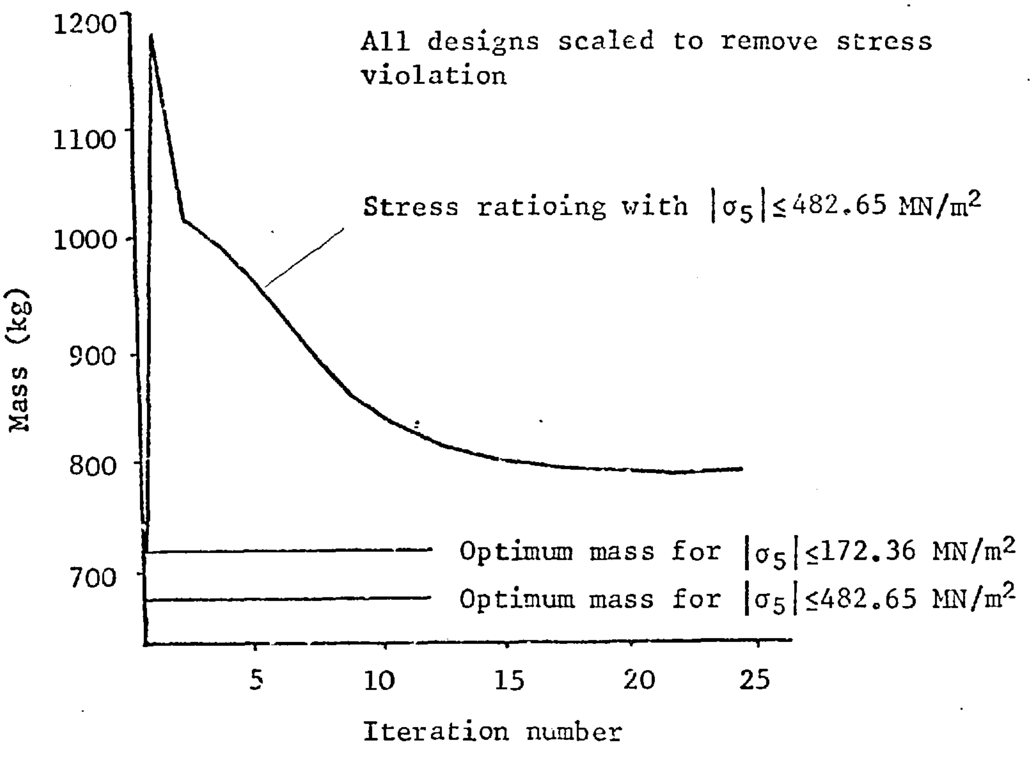



Figure 25 Iteration history for the stress ratioing design of the 10-bar truss

The constraint on the stress in member 5 has been relaxed and the new feasible region in design space for Case B therefore contains all points in the feasible region for Case A. The optimum mass for the new constraint set must therefore be at most equal to, if not lower than, the optimum mass for Case A (722.7 kg). The second fully stressed design produced by the stress ratioing method is therefore obviously not optimal.

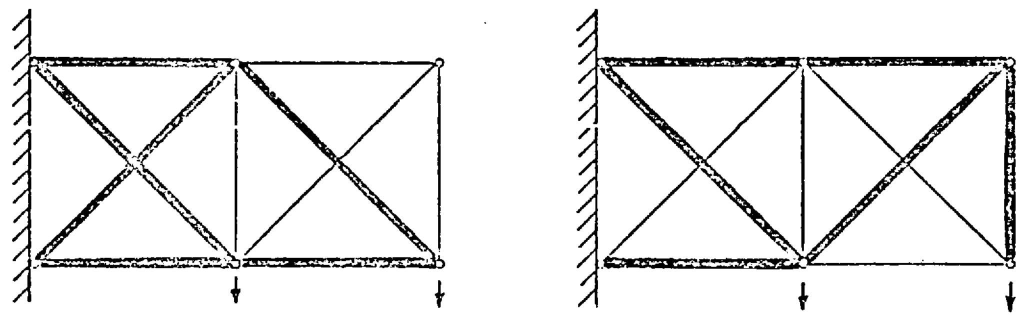



Figure 26 Load paths for the fully stressed designs for the different constraint sets

b. Application of the active set strategy proposed in Chapter 6 to the design of the truss

The active set strategy proposed in Section 6.4 was applied to design the truss for constraint Case B, starting from the fully stressed design for Case A. Single term posynomial approximations to the stress constraints were derived using (79) with the current design at each step used as the operating point for the approximations. In order to restrict the feasible region in design space to a region in which these posynomial approximations would be accurate, the limiting stress for each member was set to at most 13.8 $MN/m^2$ (2000 psi) from the value for the current design.

The iteration history, with feasible designs and bounds produced in the dual formulation, is given in Fig. 27. The bounds at each step bound the optimum within the tightened constraint set and therefore do not envelop the optimum mass. No search was conducted in the dual plane because a positive set of Lagrange multipliers was obtained at each step and no pseudo-constraints had to be defined.

The set of active, or nearest to active constraints included stress constraints in all members except members 3, 4, 6 and 10 which were set to minimum size in the initial iterations. The stress constraint in member 4 became active after the fourth cycle. The design process was terminated at the seventh step because stress constraints became active for all members but member 3 in the following cycle, and the matrix $\underset{\sim}{A}_1$ used for both dual entry and design update became singular.

This singularity will be investigated in the next section but the best design achieved has been detailed in Table 4. It can be compared to the optimum design taken from [41] also given in the table. The stress in member 5 for the final design produced by the active set strategy was 255.03 $MN/m^2$ (36988 psi), and the same stress for the optimum design is 260.29 (37750 psi). This indicates why the stress ratioing procedure, iterating towards a design with the stress limit in this member of 70000 psi, failed to produce the optimum.

Convergence to the optimum for the active set strategy could not have been predicted from the sequence of designs and bounds produced and plotted in Fig. 27. However, if the singularity can be overcome and a dual solution point found, the bound obtained should indicate the optimum has been found.

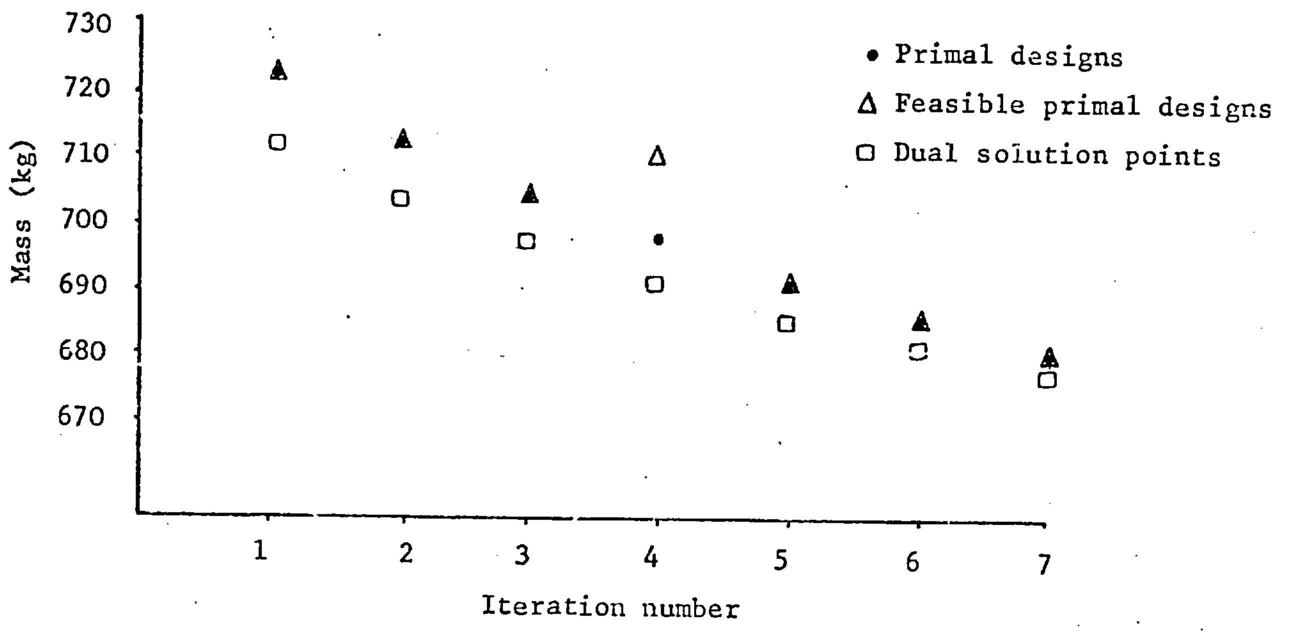



Figure 27 Iteration history for the active set strategy applied to the design of the truss

**c. An investigation of the cause of the singularity in $\underline{A}_1$**

It was observed that the singularity in the matrix $\underline{A}_1$ occurred for a design which was near to the optimum given in [41] and detailed in the last column of Table 4. The singularity is however not related to the close proximity of the optimum, but rather to the selection of the active constraint set. This fact was demonstrated by trying to enter the dual problem from a design the same as the fully stressed design Case A in Table 4, but with $x_5$ set to $12.9\ cm^2$. Stress constraints for all members except member 3 were violated or much closer to being active than the minimum size constraint. Therefore the active set in $\underline{A}_1$ was the same as that for the singular matrix encountered in the redesign cycle depicted in Fig. 27. The matrix was again singular so that the dual procedures could not be used to improve this far from optimal design unless the methods proposed in Section 6.5 could be used to determine the cause of the singularity. To ensure the generality of the application of these procedures the investigation was based on this second configuration rather than the near-optimal design.

The zero diagonal term in the Gauss reduction of the $\underline{A}_1{}^t$ matrix occurred in the tenth column. This Gauss reduction was attempting to solve the set of equations

$$\underline{A}_1{}^t\ \underline{r} = -\underline{\delta} \tag{121}$$

to define a dual solution point. Row operations were carried out on the augmented $[\underline{A}_1{}^t, \underline{\delta}]$ matrix. The corresponding term in $\underline{\delta}$ did not go to zero simultaneously indicating, as discussed in Section 6.5, that a linear dependence existed between the first ten columns of the $\underline{A}_1{}^t$ matrix. The back substitution proposed in Section 6.5 revealed the form of this dependence as

$$\begin{aligned}\underline{c}_{10} = {} & 1.49\,\underline{c}_1 - 1.54\,\underline{c}_2 - 0.001\,\underline{c}_3 - 2.83\,\underline{c}_4 + 4.41\,\underline{c}_5 \\ & - 1.0\,\underline{c}_6 - 0.76\,\underline{c}_7 + 0.74\,\underline{c}_8 + 0.75\,\underline{c}_9 \quad ,\end{aligned} \qquad \ldots (122)$$

where $\underline{c}_i$ is the ith column of the $\underline{A}_1{}^t$ matrix. The coefficients in

(122) indicate that all the columns of the matrix, except possibly column 3 depending on the accuracy of the calculations, contribute to the linear dependence.

