INTERFACE WAVES IN ANISOTROPIC MEDIA

by

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ABSTRACT

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The propagation of waves at bi-crystalline interfaces is investigated in this thesis.

The media on both sides of the interface are of the same crystalline material but differently oriented with respect to the interface axes.

The known welded boundary conditions for the propagation of generalized Stoneley waves in simple elastic media, are simplified for certain configurations with different transformations of principal crystalline axes from one medium to the other. The general forms of the displacement and stress vectors for possible interface waves are shown for each of these configurations. Under some transformations it is proved that no generalized Stoneley waves can travel. Additional information is obtained when the media involved are invariant under the transformations discussed.

The equations for interface waves in piezoelectric media are developed. Two different electric boundary conditions are investigated - that of welded half-spaces in the absence and in the presence of a grounded, infinitesimally thin, perfectly conducting electrode at the interface. The derived conditions are then simplified for different symmetric configurations for any media, and in particular for media having one of the symmetries examined within themselves.

Some numerical results are obtained for simple elastic configurations and compared with known results.

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1. INTRODUCTION.

The investigation of interface waves in anisotropic media is based on developments in elastic theory from the beginning of the 19th century up to today. Although there was an extensive interest in elastic phenomena since the 17th century (e.g. Galileo Galilei, Discorse e Dimonstrazioni matematiche, Leiden,(1638), R. Hooke, De Potentia restitutiva, London,(1678), and many others) it was limited to particular problems of vibrations of bars and plates and stability of columns.

Some of the results of early mathematicians were general, like Hooke's Law, but none of those scientists tried to obtain a set of equations describing elastic phenomena in general. The first attempt at a general theory of elasticity was made by Navier [Paris, Mem. Acad. Sciences, t.7 (1827), read May 1821]. He obtained equations of motion which, due to oversimplifications, were dependent on one elastic constant only.

Cauchy [Excercices de Mathematique, 1827 & 1828] who introduced the concepts of stress and strain tensors, arrived at the isotropic equations as we now accept them (dependent on two elastic constants) and at a later date he obtained equations for anisotropic media as well.

Poisson [Paris, Mem. de l'Acad. t.1 (1831)] showed that the solution of the equations for isotropy lead to two body waves which are, as Stokes pointed out [Phil.Soc.Trans. Vol.9 (1849)] longitudinal and transverse.

It was only natural that once the differential equations were established, solutions for various boundary value problems were sought. Navier, after obtaining his equations, derived boundary conditions that hold at a plane surface.

When the equations were corrected various boundary conditions were investigated. Lord Rayleigh [1885] investigated the problem of a wave propagating at a plane surface of an isotropic, homogeneous half-space. He imposed the condition that the waves would leave the surface stress free and their amplitudes decay exponentially with increasing distance from the free surface. He found that such waves exist (Rayleigh waves) and their velocity is less than that of the transverse body wave velocity. These waves are longitudinal in character and their displacement is elliptic.

Little has been added to Lord Rayleigh's analysis of these surface waves, but Love [1911] showed that transverse surface waves can propagate on a free surface of an infinite 'superficial' layer which has a plane interface with an infinite half-space. These waves are known as Love waves.

Stoneley [1924] published a result of a study of elastic waves at an interface between two isotropic half-spaces. He showed that under certain restrictions on the relationship between the elastic constants and densities of the two media on the two sides of the interface, there is a wave travelling with a velocity which is between Rayleigh wave velocity and the transverse wave velocity, with energy flow which is parallel to the interface. In later studies this wave was referred to as Stoneley wave.

Because of the importance of these waves to geophysics Stoneley waves were further investigated by Sezawa K. & Kanai K. [Bull. Earth Res. Inst. Tokyo U. 17, 1 (1939)] and Scholte J.G. [1947] who investigated the range of existence of Stoneley waves, and Owen [1964] searched many combinations of media for the existence of these waves and found it possible in very few combinations of media.

The equations for anisotropic elastic media were obtained by Cauchy at about the same time as the equations for isotropy. Cauchy's assumption of central force law lead to totally symmetric elastic stiffnesses ($c_{ijkl} = c_{ikjl} = c_{iljk}$). These relations, known as Cauchy relations, reduce the number of independent elastic constants from 21 to 15. This last fact, and the method of approach he used were disputed by his contemporaries. Green introduced the strain energy function [Cambridge Phil. Soc. Trans., vol.7 1839)] from which he deduced the equations for an aeolotropic medium dependent on 21 elastic constants. Lord Kelvin [Quart. J. of Math., 5, (1855)] supported Green's results and based his arguments on the first and second laws of thermodynamics.

This was not generally accepted until it was proved experimentally by Voigt [Ann. Phys. Chem (Wiedemann) Bde. 31 (1887) & 34 & 35 (1888), 38 (1889)]. By measuring the torsion and flexure of prisms of several crystals he showed that Cauchy relations do not hold in many cases.

Cauchy [Excercices de Mathematique, (1830)] and Green [Cambridge Phil. Soc. Trans. 7, (1839)] discussed the propagation of plane waves in aeolotropic media and obtained the equations for the wave velocity in terms of the direction of wave front, and showed that the wave front consists of a three sheeted closed surface.

Christoffel [Ann. di Mat. 8, 193 (1877)] and later Lord Kelvin [1904] introduced convenient notations and summed up the equations governing the propagation of elastic waves in anisotropic media but obtained no solutions. Indeed, the computational complexity of these equations was for many years an obstacle in the way of obtaining any additional results. With the advancement of technology, the introduction of computers and the apparent need for more results, mathematicians involved themselves with problems of wave propagation in aeolotropic media.

Synge [1957] and Musgrave [1954a] discussed the relation between slowness surface, velocity surface and wave surface. Later analytic and computational solutions were given for the different symmetries, e.g. Hexagonal (Musgrave [1954b]), cubic (Miller & Musgrave [1956]) and trigonal (Farnell [1961])*.

Once solutions were given for infinite media simple boundary value problems were posed, such that would lead to generalized Rayleigh, Love and Stoneley waves in anisotropy. Synge [1956] discussed surface waves in anisotropic media and conjectured that Rayleigh waves may travel only in discrete directions in anisotropic media. This was disproved by Stroh [1962], and later, independently, by Currie [1974] (see discussion at the end of chapter 2). Stoneley [1955] & [1963] and Buchwald [1961] discussed the possibility of propagation of Rayleigh waves in different directions of cubic, hexagonal and orthorhombic media. Lim & Farnell [1968] and Lim [1968] calculated Rayleigh wave velocities in various materials and directions.

* Buchwald [1959] and Duff [1960] employed Fourier integrals for the study of wave propagation in anisotropic media.

Stroh [1962] showed that when the Lagrangian, $\mathcal{L}$, of a uniformly moving straight dislocation vanishes, the velocity of the dislocation is the same as the Rayleigh velocity. His approach was further developed by Barnett et. al [1973] and Barnett & Lothe [1974], to give an integral method of calculating the Rayleigh velocity and to prove that there exists only one Rayleigh velocity in a range of velocities which can lead to an attenuating wave.

Love waves in anisotropic media were investigated by Stroh [1962], who sketched the conditions for their existence, and Stoneley [1955] & [1963] who gave the conditions for the existence of Love type waves in cubic and orthorhombic media and showed that non-dispersive Love waves can propagate only in discrete directions.

Stroh [1962] also formulated the conditions for the existence of Stoneley waves in anisotropic media. No solutions were given by Stroh to any of the conditions of existence of Rayleigh, Love or Stoneley waves. Chadwick & Currie [1974] simplified the conditions for existence of generalized Stoneley waves and showed that if there is a direction of existence there is a neighbourhood of that direction where generalized Stoneley waves exist.
Johnson [1970] showed the possibility of existence of generalized Stoneley waves at interfaces between media of similar crystallographic structure but different density and elastic stiffnesses, and examined the range of existence in configurations where the crystallographic axes in the two half-spaces had the same orientation with respect to the interface axes. Lim & Musgrave [1970a] & [1970b] have investigated the propagation of generalized Stoneley waves at interfaces between two cubic media having the same elastic constants and density but different orientation of the crystal axes with respect of the interface axes.

In this summary a general formulation of the problem of plane interface waves at a bicrystalline interface is given in chapter 2. In chapter 3 we investigate generalized Stoneley waves at interfaces where the crystalline media may be of any symmetry but are of the same material and only different in orientation with respect of the interface. In particular the relationship between the different physical characteristics of the wave in the two half-spaces is obtained when the transformation of axes from one half-space to another is that of 2-fold rotation and/or inversion with respect to one of the main interface axes. Some of these conditions were assumed by Lim & Musgrave [1970b] and here they are derived.

In chapter 4 the generalized Stoneley conditions are simplified in cases where the crystalline media are of a particular symmetry. For each of the conditions obtained the characteristics of the possible waves are investigated.

Bleustein [1968] showed the existence of a new type of transverse surface waves in piezoelectric materials. These waves depend on the piezoelectric character of the media and cannot be found in simple elastic materials. These waves are different from waves investigated in piezoelectric media, as modifications of known surface waves (Farnell [1970] and Campbell & Jones [1968]) by direct approach or by use of 'stiffened' elastic constants. These constants are modifications of the simple elastic constants which account for the piezoelectricity without calculating the electric effect.

Using a technique described by Chadwick & Currie [1974] an analysis of waves at interfaces between two piezoelectric media is made in chapter5. Chapter 6 deals with cases where the piezoelectric crystalline media involved are different only in orientation with respect of the interface axes, with emphasis on media of particular symmetries.

The numerical program used in the calculations is described in chapter 7 and the special difficulties arising in the process are explained. Numerical results are given in chapter 8 for cubic and orthorhombic symmetries.

In addition to the referred material, the historical background was obtained from Love [1934], Rayleigh [1945], Sokolnikoff [1956] and Musgrave [1970].

2. THE BASIC EQUATIONS FOR GENERALIZED STONELEY WAVES.

In order to arrive at the equations for generalized Stoneley waves, we shall first consider the propagation of a plane wave in an anisotropic medium with stresses which obey a generalized Hooke's Law, with c_{ijkl}, the elastic stiffnesses. The displacement of such a plane wave can be described by:

$$u_k = A\, p_k \exp[i\omega(s_j x_j - t)] \tag{2-1}$$

s_j being the slowness components in the direction x_j, A the amplitude and p_k the component of the displacement vector in the k direction, $(k,j=1,2,3)$, ω the frequency and t the time. Summation convention is used whenever repeated indices are in lower case letters.

The linear strains are defined as:

$$e_{k\ell} = \tfrac{1}{2}(u_{k,\ell}+u_{\ell,k}) \tag{2-2}$$

and the stress-strain relation described by a generalized Hooke's Law is:

$$\sigma_{ij} = c_{ijk\ell}e_{k\ell} \tag{2-3}$$

$c_{ijk\ell}$ is the elastic stiffnesses tensor obeying the following restrictions:

$$c_{ijk\ell} = c_{ij\ell k} = c_{ji\ell k} = c_{k\ell ij} \tag{2-4-a}$$

and

$$c_{ijk\ell}a_i a_k b_j b_\ell > 0 \text{ for all } a_i \;\&\; b_i \text{ s.t. } \|a_i\|>0 \;\&\; \|b_i\|>0 \tag{2-4-b}$$

The equation of motion in the absence of body forces is given by:

$$\rho\ddot{u}_i = \sigma_{ij,j} \tag{2-5}$$

($\cdot$ represents differentiation with respect to time, ρ the density).

Upon substitution of (2-3) in (2-5) and using the definition of the linear strains (2-2) and the symmetry of the elastic stiffnesses (2-4-a) one arrives at the equation:

$$c_{ijk\ell}u_{k,\ell j} = \rho\ddot{u}_i \tag{2-6}$$

Substitution of the expression for the plane wave (2-1) into the equation of motion (2-6) yields:

$$(c_{ijk\ell}s_j s_\ell - \rho\delta_{ik})p_k = 0 \qquad (2\text{-}7)$$

For non-trivial values of p_k one has the restriction:

$$\|c_{ijk\ell}s_j s_\ell - \rho\delta_{ik}\| = 0 \qquad (2\text{-}8)$$

which is the equation for the Slowness Surface (Musgrave [1970] and others), a three sheeted closed centrosymmetric surface of sixth degree.

One should note that $c_{ijk\ell}$ are usually quoted with respect to axes of crystal symmetry, and in general, use of the transformation law for fourth order tensors is necessary to obtain the stiffness appropriate to arbitrarily chosen reference axes.

Consider now an interface problem, in which space is divided into two by the plane $x_3=0$. We denote the medium which occupies $x_3>0$ by I, and $x_3<0$ by II. (All quantities referred to in medium I or II, will be denoted by I or II, respectively).

We shall choose the direction x_1 as the wave normal, i.e. $x_1 = 0$ is the plane of the wave, so that $x_2 = 0$ will be typical of all planes $x_2 = \text{const.}$, and will be termed the sagittal plane.

Plane waves in medium I will be of the form:

$$u_k(\mathrm{I}) = A(\mathrm{I})\, p_k(\mathrm{I}) \exp\{i\omega[s_1(\mathrm{I})x_1 + s_3(\mathrm{I})x_3 - t]\} \qquad x_3 \geq 0 \qquad (2\text{-}9\text{-}a)$$

and in medium II:

$$u_k(\mathrm{II}) = A(\mathrm{II})\, p_k(\mathrm{II}) \exp\{i\omega[s_1(\mathrm{II})x_1 + s_3(\mathrm{II})x_3 - t]\} \qquad x_3 \leq 0 \qquad (2\text{-}9\text{-}b)$$

We seek waves such that the velocity of propagation along the interface is common to the two half-spaces, therefore s_1, which describes the slowness parallel to the interface, must be the same in both media:

$$s_1(\mathrm{I}) = s_1(\mathrm{II}) = s_1 \qquad (2\text{-}10)$$

where s_1 is real. Complex s_1 will lead to either amplification or attenuation in the direction of propagation, which is not possible in a non-dissipative medium.

In each medium, (2-8) must hold (for the medium), for non-trivial $p_k(n)$, n=I,II:

$$\|c_{ijk\ell}(n)s_j(n)s_\ell(n) - \rho(n)\delta_{ik}\| = 0 \qquad (2\text{-}11)$$

Where $c_{ijk\ell}(n)$ is referred to the common set of interface coordinates. In our configuration, for each chosen value of s_1 one obtains a sextic equation (with real coefficients) in $s_3(n)$, $(n=I,II)$.

Equation (2-7) becomes:

$$\{c_{\ell 1k1}(n)s_1^2 + [c_{\ell 1k3}(n)+c_{\ell 3k1}(n)]s_1 s_3(n)+c_{\ell 3k3}(n)s_3^2(n)-\rho(n)\delta_{\ell k}\}\cdot p_k(n)=0 \qquad \text{(2-12-a)}$$

For non-trivial solution $p_k(n)$, one obtains the determinantal equation:

$$\|c_{\ell 1k1}(n)s_1^2+[c_{\ell 1k3}(n)+c_{\ell 3k1}(n)]s_1 s_3(n)+c_{\ell 3k3}(n)s_3^2(n)-\rho(n)\delta_{\ell k}\| = 0 \qquad \text{(2-12-b)}$$

Equation (2-12-b) gives two sextic equations with real coefficients, hence for each medium there are six solutions $s_3^M(I)$ or $s_3^M(II)$, which can all be real or may include pairs of complex conjugates for each medium.

Requiring that the plane wave forms an interface wave, localized to the interface, means that the displacement should attenuate with increasing distance from the plane $x_3 = 0$. Such attenuation can be obtained, in this formulation, by using in medium I the roots with positive imaginary part, and in medium II roots with negative imaginary part, so that when $|x_3|\to\infty$ the displacement tends to zero in both media. Hence, except at the interface, where we have not posed our requirements yet, the following compound wave, involving acceptable $s_3(n)$, will satisfy the requirements for an interface wave:

$$u_k(n) = \sum_{N=1}^{3} A^{(N)}(n)p_k^{(N)}(n)\ \exp\{i\omega[s_1x_1 + s_3^{(N)}(n)x_3-t]\} \qquad \text{(2-13)}$$

where $n=I,II$, $\mathscr{I}\{s_3^{(N)}(I)\}>0$, $\mathscr{I}\{s_3^{(N)}(II)\}<0$.

By substituting these results in (2-2) and (2-3), one obtains the stress vectors on a plane parallel to the interface:

$$\sigma_{3k} = i\omega\sum_{N=1}^{3}[c_{3kj1}(n)s_1 + c_{3kj3}(n)\ s_3^{(N)}(n)]A^{(N)}(n)\ p_j^{(N)}(n)\ \cdot$$

$$\cdot\exp\{i\omega[s_1x_1 + s_3^{(N)}(n)x_3 - t]\} \qquad \text{(2-14)}$$

Setting:

$$q_k^{(N)}(n) = [c_{3k1j}(n)s_1 + c_{3k3j}(n)\ s_3^{(N)}(n)]\ p_j^{(N)}(n) \qquad (2\text{-}15)$$

we may write the stress vector on a plane parallel to the interface:

$$\sigma_{3k}(n) = i\omega \sum_{N=1}^{3} A^{(N)}(n)\ q_k^{(N)}(n)\ \exp\{i\omega[s_1x_1+s_3^{(N)}(n)x_3-t]\} \qquad (2\text{-}16)$$

The welded interface requirements of a generalized Stoneley wave are that there is continuity of displacement and of stress across the interface, which means:

$$u_k(\mathrm{I})\Big|_{x_3=0} = u_k(\mathrm{II})\Big|_{x_3=0} \qquad k = 1,2,3 \qquad (2\text{-}17\text{-}a)$$

$$\sigma_{3k}(\mathrm{I})\Big|_{x_3=0} = \sigma_{3k}(\mathrm{II})\Big|_{x_3=0} \qquad k = 1,2,3 \qquad (2\text{-}17\text{-}b)$$

for all x_1 and t.

(2-17-a) yields, upon substitution of (2-13):

$$\sum_{N=1}^{3}\left[A^{(N)}(\mathrm{I})\ p_k^{(N)}(\mathrm{I}) - A^{(N)}(\mathrm{II})\ p_k^{(N)}(\mathrm{II})\right] = 0 \qquad (2\text{-}18\text{-}a)$$

and (2-17-b) becomes, upon substituting of (2-16):

$$\sum_{N=1}^{3}\left[A^{(N)}(\mathrm{I})\ q_k^{(N)}(\mathrm{I}) - A^{(N)}(\mathrm{II})\ q_k^{(N)}(\mathrm{II})\right] = 0 \qquad (2\text{-}18\text{-}b)$$

One should remember that both $p_k^{(N)}(n)$ and $q_k^{(N)}(n)$ are dependent upon s_1 and $s_3^{(N)}(n)$.

Equations (2-18-a) and (2-18-b) form a set of six linear homogeneous equations for $A^{(N)}(\mathrm{I})$ and $A^{(N)}(\mathrm{II})$ and for non-trivial solutions of $A^{(N)}(n)$ we have the requirement of the determinant of coefficients:

$$\begin{Vmatrix} p_k^{(N)}(\mathrm{I}) & -\ p_k^{(N)}(\mathrm{II}) \\ q_k^{(N)}(\mathrm{I}) & -\ q_k^{(N)}(\mathrm{II}) \end{Vmatrix} = 0 \quad k,N=1,2,3 \qquad (2\text{-}19)$$

which is the equation for the slowness component, s_1, for welded interface.

In the process of obtaining (2-19) we have not guaranteed that the body wave solutions are not included. Indeed, it is quite possible to obtain from (2-19) s_1 such that not all $s_3^{(N)}(n)$ will be complex. Such cases are either body waves which move parallel to the interface and comply with the restrictions of continuity (2-17), or 'leaky' waves, which have non-attenuating components in one medium or both, and carry energy away from the interface.

In order to obtain generalized Stoneley waves one has to further impose the restriction $\mathcal{I}\{s_3^{(N)}(n)\} \neq 0$.*

The 6x6 determinant (2-19) has in general a complex value and therefore one would expect that the vanishing of both the real and imaginary parts simultaneously is needed to obtain s_1. Chadwick & Currie [1974] have shown that the generalized Stoneley condition (2-19) can be reduced, for all cases of true Stoneley waves, i.e. $\mathcal{I}\{s_3^{(N)}(n)\} \neq 0$ (which is the region of interest) into a 3x3 determinant which can be made to be pure imaginary. The reduction is obtained in the following way: Equation (2-18-a) is multiplied by $\overline{q_k^{(M)}(II)}$ and (2-18-b) by $\overline{p_k^{(M)}(II)}$, then in each equation summation over k is carried out and the two equations obtained are added to give:

$$\sum_{N=1}^{3} \{[\overline{q_k^{(M)}(II)}\, p_k^{(N)}(I) + \overline{p_k^{(M)}(II)}\, q_k^{(N)}(I)]A^{(N)}(I) - [\overline{q_k^{(M)}(II)}\, p_k^{(N)}(II) + \overline{p_k^{(M)}(II)}\, q_k^{(N)}(II)]A^{(N)}(II)\} = 0 \quad (2\text{-}20)$$

Stroh [1958] and Currie [1974] have shown that the matrix:

$$D^{MN}(n) = \overline{q_k^{(M)}(n)}\, p_k^{(N)}(n) \quad (2\text{-}21)$$

is skew-Hermitian for the cases $s_3^{(N)}(n) - \overline{s_3^{(M)}(n)} \neq 0$ (2-22)

Since for attenuating interface waves the three $s_3^{(N)}(n)$ taken in one medium have non-zero imaginary part, of the same sign, condition (2-22) prevails and the matrix multiplying $A^{(N)}(II)$ vanishes.

Hence, one can rewrite (2-20) as:

$$\sum_{N=1}^{3} F^{MN}(I)\, A^{(N)}(I) = 0 \quad (2\text{-}23)$$

where $F^{MN}(I) = \overline{q_k^{(M)}(II)}\, p_k^{(N)}(I) + \overline{p_k^{(M)}(II)}\, q_k^{(N)}(I)$ (2-24)

In the same way, by multiplying (2-18-a) by $\overline{q_k^{(M)}(I)}$ and (2-18-b) by

*An attenuating interface wave is also possible if one or two of the slowness components are real, say $s_3^{(M)}$, if the corresponding amplitudes $A^{(M)}$ vanish as well.

$\overline{p_k^{(M)}(I)}$ and using (2-21) for medium I, one arrives at:

$$\sum_{N=1}^{3} F^{MN}(II)\ A^{(N)}(II) = 0 \qquad (2\text{-}25)$$

where $F^{MN}(II) = \overline{q_k^{(M)}(I)}\ p_k^{(N)}(II) + \overline{p_k^{(M)}(I)}\ q_k^{(N)}(II) \qquad (2\text{-}26)$

Comparing (2-24) and (2-26) one obtains the following relationship:

$$\overline{F^{NM}(II)} = F^{MN}(I) \qquad (2\text{-}27)$$

Taking the complex conjugate of (2-25) and substituting (2-27) one can see that for non-trivial solution of both $A^{(N)}(I)$ and $\overline{A^{(N)}(II)}$ one obtains the same condition:

$$\| F^{MN}(I) \| = 0 \qquad (2\text{-}28)$$

(2-28) can be taken as a simplified generalized Stoneley condition. One should remember that in the process of simplifying the Stoneley condition the restriction (2-22) was introduced. However, when we deal with 'leaky' waves (2-22) may not hold and for those cases one has to return to the original condition (2-19).

In their paper [1974] Chadwick & Currie show that $p_k^{(N)}(I)$ and $p_k^{(N)}(II)$ can be related as:

$$p_k^{(N)}(I) = \sum_{M=1}^{3} T^{NM}\ p_k^{(M)}(II) \qquad (2\text{-}29)$$

(since $p_k^{(N)}(I)$ and $p_k^{(N)}(II)$ form, or may be made to form, two bases in C^3), where T^{NM} is a non-singular matrix and by appropriate choice of $p_k^{(N)}(n)$ may be made to have real determinant.

If we substitute (2-29) into (2-18-a), the continuity of displacement equation, one obtains:

$$\sum_{M=1}^{3} p_k^{(M)}(II) \left(\sum_{N=1}^{3} A^{(N)}(I)\ T^{NM} - A^{(M)}(II) \right) = 0 \qquad (2\text{-}30)$$

since $p_k^{(M)}(II)$ is a non-singular matrix, only the trivial solution is possible for (2-30):

$$A^{(M)}(II) = \sum_{N=1}^{3} A^{(N)}(I)\ T^{NM} \qquad (2\text{-}31)$$

One can see that the amplitudes in the two media are related by the transposed transformation matrix which relates the components of the displacement vectors in the two half-spaces.

Upon substitution of (2-31) into the continuity of stress, (2-18-b), one obtains:

$$\sum_{N=1}^{3}\left(q_k^{(N)}(I) - \sum_{M=1}^{3} T^{NM} q_k^{(M)}(II)\right) A^{(N)}(I) = 0 \qquad (2\text{-}32)$$

For non-trivial solution of $A^{(N)}(I)$:

$$\left\| q_k^{(N)}(I) - \sum_{M=1}^{3} T^{NM} q_k^{(M)}(II) \right\| = 0 \qquad (2\text{-}33)$$

(2-33) can be looked upon as another alternative version of the generalized Stoneley condition, but it involves the complication of finding the transformation matrix T^{NM}. In this form one can easily see in the Stoneley condition the generalization of the Rayleigh condition, with $q_k^{(M)}(II) = 0$.

Using (2-33) as a Stoneley condition has the advantage that 'leaky' waves are not excluded, because of the skew-Hermitian character of D^{MN} (or condition (2-22)) has not been taken into consideration.

The matrix, the determinant of which vanishes in (2-33) is related easily to $F^{MN}(I)$ (using (2-24) and the skew-Hermitian properties of (2-21)):

$$F^{MN}(I) = \overline{p_k^{(M)}(II)} \left(q_k^{(N)}(I) - \sum_{L=1}^{3} T^{NL} q_k^{(L)}(II)\right) \qquad (2\text{-}34)$$

since $\overline{p_k^{(M)}(II)}$ is a non-singular matrix, one can see that $F^{MN}(I)$ and $\{q_k^{(N)}(I) - \sum_{L=1}^{3} T^{NL} q_k^{(L)}(II)\}$ are matrices of the same rank.

In their paper [1974] Chadwick and Currie have shown that $T^{MK}F^{KN}(I)$ is a skew-Hermitian matrix, in order to show that the generalized Stoneley condition can be reduced to a single real (or pure imaginary) condition. The reason for the proof is a suggestion made by Synge [1956] that Rayleigh waves would appear in discrete directions because the determinantal equation is equivalent to two separate conditions, one each for the real and pure imaginary parts.

Stroh [1962] disproved Synge's conjecture by proving that the Rayleigh determinantal equation can be made real or pure imaginary. He showed that since $p_k^{(N)}$ and $q_k^{(N)}$ contain an arbitrary complex normalizing factor, by choosing the argument of this factor suitably the dot products which are involved in the Rayleigh determinantal equation may be made real or pure imaginary and therefore the Rayleigh condition is equivalent to a real equation in the wave slowness.

3. SOME SYMMETRIC CASES.

Of special interest in the study of generalized Stoneley waves is the specification of the waves which may be freely propagated at the interface between two crystalline half-spaces of the same material as the orientation of the half spaces is altered.

In this chapter we shall investigate analytically some special cases where one can arrive at simple Stoneley conditions, the meaning of which will be studied.

We shall assume that the material throughout space has elastic stiffnesses with respect to the crystallographic axes $c'_{ijk\ell}$ and density ρ. Each half-space has its crystallographic axes oriented in a known direction so that the elastic stiffnesses, with respect to the interface axes x_j are $c_{ijk\ell}(I)$ for medium I and $c_{ijk\ell}(II)$ for medium II. The crystallographic coordinates for medium I, in the interface coordinate system, $x_j(I)$, are related to the coordinates of medium II, referred to the same system, $x_j(II)$ by:

$$x_i(II) = h_{ij}x_j(I) \tag{3-1}$$

Therefore, the elastic stiffnesses in the two half-spaces are related by:

$$c_{ijk\ell}(II) = h_{ir}h_{js}h_{kt}h_{\ell u}c_{rstu}(I) \tag{3-2}$$

We shall now consider the equations obtained for the general interface problem. (2-12-a) becomes, for medium I:

$$\{c_{i1k1}(I)s_1^2+[c_{i1k3}(I)+c_{i3k1}(I)]s_1s_3(I)+c_{i3k3}(I)s_3^2(I)-\rho\delta_{ik}\}p_k(I) = 0 \tag{3-3-a}$$

and for medium II:

$$\{h_{ir}h_{kt}[h_{1s}h_{1u}s_1^2+(h_{1s}h_{3u}+h_{3s}h_{1u})s_1s_3(II)+h_{3s}h_{3u}s_3^2(II)]c_{rstu}(I) - \rho\delta_{ik}\}\ p_k(II) = 0 \tag{3-3-b}$$

The sextic equation (2-12-b) becomes for medium I:

$$\|c_{i1k1}(I)s_1^2+[c_{i1k3}(I)+c_{i3k1}(I)]s_1s_3(I)+c_{i3k3}(I)s_3^2(I)-\rho\delta_{ik}\| = 0 \tag{3-4-a}$$

and for medium II: (3-4-b)

$$\| h_{ir}h_{kt}[h_{1s}h_{1u}s_1^2+(h_{1s}h_{3u}+h_{3s}h_{1u})s_1s_3(II)+h_{3s}h_{3u}s_3^2(II)]c_{rstu}(I) - \rho\delta_{ik}\| = 0$$

For a given material, the slowness equation referred to a given set of axes is unique. Although the set of axes to which the slowness equation is referred to in both half-spaces is the same, the crystallographic axes are differently oriented. It is this difference which accounts for the possibility of a different form of the slowness equation in each half space.

In the cases we shall consider h_{ij} was chosen to have the form:

$$h_{ij}=\begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \qquad h_k = \pm 1 \qquad (3\text{-}5)$$

This type of a matrix allows for identity (where all h_i=1), complete inversion (all h_i = -1) and reflection and two-fold rotation about each of the interface axes.

The components of the symmetric determinants in (3-4) are for these cases:

$$\left.\begin{aligned}
S_{11}(I) &= c_{11}s_1^2 + c_{55}s_3^2(I) - \rho + 2c_{15}s_1s_3(I) \\
S_{12}(I) &= c_{16}s_1^2 + c_{45}s_3^2(I) + (c_{14} + c_{56})s_1s_3(I) \\
S_{13}(I) &= c_{15}s_1^2 + c_{35}s_3^2(I) + (c_{13} + c_{55})s_1s_3(I) \\
S_{22}(I) &= c_{66}s_1^2 + c_{44}s_3^2(I) - \rho + 2c_{46}s_1s_3(I) \\
S_{23}(I) &= c_{56}s_1^2 + c_{43}s_3^2(I) + (c_{36} + c_{45})s_1s_3(I) \\
S_{33}(I) &= c_{55}s_1^2 + c_{33}s_3^2(I) - \rho + 2c_{35}s_1s_3(I)
\end{aligned}\right\} \quad (3.6a)$$

The elastic constants of medium I are given in contracted form, c_{mn}, (see e.g. Hearmon [1961]) and are referred to the interface axes. For the second medium the components of the symmetric determinant are:

$$S_{11}(II) = c_{11}s_1^2 + c_{55}s_3^2(II) - \rho + 2h_1h_3c_{15}s_1s_3(II)$$

$$\left.\begin{aligned}
S_{12}(II) &= h_1h_2[c_{16}s_1^2 + c_{45}s_3^2(II)] + h_2h_3(c_{14} + c_{56})s_1s_3(II) \\
S_{13}(II) &= h_1h_3[c_{15}s_1^2 + c_{35}s_3^2(II)] + (c_{13} + c_{55})s_1s_3(II) \\
S_{22}(II) &= c_{66}s_1^2 + c_{44}s_3^2(II) - \rho + 2h_1h_3c_{46}s_1s_3(II) \\
S_{23}(II) &= h_2h_3[c_{56}s_1^2 + c_{43}s_3^2(II)] + h_1h_2(c_{36} + c_{45})s_1s_3(II) \\
S_{33}(II) &= c_{55}s_1^2 + c_{33}s_3^2(II) - \rho + 2h_1h_3c_{35}s_1s_3(II)
\end{aligned}\right) \quad (3\text{-}6\text{-}b)$$

where h_1, h_2 & h_3 form the diagonal of the transformation matrix h_{ij} as in (3-5). The elastic stiffnesses are the same as the ones for the first medium.

Comparing the coefficients of the different powers of $s_3(I)$ and $s_3(II)$ in the two sextic equations (3-4-a) and (3-4-b), with h_{ij} given by (3-5), one finds that the coefficients of the even powers of s_3 are the same in both sextic equations, while the coefficents of odd powers of $s_3(I)$ are multiplied by a factor h_1h_3 to give the coefficients of odd powers of $s_3(II)$. Since this factor is either +1 or -1, one finds that the roots of the two sextic equations are related as:

$$s_3^{(M)}(II) = h_1h_3s_3^{(M)}(I) \qquad M = 1,\ldots,6 \qquad (3\text{-}7\text{-}a)$$

for general sextic equations. If the sextic equations become bi-cubic, the equations for both media are the same, regardless of the value of h_1h_3, and hence:

$$s_3^{(M)}(II) = s_3^{(M)}(I) \qquad M = 1,\ldots,6 \qquad (3\text{-}7\text{-}b)$$

for bi-cubic sextic equations.

Because of the nature of the waves that we are seeking the displacement should decrease with increasing distance from the interface and hence in medium I the imaginary part of s_3 should be positive and in medium II, negative. We therefore obtain the following relationship:

$$s_3^{(N)}(II) = h_1h_3\mathcal{R}\{s_3^{(N)}(I)\} - i\mathcal{I}\{s_3^{(N)}(I)\} \qquad N=1,2,3 \qquad (3\text{-}7\text{-}c)$$

$$\text{with } \mathcal{I}\{s_3^{(N)}(I)\} \geq 0\,^{*}$$

where $\mathcal{R}\{x\}$ is the real part and $\mathcal{I}\{x\}$ is the imaginary part of x.

A sextic equation which is bi-cubic has for its zeros the positive and negative square roots of the zeros of the cubic equation.

* In the appendix to their paper Eshelby et.al. [1953] showed that for sufficiently large s_1 such complex s_3 exist.

Therefore, in general, for such a medium the relation between the true roots in half-space I to those in half-space II may be given by:

$$s_3^{(N)}(II) = - s_3^{(N)}(I) \ , \ N=1,2,3, \quad \mathscr{I}\{s_3^{(N)}(I)\} \geq 0 \qquad (3\text{-}7\text{-}d)$$

regardless of the values of h_i.

When $\mathscr{I}\{s_3^{(N)}(I)\} \neq 0$ one may renumber the roots so that the numbering is consistent with (3-7-c). When $h_1h_3 = -1$, (3-7-d) and (3-7-c) are the same. However, when $\mathscr{I}\{s_3^{(N)}(I)\} = 0$ and $h_1h_3 = +1$, although (3-7-c) may hold, one has to check also the possibility that (3-7-d) holds. If this is the case, it is impossible to use (3-7-c) and one has to treat specifically this case.

In the following discussion we assume that (3-7-c) holds. Since our main interest is in attenuating waves, this assumption is not limiting. At the end of this chapter a short discussion is given about the excluded case.

Substituting (3-7-c) into (3-6-b) one obtains the relationship between the components $S_{KL}^{(N)}$ in the two half spaces:

$$S_{KL}^{(N)}(II) = h_K h_L[\mathscr{R}\{S_{KL}^{(N)}(I)\} - ih_1h_3\mathscr{I}\{S_{KL}^{(N)}(I)\}] \qquad (3\text{-}8)$$

No summation is meant by repeated upper case suffixes .

The ratios of the components $p_k^{(N)}(n)$ are given by:

$$p_1^{(N)}(n):p_2^{(N)}(n):p_3^{(N)}(n) = [S_{K2}^{(N)}(n)S_{L3}^{(N)}(n) - S_{K3}^{(N)}(n)S_{L2}^{(N)}(n)] : \qquad (3\text{-}9)$$

$$[S_{K3}^{(N)}(n)S_{L1}^{(N)}(n) - S_{K1}^{(N)}(n)S_{L3}^{(N)}(n)]:[S_{K1}^{(N)}(n)S_{L2}^{(N)}(n) - S_{K2}^{(N)}(n)S_{L1}^{(N)}(n)]$$

(where K and L are any two different rows), provided $S_{KL}^{(N)}(n)$ is a rank 2 matrix. In the particular cases where $S_{KL}^{(N)}(n)$ is of rank 1, this means that two $s_3^{(N)}(n)$ are equal and therefore one should be careful in selecting $p_k^{(N)}(n)$ in such a way that it is a regular matrix. One such possibility:

$$p_{[N]}^{(N)}(n):p_{[N+1]}^{(N)}(n):p_{[N+2]}^{(N)}(n) = 0: \ S_{K[N+2]}^{(N)}(n) \ : \ S_{K[N+1]}^{(N)}(n) \qquad (3\text{-}10)$$

where $[N+1] = (N \bmod 3)+1$, K is chosen in such a way as to have non-zero vectors $p_k^{(N)}(n)$.

In either case the following relationship is obtained by treating separately $h_1h_3 = -1$ and $h_1h_3=+1$, when (3-8) holds, for the displacement vectors:

$$p_K^{(N)}(II) = \chi^{(N)}h_Kh_3[\mathcal{R}\{p_K^{(N)}(I)\} - ih_1h_3\mathcal{I}\{p_K^{(N)}(I)\}] \qquad (3\text{-}11)$$

From (3-11) one obtains the connection between the stress vectors:

$$q_K^{(N)}(II) = \chi^{(N)}h_Kh_1[\mathcal{R}\{q_K^{(N)}(I)\} - ih_1h_3\mathcal{I}\{q_K^{(N)}(I)\}] \qquad (3\text{-}12)$$

where $\chi^{(N)}$ in (3-11) and (3-12) are arbitrary non-zero constants. Once chosen we have to be consistent.

