SUPERSYMMETRIC REPRESENTATIONS AND MANY-PARTICLE STATES

by

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A B S T R A C T

The structure of the supersymmetry algebra is investigated in the massive and massless cases, both covariantly and by the induced representation method. For the massive case, the unitary representations may be specified by the "spin" basis or the "superhelicity" basis. States are labelled as $|p^2>0, j_0; \underline{p}\, \sigma\, j\, \lambda\rangle$ or $|p^2>0, j_0; \underline{p}\, \sigma\, \kappa\, \lambda\rangle$ respectively. Here $j_0 = 0, \frac{1}{2}, 1, \ldots\ldots$ is the "superspin", j is the component of spin, with helicity $\lambda = -j, -j+1, \cdots, j$ while $\kappa = -j_0, -j_0+1, \cdots j_0$ is the "superhelicity". The quantum number $\sigma = 0, \pm\frac{1}{2}$ distinguishes the various spins and helicities, since $j = j_0 \pm \frac{1}{2} \mp |\sigma|$, and $\lambda = \kappa \pm \frac{1}{2} \mp |\sigma|$, $|\sigma| = 0, \frac{1}{2}$. For the massless case, the basis states are $|p^2 = 0, \lambda_0; \underline{p}\, \lambda\rangle$ with the superhelicity $\lambda_0 = \pm\frac{1}{2}, \pm 1, \cdots$ an invariant, and $\lambda = \lambda_0$ or $\lambda_0 - \frac{1}{2}$.

Using this formalism, the Clebsch-Gordan problem, of reducing the direct product of two unitary representations of the supersymmetry algebra, is solved for the massive case. This enables a partial wave analysis to be developed for supersymmetric scattering amplitudes. The scattering processes 1 → 2 + 3 and 1 + 2 → 3 + 4 are considered, and it is shown that the ordinary reduced partial-wave amplitudes are given in terms of a small number of supersymmetric ones. These constraints imposed by supersymmetry are worked out explicitly in a simple case, where parity is also included. If the constraints are continued to complex superspin, they are also found to relate the high-energy behaviour of the amplitudes.

The structure of the irreducible representations is also reflected in the superfield representations. Weight diagrams are

introduced which considerably simplify the analysis. For the massless case, it is shown that a superfield may or may not be gauge-dependent, depending upon a simple criterion. The supersymmetric form of such gauge transformations is worked out for one example.

Finally, the possibilities for superfields to form bound states are examined in terms of a supersymmetric generalization of the Wick-Cutkosky model. The bound states are found to be pseudoscalar and axial vector superfields, with additional O(4) labels (after a Wick rotation). The model therefore resembles rather a composite fermion-antifermion (Goldstein) model.

PREFACE

The work presented in this thesis was carried out in the Department of Theoretical Physics, Imperial College, between October 1973 and July 1976 under the supervision of Dr. R. Delbourgo. Except where otherwise stated, this work is original, and has not been submitted for a degree of this or any other university. The material of Chapter 6 was done in collaboration with Dr. Delbourgo.

I should like to thank Dr. Delbourgo for his constant guidance and encouragement during the course of the work, and for many valuable discussions and suggestions.

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To my parents

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1. INTRODUCTION

1.1 Origins

The physics and chemistry of matter is profoundly related to the division of its 'elementary' constituents (electrons, photons, nucleons and so on) into just two classes, obeying either Bose statistics, or Fermi statistics. In quantum mechanics this dichotomy appears in the form of the symmetrization postulate, that only totally symmetrical or totally antisymmetrical wavefunctions are admissible for the description of states of many identical particles (bosons or fermions, respectively). Indeed, a study of the fundamental space-time symmetry properties of matter (the quantum mechanical Poincare' group SL(2,C), including reflections) confirms that the elementary systems (that is, the unitary representations) may realize either of these possibilities, with either integral or half-integral spin (other possibilities, such as continuous spin, and parastatistics, are ruled out on observational grounds)[1]. The legendary spin statistics theorem of quantum field theory, which can be proven from basic axioms [2], establishes that the particles with integral spin are bosons, while those with half-integral spin are fermions. In fact, for ordinary groups, even reducible representations, describing several elementary systems, contain either all bosons, or all fermions, but do not mix the two.

At the subnuclear level of elementary particle reactions, the distinction in physical properties between bosons and fermions persists. The names 'mesons' and 'baryons' used in strong interactions emphasize just this division of hadrons into two classes of broadly dissimilar properties.

Nevertheless, on purely theoretical or aesthetic grounds, there seem no compelling reasons a priori for any such clear-cut groupings of the elementary particles of nature. This observation may, indeed, have implications which lead to a more satisfactory understanding of the laws of physics than exists at present. There is one practical strategy for tackling this problem, which has been successfully employed in other areas, and is in a sense the way in which any progress is made in building upon old ideas. This is to abstract from the real world, and to consider instead an ideal world in which the physical behaviour of bosons and fermions would not be completely independent. The real world would then be understood in terms of a 'breaking' of this ideal situation, leading to the observed differences between bosons and fermions.

The question of a unified theory combining bosons and fermions on an equal footing has frequently arisen in the literature. The work of Fierz [3] on anomalous angular momenta in Dirac magnetic monopoles [4] has recently been rediscovered in the context of spontaneously broken nonabelian gauge theories [5,6,7]. Parastatistics [8,9] can be regarded as raising similar questions of spin and statistics. The quest in the last decade for relativistic spin-containing internal symmetries of elementary particles [10] was motivated by a natural hope that, following the success of SU(3), bosons and fermions could be incorporated into a larger scheme. (SU(3), and also SU(6), do not mix bosons and fermions in their representations)

There is one stringent constraint on any reasonable theory of boson fermion symmetry: the necessity to incorporate, right at the very beginning, the fact that in our world the fermion number itself is very accurately conserved [11], whereas boson number suffers no such restriction. We shall be returning to this point in Sec. 4.3.

Now, it is clear that any conserved charge which transforms boson states into fermion states must itself be a half-integral spin object. The conserved current of a theory admitting such a symmetry can therefore be written in terms of an odd number of the local fermion fields. However, if the symmetry generated by such charges is to be a Lie algebra, then no information can be obtained, because the canonical anticommutation relations for fermion fields will not allow a commutator algebra of such charges to be evaluated. Some additional references to previous work along these lines can be found in Ref. 12.

The first hints at one way of overcoming this difficulty came indirectly with the work of Neveu and Schwartz [13] and Ramond [14] in dual models and Volkov and Akulov [15] in a model of the neutrino. The problem was completely resolved by Wess and Zumino [16] by a linear realization of these "supergauge" transformations in 4 dimensions: the commutator algebra of the conserved fermionic currents is simply replaced by a suitable anticommutator algebra. If this is done, then the conserved charges no longer form a Lie algebra, but a mathematical entity called a " graded Lie algebra ". The corresponding finite transformations form a "generalized Lie group". It is interesting to note that these concepts were already considered by Berezin and Kac [17] in confronting the problem of fermion field quantization by functional techniques.

We shall be concerned in this thesis with the graded Lie algebra approach to Bose-Fermi symmetry. It has come to be known colloquially to physicists as "supersymmetry". This term refers especially to the algebra underlying the work of Volkov and Akulov [15], a subalgebra of the algebra introduced by Wess and Zumino [16], and subsequently studied by Salam and Strathdee [18]. In the sequel it will be loosely referred to as "the" supersymmetry algebra, $\mathcal{S}$.

The next section is concerned with our motivations for studying the particular aspects of supersymmetry treated in this thesis. This done, a brief summary of the thesis is given, together with the main results.

1.2 Motivation and Summary of Thesis

Since its introduction more than two years ago [16], there has been a great deal of effort in developing the idea of boson-fermion "supersymmetry". We shall not attempt here to give an account of the progress which has been made in the subject; this is described in detail in some recent reviews [19-23]. Here we merely sketch some of these aspects, in order to put in context the subject-matter of this thesis.

Supersymmetry has advanced along two broad fronts. Firstly, it has been found that virtually all renormalizable field theories can be recast in a supersymmetric form [23], including nonabelian gauge theories. The supersymmetric theories have remarkable renormalization properties [23]. The introduction of "superfield" techniques (Sec. 2.2) has considerably simplified the supersymmetric formulations. Particular interest has been centred on spontaneous symmetry breaking [20], and on unified models incorporating supersymmetry[23].

The second branch of development has been in the algebriac aspects of supersymmetry theory. This derives ultimately from the fact, discussed in the last section, that supersymmetry appears to be a nontrivial relativistic spin-containing symmetry, and as such has great potential as a particle classification scheme. Therefore there have been many attempts to combine a supersymmetry algebra with a strictly internal symmetry such as SU(2), to yield a realistic scheme. A fuller discussion of this problem is given in Sec. 3.5. As we have mentioned, it is with the algebriac aspects, particularly of the supersymmetry algebra $\mathcal{S}$, that we are concerned in this thesis.

Now the supersymmetry algebra $\mathscr{S}$ is the prototype of all such boson and fermion mixing symmetries. Therefore, leaving aside the problem of obtaining the correct particle spectrum, for example until a better understanding is reached of the problems of symmetry breaking, we could already seek, for $\mathscr{S}$ itself, physical predictions such as constraints on scattering amplitudes, coupling constants, high energy (Regge) behaviour, bound states, and so on; in a more realistic scheme, the predictions should be at least qualitatively similar.

This philosophy is the basis for the remainder of this thesis. Some work along these lines has appeared already in the literature[18,24-26]. The work of this thesis is based upon three papers [27-29], the last in collaboration with Dr. R. Delbourgo. Hitherto only very preliminary results along these lines have been reported in the literature, but in the following we shall be able to make explicit many of the predictions of supersymmetry which were listed above; we shall also encounter many areas for future exploration with the formalism to be developed.

To end this introduction, we give a brief outline of the thesis. A more detailed account of each chapter can be found in its introductory paragraph. Finally, we state our main results.

It is clear from the last section that any investigation of supersymmetry in physics entails a knowledge of graded Lie algebras. These are formally introduced in Chap. 2 , where "the" supersymmetry algebra $\mathscr{S}$, which is of particular concern in this thesis, is also defined. Graded Lie algebras can also be integrated to continuous (local) groups known as "generalized Lie groups", and these are also discussed. In this context the notion of a "superfield" is introduced. Finally, the relationships between the graded Lie algebras and the "no-go" theorems for relativistic particle symmetries are briefly examined.

The review of graded Lie algebras in Chap. 2 is a necessary preliminary to a study of their representations. The superfields themselves already provide finite-dimensional, non-unitary representations; however for the analysis of scattering amplitudes, the unitary representations are required. Attention is turned to these in Chap.3, concentrating on the case of the supersymmetry algebra $\mathscr{S}$ (as is done for the remainder of the thesis). In the first two sections, all unitary irreducible representations of $\mathscr{S}$ in the massive case ($p^2 > o$) are found both by a covariant analysis and by an extension of the induced representation method. The massless case is treated in the following section. In each case convenient bases are found and the matrix elements of the "super-translations" in them are evaluated. The problem of adjoining discrete symmetries like parity to the algebra is also considered. Finally, some remarks are made about possible generalizations of $\mathscr{S}$ to include internal symmetries.

It should be emphasized here that for the purposes of our later applications, we do not need to tackle the mathematical subtleties which would be involved in a rigorous approach to the representation theory of graded Lie algebras. Our method is the physicist's one of ignoring such questions, while yet working with the formalism, until it becomes crucial to have a more complete understanding (as would be the case, in general, with the $p^2 \leqslant o$ and $p = o$ unitary irreducible representations).

In Chap. 4, the formalism of Chap. 3 is applied to solve the problem of reducing the direct product of two unitary irreducible representations of the supersymmetry algebra into a direct sum of unitary irreducible representations. Using this, it proves possible to analyse scattering amplitudes into partial waves of total "superspin", rather than spin. This supersymmetric partial wave analysis then leads to constraints upon the scattering amplitudes, and their high-energy behaviour; a particular example is worked out explicitly in the last section.

The insights gained into the structure of the irreducible representations of $\mathcal{S}$ in Chap. 3 also have applications in the finite-dimensional case, namely the superfield representations. These are considered in Chap. 5. The structure of the "superwavefunctions", and the massive and massless superfields, is analysed by means of weight-diagrams. In particular, for the massless case, the question of gauge-dependence is investigated.

In Chap. 6 the possibility for superfields to form bound states is considered with a model which is a supersymmetric generalization of the Wick-Cutkosky model. For a qualitative understanding of the problem in the simplest case (that of $p = o$), it proves sufficient to take a very special representation for the supersymmetric bound states. The type of representation needed in the general case is pointed out. Finally, some comments are made about the $p \neq o$ case (massive bound states).

In the appendices at the end of the thesis, the notational conventions are established, and various important formulae and identities called upon in the text are collected together.

For completeness, we now state the main results of the thesis.

In Chap. 3, we analyse the structure of the unitary irreducible representations of $\mathscr{A}$ from a covariant point of view, and find the most natural ways of labelling the basis states: the "spin" basis and the "superhelicity" basis. The massless case proves amenable to the same treatment. To our knowledge, the Casimir operator of the algebra in the massless case has not previously been written down [19,22,23].

In Chap. 4, we solve the Clebsch-Gordan problem of reducing the direct product of unitary irreducible representations, and use this to develop a supersymmetric partial wave analysis. This leads to supersymmetric constraints on amplitudes, coupling constants, and high-energy behaviour. This programme has only been hinted at elsewhere in the literature [24,25,30].

In Chap. 5, we are able to simplify considerably the analysis of massive and massless superfields, of arbitrary spin, by means of our weight diagrams [31]. In the massless case, we show that a superfield must be either gauge independent, or gauge-dependent, depending upon a simple condition. In the second case, we derive from general considerations the form of the supersymmetric gauge transformations, and prove in general the existence of the so-called Wess-Zumino gauge[32], which contains essentially only the physical gauge particles. Our analysis is model-independent.

The main result of Chap. 6, is to show that, at least in the simple case considered, the supersymmetric bound states obtained, namely pseudoscalar and axial vector superfields with internal labels (after Wick rotation) of O(4), form the appropriate supersymmetric and relativistic generalization of the lowest $^1S_0$ and $^3S_1$ states which may be obtained from two spin - $\frac{1}{2}$ states. Thus the supersymmetric Wick-Cutkosky model behaves rather like a Goldstein fermion-antifermion composite system.

The price for understanding some of the problems posed in the beginning of this chapter is very high, and supersymmetry is only a small deposit. However it seems that future unified theories may well need to have cognizance of its existence. Some of the most exciting current developments along these lines are in supergravity theory [33-36]. Here the graviton acquires a gauge partner of helicity $\pm\frac{3}{2}$; renormalization properties of such a supersymmetric theory could be expected to be radically different from those of quantum general relativity, with profound implications for an understanding of gravity and its relationship to matter.

2. BASIC CONCEPTS

In this chapter we give a review of the basic concepts of graded Lie algebras (GLA's) and their corresponding continuous groups, which will be needed in the remainder of the thesis. For a more sophisticated approach the work of Berezin and Kac[17] and the review of Corwin et. al.[19] should be consulted.

Sec. 2.1 introduces a GLA both as a formal algebra whose generators satisfy a certain set of commutation and anticommutation relations, and more abstractly as a realization in terms of a graded algebra of endomorphisms of a graded vector space: this realization also provides the definition of a representation of a GLA.

Some examples of simple GLA's are then given, and this leads[16] to the introduction of the spin-conformal algebra $\mathcal{W}$ of Wess and Zumino and its non-simple subalgebra, the so-called supersymmetry algebra $\mathcal{S}$ of Salam and Strathdee[18], and Volkov and Akulov[15], which is of particular concern in the sequel. $\mathcal{S}$ is also exhibited as a contraction of one of the simple "classical" GLA's.

In Sec. 2.2 it is shown how the association of a continuous "generalized Lie group" with a GLA necessarily leads to a group manifold of graded-commutative structure. Some properties of the fermionic parameters, the so-called "anticommuting c-numbers", are described. The concept of a superfield is introduced, as a representation carried by functions over the group manifold, generated by the inner automorphisms.

Finally, a discussion of the Lie algebra associated with a generalized Lie group is given. This is found by choosing a basis for the "a-number" parameters, so the infinitesimal generators form an ordinary Lie algebra over the real numbers. This procedure is applied to the supersymmetry algebra $\mathcal{S}$, and it is described how the "no-go" theorems, for relativistic particle symmetries, apply to the associated Lie algebra.

2.1 Graded Lie Algebras

This thesis is concerned with the recent developments of theories of boson-fermion symmetry utilizing the mathematical construct of graded Lie algebras (GLA's). In particular, we shall in the sequel develop the representation theory of one particular such "supersymmetry" algebra, and we shall give some applications of this work. We shall introduce this supersymmetry algebra below, and in the next section discuss some aspects of its corresponding Lie group. In the meantime, however, it is useful to give a brief sketch of some basic definitions and concepts from the general theory of GLA's, as this subject has developed under the impetus of the current interest and potential implications of these algebras as symmetries of elementary particles.

As stated in the previous chapter, a GLA has a system of generators satisfying commutation [ ] and anticommutation { } relations, which may be cast into the canonical form[17]

$$
\begin{aligned}
[X_a, X_b] &= C_{ab}{}^{c} X_c \\
[X_a, Q_\alpha] &= T_{a\alpha}{}^{\beta} Q_\beta \\
\{Q_\alpha, Q_\beta\} &= A_{\alpha\beta}{}^{c} X_c
\end{aligned}
\tag{2.1}
$$

The generators $\{Z_A\}$ are divided into two sets $\{X_a, a = 1, \cdots, M\}$ called even, or bose-type generators, and $\{Q_\alpha, \alpha = 1, \cdots, N\}$, called odd generators or fermi-type. The $C_{ab}{}^{c}$, $T_{a\alpha}{}^{\beta}$, $A_{\alpha\beta}{}^{c}$ are called the structure constants of the algebra. The X_a alone generate a Lie sub-algebra, of the GLA. The Jacobi identities imply that the $(T_a)_\alpha{}^{\beta}$ are representation matrices of the $\{X_a\}$, in an N-dimensional representation (M,N are assumed to be finite here).

The definition given here is only a particular example of a more general abstract definition in terms of graded algebras [19], which we give here for completeness. All of our work can be considered from the point of view of the above definition, but in many cases a finer grading than that used above is admitted, and a better insight into the structure of the algebra can be obtained.

We consider a graded vector space $L = \sum_{\alpha} \oplus L_{\alpha}$ where α labels elements of some additive group. On this vector space is defined an operation of algebraic product $[\ ] : L \times L \to L$, with the following properties:

$$\text{inclusion} \qquad [L_{\alpha} L_{\beta}] \subseteq L_{\alpha+\beta}$$

$$\text{graded-commutative} \qquad [xy] = (-1)^{xy}[yx]$$

$$\text{graded-associative (Jacobi)} \qquad [x[yz]] = [[xy]z] + (-1)^{xy}[y[xz]] \tag{2.2}$$

Here $(-1)^{x} = \pm 1$, for some suitably-defined mapping (typically the grading will be by the integers Z, or the integers modulo 2, Z_2 = {bose, fermi} where $(-1)^{x}$ = +1 and -1 are called even (bose), and odd (fermi), respectively) The graded vector space L, together with this product mapping, and a suitably defined operation of scalar multiplication $C \times L \to L$ (where C denotes the complex numbers), is called a graded Lie algebra.

Consider a (1 + N) - dimensional vector space $V = C + V^{1}$. Consider the linear transformations $L = \mathrm{End}\,(V) = L_{-1} + L_{0} + L_{+1}$, graded by their action on the subspaces C and V^{1}. Thus an element of L_0 is a linear mapping: $V^{1} \to V^{1}$ and a linear mapping: $C \to C$ while L_+ consists of all linear mappings: $C \to V'$, and so on. Now define, on these linear transformations, a graded product (extended to arbitrary elements of L by linearity)

$$[X\,Y] = X \circ Y - (-1)^{xy}\, Y \circ X \tag{2.3}$$

where $\circ$ denotes composition, and $(-1)^x = +1$ if $x \in L_0$ and -1 otherwise. It is easily verified that, equipped with this operation, L becomes a graded Lie algebra in the sense of Eq. (2.2) and called the GLA of End (V).

Such linear transformations may be represented in the usual way by matrices. Thus $x \in L_0$ has the form (a,A), where a is a complex number, and A is an N xN complex matrix; similarly, elements of L_{+1} and L_{-1} correspond to elements of V' (column vectors, v) and (algebraic) dual elements (row vectors, w^t). Thus arbitrary elements of L can be represented by $(N+1)^2$- matrices, and the graded Lie product similarly represented, provided that we define a graded matrix product

$$\left[\begin{pmatrix} A & \\ & a \end{pmatrix} \begin{pmatrix} B & \\ & b \end{pmatrix}\right] = \begin{pmatrix} AB - BA & \\ & 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} A & \\ & a \end{pmatrix} \begin{pmatrix} & v \\ w^t & \end{pmatrix}\right] = \begin{pmatrix} & Av - va \\ w^tA - aw^t & \end{pmatrix}$$

$$\left[\begin{pmatrix} & v \\ w^t & \end{pmatrix} \begin{pmatrix} & v' \\ w'^t & \end{pmatrix}\right] = \begin{pmatrix} vw'^t + v'w^t & \\ & w^tv' + w'^tv \end{pmatrix} \tag{2.4}$$

The above definitions can easily be extended to the $(M+N)$ - dimensional case $V = V^0 + V^1$, where the Bose sector is no longer 1-dimensional. In this case, the generators may be cast into the form of Eqns. (2.1), graded by Z_2.

To illustrate this remark, let us take as an example the (real or complex) algebra of all graded $(M+N)^2$- matrices. As usual, we can identify the generators with unit matrices (the labels locating the nonzero entry)

$$X_a{}^b = e_a{}^b \quad , \quad X_\alpha{}^\beta = e_\alpha{}^\beta ,$$

$$Q_a{}^\beta = e_a{}^\beta \quad , \quad Q_\alpha{}^b = e_\alpha{}^b ,$$

where a, b = 1, ..., M, and $\alpha, \beta = 1, \cdots, N$. In an abbreviated notation, $X_A{}^B = X_a{}^b$ or $X_\alpha{}^\beta$ and so on, we find that

$$[X_A{}^B , X_C{}^D] = \delta_A{}^D X_C{}^B - \delta_C{}^B X_A{}^D$$

$$[X_A{}^B , Q_C{}^D] = \delta_A{}^D Q_C{}^B - \delta_C{}^B Q_A{}^D$$

$$\{Q_A{}^B , Q_C{}^D\} = \delta_A{}^D X_C{}^B + \delta_C{}^B X_A{}^B \qquad (2.5)$$

In general, a representation of a graded Lie algebra $\mathcal{G}$ means a realization of $\mathcal{G}$ in terms of linear transformations of a graded vector space, or a homomorphism from $\mathcal{G}$ into the GLA (2.5).

The classification of GLA's of certain types follows to some extent along the same lines as for the classical Cartan-Weyl-Dynkin theory of the semisimple Lie algebras (but with some important exceptions which will be indicated as we proceed). Thus we call a GLA $\mathcal{G}$ simple if it has no nontrivial ideals (invariant subalgebras $\mathcal{G}$ such that

$[\mathcal{G}\,\mathcal{G}] \subseteq \mathcal{G}.$) $\mathcal{G}$ is called solvable if, for some finite n, $\mathcal{G}^{(n+1)} = 0$, where $\mathcal{G}^{(n+1)} = [\mathcal{G}^{(n)}\,\mathcal{G}^{(n)}]$ and $\mathcal{G}^{(0)} = \mathcal{G}$.

Just as in the classical case, we can identify a Killing form g (c.f. Eqns. (2.1))

$$
\begin{aligned}
g_{ab} &= h_{ab} - T_{a\alpha}{}^{\beta}\, T_{b\beta}{}^{\alpha} \\
h_{ab} &= C_{ac}{}^{d}\, C_{bd}{}^{c} \\
g_{\alpha\beta} &= -g_{\beta\alpha} = T_{a\alpha}{}^{\gamma} A_{\beta\gamma}{}^{a} - T_{a\beta}{}^{\gamma} A_{\alpha\gamma}{}^{a} \\
g_{\alpha a} &= g_{a\alpha} = 0
\end{aligned}
\tag{2.6}
$$

where h_{ab} is the metric tensor of the underlying LA of the GLA. If det $|g_{AB}| \neq 0$, then

$$K = g^{AB} Z_A Z_B = g^{ab} X_a X_b + g^{\alpha\beta} Q_\alpha Q_\beta \tag{2.7}$$

is a Casimir of $\mathcal{G}$, and we can construct higher-order Casimirs, as in the semi-simple case.

Nahm, Rittenberg and Scheunert [37], and independently Kac [38], extending the Cartan-Dynkin approach to GLA's have given a complete classification of "strictly semi-simple" GLA's (with det $|g_{AB}| \neq 0$), which prove to be direct sums of strictly simple GLA's (simple, with det $|g_{AB}| \neq 0$: both conditions are necessary).

We shall not go into the details of this here. For illustrative purposes it is sufficient merely to give some examples [39] of simple GLA's, and to introduce the GLA's which have so far found application in particle physics. All our examples have a one-dimensional bose sector, as in the example of Eqn. (2.4). Our nomenclature for such examples consists simply of a $\mathcal{G}$ affixed to the usual notation for the underlying LA, to indicate a GLA with a single multiplet of fermion-type generators.

Accordingly our example of Eqn. (2.5) may be denoted $\mathcal{G}GL(M;N)$. As in the LA case, it is not simple. The algebra SL(M) of graded-traceless matrices, where $a = tr(A)$, is simple (Eqns. (2.4)).

As a further example consider a complex graded vector space equipped with a nonsingular sesquilinear form $\langle z, w\rangle = z^{*t} J w$. We define the GLA $\mathcal{G}SU(J)$ to be the (real) graded algebra of traceless graded anti-hermitean matrices X,

$$\langle Xz, w\rangle = -(-1)^{zx}\langle z, Xw\rangle,$$

for J = I of the form

$$X = \begin{bmatrix} A & v \\ -v^{*t} & a \end{bmatrix}, \quad A = -A^{t}, \; a^{*} = -a, \; a = tr(A) \qquad (2.8)$$

Clearly, the Bose sector is $\mathcal{G}SU(M) \times U(1)$, say. The complexified form of $\mathcal{G}SU(M)$ is $\mathcal{G}SL(M)$.

Similarly we can consider a real symmetric or skew-symmetric form $z^{t}Cw$. For example, if the fermi sector has even dimension $2N$, and C has the block-diagonal form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we have the (real or complex) GLA $\mathcal{G}Sp(2N)$, of all the graded skew-symplectic matrices,

$$\begin{bmatrix} A & v \\ -v^{t}C & 0 \end{bmatrix}, \quad A^{c} = C^{-1}A^{t}C = -A. \qquad (2.9)$$

With these examples we can now identify the GLA's which have been involved so far in the recent developments in 'supergauge' symmetry in dual models, and supersymmetry in quantum field theory [19].

In the former case, the GLA's are isomorphic to $\mathcal{G}Sp(2) \cong \mathcal{G}SO(2,1)$. Another example is the so-called f/d algebra of Gell-Mann, Michel and Radicati [40]. This is a simple subalgebra of $\mathcal{G}GL(N,N)$.

The first application of GLA's in 4 dimensions came with the introduction of the 24-dimensional spin-conformal algebra $\mathcal{W} = \mathcal{G}SU(2,2) \cong \mathcal{G}SO(4,2)$ of Wess and Zumino[16]. This has generators $J_{\mu\nu}$, P_μ of the Poincaré group, K_μ, D of special conformal transformations and dilations, and two Majorana 4-spinors Q_α, R_α. Unfortunately this algebra is of limited applicability, containing the full conformal group, and may be used for massless particles only. We shall not write down the bracket relations of the generators.

The GLA with which we are almost exclusively concerned in this thesis is the non-simple subalgebra $\mathcal{S}$ of $\mathcal{W}$, generated by the poincaré algebra $\mathcal{P}$, and the 'supertranslations' S_α, a majorana spinor under Lorentz transformations (which may be taken as Q_α). (Notice that we can only define the reality properties in a representation equipped with an inner product, and for the moment we regard the S_α as independent generators). This algebra was first studied (in a nonlinear realization) by Volkov and Akulov [15], and reintroduced in a linear realization as a subalgebra of $\mathcal{W}$ by Salam and Strathdee [41], and its representations studied. We shall occasionally refer to $\mathcal{S}$ in the sequel as 'the' supersymmetry algebra.

It is an interesting fact that, although not itself simple, $\mathcal{S}$ may be derived from the simple algebra $\mathcal{G}Sp(4)$ by Inönu-Wigner contraction [42]. This may, in fact, be of physical significance and not just a mathematical trick. For example, it is known from the work of Woo,[43] that $\mathcal{S}$ cannot be obtained simply as the flat-space limit of a natural curved-spacetime formulation of supersymmetry; instead a singular limiting procedure must be devised. Also, Keck[44] has used the whole uncontracted algebra $\mathcal{G}Sp(4) \cong \mathcal{G}SO(3,2)$ to construct an alternative class of (nonlinearly realized) supersymmetries on superfields (c.f. next section). In view of these points, we shall now show in detail how the contraction $\mathcal{G}Sp(4) \rightarrow \mathcal{S}$ is carried out.

Consider the real GLA of $\mathcal{G}Sp(4)$. According to Eqns. (2.8) above, the generators satisfy the relations (with $\alpha, \beta = 1,2,3,4$)

$$
\begin{aligned}
[M_{\alpha\beta}, M_{\gamma\delta}] &= C_{\gamma\beta} M_{\alpha\delta} - C_{\alpha\delta} M_{\gamma\beta} - C_{\beta\delta} M_{\alpha\gamma} + C_{\gamma\alpha} M_{\delta\beta} \\
[M_{\alpha\beta}, Q_\gamma] &= C_{\gamma\beta} Q_\alpha + C_{\gamma\alpha} Q_\beta \\
\{Q_\alpha, Q_\beta\} &= M_{\alpha\beta}
\end{aligned}
\tag{2.10}
$$

$$M_{\alpha\beta} = M_{\beta\alpha} \; ; \quad C_{\alpha\beta} = -C_{\beta\alpha}$$

For example, we may take C to be the charge conjugation matrix in the Dirac spinor representation, and expand the $M_{\alpha\beta}$ in terms of the skew-symplectic 4 x 4 matrices $(\gamma_\mu C)_{\alpha\beta}$ and $(\sigma_{\mu\nu} C)_{\alpha\beta}$:

$$M_{\alpha\beta} = -(\gamma_\mu C)_{\alpha\beta} M^\mu - \tfrac{1}{2}(\sigma_{\mu\nu} C)_{\alpha\beta} M^{\mu\nu} \tag{2.11}$$

Using Eqs. (2.10), we find that the $M^{\mu\nu}$ generate an O(3,1) subalgebra, and the M^μ are a vector such that

$$[M_\mu, M_\nu] = -i M_{\mu\nu} \tag{2.12}$$

Comparing this with Eq. (2.10a), and defining a set of generators M^{ab} $(a, b = 0, 1, 2, 3, 4)$, where

$$M^{\mu 4} = M^\mu$$

we find that the M^{ab} generate the $SO(3,2)$ algebra of the de Sitter group, with metric η_{ab} = diag (+,-,-,-,+). Thus we have constructed the basis transformation which effects the local isomorphism of the Lie groups $Sp(4,R) \cong SO(3,2)$. For the remaining commutation relations we have

$$\begin{aligned} [M^{\mu\nu}, Q_\alpha] &= -\tfrac{1}{2}(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta \\ [M^\mu, Q_\alpha] &= -\tfrac{1}{2}(\gamma^\mu)_\alpha{}^\beta Q_\beta . \end{aligned} \tag{2.13}$$

Now let us define [39] $\bar{S}_\alpha = \frac{1}{R} Q_\alpha$, $\bar{P}^\mu = \frac{1}{R^2} M^{\mu 4}$, and $\bar{M}^{\mu\nu} = M^{\mu\nu}$, and rewrite the GLA Eq. (2.10) in terms of the barred generators, and R. We then take the limit $R \to \infty$, assuming in the limiting procedure that $\bar{M}^{\mu\nu} \to J^{\mu\nu}$, $\bar{P}^\mu \to P^\mu$, and $\bar{S}_\alpha \to S_\alpha$. It can be seen by inspecting Eqs. (2.12) and (2.13) that after the limiting process, we shall be left with a GLA

$$\begin{aligned} [J_{\mu\nu}, J_{\rho\sigma}] &= i(\eta_{\rho\nu} J_{\mu\sigma} - \eta_{\mu\sigma} J_{\rho\nu} - \eta_{\nu\sigma} J_{\mu\rho} + \eta_{\rho\mu} J_{\sigma\nu}) \\ [J_{\mu\nu}, P_\rho] &= i(P_\mu \eta_{\nu\rho} - P_\nu \eta_{\mu\rho}) \\ [J_{\mu\nu}, S_\alpha] &= \tfrac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta S_\beta \\ \{S_\alpha, S_\beta\} &= -(\gamma_\mu C)_{\alpha\beta} P^\mu \end{aligned} \tag{2.14}$$

This algebra is the GLA $\mathcal{S}$ of Salam and Strathdee, and Volkov and Akulov.

It should be pointed out that there are some pitfalls in the classification of the GLA's which do not occur for LA's. For example[30], it is possible to construct a GLA for which there are two inequivalent irreducible representations, having, however, identical second order Casimirs. Thus, although the Wigner-Eckart theorem appears to exist for simple cases, such as the $\mathcal{G}SU(2)$ studied by Pais and Rittenberg [45], it is not guaranteed. This might be expected to be the case a fortiori for the non-simple supersymmetry algebra $\mathcal{S}$. However, in Sec. 4.1 we shall explicitly carry out the reduction of the direct product of two unitary irreducible representations of $\mathcal{S}$ into a direct sum (for the massive case). Thus, in this case, also, the Wigner-Eckart theorem does hold.

We shall encounter one case where the existence of simple subalgebras of $\mathcal{S}$ is crucial: in the construction of unitary representations. For fixed $p \neq o$, the 'little algebra' of super-translations (which is adjoined to the ordinary little group) is a Clifford algebra, containing the identity (and hence simple). Thus by a theorem of Weyl on simple matrix algebras [46], there is precisely one finite-dimension al irreducible representation. Thus the irreducible representations of the little algebra are uniquely specified by those of the little group for the $p \neq o$ cases. This fact will be exploited in Sec. 3.2.

If on the other hand $p = o$, then the little algebra is not simple, and we can expect both finite and infinite-dimensional irreducible representations. Unfortunately this is precisely the starting-point in an investigation of the homogeneous Bethe-Salpeter equation. Fortunately the question is somewhat simpler after a Wick notation. This matter will be taken up in Chap. 6.

As was mentioned in Sec. 1.1, various uses had been made of the GLA concept before the advent of the "supergauge" transformations[16]. It is worth pointing out here, though, that fermion operators, with various tensor properties under rotations, and thus GLA's, have long found wide application in nuclear physics. However, it is usually some ordinary Lie subalgebra, rather than the whole GLA, which is of eventual interest.

Indeed, Freund and Kaplansky [39] give a realization of aGLA in terms of a system of boson and fermion creation and annihilation-operators. Such a mixed system resembles that used in the Green ansatz, in parastatistics [8].

2.2 Generalized Lie Groups

In the previous chapter the idea of a natural unified description of bosons and fermions led to the introduction of graded Lie algebras as one possible vehicle for such a symmetry principle. It is important, for global considerations, to be able to handle finite symmetry properties, as well as infinitesimal ones. For the present case, the graded Lie algebras, in particular the "supersymmetry" algebra of the last section, correspond to the infinitesimal symmetries. In this section we shall indicate how the graded Lie algebras can be integrated, at least formally, to "generalized Lie groups", which are still groups in the algebraic sense, and which are the desired finite symmetry transformations.

The famous theorems of Lie [47] establish the conditions under which the exponential mapping exists between a Lie algebra and a corresponding continuous group, and conversely. The group axioms of associativity, inverse and so on specify properties of the structure constants of the algebra

$$[X_A, X_B] = c_{AB}{}^{C} X_C \;, \quad c_{AB}{}^{C} = - c_{BA}{}^{C} \;, \tag{2.15}$$

where the X_A are the infinitesimal group generators. The structure constants are related to commutators $g_1 g_2 g_1^{-1} g_2^{-1}$ of group elements. Choosing $g_{1,2}$ close to the identity, e, and coordinates (g^A) on the group manifold such that e is at the origin of the coordinates,

$$\begin{aligned} (g_1 g_2 g_1^{-1} g_2^{-1})^C &= c_{AB}{}^{C} g_1{}^{A} g_2{}^{B} + o(g^3) \\ (g_2 g_1 g_2^{-1} g_1^{-1})^C &= -c_{BA}{}^{C} g_2{}^{B} g_1{}^{A} + o(g^3) \;. \end{aligned} \tag{2.16}$$

In the case of graded Lie algebras, the situation is more complicated. As we have seen, the $c_{AB}{}^{C}$ is graded-antisymmetrical,

$$[Z_A Z_B] = c_{AB}{}^{C} Z_C = -(-1)^{AB}[Z_B Z_A] \tag{2.17}$$

Thus if we assume that there is still a corresponding local group, and insist upon Eq. (2.16) we have

$$c_{AB}{}^{C}\left(g_1{}^{A} g_2{}^{B} - (-1)^{AB} g_2{}^{B} g_1{}^{A} \right) = 0$$

which suggests we take

$$g_1{}^{A}\ g_2{}^{B} = (-1)^{AB}\ g_2{}^{B}\ g_1{}^{A} \tag{2.18}$$

that is, the group manifold Ω ("parameter space") of a continuous group associated with a graded Lie algebra $\mathcal{G}$, should be graded commutative, $\Omega = \Omega^0 + \Omega^1$. For example, Ω might be the 2^n-dimensional Grassmann algebra of exterior products of an n-dimensional vector space. Then the even and odd elements of Ω would be combinations of exterior products of even and odd order, respectively. The group is $\Omega \times \mathcal{G}$, where we identify $(w,z).(w',z') = (-1)^{zw'}(\omega\omega', zz')$. Sufficiently close to the identity, we may write group elements as $g = \exp(\omega^A Z_A)$. Such a continuous group is called[17] a "generalized Lie group".

Differentiation and integration of functions over the group manifold can be defined straight-forwardly. For the odd coordinates we have [17]

$$\frac{\partial}{\partial\omega}\omega' = -\omega'\frac{\partial}{\partial\omega} + \delta_{\omega\omega'}$$

$$\int d\omega = 0, \quad \int \omega\, d\omega = 1, \quad \{\omega', d\omega\} = 0 = \{d\omega', d\omega\} \tag{2.19}$$

If we assume that Ω admits an involution

$$(\omega^*)^* = \omega, \quad (\lambda\omega)^* = \bar{\lambda}\omega^*, \quad (\omega\omega')^* = \omega'^*\omega^*, \tag{2.20}$$

then we can define an inner product with positive norm [48,49] if Ω is 2^{n+1} - dimensional,

$$\langle \varphi, \chi \rangle = \int e^{-\sum \omega_i^* \omega_i} \chi(\omega)\, \varphi(\omega)^*\, d\omega_n^*\, d\omega_n \cdots d\omega_1^* d\omega_1$$

In the sequel we shall need no more than the rules for differentiation. Other properties of a-numbers are discussed in more detail elsewhere [17, 19, 48, 49]. Using the example of GSU(2) [45], Mezinescu [50] shows how functions over the group manifold admit an invariant measure, the group volume is nonzero, and an inner product and orthogonal functions can be defined (c.f. Sec. 4.1). The Wigner-Eckart theorem applies for this particular case (c.f. Sec. 2.1).

The general question of representations of generalized Lie groups is crucial for physical applications. An obvious starting-point is the regular representation carried by functions over the group, $\rho_\omega\, \phi(\omega') = \phi(\omega'\omega)$. Alternatively, we could consider $\tau_\omega\, \phi(\omega') = \phi(\omega^{-1}\omega'\omega)$ generated by the inner automorphisms.

