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HIGH ENERGY BEHAVIOUR AND SUPERSYMMETRY

by

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PREFACE

The work presented in this thesis was carried out in the Department of Theoretical Physics, Imperial College, between October 1972 and May 1976 under the supervision of Dr. R. Delbourgo. Except where otherwise stated, this work is original and has not been submitted for a degree of this or any other university.

The author wishes to thank Dr. Delbourgo for suggesting the problems on which this thesis is based and for his advice and encouragement during completion of this work.

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ABSTRACT

This thesis is in two parts.

In Part One, we investigate the high energy behaviour of scattering amplitudes in a supersymmetric field theory involving scalar, pseudoscalar and spinor fields.

In the lowest orders in the perturbation expansion, we find that a certain class of ladder diagrams in each order give the leading logarithm behaviour of the scalar-scalar scattering amplitude. The sum to all orders of these diagrams indicates the presence of a series of fixed Regge branch cuts, coming from the increasing number of possible two-particle exchange channels in the higher orders. The spinor-scalar and spinor-spinor scattering amplitudes are investigated in the lowest orders. It is found that these have the same form as the scalar-scalar amplitude, demonstrating the preservation of the supersymmetry of the theory in the leading logarithm approximation.

Finally we reconsider the problem using a perturbation expansion in terms of superfields. Working in a manifestly supersymmetric framework throughout, we show that the class of diagrams which give the leading behaviour is the same for any scattering process, the only change needed in each case being the insertion of the appropriate external wave functions.

In Part Two, we show how the famous axial vector anomalies arise naturally in the framework of dimensional regularisation.

We consider an SU(n) symmetric theory of a spinor field, coupling to external scalar, pseudoscalar, vector and axial vector

fields. In arbitary dimensions there is an additional pseudoscalar current in the expression for the divergence of the axial vector current. This new current, which does not exist in four dimensions, produces exactly the abnormal amplitude anomalies found using other techniques of regularisation. A set of self consistent normal amplitude anomalies is also produced, which could be subtracted out using acceptable counter terms in the Lagrangian. From the anomalous terms a modified PCAC relation is derived.

CONTENTS

This thesis is dedicated to my parents,

GEORGE and CATHERINE KEE

PART ONE

HIGH ENERGY BEHAVIOUR AND SUPERSYMMETRY

PART ONE

I. INTRODUCTION

The possibility of a fundamental symmetry between bosons and fermions has attracted a lot of interest recently. The idea was first introduced in the context of dual models formulated as field theories in two dimensions[1]. This was generalised to four dimensions by Wess and Zumino[2]. They constructed a non trivial Lagrangian field theory[3] containing fields of spin zero and spin one-half, invariant under four dimensional supersymmetric transformations. The theory only has logarithmic divergences, as higher divergences expected from power counting arguments cancel among themselves. This cancellation of divergences has proved to be a feature of supersymmetric theories.

Many attempts have been made to produce more realistic supersymmetric theories, in particular in the context of non-abelian gauge theories[4]. One criterion for any realistic field theory is that its asymptotic limit should be consistent with the experimental evidence that the high energy behaviour of scattering amplitudes, and furthermore the particle states, lie on the same straight line Regge trajectory. The high energy limit of scattering amplitudes in field theories is usually obtained by finding the leading logarithm contribution in each order of perturbation theory, and summing to all orders. This is certainly not a rigorous method, but in theories where independent calculations are possible it is found not to be misleading, and so it is believed to be a reliable heuristic approach.

We shall investigate the high energy limit of the theory of Wess and Zumino[3]. As this is not a realistic model the form of the scattering amplitudes, although interesting, are not of particular importance. The really interesting questions are whether there are any cancellations in each order, between the higher logarithm terms we might expect from such a theory, which could aid possible Regge behaviour in a more realistic model; and whether the supersymmetry of the theory ismaintained in the leading logarithm approximation. By this we mean: Is the form of the amplitude the same in boson-boson, boson-fermion, and fermion-fermion scattering processes? We would then expect this to be the case in other, more realistic supersymmetric theories.

Now, it has been shown in an SU(n) symmetric non abelian gauge theory, up to sixth order, that the vector meson of the theory lies on the Regge trajectory of the fermion-fermion scattering amplitude, a desirable phenomenon known as Reggeisation[5]. In a non abelian supergauge theory we would therefore expect these Regge-isation effects to occur in other scattering processes. It would, of course be nice to show this directly, but this taskis by no means a trivial one!

We first look at the nature of the supersymmetric transformations, and of the Lagrangian of Wess and Zumino. We then consider the various scattering amplitudes in the theory. In the scalar -scalar scattering amplitude we find that a particular class of ladder diagrams gives the leading behaviour, and in the lowest orders we show that all other diagrams are non leading. We obtain an expression for the contribution to the leading behaviour from any order in perturbation theory. The amplitude turns out not to have a Regge pole form, as there are not the necessary cancellations of the higher powers of logarithms between diagrams representing the various possible types of exchanges. The sum to all orders indicates the presence of a series of fixed Regge branch cuts, coming from the increasing number of two particle exchange channels in the higher orders of the perturbation expansion.

We then investigate in the lowest orders the spinor-scalar and spinor-spinor scattering amplitudes, and find that in these orders the amplitudes obtained are of the same form, regardless of which particular scattering amplitude is considered, thus demonstrating the preservation of the supersymmetry of the theory in the leading logartihm approximation.

In the final section we reconsider the problem using a perturbation expansion in terms of superfields, an approach to supersymmetry which was initiated by Salam and Strathdee[14]; they showed that supersymmetric transformations may be viewed as a realisation of a supersymmetric 'group' on some generalised fields, which they called superfields, defined on an eight-dimensional space, where points are labelled by (x_μ, θ_α), x_μ denoting the ordinary space-time coordinates and θ_α an anticommuting Majorana spinor. We first look at some properties of superfields, the construction of the Lagrangian of Wess and Zumino using them, and the corresponding Feynman rules in perturbation theory[6]. Again we find that a class of ladder diagrams are the important ones and we obtain the same form for the scalar-scalar scattering amplitude. We would expect the equality of amplitudes to hold as we work in a manifestly supersymmetric framework throughout, and it turns out that for each different scattering process the only change we have to make is to put in the appropriate external wave functions. Thus using superfield perturbation theory enables us to evaluate the high energy behaviour of the different scattering processes just from one set of ladder graphs, showing the power of the technique in simplifying calcu lations.

II. SUPERSYMMETRY

Fermibose supersymmetry is a symmetry which connects particles of integral spin with particles of half integral spin, i.e. bosons with fermions, first introduced by Wess and Zumino[2].

The supersymmetry algebra is quite simple:

$$
\begin{aligned}
[S_\alpha, P_\mu] &= 0 \\
[S_\alpha, J_{\mu\nu}] &= \frac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta S_\beta \qquad (2.1) \\
\{S_\alpha, S_\beta\} &= -(\gamma_\mu C)_{\alpha\beta} P_\mu
\end{aligned}
$$

where P_μ, $J_{\mu\nu}$ are the usual Poincare operators, and where S_α is the generator of the supersymmetric transformation, and is a Majorana spinor.

The simplest supermultiplet consists of a scalar field A, a pseudoscalar field B, a Majorana spinor field ψ, and two auxiliary fields F and G.

Writing

$$\delta A = [\bar{\epsilon} S, A] \quad \text{,etc.}$$

we obtain for an infinitesmal supersymmetric transformation,

$$\delta A = \bar{\epsilon}\psi$$

$$\delta B = \bar{\epsilon}\gamma_5\psi$$

$$\delta\psi = \not\partial(A - \gamma_5 B)\epsilon + (F + \gamma_5 G)\epsilon \tag{2.2}$$

$$\delta F = \bar{\epsilon}\not\partial\psi$$

$$\delta G = \bar{\epsilon}\gamma_5\not\partial\psi$$

where $\bar{\varepsilon}$ is an infinitesmal Majorana spinor.

With this scalar multiplet Wess and Zumino[3] constructed a non trivial supersymmetric model with Lagrangian:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_m + \mathcal{L}_g \tag{2.3}$$

where,

$$\mathcal{L}_0 = \frac{1}{2}(\partial A)^2 + \frac{1}{2}(\partial B)^2 + \frac{i}{2}\bar{\psi}\not\partial\psi + \frac{1}{2}F^2 + \frac{1}{2}G^2 \tag{2.4}$$

$$\mathcal{L}_m = m\left(AF + BG - \frac{1}{2}\bar{\psi}\psi\right) \tag{2.5}$$

$$\mathcal{L}_g = g\left(FA^2 - FB^2 + 2GAB - \bar{\psi}(A - \gamma_5 B)\psi\right) \tag{2.6}$$

These three terms each change under the supersymmetric transformation (2.2) by a total divergence, and therefore have an invariant action integral. For example,

$$[\bar{\epsilon}S, \mathcal{L}_m] = \bar{\epsilon}\not\partial[(A - \gamma_5 B)\psi] \tag{2.7}$$

The auxiliary fields F and G could be eliminated using their own equations of motion

$$F + mA + g(A^2 - B^2) = 0 \qquad (2.8)$$

$$G + mB + 2gAB = 0 \qquad (2.9)$$

and the Lagrangian (2.3) would take on the more familiar form,

$$\mathcal{L} = \tfrac{1}{2}(\partial A)^2 + \tfrac{1}{2}(\partial B)^2 + \tfrac{i}{2}\bar{\psi}\not{\partial}\psi - \tfrac{1}{2}m^2A^2$$

$$-\tfrac{1}{2}m^2B^2 - \tfrac{m}{2}\bar{\psi}\psi - gmA(A^2+B^2) \qquad (2.10)$$

$$-\tfrac{1}{2}g^2(A^2+B^2)^2 - \bar{\psi}(A - \gamma_5 B)\psi$$

i.e. a theory involving Yukawa couplings, and tri- and quadrilinear couplings in the scalar and pseudoscalar fields, but with all the couplings expressed in terms of one coupling constant g, and with the scalar, pseudoscalar and spinor fields having the same mass.

Iliopoulos, Wess and Zumino[3,7] have investigated the renormalisation properties of this theory. They have shown that there are only logarithmic divergences in the lowest orders, any higher divergences cancelling among themselves, and that only one (wave function) renormalisation constant is needed common to all fields.

We will investigate this theory in the form (2.3), as it contains only trilinear couplings. These are much more convenient to work with in the high energy region, as the class of diagrams which give the leading behaviour are ladder diagrams. We would, of course, obtain the same results with the Lagrangian in the form (2.10).

III. SCALAR-SCALAR SCATTERING AMPLITUDE

In this section we investigate the leading logarithm behaviour of each order in perturbation theory of the scalar-scalar scattering amplitude of the supersymmetric theory of Wess and Zumino,described by the Lagrangian (2.10).

In the lowest orders, up to sixth order, we find that a class of ladder diagrams gives the leading behaviour to a particular order and we feel that these orders are sufficiently non-trivial for us to expect this to be true to any order in this theory, as has been shown to be the case in ordinary ϕ^3 theory[8].

(a) General Ladder Graph

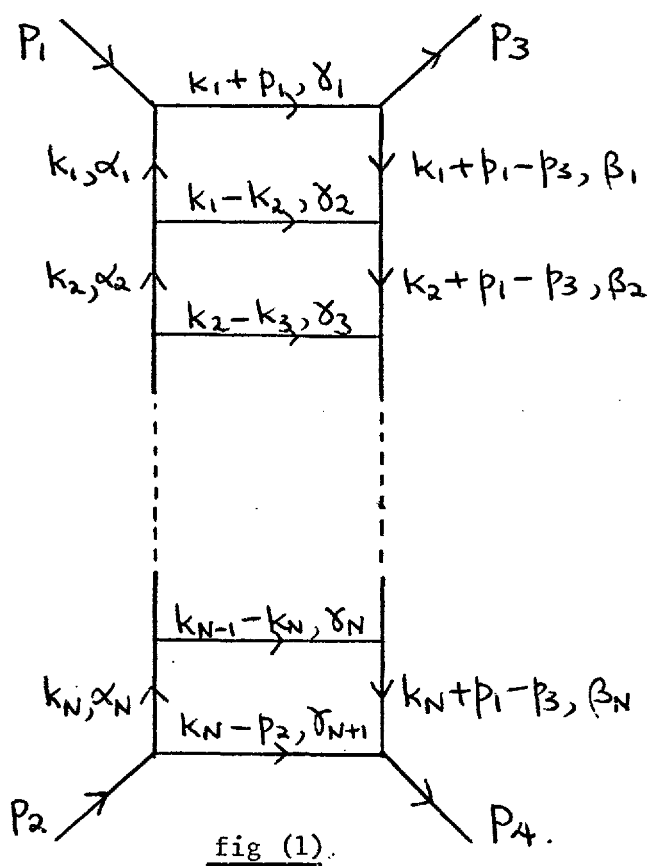


fig (1).

We first look at the general t-channel ladder graph of N loops shown in fig.(1). Referring to the momentum assignments there,we define the variables:

$$
\begin{aligned}
s &= (p_1+p_2)^2 \\
t &= (p_1-p_3)^2 \\
u &= (p_1-p_4)^2
\end{aligned}
\tag{3.1}
$$

The 3N+1 individual propagator denominators are all combined using Feynman parameters, where the parameters associated with each line are displayed in the figure. Because the sum of the diagrams does not diverge, and as each individual diagram is only logarithmically divergent at most, we can translate the 4N component integration variable,(a column vector $\underline{k}$ made up of N k_i's), so as to remove the cross terms with the external momenta. The matrix in the remaining terms is then diagonalised by an orthogonal change of variables. The resulting expression for an N loop ladder is

$$
T_N(s,t)=\Gamma(3N+1)\, i^{N-1}(2g)^{2N+2}\int\prod_{i=1}^{N}d\alpha_i\int\prod_{j=1}^{N}d\beta_j\int\prod_{k=1}^{N+1}d\gamma_k
$$

$$
\times\int\prod_{\ell=1}^{N}d^4k_\ell\left\{\frac{\delta\left(1-\sum\alpha_i-\sum\beta_j-\sum\gamma_k\right)N(k_1'\ldots k_N',s,t)}{\left(A_1k_1^2+A_2k_2^2+\ldots+A_Nk_N^2+D(N)/C(N)\right)^{3N+1}}\right\}
\tag{3.2}
$$

where $N(k_i',s,t)$ is the numerator of the particular diagram under consideration. The determinants D(N) and C(N) can be read from the table in appendix two, and

$$A_i = \alpha_i + \beta_i \text{ +terms involving } \gamma_j\text{'s} \tag{3.3}$$

$$C(N) = A_1 A_2 \cdots\cdots A_N = C \; ; \qquad D(N) = D \tag{3.4}$$

$$k_i' = k_i - \frac{1}{C}\left(\gamma_1 \ldots \gamma_i \Gamma_{i+1} \ldots \Gamma_N\right) p_1$$

$$+ \frac{1}{C}\left(\gamma_{i+1} \ldots \gamma_{N+1} \Gamma_1 \ldots \Gamma_{i-1}\right) p_2 \tag{3.5}$$

+ terms involving other k_j, p_1, or p_2, but with more γ_j parameters

+ terms involving p_1-p_3

Equation (3.5) is a consequence of the manipulations involved in obtaining a form of the denominator of (3.2) ameniable to symmetric integration. With this form we can simplify $N(k_i', s, t)$ by using the relation

$$\int k_\mu k_\nu f(k^2) d^4k = \frac{1}{4} g_{\mu\nu} \int k^2 f(k^2) d^4k \tag{3.6}$$

(b) ϕ^3 Theory

In the case of ϕ^3 scalar theory, $N(k', s't) \sim 1$, and the loop integrations in (3.2) can be performed using repeated applications of the integral:

$$\int \frac{d^4k}{(Ak^2+B)^a} = \frac{-i\pi^2}{A^2 B^{a-2}} \cdot \frac{\Gamma(a-2)}{\Gamma(a)} \tag{3.7}$$

and we obtain

$$T_N(s,t) = -ig^2\left[\frac{\pi^2 g^2}{(2\pi)^4}\right]^N N! \int \prod_{i=1}^{N} d\alpha_i \int \prod_{j=1}^{N} d\beta_j \int \prod_{k=1}^{N+1} d\gamma_k$$
$$\times \left\{ \delta\left(1-\sum\alpha_i-\sum\beta_j-\sum\gamma_k\right)\frac{C^{N-1}}{D^{N+1}} \right\} \tag{3.8}$$

From the table in appendix two we see that:

$$D = \left(\prod_{i=1}^{N+1} \gamma_i s\right) \quad \text{+ terms independent of s} \tag{3.9}$$

As $s \to \infty$, with t fixed, we expect powers of logarithms to arise from the vanishing of the coefficient of s in D. Each $\gamma_i = 0$ region implies a logarithm, and if we do the calculation[9], we find that in the limit this is indeed the case:-

$$T_N(s,t) \sim s^{-1} \ln^{N+1} s \tag{3.10}$$

(c) Wess-Zumino Theory

In this theory there will be certain differences in the calculation, due to the nature of the propagators, which can enhance the leading behaviour of $T_N(s,t)$ in the following ways:

(i) The numerator contains explicit powers of s.

(ii) Powers of the loop momenta in the numerator reduce the power of the denominator after integration.

(iii) Powers of the determinant C in the denominator produce extra logarithms because C vanishes in certain regions of Feynman parameter space.

(iv) Displacement of the loop momenta, as in (3.5), produces both explicit powers of s in the numerator and increases the power of C in the denominator.

We first consider the effects of these in the lowest orders. N = 0, corresponding to the Born term, fig.(2),

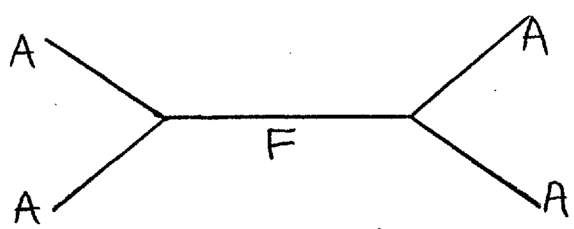


Fig.(2)

$$T_0(s,t) = i(2g)^2 \frac{s}{s-m^2} \sim -i(2g)^2 \tag{3.11}$$

N = 1, we find the diagrams of fig.(3) to be the dominant ones, and we will show in the next section that all other fourth order diagrams will be non leading. These diagrams give

$$T_1(s,t) = 3!\,(2g)^4 \int d\alpha\, d\beta\, d\gamma_1\, d\gamma_2\, \delta(1-\alpha-\beta-\gamma_1-\gamma_2) \times \int d^4k \,\frac{N(k',s,t)}{(k^2+D)^4}. \tag{3.12}$$

Here C(1) = 1, and $D = \gamma_1\gamma_2 s - m^2$ + terms independent of s, and

$$k' = k - \gamma_1 p_1 + \gamma_2 p_2 - \alpha_1 (p_1 - p_3) \tag{3.13}$$

The terms in N(k',s,t) which give the maximum enhancement are:

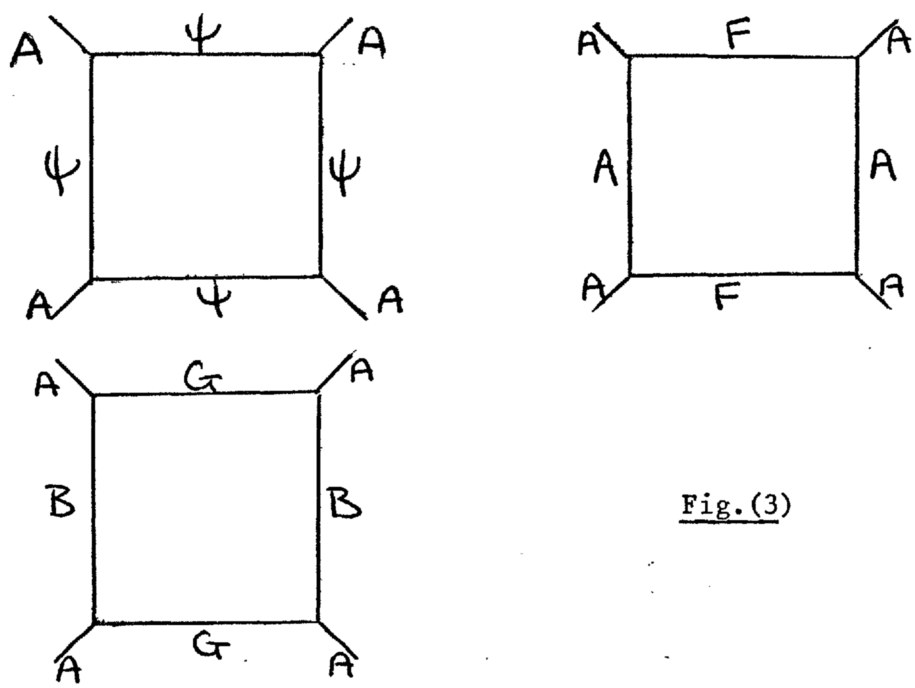


Fig.(3)

$$N(k',s,t) \sim -2\left(k^2 s - s^2 \gamma_1 \gamma_2\right) \qquad (3.14)$$

Terms in the numerator can at most involve products of four momenta. In the first term of (3.14) there is enhancement both from the explicit s term, and from the k^2 term which lowers the power of the denominator which enhances the logarithmic dependence. The second term has an explicit s^2 dependence, but this is compensated for by the $\gamma_1 \gamma_2$ coefficient which in the dominant $\gamma_1 \sim \gamma_2 \sim 0$ region will lower the power of s, although enhancing the logarithmic dependence. Other terms in the numerator, such as $k^2 s \gamma_1$; $s^2 \gamma_1^{\,2} \gamma_2$; will give a behaviour at least one power of lns down, except for the k^4 term. This is divergent, but we will show later that it cancels with similar terms in other diagrams.