When $h_1h_3 = -1$ ($h_1 = -h_3=h$, $h = \pm1$) (3-11) and (3-12) may be greatly simplified:

$$p_K^{(N)}(II) = -\chi^{(N)}hh_Kp_K^{(N)}(I) \qquad (3\text{-}13)$$

$$q_K^{(N)}(II) = \chi^{(N)}hh_Kq_K^{(N)}(I) \qquad (3\text{-}14)$$

Using the following algebraic identity:

$$ab - cd = \tfrac{1}{2}(a + c/\alpha)(b - \alpha d) + \tfrac{1}{2}(a - c/\alpha)(b+ \alpha d) \qquad (3\text{-}15)$$

we can rewrite the conditions for generalized Stoneley waves (2-18-a) and (2-18-b):

$$\tfrac{1}{2}\{\sum_{N=1}^{3}[p_k^{(N)}(I) + p_k^{(N)}(II)/\chi^{(N)}][A^{(N)}(I) - \chi^{(N)}A^{(N)}(II)] + \sum_{N=1}^{3}[p_k^{(N)}(I) - p_k^{(N)}(II)/\chi^{(N)}][A^{(N)}(I)+\chi^{(N)}A^{(N)}(II)]\} = 0 \qquad (3\text{-}16\text{-}a)$$

$k = 1,2,3$. Similar equations are obtained for the stress components:

$$\tfrac{1}{2}\{\sum_{N=1}^{3}[q_k^{(N)}(I) + q_k^{(N)}(II)/\chi^{(N)}][A^{(N)}(I) - \chi^{(N)}A^{(N)}(II)] + \sum_{N=1}^{3}[q_k^{(N)}(I) - q_k^{(N)}(II)/\chi^{(N)}][A^{(N)}(I)+\chi^{(N)}A^{(N)}(II)]\} = 0 \qquad (3\text{-}16\text{-}b)$$

Substituting (3-13) and (3-14) into (3-16) one obtains:

$$\sum_{N=1}^{3}\tfrac{1}{2}(1-hh_K)p_K^{(N)}(I)[A^{(N)}(I) - \chi^{(N)}A^{(N)}(II)] + \sum_{N=1}^{3}\tfrac{1}{2}(1+hh_K)p_K^{(N)}(I)[A^{(N)}(I) + \chi^{(N)}A^{(N)}(II)] = 0 \qquad (3\text{-}17\text{-}a)$$

and

$$\sum_{N=1}^{3} \tfrac{1}{2}(1+hh_K)q_K^{(N)}(I)[A^{(N)}(I) - \chi^{(N)}A^{(N)}(II)] + \qquad (3\text{-}17\text{-}b)$$
$$+ \sum_{N=1}^{3} \tfrac{1}{2}(1-hh_K)q_K^{(N)}(I)[A^{(N)}(I) + \chi^{(N)}A^{(N)}(II)] = 0$$

The coefficients $\frac{1}{2}(1-hh_K)$ and $\frac{1}{2}(1+hh_K)$ receive the values of either 0 or 1, when the one is 0 the other is 1. Hence we have two separate sets of three equations each, one for $A^{(N)}(I) - \chi^{(N)}A^{(N)}(II)$ and the other for $A^{(N)}(I) + \chi^{(N)}A^{(N)}(II)$. At least one of these sets has to have a non-trivial solution, otherwise $A^{(N)}(I) = A^{(N)}(II) = 0$, and there is no wave.

The equations are therefore given as:

$$\begin{pmatrix} p_1^{(N)}(I) \\ r_2^{(N)}(I) \\ q_3^{(N)}(I) \end{pmatrix} \cdot \left(A^{(N)}(I) + \chi^{(N)}\ A^{(N)}(II)\right) = 0 \qquad (3\text{-}18\text{-}a)$$

and

$$\begin{pmatrix} q_1^{(N)}(I) \\ t_2^{(N)}(I) \\ p_3^{(N)}(I) \end{pmatrix} \cdot \left(A^{(N)}(I) - \chi^{(N)}A^{(N)}(II)\right) = 0 \qquad (3\text{-}18\text{-}b)$$

where
$$r_2^{(N)}(I) = \begin{cases} p_2^{(N)}(I) & \text{if } h_2 = h \\ q_2^{(N)}(I) & \phantom{\text{if }} h_2 = -h \end{cases} \qquad (3\text{-}18\text{-}c)$$

$$t_2^{(N)}(I) = \begin{cases} q_2^{(N)}(I) & \text{if } h_2 = h \\ p_2^{(N)}(I) & \phantom{\text{if }} h_2 = -h \end{cases} \qquad (3\text{-}18\text{-}d)$$

This leads to three possible conditions:

Either:
$$\left\| \begin{matrix} p_1^{(N)}(I) \\ r_2^{(N)}(I) \\ q_3^{(N)}(I) \end{matrix} \right\| = 0 \quad \text{and} \quad \left\| \begin{matrix} q_1^{(N)}(I) \\ t_2^{(N)}(I) \\ p_3^{(N)}(I) \end{matrix} \right\| \neq 0 \qquad (3\text{-}19\text{-}a)$$

or:
$$\left\| \begin{matrix} p_1^{(N)}(I) \\ r_2^{(N)}(I) \\ q_3^{(N)}(I) \end{matrix} \right\| \neq 0 \quad \text{and} \quad \left\| \begin{matrix} q_1^{(N)}(I) \\ t_2^{(N)}(I) \\ p_3^{(N)}(I) \end{matrix} \right\| = 0 \qquad (3\text{-}19\text{-}b)$$

or both determinants vanish simultaneously:

$$\begin{Vmatrix} p_1^{(N)}(I) \\ r_2^{(N)}(I) \\ q_3^{(N)}(I) \end{Vmatrix} = 0 \quad \text{and} \quad \begin{Vmatrix} q_1^{(N)}(I) \\ t_2^{(N)}(I) \\ p_3^{(N)}(I) \end{Vmatrix} = 0 \qquad (3\text{-}19\text{-}c)$$

If we denote:

$$B_-^{(N)} = A^{(N)}(I) - \chi^{(N)} A^{(N)}(II) \qquad (3\text{-}20\text{-}a)$$

$$B_+^{(N)} = A^{(N)}(I) + \chi^{(N)} A^{(N)}(II) \qquad (3\text{-}20\text{-}b)$$

then

$$A^{(N)}(I) = \tfrac{1}{2}[B_-^{(N)} + B_+^{(N)}] \qquad (3\text{-}21\text{-}a)$$

$$A^{(N)}(II) = \tfrac{1}{2}[B_+^{(N)} - B_-^{(N)}]/\chi^{(N)} \qquad (3\text{-}21\text{-}b)$$

$B_-^{(N)}$ and $B_+^{(N)}$ are the null vectors of the matrices in (3-18).

We define the total displacement components at the interface as:

$$P_k(n) = \sum_{N=1}^{3} p_k^{(N)}(n) A^{(N)}(n) \qquad (3\text{-}22\text{-}a)$$

and the total stress vector components on the interface as:

$$Q_k(n) = \sum_{N=1}^{3} q_k^{(N)}(n) A^{(N)}(n) \qquad (3\text{-}22\text{-}b)$$

(one should remember that the actual stress vector σ_{3k} is given by $\sigma_{3k} = i\omega Q_k$)

The total displacement and stress vectors at the interface in terms of medium II are given by:

$$P_K(II) = -hh_K \sum_{N=1}^{3} p_K^{(N)}(I)\ \tfrac{1}{2}[B_+^{(N)} - B_-^{(N)}] \qquad (3\text{-}23\text{-}a)$$

$$Q_K(II) = hh_K \sum_{N=1}^{3} q_K^{(N)}(I)\ \tfrac{1}{2}[B_+^{(N)} - B_-^{(N)}] \qquad (3\text{-}23\text{-}b)$$

Using (3-18) we can rewrite (3-22-a) and (3-23-a):

$$P_1(n) = \tfrac{1}{2} \sum_{N=1}^{3} p_1^{(N)}(I) B_-^{(N)} \qquad (3\text{-}24\text{-}a)$$

$$P_2(n) = \tfrac{1}{2} \sum_{N=1}^{3} p_2^{(N)}(I) B_-^{(N)} \qquad \text{if } h_2 = h_1 \qquad (3\text{-}24\text{-}b)$$

$$P_2(n) = \tfrac{1}{2}\sum_{N=1}^{3} p_2^{(N)}(I)B_+^{(N)} \qquad \text{if } h_2 = h_3 \tag{3-24-c}$$

$$P_3(n) = \tfrac{1}{2}\sum_{N=1}^{3} p_3^{(N)}(I)B_+^{(N)} \tag{3-24-d}$$

and the total stress vector components at the interface may be rewritten as:

$$Q_1(n) = \tfrac{1}{2}\sum_{N=1}^{3} q_1^{(N)}(I)B_+^{(N)} \tag{3-25-a}$$

$$Q_2(n) = \tfrac{1}{2}\sum_{N=1}^{3} q_2^{(N)}(I)B_+^{(N)} \qquad \text{if } h_2 = h_1 \tag{3-25-b}$$

$$Q_2(n) = \tfrac{1}{2}\sum_{N=1}^{3} q_2^{(N)}(I)B_-^{(N)} \qquad \text{if } h_2 = h_3 \tag{3-25-c}$$

$$Q_3(n) = \tfrac{1}{2}\sum_{N=1}^{3} q_3^{(N)}(I)B_-^{(N)} \tag{3-25-d}$$

If each of the determinants in (3-19) vanishes separately, then two separate waves, propagating at different velocities s_1 will occur: If (3-19-a) holds, $B_-^{(N)} = 0$, and $B_+^{(N)} = 2A^{(N)}(I)$, therefore: $P_1(n) = Q_3(n) = 0$ and $P_2(n)$ vanishes if $h_2=h_1$. If $h_2=h_3$, $Q_2(n) = 0$. Similarly, when (3-19-b) holds, $B_+^{(N)} = 0$ and $B_-^{(N)} = 2A^{(N)}(I)$, which leads to: $Q_1(n)=P_3(n)=R_2(n)=0$ ($R_2(n)=Q_2(n)$ when $h_1h_2=+1$, and $R_2(n)= P_2(n)$ for $h_1h_2=-1$).

From (3-24) and (3-25) and the discussion one can see that the two wave displacements associated with (3-19-a) and (3-19-b) are normal one to the other. One total displacement vector has two non-zero components while the other has only one non-zero component, in the direction in which the first vector has a zero component. The stress vector matching the total displacement vector having two non-zero components is in the direction of the total displacement vector having only one non-zero component. The second stress vector has two non-zero components and is in the same plane with the first total displacement vector.

In the discussion we have not guaranteed that the velocity of these waves would be such that there would be attenuation of displacement and stress with increasing distance from the interface, indeed one or both of the waves may be non-attenuating.

If (3-19-c) holds this means that neither $B_-^{(N)}$ nor $B_+^{(N)}$ are the zero vectors, and therefore the total displacement and stress vectors are given by (3-24) and (3-25) where $B_{-}^{(N)}$ and $B_{+}^{(N)}$ are the null vectors of the two matrices, in (3-18). This means that the matrices in (3-18) are at most of rank 2 each, which leads to the conclusion that for this case the original matrix of the generalized Stoneley condition is at most of rank 4. Therefore, there exist two 6-dimensional null vectors $(A^{(N)}(I), A^{(N)}(II))$ of the generalized Stoneley condition which are linearly independent. For a given slowness s_1 there is only one acceptable set of solutions $s_3^{(N)}(n)$, which lead to one set of displacement components. The total displacement will therefore be a linear combination of the two solutions with each component attenuating at the same rate with increasing distance from the interface.

When $h_1 h_3 = +1$ $(h_1 = h_3 = h,\ h = \pm 1)$ and (3-8-c) holds, (3-11) and (3-12) may be simplified:

$$p_K^{(N)}(II) = \chi^{(N)} h h_K \overline{p_K^{(N)}(I)} \qquad (3\text{-}26\text{-}a)$$

$$q_K^{(N)}(II) = \chi^{(N)} h h_K \overline{q_K^{(N)}(I)} \qquad (3\text{-}26\text{-}b)$$

The fact that the displacement and stress vectors in the second medium are related to the complex conjugate displacement and stress vectors does not enable us to separate, in general, the generalized Stoneley condition into two simple decoupled conditions as in the case $h_1 h_3 = -1$.

The generalized Stoneley condition (2-18-a) and (2-18-b) may be simplified to:

$$\sum_{N=1}^{3} [p_J^{(N)}(I)\ A^{(N)}(I) - h h_J\ \overline{p_J^{(N)}(I)}\ \chi^{(N)} A^{(N)}(II)] = 0 \qquad (3\text{-}27\text{-}a)$$

$$\sum_{N=1}^{3} [q_J^{(N)}(I)\ A^{(N)}(I) - h h_J\ \overline{q_J^{(N)}(I)}\ \chi^{(N)} A^{(N)}(II)] = 0 \qquad (3\text{-}27\text{-}b)$$

(no summation on J)

or: $$\sum_{N=1}^{3} \{\mathcal{R}\{p_j^{(N)}(I)\} B_{-}^{(N)} + i\mathcal{I}\{p_j^{(N)}(I)\} B_{+}^{(N)}\} = 0\ , \quad j=1,3 \qquad (3\text{-}27\text{-}c)$$

$$\sum_{N=1}^{3} \{\mathcal{R}\{q_j^{(N)}(I)\} B_{-}^{(N)} + i\mathcal{I}\{q_j^{(N)}(I)\} B_{+}^{(N)}\} = 0\ , \quad j=1,3 \qquad (3\text{-}27\text{-}d)$$

$$\sum_{N=1}^{3}\left\{r_i^{(N)}B_-^{(N)} + i\, t_i^{(N)}B_+^{(N)}\right\} = 0 \qquad i=1,2 \qquad (3\text{-}27\text{-}e)$$

where:

$$\left.\begin{array}{ll} r_1^{(N)} = \mathcal{R}\{p_2^{(N)}(I)\} & ,\ t_1^{(N)} = \mathcal{I}\{p_2^{(N)}(I)\} \\ r_2^{(N)} = \mathcal{R}\{q_2^{(N)}(I)\} & ,\ t_2^{(N)} = \mathcal{I}\{q_2^{(N)}(I)\} \end{array}\right)\ \text{when } h_2 = h$$

$$\left.\begin{array}{ll} r_1^{(N)} = i\mathcal{I}\{p_2^{(N)}(I)\} & ,\ it_1^{(N)} = \mathcal{R}\{p_2^{(N)}(I)\} \\ r_2^{(N)} = i\mathcal{I}\{q_2^{(N)}(I)\} & ,\ it_2^{(N)} = \mathcal{R}\{q_2^{(N)}(I)\} \end{array}\right)\ \text{when } h_2 = -h$$

This can be put into a matrix form:

$$\begin{pmatrix} \mathcal{R}\{p_j^{(N)}(I)\} & i\mathcal{I}\{p_j^{(N)}(I)\} \\ r_i^{(N)} & it_i^{(N)} \\ \mathcal{R}\{q_j^{(N)}(I)\} & i\mathcal{I}\{q_j^{(N)}(I)\} \end{pmatrix} \cdot \begin{pmatrix} B_-^{(N)} \\ \\ B_+^{(N)} \end{pmatrix} = 0 \qquad (3\text{-}28\text{-}a)$$

$$j = 1,3\ ;\ i=1,2$$

If the determinant of the matrix of coefficients is non-zero $B_-^{(N)} = B_+^{(N)} = 0$, which means that $A^{(N)}(I) = A^{(N)}(II) = 0$. Hence in order to have an interface wave complying with welded conditions at the interface the determinant of the coefficients must vanish:

$$\left\| \begin{matrix} \mathcal{R}\{p_j^{(N)}(I)\} & i\mathcal{I}\{p_j^{(N)}(I)\} \\ r_i^{(N)} & it_i^{(N)} \\ \mathcal{R}\{q_j^{(N)}(I)\} & i\mathcal{I}\{q_j^{(N)}(I)\} \end{matrix} \right\| = 0 \qquad (3\text{-}28\text{-}b)$$

$$j = 1,3\ ;\ i=1,2$$

For the case $h_1 = h_2 = h_3 = h$ (identity or complete inversion) one does not expect to have an attenuating interface wave. One can however, have body waves travelling parallel to the interface, obeying the welded conditions at the interface. This expectation can be proven in the following way:

If one adds to (3-27-a) and (3-27-b) its complex conjugate (and $h_k = h$) one obtains another form of the generalized Stoneley condition:

$$\sum_{N=1}^{3} \mathcal{R}\{p_j^{(N)}(I)[A^{(N)}(I) - \overline{\chi^{(N)}A^{(N)}(II)}]\} = 0 \qquad j=1,2,3 \qquad (3\text{-}29\text{-}a)$$

$$\sum_{N=1}^{3} \mathcal{R}\{q_j^{(N)}(I)[A^{(N)}(I) - \overline{\chi^{(N)}A^{(N)}(II)}]\} = 0 \qquad j=1,2,3 \qquad (3\text{-}29\text{-}b)$$

Using the definition of q_j, (2-15) and (3-29-a) one can rewrite (3-29-b) as:

$$c_{3j3k}\sum_{N=1}^{3} \mathcal{R}\{s_3^{(N)}(I)\; p_j^{(N)}(I)\;[A^{(N)}(I) - \overline{\chi^{(N)}A^{(N)}(II)}]\} = 0 \qquad (3\text{-}29\text{-}c)$$

Since $\|c_{3j3k}\| \neq 0$ one obtains a simplified version of (3-29-b):

$$\sum_{N=1}^{3} \mathcal{R}\{s_3^{(N)}(I)\; p_j^{(N)}(I)\;[A^{(N)}(I) - \overline{\chi^{(N)}A^{(N)}(II)}]\} = 0 \qquad (3\text{-}29\text{-}d)$$

Since $p_j^{(N)}(I)$ are determined up to a multiplying constant, it is possible to find $p_J^{(N)}(I)$ ($J = 1$ or 2 or 3) such that

$$p_J^{(N)}(I)[A^{(N)}(I) - \overline{\chi^{(N)}A^{(N)}(II)}]$$

is pure imaginary of the same sign (say, non-negative) for all N. For, suppose $p_J^{(M)}[A^{(M)}(I) - \overline{\chi^{(M)}A^{(M)}(II)}] = \alpha_1 + i\alpha_2$, where $\alpha_1 \neq 0$. We can multiply $p_j^{(M)}(I)$ (holding M constant and for all values of j) by $\pm(\alpha_2 + i\alpha_1)$, where the sign is determined so that all the resulting products would be of the same sign.

For this chosen J, (3-29-d) can be rewritten as:

$$\sum_{N=1}^{3} \mathcal{I}\{s_3^{(N)}(I)\}\{p_J^{(N)}(I)\;[A^{(N)}(I) - \overline{\chi^{(N)}A^{(N)}(II)}]\} = 0 \qquad (3\text{-}30)$$

This is possible only if all $\mathcal{I}\{s_3^{(N)}(I)\} = 0$, because otherwise we would require the sum of three non-negative numbers to vanish.

When $h_1 = h_3 = h$, and $h_2 = -h$ there are three possibilities of waves:

$$B_-^{(N)} \neq 0 \;,\; B_+^{(N)} \neq 0 \qquad (3\text{-}31\text{-}a)$$

$$B_-^{(N)} \neq 0 \;,\; B_+^{(N)} = 0 \qquad (3\text{-}31\text{-}b)$$

$$B_-^{(N)} = 0 \;,\; B_+^{(N)} \neq 0 \qquad (3\text{-}31\text{-}c)$$

The total displacement at the interface is obtained from (3-26) and (3-27) for all cases $h_1h_3 = +1$:

$$P_1(n) = \tfrac{1}{2}\sum_{N=1}^{3} \mathcal{R}\{p_1^{(N)}(I)\}B_+^{(N)} + i\mathcal{I}\{p_1^{(N)}(I)\}B_-^{(N)} \qquad (3\text{-}32\text{-}a)$$

$$P_2(n) = \tfrac{1}{2}\sum_{N=1}^{3} \mathcal{R}\{p_2^{(N)}(I)\}B_+^{(N)} + i\mathcal{I}\{p_2^{(N)}(I)\}B_-^{(N)} \text{ if } h_2=h \qquad (3\text{-}32\text{-}b)$$

$$P_2(n) = \tfrac{1}{2}\sum_{N=1}^{3} \mathcal{R}\{p_2^{(N)}(I)\}B_-^{(N)} + i\mathcal{I}\{p_2^{(N)}(I)\}B_+^{(N)} \text{ if } h_2=-h \qquad (3\text{-}32\text{-}c)$$

$$P_3(n) = \tfrac{1}{2}\sum_{N=1}^{3} \mathcal{R}\{p_3^{(N)}(I)\}B_+^{(N)} + i\mathcal{I}\{p_3^{(N)}(I)\}B_-^{(N)} \qquad (3\text{-}32\text{-}d)$$

and the total stress vector components are obtained in the same way and follow the same pattern as the total displacement components.

In the case $h_2 = -h$, for (3-31-b) to hold the matrix:

$$\begin{pmatrix} \mathcal{R}\{p_j^{(N)}(I)\} \\ i\mathcal{I}\{p_2^{(N)}(I)\} \\ \mathcal{R}\{q_j^{(N)}(I)\} \\ i\mathcal{I}\{q_2^{(N)}(I)\} \end{pmatrix} \qquad j = 1,3 \tag{3-33-a}$$

is of at most of rank 2.

For (3-31-c) to hold, the matrix

$$\begin{pmatrix} i\mathcal{I}\{p_j^{(N)}(I)\} \\ \mathcal{R}\{p_2^{(N)}(I)\} \\ i\mathcal{I}\{q_j^{(N)}(I)\} \\ \mathcal{R}\{q_2^{(N)}(I)\} \end{pmatrix} \qquad j = 1,3 \tag{3-33-b}$$

is of at most of rank 2.

When either (3-31-b) or (3-31-c) hold the total displacement and total stress components are obtained by substituting $B_+^{(N)}$ or $B_-^{(N)} = 0$ respectively, in (3-32) and the similar set of equations for the total stress components.

Table (3-1) gives a summary of the relationships between the different quantities in the two media for all symmetric configurations. Note that the total displacement and total stress vectors are independent of the choice of $\chi^{(N)}$.

Table (3-2) describes the possible generalized Stoneley waves in the different symmetric configurations discussed in this chapter.

When the medium and the wave slowness give rise to a bi-cubic equation for $s_3^{(N)}(n)$ which has real roots, and $h_1h_3 = 1$, (3-7-c) does not necessarily hold and one has to check the possibility that for a bulk wave moving parallel to the interface the correct relation between the slowness of the wave on the two sides of the interface is given by (3-7-d). Substituting (3-7-d) into (3-6) does not, in general, yield a simple relation between $S_{KL}(I)$ and $S_{KL}(II)$, and it is necessary to know the form of the elastic stiffnesses which causes the sextic equation to degenerate to a bi-cubic.

Even in those cases for which one can simply relate S_{KL} in the two media, it is not always possible to relate simply the displacement and stress components. It is only for very particular cases that a simple relation can be obtained between the displacement and stress components on the two sides of the interface. One of these is the case discussed by Lim and Musgrave [1970a] & [1970b]. It is interesting that for the case they investigated (cubic media) when the transformation matrix was the identity, Lim & Musgrave found a bulk wave which has energy flux parallel to the interface with velocity which is lower than the lowest body wave velocity. This may be explained when one considers the geometry of the slowness surface (see chapter 8).

In chapter 4, treating particular cases, $h_1 h_3 = 1$ is treated, and (3-7-c) is not assumed, therefore the non-attenuating waves are included in the discussion there.

Table (3-1) - A summary of the relationship between the different physical properties in the two media, in the interface coordinate system:

Property compared	medium I	medium II
The elastic stiffnesses $c_{ijk\ell}(n)$	$c_{IJKL}(I)$	$h_I h_J h_K h_L c_{IJKL}(I) = \pm c_{IJKL}(I)$
Slowness components in the x_3 direction $s_3^{(N)}(n)$	$s_3^{(N)}(I)$ with $\mathcal{I}\{s_3^{(N)}(I)\}>0$	$h_1 h_3\ \mathcal{R}\{s_3^{(N)}(I)\} - i\ \mathcal{I}\{s_3^{(N)}(I)\}$
Elements of the secular matrix $S_{KL}^{(N)}(n)$	$S_{KL}^{(N)}(I)$	$h_K\ h_L\ [\mathcal{R}\{S_{KL}^{(N)}(I)\} - i\ h_1 h_3 \mathcal{I}\{S_{KL}^{(N)}(I)\}]$
Displacement components $p_K^{(N)}(n)$	$p_K^{(N)}(I)$	$\chi^{(N)} h_K h_3 [\mathcal{R}\{p_K^{(N)}(I)\} - i h_1 h_3 \mathcal{I}\{p_K^{(N)}(I)\}]$
Stress components $q_K^{(N)}(n)$	$q_K^{(N)}(I)$	$\chi^{(N)} h_K h_1 [\mathcal{R}\{q_K^{(N)}(I)\} - i h_1 h_3 \mathcal{I}\{q_K^{(N)}(I)\}]$
Amplitudes $A^{(N)}(n)$	$\frac{1}{2}(B_-^{(N)} + B_+^{(N)})$	$\frac{1}{2}(B_+^{(N)} - B_-^{(N)})/\chi^{(N)}$
Total displacement at interface $P_K(n)$	$P_K(I) = \sum_{N=1}^{3} p_K^{(N)}(I) A^{(N)}(I)$	$\frac{1}{2} h_K h_3 \sum_{N=1}^{3} [[\mathcal{R}\{p_K^{(N)}(I)\} - i h_1 h_3 \mathcal{I}\{p_K^{(N)}(I)\}] \cdot [B_+^{(N)} - B_-^{(N)}]]$
Total stress vector at interface $Q_K(n)$	$Q_K(I) = \sum_{N=1}^{3} q_K^{(N)}(I) A^{(N)}(I)$	$\frac{1}{2} h_K h_1 \sum_{N=1}^{3} \{[\mathcal{R}\{q_K^{(N)}(I)\} - i h_1 h_3 \mathcal{I}\{q_K^{(N)}(I)\}] \cdot [B_+^{(N)} - B_-^{(N)}]\}$

Table (3-2) - Conditions for possible generalized Stoneley waves in the different symmetric configurations (3-5).

	h_1 h_2 h_3	$B_-^{(N)}=0$ & $B_+^{(N)}\neq 0$	$B_-^{(N)}\neq 0$ & $B_+^{(N)}=0$	$B_+^{(N)}\neq 0$ & $B_-^{(N)}\neq 0$
A	h h h	No attenuating waves are possible		
B	h -h h	rank $\mathcal{M} \le 2$	rank $\mathcal{N} \le 2$	$D_5 = 0$
C	h h -h	$D_1 = 0$ & $D_2 \neq 0$	$D_2 = 0$ & $D_1 \neq 0$	$D_1 = D_2 = 0$
D	h -h -h	$D_3 = 0$ & $D_4 \neq 0$	$D_4 = 0$ & $D_3 \neq 0$	$D_3 = D_4 = 0$

where: $$\mathcal{M} = \begin{pmatrix} i\mathcal{I}\{p_j^{(N)}(I)\} \\ \mathcal{R}\{p_2^{(N)}(I)\} \\ i\mathcal{I}\{q_j^{(N)}(I)\} \\ \mathcal{R}\{q_2^{(N)}(I)\} \end{pmatrix} \quad j=1,3 \qquad \mathcal{N} = \begin{pmatrix} \mathcal{R}\{p_j^{(N)}(I)\} \\ i\mathcal{I}\{p_2^{(N)}(I)\} \\ \mathcal{R}\{q_j^{(N)}(I)\} \\ i\mathcal{I}\{q_2^{(N)}(I)\} \end{pmatrix} \quad j=i,3$$

$$D_1 = \begin{Vmatrix} p_1^{(N)}(I) \\ p_2^{(N)}(I) \\ q_3^{(N)}(I) \end{Vmatrix} \qquad D_2 = \begin{Vmatrix} q_1^{(N)}(I) \\ q_2^{(N)}(I) \\ p_3^{(N)}(I) \end{Vmatrix} \qquad D_3 = \begin{Vmatrix} p_1^{(N)}(I) \\ q_2^{(N)}(I) \\ q_3^{(N)}(I) \end{Vmatrix} \qquad D_4 = \begin{Vmatrix} q_1^{(N)}(I) \\ p_2^{(N)}(I) \\ p_3^{(N)}(I) \end{Vmatrix}$$

$$D_5 = \begin{Vmatrix} \mathcal{R}\{p_j^{(N)}(I)\} & i\mathcal{I}\{p_j^{(N)}(I)\} \\ i\mathcal{I}\{p_2^{(N)}(I)\} & \mathcal{R}\{p_2^{(N)}(I)\} \\ \mathcal{R}\{q_j^{(N)}(I)\} & i\mathcal{I}\{q_j^{(N)}(I)\} \\ i\mathcal{I}\{q_2^{(N)}(I)\} & \mathcal{R}\{q_2^{(N)}(I)\} \end{Vmatrix} \quad j=1,3$$

4. GENERALIZED STONELEY WAVES IN SYMMETRIC CONFIGURATIONS OF DIFFERENT CRYSTALLINE MEDIA.

The discussion in chapter 3 does not take into consideration the symmetries the media may have within themselves. The existence or non-existence of generalized Stoneley waves in symmetric configurations depend only on the elastic stiffnesses in the interface coordinate system and the density (which is the same in both media).

If a given medium is invariant under transformation h_{ij}, although h_{ij} may describe one of the cases B, C, or D (table (3-2)) we are actually dealing with case A. In this case no attenuating waves will propagate at the interface.

Suppose the medium in half-space I has mirror symmetry with respect to the x_3 axis (in the interface coordinate system). Then, if we use the transformation matrix of case C to obtain the elastic stiffnesses in medium II we can write:

$$h_{ij} = \begin{pmatrix} h & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & -h \end{pmatrix} = \begin{pmatrix} h & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (4\text{-}1)$$

If we first operate with the right hand side matrix, there would be no change in the elastic stiffnesses and case C would be equivalent to case A.

If the elastic stiffnesses matrix in the interface coordinate system is such that it is invariant under the symmetry operation which relates the media on the two sides of the interface, one can regard the configuration as identity or complete inversion and therefore one does not expect to find any attenuating waves.

If one deals with the different possible symmetries, one can see that for some configurations one does not expect to have any attenuating waves at the symmetric interface, and for others, one can further simplify the generalized Stoneley condition, and have some additional information about the possible waves.

The two extreme cases are those of isotropy and the triclinic systems.

In the case of isotropy one does not expect to have any generalized Stoneley waves at the interface since no discontinuity exists and the boundary conditions are identically satisfied for both the longitudinal and transverse body waves.

In the case of triclinic systems no additional symmetry is present in the medium and therefore one cannot simplify further the discussion in chapter 3.

If the medium in half-space I is invariant under $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

the elastic stiffness tensor is of the form:

$$\begin{pmatrix} * & * & * & 0 & 0 & * \\ * & * & * & 0 & 0 & * \\ * & * & * & 0 & 0 & * \\ 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & * & 0 \\ * & * & * & 0 & 0 & * \end{pmatrix} \qquad (4\text{-}2)$$

In this case the elastic stiffnesses are such that under a transformation (3-5) where $h_1h_2 = +1$, regardless of the value of h_3, the configuration is equivalent to the identity or complete inversion while if $h_1h_2 = -1$, cases B and D become identical. In this case, as far as the medium is concerned there is no difference if $h_1h_3 = \pm 1$, for a given value of h_1h_2.

The components of the symmetric matrix, the determinant of which describes the slowness surface, $S_{KL}^{(N)}$, are:

$$\left.\begin{aligned} S_{11}^{(N)}(I) &= c_{11}s_1^2 + c_{55}[s_3^{(N)}(I)]^2 - \rho \\ S_{12}^{(N)}(I) &= c_{16}s_1^2 \\ S_{13}^{(N)}(I) &= (c_{13} + c_{55})s_1s_3^{(N)}(I) \\ S_{22}^{(N)}(I) &= c_{66}s_1^2 + c_{44}[s_3^{(N)}(I)]^2 - \rho \\ S_{23}^{(N)}(I) &= (c_{36} + c_{45})s_1s_3^{(N)}(I) \\ S_{33}^{(N)}(I) &= c_{55}s_1^2 + c_{33}[s_3^{(N)}(I)]^2 - \rho \end{aligned}\right) \qquad (4\text{-}3\text{-}a)$$

Assuming that the configuration is such that generalized Stoneley waves can propagate ($h_1h_2 = -1$), cases B or D, one obtains for the second medium:

$$S_{11}^{(N)}(II) = c_{11}s_1^2 + c_{55}[s_3^{(N)}(II)]^2 - \rho \quad)$$

$$\left.\begin{aligned}
s_{12}^{(N)}(II) &= -\,c_{16}s_1^2\\
s_{13}^{(N)}(II) &= (c_{13}+c_{55})s_1s_3^{(N)}(II)\\
s_{22}^{(N)}(II) &= c_{66}s_1^2 + c_{44}[s_3^{(N)}(II)]^2 - \rho\\
s_{23}^{(N)}(II) &= -\,(c_{36}+c_{45})s_1s_3^{(N)}(II)\\
s_{33}^{(N)}(II) &= c_{55}s_1^2 + c_{33}[s_3^{(N)}(II)]^2 - \rho
\end{aligned}\right\} \qquad (4\text{-}3\text{-}b)$$

where c_{mn} in the second medium are the same as those of the first.

The sextic equations obtained are bi-cubic and are the same for both half-spaces. If the medium and configuration are such that they allow true generalized Stoneley waves to propagate at the interface the slowness components can be complex only if either the cubic equation (of the bi-cubic) has three negative real roots, in which case the slowness components would be pure imaginary, or it would have one negative real root and two complex conjugate roots. In the first case the slowness components would be pure imaginary:

$$\left.\begin{aligned}
s_3^{(1)}(I) &= i\ s^{\circ} & \qquad s_3^{(1)}(II) &= -\ i\ s^{\circ}\\
s_3^{(2)}(I) &= i\ s^{*} & \qquad s_3^{(2)}(II) &= -\ i\ s^{*}\\
s_3^{(3)}(I) &= i\ s^{\dagger} & \qquad s_3^{(3)}(II) &= -\ i\ s^{\dagger}
\end{aligned}\right\} \qquad (4\text{-}4\text{-}a)$$

If the cubic has one real root and a pair of complex conjugates the slowness components would be of the form:

$$\left.\begin{aligned}
s_3^{(1)}(I) &= is^{\circ} & \qquad s_3^{(1)}(II) &= -is^{\circ}\\
s_3^{(2)}(I) &= s^{*} + is^{\dagger} & \qquad s_3^{(2)}(II) &= -s^{*} - is^{\dagger}\\
s_3^{(3)}(I) &= -s^{*} + is^{\dagger} & \qquad s_3^{(3)}(II) &= s^{*} - is^{\dagger}
\end{aligned}\right\} \qquad (4\text{-}4\text{-}b)$$

For both cases $s_3^{(N)}(II) = -s_3^{(N)}(I)$ (4-4-c)

(4-4-c) is the same as (3-7-d) and therefore the following discussion covers cases where the interface wave does not necessarily attenuate ($\mathcal{I}\{s_3^{(N)}(n)\}$ may be zero for some or all N).

The assignment of the superscript 2 or 3 to the slowness components in (4-4-b) is quite arbitrary and is independent of h_1h_3 (compare with 3-7-c)). The moment we have chosen the numeration of the components in (4-4) we have assumed a certain relation between the components in the two media and we have to carry it through. We could have chosen different numeration which would still give us simple relations between the slowness components in the two media. One should note that different numerations lead to different relations between the displacement and stress vector components but not to different final results of the total displacement and the total stress.

Substituting the values of $S_{KL}^{(N)}(n)$ in (3-9) one obtains ratios of $p_k^{(N)}(n)$ in each medium. In medium I the ratio is:

$$p_1^{(N)}(I) : p_2^{(N)}(I) : p_3^{(N)}(I) = \tag{4-5-a}$$
$$\{s_1 s_3^{(N)}(I)[(c_{13}+c_{55})(c_{66}s_1^2 + c_{44}[s_3^{(N)}(I)]^2-\rho)-c_{16}s_1^2(c_{36}+c_{45})]\}:$$
$$:\{s_1 s_3^{(N)}(I)[(c_{11}s_1^2+c_{55}[s_3^{(N)}(I)]^2-\rho)(c_{36}+c_{45})-c_{16}s_1^2(c_{13}+c_{55})]\}:$$
$$:\{(c_{11}s_1^2+c_{55}[s_3^{(N)}(I)]^2-\rho)(c_{66}s_1^2+c_{44}[s_3^{(N)}(I)]^2-\rho)-c_{16}^2 s_1^4\}$$

and for the second medium:

$$p_1^{(N)}(II):p_2^{(N)}(II) :p_3^{(N)}(II) = \tag{4-5-b}$$
$$\{s_1 s_3^{(N)}(II)[(c_{13}+c_{55})(c_{66}s_1^2+c_{44}[s_3^{(N)}(II)]^2-\rho)-c_{16}s_1^2(c_{36}+c_{45})]\}:$$
$$:\{-s_1 s_3^{(N)}(II)[(c_{11}s_1^2+c_{55}[s_3^{(N)}(II)]^2-\rho)(c_{36}+c_{45})-c_{16}s_1^2(c_{13}+c_{55})]\}:$$
$$:\{(c_{11}s_1^2+c_{55}[s_3^{(N)}(II)]^2-\rho)(c_{66}s_1^2+c_{44}[s_3^{(N)}(II)]^2-\rho)-c_{16}^2 s_1^4\}$$

Substituting (4-4-c) into the expression for $S_{KL}^{(N)}(II)$ (4-5-b), and using (3-9), one obtains the relation between the displacement components in the two half-spaces:

$$\left.\begin{aligned} p_1^{(N)}(II) &= - p_1^{(N)}(I) \\ p_2^{(N)}(II) &= p_2^{(N)}(I) \\ p_3^{(N)}(II) &= p_3^{(N)}(I) \end{aligned}\right\} \tag{4-6-a}$$

(Choosing the proportion constants to be the same in (4-5-a) and (4-5-b)).

If the slowness components are pure imaginary the displacement components are in the first medium of the form:

$$p_k^{(N)}(I) = \begin{pmatrix} i\alpha_1^{\circ} & i\alpha_1^{*} & i\alpha_1^{\dagger} \\ i\alpha_2^{\circ} & i\alpha_2^{*} & i\alpha_2^{\dagger} \\ \alpha_3^{\circ} & \alpha_3^{*} & \alpha_3^{\dagger} \end{pmatrix} \tag{4-6-b}$$

Where for each N it may be multiplied by an arbitrary non-zero $\beta^{(N)}$.

The displacement components in the second medium are of similar form and are related through (4-6-a) to $p_k^{(N)}(I)$.