It was suggested in Section 6.5 that, if the dependence was not between the rows of (121), a row dependence should exist between the rows of the active set equation solve given by

$$\underline{A}_1 \ln \underline{x} = - \ln \underline{K} \quad . \tag{123}$$

Here the posynomial

$$K_i \prod_{j=1}^{n} x_j^{a_{ij}} = 1$$

describes the constant stress contour passing through the current design point for the stress in the ith member. In the Gauss reduction of the augmented matrix $[\underline{A}_1, -\ln\underline{K}]$ the diagonal term in $\underline{A}_1$ and the corresponding term in $[-\ln \underline{K}]$ went to zero simultaneously. Therefore a row dependence did exist and one of the variables $x_i$ could be arbitrarily set. By backward substitution it was found that this dependence could be defined as

$$\begin{aligned}
\ln x_9 &= 3.8972 - 0.0168 \ln x_{10} \\
\ln x_8 &= 3.6140 - 0.0379 \ln x_{10} \\
\ln x_7 &= 3.9834 + 0.0163 \ln x_{10} \\
\ln x_6 &= \ln x_{10} \\
\ln x_5 &= 2.4850 - 0.0342 \ln x_{10} \\
\ln x_4 &= 1.6948 + 1.00135 \ln x_{10} \\
\ln x_3 &= - 0.4385 \\
\ln x_2 &= 3.6807 + 0.0321 \ln x_{10} \\
\ln x_1 &= 3.5087 - 0.0341 \ln x_{10}
\end{aligned}$$

. . . (124)

It is interesting to note that these equations correctly indicate that the areas of members 6 and 10 will remain equal in the redesign process and one design variable could have been used to define both of them.

One of the variables, apart from $x_3$ , has to be given a definite value to define a design using (124). Therefore, in order to remove the singularity from the $\underline{A}_1$ matrix and enter the dual problem, one of the stress constraints in this matrix could be replaced by a constraint of the form

$$x_i = \bar{x}_i \quad ,$$

where $\bar{x}_i$ is the value of $x_i$ at the operating point.

It is interesting to note from (124) that a basis in design space has not been defined and multiple designs exist with the same stresses as those at the operating point (within the approximation of the posynomials). It may be possible to interpret this as the presence of alternative load paths, but this has not been investigated further. However a strong dependence between members 4, 6 and 10 , which form one of the load paths in the outer bay, is indicated in (124).

In order to enter the dual and complete the design process depicted in Fig. 27, one of the stress constraints in $\underline{A}_1$ was interchanged with a minimum size constraint in $\underline{A}_2$ . Since the current areas of members 6 and 10 at iteration number 7 were at the minimum size, one of these was selected. Member 10 was arbitrarily chosen and the stress constraint placed in $\underline{A}_2$ of (85). A search in the dual plane was then conducted to ensure that this constraint could influence the design. Positive Lagrange multipliers were then obtained for both the stress and minimum size constraints for members 6 and 10 . This indicated that both constraints were simultaneously active for these members. The Lagrange multiplier for the stress constraint in member 5 was driven to zero and a pseudo-constraint was defined for this member. The active set strategy then recovered a design close to the optimum design given in Table 4.

d. Use of the dual procedures to guide the stress ratioing method to the optimum

The similarity between the stress ratioing method and the active set strategy proposed in this thesis was pointed out in Example 4. A stress ratioing algorithm could be augmented by the solution of

$$\underline{\nabla f} + \sum_{i=1}^{n} \lambda_i \, \underline{\nabla g_i} = \underline{0} \quad , \tag{125}$$

for the Lagrange multipliers $\underline{\lambda}$ . Here the constraint set would include the $n$ constraints considered active by the stress ratio procedure. A positive set of multipliers would indicate that the constraint set would be binding if the constraint limits were locally active. However, the gradients in (125) are only locally accurate and therefore the constraint tightening procedure, proposed for the active set strategy based on posynomial approximations, should be applied to the stress ratioing procedure. For this example, investigation of the exponents in the single term posynomial approximations to the constraints showed that the truncated stress ratioing form (120) would be quite accurate and the redesign process would then follow the curve drawn in Fig. 27, rather than that in Fig. 25. However, if gradients have been evaluated to solve (125) for the Lagrange multipliers then it is logical that the redesign formula should utilize this information. If a Taylor series expansion is used, linear approximations are recovered. Alternatively posynomial forms could be used and the active set strategy used in Part (b) of this example would be recovered.

An extended version of the active set strategy was applied to this problem in the results reported in [26]. There a search for the optimum of the dual was conducted at each iteration in order to define the value of the pseudo-limit for the stress constraint in member 5. The limit so defined, starting from the fully stressed design given in Case A in Table 3, was $236.4\ MN/m^2$ and then $252.7\ MN/m^2$ from the improved design. The resulting design had converged sufficiently with the bounds generated in the dual to allow the procedure to terminate. However, considerable work in the dual was required at each step to define the pseudo-limits.

## 9.4 Example 6 - Multiple global constraints

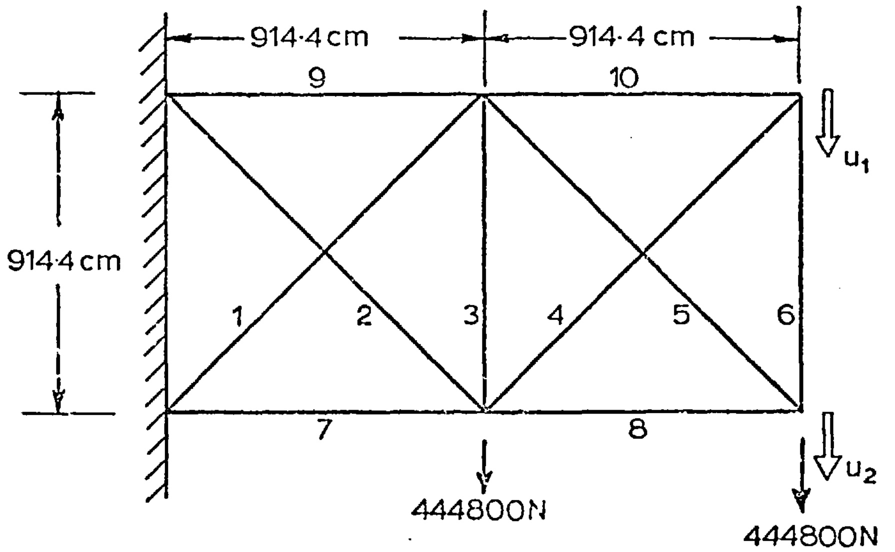
### a. The design problem

The 10-bar truss shown in Fig. 28 was designed for minimum mass under a single load condition with deflection, stress and minimum size constraints. The configuration, material properties and constraints are detailed in the figure.

b. Application of optimality criterion procedures to the design of the truss

The problem detailed in Fig. 28 was taken from [40] where the approximate envelope procedure, based on the optimality criterion approach and described in Section 4.3, was applied to the design of the truss. The iteration history for this procedure is reproduced in Fig. 29. It is reported in [40] that the minimum mass achieved for a feasible design was 5112.0 lb (2318.8 kg). However, the mass subsequently rose to 9029.1 lb (4095.6 kg) and the redesign process was not terminated until 47 iterations had been completed. The sharp rise in the mass after iteration 18 is caused by the constraint on $u_1$ becoming active as well as the constraint on $u_2$.

Improved results are reported in [41] where the design change is restricted when the mass increases. The lowest mass obtained was




Material properties: Young's modulus E = 6.895 MN/cm$^2$
Material density $\rho$ = 2.768 gm/cm$^3$

Design constraints: $u_1$, $u_2 \le 5.08$ cm
$|\sigma_i| \le 172.38$ MN/m$^2$
$x_i \ge 0.645$ cm$^2$

Figure 28 Design details for the 10-bar truss with deflection constraints.

5061.86 lb (2296.1 kg). However, no conclusions are drawn as to the best method for restricting the design change and a considerable number of attempts were made, each requiring several iterations, before this design was achieved.

It was shown in Section 4.3 that the difficulties in applying the optimality criterion approach to multiple deflection constrained problems result from the presence of multiple Lagrange multipliers which cannot be eliminated from the problem. The exact optimality criterion design equation (41) can be applied to design the truss up to iteration 18 because up to this point only a single deflection constraint is active and the "envelope" procedure is not required. A least squares solution of the Kuhn-Tucker condition

$$\underline{\nabla f} + \sum_{i=1}^{m} \lambda_i \underline{\nabla g_i} = \underline{0}$$

for the Lagrange multipliers $\underline{\lambda}$ would indicate that the design obtained at this iteration in Fig. 29 was not optimal. The relation between the least squares error in this equation and the projection direction described in Section 4.2 suggests directly the use of the projected gradient scheme to continue the design of the truss [42].

