ON THE PROPAGATION OF DISTURBANCES IN CERTAIN FLUID FLOWS

BY

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ABSTRACT

The present work is divided into three parts have three properties in common ;

(1) The medium of propagation has the property of dispersion .

(2) Each problem considered as one of initial - value problem .

(3) Each problem utilized the infinitesimal - wave theory .

Part 1 :

In this part we regard the fluid as an incompressible,inviscid, the motion is irrotational and the linearized theory is used. At the free surface we neglect the pressure and the surface tension. In its undisturbed state the fluid, which is of infinite horizontal extent , has uniform depth and resting or flowing with constant speed U . At t=0,a disturbance is initiated(suddenly or smoothly) at the bottom . The technique of Fourier and Laplace transformations are used to get the solution in the form of integral representation . This integral can be evaluated asmptotically for large x & t by the method of steepest descents. To do this we assume that:(1) x/t is fixed and let $t \rightarrow \infty$.(2) x is fixed and let t becomes large. (3) x=0 and let $t \rightarrow \infty$. i.e. we are examining the different solutions from the stand point of an observer moving with velocity = x/t , standing at a fixed position or at the origin respectively . The results can be interpreted in a striking way in terms of the notation of group velocity. We conclude this part by discussing the same problem when the disturbance at the bottom takes the form of infinite step .

Part 11 :

It is the classical problem of initial value problem which is associated with names Cauchy (1827) & Poisson (1815) . In this classical problem of water waves theory , the pressure over the free-surface is constant, say , zero. The fluid is infinitely deep, no obstructions are present. The initial displacement and the initial velocity of the free surface are given then we seek the subsequent motion. We used the theory of infinitesimal waves , by assuming the various functions entering into the problem may be expanded into power series in small parameter ϵ . The coefficients of ϵ giving the first order theory , those of ϵ^2 the second-order theory , etc. The solution will be carried through in out line through the third order. The principle mathematical tools used in solving the problem are Fourier transform , method of stationary phase and integrals in the complex domain .

Part 111 :

We consider a two layer model in which the fluid is inviscid and of uniform density , the velocity profile U(z) is continuous but the rate of shear d/dz(U(z)) has a discontinuity at the interface of the two layers. At t = 0, a disturbance is created at the lower level. The linearized theory is used and the technique of Fourier and Laplace transform is applied to obtain the solution in the integral form , the integral evaluated asymptotically for large t . The perturbation velocity tends to zero as t becomes large , also the vertical displacement $\rightarrow 0$ as $t \rightarrow \infty$, when $U(z) \neq 0$ and U(z) & $d/dk(\beta(k))_{k=k^*}$ have the same sign . The Linearized theory fails if U(z) = 0 or if the model has a width in which the mean velocity U(z) & $d/dk(\beta(k))_{k=k^*}$ have a different sign. Where $k=k^*$ is defined by ($kU(z)-\beta(k)$) = 0 .

Introduction

We are all familiar with the concept of wave motion in fluid, which is one of the oldest successful branches of fluid mechanics. For example , a breeze blowing over a river will produce waves that will move in the direction of the wind on the surface of the river , even though the current may be flowing in some other direction and , at a certain time a disturbance takeplace at a point on the surface of the fluid generating waves ; physically this would correspond to a stone being thrown into a still pond .

The subject of water waves has interested a considerable number of mathematicians beginning apparently with Lagrange, and continuing with Cauchy and Poisson in France . Later the British school of mathematical physicists gave the subject a good deal of attentions, and notable contributions were by Airy , Stokes , Kelvin , Rayleigh and Lamb .

The most striking feature of waves is , without doubt , their capability of carrying energy over long distances , as well as the energy , they carry also disturbances through the medium without giving the medium as a whole any permanent displacement . For the vast bulk of wave motions occurring in the nature , however , the phase velocity , with which the crests and troughs are propagated , and the group velocity , with which the energy is propagated , have quite different magnitude (but in some simple cases, including sound waves and waves on a flexible string , the two velocities are indeed the same). The magnitudes of the phase and group velocities are , however , not equal for any waves whose phase velocity takes different values for waves of different length. This state so affairs is usually described as dispersion , because it means that if we imagine any general disturbance split up into components of different wave length , all these components will progress at different speeds and therefore will tend to get separated out , that is "dispersed" , into a large wave train with the wave length varying rather gradually along it . In this process of dispersion , the energy assocaited with waves of a given length is propagated at the group velocity , say u , of those waves . Hence, after a time t has elapsed , waves of that length be found a distance ut farther on .

The present work lies under the category of initial value problems , i.e. we consider the motion in which the applied wave - maker begin "switch on" at time t = 0 . An excellent survey of different types of initial value problems is given by Wehausen,J.V, and Laiton , E.V, (Surface Waves 1960) . Pioneer contribution were the subjects of classic memoirs by Cauchy (1827) and Poisson(1816). Poisson consider the waves produced by an initial displacement in water of infinite depth . The general question of one - dimensional pulse propagation in dispersive medium was discussed by Rayliegh (1909) . Thompson (Lord Kelvin) (1887) presented the method of stationary phase and applied it to determines the waves produced by a concentrated elevation in water of infinite depth .

Most of the theory of water waves is concerned either with explaining some general aspects of wave motion or with predicting the behaviour of waves in the presence of some special configuration of interest to hydrolic engineers, or ship designers . Unfortunately , even some of the apparently simplest problems have proved too difficult to solve in their most complete formulation . Approximation have been necessary . The nature of the approximations used in treating a particular problem provides a natural way of classifying it . First there are the assumptions concerning the properties of the fluid : viscous or inviscid , compressible or incompressible , surface tension or not . Although assuming the fluid to be inviscid , incompressible and without surface tension simplifies the equations , they are still not easily manageable. Other approximations of a different nature are required . These are in the sense mathematical approximations . Their physical significance is not in restricting the nature of the fluid but in restricting the character of the waves . There are two principal methods of approximation ; one of two approximate theories results from the assumption that the waves amplitudes are small ; the infinitesimal - wave approximation , the other from the assumption that it is the depth of the liquid which is small ; the shallow - water approximation .

As we mention , in their exact form even the simplest problems with surface waves are difficult to solve . If one neglects viscosity and assumes irrotational motion , the problem is reduced to finding solutions of Laplace's equation , which is at least linear in the unknown. However , the problem is still difficult because of the non - linear boundary conditions at the free surface or inter face . The two principal methods of approximation may each be treated as a perturbation procedure . As is mention this procedure is not concerned with the assumptions about the nature of the fluid , but rather with the nature of the motion and its generation . The method has been applied to water - wave problems by Stokers (1957) and other .

In the present work we consider the infinitesimal - wave approximation which fits into a general scheme for approximating non - linear equations and boundary conditions by linear one . To do this, we assumed that the various functions entering the problem may be expanded into power series in small dimensionalless paramater , say ϵ . The series are substituted into the equations and the boundary conditions and grouped according to the powers of ϵ . The coefficients of each power then yield a sequence of equations and boundary conditions , the coefficions of ϵ giving the first theory, those of ϵ^2 the second - order theory , etc . Since we deal only with irrotational flows , the result is a theory based on the determination of a velocity potential in space variables and the time as a solution of the Laplace equation satisfying certain linear boundary and initial conditions .

The first part dealing with small disturbances which are created at the bottom in a stream flowing initially with uniform velocity , or into still water , and with free surface , at the time t = 0 . The technique of Laplace and Fourier transforms are used to obtain the solutions in the form of integral representations . For estimating the integral representation for the solution when t is large , we used the method of steepest descent to get an asymptotic approximation for the solution .

In the second part we consider the classical case (treated first by Cauchy and Poisson) of waves due to disturbances on the free surface into a still water at the time t = 0 . The technique of Fourier transform is used to obtain solution in the form of integral representations up to the third order theory . For this purpose it is very useful to discuss the integral representations by using an asymptotic approximation due to Kelvin and called the principle , or method , of stationary phase . These results, then, can be interpreted in a striking way in terms of the notation of group velocity .

In the third part , we consider a model of shear flow . We investigated the different perturbation functions such as the velocity components and the vertical elevation of any fluid particle due to an infinitesmal disturbance by considering the initial value problem . The technique of Laplace and Fourier transforms are used to obtain the different functions in the form of integral representation . By the method of steepest descent we had an asymptotic approximation for the solution when t becomes large .

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PART I

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FORMULATION:

Only two-dimensional flows are considered. Fig. 1 indicates the general situation: the x-axis is taken along the undisturbed free surface of flow with uniform speed U in x-direction, and y-axis is taken positive upward oppisite the force of gravity.

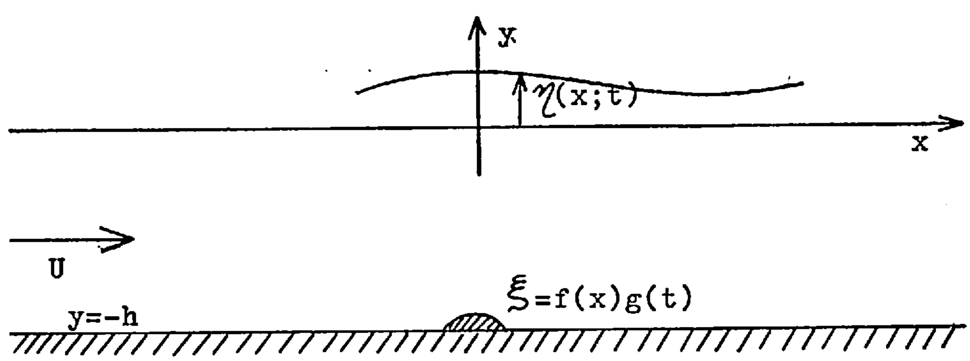


Fig. 1

For $t<0$, let the equation of the bed be given by the equation

$$y = -h,$$

when $t = 0$ a disturbance is suddenly created at the bottom which is given by

$$\xi(x;t) = f(x)g(t),$$

where g(t) is the Heaviside function and f(x) is given function, but it is better to be symmetry about the origin $x = 0$; i.e. f(x) is an even function.

Hence, for $t>0$, the bottom is given by

$$y = -h + \xi(x;t) ,$$

$$\text{i.e.} \quad y = -h + f(x)g(t).$$

Consider $F(x,y;t) = y + h - f(x)g(t)$, where $F(x,y;t) = 0$ describes the bottom for $t > o$.

We regard the fluid as incompressible , frictionless , and initially has a constant speed U (from the known laws of hydrodynamics) , then the resulting motion is irrotationally and mathematically described by the velocity potential $\Phi(x,y;t)$ which satisfies Laplace's differential equation

$$\nabla^2_{x,y} \Phi(x,y;t) = o \ , \qquad (1)$$

in the region bounded by the free surface S_f and the bottom surface S_b , where

$$\nabla^2_{x,y} = \partial^2/\partial x^2 + \partial^2/\partial y^2 \ ,$$

and

$$\Phi(x,y;t) = Ux + \phi(x,y;t) \ ,$$

where $\phi(x,y;t)$ is the potential of a small disturbance and consider it and its derivatives to be small of the same order . It is clear that $\phi(x,y;t)$ is a harmonic function . At the same time the free surface elevation $\eta(x;t)$ (also $\xi(x,y;t)$ = the vertical displacement of fluid particle) and its derivatives are also considered to be small of the same order .

THE BOUNDARY CONDITIONS:

(1) The boundary conditions at the free surface:

(a) The kinematical condition:

Let $F(x,y;t) = y - \eta(x;t) = o$ describe the free surface S_f . The velocity of a point (x,y) on the surface in the direction of the normal to the surface is given by $-F_t \,/\, (F_x^2 + F_y^2)^{\frac{1}{2}}$. Here one takes the normal in the directin (F_x, F_y). A particle of fluid at the same point of the surface at that instant will have a velocity component in the direction of the surface normal given by $\underline{v} \cdot \underline{\text{grad}}\, F = v_n$ ($\underline{v}$ the velocity vector of a fluid particle and $\underline{v} = (u,v)$),i.e.

$$v_n = (uF_x + vF_y) / (F_x^2 + F_y^2)^{\frac{1}{2}} ,$$

where $F_x = \partial F / \partial x$, $F_y = \partial F / \partial y$ & $F_t = \partial F / \partial t$.

For S_f to be a bounding surface means , of course , that there can be no transfer of matter across the surface. Consequently the following equation must be satisfied:

$$u F_x + v F_y = - F_t \quad . \qquad (2)$$

If one defines the " material derivative " by the equation

$$D F / D t = F_t + u F_x + v F_y ,$$

then (2) is the same as

$$D F / D t = o \quad . \qquad (2)'$$

(b)The dynamical condition:

The case with which we chiefly concerned is that of an inviscid fluid without surface tension . In this case the dynamic condition reduces to the single equation

$$p(x,y;t) = p_o , \qquad (3)$$

on $F(x,y;t) = o$, where p_o ; in most cases it is taken to be a constant, either an assumed atmospheric pressure or zero . But the motion is irrotational and incompressible, one may determine p explicity (from Bernoulli's equation)

$$P / \rho + g\eta + \phi_t + \tfrac{1}{2} (u^2 + v^2) = o \quad . \qquad (4)$$

In the present problem we consider $p_o = 0$. Then the dynamic condition which satisfied on $F(x,y;t) = 0$ is

$$\Phi_t + g\eta + \tfrac{1}{2}(u^2 + v^2) = 0 . \qquad (4)'$$

(2) <u>The boundary condition on the bottom</u>:

Let the equation of the bottom be given by the equation $G(x,y;t) = (y + h - \xi(x;t)) = 0$, for $t > 0$. Then in the case of an inviscid fluid (our case) the condition to be satisfied on $G(x,y;t) = 0$ is the same as the kinematic condition (2) :

$$u\, G_x + v\, G_y = - G_t \, , \qquad (5)$$

i.e., the component of velocity of the fluid normal to the surface must equal the velocity of the rigid surface in the direction of its normal .

We complete the state-ment of the boundary conditions by invoking the finiteness conditions ;

$$|\Phi| < \infty \qquad \text{as} \quad |x| \longrightarrow \infty$$

and

$$|\eta| < \infty \;,\; |\xi| < \infty \qquad \text{as} \quad |x| \longrightarrow \infty \qquad (6)$$

The vertical displacement of any fluid particle be given by

$$H(x,y;t) = y - \xi(x,y;t) = 0 \, , \qquad \text{for} \quad t > 0 \, ,$$

then we have

$$(D/Dt)H(x,y;t) = 0 \, . \qquad (7)$$

In our case the motion is irrotational, hence the velocity say $\underline{v} = (u,v)$, at a given point in the fluid may be derived from a velocity potential Φ (x,y;t) according to ;

$$\underline{v} = \underline{\text{grad}}\ \Phi\ (x,y;t)\ ,$$

i.e., $u = U + \phi_x$, $v = \phi_y$,

substituting for u and v in terms of velocity potential in the expressions (2)', (4)', (5)' and (7), one finds

$$\left(\frac{\partial}{\partial t} + (U+\phi_x) \frac{\partial}{\partial x} + \phi_y \frac{\partial}{\partial y} \right) (y - \eta(x;t)) = 0,$$

$$\phi_t + g\eta + \tfrac{1}{2} ((U + \phi_x)^2 + \phi_y^2) = 0, \qquad (8)$$

$$\left(\frac{\partial}{\partial t} + (U + \phi_x) \frac{\partial}{\partial x} + \phi_y \frac{\partial}{\partial y}\right) (y + h - \xi(x;t)) = 0, \quad (9)$$

and the expression (7) becomes

$$\left(\frac{\partial}{\partial t} + (U + \phi_x) \frac{\partial}{\partial x} + \phi_y \frac{\partial}{\partial y} \right) (y - \xi(x,y;t)) = 0\ . \quad (10)$$

As we assumed above , that the perturbation potential and the vertical displacement of the free surface $\eta(x;t)$ relative to its equilibrum position , $y = 0$, are sufficiently small to justify the neglect of all terms of second order - that is to say , we linearize the equation of the motion and the boundary conditions . This assumption permits the neglect of the second order terms such as ϕ_x^2 , $\phi_x \eta_x$, ... Hence the boundary conditions reduce to

$$g\eta + \phi_t + U\ \phi_x = 0, \qquad (11.a)$$

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right)\eta = \phi_y\ . \qquad (11.b)$$

These conditions are now to be satisfied at $y = 0$.

At the bottom the condition becomes

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \xi(x;t) = \phi_y\ , \qquad (12)$$

can be written as

$$(\partial/\partial t + U\,\partial/\partial x\,)f(x)g(t) = \phi_y \quad ; \quad \text{at } y = -h \tag{12}$$

The expression for $\xi(x,y;t)$ = the vertical displacement of fluid particle is given by

$$(\partial/\partial t + U\,\partial/\partial x\,)\xi(x,y;t) = \phi_y \;. \tag{13}$$

To gether the finiteness conditions

$$|\phi| < \infty \;,\; |\eta| < \infty \;\&\; |\xi| < \infty \quad \text{as} \quad |x| \longrightarrow \infty \tag{14}$$

THE SOLUTION:

We attack the mathematical problem posed by (11 - 14) in additional Laplace's equation $\nabla^2 \phi = 0$ by invoking a Fourier transformation with respect to x ($-\infty < x < \infty$) and a Laplace transformation with respect to t ($0 < t < \infty$) .

Let

$$\mathcal{F}\phi\,(x,y;t) = \int_{-\infty}^{\infty} \phi\,(x,y;t)\,\exp(ikx)\,dx\;,$$

$$= \bar{\phi}\,(k,y;t)\;,$$

$$\mathcal{L}\,\phi\,(x,y;t) = \int_{0}^{\infty} \phi\,(x,y;t)\,\exp(-iwt)\,dt\;,$$

$$= \bar{\phi}\,(x,y;w)\;,$$

and

$$\mathcal{L}\,\mathcal{F}\phi\,(x,y;t) = \int_{0}^{\infty} dt\,\exp(-iwt)\int_{-\infty}^{\infty} dx\,\exp(ikx)\,O(x,y;t),$$

$$= \bar{\bar{\phi}}(k,y;w)\;. \tag{15}$$

Hence, integrating by parts, we obtain

$$\mathcal{F}(\,d^n f(x)/dx^n\,) = (-ik)^n\,\bar{f}(k)\;,$$

$$\mathcal{L}(\,d\;g(t)/dt\,) = (iw)\;\bar{g}(w)\;. \tag{16}$$

where $\mathcal{F}$ implies Fourier transformation with respect to x $(-\infty < x < \infty)$ and $\mathcal{L}$ implies Laplace transfomation with respect to t $(o < t < \infty)$.

Transforming Laplace equation $\nabla^2 \phi = o$ with the aid of (16) we obtain

$$\bar{\phi}_{yy} - k^2 \bar{\phi} = o \,, \tag{17}$$

then transforming (11) , we obtain

$$(iw - ikU)\,\bar{\phi} + g\bar{\eta} = o \,, \tag{18.a}$$

and

$$(iw - ikU)\,\bar{\eta} - \bar{\phi}_y = o \,, \tag{18.b}$$

at y = o.

Eliminating $\bar{\eta}$ between (18.a & b) , we obtain

$$-(w - kU)^2 \bar{\phi} + g\,\bar{\phi}_y = o, \text{ at } y = o. \tag{19}$$

Transforming the condition (12) , we get

$$i\,(w - kU)\,\bar{f}(k)\,\bar{g}(w) = \phi_y \tag{20}$$

satisfied at y = -h .

(17) has a solution

$$\bar{\phi}\,(k,y;w) = A(k;w)\exp(ky) + B(k;w)\exp(-ky), \tag{21}$$

with A(k;w) and B(k;w) are arbitrary functions of k and w. Substituting (21) into (19) , at y = o , and (20) , at y=-h , we obtain

$$-(w - kU)^2 (A+B) + gk\,(A-B) = o \,, \tag{22}$$

$$i\,(w-kU)\,\bar{f}(k)\bar{g}(w) = k(A\exp(-kh) - B\exp(kh)) \tag{23}$$

Solving (22) and (23) for A and B , we obtain

$$A(k;w) = \frac{i\,(w-kU)\,\bar{f}(k)\,\bar{g}(w)\,(gk + (w-kU)^2)}{k\,((W-kU)^2 \cosh kh - gk \sinh kh)} ,$$

and

$$B(k;W) = \frac{i(W-kU)\,\bar{f}(k)\,\bar{g}(w)\,(gk - (w-kU)^2)}{k((w-kU)^2 \cosh kh - gk \sinh kh)} .$$

Hence , the solution (21) becomes

$$\bar{\phi}(k;y,w) = \frac{i(w-kU)\bar{f}(k)\bar{g}(w)\,(gk \cosh ky + (w-kU)^2 \sinh ky)}{k((w-kU)^2 \cosh kh - gk \sinh kh)} \qquad (24)$$

Transforming (13) , we obtain

$$(iw + (-ik)U)\,\bar{\xi}(k,y;w) = \bar{\phi}_y ,$$

by substituting for the value $\bar{\phi}$ from (24) we get

$$\bar{\xi}(k,y;w) = \frac{\bar{f}(k)\bar{g}(w)(gk \sinh ky + (w-kU)^2 \cosh ky)}{\cosh kh((w-kU)^2 - gk \tanh kh)} , \qquad (25)$$

let $(\mu(k))^2 = gk \tanh kh$.

Hence it is better to write the expression (25) as

$$\bar{\xi}(k,y;w) = \frac{\bar{f}(k)\bar{g}(w)(gk \sinh ky + (w-kU)^2 \cosh ky)}{\cosh kh\,((w-(kU+\mu))(w-(kU-\mu)))} , \qquad (25)'$$

From the definition , g(t) is the Heaviside function ,i.e.

$$g(t) = 0 , \qquad t < 0,$$
$$g(t) = 1 , \qquad t > 0.$$

This means that the disturbance is suddenly created at the bottom, hence the Laplace transformation is

$$\bar{g}(w) = \int_{-\infty}^{\infty} g(t) \exp(-iwt)\, dt$$
$$= -i / w .$$

the expression $(25)'$ now can be written as

$$\bar{\xi}(k,y;w) = \frac{-i\bar{f}(k)\ (gk \sinh ky + (w-kU)^2 \cosh ky)}{\cosh(kh)\cdot((w-(kU+\mu))(kU-\mu))\ w} , \qquad (25)''$$

Taking the inverse of $(25)''$ (for Laplace transform) , we obtain

$$\bar{\xi}(k,y;t) = \frac{1}{2i\pi}\int_L \frac{-i\bar{f}(k)(gk\sinh ky+(w-kU)^2 \cosh ky)}{\cosh(kh)((w-(kU+\mu))(w-(kU-\mu)))w} \exp(iwt)\ dw$$

in the w - plane , the path L is chosen above and parallel to the real axis (fig. 2)

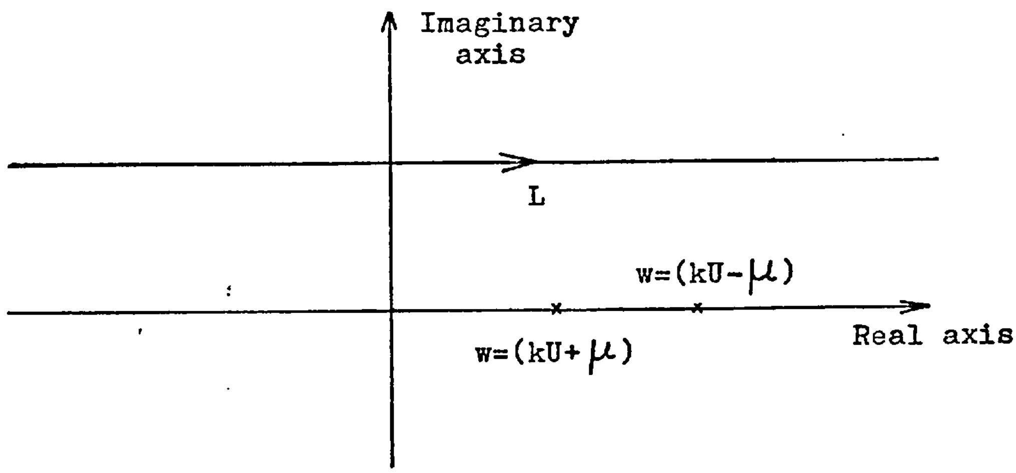


Figure 2
w - plane

By Cauchy theorem and Jordon's lemma , we obtain

$$\bar{\xi}(k,y;t) = \frac{-i\bar{f}(k)(gk \sinh ky + (kU)^2 \cosh ky)}{\cosh kh\ (kU+\mu)(kU-\mu)}$$
$$+ \frac{-i\bar{f}(k)(gk \sinh ky +(\mu(k))^2 \cosh ky)}{\cosh kh\ (2\mu)(kU+\mu)} \exp(it(kU+\mu))$$
$$+ \frac{-i\bar{f}(k)(gk \sinh ky +(\mu(k))^2 \cosh ky)}{\cosh kh\ (2\mu)(kU-\mu)} \exp(it(kU-\mu)) \qquad (26)$$

Now , the solution of the present problem is of course obtained by taking the inverse - Fourier transform of (26) ,

$$\xi(x,y;t) = (1/2\pi)\int_{-\infty}^{\infty} \bar{\xi}(k,y;t)\exp(ikx)\ dk .$$

Upon examining the integrand (the function $\bar{\xi}(k,y;t)$ given by (26) it might seem that it has singularities at zeros of the denominators, but in reality one can easily verify that the function $\overline{\xi}(k,y;t)$ has no singularities when the right hand side of (26) is taken as a whole , i.e. the integrand is non singular . Hence, we write the integrand as the sum of a singular integrals as ,

$$\overline{\xi}(k,y;t)=\left(\frac{-i\bar{f}(k)}{2\mu}\right)\left(\frac{gk\ \sinh(ky)+(kU)^2\ \cosh(ky)}{\cosh(kh)}\right)\left(\frac{-1}{KU+\mu(k)}+\frac{1}{kU-\mu(k)}\right)$$

$$+\left(\frac{-i\bar{f}(k)}{2\mu}\right)\left[\frac{gk\ \sinh(ky)+(\mu(k))^2\cosh(ky)}{\cosh(kh)}\right]\mathrm{X}$$

$$\left[\left(\frac{\exp(it(kU+\mu(k))}{kU+\mu(k)}\right)-\left(\frac{\exp(it(kU-\mu(k))}{kU-\mu(k)}\right)\right] \quad . \qquad (27)$$

What we wish to do is to consider the contribution of the separate items in the right hand side of (27) and to avoid any singularities caused by zeros in their denominators by deforming the path in the complex k-plane using indented contours along the real k-axis , and then we can use Cauchy principal value .