When the slowness components are complex the displacement components are of the form:

$$p_k^{(N)}(I) = \begin{pmatrix} i\alpha_1^o & \alpha_1^* + i\alpha_1^\dagger & -\alpha_1^* + i\alpha_1^\dagger \\ i\alpha_2^o & \alpha_2^* + i\alpha_2^\dagger & -\alpha_2^* + i\alpha_2^\dagger \\ \alpha_3^o & \alpha_3^* + i\alpha_3^\dagger & \alpha_3^* + i\alpha_3^\dagger \end{pmatrix} \qquad (4\text{-}6\text{-}c)^*$$

Where, again, for each N, $p_k^{(N)}(I)$ may be multiplied by an arbitrary non-zero constant $\beta^{(N)}$. The displacement components in this case, in the second medium are still related through (4-6-a) to $p_k^{(N)}(I)$.

If the slowness components are real, $\mathcal{I}\{s_3^{(N)}(n)\} = 0$, the relations between the slowness components in the two media are still given by (4-4-c), and (4-6-a) holds for this case as well.

One should note that the relation (4-6-a) is not absolute and is dependent on the proportion constants chosen in (4-5). If one wishes to remain consistent with the discussion in chapter 3 a multiplier $\chi^{(N)}$ should be added to each of the equations (4-6-a) on the right hand side.

Using the definition of the stress vector (2-15) and (4-6-a) one obtains the following relations:

$$\left.\begin{aligned} q_1^{(N)}(II) &= q_1^{(N)}(I) \\ q_2^{(N)}(II) &= -\, q_2^{(N)}(I) \\ q_3^{(N)}(II) &= -\, q_3^{(N)}(I) \end{aligned}\right) \qquad (4\text{-}7\text{-}a)$$

In the case of pure imaginary slowness components the stress vector components are of the form:

$$q_k^{(N)}(I) = \begin{pmatrix} \beta_1^o & \beta_1^* & \beta_1^\dagger \\ \beta_2^o & \beta_2^* & \beta_2^\dagger \\ i\beta_3^o & i\beta_3^* & i\beta_3^\dagger \end{pmatrix} \qquad (4\text{-}7\text{-}b)^*$$

and when the slowness components are given by (4-4-b), the stress

*Throughout the following discussion, α, β, ζ & η denote real numbers.

components are in the first medium of the form:

$$q_k^{(N)}(\mathrm{I}) = \begin{pmatrix} \beta_1^{\circ} & \beta_1^{*} + i\beta_1^{\dagger} & \beta_1^{*} - i\beta_1^{\dagger} \\ \beta_2^{\circ} & \beta_2^{*} + i\beta_2^{\dagger} & \beta_2^{*} - i\beta_2^{\dagger} \\ i\beta_3^{\circ} & \beta_3^{*} + i\beta_3^{\dagger} & -\beta_3^{*} + i\beta_3^{\dagger} \end{pmatrix} \qquad (4\text{-}7\text{-}c)$$

In the second half-space the stress components obey (4-7-a).

The stress vector components in the case of real slowness components are real and obey (4-7-a).

If one wishes to use the multipiers $\chi^{(N)}$ for the displacement components one has to multiply the right hand side of (4-7-a) by the same multipliers.

Substituting (4-6-a) and (4-7-a) into the generalized Stoneley conditions one obtains the following equations:

$$\left.\begin{aligned} \Sigma_{N=1}^{3} p_1^{(N)}(\mathrm{I})\ [A^{(N)}(\mathrm{I}) + A^{(N)}(\mathrm{II})] &= 0 \\ \Sigma_{N=1}^{3} p_2^{(N)}(\mathrm{I})\ [A^{(N)}(\mathrm{I}) - A^{(N)}(\mathrm{II})] &= 0 \\ \Sigma_{N=1}^{3} p_3^{(N)}(\mathrm{I})\ [A^{(N)}(\mathrm{I}) - A^{(N)}(\mathrm{II})] &= 0 \\ \Sigma_{N=1}^{3} q_1^{(N)}(\mathrm{I})\ [A^{(N)}(\mathrm{I}) - A^{(N)}(\mathrm{II})] &= 0 \\ \Sigma_{N=1}^{3} q_2^{(N)}(\mathrm{I})\ [A^{(N)}(\mathrm{I}) + A^{(N)}(\mathrm{II})] &= 0 \\ \Sigma_{N=1}^{3} q_3^{(N)}(\mathrm{I})\ [A^{(N)}(\mathrm{I}) + A^{(N)}(\mathrm{II})] &= 0 \end{aligned}\right\} \qquad (4\text{-}8)$$

(4-8) form two systems of three homogeneous linear equations in $B_+^{(N)} = A^{(N)}(\mathrm{I}) + A^{(N)}(\mathrm{II})$ and $B_-^{(N)} = A^{(N)}(\mathrm{I}) - A^{(N)}(\mathrm{II})$. If one uses throughout the multipliers $\chi^{(N)}$ they would appear in (4-8) as multipliers of $A^{(N)}(\mathrm{II})$, and one can see that (4-8) are two systems of linear homogeneous equations (as in case D) with $\chi^{(N)}=1$.

The determinants of coefficients may vanish separately or simultaneously. If they vanish separately the null vector, either $B_-^{(N)}$ or $B_+^{(N)}$, of the vanishing determinant may be calculated and the other vector, for this given slowness, is a zero vector. Calculating the null vectors and taking into consideration the special forms of $p_k^{(N)}$ and $q_k^{(N)}$, one obtains for media with pure imaginary or real slowness components:

$$B_{\pm}^{(1)} : B_{\pm}^{(2)} : B_{\pm}^{(3)} = \zeta_{\pm}^{\circ} : \zeta_{\pm}^{*} : \zeta_{\pm}^{\dagger} \qquad (4\text{-}9\text{-}a)$$

and for media with complex slowness components (given by (4-4-b)):

$$B_{\pm}^{(1)}: B_{\pm}^{(2)}: B_{\pm}^{(3)} = \zeta_{\pm}^{\circ} : \zeta_{\pm}^{*} + i\zeta_{\pm}^{\dagger} : \zeta_{\pm}^{*} - i\zeta_{\pm}^{\dagger} \qquad (4\text{-}9\text{-}b)$$

Of course, one has to remember that these are equations between ratios, and one can use an arbitrary non-zero multiplier, in each of the above equations.

The calculation of $A^{(N)}(n)$ from (4-9) shows us that regardless of which of the determinants of coefficients vanishes, the amplitudes are related in the same form as the $B^{(N)}$'s :

If $B_{\pm}^{(N)}$ are described by (4-9-a), the ratio of the amplitudes has the form:

$$A^{(1)}(n):A^{(2)}(n):A^{(3)}(n) = \eta^{\circ}(n) : \eta^{*}(n) : \eta^{\dagger}(n) \qquad (4\text{-}10\text{-}a)$$

and if $B_{\pm}^{(N)}$ are given by (4-9-b), the amplitudes are related as:

$$A^{(1)}(n):A^{(2)}(n):A^{(3)}(n) = \eta^{\circ}(n):\eta^{*}(n)+i\eta^{\dagger}(n):\eta^{*}(n)-i\eta^{\dagger}(n) \qquad (4\text{-}10\text{-}b)$$

The exact character of the interface wave is determined mathematically by whether or not one or both determinants of the coefficients in (4-8) vanish for the given slowness.

Suppose the determinant

$$\begin{Vmatrix} p_1^{(N)}(I) \\ q_2^{(N)}(I) \\ q_3^{(N)}(I) \end{Vmatrix} \qquad (4\text{-}11\text{-}a)$$

vanishes, while the determinant

$$\begin{Vmatrix} p_2^{(N)}(I) \\ p_3^{(N)}(I) \\ q_1^{(N)}(I) \end{Vmatrix} \qquad (4\text{-}11\text{-}b)$$

does not. This means that $\underline{B}^{(N)}$ is the zero vector, or that $A^{(N)}(II) = A^{(N)}(I)$. For this case $P_1(n)$ - the displacement component at the interface in the x_1 direction vanishes and the non-zero components of the total displacement are in the x_2 and x_3 directions. Hence the interface wave is transverse. When one calculates the total displacement components (see (4-12)) it is found that P_2 and P_3 are in quadrature and therefore the displacement is elliptic at the interface. (see fig. (4-1)).

If the determinants in (4-11) are such that (4-11-b) vanishes while (4-11-a) does not, $B_{+}^{(N)}$ is the zero vector, which means $A^{(N)}(II)=-A^{(N)}(I)$, and the displacement components at the interface in the x_2 and x_3 directions vanish. The only non-zero component of the total displacement in this case is P_1, in the x_1 direction, which means that the interface wave is longitudinal and rectilinear at the interface (see fig (4-2)).

When the interface wave is transverse the stress vector components in the x_2 and x_3 directions vanish, and the only non-zero component of the stress vector is Q_1. On the other hand when the total displacement vector is longitudinal, the stress vector is transverse, elliptic (see (4-13)), in the x_2-x_3 plane. In both cases it is quite obvious that the stress vector is perpendicular to the displacement vector.

When both determinants in (4-11) vanish simultaneously, the total displacement and stress vectors have three non-zero components. Using (4-8), the components of the displacement vector at the interface are of the form: (if the slowness is of the form (4-4-a)):

$$P_1(n) = \tfrac{1}{2}\,\Sigma_{N=1}^{3} p_1^{(N)}(I)B_-^{(N)} = \tfrac{1}{2}i(\alpha_1^\circ\zeta_-^\circ + \alpha_1^*\zeta_-^* + \alpha_1^\dagger\zeta^\dagger) \qquad (4\text{-}12\text{-}a)$$

$$P_2(n) = \tfrac{1}{2}\,\Sigma_{N=1}^{3} p_2^{(N)}(I)B_+^{(N)} = \tfrac{1}{2}i(\alpha_2^\circ\zeta_+^\circ + \alpha_2^*\zeta_+^* + \alpha_2^\dagger\zeta_+^\dagger) \qquad (4\text{-}12\text{-}b)$$

$$P_3(n) = \tfrac{1}{2}\,\Sigma_{N=1}^{3} p_3^{(N)}(I)B_+^{(N)} = \tfrac{1}{2}(\alpha_3^\circ\zeta_+^\circ + \alpha_3^*\zeta_+^* + \alpha_3^\dagger\zeta_+^\dagger) \qquad (4\text{-}12\text{-}c)$$

If the slowness components are given by (4-4-b) the displacement is of the form:

$$P_1(n) = \tfrac{1}{2}i(\alpha_1^\circ\zeta_-^\circ + 2\,\alpha_1^*\zeta_-^\dagger + 2\alpha_1^\dagger\zeta_-^*) \qquad (4\text{-}12\text{-}d)$$

$$P_2(n) = \tfrac{1}{2}i(\alpha_2^\circ\zeta_+^\circ + 2\alpha_2^*\zeta_+^\dagger + 2\alpha_2^\dagger\zeta_+^*) \qquad (4\text{-}12\text{-}e)$$

$$P_3(n) = \tfrac{1}{2}(\alpha_3^\circ\zeta_+^\circ + 2\alpha_3^*\zeta_+^* - 2\alpha_3^\dagger\zeta_+^\dagger) \qquad (4\text{-}12\text{-}f)$$

Therefore, independent of the slowness component pattern ((4-4-a) or (4-4-b)) if there is an attenuating wave at the interface between two media related by the symmetric transformation matrix, having a plane of symmetry perpendicular to the x_3 axis, the displacement vector components at the interface are such that the displacement in the x_1 and x_2 direction are of the same phase while the displacement in the x_3 direction is in quadrature.

The stress vector components are obtained in a similar way and give the following results:

For pure imaginary slowness components:

$$Q_1(n) = \tfrac{1}{2}(\beta^o_1\zeta^o_+ + \beta^*_1\zeta^*_+ + \beta^\dagger_1\zeta^\dagger_+) \tag{4-13-a}$$

$$Q_2(n) = \tfrac{1}{2}(\beta^o_2\zeta^o_- + \beta^*_2\zeta^*_- + \beta^\dagger_2\zeta^\dagger_-) \tag{4-13-b}$$

$$Q_3(n) = \tfrac{1}{2}i(\beta^o_3\zeta^o_- + \beta^*_3\zeta^*_- + \beta^\dagger_3\zeta^\dagger_-) \tag{4-13-c}$$

and when the slowness components are given by (4-4-b) the stress vector components are of the form:

$$Q_1(n) = \tfrac{1}{2}[\beta^o_1\zeta^o_+ + 2\ (\beta^*_1\zeta^*_+ - \beta^\dagger_1\zeta^\dagger_+)] \tag{4-13-d}$$

$$Q_2(n) = \tfrac{1}{2}[\beta^o_2\zeta^o_- + 2(\beta^*_2\zeta^*_- - \beta^\dagger_2\zeta^\dagger_-)] \tag{4-13-e}$$

$$Q_3(n) = \tfrac{1}{2}i[\beta^o_3\zeta^o_- + 2(\beta^*_3\zeta^\dagger_- + \beta^\dagger_3\zeta^*_-)] \tag{4-13-f}$$

One can see that $Q_1(n)$ and $Q_2(n)$ are of the same phase as $P_3(n)$ while $Q_3(n)$ is in quadrature with the other stress components, but of the same phase as $P_1(n)$ and $P_2(n)$. However, since the stress vector σ_{i3} at the interface is obtained by multiplication of $Q_i(n)$ by $i\omega$ (see (2-14)) the stress vector σ_{i3} components are of the same phase as $P_i(n)$.

When the slowness components are real both the total displacement and stress vectors are real, and the relation between the vectors depends on their components' actual values.

If the medium in half-space I has a symmetry plane which is perpendicular to the x_2 axis in the interface coordinate system, its elastic stiffnesses tensor is of the form:

$$\begin{pmatrix} * & * & * & 0 & * & 0 \\ * & * & * & 0 & * & 0 \\ * & * & * & 0 & * & 0 \\ 0 & 0 & 0 & * & 0 & * \\ * & * & * & 0 & * & 0 \\ 0 & 0 & 0 & * & 0 & * \end{pmatrix} \tag{4-14}$$

For such media if the transformation matrix from medium I to medium II has $h_1h_3 = +1$, it is equivalent to the identity or complete inversion, while if $h_1h_3 = -1$, cases C & D become identical. (The sign of h_2 does not play any role in the analysis)

The components of $S_{KL}^{(N)}(I)$ are:

$$\left.\begin{aligned}
S_{11}^{(N)}(I) &= c_{11}s_1^2 + c_{55}[s_3^{(N)}(I)]^2 - \rho + 2c_{15}s_1s_3^{(N)}(I) \\
S_{12}^{(N)}(I) &= 0 \\
S_{13}^{(N)}(I) &= c_{15}s_1^2 + c_{35}[s_3^{(N)}(I)]^2 + (c_{13}+c_{55})s_1s_3^{(N)}(I) \\
S_{22}^{(N)}(I) &= c_{66}s_1^2 + c_{44}[s_3^{(N)}(I)]^2 - \rho + 2c_{46}s_1s_3^{(N)}(I) \\
S_{23}^{(N)}(I) &= 0 \\
S_{33}^{(N)}(I) &= c_{55}s_1^2 + c_{33}[s_3^{(N)}(I)]^2 - \rho + 2c_{35}s_1s_3^{(N)}(I)
\end{aligned}\right) \quad (4\text{-}15\text{-a})$$

If we expect any interface waves, $h_1h_3 = -1$, and $S_{KL}^{(N)}(II)$ are given by:

$$\left.\begin{aligned}
S_{11}^{(N)}(II) &= c_{11}s_1^2 + c_{55}[s_3^{(N)}(II)]^2 - \rho - 2c_{15}s_1s_3^{(N)}(II) \\
S_{12}^{(N)}(II) &= 0 \\
S_{13}^{(N)}(II) &= -(c_{15}s_1^2+c_{35}[s_3^{(N)}(II)]^2) + (c_{13}+c_{55})s_1s_3^{(N)}(II) \\
S_{22}^{(N)}(II) &= c_{66}s_1^2 + c_{44}[s_3^{(N)}(II)]^2 - \rho - 2c_{46}s_1s_3^{(N)}(II) \\
S_{23}^{(N)}(II) &= 0 \\
S_{33}^{(N)}(II) &= c_{55}s_1^2 + c_{33}[s_3^{(N)}(II)]^2 - \rho - 2c_{35}s_1s_3^{(N)}(II)
\end{aligned}\right) \quad (4\text{-}15\text{-b})$$

The sextic equations are factorable in this case into a quadratic factor $S_{22}^{(N)}(n)$ and a quartic one $[S_{11}^{(N)}(n)\ S_{33}^{(N)}(n) - (S_{13}^{(N)}(n))^2]$. In order for the quadratic term to have a complex root the following relation must hold:

$$s_1^2(c_{46}^2 - c_{44}c_{66}) + c_{44}\rho < 0 \quad (4\text{-}16)$$

or $$s_1^2 > c_{44}\rho/(c_{44}c_{66} - c_{46}^2)$$

If this is the case, $s_3^{(1)}(I)$ is given by:

$$s_3^{(1)}(I) = \{-c_{46}s_1 + \sqrt{c_{46}^2\ s_1^2 - c_{44}(c_{66}s_1^2 - \rho)}\}/c_{44} \quad (4\text{-}17\text{-a})$$

and in the second medium:

$$s_3^{(1)}(II) = \{c_{46}s_1 - \sqrt{c_{46}^2\ s_1^2 - c_{44}(c_{66}s_1^2 - \rho)}\}/c_{44} \quad (4\text{-}17\text{-b})$$

The remaining quartic factor of the sextic equation is given by:

$$[s_3^{(N)}(I)]^4(c_{55}c_{33}-c_{35}^2) + 2s_1[s_3^{(N)}(I)]^3(c_{15}c_{33}-c_{35}c_{13}) + \tag{4-18}$$
$$[s_3^{(N)}(I)]^2[s_1^2(c_{11}c_{33}+2c_{15}c_{35}-c_{13}^2-2c_{13}c_{55}) - \rho(c_{33}+c_{55})] +$$
$$2s_3^{(N)}(I)[s_1^3(c_{11}c_{35}-c_{15}c_{13})-s_1\rho(c_{35}+c_{15})] + (c_{11}s_1^2-\rho)(c_{55}s_1^2-\rho)-c_{15}^2s_1^4=0$$

For the second medium one obtains a similar equation with the components of the odd powers of $s_3^{(N)}(II)$ having the opposite sign of the components of the odd powers of $s_3^{(N)}(I)$.

The slowness components in the two half-spaces are therefore related as:

$$s_3^{(N)}(II) = - s_3^{(N)}(I) \tag{4-19}$$

(4-19) is the same as (3-7-c) when $h_1h_3 = - 1$.

Because of the factorization of the sextic equation the displacement vector associated with $S_{22}^{(1)}(n) = 0$ is given by:

$$p_1^{(1)}(n) = p_3^{(1)}(n) = 0 \quad \text{and } p_2^{(2)}(n) = p_2^{(3)}(n) = 0 \tag{4-20}$$

Therefore $p_k^{(N)}(I)$ is given by:

$$p_k^{(N)}(I) = \begin{pmatrix} 0 & p_1^{(2)}(I) & p_1^{(3)}(I) \\ p_2^{(1)}(I) & 0 & 0 \\ 0 & p_3^{(2)}(I) & p_3^{(3)}(I) \end{pmatrix} \tag{4-21-a}$$

$$\left.\begin{array}{l} \text{where } p_1^{(N)}(I) : p_3^{(N)}(I) = -S_{13}^{(N)}(I) : S_{11}^{(N)}(I) \quad (N=2,3) \\ \quad\quad \text{if } s_3^{(2)}(I) \neq s_3^{(3)}(I) \end{array}\right\} \tag{4-22-a}$$

$$\left.\begin{array}{l} \text{and} \quad\quad \text{for } s_3^{(2)}(I) = s_3^{(3)}(I) \\ p_1^{(N)}(I) = (0,p_1^{(2)}(I),0) \quad , \; p_3^{(N)}(I) = (0,0,p_3^{(3)}(I)) \end{array}\right\} \tag{4-22-b}$$

For the second medium, since the choice of proportion constants is arbitrary, one can opt to stay consistent with (3-13) $\chi^{(N)} = 1$ (and picking an arbitrary value for h_2)

$$p_k^{(N)}(II) = \begin{pmatrix} 0 & -p_1^{(2)}(I) & -p_1^{(3)}(I) \\ p_2^{(1)}(I) & 0 & 0 \\ 0 & p_3^{(2)}(I) & p_3^{(3)}(I) \end{pmatrix} \tag{4-21-b}$$

When $s_3^{(2)}(I) = s_3^{(3)}(I)$, $p_1^{(3)}(I) = p_3^{(2)}(I) = 0$.

The stress vector components are given by:

$$q_k^{(N)}(I) = \begin{pmatrix} 0 & q_1^{(2)}(I) & q_1^{(3)}(I) \\ q_2^{(1)}(I) & 0 & 0 \\ 0 & q_3^{(2)}(I) & q_3^{(3)}(I) \end{pmatrix} \qquad (4\text{-}23\text{-}a)$$

where:

$$\left.\begin{aligned} q_1^{(N)}(I) &= [c_{15}s_1+c_{55}s_3^{(N)}(I)]p_1^{(N)}(I)+[c_{55}s_1+c_{35}s_3^{(N)}(I)]p_3^{(N)}(I) \\ q_2^{(N)}(I) &= [c_{46}s_1 + c_{44}s_3^{(N)}(I)]p_2^{(N)}(I) \\ q_3^{(N)}(I) &= [c_{13}s_1+c_{35}s_3^{(N)}(I)]p_1^{(N)}(I)+[c_{35}s_1+c_{33}s_3^{(N)}(I)]p_3^{(N)}(I) \end{aligned}\right\} \qquad (4\text{-}23\text{-}b)$$

and

$$q_k^{(N)}(II) = \begin{pmatrix} 0 & q_1^{(2)}(I) & q_1^{(3)}(I) \\ -q_2^{(1)}(I) & 0 & 0 \\ 0 & -q_3^{(2)}(I) & -q_3^{(3)}(I) \end{pmatrix} \qquad (4\text{-}23\text{-}c)$$

Substituting in the generalized Stoneley conditions one obtains the following equations:

$$\left.\begin{aligned} \Sigma_{N=1}^{3} p_1^{(N)}(I)[A^{(N)}(I) + A^{(N)}(II)] &= 0 \\ \Sigma_{N=1}^{3} p_2^{(N)}(I)[A^{(N)}(I) - A^{(N)}(II)] &= 0 \\ \Sigma_{N=1}^{3} p_3^{(N)}(I)[A^{(N)}(I) - A^{(N)}(II)] &= 0 \\ \Sigma_{N=1}^{3} q_1^{(N)}(I)[A^{(N)}(I) - A^{(N)}(II)] &= 0 \\ \Sigma_{N=1}^{3} q_2^{(N)}(I)[A^{(N)}(I) + A^{(N)}(II)] &= 0 \\ \Sigma_{N=1}^{3} q_3^{(N)}(I)[A^{(N)}(I) + A^{(N)}(II)] &= 0 \end{aligned}\right\} \qquad (4\text{-}24)$$

Notice that relations (4-6-a), (4-7-a) and therefore (4-8) hold for this symmetry as well as the symmetry with respect to the x_3 axis. However, in this case we have more information about the actual values of the components.

From the second equation of (4-24) one obtains (since $p_2^{(2)}(I) = p_2^{(3)}(I) = 0$).

$$p_2^{(1)}(I)[A^{(1)}(I) - A^{(1)}(II)] = 0 \qquad (4\text{-}25\text{-}a)$$

Since $p_2^{(1)} \neq 0$, $A^{(1)}(I) = A^{(1)}(II)$. Substituting this into the fifth equation of (4-24), if $A^{(1)}(I) \neq 0$, this means that $q_2^{(1)}(I) = 0$, or:

$$c_{46}s_1 + c_{44}s_3^{(1)}(I) = 0 \qquad (4\text{-}25\text{-}b)$$

This would mean that $s_3^{(1)}(I)$ is real, which would not lead to an attenuating wave at the interface.

When $A^{(1)}(I) = A^{(1)}(II) \neq 0$

$$s_3^{(1)}(I) = -(c_{46}/c_{44})\, s_1 \tag{4-25-c}$$

and from $S_{22}^{(1)}(I) = 0$ the slowness of this bulk wave is:

$$s_1 = \sqrt{c_{44}\rho/(c_{44}c_{66} - c_{46}^2)} \tag{4-25-d}$$

The energy flux vector $\mathfrak{F}_i$ (Musgrave [1970]) is given by: $\frac{1}{4}A^2\omega^2 c_{ijk\ell}(\bar{p}_j p_k s_\ell + p_j \bar{p}_k \bar{s}_\ell)$, and for real s_k, p_k it can be shown that:

$$\mathfrak{F}_i \propto \frac{\partial \| S_{KL} \|}{\partial s_i}$$

For the bulk wave (4-25-c), using (4-25-b) one obtains:

$$\mathfrak{F}_3 \propto \partial \| S_{KL} \| / \partial s_3 = [\partial S_{22}/\partial s_3][S_{11}S_{33} - S_{13}^2] = 2(c_{46}s_1 + c_{44}s_3)[S_{11}S_{33} - S_{13}^2]$$
$$= 0$$

Which shows that this bulk wave travels with energy flux parallel to the interface.

In most cases one may expect that neither one of the determinants for non-trivial solutions of $B_+^{(N)}$ and $B_-^{(N)}$ ($N = 2,3$) would vanish at this slowness $A^{(2)}(n) = A^{(3)}(n) = 0$. Therefore the total displacement of this non-attenuating wave is given by $P = (0, P_2, 0)$ and the total stress vector by: $Q = (0,0,0)$. This means that the interface will remain stress free and the displacement is transverse, parallel to the interface in the direction perpendicular to the sagittal plane. The amplitude of such a wave varies periodically as a function of depth. When $c_{46} = 0$ this transverse wave would have an amplitude which is constant as a function of depth. In isotropy (4-25-d) describes the transverse bulk wave slowness.

For a true Stoneley wave therefore $A^{(1)}(I) = A^{(1)}(II) = 0$.

The remaining equations of (4-24) consist of two sets of only two linear homogeneous equations each, in $B_-^{(N)}$ and $B_+^{(N)}$ respectively ($N = 2,3$). For non-trivial solution of $A^{(N)}(n)$, at least one of the determinants of the matrices:

$$\begin{pmatrix} p_1^{(2)}(I) & p_1^{(3)}(I) \\ q_3^{(2)}(I) & q_3^{(3)}(I) \end{pmatrix} \tag{4-26-a}$$

$$\begin{pmatrix} p_3^{(2)}(I) & p_3^{(3)}(I) \\ q_1^{(2)}(I) & q_1^{(3)}(I) \end{pmatrix} \tag{4-26-b}$$

must vanish. If $B_-^{(N)}$ is the null

vector of (4-26-a), and $B_+^{(N)}$ is the null vector of (4-26-b), one can write:

$$\left.\begin{aligned} B_-^{(2)} &= -ap_1^{(3)}(I) \quad , \quad B_-^{(3)} = ap_1^{(2)}(I) \\ B_+^{(2)} &= -bp_3^{(3)}(I) \quad , \quad B_+^{(3)} = bp_3^{(2)}(I) \end{aligned}\right) \qquad (4\text{-}27)$$

where a and b are proportion constants which may be zero, if $B_-^{(N)}$ or $B_+^{(N)}$ vanish. The amplitudes may now be found:

$$\left.\begin{aligned} A^{(2)}(I) &= -\tfrac{1}{2}[ap_1^{(3)}(I) + bp_3^{(3)}(I)] \\ A^{(3)}(I) &= \tfrac{1}{2}[ap_1^{(2)}(I) + bp_3^{(2)}(I)] \end{aligned}\right) \qquad (4\text{-}28)$$

With appropriate change of sign one obtains similar expressions for the amplitudes in the second medium.

The total displacement at the interface is given by:

$$\left.\begin{aligned} P_1(n) &= -\tfrac{1}{2}b[p_1^{(2)}(I)p_3^{(3)}(I) - p_1^{(3)}(I)p_3^{(2)}(I)] \\ P_2(n) &= 0 \\ P_3(n) &= \tfrac{1}{2}a[p_1^{(2)}(I)p_3^{(3)}(I) - p_1^{(3)}(I)p_3^{(2)}(I)] \end{aligned}\right) \qquad (4\text{-}29)$$

$P_1(n)$ may vanish only if b = 0, and $P_3(n)$ vanishes only when a = 0. If neither a nor b are zero then the displacement is in the sagittal plane and is elliptic. It stays in the sagittal plane for all x_3. (See fig (4-3)).

The stress vector components are:

$$\left.\begin{aligned} Q_1(n) &= -\tfrac{1}{2}a[q_1^{(2)}(I)p_1^{(3)}(I) - q_1^{(3)}(I)p_1^{(2)}(I)] \\ Q_2(n) &= 0 \\ Q_3(n) &= \tfrac{1}{2}b[p_3^{(2)}(I)q_3^{(3)}(I) - p_3^{(3)}(I)q_3^{(2)}(I)] \end{aligned}\right) \qquad (4\text{-}30)$$

Hence the stress vector lies also in the sagittal plane. When the determinant of (4-26-a) vanishes, if the determinant of (4-26-b) does not vanish, b = 0, and $P_1(n) = Q_3(n) = 0$. If the determinant of (4-26-a) does not vanish but the determinant in (4-26-b) vanishes, $Q_1(n) = P_3(n) = 0$.

Therefore when the plane of the interface is normal to either a 2-fold rotation axis or mirror symmetry plane of the medium there is a transverse bulk wave which leaves the interface stress free and moves with energy flux parallel to the interface. The slowness of this bulk wave is given by (4-25-d).

A true generalized Stoneley wave may propagate at the interface in such a configuration. The total displacement and stress vectors lie in the sagittal plane.

The third possibility for a simplification in the presence of a symmetry plane in the medium in half-space I is when this plane of symmetry is perpendicular to the x_1 axis. In this case the elastic stiffness matrix in the interface coordinate system is of the form:

$$\begin{pmatrix} * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix} \qquad (4\text{-}31)$$

For this medium the components of the secular matrix $S_{KL}^{(N)}(n)$ are given by:

$$\left.\begin{aligned}
S_{11}^{(N)}(I) &= c_{11}s_1^2 + c_{55}[s_3^{(N)}(I)]^2 - \rho \\
S_{12}^{(N)}(I) &= (c_{14} + c_{56})s_1 s_3^{(N)}(I) \\
S_{13}^{(N)}(I) &= (c_{13} + c_{55})s_1 s_3^{(N)}(I) \\
S_{22}^{(N)}(I) &= c_{66}s_1^2 + c_{44}[\,s_3^{(N)}(I)]^2 - \rho \\
S_{23}^{(N)}(I) &= c_{34}[s_3^{(N)}(I)]^2 + c_{56}s_1^2 \\
S_{33}^{(N)}(I) &= c_{55}s_1^2 + c_{33}[s_3^{(N)}(I)]^2 - \rho
\end{aligned}\right\} \qquad (4\text{-}32\text{-a})$$

and for the second medium:

$$\left.\begin{aligned}
S_{11}^{(N)}(II) &= c_{11}s_1^2 + c_{55}[s_3^{(N)}(II)]^2 - \rho \\
S_{12}^{(N)}(II) &= h_2h_3(c_{14} + c_{56})s_1 s_3^{(N)}(II) \\
S_{13}^{(N)}(II) &= (c_{13} + c_{55})s_1 s_3^{(N)}(II) \\
S_{22}^{(N)}(II) &= c_{66}s_1^2 + c_{44}[s_3^{(N)}(II)]^2 - \rho \\
S_{23}^{(N)}(II) &= h_2h_3\{c_{34}[s_3^{(N)}(II)]^2 + c_{56}s_1^2\} \\
S_{33}^{(N)}(II) &= c_{55}s_1^2 + c_{33}[s_3^{(N)}(II)]^2 - \rho
\end{aligned}\right\} \qquad (4\text{-}32\text{-b})$$

It is obvious that if $h_2h_3 = +1$, the configuration is like that of identity or complete inversion and no attenuating interface wave is expected, regardless of the value of h_1h_3. If $h_2h_3 = -1$, one may expect an interface wave. The sextic equations one obtains are bi-cubic, and the same one in both half-spaces. Therefore one would obtain for the slowness components of true generalized Stoneley waves either (4-4-a) or (4-4-b), and possible non-attenuating waves will obey (4-4-c).

The ratio of the displacement components is given by:

$$p_1^{(N)}(I) : p_2^{(N)}(I) : p_3^{(N)}(I) = \tag{4-33}$$

$$\{s_1 s_3^{(N)}(I)[(c_{34}[s_3^{(N)}(I)]^2+c_{56}s_1^2)(c_{14}+c_{56})-(c_{13}+c_{55})(c_{66}s_1^2+c_{44}[s_3^{(N)}(I)]^2-\rho)]\}:$$

$$:\{[s_3^{(N)}(I)]^2[s_1^2(c_{55}+c_{13})(c_{14}+c_{56})[s_3^{(N)}(I)]^2-(c_{34}[s_3^{(N)}(I)]^2+c_{56}s_1^2)(c_{11}s_1^2+c_{55}[s_3^{(N)}(I)]^2-\rho)]\}:$$

$$:\{(c_{11}s_1^2+c_{55}[s_3^{(N)}(I)]^2-\rho)(c_{66}s_1^2 + c_{44}[s_3^{(N)}(I)]^2 - \rho) - s_1^2[s_3^{(N)}(I)]^2(c_{14} + c_{56})^2\}$$

For the second half space, one obtains a similar relation with the appropriate changes of sign.

Apart from multipliers of proportion, one obtains for the displacement vector components, using the relations between $S_{KL}^{(N)}(n)$ and (3-9):

$$\left.\begin{aligned} p_1^{(N)}(II) &= - p_1^{(N)}(I) \\ p_2^{(N)}(II) &= - p_2^{(N)}(I) \\ p_3^{(N)}(II) &= p_3^{(N)}(I) \end{aligned}\right\} \tag{4-34-a}$$

The form of the displacement components in the case of slowness given by (4-4-a), (pure imaginary slowness components):

$$p_k^{(N)}(I) = \begin{pmatrix} i\alpha_1^{\circ} & i\alpha_1^{*} & i\alpha_1^{\dagger} \\ \alpha_2^{\circ} & \alpha_2^{*} & \alpha_2^{\dagger} \\ \alpha_3^{\circ} & \alpha_3^{*} & \alpha_3^{\dagger} \end{pmatrix} \tag{4-34-b}$$

For this case the displacement components in the second medium are given by (4-34-a) and (4-34-b).

If the slowness components are given by (4-4-b), (complex slowness components), the displacement components are of the form:

$$p_k^{(N)}(I) = \begin{pmatrix} i\alpha_1^{\circ} & \alpha_1^{*} + i\alpha_1^{\dagger} & -\alpha_1^{*} + i\alpha_1^{\dagger} \\ \alpha_2^{\circ} & \alpha_2^{*} + i\alpha_2^{\ddagger} & \alpha_2^{*} - i\alpha_2^{\ddagger} \\ \alpha_3^{\circ} & \alpha_3^{*} + i\alpha_3^{\ddagger} & \alpha_3 - i\alpha_3^{\ddagger} \end{pmatrix}. \tag{4-34-c}$$

The displacement components in this case in the second medium are given by (4-34-a) and (4-34-c).

The relation between the stress vector components in the two media is obtained from (4-34-a) and the definition of the stress vector (2-15):

$$\left.\begin{aligned} q_1^{(N)}(II) &= q_1^{(N)}(I) \\ q_2^{(N)}(II) &= q_2^{(N)}(I) \\ q_3^{(N)}(II) &= -\,q_3^{(N)}(I) \end{aligned}\right\} \qquad (4\text{-}35\text{-}a)$$

If the slowness components are all pure imaginary (4-4-a) the stress vectors components are of the form:

$$q_k^{(N)}(I) = \begin{pmatrix} \beta_1^{\circ} & \beta_1^{*} & \beta_1^{\dagger} \\ i\beta_2^{\circ} & i\beta_2^{*} & i\beta_2^{\dagger} \\ i\beta_3^{\circ} & i\beta_3^{*} & i\beta_3^{\dagger} \end{pmatrix} \qquad (4\text{-}35\text{-}b)$$

and if the slowness components are complex (given by (4-4-b)), the stress vectors in the first medium are of the form:

$$q_k^{(N)}(I) = \begin{pmatrix} \beta_1^{\circ} & \beta_1^{*} + i\beta_1^{\dagger} & \beta_1^{*} - i\beta_1^{\dagger} \\ \beta_2^{\circ} & \beta_2^{*} + i\beta_2^{\dagger} & -\beta_2^{*} + i\beta_2^{\dagger} \\ \beta_3^{\circ} & \beta_3^{*} + i\beta_3^{\dagger} & -\beta_3^{*} + i\beta_3^{\dagger} \end{pmatrix}$$

The stress components for the second medium can be easily obtained from (4-35-a).

Substituting (4-34-a) and (4-35-a) into the generalized Stoneley conditions one obtains:

$$\left.\begin{aligned} \Sigma_{N=1}^{3}\ p_1^{(N)}(I)\ [A^{(N)}(I) + A^{(N)}(II)] &= 0 \\ \Sigma_{N=1}^{3}\ p_2^{(N)}(I)\ [A^{(N)}(I) + A^{(N)}(II)] &= 0 \\ \Sigma_{N=1}^{3}\ p_3^{(N)}(I)\ [A^{(N)}(I) - A^{(N)}(II)] &= 0 \\ \Sigma_{N=1}^{3}\ q_1^{(N)}(I)\ [A^{(N)}(I) - A^{(N)}(II)] &= 0 \\ \Sigma_{N=1}^{3}\ q_2^{(N)}(I)\ [A^{(N)}(I) - A^{(N)}(II)] &= 0 \\ \Sigma_{N=1}^{3}\ q_3^{(N)}(I)\ [A^{(N)}(I) + A^{(N)}(II)] &= 0 \end{aligned}\right\} \qquad (4\text{-}36)$$

These, as in the case of a plane of symmetry which is perpendicular to the x_3-axis, gives two sets of linear homogeneous equations in $B_-^{(N)}$ and $B_+^{(N)}$ which may have non-trivial solutions at the same or at separate slownesses s_1.

The condition for non-zero $B_{+}^{(N)}$ is that the determinant of the matrix:

$$\begin{pmatrix} p_1^{(N)}(I) \\ p_2^{(N)}(I) \\ q_3^{(N)}(I) \end{pmatrix} \qquad (4\text{-}37\text{-}a)$$

vanishes, while for non-zero $B_{-}^{(N)}$ the condition is that the determinant of:

$$\begin{pmatrix} q_1^{(N)}(I) \\ q_2^{(N)}(I) \\ p_3^{(N)}(I) \end{pmatrix} \qquad (4\text{-}37\text{-}b)$$

vanishes.