We can now perform the loop integrations using (3.6) and

$$\int \frac{k^2 d^4k}{(k^2+D)^a} = \frac{-2i\pi^2}{D^{a-3}} \cdot \frac{\Gamma(a-3)}{\Gamma(a)} \qquad (3.15)$$

and we obtain

$$T_1(s,t) \sim 2i\pi^2\left(\frac{g}{\pi}\right)^4 \int d\alpha\, d\beta\, d\gamma_1\, d\gamma_2\, \delta(1-\alpha-\beta-\gamma_1-\gamma_2) \times \left\{ \frac{2s}{D} - \frac{s^2\gamma_1\gamma_2}{D^2} \right\} \tag{3.16}$$

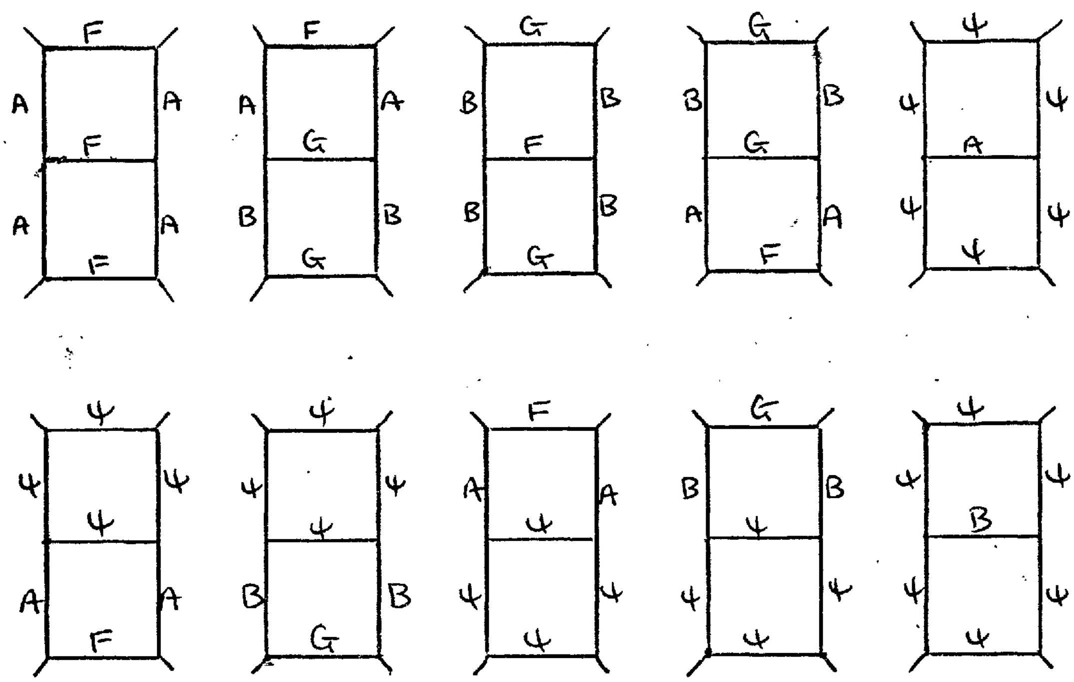
As $s \to \infty$ with t fixed, we can now do the γ_j integrations using

$$\int \prod_{i=1}^{n} d\gamma_i \frac{s(\gamma_1\gamma_2\ldots\gamma_n)^a}{(\gamma_1\gamma_2\ldots\gamma_n s + \text{constant})^{a+1}} = \frac{s^{-a}\ln^n s}{n!} \tag{3.17}$$

The α and β integrations then give unity and we obtain

$$T_1(s,t) \sim i\pi^2\left(\frac{g}{\pi}\right)^4 \ln^2 s \tag{3.18}$$

<u>N = 2</u>, the leading behaviour comes from the diagrams of fig.(4), and we show later on that no other diagrams in this order contribuute.



<u>Fig.(4)</u>

These give

$$T_2(s,t) \sim 6!(2g)^6 i \int d\alpha_1 d\alpha_2 d\beta_1 d\beta_2 d\gamma_1 d\gamma_2 d\gamma_3 \tag{3.19}$$

$$\times \frac{\delta(1-\Sigma_i\alpha-\Sigma_i\beta-\Sigma_i\gamma)\, N(k_1',k_2',s,t)}{(A_1k_1^2+A_2k_2^2+D/C)^7}$$

The maximum enhancement will come from terms in the numerator

$$N(k_1',k_2',s,t) \sim 4\Big[k_1^2k_2^2s - s\frac{\gamma_1\gamma_2\gamma_3}{C^2}\Big(k_1^2(\alpha_1+\beta_1)$$

$$+k_2^2(\alpha_2+\beta_2)\Big) + s^2\left(\frac{\gamma_1\gamma_2\gamma_3}{C^2}\right)^2(\alpha_1+\beta_1)(\alpha_2+\beta_2)\Big] \tag{3.20}$$

Here the extra powers of s and C are compensated for by the extra γ parameters, and the loss of loop momentum factors in the numerator.

We can now perform the loop integrations using repeated applications of (3.6) and (3.15):

$$T_2(s,t) \sim -2i\pi^2\left(\frac{g}{\pi}\right)^6 \int \prod_{i,j}^{2} d\alpha_i\, d\beta_j \prod_k^3 d\gamma_k$$

$$\times \frac{\delta(1-\Sigma_i\alpha-\Sigma_i\beta-\Sigma_i\gamma)\,s}{C^2}\left\{\frac{2}{D}-\frac{2\gamma_1\gamma_2\gamma_3 s}{D^2}+\frac{(\gamma_1\gamma_2\gamma_3)^2s^2}{D^3}\right\}$$

$$\sim -\frac{i\pi^2}{6}\left(\frac{g}{\pi}\right)^6 \ln^4 s \tag{3.21}$$

We should elaborate on how (3.21) arises. We make the transformation of variables

$$r_i = \alpha_i + \beta_i$$

$$\bar{\alpha}_i = \frac{\alpha_i}{\alpha_i + \beta_i} \quad ; \quad 0 \leq \bar{\alpha}_i \leq 1 \qquad (3.22)$$

This transformation is useful, as it has Jacobian $\prod_{i=1}^{N} r_i$, and C depends on r_i and γ_j parameters alone:

$$C(2) = (r_1 + \gamma_1 + \gamma_2)(r_2 + \gamma_2 + \gamma_3) - \gamma_2^2 \qquad (3.23)$$

In general we find that the t dependence is buried in the arguments of logarithms and so the leading Regge singularities will be fixed in t. We can therefore set t = 0 in our analysis, and hence

$$D = \gamma_1\gamma_2\gamma_3 s - m^2 C + \text{terms of higher order in the vanishing parameters } \gamma_j \qquad (3.24)$$

We will just consider the effect of the first term in (3.21), which reads

$$-4i\pi^2\left(\frac{g}{\pi}\right)^6 s \int \frac{r_1 dr_1 \, r_2 dr_2 \, d\gamma_1 d\gamma_2 d\gamma_3 \, \delta(1 - \sum_1 r - \sum_1 \gamma)}{C^2 \, (\gamma_1\gamma_2\gamma_3 s - m^2 C)}$$

We notice that either $r_1 = 1$ or $r_2 = 1$ in the important regions. If we choose any other end point regions, such as $\gamma_1 = 1$, these will give two powers less of lns. Using the standard procedure[10], we rescale the variables:

$$r_1 = 1 - \rho \qquad r_2 = \rho r_2' \qquad \gamma_i = \rho \gamma_i' \qquad (3.26)$$

for the $r_1 = 1$ region, where $\rho << 1$. Then (3.25) becomes

$$-4i\pi^2\left(\frac{g}{\pi}\right)^6 s\int_0^\epsilon \rho\, d\rho \int_0^1 \frac{r_2'\,dr_2'\,d\gamma_1'\,d\gamma_2'\,d\gamma_3'\,\delta(1-r_2'-\Sigma_1'\gamma')}{(r_2'+\gamma_2'+\gamma_3')^2(\rho^2\gamma_1'\gamma_2'\gamma_3's-m^2(r_2'+\gamma_2'+\gamma_3'))}$$

$$+\ (r_2=1\ \text{piece}) \qquad (3.27)$$

The two dominant regions in the first part of (3.27) are the $\gamma_1' = 1$ or $r_2' = 1$ regions. We rescale again in these regions and we are left with integrals of the form of (3.17), and on adding the $r_2 = 1$ piece , and that of the other terms in (3.21), we obtain the result there.

No other terms in the numerator will contribute, as they will lessen the enhancement effects of, or will have too many γ parameters than, those in (3.20).

So up to sixth order we have a series in $g^2\ln^2 s$. The contributing diagrams, i.e. those of figs.(2), (3) and (4), in each order are ladder diagrams with F and G rungs, or with rungs inside or part of spinor loops. This is found to be the case for any order, and we shall now consider the general term for any order.

(d) Numerator of the General Term

We have a general expression (3.2) for an N loop ladder diagram, with only the numerator function $N(k_1',..,k_N',s,t)$ dependent on which particular ladder is under consideration.

For ordinary ϕ^3 theory, $N(k_1,..,k_N,s,t) \sim 1$ and a behaviour of $s^{-1}\ln^{N+1}s$ is obtained[9].

In the Wess-Zumino theory we find that the terms in the numerator whcih give the maximum enhancement of the leading behaviour are of the form:

$N(k_1',..,k_N',s,t) \sim k_1^2 ... k_N^2 s$ + terms involving the displacement of all or some of the loop momenta (3.28)

where the displacements of the loop momenta are given by (3.5). As we found in the lowest order calculations, these terms can only come from ladder diagrams where the rungs are either F or G lines, or inside or part of a spinor loop. Any other type of ladder, e.g. one with some F lines as sides, will lose enhancement factors and hence will not contribute.

(i) We first consider the ladder graphs where all the rungs are either F or G lines. There must be an even number of G lines, due to the nature of the G interaction term, and parity conservation. In the lowest orders there are 2^N diagrams of this type and we can show by induction that this is generally true. The numerator for this class of diagrams is:

$$N(k_i', s, t) = 2^N (k_1'+p_1)^2 (k_1'-k_2')^2 \dots (k_N'-p_2)^2$$

$$\sim \frac{(-1)^N s}{C} \Big\{ k_1^2 \dots k_N^2 \; C$$

$$+ \left(-2s \frac{\gamma_1 \dots \gamma_{N+1}}{C}\right) \left[k_1^2 \dots k_{N-1}^2 \tau_1 \dots \tau_{N-1} + \text{permutations of } N-1 \text{ loops} \right] \dots$$

$$+ \left(-2s\frac{\gamma_1\ldots\gamma_{N+1}}{C}\right)^2 \left[k_1^2\ldots k_{N-2}^2\,\tau_1\ldots\tau_{N-2} + \begin{matrix}\text{permutations}\\ \text{of } N-2 \text{ loops}\end{matrix}\right]$$

$$+ \ldots\ldots\ldots + \left(-2s\frac{\gamma_1\ldots\gamma_{N+1}}{C}\right)^{N-1} \left[k_1^2\tau_1 + k_2^2\tau_2 + \ldots + k_N^2\tau_N\right]$$

$$+ \left(-2s\frac{\gamma_1\ldots\gamma_{N+1}}{C}\right)^N \Bigg\} \tag{3.29}$$

+ terms similar to these but of lower power of s or with more γ parameters

We will show that all the terms except the last contribute to the leading behaviour. The last terms do not, as the extra γ parameters lower the power of lns obtained.

(ii) Next we consider the class of diagrams where one of the loops is a fermion loop, but where all the other rungs are either F or G lines, for example the diagram of fig.(5), where the loop with momentum k_1 is a spinor loop.

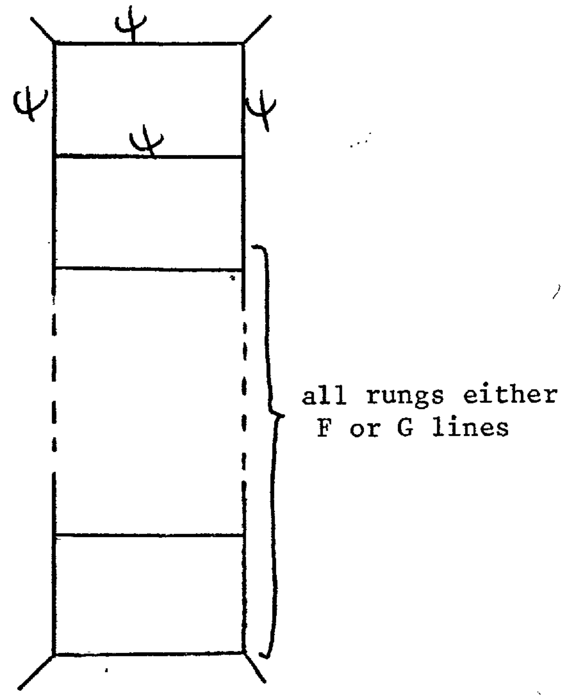


Fig.(5)

There are 2^{N-1} diagrams of this type. The contributing terms in the numerator are:

$$N(k_i', s, t) \sim \frac{(-1)^N s}{C} \Big\{ k_1^2 \ldots k_N^2 \; C$$

$$+ \left(-2s \frac{\gamma_1 \ldots \gamma_{N+1}}{C}\right)\Big[k_1^2 \ldots k_{N-1}^2 \, r_1 \ldots r_{N-1} + \begin{array}{l}\text{permutations of}\\ N-1 \text{ loops}\\ \text{involving } k_1^2\end{array} \Big]$$

$$+ \ldots\ldots + \left(-2s \frac{\gamma_1 \ldots \gamma_{N+1}}{C}\right)^{N-1} k_1^2 r_1 \Big\} \qquad (3.30)$$

In fact, (3.30) is the sum of the terms in (3.29) dependent on k_1^2. There is no term independent of the fermion loop momentum. The terms derived from the displacement of the loop momentum inside of the fermion loop do not contribute to the leading term. This is because when we take the trace of the loop momenta, we will always get extra powers of γ_i due to the ordering of the momenta inside of the trace. Here, for example we have:

$$tr\{(k_1' + p_1) k_1' (k_1' - k_2') k_1'\} \ldots\ldots \qquad (3.31)$$

compared with

$$(p_1 + k_1')^2 (k_1' - k_2')^2 \ldots\ldots \qquad (3.32)$$

from the diagrams with no fermion loop, as in (3.29).

In (3.32) the terms which contribute from the first two brackets are of the form $(k_1'.p_1)(k_1'.k_2')$ which will give a displacement term of $s^2\gamma_1\gamma_2(\gamma_3 \ldots \gamma_{N+1})^2$ which contributes to (3.29).

In (3.31), we will have a term $k_1.p_1\ k_1.k_2$, but the displacement term will be at best $s^2\gamma_1{}^2\gamma_2(\gamma_3\ldots.\gamma_{N+1})^2$, and due to the extra γ_1 factor, its contribution will be down by lns.

We find this to be the general rule, that displacement contributions form loop momenta inside a fermion loop will be non leading.

We obtain a similar result to (3.30) for all the other single fermion loop diagrams, and summing these gives a contribution similar to (3.29), but without the last term, the term independent of any loop momenta:

$$N(k_i',s,t) \sim \frac{(-1)^N s}{C}\Big\{{}^N C_1\ k_1^2\ldots k_N^2\, C$$

$$+\ {}^{N-1}C_1\left(-2s\,\frac{\gamma_1\ldots\gamma_{N+1}}{C}\right)\left[k_1^2\ldots k_{N-1}^2\,\tau_1\ldots\tau_{N-1} + \begin{matrix}\text{permutations}\\ \text{of } N-1 \text{ loops}\end{matrix}\right]$$

$$+\ldots\ldots + {}^1C_1\left(-2s\,\frac{\gamma_1\ldots\gamma_{N+1}}{C}\right)^{N-1}\left[k_1^2\tau_1 + \ldots + k_N^2\tau_N\right]\Big\} \tag{3.33}$$

(iii) The next class of graphs we will consider which contribute to the leading behaviour, are those where two of the loops are fermion loops or one fermion loop covers two loops of the ladder, where again the other rungs are either F or G lines. Examples are shown in fig.(6). There are 2^{N-1} diagrams of this type for each pair of loops. Again we find that the important terms in the numerator are of the form given by (3.29), but where the two loop momenta inside the fermion loop or loops do not contribute to the displacement terms.

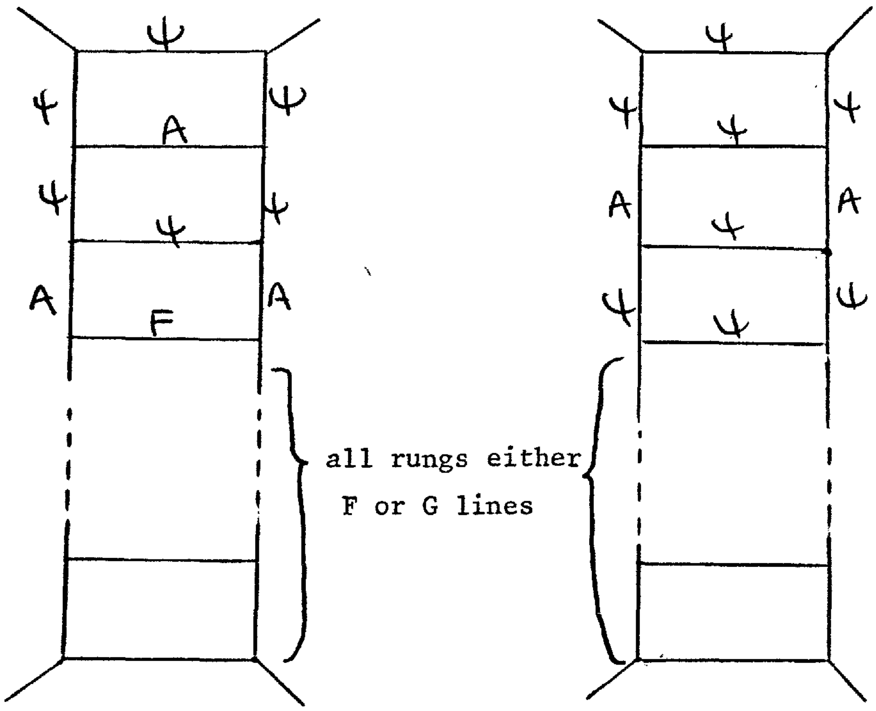


Fig.(6)

Summing all these diagrams gives us a contributing term in the numerator:

$$N(k_i', s, t) \sim \frac{(-1)^N s}{C}\left\{ {}^N C_2 \, k_1^2 \ldots k_N^2 \, C \right.$$

$$+ \, {}^{N-1}C_2 \left(-2s\frac{\gamma_1 \ldots \gamma_{N+1}}{C}\right)\left[k_1^2 \ldots k_{N-1}^2 \tau_1 \ldots \tau_{N-1} + \text{permutations of } N-1 \text{ loops}\right] \quad (3.34)$$

$$+ \ldots\ldots + \, {}^2C_2 \left(-2s\frac{\gamma_1 \ldots \gamma_{N+1}}{C}\right)^{N-2}\left[k_1^2 k_2^2 \tau_1 \tau_2 + \text{permutations of 2 loops}\right]$$

Note there are no terms dependent on less than two loop momenta.

(iv) By similar calcualations we can obtain the contributions from diagrams with more and more fermion loops.

(v) Finally we consider the diagrams where there is only one fermion loop, as in fig.(7).

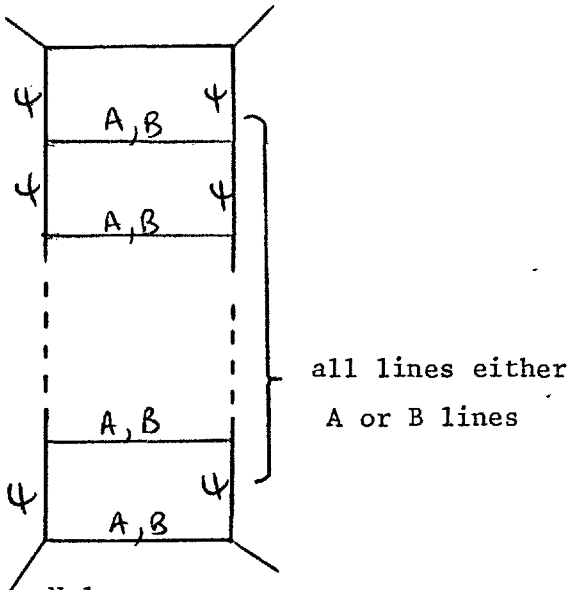


Fig.(7)

There are 2^{N-1} of this type of diagram. There will be no displacement contributions at all from these diagrams, and the important term in the numerator is just:

$$N(k_i', s, t) \sim \frac{(-1)^N s}{C}\left(k_1^2 \ldots k_N^2 C\right) \tag{3.35}$$

(vi) We are now in a position to sum up the contributions from all the diagrams.