For example, consider the supersymmetry algebra $\mathcal{S}$ of the last section. The generators P_μ and S_α generate an abelian normal subgroup of translations and "supertranslations". Therefore we can consider a representation of $\mathcal{S}$ on functions defined over the 8-dimensional manifold (x_μ, θ_α), where x_μ is a 4-vector with components in Ω^0, and is a Majorana 4-spinor with components in Ω^1 . Functions defined on this manifold are called superfields [18]. The action of finite group transformations can be derived from the graded Lie algebra of infinitesimal generators just as in the non-graded case. For supertranslations only, we have [18]

$$e^{-i\bar{\varepsilon}S}\ \Phi(x,\theta)\ e^{i\bar{\varepsilon}S} \;=\; \Phi\left(x+\tfrac{i}{2}\bar{\varepsilon}\gamma\theta,\ \theta+\varepsilon\right)$$

or
$$\left\{S_\alpha,\ \Phi(x,\theta)\right\} \;=\; i\left(\partial/\partial\bar{\theta}^\alpha + \tfrac{i}{2}\, i\not{\partial}\,\theta_\alpha\right)\Phi(x,\theta) \tag{2.20}$$

Now any smooth function over the group manifold may be expanded as a Taylor series in the coordinates. Moreover, since (as we assume) there are a finite number of a-number (Fermi) parameters, the Taylor series becomes a polynomial in the a-numbers with c-number coefficient functions.

This is true in particular of the superfields. We can take the independent linear combinations of the θ_α to be 1, θ_α, $\bar{\theta}\theta$, $\bar{\theta}\gamma_5\theta$, $\bar{\theta}i\gamma_\mu\gamma_5\theta$, $\theta\,\bar{\theta}\theta$, $(\bar{\theta}\theta)^2$. Thus [18]

$$\Phi(x,\theta) = A + \bar{\theta}\psi + \tfrac{1}{4}\bar{\theta}\theta F + \tfrac{1}{4}\bar{\theta}\gamma_5\theta G + \tfrac{1}{4}\bar{\theta}i\gamma_\mu\gamma_5\theta A^\mu + \tfrac{1}{4}\bar{\theta}\theta\,\bar{\theta}\chi + \tfrac{1}{32}(\bar{\theta}\theta)^2 D \tag{2.21}$$

The coefficient functions are called the component fields. With this expansion, the transformations Eq. (2.20) can be explicitly evaluated, and the new component fields written down. Clearly the boson components obtain parts involving ϵ and the fermion components, and vice-versa. These transformations in terms of the component fields are the famous "supergauge" transformations of Wess and Zumino [16].

The superfield transformations of Eq. (2.20) form a non-unitary, finite-dimensional and in general reducible representation of $\mathcal{S}$. We shall return to some examples in Sec. 5.1 when we consider the structure of superfields in more detail.

As we have seen, the parameters of a generalized Lie group take their values in a graded-commutative algebra Ω (for example, a Grassmann algebra). In order to determine the number of underlying real parameters of the continuous group, it is necessary to choose a basis for Ω . This process is rather arbitrary, but we can at least try to be as economical as possible.

Let us take for example the supersymmetry algebra $\mathcal{S}$. Clearly if Ω has only one generator, ω_1 , the superfields are linear in θ, and the expansion of Eq. (2.21) collapses. Rühl and Yunn [48] point out that just two generators are sufficient. In the Weyl basis, we can choose the ϵ_α independently, and the $\bar{\epsilon}_{\dot{\alpha}}$ follow from the Majorana constraint. Thus the general form must be $\epsilon_\alpha = \omega_1 \xi_\alpha + \omega_2 \eta_\alpha$, with third- and higher orders in w vanishing. Here ξ_α, η_α are complex c-number spinors, suitably restricted by the Majorana constraint upon ϵ_α (e.g. if $\omega_1^* = \omega_1$, $\omega_2 = \omega_2^*$, they are Majorana). In any case, ϵ contains 8 real parameters. Correspondingly, we can take even parameters a_μ of the form $a_\mu = a_\mu^{(0)} + \bar{\epsilon}' i \gamma_\mu \gamma_5 \epsilon'$ where ϵ' is an odd element like ε . This is equivalent to taking $a_\mu = a_\mu(0) + \omega_1\omega_2\, a_\mu(2) + \omega_2^*\omega_1^*\, a_\mu^*(2)$. In both cases a_μ is real, no higher orders appear, and a_μ contains 4 + 8 = 12 real parameters. Finally, including 6 real parameters for the homogeneous Lorentz transformations, we have [48] a total of 26 real parameters for the generalized Lie group $\mathcal{S}$.

Having found the real parameters, we can find the (local) Lie group "equivalent" to the generalized Lie group, by writing down the generators of the infinitesimal Lie algebra [19, 49]. For the above analysis, the equivalent (real) Lie algebra is 26-dimensional, generated by

$$\omega_1 S_\alpha\ ,\ \omega_1^* S_\alpha\ ;\quad P_\mu\ ;\quad \omega_1\omega_1^* P_\mu\ ,\ i\omega_1\omega_1^* P_\mu\ ;\quad J_{\mu\nu} \quad = \quad 26$$

In fact, Goddard [49] has shown that this amount of doubling is unnecessary: by exploiting the fact that the chiral projections $\frac{1}{2}(1 \pm i\gamma_5)S$ are always decoupled (eq. (2.14)), we need only take the 18-dimensional Lie algebra $\tilde{\mathcal{S}}$ generated by

$$\omega_1 S_+\ ,\ \omega_1^* S_-\ ;\quad P_\mu\ ,\ \omega_1\omega_1^* P_\mu\ ;\quad J_{\mu\nu} \quad = \quad 18$$

The comment was made in Sec. 1.2 that the supersymmetry algebra $\mathcal{S}$ may be considered as a first example of a relativistic spin-containing symmetry. It is appropriate here to consider this statement in relation to the "no-go" theorems which place severe restrictions on such a situation. Goddard [49] shows that, from the point-of-view of the "equivalent" Lie algebra $\tilde{\mathcal{S}}$, this actually corresponds to one of the possibilities distinguished by O'Raifeartaigh [51], but passed over as unpromising: namely, a Lie algebra whose solvable part (c.f. Sec. 2.1) is nonabelian, and contains the translations. On the other hand, the theorem of Coleman and Mandula [52] is not applicable, because the Hilbert space of physical states is not invariant under the group (a state $|\psi\rangle$ obtains a part proportional to $\omega_i|\psi\rangle$ under a supertranslation). Haag and others [53,54] have, in fact, extended this approach to determine all possible supersymmetries of the S-matrix. The result in general is just a graded Lie algebra of the same form as $\mathcal{S}$ (eq. (2.14)), with a multiplet of spinor fermion generators transforming under some (boson-type) Lie algebra which commutes with the Poincaré group (c.f. also the discussion of internal symmetry, Sec. 3.5).

3. UNITARY REPRESENTATIONS OF SUPERSYMMETRY

We begin our investigations of supersymmetry in the present chapter, with an analysis of the unitary irreducible representations (UIR's) and the multiplet structure of the algebra $\mathscr{S}$ in the massive and massless cases. This ground-work is essential to the development of the supersymmetric partial-wave analysis in the next chapter, and also provides valuable insights in the work on superfield representations in Chap. 5.

The GLA $\mathscr{S}$ which we study is [15, 16 18] (c.f. also Sec. 2.1)

$$
\begin{aligned}
[J_{\mu\nu}, J_{\rho\sigma}] &= i(\eta_{\rho\nu} J_{\mu\sigma} - \eta_{\mu\sigma} J_{\rho\nu} - \eta_{\nu\sigma} J_{\mu\rho} + \eta_{\rho\mu} J_{\sigma\nu}) \\
[J_{\mu\nu}, P_\rho] &= i(P_\mu \eta_{\nu\rho} - P_\nu \eta_{\mu\rho}) \\
[J_{\mu\nu}, S_\alpha] &= \tfrac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta S_\beta \;, \quad [P_\mu, S_\alpha] = 0 \\
\{S_\alpha, S_\beta\} &= -(\gamma^\mu C)_{\alpha\beta} P_\mu \;.
\end{aligned}
\tag{3.1}
$$

Here the $J_{\mu\nu}$ are the generators of Lorentz transformations, which with the translation generators (4-momentum) P_μ comprise the Poincaré subalgebra $\mathscr{P}$ (more exactly, $\mathscr{P}_+^\uparrow$). In the language of GLA's, the $J_{\mu\nu}$ and P_μ are bose generators. Adjoined to these are the fermionic "supertranslation" generators S_α.

In a unitary representation of $\mathscr{P}$ (that is, with an inner product such that group transformations are represented by unitary operators), the generators $J_{\mu\nu}$ and P_μ are hermitean. We shall define a unitary representation of the whole algebra, $\mathscr{S}$, to be one in which, in addition, the supertranslations S_α obey the reality property of a Majorana spinor, namely

$$\bar{S}^\alpha = (C^{-1})^{\alpha\beta} S_\beta$$

so that "$e^{i\bar{\xi}S}$" is unitary. However, we consider the algebra only over the complex numbers, and we do not introduce a-number parameters.

It is clear from Eq. (3.1) that irreducible representations of $\mathcal{S}$ are characterized by their mass, the eigenvalue of $P^2 = P_\mu P^\mu$, just as in the Poincaré case. Thus we can distinguish the class $P^2 > 0$ (timelike), $P^2 = 0$ (lightlike), $P = 0$ (null), and $P^2 < 0$ (spacelike). It is with the first two cases that we are concerned in this and the next two chapters. The null case occurs in Chap. 6 in connection with the bound state problem.

The work of this chapter is based mainly on Ref. 27. In Sec. 3.1, irreducible representationsof $\mathcal{S}$ in the massive case are analysed in a covariant way to determine their possible spin contents. This is repeated in Sec. 3.2 for the UIR's by an extension of the induced representation method of Wigner [24]. Convenient bases for the UIR's are found in the course of these analyses. In the "spin" basis, states are labelled $|p^2 > 0, j_0; \, \underline{p} \, \sigma \, j \, \lambda\rangle$, where $j_0 = 0, \frac{1}{2}, 1, \cdots$ is the "superspin" of the UIR, and $j = j_0, \; j_0 \pm \frac{1}{2}$ the spin component, with helicity λ. The label $\sigma = 0, \pm\frac{1}{2}$ serves to distinguish the different spins occurring since $j = j_0 \pm \frac{1}{2} \mp |\sigma|$. On the other hand, in the "superhelicity" basis, the spin is replaced by the superhelicity, $\kappa = -j_0, -j_0+1, \cdots, j_0-1, j_0$, so that states are labelled $|p^2>0, j_0;\ \not{p}\ \sigma\ \kappa\ \lambda>$, where now $\lambda = \kappa \pm \pm\frac{1}{2} \mp |\sigma|$. The basis transformations between these bases are written down. Matrix elements of the supertranslations are written down also. Weight diagrams are also introduced which display the spin structure of the UIR's.

The massless UIR's (a special case of the lightlike UIR's) are analysed similarly in Sec. 3.3. States are labelled by $|p^2=0, \lambda_0;\ ;\ \not{p}\ \lambda>$, where λ_0 is the "superhelicity", and $\lambda = \lambda_0, \lambda_0-\frac{1}{2}$ Once again, the matrix elements of supertranslations in this basis are written down.

In Sec. 3.4, it is shown how parity may be adjoined to the UIR's of $\mathscr{S}$. In the massless case it is necessary to take a direct sum of UIR's $(-\lambda_0+\frac{1}{2})\oplus(\lambda_0)$, which contains helicities of $\pm\lambda_0, \pm(\lambda_0-\frac{1}{2})$. Extending the analysis to include internal symmetries is considered in Sec. 3.5. Other progress along these lines is briefly reviewed.

Our notational conventions are established in Sec. A1. Further details of the supersymmetry algebra, including some polynomial identities, are given in Sec. A2. The spin and superhelicity bases are described in Sec. A3.

3.1 Massive Case: Covariant Analysis

In this section we consider the UIR's of the supersymmetry algebra $\mathcal{S}$, for the case when the momentum P_μ is timelike, $P^2 (= m^2) > 0$, and sign$(P_0) = \pm 1$. For massive particles in particular, the energy takes the positive sign.

As a first step in the analysis we consider the algebra of operators commuting with the supertranslations S_α. The square of the Pauli-Lubanski vector

$$W_\mu = \epsilon_{\mu\nu\rho\sigma} P^\nu J^{\rho\sigma} \tag{3.2}$$

is a Casimir of $\mathcal{P}$, but not of $\mathcal{S}$. However, if we define [18]

$$\Sigma_\mu = -\tfrac{1}{4} \bar{S} \, i \gamma_\mu \gamma_5 S \tag{3.3}$$

$$K_\mu = W_\mu - \Sigma_\mu$$

$$K^\perp_\mu = K_\mu - P_\mu P^{-2} P\cdot K , \tag{3.4}$$

then we find that $K^\perp_\mu$ commutes with S_α and P_μ, so that $(K^\perp)^2$ is a Casimir of $\mathcal{S}$, generalizing W^2 in the case of the Poincaré group. Irreducible representations of $\mathcal{S}$ are therefore labelled by the eigenvalue of $(K^\perp)^2$, as well as P^2. The physical meaning of $(K^\perp)^2$ will become clear subsequently.

The tensors

$$\begin{aligned} K_{\mu\nu} &= K_\mu P_\nu - K_\nu P_\mu \\ \mathcal{M}_{\mu\nu} &= K_{\mu\nu} + \tfrac{1}{2} \epsilon_{\mu\nu\rho\sigma} K^{\rho\sigma} \end{aligned} \tag{3.5}$$

also commute with S_α, and P_μ, and moreover

$$[M_{\mu\nu}, M_{\rho\sigma}] = iP^2(\eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\rho}M_{\nu\sigma}) \quad (3.6)$$

Thus the $M_{\mu\nu}$ generate a group, $\mathcal{L}'$, isomorphic to the Lorentz group, SL(2, C) = $\mathcal{L}$, generated by the $J_{\mu\nu}$. However, whereas the S_α and P_μ have definite transformation properties under Lorentz transformations, they are invariant under $\mathcal{L}'$.

Consider the supersymmetry transformations in more detail. According to Sec. 2.2, group elements are parametrized in the form $(a_\mu, \varepsilon_\alpha, \Lambda_\mu{}^\nu)$, acting on the 8-dimensional manifold (x_μ, θ_α)[18] by

$$(a, \varepsilon, \Lambda): (x, \theta) \to (a + \tfrac{i}{2}\bar{\varepsilon}\gamma S\theta + \Lambda x, \; S\theta + \varepsilon)$$

Here a_μ is a 4-vector, ε_α, θ_α are a-number Majorana 4-spinors, and $S(\Lambda)_\alpha{}^\beta$ is the matrix representing the Lorentz transformation $\Lambda_\mu{}^\nu$ in the 4-spinor basis. We have then for the composition of two such transformations

$$(a', \varepsilon', \Lambda') \circ (a, \varepsilon, \Lambda) = (a' + \Lambda'(a + \tfrac{i}{2}\bar{\varepsilon}\gamma\varepsilon), \; \varepsilon' + S'\varepsilon, \; \Lambda'\Lambda).$$

Hence we may describe $\mathcal{S}$ as a semidirect product of the Lorentz group, $\mathcal{L}$, and the translations and supertranslations, $\mathcal{J}$.

Comparing this with the above, we have the following descriptions of $\mathcal{S}$:

$$\mathcal{S} \cong \mathcal{L} \wedge \mathcal{J} \supset \mathcal{L}' \times \mathcal{J} \quad (3.7)$$

where Λ, $\times$ denote semi-direct, and direct products, respectively.

The subalgebra $\mathcal{L}' \times \mathcal{J}$ provides a means of analysing the UIR's of $\mathcal{S}$. In particular, since the Casimirs of L' are also Casimirs of $\mathcal{S}$, any UIR of $\mathcal{S}$ contains precisely one UIR of L'. Any irreducible representation of the latter is characterized [55] by an ordered pair (l_0, l_1), where $l_0 = 0, \frac{1}{2}, 1, \cdots$ and l_1 is an arbitrary complex number. The values of l_0 and l_1 are further restricted for the unitary representations. In each case l_0 and l_1 are defined in terms of the eigenvalues of the Casimir operators $\frac{1}{2} M_{\mu\nu} M^{\mu\nu}$ and $\frac{i}{8} \epsilon_{\mu\nu\rho\sigma} M^{\mu\nu} M^{\rho\sigma}$ in the irreducible representation. For the present case we have

$$\frac{1}{2} M_{\mu\nu} M^{\mu\nu} \equiv (P^2)^2 \left(l_0(l_0 + 2) + l_1^2 - 1 \right) = P^2 (K^2)^2$$

$$\text{and} \quad \frac{i}{8} \epsilon_{\mu\nu\rho\sigma} M^{\mu\nu} M^{\rho\sigma} \equiv (P^2)^2 l_0 l_1 = 0 \tag{3.8}$$

We shall return to Eqs. (3.8) below. The next step is to study the Poincaré subalgebra, $\mathcal{P}$, of $\mathcal{S}$.

Now, a UIR of $\mathcal{S}$ provides a representation, possibly reducible, of $\mathcal{P}$, so that we may analyse the structure of the former by determining its spin content, or the UIR's of $\mathcal{P}$ which it contains. This procedure is in any case necessary in a physical application, since it gives the particle content of the multiplet.

We are therefore led to study the algebra $\mathcal{A}$ of $\mathcal{P}$-invariants which may be constructed from the generators of $\mathcal{S}$. Each UIR of $\mathcal{P}$ subduced by a given UIR of $\mathcal{S}$ in the reduction $\mathcal{S} \supset \mathcal{P}$ should be associated with a particular irreducible representation of $\mathcal{A}$, whose dimension gives the degeneracy of the corresponding UIR of $\mathcal{P}$.

Thus the degeneracy can be removed by introducing additional quantum numbers as state labels for the irreducible representations of $\mathcal{A}$.

If we define, in addition to W_μ and Σ_μ, a four-vector

$$U_\mu \;=\; P^{-2}\,\epsilon_{\mu\nu\rho\sigma}\,P^\nu \Sigma^\rho W^\sigma , \tag{3.9}$$

then $\mathcal{A}$ is generated by the set

$$\{\, \Sigma^2,\; U^2,\; \bar{S}S_\pm\,,\; P\cdot\Sigma\,,\; \Sigma\cdot W,\; W\cdot U\,,\, \Sigma\cdot U \,\} ,$$

all of which, of course, commute with P^2 and W^2. However, these are not all algebraically independent. In fact, from the identities (eqs. (A2.8) and (A2.9)) satisfied by these invariants in any representation, it follows that the independent invariants are W^2, together with the generators $P\cdot\Sigma$, $\bar{S}S_\pm$ of $\mathcal{A}$, which has the Lie algebra:

$$\begin{aligned} [\,P\cdot\Sigma\,,\; \bar{S}S_\pm\,] \;&=\; \pm\, P^2\, \bar{S}S_\pm\,, \\ [\,\bar{S}S_+\,,\; \bar{S}S_-\,] \;&=\; 8\, P\cdot\Sigma \end{aligned} \tag{3.10}$$

Defining $T_3 \;=\; P^{-2}\,P\cdot\Sigma$ and $T_\pm = \frac{1}{2\sqrt{2}}(P^{-2})^{\frac{1}{2}}\bar{S}S_\pm$, $\mathcal{A}$ may be identified with the algebra of SU(2) (since T_1, T_2 and T_3 are hermitean, in a unitary representation). Moreover, it follows from the supersymmetry algebra [18] (eq. (A2.11)) that

$$(\,P\cdot\Sigma\,)^3 \;=\; \tfrac{1}{4}\,(P^2)^2\;P\cdot\Sigma \tag{3.11}$$

and the possible eigenvalues of T_3 are $\sigma = 0,\ \pm\frac{1}{2}$.

The supersymmetry algebra $\mathcal{S}$ therefore contains a subalgebra SU(2) x $\mathcal{P}$. A UIR of $\mathcal{S}$ breaks up into UIR's of $\mathcal{P}$ corresponding to doublets and singlets of SU(2). The eigenvalue σ of $P^{-2} P\cdot\Sigma$ acts as the third component of an internal "isospin", and it will furnish the additional quantum number required to remove the spin degeneracy. The supertranslations, $S_{\pm\alpha} = \frac{1}{2}(1 \pm i\gamma_5)_\alpha{}^\beta S_\beta$ act within this structure as spin- and "isospin" - shifting operators, since

$$[P\cdot\Sigma, \quad S_\pm] = \pm\frac{1}{2} P^2 S_\pm \tag{3.12}$$

We may now use the identities

$$\begin{aligned} (\Sigma\cdot W)^2 &= \frac{1}{4} W^2(\Sigma^2 - \frac{1}{4}P^2) - \frac{1}{2}P^2(\Sigma\cdot W) \\ (P\cdot\Sigma)(\Sigma\cdot W) &= 0 = (\Sigma\cdot W)(P\cdot\Sigma), \end{aligned} \tag{3.13}$$

and the invariance of

$$(K^\perp)^2 = W^2 - 2\Sigma\cdot W + \Sigma^2 - P^{-2}(P\cdot\Sigma)^2, \tag{3.14}$$

to determine which are the allowed values of spin associated with the "doublet" and "singlet" sectors within a given UIR of $\mathcal{S}$ in this case. We define

$$\begin{aligned} W^2 &= -P^2 j(j+1), \\ \Sigma\cdot W &= P^2 \tau, \end{aligned} \tag{3.15}$$

within each sector, where j is the spin.

If $|\sigma| = \frac{1}{2}$, then from Eqs. (3.13), (3.14) and (A2.11) we have $K^{\perp 2} = W^2$. Thus the $\sigma = \pm\frac{1}{2}$ sectors contain just one allowed spin, say j_0. If on the other hand $\sigma = 0$, then Eqs. (3.13) and (3.14) reduce to

$$(\tau + \tfrac{1}{2}(j+1))(\tau - \tfrac{1}{2}j) = 0$$

and $$-j_0(j_0+1) = -j(j+1) - 2\tau - 3/4 ,$$

respectively. The $j \geqslant 0$ solutions for $\tau = -\frac{1}{2}(j+1),\ \frac{1}{2}j$ are $j = j_0 + \frac{1}{2},\ j_0 - \frac{1}{2}$, respectively, in the latter case provided that $j_0 \geqslant \frac{1}{2}$.

To summarize, UIR's of $\mathscr{S}$ for the timelike case may be labelled $(p^2>0,\ j_0)^{\pm}$, where $j_0 = 0, \frac{1}{2}, 1, \ldots\ldots$ is the superspin, and sign $(P_0) = \pm 1$. They contain UIR's of $\mathscr{P}$ with $P^2 > 0$, and possible allowed spins $j = j_0,\ j_0,\ j_0+\frac{1}{2}$ and (provided $j_0 \geqslant \frac{1}{2}$) $j_0-\frac{1}{2}$.

Having discovered the significance of the Casimir $(K^{\perp})^2 = -P^2 j_0(j_0+1)$ and the eigenvalues $j_0 = 0, \frac{1}{2}, 1, \cdots$ we can return to Eqs. (3.8) and comment further on the UIR of L' which is associated with each UIR of $\mathscr{S}$. Now, according to Gel'fand et. al.[55], the UIR's of the Lorentz group may be classified into two series,

1. Principal Series: $\ell_0 = 0, \frac{1}{2}, 1, \cdots ;\ -\infty < \mathrm{Im}(\ell_1) < \infty,\ \mathrm{Re}(\ell_1) = 0.$
2. Supplementary Series: $\ell_0 = 0;\ \mathrm{Im}(\ell_1) = 0,\ |\ell_1| \leqslant 1.$

The values of j_0, (ℓ_0, ℓ_1), and the series, are given in Table 3.1 below, where $\ell = (j_0^2 + j_0 - 1)^{\frac{1}{2}}$. For $j_0 = 0$ and $j_0 = 1$ there appear to be two possible solutions for (ℓ_0, ℓ_1). Which of these is chosen in practice can only be decided from a detailed examination of the irreducible representations, which is beyond the scope of the present analysis, which is in any case more concerned with elucidating the structure of the subalgebra of supertranslations.

j_c	(ℓ_0, ℓ_1)	Series
0	(0, 1)	S
0	(1, 0)	P
$\frac{1}{2}$	(0, $\frac{1}{2}$)	S
1	(1, 0)	P
1	(0, i)	P
>1	(0, $i\ell$)	P

Table 3.1

The special case $j_0 = 0\ (0,1)$ is particularly interesting, since it would correspond [55] to the trivial, 1-dimensional unitary representation. Thus $M_{\mu\nu} = 0$, which implies in general that the generators $J_{\mu\nu}$ are not linearly independent. In the rest frame, for example, $\underline{J} = -P_0^{-1}\,\underline{\Sigma}$. Just such a realization of the supersymmetry algebra has in fact been proposed by Chakrabarti [56].

It should be pointed out in this connection that Nahm et. al.[30] have given an example of a simple GLA possessing two inequivalent irreducible representations, having however the same eigenvalues of the Casimir operators. The present case may furnish another example of such a phenomenon, since both GLA's are "odd reducible" (with the fermionic generators belonging to a reducible representation of the underlying Lie algebra of the Bose sector). However, too close an analogy cannot be drawn, since the GLA $\mathscr{S}$ of the present case is not simple (c.f. Chap. 2).

So far we have only determined that in a $(p^2 > 0, j_0)^{\pm}$ UIR of $\mathscr{S}$, the allowed values of the spin are $j = j_0 \pm \frac{1}{2} \mp |\sigma|$. We have not shown with what multiplicities these spins occur, if they occur at all. We now answer this question, by constructing explicit realizations of the UIR's. In particular we shall show that the $(p^2 > 0, j_0)^{\pm}$ UIR's of $\mathscr{S}$ contain $4(2j_0+1)$ helicity states, the spin content being precisely

$$4(2j_0+1) = 2(2j_0+1) + (2(j_0+\tfrac{1}{2})+1) + (2(j_0-\tfrac{1}{2})+1)$$

as in other analyses [18,24,57]. We also find the matrix elements of the supertranslations S_α in the spin basis, and verify the Majorana condition, Eq. (3.1), so that the representations are indeed unitary.

The spin content of a $(p^2 > 0, j_0)^{\pm}$ UIR of $\mathscr{S}$ may be conveniently visualized by means of a "weight diagram". This is a two-dimensional plot of j against σ, showing the values of (j, σ) or weights, which participate in the UIR. The weight diagrams for $j_0 = 0$, and arbitrary $j_0 > 0$ are given in Fig 3.1. We shall return to these diagrams subsequently, in considering direct products of representations, and also finite-dimensional (superfield) representations.

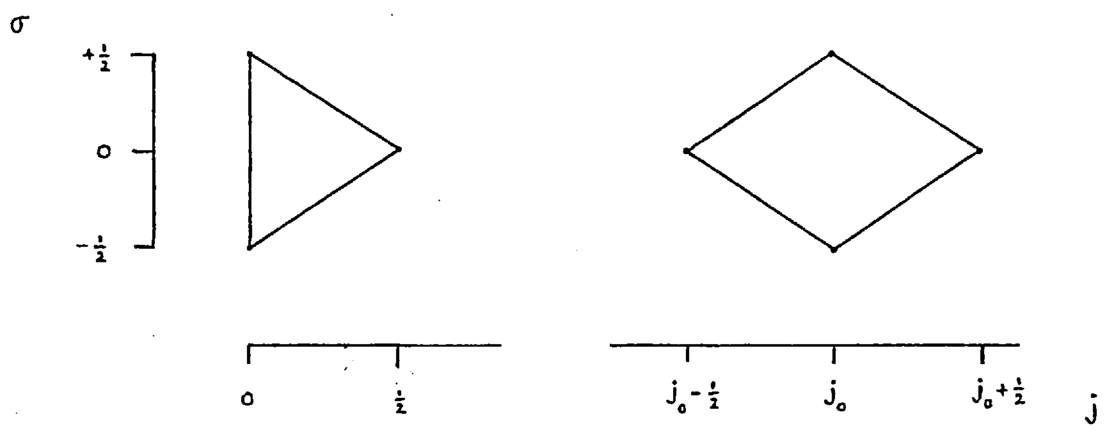

Fig. 3.1 Weight diagrams for $j_0 = 0$, and arbitrary j_0, $(p^2 > 0, j_0)^{\pm}$ UIR's.

In constructing the $(p^2 > 0, j_0)^{\pm}$ UIR's of $\mathscr{S}$, we begin with a UIR $(p^2 > 0, j_0)^{\pm}$ of $\mathscr{P}$, with weight $\sigma = -\frac{1}{2}$. Since this is the lowest participating value of σ, these states will act as vacua for the raising and lowering operators $S_{\pm\alpha}$ and $\bar{S}S_{\pm}$. In particular, we shall have

$$S_{-\alpha} | \sigma = -\tfrac{1}{2} \rangle = 0 .$$

We then use the raising operators to define covariantly normalized basis states $| p^2 > 0, j_0 ; \sigma ; p j \lambda \rangle$,

$$\langle \sigma' ; p' j' \lambda' | \sigma ; p j \lambda \rangle = \delta_{\sigma'\sigma} \delta_{j'j} \delta_{\lambda'\lambda} \langle p' | p \rangle, \tag{3.16}$$

with
$$\langle p' | p \rangle = 2 p_0 (2\pi)^3 \delta^3 (\underline{p}' - \underline{p}),$$

where the $|p\,j\,\lambda\rangle$ states of the various σ- sectors will belong to $(p^2>0, j)^{\pm}$ UIR's of $\mathcal{P}$, with spin j and helicity λ :

$$|p\,j\,\lambda\rangle = U(L_p)\,|\hat{p}\,j\,\lambda\rangle$$

$$\underline{J}^2\,|\hat{p}\,j\,\lambda\rangle = j(j+1)\,|\hat{p}\,j\,\lambda\rangle$$

$$J_3\,|\hat{p}\,j\,\lambda\rangle = \lambda\,|\hat{p}\,j\,\lambda\rangle .$$

The operators $(S_\pm)_\alpha$ have a complicated effect on helicity states, and are not the most convenient σ-shifting operators. Instead, we pass to an equivalent set $R'^{\pm}_{\kappa}$, defined by [48]

$$\begin{aligned} R'^{\pm}_{\kappa} &= \bar{u}_\kappa(p)\,S_\pm , \\ S_\pm &= \tfrac{1}{m}\sum_\kappa u_{\kappa\pm}(p)\,R'^{\pm}_{\kappa} , \end{aligned} \tag{3.17}$$

where $u_\kappa(p)_\alpha$, $\kappa=\pm\frac{1}{2}$ are normalized positive-frequency c-number spinor solutions of the Dirac equation, with mass m, and helicity k, whose properties are given in Eqs. (A3.1). The transformation properties of the $R'^{\pm}_{\kappa}$ under Lorentz transformations are

$$U(\Lambda)\,R'^{\pm}_{\kappa}(p)\,U(\Lambda)^{-1} = R'^{\pm}_{\kappa'}(\Lambda p)\,D^{\frac{1}{2}*}_{\kappa'\kappa}(\hat{\Lambda}), \tag{3.18}$$

where $\hat{\Lambda} = L^{-1}_{\Lambda p}\,\Lambda\,L_p$ is a little group rotation, the $D^{\frac{1}{2}}$ matrix represents notations for spin-$\frac{1}{2}$, and the $U(\Lambda)$ are unitary (reducible) operators representing Lorentz transformations. Using the reality property of the D-matrices [58], the operators

$$R^{\pm}_{\kappa}(p) = (-1)^{\kappa}\,R'^{\pm}_{-\kappa}(p) = (-1)^{\kappa}\,\bar{u}_{-\kappa}(p)\,S_\pm \tag{3.19}$$

will transform with $D^{\frac{1}{2}}$.

The algebra of $R_\kappa^\pm(p)$ may be derived from that of the S_α [18]. In particular, we have

$$R_\kappa^\pm \; R_{\kappa'}^\pm \;=\; \mp\,\kappa m\,\delta_{\kappa,-\kappa'}\,\bar{S}S_\pm$$

$$\{ R_\kappa^+ \; R_{\kappa'}^- \} \;=\; -\,2\kappa m^2\,\delta_{\kappa,-\kappa'} \tag{3.20}$$

and

$$(R_\kappa^\pm)^\dagger \;=\; \mp\;2\kappa\;R_{-\kappa}^\mp\;, \tag{3.21}$$

in a unitary representation.

In view of (3.18), we find that

$$U(\Lambda)\,R_\kappa^\pm(p)\,|\,p\,j\,\lambda\rangle \;=\; R_{\kappa'}^\pm(\Lambda p)\,|\,\Lambda p\;j\;\lambda'\rangle\,\mathcal{D}^{\frac{1}{2}}_{\kappa'\kappa}(\hat{\Lambda})\,\mathcal{D}_{\lambda'\lambda}(\hat{\Lambda}),$$

so that in particular, using the Wigner-Eckart Theorem,

$$R_\kappa^+\,|\,-\tfrac{1}{2};\,p\,j_0\,\lambda\rangle \;=\; \sum_j c_j\,|\,0;\;p\,j\;\lambda+\kappa\rangle\langle j\;\lambda+\kappa\,;\,\tfrac{1}{2}\,j_0\,|\,\tfrac{1}{2}\,\kappa\,;\,j\,\lambda\rangle,$$

where the $\langle j\;\;\lambda+\kappa\;;\;\tfrac{1}{2}\,j_0\,|\,\tfrac{1}{2}\;\kappa\;;\,j_0\;\lambda\rangle \;=\; \langle\tfrac{1}{2}\kappa\,;\;j_0\lambda\,|\;j\;\;\lambda+\kappa\,;\;\tfrac{1}{2}\,j_0\rangle^*$ are Clebsch-Gordan coefficients [58], and the c_j are independent of $\lambda+\kappa$.

The algebra of the $R_\kappa^\pm$ Eq. (3.20), and that of the $\bar{S}S_\pm$, and $P\cdot\Sigma$, eq. (A2.11), may now be used to determine the c_j, and hence the normalized basis vectors. The latter are given in Eq. (A3.6). We find the nonzero matrix elements of $R_\kappa^\pm$ to be

$$\langle 0\,;\,p'j\;\;\lambda+\kappa\,|\,R_\kappa^\pm\,|\mp\tfrac{1}{2}\,;\,p\,j_0\,\lambda\,\rangle \;=\; m\,\langle\, j\;\;\lambda+\kappa\,|\,\tfrac{1}{2}\;\kappa\;\;j_0\,\lambda\,\rangle\langle p'|p\rangle$$
$$\tag{3.22}$$
$$\langle\pm\tfrac{1}{2};\,p'j_0\,\lambda+\kappa\,|\,R_\kappa^\pm\,|\,0;\,p\,j\,\lambda\rangle \;=\; \mp 2m(j-j_0)\left(\frac{2j+1}{2j_0+1}\right)^{\frac{1}{2}}\langle j_0\;\lambda+\kappa\,|\,\tfrac{1}{2}\,\kappa\,j\,\lambda\rangle\langle p'|p\rangle,$$

and Eq. (3.21) holds as a matrix identity.

3.2 Massive Case: Induced Representation Method

In the previous section we analysed the UIR's of the supersymmetry algebra $\mathcal{S}$ for the massive case by a purely covariant method, without recourse to induced representation theory. However, this does provide an intuitive physical understanding of the structure of the UIR's, and it is interesting from a mathematical point of view to see that an extended form of induced representation approach can indeed be applied in this case. In this section we therefore repeat the construction of the $(p^2 > 0, j_0)^{\pm}$ UIR's of $\mathcal{S}$, but now using induced representations. We follow very closely the method of Salam and Strathdee [18, 24].

Whereas our main concern in the last section was the spin content of the UIR's of $\mathcal{S}$, in the induced representation construction we shall find another quantum number, here called the "superhelicity", k, with values $\kappa = -j_0, \cdots, +j_0$ arising much more naturally than the spin, j. Our main motivation for the induced representation approach is, in fact, to introduce the superhelicity basis, which will be used in an essential way in parallel with the spin basis in the next chapter, in solving the Clebsch-Gordan problem. Here we merely define the superhelicity basis and establish the relationship between it and the spin basis of the last section.

In the usual induced representation method for the space groups [1, 59], one considers an irreducible representation of the translation subgroup, namely a linear form $e^{ip\cdot a}$. One then induces from a UIR of the corresponding little group of transformations which leave the linear form invariant, to obtain a UIR of the whole group. In the case of $\mathcal{S}$, with timelike momentum of the form

$\hat{p} = \pm m(1, 0, 0, 0)$, the supertranslations as well as the rotations are included in a "little algebra" whose UIR's are to be found.

Salam and Strathdee [24] have observed that the subalgebra of the little algebra generated by the S_α is a 16-dimensional Clifford algebra. It follows that it has only one finite-dimensional irreducible representation: the 4-dimensional one (viz., in terms of 4 x 4 matrices). Moreover, by examining the graded Lie algebra of the little algebra generators (in the Dirac-Pauli basis, Sec. A2) in the rest frame,

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk} J_k, \qquad (i, j, k = 1,2,3), \\ [S_a, J_i] &= \tfrac{1}{2}(\sigma_i)_{ab} S_b, \qquad (a, b = 1,2) \\ \{S_a, S_b\} &= 0 = \{S_a^\dagger, S_b^\dagger\}, \\ \{S_a, S_b^\dagger\} &= m\,\delta_{ab}, \end{aligned} \tag{3.23}$$

it may be seen that the S_a have a direct realization in terms of fermion creation and annihilation operators, $S_1 = \sqrt{m}\, a_1$, $S_2 = \sqrt{m}\, a_2$, which may be used to construct a Fock representation of the algebra in a familiar way [60].

We follow this construction, but work instead with the chiral parts, writing the generators as $(S_\pm)_a$ $(a = 1, 2)$, where $(S_+)_a^\dagger = \varepsilon_{ab}(S_-)_b$. We have now

$$\begin{aligned} \{(S_\pm)_a, (S_\pm)_b\} &= 0, \\ \{(S_+)_a, (S_-)_b\} &= \tfrac{1}{2} m\, \varepsilon_{ab}. \end{aligned} \tag{3.24}$$

Now introduce a Fock vacuum space of orthonormal vectors carrying a UIR of the rotation group with spin j_0, and helicity (third component of spin) $\kappa = -j_0, -j_0+1, \cdots, j_0$, with the property that

$$(S_-)_a \,|\, j_0 \,\kappa \rangle \;=\; 0 \tag{3.25}$$

It follows that the desired irreducible representation of the algebra is spanned by the four orthonormal states which can be obtained by acting with monomials in $(S_+)_a$ on the vacuum space, taking into account Eq. (3.24a). We take these states to be

$$|j_0\kappa\rangle,\; i\left(\tfrac{2}{m}\right)^{\frac{1}{2}}(S_+)_1\,|j_0\kappa\rangle,\; i\left(\tfrac{2}{m}\right)^{\frac{1}{2}}(S_+)_2\,|j_0\kappa\rangle,\; \left(\tfrac{2}{m}\right)^{1}(S_+)_1(S_+)_2\,|j_0\kappa\rangle. \tag{3.26}$$

Let us examine the algebra in the rest frame in more detail. In view of Eq. (3.25), we have

$$\begin{aligned}
\frac{\hat{P}.\Sigma}{\hat{P}^2}\,|j_0\kappa\rangle &= \frac{\Sigma_0}{P_0}|j_0\kappa\rangle = \left(\tfrac{1}{m}(S_+)_a\,\varepsilon_{ab}(S_-)_b - \tfrac{1}{2}\right)|j_0\kappa\rangle = -\tfrac{1}{2}\,|j_0\kappa\rangle \\
J_3\,|j_0\kappa\rangle &= \kappa\,|j_0\kappa\rangle \\
\frac{\Sigma_3^{\perp}}{P_0}\,|j_0\kappa\rangle &= +\tfrac{1}{m}(S_+)_a(\sigma_1)_{ab}(S_-)_b\,|j_0\kappa\rangle = 0 \\
-\frac{K_3^{\perp}}{P_0}\,|j_0\kappa\rangle &= \left(J_3 + \frac{\Sigma_3^{\perp}}{P_0}\right)|j_0\kappa\rangle = \kappa\,|j_0\kappa\rangle .
\end{aligned} \tag{3.27}$$

Then since (Eqs. (A2.9))

$$
\begin{aligned}
[\, J_3\,, (S_+)_a\,] &= a\,(S_+)_a\,, \quad (1,2) \equiv (-\tfrac{1}{2}, \tfrac{1}{2}) \\
[\, -K_3^{\perp}, (S_+)_a\,] &= 0\,, \qquad\qquad\qquad\qquad\qquad\qquad (3.28)
\end{aligned}
$$

and
$$
[\, \frac{\Sigma_0}{P_0}\,, (S_+)_a\,] = \tfrac{1}{2}(S_+)_a\,,
$$

it follows that the basis vectors defined by Eq. (3.26) are eigenstates of the operators $\Lambda = J_3$, $P^{-2}\,P\cdot\Sigma$, and $\Delta = -P_0^{-1}\Sigma_3^{\perp}$, with eigenvalues λ, σ, and δ, respectively, where $\delta = \pm\frac{1}{2} \mp |\sigma|$. Each state has the same eigenvalue $\kappa = \lambda - \delta$ of $-K_3^{\perp}$.