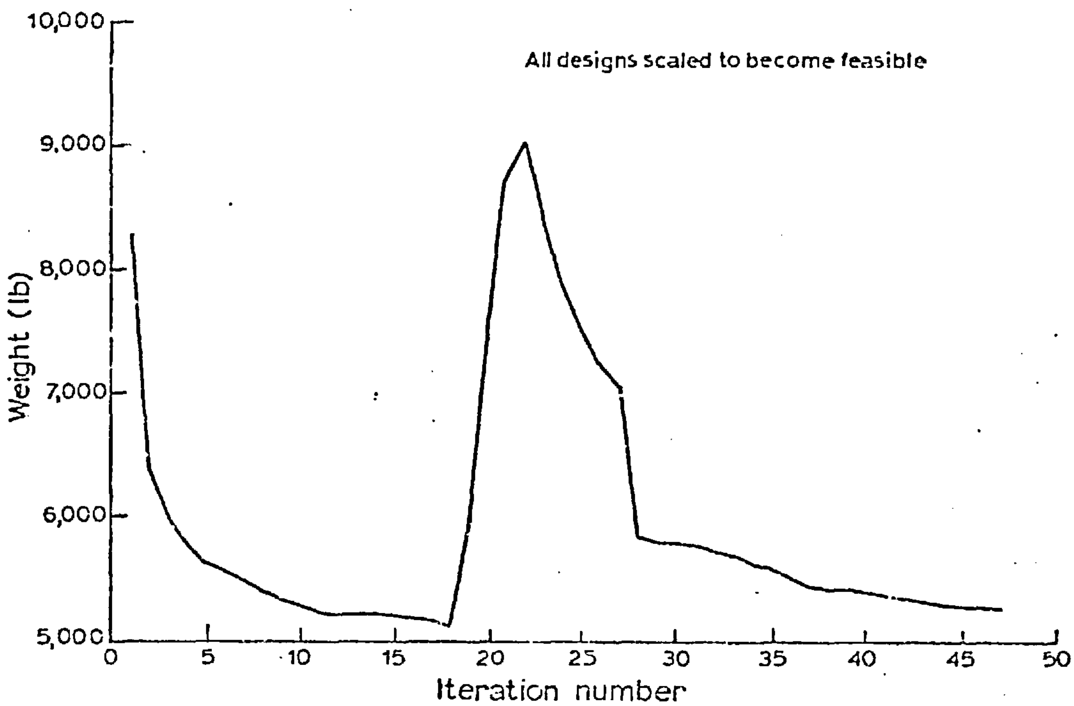



Figure 29 Iteration history for the application of the envelope procedure based on the optimality criterion approach to design the truss

A different approach also aimed at evaluating the Lagrange multipliers within the optimality criterion formulation has been proposed in [43] and use of the exact form of the geometric programming dual has also been suggested [42].

**c. A simple redesign strategy to demonstrate the use of the dual to complete the design process and generate bounds**

In order to demonstrate the use of a dual formulation to take over the redesign process from the optimality criterion algorithm and overcome the sudden increase in the weight at iteration 18 in Fig. 29, a geometric programming formulation was applied to the design of the truss. The two-term posynomials described in Appendix A, which attempt to match second derivatives, were used to approximate the deflection constraints because approximations to the second derivatives would be readily obtained. Finite difference steps can be taken and $u_i = \frac{\Delta U}{\Delta P_i}$ evaluated using the stress gradients, which are assumed constant, to evaluate $\Delta U$. Here $\Delta P_i$ is a unit load increment applied at the point and in the same direction as $u_i$, and $\Delta U$ is the increment to the strain energy in the structure.

The initial design detailed in Table 5 was selected because the design at iteration 18 in Fig. 29 was not known. However, for the single deflection $u_2$ controlling the design up to this point, the statically determinate structure defined in the initial design in Table 5 would approximate to the optimum configuration. The structural mass and deflections $u_1$ and $u_2$ for this design are given in Table 6.

Only the deflection constraint restricting $u_1$ was active in the initial design and the initial dual solution point was obtained by noting that, if the design was optimal and the deflection constraint is given by

$$K_1 \prod_{k=1}^{n} x_k^{a_{1k}} + K_2 \prod_{k=1}^{n} x_k^{a_{2k}} = 1 \quad ,$$

then (98) gives

$$K_1 \prod_{k=1}^{n} x_k^{a_{1k}} = \frac{\Gamma_1}{\Gamma_1 + \Gamma_2}$$

and

$$K_2 \prod_{k=1}^{n} x_k^{a_{2k}} = \frac{\Gamma_2}{\Gamma_1 + \Gamma_2} \quad . \qquad \ldots (126)$$

In the approximate form proposed in Appendix A both terms were given the same value so that $\Gamma_1 = \Gamma_2$ . Therefore these dual variables were set to 0.5 , all other $\Gamma_j$ were set to zero, and (73) used to define the set $\underline{\delta}$ . Scaling the dual variables then satisfied the normality condition (74). Those dual variables $\delta_i$ , related through (77) to the design variables at the minimum size in this initial design, were however negative and had to be set to zero. Equations (73) were then solved and the dual variable set scaled to again satisfy (74) to define a feasible dual solution point.

A Newton update in the dual was then used to define a single search direction. This procedure involved solving linear approximations to

$$\frac{\partial \ln v}{\partial \Gamma_j} = 0 \quad \text{for} \quad \Gamma_j \in \bar{\Lambda}$$

where

$$\bar{\Lambda} = \left\{ \Gamma_j \mid \Gamma_j^* > 0 \right\}$$

and the asterisk indicates the optimal value. The bound given in Table 6 is then the maximum found along this search direction.

| Member | Design variables $\underline{x}$ for the initial operating point for dual entry | Design variables $\underline{x}$ for the best design given in [41] |
|---|---|---|
| 1 | 150 | 136.97 |
| 2 | 100 | 48.24 |
| 3 | 5 | 0.645 |
| 4 | 0.645 | 0.645 |
| 5 | 100 | 136.08 |
| 6 | 0.645 | 3.4 |
| 7 | 200 | 151.84 |
| 8 | 100 | 96.23 |
| 9 | 200 | 199.1 |
| 10 | 0.645 | 0.645 |
| Mass | 2536.6 | 2296.0 |

Table 5 Initial and optimal designs for the 10-bar truss

A simple updating scheme was then applied to redesign the truss. A new set of design variables was obtained by defining a pseudo-limit for a minimum size constraint for each member without conducting further search in the dual. It however appeared rational to apply a restriction to the redesign at each step to ensure monotonic convergence, and only seventy-five per cent of the predicted design change at each step was taken. The following iterative redesign strategy then involved only a single Newton step in the dual, updating of the design variables and the evaluation of a new set of posynomial approximations to the constraints at each iteration. The limit on the design change at each step enabled the dual variables $\underline{\Gamma}$ from the previous iteration to be used to give an initial dual solution point and (73) was used to redefine $\underline{\delta}$ .

The iterative design sequence is given in Table 6. The bounds did not always bound the true optimum because the dual constraints in the form (65b) were not satisfied.

| Iteration Number | Mass (kg) | Constrained deflections (cm) $u_1$ | $u_2$ | Mass of feasible design (kg) | Bound |
|---|---|---|---|---|---|
| 1 | 2536.6 | 5.1782 | 4.8042 | 2585.6 | 2331.7 |
| 2 | 2371.8 | 5.0425 | 5.0687 | 2366.5 | 2277.2 |
| 3 | 2304.4 | 5.1464 | 5.0778 | 2334.5 | 2293.9 |
| 4 | 2299.4 | 5.1383 | 5.0736 | 2325.8 | 2300.8 |
| 5 | 2299.3 | 5.0904 | 5.0894 | 2304.0 | - |

Table 6 Results from application of the dual procedures to the deflection constrained design of the truss

A solution to (73) in a least squares sense was also obtained to define the initial dual solution point. The dual variables $\underline{\delta}$ were defined using (77) and the set $\underline{\Gamma}$ defined using (94). That is,

$$\underline{\Gamma} = -\left[ \underline{A}_1 \; \underline{A}_1^t \right]^{-1} \underline{A}_1 \, \underline{\delta}$$

with $\underline{A}_1$ containing only the two-term posynomial for the deflection constraint on $u_1$. The error in this solution could not be removed by using (73) to redefine $\underline{\delta}$ because some of these variables would have been driven negative. However the error was small and the dual function had a value of 2309.1. It would therefore appear possible to define a bound when as few as one constraint is active for a multi-variable problem.

This example was mainly intended however to show that an iteration history similar to that shown in Fig. 29 would not occur if correct use was made of a dual formulation or projected gradient search when multiple constraints were active. Because of the difficulty in defining a feasible dual solution point the projected gradient procedure would probably be the most suitable procedure for extending the design process.

## 9.5 Example 7 - A variable shape problem

### a. The design problem

In the three examples already considered in this chapter there has been no alternative to the bounds for assessing the merit of a given design. However, in a design problem with variable geometry and subject to stress constraints, the Michell structure gives the minimum mass which can be achieved. A variable geometry problem is therefore considered in this example to compare the bounds produced using the dual formulation and that obtained from the Michell structure. The introduction of variable shape makes the merit function non-linear and the computation becomes more difficult. The dual procedures were therefore not fully automated in this example.

When the geometry is allowed to vary the mass of a pin-ended bar structure is given by

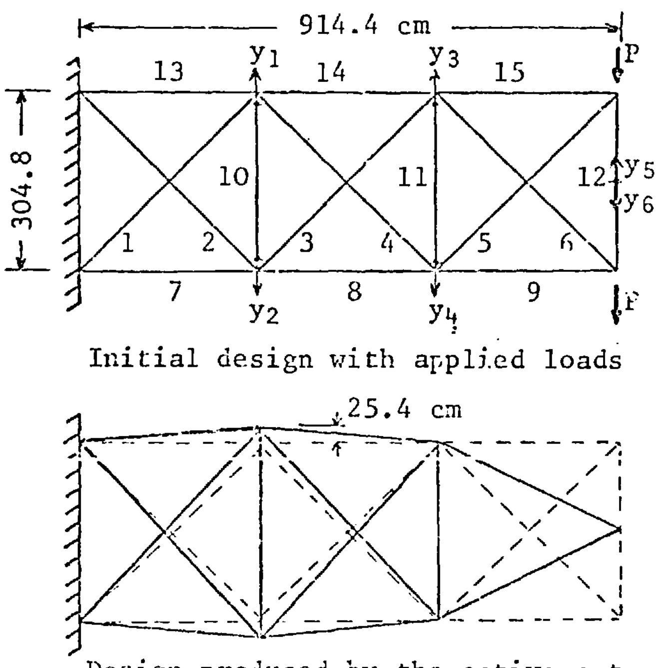
$$f(\underline{x},\underline{y}) = \sum_i \rho_i x_i l_i(\underline{y}) \quad .$$

Here $\rho_i$ is the density of the ith member, $x_i$ is the cross-sectional area of this member, $y_i$ are the coordinates of the joints connected by the bar elements and $l_i(\underline{y})$ is the length of the ith member. Since the lengths $l_i$ are functions of the positions of the joints this merit function is non-linear if the cross-sectional areas also change. The optimality criterion methods, which have already been applied to other examples, are currently confined to finding the best member sizes for a fixed geometric configuration. It has, in general, therefore been necessary to resort to mathematical programming procedures to find the optimal shape.