The coefficient of the $k_1^2 \ldots . k_N^2$ term is

$$(-1)^N s \left(1 + {}^N C_1 + {}^N C_2 + \ldots + {}^N C_N\right)$$
$$= (1+1)^N (-1)^N s = (-1)^N 2^N s. \tag{3.36}$$

The coefficient of the $k_1^2 \ldots k_{N-1}^2$ term is:

$$\frac{(-1)^N s}{C}\left(-2s\frac{\gamma_1 \ldots \gamma_{N+1}}{C}\right)\left(1 + {}^{N-1}C_1 + \ldots + {}^{N-1}C_{N-1}\right)$$

$$= (-1)^{N+1} 2^N s^2 \frac{\gamma_1 \ldots \gamma_{N+1}}{C^2} \tag{3.37}$$

Similarly we can find the coefficients of all the terms and we obtain the contributing terms of the numerator for the sum of all the N loop ladders:

$$N(k_i', s, t) \sim \frac{2^N(-1)^N s}{C}\left\{ k_1^2 \ldots k_N^2 C \right.$$

$$+\left(-\frac{s\gamma_1 \ldots \gamma_{N+1}}{C}\right)\left[k_1^2 \ldots k_{N-1}^2 \tau_1 \ldots \tau_{N-1} + \text{permutations of } {}_{N-1} \text{ loops}\right]$$

$$\left. + \ldots\ldots + \left(-\frac{s\gamma_1 \ldots \gamma_{N+1}}{C}\right)^N \right\} \tag{3.38}$$

(e) Leading Behaviour of the General Term

In principle we could calcualate the leading behaviour of $T_N(s,t)$ by looking at the dominant regions of Feynman parameter space. But as we saw in the two loop case, this involves fairly intricate manipulation, and in the higher orders it becomes too difficult as the number of scalings becomes prohibitive.

We find it is much easier when lookin at the higher orders to use Mellin transform techniques[11]. To facilitate this, instead of deriving T_N using Feynman parameters to combine the denominators, we use the integral representation:

$$\frac{i}{k^2-m^2+i\epsilon} = \int_0^\infty d\alpha \exp\left[i\alpha(k^2-m^2+i\epsilon)\right] \tag{3.39}$$

In an exactly analogous manner to that we used to derive $T_N(s,t)$ previously, we find that:

$$T_N(s,t) = (2gi)^{2N+2}\int_0^\infty \prod_1^N d\alpha_i \prod_1^N d\beta_j \prod_1^{N+1} d\gamma_k \prod_1^N d^4k_\ell$$

$$\times N(k_i',s,t)\exp\left\{i\left(A_1k_1^2+\ldots+A_Nk_N^2+\frac{D}{C}\right)\right\} \tag{3.40}$$

A_i, k_j', D and C are exactly as defined before in (3.3), (3.4) and (3.5). Terms in $N(k_i',s,t)$ cause enhancement in the same ways as previously, and so $N(k_i',s,t)$ is still given by (3.38).

We can perform the loop integrations using:

$$\int d^4k\, e^{iak^2} = \frac{i\pi^2}{a^2} \tag{3.41}$$

$$\int d^4k\, k^2 e^{iak^2} = -\frac{2\pi^2}{a^3} \tag{3.42}$$

and we obtain:

$$T_N(s,t) \sim -4\sum_{M=0}^{N} {}^N C_M (-2i)^{M-N} g^2 \left(\frac{g}{\pi}\right)^{2N} \int_0^\infty \prod_1^N d\alpha_i \quad ---$$

$$\times \int_0^\infty \prod_1^N d\beta_j \int_0^\infty \prod_1^{N+1} d\gamma_k \; \frac{(\gamma_1 \ldots \gamma_{N+1} s)^{N-M} \, s \exp\left(\frac{iD}{C}\right)}{C^{N-M+3}} \tag{3.43}$$

The form of the coefficient of s in D, $D = \gamma_1\gamma_2 \cdots \gamma_{N+1} s + J$, where J is independent of s, makes the Mellin transform method a very convenient way of determining the leading behaviour of $T_N(s,t)$. The Mellin transform $T_N(\alpha,t)$ of $T_N(\sigma,t)$ is defined by[11]:

$$T_N(\alpha, t) = \int_0^\infty d\sigma \; T_N(\sigma, t) \, \sigma^{-\alpha-1} \tag{3.44}$$

where $\sigma = -s$. The validity of the Mellin transform requires that the integral should not encounter a singularity. We cannot take the limit $s \to \infty$ as we would encounter the normal s channel thresholds. If the fixed value of t is below the lowest t channel cut, the limit $s \to -\infty$, i.e. $\sigma \to \infty$ will be taken in a singularity free region. The dominant contributions to the asymptotic behavicur will come from the rightmost singularity in the αplane. In appendix three it is shown that the singularity at $\alpha = 0$ is the important one in each term of (3.44).

In (3.44) the leading contributions will come from small α_i and β_j regions as well as small γ_k regions. So we can replace the exp (iJ/C) factor by unity, but we must keep the full parameter dependence in the $C^{-2-\alpha}$ factor. We again make the transformation of variables (3.22) and take the $\alpha \to 0$ limit, and (3.44) becomes:

$$T_N(\alpha, t) = 4ig^2\left(\frac{g}{\pi}\right)^{2N} \sum_{M=0}^{N} {}^N C_M \, 2^{M-N} (-1)^{M+1}$$

$$\times \Gamma(N-M+1) \int_0^1 \prod_1^N \tau_i \, d\tau_i \int_0^\infty \prod_1^{N+1} d\gamma_j \; \frac{(\gamma_1 \ldots \gamma_{N+1})^{\alpha-1}}{C^{2+\alpha}} \tag{3.45}$$

The upper limits should be small numbers ε_i for the dominant regions, but these have been scaled out in favour of unity.

We now define[11],

$$A(\alpha,N,p)=\int_0^1 \prod r_i\, dr_i \int_0^\infty \prod d\gamma_j \frac{(\gamma_1\cdots\gamma_{N+1})^{\alpha-1}}{C^{2+\alpha}} \ln^p\left(\frac{C}{\gamma_{N+1}C'}\right) \tag{3.46}$$

where,

$$C = C(N) \qquad \text{and} \qquad C'(N) = C(N) - \gamma_{N+1}\, C(N-1) \tag{3.47}$$

In appendix three we derive a recurrence relation for this function and we show that the rightmost singularity of (3.46) is a multiple pole:

$$A(\alpha,N,0) \longrightarrow \frac{2N!}{N!\, N+1!} \cdot \frac{1}{\alpha^{2N+1}} \tag{3.48}$$

The leading terms in (3.46) are now,

$$T_N(\alpha,t) \sim -4ig^2\left(\frac{g}{\pi}\right)^{2N} \frac{2N!}{2^N (N+1)!\, \alpha^{2N+1}} \sum_{M=0}^{N} \frac{(-1)^M 2^M}{M!} \tag{3.49}$$

The inverse Mellin Transform, from appendix three, will give:

$$T_N(s,t) \sim \frac{-4ig^2}{(N+1)!}\left[\frac{g^2 \ln^2 s}{2\pi^2}\right]^N \sum_{M=0}^{N} \frac{(-1)^M 2^M}{M!} \tag{3.50}$$

The t channel ladder sum is:

$$T(s,t) = \sum_{N=0}^{\infty} T_N(s,t) \tag{3.51}$$

The double summation in (3.51) can be rearranged to give:

$$T(s,t) \sim -4ig^2 \sum_{M=0}^{\infty} \frac{(-2)^M}{M!} \sum_{N=0}^{\infty} \frac{(\bar{g} \ln s)^{2N}}{(N+M+1)!} \tag{3.52}$$

where

$$(\bar{g}\ln s) = +\left(\frac{g^2 \ln^2 s}{2\pi^2}\right)^{1/2} > 0 \tag{3.53}$$

Using the form of the modified Bessel function of integer order[13],

$$I_\nu(z) = \left(\tfrac{1}{2}z\right)^\nu \sum_{k=0}^{\infty} \frac{(\tfrac{1}{4}z^2)^k}{k!\,\Gamma(\nu+k+1)} \tag{3.54}$$

we obtain:

$$T(s,t) \sim -4ig^2 \sum_{M=0}^{\infty} \frac{(-2)^M}{(\bar{g}\ln s)^{M+1}} I_{M+1}(2\bar{g}\ln s) \tag{3.55}$$

$$\sim \frac{-2ig^2}{\sqrt{\pi}}\, s^{2\bar{g}} \sum_{M=0}^{\infty} \frac{(-2)^M}{(\bar{g}\ln s)^{M+3/2}} \tag{3.56}$$

where we have used the asymptotic form of the Bessel function[13]:

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \quad \text{as } z \to \infty \tag{3.57}$$

The leading behaviour obtained is that of a fixed point Regge singularity, i.e. independent of t. Each increasing value of M in the series corresponds to leading contributions from diagrams of higher orders, and reflects the increasing number of possible exchanges as more and more channels open in these orders.

For completeness we must finally consider the (s,u) crossed t channel ladder graphs of the form of fig.(8). These are found to give identical contributions to the amplitude as those of the uncrossed ladders, and so the total amplitude is just twice (3.56), and there are no problems involving signature factors.

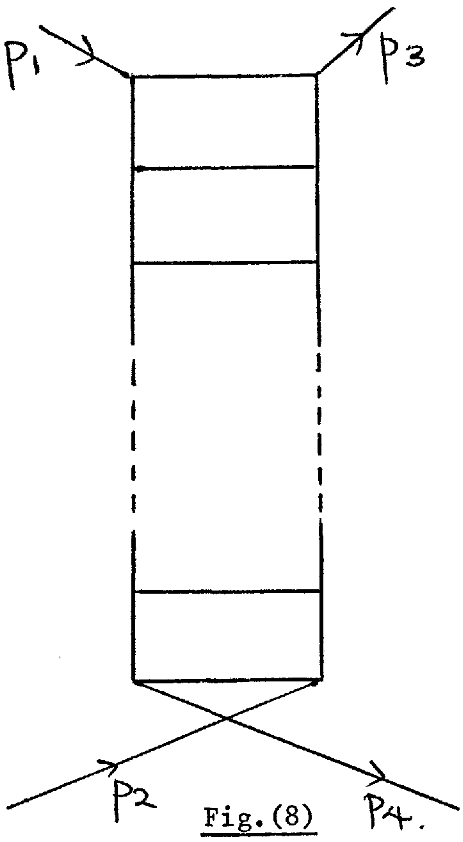


Fig.(8)

IV. NON LEADING DIAGRAMS

In the last section we claimed that the leading behaviour of this theory comes from ladder graphs where the rungs are either F or G lines, or part of a spinor loop. We will now show that this is indeed the case up to sixth order. This we feel is sufficiently non trivial for making us believe that this class of diagrams are the dominant ones to any order.

For N = 0, the Born term of fig.(2) is the only second order diagram.

For N = 1 Iliopolous and Zumino[7] have shown (to all orders) that this theory needs only one (wave function) renormalisation constant: no separate mass or coupling constant renormalisation is required.

Thus the logarithmic divergence of the diagrams of fig.(3), caused by the presence of k^4 terms in the numerator of (3.12), must be exactly cancelled by contributions from other diagrams, which turn out to be those of fig.(9).

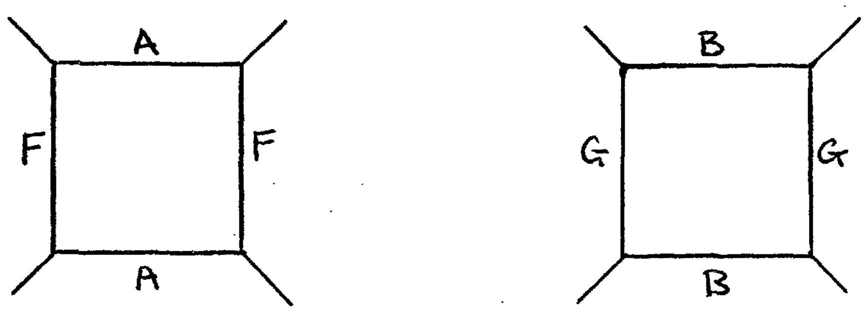


Fig.(9)

The remaining numerator factors of these diagrams are down by a factor of s, from those of fig.(3), and as such are non leading.

Due to the easier renormalisation properties of the theory, all vertices are finite, and so diagrams fo the form of fig.(10) will contribute at most a behaviour of $s^{-1}\ln s$.

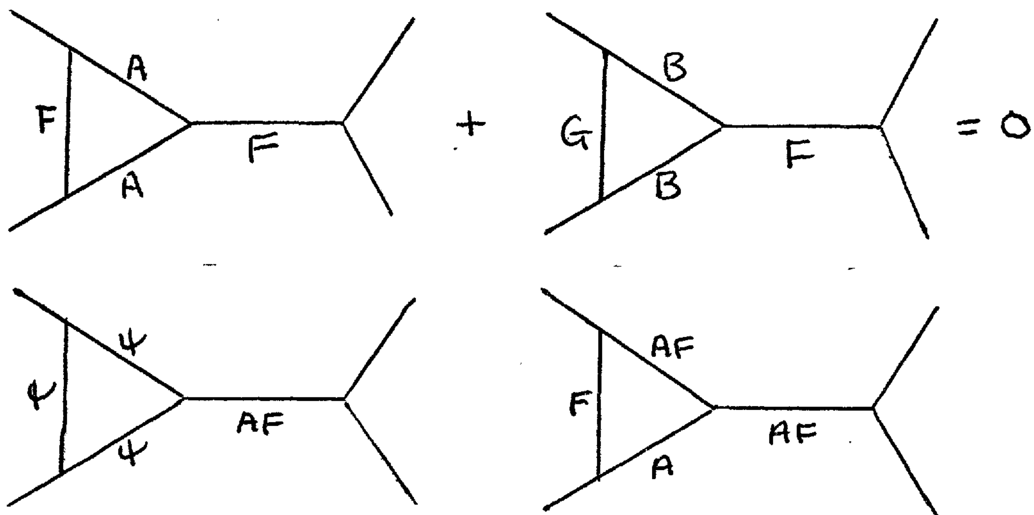


Fig.(10)

The diagrams of fig.(11a), the self energy corrections to the Born term, are logarithmically divergent. However after renormalisation we obtain only a ln s enhancement, and so these diagrams will be non leading. All diagrams of the form of fig.(11b) are zero as all tadpole graphs are zero.

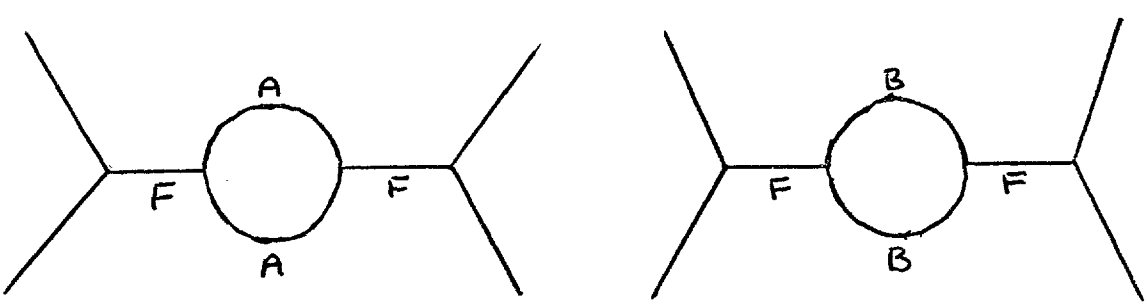


Fig.(11a)

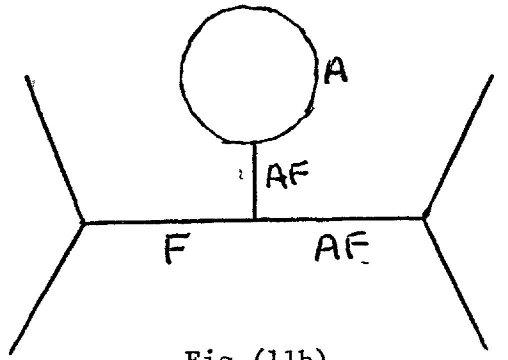


Fig.(11b)

The class of diagrams of the form of fig.(12) are non leading as there are no or not enough enhancement factors from the numerators of the propagators.

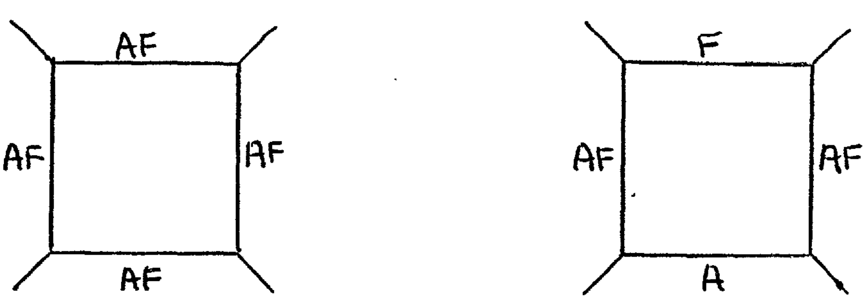


Fig.(12)

The only remaining types of diagram to consider in this order are those of fig.(13). The diagrams shown are the leading non planar graphs. With the Feynman parameterisation as illustrated, these diagrams will have a denominator, after diagonalisation;

$$(k^2 + s(xy - zw) + d)^4$$

In ordinary ϕ^3 theory[9] this type of diagram has a leading

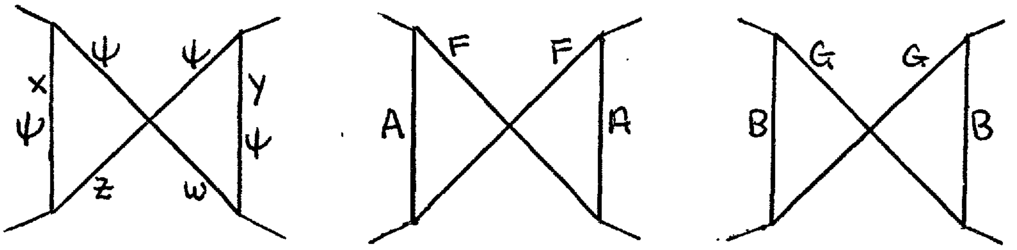


Fig.(13)

behaviour $\sim s^{-2} \ln^2 s$. The regions of integration which yield this behaviour are those given by setting one of the four parameters near one, and the other three near zero.

For the diagrams of fig.(13), we need a factor of s^2 in the numerator. There is one explicit power of s in any numerator, and another comes from the displacement of the loop momentum. But this involves the product of two Feyman parameters, and the dominant region is that where one is large and three are small. The numerator has at best therefore, a factor of s^2 multiplied by a small parameter, and this will reduce the power of ln s obtained, giving a leading behaviour of these types of diagrams of ln s.

So in fourth order we find that the diagrams of fig.(3) are the only ones to contribute to the leading behaviour.

For N = 2: The logarithmic divergences from each loop of the diagrams of fig.(4) cancel with those of diagrams of the form of fig.(14). The remaining contributions of these diagrams are down by a factor of s from those of fig.(4).

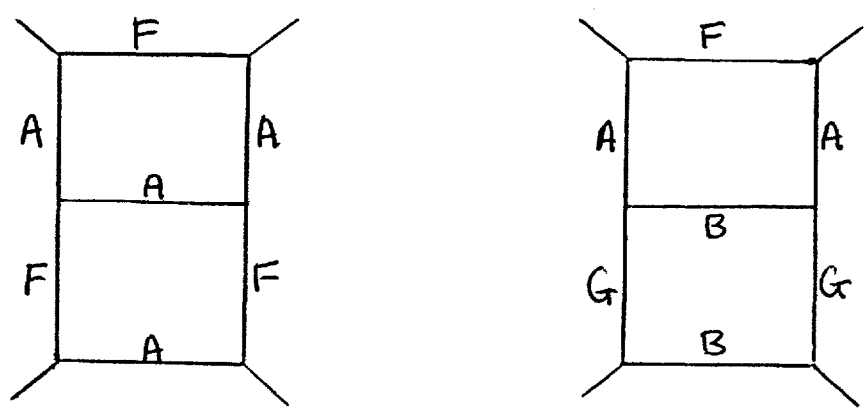


Fig.(14)

Again due to the less divergent nature of the theory we would expect diagrams involving vertex of self energy corrections to the fourth order diagrams to be non leading, as these can at best provide a ln s enhancement and so will give at best a $\ln^3 s$ leading behaviour.