If the slowness components are pure imaginary, the B's are related in the form:

$$B_{\pm}^{(1)} : B_{\pm}^{(2)} : B_{\pm}^{(3)} = \zeta_{\pm}^{o} : \zeta_{\pm}^{*} : \zeta_{\pm}^{\dagger} \qquad (4\text{-}38\text{-}a)$$

If the slowness components are given by (4-4-b) the B's have the form:

$$B_{\pm}^{(1)} : B_{\pm}^{(2)} : B_{\pm}^{(3)} = \zeta_{\pm}^{o} : \zeta_{\pm}^{*} + i\zeta_{\pm}^{\dagger} : \zeta_{\pm}^{*} - i\zeta_{\pm}^{\dagger} \qquad (4\text{-}38\text{-}b)$$

From (4-38) one can see that the amplitudes have the same form as the B's.

By use of the form of the displacement components, the B's and (4-36) one obtains the following results for the total displacement:

When the slowness components are all pure imaginary:

$$\left.\begin{aligned} P_1(n) &= \tfrac{1}{2}i(\alpha_1^{o}\zeta_{-}^{o} + \alpha_1^{*}\zeta_{-}^{*} + \alpha_1^{\dagger}\zeta_{-}^{\dagger}) \\ P_2(n) &= \tfrac{1}{2}\,(\alpha_2^{o}\zeta_{-}^{o} + \alpha_2^{*}\zeta_{-}^{*} + \alpha_2^{\dagger}\zeta_{-}^{\dagger}) \\ P_3(n) &= \tfrac{1}{2}\,(\alpha_3^{o}\zeta_{+}^{o} + \alpha_3^{*}\zeta_{+}^{*} + \alpha_3^{\dagger}\zeta_{+}^{\dagger}) \end{aligned}\right) \qquad (4\text{-}39\text{-}a)$$

If the slowness components are given by (4-4-b) the displacement components at the interface are:

$$\left.\begin{aligned} P_1(n) &= \tfrac{1}{2}i[\alpha_1^{o}\zeta_{-}^{o} + 2(\alpha_1^{*}\zeta_{-}^{\dagger} + \alpha_1^{\dagger}\zeta_{-}^{*})] \\ P_2(n) &= \tfrac{1}{2}\,[\alpha_2^{o}\zeta_{-}^{o} + 2(\alpha_2^{*}\zeta_{-}^{\dagger} - \alpha_2^{\dagger}\zeta_{-}^{*})] \\ P_3(n) &= \tfrac{1}{2}\,[\alpha_3^{o}\zeta_{+}^{o} + 2(\alpha_3^{*}\zeta_{+}^{\dagger} - \alpha_3^{\dagger}\zeta_{+}^{*})] \end{aligned}\right) \qquad (4\text{-}39\text{-}b)$$

The stress components in the case of pure imaginary slowness components are of the form:

$$\left.\begin{aligned} Q_1(n) &= \tfrac{1}{2}\,(B_1^o\zeta_+^o + B_1^*\zeta_+^* + B_1^\dagger\zeta_+^\dagger) \\ Q_2(n) &= \tfrac{1}{2}i(B_2^o\zeta_+^o + B_2^*\zeta_+^* + B_2^\dagger\zeta_+^\dagger) \\ Q_3(n) &= \tfrac{1}{2}i(B_3^o\zeta_-^o + B_3^*\zeta_-^* + B_3^\dagger\zeta_-^\dagger) \end{aligned}\right) \qquad (4\text{-}40\text{-}a)$$

and for slowness components given by (4-4-b)

$$\left.\begin{aligned} Q_1(n) &= \tfrac{1}{2}\,[B_1^o\zeta_+^o + 2(B_1^*\zeta_+^* - B_1^\dagger\zeta_+^\dagger)] \\ Q_2(n) &= \tfrac{1}{2}i[B_2^o\zeta_+^o + 2(B_2^*\zeta_+^\dagger + B_2^\dagger\zeta_+^*)] \\ Q_3(n) &= \tfrac{1}{2}i[B_3^o\zeta_-^o + 2(B_3^*\zeta_-^\dagger + B_3^\dagger\zeta_-^*)] \end{aligned}\right) \qquad (4\text{-}40\text{-}b)$$

Regardless of the form of the slowness components the form obtained for the displacement components is the same (4-39-a) and (4-39-b), and the stress vector form is independent of the form of the slowness components as well.

One can see that in the case of a medium with plane of symmetry which is perpendicular to the x_1 axis, if the transformation matrix from medium I to medium II is given by $h_2h_3 = -1$ (regardless of the value of h_1) the following waves are possible:

If the determinant of (4-37-**b**) vanishes while the one of (4-37-**a**) does not vanish, $P_3(n) = Q_1(n) = Q_2(n) = 0$, while the displacement vector will have two non-zero components, P_1 and P_2 which are in quadrature, and therefore the displacement is elliptic. The only non-zero component of the stress vector is Q_3 which is of the same phase as P_1, and therefore the actual stress σ_{33} in the x_3 direction is of the same phase as P_2. (see fig (4-4)).

If the determinant in (4-37-**b**) does not vanish while the one in (4-37-**a**) vanishes, $P_1(n) = P_2(n) = Q_3(n) = 0$. The only non-zero displacement component is in the x_3 direction, and the two non-zero components of the stress vector are in the plane of the interface. The two components of the stress vector are in quadrature, and therefore elliptic, while the displacement is rectilinear and of the same phase as Q_1. (see fig (4-5)).

If both determinants of (4-37) vanish simultaneously, one can see that the displacement components in the x_2 and x_3 directions are of the same phase while the one in the x_1 is in quadrature, while the stress components are such that Q_1 is of the same phase as P_2 and P_3 and Q_2

and Q_3 are of the same phase as P_1. (see fig (4-5)).

If the medium in half-space I exhibits additional symmetry, one may still further simplify the generalized Stoneley conditions for the possible waves, or may find out that with the additional symmetry no attenuating waves are possible at the symmetric interface.

Some of the numerical results deal with a cubic medium rotated in such a way as to obtain in the interface coordinate system an elastic stiffness matrix resembling that of the tetragonal system (crystal classes $4, \bar{4}$, & $4/m$). Some of the elastic stiffnesses become zero in the above discussion and therefore the expressions are simplified, but essentially the results are unaltered.

The discussion of the possible waves under special symmetry is summarized in table (4-1).

Table (4-1) - The possible interface waves in media with a plane of symmetry which is perpendicular to one of the axes in the interface coordinate system.

Plane of symmetry perpendicular to the axis	Requirements of transformation matrix	Total displacement vector $P_i(n)$ at interface	Total stress vector $Q_i(n)$ at interface	Condition for non-trivial $B_-^{(N)}$	Condition for non-trivial $B_+^{(N)}$
x_1	$h_2h_3 = -1$	$P_1=\frac{1}{2}\Sigma^* p_1^{(N)}(I)B_-^{(N)}$ $P_2=\frac{1}{2}\Sigma^* p_2^{(N)}(I)B_-^{(N)}$ $P_3=\frac{1}{2}\Sigma^* p_3^{(N)}(I)B_+^{(N)}$	$Q_1=\frac{1}{2}\Sigma^* q_1^{(N)}(I)B_+^{(N)}$ $Q_2=\frac{1}{2}\Sigma^* q_2^{(N)}(I)B_+^{(N)}$ $Q_3=\frac{1}{2}\Sigma^* q_3^{(N)}(I)B_-^{(N)}$	$\begin{Vmatrix} q_1^{(N)}(I) \\ q_2^{(N)}(I) \\ p_3^{(N)}(I) \end{Vmatrix}=0$	$\begin{Vmatrix} p_1^{(N)}(I) \\ p_2^{(N)}(I) \\ q_3^{(N)}(I) \end{Vmatrix}=0$
x_2	$h_1h_3 = -1$	$P_1= -\frac{1}{2}\,b\,D^{**}$ $P_2= 0$ $P_3= \frac{1}{2}\,a\,D^{**}$	$Q_1 = -\frac{1}{2}\,a\,E^{**}$ $Q_2 = 0$ $Q_3 = \frac{1}{2}\,b\,F^{**}$	$B_-^{(1)}= 0$ $B_-^{(2)}=-ap_1^{(3)}(I)$ $B_-^{(3)}= ap_1^{(2)}(I)$	$B_+^{(1)}= 0$ $B_+^{(2)}=-bp_3^{(3)}(I)$ $B_+^{(3)}= bp_3^{(2)}(I)$
x_3	$h_1h_2 = -1$	$P_1=\frac{1}{2}\Sigma^* p_1^{(N)}(I)B_-^{(N)}$ $P_2=\frac{1}{2}\Sigma^* p_2^{(N)}(I)B_+^{(N)}$ $P_3=\frac{1}{2}\Sigma^* p_3^{(N)}(I)B_+^{(N)}$	$Q_1=\frac{1}{2}\Sigma^* q_1^{(N)}(I)B_+^{(N)}$ $Q_2=\frac{1}{2}\Sigma^* q_2^{(N)}(I)B_-^{(N)}$ $Q_3=\frac{1}{2}\Sigma^* q_3^{(N)}(I)B_-^{(N)}$	$\begin{Vmatrix} q_1^{(N)}(I) \\ p_2^{(N)}(I) \\ p_3^{(N)}(I) \end{Vmatrix}=0$	$\begin{Vmatrix} p_1^{(N)}(I) \\ q_2^{(N)}(I) \\ q_3^{(N)}(I) \end{Vmatrix}=0$

$\Sigma^* = \Sigma_{N=1}^{3}$

$F^{**}=p_3^{(2)}(I)q_3^{(3)}(I) - p_3^{(3)}(I)q_3^{(2)}(I)$

$D^{**}=p_1^{(2)}(I)p_3^{(3)}(I) - p_1^{(3)}(I)p_3^{(2)}(I)$

$E^{**}=q_1^{(2)}(I)p_1^{(3)}(I) - q_1^{(3)}(I)p_1^{(2)}(I)$

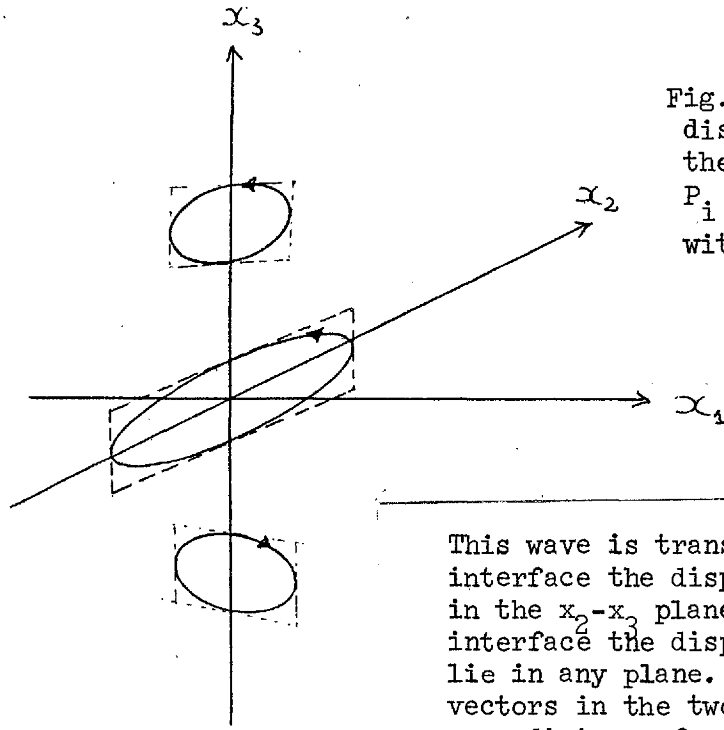


Fig. (4-1) - Particle displacement when at the interface $P_i = (0, P_2, P_3)$ with P_2 & P_3 in quadrature

This wave is transverse. At the interface the displacement vectors lie in the x_2-x_3 plane. Away from the interface the displacement vectors may lie in any plane. The displacement vectors in the two half-spaces for the same distance from the interface are related as: $P_i(x_3) = (P_1, P_2, P_3)$, and $P_i(-x_3) = (-P_1, P_2, P_3)$.

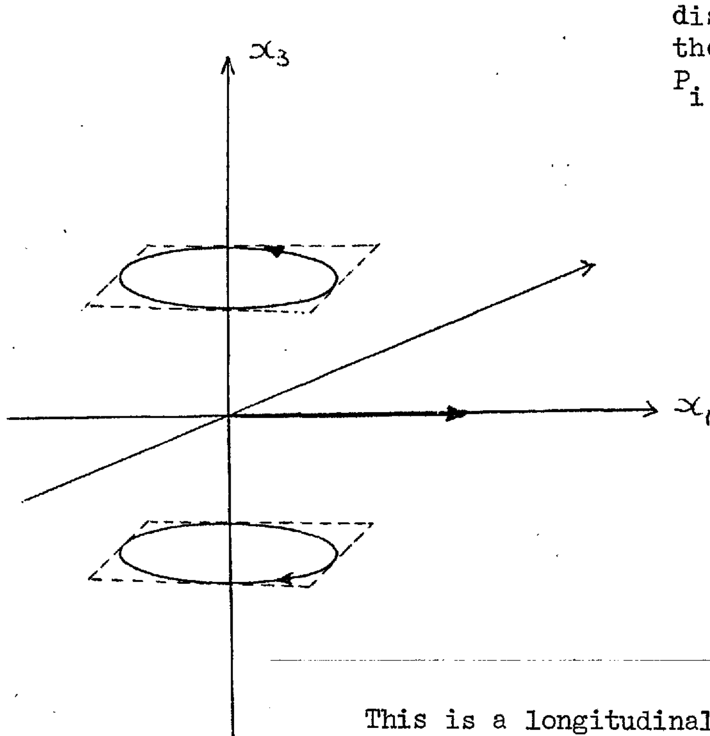


Fig. (4-2) - Particle displacement when at the interface $P_i = (P_1, 0, 0)$.

This is a longitudinal wave. At the interface the displacement is in the direction of the wave propagation. Away from the interface the displacement vectors may lie in any plane. The displacement vectors equidistant from the interface are related as: $P_i(x_3) = (P_1, P_2, P_3)$ and $P_i(-x_3) = (P_1, -P_2, -P_3)$.

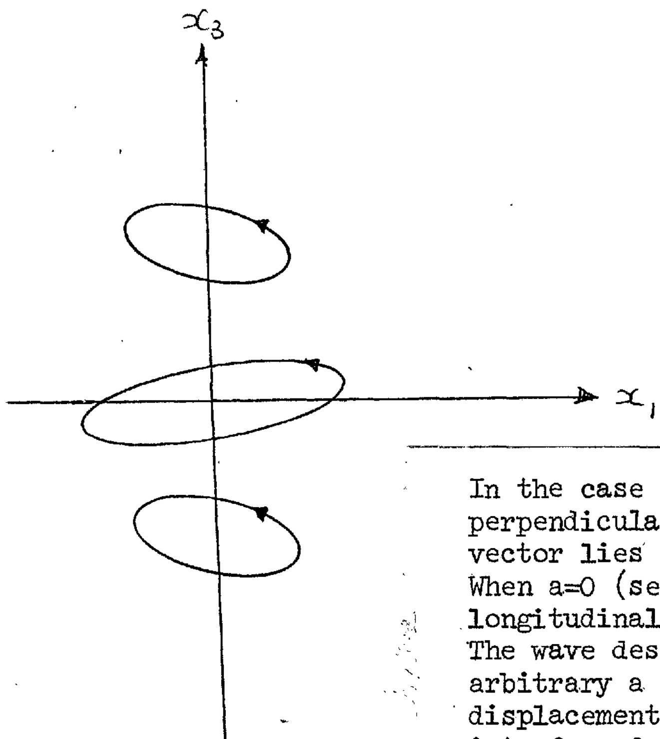


Fig. (4-3) - Particle displacement when at the interface
$P_i = (P_1, 0, P_3)$
and $A^{(1)}(n) = 0$
$p_2^{(N)}(n) = 0$, $N=2,3$
throughout.

In the case of a symmetry plane which is perpendicular to the x_2 axis the displacement vector lies in the sagittal plane throughout. When a=0 (see equation (4-29)) the wave is longitudinal, when b=0, the wave is transverse. The wave described in this figure is for an arbitrary a and b. The relation between the displacement vectors equidistant from the interface depends on a and b. When a=0 $P_1(x_3) = P_1(-x_3)$ and $P_3(x_3) = - P_3(-x_3)$. When b = 0, $P_1(x_3) = - P_1(-x_3)$ and $P_3(x_3) = P_3(-x_3)$

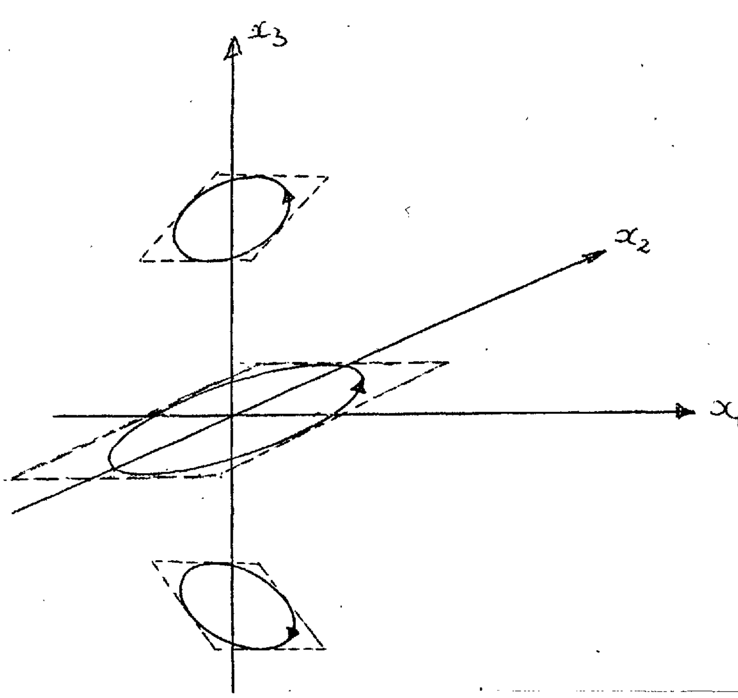


Fig. (4-4) - Particle displacement when at the interface

$P_i = (P_1,\ P_2,\ 0)$

P_1 and P_2 in quadrature.

Here the displacement vectors lie in the plane of the interface. Away from the interface the displacement vector may lie in any plane. at equidistance from the interface, the displacement vectors are related as:

$P_i(x_3) = (P_1, P_2, P_3)$ and $P_i(-x_3) = (P_1, P_2, -P_3)$.

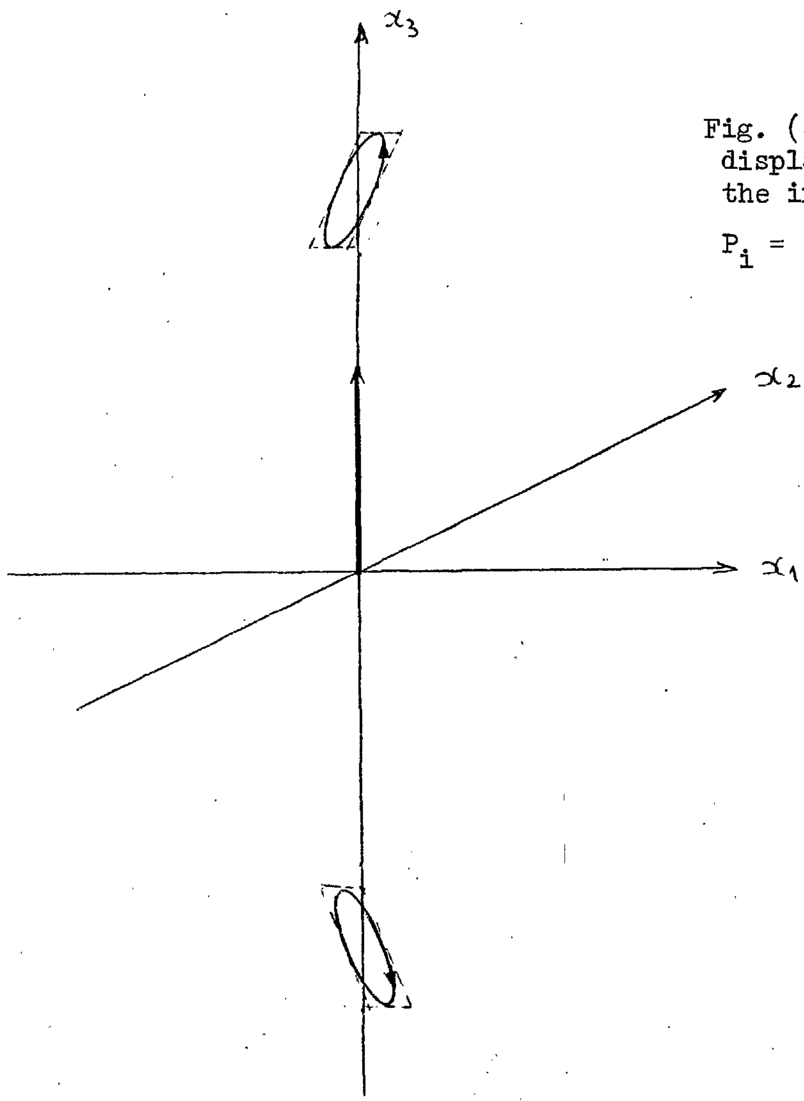


Fig. (4-5) - Particle displacement when at the interface
$P_i = (0,0,P_3)$

This wave is transverse at the interface, having a displacement component in the direction perpendicular to the interface only. Away from the interface the displacement vectors may lie in any plane. The displacement vectors in the two media equidistant from the interface are related as: $P_i(x_3) = (P_1,P_2,P_3)$ and

$$P_i(-x_3) = (-P_1,-P_2,P_3).$$

5. WAVES AT AN INTERFACE BETWEEN TWO PIEZOELECTRIC MEDIA.

5.1 GENERALIZED STONELEY CONDITIONS FOR PIEZOELECTRIC MEDIA.

When the media on the two sides of the interface exhibit piezoelectric properties, one has to take into account the stresses that arise due to the electric field in the generalized Hooke's law, and new equations should be derived.

Kraut [1969], and others have treated the piezoelectric effect in a whole space, Bleustein [1968], Farnell[1970] and others have treated the effect on elastic free surface waves. Special waves, in addition to the Rayleigh wave have been observed and are referred to in the literature as Bleustein-Gulayev Waves.

The stresses in a piezoelectric medium are given by:

$$\sigma_{ij} = c_{ijk\ell} u_{k,\ell} + e_{kij} \Phi_{,k} \tag{5-1}$$

where Φ is the scalar electric potential, and e_{kij} is a tensor which is a result of a scalar product of the piezoelectric tensor $d_{k\ell m}$ and the elastic stiffnesses (Nye [1957])

$$e_{kij} = d_{k\ell m} c_{\ell mij} \tag{5-2}$$

On substitution of (5-1) into (2-5) one obtains the equation of motion:

$$c_{ijk\ell} u_{k,\ell j} + e_{\ell ij} \Phi_{,\ell j} = \rho \ddot{u}_i \tag{5-3}$$

The electric displacement D_i is given by:

$$D_i = e_{ikj} u_{k,j} - \epsilon_{ik} \Phi_{,k} \tag{5-4}$$

where ϵ_{ik} is the dielectric permittivity tensor.

The conservation of charge is given by:

$$D_{i,i} = Q \tag{5-5}$$

where Q is the free charge density which we assume to be zero. Substituting of (5-4) into (5-5) leads to:

$$e_{ikj} u_{k,ji} - \epsilon_{ik} \Phi_{,ki} = 0 \tag{5-6}$$

By using the scalar potential we have assumed that the magnetic flux does not change in time. This assumption is correct when we are dealing with acoustic waves, which have low velocities, relative to the speed of light. In such velocities the electromagnetic part may be regarded as quasistatic.

We shall proceed in the way described by Farnell [1970] by assuming the same form of plane wave for the scalar electric potential, as that taken for the displacement:

$$\Phi = A\, p_4 \,\{\exp[i\omega(s_j x_j - t)]\} \qquad (5\text{-}7)$$

Upon substitution of (5-7) and (2-1) into (5-3) and (5-6) one obtains a set of four homogeneous equations in four unknowns, p_k:

$$S_{k\ell} p_\ell = 0 \qquad k,\ell = 1,\ldots,4 \qquad (5\text{-}8\text{-a})$$

where $S_{ik} = c_{ijk\ell} s_j s_\ell - \rho\delta_{ik}$ (5-8-b)

$i,j,k,\ell = 1,2,3$

$$S_{k4} = S_{4k} = e_{ijk} s_i s_j \qquad i,j,k=1,2,3 \qquad (5\text{-}8\text{-c})$$

$$S_{44} = -\,\epsilon_{ij} s_i s_j \qquad i,j=1,2,3 \qquad (5\text{-}8\text{-d})$$

For non-trivial solutions of (5-8) the determinant of coefficients must vanish. In this case one obtains an eighth order polynomial equation in s_3 with real coefficients, the solution of which can contain at most four pairs of complex conjugate roots.

In order to obtain waves which attenuate with increasing distance from the interface (and using the same configuration as in chapter 2) one would choose in the upper half-space the four roots with positive imaginary part. As a result the displacement and scalar potential would be described by a compound wave of four components. The stresses are obtained by differentiating the displacement and potential and substituting into (5-1):

$$\sigma_{ij} = i\omega \sum_{N=1}^{4} [c_{ijk\ell} p_k^{(N)} + e_{\ell ij} p_4^{(N)}] s_\ell^{(N)} A^{(N)} \{\exp[i\omega(s_1 x_1 + s_3^{(N)} x_3 - t)]\} \qquad (5\text{-}9)$$

and in particular, the stress vector component in the x_3 direction is given by:

$$\sigma_{i3} = i\omega \sum_{N=1}^{4} q_i'^{(N)} A^{(N)} \{\exp[i\omega(s_1 x_1 + s_3^{(N)} x_3 - t)]\} \qquad (5\text{-}10\text{-a})$$

where $q_i'^{(N)} = (c_{i3k\ell} p_k^{(N)} + e_{\ell i3} p_4^{(N)}) s_\ell^{(N)}$ (5-10-b)

If one applies the Stoneley condition for continuity of stress and displacement across the interface (welded interface) one obtains only six homogeneous equations for the eight amplitudes $A^{(N)}(n)$ in the two half-spaces. Two additional conditions can be obtained from continuity of potential and the normal component of the electric displacement.

The generalized Stoneley condition becomes an eighth order determinantal equation:

$$\begin{Vmatrix} p_m^{(N)}(I) & -p_m^{(N)}(II) \\ q_m'^{(N)}(I) & -q_m'^{(N)}(II) \end{Vmatrix} = 0 \qquad N,m = 1,\ldots,4 \tag{5-11}$$

and where

$$q_4'^{(N)}(n) = [e_{3k\ell}(n) p_k^{(N)}(n) - \epsilon_{3\ell}(n) p_4^{(N)}(n)] s_\ell^{(N)}(n) \tag{5-12}$$

$$k,\ell = 1,2,3 \qquad N=1,\ldots,4$$

The matrix $\overline{q_m'^{(M)}(n)}\, p_m^{(N)}(n)$, m=1,...,4, is not, in general, skew-Hermitian. By following a similar procedure described by Currie [1974], and using equation (5-6) as well (multiplied by $p_4^{(M)}(n)$ and $p_4^{(N)}(n)$) one arrives at the following relationship: (5-13)

$$(s_3^{(N)} - \overline{s_3^{(M)}})(\overline{D'^{NM}} + D'^{MN}) + (e_{s\ell 3} - e_{3s\ell})(\overline{p_s^{(M)}} p_4^{(N)} \overline{s_\ell^{(M)}} + p_s^{(N)} \overline{p_4^{(M)}} s_\ell^{(N)}) = 0$$

where $D'^{NM} = \overline{q_m'^{(N)}} p_m^{(M)}$ $\qquad$ N,M,m=1,...,4 (5-14)

Since $s_3^{(N)} - \overline{s_3^{(M)}} \neq 0$ for all N, M in attenuating waves, for true generalized Stoneley waves:

$$\overline{D'^{NM}} + D'^{MN} = G^{MN} \tag{5-15}$$

with

$$G^{MN} = (e_{3s\ell} - e_{s\ell 3})(\overline{p_s^{(M)}} p_4^{(N)} \overline{s_\ell^{(M)}} + p_s^{(N)} \overline{p_4^{(M)}} s_\ell^{(N)}) / (s_3^{(N)} - \overline{s_3^{(M)}}) \tag{5-16}$$

G^{MN} is obviously hermitian as a sum of a matrix and its transposed complex conjugate. In the non-piezoelectric case, $e_{3s\ell} = e_{s\ell 3} = 0$ and therefore one arrives at the skew-hermitian character of D'^{NM}. One should note that centrosymmetric media cannot be piezoelectric, and for such media $G^{MN} = 0$. G^{MN} also vanishes if $e_{3s\ell} = e_{s\ell 3}$. This happens, for instance, in cubic media.

When $e_{k\ell m} = 0$ the solutions $s_3^{(N)}$ would be the same as in the discussion of chapters 3 and 4, because the fourth equation of (5-8) would be decoupled from the rest.

If we now perform similar operations on the equations for continuity of displacement, potential and normal electric displacement and stresses as described by Chadwick and Currie [1974], we obtain the following relationships:

$$\sum_{N=1}^{4} \left[\hat{F}^{MN}(I)A^{(N)}(I) - G^{MN}(II)A^{(N)}(II)\right] = 0 \qquad (5\text{-}17\text{-}a)$$

$$\sum_{N=1}^{4} \left[-G^{MN}(I)A^{(N)}(I) + \overline{\hat{F}^{NM}(I)}\, A^{(N)}(II)\right] = 0 \qquad (5\text{-}17\text{-}b)$$

where $$\hat{F}^{MN}(I) = p_m^{(N)}(I)\overline{q_m'^{(M)}(II)} + \overline{p_m^{(M)}(II)}q_m'^{(N)}(I) \qquad (5\text{-}18)$$

For cases where both $G^{MN}(I)$ and $G^{MN}(II)$ vanish, a simplified Stoneley condition has the same form as for the non-piezoelectric case, because $\|\hat{F}^{MN}(I)\| = 0$ is a condition for non-trivial solutions of both $A^{(N)}(I)$ and $A^{(N)}(II)$. One should remember that $\hat{F}^{MN}$ may contain within it the piezoelectric constants, although $G^{MN}(n)$ may vanish.

When the configuration is such that on one side of the interface there is a centrosymmetric medium while on the other side there is a non-centrosymmetric medium, one of the equations (5-17) becomes decoupled from the other. Suppose for medium II $G^{MN}(II) = 0$. In order to have non-trivial solutions for $A^{(N)}(I)$, $\hat{F}^{MN}(I)$ must be a singular matrix. After finding the null vector of $\hat{F}^{MN}(I)$ one may substitute in (5-17-b) to obtain a set of four non-homogeneous linear equations in the four unknowns $A^{(N)}(II)$. The matrix of coefficients is singular and therefore the system will have a solution only if the rank of $\overline{\hat{F}^{MN}(I)}$ and that of the augmented matrix are the same. One should note that in this case, if $\hat{F}^{MN}(I)$ is a non-singular matrix, the trivial solution of (5-17-a) leads only to the trivial solution for $A^{(N)}(II)$.

For the case where G^{MN} does not vanish one can still reduce the generalized Stoneley condition (5-11) which is an eighth order determinant to a fourth order determinantal condition.

The displacement vectors p_k are, or may be made to be, two different bases of C^4 (being eigenvectors of the matrix $(\mathbb{S}_{kl}v^2)$)(Chadwick & Currie [1974]) and therefore there exists a regular 4x4 transformation matrix $\hat{T}$, such that:

$$p_k^{(N)}(I) = \sum_{M=1}^{4} \hat{T}^{NM} p_k^{(M)}(II) \tag{5-19}$$

By using (5-19) and the definition of $G^{MN}(n)$, (5-15), one arrives at the following result:

$$\sum_{M=1}^{4} \overline{\hat{T}^{RM}}\, \hat{F}^{MN}(I) = \sum_{M=1}^{4} \sum_{Q=1}^{4} \overline{\hat{T}^{RM}}\, \hat{T}^{NQ}\, G^{MQ}(II) + G^{RN}(I) - \sum_{Q=1}^{4} \hat{T}^{NQ}\, \overline{\hat{F}^{QR}(I)} \tag{5-20-a}$$

or:

$$\sum_{Q=1}^{4} \hat{T}^{NQ}\Big[\sum_{M=1}^{4} \overline{\hat{T}^{RM}} G^{MQ}(II) - \overline{\hat{F}^{QR}}(I)\Big] = \sum_{M=1}^{4} \overline{\hat{T}^{RM}}\, \hat{F}^{MN}(I) - G^{RN}(I) \tag{5-20-b}$$

Multiplying (5-17-a) by $\hat{T}^{RM}$, substituting from (5-20-a) and (5-17-b) one arrives at the following relationship:

$$\sum_{Q=1}^{4} \Big\{ \sum_{M=1}^{4}\Big[\overline{\hat{T}^{RM}} G^{MQ}(II) - \overline{\hat{F}^{QR}(I)}\Big]\Big[\sum_{N=1}^{4} \hat{T}^{NQ} A^{(N)}(I) - A^{(Q)}(II)\Big]\Big\} = 0 \tag{5-21}$$

The condition for this equation to hold is that the determinant of the matrix of coefficients will vanish. For, suppose the determinant does not vanish, then, the trivial solution leads to:

$$A^{(Q)}(II) = \sum_{N=1}^{4} \hat{T}^{NQ} A^{(N)}(I) \tag{5-22}$$

which, upon substitution into (5-17-a) gives:

$$\sum_{R=1}^{4} \Big[\hat{F}^{QR}(I) - \sum_{M=1}^{4} G^{QM}(II)\hat{T}^{RM}\Big] A^{(R)}(I) = 0 \tag{5-23}$$

For non-trivial solutions of $A^{(R)}(I)$ the determinant of the coefficients must vanish. The matrix in (5-23) is the complex conjugate of the one in (5-21), the therefore for equation (5-21) to hold, the following determinant must vanish:

$$\Big\| \sum_{M=1}^{4} \overline{\hat{T}^{RM}} G^{MQ}(II) - \overline{\hat{F}^{QR}(I)} \Big\| = 0 \tag{5-24}$$

One can see that if either $G^{RN}(n)$ is a zero matrix this condition leads to the condition:

$$\|\hat{F}^{MN}(I)\| = 0 \tag{5-25}$$

This can be seen also directly from equations (5-17).

We shall now see that (5-22) holds for all solutions of generalized Stoneley waves. Suppose that the null vector of the matrix in (5-21) is α^Q, which is not a zero vector, then:

$$A^{(Q)}(II) = \sum_{N=1}^{4} \hat{T}^{NQ} A^{(N)}(I) + \alpha^Q \qquad (5\text{-}26)$$

Substituting into the conditions of continuity of displacement and electric potential, one obtains:

$$\sum_{Q=1}^{4} p_m^Q(II)\, \alpha^Q = 0 \qquad (5\text{-}27)$$

For non-trivial solutions of α^Q the determinant of $p_m^Q(II)$ must vanish. But since $p_m^{(N)}(II)$ is a matrix of rank 4, its determinant does not vanish, and the only way for (5-27) to hold is for α^Q to vanish. Hence the amplitudes in the two half-spaces are related as (5-22). $A^{(N)}(I)$ is given as the null vector of (5-23), and $A^{(N)}(II)$ can be found from it by (5-22).

5.2. BLEUSTEIN WAVES AT A FREE SURFACE OF A PIEZOELECTRIC MEDIUM.

Bleustein [1968] has treated the particular case of hexagonal half space completely coated with an infinitesimally thin perfectly conducting electrode which is grounded. The equations governing the interior of the half space are the same as those obtained for piezoelectric media (5-1) to (5-10). However, this type of configuration leads to different electrical boundary conditions from the ones used traditionally (Farnell, [1970]). Rather than imposing continuity of the normal component of the electric potential and displacement one has to impose the condition of zero electric potential at the free surface. This boundary condition together with the free surface conditions (σ_{3i}=0 at x_3=0) lead to the following Bleustein condition:

$$\left\| \begin{matrix} q_k'^{(N)} \\ p_4^{(N)} \end{matrix} \right\| = 0 \qquad k=1,2,3 \quad N=1,\ldots,4 \qquad (5\text{-}28)$$

where $q'^{(N)}$ are defined by (5-10-b).

The traditional conditions for generalized Rayleigh waves in piezoelectric media may lead to a Bleustein wave in the particular case that the continuity of electric displacement lead to zero electric potential at the free surface.

5.3. BLEUSTEIN TYPE WAVES AT AN INTERFACE BETWEEN TWO PIEZOELECTRIC MEDIA.

Generalizing the Bleustein wave at a free surface to an interface, one adds to the two half-spaces configuration a coating, throughout the interface, of infinitesimally thin grounded electrode. This would cause the electric potential to be zero at the interface.

Again, the equations governing the different physical characteristics of the interior are the same as those discussed above. The welded conditions lead to six equations of continuity of mechanical displacement and stress.

The two additional equations, however are not those of continuity but:

$$\Phi(I)\Big|_{x_3=0} = \Phi(II)\Big|_{x_3=0} = 0 \qquad (5\text{-}29\text{-a})$$

which lead to:

$$\sum_{N=1}^{4} p_4^{(N)}(n)\ A^{(N)}(n) = 0 \qquad (5\text{-}29\text{-b})$$

(5-29-b) together with the welded conditions lead to:

$$\begin{pmatrix} p_k^{(N)}(I) & -p_k^{(N)}(II) \\ q_k'^{(N)}(I) & -q_k'^{(N)}(II) \\ p_4^{(N)}(I) & 0 \\ 0 & p_4^{(N)}(II) \end{pmatrix} \cdot \begin{pmatrix} A^{(N)}(I) \\ A^{(N)}(II) \end{pmatrix} = 0 \qquad (5\text{-}30)$$

$k=1,2,3$

$N=1,\dots,4$

For non-trivial solution $A^{(N)}(n)$ (5-30) leads to:

$$\left\| \begin{matrix} p_k^{(N)}(I) & -p_k^{(N)}(II) \\ q_k'^{(N)}(I) & -q_k'^{(N)}(II) \\ p_4^{(N)}(I) & 0 \\ 0 & p_4^{(N)}(II) \end{matrix} \right\| = 0 \qquad (5\text{-}31)$$

Obviously, (5-29-a) guarantees continuity of electric potential, however, it does not guarantee continuity of the normal electric displacement. When the welded conditions (5-11) lead to zero electric potential at the interface the generalized Stoneley wave coincides with the Bleustein type wave.