Hence, it is better to rewrite (27) as

$$\xi(x,y;t)=(1/2\pi)\int_{-\infty}^{\infty}\left(\frac{-i\bar{f}(k)}{2\mu}\right)\left(\frac{gk\ \sinh(ky)+(kU)^2\cosh(ky)}{\cosh(kh)}\right)\left(\frac{-\exp(-ikx)}{kU+\mu(k)}\right)dk$$

$$+(1/2\pi)\int_{-\infty}^{\infty}\left(\frac{-i\bar{f}(k)}{2\mu}\right)\left(\frac{gk\ \sinh(ky)+(kU)^2\cosh(ky)}{\cosh(kh)}\right)\left(\frac{\exp(-ikx)}{kU-\mu(k)}\right)dk$$

$$+(1/2\pi)\int_{-\infty}^{\infty}\left(\frac{-i\bar{f}(k)}{2\mu}\right)\left(\frac{gk\ \sinh(ky)+\mu^2\cosh(ky)}{\cosh(kh)}\right)\left(\frac{e^{-ikx}\ e^{it(kU+\mu(k))}}{kU+\mu(k)}\right)dk$$

$$+(1/2\pi)\int_{-\infty}^{\infty}\left(\frac{+i\bar{f}(k)}{2\mu}\right)\left(\frac{gk\ \sinh(ky)+\mu^2\cosh(ky)}{\cosh(kh)}\right)\left(\frac{e^{-ikx}\ e^{it(kU-\mu(k))}}{kU-\mu(k)}\right)dk$$

(27)'

Discussion the different integrals:

To evaluate $(27)'$ for large values of x and t , it is convenient to rewrite $(27)'$ as

$$\xi(x,y;t) = I + J \tag{28}$$

where

$$I = (1/2\pi)\int_{-\infty}^{\infty} \chi_1(k,y)\left(\frac{-\exp(ikx)}{kU+\mu(k)}\right) dk$$
$$+(1/2\pi)\int_{-\infty}^{\infty} \chi_1(k,y)\left(\frac{\exp(-ikx)}{kU-\mu(k)}\right) dk \,, \tag{29}$$

and

$$J = (1/2\pi)\int_{-\infty}^{\infty} \chi_2(k,y)\left(\frac{\exp(it(kU+\mu(k)))}{kU+\mu(k)}\right)\exp(-ikx)\, dk$$
$$+(1/2\pi)\int_{-\infty}^{\infty} \chi_2(k,y)\left(\frac{-\exp(it(kU-\mu(k)))}{kU-\mu(k)}\right)\exp(-ikx)\, dk, \tag{30}$$

where,

$$\chi_1(k,y) = \left(\frac{-i\bar{f}(k)}{2\mu(k)}\right)\left(\frac{gk\sinh ky + (kU)^2\cosh ky}{\cosh kh}\right)$$
$$\chi_2(k,y) = \left(\frac{-i\bar{f}(k)}{2\mu(k)}\right)\left(\frac{gk\sinh ky + (\mu(k))^2\cosh ky}{\cosh kh}\right) \tag{31}$$

We observe that the function $\mu(k) = (gk\tanh kh)^{\frac{1}{2}}$ can be defined as an analytic function in a neighbourhood of the real axis , and , in addition , the function has no real zero except $k = o$. Once the function $\mu(k) = (gk\tanh kh)^{\frac{1}{2}}$ has been so defined , it follows that each of $(kU - \mu(k))$ and $(kU + \mu(k))$ is an analytic function near and on the real - axis , it is important to discuss their zeros.

Roots of the equation (kU ± (k))=0 :

In the fig. 3 , we have plotted the functions $y_1(k) = \pm kU$ and $y_2(k) = \mu(k) = (gk \tanh(kh))^{\frac{1}{2}}$.
Near the origin, i.e. k is very small, the function $y_1(k)$ behaves like (kU) and the function $y_2(k)$ behaves like $(gh)^{\frac{1}{2}}k$, hence $dy_1/dk = U$, and, $dy_2/dk = (gh)^{\frac{1}{2}}$.

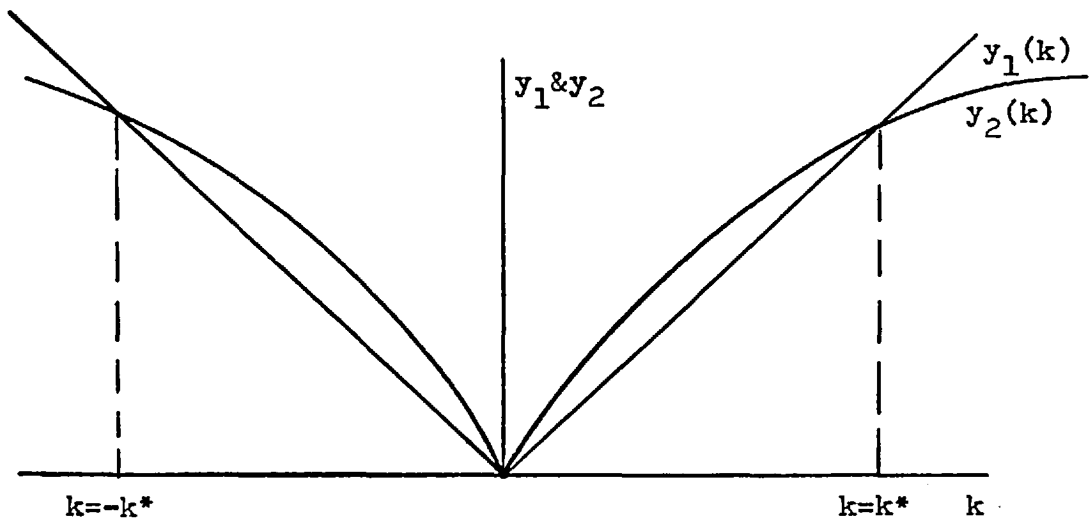


fig. 3

Roots of the equation $(kU \pm \mu(k))=0$
$(U^2/gh < 1)$

As one finds, the zeros of the function $(kU \pm \mu(k))$: k = 0, in addition real roots at $k = \pm k^*$ if the dimensionless parameter (gh/U^2) is greater than unity.

Asymptotic evaluation of the wave integral:

Case I , the infinity depth:

In this case we consider h (the depth) $\longrightarrow \infty$. But from the definition $\mu(k) = (gk \tanh kh)^{\frac{1}{2}}$, in this case $\mu(k) \simeq (gk)^{\frac{1}{2}}$.

Evaluation I :

Consider first the integral

$$(1 / 2\pi) \int_{-\infty}^{\infty} \chi_1(k,y) \left(\frac{\exp(-ikx)}{kU - \mu(k)} \right) dk \; .$$

This integral can be evaluated asymptotically for large x . But the integrand has a simple pole at $k = k^*$ on the real axis defined by $\left[kU - (gk)^{\frac{1}{2}} \right] = 0$, i.e.

$$k = k^* = (g / U^2)$$

Note: this integrand is free from singularity at the origin.

The convergence of the integral:

Near the simple pole $k = k^*$, we consider

$$k = k^* + K$$

where

$$K = K_r + i\, K_i$$

$$= \rho \exp(i\theta)$$

and K is a small complex quantity and $|K| = (K_r^2 + K_i^2)^{\frac{1}{2}}$
$= \rho \ll 1$.

$$\therefore \; dk = dK = i \rho \exp(i\theta)\, d\theta$$

i.e., $\quad dk = iK\, d\theta$,

expanding $\chi_1(k,y)$, $(kU - \mu(k))$ and $\exp(-ikx)$ about $k = k^*$, we obtain

$$\chi_1(k,y) \simeq \chi_1(k^*,y) \;,$$

$$(kU - \mu(k)) \simeq \kappa \; d/dk \; (kU-\mu(k))_{k=k^*}$$

$$\exp(-ikx) \simeq \exp(-ik^*x)\,\exp(-i\kappa x)$$

$$\exp(-ik^*x)\,\exp(-i\kappa_r x)\,\exp(\kappa_i x) \;,$$

i.e., the integrand about k^* is

$$\simeq \left[\frac{\chi_1(k^*,y)}{d/dk(kU-\mu(k))_{k=k^*}} \right] \exp(-ik^*x)\,\exp(-i\kappa_r x)\,\exp(\kappa_i x)$$

Hence , the convergence depends on the sign of x in $\exp(\kappa_i x)$,

if x is positive , then κ_i must be negative,

and, if x is negative , then κ_i must be positive.

Now to deform the path such that the singularity is avoided by making a semicircle around the simple pole , the semicircle must be above or below the real axis , this depends upon the sign of x . Then evaluate the integral as $x \longrightarrow \infty$.

(1) <u>x is positive :</u>

For convergence $\kappa_i < o$, hence , the semicircle must be in the negative half of the complex plan and the path deformed as shown in the figure,

then, $\int_{L'} \ldots = \text{p.v.}\int \ldots + \int_{\smile} \ldots$,

i.e., $\text{p.v.}\int \ldots = \int_{L'} \ldots - \int_{\smile} \ldots$,

$\int_{L'} \ldots$ for large x leads to zero like (1 / x) .

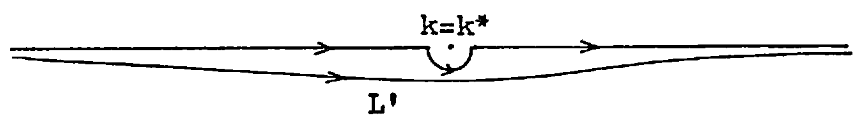
$$\therefore \quad \int_{\smile} = (1/2\pi)\int_{-\pi}^{o} \chi_1(k^*)\left[\frac{\exp(-ik^*x)}{\kappa\; d/dk(kU-\mu(k))_{k=k^*}}\right] i\,\kappa\, d\theta$$

$$=(i/2)\,\chi_1(k^*)\left[\frac{\mathrm{EXP}(-ik^*x)}{d/dk(kU-\mu(k))_{k=k^*}}\right].$$

i.e.

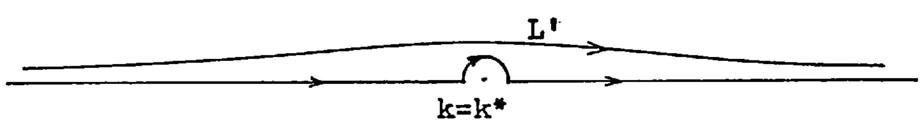
$$(1/2\pi)\int_{-\infty}^{\infty}\chi_1(k)\left[\frac{\exp(-ikx)}{(kU-\mu(k))}\right]dk \simeq$$

$$(-i/2)\,\chi_1(k^*)\left[\frac{\exp(-ik^*x)}{d/dk\,(kU-\mu(k))_{k=k^*}}\right] + O(1/x).$$



(2) x is negative :

For convergence K_i must be positive, this means that the path must be deformed as shown in the figure,



then we get,

$$(1/2\pi)\int_{-\infty}^{\infty}\chi_1(k)\left[\frac{\exp\,(-ikx)}{(kU-\mu(k))}\right]dk \simeq$$

$$(i/2)\,\chi_1(k^*)\left[\frac{\exp(-ik^*x)}{d/dk(kU-\mu(k))_{k=k^*}}\right] + O(1/x)\ .$$

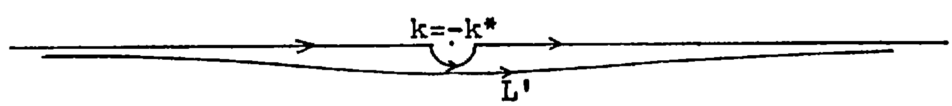
The second integral,

$$(1/2\pi)\int_{\infty}^{\infty}\chi_1(k)\left[\frac{\exp(-ikx)}{(kU+\mu(k))}\right]dk\ ,$$

has a simple pole (real) at $k=-k^*$, and free from singularity at the origin . By similar discussion about the first integral we can easly see,

(1) x is positive:

For convergence K_i must be negative, and the path must be deformed as the figure,



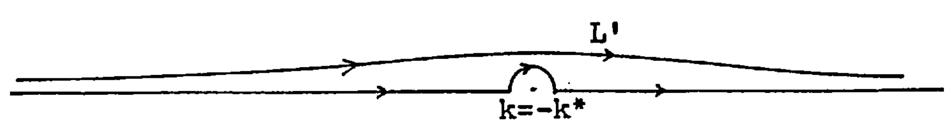
we have,

$$(1/2\pi)\int_{-\infty}^{\infty}\chi_1(k)\left[\frac{\exp(-ikx)}{(kU+\mu(k))}\right]dk \simeq$$

$$(-i/2)\,\chi_1(-k^*)\left[\frac{\exp(ik^*x)}{d/dk(kU+\mu(k))_{k=-k^*}}\right]+O(1/x)$$

(2) x is negative:

For convergence K_i must be positive , and the path must be deformed as the figure ,



we obtain,

$$(1/2\pi)\int_{-\infty}^{\infty} \chi_1(k)\left[\frac{\exp(-ikx)}{(kU + \mu(k))}\right] dk \simeq$$

$$(i/2)\ \chi_1(-k^*)\left[\frac{\exp(ik^*x)}{d/dk(kU+\mu(k))_{k=k^*}}\right] + O(1/x)\ .$$

Now, the value of the integral I,

$$I = (1/2\pi)\int_{-\infty}^{\infty}\chi_1(k,y)\ dk\ \left(\frac{-\exp(-ikx)}{(kU + \mu(k))} + \frac{\exp(-ikx)}{(kU - \mu(k))} \right),$$

when x becomes large is,

(1) <u>x is positive</u> :

$$I = (i/2)\ \chi_1(-k^*)\left[\frac{\exp\ (ik^*x\ )}{d/dk(kU+\mu(k))_{k=k^*}}\right] - (i/2)\left[\frac{\exp\ (-ik^*x)}{d/dk(kU-\ \mu(k))_{k=k^*}}\right]\chi_1(k^*)$$

$$+ O\ (1/x)\ .$$

(2) <u>x is negative :</u>

$$I = -(i/2)\chi_1(-k^*)\left[\frac{\exp\ (-ik^*x)}{d/dk(kU+\ \mu(k)_{k=-k^*}}\right] + (i/2)\chi_1(k^*)\left[\frac{\exp\ (ik^*x)}{d/dk(kU-\mu(k))_{k=k^*}}\right]$$

$$+ O\ (1/x)\ .$$

Evaluation J:

Consider first the integral

$$(1/2\pi)\int_{\infty}^{\infty} \chi_2(k,y)\left[\frac{\exp(-ikx)\ \exp(it(kU-\mu(k)))}{(kU\ -\mu(k))}\right] dk\ .$$

This integral can be evaluated asymptotically for large x and t. To do this we assume the ratio t/x or x/t is fixed, so that the resulting integral contains just one large parameter, either t orx.

Consider x = Vt, where V is fixed and representing the observer speed, hence, we can evaluate this integral for large t with x/t=V fixed. The integral can be rewriten as

$$(1/2\pi)\int_{\infty}^{\infty} \chi_2(k,y)\left[\frac{\exp(itg_1(k))}{(kU\ -\mu(k)}\right] dk\ ,$$

where ,
$$g_1(k) = (\ kU - kV - \mu(k))$$
$$= (\ kU - kV - (gk)^{\frac{1}{2}})\ .$$

The two functions $\chi_2(k,y)$ and $g_1(k)$ are analytic and well behaved in a domain containing the real axis. The integral has

(1) A simple pole at $k=k^*=(g/U^2)$.

(2) $d/dk(g_1(k))=0$, this means that , there is a saddle point at $k=k_o$, be defined by

$\therefore\quad g_1(k) = (kU - kV - (gk)^{\frac{1}{2}}),$

then $\quad d/dk(g_1(k)) = U - V - \frac{1}{2}(g/k)^{\frac{1}{2}},$

$\therefore\quad k = k_o$, the saddle point, is defined by the relation

$$U - V - \tfrac{1}{2}(g/k_o)^{\frac{1}{2}} = 0,$$

$$\therefore\quad k_o = g/(4(U-V)^2)\ .$$

and $\quad d^2/dk^2(g_1(k)) = \frac{1}{4}\ (g^{\frac{1}{2}}/k^{3/2})\ ,$

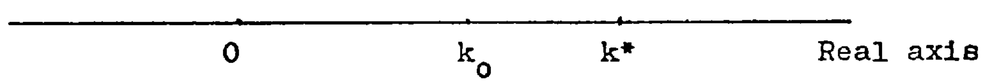
$$\therefore \quad d^2/dk^2(g_1(k_o)) = 2(U - V)^3/g \,.$$

One finds that the value of k_o (saddle point) depends on the value of $V = x/t$ (the observer speed) .

From the values of $k = k^*$ (simple pole) and $k = k_o$ (the saddle point) one easly finds that $k = k^*$ is a fixed value (real) in the same time $k = k_o$ which it is a variable value depends on $x/t=V$, i.e., by changing the value of x/t the position of k_o may be on the right or on the left of k^* as we see,

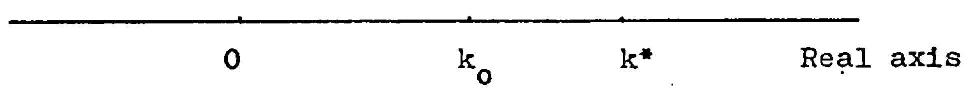
(1) V is negative , i.e. $x/t < o$, we obtain

$$k_o < k^*$$



(2) V is positive , i.e. $x/t > O$, but less than U/2 , then

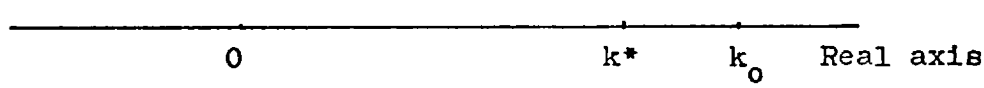
$$k_o < k^*$$



(3) V is positive , i.e. $x/t > o$, but

$$U/2 < V = x/t < U \text{ , we have}$$

$$k_o > k^*$$



One finds , that, when $V = x/t$ (the observer speed) is negative or positive but less than U/2 , hence $k_o < k^*$ i.e. k_o on the left of k^*, but when $V= x/t$ is positive and greater than U/2 and less than U ; therefore, $k_o > k^*$, i.e. k_o on the right of k^*.

The group velocity defined by $d/dk\ (kU - (gk)^{\frac{1}{2}})$

$$= U - \tfrac{1}{2}(g/k)^{\frac{1}{2}} \,,$$

i.e., the group velocity depends on k (the wave number) .

Therefore, the group velocity appropriate to the wave number k=k* is U/2 , this means that, the position of the saddle point relative to the simple pole depends on the observer speed such as,

(a) If the observer speed is less than the group velocity , the saddle point lies on the left of the simple pole.

(b) If the observer speed is greater than the group velocity, the saddle point lies on the right of the simple pole.

Return again to the integral

$$\int_{\infty}^{\infty} \chi_2(k,y) \left[\frac{\exp(itg_1(k))}{(kU - \mu(k))}\right] dk ,$$

expanding $\chi_2(k,y)$, $(kU - \mu(k))$, $\exp(itg_1(k))$ about $k = k^*$. Consider $k = k^* + K$, where $|K| < 1$, we obtain

$$\chi_2(k,y) \simeq \chi_2(k^*,y) ,$$

$$(kU - \mu(k)) \simeq d/dk\,(kU - \mu(k))_{k=k^*} ,$$

$$\exp(itg_1(k)) \simeq \exp(itg_1(k^*)) \exp(it\, g_1'(k^*))$$

$$\simeq \exp(itg_1(k)) \exp(itK_r g_1'(k^*)) \exp(-tK_i g_1'(k^*))$$

Hence, the convergence depends on the sign of $g_1'(k^*)$, in other words, the semicircle round the simple pole must be in positive or negative half according to the sign of the function $g_1'(k^*)$ is positive or negative respectively, i.e.

$$\text{if } g_1'(k^*) > 0 , K_i > 0 \quad \text{and}$$

$$\text{if } g_1'(k^*) < 0 , K_i < 0 .$$

where, accents denoting differentiation with regard to k .

From the definition $g_1'(k) = \left[U - V - \tfrac{1}{2}(g/k)^{\frac{1}{2}}\right]$

hence, $g_1'(k^*) = U - V - \frac{1}{2}U = \frac{1}{2}U - V.$

We find that, the sign of the function $g_1'(k^*)$ depends on the value and the direction of the observer velocity, i.e.

(1) If V is negative , i.e. the observer moves in the upstream direction , $g_1'(k^*)$ is positive, hence, K_i is positive, therefore the semicircle round the simple pole lies in the positive half in the complex plane as in the figure,

(2) If V is positive, i.e. the observer moves in the down stream direction with speed less than the group velcity, $g_1'(k^*)$ is positive, hence, K_i is positive also. Therefore, the semicircle round the simple pole lies in the positive half,

(3) If V is positive, i.e. the observer moves with the stream's direction with speed greater than the group velocity, $g_1'(k^*)$ is negative, hence, K_i is negative. Therefore, the semicircle round the simple pole lies in the negative half.

Now , deforming the path of the integration in every case to $L_1+L_2+L_3$ in the manner of the steepest descent , as we see in figures.

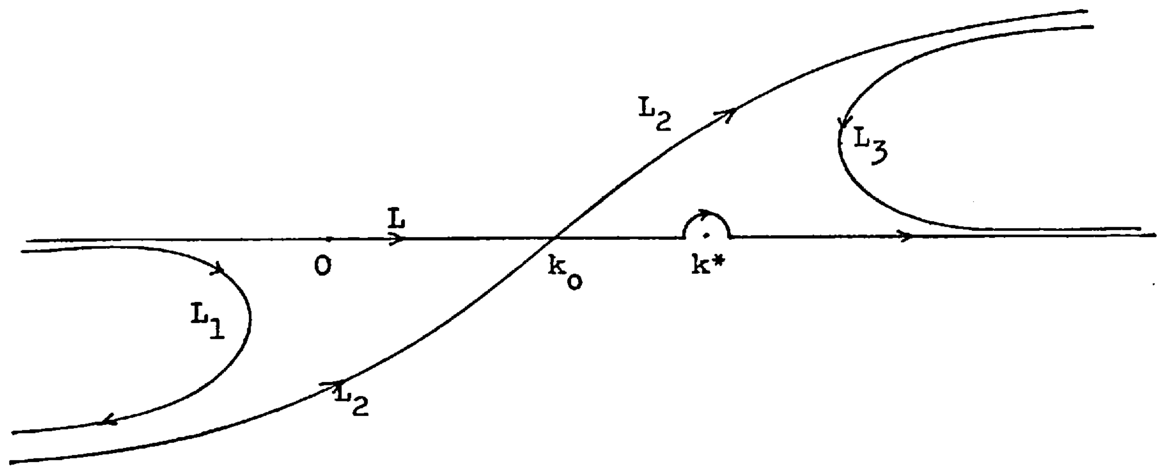


Figure 3

Deformed path when (i) $V<0$,

(ii) $0<V<\frac{1}{2}U$.