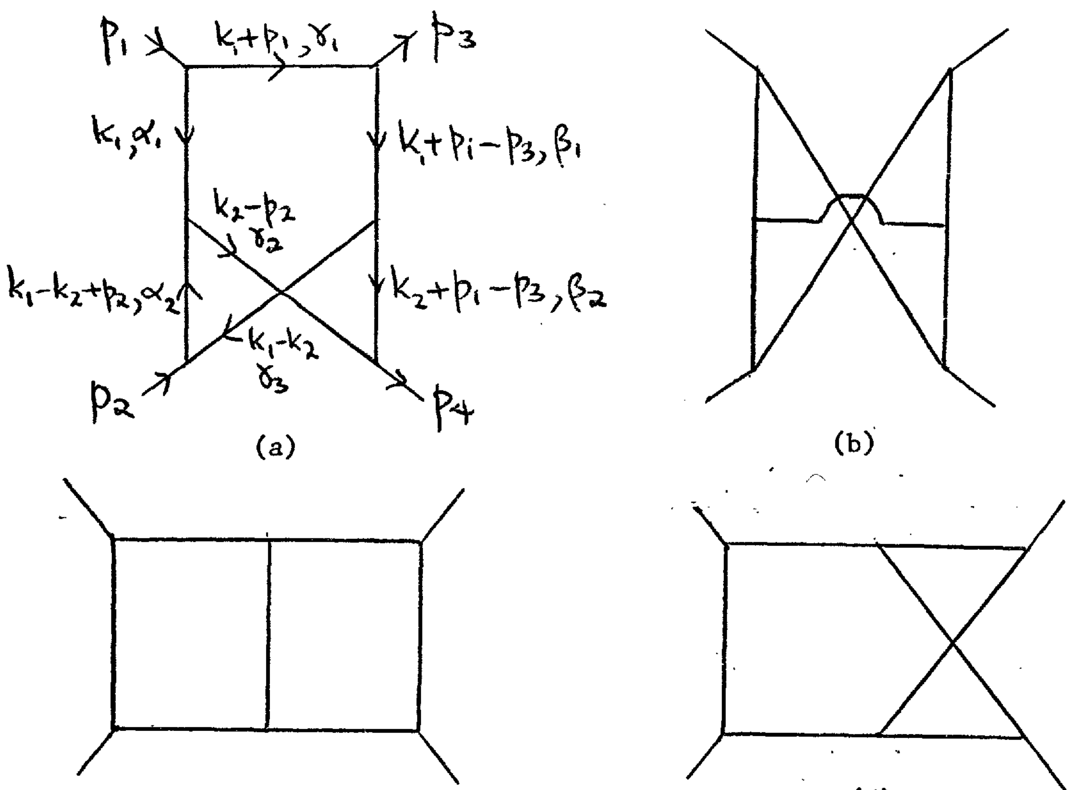


Fig.(15)

The leading behaviour of graphs of the form of those of fig.(15) can be easily worked out in ϕ^3 theory using the methods of Tiktopolous [8]. In the Wess-Zumino theory we find there are not enough enhancement effects from the numerator factors for these diagrams to contribute to the leading behaviour. To illustrate this we look at the behaviour of diagram (a) of fig.(15).

The Feynman parameters associated with each line of the diagram are displayed there. The contribution of this type of graph will have the same form as (3.2) for N = 2, but where the determinants D(2) and C(2) cannot now be read from the table in the appendix, and have to be found directly. Here:

$$D(2) = s\gamma_1(\gamma_2\gamma_3 - \alpha_2\beta_2) + \text{terms independent of } s \tag{4.1}$$

In ϕ^3 theory, $N(k_i', s, t) \sim 1$ and after the loop integrations we are left with:

$$T_2(s,t) \sim \int \frac{d\gamma_1 d\gamma_2 d\gamma_3\, d\alpha_1\, d\alpha_2\, d\beta_1 d\beta_2\, \delta(1 - \Sigma\alpha - \Sigma_1\beta - \Sigma_1\gamma)}{[s\gamma_1(\gamma_2\gamma_3 - \alpha_2\beta_2) + \ldots\ldots]^3}$$

$$\sim s^{-1} \tag{4.2}$$

In the theory of Wess and Zumino we get the maximum enhancement effects when the three rungs are F or G lines or part of a spinor loop. The numerator obtained is $k_1^2 k_2^2 s$ at best, and on doing the loop integrations we obtain:

$$T_2(s,t) \sim \int \frac{d\gamma_1 d\gamma_2 d\gamma_3 d\alpha_1 d\alpha_2 d\beta_1 d\beta_2\, \delta(1 - \Sigma_1\alpha - \Sigma_1\beta - \Sigma_1\gamma) s}{[s\gamma_1(\gamma_2\gamma_3 - \alpha_2\beta_2) + \ldots\ldots]^3\, C^2} \tag{4.3}$$

The behaviour of this type of integral has been investigated by Tiktopolous[8] and using his results we obtain enhanced behaviour from s^{-1}. A factor of ln s due to the lower power of the denominator, and another ln s from the vanishing of the determinant C are produced, as well as the explicit factor s in the numerator; these combine to give a behaviour of $\ln^2 s$, and so this diagram will be non leading.

Similarly when we consider the diagrams (b), (c) and (d) fo fig.(15) we find that they have at most a $\ln^2 s$ dependence. So we find that the diagrams of fig.(4) are the only ones that contribute to the leading behaviour in this order.

In ordinary ϕ^3 theory[8] it has been shown that the ladder graphs are indeed dominant to all orders, and as this is true in the lowest orders in the theory of Wess and Zumino it seems reasonable to assume that it is generally true to all orders.

V. OTHER SCATTERING AMPLITUDES

In a previous section we looked at the high energy behaviour of the scalar-scalar scattering amplitude. Because of the supersymmetry of the theory we would expect boson-boson, boson-fermion and fermion-fermion amplitudes to be related.

If we look at the diagrams which contribute to the scalar-pseudoscalar and pseudoscalar-pseudoscalar scattering amplitudes we find that in any order they will give the same leading behaviour as the scalar-scalar scattering amplitude.

(a) Spinor-Scalar Amplitude

We find that again in the lowest order the leading behaviour is obtained from a certain class of ladder diagrams, those of figs. (16), (17) and (18). An analysis similar to that of the scalar-scalar case shows that these are the contributing diagrams and so we will just consider these.

For N = 0, the Born term of fig.(16) gives

$$T_0(s,t) \sim \bar{u}_i(p_3)(\not{p}_1+\not{p}_2)u_j(p_1)\left(\frac{-4g^2 i}{s}\right)$$

$$\sim -4ig^2\delta_{ij} \tag{5.1}$$

where we have used the relation:

$$\bar{u}_i(p)\gamma_\mu u_j(p) = 2p_\mu \delta_{ij} \tag{5.2}$$

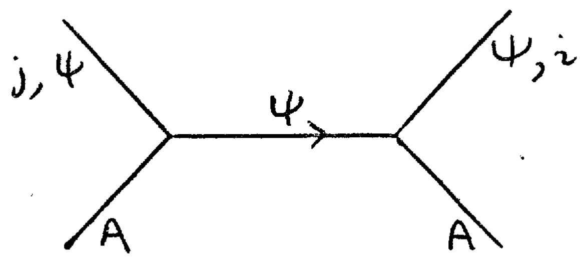


Fig.(16)

For N = 1: The leading behaviour is obtained from the diagrams of fig.(17).

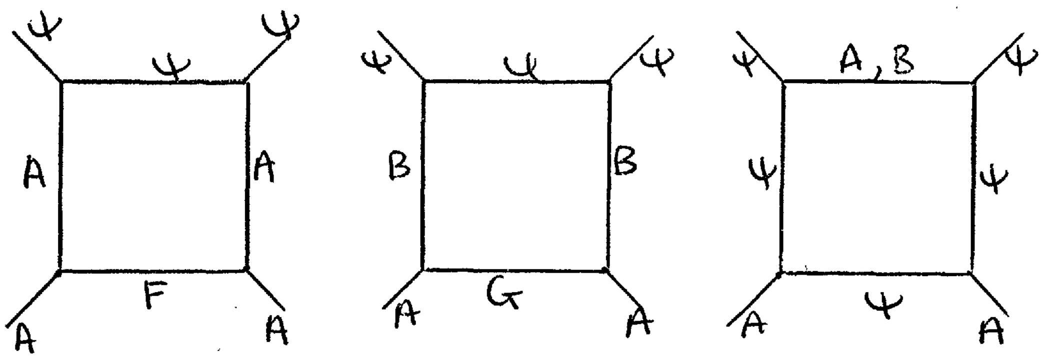


Fig.(17)

These diagrams have the same denominator as those of fig.(3), and we find that the numerator factors give the same enhancement as there; the spinor wave functions yield an explicit power of s using relation (5.2), and the loop momentum and the displacement terms are the same. There is no displacement contribution from the diagrams of type (c) due to the ordering of the spinor factors, just as we found for the spinor loop in fig.(3). We therefore obtain exactly the same numerator (3.14) and hence the same contribution (3.18).

For N = 2, the important diagrams are those of fig.(18), which give exactly the same contributions as those of fig.(4), namely (3.21).

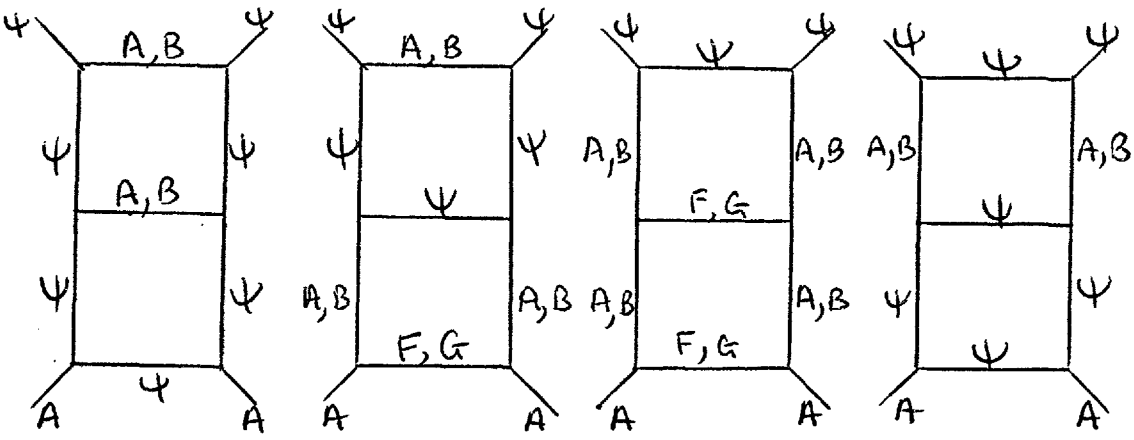


Fig.(18)

So we find that up to sixth order, the spinor-scalar scattering amplitude is exactly the same as the scalar-scalar scattering amplitude, and as there, we can see which class of diagrams are dominant to any order, and we would obtain the same general term, (3.56), by similar arguments.

(b) Spinor-Spinor Amplitude

The contributing diagrams in the lowest orders are the class of ladder diagrams of figs.(19), (20) and (21).

For N = 0

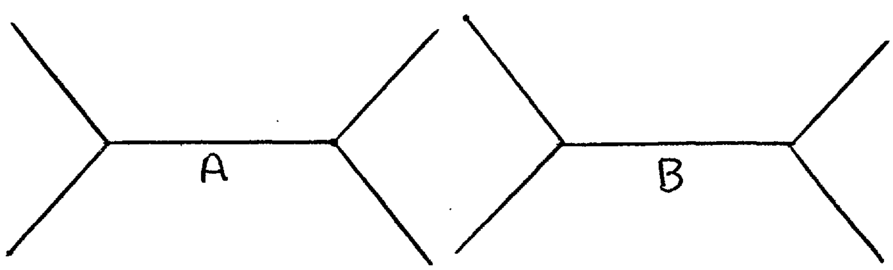


Fig.(19)

For N = 1

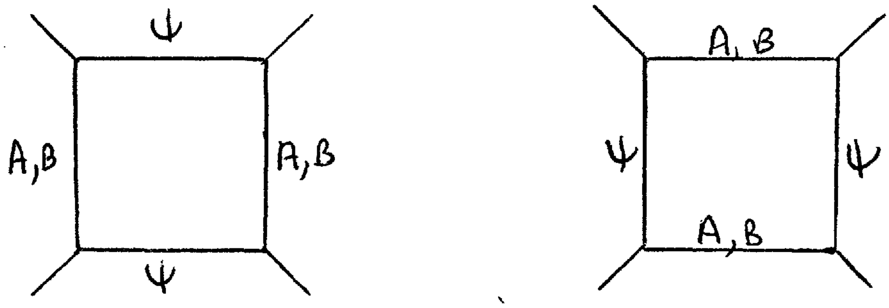


Fig.(20)

For N = 2

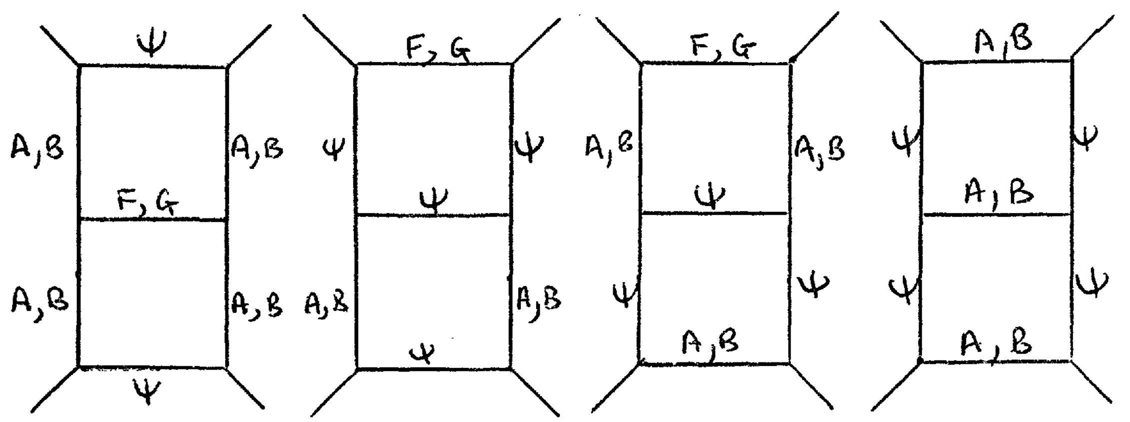


Fig.(21)

In the limit of large s we find that the helicity non-flip amplitude dominates, and that these diagrams yield the same leading behaviour as we obtained previously for the other scattering amplitudes, a result which no doubt holds to all orders.

It is again easy to show that no other diagram in these orders will contribute to the leading terms.

(c) Summary of Results

The overall picture obtained from the leading logarithm analysis is that of a series in $\ln^2 s$, whose sum indicates the presence of a series of fixed Regge branch cuts, coming from the increasing number of possible two particle exchanges in higher orders in perturbation theory.

The amplitude does not have a Regge pole form, as there are not the necessary cancellations of the $\ln^2 s$ terms between the diagrams representing the various possible types of exchanges.

The amplitude obtained is the same regardless of which particular scattering amplitude is considered, showing that the supersymmetry of the theory is preserved in the leading logarithm approximation. This would lead us to hope that this last result will be true in more realistic supersymmetric non-abelian gauge theories, so that the Reggeisation effects which we would expect there in the spinor-spinor amplitudes[5] should also occur in other amplitudes in these theories.

VI. SUPERFIELD SCATTERING AMPLITUDES

(a) Superfields

Salam and Strathdee[12] have shown that supersymmetric transformations may be viewed as a realisation of a supersymmetric 'group' on some generalised fields, called superfields, defined on an eight-dimensional space whose points are labelled by (x_μ, θ_α), where x_μ denotes the ordinary space-time coordinates and θ_α is an anticommuting Majorana spinor.

The anticommuting property of Majorana spinors implies that any superfield $\phi(x,\theta)$ is apolynomial in θ_α ,and is fully specified by sixteen ordinary functions of space-time, which are the coefficients in tis expansion in powers of θ. The transformation properties of these coefficients or components under the action of the Poincare or the supersymmetry group can be determined from those of the superfield.

i.e. for a scalar superfield,

$$\phi(x,\theta) = \phi'(x',\theta') \qquad \text{by definition}$$

where $(x_\mu, \theta_\alpha) \to (x_\mu', \theta_\alpha')$ is given by[12];

$$x_\mu' = \Lambda_{\mu\nu} x_\nu + b_\mu \quad ; \quad \theta_\alpha' = a_\alpha{}^\beta(\Lambda)\,\theta_\beta \tag{6.1}$$

for the Poincare group, and

$$x_\mu' = x_\mu + \frac{i}{2}\bar{\varepsilon}\gamma_\mu\theta \quad ; \quad \theta_\alpha' = \theta_\alpha + \epsilon_\alpha \tag{6.2}$$

for the supersymmetry group. The matrix $a_\alpha^{\ \beta}$ denotes the Dirac spinor representation of the homogeneous Lorentz transformation Λ and the parameter ε_α is an anticommuting Majorana spinor.

(b) Chiral Superfields

In the construction of supersymmetric Lagrangians one uses the so called chiral superfields[14] $\phi_+(x,\theta)$ and $\phi_-(x,\theta)$, which have eight components rather than the general sixteencomponent superfield $\phi(x,\theta)$. They are defined as the general solutions to the following two linear differential equations:

$$\left[\frac{1-i\gamma_5}{2}\cdot D\right]\phi_+(x,\theta) = 0 \tag{6.3}$$

$$\left[\frac{1+i\gamma_5}{2}\cdot D\right]\phi_-(x,\theta) = 0 \tag{6.4}$$

where

$$D_\alpha = \frac{\partial}{\partial\bar{\theta}_\alpha} - \frac{i}{2}(\gamma_\mu\theta)_\alpha\frac{\partial}{\partial x_\mu} \tag{6.5}$$

They are given in powers of θ by:

$$\phi_\pm(x,\theta) = A_\pm(x) + \bar{\theta}\psi_\pm(x) + \frac{1}{4}\bar{\theta}\theta F_\pm(x)$$

$$+\frac{1}{4}\bar{\theta}\gamma_5\theta\, i F_\pm(x) + \frac{1}{4}\bar{\theta} i\gamma^\nu\gamma_5\theta(\pm i\partial_\nu A_\pm(x)) \ldots\ldots$$

$$\ldots + \frac{1}{4}\bar{\theta}\theta\,\bar{\theta}\left(-i\not{\partial}\psi_{\pm}(x)\right) + \frac{1}{32}(\bar{\theta}\theta)^2\left(-\partial^2 A_{\pm}(x)\right)$$

$$= \exp\left[-\frac{1}{4}\bar{\theta}\not{\partial}\gamma_5\theta\right]\left\{A_{\pm}(x) + \bar{\theta}\psi_{\pm}(x) + \frac{1}{2}\bar{\theta}\left(\frac{1 \pm i\gamma_5}{2}\right)\theta F_{\pm}(x)\right\} \tag{6.6}$$

where $A_{\pm}(x)$ and $F_{\pm}(x)$ are complex boson fields, and ψ_+ and ψ_- are left and right handed Dirac spinors respectively

$$\psi_{\pm}(x) = \frac{1 \pm i\gamma_5}{2}\,\psi(x) \tag{6.7}$$

It is possible to identify ϕ_- with the complex conjugate of ϕ_+:

$$A_- = A_+^* \;;\; \psi_- = \psi_+^c \;;\; F_- = F_+^* \tag{6.8}$$

and so ψ_+ and ψ_- are identified as the left and right handed components, respectively, of a Majorana spinor.

The definitions (6.3) and (6.4) of the chiral superfields leads to their being closed under multiplication and the following laws are obtained[14]:

$$\begin{aligned} \phi_{1+}\,\phi_{2+} &= \phi_{3+} \quad \text{(chiral)} \\ \phi_{1-}\,\phi_{2-} &= \phi_{3-} \quad \text{(chiral)} \\ \phi_{1+}\,\phi_{2-} &= \phi_{3} \quad\;\; \text{(general)} \end{aligned} \tag{6.9}$$

(c) Construction of Lagrangians

The construction of supersymmetric Lagrangians is made much simpler using this formalism. It is found that the action integral of a Lagrangian density (ϕ_+,ϕ_-) is invariant under the transformations (6.2) if every θ-dependent term in (ϕ_+,ϕ_-) has the form of a spacetime divergence. i.e. from (6.2)

$$\delta\int d^4x\, \mathcal{L}(\phi_+,\phi_-) = \int d^4x\, \bar{\epsilon}\left(\frac{\partial}{\partial\bar{\theta}} + \frac{i}{2}(\gamma_\mu\theta)\frac{\partial}{\partial x^\mu}\right)\mathcal{L}(\phi_+,\phi_-)$$

$$= \bar{\epsilon}\,\frac{\partial}{\partial\bar{\theta}}\int d^4x\, \mathcal{L}(\phi_+,\phi_-) + \text{surface terms} = 0 \qquad (6.10)$$

Salam and Strathdee[14] have shown that it is possible to construct Lagrangians having any θ-dependent terms as a total divergence.

For chiral superfields:

$$-(\bar{D}D)\,\phi_{\pm} = F_{\pm} + \text{total four-divergence} \qquad (6.11)$$

Under the transformation (6.2), $F_{\pm}$ changes by a total divergence, and therefore so will $(\bar{D}D)\,\phi_{\pm}$.

For general superfields:

$$\frac{1}{64}(\bar{D}D)^2\,\phi(x,\theta) = \text{coefficient of the } (\bar{\theta}\theta)^2 \text{ term for } \phi(x,\theta)$$
$$+ \text{ total four-divergence} \qquad (6.12)$$

Again, under (6.2), $(\bar{D}D)^2\,\phi(x,\theta)$ will change by a four-divergence, and so from these terms we can construct invariant action integrals.

Using these properties we can show that the Lagrangian:

$$\mathcal{L} = \left(\frac{1}{4}\bar{D}D\right)\left(\frac{1}{2}\phi_+ \bar{D}D\phi_- + \frac{1}{2}\phi_- \bar{D}D\phi_+ - m^2\phi_+^2 - m^2\phi_-^2\right) + \left(-\frac{1}{2}\bar{D}D\right)\left(\frac{2\sqrt{2}g}{6}\right)\left(\phi_+^3 + \phi_-^3\right) \tag{6.13}$$

has an invariant action integral under the transformations (6.2). If we now expand out in components, and set:

$$A_\pm = \frac{1}{\sqrt{2}}(A \pm iB) \qquad F_\pm = \frac{1}{\sqrt{2}}(F \mp iG)$$

$$\psi_\pm = \frac{1 \mp i\gamma_5}{2}\psi \tag{6.14}$$

we find that we obtain (2.10), the Lagrangian of Wess and Zumino. Rather than working in the component fields we now do perturbation theory in terms of the superfields.