One can treat (5-30) in a similar way to that in which generalized Stoneley conditions were reduced to a 4x4 determinantal condition. However, one has to remember that here the summation in the matrix:

$$D*^{MN} = \overline{q_m'^{(M)}} p_m^{(N)} \qquad m=1,2,3 \qquad (5\text{-}32)$$

is over three components only.

Using the equations of motion (with summation over three components of the mechanical displacement and three components of the mechanical stress) one arrives at:

$$(s_3^{(N)} - \overline{s_3^{(M)}})(\overline{D*^{NM}} + D*^{MN}) = e_{\ell ij}[\overline{s_\ell^{(M)}} s_j^{(N)} p_i^{(N)} \overline{p_4^{(M)}} - s_\ell^{(N)} p_4^{(N)} \overline{s_j^{(M)}}\,\overline{p_i^{(M)}}] =$$

$$= [\overline{s_3^{(M)}} - s_3^{(N)}]G*^{MN} \qquad (5\text{-}33)$$

From the first six equations (5-30) one obtains:

$$\left.\begin{aligned} &\sum_{N=1}^{4} \{-G*^{MN}(I)A^{(N)}(I) + \overline{F*^{NM}}\, A^{(N)}(II)\} = 0 \\ &\sum_{N=1}^{4} \{F*^{MN}\, A^{(N)}(I) - G*^{MN}(II)\, A^{(N)}(II)\} = 0 \end{aligned}\right\} \qquad (5\text{-}34)$$

$$\text{where } - F*^{MN} = \overline{q_k'^{(M)}(II)}\, p_k^{(N)}(I) + \overline{p_k^{(M)}(II)}\, q_k'^{(N)}(I) \qquad (5\text{-}35)$$

Making use of boundary conditions (5-29-b) simplifies $G*^{MN}(n)A^{(N)}(n)$, since $p_4^{(N)}(n)A^{(N)}(n) = 0$. However, in general it would not vanish, and one has to treat the two equations of (5-34) with simplified $G^{*MN}(n)A^{(N)}(n)$ as (5-17), and the discussion following it, with G^{*MN} replacing G^{MN}, and F^{*MN} replacing $\hat{F}^{MN}$, bearing in mind that * matrices are in general different from the non* matrices.

6. WAVES AT AN INTERFACE BETWEEN PIEZOELECTRIC MEDIA, SOME SYMMETRIC CASES.

After obtaining the conditions for interface waves in piezoelectric media we shall obtain simplified conditions for symmetric configurations of piezoelectric media, similar to those in chapter 3, and proceed to investigate the symmetric media studied in chaper 4. In particular we shall look into the difference between interface waves in simple elastic media and piezoelectric media.

The notations used are similar to those of chapters 3 & 4. As in chapters 3 & 4 the transformation matrix (3-5) is used to obtain the different constants in medium II from those of medium I. Since $c_{ijk\ell}$ is a fourth order tensor the transformation is dependent on the sign of products of pairs h_ih_j rather than the sign of the individual h_i. Therefore, $c_{ijk\ell}$ are invariant under inversion. However, d_{ijk} is a third order tensor and is dependent on the individual sign of h_i. It therefore changes under inversion. Hence, whereas in simple elastic media complete inversion does not affect the waves propagating, it would affect the wave propagating in piezoelectric media.

Using the transformation matrix (3-5) to obtain the state tensors of medium II from those of medium I, one obtains two eighth order polynomial equations for $s_3(\mathrm{I})$ and $s_3(\mathrm{II})$, which are the conditions for non-trivial displacements $p_k(n)$. The coefficients of the odd powers of $s_3(n)$ differ by a factor h_1h_3, which means that the roots of the secular equations are related as:

$$s_3^{(M)}(\mathrm{II}) = h_1h_3s_3^{(M)}(\mathrm{I}) \qquad M = 1,\ldots,8 \qquad \text{(6-1-a)}$$

When the secular equation is bi-quartic:

$$s_3^{(M)}(\mathrm{I}) = s_3^{(M)}(\mathrm{I}) \qquad M = 1,\ldots,8 \qquad \text{(6-1-b)}$$

Since we seek interface wave solutions which attenuate with increasing distance from the interface we choose in half-space I the four roots with positive imaginary part while in half-space II the roots with negative imaginary parts are taken.

$$s_3^{(N)}(\mathrm{II}) = h_1h_3\Re\{s_3^{(N)}(\mathrm{I})\} - i\Im\{s_3^{(N)}(\mathrm{I})\} \quad N=1,\ldots,4 \quad \text{(6-2-a)}$$

When the secular equation is bi-quartic, since the roots $s_3^{(N)}(n)$ are the square roots of the zeros of the quartic equation one may write:

$$s_3^{(N)}(II) = - s_3^{(N)}(I) \qquad N = 1,\ldots,4 \tag{6-2-b}$$

regardless of the sign of h_1h_3. When the roots are complex, one may renumerate them so that they will comply with (6-2-a). However, when $h_1h_3 = +1$ and $\mathcal{I}\{s_3^{(N)}(I)\} = 0$, although (6-2-a) may hold, it is quite possible that (6-2-b) holds and one case is not equivalent to the other. Like in chapters 3 & 4 we assume in the following discussion either $\mathcal{I}\{s_3^{(N)}(I)\} \neq 0$ or $h_1h_3 = -1$. It will be pointed out when (6-2-b) holds rather than (6-2-a).

Substituting (6-2-a) into the secular equation, the elements of the secular matrix $S_{kl}^{(N)}(n)$ may then be related as:

$$S_{KL}^{(N)}(II) = h_K h_L [\mathcal{R}\{S_{KL}^{(N)}(I)\} - ih_1h_3\mathcal{I}\{S_{KL}^{(N)}(I)\}] \quad K,L=1,2,3 \tag{6-3-a}$$

$$S_{K4}^{(N)}(II) = h_K [\mathcal{R}\{S_{K4}^{(N)}(I)\} - ih_1h_3\mathcal{I}\{S_{K4}^{(N)}(I)\}] \quad K=1,2,3 \tag{6-3-b}$$

$$S_{44}^{(N)}(II) = \mathcal{R}\{S_{44}^{(N)}(I)\} - ih_1h_3\mathcal{I}\{S_{44}^{(N)}(I)\} \tag{6-3-c}$$

(6-3) can be summed up in the relationship:

$$S_{KL}^{(N)}(II) = h_K h_L [R\{S_{KL}^{(N)}(I)\} - ih_1h_3\mathcal{I}\{S_{KL}^{(N)}(I)\}] \quad K,L=1,\ldots,4 \tag{6-4}$$

$$\text{and} \quad h_4 \overset{\text{def}}{=} 1 \tag{6-5}$$

The ratios of the components $p_k^{(N)}(n)$ is given as the ratios of the cofactors:

$$p_1^{(N)}(n):p_2^{(N)}(n):p_3^{(N)}(n):p_4^{(N)}(n) = - \begin{Vmatrix} S_{14}^{(N)}(n) & S_{12}^{(N)}(n) & S_{13}^{(N)}(n) \\ S_{24}^{(N)}(n) & S_{22}^{(N)}(n) & S_{23}^{(N)}(n) \\ S_{34}^{(N)}(n) & S_{23}^{(N)}(n) & S_{33}^{(N)}(n) \end{Vmatrix} :$$

$$+ \begin{Vmatrix} S_{11}^{(N)}(n) & S_{13}^{(N)}(n) & S_{14}^{(N)}(n) \\ S_{12}^{(N)}(n) & S_{23}^{(N)}(n) & S_{24}^{(N)}(n) \\ S_{13}^{(N)}(n) & S_{33}^{(N)}(n) & S_{34}^{(N)}(n) \end{Vmatrix} : - \begin{Vmatrix} S_{11}^{(N)}(n) & S_{12}^{(N)}(n) & S_{14}^{(N)}(n) \\ S_{12}^{(N)}(n) & S_{22}^{(N)}(n) & S_{24}^{(N)}(n) \\ S_{13}^{(N)}(n) & S_{23}^{(N)}(n) & S_{34}^{(N)}(n) \end{Vmatrix} :$$

$$+ \begin{Vmatrix} S_{11}^{(N)}(n) & S_{12}^{(N)}(n) & S_{13}^{(N)}(n) \\ S_{12}^{(N)}(n) & S_{22}^{(N)}(n) & S_{23}^{(N)}(n) \\ S_{13}^{(N)}(n) & S_{23}^{(N)}(n) & S_{33}^{(N)}(n) \end{Vmatrix} \tag{6-6}$$

When one compares the 'displacement' vectors $p_k^{(N)}(n)$, $k=1,\ldots,4$, one obtains the following relationships:

$$p_K^{(N)}(II) = \chi^{(N)} h_K h_3 [\Re\{p_K^{(N)}(I)\} - i h_1 h_3 \Im\{p_K^{(N)}(I)\}] \quad K=1,\dots,4 \qquad (6\text{-}7)$$

and for the 'stress' components:

$$q_K'^{(N)}(II) = \chi^{(N)} h_K h_1 [\Re\{q_K'^{(N)}(I)\} - i h_1 h_3 \Im\{q_K'^{(N)}(I)\}] \quad K=1,\dots,4 \qquad (6\text{-}8)$$

Equations (6-7) and (6-8) appear the same as (3-11) and (3-12). However, they are the same only in form. Let us observe the electrical 'displacement' component, p_4, which describes the scalar potential (see (5-7)) and, electromechanical stress q_4' (as defined in (5-12)). Since by definition $h_4 = 1$, $p_4^{(N)}(II) = \chi^{(N)} h_3 [\Re\{p_4^{(N)}(I)\} - i h_1 h_3 \Im\{p_4^{(N)}(I)\}]$ and $q_4'^{(N)}(II) = \chi^{(N)} h_1 [\Re\{q_4'^{(N)}(I)\} - i h_1 h_3 \Im\{q_4'^{(N)}(I)\}]$.

Therefore, the electric effect in the 'displacement' component is dependent on h_3 in the same way that the 'stress' electromechanical component is dependent on h_1, both are independent of h_2.

When $h_1 h_3 = -1 \quad (h_1 = h)$

$$p_K^{(N)}(II) = -\chi^{(N)} h h_K p_K^{(N)}(I) \qquad (6\text{-}9)$$

$$q_K'^{(N)}(II) = \chi^{(N)} h h_K q_K'^{(N)}(I) \qquad (6\text{-}10)$$

Using the algebraic identity (3-15) on the generalized Stoneley condition (5-11) and substituting equations (6-9) and (6-10) one obtains two decoupled sets of linear homogeneous equations (similar to (3-17)), one for $B_+^{(N)} = A^{(N)}(I) + \chi^{(N)} A^{(N)}(II)$ and the other for $B_-^{(N)} = A^{(N)}(I) - \chi^{(N)} A^{(N)}(II)$. At least one of these has to have a non-trivial solution in order to have an interface wave. The equations may be written in the following form:

$$\begin{pmatrix} p_1^{(N)}(I) \\ r_2^{(N)}(I) \\ q_3'^{(N)}(I) \\ r_4^{(N)}(I) \end{pmatrix} \cdot B_+^{(N)} = 0 \qquad (6\text{-}11\text{-}a)$$

and:

$$\begin{pmatrix} q_1'^{(N)}(I) \\ t_2^{(N)}(I) \\ p_3^{(N)}(I) \\ t_4^{(N)}(I) \end{pmatrix} \cdot B_-^{(N)} = 0 \qquad (6\text{-}11\text{-}b)$$

where $r_2^{(N)}(I)$ and $t_2^{(N)}(I)$ are defined as in (3-18-c) and (3-18-d) and $r_4^{(N)}(I)$ and $t_4^{(N)}(I)$ are dependent on the exact value of h:

$$r_4^{(N)}(I) = \begin{cases} p_4^{(N)}(I) & \text{if } h = 1 \\ q_4'^{(N)}(I) & \text{if } h = -1 \end{cases} \qquad (6\text{-}11\text{-}c)$$

and

$$t_4^{(N)}(I) = \begin{cases} q_4'^{(N)}(I) & \text{if } h = 1 \\ p_4^{(N)}(I) & \text{if } h = -1 \end{cases} \qquad (6\text{-}11\text{-}d)$$

Comparing equations (6-11) with the corresponding equations for simple elastic media (3-18) one basic difference is apparent. Whereas in the case of simple elastic media the equations are dependent on the sign of products of pairs hh_i, in the case of piezoelectric media the dependence is on the actual value of h_1. Therefore, while in the simple elastic case there are two distinct configurations (for $h_1h_3=-1$) in the piezoelectric case there are four.

The conditions for Bleustein type waves would be of the form (6-11-a) and (6-11-b) with:

$$r_4^{(N)}(I) = p_4^{(N)}(I) \qquad (6\text{-}11\text{-}e)$$
$$t_4^{(N)}(I) = p_4^{(N)}(I)$$

These type waves do not depend on the actual value of h.

Using the values of $A^{(N)}(I)$ and $A^{(N)}(II)$ in terms of $B_\pm^{(N)}$ and (6-11) one obtains the following values for the total 'displacement' and total 'stress' vectors at the interface:

$$P_1(n) = \tfrac{1}{2}\Sigma_{N=1}^{4} p_1^{(N)}(I)B_-^{(N)} \qquad (6\text{-}12\text{-}a)$$
$$P_2(n) = \tfrac{1}{2}\Sigma_{N=1}^{4} p_2^{(N)}(I)B_-^{(N)} \quad \text{when } h_2 = h_1 \qquad (6\text{-}12\text{-}b)$$
$$P_2(n) = \tfrac{1}{2}\Sigma_{N=1}^{4} p_2^{(N)}(I)B_+^{(N)} \quad \text{when } h_2 = -h_1 \qquad (6\text{-}12\text{-}c)$$
$$P_3(n) = \tfrac{1}{2}\Sigma_{N=1}^{4} p_3^{(N)}(I)B_+^{(N)} \qquad (6\text{-}12\text{-}d)$$
$$P_4(n) = \tfrac{1}{2}\Sigma_{N=1}^{4} p_4^{(N)}(I)B_-^{(N)} \quad \text{when } h_1 = +1 \qquad (6\text{-}12\text{-}e)$$
$$P_4(n) = \tfrac{1}{2}\Sigma_{N=1}^{4} p_4^{(N)}(I)B_+^{(N)} \quad \text{when } h_1 = -1 \qquad (6\text{-}12\text{-}f)$$

$$Q_1'(n) = \tfrac{1}{2}\Sigma_{N=1}^{4} q_1'^{(N)}(I)B_+^{(N)} \qquad (6\text{-}13\text{-}a)$$
$$Q_2'(n) = \tfrac{1}{2}\Sigma_{N=1}^{4} q_2'^{(N)}(I)B_+^{(N)} \quad \text{when } h_2 = h_1 \qquad (6\text{-}13\text{-}b)$$
$$Q_2'(n) = \tfrac{1}{2}\Sigma_{N=1}^{4} q_2'^{(N)}(I)B_-^{(N)} \quad \text{when } h_2 = h_1 \qquad (6\text{-}13\text{-}c)$$
$$Q_3'(n) = \tfrac{1}{2}\Sigma_{N=1}^{4} q_3'^{(N)}(I)B_-^{(N)} \qquad (6\text{-}13\text{-}d)$$

$$Q_4'(n) = \tfrac{1}{2}\Sigma_{N=1}^{4} q_4'^{(N)}(I)B_+^{(N)} \qquad \text{when } h_1 = +1 \qquad (6\text{-}13\text{-}e)$$

$$Q_4'(n) = \tfrac{1}{2}\Sigma_{N=1}^{4} q_4'^{(N)}(I)B_-^{(N)} \qquad \text{when } h_1 = -1 \qquad (6\text{-}13\text{-}f)$$

In Bleustein type waves (6-12-a) - (6-12-d) hold, and $P_4(n) = 0$ for the 'displacement'. For the 'stress' components (6-13-a) - (6-13-d) hold while $Q_4'(n) = \Sigma_{N=1}^{4} q'^{(N)}(n)A^{(N)}(n)$.

Comparing (6-12) and (6-13) to (3-24) and (3-25), the corresponding equations for simple elastic media, one can see that the equations describing the mechanical displacement and stress in the piezoelectric media are the same as for the case of simple elastic media. Although the electrical effect would be felt in the actual values of $p_i^{(N)}(n)$, $q'^{(N)}_i(n)$, and $B_\pm^{(N)}$, the character of the wave is the same whether the media involved are simple elastic or piezoelectric and elastic. The electric potential component, P_4, and the electromechanical stress Q_4' are dependent on the actual value of h_1 in the Stoneley type wave, but not in the Bleustein type wave.

The determinants of the matrices in (6-11) may vanish separately or simultaneously, just like (3-18). Checking the possible waves for the different configurations:

When $B_-^{(N)} = 0$ (the determinant of the matrix in (6-11-a) vanishes while that in (6-11-b) does not), from (6-12) and (6-13):

$P_1(n) = Q_3'(n) = 0$

$$\text{If } h_i = (1,1,-1) \qquad \left.\begin{aligned} P_i(I) &= (0,0,P_3,0) \\ Q_i'(I) &= (Q_1',Q_2',0,Q_4') \end{aligned}\right) \qquad (6\text{-}14\text{-}a)$$

$$h_i = (1,-1,-1) \qquad \left.\begin{aligned} P_i(I) &= (0,P_2,P_3,0) \\ Q_i'(I) &= (Q_1',0,0,Q_4') \end{aligned}\right) \qquad (6\text{-}14\text{-}b)$$

$$h_i = (-1,1,1) \qquad \left.\begin{aligned} P_i(I) &= (0,P_2,P_3,P_4) \\ Q_i'(I) &= (Q_1',0,0,0) \end{aligned}\right) \qquad (6\text{-}14\text{-}c)$$

$$h_i = (-1,-1,1) \qquad \left.\begin{aligned} P_i(I) &= (0,0,P_3,P_4) \\ Q_i'(I) &= (Q_1',Q_2',0,0) \end{aligned}\right) \qquad (6\text{-}14\text{-}d)$$

Notice that for (6-14-a) and (6-14-b) the conditions for a Bleustein type wave are satisfied.

When $B_{+}^{(N)} = 0$ and $B_{-}^{(N)} \neq 0$

$P_3(n) = Q_1'(n) = 0$. The following are the forms of the different possible waves for such a case:

$$\text{If } h_i = (1,1,-1) \qquad \left.\begin{aligned} P_i(I) &= (P_1,P_2,0,P_4) \\ Q_i'(I) &= (0,0,Q_3',0) \end{aligned}\right\} \qquad (6\text{-}15\text{-}a)$$

$$h_i = (1,-1,-1) \qquad \left.\begin{aligned} P_i(I) &= (P_1,0,0,P_4) \\ Q_i'(I) &= (0,Q_2',Q_3',0) \end{aligned}\right\} \qquad (6\text{-}15\text{-}b)$$

$$h_i = (-1,1,1) \qquad \left.\begin{aligned} P_i(I) &= (P_1,0,0,0) \\ Q_i'(I) &= (0,Q_2',Q_3',Q_4') \end{aligned}\right\} \qquad (6\text{-}15\text{-}c)$$

$$h_i = (-1,-1,1) \qquad \left.\begin{aligned} P_i(I) &= (P_1,P_2,0,0) \\ Q_i'(I) &= (0,0,Q_3',Q_4') \end{aligned}\right\} \qquad (6\text{-}15\text{-}d)$$

Under these conditions in the configurations (6-15-c) and (6-15-d) Stoneley type waves and Bleustein type waves are the same.

In both (6-14) and (6-15) when one of the determinants of the matrices in (6-11) vanishes and the other does not vanish, the electrical effect at the interface is localized to either the electrical potential or the electromechanical 'stress' component, depending on the actual value of h_1 (and therefore h_3 as well) and which one of the determinants (in (6-11)) vanishes. It is independent of the value of h_2, although the mechanical components are dependent on the value of h_1h_2.

The relation between the transformations in the pairs [(6-14-a) and (6-14-d)], [(6-14-b) & (6-14-c)] , [(6-15-a) & (6-15-d)] and [(6-15-b) & (6-15-c)] is of inversion and therefore the mechanical components are of the same form in the two members of each pair. However, the electrical components in the members within a pair are different.

For each of the transformations there is a correlation between the wave for which $B_{-}^{(N)} = 0$ and the one for which $B_{+}^{(N)} = 0$. The vanishing components in the 'displacement' vector when $B_{\pm}^{(N)} = 0$ are the same as the vanishing components in the 'stress' vector when $B_{\mp}^{(N)} = 0$. For $h = +1$ and $B_{-}^{(N)} = 0$, or $h = -1$ and $B_{+}^{(N)} = 0$ the electrical effect is localized to the 'stress' and the electrical potential, P_4, vanishes at the interface. When $h = \pm 1$ and $B_{\pm}^{(N)} = 0$, the electrical effect is localized to the electric potential, and the electromechanical 'stress', Q_4', vanishes at the interface.

When $B_-^{(N)} = 0$, the mechanical displacement is transverse, the stress is purely longitudinal for transformations $(1,-1,-1)$ and $(-1,1,1)$. When $B_+^{(N)} = 0$, the mechanical stress is transverse and the displacement is purely longitudinal for these transformations.

One should note that in both these cases, either $B_-^{(N)} = 0$ or $B_+^{(N)} = 0$, if the piezoelectric effect is zero then the sign of h is not important and these cases reduce to those discussed in chapter 3.

There is always the possibility that the two determinants of the coefficient matrices in (6-11) vanish simultaneously, in which case it is possible that neither $B_-^{(N)}$ nor $B_+^{(N)}$ are zero vectors and therefore $P_i(n)$ and $Q_i'(n)$ may have four non-zero components, given by (6-12) and (6-13).

When one imposes the Bleustein type conditions $P_4(n) = 0$, the mechanical and electrical components do not depend on the actual value of h:

When the determinant of coefficients of $B_+^{(N)}$ vanishes, while that of $B_-^{(N)}$ does not vanish, $B_-^{(N)} = 0$ and:

$$\text{for } h_i = (h,h,-h) \qquad \left.\begin{array}{l} P_i(I) = (0,0,P_3,0) \\ Q_i'(I) = (Q_1',Q_2',0,Q_4') \end{array}\right) \qquad (6\text{-}14\text{-e})$$

$$h_i = (h,-h,-h) \qquad \left.\begin{array}{l} P_i(I) = (0,P_2,P_3,0) \\ Q_i'(I) = (Q_1',0,0,Q_4') \end{array}\right) \qquad (6\text{-}14\text{-f})$$

When $B_+^{(N)} = 0$ and $B_-^{(N)} \neq 0$

$$\text{for } h_i = (h,h,-h) \qquad \left.\begin{array}{l} P_i(I) = (P_1,P_2,0,0) \\ Q_i(I) = (0,0,Q_3',Q_4') \end{array}\right) \qquad (6\text{-}15\text{-e})$$

$$h_i = (h,-h,-h) \qquad \left.\begin{array}{l} P_i(I) = (P_1,0,0,0) \\ Q_i(I) = (0,Q_2',Q_3',Q_4') \end{array}\right) \qquad (6\text{-}15\text{-f})$$

$B_-^{(N)}$ and $B_+^{(N)}$ are not necessarily the same as those for the Stoneley type waves, they depend on the value of $r_4^{(N)}$ and $t_4^{(N)}$.

When $h_1 h_3 = +1$

$$p_K^{(N)}(II) = \chi^{(N)} h h_K \overline{p_K^{(N)}(I)} \qquad (6\text{-}16\text{-a})$$

$$q_K'^{(N)}(II) = \chi^{(N)} h h_K \overline{q_K'^{(N)}(I)} \qquad (6\text{-}16\text{-b})$$

One cannot simplify much further the generalized Stoneley conditions. It is possible to rewrite the Stoneley conditions for this case in terms of real and imaginary parts of $p_k^{(N)}(I)$ and $q_k'^{(N)}(I)$, as in (3-28).

If $h_1 = h_2 = h_3 = +1$, using (3-29) with N varying from 1 to 4, and i having values of 1 to 4, with q_k' replacing q_k:

$$\Sigma_{N=1}^{4} \rho\{p_i^{(N)}(I)\ [A^{(N)}(I) - \overline{\chi^{(N)}}\ \overline{A^{(N)}(II)}]\} = 0 \qquad (6\text{-}17\text{-}a)$$

$$\Sigma_{N=1}^{4} \rho\{q_i'^{(N)}(I)\ [A^{(N)}(I) - \overline{\chi^{(N)}}\ \overline{A^{(N)}(II)}]\} = 0 \qquad (6\text{-}17\text{-}b)$$

Using the definition of $q_i'^{(N)}(n)$, (5-10-b), (5-12) and (6-17-a) one obtains:

$$c_{i3k3}\ \Sigma_{N=1}^{4} \rho\{p_k^{(N)}(I)s_3^{(N)}(I)[A^{(N)}(I) - \overline{\chi^{(N)}A^{(N)}(II)}]\} + \qquad (6\text{-}17\text{-}c)$$

$$+\ e_{3i3}\Sigma_{N=1}^{4} \rho\{p_4^{(N)}(I)s_3^{(N)}(I)[A^{(N)}(I) - \overline{\chi^{(N)}A^{(N)}(II)}]\} = 0$$

$$i,k = 1,2,3$$

and

$$e_{3k3}\Sigma_{N=1}^{4} \rho\{p_k^{(N)}(I)s_3^{(N)}(I)[A^{(N)}(I) - \overline{\chi^{(N)}A^{(N)}(II)}]\} + \qquad (6\text{-}17\text{-}d)$$

$$+\ \epsilon_{33}\Sigma_{N=1}^{4} \rho\{p_4^{(N)}(I)s_3^{(N)}(I)[A^{(N)}(I) - \chi^{(N)}A^{(N)}(II)]\} = 0$$

$$k = 1,2,3$$

The 4x4 matrix
$$\begin{pmatrix} c_{i3k3} & e_{3i3} \\ e_{3k3} & \epsilon_{33} \end{pmatrix} \qquad (6\text{-}18)$$

is regular, and therefore we can follow the arguments of chapter 3 to prove that no generalized Stoneley waves are expected at an interface between two media having the same elastic and piezoelectric coefficients and the same orientation with respect to the interface coordinate system. In the case of non-piezoelectric media $e_{3i3} = 0$ and one is left with the case discussed in chapter 3. One should note that unlike the case of simple elastic media, these arguments do not hold for complete inversion ($h_i = -1$).

In a Bleustein type configuration (6-17-d) is not necessarily correct. In (6-17-c), if $e_{3i3} = 0$ the case still reduces to simple elastic media, otherwise, one has to check the possibility of a wave under the condition (5-31).

One of the configurations where the difference between simple elastic media and piezoelectric media manifests itself most is that of complete inversion. For simple elastic media complete inversion is the same as the identity and no attenuating waves at the interface are expected. But if the media are piezoelectric, this is not necessarily the case:

$$\left.\begin{aligned} p_k^{(N)}(II) &= \chi^{(N)}\overline{p_k^{(N)}(I)} \\ p_4^{(N)}(II) &= -\chi^{(N)}\overline{p_4^{(N)}(I)} \\ q_k'^{(N)}(II) &= \chi^{(N)}\overline{q_k'^{(N)}(I)} \\ q_4'^{(N)}(II) &= -\chi^{(N)}\overline{q_4'^{(N)}(I)} \end{aligned}\right) \qquad (6\text{-}19)$$

$$k = 1,2,3$$

One obtains a generalized Stoneley condition of an 8x8 determinant which has to vanish, and depending on the media characteristics one may or may not obtain attenuating waves.

The condition for Bleustein type waves is the same in case of identity or complete inversion.

We shall now try to further simplify the results for cases of particular symmetries within the media on the two sides of the interface.

Following the arguments of chapter 4: If medium I has symmetry plane which is perpendicular to the x_3 axis, its elastic stiffnesses tensor is of the form given by (4-2). The piezoelectric tensor e_{ijk} (if the symmetry is that of proper 2-fold rotation):

$$\begin{pmatrix} 0 & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & * & 0 \\ * & * & * & 0 & 0 & * \end{pmatrix} \qquad (6\text{-}20)$$

and for the same symmetry ϵ_{ij} is of the form:

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \qquad (6\text{-}21)$$

$S_{KL}^{(N)}(n)$, for K, L=1,2,3 are the same as in the non-piezoelectric case, and given by (4-3-a) and (4-3-b), $S_{K4}^{(N)}(n)$ is given by:

$$\left.\begin{aligned} S_{14}^{(N)}(I) &= (e_{131} + e_{311})s_1 s_3^{(N)}(I) \\ S_{24}^{(N)}(I) &= (e_{132} + e_{312})s_1 s_3^{(N)}(I) \\ S_{34}^{(N)}(I) &= e_{113}s_1^2 + e_{333}[s_3^{(N)}(I)]^2 \\ S_{44}^{(N)}(I) &= -\epsilon_{11}s_1^2 - \epsilon_{33}[s_3^{(N)}(I)]^2 \end{aligned}\right) \qquad (6\text{-}22\text{-a})$$

and for the second medium:

$$S_{14}^{(N)}(II) = h_3(e_{131} + e_{311})s_1 s_3^2(II) \qquad)$$

$$\left.\begin{aligned}
S_{24}^{(N)}(II) &= h_1h_2h_3(e_{132} + e_{312})s_1 s_3^{(N)}(II) \\
S_{34}^{(N)}(II) &= h_3\{e_{113}s_1^2 + e_{333}[s_3^{(N)}(I)]^2\} \\
S_{44}^{(N)}(II) &= -\epsilon_{11}s_1^2 - \epsilon_{33}[s_3^{(N)}(I)]^2
\end{aligned}\right\} \qquad (6\text{-}22\text{-}b)$$

The elements of $S_{K4}^{(N)}(II)$ are dependent on the sign of h_1h_2, like $S_{KL}^{(N)}(II)$ for K, L = 1,2,3, and in addition on the actual sign of h_3. For configurations of piezoelectric media, which have a plane of symmetry which is perpendicular to the x_3 axis, where $h_1h_2 = +1$ and $h_3 = +1$ we would not expect an attenuating Stoneley type waves. Suppose $h_1h_2 = +1$ but $h_3 = -1$. The eighth order polynomial of the secular equation is bi-quartic. The complex roots of the quartic equation may have one pair, 2 pairs or none of complex conjugates. The real roots of the quartic, if they exist and will lead to a true attenuating wave, must be all negative. There are three possible forms for the slowness of attenuating waves:

$$\left.\begin{aligned}
s_3^{(N)}(I) &= is^a,\ is^b,\ is^c,\ is^d \\
s_3^{(N)}(II) &= -is^a, -is^b, -is^c, -is^d
\end{aligned}\right\} \qquad (6\text{-}23\text{-}a)$$

$$\left.\begin{aligned}
s_3^{(N)}(I) &= is^a,\ is^b,\ s^c + is^d,\ -s^c + is^d \\
s_3^{(N)}(II) &= -is^a,\ -is^b,\ -s^c - is^d,\ s^c - is^d
\end{aligned}\right\} \qquad (6\text{-}23\text{-}b)$$

$$\left.\begin{aligned}
s_3^{(N)}(I) &= s^a + is^b,\ s^a - is^b,\ s^c + is^d,\ s^c - is^d \\
s_3^{(N)}(II) &= -s^a - is^b,\ s^a - is^b, -s^c - is^d,\ s^c - is^d
\end{aligned}\right\} \qquad (6\text{-}23\text{-}c)$$

In all these cases the same pattern appears:

$$s_3^{(N)}(II) = -\,s_3^{(N)}(I) \qquad (6\text{-}23\text{-}d)$$

One may notice that it is possible to rearrange the slowness components so that the relationship between the components in the two media will be $s_3^{(M)}(II) = \overline{s_3^{(M)}(I)}$, this in turn would cause a different order of the 'displacement' and 'stress' components, which may differ in form but lead to the same total displacement and 'stress' vectors.

Using (6-23-d), (6-22), (4-3) and (6-6), when there is a plane of symmetry perpendicular to the x_3 axis, and in this numeration, the 'displacement' components are related as follows:

$$p_1^{(N)}(II) = -\,h_3 p_1^{(N)}(I)$$

$$\left.\begin{aligned} p_2^{(N)}(II) &= -h_1h_2h_3p_2^{(N)}(I) \\ p_3^{(N)}(II) &= h_3p_3^{(N)}(I) \\ p_4^{(N)}(II) &= p_4^{(N)}(I) \end{aligned}\right\} \qquad (6\text{-}24\text{-a})$$

If the slowness components are given by (6-23-a), the 'displacement' components have the form:

$$p_k^{(N)}(I) = \begin{pmatrix} i\alpha_1^a & i\alpha_1^b & i\alpha_1^c & i\alpha_1^d \\ i\alpha_2^a & i\alpha_2^b & i\alpha_2^c & i\alpha_2^d \\ \alpha_3^a & \alpha_3^b & \alpha_3^c & \alpha_3^d \\ \alpha_4^a & \alpha_4^b & \alpha_4^c & \alpha_4^d \end{pmatrix} \qquad (6\text{-}24\text{-b})$$

and that of the second medium for this case can be obtained by use of (6-24-a).

If the slowness components are given by (6-23-c) the 'displacement' is of the form:

$$p_k^{(N)}(I) = \begin{pmatrix} \alpha_1^a+i\alpha_1^b & -\alpha_1^a+i\alpha_1^b & \alpha_1^c+i\alpha_1^d & -\alpha_1^c+i\alpha_1^d \\ \alpha_2^a+i\alpha_2^b & -\alpha_2^a+i\alpha_2^b & \alpha_2^c+i\alpha_2^d & -\alpha_2^c+i\alpha_2^d \\ \alpha_3^a+i\alpha_3^b & \alpha_3^a-i\alpha_3^b & \alpha_3^c+i\alpha_3^d & \alpha_3^c-i\alpha_3^d \\ \alpha_4^a+i\alpha_4^b & \alpha_4^a-i\alpha_4^b & \alpha_4^c+i\alpha_4^d & \alpha_4^c-i\alpha_4^d \end{pmatrix} \qquad (6\text{-}24\text{-c})$$

In the same manner one may obtain the form for $p_k^{(N)}(n)$ when the slowness components are of the form (6-23-b).

Substituting (6-24-a) in the definition of $q_k'^{(N)}(n)$, (5-10-b) and (5-12), one may obtain the form of the 'stress' vector components which correspond to the different possible forms of the slowness components. For all possible slowness forms which are related as (6-23-d) one obtains the following relations between the components of the 'stress' vectors in the two media:

$$\left.\begin{aligned} q_1'^{(N)}(II) &= h_3q_1'^{(N)}(I) \\ q_2'^{(N)}(II) &= h_1h_2h_3q_2'^{(N)}(I) \\ q_3'^{(N)}(II) &= -h_3q_3'^{(N)}(I) \\ q_4'^{(N)}(II) &= -q_4'^{(N)}(I) \end{aligned}\right\} \qquad (6\text{-}25)$$

Substituting in the generalized Stoneley conditions (6-24-d) and (6-25) one obtains a set of eight homogeneous linear equations which are decoupled, or two sets of four homogeneous linear equations each, in $B_-^{(N)}$ and $B_+^{(N)}$:

$$\Sigma_{N=1}^{4} p_1^{(N)}(I)[A^{(N)}(I) + h_3 A^{(N)}(II)] = 0 \qquad (6\text{-}26\text{-}a)$$

$$\Sigma_{N=1}^{4} p_2^{(N)}(I)[A^{(N)}(I) + h_1 h_2 h_3 A^{(N)}(II)] = 0 \qquad (6\text{-}26\text{-}b)$$

$$\Sigma_{N=1}^{4} p_3^{(N)}(I)[A^{(N)}(I) - h_3 A^{(N)}(II)] = 0 \qquad (6\text{-}26\text{-}c)$$

$$\Sigma_{N=1}^{4} p_4^{(N)}(I)[A^{(N)}(I) - A^{(N)}(II)] = 0 \qquad (6\text{-}26\text{-}d)$$

$$\Sigma_{N=1}^{4} q'^{(N)}_1(I)[A^{(N)}(I) - h_3 A^{(N)}(II)] = 0 \qquad (6\text{-}26\text{-}e)$$

$$\Sigma_{N=1}^{4} q'^{(N)}_2(I)[A^{(N)}(I) - h_1 h_2 h_3 A^{(N)}(II)] = 0 \qquad (6\text{-}26\text{-}f)$$

$$\Sigma_{N=1}^{4} q'^{(N)}_3(I)[A^{(N)}(I) + h_3 A^{(N)}(II)] = 0 \qquad (6\text{-}26\text{-}g)$$

$$\Sigma_{N=1}^{4} q'^{(N)}_4(I)[A^{(N)}(I) + A^{(N)}(II)] = 0 \qquad (6\text{-}26\text{-}h)$$

(6-26-a) to (6-26-g) hold for Bleustein type wave while instead of (6-26-h) one has to write:

$$\Sigma_{N=1}^{4} p_4^{(N)}(I)[A^{(N)}(I) + A^{(N)}(II)] = 0 \qquad (6\text{-}26\text{-}i)$$

One can see that whereas in the non-piezoelectric media the sign of h_3 is irrelevant, here it has a significance as in the values of $S_{4K}^{(N)}(II)$. Because of the different results for different values of h_3 rather than having only one possible attenuating wave, as in the simple elastic case when $h_1 h_2 = -1$, here there are three different configurations where Stoneley type attenuating waves are possible in media with a plane of symmetry perpendicular to the x_3 axis. $h_1 h_2 = +1$ and $h_3 = 1$ is the case of identity which does not lead to an attenuating wave. However there may be a non-attenuating wave travelling along the interface, the equations of which are:

$$\begin{pmatrix} p_1^{(N)}(I) \\ p_2^{(N)}(I) \\ q'^{(N)}_3(I) \\ q'^{(N)}_4(I) \end{pmatrix} \cdot \left(A^{(N)}(I) + A^{(N)}(II)\right) = 0 \qquad (6\text{-}27\text{-}a)$$

and

$$\begin{pmatrix} q'^{(N)}_1(I) \\ q'^{(N)}_2(I) \\ p_3^{(N)}(I) \\ p_4^{(N)}(I) \end{pmatrix} \cdot \left(A^{(N)}(I) - A^{(N)}(II)\right) = 0 \qquad (6\text{-}27\text{-}b)$$

For non-trivial solutions in this configuration, for this special case, one needs the determinants of the matrices (6-27) to vanish, either separately or simultaneously. For Bleustein type wave (6-27-b) holds while in (6-27-a) $p_4^{(N)}(I)$ replaces $q_4^{(N)}(I)$. The determinants of the matrices in (6-27) may be considerably simplified:

$$(c_{33}\epsilon_{33} + e_{33}^2) \begin{Vmatrix} p_1^{(N)}(I) \\ p_2^{(N)}(I) \\ s_3^{(N)}(I)\ p_3^{(N)}(I) \\ s_3^{(N)}(I)\ p_4^{(N)}(I) \end{Vmatrix} = 0 \qquad (6\text{-}28\text{-}a)$$

is equivalent to the requirements of the determinant of the matrix in (6-27-a) to vanish. Similarly, the requirement of the vanishing of the determinant of the matrix in (6-27-b) can be simplified to:

$$(c_{44}c_{55} - c_{45}^2) \begin{Vmatrix} s_3^{(N)}(I)p_1^{(N)}(I) \\ s_3^{(N)}(I)p_2^{(N)}(I) \\ p_3^{(N)}(I) \\ p_4^{(N)}(I) \end{Vmatrix} = 0 \qquad (6\text{-}28\text{-}b)$$

For Bleustein type wave, instead of (6-28-a) one has:

$$\begin{Vmatrix} p_1^{(N)}(I) \\ p_2^{(N)}(I) \\ (c_{33}p_3^{(N)}(I) + e_{33}p_4^{(N)}(I))s_3^{(N)}(I) \\ p_4^{(N)}(I) \end{Vmatrix} = 0 \qquad (6\text{-}28\text{-}c)$$

These determinants, when they vanish, would lead to non-attenuating wave solutions in the case of Stoneley type configuration. The fact that no attenuating wave solutions are possible was shown in the discussion following (6-17) and (6-18). However, non-attenuating waves may comply with the continuity conditions at the plane $x_3 = 0$ and therefore be solutions of (6-26).