The initial configuration of the pin-jointed truss given in Fig. 30 was first designed for minimum mass by varying the cross-sectional areas of the members. The design was subject only to stress and minimum cross-sectional area constraints and the stress ratioing procedure was used. Details of the optimum design for this initial configuration are given as Design 1 in Table 7. A prediction was then sought of the further reduction in mass which could be achieved by allowing the vertical positions of the unsupported nodes to become design variables

b. Application of the dual procedures

To generate the bound approximations were required for both the merit function and the stress constraints. The merit function was approximated in the form




Initial design with applied loads

Design produced by the active set strategy

Material properties:

Young's modulus = 6.895 $MN/cm^2$
Density = 2.768 $g/cm^3$

Design constraints:

$x_i \geq 0.645\ cm^2$
$|\underline{\sigma}| \leq 1.724 \times 10^4\ N/cm^2$

Applied load:

P = 0.2224 MN

Figure 30 Design details for the 15-bar cantilever

$$f(\underline{x},\underline{y}) = \sum_{i=1}^{15} c_i x_i^{d_i} + \sum_{i=1}^{6} c_{i+15}\, y_i^{d_{i+15}} + c_0 ,$$

by matching the current value, first derivatives, and second derivatives of the form $\frac{\partial^2 f}{\partial x_i^2}$ at the operating point. The geometry variables $\underline{y}$ were defined as shown in Fig. 30 to ensure the coefficients $c_i$ , $i = 11,\ldots,16$ , were positive. The stress constraints were approximated using the single term posynomial approximations (79) with the design variable set including both the variables $\underline{x}$ and $\underline{y}$ .

The configuration for the operating point for these approximate forms was the initial design shown in Fig. 30, but with the two end nodes displaced to a position 25.4 cm from the centre line of the cantilever. It was anticipated that the approximate forms would then be accurate near the optimum, and values for the member cross-sectional areas at this operating point were obtained by applying a single fully stressing step to an initially uniform design. This operating point is also detailed in Table 7 as Design 2.

The structure has 21 variables, and 21 single term constraints were included in the bound formulation. Fifteen were taken as the nearest to active of the stress or minimum area constraints for each member. The geometry variation was to be unconstrained but to limit the design change occurring in the design step six constraints,

$$y_i \le 50.8 \quad , \quad i = 1,\ldots,6 \qquad (127)$$

were added. A feasible dual solution point was defined using (77) to define $\underline{\delta}$ and solving (73) for $\underline{\Gamma}$ . Three steps in a steepest ascent direction were then sufficient to drive to zero the variables $\Gamma_j$ associated with the constraints on $y_j$ , $j = 3,\ldots,6$ . These constraints were obviously not going to be active because large negative gradients with respect to them were obtained in the dual problem.

The bound thus computed is given in Table 8 together with the mass of the initial fully stressed design. It indicated that a reduction in mass of up to 20 kg could be obtained by geometry and further member area variation. A single iteration of the active set strategy based on

| i | Design 1 ($cm^2$) $x_i$ | $y_i$ | Operating Point Design 2 $x_i$ | $y_i$ | Design 3 $x_i$ | $y_i$ | Optimum from search technique $x_i$ | $y_i$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 18.245 | 25.4 | 18.245 | 25.4 | 24.233 | 49.792 | 23.881 | 48.224 |
| 2 | 18.245 | 25.4 | 18.245 | 25.4 | 24.233 | 49.792 | 23.881 | 48.224 |
| 3 | 18.245 | 25.4 | 18.245 | 25.4 | 13.374 | 22.863 | 12.999 | 18.771 |
| 4 | 18.245 | 25.4 | 18.245 | 25.4 | 13.374 | 22.863 | 12.999 | 18.771 |
| 5 | 18.245 | 152.4 | 14.935 | 25.4 | 14.739 | 6.043 | 14.952 | 0.127 |
| 6 | 18.245 | 152.4 | 14.935 | 25.4 | 14.739 | 6.043 | 14.952 | 0.127 |
| 7 | 64.516 | | 64.516 | | 61.150 | | 61.321 | |
| 8 | 38.710 | | 38.710 | | 35.505 | | 36.091 | |
| 9 | 12.903 | | 13.981 | | 14.509 | | 14.948 | |
| 10 | 0.645 | | 0.645 | | 0.645 | | 0.645 | |
| 11 | 0.645 | | 0.645 | | 0.645 | | 0.645 | |
| 12 | 0.645 | | 0.645 | | 0.645 | | 0.645 | |
| 13 | 64.516 | | 64.516 | | 61.150 | | 61.321 | |
| 14 | 38.710 | | 38.710 | | 35.505 | | 36.091 | |
| 15 | 12.903 | | 13.981 | | 14.509 | | 14.948 | |
| Mass (kg) | 328.725 | | 317.175 | | 313.120 | | 313.032 | |

Table 7 Designs for the 15-bar cantilever

(87), with pseudo-constraints being defined for the geometry variables $\underline{y}$ , was therefore applied without further dual search. The configuration of the new design is given in Fig. 30, but when this design was analysed it was found that some of the stress constraints had been violated. This violation was therefore removed by an active set redesign step with fixed geometry. Details of the final design obtained are given in Table 7 as Design 3.

The mass of the new design has also been entered in Table 8 and is within 3 kg of the initial bound. Since only a small violation of the stress constraints had occurred, these approximations were shown to be quite accurate near the final design. The bound 310.262 should therefore be a lower bound on the optimum mass and further redesign was not attempted.

| Design | Mass (kg) | Bound |
|---|---|---|
| Initial | 328.725 | 310.262 |
| From active set strategy | 313.120 | 312.757 |
| Optimum | 313.032 | |

Table 8 Design masses and bounds for the 15-bar cantilever

To complete Table 8 a bound was computed using the final design as the operating point. This bound verified that the final design was near optimal and this design can be compared with the best design achieved after a relatively large amount of computational effort by applying the mathematical programming procedures in [16] and also given in Table 7.

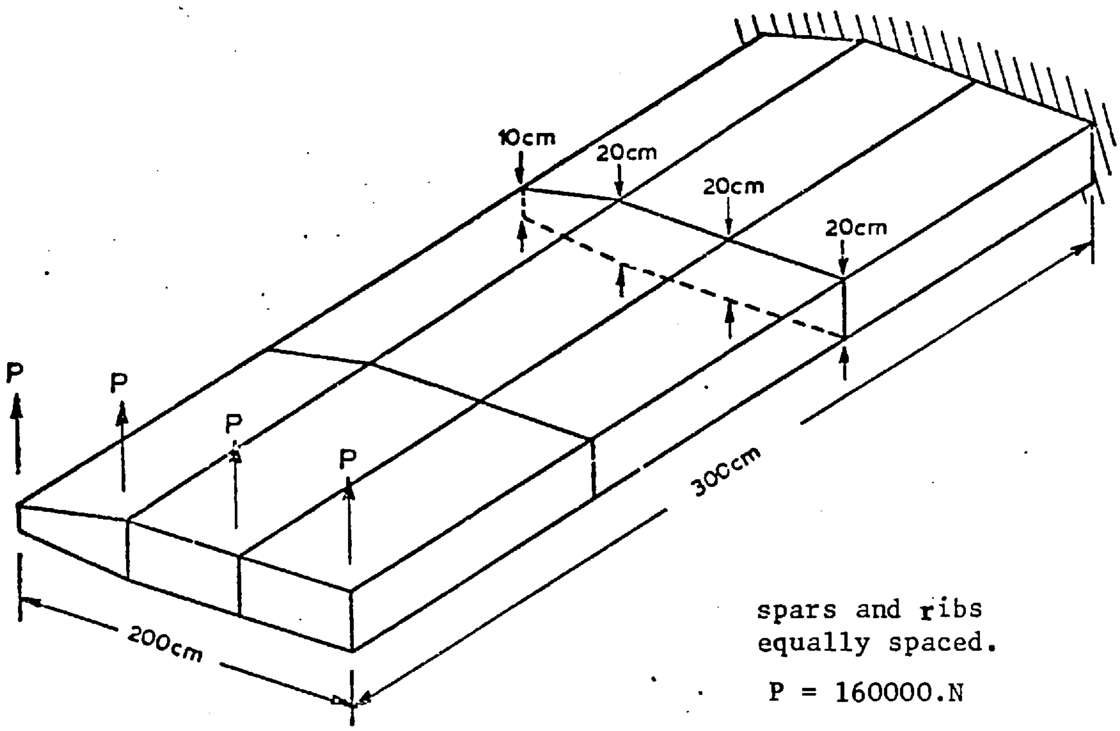
A Michell structure with a mass of 284 kg can also be generated [38] to carry the load considered in this design problem. It cannot be used as a termination criterion for this design process with the given structure having a specified topology since a gap between the current mass and the mass of the Michell structure will always exist. It can therefore only be used to assess the merit of the topology selected.

## 9.6 Example 8 - The minimum mass design of a wing box structure

### a. The design problem

In this final example the active set strategy and bound generation procedures were applied to the design of a structure representing the main load carrying members of the inboard section of a wing. One test case was constructed such that the range of accuracy of the single term posynomials was limited when compared to the design changes which occurred. The redistribution of material required an order of magnitude change in some of the design variables and the active set strategy failed to produce a converging sequence of designs even though the bounds were reliable.

The structure considered is detailed in Fig. 31 and material properties and the design constraints are given. The finite element model of the structure included 75 elements which are also detailed in




Material properties: Young's modulus $E = 6.895\ MN/cm^2$
Material density $\rho = 2.768\ gm/cm^3$
Poisson's ratio = 0.3

Design constraints: $|\sigma_i| \leq 13790\ N/cm^2$
$t_i \geq 0.2$ cm for panels
$A_i = 5.0\ cm^2$ for all bar elements

The finite element model

| Element type | Number in model | Description | Structural member |
|---|---|---|---|
| Quadrilateral panels | 18 | direct and shear stress | top and bottom covers |
| Quadrilateral shear panels | 21 | shear stress only | vertical panels of spars and ribs |
| Pin-jointed bars | 36 | axial load only | booms for spars, vertical members between shear panels of ribs to provide correct boundary conditions for these panel elements |

Figure 31 Design and structural details for the multi-cell box structure

the figure. A design variable was defined to describe the thickness $t_i$ of each panel element in the top and bottom skins, spars and ribs. The geometry of the structure was fixed and the cross-sectional areas $A_i$ of the bar elements were not allowed to vary. There were therefore 39 variables in the design problem and only a single load case was applied.

b. The design of the structure

The bound generation procedures and the active set strategy were first applied to Design A in Table 9. This initial design was obtained after some trial and error, interactively adjusting the design by hand and reanalysing the structure. The general distribution of material turned out to be almost optimal even though some stress violation occurred. To apply the dual procedures approximations were required for the stress constraints to give the problem in the correct posynomial form. Single-term posynomials were generated for them using (79).