We can therefore label the rest-frame states of Eq. (3.26) as $|p^2>0, j_0;\ \hat{p}, \sigma, \delta, \kappa\rangle$, where (σ, δ) has the values $(\pm\frac{1}{2}, 0)$ or $(0, \pm\frac{1}{2})$. We define boosted states by

$$
|p^2>0, j_0;\ p\,\sigma\,\delta\,\kappa\rangle = U(L_p)\,|p^2>0, j_0;\ \hat{p}\,\sigma\,\delta\,\kappa\rangle\,, \tag{3.29}
$$

and it is readily verified that, in an arbitrary frame, σ, δ, and k are the eigenvalues of

$$
\frac{P\cdot\Sigma}{P^2}\,, \quad \Delta = \frac{\underline{P}\cdot\underline{\Sigma}^{\perp}}{P_0\,|\underline{P}|}\,, \quad \Lambda - \Delta = \frac{\underline{P}\cdot\underline{K}^{\perp}}{P_0\,|\underline{P}|}\,. \tag{3.30}
$$

respectively. We call $\kappa = -j_0, -j_0+1, \cdots, j_0$ the "superhelicity" in analogy with the ordinary helicity.

The superhelicity basis with the labels σ, δ, and k admits a very convenient graphical notation. We represent the pair (σ, δ) by a symbol +, for example $(+\frac{1}{2}, 0) = \overset{\bullet}{+}$, $(0, \frac{1}{2}) = +\!\!\cdot$, and so forth, with four different combinations altogether. The basis states may then be written as $| p^2 > 0, j_0 ; p + \kappa \rangle$, where $+ = \dot{+}, \cdot\!+, +\!\cdot,$ or $\underset{\cdot}{+}$. For such a state, the values of σ, δ, and k may be visualized as in Fig. 3.2, where $\kappa = -j_0, -j_0+1, \cdots, j_0$ is plotted against $\sigma = 0, \pm\frac{1}{2}$. The centre of the + gives the value of k, and the dot gives the values of λ and σ. Thus there are $4(2j_0 + 1)$ helicity states for superspin j_0.

Using the simplified notation, we can easily write down the normalization of states in the superhelicity basis: it is

$$\langle p'^2 j_0' ; p' +' \kappa' | p^2 j_0 ; p + \kappa \rangle = \delta_{j_0' j_0} \delta_{+'+} \delta_{\kappa'\kappa} \langle p' | p \rangle \tag{3.31}$$

The utility of the superhelicity basis lies in the fact (Eq. (A2.9)) that k is invariant under the supertranslations, S_α, which act only on the symbol +. In fact, passing to the auxiliary generators $R_i^{\pm}$ using Eqs. (3.18) and (3.29), we have

$$\langle p +' \kappa' | R_i^{\pm}(p) | p + \kappa \rangle = \langle \hat{p} +' \kappa' | R_i^{\pm}(\hat{p}) | \hat{p} + \kappa \rangle, \tag{3.32}$$

suppressing the representation labels p^2 and j_0. The matrix elements of $R^{\pm}$ may therefore be written down by transforming to the rest frame, and using Eqs. (3.26), (A3.4), and (3.24). For fixed j_0 and k, the nonzero matrix elements are

$$
\begin{aligned}
\langle \overset{\circ}{+} \,|\, R^{+}_{+} \,|\, {}^{\circ}\!+ \rangle &= M = -\langle {}^{\circ}\!+ \,|\, R^{-}_{-} \,|\, \overset{\circ}{+} \rangle \\
\langle +^{\circ} \,|\, R^{+}_{+} \,|\, \underset{\circ}{+} \rangle &= M = -\langle \underset{\circ}{+} \,|\, R^{-}_{-} \,|\, +^{\circ} \rangle \\
\langle \overset{\circ}{+} \,|\, R^{+}_{-} \,|\, +^{\circ} \rangle &= M = \langle +^{\circ} \,|\, R^{-}_{+} \,|\, \overset{\circ}{+} \rangle \\
\langle {}^{\circ}\!+ \,|\, R^{+}_{-} \,|\, \underset{\circ}{+} \rangle &= -M = \langle \underset{\circ}{+} \,|\, R^{-}_{+} \,|\, {}^{\circ}\!+ \rangle
\end{aligned}
\tag{3.33}
$$

where $\quad R_{\pm} \equiv R_{\pm\frac{1}{2}}$.

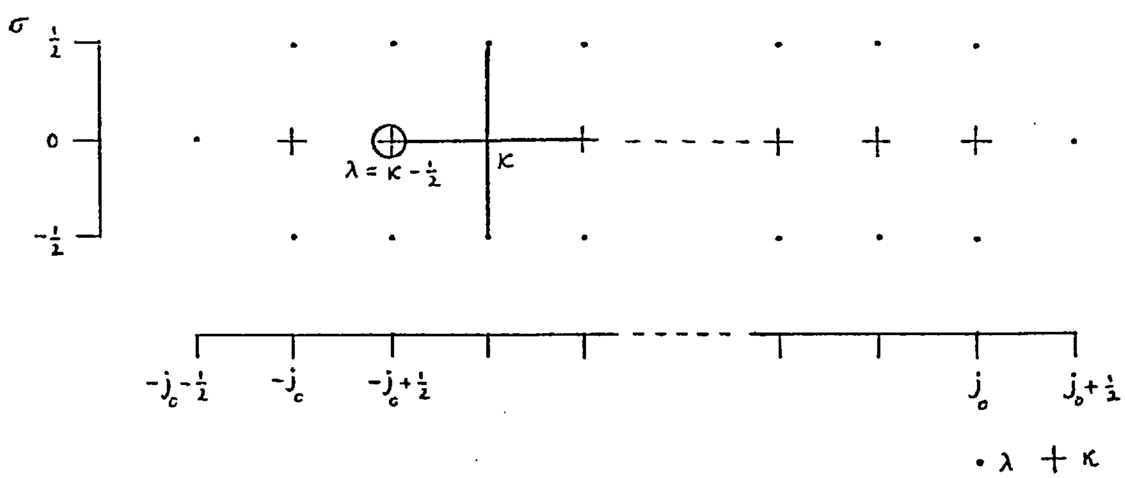

Fig. 3.2 Visualization of helicity states in the superhelicity basis

The total number of labels in the superhelicity basis $|p^2 > 0, j_0;\ \underline{p}\ \sigma\ \delta\ \kappa\rangle$ is 8, which is the same as in the spin basis, $|p^2 > 0, j_c;\ \underline{p}\ \sigma\ j\ \lambda\rangle$. The helicity $\lambda\ (= \kappa + \delta)$ is diagonal in both bases. From the way in which the superhelicity basis was constructed, Eq. (3.26), it is clear that here the spin cannot be diagonal, and it is easily verified that $[K_0^{\perp}, W^2] \neq 0$. The transformation coefficients relating the two bases may be deduced by comparing Eqs. (3.22), (3.33). We find, for fixed j_0 and p,

$$|\overset{\circ}{+}\,\kappa\rangle = |\tfrac{1}{2}\, j_0\, \kappa\rangle$$

$$|{\cdot}\!+\,\kappa\rangle = -\left(\frac{j_0-\kappa+1}{2j_0+1}\right)^{\frac{1}{2}} |0\; j_0+\tfrac{1}{2}\;\kappa-\tfrac{1}{2}\rangle + \left(\frac{j_0+\kappa}{2j_0+1}\right)^{\frac{1}{2}} |0\; j_0-\tfrac{1}{2}\;\kappa-\tfrac{1}{2}\rangle \qquad (3.34)$$

$$|+\!{\cdot}\,\kappa\rangle = \left(\frac{j_0+\kappa+1}{2j_0+1}\right)^{\frac{1}{2}} |0\; j_0+\tfrac{1}{2}\;\kappa+\tfrac{1}{2}\rangle + \left(\frac{j_0-\kappa}{2j_0+1}\right)^{\frac{1}{2}} |0\; j_0-\tfrac{1}{2}\;\kappa+\tfrac{1}{2}\rangle$$

$$|\underset{\circ}{+}\,\kappa\rangle = |-\tfrac{1}{2}\, j_0\, \kappa\rangle$$

Since the spin is not diagonal, Lorentz transformations in the superhelicity basis are represented by linear combinations of Wigner rotation matrices for different spins. We have

$$U(\Lambda)\,|p + \kappa\rangle = \sum_{+',\kappa'} |\Lambda p\; +'\,\kappa'\rangle \{+'+\}^{j_0}_{\kappa'\kappa}(\hat{\Lambda}) \qquad (3.35)$$

where $\hat{\Lambda} = L^{-1}_{\Lambda p}\,\Lambda\, L_p$ and $\sigma' = \sigma$ is Lorentz invariant. Hence there are only six nonzero $\{+'+\}$ matrices, namely $\{\overset{\circ}{+}\overset{\circ}{+}\}$, $\{{\cdot}\!+{\cdot}\!+\}$, $\{{\cdot}\!+\;+\!{\cdot}\}$, $\{+\!{\cdot}\;{\cdot}\!+\}$, $\{+\!{\cdot}\;+\!{\cdot}\}$, $\{\underset{\circ}{+}\underset{\circ}{+}\}$. Using Eqs. (3.34) gives for example

$$\{\overset{\circ}{+}{}'\overset{\circ}{+}\}^{j_0}_{\kappa'\kappa} = \mathcal{D}^{j_0}_{\kappa'\kappa} = \{\underset{\circ}{+}{}'\underset{\circ}{+}\}^{j_0}_{\kappa'\kappa} \qquad (3.36)$$

$$\{{\cdot}\!+'{\cdot}\!+\}^{j_0}_{\kappa'\kappa} = \left[\frac{j_0-\kappa'+1}{2j_0+1}\right]^{\frac{1}{2}}\left[\frac{j_0-\kappa+1}{2j_0+1}\right]^{\frac{1}{2}} \mathcal{D}^{j_0+\frac{1}{2}}_{\kappa'-\frac{1}{2}\,\kappa-\frac{1}{2}} + \left[\frac{j_0+\kappa'}{2j_0+1}\right]^{\frac{1}{2}}\left[\frac{j_0+\kappa}{2j_0+1}\right]^{\frac{1}{2}} \mathcal{D}^{j_0-\frac{1}{2}}_{\kappa'-\frac{1}{2}\,\kappa-\frac{1}{2}}$$

where the $\mathcal{D}^{j}_{\kappa'\kappa}(\hat{\Lambda})$ are ordinary spin-j Wigner rotation matrices. The $\{+'+\}$ matrices and their symmetry properties [58] are summarized in Eqs. (A3.8).

3.3 Massless Case.

As was noted at the beginning of this chapter, the irreducible representations of $\mathcal{S}$ may be classified according to the eigenvalue of the Casimir P^2. In the last sections, we treated the timelike case $P^2 > 0$. There remain the spacelike, $P^2 < 0$ lightlike, $P^2 = 0$, and null, $P = 0$, cases. In this preliminary study, we do no attempt an exhaustive account of these cases, which would confront the pathologies resident in a rigorous definition of the operators S_α and their domain (that is, the carrier space of the representations). Rather, we are concerned with elucidating the UIR's of $\mathcal{S}$ which are of most relevance to the classification of elementary particles. (This criterion of physical relevance is open to question, since little algebra expansions of crossed channel scattering amplitudes would necessitate a knowledge of UIR's for all types of momentum transfer. However, this application is not considered here). The only remaining such class for the Poincaré group is then the massless case, a subset of the lightlike case where the noncompact Euclidean generators are trivially represented, with the helicity an invariant. Correspondingly, we there fore restrict our attention to the massless UIR's of $\mathcal{S}$, containing only such massless particles.

It is again possible to give a covariant analysis of the spin content of the massless UIR's. Massless UIR's of $\mathcal{P}$ have $P^2 = W^2 = 0$, the helicity, Λ, becomes a Casimir of $\mathcal{P}$, and the generators satisfy the constraint

$$
\begin{aligned}
W_\mu &= \Lambda P_\mu , \\
\Lambda &= P_0^{-1} \epsilon_{0\lambda\mu\nu} P^\lambda J^{\mu\nu} .
\end{aligned} \tag{3.37}
$$

This will therefore hold in each spin sector of a massless UIR of $\mathscr{A}$. In fact, this constraint implies a restriction also on the supertranslations S_α . For (Eq. (A2.9))

$$[S_\alpha, W_\mu] = \tfrac{1}{4}\epsilon_{\mu\nu\rho\sigma} P^\nu (\sigma^{\rho\sigma} S)_\alpha$$

$$= [S_\alpha, \Lambda P_\mu] = \tfrac{1}{4} P_0^{-1} P_\mu \epsilon_{0\nu\rho\sigma} P^\nu (\sigma^{\rho\sigma} S)_\alpha ,$$

and using Eq. (A1.6), we find that (3.37) implies

$$\not{P}_\alpha{}^\beta S_\beta = 0 . \tag{3.38}$$

Also, since

$$[S_\alpha, W^2] = (i\gamma_5 \not{W}\not{P})_\alpha{}^\beta S_\beta - \tfrac{3}{4} P^2 S_\alpha ,$$

this is consistent with the condition $W^2 = 0$

It follows from Eq. (3.38) that S_α may be represented as $S = i\not{P} S'$ for some Majorana spinor S'. Then we have explicitly, for the Poincaré invariants, Eq. (3.10), that $\bar{S}S_\pm = 0$, and $P\cdot\Sigma = 0$. Furthermore, we have

$$\Sigma_\mu = N P_\mu , \tag{3.39}$$

where $N = -\frac{1}{2}\bar{S}' i\not{P}\gamma_5 S'$ must be a Poincaré invariant. From Eq. (A2.8) it follows that $N^2 = \frac{1}{4}$ whence N has eigenvalues $\nu = \pm\frac{1}{2}$

Furthermore, from Eqs. (A2.9) , it follows that

$$[N, (S_\pm)_\alpha] = \pm (S_\pm)_\alpha$$

$$[\Lambda, (S_\pm)_\alpha] = \pm \tfrac{1}{2}(S_\pm)_\alpha \tag{3.40}$$

so that the operator $\Lambda - \frac{1}{2}N$, or

$$K = \Lambda - \tfrac{1}{2}N + \tfrac{1}{4} \tag{3.41}$$

is a Casimir of $\mathscr{S}$, with eigenvalue λ_0.

To summarize, the massless UIR's of $\mathscr{S}$ may be labelled $(p^2 = 0, \lambda_0)$, where $\lambda_0 = 0, \pm\frac{1}{2}, \pm 1, \cdots$ is the superhelicity. They contain two spin sectors, carrying massless UIR's of $\mathscr{P}$ with invariant helicities $\lambda_0, \lambda_0 - \frac{1}{2}$ corresponding to eigenvalues $\nu = +\frac{1}{2}, -\frac{1}{2}$ respectively. As in Sec. 3.1, by constructing explicit realizations, we shall verify that the $(p^2 = 0, \lambda_0)$ UIR of $\mathscr{S}$ contains precisely one massless UIR of $\mathscr{P}$ of each allowed helicity. We shall also verify the Majorana constraint.

Just as in the massive case, we can visualize the spin content of the $(p^2 = 0, \lambda_0)$ UIR by means of a "weight diagram" indicating the participating weights (λ, ν). In analogy with the massive case, we still draw the diagram in two dimensions, even though $\lambda = \lambda_0 + \frac{1}{2}(\nu - \frac{1}{2})$, and ν is redundant. Such a diagram is shown in Fig. 3.3 below. These diagrams will be useful later when we come to consider massless superfields and massless super-wavefunctions.

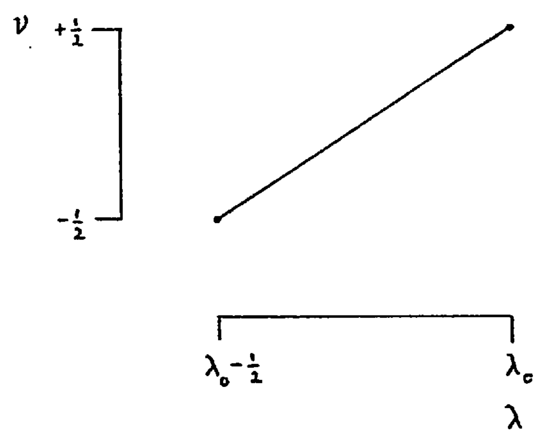

Fig. 3.3 Weight Diagram for the $(p^2 = 0, \lambda_0)$ UIR of $\mathscr{S}$.

We can now follow the method of Sec. 3.1 and construct the $(p^2 = 0, \lambda_0)$ UIR of $\mathscr{S}$ explicitly. As cyclic states we introduce a massless UIR of $\mathscr{P}$ with lowest weight $\nu = -\frac{1}{2}$, and invariant helicity $\lambda = \lambda_0 - \frac{1}{2}$. In view of Eq. (3.40), we have

$$(S_-)_\alpha \, | p^2 = 0, \lambda_0 ; \underline{p} \; \lambda_0 - \tfrac{1}{2} \rangle = 0 \tag{3.42}$$

By acting with the raising operator $(S_+)_\alpha$ we can then define covariantly-normalized basis vectors of the form $| p^2 = 0, \lambda_0 ; \underline{p} \, \lambda \rangle$, where

$$\langle p' \lambda' | p \lambda \rangle = \delta_{\lambda'\lambda} \langle p' | p \rangle \tag{3.43}$$

We cannot here define shift operators analogous to the $R_\kappa^\pm$ introduced in the massive case, since the condition $\not{p} S_\pm = 0$ ensures the vanishing of combinations like $\bar{u} S_\pm$. However, $(S_\pm)_\alpha$ changes the helicity by $\pm\frac{1}{2}$, and commutes with P_μ , so we must have

$$(S_+)_\alpha \,|p\, \lambda_0 - \tfrac{1}{2}\rangle = u_{+\alpha}(p)\,|p\,\lambda_0\rangle ,$$

and

$$(S_-)_\alpha \,|p\, \lambda_0 \;\rangle = u_{-\alpha}(p)\,|p\,\lambda_0 - \tfrac{1}{2}\rangle , \tag{3.44}$$

where the spinor normalization constants must also satisfy

$$\not{p}\, u_\pm(p) = 0 = \Gamma_\mp u_\pm(p) \tag{3.45}$$

and so may be written $u_\pm(p) = \Gamma_\pm u(p)$, as the chiral projections of a c-number solution of the massless Dirac equation.

The algebra of the $(S_\pm)_\alpha$ in the massless case may be deduced from Sec. A2 and Eq. (3.38). One finds

$$(S_\pm)_\alpha (S_\pm)_\beta = 0$$

$$(S_\pm)_\alpha (S_\mp)_\beta = \mp(\not{p}\,\Gamma_\mp C)_{\alpha\beta}(N \pm \tfrac{1}{2}) , \tag{3.46}$$

and applying Eqs. (3.45) and (3.46) to states $|p\,\lambda_0 - \frac{1}{2}\rangle$,

$$(S_-)_\alpha (S_+)_\beta \;|p\,\lambda_0 - \tfrac{1}{2}\rangle = (u_-)_\alpha(p)(u_+)_\beta(p)\;|p\,\lambda_0 - \tfrac{1}{2}\rangle$$

$$= -(\not{p}\,\Gamma_+ C)_{\alpha\beta}\;|p\,\lambda_0 - \tfrac{1}{2}\rangle$$

Using the completeness of the algebra of γ-matrices, we deduce

$$\bar{u}^c \gamma_\mu \Gamma_+ u = 2 p_\mu \tag{3.47}$$

Finally, the representation will be unitary, satisfying

$$\overline{(S_+)}^{\alpha} = (C^{-1})^{\alpha\beta}\,(S_-)_\beta\,, \tag{3.48}$$

if

$$u(p) = u^c(p) = C\bar{u}(p)\,,$$

that is, if $u(p)$ is a Majorana spinor.

Conditions (3.45), (3.47) and (3.48) are sufficient to fix $u(p)$, and hence $|\,p\,\lambda_0\,\rangle$, up to a phase. The nonzero matrix elements in this basis are

$$\langle\, p'\,\lambda_0\,|\,(S_+)_\alpha\,|\,p\,\lambda_0-\tfrac{1}{2}\,\rangle = (u_+)_\alpha(p)\,\langle\, p'\,|\,p\,\rangle\,, \tag{3.49}$$

and

$$\langle\, p'\,\lambda_0-\tfrac{1}{2}\,|\,(S_-)_\alpha\,|\,p\,\lambda_0\,\rangle = (u_-)_\alpha(p)\,\langle\, p'\,|\,p\,\rangle\,.$$

3.4 Parity

It is straightforward to extend the representations of the last three sections to UIR's of $\mathscr{S}$ augmented by parity, $\mathbb{P}$. For this purpose we shall take the definition [18, 61, 62]

$$U_{\mathbb{P}}\, S_\alpha\, U_{\mathbb{P}}^{-1} = (i\gamma_0 S)_\alpha \tag{3.50}$$

or

$$U_{\mathbb{P}}\,(S_\pm)_a\, U_{\mathbb{P}}^{-1} = (i\, S_\mp)_a$$

Consider firstly the massive case. From Eq. (3.3), σ is a pseudoscalar, so we have (suppressing the labels p^2 and j_0):

$$U_{\mathbb{P}} \,|\, \sigma\, p \;\; j \;\; \lambda \rangle \;\propto\; |\, -\sigma \;\; \mathbb{P}p \;\; j \;\; -\lambda \rangle$$

Thus $\mathbb{P}$ may be adjoined to each $\sigma = 0$ spin sector in a familiar manner [63]. For example, if

$$U_{\mathbb{P}} \,|\, 0 \; p \; j_0 + \tfrac{1}{2} \; \lambda \rangle = i\eta_0 (-1)^{-j_0 - \frac{1}{2}} \,|\, 0 \; \mathbb{P}p \; j_0 + \tfrac{1}{2} \; -\lambda \rangle, \tag{3.51}$$

then the remaining parities in the multiplet are completely determined by the supersymmetry. For, from Eqs. (3.50), (3.19) and the matrix element (3.22a), we conclude that

$$\begin{aligned} U_{\mathbb{P}} \,|\, \pm\tfrac{1}{2} \; p \; j_0 \; \lambda \rangle &= -\eta_0 (-1)^{-j_0} \,|\, \mp\tfrac{1}{2} \; \mathbb{P}p \; j_0 \; -\lambda \rangle \\ U_{\mathbb{P}} \,|\, 0 \; p \; j_0 - \tfrac{1}{2} \; \lambda \rangle &= i\eta_0 (-1)^{-j_0 + \frac{1}{2}} \,|\, 0 \; \mathbb{P}p \; j_0 - \tfrac{1}{2} \; -\lambda \rangle \end{aligned} \tag{3.52}$$

By taking combinations $\frac{1}{\sqrt{2}} \left(|+\tfrac{1}{2} \; p \; j_0 \; \lambda \rangle \mp |-\tfrac{1}{2} \; p \; j_0 \; \lambda \rangle \right)$ of definite parity $\pm\eta_0$, we can use the parity in place of σ in the labelling of states. We thus arrive at the spin parity basis $|p^2>0, j_0, \eta_0;\ p\ \eta\ j\ \lambda\rangle$, where

$$U_{\mathbb{P}}\,|\,p\,\eta\,j\ \lambda\rangle = \eta(-1)^{-j}\,|\,\mathbb{P}p\ \eta\ j\ -\lambda\rangle\,, \tag{3.53}$$

and the spin content is [18] $j_0^{\pm\eta_0},\ (j_0\pm\frac{1}{2})^{i\eta_0} \equiv j^P$. If we define $|\sigma| = \frac{1}{2} - |j-j_0|$, then

$$\eta = (-1)^{\pm|\sigma|}\,\eta_0\,. \tag{3.54}$$

This discussion could have been formulated also in the superhelicity basis. Herewe merely quote the results. We find

$$U_{\mathbb{P}}\,|\,p\ +\ \kappa\rangle = \eta_0(-1)^{2|\varepsilon|-j_0}\,|\,\mathbb{P}p\ \ \mathbb{P}+\ \ -\kappa\rangle\,, \tag{3.55}$$

where, for example, $\mathbb{P}\,\dot{+} = \dot{+}$, $\mathbb{P}{\cdot}{+} = {+}{\cdot}$.

The treatment of the massless case proceeds similarly. The labels Λ and N are both pseudoscalar and so (Eq. (3.41))

$$U_{\mathbb{P}}\,K\,U_{\mathbb{P}}^{-1} = -K + \tfrac{1}{2} \tag{3.56}$$

Therefore the massless UIR's of $\mathscr{A}$ augmented by $\mathbb{P}$ must have the form of a direct sum $(p^2=0,\ \lambda_0) + (p^2=0,\ -\lambda_0+\frac{1}{2})$ of massless UIR's, with helicity content $\pm\lambda_0$, $\pm(\lambda_0-\frac{1}{2})$, viz., 4 helicity states. This result was quoted in Ref. 18.

In order to see how the parity factors combine for the massless case, we take the required direct sum of UIR's, and fix the phase of the spinor normalization constant through $u(\mathbb{P}p) = i\gamma_0 u(p)$. If

$$U_{\mathbb{P}} \; | \, p, \lambda_0 - \tfrac{1}{2} \rangle \;=\; -\eta_0 \; | \, \mathbb{P}p\,, -\lambda_0 + \tfrac{1}{2} \rangle \; ,$$

then by considering $(S_+)_\alpha \, | \, p, \lambda_0 - \frac{1}{2} \rangle$, Eq. (3.44), it follows that the parities for the λ_0, $\lambda_0 - \frac{1}{2}$ components are $\eta_0 (-1)^{|\lambda_0|}$, $-\eta_0 (-1)^{|\lambda_0 - \frac{1}{2}|}$, respectively.

3.5 Internal Symmetry

As was indicated in Sec. 1.1, the concept of "supersymmetry" in elementary particle physics, that is to say the utilization of the rich mathematical structure of graded Lie algebras in a fundamental way, is a powerful one. There are inherent possibilities for incorporating bosons and fermions into a natural unified framework. Conventional globally supersymmetric Lagrangian field theories have remarkable properties [16, 18, 64-66]. The supersymmetry of Wess and Zumino, which we have been studying in this chapter, can be regarded as a first example of a unitary, relativistic spin-containing symmetry, in which the Poincaré group is embedded in a larger group in a nontrivial manner but avoiding proscription by the famous "no-go" theorems [51,52]. (This point was discussed in detail in Sec. 2.2). As the original name of "supergauge" symmetry implies, the theory has deep possibilities as a local space-time symmetry [32,67].

All of these aspects motivate the search for algebraic structures which generalize and exploit the supersymmetry idea. To complement our detailed study in the foregoing of the simplest example of the Wess-Zumino algebra, in this section we briefly review some of the developments along these lines.

The so called "internal degrees of freedom" which are a fundamental attribute of matter in the subnuclear regime are conventionally described by internal symmetry groups, independent of the space-time aspect. As mentioned above, supersymmetry is unconventional in that it is in a sense a fusion of internal and geometrical symmetries. Indeed, in Sec. 3.1, in investigating the structure of the Wess-Zumino algebra, we found an $SU(2) \times \mathcal{P}$ subalgebra. However, this SU(2) is a manifestation purely of the supersymmetry structure, and not an internal symmetry group. This will become transparent when we discuss the finite-dimensional representations in Chap. 5. For example, in the $j_0 = \frac{1}{2}$ case (the so-called "vector supermultiplet") the doublet sector of the representation is associated with (different chiral projections of) a single fermion field. In contradistinction an internal SU(2) doublet entails two fields, for example the nucleon doublet.

This observation means that the incorporation of conventional internal symmetries into a supersymmetry scheme requires a larger structure. The possibilities are not arbitrary, but are limited by consistency with certain properties of the S-matrix [53]. In fact, for the massive case, and in some circumstances for the massless case, this constraint is strong enough to determine the form of the algebra [53]. The result (see Sec. 2.1) is a graded Lie algebra containing the Poincaré algebra, which commutes with the internal Bose generators, as in the older "no-go" theorems.

Many attempts have been made to produce a fruitful generalized supersymmetry scheme consistent with the above requirements [24,25,68-72]. The results for the simplest ansätze are now well established, and we shall review these, rather than the more exotic possibilities.

The most obvious enlargement of the supersymmetry algebra we have been considering is to replace the S_α with generators $S_{\alpha i}$ carrying an additional degree of freedom. For example [24,69,70] $S_{\alpha i}$ could be an isospinor ($i = 1, 2$), with a modified anticommutation relation [24]

$$\{ S_{\alpha i} , S_{\beta j} \} = \epsilon_{ij} (i\gamma_\mu \gamma_5 C)_{\alpha\beta} \bar{S}^{\beta j} \tag{3.57}$$

and Majorana property

$$S_{\alpha i} = i \epsilon_{ij} (\gamma_5 C)_{\alpha\beta} \bar{S}^{\beta j} \tag{3.58}$$

The unitary representations for the massive case [24] can be constructed by the induced representation technique used in Sec. 3.2. They are labelled ($P^2 > 0$, J_0 , I_0) and contain UIR's of different (spin, isospin) content. For ($m, 0, 0$) one has [24]

$$\begin{aligned} 16 &= 1 + 4 + 6 + \bar{4} + 1 \\ &= (0,0)^2 + (\tfrac{1}{2},\tfrac{1}{2})^2 + (1,0) + (0,1) \end{aligned} \tag{3.59}$$

For n > 2 the fundamental representation of SU(n) is not equivalent to its conjugate, and the Majorana constraint cannot be applied. This implies a rapid increase with n of the number of generators. However for O(n) this difficulty is avoided, and we can take an algebra [25]

$$\{ S_{\alpha \ell} , S_{\beta m} \} = - \delta_{\ell m} (\gamma_\mu C)_{\alpha\beta} P^\mu \tag{3.60}$$

Once again, one can construct a Fock representation of the algebra in the rest frame, which can be decomposed with respect to the bilinear generators $[S_{a i} , S^{\dagger}_{bj}]$ of SU(2n). Obviously, by this method one constructs purely antisymmetric tensors. For example, if $n = 3$,

$$64 = 1 + 6 + 15 + 20 + \overline{15} + \bar{6} + 1 . \tag{3.61}$$

Unfortunately, results like Eq. (3.61) are physically unacceptable. The fundamental representation contains the 6-fold q, but also diquarks qq, rather than mesons $\bar{q}q$ and baryons qqq, and furthermore the SU(6) structure persists in the higher-dimensional representations [25]. The same results can also be obtained from a superfield approach [69], where there is the additional fact that interactions cannot be introduced in a renormalizable way. These problems have led to alternative proposals [68], for example incorporating colour [73], but so far with no positive results.

Clearly schemes such as (3.57) and (3.60) are amenable to the covariant methods of Sec. 3.1. In general we should have a Poincaré-invariant algebra of generators of the internal group, G, like $\bar{S}S^{ij}_{\pm}$ and $P\cdot\Sigma^{ij}$. The algebra also includes vector and axial-vector generators $\bar{S}\gamma_\mu S^{ij}$ and $\bar{S}i\gamma_\mu\gamma_5 S^{ij}$. We should thus be able to identify a subgroup like $SU(2)\times G\times G'\times \mathcal{P}$. Also, because of the anticommutation relations, the bilinear generators satisfy matrix polynomial identities of finite order [74], which can be used to identify the allowed values of spin for each internal sector, in direct analogy with Eq. (3.14).

However, this programme is not likely to bear fruit until the means are found of circumventing the shortcomings outlined above. Perhaps more information in the way of physical motivation and mathematical knowledge is required. In any case it seems that these ideas could benefit if some of the more sophisticated techniques of the fermion calculus [75] are brought to bear than have hitherto been used.

One possible alternative, which may avoid the unphysical multiplets encountered above, is to replace the GLA system by a variant involving parastatistics for the fermionic generators [8, 76] (a so called "graded triple system"[77]). Fock representations of the algebra can certainly be constructed [78]. It may also be possible to retain a group interpretation by introducing a-number parameters, as for GLA's (Sec. 2.2). Further investigations along these lines are merited.

We shall not reconsider internal symmetry in the sequel. For the purposes of this study the Wess-Zumino supersymmetry already has the essential feature of boson-fermion symmetry which enables us to regard it as the prototype of such schemes, and its predictions as qualitatively typical of the multitude of possible alternative schemes.

4. PARTIAL-WAVE ANALYSIS FOR SUPERSYMMETRIC SCATTERING AMPLITUDES

In this chapter the previous formalism (for the massive case) is directly applied in developing a partial-wave analysis for supersymmetric scattering amplitudes. The work is based on Ref. 28.

In Sec. 4.1 the direct product of two massive UIR's of is reduced into a direct sum of UIR's, and the Clebsch-Gordan coefficients are written down. The consideration of degeneracy labels for the reduced states is shown to favour the use of the superhelicity basis, in which the spin is not diagonal, as the most appropriate one. Precisely such a mixing is, indeed, characteristic of supersymmetry as a spin-containing symmetry. This enables the angular dependence of helicity amplitudes for the processes $1 \to 2+3$, and $1+2 \to 3+4$, to be extracted in Sec. 4.2 as a series in certain generalized Wigner matrices (the matrices of Lorentz transformations in the superhelicity basis), together with some supersymmetric reduced amplitudes $\langle 2\,3 \,\|\, T^{j_0}(s) \,\|\, 1 \rangle$ and $\langle 3\,4 \,\|\, T^{j_0}(s) \,\|\, 1\,2 \rangle$, corresponding to partial waves of total superspin j_0. Symmetry properties of these reduced amplitudes (especially under parity) are considered in Sec. 4.3.

In Sec. 4.4 an interpretation is given of these supersymmetric reduced amplitudes by comparing the supersymmetric with the ordinary partial wave expansions of the scattering amplitudes. It is found that, after cancelling off the common angular dependence, the ordinary reduced amplitudes , $\langle 2\,3 \| A^j(s) \| 1 \rangle$, and $\langle 3\,4 \| A^j(s) \| 1\,2 \rangle$, can be expressed directly in terms of a much smaller number of supersymmetric ones. A careful count is given of the numbers of independent reduced amplitudes involved. The same supersymmetric constraints on the reduced amplitudes, if continued to complex superspin (and hence complex spin), are shown also to imply stringent constraints upon the high energy behaviour.

Finally in Sec. 4.5 a simple case study, that of three-point couplings for $j_0 = 0^+$, thus involving particles $0^{\pm}$ and $\frac{1}{2}^{i}$, is given to illustrate the general arguments of Sec. 4.4. It is found that the 7 independent couplings $g^{231}(s)$ (including parity) are given in terms of just one supersymmetric one G(s), plus kinematical factors.

In Secs. A4 and A5, some details are given of the algebra of the direct product, and the notation used for the labelling operators, and some of the kinematics, is introduced.

4.1 Reduction of the Direct Product

In this section we consider the reduction of the direct product of two UIR's of the supersymmetry algebra in the massive case, $(m_1, j_{01})^+ \times (m_2, j_{02})^+$, where $p_1^2 = m_1^2 > 0$, $p_2^2 = m_2^2 > 0$, and sign $(p_{01}) = +1 =$ sign (p_{02}), into a direct sum of UIR's. In applications we shall be describing each factor of the direct product by means of the physical spin-basis, so that the problem is to reduce the two-particle states

$$| j_{01}\, p_1\, \sigma_1\, j_1\, \lambda_1 \,;\, j_{02}\, p_2\, \sigma_2\, j_2\, \lambda_2 \rangle = | j_{01}\, p_1\, \sigma_1\, j_1\, \lambda_1 \rangle | j_{02}\, p_2\, \sigma_2\, j_2\, \lambda_2 \rangle$$

into a direct sum of states of the form

$$| j_0\, p\, \sigma\, j\, \lambda \,;\, \cdots \rangle$$

where we have indicated that, since each massive UIR requires 8 labels for its states (two representation labels, m and j_0, and 6 state labels $\underline{p}$, σ, j, λ), the reduced states still lack at least 8 degeneracy labels.

The total supertranslation, momentum, and angular-momentum generators for the reduced states are defined by

$$\begin{aligned} S_\alpha &= S_\alpha^1 + S_\alpha^2 \\ P_\mu &= P_\mu^1 + P_\mu^2 \\ J_{\mu\nu} &= J_{\mu\nu}^1 + J_{\mu\nu}^2 \end{aligned} \tag{4.1}$$

where we must assume

$$\{ S^1_\alpha , S^2_\beta \} = 0 , \tag{4.2}$$

in order to have

$$\{ S_\alpha , S_\beta \} = -(\not{p} C)_{\alpha\beta} \tag{4.3}$$

For consistency it is also necessary to adopt the definition [24]

$$S^2_\alpha \mid 1 ; 2 \rangle = \sum_{2'} (-1)^{2j_1} \mid 1 ; 2' \rangle \langle 2' \mid S^2_\alpha \mid 2 \rangle \tag{4.4}$$

for the generators S^2_α acting in the product space.

With these definitions, by considering the translation invariance of the inner product

$$\langle j_0 \, p \cdots \mid j_{01} \, p_1 \cdots ; j_{02} \, p_2 \cdots \rangle \propto \delta^4 (p - p_1 - p_2) , \tag{4.5}$$

we may assume

$$p = p^1 + p^2 . \tag{4.6}$$

Furthermore, sign $(p_0) = +1$, and $p^2 = (p_1 + p_2)^2 > 0$. Thus the two-particle states reduce into a sum of $(p^2 > 0 , j_0)^+$ UIR's, just as in the case of the Poincaré group.

In general we can write

$$p^1_\mu = m^1 (\cosh \zeta_1 , \sinh \zeta_1 (\sin \theta_1 \cos \phi_1 , \sin \theta_1 \sin \phi_1 , \cos \theta_1)) ,$$

where $\quad -\infty < \zeta_1 < +\infty, \quad 0 \leq \theta_1 < \pi, \quad 0 \leq \phi_1 < 2\pi,$

with a similar parametrization for p^2_μ. However, by considering the Lorentz invariance of (4.6), it is sufficient to consider the particular choice of orientations given by

$$\begin{aligned} p^{1\mu} &= m^1 (\cosh \zeta_1, 0, 0, \sinh \zeta_1) \\ p^{2\mu} &= m^2 (\cosh \zeta_2, 0, 0, -\sinh \zeta_2) \\ p^\mu &= M (1, 0, 0, 0) \end{aligned} \tag{4.7}$$

where $M = \sqrt{p^2} = \sqrt{s}$ is the total energy in the centre-of-mass frame. With this choice, we see that $\underline{p}^1$ points in the $+z$-direction, and $\underline{p}^2$ in the $-z$-direction, in the rest frame of $p^1 + p^2$. The Clebsch-Gordan coefficients for arbitrary orientations of p, p^1 and p^2 may be obtained from those for this special orientation by applying the appropriate Lorentz transformation.

Before proceeding we must settle the question of the degeneracy labels for the reduced states. Adopting the spin-basis labelling suggested above, it is readily verified using the commutation relations (Eqs (A2.9)) and the definitions of these labels (Eqs. (A4.3)) that

$$[j_0, j_1] \neq 0$$

$$[j_0, \lambda_1] \neq 0$$

and

$$[j_0, \sigma_1] \neq 0 .$$

Further identities in the algebra of the direct product are collected in Sec. A4. Thus in particular, since j_0 must be used as a representation label for the reduced states, we are forced to abandon the use of j_1, j_2, λ_1, λ_2 and σ_1, σ_2 as degeneracy labels, and look for other invariant combinations. Two such (Table A4.1) are $\lambda_1 - \lambda_2$ and $\sigma^1 + \sigma^2$, but counting j_{01}, j_{02}, m_1 and m_2, which are still good degeneracy labels, we still lack two labels.

Since we can in any case no longer retain the usual labels for the reduced states, we may also consider the possibility of choosing different bases for the product states, such that the degeneracy problem is simplified. As suggested in the introduction to this chapter, the philosophy is that the results of the supersymmetric analysis may always be re-interpreted in terms of the physical spin-basis by transforming back to it at the end.

We find that the superhelicity basis introduced in Sec. 3.2 is a very natural choice from this point of view. The superhelicity commutes with the supertranslations, and hence with any labels constructed from them, including the total j_0, σ, δ and κ of the reduced states. Thus in the superhelicity basis with k^1 and k^2 as good degeneracy labels (Table A4.1), the labelling problem for the reduced states in the direct product is now solved. Including the labels suggested above, the degeneracy labels are

$$m_1,\ j_{01},\ m_2,\ j_{02}\ ;\ \kappa_1\ \kappa_2\ ;\ \sigma_1+\sigma_2,\ \lambda_1-\lambda_2\ =\ 8\ . \tag{4.8}$$

However, the last two labels, while Lorentz invariant, are not super-translation invariant, since

$$\begin{aligned} [\,\sigma^1+\sigma^2,\ (S_\pm)_a\,] &= \pm\tfrac{1}{2}(S_\pm)_a \\ [\,\lambda^1-\lambda^2,\ (S_\pm)_a\,] &= a(S_\pm)_a\ , \end{aligned} \tag{4.9}$$

where the values a = 1,2 are replaced by a = $-\frac{1}{2},\frac{1}{2}$ on the right-hand side. On the other hand, in the direct product algebra $\mathcal{S}_1 \times \mathcal{S}_2$ the number of labels commuting with, for example, the algebra $\mathcal{S}_1$, is 10, namely the two representation labels (Casimirs) of $\mathcal{S}_1$, and any of the 8 independent labels of $\mathcal{S}_2$. We might therefore expect a similar accounting to be valid also if we choose to reduce $\mathcal{S}_1 \times \mathcal{S}_2$ with respect to the diagonal generators $\mathcal{S}_1 + \mathcal{S}_2$, namely the total supertranslation, momentum, and angular momentum. We therefore seek to replace $\sigma_1 + \sigma_2$ and $\lambda_1 - \lambda_2$ by invariant combinations.