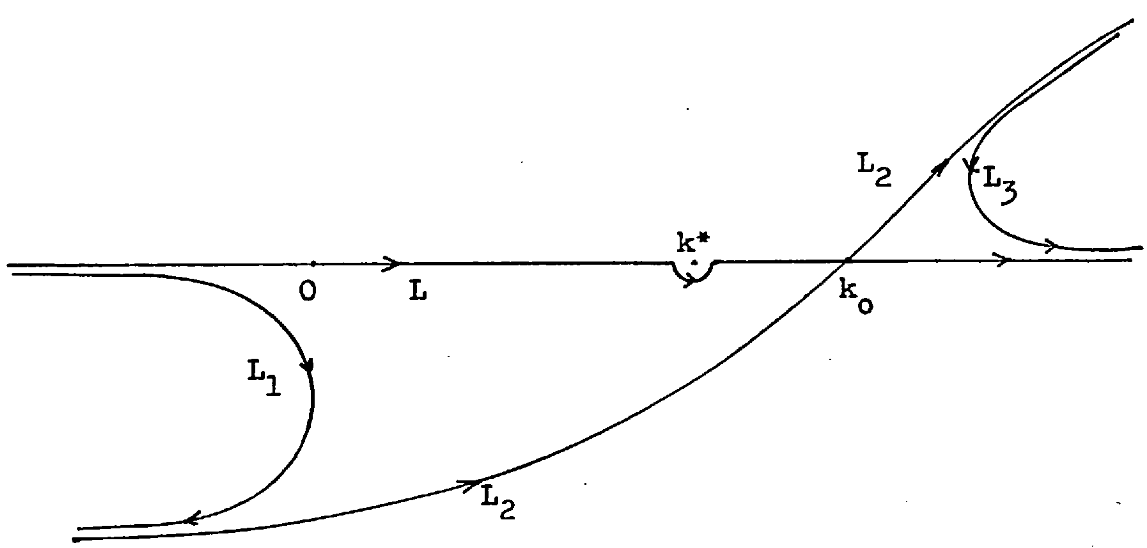


Figure 4

Deformed path when $V>\frac{1}{2}U$.

Corresponding to fig(3) we have

$$\int_L \cdots + \int_{\frown} \cdots = \int_{L_1} \cdots + \int_{L_2} \cdots + \int_{L_3} \cdots ,$$

i.e., $$\int_L \cdots = -\int_{\frown} \cdots + \int_{L_2} \cdots + O(1/t) .$$

where the contributions from L_1 and L_3 are of order $(1/t)$.

Then, $$\int_L \cdots = \tfrac{1}{2}i\chi_2(k^*)\left[\frac{\exp(-ik^*Vt)}{d/dk(kU-\mu(k))_{k=k^*}}\right] + \int_{L_2} \cdots + O(1/t),$$

to evaluate the integral $\int_{L_2}$, we use the result (Jeffreys and Jeffreys) where the major contribution comes from the region about the saddle point defined by $g_1'(k_0) = 0$, hence,

$$(1/2\pi)\int_{-\infty}^{\infty}\chi_2(k)\left[\frac{\exp(ig_1(k)t)}{(kU-\mu(k))}\right]dk \approx \tfrac{1}{2}i\,\chi_2(k^*)\left[\frac{\exp(-ik^*Vt)}{d/dk(kU-\mu(k))_{k=k^*}}\right]$$

$$+(1/2\pi)\chi_2(k_0)\left[\left(\frac{\exp(ig_1(k_0)t)\exp(\tfrac{1}{4}i\pi)}{(k_0U-\mu(k_0))}\right)\left(\frac{2\pi}{t\,g_1''(k_0)}\right)^{\frac{1}{2}}\right]$$

$$+ O(1/t) .$$

when ,

(1) V is negative.

(2) V is positive and lessthan $\tfrac{1}{2}U$.

By the similar manner , the case in which V is positive and greater than $\tfrac{1}{2}U$, we obtain

$$(1/2\pi)\int_{-\infty}^{\infty}\chi_2(k)\left[\frac{\exp(ig_1(k)t)}{(kU-\mu(k))}\right]dk \approx -\tfrac{1}{2}i\,\chi_2(k^*)\left[\frac{\exp(-ik^*Vt)}{d/dk(kU-\mu(k))_{k=k^*}}\right]$$

$$+(1/2\pi)\chi_2(k_0)\left[\left(\frac{\exp(ig_1(k_0)t)\exp(\tfrac{1}{4})}{(k_0U-\mu(k_0))}\right)\left(\frac{2\pi}{t\,g_1''(k_0)}\right)^{\frac{1}{2}}\right]+O(1/t)$$

Similarly , for the integral ,

$$(1/2\pi)\int_{-\infty}^{\infty}\chi_2(k,y)\left[\left(\frac{\exp(-ikx)\exp(it(kU+\mu(k))}{(kU+\mu(k))}\right)\right]dk ,$$

it is better to rewrite the above integral as

$$(1/2\pi)\int_{-\infty}^{\infty}\chi_2(k,y)\left[\left(\frac{\exp(it(g_2(k))}{(kU+\mu(k))}\right)\right]dk ,$$

where , $g_2(k) = (-kV + kU + (gk)^{\frac{1}{2}})$, the simple pole is defined by $kU + \mu(k) = o$, leads to $k = -k^*$ & the saddle point is defined by $g_2'(k_{oo}) = o$.

$$\therefore (1/2\pi)\int_{-\infty}^{\infty}\chi_2(k)\left[\left(\frac{\exp(itg_2(k))}{(kU+\mu(k))}\right)\right]dk \simeq \tfrac{1}{2}i\,\chi_2(-k^*)\left[\left(\frac{\exp(ik^*Vt)}{d/dk(kU-\mu(k))_{k=-k^*}}\right)\right]$$

$$+(1/2\pi)\chi_2(k_{oo})\left[\left(\frac{\exp(ig_2(k_{oo})t)\exp(\tfrac{1}{4}\ )}{(k_{oo}U-\mu(k_{oo}))}\right)\left(\frac{2\pi}{tg''_2(k_{oo})}\right)^{\frac{1}{2}}\right]+O(1/t)$$

when ,

(1) V is negative .

(2) V is positive and less than $\frac{1}{2}U$.

and,

$$(1/2\pi)\int_{-\infty}^{\infty}\chi_2(k,y)\left[\left(\frac{\exp(itg_2(k))}{(kU+\mu(k))}\right)\right]dk \simeq -\tfrac{1}{2}i\,\chi_2(-k^*)\left[\left(\frac{\exp(ik^*Vt)}{d/dk(kU+\mu(k))_{k=-k^*}}\right)\right]$$

$$+(1/2\pi)\chi_2(k_{oo})\left[\left(\frac{\exp(ig_2(k_{oo})t)\exp(\tfrac{1}{4}\ )}{(kU+\mu(k))_{k=k_{oo}}}\right)\left(\frac{2\pi}{tg''_2(k_{oo})}\right)^{\frac{1}{2}}\right]+O(1/t)$$

when V is positive and greater than U/2 .

We have ,

$$J=(1/2\pi)\left(\int_{-\infty}^{\infty}\chi_2(k,y)\left[\left(\frac{-\exp(itg_1(k)t)}{(kU-\mu(k))}\right)+\left(\frac{\exp(itg_2(k))}{(kU+\mu(k))}\right)\right]dk\right.$$

The values of J corresponding the different cases,

(1) $V<o$, i.e., the observer moves in the upstream direction

$$J\simeq -\tfrac{1}{2}i\,\chi_2(k^*,y)\left[\left(\frac{\exp(-ik^*Vt)}{d/dk(kU-\mu(k))_{k=k^*}}\right)\right]+\tfrac{1}{2}i\,\chi_2(-k^*,y)\left[\left(\frac{\exp(itg_2(k))}{d/dk(kU+\mu(k))_{k=-k^*}}\right)\right]$$

$$-(1/2\pi)\,\chi_2(k_o,y)\left[\left(\frac{\exp(ig_1(k_o)t)\exp(\frac{1}{4}\pi i)}{(kU+\mu(k))_{k=k_o}}\right)\left(\frac{2\pi}{t\,g_1''(k_o)}\right)^{\frac{1}{2}}\right]$$

$$+(1/2\pi)\,\chi_2(k_{oo},y)\left[\left(\frac{\exp(ig_2(k_{oo})t)\exp(\frac{1}{4}\pi)}{(kU+\mu(k))_{k=k_{oo}}}\right)\left(\frac{2\pi}{t\,g_2''(k_{oo})}\right)^{\frac{1}{2}}\right]+O(1/t).$$

(2) The observer moves with speed $V<\frac{1}{2}U$ in the downstream direction

$$J\simeq -\tfrac{1}{2}i\,\chi_2(k^*,y)\left[\left(\frac{\exp(-ik^*Vt)}{d/dk(kU-\mu(k))_{k=k^*}}\right)\right]+\tfrac{1}{2}i\,\chi_2(-k^*,y)\left[\left(\frac{\exp(ik^*Vt)}{d/dk(kU+\mu(k))_{k=-k^*}}\right)\right]$$

$$-(1/2\pi)\,\chi_2(k_o,y)\left[\left(\frac{\exp(ig_1(k_o)t)\exp(\frac{1}{4}\pi i)}{(kU-\mu(k))_{k=k_o}}\right)\left(\frac{2\pi}{t\,g''_1(k_o)}\right)^{\frac{1}{2}}\right]$$

$$+(1/2\pi)\,\chi_2(k_{oo},y)\left[\left(\frac{\exp(ig_2(k_{oo})t)\exp(\frac{1}{4}\pi i)}{(kU+\mu(k))_{k=k_{oo}}}\right)\left(\frac{2\pi}{t\,g_2''(k_{oo})}\right)^{\frac{1}{2}}\right]+O(1/t).$$

(3) $V>o$, i.e., the observer moves with the stream direction with speed greater than $\frac{1}{2}U$ and less than U (the mean velocity),

$$J\simeq \tfrac{1}{2}i\,\chi_2(k^*,y)\left[\left(\frac{\exp(-ik^*Vt)}{d/dk(kU-\mu(k))_{k=k^*}}\right)\right]-\tfrac{1}{2}i\,\chi_2(-k^*,y)\left[\left(\frac{\exp(ik^*Vt)}{d/dk(kU+\mu(k))_{k=-k^*}}\right)\right]$$

$$+(1/2\pi)\,\chi_2(k_o,y)\left[\left(\frac{\exp(ig_1(k_o)t)\exp(\frac{1}{4}\pi i)}{(kU-\mu(k))_{k=k_o}}\right)\left(\frac{2\pi}{t\,g_1''(k_o)}\right)^{\frac{1}{2}}\right]$$

$$-(1/2\pi)\,\chi_2(k_{oo},y)\left[\left(\frac{\exp(ig_2(k_{oo})t)\exp(\frac{1}{4}i\pi)}{(kU+\mu(k))_{k=k_{oo}}}\right)\left(\frac{2\pi}{t\,g_2''(k_{oo})}\right)^{\frac{1}{2}}\right]+O(1/t).$$

The simple pole $k = \pm k^*$ is defined by the relation

$$kU \mp \mu(k) = 0 \quad ,$$

this leads to,

$$\chi_1(\pm k^*,y) = \chi_2(\pm k^*,y)$$

Hence , the solutions for large t and x , in different cases, are

Case (1) : the observer moving in the upstream direction, i.e., $V<0$ and $x<0$, the corresponding solution is

$$\xi(x,y;t) = -(1/2\pi)\,\chi_2(k_o,y)\left[\left(\frac{\exp(ig_1(k_o)t)\exp(\frac{1}{4}i\pi)}{(kU-\mu(k))_{k=k_o}}\right)\left(\frac{2\pi}{t\,g_1''(k_o)}\right)^{\frac{1}{2}}\right]$$

$$+(1/2\pi)\,\chi_2(k_{oo},y)\left[\left(\frac{\exp(ig_2(k_{oo})t)\exp(\frac{1}{4}i\pi)}{(kU+\mu(k))_{k=k_{oo}}}\right)\left(\frac{2\pi}{t\,g_2''(k_{oo})}\right)^{\frac{1}{2}}\right]$$

$$+\,O(\,1/t\,)\;.$$

i.e.the solution from the stand point of an observer moving opossitely to the stream , will decrease in amplitude because of $(1/t)^{\frac{1}{2}}$.

Case (2) : the observer moving with the speed $V<\frac{1}{2}U$ in the stream direction, the solution for large t & x takes the form

$$\xi(x,y;t)\simeq -i\,\chi_1(k^*,y)\left[\left(\frac{\exp\,(i(-k^*)x)}{d/dk(kU-\mu(k))_{k=k^*}}\right)\right]$$

$$+i\,\chi_1(-k^*,y)\left[\left(\frac{\exp(ik^*x)}{d/dk(kU+\mu(k))_{k=-k^*}}\right)\right]$$

$$-(1/2\pi)\,\chi_2(k_o,y)\left[\left(\frac{\exp(ig_1(k_o)t)\exp(\frac{1}{4}i\pi)}{(kU-\mu(k))_{k=k_o}}\right)\left(\frac{2\pi}{t\,g_1''(k_o)}\right)^{\frac{1}{2}}\right]$$

$$+(1/2\pi)\chi_2(k_{oo},y)\left[\left(\frac{\exp(ig_2(k_{oo})t)\exp(\frac{1}{4}i\pi)}{(kU+\mu(k))_{k=k_{oo}}}\right)\left(\frac{2\pi}{t\ g_2''(k_{oo})}\right)^{\frac{1}{2}}\right]$$

$$+\ O\ (1/t)\ .$$

The two first terms represent the steady-state solution and the next two terms represent the transient solution which behaves like $(1/t)^{\frac{1}{2}}$.

Case (3) : the observer moving with the speed $V > \frac{1}{2}U$, but $< U$ in the stream direction , hence ,for t&x are large the solution is given by

$$\xi(x,y;t)\simeq(1/2\pi)\chi_2(k_o,y)\left[\left(\frac{\exp(ig_1(k_o)t)\exp(\frac{1}{4}i\pi)}{(kU-\mu(k))_{k=k_o}}\right)\left(\frac{2\pi}{t\ g_1''(k_o)}\right)^{\frac{1}{2}}\right]$$

$$-(1/2\pi)\chi_2(k_{oo},y)\left[\left(\frac{\exp(ig_2(k_{oo})t)\exp(\frac{1}{4}i\pi)}{(kU+\mu(k))_{k=k_{oo}}}\right)\left(\frac{2\pi}{t\ g_2''(k_{oo})}\right)^{\frac{1}{2}}\right]$$

$$+\ O(1/t)\ ,$$

which is represented a transient solution decays like $(1/t)^{\frac{1}{2}}$.

By examining the solution we find that;

(1) If V is negative , i.e. the observer moving in the oposite direction of the mean stream, he observing only a system of a transient waves with an amplitude modulation which behaves like $(1/t)^{\frac{1}{2}}$. The same result obtained for an observer moving in the stream direction with speed greater than $\frac{1}{2}U$ and less than U.

(2) The steady-state solution occures only when the observer moving with speed less than $\frac{1}{2}U$ in the same direction with the mean stream.

Asymptotic evaluation of the wave integrals in a general case

The solution for the present problem given by the expression

$$\xi(x,y;t) = (1/2\pi)\int_{-\infty}^{\infty}\chi_1(k,y)\left[\left(\frac{-\exp(-ikx)}{(kU+\mu(k))}\right)\right]dk$$
$$+ (1/2\pi)\int_{-\infty}^{\infty}\chi_1(k,y)\left[\left(\frac{\exp(-ikx)}{(kU-\mu(k))}\right)\right]dk$$
$$+ (1/2\pi)\int_{-\infty}^{\infty}\chi_2(k,y)\left[\left(\frac{\exp(it(kU+\mu(k))\exp(-ikx)}{(kU+\mu(k))}\right)\right]dk$$
$$+ (1/2\pi)\int_{-\infty}^{\infty}\chi_2(k,y)\left[\left(\frac{-\exp(it(kU-\mu(k))\exp(-ikx)}{(kU-\mu(k))}\right)\right]dk \,.$$

Through the previous discussion , we evaluated this integral asymptotically for large t & x , in the case of infinity depth i.e., $h \longrightarrow \infty$. We have $\mu(k) = (gk \tanh kh)^{\frac{1}{2}}$, in this case we consider $\mu(k) \approx (gk)^{\frac{1}{2}}$ .But , we like to evaluate this integral when h has a finite value.

Evaluation the different integrals:

We consider,

$$I_1 = \int_{-\infty}^{\infty}\chi_1(k,y)\left[\left(\frac{\exp(-ikx)}{(kU-\mu(k))}\right)\right]dk \quad ,$$

The integrand has a simple pole $(k = k^*)$ on the real axis which is defined by $kU - \mu(k) = o$,i.e., $kU = (gk \tanh kh)^{\frac{1}{2}}$,in the same time it is free from any singularity at the origin .

For the convergence , consider

$$k = k^* + K \quad ,$$

where K is a complex quantity such as $|K|<1$ and $K = \rho\exp(i\theta)$ then $dk = dK = iK\,d\theta$.

Expanding $\chi_1(k)$, $(kU - \mu(k))$ and exp(-ikx) about $k = k^*$, we obtain

$$\chi_1(k,y) \simeq \chi_1(k^*,y)$$

$$(kU - \mu(k)) \simeq K \; d/dk \; (kU - \mu(k))_{k=k^*} \quad ,$$

$$\exp(-ikx) \simeq \exp(-ik^*x) \; \exp(-iK_r x) \; \exp(K_i x) \quad ,$$

i.e.,the integrand about $k=K^*$ can be rewritten as

$$\chi_1(k^*,y) \; \left(\frac{\exp(-ik^*x)\exp(-iK_r x)\exp(K_i x)}{d/dk(kU - \mu(k))_{k=k^*}}\right)$$

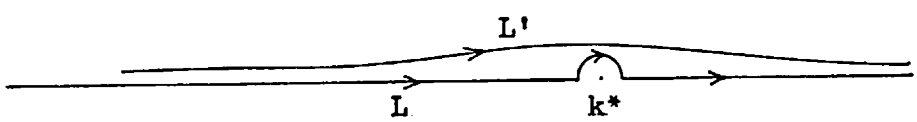
we like to evaluate this integral for x is large , for convergence

(1) if $x > o$, then $K_i < o$,

(2) if $x < o$, then $K_i < o$,

deforming the path to avoid the singularity at $k = k^*$ by using indented contours , the semicircle round k^* lies in positive or negative half of a complex plane depending on the sign of x:

(1) if $x < o$, hence , for convergence $K_i > o$,i.e. the semicircle lies in the positive half and the path deformed as in the figure



$$\therefore \int_{L'} \cdots = \int_{L} \cdots + \int_{\frown} \cdots \quad ,$$

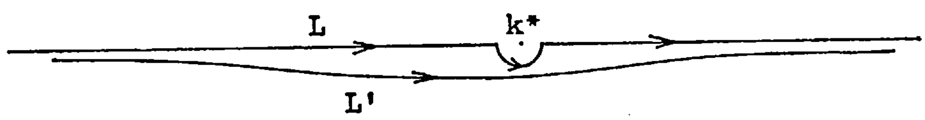
$$\text{i.e. } (1/2\pi) \int_{-\infty}^{\infty} \chi_1(k,y) \left[\left(\frac{\exp(-ikx)}{(kU - \mu(k))} \right) \right] dk = \int_{L'} \cdots - \int_{\frown} \cdots ,$$

but , $\int_{L'} \cdots$ its contribution is of order O(1/t) , and $\int_{\frown} \cdots \; = -\tfrac{1}{2}i \; \chi_1(k^*,y)\left[\left(\frac{\exp(-ik^*x)}{d/dk(kU\; -\mu(k))}\right)_{k=k^*}\right]$,

hence,

$$I_1 \simeq \tfrac{1}{2}i \cdot \; \chi_1(k^*,y)\left[\left(\frac{\exp\;(-ik^*x)}{d/dk(kU\; -\mu(k))}\right)_{k=k^*}\right] + O(1/t)\;.$$

(2) $x > o$, for convergence $K_i < o$, i.e., the semicircle must be in negative half plane and the path deformed as



$$\therefore \quad \int_{L'} \cdots = \int_{L} \cdots \; + \int_{\smile} \cdots \quad ,$$

i.e. in this case ,

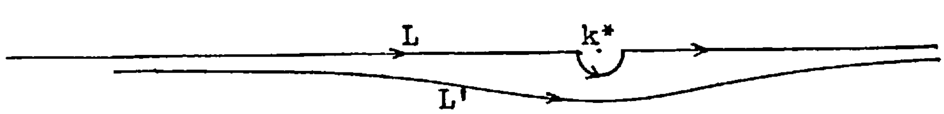
$$I_1 \; = -\tfrac{1}{2}i\,\chi_1(k^*,y)\left[\left(\frac{\exp(-ik^*x)}{d/dk(kU\; -\mu\,(k))}\right)_{k=k^*}\right] + O(1/x)\;.$$

The second integral,

$$I_2 \; = (1/2\pi)\int_{-\infty}^{\infty}\chi_1(k,y)\left[\left(\frac{\exp(-ikx)}{(kU\; +\mu\,(k))}\right)\right]dk\;,$$

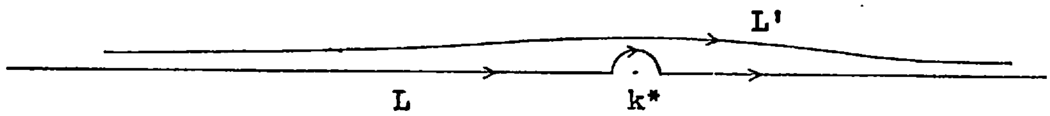
has a simple pole (real) at $k=-K^*$, be defined by $(kU + (gk \tanh hk)^{\frac{1}{2}})$ $= o$, but free from any singularity at $k = o$. By the similar investigation for I_1 one finds :

(1) $x > o$, for convergence $K_i < o$, the deformation path is shown in the figure



$$I_2 \simeq \tfrac{1}{2}i\ \chi_1(-k^*,y)\left[\left(\frac{\exp(ik^*x)}{d/dk(kU+\mu(k))_{k=-k^*}}\right)\right] + O(1/x)\ .$$

(2) $x < o$, this means $K_i > o$ for convergence, and the path must be deformed as



$$I_2 \simeq -\tfrac{1}{2}i\ \chi_1(-k^*,y)\left[\left(\frac{\exp(-ik^*x)}{d/dk(kU+\mu(k))_{k=-k^*}}\right)\right] + O(1/x).$$

The third integral,

$$I_3 = (1/2\pi)\int_{-\infty}^{\infty}\chi_2(k,y)\left[\left(\frac{\exp(-ikx+it(kU-\mu(k)))}{(kU-\mu(k))}\right)\right]dk\ .$$

To evaluate this integral asymptotically for large value of x & t , to carry this we consider the ratio t/x or x/t is fixed, i.e. the resulting integral contains just one large parameter , either t or x . Consider $x = Vt$, where , as before , V representing the speed of the moving observer . It is better to rewrite the integral as,

$$I_3 = (1/2\pi)\int_{-\infty}^{\infty}\chi_2(k,y)\left[\left(\frac{\exp(itg_1(k))}{(kU-\mu(k))}\right)\right]dk\ ,$$

where , $g_1(k) = -kV + kU - (gk\tanh kh)^{\frac{1}{2}}$.

The two functions $\chi_2(k,y)$ and $g_1(k)$ are analytic and well behaved in a domain containing the real axis,in the same time the integral has:

(i) a simple pole at $k=k^*$ (real),be defined by the relation $[kU - (gk\tanh kh)^{\frac{1}{2}}] = o$, then the principal value of the integral is applied.

(ii) a saddle point at $k = K_o$, be defined by the relation

$$d/dk(g_1(k)) = o \ .$$

As the previous discussion , the deformation of the path in the manner of the steepest descent depends on the position of the saddle point ($k = k_o$) relative to the position of the simple pole ($k = k^*$) . For the convergence , we expand the function $\exp(itg_1(k))$ about $k = k^*$. To do this consider $k = k^* + K$, where K is small complex quantity $|K| < 1$, $K = \rho \exp(i\theta)$, by Taylor's theorem we obtain

$$\exp(itg_1(k)) \simeq \exp(itg_1(k^*))\exp(iK_r g_1'(k^*)t)\exp(-K_i g_1'(k^*)t) \ ,$$

but $t > o$, then the convergence (and the position of the semicircle about the simple pole) depends on the sign of $g_1'(k^*)$, where accent denoting differentiation with regard to k . Considerring the different cases:

(1) $g_1'(k^*) > o$, for convergence $K_i > o$, i.e. the semicircle lies in the positive half of the complex plane. This means that

$$U - \mu'(k^*) - V > o \ ,$$

k*

$\therefore \quad V < U - \mu'(k^*)$,

i.e. $\quad V < d/dk(kU - \mu(k))_{k=k^*}$,

$\therefore \quad V <$ the group velocity appropriate to the wave number $k = k^*$,

this inequality leads to :

(a) V (the observer speed) is negative , i.e. the observer moves in the upstream direction.

or (b) V is positive and less than the group velocity appropriate to the wave number $k=k^*$, i.e. the observer moves in the downstream direction .

(2) $g'_1(k^*) < o$,then $K_i < o$ for convergence ,i.e. the semi-circle lies in the negative half of the complex plane,

this means that,

$$U - \mu'(k^*) - V < o,$$

$$\therefore \quad V > (\text{group velocity})_{k=k^*} \; ,$$

this means that the observer moves with a velocity its value greater than the $(\text{group velocity})_{k=k^*}$ in the dowenstream direction.