The iteration history is given in Fig. 32 where the mass at each iteration is for a feasible design found by scaling the structure to remove stress violation. No constraint tightening procedures were used and no search was conducted in the dual plane because negative Lagrange multipliers were not encountered and no pseudo-constraints had to be defined.

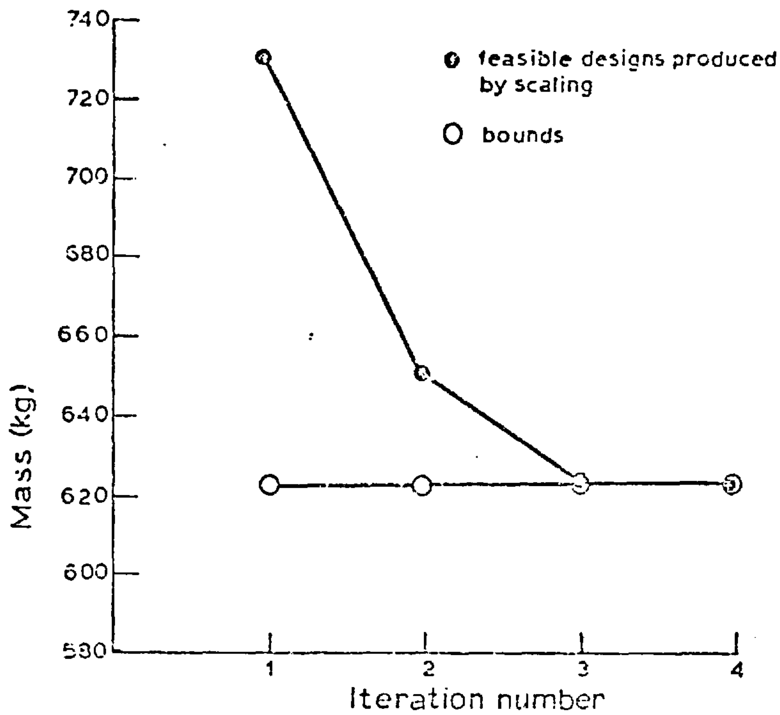
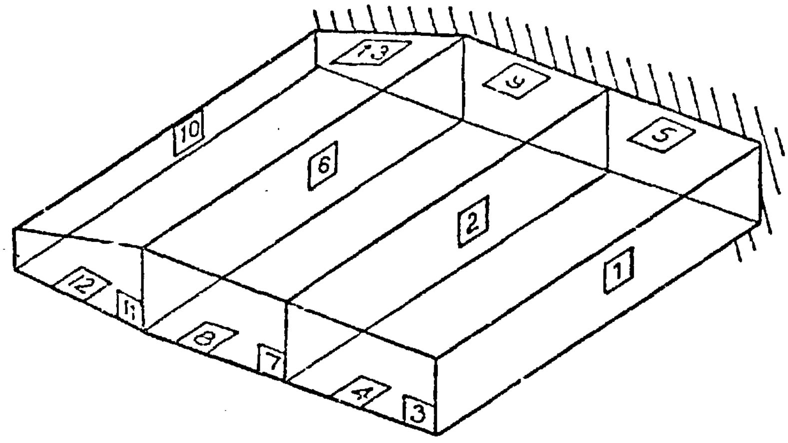



Figure 32 Application of the active set strategy to Design A




Element numbering sequence for Table 9, outboard bays follow sequentially.

| Design A | | | | | | Design B | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| i | initial design | final design | i | initial design | final design | i | initial design | final design | i | initial design | final design |
| 1 | 0.2 | 0.2 | 21 | 7.5 | 6.695 | 1 | 0.22 | 0.2 | 21 | 8.0 | 6.766 |
| 2 | 1.0 | 1.076 | 22 | 7.5 | 6.695 | 2 | 1.0 | 1.077 | 22 | 8.0 | 6.768 |
| 3 | 0.2 | 0.2 | 23 | 0.2 | 0.2 | 3 | 0.2 | 0.2 | 23 | 0.2 | 0.2 |
| 4 | 2.5 | 2.648 | 24 | 0.2 | 0.2 | 4 | 3.0 | 2.566 | 24 | 0.2 | 0.2 |
| 5 | 2.5 | 2.648 | 25 | 0.2 | 0.2 | 5 | 3.0 | 2.566 | 25 | 0.6 | 0.2 |
| 6 | 1.0 | 1.027 | 26 | 0.2 | 0.2 | 6 | 1.0 | 1.028 | 26 | 0.6 | 0.2 |
| 7 | 0.2 | 0.2 | 27 | 0.4 | 0.299 | 7 | 0.2 | 0.2 | 27 | 0.4 | 0.287 |
| 8 | 12.0 | 10.605 | 28 | 1.0 | 0.875 | 8 | 10.0 | 10.687 | 28 | 1.0 | 0.886 |
| 9 | 12.0 | 10.606 | 29 | 0.3 | 0.281 | 9 | 10.0 | 10.687 | 29 | 0.4 | 0.345 |
| 10 | 0.2 | 0.2 | 30 | 0.5 | 0.412 | 10 | 0.2 | 0.2 | 30 | 0.5 | 0.383 |
| 11 | 0.2 | 0.2 | 31 | 0.5 | 0.411 | 11 | 0.2 | 0.2 | 31 | 0.5 | 0.382 |
| 12 | 0.2 | 0.2 | 32 | 1.0 | 1.011 | 12 | 1.0 | 0.2 | 32 | 1.0 | 1.016 |
| 13 | 0.2 | 0.2 | 33 | 0.2 | 0.2 | 13 | 1.0 | 0.2 | 33 | 0.2 | 0.2 |
| 14 | 0.2 | 0.216 | 34 | 2.0 | 1.828 | 14 | 0.22 | 0.211 | 34 | 2.0 | 1.855 |
| 15 | 1.0 | 1.0 | 35 | 2.0 | 1.829 | 15 | 1.0 | 1.005 | 35 | 2.0 | 1.856 |
| 16 | 0.2 | 0.2 | 36 | 0.2 | 0.217 | 16 | 0.2 | 0.2 | 36 | 0.2 | 0.2 |
| 17 | 1.75 | 1.654 | 37 | 0.5 | 0.490 | 17 | 2.0 | 1.588 | 37 | 0.5 | 0.493 |
| 18 | 1.75 | 1.653 | 38 | 0.3 | 0.220 | 18 | 2.0 | 1.587 | 38 | 0.4 | 0.214 |
| 19 | 1.0 | 1.073 | 39 | 0.3 | 0.220 | 19 | 1.0 | 1.074 | 39 | 0.4 | 0.215 |
| 20 | 0.2 | 0.2 | | | | 20 | 0.2 | 0.2 | | | |

Table 9 Designs for the wing structure

The dual procedures were also applied to Design B in Table 9. Significant changes in the design were expected so that a maximum stress variation of 1000 $N/cm^2$ and a maximum panel thickness change of 1 cm were allowed at each redesign step. These pseudo-limits were applied immediately after the bounds were found before the active set strategy was applied to design the structure.

The iteration history is given in Fig. 33. Up to eight negative Lagrange multipliers were found on each of the early design steps but bounds on the optimum were still found. This result is encouraging because it has already been pointed out on a number of occasions that the bound cannot be guaranteed to envelop the optimum when the dual solution point has to be modified to remove negative Lagrange multipliers. The relatively slow convergence of the active set strategy on the other hand was controlled by the pseudo-limits which limited the design change at each step up to the eighth design iteration. Some further updating to define pseudo-constraints was required for some of the constraints associated with the negative Lagrange multipliers. Only 50 steps were

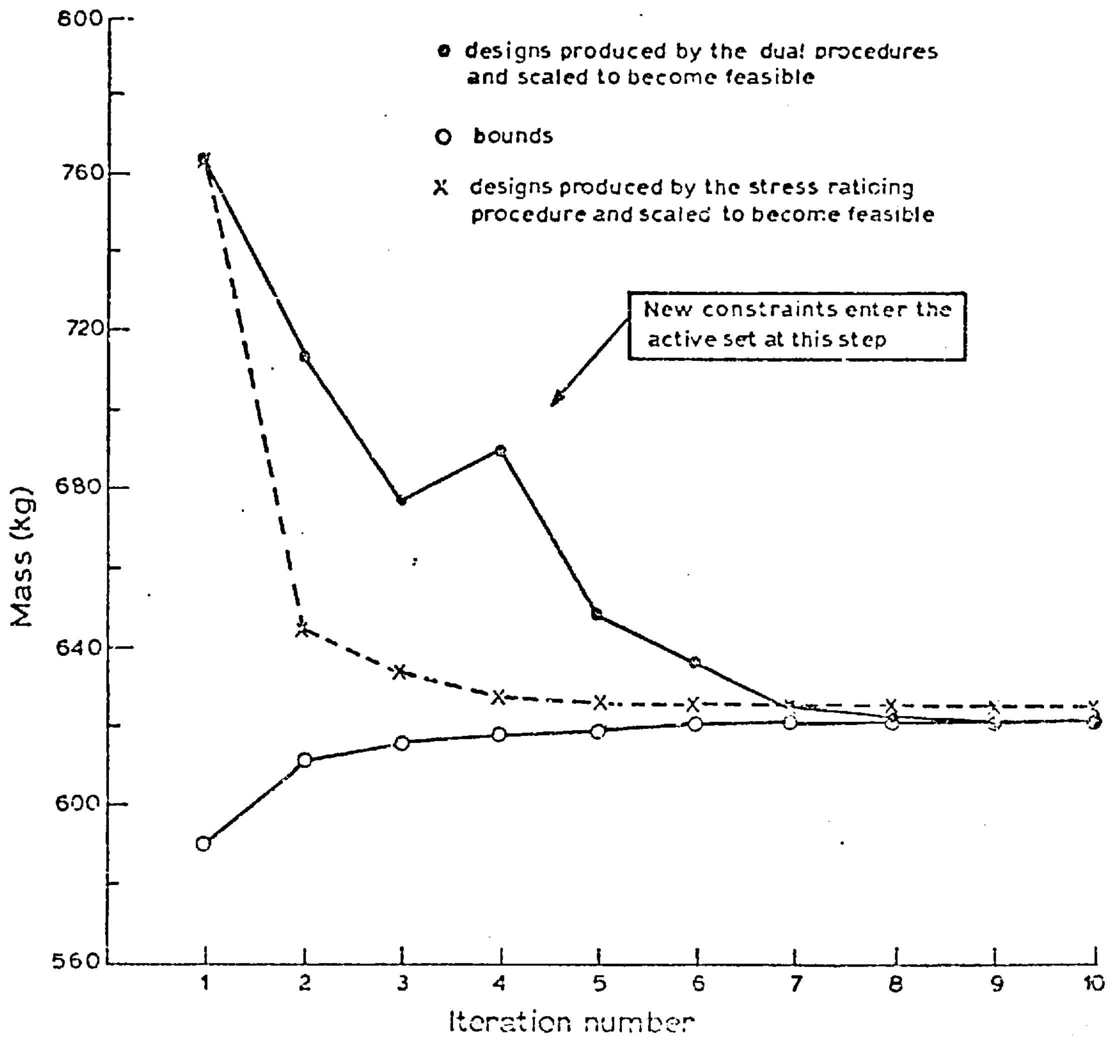



Figure 33 Application of the active set strategy to Design B

allowed in the dual search leading to, at most, two line searches in steepest ascent directions.