Such combinations may, indeed, be constructed: for, if from the supertranslation generators S^1_α and S^2_α, which anticommute, we construct some nontrivial linear combination, then there exists another linear combination which anticommutes with the first. We define, corresponding to the total supertranslation $S_\alpha = S^1_\alpha + S^2_\alpha$, the "relative supertranslation"

$$\tilde{S}_\alpha = (\not{p}^1)^{-1} S^1_\alpha - (\not{p}^2)^{-1} S^2_\alpha , \tag{4.10}$$

(which is anti-Majorana), and observe that

$$\begin{aligned} \{ S_\alpha , \tilde{S}_\beta \} &= 0 , \\ \{ \tilde{S}_\alpha , \tilde{S}_\beta \} &= -(\gamma_\mu C)_{\alpha\beta} \tilde{P}^\mu \end{aligned} \tag{4.11}$$

where $$\tilde{P}_\mu = \frac{1}{m_1^2} P^1_\mu + \frac{1}{m_2^2} P^2_\mu \tag{4.12}$$

Moreover, $\tilde{S}_\alpha$ is a spinor under the total angular momentum, and commutes with $\tilde{P}_\mu$. It follows that we may define labels $\tilde{\sigma}$, $\tilde{\delta}$ and $\tilde{\kappa}$ for $\tilde{S}_\alpha$ in precisely the same way as for S_α (Eq. (A4.3)), replacing S_α by $\tilde{S}_\alpha$, P_μ by $\tilde{P}_\mu$, and with

$$\tilde{P}^2 = \tilde{M}^2 = \left(\frac{M}{m_1 m_2}\right)^2 . \tag{4.13}$$

We find, for example, $\tilde{\lambda} = \lambda$, $\tilde{j}_0 = j_0$, $\tilde{\kappa} = \lambda - \tilde{\delta}$ and

$$\tilde{\sigma} = \sigma - \sigma_1 - \sigma_2 \tag{4.14a}$$

In general $\tilde{\delta}$ is a complicated combination of $\delta_1, \delta_2, \delta$ and $\tilde{\sigma}$, but for the reduced states in the centre-of-mass frame, the formula (c.f. Eq. (A4.4))

$$\tilde{\delta} = \delta^1 - \delta^2 - \delta \tag{4.14b}$$

(which is obviously not Lorentz invariant) is true, and we shall use this henceforth. The $\tilde{S}_\alpha$ and the labels constructed from them are discussed in more detail in Sec. A4.

The invariant labels ($\tilde{\sigma}$, $\tilde{\delta}$), for which we use the notation $\tilde{+}$ in analogy with Sec. 3.2, replace ($\tilde{\sigma}$, $\tilde{\delta}$) $\leftrightarrow$ ($\sigma^1 + \sigma^2$, $\lambda^1 - \lambda^2$) of Eq. (4.8). The labels for the reduced states are finally

$$| j_0 \, p + \kappa \, ; m_1 \, j_{01} \, m_2 \, j_{02} \, ; \kappa_1 \, \kappa_2 \, ; \tilde{+} \rangle = 16 , \tag{4.15}$$

where $+$ specifies (σ, δ) and $\tilde{+}$ specifies ($\tilde{\sigma}$, $\tilde{\delta}$). The problem at hand is therefore to find the Clebsch-Gordan coefficients

$$\langle j_0 \; p + \kappa \; ; \; m_1 j_{01} m_2 j_{02} \; \kappa_1 \kappa_2 \; ; \tilde{\mp} \; | \; j_{01} \, p_1 +_1 \kappa_1 \; ; \; j_{02} \, p_2 +_2 \kappa_2 \rangle \tag{4.16}$$

which we shall often simplify, omitting inessential labels, to

$$\langle + \kappa \; ; \; \kappa_1 \kappa_2 \; ; \; \tilde{\mp} \; | \; +_1 \kappa_1 \; ; \; +_2 \kappa_2 \rangle \; . \tag{4.17}$$

The evaluation of the Clebsch-Gordan coefficients is straightforward. Using the operators $(S_\pm)_a$ as shifting operators, in particular

$$[\sigma, \; (S_\pm)_a] \;=\; \pm \tfrac{1}{2} (S_\pm)_a$$

$$[\delta, \; (S_\pm)_a] \;=\; a (S_\pm)_a \; ,$$

we apply the operators

$$\bar{S} S_\pm \;=\; \pm \; 4 (S_+)_1 (S_+)_2$$

to the two-particle states, to construct states of total $\sigma = \pm \frac{1}{2}$ which may be further resolved into eigenstates of δ. Once + is given, $\tilde{\mp}$ follows from Eq. (4.14). As usual for the two-particle states we have

$$\Lambda \; | +_1 \kappa_1 \; ; \; +_2 \kappa_2 \rangle \;=\; (\lambda_1 - \lambda_2) \, | +_1 \kappa_1 \; ; \; +_2 \kappa_2 \rangle \; .$$

Hence for the total superhelicity we have by definition

$$\kappa \;=\; \lambda - \delta \;=\; \kappa_1 - \kappa_2 + \delta_1 - \delta_2 - \delta \;=\; \kappa_1 - \kappa_2 + \tilde{\delta} \;\equiv\; \kappa_{12} \; . \tag{4.18}$$

By this means we can construct, for $\sigma = \pm\frac{1}{2}$, eigenstates of the form $|+\kappa;\ \kappa_1\kappa_2;\ \tilde{\mp}\rangle_t$. Other total eigenstates may be constructed by applying the $(S\pm)a$. Notice that these states are not yet eigenstates of total superspin, j_0.

The distribution of such $|+;\tilde{\mp}\rangle$ states is plotted against $\sigma_1+\sigma_2$ and $\delta_1-\delta_2$ in Fig. 4.1, together with the $|+_1\,+_2\rangle$ states with which they couple. For example, the state $|\dot{+}\,+\rangle$ has $\sigma_1+\sigma_2 = 0$ and $\delta_1-\delta_2 = 0$ and so is coupled to $|+;\tilde{\mp}\rangle$, $|+;\tilde{\mp}\rangle$, $|\pm;\tilde{\mp}\rangle$ and $|\pm;\tilde{\mp}\rangle$. The counting of states is consistent: for fixed k_1 and k_2, the 16 $|+_1\,+_2\rangle$ states have been resolved into 16 $|+\tilde{\mp}\rangle$ states.

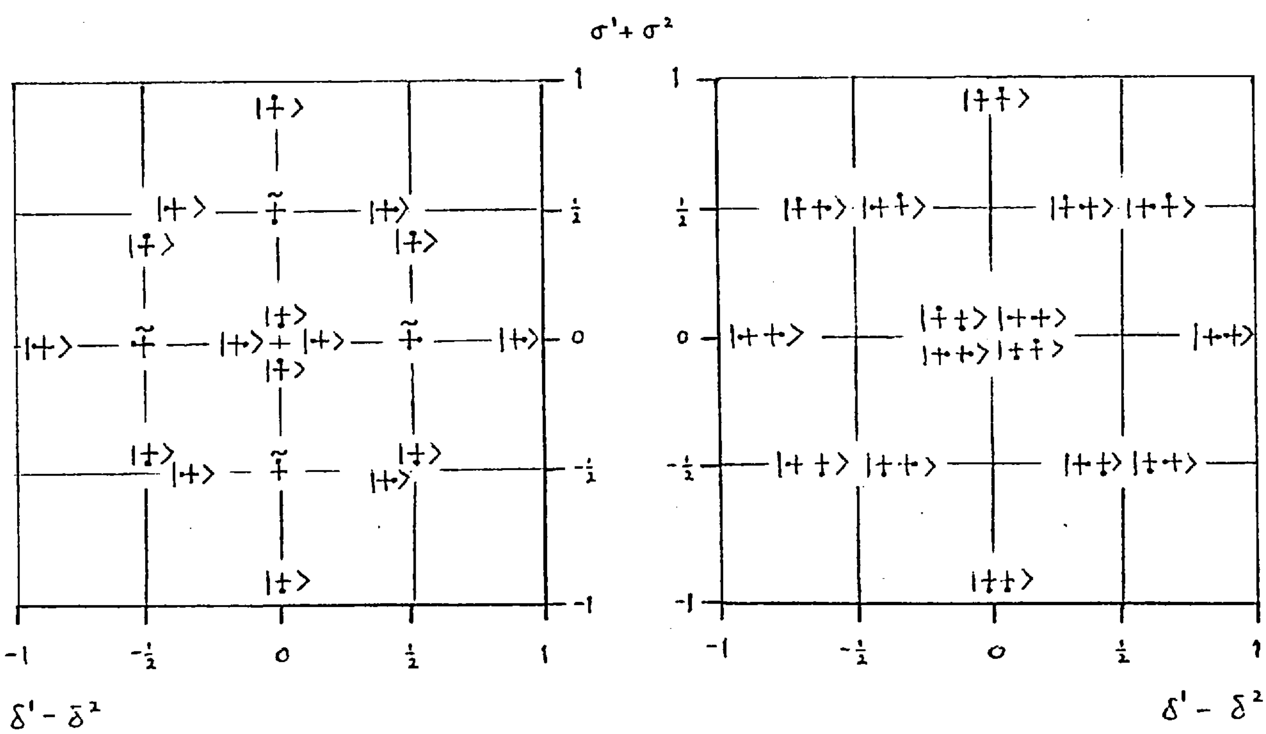

Fig. 4.1. Plot of the Distribution of $|+\tilde{\mp}\rangle$ and $|+_1\,+_2\rangle$ against $\sigma_1+\sigma_2$ and $\delta_1-\delta_2$.

These considerations already allow us to deduce the Clebsch-Gordan series. For fixed superhelicities k_1 and k_2, but allowing the values of $+_1$ and $+_2$ to vary, we have

$$\mathcal{D}^{j_{01}} \times \mathcal{D}^{j_{02}} = \sum_{j_0=|\kappa_1-\kappa_2-\frac{1}{2}|}^{\infty} \mathcal{D}^{j_0} + \sum_{j_0=|\kappa_1-\kappa_2|}^{\infty} (\mathcal{D}^{j_0})^2 + \sum_{j_0=|\kappa_1-\kappa_2+\frac{1}{2}|}^{\infty} \mathcal{D}^{j_0} , \tag{4.19}$$

giving for example

$$0 \times 0 = 0^2 + \tfrac{1}{2}^2 + 1^2 + \tfrac{3}{2}^2 + \cdots .$$

This is to be compared with the corresponding series for the Poincaré group (the product of two UIR's of spin zero),

$$0 \times 0 = 0 + 1 + 2 + 3 + \cdots .$$

In general, for a given state $|+_1 +_2\rangle$, there will be one or more different UIR's with total superspin j_0 coupling to it, distinguished by different values of $\tilde{+}$ (Fig. (4.1)). Altogether there will be four such UIR's. For example, we can construct the following four special states $|+\tilde{+}\rangle_t$ (the assignment of labels can be verified directly from the definitions given in Sec. A5):

$$\begin{aligned}
|\dot{+}\,\kappa\,;\,\tilde{\underset{\bullet}{+}}\rangle_t &= |\dot{+}\,\dot{+}\rangle \\
|\underset{\bullet}{+}\,\kappa-\tfrac{1}{2}\,;\,\tilde{{\cdot}{+}}\rangle_t &= \left(\tfrac{m_2}{M}\right)^{\frac{1}{2}} e^{\frac{i}{2}\gamma_2}\,|{\cdot}{+}\,\underset{\bullet}{+}\rangle + i\left(\tfrac{m_1}{M}\right)^{\frac{1}{2}} e^{-\frac{i}{2}\gamma_1}\,|\underset{\bullet}{+}\,{+}{\cdot}\rangle \\
|\dot{+}\,\kappa+\tfrac{1}{2}\,;\,\tilde{{+}{\cdot}}\rangle_t &= \left(\tfrac{m_2}{M}\right)^{\frac{1}{2}} e^{\frac{i}{2}\gamma_2}\,|{+}{\cdot}\,\dot{+}\rangle + i\left(\tfrac{m_1}{M}\right)^{\frac{1}{2}} e^{-\frac{i}{2}\gamma_1}\,|\dot{+}\,{\cdot}{+}\rangle \\
|\underset{\bullet}{+}\,\kappa\,;\,\tilde{\dot{+}}\rangle_t &= |\underset{\bullet}{+}\,\underset{\bullet}{+}\rangle .
\end{aligned} \tag{4.20}$$

Here $\kappa = \kappa_1 - \kappa_2$. The distribution of these special states can be read off from Fig. 4.1. Their projections onto total superspin j_0 are easily obtained: since $|\sigma| = \frac{1}{2}$ and $j = j_c \pm \frac{1}{2} \mp |\sigma| = j_0$, the superspin coincides with the spin, and the projections simply involve the Clebsch-Gordan coefficients for the Poincaré group, on passing back to the spin basis on the right-hand sides of Eqs. (4.20), using Eqs. (3.34). Consider for example

$$|\dot{+}\,;\tilde{\dot{\mp}}\rangle_t = \alpha\,|{+}{\cdot}\,\dot{+}\rangle + i\beta\,|\dot{+}\,{\cdot}{+}\rangle .$$

If the projection onto spin (= superspin) $-j_0$ is denoted by Π^{j_0} , then the reduced state will be given by

$$|j_c\,\dot{+}\,;\tilde{\dot{\mp}}\rangle \propto \Pi^{j_0}\,|\dot{+}\,;\tilde{\dot{\mp}}\rangle_t$$

If $\quad D = \left(\alpha^2\langle{+}{\cdot}\,\dot{+}\,|\,\Pi^{j_0}\,|{+}{\cdot}\,\dot{+}\rangle + \beta^2\langle\dot{+}\,{\cdot}{+}\,|\,\Pi^{j_0}\,|\dot{+}\,{\cdot}{+}\rangle\right)^{\frac{1}{2}}$,

we define $\quad |\dot{+}\,;\tilde{\dot{\mp}}\rangle = D^{-1}\,\Pi^{j_0}\,|\dot{+}\,;\tilde{\dot{\mp}}\rangle_t$,

so
$$\langle{+}{\cdot}_1\,\dot{+}_2\,|\,\dot{+}\,;\tilde{\dot{\mp}}\rangle = \frac{\alpha}{D}\,\langle{+}{\cdot}_1\,\dot{+}_2\,|\,\Pi^{j_0}\,|\,{+}{\cdot}_1\,\dot{+}_2\rangle$$

$$\langle\dot{+}_1\,{\cdot}{+}_2\,|\,\dot{+}\,;\tilde{\dot{\mp}}\rangle = i\frac{\beta}{D}\,\langle\dot{+}_1\,{\cdot}{+}_2\,|\,\Pi^{j_0}\,|\,\dot{+}_1\,{\cdot}{+}_2\rangle . \qquad (4.21)$$

The phase of $|\dot{+}\,;\tilde{\dot{\mp}}\rangle$ is fixed by this definition. The second factors on the right-hand sides involve linear combinations of Poincaré group Clebsch-Gordan coefficients; we take these to be

$$\langle j\,\lambda\,;\,\lambda_1\lambda_2\,|\,j_1\,\lambda_1\,;\,j_2\,\lambda_2\rangle = \delta_{\lambda,\lambda_1-\lambda_2}\,(2j+1), \qquad (4.22)$$

where $p = \hat{p} = p_1 + p_2$, $j \geq |\lambda_1 - \lambda_2|$, and p_1 and p_2 are in the special orientation of Eq. (4.7).

The combinations needed in Eqs. (4.20) and (4.21) are given in Sec. A5, where notations for the various kinematical factors are introduced. In this notation, and using the method which led to Eqs. (4.21) we have for example

$$\begin{aligned} \langle +\!\cdot_1\ \dot{+}_2 \,|\, \dot{+}\,;\, \widetilde{+\!\cdot} \rangle &= \delta_{\kappa,\kappa_1-\kappa_2+\frac{1}{2}}\,(2j_0+1)\ \frac{N_+(\kappa_1)}{D_{++}(\kappa_1,\kappa_2)} \\ \langle \dot{+}_1\ \dot{+}_2 \,|\, \dot{+}\,;\, \tilde{\dot{\underset{\cdot}{+}}} \rangle &= \delta_{\kappa,\kappa_1-\kappa_2}\,(2j_0+1) \end{aligned} \tag{4.23}$$

Once the Clebsch-Gordan coefficients for these four special states have been found, the remainder are determined by supersymmetry, by acting on each side with the shift operators $(S_\pm)_a$. Thus, for example,

$$|\cdot\!+\rangle = i\left(\tfrac{2}{M}\right)^{\frac{1}{2}}(S_+)_1\,|\underset{\cdot}{+}\rangle$$

and $$(S_+)_1^\dagger = (S_-)_2 = (S^1_-)_2 + (S^2_-)_2\,, \tag{4.24}$$

so $$\langle +_1 +_2 \,|\, \cdot\!+\,;\,\tilde{+} \rangle = i\left(\tfrac{2}{M}\right)^{\frac{1}{2}}\left(\langle \underset{\cdot}{+}\,;\,\tilde{+}\,|\,(s^1_-)_2 + (s^2_-)_2\,|\,+_1 +_2\rangle\right)^*,$$

where we have used the matrix elements of $(S_\pm)_a$ in the rest frame (Eqs. (A3.4) and (3.33)).

Using this method, the remaining Clebsch-Gordan coefficients (altogether 36 nonzero ones) may be written down. They are given in Table 4.1, which is to be consulted in conjunction with Fig. 4.1. Notation for Table 4.1 is established in Sec. A5.

As mentioned above, the Clebsch-Gordan coefficients for an arbitrary frame are related to the above coefficients in the special frame by means of Lorentz transformations. In particular, we shall restrict ourselves to the case of the centre of mass frame, with orientations of p_1 and p_2 with zero azimuthal angle, but with arbitrary polar angle (Eq. (4.7)), defining

$$|t_1 t_2\rangle^{\theta} = e^{-i\theta J_2}\,|t_1 t_2\rangle^{0} \tag{4.25}$$

where the $|t_1 t_2\rangle^{0}$ refers to the states with the z-axis orientation which we have been considering. The matrix elements of Lorentz transformations in the superhelicity basis have been discussed in Sec. 3.2, where the $\{t' t\}^{j_0}_{\kappa'\kappa}$ matrices were introduced. In terms of these matrices, we therefore have

$$^{\theta}\langle t_1 t_2 | t\kappa ; \tilde{\mp}\rangle \equiv \{t_1 t_2 | t\tilde{\mp}\}^{j_0}(\theta) = {}^{0}\langle t_1 t_2 | t_{12}\,\kappa_{12} ; \tilde{\mp}\rangle \{t_{12} t\}^{j_0}_{\kappa_{12}\kappa}(\theta), \tag{4.26}$$

where $\quad j_0 \geqslant \max(|\kappa|, |\kappa_{12}|)$,

where k_{12} is given by Eq. (4.18), and t_{12} is given in terms of t_1, t_2 and $\tilde{\mp}$ by Eqs. (4.14).

Finally, we may use the completeness property of the reduced states to expand any state $^{\theta}\langle t_1 t_2|$ as:

$$^{\theta}\langle t_1 t_2| = \sum_{\tilde{\mp}} \sum_{j_0=|\kappa_{12}|}^{\infty} \{t_1 t_2 | t\tilde{\mp}\}^{j_0}(\theta)\,\langle t\kappa ; \kappa_1\kappa_2\,\tilde{\mp}| \;,\quad t \equiv t_{12}. \tag{4.27}$$

Once again, notice that t is given once t_1, t_2 and $\tilde{\mp}$ are specified. The $\{t' t\}^{j_0}_{\kappa'\kappa}(\theta)$ matrices are clearly the analogues in the superhelicity basis of the ordinary Wigner notation matrices, $d^{j}_{\lambda'\lambda}(\theta)$, in the spin basis. They will play a crucial role in the supersymmetric partial wave analysis to be developed in the next three sections.

The Clebsch-Gordan coefficients of Eq. (4.26) may also be obtained by the rather more aesthetically appealing technique of harmonic analysis over the little group via the Plancherel theorem. In the case of the Poincaré group [79], (matrix elements of) the two-particle states are exhibited as (square-integrable) functions over the appropriate little group, which are expanded as an integral or summation over UIR's of the little group, thereby effecting the decomposition into reduced states. In the present case, the little algebra of Eq. (3.23) integrates to a group by means of a-number Majorana spinor parameters ε_α, as explained in Sec. 2.2. Arbitrary little group transformations have the form

$$U(\varepsilon, \varphi, \theta) = e^{-i\bar{\varepsilon}S}\, e^{-i\varphi J_3}\, e^{-i\theta J_2}$$

with matrix elements

$$\langle +' \kappa' | U(\varepsilon, \varphi, \theta) | + \kappa \rangle = \sum_{+''} \{+'+''\}(\varepsilon) \{+''+\}^{j_0}_{\kappa'\kappa}(\varphi\theta)$$

where $\quad \{+'+''\}(\varepsilon) = \langle +' | e^{-i\bar{\varepsilon}S} | +'' \rangle$

is independent of the superhelicity. For example, if we label the basis vectors in the order $\dot{+}, \cdot\!\!+, +\!\cdot, \underset{\cdot}{+}$, and write (c.f. Eq. (A3.3))

$$\bar{\varepsilon}S = \frac{1}{m}\bar{\varepsilon}v^+_\kappa R^+_\kappa + \frac{1}{m}\bar{\varepsilon}v^-_\kappa R^-_\kappa$$

then $\quad (\bar{\varepsilon}S) = \begin{pmatrix} 0 & -\bar{\varepsilon}v^-_- & \bar{\varepsilon}v^-_+ & 0 \\ \bar{\varepsilon}v^+_+ & 0 & 0 & -\bar{\varepsilon}v^-_+ \\ \bar{\varepsilon}v^+_- & 0 & 0 & -\bar{\varepsilon}v^-_- \\ 0 & -\bar{\varepsilon}v^+_- & \bar{\varepsilon}v^+_+ & 0 \end{pmatrix}.$

In particular, in the rest frame, according to Eq. (A3.4), we have $(v_{-\frac{1}{2}})_\alpha = i\sqrt{2m}\,\delta_{1\alpha}$, $(v_{\frac{1}{2}})_\alpha = -i\sqrt{2m}\,\delta_{2\alpha}$. From this we can write down the unitary matrix $\{+'+''\}(\varepsilon)$.

The next step is to use the method of Mezinescu [50] to find the invariant group measure $\int d^4\varepsilon\,\rho(\varepsilon)$, just as in the case of $\mathcal{G}\,SU(2)$ described in Sec. 2.2. Since

$$|+_1 +_2\rangle^\theta = e^{-i\theta J_2}\,|+_1 +_2\rangle^0 ,$$

the reduced states are simply given in terms of the projections [79]

$$|j_0 + \kappa\, ;\, +'\rangle\,\delta_{\kappa'\kappa_{12}} = \int d^4\varepsilon\,\rho(\varepsilon)\int d(\cos\theta)\,d\varphi\,\{++'\}^{j_0}_{\kappa\kappa'}(\varepsilon\varphi\theta)^*\,|+_1 +_2\rangle^\theta .$$

We shall not develop this approach to the analysis of the UIR's in detail here. The algebraic approach to the analysis of the representations, which we have adopted so far, has solved the Clebsch-Gordan problem, and indeed provides a useful reference point for the development of the global method.

Table 4.1 Clebsch-Gordan coefficients. Here $\mu \equiv \frac{m}{M}$, and a common factor $(2j_0+1)^{1-|\tilde{\varepsilon}|}(-1)^{4|\varepsilon|j_{c1}}\,\delta_{\kappa,\kappa_1-\kappa_2+\tilde{\delta}}$ is understood.

$$\{++|+\} = i(\rho+1)^{-\frac{1}{2}}(D_{--})^2/D_{-+}$$

$$\{++|+\} = N_-(1)/D_{-+} \qquad \{++|+\} = -i\mu_1\rho^{-\frac{1}{2}}D_{-+}$$

$$\{++|+\} = i\rho^{\frac{1}{2}}N_+(2)/D_{-+} \qquad \{++|+\} = \mu_2 D_{-+}$$

$$\{++|+\} = \left(\frac{\rho}{\rho+1}\right)^{-\frac{1}{2}} \qquad \{++|+\} = i(\rho+1)^{-\frac{1}{2}}$$

$$\{++|+\} = -i\mu_2(\rho+1)^{-\frac{1}{2}} \qquad \{++|+\} = \mu_1\left(\frac{\rho+1}{\rho}\right)^{-\frac{1}{2}}$$

$$\{++|+\} = \mu_1\left(\frac{\rho+1}{\rho}\right)^{\frac{1}{2}} N_-(1)/D_{-+}$$

$$\{++|+\} = \mu_2\left(\frac{\rho+1}{\rho}\right)^{\frac{1}{2}}\rho\, N_+(2)/D_{-+}$$

$$\{++|+\} = i\mu_2(\rho+1)^{\frac{1}{2}} N_-(1)/D_{-+}$$

$$\{++|+\} = -i\mu_1(\rho+1)^{\frac{1}{2}} N_+(2)/D_{-+}$$

$$\{++|+\} = 1$$

$$\{++|+\} = -i\mu_1\rho^{-\frac{1}{2}} \quad \{++|+\} = i\mu_2\rho^{-\frac{1}{2}} \quad \{++|+\} = \mu_2 \quad \{++|+\} = \mu_1$$

$$\{++|+\} = i\mu_2\rho^{-\frac{1}{2}} \quad \{++|+\} = -i\mu_1\rho^{-\frac{1}{2}} \quad \{++|+\} = \mu_1 \quad \{++|+\} = \mu_2$$

$$\{++|+\} = 1$$

$$\{++|+\} = -i\mu_2(\rho+1)^{-\frac{1}{2}} \qquad \{++|+\} = \mu_1\left(\frac{\rho+1}{\rho}\right)^{-\frac{1}{2}}$$

$$\{++|+\} = \mu_2\left(\frac{\rho+1}{\rho}\right)^{\frac{1}{2}}\rho\, N_-(2)/D_{++}$$

$$\{++|+\} = \mu_1\left(\frac{\rho+1}{\rho}\right)^{\frac{1}{2}} N_+(1)/D_{++}$$

$$\{++|+\} = -i\mu_1(\rho+1)^{\frac{1}{2}} N_-(2)/D_{++}$$

$$\{++|+\} = i\mu_2(\rho+1)^{\frac{1}{2}} N_+(1)/D_{++}$$

$$\{++|+\} = -i(\rho+1)^{-\frac{1}{2}} \qquad \{++|+\} = \left(\frac{\rho}{\rho+1}\right)^{-\frac{1}{2}}$$

$$\{++|+\} = \mu_2 D_{++} \qquad \{++|+\} = N_+(1)/D_{++}$$

$$\{++|+\} = -i\mu_1\rho^{-\frac{1}{2}}D_{++} \qquad \{++|+\} = i\rho^{\frac{1}{2}}N_-(2)/D_{++}$$

$$\{++|+\} = -i(\rho+1)^{-\frac{1}{2}}(D_{+-})^2/D_{++}$$

4.2 Partial-Wave Series

We now use the formalism of the last section to analyse the angular dependence of three- and four- particle scattering in the superhelicity basis, by making a partial-wave expansion in terms of the total superspin, j_0. We take

$$i \langle p'; f \mid S - 1 \mid p; i \rangle = (2\pi)^4 \delta^4(p'-p) \langle f \mid T(s) \mid i \rangle, \tag{4.28}$$

where $s = p^2$, and as before adopt the centre-of-mass frame so that the total 4 - momentum is $p_\mu = \hat{p}_\mu$. Assuming T to be invariant under supertranslations as well as Lorentz transformations, we have by a generalization of the Wigner-Eckart Theorem:

$$\langle +' \kappa'; \kappa_2 \kappa_3 \mp' \mid T \mid +_1 \kappa_1 \rangle = \delta_{j_c' j_{c1}} \delta_{+' +_1} \delta_{\kappa' \kappa_1} \langle \kappa_2 \kappa_3 \mp' \| T^{j_{c1}}(s) \| \rangle$$

and

$$\langle +' \kappa'; \kappa_3 \kappa_4 \mp' \mid T \mid + \kappa; \kappa_1 \kappa_2 \mp \rangle = \delta_{j_c' j_0} \delta_{+' +} \delta_{\kappa' \kappa} \langle \kappa_3 \kappa_4 \mp' \| T^{j_0}(s) \| \kappa_1 \kappa_2 \mp \rangle. \tag{4.29}$$

for the decay process $1 \to 2 + 3$ and the scattering process $1 + 2 \to 3 + 4$, respectively.

The partial-wave expansion is simply obtained by writing the two-particle (final) states ${}^{\theta}\langle +_1 +_2 |$ in terms of the reduced states $\langle + \kappa ; \kappa_1 \kappa_2 \mp |$, Eq. (4.27). The angular dependence for the process $1 \to 2 + 3$ is therefore (Eq. (4.26))

$$ {}^{\theta}\langle +_2 \kappa_2 +_3 \kappa_3 \mid T \mid +_1 \kappa_1 \rangle = \sum_{\mp} \{ +_2 +_3 \mid + \mp \}^{j_{01}}_{\kappa_{23} \kappa_1} (\theta) \langle \kappa_2 \kappa_3 \mp \| T^{j_{01}} \| \rangle, \tag{4.30}$$

where $2 j_{01} = 2 |\kappa_{23}| \pmod 2$, $j_{01} \geq |\kappa_{23}|$.

Here, as in Eq. (4.18), $\kappa_{23} = \kappa_2 - \kappa_3 + \tilde{\delta}$. For the process 1 + 2 → 3 + 4 we have similarly

$$^{\theta}\langle t_3\kappa_3 ; t_4\kappa_4 | T | t_1\kappa_1 ; t_2\kappa_2\rangle = \tag{4.31}$$
$$= \sum_{\tilde{\mp}',\tilde{\mp}} \sum_{j_0=J_0}^{\infty} \{t_3 t_4 | \tilde{\mp}'\tilde{\mp} | t_1 t_2\}^{j_0}(\theta) \langle \kappa_3\kappa_4\tilde{\mp}' \| T^{j_0} \| \kappa_1\kappa_2\tilde{\mp}\rangle,$$

where $2|\kappa_{12}| = 2|\kappa_{34}|$ (mod 2), $J_0 = \max(|\kappa_{12}|, |\kappa_{34}|)$, and we define (Eq. (4.26))

$$\{t_3 t_4 | \tilde{\mp}'\tilde{\mp} | t_1 t_2\}^{j_0}(\theta) = {}^{c}\langle t_3 t_4 | t'\kappa_{34} ; \tilde{\mp}'\rangle \{t' t\}^{j_0}_{\kappa_{34}\kappa_{12}}(\theta) \langle t\kappa_{12} ; \tilde{\mp} | t_1 t_2\rangle^{c} , \tag{4.32}$$

As in Eqs. (4.26) and (4.27), t is fixed by t_1, t_2 and $\tilde{\mp}$ through Eq. (4.14), and similarly for t'.

The angular functions of Eqs. (4.30) and (4.31) are essentially the $\{t' t\}^{j_0}_{\kappa'\kappa}(\theta)$ functions of Eq. (3.35), multiplied by the appropriate Clebsch-Gordan coefficients for the special z-axis orientation of p_1 and p_2. For example, from Table 4.1, we have

$$\{t_1 t_2 | t \tilde{\mp}\}^{j_0}(\theta) = i \left(\frac{m_1 m_2}{s}\right)^{\frac{1}{2}} e^{-\frac{i}{2}(\zeta_1+\zeta_2)} \{t\, t\}^{j_0}_{\kappa_1-\kappa_2,\kappa}(\theta). \tag{4.33}$$

Thus they are linear combinations of $d^{j_0}(\theta)$ and $d^{j_0\pm\frac{1}{2}}(\theta)$ (Eq. (A3.8)), with coefficients depending upon k_1, k_2, k_3 and k_4, and invariants such as m_1 and s. Here $\sqrt{s} = M$ and θ are the centre-of-mass energy, and scattering angle, respectively.

We defer for the moment the questions of convergence, inversion and interpretation of the expansions (4.30) and (4.31). These will be dealt with in Sec. 4.4, in the context of the relationship between the supersymmetric reduced amplitudes, $< \kappa_3 \kappa_4 \tilde{\mp}' \| T^{j_0} \| \| \kappa_1 \kappa_2 \tilde{\mp} >$ and the ordinary partial wave amplitudes $< \lambda_3 \lambda_4 \| A^j \| \| \lambda_1 \lambda_2 >$, with total angular momentum j. A careful count of the numbers of these amplitudes of each type will also be given.

In the meantime, we shall investigate what further constraints are imposed upon our reduced amplitudes by symmetries such as parity.

4.3 Symmetry Properties

In the previous section we exploited the invariance of T under Lorentz transformations and supertranslations, namely

$$[T, J_{\mu\nu}] = 0 = [T, S_\alpha]$$

to separate out the angular dependence of the amplitudes and derive the partial wave series in the reduced amplitudes. A priori there are $4^2 \prod_{i=1}^{4} (2j_{ci} + 1)$ supersymmetric reduced amplitudes for the process $1 + 2 \rightarrow 3 + 4$, and $4 \prod_{i=2}^{3} (2j_{ci} + 1)$ reduced amplitudes for the process $1 \longrightarrow 2 + 3$. A more careful count of the number of reduced amplitudes will be given in the next section. These amplitudes are further constrained if we assume that T has additional symmetry properties. In particular, we shall investigate here the consequences of parity conservation,

$$U_{\mathbb{P}} \; T \; U_{\mathbb{P}}^{-1} = T . \tag{4.34}$$

In Sec. 3.4 it was shown how to extend the UIR's of supersymmetry to include parity. This was taken to act on the supertranslations as [24]

$$U_{\mathbb{P}} (S_\pm)_a \; U_{\mathbb{P}}^{-1} = i (S_\mp)_a . \tag{4.35}$$

In the spin basis, it was found that the multiplets could be described in terms of an intrinsic parity η_0, in addition to the superspin j_0, with spin parity content $j_0^{\pm\eta_0}$, $(j_0 \pm \frac{1}{2})^{i\eta_0}$ (Eq. (3.53)).

In the superhelicity basis, this becomes (Eq. (3.55))

$$U_{\mathbb{P}}\,|\,p + \kappa\rangle = \eta_0 (-1)^{2|\delta| - j_0}\,|\,\mathbb{P}p \;\; \mathbb{P}+ \; -\kappa\rangle . \qquad (4.36)$$

The action of $U_{\mathbb{P}}$ on the reduced states may be derived similarly. We consider firstly the projections $|\,p\,j\,\lambda\,;\,\lambda_1\,\lambda_2\rangle$ of states $|\,+_1\,\kappa_1\;+_2\,\kappa_2\rangle$ onto total spin j (not superspin). These projections may be expanded in terms of the $|\,p\,j\,\lambda\,;\,j_1\,\lambda_1\,j_2\,\lambda_2\rangle$ by means of the basis transformations, Eq. (3.34). These latter have the property [63]

$$U_{\mathbb{P}}\,|\,p\,j\,\lambda;\; j_1\lambda_1\,j_2\lambda_2\rangle = \eta_{01}\eta_{02}(-1)^{j_1+j_2-2j}\,|\,\mathbb{P}p\;j\;-\lambda\,;\,j_1 -\lambda_1\;j_2-\lambda_2\rangle.$$

This procedure results in

$$U_{\mathbb{P}}\,|\,p\,j\,\lambda\,;\,+_1\kappa_1\,+_2\kappa_2\rangle = \eta_{01}\eta_{02}(-1)^{j_{01}+j_{02}-2j}\,|\,\mathbb{P}p\;j\;-\lambda\,;\;\mathbb{P}+_1\,-\kappa_1\,;\,\mathbb{P}+_2\,-\kappa_2\rangle \qquad (4.37)$$

Finally, the states $|\,p\,+\,\kappa\,;\;\kappa_1\,\kappa_2\,\tilde{\mp}\rangle$ may be written as a sum of spin $-j_0$ and $j_0 \pm \frac{1}{2}$ projections on combinations of $|\,+_1\,\kappa_1\,;\,+_2\,\kappa_2\rangle$ states (Eq. (4.20)). The result is

$$U_{\mathbb{P}}\,|\,p+\kappa\,;\;\kappa_1\kappa_2\tilde{\mp}\rangle = \eta_{01}\eta_{02}(-1)^{j_{01}+j_{02}-2(j_0-|\delta|)}\,|\,\mathbb{P}p\;\mathbb{P}+\;-\kappa\,;\;-\kappa_1\,-\kappa_2\;\mathbb{P}\tilde{\mp}\rangle \qquad (4.38)$$

Eqs. (4.34),(4.36) and (4.38) give for the reduced amplitudes (4.30) and (4.31) the required constraints following from parity conservation.

The results are simply

$$\langle \kappa_2 \kappa_3 \tilde{\mp} \| T^{j_{01}}(s) \| \rangle = \eta^*_{02} \eta^*_{03} \eta_{01} (-1)^{j_{01}-j_{02}-j_{03}} \langle -\kappa_2 -\kappa_3\, \mathbb{P}\tilde{\mp} \| T^{j_{01}} \| \rangle ,$$

$$\langle \kappa_3 \kappa_4 \tilde{\mp}' \| T^{j_0}(s) \| \kappa_1 \kappa_2 \tilde{\mp} \rangle = \eta^*_{03} \eta^*_{04} \eta_{01} \eta_{02} (-1)^{j_{01}+j_{02}-j_{03}-j_{04}} \langle -\kappa_3 -\kappa_4\, \mathbb{P}\tilde{\mp}' \| T^{j_0} \| -\kappa_1 -\kappa_2\, \mathbb{P}\tilde{\mp} \rangle . \tag{4.39}$$

Thus under parity conservation the number of independent amplitudes is reduced by a factor of (roughly) 2.

It would clearly be possible to study analogously the consequences of other symmetries of the T-matrix, such as time reversal invariance, and charge conjugation (crossing) symmetry, in the supersymmetry framework. In general, however, the results are no more than the transferral to the superhelicity basis of the corresponding well-known properties [63] of the ordinary reduced amplitudes (in the spin basis). We therefore do not develop this formalism here.

The frequent use of superspin-zero models in the literature lends some interest to the general question of identical-particle states in supersymmetry. In the spin basis it is well known that the physical states with permutation properties appropriate to the connection between spin and statistics are of the form [63]

$$|j \lambda\, (12)\rangle = \frac{1}{\sqrt{2}} \left(|j\lambda ; 12\rangle + (-1)^j\, |j -\lambda ; 21\rangle \right) , \tag{4.40}$$

for total spin-j. If we construct these states for all possible spins j_0, $j_0 \pm \frac{1}{2}$ in the multiplet, using the basis transformations, Eqs. (3.34), we find that they may be expressed in terms of symmetrised states of the form $| j_0 + (12)\, \tilde{\mp} \rangle$, with the same basis transformations, Eqs. (3.34), if we define

$$|j_0\, \sigma\, \delta\, \kappa\, (\kappa_1\, \kappa_2)\, \tilde{\sigma}\, \tilde{\delta} \rangle =$$

$$= \frac{1}{\sqrt{2}} \left(|j_0\, \sigma\, \delta\, \kappa\, ;\, \kappa_1\, \kappa_2\, \tilde{\sigma}\, \tilde{\delta} \rangle + (-1)^{j_0 - |\delta|} |j_0\, \sigma\, -\delta\, -\kappa\, ;\, \kappa_2\, \kappa_1\, \tilde{\sigma}\, -\tilde{\delta} \rangle \right). \quad (4.41)$$

Thus these $|j_0 + (\kappa_1 \kappa_2) \tilde{\mp} \rangle$ states are the appropriately symmetrized identical particle states in the superhelicity basis. In particular, these states vanish for odd j_0 if $\kappa_1 = \kappa_2$ and $\delta = 0 = \tilde{\delta}$, viz. the states $|\mp \tilde{\mp} \rangle$, $|\mp \tilde{\mp} \rangle$, $|\mp \tilde{\mp} \rangle$, and $|\mp \tilde{\mp} \rangle$ (c.f. Fig. 3.1). This is the analogue of the result [63] that the states (4.40) vanish for odd j, and $\lambda_1 = \lambda_2$

It should be pointed out here that the assumption of the spin-statistics theorem for a supersymmetric formalism is itself nontrivial (for example, there is as yet no axiomatic system which encompasses supersymmetry). However, since it does seem to be consistent in all the models which have been used so far, we shall accept it here also.