(d) Feynman Rules

Salam and Strathdee[6] have obtained the following Feynman rules for superfields from the Lagrangian (6.13). They obtain the following effective momentum space propagators:

For a line joining a pair of vertices with the same chirality ($\pm$),

$$\Delta_{\pm\pm} = -\frac{m}{2}(\bar{\theta}-\bar{\theta}')(\theta-\theta')_\pm \Delta_c \tag{6.15}$$

For a line joining vertices of opposite chirality,

$$\Delta_{\substack{+-\\-+}} = \exp\left(\bar{\Theta}\not{p}\,\Theta'_{\mp}\right)\Delta_c \tag{6.16}$$

where the four momentum is directed from the + vertex to the - vertex. In these expressions,

$$\Delta_c = \frac{i}{p^2 - m^2 + i\epsilon} \tag{6.17}$$

$$\Theta_\pm = \frac{1}{2}\left(1 \pm i\gamma_5\right)\Theta \tag{6.18}$$

Ateach vertex there will be a term $2\sqrt{2}g$, and we must apply the operator:

$$\left(-\frac{1}{2}\bar{D}D\right) \sim -\frac{1}{2}(C^{-1})^{\beta\alpha}\frac{\partial}{\partial\bar{\Theta}^\alpha}\frac{\partial}{\partial\bar{\Theta}^\beta} \tag{6.19}$$

With each external line we associate the wave function:

$$\begin{aligned}\phi_\pm^{ext}(p) = \Big\{ & A_\pm^{ext}(p) + \bar{\Theta}^\pm\psi^{ext}(p) \\ & + \frac{1}{2}F_\pm^{ext}(p)\bar{\Theta}\Theta_\pm\Big\}\exp\left[\pm\frac{i}{4}\bar{\Theta}\not{p}\gamma_5\Theta\right]\end{aligned} \tag{6.20}$$

where the notation is obvious, with u(p) the spinor wave function.

(e) Ladder Graphs

We now look again at the t-channel ladder graphs. In the lowest orders it is again simple to show that it is the ladder graphs which are dominant, and so we will look at the general ladder graphs, assuming those to be dominant in any given order.

In any given ladder graph, the denominator factors will be exactly the same as we had before, and so we will investigate the possible numerator factors to see which ones will give the maximum enhancement to the leading behaviour.

We first look at the diagrams of figs.(22) and (23).

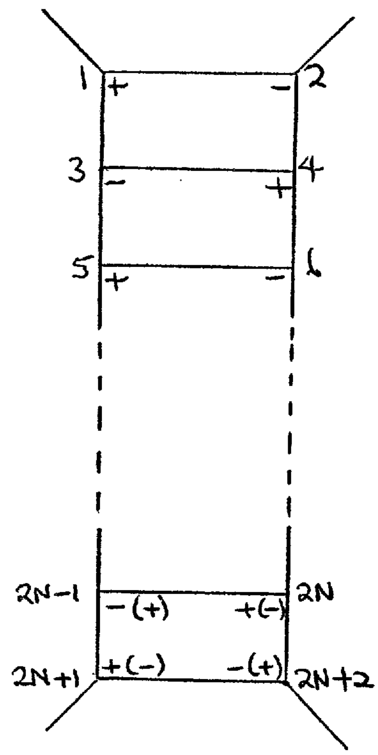


Fig.(22)

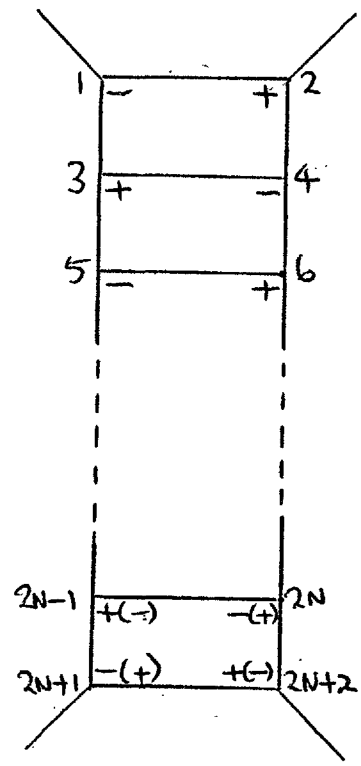


Fig.(23)

These are the important ladder diagrams for the case when N is even. The chiralities in brackets are for the case when N is odd. For each loop, e.g.fig.(24), we will have a term in the numerator:

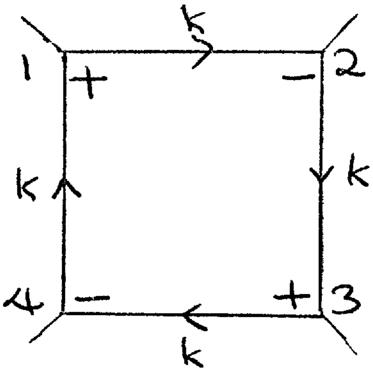


Fig.(24)

$$\exp\{\bar\theta_1 \not{k}\, \theta_{2-} + \bar\theta_2 \not{k}\, \theta_{3+} + \bar\theta_3 \not{k}\, \theta_{4-} + \bar\theta_4 \not{k}\, \theta_{1+}\}$$

$$= \exp\{\bar\theta_{13} \not{k}\, \theta_{24-}\} \tag{6.21}$$

where $\theta_{13} = \theta_1 - \theta_3$, and we have used the property of Majorana spinors:

$$\bar\theta \not{k} \chi_+ = - \bar\chi \not{k}\, \theta_- \tag{6.22}$$

We can write down the numerator for fig.(22), for an even number of loops, with the notation as in equation (3.2).

$$\begin{aligned} N(k_i, s, t) \sim 2^{N+1} \hat{O} \Big[&\exp\{\bar\theta_{14} \not{K}_1 \theta_{32-} + \bar\theta_{36} \not{K}_2 \theta_{54+} \\ &+ \ldots + \bar\theta_{2N-1,2N+2} \not{K}_N \theta_{2N,2N+1+}\} \exp\{\bar\theta_2 (\not{p}_1 - \not{p}_3) \theta_{4+} \\ &+ \bar\theta_4 (\not{p}_1 - \not{p}_3) \theta_{6-} + \ldots + \bar\theta_{2N} (\not{p}_1 - \not{p}_3) \theta_{2N+2-}\} \\ &\times \exp\{\bar\theta_1 \not{p}_1 \theta_{2-} - \bar\theta_{2N+2} \not{p}_2 \theta_{2N+1+}\} \\ &\times \phi_+(p_1)\, \phi_-(p_2)\, \phi_-(p_3)\, \phi_+(p_4) \Big] \end{aligned} \tag{6.23}$$

$\hat{O}$ is the vertex operator:

$$\hat{O} = \left(-\tfrac{1}{2}\bar{D}D\right)_1 \left(-\tfrac{1}{2}\bar{D}D\right)_2 \cdots\cdots \left(-\tfrac{1}{2}\bar{D}D\right)_{2N+2} \qquad (6.24)$$

where

$$\left(-\tfrac{1}{2}\bar{D}D\right)_i = -\tfrac{1}{2}\left(C^{-1}\right)^{\beta\alpha} \frac{\partial}{\partial \bar{\theta}^{\alpha}{}_i} \frac{\partial}{\partial \bar{\theta}^{\beta}_i} \qquad (6.25)$$

The factor 2^{N+1} arises from the extra $\sqrt{2}$ from each vertex term, over that from the vertices in (3.2). The operator $\hat{O}$ picks out the coefficient of the term:

$$\frac{1}{2} \prod_i^{\text{vertices}} \bar{\theta}_i \theta_{i\pm}$$

since,

$$\left(-\tfrac{1}{2}\bar{D}D\right)_i \left(\tfrac{1}{2}\bar{\theta}_i \theta_{i\pm}\right) = 1 \qquad (6.26)$$

We are looking for the coefficient of this term in (6.23) which gives the maximum enhancement to the leading behaviour. The second exponential in (6.23) can be set to unity as $p_1 - p_3$ terms cannot aid enhancement, and any θ factors involved will hinder the effects of other terms.

Two relations of Majorana spinors which can be easily derived are :

$$\left(\bar{\theta}\not{p}\chi_{\pm}\right)\left(\bar{\theta}\not{q}\chi_{\pm}\right) = \tfrac{1}{2}p\cdot q\left(\bar{\theta}\theta_{\mp}\right)\left(\bar{\chi}\chi_{\pm}\right) \qquad (6.27)$$

$$\left(\bar{\theta}\theta_{\pm}\right)\theta_{\pm} = 0 \qquad (6.28)$$

We use these to simplify the first exponential. The important terms from there are:

$$\left(\frac{1}{4}\right)^N k_1^2 \ldots\ldots k_N^2 \Big[(\bar{\theta}_{14}\theta_{14+})(\bar{\theta}_{45}\theta_{45+}) \ldots\ldots$$

$$(\bar{\theta}_{2N-1,2N+2}\,\theta_{2N-1,2N+2+})(\bar{\theta}_{23}\theta_{23-}) \ldots (\bar{\theta}_{2N,2N+1}\,\theta_{2N,2N+1}) \Big] \tag{6.29}$$

Any other term in this exponential will have less enhancement effects from the loop momenta, but will have no extra effects due to fewer θ factors.

The third exponential in (6.23) can be rewritten as:

$$\exp\Big\{ \bar{\theta}_4 \not{p}_1 \theta_{2-} + \bar{\theta}_{45} \not{p}_1 \theta_{2-} + \ldots + \bar{\theta}_{2N,2N+1} \not{p}_1 \theta_{2-}$$

$$+ \bar{\theta}_{2N+1} \not{p}_1 \theta_{23-} + \ldots + \bar{\theta}_{2N+1} \not{p}_1 \theta_{2N-1,2N+2-} \Big\}$$

$$\times \exp\Big\{ \theta_{2N+1} (\not{p}_1 + \not{p}_2) \theta_{2N+2-} \Big\} \tag{6.30}$$

Using (6.28) we can see that terms from the first part of the exponential will not contribute and so the remaining terms in (6.23) are:

$$N(k_i, s, t) \sim \hat{O}\Big[\left(\frac{1}{2}\right)^{N-1} k_1^2 \ldots k_N^2 (\bar{\theta}_{14}\theta_{14+}) \ldots$$

$$\ldots (\bar{\theta}_{2N-1,2N+2}\,\theta_{2N-1,2N+2+})(\bar{\theta}_{23}\theta_{23-}) \ldots\ldots$$

$$(\bar{\theta}_{2N,2N+1}\,\theta_{2N,2N+1}) \exp\Big\{ \bar{\theta}_{2N+1} (\not{p}_1 + \not{p}_2) \theta_{2N+2-} \Big\}$$

$$\phi_+(p_1)\, \phi_-(p_2)\, \phi_-(p_3)\, \phi_+(p_4) \Big] \tag{6.31}$$

If N is odd, we have a similar expression, the important difference being a $p_1 - p_2$ term in the exponential, which yields a factor of $(-1)^N$ in the final amplitude.

The diagram of fig.(23) will give exactly the same numerator, except for the chiralities being reversed. Any other ladder diagram, e.g. fig.(25), will involve propagators of the $\Delta_{\substack{++\\--}}$ type, and these will introduce $m\theta^2$ factors in the numerator without accompanying momentum factors, and so enhancement effects will be reduced and hence these diagrams will be non leading.

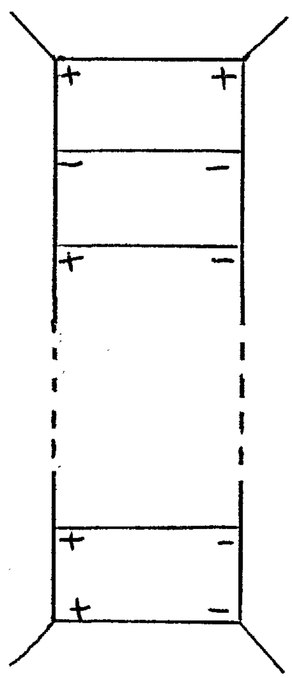


Fig.(25)

So for the study of the leading behaviour, we need only consider the ladder graphs of figs.(22) and (23).

(f) Scalar and Pseudoscalar Scattering Amplitudes

We first look at the scattering amplitude when all the external legs are scalars. $\phi_{\pm}(p)$ is replaced by its scalar component $A_{\pm}(p)$, which in turn is replaced by $(1/\sqrt{2})\, A(p)$. The scalar wave functions $A(p)$ are equal to one.

In (6.31) $\hat{O}$ is the product of 2N+2 operators $(-\frac{1}{2}\bar{D}D)_i$ from each of the 2N+2 vertices i. There are 2N products of $\bar{\theta}\theta$ outside the exponential, and so we need two more products from it. Hence using (6.27) we obtain:

$$N(k_i,s,t) \sim \hat{O}\Big[\left(\tfrac{1}{2}\right)^{N+3} k_1^2 \ldots k_N^2 (\bar{\theta}_{14}\theta_{14+}) \ldots\ldots (\bar{\theta}_{2N-1\,2N+2}\theta_{2N-1\,2N+2})(\bar{\theta}_{23}\theta_{23-}) \ldots (\bar{\theta}_{2N\,2N+1}\theta_{2N\,2N+1-})(\bar{\theta}_{2N+1}\theta_{2N+1+})(\bar{\theta}_{2N+2}\theta_{2N+2-})\Big]$$

$$= 2^{N-1}\, k_1^2 \ldots\ldots k_N^2\, s \tag{6.32}$$

For general N we will have a $(-1)^N$ factor. From the diagram of fig.(23) we obtain exactly the same result, and so the numerator for the scalar-scalar scattering amplitude is:

$$N(k_i,s,t) \sim 2^N (-1)^N k_1^2 \ldots k_N^2\, s \tag{6.33}$$

The displacement of the loop momenta will give $N(k',s,t)$ exactly as in (3.38) which was derived by considering the problem in terms of the individual fields. We have reproduced the tedious sum of graphs there by the calculation of just one type of diagram.

By making the substitution in the wave functions of (6.31), $A_{\pm}(p) \to \pm(1/\sqrt{2})\, B(p)$, we also find that the scalar-pseudoscalar and pseudoscalar-pseudoscalar amplitudes will be identical.

(g) Other Scattering Amplitudes

Equation (6.31) will also give us the scattering amplitudes for processes involving spinors. We first look at the spinor -scalar amplitude. In (6.31) the wave functions become:

$$\phi_\pm(p_1) \to \bar{\theta}_1^{\pm} u_i(p_1)$$

$$\phi_\pm(p_3) \to \bar{u}_j(p_3)\,\theta_{2\pm} \tag{6.34}$$

$$\phi_\pm(p_2) \to \frac{1}{\sqrt{2}} A(p_2) \to \frac{1}{\sqrt{2}}\ ;\ \phi_\pm(p_4) \to \frac{1}{\sqrt{2}}.$$

Since,

$$\bar{\theta}_1^{+} = \bar{\theta}_{14}^{+} + \bar{\theta}_{45}^{+} + \ldots + \bar{\theta}_{2N+1}^{+}$$

$$\theta_{2-} = \theta_{23-} + \theta_{36-} + \ldots + \theta_{2N-} \tag{6.35}$$

we can use (6.28) and obtain:

$$\phi_\pm(p_1) \to \bar{\theta}_{2N+1}^{\pm} u_i(p_1)$$

$$\phi_\pm(p_2) \to \bar{u}_j(p_3)\,\theta_{2N+2\pm} \tag{6.36}$$

We now require only the term involving one θ_{2N+2}, and one θ_{2N+1} term from the exponential, and so (6.31) now reads,

$$N(k_i,s,t) \sim \hat{O}\left[\left(\frac{1}{2}\right)^N k_1^2 \ldots k_N^2 \left(\bar{\theta}_{14}\theta_{14+}\right)\ldots\right.$$

$$---\left(\bar{\theta}_{2N\,2N+1}\,\theta_{2N\,2N+1\,-}\right)\left(\bar{\theta}_{2N+1}(\not{p}_1+\not{p}_2)\,\theta_{2N+2\,-}\right)$$

$$\left(\bar{\theta}_{2N+1}^{\;+}\,u_i(p_1)\right)\left(\bar{u}_j(p_3)\,\theta_{2N+2\,-}\right) \tag{6.37}$$

After operating by $\hat{O}$, putting in the asymtotic form of the spinor wave functions, including the $(-1)^N$ factor for the diagrams with an odd number of loops, and adding the contribution from the diagram of fig.(23), we obtain (6.33). So we find that the numerators, and hence the scattering amplitudes are the same in the spinor-scalar and scalar-scalar cases.

It is now just as easy to find the spinor-spinor numerator, by putting the spinor wave functions into (6.31). This forces the exponential to unity, and the asymptotic form of the wave functions will give us the same form again, (6.33).

Thus using the powerful technique of superfield perturbation theory we have reproduced our results for both the form, and the equality for different scattering processes, of the scattering amplitude. We would expect the equality to hold as we work in a manifestly supersymmetric framework throughout. For each different procees the only change we have to make is to put in the appropriate external wave functions.

APPENDIX ONE

(a) Notation:

Our notation is that used by Salam and Strathdee[14]. The metric tensor is:

$$g_{\mu\nu} = \text{diag}\ (+1, -1, -1, -1)$$

The Dirac matrices are given by:

$$\{\gamma_\mu,\gamma_\nu\} = 2g_{\mu\nu}$$

and γ_5 is defined by:

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$$

The matrices γ_0, $\gamma_0\gamma_\mu$, $\gamma_0\sigma_{\mu\nu} = \frac{1}{2}i\gamma_0\ \gamma_\mu,\gamma_\nu$, $\gamma_0 i\gamma_\mu\gamma_5$, $\gamma_0\gamma_5$ are all hermitian. The adjoint spinors are defined by $\bar{\psi} = \psi^\dagger\gamma_0$, and the charge conjugate of ψ by $\psi^C = C\,\bar{\psi}^T$, where $C^T = -C$ and $C^{-1}\gamma_\mu C = -\gamma_\mu{}^T$.

We define a Majorana spinor by $\psi = \psi^C$. If ψ and χ are Majorana spinors then:

$$\bar{\psi}\chi = \bar{\chi}\psi$$

$$\bar{\psi}\gamma_\mu\chi = -\bar{\chi}\gamma_\mu\psi$$

$$\bar{\psi}i\gamma_\mu\gamma_5\chi = \bar{\chi}i\gamma_\mu\gamma_5\psi$$

$$\bar{\psi}\gamma_5\chi = \bar{\chi}\gamma_5\psi$$

$$\bar{\psi}\sigma_{\mu\nu}\chi = -\bar{\chi}\sigma_{\mu\nu}\psi$$

(b) Feynman Rules

For the Lagrangian (2.3).

$$\int(2\pi)^{-4}\, d^4k_i \quad \text{for each closed loop.}$$

Propagators:

$$\langle AA\rangle = i\ (p^2 - m^2 + i\varepsilon)^{-1} = \langle BB\rangle$$

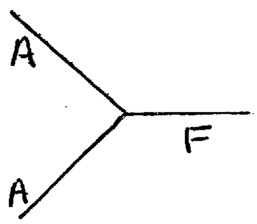
$$\langle AF\rangle = -im\,(p^2 - m^2 + i\varepsilon)^{-1} = \langle BG\rangle$$

$$\langle FF\rangle = ip^2\,(p^2 - m^2 + i\varepsilon)^{-1} = \langle GG\rangle$$

$$\langle \psi\psi\rangle = i\,(\not p + m)\,(p^2 - m^2 + i\varepsilon)^{-1}$$

<u>Vertices</u>:

Internal:



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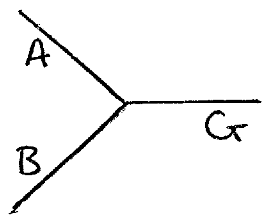

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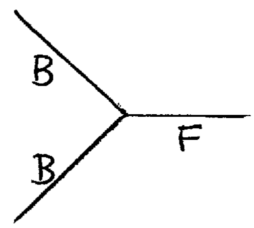
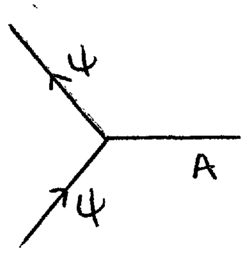
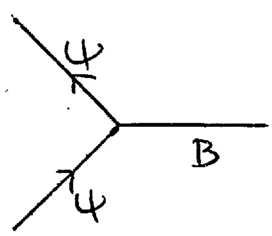


$= 2ig$

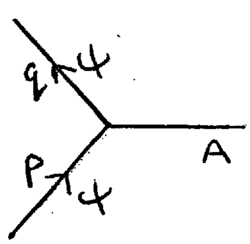


$= -2ig$

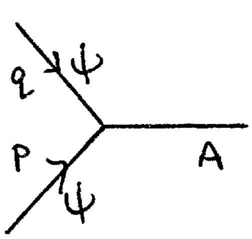


$= 2ig\gamma_5$

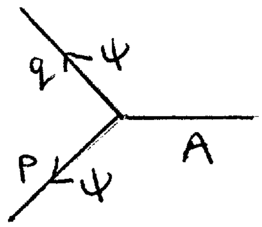
External:



$= -2ig\ \bar{u}_i^{\alpha}(q)\ u_{\alpha}^{j}(p)$



$= -2ig\ u_{\alpha}^{i}(p)\ u_{\beta}^{j}(q)\ C^{\alpha\beta}$



$$= -2ig\ \bar{u}_i^{\alpha}(p)\ \bar{u}_j^{\beta}(q)\ C_{\alpha\beta}$$

Here u^i, $\bar{u}_j$ are the incoming and outgoing spinor wave functions with their respective helicity labels; and C is the charge conjugation matrix. The last two types of vertices arise as we are dealing with Majorana spinors.