The saddle point ($k = k_o$),be defined by the relation:

$$d/dk(\ g_1(k)\ ) = o \; ,$$

$$\therefore \quad d/dk(-kV + kU - \mu(k)) = o \; ,$$

i.e. the saddle point must satisfy the relation

$$V = d/dk(Uk - \mu(k))_{k=k_o} ,$$

this means that the value of the saddle point ($k=k_o$) depends on the value of $V = x/t$ (the observer speed) . There is a connection between V and the$(\text{group velocity})_{k=k^*}$, hence we have two cases:

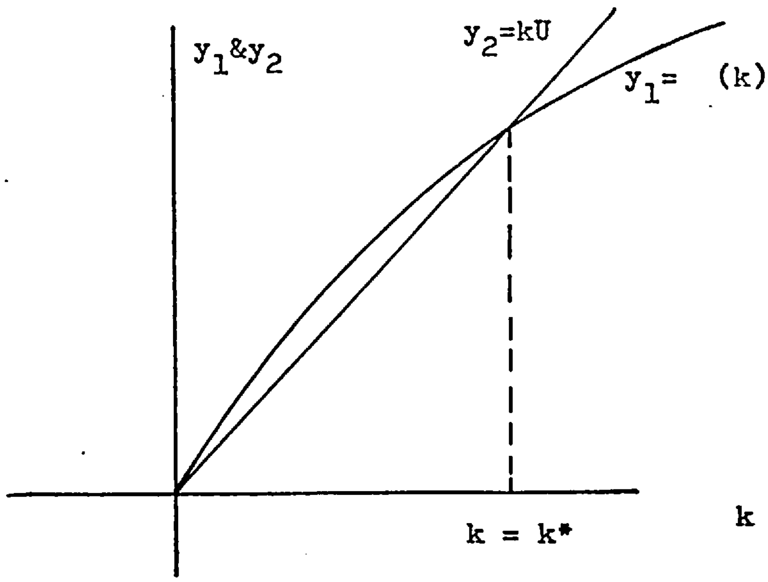
(a) $V < (\text{groub velocity})_{k=k^*}$,

$$V < d/dk(Uk - \mu(k))_{k=k^*} ,$$

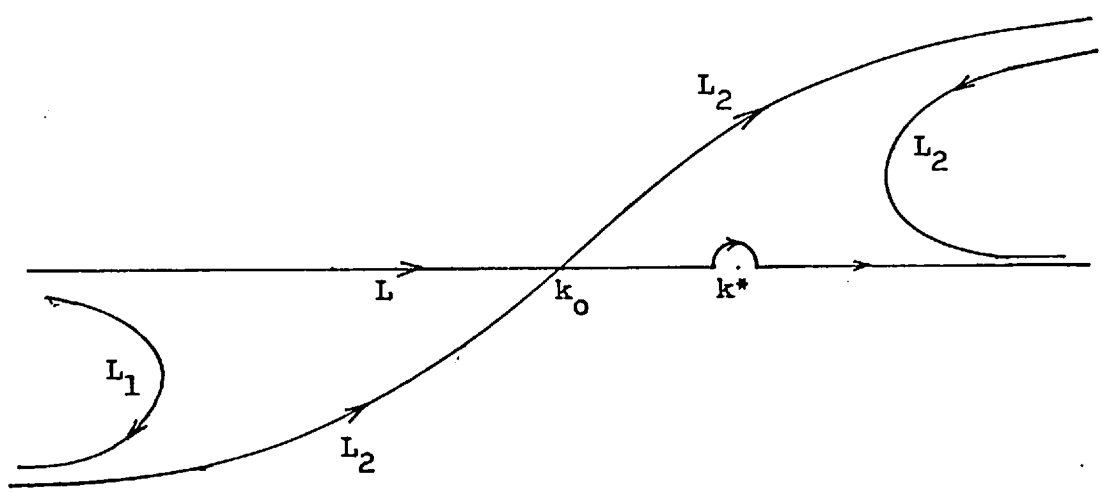
$$\therefore \; d/dk(kU - \mu(k))_{k=k_o} < d/dk(\ kU - \mu(k))_{K=k^*} ,$$

$$\mu'(k_o) > \mu'(k^*) ,$$

From the figure, $d/dk(\ gk \tanh kh\ )^{\frac{1}{2}}$ decreases monotonically from $(gh)^{\frac{1}{2}}$ near the origin ($k \simeq o$) , to zero as k becomes large ($k \longrightarrow \infty$) , this leads to the result: the saddle point lies on the left of the simple pole ,i.e. $k_o < k^*$.



From this discussion , the path must be deformed by the manner of the steepest descent as in the figure,



The integral I_3 has a contribution from its pole to-gether the contribution from the saddle point k_0 , hence , the result is

$$I_3 \simeq \tfrac{1}{2}i\,\chi_2(k^*,y)\left[\left(\frac{\exp(-ik^*Vt)}{d/dk(kU-\mu(k))_{k=k^*}}\right)\right]$$

$$+\ (1/2\pi)\,\chi_2(k_o,y)\left[\left(\frac{\exp(itg_1(k_o)\ \exp(\tfrac{1}{4}i\pi)}{(kU-\mu(k))_{k=k_o}}\right)\left(\frac{2\pi}{t\ g_1''(k_o)}\right)^{\frac{1}{2}}\right]$$

$$+\ O\ (1/t)\ .$$

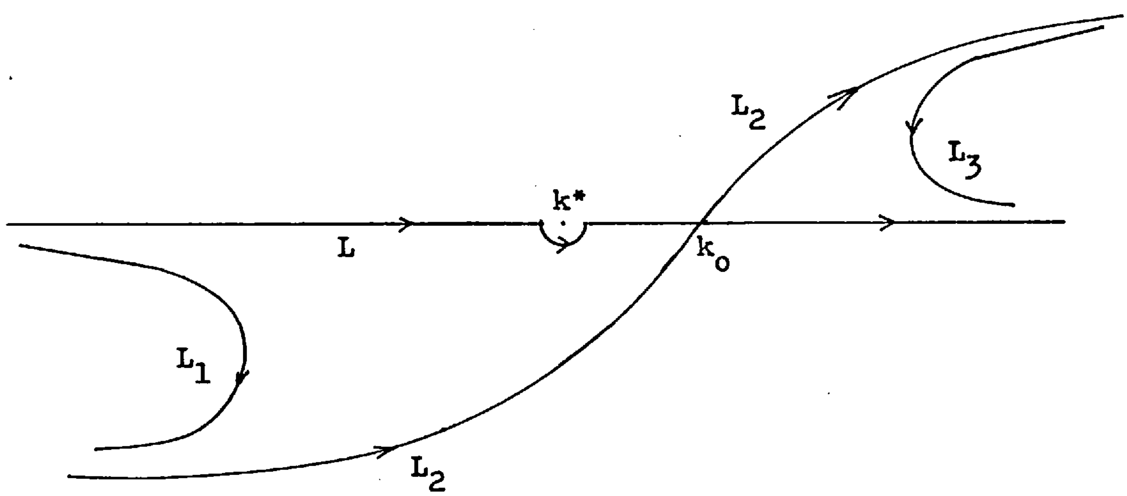
(b) $V >(\text{group velocity})_{k=k^*}$

i.e., $d/dk(kU-\mu(k))_{k=k_o} > d/dk(kU-\mu(k))_{k=k^*}$

$\therefore\ \mu'(k_o) < \mu'(k^*)$,

this means that $k_o > k^*$.

Then the path deformed as



The contributions from the simple pole ($k = k^*$) and the saddle point ($k = k_o$) are given by

$$I_3 \simeq -\tfrac{1}{2}i\ \chi_2(k^*,y)\left[\left(\frac{\exp(-ik^*Vt)}{d/dk(kU-\mu(k))_{k=k^*}}\right)\right]$$

$$+ (1/2\pi)\, \chi_2(k_o,y)\left[\left(\frac{\exp(itg_1 k_o))\exp(\frac{1}{4}i\pi)}{(kU-\mu(k))_{k=k_o}}\right)\left(\frac{2\pi}{t\; g_1''(k_o)}\right)^{\frac{1}{2}}\right]$$

$$+ O\,(\,1/t\,)\,.$$

By the similar manner , we can evaluate the fourth integral I_4 ,

$$I_4 = (1/2\pi)\int_{-\infty}^{\infty}\chi_2(k,y)\left[\left(\frac{\exp(-ikx)\exp(it(kU+\mu(k))}{(kU+\mu(k))}\right)\right]dk \quad ,$$

which can be written as

$$I_4 = (1/2\pi)\int_{-\infty}^{\infty}\chi_2(k,y)\left[\left(\frac{\exp\,(itg_2(k))}{(\,kU+\mu(k))}\right)\right]dk \;,$$

for large t , where

$$g_2(k) = (\,-kV + kU + \mu(k))\;.$$

The integral has;

(1) simple pole ($k=-k^*$), be defined by the relation $kU+\mu(k) = o$.

(2) saddle point ($k=k_{oo}$) be defined by the relation $g_2'(k_{oo}) = o$.

We obtain,

$$I_4 \simeq \tfrac{1}{2}i\,\chi_2(-k^*,y)\left[\left(\frac{\exp(ik^*Vt)}{d/dk(kU+\mu(k))_{k=-k^*}}\right)\right]$$

$$+ (1/2\pi)\,\chi_2(k_{oo},y)\left[\left(\frac{\exp(ig_2(k_{oo})t)\exp(\frac{1}{4}i\pi)}{(kU+\mu(k))_{K=k_{oo}}}\right)\left(\frac{2\pi}{t\; g_2''(k_{oo})}\right)^{\frac{1}{2}}\right]$$

$$+ O\,(\,1/t\,)\;,$$

corresponding to the case : V is negative (i.e. the observer moving in the upstream direction) or V is positive and less than the group velocity appropriate to the wave number $k = -k^*$.

Next, in the case V is positive and greater than the (group velocity)$_{k=-k^*}$, we get

$$I_4 \simeq -\tfrac{1}{2}i\,\chi_2(-k^*,y)\left[\left(\frac{\exp(ik^*Vt)}{d/dk(kU+\mu(k))_{k=-k^*}}\right)\right]$$

$$+\ (1/2\pi)\,\chi_2(k_{oo},y)\left[\left(\frac{\exp(ig_2(k_{oo})t)\exp(\tfrac{1}{4}i\pi)}{(kU+\mu(k))_{k=k_{oo}}}\right)\left(\frac{2\pi}{t\ g_2''(k_{oo})}\right)\right]$$

$$+\ O\ (\ 1/t\ )\ .$$

Now, by adding the values of I_1 , I_2 , I_3 and I_4 in the different cases , we get solutions corresponding the different cases ,

Case 1 :

In this case, the observer moving in the upstream direction, the corresponding solution is

$$\xi(x,y;t) \simeq -\ (1/2\pi)\,\chi_2(k_o,y)\left[\left(\frac{\exp(itg_1(k))\exp(\tfrac{1}{4}i\pi)}{(kU-\mu(k))_{k=k_o}}\right)\left(\frac{2\pi}{t\ g_1''(k_o)}\right)^{\frac{1}{2}}\right]$$

$$+(1/2\pi)\,\chi_2(k_{oo},y)\left[\left(\frac{\exp(itg_2(k_{oo}))\exp(\tfrac{1}{4}i\pi)}{(kU+\mu(k))_{k=k_{oo}}}\right)\left(\frac{2\pi}{t\ g_2''(k_{oo})}\right)^{\frac{1}{2}}\right]$$

$$+\ O\ (1/t)\ .$$

i.e. any one moves in the upstream direction , he will observe a system of transient waves tends to zero like $(1/t)^{\frac{1}{2}}$.

Case 2 :

The observer moving with the stream direction with a speed less than $(\text{group velocity})_{k=k^*}$, the corresponding solution is,

$$\xi(x,y;t) = -i\,\chi_1(k^*,y)\left[\left(\frac{\exp(-ik^*x)}{d/dk(kU-\mu(k))_{k=k^*}}\right)\right]$$

$$+ i\chi_2(-k^*,y)\left[\left(\frac{\exp(ik^*Vt)}{d/dk(kU+\mu(k))_{k=-k^*}}\right)\right]$$

$$-(1/2\pi)\chi_2(k_o,y)\left[\left(\frac{\exp(itg_1(k_o))\exp(\frac{1}{4}i\pi)}{(kU-\mu(k))_{k=k_o}}\right)\left(\frac{2\pi}{t\ g_1''(k_o)}\right)^{\frac{1}{2}}\right]$$

$$+(1/2\pi)\chi_2(k_{oo},y)\left[\left(\frac{\exp(itg_2(k_{oo})\exp(\frac{1}{4}i\pi)}{(kU+\mu(k))_{k=k_{oo}}}\right)\left(\frac{2\pi}{t\ g_2''(k_{oo})}\right)^{\frac{1}{2}}\right]$$

$$+ O\ (1/t)\ .$$

We find in this case , the solution corresponding the contributions from the simple poles gives the Steady State Solution, in the same time the Transient Solution comes from the contributions of saddle points and it behaves like $(1/t)^{\frac{1}{2}}$. Hence , any one moves with a speed less than the group velocity appropriate to the wave number $k=k^*$, in the downstream direction , he always watching a system of Steady waves .

Case 3 :

The moving observer has the stream direction and a velocity greater than the (group velocity $)_{k=k^*}$, the solution is

$$\xi(x,y;t) = -(1/2\pi)\chi_2(k_o,y)\left[\left(\frac{\exp(itg_2(k_o))\exp(\frac{1}{4}i\pi)}{(kU-\mu(k))_{k=k_o}}\right)\left(\frac{2\pi}{t\ g_1''(k_o)}\right)^{\frac{1}{2}}\right]$$

$$+(1/2\pi)\chi_2(k_{oo},y)\left[\left(\frac{\exp(itg_2(k_{oo}))\exp(\frac{1}{4}i\pi)}{(kU+\mu(k))_{k=k_{oo}}}\right)\left(\frac{2\pi}{t\ g_2''(k_{oo})}\right)^{\frac{1}{2}}\right]$$

$$+ O\ (\ 1/t\ )\ .$$

representing a system of a transient waves decaying like $(1/t)^{\frac{1}{2}}$.

By examining the solution we find that;

(1) If V is negative , i.e. the observer moving in the oposite direction with the mean stream , he observing only a system of a transient waves with an amplitude modulatin which behaves like $(1/t)^{\frac{1}{2}}$. The same result obtained for an observer moving in the stream direction with speed greater than the group velocity.

(2) The steady-state solution occures only when the observer moving with speed less than the group velocity in the same direction with the mean stream.

The Transition Case

From the previous discussion we find that :

(1) When the observer moves with a velocity less than the group velocity appropriate to the wave number $k = k^*$ and has the stream direction (in this case the saddle point $k = k_0$ lies on the left of the simple pole $k = k^*$) , he always watching a system of a Steady waves .

(2) If he moves with a velocity greater than the group velocity with the stream direction (in this case the saddle point lies on the right of the simple pole) , he will watch a system of a transient waves behaves like ($1/t^{\frac{1}{2}}$) .

The question we now answer is of the nature of the transition of the wave train when the saddle point becomes close as we like the simple pole from the two sides , in other words , when the observer moves with a velocity its magnitude nearly equal the group velocity ?

We have the solution

$$\xi(x,y;t)=(1/2\pi)\left[\int_{-\infty}^{\infty}\chi_1(k,y)\,\frac{e^{-ikx}}{(kU-\mu(k))}\,dk\;-\int_{-\infty}^{\infty}\chi_2(k,y)\,\frac{\exp(itg_1(k))}{(kU-\mu(k))}\,dk\right]$$

$$+(1/2\pi)\left[\int_{-\infty}^{\infty}\chi_1(k,y)\,\frac{-\,e^{-ikx}}{(kU+\mu(k))}\,dk\;+\int_{-\infty}^{\infty}\chi_2(k,y)\,\frac{\exp(itg_2(k))}{(kU+\mu(k))}\,dk\right]$$

where , $g_1(k) = (-kV +kU -\mu(k))$ and

$g_2(k) = (-kV +kU +\mu(k))$.

<u>Consider first ($I_1 - J_1$) :</u>

where , $$I_1 = (1/2\pi)\int_{-\infty}^{\infty} \chi_1(k,y)\,\frac{e^{-ikx}}{(kU-\mu(k))}\,dk \quad ,$$

the function ($kU - \mu(k)$) has a simple zero at $k = k^*$, while $\chi_1(k,y)$ is analytic and not zero at $k = k^*$, hence we can write

$$\frac{\chi_1(k,y)}{(kU-\mu(k))} = \frac{\chi_1(k^*,y)}{(k-k^*)\ d/dk(kU-\mu(k))_{k=k^*}} + \phi_1(k,y) \quad ,$$

where $\phi_1(k,y)$ is analytic function at $k = k^*$.

$$\therefore \quad I_1 = (1/2\pi)\,\frac{\chi_1(k^*,y)}{d/dk(kU-\mu(k))_{k=k^*}} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{(k-k^*)}\,dk$$

$$+(1/2\pi)\int_{\infty}^{\infty} \phi_1(k,y)\,e^{-ikx}\,dk \ .$$

For the integral $\int_{-\infty}^{\infty} \frac{e^{-ikx}}{(k-k^*)}\,dk$, near the simple pole $k=k^*$

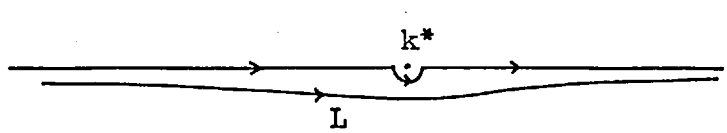
we put $k = k^* + K$, where $K = K_r + i\,K_i = \rho\, e^{i\theta}$ and $\rho = |K| < 1$,

$$\therefore \quad dk = dK = iK\,d\theta$$

$$\therefore \ \exp(-ikx) = \exp(-ik^*x)\,\exp(-iKx)$$
$$= \exp(ik^*x)\,\exp(-iK_r x)\,\exp(K_i x) \ ,$$

it is clear that the convergence depends on the sign of x ; i.e.

(1) If x is positive , then K_i must be negative , hence we deform the path as in the figure

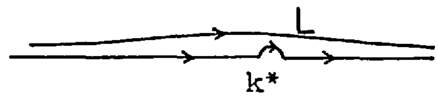


Then $\int_L \ldots = P\int_{-\infty}^{\infty} \ldots + \int_{\smile} \ldots$,

i.e. $P\int_{-\infty}^{\infty} \ldots = -\int_{\smile} \ldots + \int_L \ldots = -i\theta \,\Big|_{-\pi}^{0} e^{-ik^*x} + O(1/x)$.

$$= -i\pi e^{-ik^*x} + O(1/x)$$

(2) If x is negative , then K_1 must be positive , hence we deform the path as in the figure



$$\therefore \int_L \ldots = P\int_{-\infty}^{\infty} \ldots + \int_{\frown} \ldots ,$$

then, $P\int_{-\infty}^{\infty} \ldots = -\int_{\frown} \ldots + \int_L \ldots = i\theta \,\Big|_{\pi}^{0} e^{ikx} + O(1/x)$

$$= i\pi e^{-ik^*x} + O(1/x) .$$

Hence , $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-ikx}}{k-k^*}\, dk = -i\pi\, \mathrm{sgn}(x)\, e^{-ik^*x}$.

Therefore , $I_1 \simeq -i\pi\, \mathrm{sgn}(x) \dfrac{\chi_1(k^*,y)}{d/dk(kU-\mu(k))_{k=k^*}} e^{-ik^*x} + O(1/x)$

since the integral $\int_{-\infty}^{\infty} \phi_1(k,y)\, e^{-ikx}\, dk$ behaves like $(1/x)$.

Now , the integral, $J_1 = (1/2\pi)\displaystyle\int_{-\infty}^{\infty} \chi_2(k,y) \frac{\exp(itg_1(k))}{(kU - \mu(k))}\, dk$, has

(1) a simple pole at $k = k^*$, be defined by the relation $(kU-\mu(k)) = o$,

(2) a saddle point at $k = k_o$, be defined by $d/dk(g_1(k)) = o$,

where , $g_1(k) = (-kV + kU - \mu(k))$ and $\mu(k) = (gk \tanh(kh))^{\frac{1}{2}}$;

for inifinity depth $\mu(k) \simeq (gk)^{\frac{1}{2}}$.

$$\therefore \quad g_1(k) = (-kV + kU - (kg)^{\frac{1}{2}})$$

$$\therefore \quad d/dk(g_1(k)) = g_1'(k) = (-V+U-\tfrac{1}{2}(g/k)^{\frac{1}{2}}) ,$$

and $d^2/dk^2(g_1(k)) = g_1''(k) = \frac{1}{4}(g)^{\frac{1}{2}}(k)^{-3/2}$.

In this case $k_o = (g/4(U-V)^2)$ and $k^* = g/U^2$.

The integrand $\frac{\chi_2(k,y)}{(kU-\mu(k))}$, where $\chi_2(k,y)$ is analytic and not zero at $k = k^*$, while $(kU-\mu(k))$ has a simple zero there , can be written as

$$\frac{\chi_2(k,y)}{(kU-\mu(k))} = \frac{\chi_2(k^*,y)}{d/dk(kU-\mu(k))_{k=k^*}} \frac{1}{(k-K^*)} + \phi_2(k,y) ,$$

where $\phi_2(k,y)$ is analytic function .

$$\text{Now , } J_1 = (1/2\pi)\frac{\chi_2(k^*,y)}{d/dk(kU-\mu(k))_{k=k^*}}\int_{-\infty}^{\infty}\frac{\exp(itg_1(k))}{(k-k^*)}\,dk$$

$$+ (1/2\pi)\int_{-\infty}^{\infty}\phi_2(k,y)\exp(itg_1(k))\,dk .$$

First , we consider the integral $J' = \int_{-\infty}^{\infty} dk \frac{\exp(itg_1(k))}{(k-k^*)}$, where the function $g_1(k)$ is analytic and well behaved in a domain containing the real axis . k^* is real and the principal value of the integral implied . The function $g_1(k)$ has a saddle point . The problem is to estimate the integral (for large t) . Then we deform the path in the manner of the steapest descent,

$$\therefore \quad J' = -\int_{\circlearrowleft}\cdots + \int_{L_1}\cdots + \int_{L_2}\cdots + \int_{L_3}\cdots$$

The contribution from the simple pole :

Near the simple pole , let $k = k^* + \kappa$,

then , $\exp(itg_1(k)) = \exp(itg_1(k^*)\exp(it\kappa_r g'_1(k^*))\exp(-t\kappa_i g'_1(k^*))$

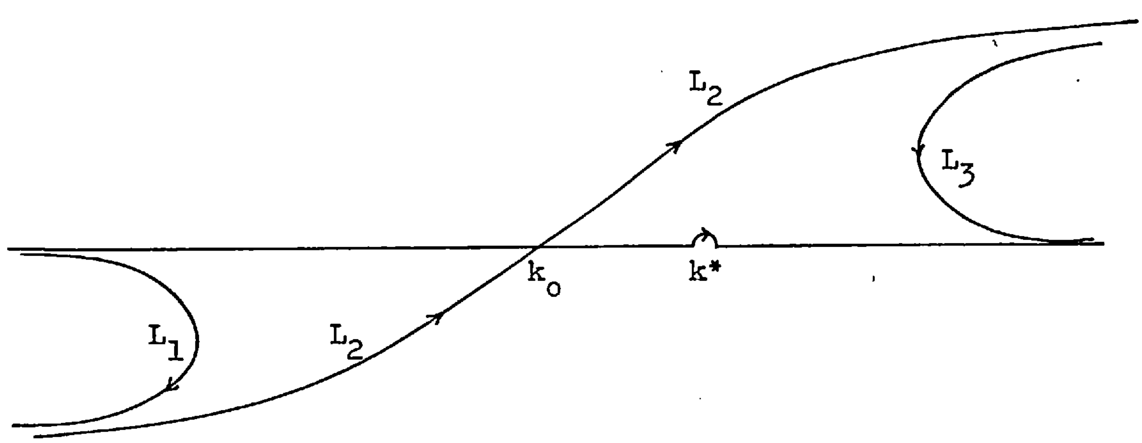
It is clear that the convergence depends on the sign of $g_1'(k^*)$;

(1) If $g_1'(k^*)$ is positive , then K_i must be positive , i.e. the semicircle round the the simple pole lies in the positive half. But

$$g_1'(k^*) = \left[-V + d/dk(kU- \ (k))_{k=k^*}\right] = \left[-V + \text{the group velocity appropriate to wave number } k=k^*\right],$$

$$== -V + \tfrac{1}{2}U \ ,$$

in this case V is less than $\frac{1}{2}U$. From the values of k^* and k_o , we find that k^* is greater than k_o . Since $g_1''(k_o)$ is positive , then the direction of the path through the saddle point is $\pi/4$ with the real axis . Hence , the deformed path in the complex k-plane is