When $h_1h_2 = +1$ one does not expect for simple elastic media, in this symmetry, an attenuating interface wave, regardless of the value of h (chapter 4). However, when the medium is piezoelectric one does expect some waves when $h_3 = -1$. This covers two cases: complete inversion $(h_1 = h_2 = h_3 = -1)$, and rotation about the x_3 axis, with inversion (improper rotation).

Two sets of equations obtained from (6-26) are:

$$\begin{pmatrix} p_1^{(N)}(I) \\ p_2^{(N)}(I) \\ q_3'^{(N)}(I) \\ p_4^{(N)}(I) \end{pmatrix} \cdot \left(A^{(N)}(I) - A^{(N)}(II) \right) = 0 \qquad (6\text{-}29\text{-}a)$$

and

$$\begin{pmatrix} q_1'^{(N)}(I) \\ q_2'^{(N)}(I) \\ p_3^{(N)}(I) \\ q_4'^{(N)}(I) \end{pmatrix} \cdot \left(A^{(N)}(I) + A^{(N)}(II) \right) = 0 \qquad (6\text{-}29\text{-}b)$$

Notice that (6-29-a) is identical to the condition for the existence of Bleustein wave in the configuration $h_1 h_2 = +1$, $h_3 = +1$. For Bleustein type wave $(h_3 = -1)$ $p_4^{(N)}(I)$ replaces $q_4'^{(N)}(I)$ in (6-29-b) which makes it identical to (6-27-b).

The two determinants of the matrices (6-29) may vanish simultaneously or separately. From (6-24) and (6-25) one should notice that for $h_1 h_2 = +1$ and $h_3 = -1$ two forms of components are present: those of $p_1^{(N)}(I)$, $p_2^{(N)}(I)$, $q_3'^{(N)}(I)$ & $q_4'^{(N)}(I)$ (group 1), and that of $q_1'^{(N)}(I)$, $q_2'^{(N)}(I)$, $p_3^{(N)}(I)$ & $p_4^{(N)}(I)$ (group 2). If the slowness components are pure imaginary the form of the elements of group 1 are pure imaginary (multiplied by some complex coefficients) while the elements of group 2 are real (multiplied by the same coefficients). If the slowness components are complex they appear in conjugate pairs. The corresponding elements of group 1 appear as anti-conjugate pairs, and those of group 2 as conjugate pairs.

As a result the vector components of $B_+^{(N)}$ and $B_-^{(N)}$, for slowness components of the form (6-23-a):

$$B_\pm^{(1)} : B_\pm^{(2)} : B_\pm^{(3)} : B_\pm^{(4)} = i\zeta_\pm^a : i\zeta_\pm^b : i\zeta_\pm^c : i\zeta_\pm^d \qquad (6\text{-}30\text{-}a)$$

For slowness components of the form: (6-23-c):

$$B_\pm^{(1)} : B_\pm^{(2)} : B_\pm^{(3)} : B_\pm^{(4)} = \zeta_\pm^a + i\zeta_\pm^b : -\zeta_\pm^a + i\zeta_\pm^b : \zeta_\pm^c + i\zeta_\pm^d : -\zeta_\pm^c + i\zeta_\pm^d \qquad (6\text{-}30\text{-}b)$$

The total displacement components in the x_1 and x_2 directions and the total stress component in the x_3 direction and Q'_4 would all be real (multiplied by the same arbitrary complex constant), regardless of the form of the slowness components. The total displacement component in the x_3 direction and the electric potential and the total stress components in the x_1 and x_2 directions are all pure imaginary (multiplied by the same complex constant).

The general expressions for the total 'displacement' and total 'stress' vectors at the interface, for Stoneley type waves, are:

$$\left.\begin{aligned} P_i &= \tfrac{1}{2}\{\sum_{N=1}^{4} p_1^{(N)}(I)B_+^{(N)}, \sum_{N=1}^{4} p_2^{(N)}(I)B_+^{(N)}, \sum_{N=1}^{4} p_3^{(N)}(I)B_-^{(N)}, \sum_{N=1}^{4} p_4^{(N)}B_+^{(N)}\} \\ Q_i &= \tfrac{1}{2}\{\sum_{N=1}^{4} q_1^{(N)}(I)B_-^{(N)}, \sum_{N=1}^{4} q_2^{(N)}(I)B_-^{(N)}, \sum_{N=1}^{4} q_3^{(N)}(I)B_+^{(N)}, \sum_{N=1}^{4} q_4^{(N)}B_-^{(N)}\} \end{aligned}\right\} \quad (6\text{-}31\text{-}a)$$

If $B_-^{(N)} = 0$, $B_+^{(N)} = 2A^{(N)}(I)$

$$P_i = (P_1, P_2, 0, P_4) \quad , \quad Q_i' = (0, 0, Q_3', 0) \qquad (6\text{-}31\text{-}b)$$

If $B_+^{(N)} = 0$, $B_-^{(N)} = 2A^{(N)}(I)$

$$P_i = (0, 0, P_3, 0) \quad , \quad Q_i' = (Q_1', Q_2', 0, Q_4') \qquad (6\text{-}31\text{-}c)$$

For Bleustein type waves $P_4(n) = 0$ and the mechanical components are of the same form as in (6-31) although $B_+^{(N)}$ is a null vector of a different matrix from that of (6-29-b).

The above analysis dealt with two possible transformations when $h_1h_2 = +1$ and $h_3 = -1$ $(-1,-1,-1)$ and $(1,1,-1)$. In the case of improper rotation about the x_3 axis, $h_1h_3 = -1$, and therefore the general discussion and (6-9) to (6-15) hold. Since the numeration is not the same, one should notice that the different components do not correspond. However, the results are not contradictory as they may seem at first sight, and the two possible waves (6-31-b) and (6-31-c) represent the waves in (6-15-a) and (6-14-a) respectively.

Similar analysis may be done for the configurations: $h_1h_2 = -1$, $h_3=1$, where the generalized Stoneley conditions obtained from (6-26) lead to:

$$\begin{pmatrix} q_1'^{(N)}(I) \\ p_2^{(N)}(I) \\ p_3^{(N)}(I) \\ p_4^{(N)}(I) \end{pmatrix} \cdot (A^{(N)}(I) - A^{(N)}(II)) = 0 \; ; \begin{pmatrix} p_1^{(N)}(I) \\ q_2'^{(N)}(I) \\ q_3'^{(N)}(I) \\ q_4'^{(N)}(I) \end{pmatrix} \cdot (A^{(N)}(I) + A^{(N)}(II)) = 0 \qquad (6\text{-}32\text{-}a)$$

For Bleustein type wave $q_4'^{(N)}(I)$ is replaced by $p_4^{(N)}(I)$.

The total 'displacement' and 'stress' vectors are given by:

$$\left.\begin{aligned} P_i &= \tfrac{1}{2}\{\sum_{N=1}^{4} p_1^{(N)}(I)B_-^{(N)}, \sum_{N=1}^{4} p_2^{(N)}(I)B_+^{(N)}, \sum_{N=1}^{4} p_3^{(N)}(I)B_+^{(N)}, \sum_{N=1}^{4} p_4^{(N)}(I)B_+^{(N)}\} \\ Q_i' &= \tfrac{1}{2}\{\sum_{N=1}^{4} q_1'^{(N)}(I)B_+^{(N)}, \sum_{N=1}^{4} q_2'^{(N)}(I)B_-^{(N)}, \sum_{N=1}^{4} q_3'^{(N)}(I)B_-^{(N)}, \sum_{N=1}^{4} q_4'^{(N)}(I)B_-^{(N)}\} \end{aligned}\right\} \quad (6\text{-}32\text{-}b)$$

For Bleustein type wave the mechanichal components are of the same form and $P_4(n) = 0$.

When $h_1h_2 = -1$, $h_3 = -1$, one obtains:

$$\begin{pmatrix} p_1^{(N)}(I) \\ q_2'^{(N)}(I) \\ q_3'^{(N)}(I) \\ p_4^{(N)}(I) \end{pmatrix} \cdot B_-^{(N)} = 0 \quad ; \quad \begin{pmatrix} q_1'^{(N)}(I) \\ p_2^{(N)}(I) \\ p_3^{(N)}(I) \\ q_4'^{(N)}(I) \end{pmatrix} \cdot B_+^{(N)} = 0 \qquad (6\text{-}32\text{-}c)$$

The total displacement and 'stress' vectors are given by:

$$P_i = \begin{pmatrix} \Sigma_{N=1}^{4} p_1^{(N)} B_+^{(N)} \\ \Sigma_{N=1}^{4} p_2^{(N)} B_-^{(N)} \\ \Sigma_{N=1}^{4} p_3^{(N)} B_-^{(N)} \\ \Sigma_{N=1}^{4} p_4^{(N)} B_+^{(N)} \end{pmatrix} \qquad Q_i' = \begin{pmatrix} \Sigma_{N=1}^{4} q_1'^{(N)} B_-^{(N)} \\ \Sigma_{N=1}^{4} q_2'^{(N)} B_+^{(N)} \\ \Sigma_{N=1}^{4} q_3'^{(N)} B_+^{(N)} \\ \Sigma_{N=1}^{4} q_4'^{(N)} B_-^{(N)} \end{pmatrix} \qquad (6\text{-}32\text{-}d)$$

Again, for Bleustein type wave $p_4^{(N)}(I)$ replaces $q_4'^{(N)}(I)$ in (6-32-c) and the mechanical components are of the same form as in the Stoneley type wave while $P_4(n) = 0$.

When the symmetry in the medium is not that of proper rotation with respect to the x_3 axis but or rotation inversion, the elastic stiffnesses and dielectric permittivity coefficients are the same as in the above discussion but the piezoelectric coefficients tensor is of the form:

$$\begin{pmatrix} * & * & * & 0 & 0 & * \\ * & * & * & 0 & 0 & * \\ 0 & 0 & 0 & * & * & 0 \end{pmatrix} \qquad (6\text{-}33)$$

The contributions $S_{ij}^{(N)}(n)$ which are different from (4-3) and (6-22) are:

$$
\left.\begin{aligned}
S_{14}^{(N)}(I) &= e_{11}s_1^2 + e_{35}[s_3^{(N)}(I)]^2 \\
S_{24}^{(N)}(I) &= e_{16}s_1^2 + e_{34}[s_3^{(N)}(I)]^2 \\
S_{34}^{(N)}(I) &= (e_{13} + e_{35})s_1 s_3^{(N)}(I)
\end{aligned}\right) \qquad (6\text{-}34\text{-}a)
$$

$$
\left.\begin{aligned}
S_{14}^{(N)}(II) &= h_1\{e_{11}s_1^2 + e_{35}[s_3^{(N)}(II)]^2\} \\
S_{24}^{(N)}(II) &= h_2\{e_{16}s_1^2 + e_{34}[s_3^{(N)}(II)]^2\} \\
S_{34}^{(N)}(II) &= h_1(e_{13} + e_{35})s_1 s_3^{(N)}(II)
\end{aligned}\right) \qquad (6\text{-}34\text{-}b)
$$

There is no dependence on the value of h_3 but there is a difference in the secular matrix components if h_1 and h_2 change their sign. The secular equation is independent of either h_1h_2 or h_3, so that the bi-quartic equation is the same in both half-spaces and relation (6-23-d) holds for the slowness components. The relations between the displacement components in the two half-spaces ~~is~~ are:

$$
\left.\begin{aligned}
p_1^{(N)}(II) &= h_1 p_1^{(N)}(I) \\
p_2^{(N)}(II) &= h_2 p_2^{(N)}(I) \\
p_3^{(N)}(II) &= -\, h_1 p_3^{(N)}(I) \\
p_4^{(N)}(II) &= p_4^{(N)}(I)
\end{aligned}\right) \qquad (6\text{-}35\text{-}a)
$$

$$
\left.\begin{aligned}
q_1'^{(N)}(II) &= -\, h_1 q_1'^{(N)}(I) \\
q_2'^{(N)}(II) &= -\, h_2 q_2'^{(N)}(I) \\
q_3'^{(N)}(II) &= h_1 q_3'^{(N)}(I) \\
q_4'^{(N)}(II) &= -\, q_4'^{(N)}(I)
\end{aligned}\right) \qquad (6\text{-}35\text{-}b)
$$

The resulting equations of continuity across the interface are:

$$
\left.\begin{aligned}
\Sigma_{N=1}^{4} p_1^{(N)}(I)[A^{(N)}(I) - h_1 A^{(N)}(II)] &= 0 \\
\Sigma_{N=1}^{4} p_2^{(N)}(I)[A^{(N)}(I) - h_2 A^{(N)}(II)] &= 0 \\
\Sigma_{N=1}^{4} p_3^{(N)}(I)[A^{(N)}(I) + h_1 A^{(N)}(II)] &= 0 \\
\Sigma_{N=1}^{4} p_4^{(N)}(I)[A^{(N)}(I) - A^{(N)}(II)] &= 0 \\
\Sigma_{N=1}^{4} q_1'^{(N)}(I)[A^{(N)}(I) + h_1 A^{(N)}(II)] &= 0 \\
\Sigma_{N=1}^{4} q_2'^{(N)}(I)[A^{(N)}(I) + h_2 A^{(N)}(II)] &= 0
\end{aligned}\right) \qquad (6\text{-}36\text{-}a)
$$

$$\left.\begin{array}{l}\Sigma_{N=1}^{4} q_3'^{(N)}(I)[A^{(N)}(I) - h_1 A^{(N)}(II)] = 0 \\ \Sigma_{N=1}^{4} q_4'^{(N)}(I)[A^{(N)}(I) + A^{(N)}(II)] = 0\end{array}\right\}$$

which lead to four different conditions depending on the individual values of h_1 and h_2 in the transformation. Therefore:

for $h_i = (1,1,h_3)$

$$P_i = \tfrac{1}{2}\begin{pmatrix}\Sigma_{N=1}^{4} p_1^{(N)}(I)B_+^{(N)} \\ \Sigma_{N=1}^{4} p_2^{(N)}(I)B_+^{(N)} \\ \Sigma_{N=1}^{4} p_3^{(N)}(I)B_-^{(N)} \\ \Sigma_{N=1}^{4} p_4^{(N)}(I)B_+^{(N)}\end{pmatrix} \qquad Q_i' = \tfrac{1}{2}\begin{pmatrix}\Sigma_{N=1}^{4} q_1'^{(N)}(I)B_-^{(N)} \\ \Sigma_{N=1}^{4} q_2'^{(N)}(I)B_-^{(N)} \\ \Sigma_{N=1}^{4} q_3'^{(N)}(I)B_+^{(N)} \\ \Sigma_{N=1}^{4} q_4'^{(N)}(I)B_-^{(N)}\end{pmatrix} \qquad (6\text{-}36\text{-}b)$$

for $h_i = (1,-1,h_3)$

$$P_i = \tfrac{1}{2}\begin{pmatrix}\Sigma_{N=1}^{4} p_1^{(N)}(I)B_+^{(N)} \\ \Sigma_{N=1}^{4} p_2^{(N)}(I)B_-^{(N)} \\ \Sigma_{N=1}^{4} p_3^{(N)}(I)B_-^{(N)} \\ \Sigma_{N=1}^{4} p_4^{(N)}(I)B_+^{(N)}\end{pmatrix} \qquad Q_i' = \tfrac{1}{2}\begin{pmatrix}\Sigma_{N=1}^{4} q_1'^{(N)}(I)B_-^{(N)} \\ \Sigma_{N=1}^{4} q_2'^{(N)}(I)B_+^{(N)} \\ \Sigma_{N=1}^{4} q_3'^{(N)}(I)B_+^{(N)} \\ \Sigma_{N=1}^{4} q_4'^{(N)}(I)B_-^{(N)}\end{pmatrix} \qquad (6\text{-}36\text{-}c)$$

for $h_i = (-1,1,h_3)$

$$P_i = \tfrac{1}{2}\begin{pmatrix}\Sigma_{N=1}^{4} p_1^{(N)}(I)B_-^{(N)} \\ \Sigma_{N=1}^{4} p_2^{(N)}(I)B_+^{(N)} \\ \Sigma_{N=1}^{4} p_3^{(N)}(I)B_+^{(N)} \\ \Sigma_{N=1}^{4} p_4^{(N)}(I)B_+^{(N)}\end{pmatrix} \qquad Q_i' = \tfrac{1}{2}\begin{pmatrix}\Sigma_{N=1}^{4} q_1'^{(N)}(I)B_+^{(N)} \\ \Sigma_{N=1}^{4} q_2'^{(N)}(I)B_-^{(N)} \\ \Sigma_{N=1}^{4} q_3'^{(N)}(I)B_-^{(N)} \\ \Sigma_{N=1}^{4} q_4'^{(N)}(I)B_-^{(N)}\end{pmatrix} \qquad (6\text{-}36\text{-}d)$$

for $h_i = (-1,-1,h_3)$

$$P_i = \tfrac{1}{2}\begin{pmatrix}\Sigma_{N=1}^{4} p_1^{(N)}(I)B_-^{(N)} \\ \Sigma_{N=1}^{4} p_2^{(N)}(I)B_-^{(N)} \\ \Sigma_{N=1}^{4} p_3^{(N)}(I)B_+^{(N)} \\ \Sigma_{N=1}^{4} p_4^{(N)}(I)B_+^{(N)}\end{pmatrix} \qquad Q_i' = \tfrac{1}{2}\begin{pmatrix}\Sigma_{N=1}^{4} q_1'^{(N)}(I)B_+^{(N)} \\ \Sigma_{N=1}^{4} q_2'^{(N)}(I)B_+^{(N)} \\ \Sigma_{N=1}^{4} q_3'^{(N)}(I)B_-^{(N)} \\ \Sigma_{N=1}^{4} q_4'^{(N)}(I)B_-^{(N)}\end{pmatrix} \qquad (6\text{-}36\text{-}e)$$

For Bleustein type waves, in the eighth equation of (6-36-a) $q_4'^{(N)}(I)$ is replaced by $p_4^{(N)}(I)$. The mechanical 'displacement' and 'stress' are of the same form as in (6-36-b) to (6-36-e) while $P_4(n) = 0$.

$B_+^{(N)}$ and $B_-^{(N)}$ are the null vectors of different matrices, depending on the transformation matrix:

for $h_i = (1,1,h_3)$

$$\begin{pmatrix} p_1^{(N)}(I) \\ p_2^{(N)}(I) \\ q_3'^{(N)}(I) \\ p_4^{(N)}(I) \end{pmatrix} \cdot B_-^{(N)} = 0 \quad ; \qquad \begin{pmatrix} q_1'^{(N)}(I) \\ q_2'^{(N)}(I) \\ p_3^{(N)}(I) \\ q_4'^{(N)}(I) \end{pmatrix} \cdot B_+^{(N)} = 0 \tag{6-37-a}$$

for $h_i = (1,-1,h_3)$

$$\begin{pmatrix} p_1^{(N)}(I) \\ q_2'^{(N)}(I) \\ q_3'^{(N)}(I) \\ p_4^{(N)}(I) \end{pmatrix} \cdot B_-^{(N)} = 0 \quad ; \qquad \begin{pmatrix} q_1'^{(N)}(I) \\ p_2^{(N)}(I) \\ p_3^{(N)}(I) \\ q_4'^{(N)}(I) \end{pmatrix} \cdot B_+^{(N)} = 0 \tag{6-37-b}$$

for $h_i = (-1,1,h_3)$

$$\begin{pmatrix} q_1'^{(N)}(I) \\ p_2^{(N)}(I) \\ p_3^{(N)}(I) \\ p_4^{(N)}(I) \end{pmatrix} \cdot B_-^{(N)} = 0 \quad ; \qquad \begin{pmatrix} p_1^{(N)}(I) \\ q_2'^{(N)}(I) \\ q_3'^{(N)}(I) \\ q_4'^{(N)}(I) \end{pmatrix} \cdot B_+^{(N)} = 0 \tag{6-37-c}$$

for $h_i = (-1,-1,h_3)$

$$\begin{pmatrix} q_1'^{(N)}(I) \\ q_2'^{(N)}(I) \\ p_3^{(N)}(I) \\ p_4^{(N)}(I) \end{pmatrix} \cdot B_-^{(N)} = 0 \quad ; \qquad \begin{pmatrix} p_1^{(N)}(I) \\ p_2^{(N)}(I) \\ q_3'^{(N)}(I) \\ q_4'^{(N)}(I) \end{pmatrix} \cdot B_+^{(N)} = 0 \tag{6-37-d}$$

Therefore if the determinants for non-trivial solutions vanish separately one obtains the following possible 'displacement' and 'stress' at the interface:

$h_i = (1,1,h_3)$

$$B_-^{(N)} = 0 \ \& \ B_+^{(N)} \neq 0 \ : \ P_i = (P_1, P_2, 0, P_4) \ ; \ Q_i' = (0,0,Q_3',0) \qquad (6\text{-}38\text{-}a)$$

$$B_{+}^{(N)} = 0 \ \& \ B_{+}^{(N)} \neq 0 \ : \ P_i=(0,0,P_3,0) \ ; \ Q_i'=(Q_1',Q_2',0,Q_4') \qquad)$$

$h_i=(1,-1,h_3)$

$$\left.\begin{array}{l} B_{-}^{(N)} = 0 \ \& \ B_{+}^{(N)} \neq 0 \ : \ P_i=(P_1,0,0,P_4) \ ; \ Q_i'=(0,Q_2',Q_3',0) \\ B_{+}^{(N)} = 0 \ \& \ B_{-}^{(N)} \neq 0 \ : \ P_i=(0,P_2,P_3,0) \ ; \ Q_i'=(Q_1',0,0,Q_4') \end{array}\right) \quad (6\text{-}38\text{-}b)$$

$h_i=(-1,1,h_3)$

$$\left.\begin{array}{l} B_{-}^{(N)} = 0 \ \& \ B_{+}^{(N)} \neq 0 \ : \ P_i=(0,P_2,P_3,P_4) \ ; \ Q_i'=(Q_1,0,0,0) \\ B_{+}^{(N)} = 0 \ \& \ B_{-}^{(N)} \neq 0 \ : \ P_i=(P_1,0,0,0) \ ; \ Q_i'=(0,Q_2',Q_3',Q_4') \end{array}\right) \quad (6\text{-}38\text{-}c)$$

$h_i=(-1,-1,h_3)$

$$\left.\begin{array}{l} B_{-}^{(N)} = 0 \ \& \ B_{+}^{(N)} \neq 0 \ : \ P_i=(0,0,P_3,P_4) \ ; \ Q_i'=(Q_1',Q_2',0,0) \\ B_{+}^{(N)} = 0 \ \& \ B_{-}^{(N)} \neq 0 \ : \ P_i=(P_1,P_2,0,0) \ ; \ Q_i'=(0,0,Q_3',Q_4') \end{array}\right) \quad (6\text{-}38\text{-}d)$$

Since the whole discussion is independent of the value of h_3 we can choose a value for h_3 so that $h_1h_3 = -1$ and we can compare (6-38) with (6-14) & (6-15). (6-38-a) corresponds to (6-15-a) & (6-14-a) (6-38-b) corresponds to (6-15-b) & (6-14-b), (6-38-c) corresponds to (6-14-c) & (6-15-c) and (6-38-d) corresponds to (6-14-d) & (6-15-d).

The transformation in (6-36-b) (and (6-37-a) & (6-38-a)) may describe the identity, if $h_3 = +1$. In such a case, if there is a solution, it would describe a non-attenuating wave travelling parallel to the plane $x_3 = 0$.

For Bleustein type waves $q_4'^{(N)}(I)$ is replaced by $p_4^{(N)}(I)$ and the mechanical components are of the same form as (6-38) while $P_4(n)=0$ for all configurations.

When the plane of symmetry (of proper rotation) is perpendicular to the x_1 axis, the elastic stiffnesses are given by (4-31), the dielectric permittivity coefficients are of the form:

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \qquad (6\text{-}39\text{-}a)$$

the piezoelectric constants are of the form:

$$\begin{pmatrix} * & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix} \qquad (6\text{-}39\text{-}b)$$

When the plane of symmetry is of rotation inversion the only coefficients which are different in form are those of the piezo-electric tensor. Instead of (6-39-b) this tensor has the form:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & * & * \\ * & * & * & * & 0 & 0 \\ * & * & * & * & 0 & 0 \end{pmatrix} \qquad (6\text{-}39\text{-}c)$$

For each of these symmetries one obtains a bi-quartic secular equation. In the case of (6-39-b) it is dependent on the sign of both h_2h_3 and h_1, while in the case (6-39-c) it depends on the signs of h_2 and h_3 only, and is independent of h_1.

For the case (6-39-b) the equations of continuity are simplified to:

$$\Sigma_{N=1}^{4} p_1^{(N)}(I)[A^{(I)}(I) - h_1A^{(N)}(II)] = 0 \qquad (6\text{-}40\text{-}a)$$
$$\Sigma_{N=1}^{4} p_2^{(N)}(I)[A^{(N)}(I) + h_1h_2h_3A^{(N)}(II)] = 0 \qquad (6\text{-}40\text{-}b)$$
$$\Sigma_{N=1}^{4} p_3^{(N)}(I)[A^{(N)}(I) + h_1A^{(N)}(II)] = 0 \qquad (6\text{-}40\text{-}c)$$
$$\Sigma_{N=1}^{4} p_4^{(N)}(I)[A^{(N)}(I) - A^{(N)}(II)] = 0 \qquad (6\text{-}40\text{-}d)$$
$$\Sigma_{N=1}^{4} q_1'^{(N)}(I)[A^{(N)}(I) + h_1A^{(N)}(II)] = 0 \qquad (6\text{-}40\text{-}e)$$
$$\Sigma_{N=1}^{4} q_2'^{(N)}(I)[A^{(N)}(I) - h_1h_2h_3A^{(N)}(II)] = 0 \qquad (6\text{-}40\text{-}f)$$
$$\Sigma_{N=1}^{4} q_3'^{(N)}(I)[A^{(N)}(I) - h_1A^{(N)}(II)] = 0 \qquad (6\text{-}40\text{-}g)$$
$$\Sigma_{N=1}^{4} q_4'^{(N)}(N)[A^{(N)}(I) + A^{(N)}(II)] = 0 \qquad (6\text{-}40\text{-}h)$$

For the case (6-39-c) the equations of continuity are:

$$\Sigma_{N=1}^{4} p_1^{(N)}(I)[A^{(N)}(I) + h_3A^{(N)}(II)] = 0 \qquad (6\text{-}41\text{-}a)$$
$$\Sigma_{N=1}^{4} p_2^{(N)}(I)[A^{(N)}(I) - h_2A^{(N)}(II)] = 0 \qquad (6\text{-}41\text{-}b)$$
$$\Sigma_{N=1}^{4} p_3^{(N)}(I)[A^{(N)}(I) - h_3A^{(N)}(II)] = 0 \qquad (6\text{-}41\text{-}c)$$
$$\Sigma_{N=1}^{4} p_4^{(N)}(I)[A^{(N)}(I) - A^{(N)}(II)] = 0 \qquad (6\text{-}41\text{-}d)$$
$$\Sigma_{N=1}^{4} q_1'^{(N)}(I)[A^{(N)}(I) - h_3A^{(N)}(II)] = 0 \qquad (6\text{-}41\text{-}e)$$
$$\Sigma_{N=1}^{4} q_2'^{(N)}(I)[A^{(N)}(I) + h_2A^{(N)}(II)] = 0 \qquad (6\text{-}41\text{-}f)$$
$$\Sigma_{N=1}^{4} q_3'^{(N)}(I)[A^{(N)}(I) + h_3A^{(N)}(II)] = 0 \qquad (6\text{-}41\text{-}g)$$
$$\Sigma_{N=1}^{4} q_4'^{(N)}(I)[A^{(N)}(I) + A^{(N)}(II)] = 0 \qquad (6\text{-}41\text{-}h)$$

The analysis for each of the different transformations in this symmetry is similar to the cases of media having a plane of symmetry perpendicular to the x_3 axis.

For Bleustein type waves $p_4^{(N)}(I)$ replace $q_4'^{(N)}(I)$ in (6-40-h) and (6-41-h).

When the plane of symmetry is perpendicular to the x_2 axis the secular equations are not bi-quartic, but similar to the non-piezoelectric case they are separable.

The piezoelectric constants are of the following forms:

If the symmetry is of rotation:

$$\begin{pmatrix} 0 & 0 & 0 & * & 0 & * \\ * & * & * & 0 & * & 0 \\ 0 & 0 & 0 & * & 0 & * \end{pmatrix} \tag{6-42-a}$$

and if it is symmetry of inversion rotation:

$$\begin{pmatrix} * & * & * & 0 & * & 0 \\ 0 & 0 & 0 & * & 0 & * \\ * & * & * & 0 & * & 0 \end{pmatrix} \tag{6-42-b}$$

The dielectric coefficients are in both cases of the form:

$$\begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix} \tag{6-42-c}$$

If one calculates the elements of the secular matrices in the case (6-42-a)

$$S_{12}^{(N)}(n) = S_{23}^{(N)}(n) = S_{14}^{(N)}(n) = S_{34}^{(N)}(n) = 0 \tag{6-43-a}$$

and in the case (6-42-b) :

$$S_{12}^{(N)}(n) = S_{23}^{(N)}(n) = S_{24}^{(N)}(n) = 0 \tag{6-43-b}$$

Hence in (6-42-a) case one obtains a secular equation which may be written as:

$$\begin{Vmatrix} S_{11}^{(N)}(n) & S_{13}^{(N)}(n) \\ S_{13}^{(N)}(n) & S_{33}^{(N)}(n) \end{Vmatrix} \cdot \begin{Vmatrix} S_{22}^{(N)}(n) & S_{24}^{(N)}(n) \\ S_{24}^{(N)}(n) & S_{44}^{(N)}(n) \end{Vmatrix} = 0 \tag{6-44-a}$$

and in the case (6-42-b):

$$\left\| S_{22}^{(N)}(n) \right\| \cdot \begin{Vmatrix} S_{11}^{(N)}(n) & S_{13}^{(N)}(n) & S_{14}^{(N)}(n) \\ S_{13}^{(N)}(n) & S_{33}^{(N)}(n) & S_{34}^{(N)}(n) \\ S_{14}^{(N)}(n) & S_{34}^{(N)}(n) & S_{44}^{(N)}(n) \end{Vmatrix} = 0 \tag{6-44-b}$$

(6-44-a) leads to two quartic equations, the first of which is bi-quadratic, and the second one having third and first order terms. These two equations may or may not have complex roots, depending on s_1, the elastic stiffnesses and the density. The 'displacement' components associated with the vanishing of the first determinant of (6-44-a) cannot be obtained from (6-6) since all the cofactors vanish, however if one uses different cofactors, one obtains for the 'displacement' in the case (6-42-a): (6-45-a)

$$p_1^{(N)}(n):p_2^{(N)}(n):p_3^{(N)}(n):p_4^{(N)}(n) = S_{33}^{(N)}(n)D_1(N,n):0:-S_{13}^{(N)}D_1(N,n):0$$

$$N=1,2 \quad D_1(N,n)=[S_{24}^{(N)}(n)]^2-S_{22}^{(N)}(n)S_{44}^{(N)}(n)$$

$$p_1^{(N)}(n):p_2^{(N)}(n):p_3^{(N)}(n):p_4^{(N)}(n) = 0:-S_{24}^{(N)}(n)D_2(N,n):0:S_{22}^{(N)}(n)D_2(N,n) \tag{6-45-b}$$

$$N=3,4 \quad D_2(N,n)=S_{11}^{(N)}(n)S_{33}^{(N)}(n) - [S_{13}^{(N)}(n)]^2$$

The decoupling of the displacement components would cause similar decoupling in the solutions for the amplitudes, analogous to the non-piezoelectric case (with similar symmetry).

In (6-44-b) one obtains a quadratic equation from the factor $S_{22}^{(N)}(n)$, the solution of which is given by (4-17). Hence under the restriction (4-16) one slowness component is pure imaginary. The rest of the secular equation is a sextic equation. The treatment from here on is the same as for non-piezoelectric media. The 'displacement' components for this case are decoupled in a different way (stemming from the decoupling of the secular equation:

$$p_1^{(1)}(n):p_2^{(1)}(n):p_3^{(1)}(n):p_4^{(1)}(n) = 0:p_2^{(1)}(n):0:0 \tag{6-46-a}$$

$$p_1^{(N)}(n):p_2^{(N)}(n):p_3^{(N)}(n):p_4^{(N)}(n) = -\begin{Vmatrix} S_{14}^{(N)}(n) & S_{13}^{(N)}(n) \\ S_{34}^{(N)}(n) & S_{33}^{(N)}(n) \end{Vmatrix}:0:$$

$$-\begin{Vmatrix} S_{11}^{(N)}(n) & S_{14}^{(N)}(n) \\ S_{13}^{(N)}(n) & S_{34}^{(N)}(n) \end{Vmatrix} : \begin{Vmatrix} S_{11}^{(N)}(n) & S_{13}^{(N)}(n) \\ S_{13}^{(N)}(n) & S_{33}^{(N)}(n) \end{Vmatrix} \quad N=2,3,4 \tag{6-46-b}$$

This decoupling leads again to a decoupling of the equations for the amplitudes.

The results obtained for proper rotations are dependent on both the signs of h_1h_3 and h_2, while in the case of symmetry of rotation-inversion the results are dependent only on the signs of h_1 and h_3.

In all these cases piezoelectricity has contributed to the modification of the mechanical results. One can see that the possible forms of the mechanical waves do not change, although the wave parameters do.

7. THE NUMERICAL CALCULATIONS.

In order to calculate the generalized Stoneley wave velocity in a given configuration a program was written in FORTRAN IV to be used on the CDC 6400 at the Imperial College, and later modified to be run on IBM 360/75 at UCSB (University of California, Santa Barbara). The program is based partially on a program written by T.C. Lim [1968] & Lim & Musgrave [1970].

The program is written so that one can calculate either the slowness (velocity) of a generalized Rayleigh wave in a given direction of an anisotropic medium, or, one can find the slowness (velocity) of a generalized Stoneley wave in a given direction at an interface between two anisotropic media. The two media on the two sides of the interface can differ in any or all of their properties.

Besides the slowness, the output of the program gives other information about the generalized Rayleigh or Stoneley waves, such as displacement and stress components at the free surface or interface, respectively.

The input to the program includes the physical parameters of the medium or media involved, its orientation with respect to the free surface or interface coordinate system, and the choice of either Rayleigh or Stoneley waves.

In the first part of the program the appropriate transformations are done so that the elastic stiffnesses of the media involved would be given in the interface coordinate system.

The program then goes through the following stages:

1. Calculation of the body velocities in the x_1 direction at the free surface or interface. This involves the solution of the secular equation setting $s_3=0$.

For velocities less than the lowest body wave velocity:

2. Calculation of the slowness components $s_3^{(N)}(n)$. This involves solution of the secular equation for a given s_1, and choosing the appropriate three roots by the sign of the imaginary part of the solution.

3. Calculation of the displacement vector components $p_k^{(N)}(n)$. These are the null vectors of the matrices $S_{\ell k}^{(N)}(n)$.

4. Calculation of the stress vector components $q_k^{(N)}(n)$. This is done by using the definition (2-15).

5. Calculation of the determinant (2-19) for the generalized Stoneley waves (6x6), or a similar one for the generalized Rayleigh waves (3x3).

6. Minimization of the absolute value of the determinant for the generalized Rayleigh or Stoneley waves. The value of the velocity for which the determinant is minimum is taken to be the generalized Rayleigh or Stoneley wave velocity. The interval of search is either dictated with the input or decided automatically as a function of the lowest body wave velocity.

7. The amplitudes are calculated as the null vectors of the matrix of the generalized Rayleigh or Stoneley condition.

8. Calculation of the total displacement and stress at the interface.

There are four main numerical problems in this process:

1. The solution of a sixth order polynomial for its roots.
2. The calculation of 6th order determinant.
3. The calculation of null vectors of 3x3 and 6x6 matrices.
4. The minimization of the function obtained by the determinant, since one has to find the tips of very narrow minima (which may be cusps). Sometimes the minima are very close, and are difficult to distinguish.

The problems were solved as follows:

1. The sixth order polynomial is checked if it is bi-cubic. When it is one can solve the cubic equation analytically and improve the result by use of Newton-Raphson process, and then take the square root of the solutions of the cubic. The formulae* used were taken so as to reduce the numerical error:

If $x^3 + ax + b = 0$ is the reduced cubic equation to be solved, and if $b^2/4 >> a^3/27$, then in obtaining the auxiliary variables:

$A = \sqrt[3]{-b/2 + \sqrt{b^2/4 + a^3/27}}$, $B = \sqrt[3]{-b/2 - \sqrt{b^2/4 + a^3/27}}$ one faces the problem of loosing accuracy due to subtraction of like numbers. For this reason one multiplies and divides by the conjugate to avoid subtraction. For instance, if $b>0$, A would be very inaccurate in its present form but would be more accurate if we take: $A = AXB/B$, since $AXB = -a/3$, and B involves addition rather than subtraction of two like numbers. When $b<0$ and $b^2/4 \gg a^3/27$ one uses $B=BxA/A=-a/(3A)$ for better accuracy.