APPENDIX TWO

The determinants C(N) and D(N), in the denominator of (3.2) are obtained from the table below[8]. D(N) is the determinant of the whole matrix, and the determinant of the matrix made up of the first N rows and columns is C(N).

row \ column	1	2	3	.....	N + 1
1	$(r_1 + \gamma_1 + \gamma_2)$	$-\gamma_2$	0		$\beta_1 p + \gamma_1 p_1$
2	$-\gamma_2$	$(r_2 + \gamma_2 + \gamma_3)$	$-\gamma_3$		$\beta_2 p$
3	0	$-\gamma_3$	$(r_3 + \gamma_3 + \gamma_4)$		$\beta_3 p$
⋮					
N	0	0	0		$\beta_N p - \gamma_{N+1} p_2$
N + 1	$\beta_1 p + \gamma_1 p_1$	$\beta_2 p$	$\beta_3 p$		X

where: $X = -m^2 (1 - \gamma_1 - \gamma_{N+1}) + t \sum_{i=1}^{N} \beta_i$

$r_i = \alpha_i + \beta_i$ and $p = p_1 - p_3$

APPENDIX THREE

Mellin Transforms

The Mellin Transform $F(\beta)$ of a function $f(x)$ is defined by the relation:

$$F(\beta) = \int_0^\infty f(x)\, x^{-\beta-1}\, dx \tag{A.1}$$

It possesses an inversion formula:

$$f(x) = \frac{1}{2\pi i} \int_C F(\beta)\, x^{\beta}\, d\beta \tag{A.2}$$

where the contour C is parallel to the imaginary β axis, and $F(\beta)$ is analytic along C. In particular functions of the form

$$f(x) = \frac{x^{-\beta_0}\, \ln^{n-1} x}{(n-1)!} \tag{A.3}$$

have Mellin transforms which are multiple poles,

$$F(\beta) = \frac{1}{(\beta+\beta_0)^n} \tag{A.4}$$

If $F(\beta)$ is analytic except for multiple poles in some region, then we can displace the contour to the left and obtain a series of contributions from the multiple poles that have been crossed over. The dominant contributions will come from the rightmost pole in the β plane; when $x \to \infty$ in (A.3) the smallest value of β_0 at the poles will give the leading term, and this corresponds to the rightmost pole.

In order to obtain the behaviour of (3.45), we define:

$$A(\alpha,N,p) = \int_0^1 \prod_i^N r_i\, dr_i \prod_j^{N+1} d\gamma_j \frac{(\gamma_1 \dots \gamma_{N+1})^{\alpha-1}}{C^{2+\alpha}} \ln^p \left[\frac{C}{\gamma_{N+1} C'}\right] \tag{A.5}$$

where $C = C(N)$ and:

$$C' = C(N) - \gamma_{N+1} C(N-1) = r_N C(N-1) + \gamma_N C'(N-1) \tag{A.6}$$

$A(\alpha,N,p)$ can be rearranged:

$$A(\alpha,N,p) = \int_0^1 \prod^N r_i \, dr_i \prod^N d\gamma_j (\gamma_1 \ldots \gamma_N)^{\alpha-1}$$

$$\times \int_0^1 \frac{d\gamma_{N+1} \, \gamma_{N+1}^{\alpha-1}}{[C'(N) + \gamma_{N+1} C(N-1)]^{2+\alpha}} \ln^p \left[\frac{\gamma_{N+1} C(N-1) + C'(N)}{\gamma_{N+1} C'(N)} \right] \tag{A.7}$$

We perform the γ_{N+1} integration in the region, $1 > \mathrm{Re}\alpha > 0$:

$$A(\alpha,N,p) \sim \int_0^1 \prod^N r_i \, dr_i \prod^N d\gamma_j \frac{(\gamma_1 \ldots \gamma_N)^{\alpha-1}}{C'(N)^2 \, C(N-1)^\alpha} p!$$

$$\times \sum_{k=0}^{p} \frac{1}{k!} \ln^k \left[\frac{C(N-1)}{C'(N)} \right] \left\{ \frac{1}{\alpha^{p-k+1}} - \frac{1}{(\alpha+1)^{p-k+1}} \right\} \tag{A.8}$$

This can be reexpressed as:

$$A(\alpha,N,p) \sim \int_0^1 \prod^{N-1} r_i \, dr_i \prod^N d\gamma_j (\gamma_1 \ldots \gamma_N)^{\alpha-1}$$

$$\times p! \, C(N-1)^{-\alpha} \sum_{k=0}^{p} \frac{1}{k!} \left\{ \frac{1}{\alpha^{p-k+1}} - \frac{1}{(\alpha+1)^{p-k+1}} \right\}$$

$$\times \int_0^1 \frac{r_N \, dr_N}{[r_N C(N-1) + \gamma_N C'(N-1)]^2} \ln^k \left[\frac{C(N-1)}{r_N C(N-1) + \gamma_{N+1} C'(N-1)} \right] \tag{A.9}$$

The r_N integration is now fairly straightforward:

$$A(\alpha,N,p) \sim \int_0^1 \prod^{N-1} r_i\,dr_i \prod^{N} d\gamma_j (\gamma_1 \ldots \gamma_N)^{\alpha-1}$$

$$\times\, p!\, C(N-1)^{-\alpha-2} \sum_{k=0}^{p} \frac{1}{(k+1)!} \left\{ \frac{1}{\alpha^{p-k+1}} - \frac{1}{(\alpha+1)^{p-k+1}} \right\}$$

$$\times\, \ln^{k+1} \left[\frac{C(N-1) + \gamma_N C'(N-1)}{\gamma_N\, C'(N-1)} \right] \tag{A.10}$$

This contains $A(\alpha,N-1,k)$, and so we have a recurrence relation:

$$A(\alpha,N,p) \sim \sum_{k=1}^{p+1} \frac{p!}{k!} \left\{ \frac{1}{\alpha^{p-k+2}} - \frac{1}{(\alpha+1)^{p-k+2}} \right\} A(\alpha,N-1,k) \tag{A.11}$$

This recurrence relation is valid in the region $0 < \mathrm{Re}\alpha < 1$. We are interesed in the rightmost poles of $A(\alpha,N,p)$ in the αplane and the important term of (A.11) is:

$$A(\alpha,N,p) \sim \sum_{k=1}^{p+1} \frac{p!}{k!} \frac{1}{\alpha^{p-k+2}} A(\alpha,N-1,k)$$

$$\sim \sum_{m_1=1}^{p+1} \frac{p!}{m_1!} \frac{1}{\alpha^{p-m_1+2}} \sum_{m_2=1}^{m_1+1} \frac{m_1!}{m_2!} \frac{1}{\alpha^{m_1-m_2+2}} \cdots$$

$$\cdots\cdots\cdots \times \sum_{m_N=1}^{m_{N-1}+1} \frac{m_{N-1}!}{m_N!} \frac{1}{\alpha^{m_{N-1}-m_N+2}} A(\alpha,0,m_N) \tag{A.12}$$

Now since $C(0) = C'(0) = 1$,

$$A(\alpha,0,p) = \int_0^1 dx\, x^{\alpha-1} (-\ln x)$$

$$= \frac{p!}{\alpha^{p+1}} \tag{A.13}$$

Therefore we have:

$$A(\alpha, N, p) \sim \frac{p!\, B(N,p)}{\alpha^{p+2N+1}} \qquad \text{(A.14)}$$

where:

$$B(N,p) = \sum_{m_1=1}^{p+1} \sum_{m_2=1}^{m_1+1} \cdots\cdots \sum_{m_N=1}^{m_{N-1}+1} \qquad \text{(A.15)}$$

In order to evaluate (3.45) we require $A(\alpha, N, 0)$:

$$A(\alpha, N, 0) \sim \frac{B(N,0)}{\alpha^{2N+1}}$$

$$= \frac{2N!}{N!(N+1)!} \times \frac{1}{\alpha^{2N+1}} \qquad \text{(A.16)}$$

Hence (3.45) becomes

$$T_N(\alpha, t) \sim \text{constant} \times \frac{1}{\alpha^{2N+1}} \qquad \text{(A.17)}$$

and it is analytic in the region $0 < \text{Re}\alpha < 1$. The inverse Mellin transorm requires that $T_N(\alpha,t)$ is analytic along the contour, so we take as the contour in (A.2), the line AB in fig.(26).

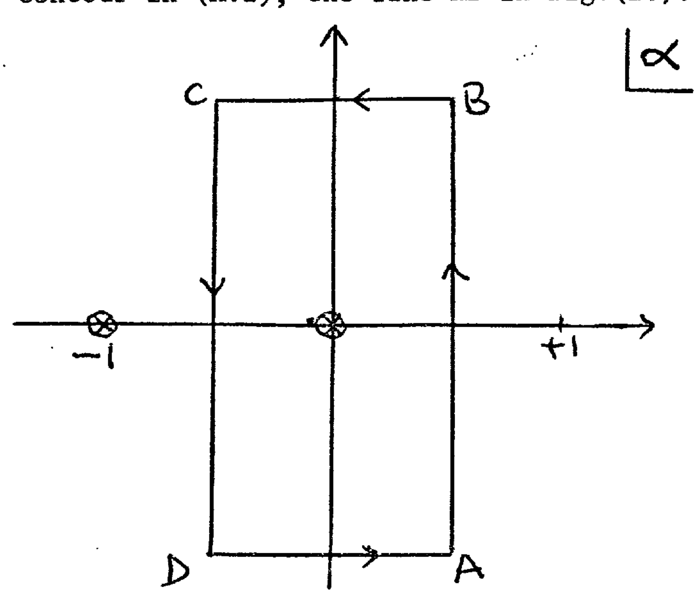


Fig.(26)

We analytically continue $T_N(\alpha,t)$ as defined by (A.17) into the region $-1 < \mathrm{Re}\alpha < 0$, except for multiple poles at $\alpha = 0$. We can now evaluate:

$$\frac{1}{2\pi i}\oint_C T_N(\alpha,t)\, s^{\alpha}\, d\alpha \sim \text{constant} \times \ln^{2N} s$$

$$= \frac{1}{2\pi i}\left[\int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}\right] s^{\alpha}\, T_N(\alpha,t)\, d\alpha \qquad \text{(A.18)}$$

The contribution from the horizontal lines BC and DA vanish as the imaginary coordinate goes to infinity, and the contribution from the left hand vertical goes to zero as $s \to \infty$. The remaining integral is the inverse Mellin transform we require, so we have:

$$T_N(s,t) \sim \text{constant} \times \ln^{2N} s \qquad \text{(A.19)}$$

which is the form of equation (3.56).

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PART TWO

ANOMALIES VIA DIMENSIONAL REGULARISATION

PART TWO

I. INTRODUCTION

The validity of the manipulation of highly divergent quantities is a question which arises often in quantum field theories. The usual technique to overcome such problems is to regularise the divergent integrals by some method, and the most usual way now is the method of dimensional regularisation[1]. This has the advantage over other previously used means of regularisation in that it preserves all the symmetries, like gauge invariance (abelian or not) of the original field theories.

When using dimensional regularisation we define the theory in arbitary dimensions. On evaluating Feynman integrals, the divergences of the theory appear as poles in the number of dimensions, and can be removed easily. The limit of four dimensions is taken at the end of the calcualation. The actual continuation to arbitary dimensions is trivial for theories with only scalar and vector fields. However, in theories involving spinors or abnormal parity objects, the generalisation requires considerable care.

One particular area in which the manipulation of divergent quantities turns out to be invalid, is in the divergence of the axial vector current, defined in a number of spinor field theories. This has been investigated by several authors[2,3], who have shown the presence of anomalous terms in the corresponding Ward identities, absent from the identities obtained formally from the equation of motion of the spinor field.

Akyeampong and Delbourgo[4] have shown how these anomalous terms arise naturally in the framework of dimensional regularisation. Identifying axial vectors and pseudoscalars as three component and four component tensors in arbitary dimensions, an expression for the divergence of the axial vector current is found:

$$\partial_K A_{LMN} = 2miP_{KLMN} + P'_{KLMN} \tag{1.1}$$

where P is the usual pseudoscalar current. The new current P', which is overall pseudoscalar, involves the antisymmetric product of five gamma matrices and so is zero in four dimensions. However, the contributions from this current produce the anomalous terms in the limit of four dimensions [4].

We consider an SU(n) symmetric theory of a spinor field, coupling to external scalar, pseudoscalar, vector and axial vector fields. We first look at the divergence of the axial vector current in this theory, in four dimensions[5]. The existence of a minimal set of anomalous terms in the abnormal Ward identities for the axial vector current has been demonstrated using various techniques of regularisation[2], and the ε-separation method of defining local operator products[3]. We show that the contribution from the new current P', produces exactly this minimal set. We find that this current also produces a set of other anomalies, associated with processes involving other external fields. These are self consistent, and we can remove them using acceptable gauge invariant counter terms in the Lagrangian, although whether we should do so is uncertain. We certainly cannot do this with the minimal set as it is not possible to include counter terms to remove these, without affecting the vector Ward identities, and hence the gauge invariance and renormalisation of the theory.

Finally from the anomalous terms we derive a modified PCAC relation, and we discuss possible implications of it, with or without the counterterms.

II. SU(n) MODEL IN FOUR DIMENSIONS

We consider an SU(n) symmetric theory of a spinor field coupled to external scalar, pseudoscalar, vector and axial vector fields, described by the Lagrangian:

$$\mathcal{L} = \bar{\psi}(x)\left(i\not{\partial} - m\right)\psi(x) + g_a j^i_a(x)\,\phi^a_i(x) \tag{2.1}$$

where the currents $j^i_a(x)$ are constructed from the free fermion fields:

$$j^i_a(x) = \bar{\psi}(x)\,\frac{1}{2}\lambda^i\,\Gamma_a\,\psi(x) \tag{2.2}$$

The $\frac{1}{2}\lambda_i$ are a representation of SU(n); $\Gamma_a = 1, \gamma_5, \gamma_\mu, i\gamma_\mu\gamma_5$ correspond to the scalar, pseudoscalar, vector and axial vector currents respectively; $\phi^i_a = S^i, P^i, V_\mu{}^i, A_\mu{}^i$ are their respective external fields; and g_a their respective coupling constants which we will absorb into the definition of the fields, and so can be set to unity in (2.1).

Using the equations of motion which follow naively from (2.1), the divergence of the axial vector current is:

$$\partial_\mu j^{i\mu}_5(x) = -2m\, j^i_5(x) \tag{2.3}$$

In perturbation theory, to the lowest order in the coupling constants, we can reexpress (2.3) as the Ward identity:

$$(\not{p}+\not{k}-m)^{-1}\, i k^\mu \gamma_\mu \gamma_5 (\not{p}-m)^{-1} \; \text{- - - -}$$

$$= (\not{p}+\not{k}-m)^{-1} 2m\gamma_5 (\not{p}-m)^{-1} - \gamma_5 (\not{p}-m)^{-1} - (\not{p}+\not{k}-m)^{-1}\gamma_5 \tag{2.4}$$

where k is the four momentum carried by the axial current, and p is that carried by the incoming fermion line at the vertex.

Eqation (2.4) has been derived from the rather naive use of the field equations, involving the formal manipulation of highly divergent quantities. The question arises therefore as to the validity of these manipulations.

To illustrate that in fact these manipulations are not justified in certain cases, we consider the Ward identity for the divergence of the axial current $j^{i\,5}_{\mu}(k)$ to two vector currents $j_{\nu}^{j}(k_1)$ and $j_{\sigma}^{k}(k_2)$. In the lowest order of perturbation theory the contribution from the left hand side of equation (2.4) is that of the diagram of fig.(1), together with the diagram with the vector currents interchanged.

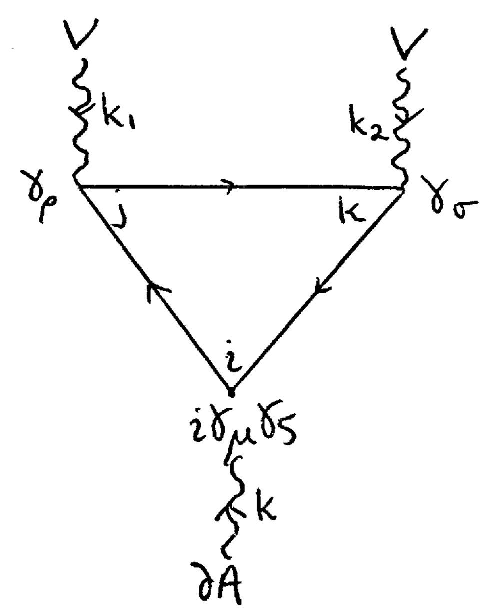

Fig.(1)

The total contribution we denote by $R_{\nu\sigma\mu}d^{ijk}$, where the d^{ijk} are the completely symmetric SU(n) structure constants. The first term on the right hand side of equation (2.4) gives a contribution which is that of a similar pair of diagrams with $i\gamma_\mu\gamma_5$ replaced by $2m\gamma_5$, and we denote it by $2mR_{\nu\sigma}d^{ijk}$. The remaining terms in (2.4) do not contribute in this order. If (2.4) holds then:

$$k^\mu R_{\nu\sigma\mu} = 2m R_{\nu\sigma} \tag{2.5}$$

should also be true.

Using the explicit calcualation of Rosenberg[6], Adler has shown that equation (2.5) is not valid, and that instead:

$$k^\mu R_{\nu\sigma\mu} = 2m R_{\nu\sigma} + 8\pi^2 k_1^\alpha k_2^\beta \epsilon_{\alpha\beta\nu\sigma} \tag{2.6}$$

i.e. the axial vector Ward identity fails in the case of the triangle graph[7]. This failure can be traced back to the illicit operation of the translation of an integration variable in a linearly divergent Feynman integral in the derivation of (2.4).

By several techniques of regularisation[2] and by the ε-separation method of defining local operator products[3], it has been shown that there are other extra terms, "anomalies", in the divergence of the axial vector current which are not obtained when the divergence is calcualated using formal manipulations of the equations of motion of the spinor field. These arise from the couplings, via closed fermion loops, to various other combinations of currents. Their results can be summarised as follows:

(a) No loops involving scalar or pseudoscalar couplings have

Ward identity anomalies which cannot be removed by appropriately chosen, and acceptable gauge invariant counter terms in the Lagrangian.

(b) The only loops with'true'anomalies, i.e. those which cannot be removed in this manner, are those with vector and axial vector vertices, with an odd number of axial vector vertices. If subtraction terms are chosen so that all the vector current Ward identities are satisfied then the following loops have anomalous axial vector Ward identities:- the AVV and AAA triangles; the AAAV and AVVV squares; and the AVVVV, AAAVV and AAAAA pentagons. The triangle anomalies are the only ones obtained when there are no internal degrees of freedom present. We will see that these anomalies emerge naturally in the framework of dimensional regularisation.

Adler has shown that, as a consequence of these extra terms, the axial vector divergence is not multiplicatively renormalisable[7]. One effect of this is that in the usual local current-current theory of the leptonic weak interactions:

$$\mathcal{L}_{eff} \sim \frac{G}{\sqrt{2}}\, j_\lambda^{+}\, j^{\lambda} \tag{2.7}$$

where:

$$j^{\lambda} = \bar{\nu}_\mu \gamma^\lambda (1 - i\gamma_5)\mu + \bar{\nu}_e \gamma^\lambda (1 - i\gamma_5) e \tag{2.8}$$

is the leptonic current, the radiative corrections to $\nu_e e$ and $\nu_\mu \mu$ scattering are divergent in fourth and higher orders of perturbation theory.

So we see that the presence of anomalies in a theory destroys its renormalisability for abnormal amplitudes, and possibly its validity as a description of the weak interactions, and so it is important to know whether a spinor field theory is anomaly free or not.

III. SOME PROPERTIES OF ARBITARY DIMENSIONS

(a) Generalised Gamma Matrices

That one can generalise the gamma matrices to an n-dimensional space is well known. We note some of the relevant properties beginning with the Clifford algebra:

$$\{ \Gamma_M, \Gamma_N \} = 2g_{MN} \tag{3.1}$$

wherethe indices M,N run from 0,1,2,....,n-1. The metric g is that appropriate to the $SO_{n-1,1}$ group:

$$g_{00} = 1, \quad g_{0N} = 0, \quad g_{MN} = -\delta_{MN} \quad \text{where } M,N \geq 1$$

Aa usual we can lower and raise indices:

$$\Gamma^M = g^{MN}\Gamma_N \tag{3.2}$$

There are some differences for even and odd dimensional spaces, but these are not relevant to our work. We work in an even dimensional, $n = 2\ell$, space and dimensional regularisation will correspond to analytic continuation in ℓ. In such a space the Γ-matrices are of dimension $2^\ell \times 2^\ell$, and there are $n^2 - 1 = 2^{2\ell} - 1$ of them obtained by multiplication and these form a complete set. If we define the antisymmetric product of Γ-matrices:

$$\Gamma_{[M_1}\Gamma_{M_2} \cdots\cdots \Gamma_{M_r]} = \sum_{\substack{\text{permutations} \\ \text{of } M_r}} (-1)^\tau \, \Gamma_{M_1}\Gamma_{M_2} \cdots\cdots \Gamma_{M_r} \tag{3.3}$$

then the complete set of matrices are:

i. the unit matrix

ii. the vector matrices Γ_M

iii. the matrices $\Gamma_{[M}\Gamma_{N]}$

iv. the "axial" matrices $\Gamma_{[K}\Gamma_L\Gamma_{M]}$

and v. the pseudoscalar matrices $\Gamma_{[K}\Gamma_L\Gamma_M\Gamma_{N]}$.