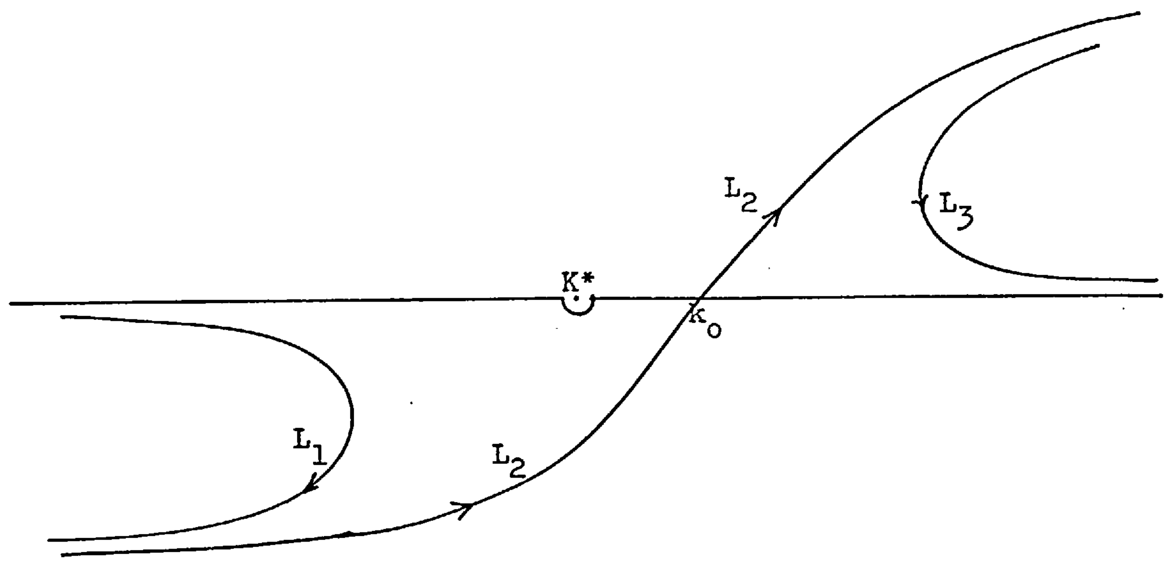


$$\therefore \ J' = -\int_{\curvearrowright} \cdots \ + \int_{L_1} \cdots \ + \int_{L_2} \cdots \ + \int_{L_3} \cdots \ .$$

Hence , the contribution from the simple pole is

$$-i\theta \Big|_{\pi}^{0} \ \exp(itg_1(k^*))$$
$$= i\pi \exp(-ik^*x)$$

(2) If $g_1'(k^*)$ is negative , then K_1 must be negative , i.e. the semicircle round the simple pole lies in the negative half . Here $g_1'(k^*)$ is negative , then $V > \frac{1}{2}U$ and $k_o > k^*$. Also $g_1''(k_o)$ is positive , hence the deformed path in the complex k-plane is



$$\therefore \quad J' = -\int_{\cup} \dots + \int_{L_1} \dots + \int_{L_2} \dots + \int_{L_3} \dots .$$

The contribution from the simple pole is

$$- i\theta \Big|_{-\pi}^{0} \exp(itg_1(k^*))$$

$$= -i\pi \exp(-ik^*x) .$$

In both cases , the contributions from L_1 and L_3 are $O(1/t)$.

hence,
$$J' = i\pi\,\mathrm{sgn}(k^*-k_o)\, e^{-ik^*x} + \int_{L_2} dk \frac{\exp(itg_1(k))}{(k-k^*)} + O(1/t).$$

Now , we consider the integral $\displaystyle\int_{L_2} \frac{\exp(itg_1(k))}{(k - k^*)}\, dk$,

since $g_1''(k_o) \neq 0$, then $g_1(k)$ near the saddle point can be expanded in the form $g_1(k) \simeq g_1(k_o) + \frac{1}{2}(k-k_o)^2 g_1''(k_o)$, let $k - k_o = \lambda e^{i\alpha}$, where λ is real and small , and

$g_1''(k_o) = |g_1''(k_o)| e^{i\beta}$, but $g_1''(k_o)$ is real and positive ,

then $\beta = o$.

$$\therefore \quad g_1(k) \simeq g_1(k_o) + \tfrac{1}{2} \lambda^2 \; |g_1''(k_o)| \; e^{i2\alpha} ,$$

if we write $g_1(k) = u + iv$,

then , the path through the saddle point is defined by;

$$u - u_o = \tfrac{1}{2} \lambda^2 \; |g_1''(k_o)| \cos(2\alpha) = o ,$$

$$v - v_o = \tfrac{1}{2} \lambda^2 \; |g_1''(k_o)| \; \sin(2\alpha) > o$$

i . e . $(2\alpha) = \pi/2$.

$$g_1(k) \simeq g_1(k_o) + \tfrac{1}{2} \lambda^2 \; |g_1''(k_o)| \; e^{\frac{1}{2}i\pi}$$

$$g_1(k_o) + \tfrac{1}{2}i \; \lambda^2 \; |g_1''(k_o)| ,$$

and $\lambda = (k-k_o)e^{-\frac{1}{4}i\pi}$.

Then , the integral can be written as

$$= \exp(itg_1(k_o)) \int_A^B d\lambda \; \frac{\exp(-\frac{1}{2}t \; g_1''(k_o)\lambda^2)}{(k(\lambda) - k^*)} \; (dk(\lambda)/d\lambda)$$

where λ is a point of L_2 . The function $\frac{(dk(\lambda)/d\lambda)}{(k(\lambda)-k^*)}$ can be written in the form

$$\frac{(dk(\lambda)/d\lambda)}{(k(\lambda) - k^*)} = (2i\pi)^{-1} \int_{C_1} \frac{(dk(s)/ds)}{(k(s)-k^*)} \; \frac{ds}{s - \lambda} ,$$

where C_1 is a contour in the s-plane which encloses λ only .

By enlarging the contour C_1 to C_2 to enclose all singularities , we have

$$\frac{(dk(\lambda)/d\lambda)}{(k(\lambda) - k^*)} = - \left[\frac{(dk(s)/ds)}{dk/ds \; (s-\lambda)}\right]_{s=s^*} + (2i\pi)^{-1} \int_{C_2} \frac{(dk(s)/ds)}{(k(s)-k^*)} \; \frac{ds}{(s-\lambda)}$$

where s* is defined by $k(s^*)=k^*$; since $k(\lambda^*)=k^*$ (∴ $s^*=\lambda^*$).
Hence, the integral can be written as

$$= \exp(itg_1(k_o)) \int_A^B d\lambda \ \frac{\exp(-\frac{1}{2}tg_1''(k_o)\lambda^2)}{(\lambda - \lambda^*)}$$

$$+(2i\pi)^{-1} \exp(itg_1(k_o)) \int_A^B d\lambda \ \exp(-\tfrac{1}{2}tg_1''(k_o)\lambda^2) \int_{C_2} \frac{dk/ds}{(k(s)-k^*)} \ \frac{ds}{(s-\lambda)} \ .$$

Since $(k(s)-k^*)$ is bounded away from zero on C_2 , the second term here is $O(1/t^{\frac{1}{2}})$. i.e. the first term is the most important part, and it can be written as

$$\exp(itg_1(k_o)) \int_{-\infty}^{\infty} d\lambda \ \frac{\exp(-\frac{1}{2}tg_1''(k_o)\lambda^2)}{(\lambda - \lambda^*)} \quad ,$$

since the contributions $\int_{-\infty}^{A} + \int_{B}^{\infty}$ are $O(1/t)$.

Hence , returning to the original integral ($I_1 - J_1$) , we find

$$(I_1-J_1) = -i\pi \left[\mathrm{sgn}(x)+\mathrm{sgn}(k^*-k_o)\right] \frac{\chi_1(k^*,y)}{d/dk(kU-\mu(k))_{k=k^*}} \ e^{-ik^*x}$$

$$- \frac{\chi_1(k^*,y)}{d/dk(kU-\mu(k))_{k=k^*}} \ \exp(itg_1(k_o)) \int_{\infty}^{\infty} d\lambda \ \frac{\exp(-\frac{1}{2}tg_1''(k_o)\lambda^2)}{\lambda - \lambda^*}$$

$$+ \int_{-\infty}^{\infty} \phi_1(k,y) \ e^{-ikx} \ dk - \int_{-\infty}^{\infty} \phi_2(k,y) \ \exp(itg_1(k)) \ dk$$

since $\chi_1(k^*,y) = \chi_2(k^*,y)$. Here , the controbution from the third term behaves like $(1/x)$ and from the forth term behaves like $(1/t^{\frac{1}{2}})$.

Now consider the integral $\int_{\infty}^{\infty} d\lambda \, \frac{\exp(-\frac{1}{2}tg_1''(k_o)\lambda^2)}{(\lambda - \lambda^*)}$,

put $\quad \frac{1}{2}tg_1''(k_o)\,\lambda^2 = \mu^2 \quad$, i.e. $\quad \lambda = (2/tg_1'')^{\frac{1}{2}}\,\mu$

$$\therefore \quad \int_{\infty}^{\infty} d\lambda \, \frac{\exp(-\frac{1}{2}tg_1''\,\lambda^2)}{(\lambda-\lambda^*)} \quad = \quad \int_{\infty}^{\infty} d\,\mu \, \frac{\exp(-\mu^2)}{(\mu-\mu^*)}$$

where ,
$$\mu^* = (\tfrac{1}{2}tg_1''(k_o))^{\frac{1}{2}}\,\lambda^*$$
$$= (t(g_1(k^*)-g_1(k_o)))^{\frac{1}{2}}\, e^{-i\pi/4} \quad ,$$

since λ^* defined by the relation
$$g_1(k^*) = g_1(k_o) + \tfrac{1}{2}ig_1''(k_o)\lambda^{*2} \quad .$$

Define $W(\mu^*) = (2\pi)^{-1} \int_{\infty}^{\infty} d\mu \, \frac{\exp(-\mu^2)}{(\mu-\mu^*)}$, which is tabulated for complex values of the argument μ^* . In terms of errer fonctin we have,

$$W(\mu^*) = i\, e^{-\mu^{*2}}\, \mathrm{erf}(i\mu^*) \qquad \text{when } \mathrm{Im}\mu^* > 0$$
$$= i\, e^{-\mu^{*2}}(\,\mathrm{erf}(i\mu^*) - 1\,) \qquad \text{when } \mathrm{Im}\mu^* < 0 \;.$$

It is clear that $W(\mu^*)$ is discontinous across the real axis .

Hence , the integral ($I_1 - J_1$) can be written as

$$I_1 - J_1 = \frac{\chi_1(k^*,y)}{d/dk(kU-\ (k))_{k=k^*}} \left[-i\pi \left[\mathrm{sgn}(x)+\mathrm{sgn}(k^*-k_o) \right] e^{-ik^*x} \right.$$
$$\left. - 2\pi\, e^{itg_1(k_o)}\; W\left[t^{\frac{1}{2}}\, e^{-\frac{1}{4}i\pi}\, (g_1(k^*)-g_1(k_o))^{\frac{1}{2}} \right] \right]$$

We can write (k^*-k_o) and $(g_1(k^*)-g_1(k_o))$ as a function of a parameter ϵ , where $o < |\epsilon| < 1$, if $V = \frac{1}{2}U(1+\epsilon)$, then

$$k^*-k_o = g/U^2 - (\tfrac{1}{4}g/(U-V)^2)$$
$$=(2g/U^2)(-\epsilon) ,$$

and $$g_1(k^*) - g_1(k_o) = (g/2U)(\epsilon^2) .$$

Now ,

$$I_1-J_1 = \frac{\chi_1(k^*,y)}{d/dk(kU-\mu(k))_{k=k^*}} \Big[-i\pi\,\mathrm{sgn}(x)+\mathrm{sgn}(-\epsilon) \quad e^{-ik^*x}$$
$$- 2\pi\, e^{itg_1(k_o)}\, W\left[e^{-\frac{1}{4}i\pi}\, t^{\frac{1}{2}}\epsilon\, (\tfrac{1}{2}g/U)^{\frac{1}{2}}\right]\Big].$$

Similarly, we can get a similar expression for (I_2-J_2) , where

$$(I_2 - J_2) = \int_{-\infty}^{\infty}\chi_1(k,y)\,\frac{e^{-ikx}}{(kU+\mu(k))}dk - \int_{-\infty}^{\infty}\chi_2(k,y)\,\frac{\exp(itg_2(k))}{(kU+\mu(k))}\,dk$$

If $\epsilon \neq 0$, and t becomes large , i.e. μ^* is large , then the function $W(\mu^*)$ tends to zero and so we find the solution is

$$\xi(x,y;t) \simeq \frac{\chi_1(k^*,y)}{d/dk(kU-\mu(k))_{k=k^*}} \quad -2i\pi\, e^{-ik^*x} \qquad \text{if } \epsilon < o, (\text{i.e.} V < \tfrac{1}{2}U)$$

$$\simeq 0 \qquad\qquad \text{if } \epsilon > 0 , (\text{i.e. } V > \tfrac{1}{2}U) \quad (\text{as before})$$

If the difference (k^*-k_o) is sufficiently small , the value of

$$\mu^* = \sqrt{(\tfrac{1}{2}tg_1''(k_o))}\quad (k^*-k_o)\,\exp(-\tfrac{1}{4}i\pi)$$

is small , The the solution can be expressed as ,

$$\xi(x,y;t) \simeq \frac{\chi_1(k^*,y)}{d/dk(kU-\mu(k))_{k=k^*}} \quad -i\pi\left[sgn(x)+sgn(-\epsilon)\right] e^{-ik^*x}$$

$$- 2\pi \; e^{itg_1(k_o)} \quad W\left[e^{-\frac{1}{4}i\pi} t^{\frac{1}{2}} \epsilon \, (\tfrac{1}{2}g/U)^{\frac{1}{2}}\right]$$

i.e. in terms of the function $W(\mu^*)$, which is tabulated , or in terms of error function if we write $W(\mu^*)$ in terms of the error function . This provides a smooth transition from negative ϵ (i.e. when $V<\frac{1}{2}U$) to positive ϵ (i.e. when $V>\frac{1}{2}U$) .

The Asymptotic Solution When t becomes large and x has one value

Let us fix our attention upon one value of x and let t increases , then x/t will decrease , i.e. we examining the solution $\xi(x;y;t)$ from the stand point of an observer standing at a given value of x. The solution is given by

$$\xi(x,y;t) = I + J$$

where

$$I = (1/2\pi)\int_{-\infty}^{\infty}\chi_1(k,y)\left[\left(\frac{-\exp(-ikx)}{kU+\mu(k)}\right)\right]dk + (1/2\pi)\int_{-\infty}^{\infty}\chi_1(k,y)\left[\left(\frac{\exp(-ikx)}{kU-\mu(k)}\right)\right]dk .$$

and,

$$J = (1/2\pi)\int_{-\infty}^{\infty}\chi_2(k,y)\left[\left(\frac{\exp(-ikx)\exp(it(kU+\mu(k)))}{kU+\mu(k)}\right)\right]dk + (1/2\pi)\int_{-\infty}^{\infty}\chi_2(k,y)\left[\left(\frac{-\exp(-ikx)\exp(it(kU-\mu(k)))}{kU-\mu(k)}\right)\right]dk .$$

Evaluation the different integrals:

Consider

$$I_1 = (1/2\pi)\int_{-\infty}^{\infty}\chi_1(k,y)\left[\left(\frac{\exp(-ikx)}{kU-\mu(k)}\right)\right]dk = (1/2\pi)\int_{-\infty}^{\infty}\chi_1(k,y)\left[\left(\frac{\exp(-ik(x/t)t)}{kU-\mu(k)}\right)\right]dk$$

Firstly, the fixed value of x is positive:

The function $\left[\chi_1(k,y) / (kU-\mu(k))\right]$ is analytic and well behave near the real axis except at a simple pole (real) $k = k^*$, be defined by $[k^*U - \mu(k^*)] = 0$; where $\mu(k) = (gk \tanh kh)^{\frac{1}{2}}$. Consider $k = k^* + K$, where K is a small complex quantity, i.e. $|K| < 1$ and in polar coordinate $K = \rho\exp(i\theta)$.

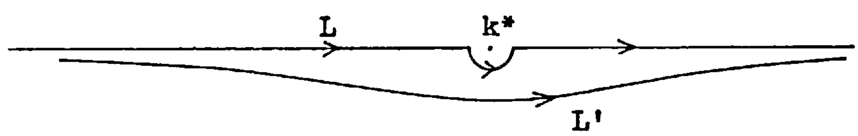
Expanding by Taylor's theorem the functions $\chi_1(k,y)$,$(kU-\mu(k))$ and $\exp(-ik(x/t)t)$ about the simple pole $k = k^*$, we obtain

$$\chi_1(k,y) \simeq \chi_1(k^*,Y) \,,$$

$$(kU-\mu(k)) \simeq K \, d/dk(kU - \mu(k))_{k=k^*} \,,$$

and $\exp(-ik(x/t)t) \simeq \exp(-ik^*(x/t)t)\exp(-i K_r(x/t)t)\exp(K_i(x/t)t)$.

(x/t) is positive and $t > o$, then for the convergence K_i must be negative . The integrand has a simple pole (real) , then the principal value is implied and surrounded the simple pole by semicircle in the negative half and the deformedpath is



$$\therefore \quad \int_{L'} \ldots = \int_{L} \ldots + \int_{\smile} \ldots \,,$$

i.e.

$$(1/2\pi)\int_{-\infty}^{\infty} \chi_1(k,y)\left[\left(\frac{\exp(-ik(x/t)t)}{kU - \mu(k)}\right)\right]dk = \int_{L'} \ldots - (1/2\pi)\int_{\smile} \ldots$$

But , $\int_{L'} \ldots$ tends to zero like $1/t$ and

$$(1/2\pi)\int_{\smile} \chi_1(k,y)\left[\left(\frac{\exp(-ik(x/t)t)}{(kU - \mu(k))}\right)\right]dk = \tfrac{1}{2}i\,\chi_1(k,y)\left[\left(\frac{\exp(-ik^*x)}{d/dk(kU-\mu(k))_{k=k^*}}\right)\right]$$

Then we have, $I_1 \simeq -\tfrac{1}{2}i\,\chi_1(k,y)\left[\left(\frac{\exp(-ik^*x)}{d/dk(kU-\mu(k))_{k=k^*}}\right)\right] + O(1/t)$.

<u>Secondly</u>,the fixed value of x is negative:

In this case x/t approaches zero through negative values as t becomes large , we obtain

$$I_1 \simeq \tfrac{1}{2}i\,\chi_1(k,y)\left[\left(\frac{\exp(-ik^*x)}{d/dk(kU - \mu(k))_{k=k^*}}\right)\right] + O(1/t) \,.$$

Consider $I_2 = (1/2\pi)\int_{-\infty}^{\infty} \chi_1(k,y)\left(\frac{\exp(-ikx)}{kU+\mu(k)}\right)dk$,

by the same manner we can evaluate I_2 as t becomes large, we get

For $x > o$,

$$I_2 \simeq -\tfrac{1}{2}i\,\chi_1(k,y)\left[\left(\frac{\exp(ik^*x)}{d/dk(kU+\mu(k))}\right)_{k=k^*}\right] + O(1/t) ,$$

for $x < o$,

$$I_2 \simeq \tfrac{1}{2}i\,\chi_2(-k^*,Y)\left[\left(\frac{\exp(+ik^*x)}{d/dk(kU+\mu(k))}\right)_{k=-k^*}\right] + O(1/t) .$$

Let, $J_1 = -(1/2\pi)\int_{-\infty}^{\infty} f(k,y)\exp(ig_1(k)t)\,dk$,

where $f(k,y) = \left[\chi_2(k,y)/(kU-\mu(k))\right]$ is analytic round and on the real axis except at the zero of $(kU-\mu(k)) = o$, also $g_1(k)$ is analytic and has a saddle point, defined by $d/dk(g_1(k)) = o$, where $g_1(k) = (kU-\mu(k)-x/t\; k)$, then

$$d/dk\,(g_1(k)) = d/dk\,(kU-\mu(k)) - x/t ,$$
$$= (U-\mu'(k)) - x/t ,$$
$$d^2/dk^2(g_1(k)) = -\mu''(k) .$$

here, the primes denoting the differentiation with respect to k. The saddle point defined by the relation

$$d/dk\,(g_1(k)) = o ,$$

i.e. $[U-\mu'(k_o)] = x/t$.

<u>(1) The fixed value of x is positive</u>:

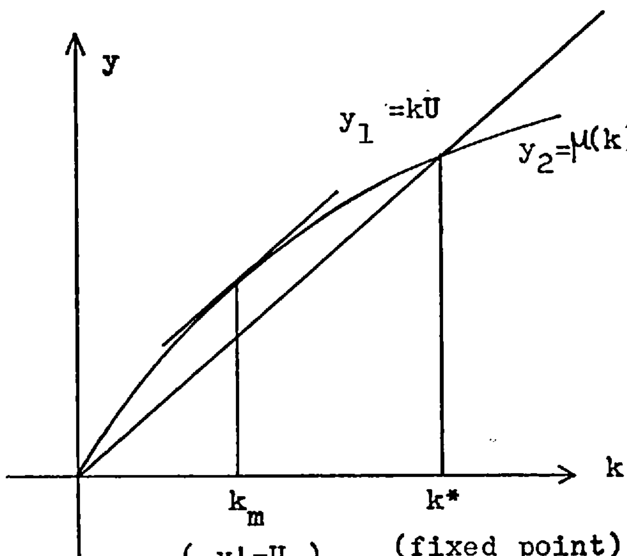
As the time t increasing as we like , then x/t approaching to zero through the positive values. This means , as $t \longrightarrow \infty$, $(U-\mu'(k_o))$ decreasing through positive quantities , from the definition, U has a constant value and positive (the stream velocity), hence , $\mu'(k_o)$ increasing with the time .

<u>In the case of infinity depth</u> :

$$\mu(k) = (gk)^{\frac{1}{2}} ,$$

$$\therefore \quad \mu'(k) = \tfrac{1}{2} (g/k)^{\frac{1}{2}} .$$

But , $\mu'(k_o) = \frac{1}{2} (g/k_o)^{\frac{1}{2}}$ increasing as the time t increasing, this means that the value k_o assocaited with the fixed value x will decrease, passing with k* (the real simple pole) and the mimi-mum value for k_o is k_m(where k_m is given by the relation $\mu'(k_m)=U$ then $k_m=g/4U^2$), hence, the range of k_o is (k_m,∞) .



Consider k = k* + K, where K is a small complex quantity , expanding the function exp(it(kU-μ(k)-x/t k)) about k = k* by Taylor, we get

$$\exp(it(k^*U-\mu(k^*)-x/tk^*)\exp(itK_r(U-\mu'(k^*))-X/t)\exp(-K_i(U-\mu'(k^*))-x/t)t)$$

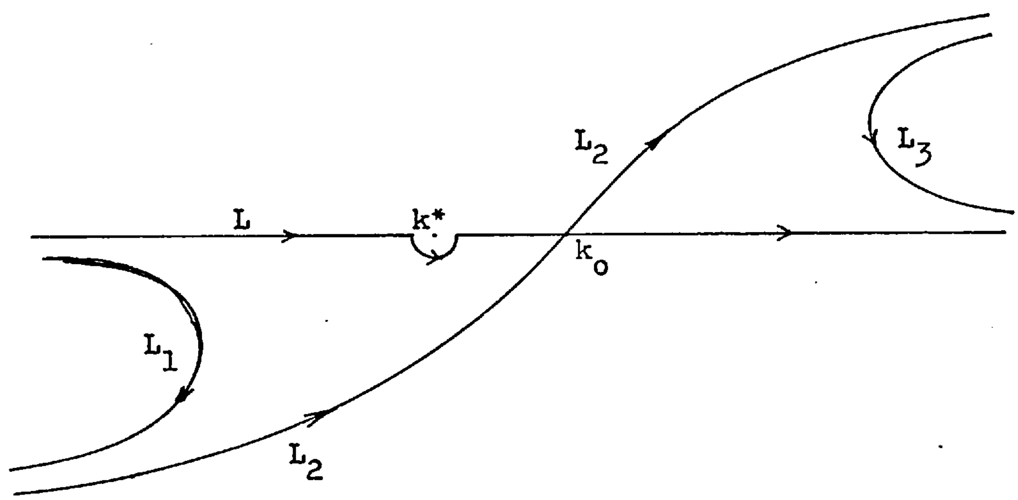
The convergence depends on the sign of (U-μ'(k*)) -x/t) in the function exp(-tK_i(U-μ'(k*)-x/t)), t is positive , then wehave two cases ;

(1) $d/dk(Uk-\mu(k))_{k=k^*} < x/t$

i.e. the group velocity appropriate to the wave number k = k*less than (x/t).

$$\therefore \quad U - \mu'(k^*) < U - \mu'(k_0) \quad ,$$
$$\text{i.e.,} \quad \mu'(k^*) > \mu'(k_0) \ .$$

then , the saddle point ($k = k_0$) lies on the right of the simple pole ($k = K^*$) . For the convergence corresponding to this case K_i must be negative , then the semicircle round the simple pole lies in the negative half of the complex - plane . By deforming the path in the manner of the steepest descent as shown inthe figure



Now , we can write

$$\int_L \cdots = \int_{L_1} \cdots + \int_{L_2} \cdots + \int_{L_3} \cdots - \int_{\smile} \quad ,$$

the contributions from $\int_{L_1} \cdots$ and $\int_{L_3} \cdots$ lead to zero like(1/t),

the contribution from $\int_{\smile} \cdots$ comes from the simple pole,

the contribution from $\int_{L_2} \cdots$ comes from the saddle point .