The stress ratioing procedure was also applied to the design of the wing structure using Design B in Table 9 as the initial design. The iteration history is superimposed on Fig. 33. While the initial improvement to the design was rapid, the final convergence to the optimum was slow. However, the stress ratioing procedure does not require the evaluation of gradients which can be computationally expensive. The best design strategy for this problem would therefore have been to apply this approximate redesign procedure and use the active set strategy based on posynomials on those steps when the gradients are found to generate bounds and to aid the final convergence. Time was unfortunately not available in this research project to determine the effect of varying the pseudo-limits, or replacing the single-term posynomials with linear approximations in the redesign process.

c. The limited accuracy of the single-term posynomial approximations

One difficulty which was investigated in detail results directly from the product form of the posynomials,

$$f = c \prod_{i=1}^{39} x_i^{a_i} \quad . \qquad (128)$$

If $f$ is to remain constant as the design variables are changed and one is driven small, then significant overprediction of the other variables can occur. This effect can be demonstrated by considering the stress in member 13 of the wing structure. For an operating point with $x_5 = 7.5$, $x_9 = 7.5$ and $x_{13} = 3.0$, the exponents $a_i$ in the expression (128) for the stress in member 13 included $a_5 = -0.035$, $a_9 = -0.516$ and $a_{13} = -0.395$. A solution for the intercept of a set of tightened pseudo-constraints including stress constraints for members 5, 9 and 13 gave

$$x_5 = 0.22 \quad , \quad x_9 = 1.25 \times 10^4 \quad \text{and} \quad x_{13} = 1.17 \times 10^{-7} \ .$$

In fact the stress constraint for member 13 is not active at the optimum and $x_{13}$ is correctly being driven to zero because the minimum size constraint for $x_{13}$ was not included in this redesign solution. However if $x_{13}$ is reduced from 3.0 to 0.3 investigation of (79) indicates that $a_{13}$ would be reduced by a similar order of magnitude. However, the exponents of course remain constant during each design step and a large change was required in $x_9$ to keep the stress in member 13 constant at its limiting value.

Gross overprediction of this kind did not occur when the active set strategy was applied to Design B in Table 9. However some of the initial design changes were large and these were truncated by imposing limits on the design change at each step.

The procedures for generating bounds on the minimum mass were however again successful for this problem. The reason for this success even when the redesign strategy encountered difficulties is that the solution of (65b) or (73) is attempting to satisfy conditions at the operating point and form the geometric construction depicted in Fig. 2. The redesign step on the other hand solves for the intercept of the posynomial approximations to the constraints in design space and thus requires extrapolation away from the current operating point.

## 9.7 Summary

The five examples considered in this chapter have been intended to extend the investigation which started with the preliminary examples in Section 7.3. An attempt has been made to present a range of problems each demonstrating and testing different aspects of the primal-dual formulation and the active set strategy suggested in Chapter 6.

The first example in this chapter, Example 4, was straight forward and no difficulty was encountered in either generating a feasible dual solution point or using the active set strategy to design the structure. It is considered that this example, together with the first application of the procedures to the design of the multi-cell structure in Fig. 32, reflect a practical application of the automated procedures in the sense that a good starting design is defined.

Example 5 demonstrated that it is possible to obtain a singular dual entry matrix and considered the difficulties this can cause in the procedures proposed. This example and Example 6 then showed that the dual procedures could overcome difficulties encountered by approximate redesign strategies.

Example 7 demonstrated that the methods are not restricted to problems with linear merit functions by considering a variable shape structure. The bounds also provided a better measure of merit than comparison with a Michell structure.

Finally the last example gave a warning that difficulties can be encountered in the active set strategy based on the single term polynomials when large changes can occur in the relative magnitude of the design variables.

As examples of structural design for minimum weight these results do illustrate how well the new procedures may be expected to perform. However, the examples themselves do not reflect real design problems. In practice a large number of load cases may have to be considered and additional constraints, such as to prevent buckling, may have to be imposed. It was also in many respects unrealistic to make the mass of the structure the sole contributor to the merit function since the cost must also be considered. These examples would therefore fit the description of automated rather than optimal design as feasible designs are automatically generated with the minimum mass criterion giving the algorithm some indication of what will characterize a good design. In this context however the tables of design variables quoted to four decimal places would appear pointless. However these results are quoted accurately so that they can be checked and the performance of the active set strategies compared to other methods in a mathematical rather than a practical sense.

The most notable feature common to the results is the fact that the bounds converged to the optimum value much faster then the feasible solutions to the primal problem, and provided a good prediction of the optimum mass. It is worth noting that, when the dual is derived from the Legendre transformation [35], an arbitrary condition can be satisfied. It is chosen to satisfy the zero derivative optimality condition for one of the variables and this could further help to explain why the bounds are near optimal.

Finally, the computational expenditure for these examples has not been discussed in detail. The aim has not been to present a working algorithm, and the research programs are in many respects inefficient. A detailed consideration of the operations involved should therefore give better insight into the computational aspect.

# CHAPTER 10

## DISCUSSION AND CONCLUSIONS

The early chapters in this thesis attempted to isolate a unifying theory common to numerical procedures for optimization. The Lagrangean saddle-function, duality and the Kuhn-Tucker conditions are strongly interdependent and form the basis of this theory. The main numerical procedures of mathematical programming for the type of problem being considered were shown to base their search strategies either on the Kuhn-Tucker conditions or to work directly with the Lagrangean function itself.

While none of the theory presented is new, the particular emphasis has been placed on those features of the algorithms which conform to the unifying basis. What is also significant is that an attempt has been made to show that the saddle function and duality can be used to extend certain approximate redesign strategies and include them in this framework. In particular, the fully stressing algorithm has the same basic strategy as the active set algorithm herein except that the constraint set defining the optimum is pre-assumed rather than evaluated as the redesign process proceeds. The "envelope" strategy for applying optimality criteria to multiple constrained problems should also be modified by a dual search to evaluate the Lagrange multipliers appearing in the formulation. Both procedures could then be made mathematically rigorous, the active set algorithm degenerating to a projected gradient scheme if necessary and a geometric programming dual formulation has been proposed [42] for the optimality criterion approach. The approximations in these procedures would then only relate to the assumptions about structural behaviour.

The main aim of the research has been however to look for a method for assessing the merit of a given design because the redesign process can often be expensive. The study of the saddle function and duality led to the proposal that procedures could be developed which allow feasible solutions to be found for both primal and dual problems, and the use of these procedures for generating bounds on the optimum value of the cost function is immediately obvious. In Chapter 6 it was shown that releasing the Kuhn-Tucker condition requiring that either a constraint be active or the corresponding Lagrange multiplier must be zero, leaves the remaining condition defining feasible (but not optimal) dual points if the non-negativity conditions on the Lagrange multipliers are satisfied. If a negative Lagrange multiplier is encountered a method is required to move from the given non-feasible dual point to a feasible dual point. The procedure proposed, which is based on a simple form of the geometric programming primal-dual formulation, is extremely simple compared to any numerical search scheme and has be shown to be successful in the examples considered. This procedure is based on a particular form of approximation to the given primal problem based on posynomials, and the effect of inaccuracy in these approximations has been considered. However, the bound will converge to the optimum as the design in primal space improves and it was pointed out that this approximate procedure deals with the same equation set as the standard method for defining a dual point for the Lagrangean function. In fact the dual point is a bound on the true optimum if the initial dual variable set obeys the non-negativity condition, in spite of the approximations.

The interface between the primal and dual problems has been considered in some detail from a practical point of view, and the dual role of the matrix of exponents of the posynomials in defining primal and dual solution points was pointed out. Singularity in this matrix has been given particular attention because practically it would prevent the transition between primal and dual problems. However, it would appear that sufficient information can be generated to determine the cause of the singularity and allow the design process to proceed.

The development of procedures to enter the dual problem were of course soon followed by the recognition that the dual can provide significant information about constraint activity levels in the primal problem. It is considered that the general format of the design algorithms proposed in this thesis represent a maximum use of the information that can be obtained from the saddle function. The bound of course allows early termination of the redesign process, but in addition the procedures proposed are oriented towards minimizing the work required. If many constraints are active the dual is only used to ensure an active set can be found defining the optimum point, and a solution for the intercept of these constraints is used for redesign. However, if few constraints are active the dimension of the dual problem is low and it is then efficient to conduct an extended search in the dual problem. The accuracy of the posynomial approximations required for the procedures proposed in this thesis may severely restrict their usefulness, but this is the fault of the particular formulation rather than the overall strategy based on the saddle function.

When considering the mathematical algorithms themselves it would appear that the penalty function procedure is very attractive because of the availability of bounds after a search at each penalty level. A feasible solution to the dual is only obtained at the optimum with a projected gradient method which moves on the surface of constraints in the feasible region.

The examples presented adequately demonstrate that it is possible to propose algorithms which work simultaneously with feasible solutions to both primal and dual problems. A wide range of examples were presented aimed at illustrating different features of the interface between the primal and dual problems. The preliminary examples are of particular interest because the calculations can be carried out with the aid of a desk calculator, and they illustrate very simply all the procedures proposed. On the other hand the two examples relating to the 10-bar truss in the detailed examples represent more difficult problems on which the stress ratioing and optimality criterion approaches do not perform satisfactorily. In particular the stress limited design problem for the 10-bar truss can be made to produce a singular dual entry matrix.