As in four dimensions the product of an odd number of Γ-matrices has vanishing trace:

$$2^{-\ell}\,\mathrm{Tr}\{\Gamma_{M_1}\cdots\cdots\Gamma_{M_r}\} = 0 \qquad \text{if r is odd} \tag{3.4}$$

For r even:

$$2^{-\ell}\,\mathrm{Tr}\{\Gamma_M\Gamma_N\} = g_{MN} \tag{3.5}$$

$$2^{-\ell}\,\mathrm{Tr}\{\Gamma_K\Gamma_L\Gamma_M\Gamma_N\} \tag{3.6}$$

$$= g_{KL}g_{MN} - g_{KM}g_{LN} + g_{KN}g_{LM} \quad \text{etc.}$$

Other trace form ula we find useful are:

$$\mathrm{Tr}\{\Gamma_{[M_1}\Gamma_{M_2}\cdots\Gamma_{M_r]}(\Gamma)^s\} \qquad \text{unless } s = r \tag{3.7}$$

$$2^{-\ell}\,\mathrm{Tr}\{\Gamma_{[M_1}\cdots\cdots\Gamma_{M_r]}\Gamma^{N_1}\cdots\cdots\Gamma^{N_r}\}$$

$$= (-1)^{\frac{1}{2}r}\,\delta^{[N_1}_{[M_1}\cdots\cdots\delta^{N_r]}_{M_r]} \tag{3.8}$$

$$2^{-\ell}\,\mathrm{Tr}\{\Gamma_{[M_1}\Gamma_{M_2}\cdots\Gamma_{M_r]}\Gamma_K\,\Gamma^{[N_1}\cdots\Gamma^{N_r]}\Gamma_L\}$$

$$=(-1)^{\frac{3r}{2}}\Big[-\delta^{[N_1}_{K}\,g_{L[M_1}\,\delta^{N_2}_{M_2}\cdots\delta^{N_r]}_{M_r]}$$

$$-\,\delta^{[N_1}_{L}\,g_{K[M_1}\delta^{N_2}_{M_2}\cdots\delta^{N_r]}_{M_r]}+g_{KL}\,\delta^{[N_1}_{[M_1}\cdots\delta^{N_r]}_{M_r]} \qquad (3.9)$$

We also need the multiplication rules:

$$\Gamma_N\Gamma_{[M_1}\cdots\Gamma_{M_r]}\Gamma^N=(n-2r)(-1)^r\,\Gamma_{[M_1}\cdots\Gamma_{M_r]} \qquad (3.10)$$

$$[\Gamma_N,\Gamma_{[M_1}\cdots\Gamma_{M_r]}]=2g_{N[M_1}\Gamma_{M_2}\cdots\Gamma_{M_r]} \qquad (3.11)$$

$$\{\Gamma_N,\Gamma_{[M_1}\cdots\Gamma_{M_r]}\}=\frac{2}{r+1}\Gamma_{[N}\Gamma_{M_1}\cdots\Gamma_{M_r]} \qquad (3.12)$$

and the contraction formula:

$$\delta^{N}_{[N}\,\delta^{N_1}_{M_1}\cdots\cdots\delta^{N_r}_{M_r]}=(n-r)\,\delta^{N_1}_{[M_1}\cdots\cdots\delta^{N_r}_{M_r]} \qquad (3.13)$$

(b) Loop Integrations

The other properties of n dimensional spaces we need are the loop integrals. It is easily shown that[1].

$$\int \frac{d^{2\ell}p\,(p^2)^\beta}{(p^2+D)^\alpha} = \frac{i(-1)^{\beta+\ell+1}}{2^{2\ell+1}\pi^\ell} \times \frac{\Gamma(\alpha-\beta-\ell)}{\Gamma(\alpha)} D^{\ell-\alpha+\beta}. \tag{3.14}$$

The important cases of this integral are those when $\alpha = 2,3,$ 4 and 5 and $\beta = 0$, 1, 2 and 3 respectively, when this integral has the value:

$$\frac{i}{16\pi^2}\Gamma(2-\ell)$$

where we have put $\ell = 2$ in the factor which multiplies the gamma function.

The symmetric integration relations in n dimensions also prove valuable:

$$\int d^{2\ell}p\, f(p^2)\, p_\alpha p_\beta = \frac{g_{\alpha\beta}}{2\ell}\int d^{2\ell}p\; p^2 f(p^2) \tag{3.15}$$

$$\int d^{2\ell}p\; p_\alpha p_\beta p_\gamma p_\delta f(p^2) = \int d^{2\ell}p\; p^4 f(p^2) \times \frac{g_{\alpha\beta}g_{\gamma\delta} + g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\gamma}}{2\ell(2\ell+2)} \tag{3.16}$$

(c) Axial Current in 2ℓ Dimensions

Before we can consider abnormal amplitudes in the theory, we must first decide what we mean by the axial vector current in 2ℓ dimensions. One possible alternative is[4] where $\Gamma_{-1} = \Gamma_0\Gamma_1\Gamma_2\ldots\Gamma_{2\ell-1}$, which in four dimensions reduces to γ_5, and so will give the correct axial vector form. But this also involves identifying the pseudoscalr current as $\bar{\psi}(x)\frac{1}{2}\lambda^i\Gamma_{-1}\psi(x)$. This does not give the usual PCAC relation, and we obtain a zero answer in the computation of the $\pi^0 \to 2\gamma$ amplitude for $\ell \geq 3$.

Another possibility would be to identify the pion current with the product $\Gamma_0\Gamma_1\Gamma_2\Gamma_3$, which again is γ_5 in four dimensions. But this is non covariant, and as such unattractive.

Akyeampong and Delbourgo[4] introduced a pseudoscalar tensor current :

$$\frac{1}{4!}\,\overline{\psi}(x)\,\frac{1}{2}\lambda^i\,\Gamma_{[K}\Gamma_L\Gamma_M\Gamma_{N]}\psi(x)$$

which is correct in four dimensions, and overcomes the problems of the current involving Γ_{-1}. Identifying the axial vector current as:

$$\frac{1}{3!}\,\overline{\psi}(x)\,\frac{1}{2}\lambda^i\,\Gamma_{[L}\Gamma_M\Gamma_{N]}\,\psi(x).$$

and using the equations of motion for the spinor field, the divergence of the axial vector current is[4],

$$-i\,\partial_{[K}\overline{\psi}(x)\Gamma_L\Gamma_M\Gamma_{N]}\psi(x) = 2m\,\overline{\psi}(x)\Gamma_{[K}\Gamma_L\Gamma_M\Gamma_{N]}\psi(x)$$

$$-i\,\overline{\psi}(x)\overleftrightarrow{\partial}^J\Gamma_{[J}\Gamma_K\Gamma_L\Gamma_M\Gamma_{N]}\psi(x) \quad + \text{ terms of } O(g) \qquad (3.17)$$

This can be reexpressed as a Ward identity, which reads to the lowest order in the coupling constants,(using the notation of (2.4)),

$$\left(\Gamma\cdot(p+k)-m\right)^{-1} k_{[K}\Gamma_L\Gamma_M\Gamma_{N]}\left(\Gamma\cdot p-m\right)^{-1}$$

$$= \left(\Gamma\cdot(p+k)-m\right)^{-1}\Big[2m\,\Gamma_{[K}\Gamma_L\Gamma_M\Gamma_{N]} - \frac{1}{5}(2p+k)^J_{\times}$$

$$\times \Gamma_{[J}\Gamma_K\Gamma_L\Gamma_M\Gamma_{N]}\Big]\left(\Gamma\cdot p-m\right)^{-1} + \Gamma_{[K}\Gamma_L\Gamma_M\Gamma_{N]}\left(\Gamma\cdot p-m\right)^{-1}$$

$$+ \left(\Gamma\cdot(p+k)-m\right)^{-1}\Gamma_{[K}\Gamma_L\Gamma_M\Gamma_{N]} \tag{3.18}$$

In four dimensions, since

$$\gamma_{[\lambda}\gamma_\mu\gamma_{\nu]} = -6\varepsilon_{\lambda\mu\nu\rho}\gamma^\rho\gamma_5, \qquad \gamma_{[\lambda}\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_{\rho]} = 0,$$

$$\gamma_{[\lambda}\gamma_\mu\gamma_\nu\gamma_{\sigma]} = 4!\ \gamma_5\varepsilon_{\lambda\mu\nu\sigma}, \text{ and } \partial_{[\lambda}\varepsilon_{\mu\nu\sigma]\rho} = -\partial_\rho\varepsilon_{\lambda\mu\nu\sigma} \tag{3.19}$$

we find that (3.17) and (3.18) formally reduce to the usual PCAC identities, (2.3) and (2.4) respectively.

However in arbitary dimensions PCAC is no longer quite true. Here the divergence of the axial vector current is the sum of the pseudoscalar current and a new current $\bar{\psi}(x)\overleftrightarrow{\partial}^J\Gamma_{[J}\Gamma_K\Gamma_L\Gamma_M\Gamma_{N]}\psi(x)$, which is overall pseudoscalar, and it is this new current which gives rise to the anomalous terms when we go to four dimensions.

IV. THE ANOMALIES[8]

(a) Abnormal Amplitudes

We first consider the Ward identities for amplitudes which are overall abnormal, taking as an example that for the divergence of the axial vector current coupled via a closed fermion loop to two vector currents. The right hand side of (3.18) yields three types of graphs:

i. the usual pseudoscalar-vector-vector, <PVV>, vertex.

ii. two self-energy like graphs due to the contraction of propagators.

iii. the anomalous term involving the antisymmetric product of five Γ-matrices, <P'VV>.

It is not too difficult to carry out the calcualation in spite of the profusion of indices if we recognise that the four dimensional limit $\ell \to 2$ is to be taken at the end, and that the anomalies emerge as the product of an integral which diverges in this limit multiplied by a kinematic term with a factor $(\ell-2)$ from the trace of the Γ-matrices, which vanishes in this limit. Other kinematic factors like $\delta^{[P}_{[J}\delta^{Q}_{K}\delta^{R}_{L}\delta^{S}_{M}\delta^{T]}_{N]}$ which are zero in four dimensions must be multiplied by integrals which are not singular as $\ell \to 2$. These important technical aspects of dimensional regularisation prove to be well substantiated by detailed computation.

i. The $\langle P(k)V(k_1)V(k_2)\rangle$ vertex is described by the tensor:

$$T^{[4]}_{[1][1]} = 2m\int d^{2\ell}p\ \mathrm{Tr}\left\{\Gamma_{[4]}(\Gamma\cdot p+m)\Gamma_A(\Gamma\cdot(p+k_1)+m)\right.$$
$$\left.\Gamma_B(\Gamma\cdot(p+k_1+k_2)+m)\right\}\ldots\ldots$$

$$\times\left[(p^2-m^2)((p+k_1)^2-m^2)((p+k_1+k_2)^2-m^2)\right]^{-1} \tag{4.1}$$

where we have used the notation $[n]$ to denote an antisymmetric product with n indices. The p^2 term in the numerator of the integrand gives a $1/(\ell-2)$ factor using equations (3.14) and (3.15), but it multiplies a trace factor $\mathrm{tr}(\Gamma_{[4]}\Gamma_A\Gamma_B) \equiv 0$ for all ℓ, and so does not contribute. The remaining terms give a finite integral, and the usual answer when the limit $\ell \to 2$ is taken.

ii. The self energy like graphs give zero for all ℓ, and so do not contribute.

iii. The anomalous term associated with the new current, $\langle P'(k)V(k_1)V(k_2)\rangle$, neglecting the SU(n) indices is:

$$T^{[5]}_{[1][1]} = \frac{1}{5}\int d^{2\ell}p\,(2p+k)^J \,\mathrm{Tr}\left\{\Gamma_{[J4]}(\Gamma\cdot p+m)\right.$$

$$\left.\times\Gamma_A(\Gamma\cdot(p+k_1)+m)\Gamma_B(\Gamma\cdot(p+k_1+k_2)+m)\right\}$$

$$\times\left[(p^2-m^2)((p+k_1)^2-m^2)((p+k_1+k_2)^2-m^2)\right]^{-1} \tag{4.2}$$

With the usual Feynman parameterisation this becomes:

$$T^{[5]}_{[1][1]} = \frac{1}{5}\int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 \int d^{2\ell}p\,(p^2-D)^{-3}$$

$$\times(2p+k')^J\,\mathrm{Tr}\left\{\Gamma_{[J4]}(\Gamma\cdot(p+k_0')+m)\ldots\ldots\right\} \tag{4.3}$$

where D, k', k_0' etc. are functions of the α's and the k's.

After symmetric integration the p^4 term multiplies tr $(\Gamma_{[5]}\Gamma_A\Gamma_B) \equiv 0$, for all ℓ, and so these terms do not contribute.

To simplify the p^2 terms the relation

$$\frac{1}{5} 2p^J \Gamma_{[J}\Gamma_K\Gamma_L\Gamma_M\Gamma_{N]} = \{\Gamma\cdot p, \Gamma_{[4]}\} \tag{4.4}$$

proves useful. We have traces of the form:

$$\mathrm{Tr}\left[\{\Gamma\cdot p, \Gamma_{[4]}\}\Gamma\cdot p\,\Gamma_A\,\Gamma\cdot k_1'\,\Gamma_B\,\Gamma\cdot k_2'\right] \tag{4.5}$$

Remembering that, (summing over repeated indices),

$$\{\Gamma_x, \Gamma_{[4]}\}\Gamma^x = 4(\ell-2)\,\Gamma_{[4]} \tag{4.6}$$

this trace yields, using equation (3.15), a factor:

$$\frac{4p^2(\ell-2)}{2\ell}\mathrm{Tr}\left[\Gamma_{[4]}\Gamma_A\,\Gamma\cdot k_1'\,\Gamma_B\,\Gamma\cdot k_2'\right]$$

The p^2 integration gives a $1/(\ell-2)$ factor from equation (3.14), and so the coefficient of the trace is finite. We can now take the limit $\ell \to 2$ and make the substitution $\Gamma_{[4]} \to \gamma_5$, $\Gamma_A \to \gamma_\alpha$, $\Gamma_B \to \gamma_\beta$; the trace becomes:

$$\mathrm{Tr}(\gamma_5\gamma_\alpha \not{k}_1'\gamma_\beta\not{k}_2') = -\epsilon_{\alpha\beta\gamma\delta}\,k_1'^{\gamma}\,k_2'^{\delta} \tag{4.7}$$

and so we can evaluate the total contribution from these terms which is:

$$-\frac{i}{8\pi^2}\epsilon_{\alpha\beta\gamma\delta}\,k_1^{\gamma}\,k_2^{\delta}$$

Other terms in equation (4.2), whose numerator is independent of the loop momenta vanish for all ℓ.

The diagram with the vector currents interchanged gives a similar contribution, and if we include the internal symmetry factors we obtain

$$\langle P'^{i}(k) V^{j}_{\alpha}(k_1) V^{k}_{\beta}(k_2)\rangle = \frac{i}{8\pi^2}\epsilon_{\alpha\beta\gamma\delta} k_1^{\gamma} k_2^{\delta} d^{ijk} \tag{4.8}$$

where the d^{ijk} are the totally symmetric SU(n) structure constants. This is just the famous axial vector anomaly and it has arisen quite naturally in the framework of dimensional regularisation as the contribution from the new overall pseudoscalar current P'.

If we examine the left hand side of (3.18) we can see that we obtain this extra new term. The relation

$$k_{[K}\Gamma_{L}\Gamma_{M}\Gamma_{N]} = [\Gamma\cdot k, \Gamma_{[3]}] \tag{4.9}$$

simplifies this calcualation. So we have demonstrated the validity of (3.18) to the lowest order of the perturbation expansion, for the coupling to two vector currents.

By similar manipulations the other abnormal amplitudes can be evaluated relatively simply, despite the awesome number of indices involved. To demonstrate the ease of the method we look at the important features of the <AAAAA>anomaly in the SU(n) gauge theory.

The anomalous term again comes from the new current, so we just consider the <P'AAAA> amplitude. The important term after Feynman parameterisation and diagonalisation of the denominator is:

$$T^{[5]}_{[3][3][3][3]} = \int d^{2\ell}p \; (p^2 - D)^{-5} 2p^{J}$$

$$\times \mathrm{Tr}\left\{ \Gamma_{[34]} \Gamma\cdot p \, \Gamma_{[3]} \Gamma\cdot p \, \Gamma_{[3]} \Gamma\cdot p \, \Gamma_{[3]} \Gamma\cdot p \, \Gamma_{[3]} \Gamma\cdot p \right\} \qquad (4.10)$$

where D is a function of the external momenta and the Feynman parameters. Other terms give at most a p^4 factor in the numerator and, from equation (3.14), these integrals are non singular. They are multiplied by kinematic factors which are zero in four dimensions, and so do not contribute. So we need only consider the p^6 integration. The trace involved in the integration is :

$$\mathrm{Tr}\left[\left\{ \Gamma\cdot p, \Gamma_{[4]} \right\} \Gamma\cdot p \, \Gamma_{[3]} \Gamma\cdot p \, \Gamma_{[3]} \Gamma\cdot p \, \Gamma_{[3]} \Gamma\cdot p \, \Gamma_{[3]} \Gamma\cdot p \right]$$

$$= p^2 \, \mathrm{Tr}\left[\left\{ \Gamma\cdot p, \Gamma_{[4]} \right\} \Gamma_{[3]} \Gamma\cdot p \, \Gamma_{[3]} \Gamma\cdot p \, \Gamma_{[3]} \Gamma\cdot p \, \Gamma_{[3]} \right] \qquad (4.11)$$

Using the symmetric integration relation (3.16) this produces three terms:

$$\frac{p^6}{2\ell(2\ell+2)} \mathrm{Tr}\left[\left\{ \Gamma_x, \Gamma_{[4]} \right\} \Gamma_{[3]} \Gamma^x \Gamma_{[3]} \Gamma_z \Gamma_{[3]} \Gamma^z \Gamma_{[3]} \right]$$

$$\frac{p^6}{2\ell(2\ell+2)} \mathrm{Tr}\left[\left\{ \Gamma_x, \Gamma_{[4]} \right\} \Gamma_{[3]} \Gamma_z \Gamma_{[3]} \Gamma^x \Gamma_{[3]} \Gamma^z \Gamma_{[3]} \right]$$

$$\frac{p^6}{2\ell(2\ell+2)} \mathrm{Tr}\left[\left\{ \Gamma_x, \Gamma_{[4]} \right\} \Gamma_{[3]} \Gamma_z \Gamma_{[3]} \Gamma^z \Gamma_{[3]} \Gamma^x \Gamma_{[3]} \right]$$

From each of these we get a kinematic term of the form $A\ (\ell-2) + B$, which multiplies a $1/(\ell-2)$ factor from the p^6 integration. The terms B, which do not exist in four dimensions, appear also in other parts of the equation (the self energy parts in this case), and they cancel for all . It is A therefore which is of primary interest.

We will look at the most complicated case, the second term, the others are similar and simpler. Making use of the identities in section III.(a) throughout this term becomes:

$$\frac{p^6}{2\ell(2\ell+2)} \mathrm{Tr}\Big[\{\Gamma_x, \Gamma_{[4]}\} \Gamma_{[3]} \{\Gamma_z, \Gamma_{[3]}\} \Gamma^x \Gamma_{[3]} \Gamma^z \Gamma_{[3]} - \{\Gamma_x, \Gamma_4\} \Gamma_{[3]} \Gamma_{[3]} \Gamma_z \Gamma^x \Gamma_{[3]} \Gamma^z \Gamma_{[3]} \Big]$$

$$= \frac{p^6}{2\ell(2\ell+2)} \mathrm{Tr}\Big[2\{\Gamma_x, \Gamma_{[4]}\} \Gamma_{[3]} \Gamma_{1B} \Gamma_{C]} \Gamma^x \Gamma_{[3]} \Gamma_{1A]} \Gamma_3 - 2\{\Gamma_x, \Gamma_{[4]}\} \Gamma_{[3]} \Gamma_{[3]} \Gamma_{[3]} \Gamma^x \Gamma_{[3]} + \{\Gamma_x, \Gamma_{[4]}\} \Gamma_{[3]} \Gamma_{[3]} \Gamma^x \Gamma_z \Gamma_{[3]} \Gamma^z \Gamma_{[3]} \Big] \tag{4.12}$$

We can now move the Γ^x's through the Γ's to their left, as each g_{AB} term will give rise to a $\{\Gamma_A, \Gamma_{[4]}\}$ type term with no $\ell-2$ factor, which will add to the uninteresting term B. In A one is left with:

$$\frac{p^6}{2\ell(2\ell+2)} \Big[-4(\ell-2)\,\mathrm{Tr}\big(\Gamma_{[4]} \Gamma_{[3]} \{\Gamma_z, \Gamma_{[3]}\} \Gamma_{[3]} \Gamma^z \Gamma_{[3]}\big) + 8(\ell-2)\,\mathrm{Tr}\big(\Gamma_{[4]} \Gamma_{[3]} \Gamma_{[3]} \Gamma_{[3]} \Gamma_{[3]}\big) + 4(6-2\ell)(\ell-2)\,\mathrm{Tr}\big(\Gamma_{[4]} \Gamma_{[3]} \Gamma_{[3]} \Gamma_{[3]} \Gamma_{[3]}\big) \Big]$$

$$= \frac{p^6}{2\ell(2\ell+2)}\Big[-4(\ell-2)\,\mathrm{Tr}\left(\Gamma_{[4]}\Gamma_{[3]}\Gamma_z\Gamma_{[3]}\Gamma_{[3]}\Gamma^z\Gamma_{[3]}\right)$$

$$+8(\ell-2)\,\mathrm{Tr}\left(\Gamma_{[4]}\Gamma_{[3]}\Gamma_{[3]}\Gamma_{[3]}\Gamma_{[3]}\right)\Big] \quad (4.13)$$

As there is an (ℓ-2) factor in front of both traces we can now evaluate them with ℓ set equal to two and can make the substitution $\Gamma_{[4]} \to \gamma_5$, $\Gamma_{[3]} \to i\gamma_\sigma\gamma_5$. The first trace is zero, and the second gives a contribution:

$$\frac{-4p^6(\ell-2)}{3}\,\epsilon_{\alpha\beta\gamma\delta}$$

The total contribution from all the traces after doing the integration using (3.14), is:

$$\frac{i}{12\pi^2}\,\epsilon_{\alpha\beta\gamma\delta}$$

The anomalous <P'AAAA> amplitude is this term multiplied by the appropriate SU(n) tensors. If there were no internal symmetry this term would cancel with terms from other diagrams with the axial vector currents interchanged, and there would be no anomalous contribution from the <AAAAA> diagram.