We have ,

$$J_1 \simeq -\tfrac{1}{2}i\,\chi_2(k^*,y)\left[\left(\frac{\exp(i(-K^*)x)}{d/dk(kU-\mu(k))_{k=k^*}}\right)\right]$$

$$+ (1/2\pi)\,\chi_2(k_o,y)\left[\left(\frac{\exp(itg_1(k_o))\exp(\tfrac{1}{4}i\pi)}{(kU-\mu(k))_{k=k_o}}\right)\left(\frac{2\pi}{t\,g_1''(k_o)}\right)^{\frac{1}{2}}\right]$$

$$+ O\,(1/t) .$$

(2) $\left[d/dk(KU-\mu(k))_{k=k^*}\right] > x/t$,

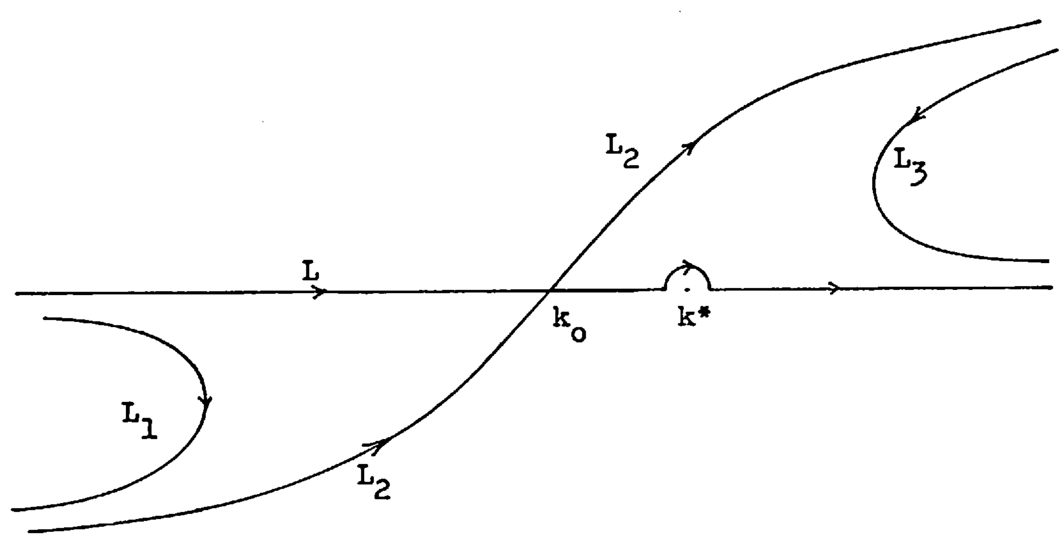
i.e., the group velocity appropriate to the wave number $k = k^*$ greater than x/t .

$$\therefore\quad U - \mu'(k^*) > U - \mu'(k_o) ,$$

then , $\mu'(k^*) < \mu'(k_o)$

$(k_m < k_o < k^*)$

For convergence, K_i must be positive ,then , the deformation pathis given in the figure



then, we get

$$J_1 \approx \tfrac{1}{2}i\,\chi_2(k^*,y)\left[\left(\frac{\exp(-ik^*x)}{d/dk(kU-\mu(k))_{k=k^*}}\right)\right]$$

$$+(1/2\pi)\,\chi_2(k_o,y)\left[\left(\frac{\exp(itg_1(k_o)+\frac{1}{4}i\pi)}{(kU-\mu(k))_{k=k_o}}\right)\left(\frac{2\pi}{t\,g_1''(k_o)}\right)^{\frac{1}{2}}\right]$$

$$+\ O(\ 1/t\ )\ .$$

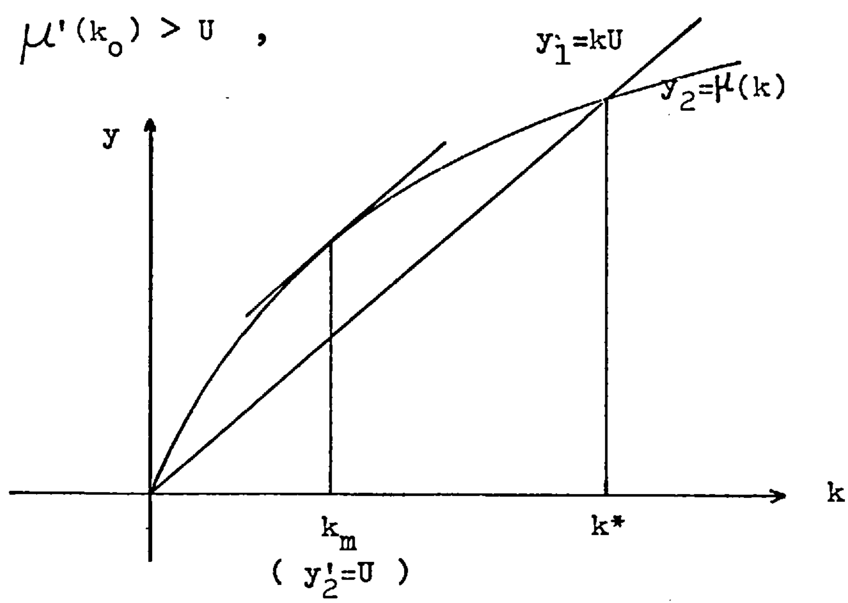
(2) <u>The fixed value of x is negative</u>:

From the definition of the saddle point we have

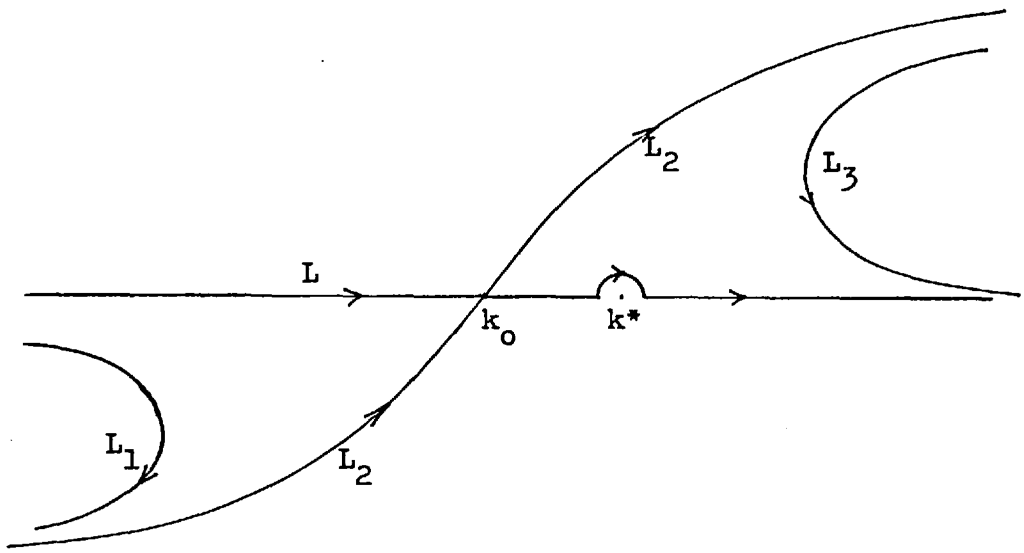
$$U-\mu'(k_o) = (x/t)\quad ,$$

but , x/t is negative , $\therefore\ \left[U-\mu'(k_o)\right] < o$,

i.e. , $\mu'(k_o) > U$,



The values of k_o corresponding this case are less than the value of k* and the path deforming as in the figure



$$J_1 \approx \tfrac{1}{2} i \chi_2(k^*,y)\left[\left(\frac{\exp(k^*(-ix))}{d/dk(kU-\mu(k))_{k=k^*}}\right)\right]$$

$$+ (1/2\pi)\chi_2(k_o,y)\left[\left(\frac{\exp(itg_1(k_o)+\frac{1}{4}i\pi)}{(kU-\mu(k))_{k=k_o}}\right)\left(\frac{2\pi}{t\, g_1''(k_o)}\right)^{\frac{1}{2}}\right]$$

$$+ O(1/t) .$$

By similar discussion we can evaluate the integral J_2

$$J_2 = (1/2\pi)\int_{-\infty}^{\infty}\chi_2(k,y)\left[\left(\frac{\exp(-ikx + it(kU+\mu(k)))}{(\mu(k) + kU)}\right)\right] dk ,$$

the integral has a simple pole $(k=-k^*)$, be defined by the relation $(kU + \mu(k)) = o$, to gether a saddle point, be given by

$$\left[d/dk(kU + \mu(k) - x/t)\right] = o .$$

The Asymptotic Solution in Different Cases:

Case 1 :

In this case , we consider

x/t is positive, and the group velocity, appropriate to the wave number equal the simple pole, is less than x/t , i.e., $\left[U - \mu'(k^*)\right] < X/t$ .The corresponding solution is

$$\xi \simeq -(1/2\pi)\chi_2(k_o,y)\left[\left(\frac{\exp(itg_2(k_o) + \frac{1}{4}i\pi)}{(kU - \mu(k))_{k=k_o}}\right)\left(\frac{2\pi}{t\ g_1''(k_o)}\right)^{\frac{1}{2}}\right]$$

$$+ (1/2\pi)\chi_2(k_{oo},y)\left[\left(\frac{\exp(itg_2(k_{oo})+\frac{1}{4}i\pi)}{(kU - \mu(k))_{k=k_{oo}}}\right)\left(\frac{2\pi}{t\ g_2''(k_{oo})}\right)^{\frac{1}{2}}\right]$$

$$+ O(1/t) .$$

In this case the contirbutions from the simple poles are cancel , then by examining the given solutionfrom standpoint of an observer stationed at a fixed value of x will then observe waves of continually decreasing wave number (increasing wave length) moving by with phase velocities appropraite to their lengths . The gross outline of the waves will pass the observer at the group velocity appropriate to the wave number present at the moment , and , of course , the amplitude is decreasing as $(1/t)^{\frac{1}{2}}$.

Case 2 :

In this case x/t is positive but $(U - \mu'(k^*)) > x/t$ and the wave number k decreasing through the interval(k_m, k^*) i.e.$(g/4U^2) < k < (g/U^2)$, the solution is given by

$$\xi(x,y;t) \approx -i\,\chi_1(k^*,y)\left[\left(\frac{\exp(-ik^*x)}{d/dk(kU-\mu(k))_{k=k^*}}\right)\right]$$

$$+i\,\chi_1(-k^*,y)\left[\left(\frac{\exp(ik^*x)}{d/dk(kU-\mu(k))_{k=-k^*}}\right)\right]$$

$$-(1/2\pi)\,\chi_2(k_o,y)\left[\left(\frac{\exp(ig_1(k_o)t+\tfrac{1}{4}i\pi)}{(kU-\mu(k))_{k=k_o}}\right)\left(\frac{2\pi}{t\,g''_1(k_o)}\right)^{\frac{1}{2}}\right]$$

$$+(1/2\pi)\,\chi_2(k_{oo},y)\left[\left(\frac{\exp(ig_2(k_{oo})t+\tfrac{1}{4}i\pi)}{(kU+\mu(k))_{k=k_{oo}}}\right)\left(\frac{2\pi}{t\,g''_2(k_{oo})}\right)^{\frac{1}{2}}\right]$$

$$+\,0\,(\,1/t\,)\,.$$

In this case the contirbutions from the simple poles representing a system of Steady waves , then by examining the above solution from the standpoint of an observer stationed at a fixed value of x will then observe a system of steady waves .

Case 3 :

The fixed value of x is negative , the corresponding solution is given by

$$\xi(x,y;t) \approx -(1/2\pi)\chi_2(k_o,y)\left[\left(\frac{\exp(itg_1(k_o)+\tfrac{1}{4}i\pi)}{(kU-\mu(k)_{k=k_o}}\right)\left(\frac{2\pi}{t\,g''_1(k_o)}\right)^{\frac{1}{2}}\right]$$

$$+(1/2\pi)\chi_2(k_{oo},y)\left[\left(\frac{\exp(itg_2(k_{oo})+\tfrac{1}{4}i\pi)}{(kU+\mu(k))_{k=k_{oo}}}\right)\left(\frac{2\pi}{t\,g''_2(k_{oo})}\right)^{\frac{1}{2}}\right]$$

$$+\,0\,(\,1/t\,)\,.$$

In this case the fixed value of x is negative , as t increase , then x/t will tend to zero through negative values , but $\left[U-\mu'(k_o)\right]= x/t$,i.e., $\mu'(k_o) > U$. Then , as the value of

$x/t \longrightarrow 0$, the value of k_o will increase in the interval $(o , g/4U^2)$. By examining the solution from standpoint of an observer stationed at a fixed value of x (negative in this case), the observer will then observe waves of continually increasing wave number (decreasing wave length) moving by with phase velocities appropriate to their lengths. The gross outline of the waves will pass the observer at the group velocity appropriate to the wave number present at the moment , and , of course , the amplitude is decreasing at $(1/t)^{\frac{1}{2}}$.

In this problem we fixed our attention on one value of x and let the time t increases,then we investigated the problem in different cases ;

Case 1: x/t is positive and greater than $(U - \mu'(k^*))$, from standpoint of an observer stationed at a given value of x , the observer will then observe waves of continually decreasing wave number through the interval (k_m , ∞) i.e. increasing wave length, and the amplitude is decreasing as $(1/t)^{\frac{1}{2}}$.

Case 2: x/t is positive and less than $(U - \mu'(k^*))$, from standpoint of an observer stationed at a fixed value of x, the observer will then observe a system of steady waves.

Case 3: x/t is negative . An observer stationed at x will then observe waves of continually increasing wave number through the interval $(0 < k < g/4U^2)$, i.e. decreasing wave length , and the amplitude is decreasing like $(1/t)^{\frac{1}{2}}$.

Generation Of Waves In Rest Fluid (U=o) Due To Initial Disturbance At The Bottom

In the previous discussion , we investigated the generation of waves on a running stream ($U \neq o$) due to initial disturbance (suddenly) at the bottom . Now we like to discuss the same problem by considering U= o ,i.e. we can estimate our solution from the general case by putting U = o . The corresponding solution is given by

$$\xi(x,y;t) = (1/2\pi)\int_{-\infty}^{\infty}\left[\left(\frac{-i\bar{f}(k)}{(\mu(k))^2}\right)\left(\frac{(gk \sinh ky)}{\cosh kh}\right)\right]\exp(-ikx)\,dk$$
$$+ (1/2\pi)\int_{-\infty}^{\infty}\left[\left(\frac{i\bar{f}(k)}{2(\mu(k))^2}\right)\left(\frac{gk\sinh ky + \mu^2\cosh ky}{\cosh kh}\right)\right]\exp(it\mu(k)-ikx)\,dk$$
$$+ (1/2\pi)\int_{-\infty}^{\infty}\left[\left(\frac{i\bar{f}(k)}{2(\mu(k))^2}\right)\left(\frac{gk\sinh ky + \mu^2\cosh ky}{\cosh kh}\right)\right]\exp(-it\mu(k)-ikx)\,dk$$

Evaluating these integrales for large x and t , it is clear that all integrands are well behaved functions and all free from any singularity at the origin .

The first integral , I_1

$$I_1 = (1/2\pi)\int_{-\infty}^{\infty}\left[\left(\frac{-i\bar{f}(k)}{(\mu(k))^2}\right)\left(\frac{gk \sinh ky}{\cosh kh}\right)\right]\exp(-ikx)\,dk$$

has not a saddle point , and integration by parts , one can find I_1 behaves like ($1/x$) .

The second integral I_2 ,

$$I_2 = (1/2\pi)\int_{-\infty}^{\infty}\left[\left(\frac{i\bar{f}(k)}{2(\mu(k))^2}\right)\left(\frac{gk\ \sinh\ ky + \mu^2\cosh\ ky}{\cosh\ kh}\right)\right]\exp(it\mu(k)-ikx)\ dk,$$

here , consider $x = Vt$, where V is held constant equal the observer speed . Hence , the integral has a saddle point,be defined by $\left[d/dk(\mu(k) - Vk)\right] = o$, i.e. the saddle point $K=k_o$ must satisfy the relation $\mu'(k_o) = V = x/t$, the prime denotes the differentiation with respect to k , here , $\mu'(k_o)$ = the group velocity appropriate to the wave number $k = k_0$. For t is large , the asymptotic expression for I_2 given by

$$\simeq(1/2\pi)\left[\left(\frac{i\bar{f}(k)}{2(\mu(k))}\right)\left(\frac{gk_o\sinh\ k_oy+\mu^2(k_o)\cosh\ k_oy}{\cosh\ k_oh}\right)\right]\exp\left[itg_1(k_o)+\tfrac{1}{4}i\pi\right]\left(\frac{2\pi}{t\ g_1''(k_o)}\right)^{\frac{1}{2}}$$

where , $g_1(k) = (\mu(k) - Vk)$.

By similar manner , for the third integral I_3

$$I_3 = (1/2\pi)\int_{-\infty}^{\infty}\left[\left(\frac{i\bar{f}(k)}{2(\mu(k))^2}\right)\left(\frac{gk\ \sinh\ ky + \mu^2\cosh\ ky}{\cosh\ kh}\right)\right]\exp(-i\mu(k)-ikx)\ dk,$$

we have the asymptotic expression,be given by

$$\approx(1/2\pi)\left[\left(\frac{i\bar{f}(k)}{2(\mu(k))^2}\cdot\frac{gk\ \sinh\ ky + \mu^2\cosh\ ky}{\cosh\ kh}\right)_{k=k_{oo}}\right]\exp\left[ig_2(k_{oo})+\tfrac{1}{4}i\pi\right]\left(\frac{2\pi}{t\ g_2''(k_{oo})}\right)$$

where , $g_2(k) = \mu(k) + Vk$.

Then , the asymptotic expression for the solution $\xi(x,y;t)$ as t and x become large, is given by

$$\xi(x;y;t) \simeq (1/2\pi)\left[\left(\frac{i\bar{f}(k)}{2(\mu(k))^2}\frac{(gk\sinh ky+\mu^2\cosh ky}{\cosh kh}\right)_{k=k_o}\right] x$$

$$x\left[\exp(itg_1(k_o)+\tfrac{1}{4}i\pi)\cdot\left(\frac{2\pi}{t\,g_1''(k_o)}\right)^{\frac{1}{2}}\right]$$

$$+(1/2\pi)\left[\left(\frac{i\bar{f}(k)}{2(\mu(k))^2}\frac{(gk\sinh ky+\mu^2\cosh ky}{\cosh kh}\right)_{k=k_{oo}}\right] x$$

$$x\left[\exp(itg_2(k_{oo})+\tfrac{1}{4}i\pi)\cdot\left(\frac{2\pi}{t\,g_2''(k_{oo})}\right)^{\frac{1}{2}}\right]$$

$$+ O(1/x).$$

Here , we are examining ξ from the standpoint of an observer moving with group velocity $\mu'(k_o)$, the gross outline of waves will appeareconstant in form , but decreasing in amplitude because of $t^{-\frac{1}{2}}$.

For the values of V (= the observer speed) for which no solution to $\left[d/dk\ (\mu(k) - Vk)\right] = o$ exists , i.e. there is no saddle point , it is easy to find , by integration by parts that the asymptotic expression for ξ behaves like (1/t) .

The solution at the free surface when U = o, be given by putting, in the above expression for $\xi(x;y;t)$ when U = o , y=o we obtain

$$\eta(x;t) = \xi(x,o;t) = (1/2\pi)\int_{-\infty}^{\infty}\left(\frac{i\bar{f}(k)}{2\cosh kh}\right)\exp[(it\mu(k) - ikx)]dk$$
$$+(1/2\pi)\int_{-\infty}^{\infty}\left(\frac{i\bar{f}(k)}{2\cosh kh}\right)\exp[(-it\mu(k)-ikx)]\,dk \,.$$

The asymptotic expression for $\eta(x;t)$ when t and x become large is given by holding x/t constant , then we have

(1) If x/t satisfied the relation $[d/dk(\mu(k) - x/tk)] = o$, in this case The solution behaves like $(1/t)^{\frac{1}{2}}$.

(2) For values of x/t which no solution to $[d/dk(\mu(k) - x/t\,k)] = o$ exists , the corresponding solution behaves like 1/t .

The Vertical Displacement Of The Free Surface

At The Origin X=o and $t \longrightarrow \infty$

The corresponding solution is given by

$$\eta(o;t) = (1/2\pi) \int_{-\infty}^{\infty} \left(\frac{i\bar{f}(k)}{2\cosh kh}\right) \exp(it\mu(k))\, dk$$

$$+ (1/2\pi) \int_{-\infty}^{\infty} \left(\frac{\overline{i\bar{f}(k)}}{2\cosh kh}\right) \exp(-it\mu(k))\, dk \,.$$

where , $\mu(k) = (gk \tanh kh)^{\frac{1}{2}}$ is a monotonic function as we know , i.e. $d/dk(\mu(k)) \neq o$. Therefore, the asymptotic expression for the vertical displacement of the free surface at the origin behaves like $1/t$.

Generation Of Waves In Still Fluid Due To Initial &Smoothly Disturbance At The Bottom

In all previous discussion we have investigated the wavy motions creating from initially and suddenly disturbance in the bed of the fluid, we used the Heaviside function to define the initially suddenly deformation at the bottom. We like here to discuss the same problem , when the initial disturbance occurs smoothly,in this case we replace the Heaviside function by other suitable function ($g_1(t) - g_2(t)$) , combination of two functions

(1) $g_1(t)$ is the Heaviside function, i.e.

$$g_1(t) = o \qquad t < o$$

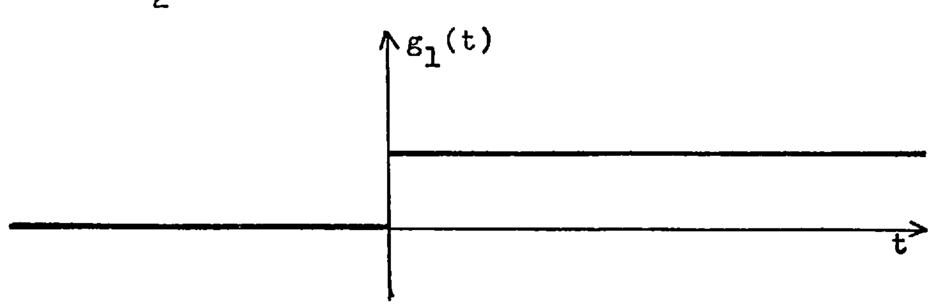
$$g_1(t) = 1 \qquad t > o \quad ,$$

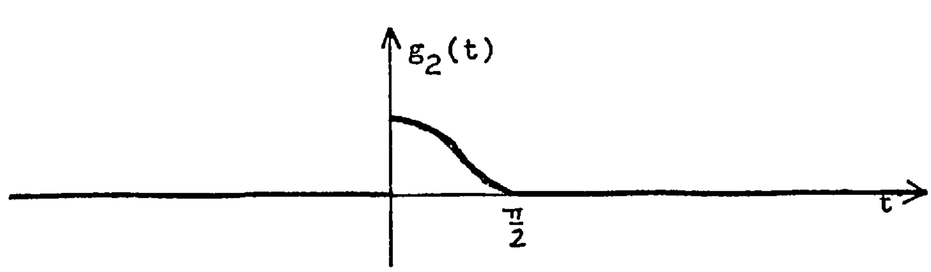
(2) $g_2(t)$ is defined by

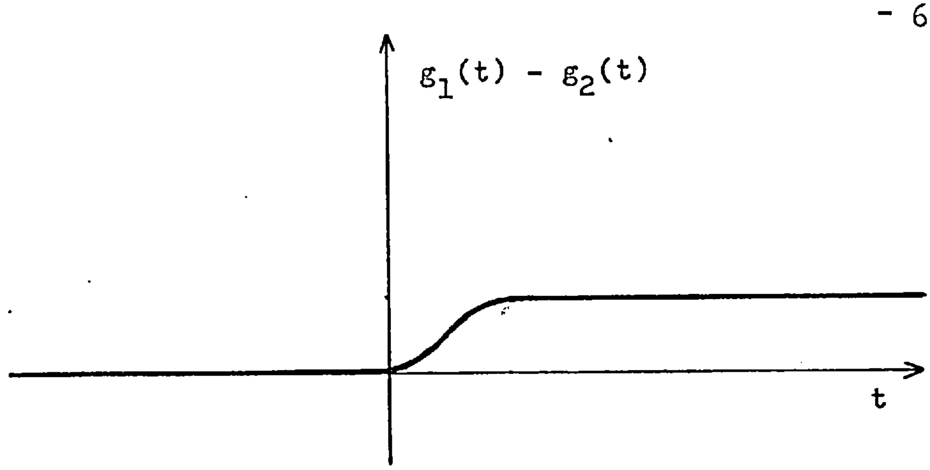
$$g_2(t) = o \qquad t < o \, ,$$

$$g_2(t) = \tfrac{1}{2}(1+\cos 2t) \qquad o < t < \pi/2 \, ,$$

$$g_2(t) = o \qquad t > \pi/2 \, .$$







Formulation:

We consider that the density and the viscosity in the fluid are neglected . The motion - originally started from rest- is irrotational and can be described by a velocity potential $\phi(x,y;t)$, therefore the governed equation is

$$\nabla^2_{x,y}\, \phi(x,y) = 0 \ .$$

The Boundary Conditions:

Due to the linearization theorem , the boundary conditions become,

(1) At the free surface ($y = o$) we have

(a) $\partial\eta(x,t)/\partial t = \phi_y(x,o;t)$,

(b) $\partial \phi(x,o;t)/\partial t + g\eta(x;t) = o$.