When the sextic equation is not bi-cubic one has to use one of the numerical methods available. The one method found to be most suited is the Lin-Bairstow method (Young & Gregory [1972]). In this method one seeks quadratic factors of the polynomial with real coefficients to be solved. The quadratic factors are then solved analytically by formulae which minimize the numerical error (similar to those described for the cubic equation).

Using:

$$\sum_{i=1}^{n+1} a_i x^{n-i+1} = (x^2 + px + q) \sum_{i=1}^{n-1} b_i x^{n-i-1} + b_n x + b_{n+1} \qquad (7\text{-}1)$$

b_n and b_{n+1} are looked upon as functions of p and q, and one seeks the roots of these functions by a two variable Newton-Raphson method. The Lin-Bairstow method succeeds if the initial guess for p and q is sufficiently close to the right value. Once one quadratic factor is found one looks for a quadratic factor of the polynomial of the (n-2)th degree, unless it is either a first or a second order polynomial. This repetitive division may give rise to a serious loss of accuracy in the value of the coefficients of the polynomials in the process. This problem is by-passed by taking several iterations of a Newton-Raphson process with initial guess of the roots found. One has to modify the N-R method when the roots are very close, approaching a double root solution.

The initial values for the quadratic factors are taken to be the elements on the diagonal of the matrix, the determinant of which forms the secular equation. This guarantees that if the secular equation is factorable (as in the case of symmetry with respect to the x_2 axis, dealt with in chapter 4), no iteration is needed. In those cases where the secular equation is not immediately factorized, these are still good initial values because the elements on the off diagonal have in most cases less weight than the diagonal elements.

2. It was quite tempting to try and use the F^{MN} matrix (2-24) as the simplified generalized Stoneley condition rather than the matrix of coefficients (2-18) which is a 6x6 matrix. However, besides the reason given in chapter 2, namely that calculation of F^{MN} does not allow for 'leaky' waves, there is a numerical reason for working with the 6x6 matrix. In the calculation of each element of F^{MN} one has to have 6 multiplications and 3 additions of elements of the 6x6 matrix (the total of 54 multiplications and 27 additions). These calculations done in floating point arithmetic greatly reduce the accuracy of the elements of F^{MN}, so that when one calculates the determinant of F^{MN} it would have a very large error in it.

Using the Gauss elimination process with total pivoting strategy (Conte & de Boor [1972]) on the 6x6 matrix assures us of least errors in the calculations and the matrix is diagonalized with 54 multiplications/divisions and 54 additions/subtractions. The determinant is the product of the elements on the diagonal.

It is a very good policy to use partial double-precision (Conte & de Boor [1972]) in the calculation of sums of products either in the calculation of the elements of F^{MN}, if one chooses to do so, or in the process of back-substitution in the Gauss elimination process. This method reduces considerably the errors due to the fact that the number of digits in the mantissa of an exact product is the sum of the digits in the mantissas of the multipliers, since in this method the double-precision does not round-off after each multiplication but after the addition of all the products. This, however, has not been implemented in the program. The original program was written for running on CDC 6400 which has a single precision word length of 64 bits. This was accurate enough for most of the calculations and a partial double precision would have improved the results and maybe would have allowed some results which could not be obtained otherwise. However, when the program was run on IBM 360/75, whose word length is 32 bits, it was found that all calculations had to be done in double-precision in order to obtain any meaningful results. The improvement of the addition of products would now require a special subroutine which will do the calculations in two double-precision words. This seemed unjustified.

3. Once a matrix is triangularized (by a Gauss elimination process) it is quite simple to find a null vector of the matrix and to determine if there is one orthogonal direction to the matrix, or, if the rank of the n x n matrix is less than n-1, i.e. n-m, then we should look for the m independent null vectors of the matrix. This is done by back-substitution, and assigning an arbitrary value to x_n.

There are iterative methods which may calculate the null vectors more accurately than this direct method (Wilkinson [1970]) but they involve considerable calculations. The finding of the null vectors of the matrix $S_{\ell k}^{(N)}(n)$ is done many times in the process of seeking the interface or free surface velocity and it seems like the cofactor method is sufficient.

There are problems which had to be resolved of how close two roots should be one to the other in order to be considered equal, in which case we are looking for two independent null vectors of the same matrix. These problems were solved by choosing an arbitrary value: If : $|s_3^{(N)}(n) - s_3^{(M)}(n)| < 10^{-5}$. Since the accuracy to which we calculate the slowness components is less than this number, it may seem too strict a value. But when higher values were taken for the difference the function which described the absolute value of the determinant had a discontinuity which seemed numerical and was eliminated once the value for closeness of roots was lowered.

4. The minimization method is essentially the Golden Section method described by Lim [1968] and Guilfoyle et.al [1967]. A use was made of the properties of the function involved. It was observed, and for generalized Rayleigh wave proven (Barnett et al [1973]) that for velocities greater than the interface/surface wave velocities, the function is monotonic decreasing. Therefore the slope of the function is of the same sign and changes at a rate which varies very slowly up to the value of the Rayleigh or Stoneley wave velocity.

One should note that at the minima involved there is, for most cases, a discontinuity in the derivative of the function, since we are looking for the minima of a function which is the absolute value of the Rayleigh/Stoneley condition function. The interval over which one looks for the minimum is found by checking the slope of the function. This guarantees that if there is a narrow minimum, which falls between two points of calculations, the program would at least sense that there is a change in slope. In Lim's program the indication of a root was the minimum value of the function at the calculation points. The change was made because this minimum value often happens to be the body wave velocity or near it, because of the narrowness of the minima at the surface/interface wave velocity. There are very few restrictions for use of the Golden Section method for finding the minimum of a function but a necessary condition for this method to work is that over the interval in which one searches for the minimum the function is unimodal. A continuous function $f(x)$ is unimodal over an interval $[A,D]$ if there exists a point $x_o \in [A,D]$ such that the function is strictly decreasing (increasing) on $[A,x_o)$ and strictly increasing (decreasing) on $(x_o,D]$. When the difference between Rayleigh or Stoneley velocities and the lowest body wave velocity is larger than the intervals over which the first rough search is done, the function is unimodal. But if this difference in velocities becomes smaller than the interval of search the function may not be unimodal in any of the intervals and therefore the Golden Section method does not work very well. For such an interval it is advisable to check the square of the absolute value of the determinant rather than the absolute value. Although one loses in accuracy by taking the square of the function one obtains a smoother curve which is more suitable for a cubic fit method (Guilfoyle [1967]) of minization of a function.

The Golden Section method is based on the theorem on optimal one-dimensional maximization (or minimization) (Bellman & Dreyfus [1962]). This theorem states that if F_n represents the interval of maximum length over which it is possible to locate the minimum of a unimodal function $f(x)$ by calculating the value of $f(x)$ at most n times, $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$. F_n are Fibonacci numbers. For instance, $F_{20} > 10{,}000$. Therefore, the position of the minimum can always be located within 10^{-4} of the original interval in at most 20 calculations. The connection between the Golden Section and Fibonacci numbers is given in Binet's formula: $F_n = [1/\sqrt{5}][(1+\sqrt{5})/2]^n - [1/\sqrt{5}][(1-\sqrt{5})/2]^n$. For large values of n the second term may be disregarded and one may approximate $F_n \approx [1/\sqrt{5}][(1+\sqrt{5})/2]^n$, and therefore $F_{n-1} \approx \frac{1}{2}(\sqrt{5}-1)F_n = G*F_n$ (with $G \approx .618034$). This dictates the next two points of checking B and C, the values of the function within the interval [A,D] : $F_{n-1} = [A,C] = [B,D]$, $B = (1 - G)(D - A) + A$, $C = A + G(D - A)$.

If we are not sure that the function is unimodal within the initial interval of search then we cannot be sure of blocking the minimum. If this is the case, the method of cubic fit may be more suitable. We still have to have only one minimum within the interval of search but the function may have one maximum as well. This slackening of restriction of unimodality is very important, especially in Stoneley wave velocity calculations where the velocity searched for is not very different from one of the body wave velocities. The idea of a cubic fit is a regular curve fitting, in this case to a cubic polynomial, which may be done with only 4 points - and then one obtains the interpolation polynomial (perfect fit), or, best fit, which is done with the least square method (5 - 10 points). One then finds the minimum of the cubic $a_o + a_1x + a_2x^2 + a_3x^3$ by:

$$\text{If } a_3 \not\approx 0 \quad x_{min} = [\, - a_2 + \sqrt{(a_2)^2 - 3a_1a_3}\,]/(3a_3) \qquad (7\text{-}2\text{-}a)$$

$$\text{If } a_3 \approx 0 \,\&\, a_2 > 0 \quad x_{min} = - a_3/[\, a_2 + \sqrt{(a_2)^2 - 3a_1a_3}\,] \qquad (7\text{-}2\text{-}b)$$

Otherwise no cubic minimum can be determined.
The cubic fit method involves solution of a system of four linear equations for each approximation. This may be a much more lengthy operation than the Golden Section method and is resorted to only when the unimodality of the function is in doubt - i.e. - if the initial search interval is close to a body wave velocity.

The way the program is written it may easily be converted to the calculation~~s~~ of different conditions at the interface from the generalized Stoneley conditions - conditions of continuity of displacement and stress across the interface. Dr. C. Atkinson has suggested the use of this program for the calculation of the rate at which a crack would freely propagate along a plane. This however is not the subject of this present work and may be done at a later date.

8. NUMERICAL RESULTS.

Calculations were done with the program described in chapter 7 to obtain the generalized Stoneley wave velocities in different configurations, and different directions.

The program is designed to take any two media for the two half-spaces. By checking the results one may obtain the generalized Stoneley wave velocity, if such a wave exists. One may also obtain waves which comply with the welded conditions at the interface but for which there is no attenuation, or attenuation of some of the components, in one or both media.

Problems arise when the imaginary part of $s_3^{(N)}(n)$ is much smaller than the real part of the slowness components in the x_3 direction. These cases, however, exhibit little attenuation with increasing distance from the interface, and therefore do not give rise to generalized Stoneley waves localized to the interface.

Although the analysis in chapters 3 and 4 has a significance of its own, it serves as an excellent check on the numerical results. Since the program is independent of the symmetries in the media, or of h_{ih}, one expects that in the particular cases where these symmetries exist, the patterns of results, consistent with the analysis, should be obtained.

Other checkes on the program were made by comparison with known calculated results by W.W. Johnson [1970] and Lim & Musgrave [1970a] and [1970b].

W.W. Johnson gave ranges of existence of generalized Stoneley waves when the media on the two sides of the interface are cubic, orthorhombic and monoclinic, of the same orientation with respect to the interface axis but having different elastic parameters. He showed that the range varies with direction. The ranges are given in terms of $c_{11}^{(1)}/c_{11}^{(2)}$ as a function of $\rho^{(1)}/\rho^{(2)}$ for specific ratios of elastic stiffnesses $c_{ij}^{(1)}/c_{11}^{(1)}$ and $c_{ij}^{(2)}/c_{11}^{(2)}$.

Lim & Musgrave reported calculations of generalized Stoneley waves at interfaces between cubic media of the same elastic parameters but different orientation with respect to the interface axes. The calculations were done on a hypothetical cubic elastic medium having the following elastic constants referred to the principal axes of crystal symmetry:

$c_{11} = 17.1\text{x}10^{10}\text{N/M}^2$, $c_{12}=12.39\text{x}10^{10}\text{N/M}^2$ and $c_{44}=3.56\text{x}10^{10}\text{N/M}^2$
(anisotropy factor $c = c_{11}-c_{12}-2c_{44} = -2.41\text{x}10^{10}\text{N/M}^2$)

The density $\rho = 8.95\text{gr/cm}^3$. Using the notation of chapter 3, $x_i(n)$ being the crystallographic coordinate system of medium n (n=I, II) as referred to in the interface coordinate system, x_i. The transformation matrices relating the coordinate systems are in medium I:

$$x_i = \begin{pmatrix} \cos\varphi(\text{I}) & \sin\varphi(\text{I}) & 0 \\ -\sin\varphi(\text{I}) & \cos\varphi(\text{I}) & 0 \\ 0 & 0 & 1 \end{pmatrix} x_i(\text{I}) \qquad (8\text{-}1)$$

and for the second medium:

$$x_i = \begin{pmatrix} \cos\varphi(\text{II}) & \sin\varphi(\text{II}) & 0 \\ \sin\varphi(\text{II}) & -\cos\varphi(\text{II}) & 0 \\ 0 & 0 & -1 \end{pmatrix} x_i(\text{II}) \qquad (8\text{-}2)$$

where $\varphi(n)$ is a specified angle of rotation.

The generalized Stoneley wave velocities are given as a function of $\varphi(\text{II})$ for different constant $\varphi(\text{I})$.

One should note that the equations of generalized Stoneley waves in anisotropic media are dependent on each of the elastic stiffnesses and densities in the two media, which in general involve 44 parameters. Therefore, for any instructive investigation of the variation in velocity and range of existence of generalized Stoneley waves one needs to hold most of the parameters constant. One obvious way to reduce the number of parameters is to have the same crystallographic structure on both sides of the interface with known relation between the two media involved.

Johnson kept the orientation of the media constant and varied the ratios of only one of the elastic parameters and densities. This is a continuation of Scholte's [1947] approach for isotropic media and does not take into account the main difference between isotropy and anisotropy, namely, that of change in physical properties of a medium with direction.

It is this difference between isotropy and anisotropy which is the basis to Lim & Musgrave's work - they investigated the existence of generalized Stoneley waves as a function of change in relative orientation only. In the extreme case of isotropy both the isotropic bulk waves comply identically with the welded conditions, but no attenuating wave would propagate. The introduction of anisotropy accounts for the existence of the interface waves.

One of the questions Johnson's report raises is whether the same ranges of existence hold for the ratios quoted but different elastic constants $c_{ij}^{(n)}/c_{11}^{(n)}$ in the media involved. A set of calculations was done with the elastic parameters quoted in the paper. The calculated results correspond with those obtained by Johnson. Another set of calculations was done with aluminum on one side and a hypothetical medium on the other side of the interface with $\rho^{(2)}/\rho^{(1)} = 3$ and $c_{11}^{(2)}/c_{11}^{(1)} = 2.2$. This represents a point which is well inside the range of existence for 0° and 15° angles of rotation. $c_{ij}^{(2)}/c_{11}^{(2)}$ was chosen arbitrarily to be different from the ones given. No generalized Stoneley wave was found, which emphasizes the need for more comprehensive investigation of the dependence of range of existence on variation in the various elastic parameters.

The main concern of the present work was the understanding of the dependence of interface waves on the relative orientation of the media involved. For this purpose several sets of computations were made, the first of which was similar to Lim & Musgrave's set of computations.

The transformation matrices relating the principal crystallographic axes, $x_i(n)$, and the interface axes, x_i, are given by:

$$x_i = \begin{pmatrix} \cos\varphi(n) & \sin\varphi(n) & 0 \\ -\sin\varphi(n) & \cos\varphi(n) & 0 \\ 0 & 0 & 1 \end{pmatrix} x_i(n) \qquad n=I, II \tag{8-3}$$

For medium I (8-3) is the same as (8-1), but, in general, the transformation (8-2) is different from (8-3) for n=II, and they are related as:

$$\begin{pmatrix} \cos\varphi(II) & \sin\varphi(II) & 0 \\ \sin\varphi(II) & -\cos\varphi(II) & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\varphi(II) & \sin\varphi(II) & 0 \\ -\sin\varphi(II) & \cos\varphi(II) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8-4}$$

Therefore, the Lim-Musgrave configuration may be obtained from the configuration used in the program described by a 2-fold rotation about the x_1 axis. The two configurations coincide when medium II in the configuration used is invariant under 2-fold rotation about the x_1 axis. This is the case for the medium used in both Lim-Musgrave's and the present work.* While L-M obtained the longitudinal waves, corresponding to $B_+^{(N)}=0$ (Fig (4-2)), the waves calculated here (described in figs. (8-1)-(8-5) are transverse and correspond to $B_-^{(N)}=0$ (fig. (4-1)).

Fig. (8-1) shows the results obtained for the different configurations with the lowest body wave velocities and the Rayleigh velocities given in each direction. The configurations checked were such that half space II was rotated at angles $\varphi(II) = 0^\circ$ to -45° (at intervals of -5°) and in half space I the angles $\varphi(I) = 5^\circ$, 10° and 20° were taken.

Each curve of constant $\varphi(I)$ merges with the slowest Bulk wave velocity curve. Results for configurations where the continuation of the generalized Stoneley waves beyond the bulk wave velocity were not conclusive, although it seems that there exist configurations for which one can find 'pseudo'generalized Stoneley wave similar to the pseudo generalized Rayleigh waves described by Lim [1968] and Farnell [1970].

Fig. (8-2) describes the imaginary parts of the slowness components in the two half-spaces in the 20° configurations. The larger the imaginary part in absolute value the stronger the attenuation. The equations for the slowness components in the x_3 direction are bi-cubic which give rise in most attenuation cases to one pure imaginary and a pair of anti-conjugate components, having the same imaginary parts: $\mathscr{I}\{s_3^{(N)}(II)\} = -\mathscr{I}\{s_3^{(N)}(I)\}$, $\mathscr{I}\{s_3^{(2)}(n)\} = \mathscr{I}\{s_3^{(3)}(n)\}$. As the angle of rotation increases beyond 30° one of the slowness components in medium II is real and therefore there is one non-attenuating component in medium II. For angles less than 15° there is one non-attenuating component in medium I. Therefore, the range

*In the interface coordinate system the elastic stiffness matrix for a cubic medium rotated about the x_3 axis has a tetragonal form (see p.53).

of existence of the generalized Stoneley waves, with $\varphi(I) = 20^{\circ}$ is approximately $-30^{\circ} < \varphi(II) < -15^{\circ}$. This range is in the neighborhood of the symmetric configuration $\varphi(II) = -20^{\circ}$.

An auxiliary program was written for symmetric cases only, in which the input, besides the elastic components of the medium I investigated, includes the transformation matrix h_{ij}. For the generalized Stoneley wave velocity calculated, the values of two other determinants are given, those of the simplified generalized conditions (chapter 3, table (3-2)). In this way one can find out the character of the generalized Stoneley wave obtained. In each determinant only three vectors are involved, rather than six in the general program, therefore one expects more accuracy in the calculations done with the auxiliary program. The results of a set of symmetric calculations for the hypothetic medium is summarized in fig. (8-3), together with the lowest bulk wave velocity and the Rayleigh velocity for each direction. Fig. (8-4) shows the real and imaginary parts of the slowness components for the symmetric cases $\varphi(I) = -\varphi(II)$ as a function of the angle of rotation $\varphi(n)$. For the hypothetic cubic medium used in the calculations, $\Im\{s_3^{(1)}(I)\} = -\Im\{s_3^{(1)}(II)\}$ is a decreasing function of the angle (in the interval $0^{\circ} \leq \varphi \leq 45^{\circ}$) while $\Im\{s_3^{(2)}(I)\} = \Im\{s_3^{(3)}(I)\} = -\Im\{s_3^{(2)}(II)\} = -\Im\{s_3^{(3)}(II)\}$ is an increasing function of the angle. The range of existence is much larger than in the case discussed in fig. (8-2) and includes the open range $0^{\circ} < \varphi < 45^{\circ}$.

The attenuation of the total displacement and stress depend on the relative size of the displacement components as well as the magnitude of the imaginary part of the matching slowness components. In fig. (8-5) the attenuation of the (normalized) displacement components is given as a function of distance from the interface for the configuration when $\varphi(I) = -\varphi(II) = 20^{\circ}$.

It is interesting to note that although one does not expect to obtain a generalized Stoneley wave for the case of no rotation, since this represents an infinite medium without an interface, one does obtain a pseudo-Stoneley wave velocity with one non-attenuating slowness component which is lower than the lowest bulk wave velocity. The explanation for this is in the shape of the slowness surface for cubic media with negative factor of anisotropy (fig. (8-6)). In

(8-6) the lowest bulk wave velocity is obtained where the outermost sheet of the slowness surface intersects the s_1 axis, (at (1)). The other root obtained is the intersection of the slowness surface with the line $s=s_1$ (2), which has two real intersections and four imaginary ones. The energy flux of this wave is parallel to the interface.

When the cubic medium has properties such that the outermost sheet of the slowness surface is the circle $s_1^2 + s_3^2 = s_{T_1}^2$ there is a bulk wave with slowness s_{T_1} which complies with the conditions for a Rayleigh wave and generalized Stoneley wave in all directions. Both the Rayleigh and Stoneley waves would have at least one non-attenuating component. An example of such a medium was calculated, fig. (8-7). The medium taken was KF (Potassium fluoride) with $c_{11} = 6.58\times10^{10}N/M^2$, $c_{12} = 1.49\times10^{10}N/M^2$, $c_{44} = 1.28\times10^{10}\ N/M^2$ (anisotropy factor $c = 2.53N/M^2$), and density $\rho = 2.48gr/cm^3$.

Since symmetric configurations seem to have a wider range of existence than non-symmetric configurations, additional calculations were done in symmetric configurations of another medium.
New results were obtained for spruce, which is orthorhombic and very highly anisotropic. The choice was made because of the high anisotropy. The elastic stiffnesses taken for the spruce are:

$c_{11}=0.078\times10^{10}N/M^2$, $c_{22} = 0.044\times10^{10}N/M^2$, $c_{33} = 16.3\times10^{10}N/M^2$,
$c_{12}=0.020\times10^{10}N/M^2$, $c_{13} = 0.043\times10^{10}N/M^2$, $c_{23} = 0.031\times10^{10}N/M^2$,
$c_{44}=0.077\times10^{10}N/M^2$, $c_{55} = 0.062\times10^{10}N/M^2$, $c_{66} = 0.004\times10^{10}N/M^2$.
The density taken is $\rho = 0.431\ gr/cm^3$.

In fig (8-8) the following results are summarized: The lowest bulk wave velocity is given in the x_1 direction when the medium principal axes are rotated with transformation (8-3), $\varphi(I)$ from 0° to 90° at intervals of 5°. The Rayleigh wave velocity is plotted, as well as the generalized Stoneley wave velocity where the medium in the second half-space is spruce as well, and the transformation matrix is given by:

$$h_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad (8\text{-}4)$$

Here, again the auxiliary program was used in order to calculate the value of the simplified generalized Stoneley condition determinants, as well as condition (2-19).

In the calculations done generalized Stoneley waves were found, when present, to exist between the Rayleigh wave velocity (shown to be unique by Barnett et.al [1973]) and the lowest bulk wave velocity, in a narrow band, closer to the bulk velocity than to the Rayleigh velocity. Since we are looking for attenuating waves when we search for generalized Stoneley waves, we want complex intersections of real lines $s=s_1$ with the slowness surface. This type of intersection is possible only when the slowness s_1 is outside of all the slowness sheets of the slowness surface, or the generalized Stoneley wave velocity has to be lower than the lowest body wave velocity. On the other hand it is not self-evident that generalized Stoneley wave velocities should be higher than generalized Rayleigh velocity.

For cases explored the general behaviour of the determinant of the generalized Stoneley condition as a function of the wave velocity, is consistently very similar to that of the determinant of the generalized Rayleigh condition.

In fig. (8-9) the logarithm of the function describing the Rayleigh condition for the hypothetical material rotated with transformation matrix (8-3), $\varphi = 5^{\circ}$. Fig (8-10) describes the behaviour of the logarithm of the generalized Stoneley condition determinant when a symmetric configuration was taken with $\varphi(I) = -\varphi(II) = 5^{\circ}$.

The simplified Stoneley wave condition determinants calculated exhibit behaviour which is not always exactly the same as the generalized Stoneley wave condition (2-19). While the determinant for the non-trivial values of $B_{+}^{(N)}$, with $B_{-}^{(N)}=0$, (fig. (8-11)) exhibits exactly the same behaviour as that of the generalized Stoneley condition (8-10) for the cubic medium investigated, the determinant for non-trivial $B_{-}^{(N)}$ with $B_{+}^{(N)}=0$ (fig. (8-12)) shows a monotonous behaviour.

For the orthorhombic medium taken, spruce, both determinants minimize simultaneously, but the determinant associated with non-zero $B_{+}^{(N)}$ is several orders of magnitude less than that for the non-trivial $\underline{B}^{(N)}$ (characteristically 7 orders of magnitude difference).

Many more computations are needed for the complete understanding of the ranges of existence of generalized Stoneley waves and the dependence of the velocity on the configuration. For Lim & Musgrave configurations some degree of misorientation is necessary for the existence of generalized Stoneley waves. However, there is, in all cases tested, a maximal degree of misorientation beyond which no such waves exist. Symmetric configurations seem to have a larger range of existence than non-symmetric configurations. Additional calculations should be illuminating.

Further investigation is still needed to find the dependence of the range of existence on the degree of anisotropy both in Johnson's and Lim & Musgrave's approaches. In both approaches, as the degree of anisotropy increases so does the range of existence. But there is a degree of anisotropy beyond which the range of existence diminishes.

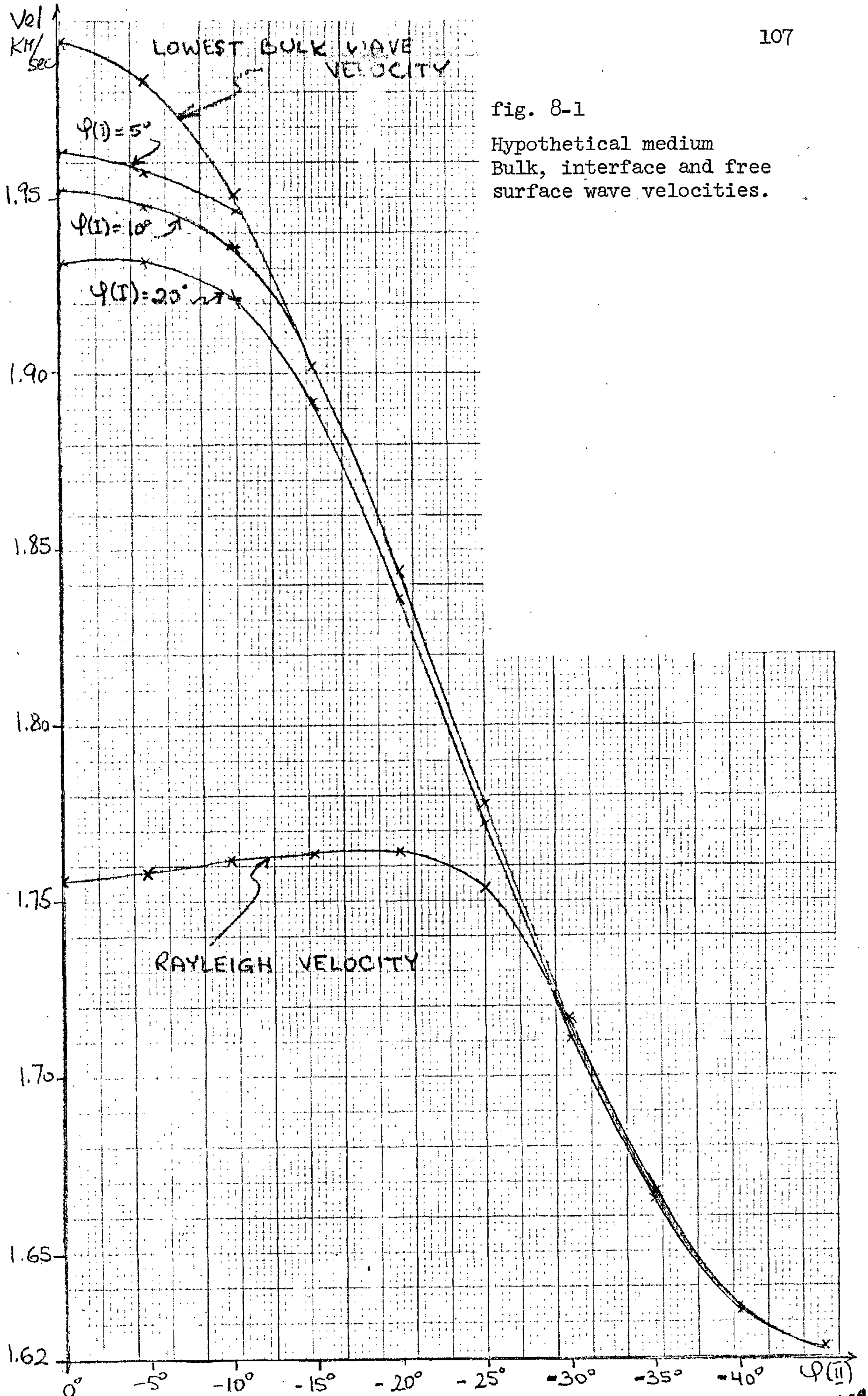
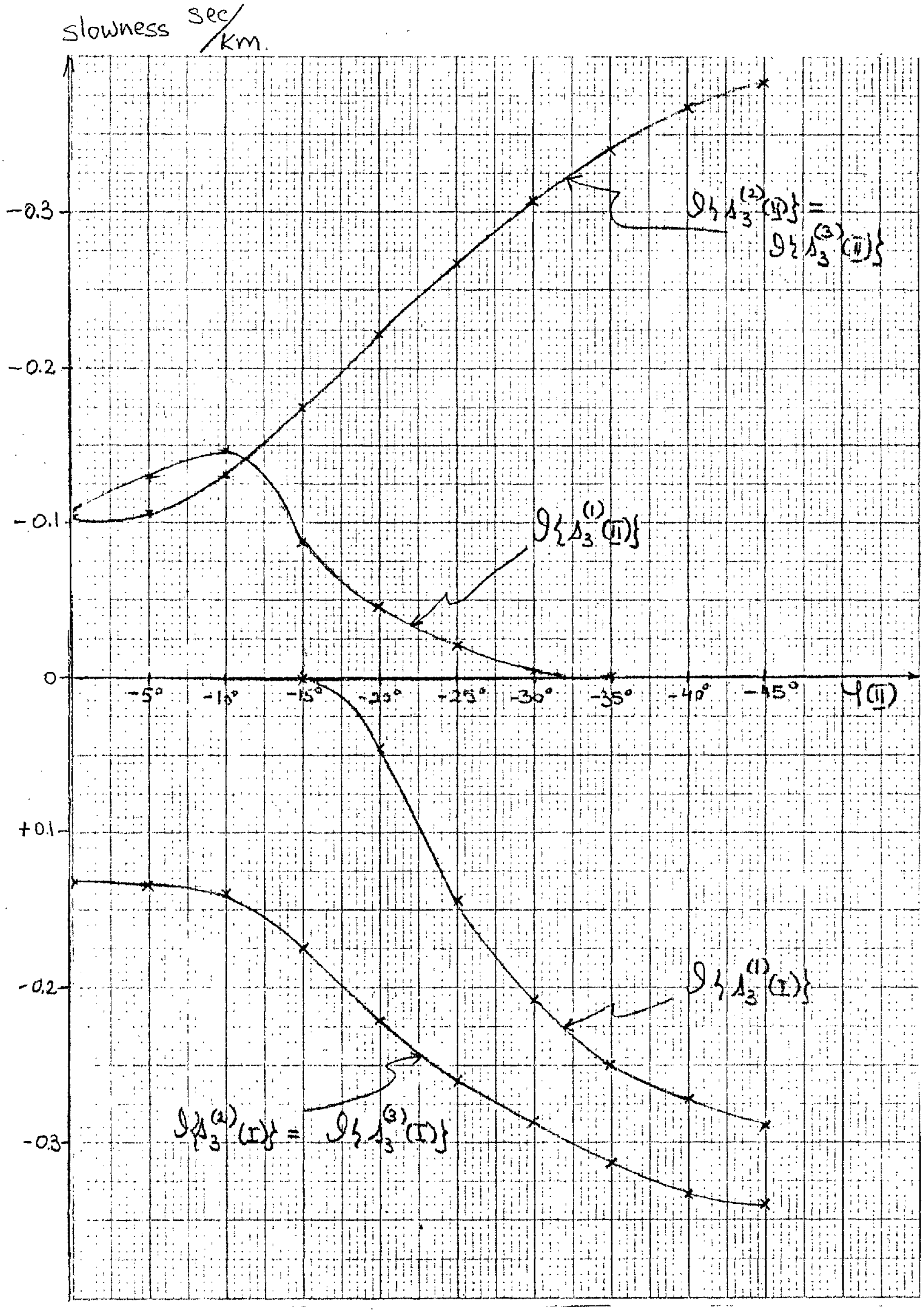


fig. 8-1

Hypothetical medium
Bulk, interface and free
surface wave velocities.

fig. (8-2) Hypothetical medium, imaginary parts of the slowness components of interface waves $\varphi(I) = 20^{\circ}$.



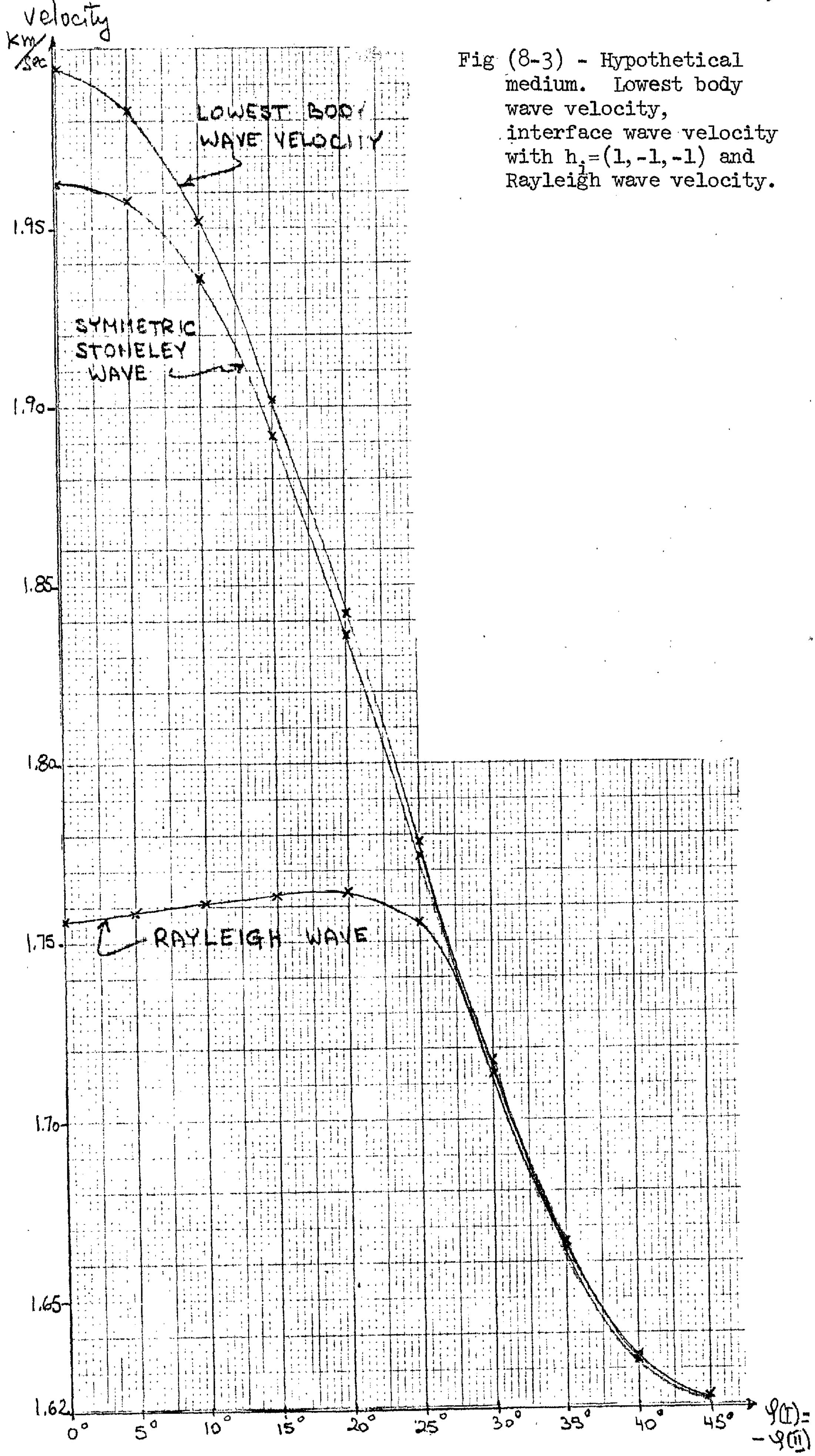


Fig (8-3) - Hypothetical medium. Lowest body wave velocity, interface wave velocity with $h_j=(1,-1,-1)$ and Rayleigh wave velocity.

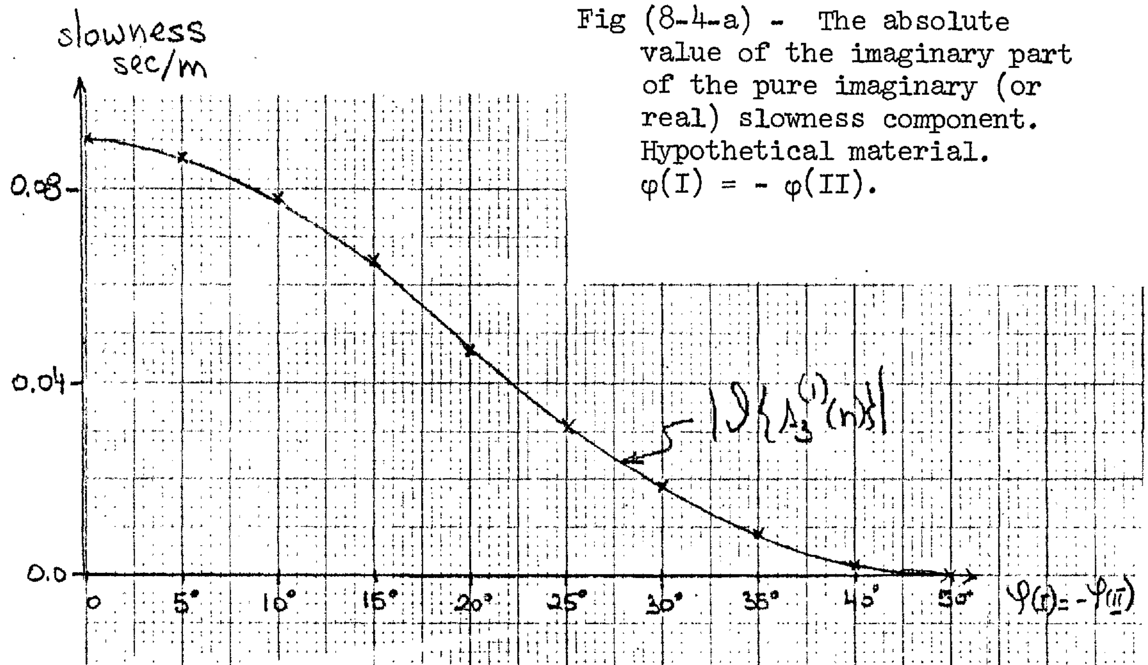


Fig (8-4-a) - The absolute value of the imaginary part of the pure imaginary (or real) slowness component. Hypothetical material. $\varphi(I) = -\varphi(II)$.