It can be easily shown that the self energy type diagrams have the same traces, and the p^4 integrations yield the same structure; i.e. the calcualations are as before with identically the same uninteresting terms B.

We can evaluate all the abnormal amplitude anomalies in a similar manner, and we obtain the obtain the minimal Bardeen set[2], where we denote the anomalies by <P'.....>, as they are due to the new current P':

$$\langle P'VV\rangle = -(i/8\pi^2)\, d^{ijk}\varepsilon_{\alpha\beta\gamma\delta}\, k_1{}^{\gamma}k_2{}^{\delta}$$

$$\langle P'AA\rangle = -(i/24\pi^2)\, d^{ijk}\varepsilon_{\alpha\beta\gamma\delta}\, k_1{}^{\gamma}k_2{}^{\delta}$$

$$\langle P'VVV\rangle = (i/64\pi^2)\varepsilon_{\alpha\beta\gamma\delta}\,(-\bar{W}_1(k_3 + k_1)^{\delta} + \bar{W}_2(k_1 + k_2)^{\delta} + \bar{W}_3(k_2 + k_3)^{\delta})$$

$$\langle P'AAV\rangle = (i/192\pi^2)\varepsilon_{\alpha\beta\gamma\delta}\,(\bar{W}_1(k_3 - k_1)^{\delta} + \bar{W}_2(k_1 + k_2 + 4k_3)^{\delta} + \bar{W}_3(k_2 + k_3)^{\delta})$$

$$\langle P'VVVV\rangle = -(i/256\pi^2)\varepsilon_{\alpha\beta\gamma\delta}\ \Sigma'\varepsilon(jklm)\, Z^{ijklm}$$

$$\langle P'AAVV\rangle = (i/384\pi^2)\varepsilon_{\alpha\beta\gamma\delta}\,(Z^{ijklm} + Z^{ikmjl} + Z^{ijlkm} + Z^{ikjml} + Z^{ilkjm} + 3Z^{ijlmk} - (l \leftrightarrow m))$$

$$\langle P'AAAA\rangle = (i/768\pi^2)\varepsilon_{\alpha\beta\gamma\delta}\ \Sigma'\ \varepsilon(jklm)\, Z^{ijklm} \tag{4.14}$$

Our notation is the following; the momenta from left to right are k, k_1, k_2, k_3 and k_4; the Lorentz indices are ε, α, β, γ and δ; and the SU(n) indices are i, j, k, l and m. $\varepsilon(jklm) = +1$ (-1) for even(odd) permutations of j, k, l and m, Σ' denotes the sum over all the permutations of j, k, l and m, and similarly Σ'' denotes the sum over all permutations of j, k and l. The SU(n) factors

δ, d, f, W, $\bar{W}$, Z and $\bar{Z}$ are as defined in the appendix.

All the other possible abnormal amplitude anomalies, i.e. <P'V>, <P'VS>, etc. are zero.

(b) Normal Amplitudes

The procedure outlined previously also yields unambiguously a host of other anomalies in the Ward identities for amplitudes which are overall normal. In the same notation as before these are:

$$\langle P'A\rangle = -k_\alpha (1/12\pi^2)\ (k^2 - 6m^2)\delta^{ij}$$

$$\langle P'P\rangle = im\ (1/12\pi^2)\ (k^2 - 6m^2)\ \delta^{ij}$$

$$\langle P'PS\rangle = i(1/24\pi^2)\ (3k^2 + 3k.k_1 + k_1{}^2 - 18m^2)\ d^{ijk}$$

$$\langle P'AS\rangle = -m\ (1/4\pi^2)\ k_{1\alpha}\ d^{ijk}$$

$$\langle P'AV\rangle = i(1/24\pi^2)\ ((k_\alpha k_{1\beta} + 2k_\alpha k_\beta + 3k_{1\alpha}k_{2\beta}) - g_{\alpha\beta}(3k^2 + k_1{}^2 + 3k.k_1 - 6m^2))\ f^{ijk}$$

$$\begin{aligned}\langle P'AAA\rangle = -(1/192\ {}^2)\ (& g_{\alpha\beta}g_{\gamma\delta}\{(-3k_1 - 2k_2 - k_3)^\delta W_1 + (k_1 + k_2 \\ & -2k_3)^\delta W_2 + (-2k_1 - 3k_2 - k_3)^\delta W_3\} + g_{\alpha\gamma}g_{\beta\delta}\{(k_1 \\ & - 2k_2 + k_3)^\delta W_1 + (-3k_1 - k_2 - 2k_3)^\delta W_2 + (-2k_1 \\ & -k_2 - 3k_3)^\delta W_3\} + g_{\alpha\delta}g_{\beta\gamma}\{(-k_1 - 2k_2 - 3k_3)^\delta W_1 \\ & + (-k_1 - 3k_2 - 2k_3)^\delta W_2 + (-2k_1 + k_2 + k_3)^\delta W_3\})\end{aligned}$$

$$\langle P'VP \rangle = -m(1/12\pi^2)\ (\ k_1 + 2k_2\ )_\alpha\ f^{ijk}$$

$$\langle P'AAP \rangle = im(1/32\pi^2)\ g_{\alpha\beta}(\ W_1 + 2W_2 + W_3\ )$$

$$\langle P'APP \rangle = -(1/192\pi^2)\ \{(3k_1 + 4k_2 + 5k_3)_\alpha W_1 + (3k_1 + 5k_2 + 4k_3)_\alpha W_2 + (3k_2 + 3k_3)_\alpha W_3\}$$

$$\langle P'PPP \rangle = -im(i/48\pi^2)\ (\ W_1 + W_2 + W_3\ )$$

$$\langle P'PSS \rangle = -im(1/16\pi^2)\ (\ W_1 + W_2 + W_3\ )$$

$$\langle P'ASS \rangle = -(1/64\pi^2)\ \{\ W_1(k_1 + k_3)_\alpha + W_2(k_1 + k_2)_\alpha + W_3(k_2 + k_3)_\alpha\ \}$$

$$\langle P'AVS \rangle = m(1/32\pi^2)\ g_{\alpha\beta}\ (\ \bar{W}_1 - \bar{W}_3\ )$$

$$\begin{aligned}\langle P'AVV \rangle = {} & (1/192\pi^2)\ (\ g_{\alpha\beta}g_{\gamma\delta}\{\ 3(k_1 + 2k_2 + k_3)^\delta W_1 - (k_1 + 3k_2 \\ & + 2k_3)^\delta W_2 - (2k_1 + 3k_2 + k_3)^\delta W_3\ \} + g_{\alpha\gamma}g_{\beta\delta}\{\ -(k_1 \\ & + 2k_2 + 3k_3)^\delta W_1 + 3(k_1 + k_2 + 2k_3)^\delta W_2 - (2k_1 + k_2 \\ & + 3k_3)^\delta W_3\ \} + g_{\alpha\delta}g_{\beta\gamma}\{\ (k_1 - 2k_2 + k_3)^\delta W_1 \\ & + (k_1 + k_2 - 2k_3)^\delta W_2 + (2k_1 - k_2 - k_3)^\delta W_3\ \})\end{aligned}$$

$$\langle P'VPS \rangle = i(1/192\pi^2)\ \{\ (k_1 + 2k_2 + 3k_3)_\alpha \bar{W}_1 + 3(k_1 + k_2 + 2k_3)_\alpha \bar{W}_2 + (2k_1 + k_2 + 3k_3)_\alpha \bar{W}_3\ \}$$

$$\langle P'VVP \rangle = im(1/96\pi^2)\ g_{\alpha\beta}\ (\ W_1 - 2W_2 + W_3\ )$$

$$\langle P'AVPP\rangle = (1/384\pi^2)\ g_{\alpha\beta}\{\ 5\bar{Z}^{ijlmk} + 3\bar{Z}^{imkjl} + 3\bar{Z}^{iljmk} + \bar{Z}^{ilkmj} + \bar{Z}^{ilmkj} + 3\bar{Z}^{ilmjk} + (\ l \leftrightarrow m\ )\ \}$$

$$\langle P'AAPS\rangle = (i/144\pi^2)\ g_{\alpha\beta}\{\ Z^{ijklm} + Z^{ijlkm} + Z^{iljkm} + Z^{iljmk} + Z^{ijmlk} - Z^{ijkml} + (\ j \leftrightarrow k\ )\ \}$$

$$\langle P'VVPS\rangle = i(1/384\pi^2)\ g_{\alpha\beta}\{\ Z^{ijklm} - 3Z^{iljmk} + 3Z^{ilmjk} + Z^{iljkm} + Z^{imjlk} - 3Z^{ijmlk} + (\ j \leftrightarrow k\ )\ \}$$

$$\langle P'PPPS\rangle = -i(1/384\pi^2)\ \Sigma''\ (\ 5Z^{ijklm} - 3Z^{ijkml}\ )$$

$$\langle P'PSSS\rangle = -i(1/256\pi^2)\ \Sigma'\ Z^{ijklm}$$

$$\langle P'AAAV\rangle = -(1/384\pi^2)\ \{\ g_{\alpha\beta}g_{\gamma\delta}(\ \bar{Z}^{ijklm} + 3\bar{Z}^{iljkm} + \bar{Z}^{ijkml} + \bar{Z}^{ijmkl} + 2\bar{Z}^{ijlmk} + \bar{Z}^{imjlk} + (\ j \leftrightarrow k)\ ) + g_{\alpha\gamma}g_{\beta\delta}(\ \bar{Z}^{ijlkm} + 3\bar{Z}^{ikjlm} + \bar{Z}^{ijlmk} + 2\bar{Z}^{ijkml} + \bar{Z}^{ilmjk} + \bar{Z}^{imlkj} + (\ j \leftrightarrow l\ )\ ) + g_{\alpha\delta}g_{\beta\gamma}(\ \bar{Z}^{ilkjm} + 3\bar{Z}^{ijklm} + \bar{Z}^{ilkmj} + 2\bar{Z}^{ikjml} + \bar{Z}^{ilmkj} + \bar{Z}^{imljk} + (\ l \leftrightarrow k\ )\ )\ \}$$

$$\langle P'AVSS\rangle = (1/144\pi^2)\ g_{\alpha\beta}\{\ \bar{Z}^{ijklm} + \bar{Z}^{imklj} + \bar{Z}^{imljk} + \bar{Z}^{ikjlm} + \bar{Z}^{ikljm} + \bar{Z}^{iljkm} + (\ l \leftrightarrow m\ )\ \}$$

$$\langle P'AVVV\rangle = -(1/384\pi^2)\ \{\ g_{\alpha\beta}g_{\gamma\delta}(\ 3\bar{Z}^{ijklm} + 3\bar{Z}^{imklj} + \bar{Z}^{ijlmk} + 3\bar{Z}^{imkjl} + \bar{Z}^{imljk} + \bar{Z}^{iljmk} + (\ l \leftrightarrow m\ )\ ) + \ldots$$

$$+ g_{\alpha\gamma}g_{\beta\delta}(3\bar{Z}^{imlkj} + 3\bar{Z}^{ijlkm} + \bar{Z}^{ijkml} + \bar{Z}^{imkjl}$$

$$+ 3\bar{Z}^{imljk} + \bar{Z}^{ikjml} + (k \leftrightarrow m)) + g_{\alpha\delta}g_{\beta\gamma}(\bar{Z}^{ijklm}$$

$$+ \bar{Z}^{ikjlm} + 3\bar{Z}^{ijkml} + 3\bar{Z}^{ijmlk} + 3\bar{Z}^{ilmjk} + \bar{Z}^{ilkjm}$$

$$+ (k \leftrightarrow l)) \} \qquad (4.15)$$

It can be seen that the consistency conditions between these amplitudes :

$$\frac{\partial}{\partial m} \langle P' P \rangle = \langle P' P\, S(0) \rangle$$

$$\frac{\partial}{\partial k_{1\mu}} \langle P' P \rangle = - \langle P' P\, V_\mu(0) \rangle \qquad (4.16)$$

are all obeyed.

(c) The Modified PCAC Relation

The PCAC relation, equation (2.3), is no longer true, and the extra terms needed can be evaluated from the anomalies. The new relation will be of the form:

$$\partial_\mu j^\mu{}_5(x) = -2m\ j_5(x) + \text{Abnormal} + \text{Normal} \qquad (4.17)$$

The abnormal term is that of Bardeen[2]:

$$-i(1/4\pi^2)\varepsilon_{\alpha\beta\gamma\delta}\ \mathrm{Tr}_I(\tfrac{1}{2}\lambda^i)\ (\tfrac{1}{4}F_V^{\alpha\beta}(x)\ F_V^{\gamma\delta}(x) + (1/12)F_A^{\alpha\beta}(x)\ F_A^{\gamma\delta}(x)$$

$$+ (2/3)iA^\alpha(x)\ A^\beta(x)\ F_V^{\gamma\delta}(x) + (2/3)iF_V^{\alpha\beta}(x)\ A^\gamma(x)\ A^\delta(x)$$

$$+ (2/3)iA^{\alpha}(x)\ F_V^{\beta\gamma}(x)\ A^{\delta}(x) - (8/3)A^{\alpha}(x)\ A^{\beta}(x)\ A^{\gamma}(x)\ A^{\delta}(x)\ ) \tag{4.18}$$

where Tr_I means the trace only over the internal SU(n) matrices. The vector and axial vector field strength tensors are defined by:

$$F_V^{\alpha\beta}(x) = \nabla^{\alpha}V^{\beta}(x) - \nabla^{\beta}V^{\alpha}(x) \tag{4.19}$$

$$F_A^{\alpha\beta}(x) = \nabla^{\alpha}A^{\beta}(x) - \nabla^{\beta}A^{\alpha}(x) \tag{4.20}$$

where

$$\nabla^{\alpha}\phi_i(x) = \partial^{\alpha}\phi_i(x) - if_{ijk}\ V^{\alpha j}(x)\ \phi^k(x)\ . \tag{4.21}$$

is the gauge covariant derivative.

The extra normal term in equation (4.17) is:

$$\begin{aligned}
-i(1/\pi^2)\ Tr_I(\ \tfrac{1}{2}\lambda^i\ )\ (\ & (1/6)m\nabla^{\mu}\nabla_{\mu}P(x) + m^3P(x) + (1/12)\nabla^{\nu}\nabla^{\mu}\nabla_{\mu}A_{\nu}(x) \\
& -(1/4)\nabla_{\mu}\nabla^{\nu}\nabla^{\mu}A_{\nu}(x) + (1/4)\nabla_{\mu}\nabla^{\mu}\nabla^{\nu}A_{\nu}(x) + (1/2)m^2\nabla^{\mu}A_{\mu}(x) \\
& - (1/6)g_{\mu\nu}g_{\rho\sigma}\ \{3\nabla^{\sigma}A^{\mu}(x)A^{\rho}(x)A^{\nu}(x) + A^{\mu}(x)A^{\nu}(x)\nabla^{\sigma}A^{\rho}(x) \\
& + 2A^{\mu}(x)\nabla^{\sigma}A^{\rho}(x)A^{\nu}(x) - \nabla^{\sigma}A^{\mu}(x)A^{\rho}(x)A^{\nu}(x) \\
& + A^{\mu}(x)\nabla^{\sigma}A^{\rho}(x)A^{\nu}(x)\ \} - mg_{\mu\nu}\{\ A^{\mu}(x)A^{\nu}(x)P(x) \\
& + A^{\mu}(x)P(x)A^{\nu}(x)\ \} + (1/3)mP(x)P(x)P(x) \\
& + (1/6)\ \{\ 3\nabla^{\mu}A_{\mu}(x)P(x)P(x) + 4A_{\mu}(x)\nabla^{\mu}P(x)P(x) \\
& + 5A_{\mu}(x)P(x)\nabla^{\mu}P(x) + 3\nabla^{\mu}P(x)A_{\mu}(x)P(x)\ \}\ )
\end{aligned} \tag{4.22}$$

This has to be modified to include the scalar field S(x). One makes the substitutions:

$$m \to m + S(x)$$

$$m\nabla_\mu \nabla^\mu P(x) \to \tfrac{1}{2}\ (\ 3\nabla^\mu \nabla_\mu S(x)P(x) - \nabla^\mu \nabla_\mu P(x)S(x) + 3\nabla^\mu \nabla_\mu (S(x)P(x)))$$

$$m^2\nabla_\mu A^\mu(x) \to \nabla^\mu A_\mu(x)S(x)S(x) + A_\mu(x)S(x)\nabla^\mu S(x) - \nabla^\mu S(x)A_\mu(x)S(x)$$

$$mP(x)P(x)P(x) \to \tfrac{1}{2}\ (\ 5P(x)P(x)P(x)S(x) - 3P(x)P(x)S(x)P(x)\ )$$

$$\begin{aligned} mA_\mu(x)A_\nu(x)P(x) \to \tfrac{1}{4}\ (\ & A_\mu(x)A_\nu(x)P(x)S(x) - A_\mu(x)A_\nu(x)S(x)P(x) \\ & + A_\mu(x)P(x)A_\nu(x)S(x) + P(x)A_\mu(x)S(x)A_\nu(x) \\ & + P(x)A_\mu(x)A_\nu(x)S(x) + A\ (x)S(x)P(x)A\ (x)\ ) \end{aligned}$$

and every other term like:

$$mP(x) \to \tfrac{1}{2}\ (\ S(x)P(x) + P(x)S(x)\ ),\quad \text{etc.} \tag{4.23}$$

These extra polynomial terms can be subtracted out using acceptable gauge invariant counter terms in the Lagrangian. These will be of the same form as the extra normal terms in the divergence equation, (4.22), and will leave the minimal Bardeen term. If we do so, then the fact that no loops involving scalar and pseudoscalar couplings have Ward identity anomalies means that $\pi^0 \to 2\gamma$ and the SU(3) related processes of $\eta \to 2\gamma$ and $X^0 \to 2\gamma$, are the only cases in which the anomalies alter the predictions of the usual PCAC current algebra[5] . In particular they would not alter the predictions of current algebra in the troublesome $\eta \rightarrow 3\pi$ decays. However, if we do not subtract out these terms, the normal anomaly <P'PPP> which is non zero, would alter the current algebra predictions for this process. So apart from the interests of simplicity and possible renormalisability, it is not certain that one should subtract out the normal anomalies, and whether we should include the expression (4.22) in the modified PCAC relation (4.17)

APPENDIX

SU(n) Tensors

We choose as a representation of SU(n), the usual matrices $\frac{1}{2}\lambda^{i}$. Then the tensors used in the anomalies are defined by:

$$\delta^{ij} = \tfrac{1}{2}\,\mathrm{Tr}(\lambda^{i},\lambda^{j})$$

$$\lambda^{i},\lambda^{j} = 2if^{ijk}$$

$$\{\lambda^{i},\lambda^{j}\} = 2d^{ijk}$$

$$W_1 = W^{ijkl} = \mathrm{Tr}(\ \lambda^{i}\lambda^{j}\lambda^{k}\lambda^{l} + \lambda^{iT}\lambda^{jT}\lambda^{kT}\lambda^{lT}\ )$$

$$\bar{W}_1 = \qquad = \mathrm{Tr}(\ \lambda^{i}\lambda^{j}\lambda^{k}\lambda^{l} - \lambda^{iT}\lambda^{jT}\lambda^{kT}\lambda^{lT}\ )$$

$$W_2 = W^{ijlk}, \qquad W_3 = W^{ikjl}, \text{ etc.}$$

$$Z^{ijklm} = \mathrm{Tr}(\lambda^{i}\lambda^{j}\lambda^{k}\lambda^{l}\lambda^{m} + \lambda^{iT}\lambda^{jT}\lambda^{kT}\lambda^{lT}\lambda^{mT})$$

$$\bar{Z}^{ijklm} = \mathrm{Tr}(\lambda^{i}\lambda^{j}\lambda^{k}\lambda^{l}\lambda^{m} - \lambda^{iT}\lambda^{jT}\lambda^{kT}\lambda^{lT}\lambda^{mT})$$

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