(2) At the bottom ($y = -h$) we have

$$\partial \xi/\partial t = \phi_y \ .$$

where $\xi(x,t) = f(x)g(t)$,

$f(x)$ is defined function ,

and $g(t) = g_1(t) - g_2(t)$.

To gether the finiteness conditions

$|\phi|$ as $|x| \longrightarrow \infty$

$|\eta|$ & $|\xi|$ as $|x| \longrightarrow \infty$.

We note that $\eta = \eta(x;t)$ representes the vertical displacement of the free surface.

and $\xi = \xi(x,y;t)$ representes the vertical displacement of any fluid particle at any depth .

Using the techinque of Fourier transformation and Laplace transformation ,

$$\left(\int_0^{\infty} dt \exp(-itw) \int_{-\infty}^{\infty} dx \exp(ikx) \right) .$$

Applying this to the Laplace equation , we obtain

$$\bar{\phi}_{yy} - k^2 \bar{\phi} = o ,$$

provided ϕ & $\phi_x \longrightarrow o$, as $|x| \longrightarrow \infty$.

This ordinary differential equation in y has a solution

$$\bar{\phi}(k,y;w) = A(k;w) \exp(ky) + B(k;w) \exp(-ky) ,$$

where A(k;w) and B(k;w) are arbitrary constants .

Then the transform is applied to the boundary conditions at the free surface (y = o) and at the bottom (y=-h) , we get

$$iw \bar{\phi}(k,o;w) + g\bar{\eta}(k,w) = o , \qquad \text{at } y = o$$

$$iw\bar{\eta}(k;w) - \bar{\phi}_y(k,o;w) = o ,$$

$$iw \bar{f}(k)\bar{g}(w) = \bar{\phi}(k,-h;w) \qquad \text{at } y = -h$$

Eliminating $\bar{\eta}$ between the two conditions at y = o, we obtain

$$(iw)^2 \bar{\phi} + g \bar{\phi}_y = o \quad .$$

Inserting the value of $\bar{\phi}$ in the single condition at the free surface and the condition at the bottom , we get

$$- w^2 (A + B) + gk (A - B) = o ,$$

and

$$iw \ \bar{f}(k)\bar{g}(w) = k (A e^{-kh} - B e^{kh}) \quad .$$

by solving for A(k,w) & B(k;w) , we obtain

$$A(k;w) = \left[\left(\frac{iw\bar{f}(k)\bar{g}(w)}{2k} \right) \left(\frac{(gk + w^2)}{(w^2 \cosh(kh) - gk \sinh kh)} \right) \right] ,$$

and

$$B(k;w) = \left[\left(\frac{iw\bar{f}(k)\bar{g}(w)}{2k} \right) \left(\frac{(gk - w^2)}{(w^2 \cosh kh - gk \sinh kh)} \right) \right] .$$

Hence , the expression for $\bar{\phi}(k,y;w)$ is

$$\bar{\phi}(k,y;w) = \left[\frac{iw\bar{f}(k)\bar{g}(w) \ (gk \cosh ky + w^2 \sinh ky)}{k \cosh(kh) . \ (w^2 - gk \tanh kh)} \right] .$$

The vertical displacement for any fluid particle is given by

$$\xi_t \ (x,y;t) = \phi_y \ (x,y;t) \quad .$$

By Fourier - Laplace transformation , we obtain

$$iw \bar{\xi}(k,y;w) = \bar{\phi}_y \ (k,y;w) \quad .$$

Then ,

$$\bar{\xi}(k,y;w) = \left[\frac{\bar{f}(k)\bar{g}(w) \ (gk \sinh ky + w^2 \cosh ky)}{\cosh kh. \ (w^2 - gk \tanh kh)} \right] ,$$

we have , $g(t) = g_1(t) - g_2(t)$

by Laplace transformation we get

$$\bar{g}(w) = \bar{g}_1(w) - \bar{g}_2(w) \quad ,$$

$$\therefore \quad \bar{\xi}(k,y;w) = \left[\frac{\bar{f}(k)\bar{g}_1(w)\ (gk \sinh ky + w^2 \cosh ky)}{(w^2 - \mu^2(k)) \cosh kh}\right] + \left[\frac{\bar{f}(k)\bar{g}_2(w)\ (gk \sinh ky + w^2 \cosh ky)}{(w^2 - \mu^2(k)) \cosh kh}\right]$$

where $\mu(k) = (gk \tanh kh)^{\frac{1}{2}}$.

One can easy find , $\bar{g}_1(w) = -i/w$, and

$$\bar{g}_2(w) = \left[(i/2w)(e^{-iw\pi/2} - 1) + (iw/2)\left(\frac{1 + e^{-iw\pi/2}}{4 + (iw)^2}\right)\right].$$

Rewritting the expression of $\bar{\xi}(k,y;w)$, we have

$$\bar{\xi}(k,y;w) = \left[\frac{-i\bar{f}(k)\ (gk \sinh ky + w^2 \cosh ky)}{w\ (w-\mu)\ (w+\mu) \cosh kh}\right] + \left[\frac{i\bar{f}(k)\ (e^{-iw\pi/2} - 1)\ (gk \sinh ky + w^2 \cosh ky)}{2\ w\ (w-\mu)\ (w+\mu) \cosh kh}\right] + \left[\frac{iw\bar{f}(k)\ (1 + e^{-iw\pi/2})\ (gk \sinh ky + w^2 \cosh ky}{2(2 - w)(2 + w)(w - \mu)(w + \mu) \cosh kh}\right]$$

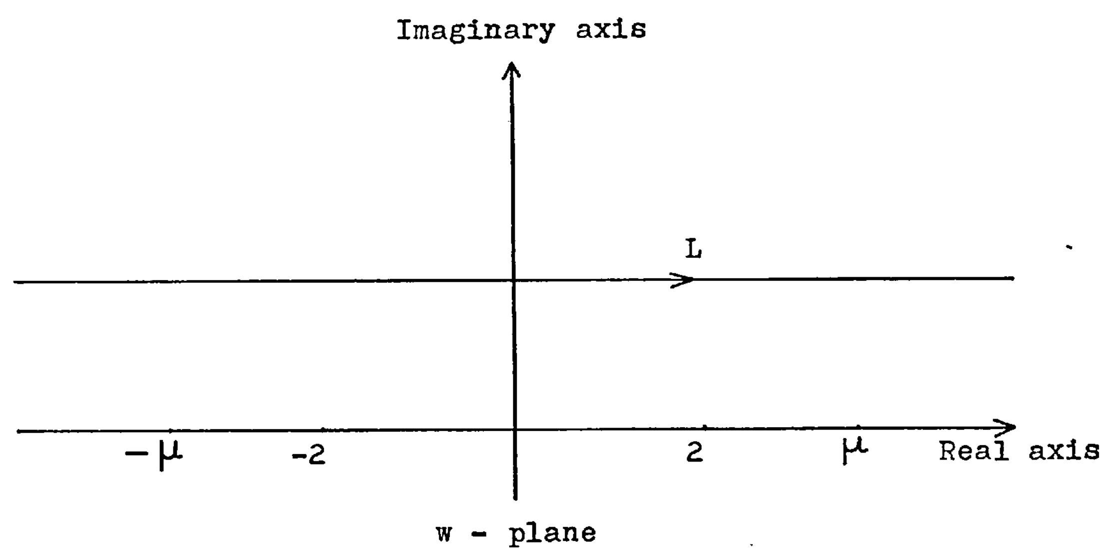
Taking the inverse of Laplace transformation, we obtain

$$\xi(k,y;t)=(1/2i\pi)\int_L \frac{-\bar{f}(k)\ (gk\ \sinh(ky) + w^2\cosh(ky))\ e^{iwt}}{w(w-\mu)\ (w+\mu)\ \cosh(kh)}\, dw$$

$$+(1/2i\pi)\int_L \frac{i\bar{f}(k)\ (e^{-iw\pi/2}-1)\ (gk\ \sinh(ky)+w^2\cosh(ky)}{2w\ (w-\mu)\ (w+\mu)\ \cosh(kh)}\, e^{iwt}\, dw$$

$$+(1/2i\pi)\int_L \frac{i\bar{f}(k)\ (1+e^{-iw\pi/2})w(gk\ \sinh(ky)+w^2\cosh(ky)}{2(2-w)\ (2+w)\ (w-\mu)\ (w+\mu)\ \cosh(kh)}\, e^{iwt}\, dw$$

In w-plane, the path L is taken above and parallel to the real axis as in the figure



By Cauchy theorem and Jordon's lemma we obtain

$$\bar{\xi}(k,y;t) = \left[\frac{-i\bar{f}(k)\ (gk\ \sinh kh)}{-\mu^2\ \cosh kh}\right]$$

$$+\left[\frac{-i\bar{f}(k)\ (gk\ \sinh ky + (-\mu)^2 \cosh ky)}{(-\mu)(-2\mu)\ \cosh kh}\right] e^{-it\mu}$$

$$+\left[\frac{-i\bar{f}(k)\ (gk\ \sinh ky + \mu^2\ \cosh ky\ )}{(2\mu)\ \cosh kh}\right] e^{-it\mu}$$

$$+\left[\frac{i\bar{f}(k)(e^{-i\pi\mu/2} - 1)(gk\ \sinh ky + \mu^2 \cosh ky)}{(2\mu)(2\ \cosh kh)}\right] e^{i\mu t}$$

$$+\left[\frac{i\bar{f}(k)(e^{i\pi\mu/2} - 1)(gk\ \sinh ky\ +\ (-\mu)^2 \cosh ky)}{2\ (-\mu)(-2\mu)\ \cosh kh}\right] e^{-it\mu}$$

$$+\left[\frac{-i\bar{f}(k)(\mu)(1+e^{-i\pi\mu/2})(gk\ \sinh ky + \mu^2 \cosh ky)}{2\ \ (2\mu)(2-\mu)(2+\mu)\ \cosh kh}\right] e^{it\mu}$$

$$+\left[\frac{-i\bar{f}(k)(-\mu)(1+e^{i\pi\mu/2})(gk\ \sinh ky + \mu^2 \cosh ky)}{2\ (-2\mu)(2-\mu)(2+\mu)\ \cosh kh}\right] e^{-it\mu}$$

It is better to rewrite the above expression as

$$\bar{\xi}(k,y;t) = \left[\frac{-i\bar{f}(k)\ (gk\ \sinh ky)}{(-\mu^2)\ \cosh kh}\right]$$

$$+\left[\frac{-i\bar{f}(k)\ (gk\ \sinh ky + \mu^2 \cosh kh)}{2\mu^2\ \cosh kh}\right]\left[\left(e^{it\mu} + e^{-it\mu}\right)\right]$$

$$+\left[\frac{i\bar{f}(k)(gk\ \sinh kh + \mu^2 \cosh ky)}{4\ \mu^2\ \cosh kh}\right]\left(\left[e^{it\mu}\ (-1+e^{-i\pi\mu/2}) + e^{it\mu}\ (-1 + e^{i\pi\mu/2})\right]\right)$$

$$+\left[\frac{-i\bar{f}(k)(gk\ \sinh ky + \mu^2 \cosh ky)}{4(2+\mu)(2-\mu)\ \cosh kh}\right]\left(\left[e^{it\mu}\ (1 + e^{-i\pi\mu/2}) + e^{-it\mu}\ (1 + e^{i\pi\mu/2})\right]\right).$$

One can easy find that the expression for the solution $\xi(k,y;t)$ free from singularties when taken as a whole . It is suitable to write the above expression for $\bar{\xi}(k,y;t)$ as

$$\bar{\xi}(k,y;t) = (1/\mu^2)\Bigg[\Big(\frac{i\bar{f}(k)\ (gk \sinh ky)}{\cosh kh}\Big) - \Big(\frac{i\bar{f}(k)(gk \sinh ky + \mu^2 \cosh ky)}{2 \cosh kh}\Big)\big(e^{it\mu} - e^{-it\mu}\big)$$

$$+\Big(\frac{i\bar{f}(k)(gk \sinh ky + \mu^2 \cosh ky)}{4 \cosh kh}\Big)\big(e^{it\mu}(-1+e^{-i\mu\pi/2}) + e^{-it\mu}(-1 + e^{i\mu\pi/2})\big)\Bigg]$$

$$+\Bigg[\Big(\frac{-i\bar{f}(k)(gk \sinh ky + \mu^2 \cosh ky)}{16 \cosh kh}\Big)\Big(\frac{1}{\mu+2} - \frac{1}{\mu-2}\Big)\Bigg] \times$$

$$\times \Big[\big(e^{it\mu}(1 + e^{-i\mu\pi/2}) + e^{-it\mu}(1 + e^{i\mu\pi/2})\big)\Big].$$

Taking the inverse of the Fourier transform , we obtain the solution $\xi(x,y;t)$

$$\xi(x,y;t) = (1/2\pi)\int_{-\infty}^{\infty} \frac{i\bar{f}(k)(gk \sinh ky)}{\cosh kh}\Big(\frac{e^{-ikx}}{\mu^2}\Big)\, dk$$

$$+(1/2\pi)\int_{-\infty}^{\infty} \frac{i\bar{f}(k)(gk \sinh ky + {}^2 \cosh ky)}{2\mu^2 \cosh kh}\Bigg[-(e^{it\mu} + e^{-it\mu}) +$$

$$\tfrac{1}{2}\Big[e^{it\mu}(-1 + e^{-i\mu\pi/2}) + e^{-it\mu}(-1 + e^{i\mu\pi/2})\Big]\Bigg] e^{-ikx}\, dk$$

$$+(1/2\pi)\int_{-\infty}^{\infty}\Big[\frac{-i\bar{f}(k)(gk \sinh ky + \mu^2 \cosh ky)}{16 \cosh kh}\Big] \times$$

$$\left[\frac{e^{it\mu}(1 + e^{-i\mu\pi/2})\, e^{-ikx}}{(2+\mu)}\right] dk$$

$$+ (1/2\pi)\int_{-\infty}^{\infty}\left[\frac{i\bar{f}(k)(gk \sinh ky + \mu^2 \cosh ky)}{16 \cosh kh}\right] X$$

$$\left[\frac{e^{-it\mu}(1 + e^{\frac{1}{2}i\mu\pi})\, e^{-ikx}}{(\mu - 2)}\right] dk ,$$

all integrands are well behaved near and on the real axis and free from any singularities at the origin as well as at $\mu(k) = \pm 2$.

The corresponding expression for the free surface $\eta(x;t)$, is derived from the expression for $\xi(x,y;t)$ by putting y = o ,

$$\eta(x,t) = \xi(x,o;t) = (1/2\pi)\int_{-\infty}^{\infty}\left(\frac{i\bar{f}(k)}{2 \cosh kh}\right)\Big[-(e^{it\mu} + e^{-it\mu})$$
$$\tfrac{1}{2}(e^{it\mu}(-1+e^{-\frac{1}{2}i\mu\pi}) + e^{-it\mu}(e^{\frac{1}{2}i\mu\pi}-1))\, e^{-ikx}\, dk\Big]$$

$$+(1/2\pi)\int_{-\infty}^{\infty}\left(\frac{i\bar{f}(k)\mu^2}{16 \cosh kh}\right)\left(\frac{e^{it\mu}(1+e^{-\frac{1}{2}i\mu\pi})\, e^{-ikx}}{(2+\mu)}\right) dk$$

$$+(1/2\pi)\int_{-\infty}^{\infty}\left(\frac{i\bar{f}(k)\mu^2}{16 \cosh kh}\right)\left(\frac{e^{-it\mu}(1+e^{\frac{1}{2}i\mu\pi})\, e^{-ikx}}{(-2+\mu)}\right) dk .$$

This integral can be evaluated asymptotically for large x and t . To do this we assume that the ratio t/x or x/t is fixed , so that the resulting integral contians just one large parameter , either t or x . Carrying this program , we consider x = Vt , where V is held constant representing the observer velocity , the different integrals contain exponential factors like $\exp(ig_1(t))$ and $\exp(ig_2(k)t$, where $g_1(k) = [\mu(k) - Vk]$ and $g_2(k) = [\mu(k) + Vk]$.

In the complex k-plane , we must deform the paths of the different integrals in the the manner of the steepest descent passing through the saddle points which are defined by

$$d/dk\ (g_1(k)) = d/dk\ (\mu(k) - Vk) = 0\ ,$$

and
$$d/dk\ (g_2(k)) = d/dk\ (\mu(k) + Vk) = 0\ ,$$

hence , the saddle point $k = k_o$ must satisfy the relation $d/dk\ (\mu(k_o)) = V$ and the other saddle point also must stisfy the relation $d/dk\ (\mu(k_{oo})) = -V$. This means , the observer velocity equal the group velocity appropriate to the wave number $k = \pm k_o$ and $k = k_{oo}$. The asymptotic expression for $\eta(x;t)$ is given by

$$\eta(x,t) = (1/2\pi)\Big[\Big(\frac{i\bar{f}(k)}{2\cosh kh}\Big)\Big(-1 + \big(-\tfrac{1}{2} + \tfrac{1}{2}e^{-\frac{1}{2}i\mu\pi}\big)\Big) + \Big(\frac{-i\bar{f}(k)\mu^2}{16\cosh kh}\Big)\Big(\frac{1 + e^{-\frac{1}{2}i\mu\pi}}{2 + \mu}\Big)\Big]_{k=k_o} \cdot e^{ig_1(k_o)t + \frac{1}{4}i\pi} \cdot \Big(\frac{2\pi}{t\, g_1''(k_o)}\Big)^{\frac{1}{2}}$$

$$+ (1/2\pi)\Big[\Big(\frac{i\bar{f}(k)}{2\cosh kh}\Big)\Big(-1 + \big(-\tfrac{1}{2} + \tfrac{1}{2}e^{\frac{1}{2}i\mu\pi}\big)\Big) + \Big(\frac{i\bar{f}(k)\mu^2}{16\cosh kh}\Big)\Big(\frac{1 + e^{\frac{1}{2}i\mu\pi}}{\mu - 2}\Big)\Big]_{k=k_{oo}} \cdot e^{ig_2(k_{oo})t + \frac{1}{4}i\pi} \cdot \Big(\frac{2\pi}{t\, g_2''(k_{oo})}\Big)^{\frac{1}{2}}$$

$$+ O(1/t)\ .$$

Let us examine the asymptotic expression ; if x/t is held constant while t increases ,then clearly one must set $x = \mu'(k_s)t$, where $k_s = k_o$ or k_{oo}, i.e. we are examining η from the standpoint of an observer moving with the group velocity $\mu'(k_s)$. The gross outline decreasing in amplitude like $(1/t)^{\frac{1}{2}}$.

For values of $x/t = V$ which no solution to the relation $\mu'(k) = V = x/t$ exists , it is easy to show, by integration by parts, that $\eta(x;t)$ behaves like $(1/t)$.

These results are exactly identical with the previous results which we have obtained before , in the case of suddenly disturbance at the bottom .

The Behavior Of The Vertical Displacement Of The Free Surface At The Origin X = o As t Becomes Large

The corresponding solution is

$$\eta(o;t) = \xi(o,o;t) = (1/2\pi)\int_{-\infty}^{\infty}\left(\frac{i\bar{g}(k)}{2\cosh kh}\right)\Big[-(e^{it\mu} + e^{-it\mu}) + e^{it\mu}(-\tfrac{1}{2} + \tfrac{1}{2}e^{-\frac{1}{2}i\mu\pi}) + e^{-it\mu}(-\tfrac{1}{2} + e^{\frac{1}{2}i\mu\pi})\Big]dk$$

$$+(1/2\pi)\int_{-\infty}^{\infty}\left[\left(\frac{-i\bar{f}(k)\mu^2}{16\cosh kh}\right)\left(\frac{e^{it\mu}\,(1 + e^{-\frac{1}{2}i\mu\pi})}{2+\mu}\right)\right]dk$$

$$+(1/2\pi)\int_{-\infty}^{\infty}\left[\left(\frac{i\bar{f}(K)\mu^2}{16\cosh kh}\right)\left(\frac{e^{-it\mu}\,(1 + e^{\frac{1}{2}i\mu\pi})}{-2+\mu}\right)\right]dk \ .$$

The integrands are well behaved near the real axis and on it in the k - plane , also the function $\mu(k) = (gk \tanh kh)^{\frac{1}{2}}$ is monotonic in k , then we have $d/dk\,\mu(k) \neq o$, by integration by parts , one finds that $\eta(o;t)$ behaves like $(1/t)$.

Conclusion:

When a disturbance is initiated at the bottom of still fluid by suddenly or smoothely deformation of the bed , the effect is in general the creation of waves in the fluid and on the free surface . By , examining the asymptotic expression for the solution in both cases from the standpoint of :

(1) An observer moving with group velocity , the velocity which satisfied the relation $d/dk\,\mu(k) = x/t$.

(2) An observer moving with the velocity for which the relation $d/dk\,\mu(k) = x/t$ has not a solution.

(3) An observer standing at the origin .

the given results in both cases are basically identical .

Flow Over An Infinite Step At The Bed Of A Uniform Stream With Free Surface

In the present problem (as previous problems) we like to examine the creation of waves through and at the free surface of the fluid , due to a suddenly appearance of a step at the bottom of a stream flowing with a uniform velocity U . The problem is based upon the usual assumptions of classical hydrodynamics , i.e. the fluid is invistid and of uniform density , and the motion is irrotational -can be described by the velocity potential - , nonlinear terms in the equations of the motions are neglected . It is also assumed that the motion is twd-dimensional.

The governed equation;

The governed equation is

$$\nabla^2_{x,y} \Phi(x,y;t) = o \ ,$$

where the potential $\Phi(x,y;t) = Ux + \phi(x,y;t)$ % the potential $\phi(x,y;t)$ corresponding the disturbance velocity potential . Then the problem reduces to find the solution of

$$\nabla^2_{x,y} \phi(x,y;t) = o \ , \quad \text{to-gether,}$$

The Boundary Conditions:

(a) At the free surface (y=o) ;

(i) the dynamical condition is

$$\phi_t + U\phi_x + g\eta = o \ ,$$

where $\eta = \eta(x,t)$ is the elavation of the free surface .

(ii) the kinematic condition is

$$(\partial/\partial t + U\partial/\partial x)\eta(x;t) = \phi_y .$$

The disturbance at the bed is defined by

$$\xi(x;t) = b\, f(x)g(t) ,$$

where , $f(x) = o , \quad x < o$

$= 1 , \quad o < x < L$

$= o , \quad x > L$

and , the function g(t) is the Heaviside function (to represent the suddenly effect) .