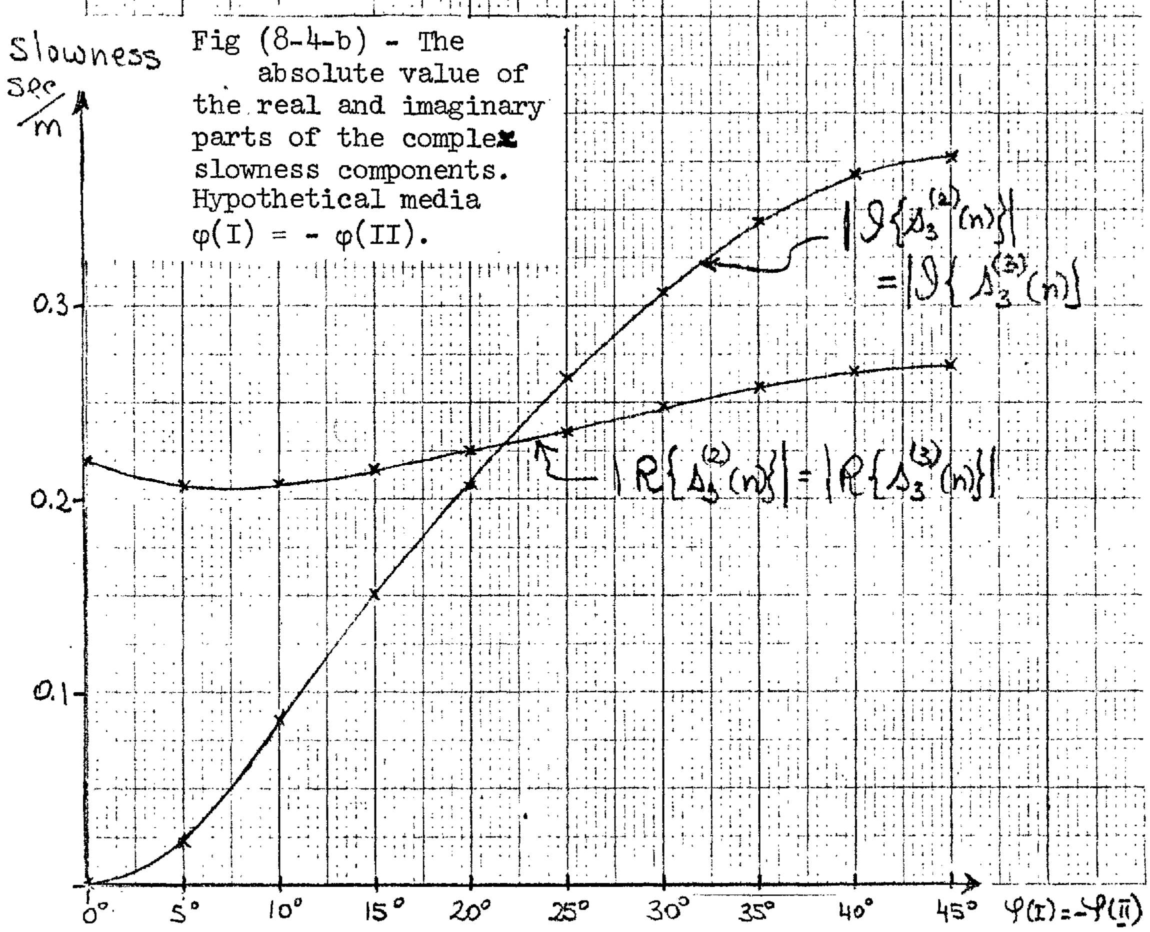


Fig (8-4-b) - The absolute value of the real and imaginary parts of the complex slowness components. Hypothetical media $\varphi(I) = -\varphi(II)$.

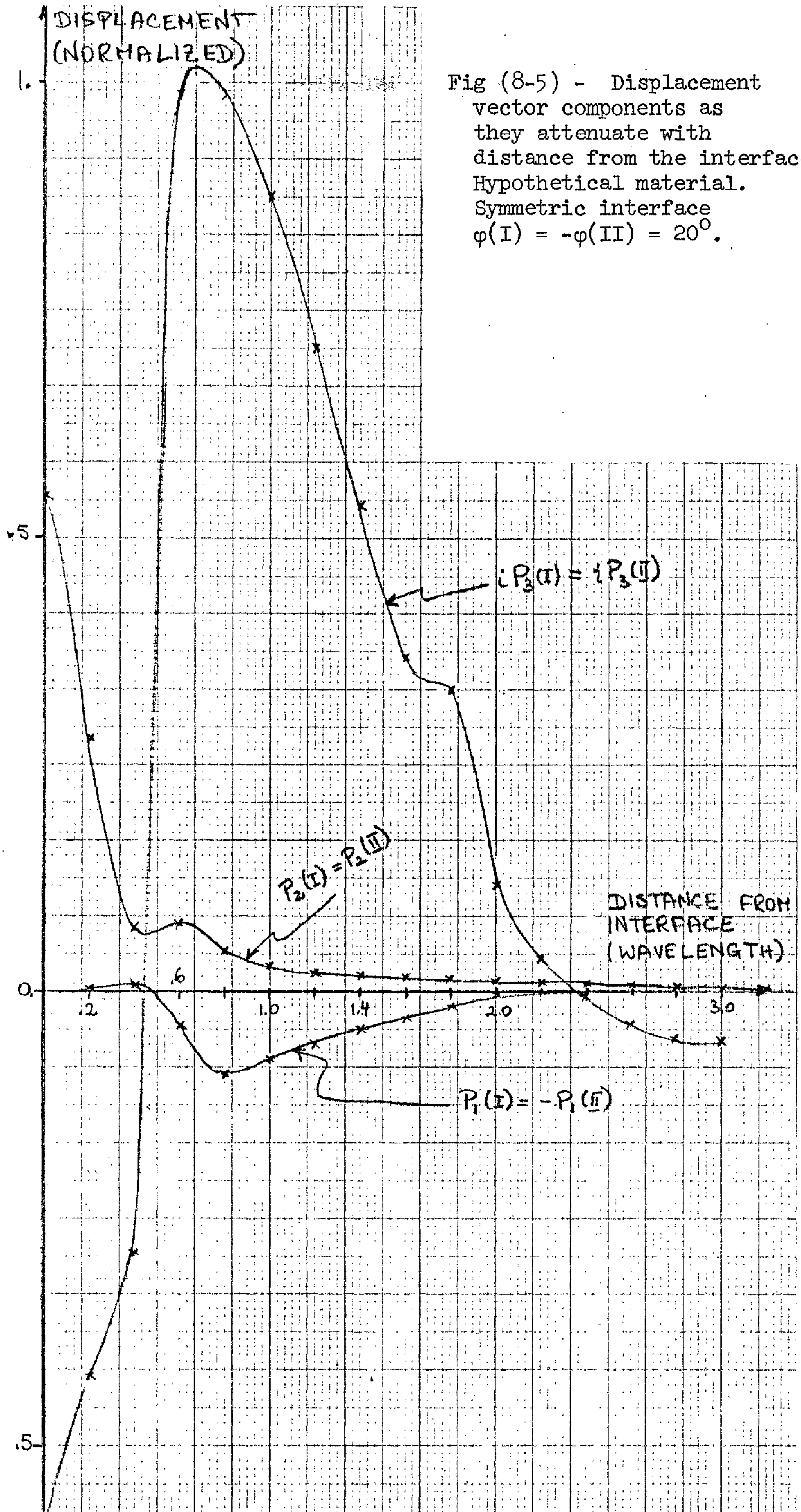


Fig (8-5) - Displacement vector components as they attenuate with distance from the interface. Hypothetical material. Symmetric interface $\varphi(I) = -\varphi(II) = 20^0$.

Fig (8-6)- Hypothetical medium, Sloness Surface, intersection with $s_1 = s_{T_2}$.

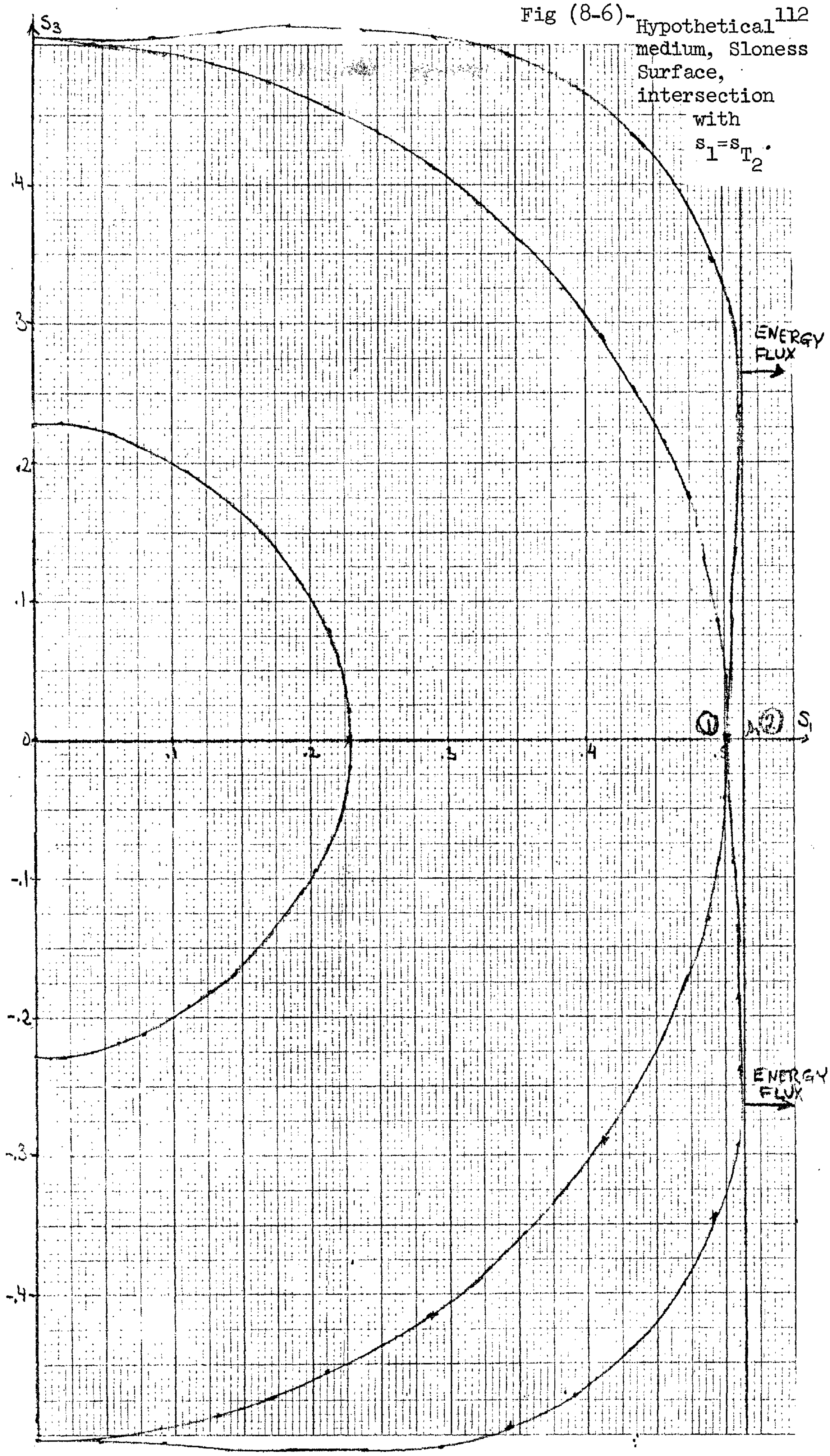


Fig. (8-7) - KF - Body wave velocities.
L.B.V. = Rayleigh velocity =
Symmetric interface velocity =
7.184Km/sec.

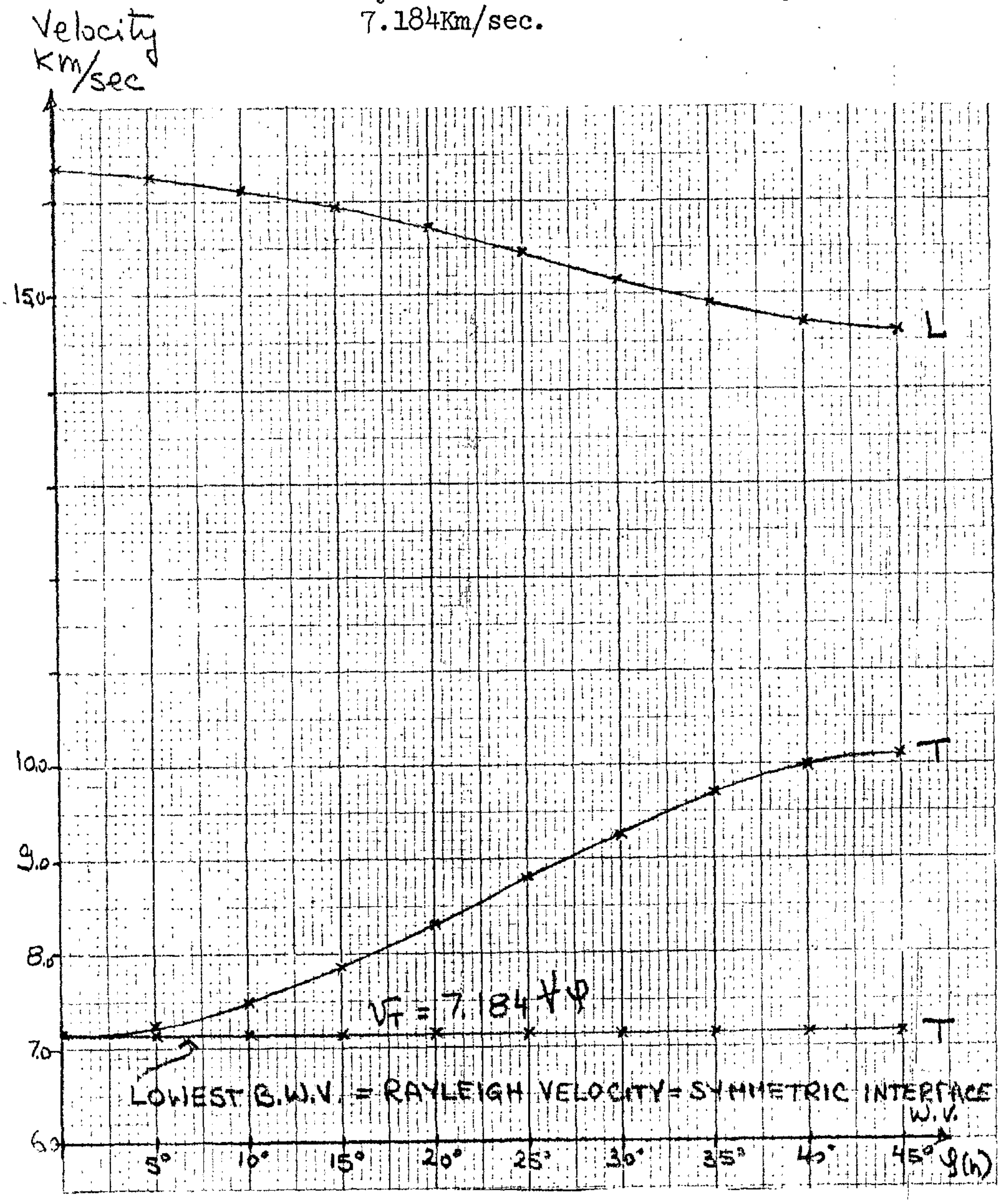


Fig. (8-8) - Lowest bulk wave velocity, Rayleigh velocity and symmetric interface wave velocity for spruce.

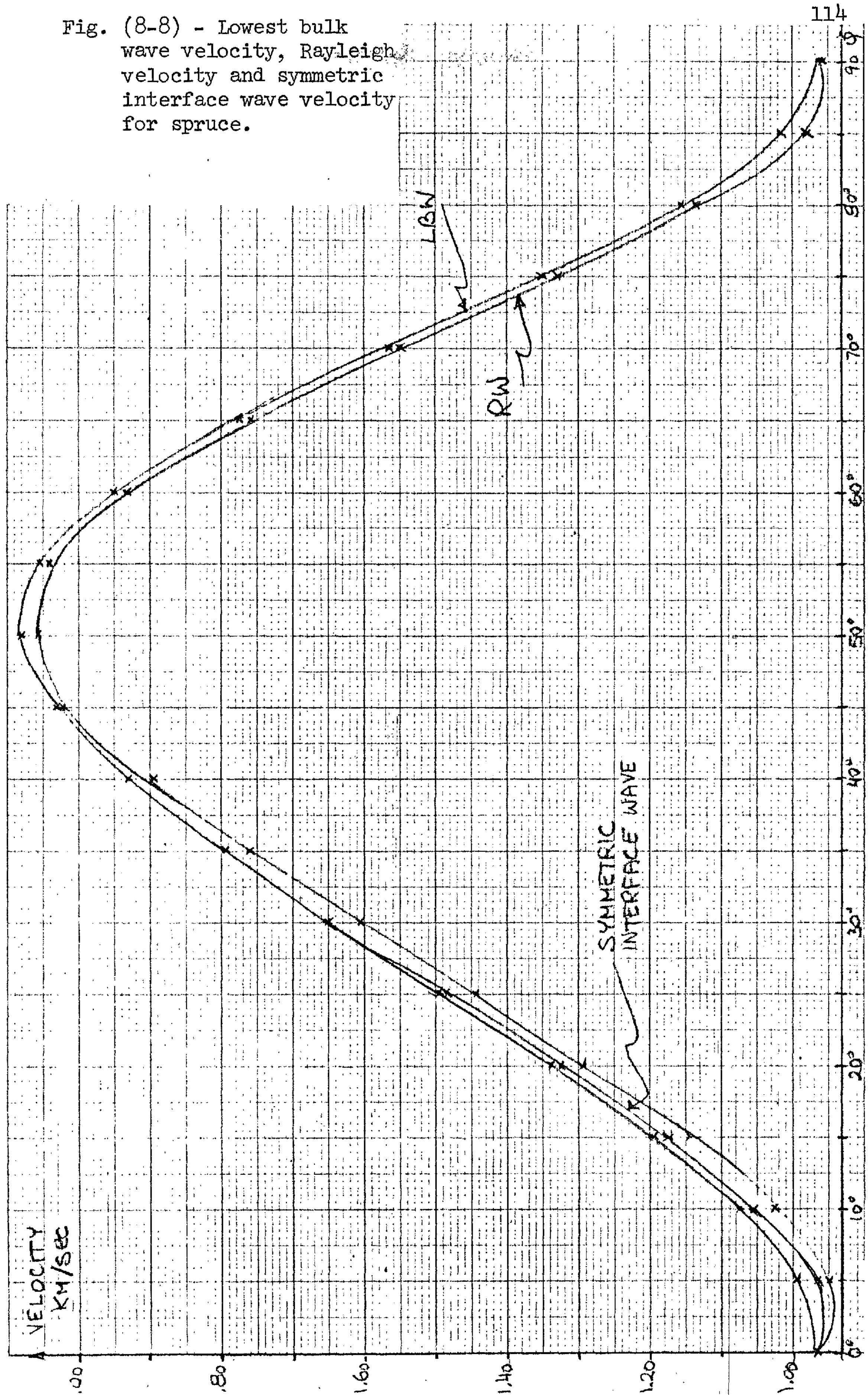


Fig. (8-9) - Hypothetical medium. Rayleigh condition, D, as a function of velocity. Rotation - 5°.

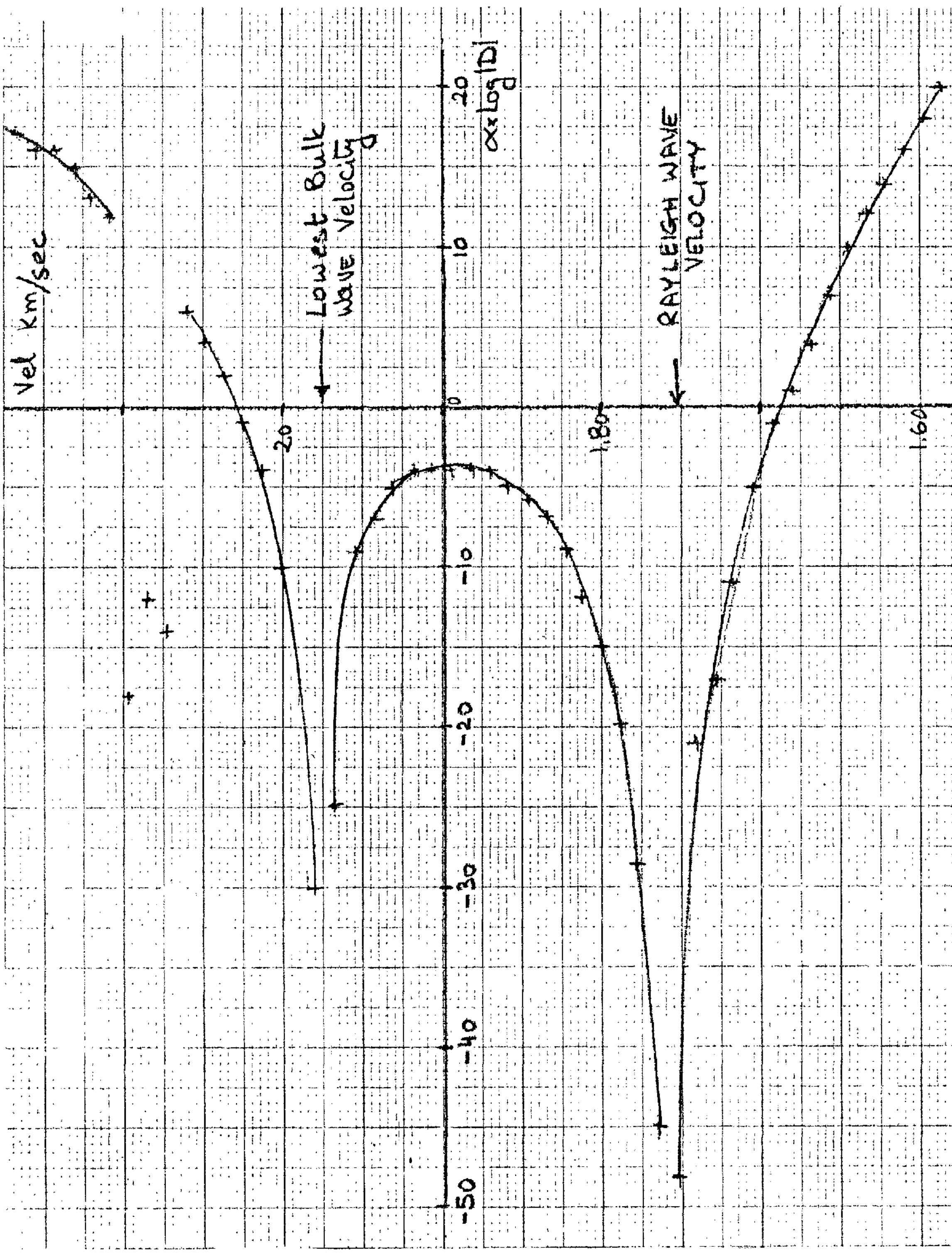


Fig. (8-10) - Hypothetical medium. Symmetric interface wave condition, D, as a function of velocity. Rotation: $\varphi(I) = -\varphi(II) = 5^{\circ}$.

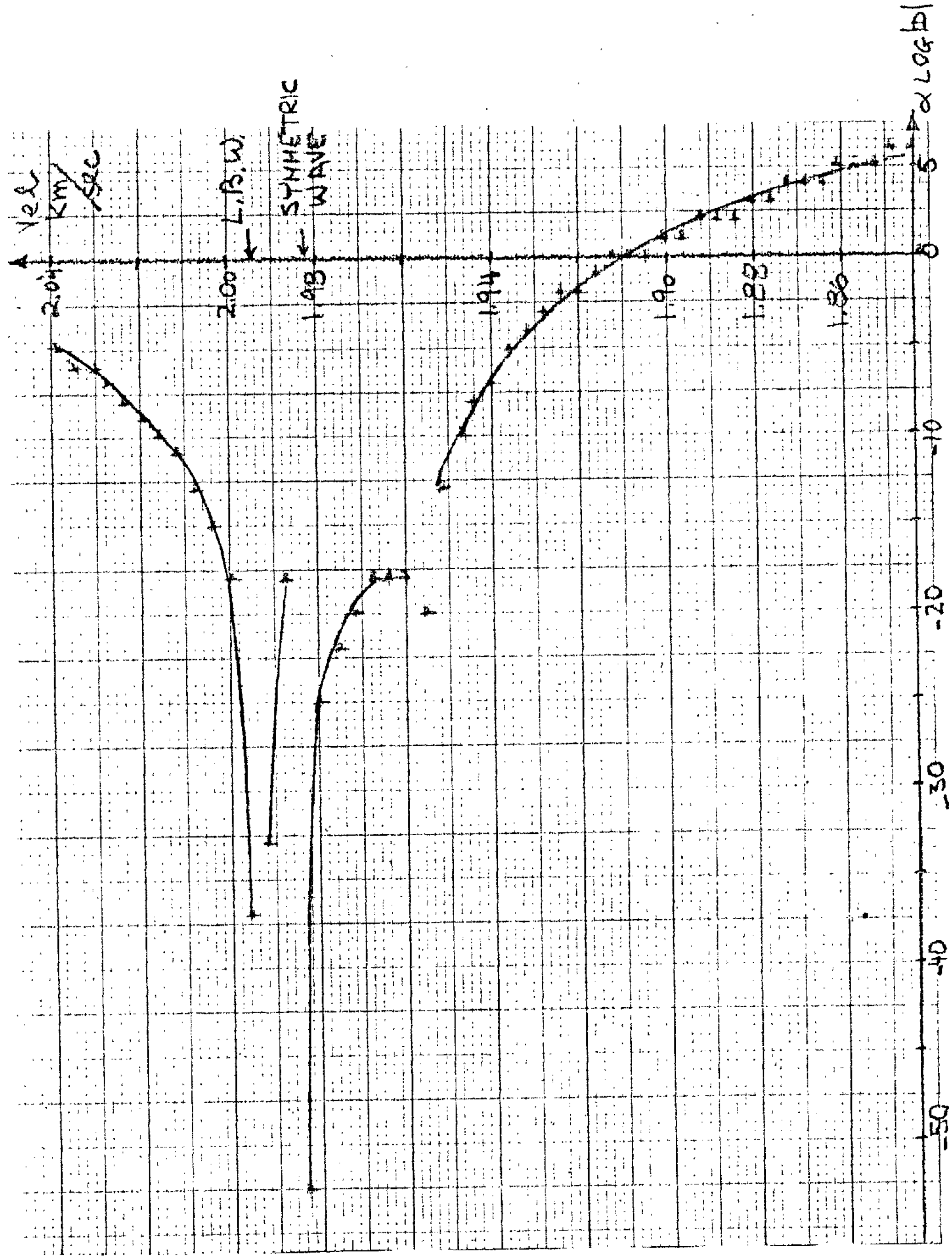


Fig. (8-11) - Hypothetical medium. Symmetric configuration $\varphi(I) = -\varphi(II) = 20^{o}$. $D^{(N)}$ is the condition for non-trivial $B_{+}^{(N)}$, $B_{-}^{(N)}=0$.

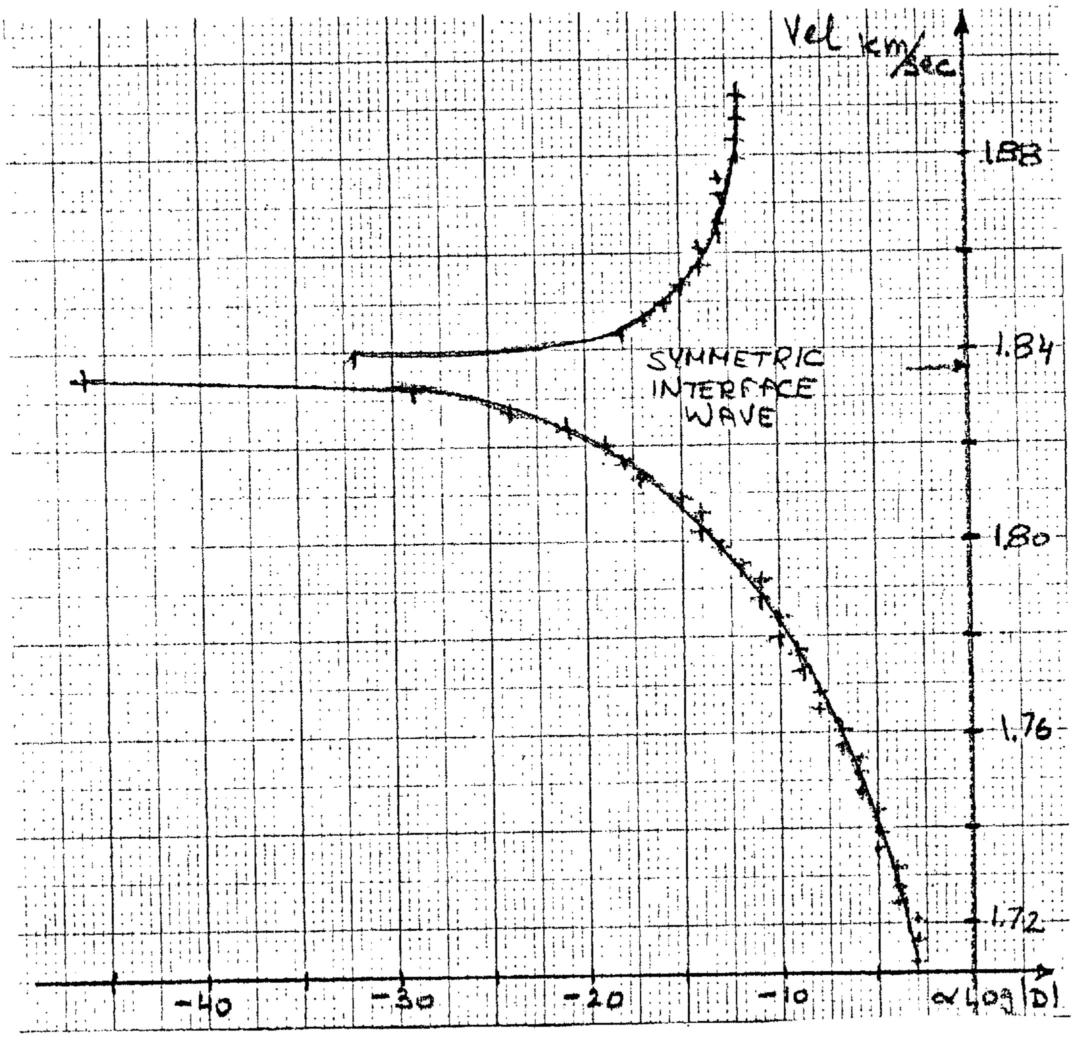
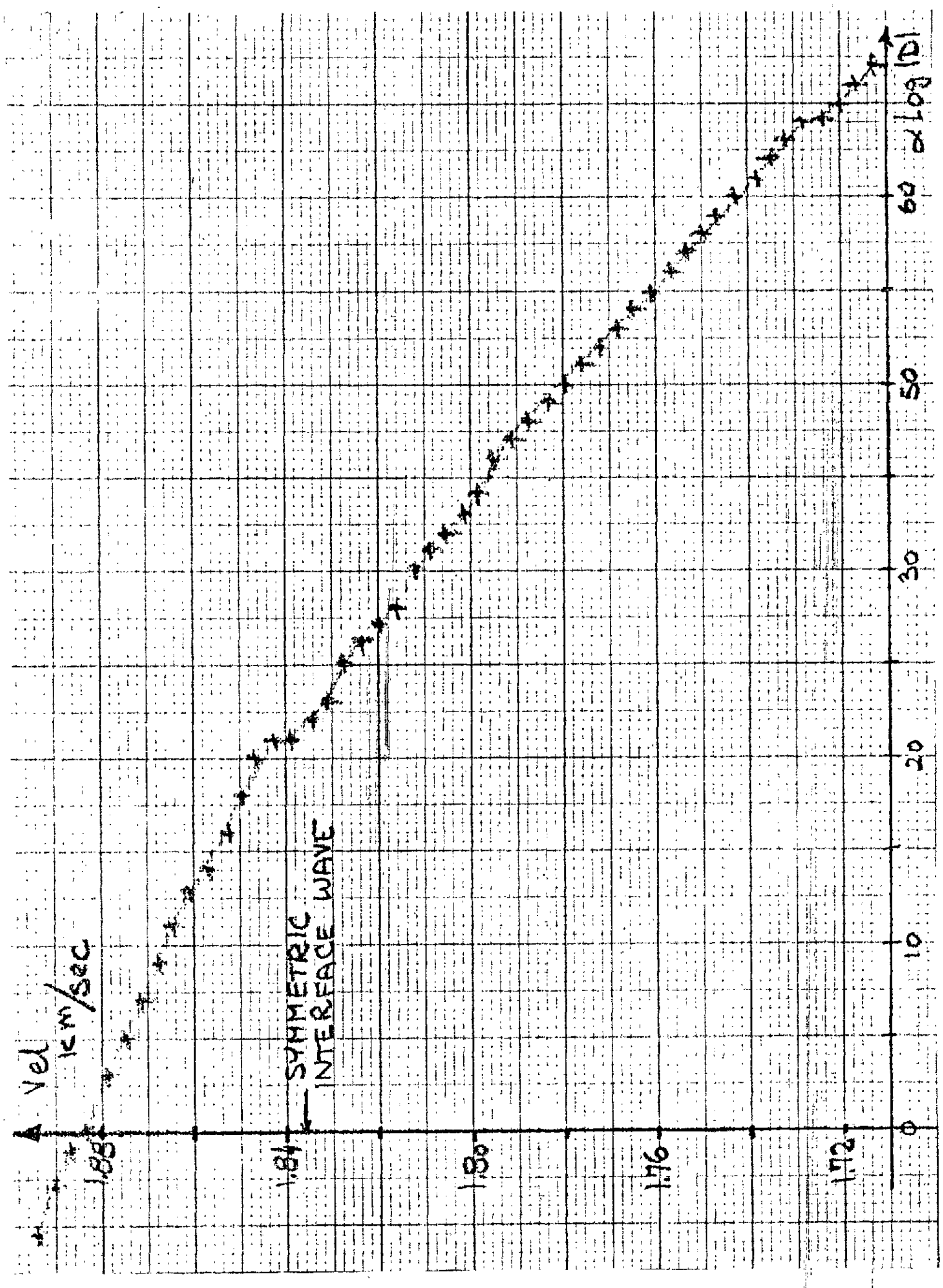


Fig. (8-12) - Hypothetical medium. Symmetric configuration: $\varphi(I)=-\varphi(II) = 20^{\circ}$. D is the condition for non-trivial $B_-^{(N)}$. $B_+^{(N)} = 0$.



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C O R R I G E N D A

p.3	line 31	Hypothetic	should read	Hypothetical
p.11	1st line	BASIS	" "	BASIC
p.13	(2-12-a)	$c_{\ell 3kl}$	" "	$c_{\ell 3kl}^{(n)}$
	line 6	obtaines	" "	obtains
p.14	(2-18-a)	$\sum\limits_{N=1}^{3} A^{(N)}$	" "	$\sum\limits_{N=1}^{3} (A^{(N)}$
		$(II) = 0$	" "	$(II)) = 0$
	(2-18-b)	$\sum\limits_{N=1}^{3} A^{(N)}$	" "	$\sum\limits_{N=1}^{3}(A^{(N)}$
		$(II) = 0$	" "	$(II)) = 0$
p.16	(2-25)	$\sum\limits_{n=1}^{3}$	" "	$\sum\limits_{N=1}^{3}$
	line 15	Cadwick	" "	Chadwick
	(2-29)	$\sum\limits_{N=1}^{3}$	" "	$\sum\limits_{M=1}^{3}$
	line 21	(2-28-a)	" "	(2-18-a)
	(2-30)	$p_k^{(M)}$	" "	$p_k^{(M)}(II)$
p.17	line 19	$\Sigma^3_{N=1}$	" "	$\Sigma^3_{L=1}$
P.23	line 19	componenents	" "	components
p.26	line 19	insert has after while the other		
p.32	2nd line from bottom	intervace	should read	interface
p.34	line 13 from bottom	stiffnesses	" "	stiffness
p.38	line 6	Where	" "	where
	line 8	though	" "	through
p.39	line 11	delete co at the end of the line		
p.47	line 17	x	should read	x_3
	2nd line from bottom	insert after moves: with energy flux		

p.47	line 6	stiffnesses	should read	stiffness
p.58	line 5	Hook	" "	Hooke
	line 7	[1959]	" "	[1969]
	line 11	Gulayev	" "	Gulyaev
p.61	(5-17-a)	$\sum_{N=1}^{4} \hat{F}/$	" "	$\sum_{N=1}^{4}(\hat{F}$
		$(II) = 0$	" "	$(II)) = 0$
	(5-17-b)	$\sum_{N=1}^{4} -G$	" "	$\sum_{N=1}^{4}(-G$
		$(II) = 0$	" "	$(II)) = 0$
	(5-18)	$q'^{(N)}(I)$	" "	$q'^{(N)}_{m}(I)$
p.64	line 13	iterior	" "	interior
p.65	(5-35)	$q'^{(N)}(I)$	" "	$q'^{(N)}_{k}(I)$
p.74	2nd line from bottom	comlete	" "	complete
p.76	line 11	non	" "	none
	line 12	to true	" "	to a true
p.83	line 13	is	" "	are
p.92	line 28	formulaes	" "	formulae
p.93	1st line	subtration	" "	subtraction
p.96	line 27	sbsolute	" "	absolute

p.107 on the $\varphi(II)$ axis add -45° below the last point
add - sign to 30°, 35° and 40°

p.110 line 4 in Fig (8-4-b) - comples should read complex

p.115	line 1	Hypothetic	should read	Hypothetical