The attempt at designing the 15-bar truss with variable shape allowed the bound generated to be compared with that which could be obtained by evaluating the weight of a Michell structure to carry the applied loads. The Michell weight could not provide a termination criterion for the design process but could be used to assess the merit of the configuration selected for the design. The results for the 252-bar tower and the first of the wing-box problems however reflect the extremely good performance of the bound generation and redesign procedures which was encountered in the majority of the problems attempted, but not presented here. In fact the results presented do not correctly indicate this performance unless it is recognised that an attempt has been made to present results pointing out different features of the saddle function and the new methods proposed, and this required the consideration of a number of potentially difficult problems. In the two examples just mentioned however the initial designs are near-optimal, as would be most initial designs produced by design engineers and to which the automated procedures are to be applied.

In all of these problems the posynomial approximations were sufficiently accurate for the convergence of the redesign algorithm to be rapid. Considerable design change did occur in the final example and it is obvious from the product form of the posynomial approximations to the stress constraints, that they will not be accurate over such a wide range. However, the procedures based on tightening the constraint limits to reduce the size of the feasible region centred on the current design point successfully overcame this problem. This was only at the expense of an increased number of redesign iterations and other forms of approximation may be more suited to the problem where large changes can occur in the design. However this example does demonstrate the fact that if the initial set of Lagrange multipliers in the dual are positive the bound will bound the true optimum in spite of the inaccuracy in the posynomial approximations.

The computational expenditure involved in applying the new methods has not been detailed in terms of actual computing time for each run because emphasis has not been placed on developing efficient programs. Rather the intention, in the time available, has been to develop programs to investigate all the proposals about dual entry, and also consider a wide range of problems. However, in Chapter 6 it was shown that only a single triangularization of the matrix $\underline{A}$ is required in the dual if a Choleski method is used to solve the equation sets. This single decomposition enabling both the dual entry and redesign equation solutions to be performed by only the relatively inexpensive forward and backward substitutions of the method. All other operations in the dual are linear, apart from the evaluation of the dual function itself so that the search and pseudo-limit updating procedures are extremely simple. In practical problems in structural design, where many load cases are often considered, the evaluation of the gradients for the posynomial approximations to the constraints may require the most computational effort. However little can be ascertained about the optimality of a given design without the gradients of the merit function and constraints being found. A check of whether the Kuhn-Tucker conditions are satisfied by the given design requires a knowledge of these gradients so that this computational expenditure seems unavoidable.

Convergence of redesign algorithms was also considered briefly in Chapter 8 because it has been suggested that the new methods proposed in this thesis might be used to supplement approximate redesign algorithms. The dual procedures could be used to check the convergence of an approximate algorithm by generating bounds, and providing sufficient refinement to ensure convergence of the redesign process to the optimum when the approximate algorithm fails to produce this design. A good understanding of the minimal requirements for convergence, and the limitations in this respect of the particular form of the saddle function procedures proposed in this thesis, would be essential in any such application.

It is recognised that this thesis is basically an exercise in convex analysis. However, even though convexity forms the basis of the primal-dual procedures, there is no guarantee that they will produce bounds on the global optimum or find this design rather than a local optimum. When locally accurate convex approximations in the form of the posynomials are made for the merit function and constraints, a local optimum near the given design could become the global optimum for the convex problem. However convex analysis does form the basis for numerical procedures for optimization, and the ability to include approximate design procedures in the same framework as the rigorous methods would appear to offer a useful extension to these procedures.

In addition to the identification of this unifying theory, the generation of corresponding feasible solutions to both primal and dual problems has formed the basis of the research carried out for this thesis. The resulting bounds can be used to check convergence and even decide whether design optimization is worthwhile. It should finally be noted that the Lagrange multipliers evaluated for the final design provide the sensitivity coefficients for post-optimal analysis. The extension of the design process to include sensitivity studies is then a logical step. Indeed it should be possible to treat the design problem in an integrated way incorporating special redesign strategies, mathematical programming procedures, dual formulations and sensitivity studies into the same framework.

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## APPENDIX A

## POSYNOMIAL APPROXIMATIONS

The dual procedures based on geometric programming require both the cost function and constraints to be in posynomial form. That is, they must be functions of the form

$$h(\underline{x}) = \sum_{i=1}^{n} c_i \prod_{j=1}^{n} x_j^{a_{ij}} \qquad (A1)$$

with $c_i > 0$ , $i = 1,\ldots,n$ but with $a_{ij}$ arbitrary constants. The cost function and constraints may not in general have this form and the purpose of this appendix is to present methods for generating posynomial approximations to general functions.

**a.** The basic approximation to the cost function

The form required for the cost function is in general a linear posynomial given by

$$f'(\underline{x}) = \sum_{i=1}^{n} c_i x_i + c_{n+1} \quad \text{with} \quad c_i > 0 \ , \quad i = 1,\ldots,n \ . \qquad \ldots (A2)$$

When structural mass forms the cost function for fixed geometry no approximation is required. However in general it is possible to set

$$c_i = \frac{\partial f}{\partial x_i} \ ;$$

where $f(\underline{x})$ is the function being approximated. If any of the coefficients in (A2) are now negative, the transformation

$$x_i' = (k - x_i)$$

can be used to remove the negative sign. Here k is a suitably defined constant such that $x_i' > 0$ for the anticipated changes to $x_i$ .

The current value of the function is matched if a new constant $c_{n+1}$ is defined by

$$c_{n+1} = f(\underline{x}) - \sum_{i=1}^{n} c_i x_i \quad .$$

Bounds are then formed on

$$(f'(\underline{x}) - c_{n+1}) = \sum_{i=1}^{n} c_i x_i \quad .$$

b. The basic approximation to the constraints

A single term posynomial approximation is in general required for the constraints. The dual formulation will then correspond to an extension of the dual given by (65) to allow negative Lagrange multipliers to be removed. These approximations are given in [28] as

$$g_i'(\underline{x},\underline{x}') = g_i(\underline{x}') \prod_{i=1}^{n} \left(\frac{x_i}{x_i'}\right)^{a_i} \quad ,$$

$$a_i = \left[\frac{x_i}{g_i} \frac{\partial g_i}{\partial x_i}\right]_{\underline{x}=\underline{x}'} \quad , \qquad i = 1,\ldots,n \quad .$$

. . . (A3)

This approximation is equivalent to expanding $\ln g$ in a power series in terms of the variables $z_j = \ln\left(\frac{x_j}{x_j'}\right)$ and neglecting all but the linear terms. Here $\underline{x}'$ is the operating point and $g_i'(\underline{x},\underline{x}')$ is a posynomial if $g_i(\underline{x}')$ is positive. The approximation is such that $g_i(\underline{x})$ and the posynomial have the same value and the same first partial derivatives at the operating point.

c. Extensions to the approximation to the cost function

A posynomial approximation to the cost function matching second derivatives of the form $\frac{\partial^2 f}{\partial x_i^2}$ can be obtained if the approximation has the form

$$f'(\underline{x}) = \sum_{i=1}^{n} c_i x_i^{d_i} + c_{n+1} \quad .$$

Here $$\frac{\partial f'}{\partial x_i} = c_i d_i x_i^{d_i - 1}$$

and $$\frac{\partial^2 f'}{\partial x_i^2} = c_i d_i (d_i - 1) x_i^{d_i - 2} \quad .$$

Defining $$c_i d_i x_i^{d_i - 1} = \frac{\partial f}{\partial x_i} \tag{A4}$$

and $$c_i d_i (d_i - 1) x_i^{d_i - 2} = \frac{\partial^2 f}{\partial x_i^2}$$

gives by division $$(d_i - 1) x_i = \frac{\partial^2 f}{\partial x_i^2} \Big/ \frac{\partial f}{\partial x_i}$$

so that $$d_i = \frac{1}{x_i} \left[ \frac{\partial^2 f}{\partial x_i^2} \Big/ \frac{\partial f}{\partial x_i} \right] - 1$$

and substitution into (A4) gives $c_i$ .

A difficulty now arises in ensuring the $c_i$ are positive for $i=1,\ldots,n$ although this form of approximation to the cost function was used successfully in Examples 1 and 7. In general if a negative sign persists, even after a variable transformation of the form $x_i' = (k-x_i)$ is tried, it would be necessary to revert to the linear form for this particular variable. It is guaranteed that in the linear form a posynomial can be generated.

d. The generation of two term posynomial approximations for the constraints

The extension of the posynomial approximations to the constraints to match second derivatives at the operating point is more difficult and requires the definition of a two term posynomial. Consider the posynomial form

$$g(\underline{x},\underline{x}') = \sum_{i=1}^{2} c_i \prod_{j=1}^{n} \left( \frac{x_j}{x_j'} \right)^{a_{ij}} \tag{A5}$$

where $\underline{x}'$ is the operating point about which the approximation is required.

Now $$\left.\frac{\partial g}{\partial x_k}\right|_{\underline{X}=\underline{X}'} = \sum_{i=1}^{2} c_i a_{ik} \tag{A6}$$

and $$\left.\frac{\partial^2 g}{\partial x_k^2}\right|_{\underline{X}=\underline{X}'} = \sum_{i=1}^{2} c_i a_{ik}(a_{ik} - 1) \quad . \tag{A7}$$

The posynomial (A5) has $2n+2$ constants $c_i$ and $a_{ij}$ to be evaluated. These will be defined by requiring that the current value and derivatives of the form of (A6) and (A7) be matched at the operating point. Therefore,

$$c_1 + c_2 = g(\underline{X})\Big|_{\underline{X}=\underline{X}'} \tag{A8}$$

$$c_1 a_{1k} + c_2 a_{2k} = \left.\frac{\partial g(\underline{X})}{\partial x_k}\right|_{\underline{X}=\underline{X}'} , \quad k = 1,\ldots,n \tag{A9}$$

and $$c_1 a_{1k}(a_{1k} - 1) + c_2 a_{2k}(a_{2k} - 1) = \left.\frac{\partial^2 g(\underline{X})}{\partial x_k^2}\right|_{\underline{X}=\underline{X}'}$$

$$k = 1,\ldots,n$$

. . . (A10)

To guarantee the matching of the current value and the first derivatives by the approximation define

$$c_2 = g(\underline{X})\Big|_{\underline{X}=\underline{X}'} - c_1 \tag{A11}$$

and $$a_{2k} = \frac{1}{c_2}\left[\left.\frac{\partial g(\underline{X})}{\partial x_k}\right|_{\underline{X}=\underline{X}'} - c_1 a_{1k}\right] , \quad k = 1,\ldots,n.$$

. . . (A12)

For ease of notation define

$$g(\underline{X})\Big|_{\underline{X}=\underline{X}'} = g \quad , \quad \left.\frac{\partial g(\underline{X})}{\partial x_k}\right|_{\underline{X}=\underline{X}'} = g^k$$

and $$\left.\frac{\partial^2 g(\underline{X})}{\partial x_k^2}\right|_{\underline{X}=\underline{X}'} = g^{kk} \quad .$$