There is one further point to be emphasized, in connection with the conservation of total superspin, with regard to the total number of fermions occurring in the multiparticle states in matrix elements of T, such as $\langle +_3\, +_4 \,|\, T \,|\, +_1\, +_2 \rangle$. Consider for example $\langle + + | T | \mp \mp \rangle$ and suppose the superspins j_{0i} are all integral. The only possible term in the partial wave expansion, Eq. (4.31), is that involving the reduced amplitude $\langle \tilde{\mp} \| T^{j_0} \| \tilde{\mp} \rangle$ (c.f. Fig. 3.1). However, for this case, $|\kappa_{12}| = |\kappa_1 - \kappa_2|$ is integral, but $|\kappa_{34}| = |\kappa_3 - \kappa_4 + \frac{1}{2}|$

is half-integral (c.f. Eq. (4.18)). This means that, in the amplitudes $\langle + ' \kappa ' ; \tilde{+} | T | + \kappa ; \tilde{+} \rangle$, leading to the partial-wave series (before the Wigner-Eckart Theorem, Eq. (4.29), is applied), we have j_0' half-integral, and j_0 integral. Thus, in view of Eq. (4.29), and (4.27), this amplitude must vanish, $\langle + + | T | + + \rangle = 0$. Of course, referring to the spin basis via Eqs. (3.34), this is no more than the statement that $\langle j_{03} \pm \frac{1}{2} \quad j_{04} | T | j_{01} \ j_{02} \rangle = 0$ for j_{0i} integral, and we expect conservation of total superspin, j_0, to be consistent with this. It does, at least, show that supersymmetry is consistent with conservation of fermion number modulo 2, and it devolves upon the model in question to ensure that fermion number is conserved (for example, by giving the correct antifermion assignments). This question, and in general the question of discrete transformations and supersymmetry, has received considerable attention in the literature [62, 80-83], with solutions proposed in some cases requiring states with rather exo tic assignments of fermion number [61]. It is to be hoped that, in a future application of our present formalism, either with the Wess-Zumino supersymmetry, or a generalization of it, these problems may be resolved.

4.4 Interpretation of the Reduced Amplitudes

We take up in this section the questions of interpretation of the supersymmetric reduced amplitudes, introduced in the partial wave expansions, Eqs. (4.30) and (4.31). We shall not develop a full theory of supersymmetric partial wave analysis (for example of complex superspin) but will be content with the philosophy that the results of our supersymmetric approach are to be regarded in parallel with the standard, and well-developed, formalism for ordinary spin, with all its ramifications for particle physics [63, 79, 84, 85]. Thus the more delicate questions of convergence, analyticity and so on, which must eventually be answered in a thorough analysis, will here be transferred back to corresponding questions in the conventional partial wave formalism, despite some loss of structure in doing so.

In any case, as we have already found, our supersymmetric analysis certainly has the effect of imposing strong restrictions upon the ordinary partial wave analysis, and one of our natural aims is to investigate this issue in detail. As we shall see, our present attitude will prove sufficient to do this. We shall investigate firstly the explicit relationship between the ordinary and reduced partial waves, and then make some preliminary qualitative remarks on the continuation to complex superspin, and the implications which it has for the more conventional Regge pole hypothesis (or at least the assumption of analytic continuation in angular momentum).

The first point we should clarify relates to the simple counting of amplitudes. Consider a decay process $1 \to 2 + 3$ for example, involving supersymmetric multiplets with superspins j_{o1}, j_{o2} and j_{o3}. According to our analysis of Sec. 3.1 of the spin content of these multiplets (viz. 4 different spins $j_o^2, j_o \pm \frac{1}{2}$ for superspin j_o), for this process there could be in principle 4^3 reactions with different combinations of spins; and, since each supersymmetry multiplet contains a total of $4(2j_o + 1)$ helicity states for superspin j_o (Sec. 3.1), this gives a possible $4^3 \prod_{i=1}^{3}(2j_{oi} + 1)$ different helicity amplitudes. However, half of these will be forbidden by conservation of total angular momentum, leaving $32 \prod_{i=1}^{3} (2j_{oi} + 1)$ possible helicity amplitudes. This is of course also the number of possible superhelicity amplitudes, since the helicity and superhelicity are both diagonal in the superhelicity basis.

If we now partial wave analyse these amplitudes [63] in the standard way into partial waves of definite total spin j, we find that they may all be expressed in terms of reduced partial wave amplitudes $\langle j_2 j_3 ; \lambda_2 \lambda_3 \| A^{j_1}(s) \| \rangle$, a priori a total of $4 \times 4^2 \prod_{i=2}^{3}(2j_{oi} + 1)$, since the λ_1 - dependence is given explicitly through the Wigner-Eckart theorem. Once again, allowing for conservation of total angular momentum, the final number of independent invariants is reduced to $32 \prod_{i=2}^{3} (2j_{oi} + 1)$.

If we also perform a supersymmetric partial wave analysis of all of these helicity amplitudes via Eqs. (3.34) and (4.30), we find that these may all be described in terms of reduced supersymmetric partial waves $\langle j_{o2} j_{o3} \; \kappa_2 \kappa_3 \; \tilde{\mp} \| T^{j_{o1}} \| \rangle$, where once again the κ_1 - dependence is completely specified by the Wigner-Eckart Theorem, Eq. (4.29). This gives a priori $4 \prod_{i=2}^{3}(2j_{oi} + 1)$

reduced amplitudes; allowing for conservation of total superspin, the final number of supersymmetric independent invariants is $2 \prod_{i=2}^{3} (2j_{oi} + 1)$.

Comparing the above observations, we conclude that the existence of supersymmetry effects a 16-fold reduction in the number of independent invariants ! This is a strong constraint indeed. Moreover, as was stressed in the first introduction, Sec. 1.2, supersymmetry typically connects reactions in which different numbers of bosons and fermions participate: in this respect, it is, truly on the level of a relativistic spin-containing symmetry. It remains to be seen, however, whether such predictions will entail any physics, in a realistic scheme. This point will be returned to in the next section, where we present a detailed case study of three-particle couplings in a simple example.

We could carry out precisely the same analysis as the foregoing for the process $1 + 2 \rightarrow 3 + 4$. The results for both cases are summarized in Table 4.2, where we give the total number of helicity amplitudes, $\langle \mid T \mid \rangle$, partial waves of total spin j, $\langle \parallel A^j \parallel \rangle$, and partial waves of total superspin j_0, $\langle \parallel T^{j_0} \parallel \rangle$.

Type	1. $\langle \mid T \mid \rangle$	2. $\langle \parallel A^j \parallel \rangle$	3. $\langle \parallel T^{j_0} \parallel \rangle$
$1 \rightarrow 2+3$	$32 \prod_{i=1}^{3} (2j_{oi} + 1)$	$32 \prod_{i=2}^{3} (2j_{oi} + 1)$	$2 \prod_{i=2}^{3} (2j_{oi} + 1)$
$1+2 \rightarrow 3+4$	$128 \prod_{i=1}^{4} (2j_{oi} + 1)$	$128 \prod_{i=1}^{4} (2j_{oi} + 1)$	$8 \prod_{i=1}^{4} (2j_{oi} + 1)$

Table 4.2 Number of 1. helicity amplitudes, 2. partial waves of total spin j, and 3. partial waves of total superspin j_0, in (supersymmetric) two-particle processes. If, in addition, parity is conserved, all numbers must be halved.

In the last section, we analysed the requirements of spin and statistics for two-particle states of two identical particles, and found that for odd j_o, certain of these states vanished (Eq. (4.41)). This provides further restrictions on the amplitudes, but since this is not essentially a supersymmetric effect (subject to the qualifications made in deriving Eq. (3.41)), we shall not develop this here.

In the previous section we also found consequences of parity conservation for the supersymmetric amplitudes. The result (Eq. (4.39)) has precisely the same form as the ordinary case, but involving the intrinsic supermultiplet "super parities" and superspins, rather than individual parities and spins. The effect of parity conservation, therefore, both for the ordinary and the supersymmetric partial waves, is (approximately) to halve the number of independent invariants, in Table 4.2.

We turn now to the examination of the explicit constraints relating the spin partial waves to the superspin partial waves. For the general case we do not bother to give a complete analysis (reserving this for the case study of the next section), but merely give some illustrative examples of the types of constraint equations which emerge.

Consider once again the three-particle process $1 \longrightarrow 2 + 3$, and assume for simplicity that j_{01}, j_{02} and j_{03} are all integral. Consider a reaction involving the spin components $(j_{01} - \frac{1}{2}) \rightarrow (j_{02} + \frac{1}{2}) + j_{03}$. The angular dependence in the spin basis is given by [63]

$$^{\theta}\langle 0\, j_{02}+\tfrac{1}{2}\, \lambda_2 ; \tfrac{1}{2}\, j_{03}\, \lambda_3 \,|\, T \,|\, 0\, j_{01} - \tfrac{1}{2}\, \lambda_1 \rangle = \langle \lambda_2 \lambda_3 \| A^{j_{01}-\frac{1}{2}}(s) \| \rangle (2 j_{01})\, d^{j_{01}-\frac{1}{2}}_{\lambda \lambda_1}(\theta) ,$$

where $\lambda = \lambda_2 - \lambda_3$ (4.42)

On the other hand, performing the corresponding supersymmetric decomposition, we have, on transforming to the superhelicity basis and using Eqs. (3.34), (4.26) and (4.30), in the notation introduced in Sec. A3.5

$$
\begin{aligned}
&{}^{\theta}\langle 0\ j_{02}+\tfrac{1}{2}\ \lambda_2\ ;\ \tfrac{1}{2}\ j_{c3}\ \lambda_3\ |T|\ 0\ j_{c1}-\tfrac{1}{2}\ \lambda_1\rangle = \\
&= \langle\lambda_2-\tfrac{1}{2}\ \lambda_3\ \tilde{\mp}\ \|T^{j_{01}}\|\rangle\{+\mp|+\tilde{\mp}\}^{j_c}\Big[c_1c_2\{++\}^{j_{c1}}_{\lambda-\frac{1}{2}\ \lambda_1+\frac{1}{2}}(\theta) + s_1c_2\{++\}^{j_{c1}}_{\lambda-\frac{1}{2}\ \lambda_1-\frac{1}{2}}(\theta)\Big] \\
&\quad - \langle\lambda_2+\tfrac{1}{2}\ \lambda_3\ \tilde{\mp}\ \|T^{j_{c1}}\|\rangle\{+\mp|+\tilde{\mp}\}^{j_c}\Big[c_1s_2\{++\}^{j_{c1}}_{\lambda+\frac{1}{2}\ \lambda_1+\frac{1}{2}}(\theta) + s_1s_2\{++\}^{j_{c1}}_{\lambda+\frac{1}{2}\ \lambda_1-\frac{1}{2}}(\theta)\Big],
\end{aligned}
$$

where $c \equiv c(\lambda_1)$, and so on (Eq. (A3.7)). Using Eqs. (A3.8) and Table 4.1, we can write this angular dependence simply in terms of $d^{\,j_{01}\pm\frac{1}{2}}$ functions. In fact the $d^{\,j_{01}+\frac{1}{2}}(\theta)$ dependence vanishes (as must happen if the supersymmetric formalism is to be consistent with standard partial-wave analysis). The result is simply

$$
\begin{aligned}
&{}^{\theta}\langle 0\ j_{01}+\tfrac{1}{2}\ \lambda_2\ ;\ \tfrac{1}{2}\ j_{c3}\ \lambda_3\ |T|\ 0\ j_{c1}-\tfrac{1}{2}\ \lambda_1\rangle = \\
&= \Big(e^{\frac{1}{2}\gamma_1}c_\lambda s_2\langle\lambda_2+\tfrac{1}{2}\ \lambda_3\ \tilde{\mp}\ \|T^{j_{01}}\|\rangle - \big(\tfrac{m_1}{\sqrt{s}}\big)^{\frac{1}{2}}e^{-\frac{1}{2}\gamma_1}s_\lambda c_2\langle\lambda_2-\tfrac{1}{2}\ \lambda_3\ \tilde{\mp}\ \|T^{j_{c1}}\|\rangle\Big)(2j_{c1}+1)\, d^{\,j_{c1}-\frac{1}{2}}_{\lambda\lambda_1}(\theta).
\end{aligned} \tag{4.43}
$$

Thus, comparing Eqs. (4.42) and (4.43) we have, in a simplified notation

$$
(2j)\,A^{j-\frac{1}{2}}_{\lambda_2\lambda_3}(s) = (2j+1)\Big[e^{\frac{1}{2}\gamma_1}c_\lambda s_2\ T^{j}_{\lambda_2+\frac{1}{2}\ \lambda_3\ \mp}(s) - \big(\tfrac{m_1}{\sqrt{s}}\big)^{\frac{1}{2}}e^{-\frac{1}{2}\gamma_1}s_\lambda c_2\ T^{j}_{\lambda_2-\frac{1}{2}\ \lambda_3\ \mp}(s)\Big], \tag{4.44}
$$

that is, in this example $A^{j-\frac{1}{2}}_{\lambda_2\lambda_3}(s)$ is given as a linear combination of $T^{j}_{\lambda_2+\frac{1}{2}\ \lambda_3\ \mp}(s)$ and $T^{j}_{\lambda_2-\frac{1}{2}\ \lambda_3\ \mp}(s)$.

Obviously similar identifications could be made for any of these helicity amplitudes. According to Table 4.2, the $32\prod_{i=2}^{3}(2j_{ci}+1)$ reduced partial waves will thereby be given in terms of just $2\prod_{i=2}^{3}\cdot(2j_{ci}+1)$ supersymmetric reduced partial waves. These relations may alternatively be viewed simply as additional helicity constraints, arising from supersymmetry.

The analysis of the scattering processes $1 + 2 \rightarrow 3 + 4$, for components of supersymmetric multiplets, proceeds similarly. Let us assume as before that the j_{oi} are integral, and take for example the process involving spins $(j_{o1}+\frac{1}{2}) + j_{o2} \rightarrow (j_{o3}+\frac{1}{2}) + j_{o4}$ The angular dependence in the spin basis is given by a conventional partial wave expansion [63],

$$^{\theta}\langle 0\ j_{o3}+\tfrac{1}{2}\ \lambda_3\ ;\ \tfrac{1}{2}\ j_{o4}\ \lambda_4\ |\ T\ |\ 0\ j_{o1}+\tfrac{1}{2}\ \lambda_1\ ;\ -\tfrac{1}{2}\ j_{o2}\ \lambda_2\rangle =$$

$$= \sum_{j=J}^{\infty} (2j+1)\, d^{j}_{\mu j}(\theta)\ \langle \lambda_3\lambda_4 \| A^{j}(s) \| \lambda_1\lambda_2\rangle\ , \qquad (4.45)$$

where

$$\mu = \lambda_3 - \lambda_4\ ,\quad \lambda = \lambda_1 - \lambda_2\ ,\quad J = \max(|\mu|, |\lambda|)\ .$$

On the other hand, we can also exhibit the angular dependence by means of a supersymmetric partial wave expansion. The first step is to transform to the superhelicity basis, using Eqs. (3.34) and the notation of Sec. A3:

$$^{\theta}\langle 0\ j_{o3}+\tfrac{1}{2}\ \lambda_3\ ;\ \tfrac{1}{2}\ j_{o4}\ \lambda_4\ |\ T\ |\ 0\ j_{o1}+\tfrac{1}{2}\ \lambda_1\ ;\ -\tfrac{1}{2}\ j_{o}\ \lambda_{\ }\rangle =$$

$$= S_1S_3\,^{\theta}\langle + \lambda_3+\tfrac{1}{2}\,;\,\mp\lambda_4\,|T|\,+\lambda_1+\tfrac{1}{2}\,;\,\mp\lambda_2\rangle \ -\ C_1S_3\,^{\theta}\langle + \lambda_3+\tfrac{1}{2}\,;\,\mp\lambda_4\,|T|\,+\lambda_1-\tfrac{1}{2}\,;\,\mp\lambda_2\rangle$$

$$- S_1C_3\,^{\theta}\langle + \lambda_3-\tfrac{1}{2}\,;\,\mp\lambda_4\,|T|\,+\lambda_1+\tfrac{1}{2}\,;\,\mp\lambda_2\rangle \ +\ C_1C_3\,^{\theta}\langle + \lambda_3-\tfrac{1}{2}\,;\,\mp\lambda_4\,|T|\,+\lambda_1-\tfrac{1}{2}\,;\,\mp\lambda_2\rangle$$

$$(4.46)$$

Each term may now be partial-wave analysed by means of Eq. (4.30). For example, using Eq. (4.32) and Table 4.1, we have

$$^{\theta}\langle + \lambda_3 + \tfrac{1}{2} ; \mp \lambda_4 | T | + \lambda_1 + \tfrac{1}{2} ; \mp \lambda_2 \rangle =$$

$$= \sum_{j_0 = J_0}^{\infty} \left(\tfrac{m_i}{\sqrt{s}}\right)^{\frac{1}{2}} e^{\frac{i}{2}(\gamma_3 - \gamma_1)} \{++\}^{j_0}_{\mu+\frac{1}{2}\,\lambda+\frac{1}{2}}(\theta) \langle \lambda_3 + \tfrac{1}{2}\, \lambda_4 \mp \| T^{j_0} \| \lambda_1 + \tfrac{1}{2}\, \lambda_2 \mp \rangle ,$$

where

$$J_0 = \max(|\mu + \tfrac{1}{2}|, |\lambda + \tfrac{1}{2}|) . \tag{4.47}$$

A similar expansion can be written down for the other components, with different lower limits for the summations. By substituting these expressions into Eq. (4.46), writing the $\{++\}^{j_0}(\theta)$, $\{++\}^{j_0}(\theta)$, in terms of the $d^{j_0 \pm \frac{1}{2}}(\theta)$, (Eq. (A3.8)), and rearranging, we obtain a form comparable with the right-hand side of Eq. (4.45). For ease of writing, we specialize to the case $m_i = m$ and $j_{0i} = 0$. Then $\lambda_2 = \lambda_4 = 0$, and the angular dependence, and remaining dependence on λ_1, λ_3, is given explicitly (in a simplified notation) by

$$\sum_{j=\frac{1}{2}}^{\infty} (2j+1)\, d^{j}_{\mu\lambda}(\theta)\, A^{j}_{\lambda_3\lambda_1}(s) =$$

$$= \sum_{j_0 = \frac{1}{2}}^{\infty} d^{j_0}_{\mu\lambda}(\theta) \left[(2j_0+2)^2 \tfrac{m}{\sqrt{s}}\, T^{j_0+\frac{1}{2}}_{++}(s)\, F^{j_0+\frac{1}{2}}_{\lambda_3\lambda_1} + (2j_0)^2 \tfrac{m}{\sqrt{s}}\, T^{j_0-\frac{1}{2}}_{++}(s)\, G^{j_0-\frac{1}{2}}_{\lambda_3\lambda_1} \right] . \tag{4.48}$$

where

$$G^{0}_{\lambda_3\lambda_1} = 0,$$

$$F^{j_0}_{\lambda_3\lambda_1} = (e^{-\frac{i}{2}\gamma} c_3 s_\mu - s_3 c_\mu)(e^{\frac{i}{2}\gamma} c_1 s_\lambda - s_1 c_\lambda), \tag{4.49}$$

$$G^{j_0}_{\lambda_3\lambda_1} = (e^{-\frac{i}{2}\gamma} c_3 c_\mu + s_3 s_\mu)(e^{\frac{i}{2}\gamma} c_1 c_\lambda + s_1 s_\lambda),$$

for example

$$F^{j_0}_{++} = \left(\frac{j_0}{2j_0+1}\right), \qquad G^{j_0}_{+-} = e^{-\frac{i}{2}\gamma}\left(\frac{j_0+1}{2j_0+1}\right) .$$

Note that in the general case of Eq. (4.48), the terms of the series on the right-hand side near the lower limit of the summation must be evaluated separately. In general, however, this example shows that $A^{j}_{\lambda_3\lambda_1}(s)$ is given as a linear combination of $T^{j_o+\frac{1}{2}}_{++}(s)$ and $T^{j_o-\frac{1}{2}}_{++}(s)$.

An example of another process in the same reaction 1 + 2 → 3 + 4 is that involving the spins $j_{o1} + j_{o2} \rightarrow j_{o3} + j_{o4}$. we can analyse this in just the same way as before: it is simpler than the above, since it must involve only $|\sigma| = \frac{1}{2}$ states of the spin basis (Eq. (3.34)). The same procedure as above, for the case $\langle \frac{1}{2}\, j_{o3}\, \lambda_3\, ;\, \frac{1}{2}\, j_{o4}\, \lambda_4 \,|T|\, -\frac{1}{2}\, j_{o1}\, \lambda_1\, ;\, \frac{1}{2}\, j_{o2}\, \lambda_2 \rangle$ leads (in a simplified notation) to

$$\sum_{j=J}^{\infty} (2j+1)\, d^{j}_{\mu\lambda}(\theta)\, A^{j}_{\lambda_3\lambda_4\lambda_1\lambda_2}(s) = \sum_{j_o=J}^{\infty} (2j_o+1)^2 \frac{m_2}{\sqrt{s}}\, d^{j_o}_{\mu\lambda}(\theta)\; T^{j_o}_{++\,\lambda_3\lambda_4\lambda_1\lambda_2}(s)\,, \tag{4.50}$$

and in this case, for the same conditions of $j_{oi} = 0$, $m_i = m$, as in Eq. (4.48), $A^{j}(s)$ is proportional to $T^{j}_{++}(s)$.

Similar identifications may be made for any of the amplitudes of the reaction. According to Table 4.1, the $128 \prod_{i=1}^{4} (2j_{oi}+1)$ reduced portial waves are given in terms of only $8 \prod_{i=1}^{4}(2j_{oi}+1)$ super-symmetric ones. These relations may alternatively be viewed simply as additional helicity constraints arising from supersymmetry. The way in which this arises is shown by the cases considered above, Eqs. (4.48) and (4.50). As we see, the $A^{j}_{\lambda_3\lambda_1}(s)$ amplitudes for the spinor process $0+\frac{1}{2} \rightarrow 0+\frac{1}{2}$ is given in terms of the amplitude $A^{j}(s)$ for the scalar process 0 + 0 → 0 + 0; in particular the helicity dependence is explicitly specified.

Particular cases of the above relationships between the ordinary and the supersymmetric partial waves occur in the asymptotic regimes near threshold, and in the high-energy limit. In particular, the number of independent (supersymmetric) amplitudes quoted in Table 4.2 for the case when parity is conserved, is also just the number of nonvanishing amplitudes in the forward direction at threshold. However, to make any further predictions of the threshold behaviour, for example the angular momentum barrier effects, we would need to develop the formalism of supersymmetric representations in the canonical LS basis. We leave this however as a future technical application of our general formalism. As to the high-energy limit, one way of proceeding might be to take a form $A(s,t) \propto g(t)\, s^{\alpha(t)}$, for the amplitude for one particular reaction, and then substitute this into the helicity constraint equations derived from the supersymmetry, so obtaining asymptotic forms for amplitudes of several related reactions in the same process, but with different spin components. This could for example be done directly, in the case study to be treated in the next section.

There is another approach to the implications of supersymmetry for the high-energy behaviour of the amplitudes, which involves a somewhat stronger assumption than the above: we extend the analyticity postulates in a natural way, and assume that the S-matrix can be continued to complex superspin, j_0, with only isolated singularities [85]. We shall also assume that the above identifications, such as (from Eqs. (4.48) and (4.50)):

$$
\begin{aligned}
(2j+1)\, A_{\lambda_3\lambda_1}(j,s) = {} & (2j+1)^2 \tfrac{m}{\sqrt{s}}\, T_{\ddagger\ddagger}(j+\tfrac{1}{2},s)\, F_{\lambda_3\lambda_1}(j+\tfrac{1}{2},s) \\
& + (2j)^2 \tfrac{m}{\sqrt{s}}\, T_{\ddagger\ddagger}(j-\tfrac{1}{2},s)\, G_{\lambda_3\lambda_1}(j-\tfrac{1}{2},s),
\end{aligned} \tag{4.51}
$$

$$
(2j+1)\, A(j,s) = (2j+1)^2 \tfrac{m}{\sqrt{s}}\, T_{\ddagger\ddagger}(j,s),
$$

continue to be valid. Thus the analyticity postulate applies in particular to continuation in complex angular momentum. This may be performed separately, for each combination of spins in the process, by established techniques [86]. Supersymmetry then demands that all of these continued amplitudes be given in terms of a small number of supersymmetric amplitudes.

Thus, in the example of Eq. (4.51), the leading singularities of $T_{\ddagger\ddagger}(j_c, s)$ dominate the high-energy behaviour in the crossed channels of both $A(j, s)$ and $A_{\lambda_3\lambda_1}(j, s)$. For example, a pole at $j_0 = \alpha_0(s)$ will generate, in $A(j,s)$, a dependence $\propto t^{\alpha_0(s)}$ and will occur in $A_{\lambda_3\lambda_1}(j,s)$ when $j \pm \frac{1}{2} = \alpha_0(s)$, giving a dependence of $t^{\alpha_0(s) \mp \frac{1}{2}}$. A "super-Regge-trajectory" $\alpha_0(s)$ in general corresponds therefore to ordinary Regge trajectories $\alpha_0(s)$ and $\alpha_0(s) \pm \frac{1}{2}$, and the residues of the various contributions would be given in terms of the residue of $T(\alpha_0(s), s)$.

It is clear from these examples that there is a need to give a more thorough treatment of the complex superspin formalism, for application to these questions of high-energy behaviour. Once again, we must pass over this subject here, leaving it as an area for future development.

It should be pointed out that recent work on the high-energy behaviour of the Wess-Zumino model [87] indicates that the leading singularity is not a simple pole. Nevertheless the effect of supersymmetry is still to tie together the asymptotic behaviour of the various spin channels. However, it may be that in a gauge theory model [88,89] Regge pole behaviour will be reinstated also in the supersymmetric case.

4.5 Case Study: Three Point Couplings for Zero Superspin

In this section we shall illustrate the foregoing general discuss ion by working out in detail some of the supersymmetric constraints. The simplest case is that of a superspin zero multiplet. According to Eq. (3. 4), for intrinsic parity $\eta_0 = +$ the spin parity content must be $0^{\pm}, \frac{1}{2}^{i}$. We might therefore label the weight diagram $\begin{smallmatrix}\sigma\\ 1\\ \pi\end{smallmatrix}\!>\!N$. Since we are considering supersymmetry alone, with internal symmetry only as a direct product, for comparison with experiment we should for consistency consider the labels σ, π, N, to refer to the entire SU(3) octets to which these particles belong, rather than the particles themselves. Another possible supermultiplet, again with SU(3) octets, might be the $j_0^{\eta_0} = \frac{1}{2}^{i}$ case $\pi < \begin{smallmatrix}N'\\ N\end{smallmatrix} > \rho$, where N' is one of the $\frac{1}{2}^{-i}(\sim 1500)$ octets. However, such multiplets can never mix decuplets and octets. Thus a realistic comparison of supersymmetry with experiment must await a satisfactory solution of the problem of incorporating supersymmetry and internal symmetry (see Sec. 3.5). The main point here is to see the power of the assumption that the scattering process preserves supersymmetry.

We therefore take as our example the three-point couplings of the $\begin{smallmatrix}\sigma\\ 1\\ \pi\end{smallmatrix}\!>\!N$ supermultiplet. Of course, since we have not introduced any mass breaking, the processes $1 \rightarrow 2 + 3$ cannot be regarded as physical: the particles 2 and 3 are off mass-shell with $\cosh \zeta = \frac{1}{2}$.

According to the amplitude count given in the last section (Table 3.2), for this case there should be 16 independent reduced partial-wave amplitudes, including conservation of total angular momentum and parity. However only one supermultiplet is involved, so the symmetry between particles 2 and 3 reduces this to altogether 7 reduced couplings $g^{231}(s)$. The independent helicity amplitudes in the spin parity basis are given in Table 4.3 together with their reduced forms with the angular dependence made explicit. Some constraints arising from parity (P), conservation of total angular momentum (J) and 2 ↔ 3 symmetry (s) are also given.

Also according to Table 4.1, for this case there should be just one supersymmetric reduced partial-wave amplitude, if parity is conserved. This is indeed the case: the 2-particle states $|+_2\, o\, ;\, +_3\, o\rangle$ couple only to $j_0 = o = \kappa$ so the corresponding Clebsch-Gordan coefficients must be of the form $\{+_2 +_3 | +' \tilde{\mp}\}^0 \{+' +\}^0_{\kappa' o}$, with $\kappa' = o = o - o + \tilde{\delta}$. Thus, including parity (Eq. (4.39)), there is only one reduced supersymmetric amplitude, namely $\langle \tilde{\mp} \| T^0(s) \| \rangle = \langle \tilde{\mp} \| T^0(s) \| \rangle = G(s)$, say.

The $g^{231}(s)$ are given uniquely by G(s) and some kinematical factors. The explicit relationships are given in Table 4.4. To derive this table, we simply write out the spin parity states in terms of the superhelicity states, using Eqs. (4.39) and (3.34). The latter are simply $|+\frac{1}{2}^i\rangle = |+\rangle$, $|-\frac{1}{2}^i\rangle = -|+\rangle$, and $|o^\pm\rangle = \frac{1}{\sqrt{2}}(|\mp\rangle \pm |\mp\rangle)$, for $j_0 = o$. Thus, for example,

$$\langle \tfrac{1}{2}^i\, o^\pm | T | -\tfrac{1}{2}^i \rangle = -\tfrac{1}{\sqrt{2}} \langle + \mp | T | + \rangle \mp \tfrac{1}{\sqrt{2}} \langle + \mp | T | + \rangle$$

We then use Eqs. (4.30), (4.26), (A3.8) and Table 4.1. The angular functions $\{+' +\}^{j_0}_{\kappa' \kappa}(\theta)$ simplify greatly for $j_0 = o$. For example

$$\{+ +\}^{o}_{oo}(\theta) = d^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}}(\theta).$$

After carrying out this process, each helicity state is given in terms of G(s). The $g^{231}(s)$ (from Table 4.3)can then be directly compared with G(s), after cancelling off the angular dependence (the same from both the ordinary and the supersymmetric partial-wave analysis).

Amplitude	Reduced Form	Independent couplings	Symmetry
$\langle \frac{1}{2}^i\ \frac{1}{2}^i \| T \| 0^{\pm} \rangle$	$\langle \frac{1}{2}^i\ \frac{1}{2}^i \\| A^0 \\| 0^{\pm} \rangle$	$g_{--}^{NN\frac{\sigma}{\pi}} = \pm\ \underline{g}_{++}^{NN\frac{\sigma}{\pi}}$	P
$\langle \frac{1}{2}^i\ -\frac{1}{2}^i \| T \| 0^{\pm} \rangle$	$d^0_{10}(\theta) \langle \frac{1}{2}^i -\frac{1}{2}^i \\| A^0 \\| 0^{\pm} \rangle$	$g_{+-}^{NN\frac{\sigma}{\pi}} = g_{-+}^{NN\frac{\sigma}{\pi}} = 0$	P, J
$\langle \frac{1}{2}^i\ 0^{\pm} \| T \| \pm\frac{1}{2}^i \rangle$	$d^{\frac{1}{2}}_{\frac{1}{2}\pm\frac{1}{2}}(\theta) \langle \frac{1}{2}^i 0^{\pm} \\| A^{\frac{1}{2}} \\| i \rangle$	$\underline{g}_{+}^{N\frac{\sigma}{\pi}N}$	
$\langle -\frac{1}{2}^i\ 0^{\pm} \| T \| \pm\frac{1}{2}^i \rangle$	$d^{\frac{1}{2}}_{-\frac{1}{2}\pm\frac{1}{2}}(\theta) \langle -\frac{1}{2}^i 0^{\pm} \\| A^{\frac{1}{2}} \\| i \rangle$	$g_{-}^{N\frac{\sigma}{\pi}N} = \pm\ g_{+}^{N\frac{\sigma}{\pi}N}$	P
$\langle 0^{\pm}\ 0^{\pm} \| T \| 0^{+} \rangle$	$\langle 0^{\pm} 0^{\pm} \\| A^0 \\| + \rangle$	$\underline{g}^{\sigma\sigma\sigma}$, $\underline{g}^{\pi\pi\sigma}$	
$\langle 0^{+}\ 0^{-} \| T \| 0^{-} \rangle$	$\langle 0^{+} 0^{-} \\| A^0 \\| - \rangle$	$g^{\pi\sigma\pi} = \underline{g}^{\sigma\pi\pi}$	S
$\langle 0^{\pm}\ 0^{\pm} \| T \| 0^{-} \rangle$	$\langle 0^{\pm} 0^{\pm} \\| A^0 \\| - \rangle$	0	P

Table 4.3 Independent helicity amplitudes for three-point couplings of a superspin-0 multiplet

Amplitude	Reduced form	Supersymmetric form
$\langle \frac{1}{2}^i\ \frac{1}{2}^i \| T \| 0^{\pm} \rangle$	$g_{++}^{NN\frac{\sigma}{\pi}}$	$i\sqrt{2} \left\{ \begin{matrix} -\sinh \zeta \\ \cosh \zeta \end{matrix} \right\} G(s)$
$\langle \frac{1}{2}^i\ 0^{\pm} \| T \| \pm\frac{1}{2}^i \rangle$	$d^{\frac{1}{2}}_{\frac{1}{2}\pm\frac{1}{2}}(\theta)\ g_{+}^{N\frac{\sigma}{\pi}N}$	$d^{\frac{1}{2}}_{\frac{1}{2}\pm\frac{1}{2}}(\theta) \sqrt{2} \left\{ \begin{matrix} \cosh \zeta \\ -\sinh \zeta \end{matrix} \right\} G(s)$
$\langle 0^{\pm}\ 0^{\pm} \| T \| 0^{+} \rangle$	$g^{\frac{\sigma\sigma}{\pi\pi}\sigma}$	$\frac{1}{2}(1 \pm 2)\ G(s)$
$\langle 0^{+}\ 0^{-} \| T \| 0^{-} \rangle$	$g^{\sigma\pi\pi}$	$\frac{1}{\sqrt{2}} G(s)$

Table 4.4 Comparison of ordinary and supersymmetric reduced partial waves

5. STRUCTURE OF SUPERFIELDS

The analysis of the structure of $\mathscr{A}$ carried out in Secs. 3.1 and 3.3 applies to the irreducible representations in general, as well as to the UIR's, and so may be used in the finite-dimensional, non-unitary superfield representations (Sec. 2.2). This chapter is based upon this remark, and applies our formalism to analyse the structure of superfields, both in the massive and massless cases. The work extends and develops some of the ideas of Ref. 27.

These ideas are explored firstly in Sec. 5.1 in the context of "super-wavefunctions", which may be labelled $\Phi_A^{\eta j \lambda}(p)$, where η , j and λ are the parity, spin, and helicity, respectively (for given superspin, j_0, and intrinsic parity, η_0), and A runs over a set of Lorentz indices appropriate to the various spins occurring. The method of Salam and Strathdee[18] for constructing the super-wavefunctions is carried out explicitly for $j_0 = 0$ and $j_0 = \frac{1}{2}$. In the latter case the super-wavefunction is reducible, and its reduction is given.

In Sec. 5.2 the reduction of a general massive scalar superfield is reviewed[18]. It is shown that, if the superfield is rewritten in a basis in which the label σ is diagonal (with θ , $\bar{\theta}\theta$, $\bar{\theta} i \gamma_\mu \gamma_5 \theta$, ... replaced by σ-eigenfunctions $\Omega_{p\sigma}(\theta)$), it is already in manifestly reduced form. This process is viewed in the light of the weight diagrams (Sec. 3.1) whose nodes are labelled by the corresponding component fields. It is shown how this procedure naturally generalizes to the case of higher-spin superfields, and one example is given.

Massless superfields are investigated in Sec. 5.3. Two cases arise, depending upon whether the triviality condition $\not{p}S = 0$ acting on the superfield (ensuring the correct physical helicity components: Sec. 3.3) is satisfied. If not, then it must be implemented via gauge invariance of the corresponding $\mathcal{M}$-function. The required property of the $\mathcal{M}$-function, and the supersymmetric form of the corresponding gauge transformations, are worked out explicitly for the case of the massless, non-chiral superfield $\bar{\Phi}_1(x, \theta)$.

Some properties of spin-$\frac{1}{2}$ and -1 helicity projectors, needed in this analysis, are collected in Sec. A6.

5.1 Super-Wavefunctions

A superfield $\Phi(x,\theta)$ (Sec. 2.2) provides a finite-dimensional, non-unitary representation of the supersymmetry algebra. It may be reduced into its component fields (of different spins and parities) and each of these must similarly be associated with a finite-dimensional non-unitary irreducible representation of the Poincaré group. As we found in Sec. 3.4, momentum states (of the field) could be labelled in the massive case by their spins, parities, and helicities in the form $|p^2>0,\, j_0,\, \eta_0\, ;\, p\, j\, \eta\, \lambda\rangle$, where j_0 is the superspin, η_0 the intrinsic parity of the superfield, $j = j_0$ or $j_0 \pm \frac{1}{2}$, and $\eta = (-1)^{\pm|\sigma|}\eta_0$, where $|\sigma| = \frac{1}{2} - |j-j_0|$ (we treat here only the massive case, and defer the massless case until Sec. 5.3). Therefore, for each of the component fields, say $\Psi_m(x)$, where m is the label of the finite-dimensional representation, we can associate a momentum-space wavefunction

$$\Psi_m^{\eta j \lambda}(p) = \langle 0 |\, \Psi_m(0)\, | p^2 > 0\; j_0\, \eta_0\, ;\; p\, \eta\, j\, \lambda \rangle . \tag{5.1}$$

By the super-wavefunction $\Phi_A^{\eta j \lambda}(p)$ we shall mean the set of all such wavefunctions, where $\eta\, j\, \lambda$ runs over all the spin components, and A runs over all the corresponding finite-dimensional representations (such that $\Psi_A^j(p) = 0$ unless A belongs to the index set of the spin-j component).

There is an alternative means of defining the super-wavefunction as a function of θ ,

$$\Phi^{\sigma j \lambda}(p, \theta) = \langle 0 |\, \Phi(0,\theta)\, | p^2 > 0, j_0\, ;\, p\, \sigma\, j\, \lambda \rangle, \tag{5.2}$$

which is a function only of our wavefunction $\Psi_m(p)$ since the spin-j momentum states will be orthogonal to any other components created by the superfield $\bar{\Phi}$. We do not use this definition here because the spin transformation properties are obscured by the θ-dependence. However, this wavefunction is appropriate for discussing the reduction of superfields in the σ-basis (c.f. Sec. 5.2).

As we expect, on the super-wavefunction $\bar{\Phi}_A(p)$ we have a finite-dimensional representation of supersymmetry:

$$e^{-i\bar{\varepsilon}S}\ \bar{\Phi}_A(p)\ e^{i\bar{\varepsilon}S} = D_A{}^B(\varepsilon)\ \bar{\Phi}_B(p)$$

$$\{S_\alpha,\ \bar{\Phi}_A(p)\} = (S_\alpha)_A{}^B\ \bar{\Phi}_B(p)\ . \tag{5.3}$$

We begin our study of the finite-dimensional representations with these super-wavefunctions, rather than the superfields.

It might be asked here whether or not the spin composition of the superfields should be the same as for the UIR's. For fixed p, the finite-dimensional matrices $(S_\alpha)_A{}^B$ are irreducible representations of the Clifford algebra of the generators S_α, which for $p \neq 0$ is simple (Sec. 2.1). Therefore by the theorem of Weyl[46] it follows that there is a unique finite-dimensional irreducible representation, which is the one already used in the construction of the $(p^2 > 0, j_0)$ UIR's of the supersymmetry algebra $\mathscr{S}$ (Sec. 3.2). Where parity is included (as it is in the sequel) this is no longer the case. This is illustrated below for the $j_0 = \frac{1}{2}$ case (the so-called "vector" superfield[18]).

As in the case of ordinary wavefunctions, there is considerable freedom as to what sort of finite dimensional representations may be used and grouped together into a super-wavefunction. The most economical choice[90] involves (2J + 1) -component spinor representations D(J,o) of the Poincare group $SL(2,C)$. We shall briefly describe how to construct such superwavefunctions using the general method of Salam and Strathdee[18]. The method is to compute the change in one particular spin component under a supertranslation of the whole super-wavefunction. This introduces in the original component an admixture of other components, whose spins and wavefunctions may be deduced by covariance considerations. The procedure is repeated for these new components until all the participating wavefunctions have been found.