Substitution into (A10) gives

$$c_1 a_{1k}(a_{1k} - 1) + (g - c_1) \cdot \frac{1}{(g - c_1)} \left[g^k - c_1 a_{1k}\right] \cdot$$

$$\left[\frac{1}{(g - c_1)} (g^k - c_1 a_{1k}) - 1\right] = g^{kk}$$

Solving for $a_{1k}$ now gives

$$a_{1k} = g^k \pm \frac{1}{c_1 g}\left[ (g^k)^2 c_1{}^2 - c_1 g (g^k)^2 + c_1 g^2 g^k - c_1{}^2 g g^k + c_1 g g^{kk}(g - c_1) \right]^{\frac{1}{2}} , \quad k = 1,\ldots,n \quad . \qquad \ldots (A13)$$

The definition of $a_{1k}$ therefore depends on the expression within the square bracket in (A13) being non-negative. That is

$$(g^k)^2 c_1{}^2 - c_1 g (g^k)^2 + c_1 g^2 g^k - c_1{}^2 g g^k + c_1 g g^{kk}(g - c_1) \geq 0$$

or for $c_1 \geq 0$

$$-(g^k)^2 (g - c_1) + g^k g (g - c_1) + g g^{kk} (g - c_1) \geq 0 \quad .$$

This requires

$$g(g^k + g^{kk}) \geq (g^k)^2 \quad \text{if} \quad g > c_1 \qquad \text{(A14)}$$

or $\quad g(g^k + g^{kk}) \leq (g^k)^2 \quad \text{if} \quad g < c_1$ .

Since equations (A8), (A9) and (A10) only define $2n+1$ of the constants, one can be arbitrarily set. If $c_1$ and $c_2$ are set such that

$$c_1 = c_2 = \frac{g}{2}$$

condition (A14) operates. It can be seen that $a_{1k}$ can be defined if $g^{kk} >> 0$ . This positive second derivative relates to the condition that the Hessian be positive semi-definite for convexity given in Section 5.2.

It would therefore appear that it will be possible to define $a_{1k}$ if the constraint function $g$ is highly curved and convex. However it will not always be possible to generate the two term posynomials matching the second derivatives defined in (A10). When the term within the square brackets of (A13) is negative $a_{1k}$ can be set to zero. Equations (A11) and (A12) will still ensure that the current value and first derivative will be matched at the operating point $\underline{x}'$ .

# APPENDIX B

## COMPUTATION ASPECTS FOR STRUCTURAL DESIGN

The computer programs used to generate the results for the detailed structural examples presented in Chapter 9 are described in detail in [44] . Only a brief description of the computational procedures will therefore be given in this appendix.

The finite element analysis procedure was used to analyse the structures for deflections and stresses. The idealization into finite elements involved taking each pin-ended bar element, or the panels bounded by the spars and the ribs in the wing box, as different elements. For these simple elements and fixed geometry design, it is then possible to write the element stiffness matrix $\underline{k}_{e_i}$ as

$$\underline{k}_{e_i} = x_i . \underline{k}'_{e_i} \tag{A15}$$

where $x_i$ is the design variable associated with the ith element and $\underline{k}'_{e_i}$ can be found by generating the normal element matrix for $x_i=1$ . The matrix $\underline{k}'_{e_i}$ need then only be found once during the design process. The element matrices are then assembled to give the global stiffness matrix $\underline{K}$ with

$$\underline{K}\,\underline{r} = \underline{P} \tag{A16}$$

and

$$\underline{\sigma} = \underline{B}\,\underline{r} \tag{A17}$$

where $\underline{P}$ are the applied loads, $\underline{r}$ the nodal displacements and $\underline{B}$ is the stress matrix derived for each element. The Cholesky scheme was then used to solve (A16) by triangularization of the symmetric matrix $\underline{K}$ such that

$$\underline{K} = \underline{L}\,\underline{L}^t \quad . \tag{A18}$$

Equations (A16) now become

$$\underline{L}\,\underline{L}^t\,\underline{r} = \underline{P} \quad .$$

Defining $$\underline{L}^t\,\underline{r} = \underline{u} \quad . \qquad (A19)$$

gives $$\underline{L}\,\underline{u} = \underline{P} \qquad (A20)$$

with $\underline{L}$ and $\underline{L}^t$ lower and upper triangular matrices respectively. Equations (A20) and (A19) are the forward and backward substitution operations to define $\underline{r}$ . These operations are relatively inexpensive for banded equations when compared to the effort required for the triangularization in (A18).

Some special purpose techniques have been developed [25] to evaluate the gradients $\frac{\partial \underline{\sigma}}{\partial x_j}$ for pin-ended bar structures. However, these procedures were not incorporated into the computer programs. Instead a general purpose procedure based on differentiation of the finite element equations was used. From (A16)

$$\underline{K}\,\frac{\partial \underline{r}}{\partial x_j} = -\frac{\partial \underline{K}}{\partial x_j} \cdot \underline{r} \quad \text{for} \quad \frac{\partial \underline{P}}{\partial x_j} = \underline{0} \, . \qquad (A21)$$

However (A15) gives

$$\frac{\partial \underline{K}}{\partial x_j} = \underline{k}'_{ei}$$

with $\underline{k}'_{ei}$ assembled alone into the global matrix. Equations (A17) then give

$$\frac{\partial \underline{\sigma}}{\partial x_j} = \underline{B}\,\frac{\partial \underline{r}}{\partial x_j} \quad \text{with} \quad \frac{\partial \underline{B}}{\partial x_j} = \underline{0} \quad \text{for fixed geometry design.}$$

Since a static analysis always precedes the design cycle the matrix $\underline{L}$ from (A18) can be stored and used to solve (A21). The solution for the gradients in (A21) therefore requires only the application of the forward and backward substitution operations defined by (A19) and (A20).

A number of procedures are discussed in [39] for reducing the computational expense when repeated analysis of structures with only minor design changes are required. These procedures were again not incorporated in the design program since the research was directed towards the investigation of different features of the saddle function. However they could form the basis for future research and improvements to the computer programs.

Details of the operations for the dual procedures will not be considered here because they are described in detail in the text of this thesis. In particular the individual steps were detailed in Section 7.2 and this sequence was followed in the computer programs. Particular use of the linearity of the dual constraints was made in the search in the dual plane and in the definition of the pseudo-constraints, and led to the conclusion that highly efficient algorithms could be developed.

A simple flow chart showing the modular nature of the program and the flow of the operations is given in Fig. A1. Reference [44] should be consulted for more details of the layout and operations of the program.

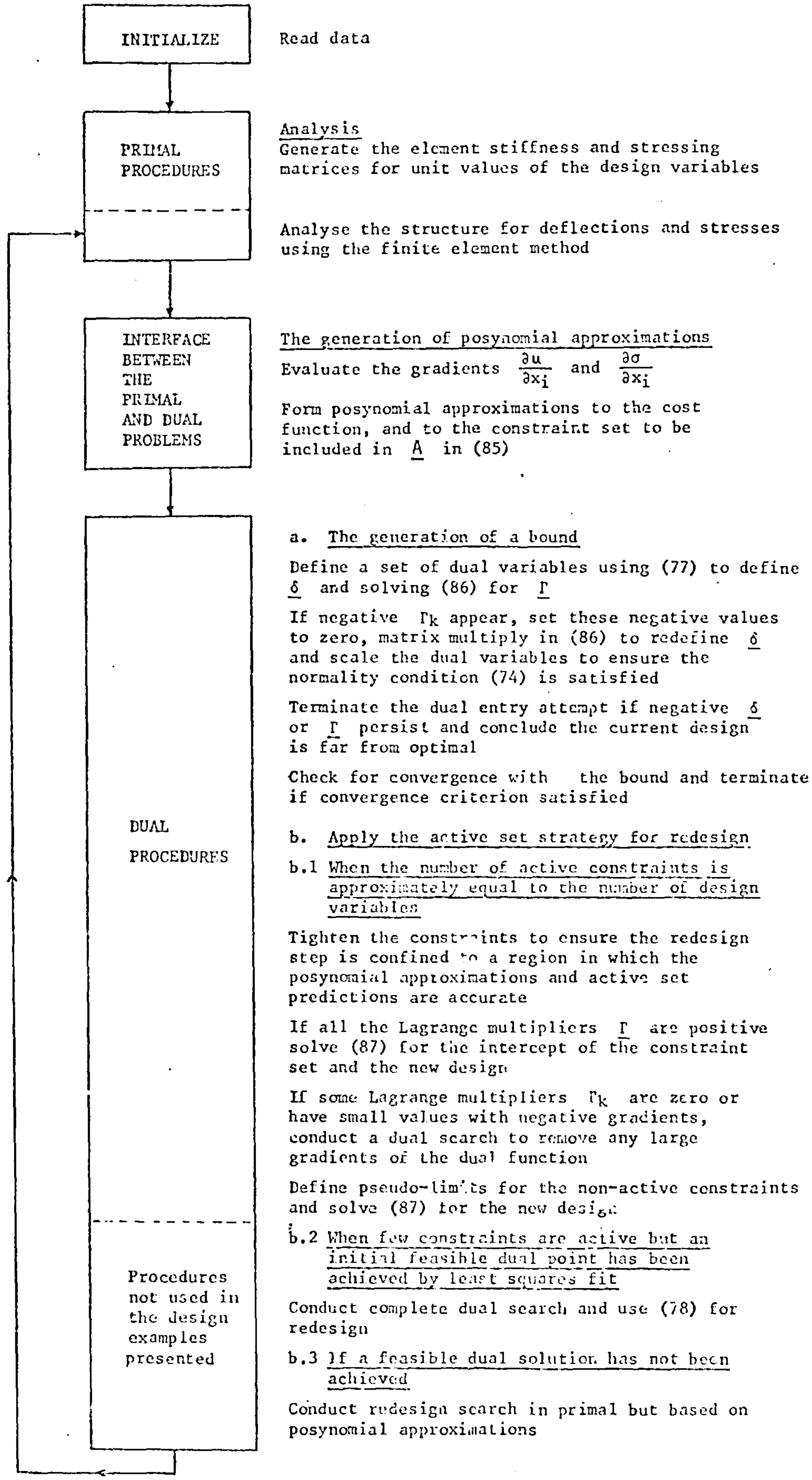


Figure A1 Operations flow chart for the computer program used to generate the results for the examples in Chapter 9