For this analysis it is necessary to treat the supertranslations S_α as a pair of Weyl spinors S_a and $\bar{S}_{\dot{a}}$, where $a, \dot{a} = -\frac{1}{2}, +\frac{1}{2}$. The algebra of the supertranslations in the Weyl basis is given in Sec. A2. In particular, the Majorana constraint becomes $\bar{S}_{\dot{a}} = \varepsilon_{\dot{a}\dot{b}}(S^\dagger)_{\dot{b}}$.

Thus suppose the superwavefunction Φ has a component Ψ_m of spin $-J_0$, belonging to the $D(J_0,o)$ representation, where $m = -J_0, -J_0+1, \dots J_0$. Let us suppose that Ψ_m is a "vacuum" state for $\bar{S}_{\dot{a}}$, in the sense that

$$(\bar{S}_{\dot{a}}\,\Phi)_m = 0 \tag{5.4}$$

Now consider $\delta\Psi_m = (S_a\,\Phi)_m$ under a supertranslation by S_a. The right-hand side must transform as direct product of spins, $\frac{1}{2} \times J_0$, so we have

$$(S_a\,\Phi)_m = \langle \tfrac{1}{2}\, a\; J_0\, m \,|\, J_0 - \tfrac{1}{2}\, \mu \rangle\; U_\mu + \langle \tfrac{1}{2}\, a\; J_0\, m \,|\, J_0 + \tfrac{1}{2}\, M \rangle\, V_M , \tag{5.5}$$

where $U_\mu = \langle J_0 - \frac{1}{2}\, \mu \,|\, \frac{1}{2}\, a\; J_0\, m \rangle\, \delta\Psi_m$ and similarly for V_M. Here the

$\langle J_0 \pm \frac{1}{2}\ \mu \mid \frac{1}{2}\ a\ J_0 m\rangle$ are Clebsch-Gordan coefficients. Thus by considering the spin-J_0 component of $\delta\bar{\Phi}$, we learn that $\bar{\Phi}$ contains components of spins $J_0 \pm \frac{1}{2}$. Considering in turn these components, we have

$$(S_a \bar{\Phi})_\mu = \langle J_0 - \tfrac{1}{2}\ \mu \mid \tfrac{1}{2}\ b\ J_0\ m'\rangle\ \varepsilon_{ab}\ \chi_{m'} \tag{5.6}$$

$$(S_a \bar{\Phi})_M = \langle J_0 + \tfrac{1}{2}\ M \mid \tfrac{1}{2}\ b\ J_0\ m'\rangle\ \varepsilon_{ab}\ \chi_{m'}\ , \tag{5.7}$$

where $\chi_{m'} = \frac{1}{2}(S_a \varepsilon_{ab} S_b \bar{\Phi})_{m'}$, $m' = -J_0, \ldots, J_0$. The ε_{ab} tensor arises again because $\{S_a, S_b\} = 0$. For the same reason, there is no $\delta\chi$ component of $S_a \bar{\Phi}$:

$$(S_a \bar{\Phi})_{m'} = 0\ . \tag{5.8}$$

The simplest example of this process arises when we start with a scalar component of the superwavefunctions, say A. Then we have simply

$$(\bar{S}_{\dot{a}} \bar{\Phi})_A = 0$$

$$(S_a \bar{\Phi})_A = \psi_a$$

$$(S_a \bar{\Phi})_b = \varepsilon_{ab}\ B$$

$$(S_a \bar{\Phi})_B = 0\ . \tag{5.9}$$

Here the wavefunction is simply $\bar{\Phi} = \{A, \psi, B\}$: the so-called "chiral scalar" superfield[18].

Returning to the general case, Eqs. (5.5) to (5.8) give the matrix elements of S_a acting on the superwavefunction $\{U_\mu, \psi_m, \chi_{m'}, V_M\}$ where the components have spins $J_0 - \frac{1}{2}$, J_0, J_0, and $J_0 + \frac{1}{2}$. The superwavefunction is irreducible, with $4\,(2J_0 + 1)$ components. By comparison with Sec. 3.3, we see that this method of construction parallels the construction of a UIR of supersymmetry for superspin $-J_0$, by the induced representation method.

We could easily write down the matrix elements of $\bar{S}_{\dot{a}}$ in this basis, by using the definitions of the various components of Φ given above, and using the algebra of Eq. (A2.6).

If we consider representations including parity, we must adjoin the conjugate wavefunction $\{U_{\dot{\mu}}, \psi_{\dot{m}}, \chi_{\dot{m}'}, V_{\dot{M}}\}$. If we assume that the ψ compoennt has a parity factor $i\eta_0$, so that

$$U_{\mathbb{P}} \psi_m U_{\mathbb{P}}^{-1} = i\eta_0 \psi_{\dot{m}}, \tag{5.10}$$

then by considering the transformation properties under S_a, in the same way as above (with $U_{\mathbb{P}} S_a U_{\mathbb{P}}^{-1} = i\bar{S}_{\dot{a}}$ (Eq. (A2.7)), then we find that the spin and parity content must be $J_0^{\pm i\eta_0}$, $(J_0 \pm \frac{1}{2})^{-\eta_0}$, just as in Sec.3.3. The dimension of the parity-doubled representation is $8(2J_0+1)$. As mentioned above, this may be reducible.

Using the super-wavefunction formalism, we could also develop an $\mathcal{M}$-function approach to supersymmetric scattering amplitudes. Thus we could write in general

$$\langle f \,|\, S \,|\, \cdots ; p\, \eta\, j\, \lambda \rangle = \mathcal{M}^{\cdots A}\, \bar{\Phi}_A^{\eta j \lambda}(p). \tag{5.11}$$

We shall not pursue this idea further here. However, the notion will be used in Sec. 5.3 below when we consider the massless superfield.

The multispinor superwavefunctions considered above, while economical, are not usually the ones encountered in association with superfields. We should also consider superwavefunctions constructed by the above method from reducible component fields, containing for example a wavefunction $\psi_{m\dot{n}}$ belonging to a representation $D(J_1, J_2)$ of $SL(2,C)$.

As an example of this, we give here an explicit construction of a superwavefunction Φ containing a 2-component spinor ψ_a, but with $(S_a\Phi)_b = 0$. In this case we shall see explicitly how parity-doubling may be avoided.

It is straightforward to write down the matrix elements of S_a and $S_{\dot{a}}$ in this basis, using the construction given above. We have simply (c.f. Sec.A2)

$$
\begin{aligned}
S_{\dot{a}}\psi_b &= V_{\dot{a}b} \qquad & S_a\psi_b &= 0 \\
S_{\dot{a}}V_{\dot{b}b} &= \varepsilon_{\dot{a}\dot{b}}\chi_b \qquad & S_aV_{\dot{b}b} &= -(p\cdot\sigma\varepsilon)_{a\dot{b}}\psi_b \\
S_{\dot{a}}\chi_b &= 0 \qquad & S_a\chi_b &= (p\cdot\sigma)_{a\dot{b}}V_{\dot{b}b} ,
\end{aligned}
\tag{5.12}
$$

an 8-dimensional representation in terms of components $\{\psi_a, \chi_a, V_{\dot{a}a}\}$.

If parity is included, then corresponding to Eq.(5.12) we introduce another 8-dimensional representation in terms of (conjugate) components $\{\psi_{\dot{a}}, \chi_{\dot{a}}, V_{a\dot{a}}\}$. Let us suppose that, under parity, the ψ component transforms as

$$
U_{\mathbb{P}}\psi_a(p)U_{\mathbb{P}}^{-1} = i\eta_0\psi_{\dot{a}}(\mathbb{P}p)
\tag{5.13}
$$

with parity factor $i\eta_0$ (we assume here that the components will transform as their corresponding fields). If $U_{\mathbb{P}}S_aU_{\mathbb{P}}^{-1} = iS_{\dot{a}}$ holds in this representation, it is obvious from the definitions of V and χ (eqs. (5.12)) that

$$
U_{\mathbb{P}}\chi_aU_{\mathbb{P}}^{-1} = -i\eta_0\chi_{\dot{a}}
$$

and

$$
U_{\mathbb{P}}V_{\dot{a}b}U_{\mathbb{P}}^{-1} = -\eta_0 V_{a\dot{b}}
\tag{5.14}
$$

We may write the components $V_{\dot{a}b}, V_{a\dot{b}}$ in the forms

$$
V_{\dot{a}a} = (\bar{V}^\mu\bar{\sigma}_\mu\varepsilon)_{\dot{a}a} , \quad V_{a\dot{a}} = (V^\mu\sigma_\mu\varepsilon)_{a\dot{a}} .
\tag{5.15}
$$

Then using Eq. (5.14) we have alternatively

$$U_{\mathbb{P}} V^{\mu} U_{\mathbb{P}}^{-1} = -\eta_0 \bar{V}^{\mu} \tag{5.16}$$

Thus, when parity is included, we have a 16-dimensional representation in terms of Dirac spinors ψ_α, χ_α and (reducible) vectors $V^\mu, \bar{V}^\mu$. Moreover, if we assume (for example) that ψ_α is Majorana, then so also is χ_α, and V_μ and $\bar{V}_\mu$ are hermitean.

The question arises as to whether this 16-dimensional representation is reducible. For example, we may try to relate the spinors ψ and χ in some way, consistently with the transformation properties. For example, we may set

$$\chi_a = X_{a\dot{b}} \psi_{\dot{b}} \quad , \quad \chi_{\dot{a}} = \bar{X}_{\dot{a}b} \psi_b \tag{5.17}$$

thus enabling us to eliminate χ_a and $\chi_{\dot{a}}$ in favour of ψ. Examining $S_a \chi_b$ from Eq. (5.12) and $S_{\dot{a}} \chi_{\dot{b}}$ from its parity conjugate equation, we find that, for consistency, X, $\bar{X}$ must be such that

$$V_{\dot{a}b} \bar{X}^t{}_{b\dot{c}} = -(p\cdot\bar{\sigma})_{\dot{a}b} V_{b\dot{c}}$$

and

$$V_{a\dot{b}} X^t{}_{\dot{b}c} = (p\cdot\sigma)_{a\dot{b}} V_{\dot{b}b}$$

i.e. $\qquad X\bar{X} = -p^2$

or $\qquad X = \pm(p\cdot\sigma) \ , \quad \bar{X} = \mp(p\cdot\bar{\sigma}) \ . \qquad (5.18)$

It is easy to see that these solu tions respect Eq. (5.14a).

Having obtained χ in terms of ψ, we can immediately find $\bar{V}_\mu$ in terms of V_μ. Substituting $X = \pm(p\cdot\sigma)$ into Eq. (5.18b) and using Eqs. (A1.1) and (A1.3) leads to

$$p\cdot V = \pm\, p\cdot\bar{V} \ , \quad V_\mu = \mp \bar{V}_\mu \pm 2p_\mu \frac{p\cdot\bar{V}}{p^2} \ .$$

Finally, we may always write $\bar{V}, V$ in terms of a scalar and a divergence-free vector. We find, for $X = \pm(p\cdot\sigma)$,

$$\bar{V} = \pm \tfrac{1}{2} A_\mu \pm \tfrac{1}{2} p_\mu A ,$$

$$V = -\tfrac{1}{2} A_\mu + \tfrac{1}{2} p_\mu A , \tag{5.19}$$

where $p \cdot A = 0,$

respectively. The spin parity content, for $\chi = \pm(p\cdot\sigma)$, is therefore

$$J^P = \{ 0^{\mp\eta_0}, \tfrac{1}{2}^{i\eta_0}, 1^{\mp\eta_0} \} \quad (j_0 = \tfrac{1}{2}) . \tag{5.20}$$

Thus we have shown how the 16-dimensional parity-doubled system, Eqs. (5.12) reduces to two equivalent systems, say $\bar{\Phi}_1^{(\pm)}$, by means of the decomposition

$$\begin{aligned}
\Psi &= \Psi^{(+)} + \Psi^{(-)} \\
\chi &= (p\cdot\sigma)\,\Psi^{(+)} - (p\cdot\sigma)\,\Psi^{(-)} \\
\bar{V}_\mu &= +\tfrac{1}{2}(A_\mu^{(+)} - A_\mu^{(-)}) + \tfrac{1}{2} p_\mu (A^{(+)} - A^{(-)}) \\
V_\mu &= -\tfrac{1}{2}(A_\mu^{(+)} + A_\mu^{(-)}) + \tfrac{1}{2} p_\mu (A^{(+)} + A^{(-)}) .
\end{aligned} \tag{5.21}$$

Finally, we write out the complete set of transformations of the super-wavefunction $\Phi_1^{(-)}$, for the case $\chi = -p\cdot\sigma\psi$, in the basis $\bar{\Phi}_{1A}^{(-)} = \{ A , \Psi_a , \Psi_{\dot a} , A_\mu \}$:

$$\begin{aligned}
S_a A &= \psi_a & S_{\dot a} A &= \psi_{\dot a} \\
S_a \psi_b &= 0 & S_{\dot a} \psi_b &= -\tfrac{1}{2}(A^\mu + p^\mu A)(\bar{\sigma}_\mu \varepsilon)_{\dot a b} \\
S_a \psi_{\dot b} &= -\tfrac{1}{2}(A^\mu - p^\mu A)(\sigma_\mu \varepsilon)_{a \dot b} & S_{\dot a} \psi_{\dot b} &= 0 \\
S_a A_\mu &= (p_\mu - \sigma_\mu\, p\cdot\bar{\sigma})_{ab}\, \psi_b & S_{\dot a} A_\mu &= (-p_\mu + \bar{\sigma}_\mu\, p\cdot\sigma)_{\dot a \dot b}\, \psi_{\dot b} .
\end{aligned} \tag{5.22}$$

Comparisons with Eqs. (4.30b) of the next section shows that these transformations are precisely those in the Weyl basis of the "vector supermultiplet" Φ_1 which occurs in the reduction of the general scalar superfield.

This example shows explicitly how the parity-doubling which is generally necessary for the super-wavefunctions, may be avoided in some cases.

5.2 Massive Superfields

Armed with our findings of the previous section concerning the superwavefunctions, we can now take up the study of massive superfields. The concept of a superfield as a function over an 8-dimensional manifold (x_μ, θ_α) was introduced in Sec. 2.2. Thus a scalar superfield $\Phi(x,\theta)$ may be written as a polynomial in the a-number Majorana spinor θ_α, whose coefficients are ordinary fields[18]:

$$\Phi(x,\theta) = A + \bar{\theta}\psi + \frac{1}{4}\bar{\theta}(F + \gamma_5 G + i\not{A}\gamma_5)\theta + \frac{1}{4}\bar{\theta}\theta\,\bar{\theta}\chi + \frac{1}{32}\bar{\theta}\theta^2 D \tag{5.23}$$

On such a function the supertranslations S_α have a differential representation (Eq. (2.20)):

$$\{S_\alpha, \Phi(x,\theta)\} = i(\underline{\partial}_\alpha + \tfrac{i}{2}\not{\partial}\theta_\alpha)\Phi(x,\theta), \tag{5.24}$$

where $\underline{\partial}_\alpha = \partial/\partial\bar{\theta}^\alpha$, $\bar{\partial}^\alpha = \partial/\partial\theta_\alpha$.

From Eq. (5.23) it can be seen that a general scalar superfield provides a 16-dimensional representation of the supersymmetry algebra, and (in view of the discussion of the previous section) may be reducible. It was, in fact, shown by Salam and Strathdee[18,91,92] that one can define "covariant derivative" operators

$$\{D_\alpha, \Phi(x,\theta)\} = i(\underline{\partial}_\alpha - \tfrac{i}{2}\not{\partial}\theta_\alpha)\Phi(x,\theta) \tag{5.25}$$

which anticommute with the S_α, and satisfy an algebra isomorphic to that of the S_α, Eq. (3.1). From the D_α one can construct three orthogonal projection operators[18],

$$E_{\pm} = (i\partial)^{-2}\, \bar{D}\Gamma_{\mp} D\, \bar{D}\Gamma_{\pm} D$$

$$E_{1} = 1 - \tfrac{1}{4}(i\partial)^{-2}(\bar{D}D)^{2}, \tag{5.26}$$

and hence decompose $\Phi(x, \theta)$ in an invariant manner into three irreducible parts $\Phi_{\pm}(x,\theta)$ and $\Phi_{1}(x, \theta)$:

$$\Phi_{\pm}(x,\theta) = A_{\pm}(x) + \bar{\theta}\psi_{\pm}(x) + \tfrac{1}{4}\bar{\theta}\theta_{\pm} F_{\pm}$$

$$\pm \tfrac{1}{4}\bar{\theta} i\gamma_{\mu}\gamma_{5}\theta\, i\partial^{\mu}A_{\pm} + \tfrac{1}{32}(\bar{\theta}\theta)^{2}(i\partial)^{2}A_{\pm},$$

$$\Phi_{1}(x,\theta) = A_{1}(x) + \bar{\theta}\psi_{1} + \tfrac{1}{4}\bar{\theta}\, i\gamma_{\mu}\gamma_{5}\theta\, A'^{\mu}$$

$$+ \tfrac{1}{4}\bar{\theta}\theta\;\bar{\theta}\, i\not{\partial}\psi_{1} - \tfrac{1}{32}(\bar{\theta}\theta)^{2}(i\partial)^{2}A_{1},$$

where

$$i\partial^{\mu} A_{1\mu} = 0 \tag{5.27}$$

Salam and Strathdee[18] showed that necessary and sufficient conditions for the fields to be eigenfunctions of the projection operators are given by the supplementary conditions

$$D_{\pm\alpha}\, \Phi_{\mp}(x, \theta) = 0$$

and

$$\bar{D}D_{\pm}\, \Phi_{1}(x,\theta) = 0 \tag{5.28}$$

The $\Phi_{\pm}(x,\theta)$ are called the chiral supermultiplets, and the $\Phi_{1}(x,\theta)$ the non-chiral, or vector, supermultiplet.

The transformation properties of the components of $\Phi_{\pm}$ and Φ_{1} under supertranslations can be found by using Eq. (5.24), and the rules for the differentiation of anticommuting quantities (Eq. (2.19)). If we write the components as

$$(\Phi_{\pm})_{B} = \{ A_{\pm}\, ,\, \psi_{\pm\beta}\, ,\, F_{\pm} \}$$

$$(\Phi_{1})_{B} = \{ A_{1}\, ,\, \psi_{1\beta}\, ,\, A_{1\mu} \} \tag{5.29}$$

then the transformations may be written $(i\partial_{\mu} = p_{\mu})$:

$$(S_\alpha \bar{\Phi}_\pm)_B = \{ \psi_{\pm\alpha} , (F_\pm + \not{p} A_\pm) C_{\alpha\beta} , - \not{p}\psi_{\pm\alpha} \}$$

$$(S_\alpha \bar{\Phi}_1)_B = \{ \psi_{1\alpha} , \tfrac{1}{2}(\not{p}A_1 + i\not{A}_1\gamma_5) C_{\alpha\beta} , i\gamma_5(p_\mu - \gamma_\mu \not{p})\psi_{1\alpha} \} . \tag{5.30}$$

They have already been given, as examples of irreducible super-wave-functions, in the last section, Eqs. (5.9) and (5.22), respectively.

Notice that, in the Weyl basis, the conditions (5.28) mean that $\bar{\Phi}_+$, $\bar{\Phi}_-$ is a function only of $\theta_{\dot{\alpha}}$, θ_α respectively. This is perhaps not surprising in view of the manner in which Eqs. (5.9) were constructed.

If we assume that $\bar{\Phi}(x,\theta)$ has intrinsic parity η_0 , say (c.f. Eq. (A2.4))

$$U_{\mathbb{P}} \bar{\Phi}(x,\theta) U_{\mathbb{P}}^{-1} = \eta_0 \bar{\Phi}(\mathbb{P}x , i\gamma_0\theta) \tag{5.31}$$

then we find $$U_{\mathbb{P}} \bar{\Phi}_+(x,\theta) U_{\mathbb{P}}^{-1} = \eta_0 \bar{\Phi}_-(\mathbb{P}x , i\gamma_0 \theta)$$

and $$U_{\mathbb{P}} \bar{\Phi}_1 (x,\theta) U_{\mathbb{P}}^{-1} = \eta_0 \bar{\Phi}_1(\mathbb{P}x , i\gamma_0\theta) .$$

Thus under parity the irreducible multiplets are $\bar{\Phi}_+ + \bar{\Phi}_-$, and $\bar{\Phi}_1$. By considering $\Psi_1 = A_1 + \bar{\theta}\psi_1 + \ldots$, we deduce that the correct spin parity assignment is $\bar{\Phi}_1^{(-)} \sim (0^{\eta_0}, \tfrac{1}{2}^{i\eta_0}, 1^{\eta_0})$, as already observed in the last section (Eqs. (5.20) and (5.22)).

In the previous section it was explained (and we found in practice) that the same spin composition is expected for the finite-dimensional representations of supersymmetry, namely, the massive superfields, as for the massive UIR's of Secs. 3.1 and 3.2. In Sec. 3.1 it was seen how this structure could be conveniently represented for the massive case by means of a two-dimensional weight diagram giving a plot of the weights (σ, j) participating in a UIR. In view of the above remark, we might also expect to be able to represent the structure of superfields by means of weight diagrams.

The first step is to discover which parts of the superfields are eigenfunctions of the operator $(i\partial)^{-2}\, i\partial^\mu \Sigma_\mu$ (Eq. (3.3)),

$$[(i\partial)^{-2}\partial^\mu \Sigma_\mu \,,\ \Phi(x,\theta)] = \frac{i}{4}\left(\bar{\partial}(i\partial\!\!\!/)^{-1}\gamma_5\partial + \frac{1}{4}\bar{\theta}(i\partial\!\!\!/)\gamma_5\theta\right)\Phi = \sigma\Phi$$

with $\sigma = 0, \pm\frac{1}{2}$. For an arbitrary field $\Phi(x,\theta)$ it is easy to verify that there are 9 different possible eigenfunctions. Each of these has the form of a polynomial in θ, multiplied by a single arbitrary function of x (c.f. Eq. (5.2)). For polynomials of lowest order $p = 0, 1, 2$, we can find in each case eigenfunctions with $\sigma = 0, \pm\frac{1}{2}$. We can therefore label the σ-eigenfunctions as $\Omega_{p\sigma}(\theta)$. The normalized eigenfunctions are given in Table 5.1 (the arbitrary function of x is taken to be a constant).

p	$\sigma = 0$	$\sigma = \pm\frac{1}{2}$
0	$\Omega_{00}(\theta) = 1 + \frac{1}{32}\bar{\theta}\theta^2\partial^2$	$\Omega_{0\pm}(\theta) = 1 \mp \frac{1}{4}\bar{\theta}\partial\!\!\!/\gamma_5\theta - \frac{1}{32}\bar{\theta}\theta^2\partial^2$
1	$\bar{\Omega}_{10}(\theta) = \bar{\theta} - \frac{1}{4}\bar{\theta}\theta\,\bar{\theta}\, i\partial\!\!\!/$	$\bar{\Omega}_1(\theta)\Gamma_\mp = (\bar{\theta} + \frac{1}{4}\bar{\theta}\theta\,\bar{\theta}\, i\partial\!\!\!/)\Gamma_\mp$
2	$\bar{\Omega}^\mu_{20}(\theta) = \frac{1}{4}\bar{\theta}\, i\gamma^\mu_\perp\gamma_5\theta$	$\Omega_{2\pm}(\theta) = \frac{1}{4}\bar{\theta}\theta_\mp$

Table 5.1 Normalized $\Omega_{p\sigma}(\theta)$ Eigenfunctions.

The functions $\Omega_{p\sigma}(\theta)$ are in 1-to-1 correspondence with the 8 independent monomials 1, $\bar{\theta}$, $\bar{\theta}\theta_{\pm}$, $\bar{\theta}i\gamma_{\mu}^{\perp}\gamma_5\theta$, $\bar{\theta}i\not{\partial}\gamma_5\theta$, $\bar{\theta}\theta\,\bar{\theta}$ and $(\bar{\theta}\theta)^2$ which can be constructed from θ_{α} and $i\partial^{\mu}$, and which in Eq. (5.23) are used as a basis for the Grassmann algebra of the a-number Majorana spinor θ (Here matrix multiplication of the spinor (fermion) monomials merely redefines the spinor coefficient by a matrix factor, so we are counting $\bar{\Omega}_{1\pm}$ as one function $\bar{\Omega}_1(\theta)$). Therefore we may just as well expand any superfield in terms of the $\Omega_{p\sigma}(\theta)$ basis, with suitable coefficient fields. Table 5.1 implies that these new fields will be related to the old fields, Eq. (5.23), by a nonsingular (but possibly nonlocal[18]) basis transformation. We may in any case write

$$\begin{aligned}\Phi(x,\theta) \;=\;& \Omega_{0+}(\theta)\,A_+(x) + \bar{\Omega}_{10}(\theta)\,\psi_+(x) + \Omega_{2-}(\theta)\,F_+(x)\\ &+ \Omega_{0-}(\theta)\,A_-(x) + \bar{\Omega}_{10}(\theta)\,\psi_-(x) + \Omega_{2+}(\theta)\,F_-(x)\\ &+ \Omega_{c0}(\theta)\,A_1(x) + \bar{\Omega}_1(\theta)\,\psi_1(x) + \Omega^{\mu}_{20}(\theta)\,A_{1\mu}(x)\end{aligned}$$

where
$$i\partial^{\mu}A_{1\mu} = 0 \tag{5.32}$$

Comparison with Eq. (5.27) shows that, written in the $\Omega_{p\sigma}$ basis, the scalar superfield $\Phi(x,\theta)$ is already in completely reduced form. The chiral supermultiplets $\Phi_{\pm}$ are associated with the $p = 1 \pm 2|\sigma|$ eigenfunctions, and the non-chiral supermultiplet with the $p = 1 \pm 2|\delta|$ eigenfunctions, with $|\delta| = \frac{1}{2} - |\sigma|$ (c.f. Sec. 3.2).

We can immediately write down the weight diagram for the scalar superfield, giving the values of the weights (j,σ) which participate, and labelling each weight with the associated component field. This is given in Fig. 5.1.

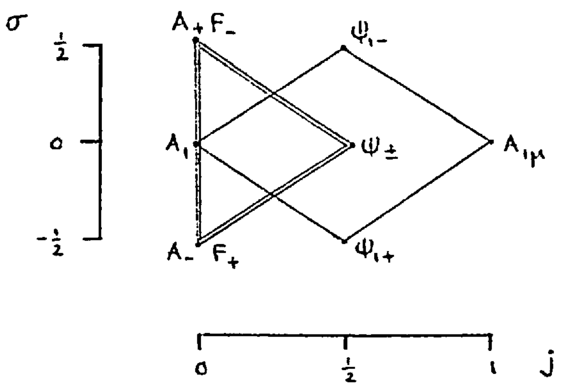

Fig. 5.1 Weight Diagram for Scalar Superfield.

O'Raifeartaigh[31] has discussed similar weight diagrams for superfields. However, those diagrams involve only the order in of the various monomial coefficients of the fields. As such, the structure is not so transparent as with the (j,σ) weight diagrams.

It is worth noticing also that the tensor calculus[31] of superfields, for example the question of reducing the product of superfields, can profitably be carried out in the $\Omega_{p\sigma}(\theta)$ basis, using their multiplication table; for example,

$$\Omega_{10}(\theta)\bar{\Omega}_1(\theta) = 2(i\not\partial)^{-1}\Omega_{00} - (i\not\partial)^{-1}\Omega_{0+} - (i\not\partial)^{-1}\Omega_{0-} - \Omega_{2+} - \Omega_{2-} .$$

Thus it is evident that the subsets $\Omega_{0\pm}$, $\bar{\Omega}_{10}\Gamma_{\pm}$ and $\Omega_{2\pm}$ are bases for two subspaces of the complete algebra which are closed under multiplication: the products of chiral superfields are again chiral superfields.

The use of the $\Omega_{p\sigma}$ basis is obviously not restricted to the scalar superfield case. Higher-spin superfields have, in fact, received much attention in the literature[20,93-99]. In the present approach, a superfield $\Phi_A(x,\theta)$ of arbitrary spin (where A labels an arbitrary representation, possibly reducible, of $SL(2,C)$ may still be written in the $\Omega_{p\sigma}(\theta)$ basis, as in Eq. (5.32), with components labelled in exactly the same way. The problem is simply to decompose the reducible representations A , αA and μA , in each specific case. The irreducible supermultiplets are found by inspection, after drawing the weight diagram and labelling each point (j, σ) by the component carrying that particular value of spin. For each component $A_{\pm}^{(J)}$ of spin-J, there will be a pair of chiral-type supermultiplets $\bar{\Phi}_{\pm}^{(J)}$, and for each component $\psi_1^{(J)}$ of spin-J, there will be a non-chiral-type supermultiplet $\Phi_1^{(J)}$, in each case with superspin equal to J. The observation that there is a lowest weight uniquely associated with each irreducible supermultiplet was made by O'Raifeartaigh[31], and is implicit in the discussion of superwavefunctions given by Salam and Strathdee[18].

An elegant higher-spin formalism for superfields which illustrates the above remarks has been developed by Sokatchev[99]. We conclude our study of the massive superfields with a brief review of this work.

It is straightforward to verify that the scalar superfield, while reducible as a representation of the supersymmetry algebra $\mathcal{S}$, is irreducible as a representation of the larger algebra $\mathcal{G}$ generated by S_α and the covariant derivatives D_α which anticommute with them. The projection operators (Eqs. (5.26)) then project onto eigenstates of the invariants of $\mathcal{S}$ (including the superspin) which can be constructed in $\mathcal{G}$.

By the Fock space method of Sec. 3.2[18], it transpires that arbitrary irreducible representations of $\mathcal{G}$ (of which the scalar superfield is the simplest case) are labelled by a spin $Y_0 = 0, \frac{1}{2}, 1, \ldots,$

containing superspins $J_0 = Y_0 - \frac{1}{2}$, Y_0, Y_0, $Y_0 + \frac{1}{2}$ (with another degeneracy label, $Q = 0, \pm 1$). Furthermore, the spin Y_0 just corresponds to the external spin of the superfield: thus such an irreducible representation may be realized for example as a Rarita-Schwinger superfield of spin- Y_0, $\Phi_{(\mu_1 \cdots \mu_{Y_0})}(x, \theta)$ or $\Phi_{(\mu_1 \cdots \mu_Y)\alpha}(x, \theta)$. Also, just as in the scalar superfield case, the projections onto the irreducible supermultiplets prove to be equivalent to (in general) much simpler supplementary conditions on the superfields.

The weight diagram for this case is given in Fig. 5.2. As can be expected in the light of the above discussion, here the superfield $\Phi_{(\mu)}$ has one component of $\psi_{1\alpha(\mu)}$ of spin $Y_0 - \frac{1}{2}$, and one of spin $Y_0 + \frac{1}{2}$, corresponding to the two vector-type supermultiplets; and a pair of components $A_{\pm(\mu)}$ of spin - Y_0, giving the two chiral supermultiplets.

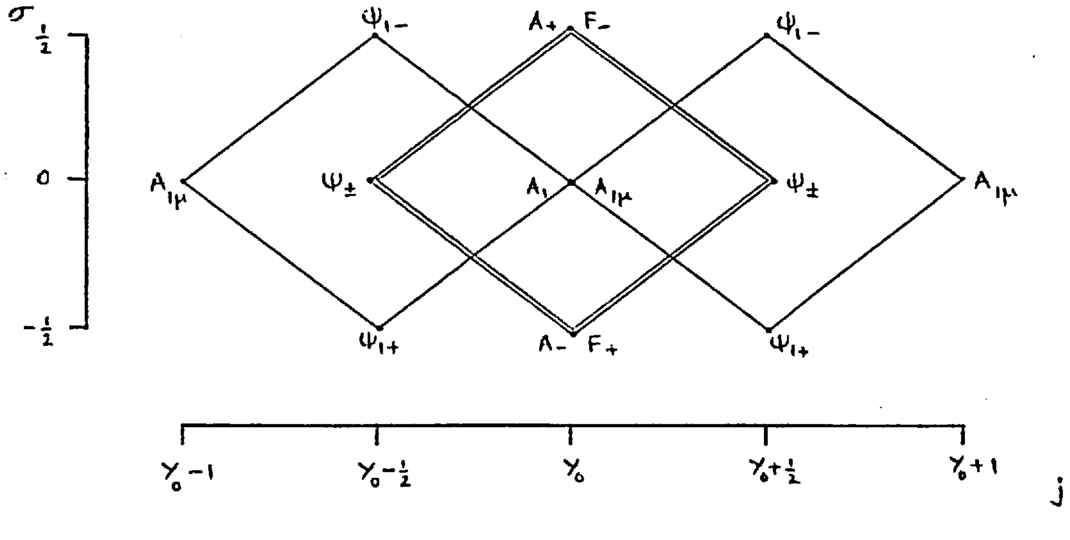

Fig. 5.2 Rarita-Schwinger Superfield of Spin $-Y_0$.

5.3 Massless Superfields

In our partial treatment in Sec. 3.3 of the representations of the supersymmetry algebra, $\mathscr{A}$, in the lightlike case, we found all the massless UIR's, that is, those with components which are physical massless particles. For the latter, there is the requirement that the noncompact Euclidean group generators be trivially represented, so as to ensure that the helicity Λ becomes an invariant (Eq. (3.37))

$$W_\mu = \Lambda P_\mu \tag{5.33}$$

The condition (Eq. (3.38))

$$\not{P} S = 0 \tag{5.34}$$

was then found to be necessary for (5.33). In this case there is only one independent supertranslation generator, and its conjugate. The UIR's were therefore found to contain only two different spin sectors, with invariant helicities λ_0 and $\lambda_0 - \frac{1}{2}$.

When we turn to the finite dimensional representations (superfields) in the massless case, we must obtain the same spin content as for the UIR's. Now, with ordinary massless fields, this can be ensured in two distinct ways. The first way, the "gauge-independent" method, is to choose the finite-dimensional representation of $SL(2,C)$ carefully so that Eq. (5.33) is satisfied. An example of this is the electromagnetic tensor, $F_{\mu\nu} \sim D(1,0) + D(0,1)$. In the second, or "gauge-dependent" way, when Eq. (5.33) is not satisfied, the field is defined only up to a "gauge transformation", which can be chosen to remove the unphysical components of the fields, while the physical amplitudes remain gauge-invariant. For example, the electromagnetic vector potential is defined classically up to a gauge transformation $\delta A_\mu = \partial_\mu \Lambda$. It may happen, as well, that other components of the field are dependent variables, and can be eliminated: this is true for example of the gravitational metric tensor $g_{\mu\nu}(x)$.

In studying the supersymmetric massless case, we shall be able to find direct analogies for both of these methods. Eq. (5.34) takes the corresponding role to Eq. (5.33) in the ordinary case.

Consider firstly massless superfields, all of whose components are gauge-independent in the sense described above. Thus necessarily Eq. (5.34) holds, or, in the superfield representation (Eq. (5.3))

$$\not{P} S \, \Phi(x, \theta) = 0 \tag{5.35}$$

This condition is Lorentz-covariant. Applying an infinitesimal supertranslation,

$$U(\varepsilon)^{-1} \not{P} S \, \Phi(x,\theta) U(\varepsilon) = (\not{P} S + 2i P^2 \varepsilon) \, \Phi(x,\theta) + O(\varepsilon^2)$$

so that it is also supercovariant, provided $P^2 = 0$. Moreover, it commutes with the supplementary conditions, Eqs. (5.28), in the D_α , which project out the irreducible parts of a superfield.

We turn for examples to the chiral scalar supermultiplets $\Phi_\pm$, and the vector supermultiplet $\Phi_\perp$, treating these as massless, gauge-independent superfields, and imposing the condition (5.35). The action of supertranslations on their components is given in Eqs. (5.29) and (5.30).

Strictly speaking, the separation of Φ into $\Phi_\pm$ and $\Phi_\perp$ cannot be performed for arbitrary superfields Φ in the $P^2 = 0$ case:

in making such a separation, we are assuming that the field is already "locally reducible" in the sense of Ref. 18. However, our results are not essentially affected by this specialization.

For the chiral supermultiplets we find that Eq. (5.35) implies

$$i\not{\partial}\,\psi_\pm = 0 = F_\pm\,, \qquad \partial^2\psi_\pm = 0 = \partial^2 A_\pm\,,$$

and $$\Phi_\pm(x,\theta) = A_\pm + \bar\theta\psi_\pm \pm \tfrac{1}{4}\,\bar\theta i\gamma_\mu\gamma_5\theta\, i\partial^\mu A_\pm\,. \qquad (5.36)$$

For the vector supermultiplet we find that Eq. (5.35) implies

$$i\not{\partial}\,\psi_1 = 0\,, \quad A_{1\mu} = i\partial_\mu \chi_1\,, \quad \partial^2\psi = 0 = \partial^2\chi_1\,, \qquad (5.37)$$

whence, defining $A_{1\pm} = (A_1 \pm \chi_1)$, we find $\Phi_1 = \Phi_{1+} + \Phi_{1-}$, where $\Phi_{1\pm}$ is as in Eq. (5.36). Thus in both cases, the irreducible, gauge-independent massless superfields are of chiral type, $\Phi_\pm$ or $\Phi_{1\pm}$, and contain a pair of free, massless fields $(A_\pm, \psi_\pm)$.

We can introduce weight diagrams for these superfields in the same way as for the massive case in the previous section. Here the weights are of the form (λ, ν), where $\nu = \pm\frac{1}{2}$ is the eigen-value of $\Sigma_\mu = N P_\mu$ (Sec. 3.3). There are now only four basis functions, $\omega_{p\nu}(\theta)$, where $p = 0, 1$. These are given in Table 5.2.

p	$\nu = \pm\frac{1}{2}$
0	$\omega_{0\pm}(\theta) = 1 \mp \frac{1}{4}\bar\theta\not{P}\gamma_5\theta$
1	$\bar\omega_1\Gamma_\mp(\theta) = \bar\theta\,\Gamma_\mp$

Table 5.2 $\omega_{p\nu}(\theta)$ Eigenfunctions for $\not{P}S = 0$.

When $\not{P}S = 0$, the supertranslations act effectively on a 4-dimensional space of a single a-number θ, and its conjugate $\bar{\theta}$, with basis $1, \theta, \bar{\theta}, \theta\bar{\theta}$. The $\omega_{p\nu}(\theta)$ eigenfunctions are in one to one correspondence with these, so any gauge-independent superfield may be expanded in terms of them. For example, for the case of the general scalar superfield, we have from the above

$$\Phi = \Phi_+ + \Phi_- + \Phi_{1+} + \Phi_{1-} ,$$

where each massless chiral part is in reduced form,

$$\Phi_\pm = \omega_{0\pm}(\theta)\, A_\pm + \bar{\omega}_1(\theta)\, \Psi_\pm . \tag{5.38}$$

The weight diagram for this case, labelled by the components, is given in Fig. 5.3 below.

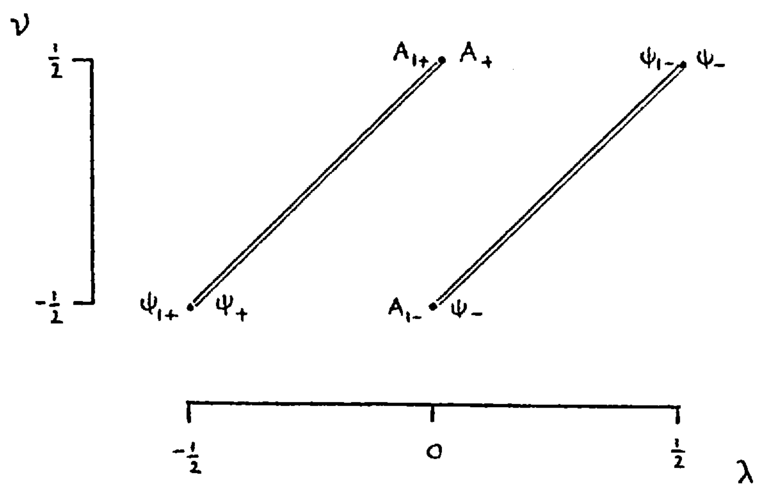

Fig. 5.3 Weight Diagram for Gauge-Independent, Massless Scalar Superfield.

We now take up the case where at least one component of the massless superfield is gauge dependent. Here the condition $\not{P}S\,\Phi = 0$ cannot be imposed directly, but it is still implemented through the gauge invariance of the amplitudes. Specifically, suppose that these are expressed in terms of $\mathcal{M}$-functions multiplying external super-wavefunctions (Eq. (5.11)),

$$\langle f \mid S \mid p j \eta \lambda \rangle = \mathcal{M}^{A}\, \Phi_{A}^{\eta j \lambda}(p)$$

Consider the change $\delta\Phi_A$ in Φ_A induced by an infinitesimal super-translation by $i\not{p}\varepsilon$. To guarantee Eq. (5.34) and thus the correct components of the superfield, we should have

$$\mathcal{M}^{A}\, \delta\Phi_{A} = 0 \tag{5.39}$$

This will impose certain constraints on $\mathcal{M}^{A}$, but such that the amplitudes are invariant with respect to a larger class of transformations,

$$\Phi_A \rightarrow \Phi_A + \chi_A \tag{5.40}$$

of the external super-wavefunction. It is these that we shall identify as the gauge transformations of the corresponding superfields.

The simplest example of a gauge-dependent $\Phi(x,\theta)$ is the massless vector supermultiplet, Eq. (5.27). The action of super-translations on the components is given by (Eqs. (5.29), (5.30))

$$\Phi_{1A} \equiv (\Phi_{10}, \Phi_{1\alpha}, \Phi_{1\lambda}) = (A_1, \psi_{1\alpha}, A_{1\lambda}), \quad p\cdot A_1 = 0,$$

$$(S_\alpha \Phi_1)_B = \left(\psi_\alpha, \tfrac{1}{2}(\not{p}A + i\not{\alpha}\gamma_5)C_{\alpha\beta}, i\gamma_5(p_\mu - \gamma_\mu \not{p})\psi_\alpha\right). \tag{5.41}$$

Let us examine in more detail the way in which the different helicity components of the fields transform under the $S_{\pm\alpha}$ in the massless case. We can do this by using the helicity projectors Γ_κ and Π^0, Π^ℓ ($\kappa, \ell = \pm$) for spins $\frac{1}{2}$ and 1, respectively.

The helicity - $\pm\frac{1}{2}$ component of a massless spinor is just the chiral component,

$$\psi^{(\pm)} = \Gamma_{\mp}\psi$$

while for a vector ($\hat{\underline{p}} = |\underline{p}|^{-1}\underline{p}$)

$$\underline{A}^{(\pm)} = \underline{A} - \hat{\underline{p}}\,\hat{\underline{p}}\cdot\underline{A} \pm i\,\hat{\underline{p}}\times\underline{A}$$

and $$\underline{A}^{(0)} = \hat{\underline{p}}\,\hat{\underline{p}}\cdot\underline{A}$$

The projectors are introduced in Sec. A6, and some of their important algebraic properties are summarized there.

Using these projectors, and especially Eq. (A6.3b), we may rewrite the transformations of $\underline{\Phi}_1$ as

$$\begin{aligned}
(S_{\pm\alpha}\,\underline{\Phi}_1)_0 &= \psi_{\pm\alpha} \\
(S_{\pm\alpha}\,\underline{\Phi}_1)^{(\kappa)}_{\beta} &= \tfrac{1}{2}\Gamma_{\pm}(\not{p} \mp \not{K})\Gamma_{\kappa}C_{\alpha\beta} \\
\underline{(S_{\pm\alpha}\,\underline{\Phi}_1)}^{(0)} &= \mp\,\hat{\underline{p}}\,(|\underline{p}| + \gamma_0\not{p})\,\psi_{\mp\alpha} \\
\underline{(S_{\pm\alpha}\,\underline{\Phi}_1)}^{(\ell)} &= \mp 2\ell\,|\underline{p}|\,\underline{\sigma}^{(\ell)}\,\Gamma_{-\ell}\,\psi_{\pm\alpha}
\end{aligned} \tag{5.42}$$

It is important to note here that, since we could be applying these considerations to interacting fields, we do not impose equations of motion on the components.

Thus from Eqs. (5.42b,d) we have

$$\begin{aligned}
(S_{\pm}\,\underline{\Phi}_1)^{(\pm)}_{\alpha} &= 0 = \underline{(S_{\mp}\,\underline{\Phi}_1)}^{(\mp)} \\
(S_{\pm}\,\underline{\Phi}_1)^{(\mp)}_{\alpha} &\neq 0 \neq \underline{(S_{\mp}\,\underline{\Phi}_1)}^{(\pm)}
\end{aligned} \tag{5.43}$$

This suggests (c.f. Eqs. (5.22b,g) and Fig. 5.3) that possible candidates for the irreducible massless constituents of Φ_1 are the pairs $\Phi_1^{(\pm)} = (\psi_{1\mp}, \underline{A}^{(\pm)})$. Unfortunately under $S_{\pm\alpha}$ the other components A and $\underline{A}^{\circ}$ of Φ_1 also pick up parts (depending upon $\psi_{\mp}$ and $\psi_{\pm}$, respectively), and the system does not close on the $\Phi_1^{(\pm)}$ parts. It is the role of the gauge dependence of Φ_1 to remove the unwanted components.

Following Eqs. (5.39) and (5.40) consider the change $\delta\Phi_1$ of Φ_1 under an infinitesimal supertranslation by $i\not{p}\varepsilon$. Firstly note that from Eq. (5.4) the components A, $\underline{A}^{(0)}$ do not contribute to $\delta\Phi_1$. Further, suppose that Φ_1 contains in addition only the $+\frac{1}{2}$, $+1$ helicity states. Then from Eq. (5.41), using Eqs. (A6.3), we have

$$(\Phi_1) = (0, \psi_{1-}, \underline{A}_1^{(+)})$$

$$(\delta\Phi_1) = (X, \Gamma_+\chi, p_\mu X)$$

where $$X = \not{p}\psi_{1-}, \quad \chi = -|p|(\underline{A}^{(+)}\cdot\underline{\sigma}^{(-)}C). \tag{5.44}$$

Thus substituting Eq. (5.44) into Eq. (5.39), the M-function $(M) = (M^{\circ}, M^{\alpha}, M^{\lambda})$ must be such that

$$(M^{\circ} + M^{\lambda}p_\lambda)X + M^{\alpha}(\Gamma_+\chi)_\alpha = 0 \tag{5.45}$$

Moreover, since this equation is to be super-covariant, the individual coefficients must vanish. Thus

$$(M^{\circ} + M^{\lambda}p_\lambda) = 0 = M^{\alpha} \tag{5.46}$$

But this means that the amplitude is invariant for any transformation $\Phi_A \to \Phi_A + \chi_A$, where χ_A has the form of Eq. (5.44b), but where X and χ are now arbitrary. A similar set of gauge transformations is found for the $\bar{\Phi}_1^{(-)}$ - component, involving of the same form as (5.44b) but with the opposite sign of $p_\mu X$.

It only remains to transpose the above findings back into the language of superfields. The conclusion is that the massless vector supermultiplet contains irreducible, gauge-dependent parts $\bar{\Phi}_1^{(\pm)}$, the condition (5.34) being ensured by invariance of the amplitudes under the gauge transformations

$$\bar{\Phi}_1^{(\pm)} \rightarrow \bar{\Phi}_1^{(\pm)} + \mathcal{X}_{1\pm} , \tag{5.47}$$

where
$$\bar{\Phi}_1^{(\pm)} = A_1^{(\pm)} + \bar{\theta}\psi_{1\mp} + \tfrac{1}{4}\bar{\theta}i\gamma^\mu\gamma_5\theta A_{1\mu}^{(\pm)} + \tfrac{1}{4}\bar{\theta}\theta\,\bar{\theta}\not{p}\psi_\mp ,$$

and
$$\mathcal{X}_{1\pm} = X_{1\pm} + \bar{\theta}\chi_{1\pm} \pm \tfrac{1}{4}\bar{\theta}i\not{p}\gamma_5\theta X_{1\pm} + \tfrac{1}{4}\bar{\theta}\theta\,\bar{\theta}\not{p}\chi_{1\pm} , \tag{5.48}$$

so
$$A_1^{(\pm)} \rightarrow A_1^{(\pm)} + X_{1\pm} ,$$

$$A_{1\mu}^{(\pm)} \rightarrow A_{1\mu}^{(\pm)} \pm p_\mu X_{1\pm} ,$$

$$\psi_{1\mp} \rightarrow \psi_{1\mp} + \Gamma_\pm \chi \tag{5.49}$$

where $X_{1\pm}$ and χ are arbitrary.

Note that the form Eq. (5.49) of gauge transformation is only supercovariant if $\not{P}\mathcal{X}_{1\pm} = 0$. Thus the massless vector superfield is subject to gauge transformations

$$\bar{\Phi}_1^{(\pm)} \rightarrow \bar{\Phi}_1^{(\pm)} + \mathcal{X}_{1\pm}$$

where the gauge function $\mathcal{X}_{1\pm}(x, \theta)$ is an arbitrary (gauge-independent) massless chiral superfield.

Under parity, the constituents $\bar{\Phi}_1^{(\pm)}$ transform into one another (Sec. 3.3). The massless vector supermultiplet $\bar{\Phi}_1$ is now defined up to gauge transformations

$$\mathcal{X}_1 = X' + \bar{\theta}\chi + \tfrac{1}{4}\bar{\theta} i \gamma^\mu \gamma_5 \theta \partial_\mu X + \tfrac{1}{4}\bar{\theta}\theta\bar{\theta}\chi ,$$

$$A_1 \rightarrow A_1 + X'$$

$$A_{1\mu} \rightarrow A_{1\mu} + \partial_\mu X$$

$$\psi_{1\pm} \rightarrow \psi_{1\pm} + \Gamma_{\mp}\chi .$$

(5.50)

Thus in this case there exists a gauge in which $A_1 = 0$, and the massless superfield manifestly involves only the physical components ψ_1 and $A_{1\mu}$, which are subject to the ordinary types of gauge transformations of massless fields.

The weight diagram for the massless vector supermultiplet is given in Fig. 5.4 below.

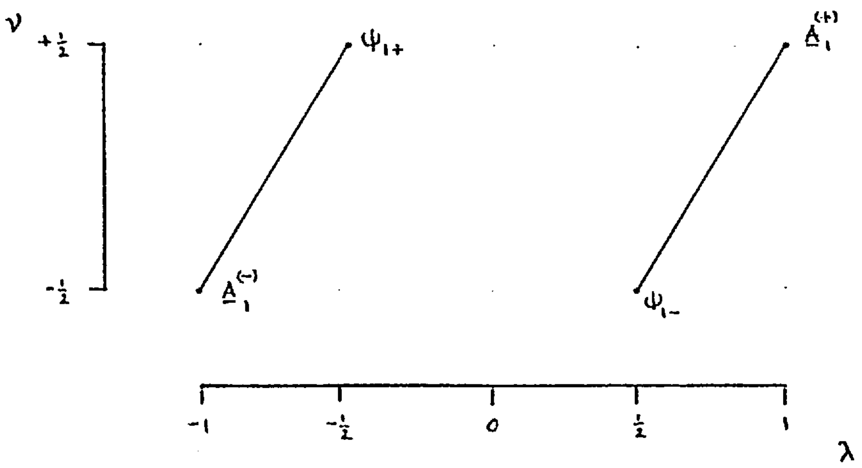

Fig. 5.4 Weight Diagram for Massless Vector Supermultiplet.

The gauge fields which have been used in the literature[32,67] in supersymmetric abelian and non abelian gauge theories bear some resemblance to the case of Φ_1 that we have studied here. There the gauge field is a (reducible) scalar superfield, but with gauge dependence such that in a special gauge (the "Wess-Zumino" gauge[32]) the superfield is left with components

$$\Psi(x, \theta) = \frac{1}{4}\bar{\theta} i \gamma_\mu \gamma_5 \theta A^\mu + \frac{1}{4}\bar{\theta}\theta\bar{\theta}\gamma_5 \chi + \frac{1}{4}\bar{\theta}\theta^2 D_5 ,$$

so that $\Psi(x,\theta)$ is nilpotent, just as Φ_1 is in the gauge with $A_1 = 0$. However, in the special gauge the physical components of $\Psi(x,\theta)$ are accompanied by a pseudoscalar auxiliary field, whereas the special gauge of Φ_1 contains only the physical components.

The generalization of this work to higher-spin superfields is straightforward in principle. The general result survives that the only possible gauge-independent parts of a massless superfield are the massless chiral-type parts (Eq. (5.36)). It may be possible to utilize the chiral parts, instead of the non-chiral parts, as gauge fields, provided the external spin is at least one-half. Some attempts have already been made in this direction[97,98].

6. THE BETHE-SALPETER EQUATION IN SUPERFIELD THEORY

In the preceding chapters we have used our formalism for the unitary representations to explore the consequences of supersymmetry as it applies to scattering amplitudes, partial wave analysis, and so on. In the present chapter we shall be concerned with another area where the power of supersymmetry in tying together many different processes is also manifest: the bound state problem. The primary vehicle for attempting a relativistically covariant description is provided by the Bethe-Salpeter equation, and here we deal with its supersymmetric generalization. The work to be described was developed from a study by Delbourgo[100] of supersymmetric composite states, and is based upon Ref. 29.

Clearly it is quite impossible to classify all supersymmetric bound state equations, which depend both upon the interaction kernels used, and the participating fields. We shall therefore focus on a generalization of one of the simplest models, where the elementary fields are scalar, and the interaction corresponds to the exchange of a massless scalar: namely, the Wick-Cutkosky[101,102] problem. Of course, even in this case, the supersymmetric version is far from trivial, since there are a great number of channels to contend with, all coupled together by supersymmetry (Fig. 6.1). In this situation, it is mandatory to use the formalism of superfields, rather than treat the various components and coupled channels separately.

Superfields were introduced in Sec. 2.2, and investigated in the last chapter. We shall not describe in detail here how they are applied in supersymmetric models; further information is to be found, for example, in Ref. 18.

For our model, then, we consider a theory involving only elementary scalar fields $\Phi_\pm(x,\theta)$. The potential which drives the dynamics will also be assumed to be generated by the exchange of another scalar superfield χ . Thus we take the Lagrangian

$$\begin{aligned}\mathcal{L} = {} & \tfrac{1}{4}(\bar{D}D)\left(\tfrac{1}{2}\Phi_+\bar{D}D\Phi_- + \tfrac{1}{2}\Phi_-\bar{D}D\Phi_+ - m\Phi_+^2 - m\Phi_-^2\right) + \\ & + \tfrac{1}{4}(\bar{D}D)\left(\tfrac{1}{2}\chi_+\bar{D}D\chi_- + \tfrac{1}{2}\chi_-\bar{D}D\chi_+ - \mu\chi_+^2 - \mu\chi_-^2\right) - \\ & - \tfrac{1}{4}g(\bar{D}D)\left(\Phi_+^2\chi_+ + \Phi_-^2\chi_-\right) \end{aligned} \tag{6.1}$$

corresponding to two separate superscalars Φ and χ of masses m and μ interacting trilinearly (D is the "covariant derivative", Eq. (5.25)). For $\mu = 0$ this is the supersymmetric generalization of the Wick-Cutkosky model. The methods we use are straightforwardly extended to other more intricate cases, which may include gauge vector superfields and more complicated kernels, so in that respect , ours is the prototype theory.

In Sec. 6.1 the superfield equations for our model are set up, giving the reduced equationd for the Bethe-Salpeter wavefunctions, after all the extraneous kinematical factors have been extracted. In Sec. 6.2, the simplest case p = 0 is considered, and after choosing a special supersymmetric representation for the bound states, valid in this case, the equations prove tractable. In the $p = 0$ limit it is found that the equations admit a continuum of possible eigenvalues of the coupling constant g . This difficulty is familiar from studies of the fermion-antifermion (Goldstein) composite problem[103]; the model does, indeed, include a two-fermion channel.(Fig. 6.1) The ambiguity is removed by other means.[103,104] Finally, in Sec. 6.3 some comments are made about the massive bound state problem.

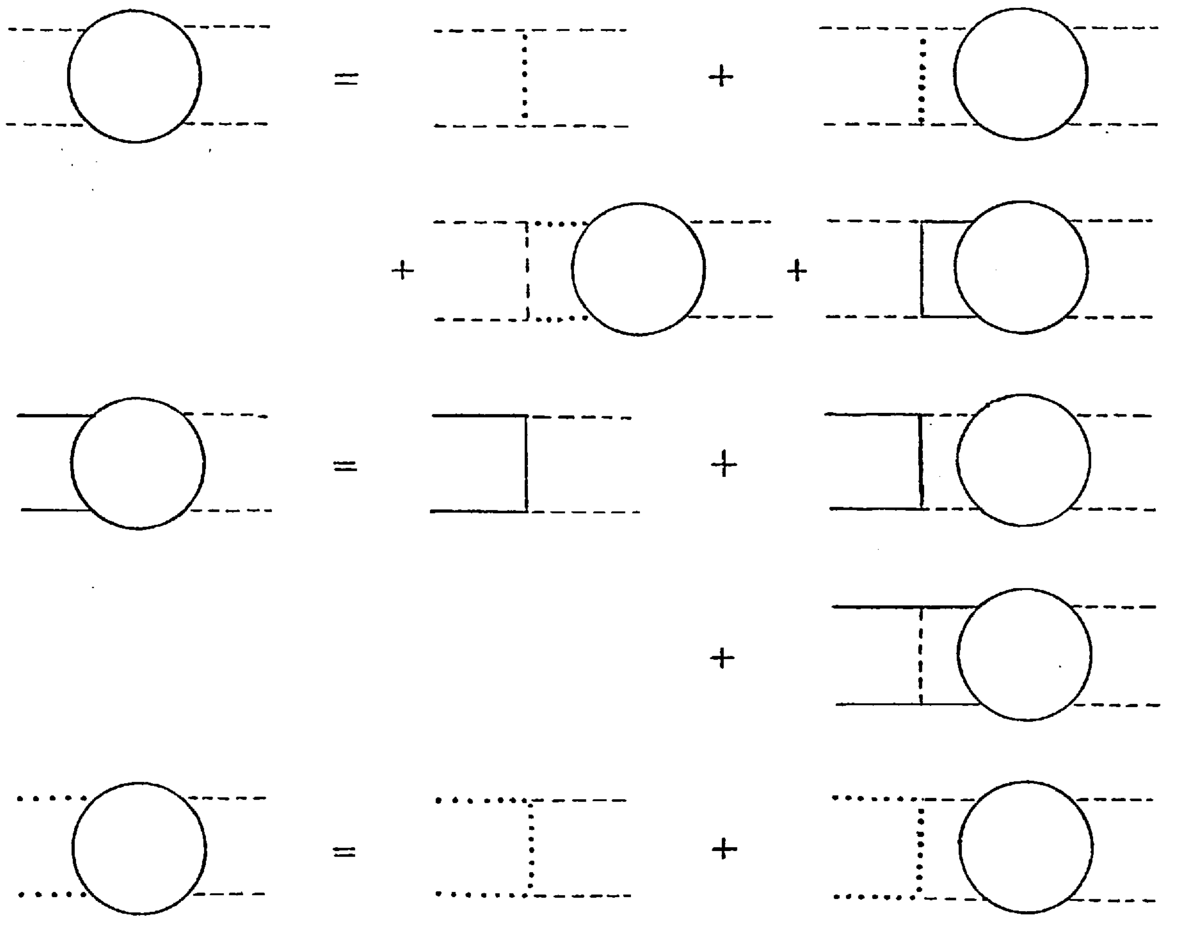

Fig. 6.1 Typical coupled Bethe-Salpeter equations for the field components in supersymmetric theories. The solid, broken and dotted lines represent respectively the Majorana fermion, physical scalars and auxiliary scalars.

6.1 The Bethe-Salpeter Wavefunctions for Superfields

We are interested in the bound states $|\Psi\rangle$ occurring in the Φ^2-channel due to the exchange of the χ field, and accordingly there are four kinds of Bethe-Salpeter wavefunction to be analysed, depending upon the chirality of the Φ fields (c.f. Eq.(5.1)):

$$\Psi_{\pm\pm}(x_1\theta_1, x_2\theta_2) = \langle 0 | \Phi_\pm(x_1\theta_1)\, \Phi_\pm(x_2\theta_2) | \Psi \rangle$$

$$\Psi_{\pm\mp}(x_1\theta_1, x_2\theta_2) = \langle 0 | \Phi_\pm(x_1\theta_1)\, \bar{\Phi}_\mp(x_2\theta_2) | \Psi \rangle \tag{6.2}$$

The free propagators occurring in the Bethe-Salpeter kernel are[105,106,107]

$$\langle 0 | T[\chi_\pm(x_1\theta_1)\chi_\pm(x_2\theta_2)] | 0 \rangle \equiv i\Delta_{\pm\pm}(x_1\theta_1, x_2\theta_2)$$

$$= -\tfrac{1}{4}\, i\mu\, \bar{\theta}\Gamma_\pm\theta \; \exp\left(\tfrac{1}{2} i \bar{\theta}_1 \not{\partial} \theta_2\right) \Delta_c(x)$$

$$\langle 0 | T[\chi_\pm(x_1\theta_1)\chi_\mp(x_2\theta_2)] | 0 \rangle \equiv i\Delta_{\pm\mp}(x_1\theta_1, x_2\theta_2)$$

$$= i \exp\left(\tfrac{1}{2} i \bar{\theta}_1 \not{\partial} \theta_2 \mp \tfrac{1}{4}\bar{\theta}\not{\partial}\gamma_5\theta\right) \Delta_c(x) \tag{6.3}$$

where $\theta = \theta_1 - \theta_2$ and $x = x_1 - x_2$ are the relative coordinates. The equations of motion are

$$\bar{D}D\,\Phi_\pm = 2m\,\bar{\Phi}_\mp + 2g\,\bar{\Phi}_\mp \chi_\mp \tag{6.4}$$

and if we apply these to Eqs. (6.2) we arrive at the homogeneous Bethe-Salpeter equations

$$\bar{D}D_1\, \bar{D}D_2\, \Psi_{\pm\pm}(1,2) = 4\left(m^2 + g^2\Delta_{\mp\mp}(1,2)\right)\Psi_{\mp\mp}(1,2)$$

$$\bar{D}D_1\, \bar{D}D_2\, \Psi_{\pm\mp}(1,2) = 4\left(m^2 + g^2\Delta_{\pm\mp}(1,2)\right)\Psi_{\mp\pm}(1,2) \tag{6.5}$$

in an obviously abbreviated notation. The covariant derivatives appearing here commute as usual with the supertransformations.

As they stand, Eqs. (6.5) are not very useful: we need to extract out all irrelevant kinematical factors from the Ψ associated with the "centre of mass" variables. To do this, we recall the action of the supertranslations (Eq. (2.20)),

$$e^{i(a\cdot P + \bar{\eta}S)}\,\Phi(x,\theta)\,e^{-i(a\cdot P + \bar{\eta}S)} = \Phi(x - a - \tfrac{1}{2}i\bar{\eta}\gamma\theta,\ \theta - \eta)$$

providing the representations

$$P_\mu \to i\partial_\mu\ ,\quad S_\alpha \to i(\underline{\partial}_\alpha + \tfrac{1}{2}i\not{\partial}\theta_\alpha)$$

If therefore we choose the coordinate transformations to be the mean values

$$a = \tfrac{1}{2}(x_1 + x_2)\ ,\qquad \eta = \tfrac{1}{2}(\theta_1 + \theta_2)$$

then we find

$$\Psi(1,2) = \langle 0|\,\Phi(\tfrac{1}{2}x - \tfrac{1}{4}i\bar{\theta}_2\gamma\theta_1,\ \tfrac{1}{2}\theta)\ \Phi(-\tfrac{1}{2}x - \tfrac{1}{4}i\bar{\theta}_1\gamma\theta_2,\ -\tfrac{1}{2}\theta)\ \times$$
$$\times\ e^{\frac{i}{2}((x_1+x_2)\cdot P\ +\ (\bar{\theta}_1+\bar{\theta}_2)S)}\,|\Psi\rangle \tag{6.6}$$

Evidently our bound state is to be chosen as an eigenstate of total four-momentum $p = p_1 + p_2$, and the action of S_α is specified by the commutation rule (Eq. (3.1))

$$\{S, \bar{S}\} = \not{p} \tag{6.7}$$

Besides this, $|\Psi\rangle$ will carry a label corresponding to internal (relative) degrees of freedom, like the momentum difference $2q = p_1 - p_2$. We shall attend to the choice of representation for $|\Psi\rangle$ in the next section; for the moment we merely assume that there is one, and leave it unspecified. From Eq. (6.6) it follows that each wavefunction occurring in Eq. (6.5) must be of the factorized form

$$\Psi_p(x_1\theta_1, x_2\theta_2) = e^{\frac{i}{2}((x_1+x_2)p + \bar{\theta}_1\not{\partial}\theta_2 + (\bar{\theta}_1+\bar{\theta}_2)S)} W_p(x,\theta) \tag{6.8}$$

where W is only a function of the relative variables x and θ . We easily find that the covariant derivatives act on W as follows:

$$\begin{aligned} D_1\Psi(1,2) &= e^{\frac{i}{2}(\cdot\cdot)}\, i\left(\underline{\partial} + \tfrac{1}{8}\not{p}\theta - \tfrac{i}{2}(\not{\partial}\theta - S)\right) W \\ D_2\Psi(1,2) &= -e^{\frac{i}{2}(\cdot\cdot)}\, i\left(\underline{\partial} + \tfrac{1}{8}\not{p}\theta + \tfrac{i}{2}(\not{\partial}\theta - S)\right) W \end{aligned} \tag{6.9}$$

This permits us to investigate the chirality constraints, Eqs. (5.28), on the individual fields, and we obtain the conditions

$$\begin{aligned} &\Gamma_{\mp}\left(\underline{\partial} + \tfrac{1}{8}\not{p}\theta\right) W_{\pm\pm} = \Gamma_{\mp}\left(\not{\partial}\theta - S\right) W_{\pm\pm} = 0 \\ &\left(\underline{\partial} + \tfrac{1}{8}\not{p}\theta \mp \tfrac{1}{2}\gamma_5(\not{\partial}\theta - S)\right) W_{\pm\mp} = 0 \end{aligned} \tag{6.10}$$

on the wavefunctions. The solutions of these equations lead to a further set of reduced wavefunctions with

$$\begin{aligned} W_{p\,\pm\pm}(x\theta) &= e^{\mp i\bar{\theta}\not{p}\gamma_5\theta}\, \tfrac{1}{2}(\bar{S} + \bar{\theta}\not{\partial})\,\Gamma_{\mp}(S - \not{\partial}\theta)\,\mathcal{W}_{\pm\pm}(x) \\ W_{p\pm\mp}(x\theta) &= e^{\mp\frac{1}{4}\bar{\theta}\not{\partial}\gamma_5\theta \mp \frac{1}{2}\bar{\theta}\gamma_5 S}\,\mathcal{W}_{\pm\mp}(x) \end{aligned} \tag{6.11}$$

The next step is to substitute Eqs. (6.11) and (6.3) into (6.5) to get the reduced equations (see Sec. A7, especially Eqs. (A7.1,2,3)). After some algebra one arrives at

$$\begin{aligned} -\partial^2 E_{\pm}(S,-\partial)\,E_{\mp}(S,\partial)\,\mathcal{W}_{\pm\pm} &= \left(m^2 + \tfrac{i}{2}g^2\mu\,\Delta_c\,\bar{S}S_{\pm}\,\partial^{-2}\right)\mathcal{W}_{\pm\pm} \\ -\partial^2 E_{\pm}(S,-\partial)\,E_{\pm}(S,\partial)\,\mathcal{W}_{\pm\mp} &= \left(m^2 + ig^2\Delta_c\right)\mathcal{W}_{\mp\pm} \end{aligned} \tag{6.12}$$

where the operators E are defined by

$$E_{\pm}(S,\partial) = 1 + \frac{i}{2}\bar{S}\not{\partial}^{-1}S_{\pm} - \frac{1}{8}(\bar{S}\not{\partial}^{-1}S_{\mp})^{2} + \frac{1}{8}\bar{S}\not{\partial}^{-1}\not{p}\,\not{\partial}^{-1}S_{\mp} \tag{6.13}$$

Eqs. (6.12) are the distillate of our work thus far and contain the essence of all the dynamics.

6.2 Zero-Momentum Bound States

So far we have avoided committing ourselves to a choice of supersymmetric representation carried by the bound states $|\Psi\rangle$. If, for example, $p^2 > 0$, then we could directly apply the results of Secs. 3.1 and 3.2, and use, for example, the spin basis for $|\Psi\rangle$. We could then, in principle, write down Eqs. (6.12) explicitly, knowing the matrix elements of S. However, this is rather complicated, and in any case the resulting equations would be practically insoluble.

We therefore choose instead to look at a special case in which Eqs. (6.12) are tractable: namely, the limit p=0 . This case represents a minimal requirement on our understanding of the problems involved, and sets the scene for the $p^2 > 0$ case. Unfortunately, although the equations simplify, the choice of representations becomes more difficult (c.f. Sec. 2.1), and our results from Chap. 3 are inapplicable. Instead, we must use the correct UIR's of the algebra for the null case.

In the covariant approach, it is appropriate in this case to analyse such UIR's in terms of their Lorentz group content (c.f. Sec. 3.1). By using the supersymmetry algebra, Eq. (3.1), we can evaluate $[S_\alpha, \frac{1}{2}J_{\mu\nu}J^{\mu\nu}]$ and $[S_\alpha, \frac{i}{8}\epsilon_{\mu\nu\rho\sigma}J^{\mu\nu}J^{\rho\sigma}]$ explicitly, and split S_α into raising and lowering operators for the Casimirs (ℓ_0, ℓ_1). Thus we can deduce which representations (ℓ_0, ℓ_1) will participate in a null UIR of $\mathcal{S}$. Because of the anticommutation relations, there is the possibility that only a finite number of such weights occurs.

The situation is similar if (as we do eventually) we perform a Wick rotation. The supersymmetry algebra now becomes a graded generalization of $O(4)$ (Sec. 2.1), with the simplifying fact that the unitary representations of the participating weights become finite-dimensional.

Fortunately, there is an elegant trick by which we can proceed to the essential qualitative results of the model, and avoid the details of the representations, at least for this case. This follows from the observation that, for $p = 0$, the supertranslations S_α anticommute, and so can be 'diagonalized' in the sense that we can consider $|\zeta\rangle$ such that

$$S_\alpha |\zeta\rangle \quad = \quad \zeta_\alpha |\zeta\rangle$$

where ζ is an a-number Majorana spinor. In fact, for later simplification, we shall choose $\zeta = \not{\partial}\xi$. In this case the wavefunctions become functions of x and ξ, $\mathcal{W}(x\xi)$, and are eigenfunctions of all supertranslations, so

$$S_\alpha \mathcal{W} = \not{\partial}\xi\,\mathcal{W}, \quad E_\pm(S,\partial)\mathcal{W} = e^{\pm\frac{1}{4}\bar{\xi}\not{\partial}\gamma_5\xi}\,\mathcal{W}, \tag{6.14}$$

making for a number of simplifications. One obtains from Eqs. (6.12)

$$\begin{aligned} -\partial^2 \mathcal{W}_{\pm\pm}(x,\xi) &= (m^2 + \tfrac{i}{2} g^2 \mu\,\bar{\xi}\xi \mp \Delta_c(x))\,\mathcal{W}_{\mp\mp}(x,\xi) \\ -\partial^2 e^{\pm\frac{1}{2}\bar{\xi}\not{\partial}\gamma_5\xi}\,\mathcal{W}_{\pm\mp}(x,\xi) &= (m^2 + ig^2\Delta_c(x))\,\mathcal{W}_{\mp\pm}(x,\xi) \end{aligned} \tag{6.15}$$

Let us now expand $\mathcal{W}$ in terms of ξ (Eq. (2.21))

$$\mathcal{W} = A + \bar{\xi}\psi + \tfrac{1}{2}\bar{\xi}(F + \gamma_5 G + i\not{V}\gamma_5)\xi + \tfrac{1}{2}\bar{\xi}\xi\,\bar{\xi}\gamma_5\lambda + \tfrac{1}{8}(\bar{\xi}\xi)^2 D$$

for each of the wavefunctions, and substitute into Eqs. (6.15), obtaining coupled field equations (some of which are effectively inhomogeneous). We shall write down these equations just for $\mu = 0$, as it is only in this limit anyway that one can derive analytic solutions. As $\mu \to 0$ one set of equations is trivial,

$$\partial^2 \mathcal{W}_{\pm\pm} = -m^2 \mathcal{W}_{\mp\mp}$$

corresponding to the fact that in equal chirality situations, there is no potential ($\Delta_{\pm\pm} \to 0$) to assist the binding. In the opposite chirality case, the content of the equations

$$-\partial^2 (A_{\pm\mp}, \psi_{\pm\mp}, F_{\pm\mp}, G_{\pm\mp}) = (m^2 - g^2/4\pi^2 x^2)(A_{\mp\pm}, \psi_{\mp\pm}, F_{\mp\pm}, G_{\mp\pm})$$

$$-\partial^2 (V^\mu_{\pm\mp} \mp i\partial^\mu A_{\pm\mp}) = (m^2 - g^2/4\pi^2 x^2) V^\mu_{\mp\pm}$$

$$-\partial^2 (\lambda_{\pm\mp} \pm \not{\partial} \psi_{\pm\mp}) = (m^2 - g^2/4\pi^2 x^2) \lambda_{\mp\pm}$$

$$-\partial^2 (D_{\pm\mp} - \partial^2 A_{\pm\mp} \mp 2i\partial_\mu V^\mu_{\pm\mp}) = (m^2 - g^2/4\pi^2 x^2) D_{\mp\pm} \qquad (6.16)$$

is easier to appreciate if we take even and odd parity combinations

$$A^{(\pm)} = A_{+-} \pm A_{-+}, \quad F^{(\pm)} = F_{+-} \pm F_{-+}, \cdots,$$

and break up V into a longitudinal and a transverse part:

$$V^\mu = V^\mu_\perp + \partial^\mu B \qquad \partial_\mu V^\mu_\perp = 0$$

We then get partly decoupled sets of homogeneous and inhomogeneous equations:

$$(\partial^2 \pm (m^2 - g^2/4\pi^2 x^2))(A^{(\pm)}, \psi^{(\pm)}, F^{(\pm)}, G^{(\pm)}, V^{(\pm)}_\perp) = 0$$

$$\partial^2 (B^{(\pm)} - i A^{(\mp)}) \pm (m^2 - g^2/4\pi^2 x^2) B^{(\pm)} = 0$$

$$\partial^2 (\lambda^{(\pm)} + \not{\partial} \psi^{(\mp)}) \pm (m^2 - g^2/4\pi^2 x^2) \lambda^{(\pm)} = 0$$

$$\partial^2 (D^{(\pm)} - \partial^2 A^{(\pm)} - 2i\partial^2 B^{(\mp)}) \pm (m^2 - g^2/4\pi^2 x^2) D^{(\pm)} = 0. \qquad (6.17)$$

To discuss the solutions, consider typically the scalar sector in Eq. (6.17). Perform a Wick rotation $r^2 = -x^2$ and make the usual expansion into 4-dimensional spherical harmonics:

$$A(x) = \sum_{n\,\ell\,m} a_n(r)\, Y_{n\ell m}(\hat{r})/r$$

$$n = 1, 2, \cdots ; \quad \ell = 0, 1, \cdots n-1 \; ; \quad m = -\ell, \cdots, \ell$$

Then

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2} \mp \left(m^2 + \frac{g^2}{4\pi^2 r^2} \right) \right) a_n^{(\pm)}(r) = 0 \tag{6.18}$$

which has a direct analogue in the fermion-antifermion bound state problem. The only acceptable solutions, bounded as $r \to \infty$ are for $a^{(-)}$:

$$a_n^{(-)}(r) = J_{(n^2 - g^2/4\pi^2)^{\frac{1}{2}}}(mr)$$

$$a_n^{(+)}(r) = 0 . \tag{6.19}$$

As is well known, the nonzero solutions, Eq. (6.19), would seem to suggest a continuum set of eigenvalues $0 \leq g \leq 2\pi n$. There are ways of curing this unpleasant feature which tell us to pick the least singular solution[103,104] :

$$g = 2\pi n \; ; \quad a_n^{(-)}(r) = J_0(mr) \tag{6.20}$$

Parallelstatements apply to the components $\psi^{(-)}$, $F^{(-)}$, $G^{(-)}$ and $V_\perp^{(-)}$. Carrying the argument over to the inhomogeneous equations, we find that $B^{(-)}$, $\lambda^{(-)}$ and $D^{(-)}$ have a similar character, and for the rest, in view of Eq. (6.20), we discover that

$$B^{(+)} = \frac{i}{2} A^{(-)} , \quad \text{or} \quad b_n^{(+)} = \frac{i}{2} a_n^{(-)}$$

and

$$\partial^2 \left(D^{(+)} - 2i\,\partial^2 B^{(-)} \right) + \left(m^2 - n^2/x^2 \right) D^{(+)} = 0$$

$$\partial^2 \left(\lambda^{(+)} + \not\partial \psi^{(-)} \right) + \left(m^2 - n^2/x^2 \right) \lambda^{(+)} = 0$$

as the final inhomogeneous set, giving $D^{(+)}$ and $\lambda^{(+)}$ in terms of $B^{(-)}$ and $\psi^{(-)}$.

In the final analysis then we see that it is the odd parity wave-functions which represent the only bound systems for $p=0$ and that these correspond to a 16-fold complete tensor superfield $\Phi(-)$, made up of a pseudoscalar superfield and an axial vector superfield, to which are associated internal $O(4)$ excitation quantum numbers n, ℓ, m providing of course that the coupling constant equals the value $g = 2\pi n$. The result is rather pleasing as it represents the appropriate relativistic and supersymmetric generalization of the lowest ${}^1S_0$ and ${}^3S_1$ states that can be obtained from a fermion-antifermion system.

6.3 Massive Bound States

Let us now imagine a slow variation of g from the maximal $p = 0$ eigenvalue to its physical value. In the course of this variation we expect to get a splitting of $O(4)$ vectors into $O(3)$ vectors and a splitting between the pseudoscalar and axial vector superstates. To follow algebraically what happens, we need to extend our special representations to the massive case, which we can indeed do, by positing $S \to (\not{\gamma}\xi - \frac{1}{2}\not{p}\not{\gamma}^{-1}\underline{\partial})$. We could then treat the $p \neq 0$ case as a perturbation of the $p=0$ case, making an expansion in powers of p, retaining the lowest order, and looking for example at the equations for the δF and $\delta V_\mu^\perp$ components of the wavefunctions, which belong uniquely to the scalar and vector supermultiplets, respectively. By this means the ℓ degeneracy is removed.

However, we already know from Chap. 3 the correct representations to choose in the massive case. It is therefore not appropriate to extend the special representation of the last section. Our effort to understand the level splitting should rather be devoted to reworking the $p=0$ case using the null representations outlined in the last section. For the present, the subject must be left as an area for future investigation.

It should be pointed out here that our study of the homogeneous Bethe-Salpeter equation is rather limited, for two very good reasons. Firstly, we have taken a very special model where the kernel is described by massless supersinglet exchange; the results could look very different if we alter the kernel; e.g. if the exchanged particle corresponds to the supergauge field. Secondly, the problem is far from realistic as yet; if by some stretch of the imagination, the physical particles do organize themselves into supermultiplets, the supersymmetry must certainly be badly broken (spontaneously or otherwise) and therefore it is essential to take proper account of the external mass-breaking in the Bethe-Salpeter equation. In spite of these criticisms, our calculations have value in so far as they give a proper count of the bound states to be expected in problems of this type, and in any case they represent the first step of a more exact treatment.

APPENDICES

A1 Notation

Conventions

Three-vectors are covariant, $\underline{a} = (a_i)$, $i = 1,2,3$, unless otherwise specified. Four-vectors are written in component form $a_\mu = (a_0, a_i)$, $\mu = 0,1,2,3$. We use a Lorentz metric $\eta_{\mu\nu} = (+\,-\,-\,-)$, and $\epsilon_{0123} = +1$. Thus $a^2 = a^\mu a_\mu = a_0^2 - \underline{a}\cdot\underline{a}$.

Pauli Matrices $\sigma_\mu = (\sigma_0, \underline{\sigma})$

$$(\sigma_0)_{a\dot{b}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (\sigma_i)_{a\dot{b}} = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$$

$$\varepsilon = i\sigma_2 = -\varepsilon^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$(\bar{\sigma}_\mu)_{\dot{a}b} = (\varepsilon^{-1}\sigma_\mu^t\,\varepsilon)_{\dot{a}b} \tag{A1.1}$$

$$\mathrm{Trace}\,(\sigma_\mu\bar{\sigma}_\nu) = 2\eta_{\mu\nu} \tag{A1.2}$$

$$\sigma_\mu\bar{\sigma}_\nu = \eta_{\mu\nu} - \tfrac{i}{2}\epsilon_{\mu\nu\rho\sigma}\,\sigma^\rho\bar{\sigma}^\sigma \tag{A1.3}$$

$$(\sigma_\mu)_{a\dot{b}}(\sigma^\mu)_{c\dot{d}} = 2\left((\sigma_0)_{a\dot{b}}(\sigma_0)_{c\dot{d}} - (\sigma_0)_{a\dot{d}}(\sigma_0)_{c\dot{b}}\right) \tag{A1.4}$$

$$\underline{a}\cdot\underline{\sigma}\;\underline{\sigma} = \underline{a} - i\underline{a}\times\underline{\sigma}$$

$$\underline{a}\times\underline{\sigma}\cdot\underline{\sigma} = 2i\,\underline{a}\cdot\underline{\sigma}$$

$$(\underline{a}\times\underline{\sigma})\cdot(\underline{b}\times\underline{\sigma}) = 2\underline{a}\cdot\underline{b} + i\,\underline{a}\times\underline{b}\cdot\underline{\sigma} \tag{A1.5}$$

γ-Matrices

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \qquad \sigma_{\mu\nu} = \tfrac{i}{2}[\gamma_\mu, \gamma_\nu] \qquad \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$$

$$\epsilon_{\mu\nu\rho\sigma}\,\sigma^{\rho\sigma} = 2\sigma_{\mu\nu}\gamma_5 \tag{A1.6}$$

$$\epsilon_{\mu\nu\rho\sigma}\,\gamma^\sigma = i\gamma_5\gamma_\mu\sigma_{\nu\rho} \tag{A1.7}$$

Dirac-Pauli Basis:

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \underline{\gamma} = \begin{pmatrix} 0 & \underline{\sigma} \\ -\underline{\sigma} & 0 \end{pmatrix}, \quad i\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Weyl Basis:

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \underline{\gamma} = \begin{pmatrix} 0 & \underline{\sigma} \\ -\underline{\sigma} & 0 \end{pmatrix}, \quad i\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

4-Spinors

Adjoint $\bar{S}^{\alpha} = (A^{-1})^{\alpha\beta} S_{\beta}^{\dagger}$

Conjugate $\bar{S}^{c\,\alpha} = (C^{-1})^{\alpha\beta} S_{\beta}$

Chiral Parts $(S_{\pm})_{\alpha} = (\Gamma_{\pm})_{\alpha}{}^{\beta} S_{\beta} = \frac{1}{2}(1 \pm i\gamma_5)_{\alpha}{}^{\beta} S_{\beta}$ (A1.8)

$$C^{-1} = -C = C^{t} \equiv i\gamma_0\gamma_2$$

$$A = A^{-1} = A^{t} \equiv \gamma_0$$

A2 Supersymmetry Algebra and Identities

Dirac-Pauli Basis:

Majorana Constraint: $S = \begin{pmatrix} S_a \\ \varepsilon_{ab} S_b^{\dagger} \end{pmatrix}, \quad \bar{S} = (S_a^{\dagger}, S_b \varepsilon_{ba})$ (A2.1)

Chiral Parts: $(S_{\pm})_a = \frac{1}{2}(S_a \pm \varepsilon_{ab} S_b^{\dagger}), \; (S_{\pm})_a^{\dagger} = \pm \varepsilon_{ab} (S_{\mp})_b$ (A2.2)

Anticommutation Relations:

$$\{ (S_{\pm})_a (S_{\pm})_b \} = 0, \quad \{ (S_+)_a (S_-)_b \} = \frac{1}{2}(p\cdot\bar{\sigma}\varepsilon)_{ab} \tag{A2.3}$$

Parity: $U_P S_{\alpha} U_P^{-1} = (i\gamma_0 S)_{\alpha}, \quad U_P (S_{\pm})_a U_P^{-1} = i(S_{\mp})_a$ (A2.4)

Weyl Basis:

Majorana Constraint: $S = \begin{pmatrix} S_a \\ \bar{S}_{\dot{a}} \end{pmatrix}, \quad \bar{S}_{\dot{a}} = \varepsilon_{\dot{a}\dot{b}} S_{\dot{b}}^{\dagger}$ (A2.5)

Anticommutation Relations:

$$\{ S_a, S_b \} = 0 = \{ \bar{S}_{\dot{a}}, \bar{S}_{\dot{b}} \}$$

$$\{ \bar{S}_{\dot{a}}, S_b \} = (p\cdot\bar{\sigma}\varepsilon)_{\dot{a}b} \quad \{ S_a, \bar{S}_{\dot{b}} \} = -(p\cdot\sigma\varepsilon)_{a\dot{b}} \tag{A2.6}$$

Parity: $U_P S_a U_P^{-1} = i \bar{S}_{\dot{a}}$ (A2.7)

Commutation Relations and Shift Operators[18]:

$$
\begin{aligned}
[W_\mu, W_\nu] &= i\,\epsilon_{\mu\nu\rho\sigma}\, P^\rho W^\sigma \\
[\Sigma_\mu, W_\nu] &= i\,\epsilon_{\mu\nu\rho\sigma}\, P^\rho \Sigma^\sigma \\
[\Sigma_\mu, \Sigma_\nu] &= i\,\epsilon_{\mu\nu\rho\sigma}\, P^\rho \Sigma^\sigma \\
\{\Sigma_\mu, \Sigma_\nu\} &= \tfrac{1}{2}\eta_{\mu\nu}(\Sigma^2 - \tfrac{1}{4}P^2) + \tfrac{1}{2}P_\mu P_\nu
\end{aligned}
\qquad \text{(A2.8)}
$$

$$
\begin{aligned}
[S_\alpha, W_\mu] &= \tfrac{1}{2}(\sigma_{\mu\nu}P^\nu)\gamma_5 S_\alpha \\
[S_\alpha, \Lambda] &= \tfrac{1}{2}P_0^{-1}(\sigma_{0\nu}P^\nu)\gamma_5 S_\alpha \\
[S_\alpha, \Sigma_\mu] &= -\tfrac{i}{2}(\not{P}\gamma_\mu\gamma_5 S)_\alpha
\end{aligned}
\qquad \text{(A2.9)}
$$

$$
\begin{aligned}
[P\cdot\Sigma, S_\pm] &= \pm\tfrac{1}{2}P^2 S_\pm \\
[P\cdot\Sigma, \bar{S}S_\pm] &= \pm P^2\, \bar{S}S_\pm
\end{aligned}
\qquad \text{(A2.10)}
$$

Polynomial Identies[18]:

$$
\begin{aligned}
(\bar{S}S_\pm)^3 &= 4P^2\,\bar{S}S_\pm \\
(P\cdot\Sigma)^3 &= \tfrac{1}{4}(P^2)^2\, P\cdot\Sigma \\
\Sigma\cdot\Sigma &= 4P^{-2}(P\cdot\Sigma)^2 - \tfrac{3}{4}P^2 \\
(P\cdot\Sigma)^2 &= \tfrac{1}{16}P^2\,\bar{S}S^2
\end{aligned}
\qquad \text{(A2.11)}
$$

$$
\begin{aligned}
(P\cdot\Sigma)(\Sigma\cdot W) &= 0 = (\Sigma\cdot W)(P\cdot\Sigma) \\
(\Sigma\cdot W)^2 &= \tfrac{1}{4}W^2(\Sigma^2 - \tfrac{1}{4}P^2) - \tfrac{1}{2}P^2(\Sigma\cdot W)
\end{aligned}
\qquad \text{(A2.12)}
$$

A3 Spin and Superhelicity Bases:

Normalized Dirac Wavefunctions

$$(\not{p} - m)\,u_\kappa(p) = 0 = \bar{u}_\kappa(p)(\not{p} - m)$$

$$\tfrac{1}{2}(\hat{p}\cdot\underline{\sigma})\,u_\kappa(p) = \kappa\, u_\kappa(p)$$

$$\bar{u}_\kappa(p)\,\bar{u}_{\kappa'}(p) = 2m\,\delta_{\kappa,\kappa'}$$

$$\bar{u}_\kappa \gamma_\mu u_{\kappa'} = 2p_\mu\,\delta_{\kappa,\kappa'}$$

$$\Sigma_\kappa\, u_\kappa(p)\,\bar{u}_\kappa(p) = (\not{p} + m)$$

$$C\bar{u}_\kappa = u^c_\kappa = (-1)^{-\kappa}\gamma_5 u_{-\kappa} \qquad \text{(A3.1)}$$

Algebra of Shift Operators:

$$R^\pm_\kappa(p) = (-1)^\kappa\,\bar{u}_{-\kappa}(p)\,S_\pm$$

$$(R^\pm_\kappa)^\dagger = \mp 2\kappa\,R^\mp_{-\kappa} \qquad \text{(A3.2)}$$

$$R^\pm_\kappa R^\pm_{\kappa'} = \mp \kappa m\,\delta_{\kappa,-\kappa'}\,\bar{S}S_\pm$$

$$\{R^+_\kappa\, R^-_{\kappa'}\} = -2\kappa m^2\,\delta_{\kappa,-\kappa'}$$

$$(\bar{S}S_\pm)^2 = 0$$

$$\bar{S}S_\pm\,\bar{S}S_\mp = 4P^2\,(P^{-2}P\cdot\Sigma)(2(P^{-2}P\cdot\Sigma) \pm 1)$$

$$\bar{S}S_\pm\,P\cdot\Sigma = -\tfrac{1}{2}P^2\,\bar{S}S_\pm = -P\cdot\Sigma\,\bar{S}S_\pm \qquad \text{(A3.3)}$$

Relation Between S_α and $R^\pm_\kappa$:

Rest Frame $\quad p^\mu = m(1, 0, 0, 0)$:

$$(S_\pm)_1 = \frac{i}{\sqrt{2m}}\,R^\pm_{-\frac{1}{2}} \qquad (S_\pm)_2 = -\frac{i}{\sqrt{2m}}\,R^\pm_{\frac{1}{2}} \qquad \text{(A3.4)}$$

Boosted Frame $\quad p^\mu = m(\cosh\zeta, 0, 0, \sinh\zeta)$:

$$(S_\pm)_1 = \frac{i}{\sqrt{2m}}\,e^{\mp\frac{\zeta}{2}}\,R^\pm_{-\frac{1}{2}} \qquad (S_\pm)_2 = -\frac{i}{\sqrt{2m}}\,e^{\pm\frac{\zeta}{2}}\,R^\pm_{\frac{1}{2}}$$

$$S_1 = \frac{i}{\sqrt{2}\,m}\left(e^{-\frac{\gamma}{2}} R^{+}_{-\frac{1}{2}} + e^{\frac{\gamma}{2}} R^{-}_{-\frac{1}{2}}\right)$$

$$S_2 = -\frac{i}{\sqrt{2}\,m}\left(e^{\frac{\gamma}{2}} R^{+}_{\frac{1}{2}} + e^{-\frac{\gamma}{2}} R^{-}_{\frac{1}{2}}\right) \qquad \text{(A3.5)}$$

Normalized Spin-Basis Vectors:

$$|+\tfrac{1}{2}\ j_0\ \lambda\rangle = \frac{1}{2m}\,\bar{S} S_{+}\,|-\tfrac{1}{2}\ j_0\ \lambda\rangle$$

$$|\ 0\ j_0+\tfrac{1}{2}\ \lambda\rangle = \sum_{\kappa}\left(\frac{j_0+2\kappa\lambda+\frac{1}{2}}{2j_0+1}\right)^{\frac{1}{2}} \frac{R^{+}_{\kappa}}{m}\,|-\tfrac{1}{2}\ j_0\ \lambda-\kappa\rangle$$

$$|\ 0\ j_0-\tfrac{1}{2}\ \lambda\rangle = \sum_{\kappa} 2\kappa\left(\frac{j_0-2\kappa\lambda+\frac{1}{2}}{2j_0+1}\right)^{\frac{1}{2}} \frac{R^{+}_{\kappa}}{m}\,|-\tfrac{1}{2}\ j_0\ \lambda-\kappa\rangle \qquad \text{(A3.6)}$$

$\{+'+\}^{j_0}_{\kappa'\kappa}(\theta)$ Functions, and Symmetry Properties:

$$C(\kappa) = \left(\frac{j_0+\kappa+\frac{1}{2}}{2j_0+1}\right)^{\frac{1}{2}} = S(-\kappa) \qquad \text{(A3.7)}$$

$$\{\dot{+}\dot{+}\}^{j_0}_{\kappa'\kappa}(\theta) = d^{j_0}_{\kappa'\kappa}(\theta) = \{\underset{\cdot}{+}\underset{\cdot}{+}\}^{j_0}_{\kappa'\kappa}(\theta)$$

$$\{{\cdot}{+}{\cdot}{+}\}^{j_0}_{\kappa'\kappa}(\theta) = S(\kappa'-\tfrac{1}{2})\,S(\kappa-\tfrac{1}{2})\,d^{j_0+\frac{1}{2}}_{\kappa'-\frac{1}{2}\,\kappa-\frac{1}{2}}(\theta) + C(\kappa'-\tfrac{1}{2})\,C(\kappa-\tfrac{1}{2})\,d^{j_0-\frac{1}{2}}_{\kappa'-\frac{1}{2}\,\kappa-\frac{1}{2}}$$

$$\{{\cdot}{+}{+}{\cdot}\}^{j_0}_{\kappa'\kappa}(\theta) = -S(\kappa'-\tfrac{1}{2})\,C(\kappa+\tfrac{1}{2})\,d^{j_0+\frac{1}{2}}_{\kappa'-\frac{1}{2}\,\kappa+\frac{1}{2}}(\theta) + C(\kappa'-\tfrac{1}{2})\,S(\kappa+\tfrac{1}{2})\,d^{j_0-\frac{1}{2}}_{\kappa'-\frac{1}{2}\,\kappa+\frac{1}{2}}$$

$$\{{+}{\cdot}{\cdot}{+}\}^{j_0}_{\kappa'\kappa}(\theta) = -C(\kappa'+\tfrac{1}{2})\,S(\kappa-\tfrac{1}{2})\,d^{j_0+\frac{1}{2}}_{\kappa'+\frac{1}{2}\,\kappa-\frac{1}{2}}(\theta) + S(\kappa'+\tfrac{1}{2})\,C(\kappa-\tfrac{1}{2})\,d^{j_0-\frac{1}{2}}_{\kappa'+\frac{1}{2}\,\kappa-\frac{1}{2}}$$

$$\{{+}{\cdot}{+}{\cdot}\}^{j_0}_{\kappa'\kappa}(\theta) = C(\kappa'+\tfrac{1}{2})\,C(\kappa+\tfrac{1}{2})\,d^{j_0+\frac{1}{2}}_{\kappa'+\frac{1}{2}\,\kappa+\frac{1}{2}}(\theta) + S(\kappa'+\tfrac{1}{2})\,S(\kappa+\tfrac{1}{2})\,d^{j_0-\frac{1}{2}}_{\kappa'+\frac{1}{2}\,\kappa+\frac{1}{2}}$$

$$\{++\}^{j_0}_{\kappa'\kappa}(\theta) = (-1)^{\kappa'-\kappa}\,\{{+}{\cdot}{+}{\cdot}\}^{j_0}_{-\kappa'\,-\kappa}(\theta)$$

$$\{{\cdot}{+}{+}{\cdot}\}^{j_0}_{\kappa'\kappa}(\theta) = -(-1)^{j_0-\kappa'}\,\{{+}{\cdot}{+}{\cdot}\}^{j_0}_{-\kappa'\,\kappa}(\theta-\pi)$$

$$\{{+}{\cdot}{\cdot}{+}\}^{j_0}_{\kappa'\kappa}(\theta) = (-1)^{j_0+\kappa}\,\{{+}{\cdot}{+}{\cdot}\}^{j_0}_{\kappa'\,-\kappa}(\theta-\pi) \qquad \text{(A3.8)}$$

A4 Algebra of the Direct Product

Definitions of Labels

$$Q(m_1^2, m_2^2, M^2) = [(M^2 - (m_1+m_2)^2)(M^2 - (m_1-m_2)^2)]^{\frac{1}{2}} \tag{A4.1}$$

$$S_\alpha = S_\alpha^1 + S_\alpha^2 \qquad \tilde{S}_\alpha = (P^1)^{-1} S_\alpha^1 - (P^2)^{-1} S_\alpha^2$$

$$P_\mu = P_\mu^1 + P_\mu^2 \qquad \tilde{P}_\mu = (P^1)^{-2} P_\mu^1 + (P^2)^{-2} P_\mu^2 \tag{A4.2}$$

$$(K^\perp)^2 = -P^2 j_0 (j_0 + 1) \qquad (K^{1\perp})^2 = -P^{12} j_{01}(j_{01}+1)$$

$$W^2 = -P^2 j(j+1) \qquad (W^1)^2 = -P^{12} j_1 (j_1 + 1)$$

$$\lambda = |\underline{P}|^{-1} W_0 \qquad \lambda^1 = \frac{2}{Q} P\cdot W^1$$

$$\kappa = |\underline{P}|^{-1} K_0^\perp \qquad \kappa^1 = \frac{2}{Q} P\cdot K^{1\perp}$$

$$\sigma = P^{-2} P\cdot\Sigma \qquad \sigma^1 = (P^1)^{-2} P^1\cdot\Sigma^1$$

$$\tilde{\sigma} = \tilde{P}^{-2}\tilde{P}\cdot\tilde{\Sigma} = \sigma - \sigma^1 - \sigma^2$$

$$\tilde{\delta} = \frac{2}{Q}\frac{m_1^2 m_2^2}{m_2^2 - m_1^2} P\cdot\tilde{\Sigma}^\perp \quad (= \delta^1 - \delta^2 - \delta)$$

$$\tilde{\lambda} = \frac{2}{Q}\frac{m_1^2 m_2^2}{m_2^2 - m_1^2} P\cdot\tilde{W} = \lambda \tag{A4.3}$$

$$\tilde{\delta} = \frac{2}{\cosh(\zeta_1-\zeta_2)}\Big[\frac{2}{\tilde{M}}\Big(\frac{\cosh\zeta_1}{m_1}\delta^1 - \frac{\cosh\zeta_2}{m_2}\delta^2\Big) - \sinh\zeta_1 \sinh\zeta_2 (\delta^1 - \delta^2 - 2\delta)$$
$$- \big(\sinh\zeta_1\cosh\zeta_2\,\sigma^2 - \sinh\zeta_2\cosh\zeta_1\,\sigma^1 + \sinh(\zeta_1-\zeta_2)\,\tilde{\sigma}\big)\Big] \tag{A4.4}$$

Total and Relative Supertranslation:

$$S_\alpha = S_\alpha^1 + S_\alpha^2, \qquad \tilde{S}_\alpha = (P^1)^{-1} S_\alpha^1 - (P^2)^{-1} S_\alpha^2$$

$$\tilde{P}_\mu = (P^1)^{-2} P^1_\mu + (P^2)^{-2} P^2_\mu \,, \qquad (\tilde{P})^2 = \tilde{M}^2 = \left(\frac{M}{m_1 m_2}\right)^2 \tag{A4.5}$$

$$(S_+)_1 = \frac{i}{\sqrt{2m_1}}\, e^{-\frac{1}{2}\tau_1} R^{1\,+}_{\;-} - \frac{1}{\sqrt{2m_2}}\, e^{\frac{1}{2}\tau_2} R^{2\,+}_{\;+}$$

$$(S_+)_2 = -\frac{i}{\sqrt{2m_1}}\, e^{\frac{1}{2}\tau_1} R^{1\,+}_{\;+} + \frac{1}{\sqrt{2m_2}}\, e^{-\frac{1}{2}\tau_2} R^{2\,+}_{\;-}$$

$$(S_-)_1 = \frac{i}{\sqrt{2m_1}}\, e^{\frac{1}{2}\tau_1} R^{1\,-}_{\;-} - \frac{1}{\sqrt{2m_2}}\, e^{-\frac{1}{2}\tau_2} R^{2\,-}_{\;+}$$

$$(S_-)_2 = -\frac{i}{\sqrt{2m_1}}\, e^{-\frac{1}{2}\tau_1} R^{1\,-}_{\;+} + \frac{1}{\sqrt{2m_2}}\, e^{\frac{1}{2}\tau_2} R^{2\,-}_{\;-} \tag{A4.6}$$

$$(\tilde{S}_+) = \left(\frac{\cosh\tau_1}{m_1}(S^1_+) - \frac{\cosh\tau_2}{m_2}(S^2_+)\right) - \sigma_3\left(\frac{\sinh\tau_1}{m_1}(S^1_-) + \frac{\sinh\tau_2}{m_2}(S^2_-)\right)$$

$$(\tilde{S}_-) = \sigma_3\left(\frac{\sinh\tau_1}{m_1}(S^1_+) + \frac{\sinh\tau_2}{m_2}(S^2_+)\right) + \left(\frac{\cosh\tau_1}{m_1}(S^1_-) - \frac{\cosh\tau_2}{m_2}(S^2_-)\right) \tag{A4.7}$$

	j_0'	j'	σ'	λ'	κ'	j_0	j	σ	λ	κ	$\sigma^1+\sigma^2$	$\tilde{\sigma}$	$\delta^1-\delta^2$	$\tilde{\delta}$
j_0'														
j'					X	X		X		X		X	X	
σ'						X		X		X		X		
λ'						X		X		X		X		
κ'														
j_0														
j										X				
σ														
λ														
κ														
$\sigma^1+\sigma^2$														
$\tilde{\sigma}$														
$\delta^1-\delta^2$														
$\tilde{\delta}$														

X [ROW, COL] $\neq 0$

Table A4.1 Commutation properties of labels in the direct product algebra

Evaluations of some Labels ($R'^{\pm}_{-} \equiv 1^{\pm}_{-}$, etc.):

$$\frac{\Sigma_0^1}{m_1} = \frac{1}{2m_1^2}\left(e^{-\tau_1}\, 1^{\pm}_{-} 1^{-}_{+} - e^{\tau_1}\, 1^{\pm}_{+} 1^{-}_{-}\right) - \frac{1}{2}\cosh\tau_1 \qquad (1 \leftrightarrow 2)$$

$$\frac{\Sigma_3^1}{m_1} = \frac{1}{2m_1^2}\left(e^{-\tau_1}\, 1^{\pm}_{-} 1^{-}_{+} + e^{\tau_1}\, 1^{\pm}_{+} 1^{-}_{-}\right) + \frac{1}{2}\sinh\tau_1 \qquad (1 \leftrightarrow -2)$$

$$\frac{\tilde{\Sigma}_3}{\tilde{M}} = \frac{1}{\tilde{M}}\left((\tilde{S}_+)_1 (\tilde{S}_-)_2 + (\tilde{S}_+)_2 (\tilde{S}_-)_1 - \frac{1}{2}\tilde{P}_3 \right)$$

$$\sigma = P^{-2}\, P\cdot\Sigma = \frac{m_1}{M}\frac{\Sigma_0^1}{m_1} + \frac{m_2}{M}\frac{\Sigma_0^2}{m_2} + \frac{i}{2}\frac{\sqrt{m_1 m_2}}{M}\, X^{12}$$

$$X^{12} = \frac{1}{m_1 m_2}\left[e^{-\frac{1}{2}(1-2)} 1^{\pm}_{-} 2^{-}_{-} - e^{-\frac{1}{2}(1-2)} 1^{-}_{+} 2^{+}_{+} - e^{\frac{1}{2}(1-2)} 1^{\pm}_{+} 2^{-}_{+} + e^{\frac{1}{2}(1-2)} 1^{-}_{-} 2^{\pm}_{-} \right]$$

$$k = -\frac{K_3^1}{M} = J_3 + \frac{m_1}{M}\frac{\Sigma_3^1}{m_1} + \frac{m_2}{M}\frac{\Sigma_3^2}{m_2} + \frac{i}{2}\frac{\sqrt{m_1 m_2}}{M}\, Y^{12}$$

$$Y^{12} = \frac{1}{m_1 m_2}\left[e^{-\frac{1}{2}(1-2)} 1^{\pm}_{-} 2^{-}_{-} - e^{-\frac{1}{2}(1-2)} 1^{-}_{+} 2^{+}_{+} + e^{\frac{1}{2}(1-2)} 1^{\pm}_{+} 2^{-}_{+} - e^{\frac{1}{2}(1-2)} 1^{-}_{-} 2^{\pm}_{-} \right]$$

$$\tilde{\delta} = \frac{1}{\cosh(\tau_1 - \tau_2)}\left[\frac{\tilde{\Sigma}_3}{\tilde{M}} + \tilde{\sigma}\sinh(\tau_1 - \tau_2) \right]$$

(A4.8)

A5 Clebsch-Gordan Coefficients $\{+_1 +_2 \mid +\tilde{+}\}^{j_0}(\theta)$:

Notation:

$$\cosh\tau_1 = \frac{M^2 + m_1^2 - m_2^2}{2Mm_1}$$

$$m_1 \sinh\tau_1 = \frac{Q(m_1^2, m_2^2, M^2)}{2M} = m_2 \sinh\tau_2 \qquad \text{(A5.1)}$$

$$\rho = \frac{m_1}{m_2}\, e^{-(\tau_1 + \tau_2)} = \frac{1}{2m_2^2}\left(M^2 - m_1^2 - m_2^2 - Q\right)$$

(A5.2)

$$C(\kappa) = \left[\frac{j_0 + \kappa + \frac{1}{2}}{2j_0 + 1}\right]^{\frac{1}{2}} = S(-\kappa)$$

$$N_{\pm}(\kappa) = C(\kappa \pm \tfrac{1}{2}) \pm S(\kappa \pm \tfrac{1}{2})$$

$$D_{\pm\pm}(1,2) = (N_{\pm}(1) \pm \rho\, N_{\mp}(2))^{\frac{1}{2}}$$

$$D_{\pm\mp}(1,2) = (N_{\pm}(1) \mp \rho\, N_{\mp}(2))^{\frac{1}{2}} \qquad \text{(A5.3)}$$

$$\langle +' | e^{i\theta J_2} | + \rangle = \{+'+\}^{j_0}_{\kappa'\kappa}(\theta) \qquad \text{(A5.4)}$$

Combinations of Poincaré group Clebsch-Gordan Coefficients:

$$\langle \dot{+}\dot{+} | \Gamma_{j_0} | \dot{+}\dot{+} \rangle = \delta_{\lambda,\kappa_1-\kappa_2}(2j_0+1) = \langle \underset{\cdot}{+}\underset{\cdot}{+} | \Gamma_{j_0} | \underset{\cdot}{+}\underset{\cdot}{+} \rangle$$

$$\langle \dot{+}\,{\cdot}\!+ | \Gamma_{j_0} | \dot{+}\,{\cdot}\!+ \rangle = \delta_{\lambda,\kappa_1-\kappa_2+\frac{1}{2}}(2j_0+1)\, N_-(\kappa_2-\tfrac{1}{2})$$

$$\langle +\!{\cdot}\,\dot{+} | \Gamma_{j_0} | +\!{\cdot}\,\dot{+} \rangle = \delta_{\lambda,\kappa_1-\kappa_2+\frac{1}{2}}(2j_0+1)\, N_+(\kappa_1+\tfrac{1}{2})$$

$$\langle {\cdot}\!+\,\underset{\cdot}{+} | \Gamma_{j_0} | {\cdot}\!+\,\underset{\cdot}{+} \rangle = \delta_{\lambda,\kappa_1-\kappa_2-\frac{1}{2}}(2j_0+1)\, N_-(\kappa_1-\tfrac{1}{2})$$

$$\langle \underset{\cdot}{+}\,+\!{\cdot} | \Gamma_{j_0} | \underset{\cdot}{+}\,+\!{\cdot} \rangle = \delta_{\lambda,\kappa_1-\kappa_2-\frac{1}{2}}(2j_0+1)\, N_+(\kappa_2+\tfrac{1}{2}) \qquad \text{(A5.5)}$$

A6 Helicity Projectors for Spin ½ and Spin 1

Spin ½ ($p^2 = 0$)

$$(J_{\mu\nu})_\alpha{}^\beta = \tfrac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta$$

$$(\Sigma^{\pm})_\alpha{}^\beta = \tfrac{1}{2}(1 \mp \hat{\underline{p}}\cdot\underline{\sigma})_\alpha{}^\beta, \quad \hat{\underline{p}} = |\underline{p}|^{-1}\underline{p} \qquad \text{(A6.1)}$$

Spin 1

$$(J_{\mu\nu})_\rho{}^\sigma = i(\delta_\mu{}^\sigma \eta_{\nu\rho} - \delta_\nu{}^\sigma \eta_{\mu\rho})$$

$$\Pi^0 = (1-\Lambda^2), \quad \Pi^{\pm} = \tfrac{1}{2}\Lambda(\Lambda \pm 1)$$

$$\underline{\Pi}^{\circ}(A) = \hat{\underline{p}}\,\hat{\underline{p}}\cdot\underline{A} \quad , \quad \underline{\Pi}^{\pm}(A) = \underline{A} - \hat{\underline{p}}\,\hat{\underline{p}}\cdot\underline{A} \pm i\,\hat{\underline{p}}\times\underline{A} \tag{A6.2}$$

Algebraic Identities:

$$\underline{\sigma}^{\ell}\,\hat{\underline{p}}\cdot\underline{\sigma} = \ell\,\underline{\sigma}^{\ell} = -\hat{\underline{p}}\cdot\underline{\sigma}\;\underline{\sigma}^{\ell}$$

$$\underline{\gamma}^{\ell}\,\not{p} = 2\ell\,|\underline{p}|\,\underline{\sigma}^{\ell}\,\Gamma_{-\ell} = -\not{p}\,\underline{\gamma}^{\ell}$$

$$\underline{\gamma}^{\ell}\,\Sigma^{\kappa} = \tfrac{1}{2}(1-\kappa\ell)\,\underline{\gamma}^{\ell}$$

$$\Sigma^{\kappa}\,\underline{\gamma}^{\ell} = \tfrac{1}{2}(1+\kappa\ell)\,\underline{\gamma}^{\ell}$$

$$\not{p}\,\Sigma^{\kappa} = 2|\underline{p}|\,\gamma_0\,\Gamma_{\kappa}\,\Sigma^{\kappa} = \Sigma^{\kappa}\,\not{p}$$

$$\Sigma^{\kappa}\,\hat{\underline{p}}\cdot\underline{\sigma} = -\kappa\,\Sigma^{\kappa} = \hat{\underline{p}}\cdot\underline{\sigma}\;\Sigma^{\kappa}$$

$$\not{A}^{\ell} = -\underline{A}^{\ell}\cdot\underline{\gamma}^{-\ell}$$

$$\underline{\sigma}^{\ell}\cdot\underline{\sigma}^{m} = 2\left(1 - \ell m - (\ell - m)\,\hat{\underline{p}}\cdot\underline{\sigma}\right) \tag{A6.3}$$

A7 Covariant Derivatives of the Bethe-Salpeter Wavefunctions

It was established in the text that

$$\Psi(1,2) = e^{\frac{i}{2}\left((x_1+x_2)p + \bar{\theta}_1\not{p}\,\theta_2 + (\bar{\theta}_1+\bar{\theta}_2)S\right)}\,W$$

and

$$D_{1,2}\,\Psi = e^{\frac{i}{2}(\;)}\left(\pm i\left(\partial + \tfrac{1}{8}\not{p}\,\theta\right) - \tfrac{1}{2}\left(S - \not{\partial}\theta\right)\right)W$$

$$\bar{D}_{1,2}\,\Psi = e^{\frac{i}{2}(\;)}\left(\mp i\left(\bar{\partial} + \tfrac{1}{8}\bar{\theta}\not{p}\right) - \tfrac{1}{2}\left(\bar{S} + \bar{\theta}\not{\partial}\right)\right)W$$

If we substitute the solutions of the auxiliary conditions

$$W_{\pm\pm} = e^{\mp\frac{1}{16}i\,\bar{\theta}\not{p}\gamma_5\theta}\left(\bar{S} + \bar{\theta}\not{\partial}\right)\left(S - \not{\partial}\theta\right)_{\mp}\mathcal{W}_{\pm\pm}$$

$$W_{\pm\mp} = e^{\mp\left(\frac{1}{4}\bar{\theta}\not{\partial}\gamma_5\theta + \frac{1}{2}\bar{\theta}\gamma_5 S\right)}\mathcal{W}_{\pm\mp}$$

using such formulae as

$$S\, e^{-\frac{1}{2}i\bar{\theta}\gamma_5 S} = e^{-\frac{1}{2}\bar{\theta}\gamma_5 S}\left(S - \tfrac{1}{2}\not{p}\gamma_5\theta\right)$$

$$\underline{\partial}\, e^{-\frac{1}{2}\bar{\theta}\gamma_5 S} = e^{-\frac{1}{2}\bar{\theta}\gamma_5 S}\left(-\tfrac{1}{2}\gamma_5 S + \tfrac{1}{8}\not{p}\theta\right)$$

we can pass the derivatives through all the exponents to get

$$D_{1,2}\Psi_{\pm\pm} = e^{\frac{i}{2}(\cdot\cdot \mp \frac{1}{8}\bar{\theta}\not{p}\gamma_5\theta)}\left(\pm i(\underline{\partial} + \tfrac{1}{4}\not{p}\theta) - \tfrac{1}{2}(S - \not{\partial}\theta)\right)_{\pm}(\bar{S}+\bar{\theta}\not{\partial}).(S-\not{\partial}\theta)_{\mp}\mathcal{W}_{\pm\pm}$$

$$\bar{D}_{1,2}\Psi_{\pm\pm} = e^{\frac{i}{2}(\cdot\cdot \mp \frac{1}{8}\bar{\theta}\not{p}\gamma_5\theta)}\left(\mp i(\bar{\partial} + \tfrac{1}{4}\bar{\theta}\not{p}) - \tfrac{1}{2}(\bar{S} + \bar{\theta}\not{\partial})\right)_{\pm}(\bar{S}+\bar{\theta}\not{\partial}).(S-\not{\partial}\theta)_{\mp}\mathcal{W}_{\pm\pm}$$

$$D_{1,2}\Psi_{\pm\mp} = e^{\frac{i}{2}(\cdot\cdot \pm i\bar{\theta}\gamma_5 S)}\left(\pm i\underline{\partial} + (\pm\tfrac{1}{2}i\not{p}\theta - S + \not{\partial}\theta)_{\pm}\right)\mathcal{W}_{\pm\mp}$$

$$\bar{D}_{1,2}\Psi_{\pm\mp} = e^{\frac{i}{2}(\cdot\cdot \pm i\bar{\theta}\gamma_5 S)}\left(\mp i\bar{\partial} + (\mp\tfrac{1}{2}i\bar{\theta}\not{p} - \bar{S} - \bar{\theta}\not{\partial})_{\pm}\right)\mathcal{W}_{\pm\mp}$$

We are now in a position to work out the action of the $\bar{D}D_{1,2}$; thus

$$\bar{D}D_1\Psi_{+-} = e^{\frac{i}{2}(\cdot\cdot)}\left(\tfrac{i}{2}\bar{\theta}\not{p} + \bar{S} + \bar{\theta}\not{\partial}\right)\left(-\tfrac{i}{2}\not{p}\theta + S - \not{\partial}\theta\right)_{+}\mathcal{W}_{+-}$$

Since the right-hand side can be compared with Ψ_{--}, the following identity will prove of assistance:

$$e^{-\frac{i}{4}\bar{\theta}\not{\partial}\gamma_5\theta - \frac{1}{2}\bar{\theta}\gamma_5 S}\left(\tfrac{i}{2}\bar{\theta}\not{p} + \bar{S} + \bar{\theta}\not{\partial}\right)\left(-\tfrac{i}{2}\not{p}\theta + S - \not{\partial}\theta\right)_{+} =$$

$$= e^{-\frac{i}{4}\bar{\theta}\not{\partial}\gamma_5\theta}\left(\bar{S} + \bar{\theta}\not{\partial}\right)\left(S - \not{\partial}\theta\right)_{+} e^{-\frac{1}{2}\bar{\theta}\gamma_5 S}$$

$$= \left(\bar{S} + \bar{\theta}\not{\partial}\right)\left(S - \not{\partial}\theta\right)_{+} e^{-\frac{i}{4}i\bar{\theta}S_{+}} e^{-\frac{1}{2}\bar{\theta}\gamma_5 S}$$

$$= \left(\bar{S} + \bar{\theta}\not{\partial}\right)\left(S - \not{\partial}\theta\right)_{+} e^{-\frac{i}{4}i\bar{\theta}\not{\partial}\Gamma_{+}\not{\partial}^{-1}S} e^{\frac{i}{16}[\bar{\theta}\Gamma_{+}S,\, \bar{S}\gamma_5\theta]}$$

$$= e^{\frac{i}{16}\bar{\theta}\not{p}\gamma_5\theta}\left(\bar{S} + \bar{\theta}\not{\partial}\right)\left(S - \not{\partial}\theta\right)_{+} E_{+}(S,\partial) \tag{A7.1}$$

where E_{+} has been defined in Eq. (6.13). Thus

$$\bar{D}D_1\Psi_{+-} = e^{\frac{i}{2}(\cdot\cdot)} e^{\frac{i}{16}\bar{\theta}\not{p}\gamma_5\theta}\left(\bar{S} + \bar{\theta}\not{\partial}\right)\left(S - \not{\partial}\theta\right)_{+} E_{+}(S,\partial)\,\mathcal{W}_{+-}$$

We know furthermore that

$$\bar{D}D_1^2\,\bar{\Psi}_{+-} = -4\partial_1^2\,\bar{\Psi}_{+-} = e^{\frac{i}{2}(\cdot\cdot)}(p-2i\partial)^2 e^{-\frac{1}{4}\bar{\theta}\not\partial\gamma_5\theta - \frac{1}{2}\bar{\theta}\gamma_5 S}\,\mathcal{W}_{+-}$$

By comparing the right-hand side with explicit calculation, we deduce the second identity

$$(-i(\bar{\partial}+\tfrac{1}{4}\bar{\theta}\not p) - \tfrac{1}{2}(\bar{S}+\bar{\theta}\not\partial))(i(\underline{\partial}+\tfrac{1}{4}\not p\theta) - \tfrac{1}{2}(S-\not\partial\theta))_-(\bar{S}+\bar{\theta}\not\partial)(S-\not\partial\theta)_+$$

$$= e^{-\frac{i}{16}\bar{\theta}\not p\gamma_5\theta}(p-2i\partial)^2 e^{-\frac{1}{4}\bar{\theta}\not\partial\gamma_5\theta - \frac{1}{2}\bar{\theta}\gamma_5 S}\,E_+^{-1}(S,\partial) \qquad \text{(A7.2)}$$

and other identities which may be obtained either by reversing γ_5 or by making the change of variable $\partial \to -\partial$, $\theta \to -\theta$. We easily deduce that

$$(\bar{D}D_2)(\bar{D}D_1)\,\bar{\Psi}_{+-} = e^{\frac{i}{2}(\cdot\cdot)}\,e^{\frac{i}{16}\bar{\theta}\not p\gamma_5\theta}\,(i(\bar{\partial}+\tfrac{1}{4}\bar{\theta}\not p) - \tfrac{1}{2}(\bar{S}+\bar{\theta}\not\partial))\times$$

$$\times(-i(\underline{\partial}+\tfrac{1}{4}\not p\theta) - \tfrac{1}{2}(S-\not\partial\theta))(\bar{S}+\bar{\theta}\not\partial)(S-\not\partial\theta)_+\,E_+(S,\partial)\,\mathcal{W}_{+-}$$

$$= e^{\frac{i}{2}(\cdot\cdot)}(p+2i\partial)^2\,e^{\frac{1}{4}\bar{\theta}\not\partial\gamma_5\theta + \frac{1}{2}\bar{\theta}\gamma_5 S}\,E_+^{-1}(S,-\partial)\,E_+(S,\partial)\,\mathcal{W}_{+-}$$

In similar vein we discover the companion formulae

$$\bar{D}D_1\,\bar{\Psi}_{++} = e^{\frac{i}{2}(\cdot\cdot)}(p-2i\partial)^2\,e^{\frac{1}{4}\bar{\theta}\not\partial\gamma_5\theta + \frac{1}{2}\bar{\theta}\gamma_5 S}\,E_-^{-1}(S,\partial)\,\mathcal{W}_{++}$$

$$\bar{D}D_2\,\bar{D}D_1\,\bar{\Psi}_{++} = e^{\frac{i}{2}(\cdot\cdot)}\,e^{\frac{i}{16}\bar{\theta}\not p\gamma_5\theta}(p-2i\partial)^2(\bar{S}+\bar{\theta}\partial)(S-\not\partial\theta)_+\times$$

$$\times\,E_+(S,-\partial)\,E_-^{-1}(S,\partial)\,\mathcal{W}_{++}$$

Finally, if we make use of the supersymmetry algebra[18], one can establish that

$$E_\pm^{-1}(S,\partial) = (\partial+\tfrac{i}{2}p)^{-2}\,\partial^2\,E_\mp(S,\partial) \qquad \text{(A7.3)}$$

and that the $E_\pm$ commute. Putting all these things together, we arrive at Eq. (6.12).

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