Hence , the bottom is described by

$$y = -h + b\, f(x)\, g(t) , \qquad \text{where } b \ll h$$

therefore , the condition at the bottom ($y = -h$) is

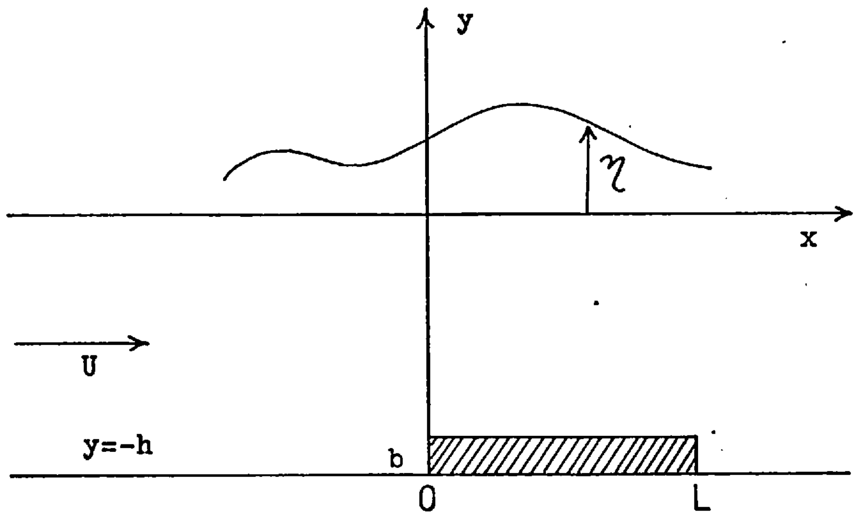
$$(\partial/\partial t + U\partial/\partial x)\, bf(x)g(t) = \phi_y .$$

The vertical displacemant for any fluid particle is given by

$$y = \xi(x,y;t) ,$$

then , we have the relation

$$(\partial/\partial t + U\partial/\partial x)\,\xi(x,y;t) = \phi_y(x;y,t)$$



Applying Fourier - Laplace transform

$$\left(\int_0^{\infty} dt\ e^{-itw} \int_{-\infty}^{\infty} dx\ e^{ikx} \right),$$

on the problem , we note , the function f(x) is defined on the interval ($-\infty < x < \infty$) and its absolute integral is convergent, i.e. $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$, this leads to the existence of the integral $\int_{-\infty}^{\infty} f(x) \exp(ikx)\, dx = \bar{f}(k) = (1/ik)(-1 + \exp(ikL))$, then we get an expression for $\bar{\phi}(k,y;w)$ is given by

$$\bar{\phi}(k,y;w) = \left[\frac{i(w-kU)\, b\bar{f}(k)\bar{g}(w)\, (\, gk \sinh ky + (w-kU)^2 \cosh ky\,)}{k(\, (w-kU)^2 - gk \tanh kh\,) \cosh kh} \right],$$

and the expression for $\bar{\xi}(k,y;w)$ is given by

$$\bar{\xi}(k,y;w) = \left[\frac{b\bar{f}(k)\bar{g}(w)\, (\, gk \sinh ky + (w-kU)^2 \cosh ky\,)}{(\, (w-kU)^2 - gk \tanh kh\,) \cosh kh} \right]$$

Substituting for $\bar{f}(k)$ & $\bar{g}(w)$, then by the inverse theorem , we obtain the solution

$$\xi(x,y;t) = (1/2\pi)\int_{-\infty}^{\infty}\left(\frac{-ib(gk\ \sinh ky + (kU)^2\cosh ky)}{-ik(-2\mu)(kU+\mu)\cosh kh}\right) e^{-ikx}\, dk$$

$$+(1/2\pi)\int_{-\infty}^{\infty}\left(\frac{-ib(gk\ \sinh ky + (kU)^2\cosh ky)}{-ik(2\mu)(kU-\mu)\cosh kh}\right) e^{-ikx}\, dk$$

$$+(1/2\pi)\int_{-\infty}^{\infty}\left(\frac{-ib(gk\ \sinh ky+(kU)^2\cosh ky)}{-ik(2\mu)(kU+\mu)\cosh kh}\right)e^{-ikx+it(kU+\mu)}\, dk$$

$$+(1/2\pi)\int_{-\infty}^{\infty}\left(\frac{-ib(gk\ \sinh ky+\mu^2\cosh ky)}{-ik(-2\mu)(kU-\mu)\cosh kh}\right)e^{-ikx+it(kU-\mu)}\, dk$$

$$+(1/2\pi)\int_{-\infty}^{\infty}\left(\frac{-ib(gk\ \sinh ky+(kU)^2\cosh ky)}{ik(-2\mu)(kU+\mu)\cosh kh}\right)e^{ik(L-x)}\, dk$$

$$+(1/2\pi)\int_{-\infty}^{\infty}\left(\frac{-ib(gk\ \sinh ky+(kU)^2\cosh ky)}{ik(2\mu)(kU-\mu)\cosh kh}\right)e^{ik(L-x)}\, dk$$

$$+(1/2\pi)\int_{-\infty}^{\infty}\left(\frac{-ib(gk\ \sinh ky+\mu^2\cosh ky)}{ik(2\mu)(kU+\mu)\cosh kh}\right)e^{ik(L-x)+it(kU+\mu)}\, dk$$

$$+(1/2\pi)\int_{-\infty}^{\infty}\left(\frac{-ib(gk\ \sinh ky+\mu^2\cosh ky)}{ik(-2\mu)(kU-\mu)\cosh kh}\right)e^{ik(L-x)+it(kU-\mu)}\, dk$$

<u>First</u> :

The representation integral for the solution $\xi(x,y;t)$ can be evaluated asymptotically for large L,t&x . Todo this we assume that L tends to infinity before t and x . Discussing each integral seperately ,

$$(1) \quad I_1 = (1/2\pi)\int_{-\infty}^{\infty}\left[\left(\frac{-ib(gk\ \mathrm{Sinh}\ ky+(kU)^2\cosh ky)}{-ik(2\mu)(kU-\mu)\cosh kh}\right)\right] e^{-ikx}\, dk$$

The integral is free from any singularity at the origin and the integrand analytic about the real axis except at the real pole $k = k^*$ (be defined by the relation $[kU - \mu(k)] = o$, provided U^2 is less than gh) . Near the simple pole , consider $k = k^* + K$, where K is a small complex quantity $= K_r + i\, K_i = \rho \exp(i\theta)$. The integrand can be written as $(\chi_1(k,y)/(kU-\mu))e^{-ikx}$, where

$$\chi_1(k,y) = \left(\frac{-ib(gk \sinh ky + (kU)^2 \cosh ky)}{-ik\,(2\mu(k))\cosh kh} \right),$$

Expanding the different functions about $k = k^*$ by Taylor , we obtain

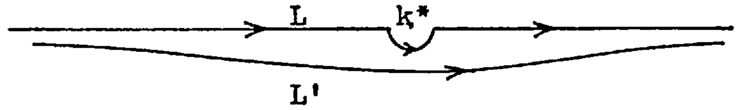
$$\chi_1(k,y) \approx \chi_1(k^*,y) \quad ,$$

$$(kU - \mu(k)) \simeq K[d/dk\,(kU - \mu(k))]_{k=k^*}$$

$$\exp(-ikx) \approx \exp(-ik^*x)\exp(-iK_r x)\exp(K_i x) \quad ,$$

for convergence , we have

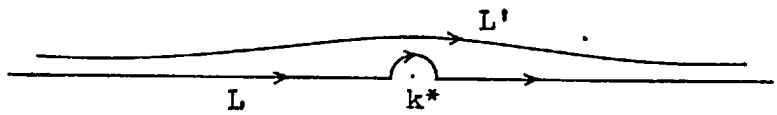
(i) x is positive , then for convergence K_i must be negative, therefore , the semicircle about the simple pole lies in the negative half of the k-plane . Then the path deformdd as in the figure



$$\therefore \int_L \ldots = P.V\int \ldots + \int_{\smile} \ldots \quad , \text{ we have}$$

$$I_1 = -i/2\left[\left(\frac{-ib(gk \sinh ky + (kU)^2 \cosh ky)}{-ik(2\mu).\cosh(kh).d/dk(kU-\mu)} \right)\right]_{k=k^*} e^{-ixk^*}$$

(ii) x is negative, then for convergence K_i must be positive, therefore, the semicircle about the simple pole lies in the positive half of the k-plane . Then the path deformed as in the figure



$$\therefore \quad I_1 = (\tfrac{1}{2}i)\left[\left(\frac{-ib(gk \sinh ky + (kU)^2 \cosh ky)}{-ik(2\mu).\cosh(kh).d/dk(kU-\mu)}\right)\right]_{k=k^*} e^{-ixk^*}$$

(2) The second integral I_2 is

$$I_2 = (1/2\pi)\int_{-\infty}^{\infty}\left(\frac{-ib(gk \sinh ky + (kU)^2 \cosh ky)}{-ik(-2\mu)(kU+\mu)\cosh kh}\right) e^{-ikx}\, dk ,$$

is similar as I_1 .

(3) The third integral I_3 is

$$I_3 = (1/2\pi)\int_{-\infty}^{\infty}\left(\frac{-ib(gk \sinh ky + \mu^2 \cosh ky)}{-ik(-2\mu)(kU-\mu).\cosh(kh)}\right) e^{i(-kx+(kU-\mu))t}\, dk,$$

to evaluate I_3 asymptotically for large t and x . To do this we assume the x/t is fixed , i.e. the integral contains just one large parameter , it is better to rewrite I_3 as

$$I_3 = (1/2\pi)\int_{-\infty}^{\infty}\left(\frac{\chi_3(k,y)}{kU - \mu(k)}\right) e^{itg_1(k)}\, dk$$

where , $\chi_3(k,y) = \left(\frac{-ib(gk\sinh ky + \mu^2\cosh ky)}{-ik(-2\mu)\cosh(kh)}\right)$,

and $\quad g_1(k) = [(-x/t)k + Uk - \mu(k)]$.

The integrand of I_3 has a simple pole at $k = K^*$, be defined by the relation $[kU - \mu(k)] = o$ and a saddle point at $k = K_o$, be defined by the reation $[d/dk(g_1(k))] = o$, i.e. $K = K_o$ must satisfy the relation $[U - \mu'(k_o)] = x/t$. Expanding the different functions of the integrand about $k = k^*$ by Taylor , we obtain an important factor , $\exp(-K_i(-x/t + U - \mu'(k^*))t)$, then for the convergence we have

(i) $[d/dk(g_1(k))]_{k=k^*}$ is positive , then for convergence K_i must be also positive , i.e. the semicircle round the simple pole on the real axis lies in the positive half in the k - plane , as in the figure

k=k*

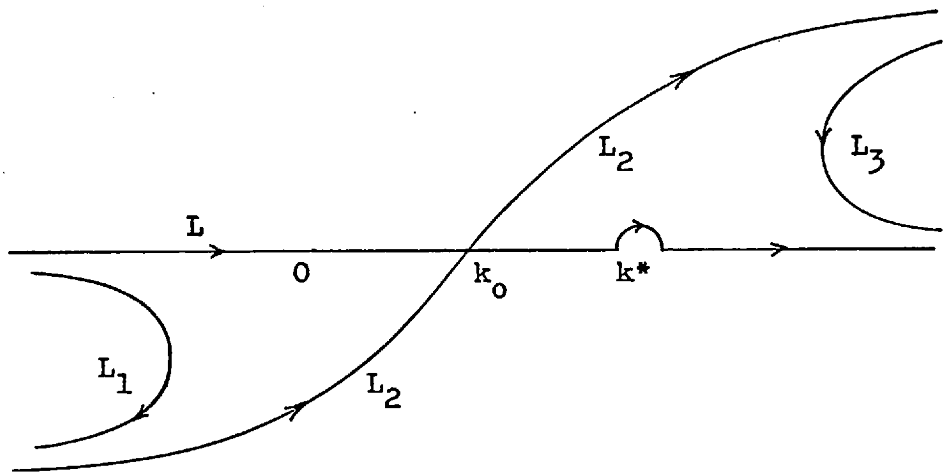
$\therefore$ $[d/dk(g_1(k))]_{k=k^*} = [(U - \mu'(k^*)) - x/t]$,

then we have two cases ;

(a) x/t is positive (i.e. the observer speed is positive, i.e. in the downstream direction) in this case we find that

$$[d/dk(\mu(k))]_{k=k_o} > [d/dk(\mu(k))]_{k=k^*}$$

here , the saddle point lies on the left of the simple pole ($k^* > k_o$) , hence , we deformed the path in the manner of the steepest descent as in the figure



(b) x/t is negative (i.e. the observer speed is negative , in the upstream direction) in this case we have

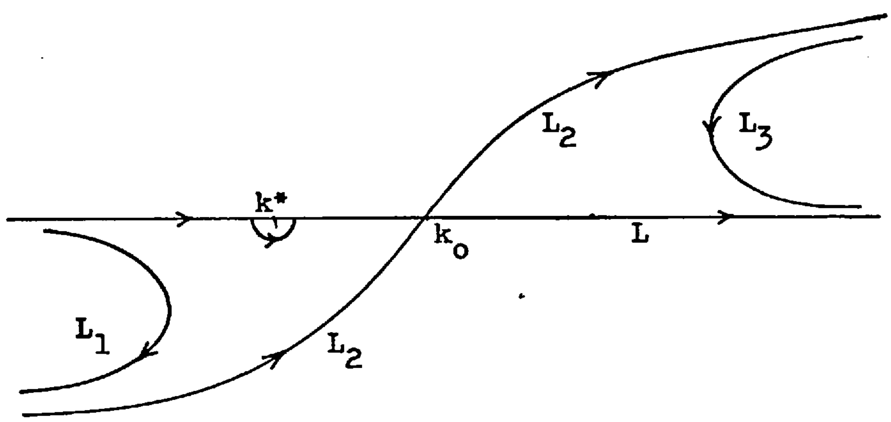
$$[d/dk\ (\mu(k))]_{k=k_o} > [d/dk\ (\mu(k))]_{k=k^*}$$

here , the saddle point ($k = k_o$) lies on the left of the simple pole ($k = k^*$) , hence ,the path deformed as in the case (a) .

(ii) $[d/dk\ (g_1(k))]_{k=k^*}$ is negative , for the convergence K_i must be negative , i.e. the semicircle round the simple pole on the real axis lies in the negative half in the k-plane . Since $\mu'(k^*)$ less than U , hence , x/t is positive ,

$$\therefore [d/dk\ (\mu(k))]_{k=k_o} < [d/dk\ (\mu(k))]_{k=k^*}$$

i.e. the saddle point lies on the right of the simple pole , then the path deformed by the steepest descent manner as in the figure



The corresponding values ;

(1) The case (i ; a&b) , we have

$$I_3 = (\tfrac{1}{2}i)\left(\frac{-ib(gk\ \sinh(ky)+\mu^2\cosh ky)}{-ik(-2\mu)d/dk(kU-\mu)\cosh(kh)}\right)_{k=k^*} e^{-ik^*x}$$

$$+(1/2\pi)\left(\frac{-ib(gk\ \sinh ky+\mu^2\cosh ky)}{-ik(-2\mu)(kU-\mu)\cosh(kh)}\right)_{k=k_o} \cdot e^{i(tg_1(k_o)+\frac{1}{4}\pi)} \cdot \left(\frac{2\pi}{t\ g_1''(k_o)}\right)^{\frac{1}{2}}$$

(2) The case (ii)

$$I_3 =(-\tfrac{1}{2}i)\left(\frac{-ib(gk\ \sinh ky+\mu^2\cosh ky)}{-ik(-2\mu)d/dk(kU-\mu)\cosh(kh)}\right)_{k=k^*} e^{-ik^*x}$$

$$+(1/2\pi)\left(\frac{-ib(kg\ \sinh ky+\mu^2\cosh ky)}{-ik(-2\mu)(kU-\mu)\cosh kh}\right)_{k=k_o} e^{i(tg_1(k_o)+\frac{1}{4}\pi)} \left(\frac{2}{t\ g_1''(k_o)}\right)^{\frac{1}{2}}$$

(4) The fourth integral I_4 is

$$I_4 = (1/2\pi)\int_{-\infty}^{\infty} \left(\frac{-ib(kg \sinh ky + \mu^2 \cosh ky)}{-ik(2\mu)(kU + \mu) \cosh kh} \right) e^{i(-kx + t(kU + \mu))}\, dk$$

similar to I_3 .

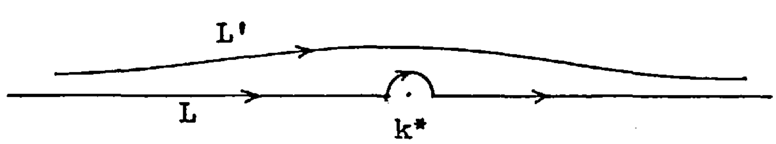
(5) The fifth integral I_5 is

$$I_5 = (1/2\pi)\int_{-\infty}^{\infty} \left(\frac{-ib(gk \sinh ky + (kU)^2 \cosh ky)}{ik(2\mu)(kU + \mu) \cosh kh} \right) e^{ikL}\, e^{-ikx}\, dk$$

it is better to write I_5 as $(1/2\pi)\int_{-\infty}^{\infty} \left(\frac{\chi_5(k,y)}{(kU - \mu)} \right) e^{ikL}\, dk$,

where , $$\chi_5(k,y) = \left(\frac{-ib(gk \sinh ky + (kU)^2 \cosh ky)}{ik(2\mu) \cosh kh} \right) e^{-ikx}$$

The integral has a simple pole at $k = k^*$, be defined by the relation $[kU - \mu(k)] = 0$. by expanding the integrand about $k = k^*$, we get the important factor , $\exp(-K_i L)$, L is positive ,then for convergence K_i must be positive ,i.e. the semicircle round the simple pole on the real axis lies in the positive half , the path deformed in the k - plane as in the figure



$$\therefore\ I_5 = \tfrac{1}{2} i \left(\frac{-ib(gk \sinh ky + (kU)^2 \cosh ky)}{ik(2\mu) d/dk(kU - \mu) \cosh kh} \right)_{k=k^*} e^{ikL}\, e^{-ixk^*} + O(1/L)$$

the value of I_5 independent on the sign of x

(6) The sixth integral I_6 is

$$I_6 = (1/2\pi)\int_{-\infty}^{\infty} \left(\frac{-ib(gk \sinh ky + (kU)^2 \cosh ky)}{ik(-2\mu)(kU + \mu) \cosh kh} \right) e^{ik(L-x)}\, dk$$

similar to I_5 .

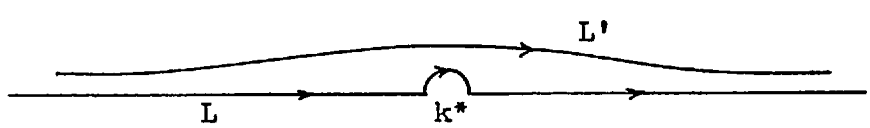
(7) The seventh integral I_7 is

$$I_7 = (1/2\pi)\int_{-\infty}^{\infty} \left(\frac{-ib(gk \sinh ky + (kU)^2 \cosh ky)}{ik(-2\mu)(kU - \mu) \cosh kh} \right) e^{it(kU-\mu)}\, e^{-ikx}\, e^{ikL}\, dk$$

it is better to rewrite I_7 as $(1/2\pi)\int_{-\infty}^{\infty} \frac{\chi_7(k,y)}{(kU - \mu(k))}\, e^{ikL}\, dk$,

where , $\chi_7 = \left(\frac{-ib(gk \sinh ky + (kU)^2 \cosh ky)}{ik(-2\mu) \cosh kh} \right) e^{it(kU-\mu)-ikx}$

it is clear that the integral has a simple pole at $k = k^*$, by expanding the integrand about $k = k^*$, we obtain the important factor $\exp(- K_i L)$, where L is positive , for convergence K_i must be positive, then the semicircle round the simple pole lies in the positive half in the k - plane as in the figure



$$I_7 = +\tfrac{1}{2}i \left(\frac{-ib(gk \sinh ky + \mu^2 \cosh ky)}{ik(-2\mu)\, d/dk(kU-\mu) \cosh kh} \right)_{k=k^*} e^{-ixk^*}\, e^{iLk^*}$$

(8) The eighth integral I_8 is

$$I_8 = (1/2\pi)\int_{-\infty}^{\infty} \left(\frac{-ib(gk \sinh ky+\mu^2\cosh ky)}{ik(-2\mu)(kU+\mu) \cosh kh} \right) e^{it(kU+\mu)}\, e^{-ikx}\, e^{ikL}\, dk$$

it is similar to I_7 .

Now , the asymptotic expression for the solution $\xi(x,y;t)$ in the different cases is given by

(a) The case in which x/t is positive and $g_1'(k^*)$ also positive, where $g_1'(k^*) = [-x/t + U - \mu'(k^*)]$, i.e. the case in which the observer moves with a velocity less than the group velocity appropraite to wave number $k = k^*$, the solution is

$$\xi(x,y;t;L) = -i \left(\frac{-ib(gk \sinh ky+(kU)^2\cosh ky)}{-ik(-2\mu)(kU - \mu)\cosh kh} \right)_{k=*k^*} e^{-ik^*x}$$

$$+(1/2\pi)\left[\frac{-ib(gk \sinh ky+\mu^2\cosh ky)}{-ik(-2\mu)(kU-\mu)\cosh hk}\right]_{k=k_o} e^{i(tg_1(k_o)+\frac{1}{4}\pi)}\left(\frac{2\pi}{t\, g_1''(k_o)}\right)^{\frac{1}{2}}$$

representing a system of a steady waves .

(b) The case in which x/t is postive but $g'_1(k^*)$ is negative , i.e. the case in which the observer moves with velocity greater than the group velocity , the corresponding solution is

$$\xi(x,y;t;L) = (1/2)\left[\frac{-ib(gk \sinh ky+\mu^2\cosh ky)}{-ik(-2\mu)(kU-\mu)\cosh hk}\right]_{k=k_o} e^{i(tg_1(k_o)+\frac{1}{4}\pi)}\left(\frac{2\pi}{t\, g_1''(k_o)}\right)^{\frac{1}{2}}$$

representing a transient solution behaves like $t^{-\frac{1}{2}}$.

(c) The case in which x/t and $g_1'(k^*)$ are negative , i.e. the observer moving towards upstream , the solution is

$$\xi(x,y;t;L)=(1/2\pi)\left[\frac{-ib(gk\ \sinh\ ky+\mu^2\cosh\ ky)}{-ik(-2\mu)(kU-\mu)\cosh\ kh}\right]_{k=k_o} e^{i(tg_1(k_o)+\frac{1}{4}\pi)}\left(\frac{2\pi}{t\,g_1''(k_o)}\right)^{\frac{1}{2}}$$

representing a transient solution behaves like $t^{-\frac{1}{2}}$.

We evaluated the solution $\xi(x,y;t;L)$ asymptotically for large t,x&L , when L becomes large before t & x . Now , we like to evaluate the solution asymptotically when t ,the time, becomes large before x and L . Considering $x = Vt$, where V is the observer speed . By the same analysis , we obtain different expressions for the solution corresponding to the different cases.

Case 1 :

The case in which x/t and $g'_1(k^*)$ are positive , i.e. the observer velocity is positive and less than the group velocity , we obtain the solution

$$\xi(x,y;t;L)=-i\left[\frac{-ib(gk\sinh ky+(kU)^2\cosh ky)}{-ik(2\mu)d/dk(kU-\mu)\cosh kh}\right]_{k=\pm k^*}(1+e^{\pm ik^*L})\,e^{-ik^*x}$$

$$+\,O\,(1/t^{\frac{1}{2}})\,.$$

representing a system of a steady waves .

Case 2 :

The case in which x/t is positive but $g'(k^*)$ is negative, therefore x/t is less than $d/dk\,(kU-\mu(k))_{k=k^*}$,i.e. the observer moves with a positive velocity greater than the group velocity .

Case 3 :

The case in which x/t is negative , i.e. the observer moving in the upstream direction .

The corresponding solution to the cases2 & 3 is a transient solution behaves like $1/t^{\frac{1}{2}}$.

The present problem in the case of no current U = o :

The corresponding expression for the solution is

$$\xi(x,y;t;L) = (1/\pi)\int_{-\infty}^{\infty}\left(\frac{-ib(gk\sinh ky)}{-ik(-2\mu^2)\cosh(kh)}\right)e^{-ikx}\,dk$$

$$+(1/2\pi)\int_{-\infty}^{\infty}\left(\frac{-ib(gk\sinh ky+\mu^2\cosh ky)}{-ik\,.2\mu^2\cosh kh}\right)e^{it\mu}e^{-ikx}\,dk$$

$$+(1/2\pi)\int_{-\infty}^{\infty}\left(\frac{-ib(gk\sinh ky+\mu^2\cosh ky)}{-ik.\,2\mu^2\cosh kh}\right)e^{-it\mu}e^{-ikx}\,dk$$

$$+(1/\pi)\int_{-\infty}^{\infty}\left(\frac{-ib(\sinh ky)}{ik(-2\mu^2)\cosh kh}\right)e^{ik(L-x)}\,dk$$

$$+(1/2\pi)\int_{-\infty}^{\infty}\left(\frac{-ib(gk\sinh ky+\mu^2\cosh ky)}{ik(2\mu^2)\cosh kh}\right)e^{it\mu}e^{ik(L-x)}\,dk$$

$$+(1/2\pi)\int_{-\infty}^{\infty}\left(\frac{-ib(gk\sinh ky+\mu^2\cosh ky)}{ik(2\mu^2)\cosh kh}\right)e^{-it\mu}e^{ik(L-x)}\,dk$$

the different integrals are free from any real simple poles and well behave at the origin .

The solution can be evaluated asymptotically for large L , t & x . To do this , let L becomes large before t & x and assume x/t fixed first . Secondly , let t becomes large before L & x and assum x/t fixed . In both cases the most contributions come from the saddle points be defined by $[d/dk(\mu(k)-x/t\ k)] = o$ and $[d/dk(\mu(k)+k\ x/t\ ] = o$, hence , by examining the solution $\xi(x,y;t;L)$ from standpoint of an observer moves with the group velocity appropraite to wave number k = the saddle point , then the solution behaves like $1/t^{\frac{1}{2}}$. When the observer moves with velocity x/t such that the relation $[d/dk\ (\mu(k))] = \pm\ x/t$ has no solution , by integrating by parts , the solution behaves like 1/t .

At the time $t = 0$ a step, of length L, is created at the bottom, of a running fluid or still one with free surface, suddenly. By investigating the behaviour of the vertical displacement of every fluid particle for large t, x & L, we obtain the asymptotic expression for the solution in different cases;

(1) From the standpoint of an observer moving with a velocity less than the group velocity, in the dowenstream direction. in this case we obtain a Steady Solution.

(2) From the standpoint of an observer moving with a velocity greater than the group velocity, we get a Transient Solution behaves like $1/t^{\frac{1}{2}}$.

(3) In the case in which $U = 0$ (still fluid), from the standpoint of an observer moving with group velocity, the solution behaves like $1/t^{\frac{1}{2}}$, but when he moves with velocity for which no solution to the relation $d/dk((k)) = \pm x/t$ exists, the solution behaves like $1/t$.

Conclusion:

In this part we considered the creation of waves in a uniform stream ($U \neq 0$) or in still fluid ($U=0$), due to a disturbance created at t=o; suddenly or smoothely at the bottom. We considered a symmetry disturbance about the point beneath the origin and a flow over an infinite step at the bed. By Fourier-Laplace technique we got a solution in the integral expression. This integral can be evaluated asymptotically for large x and t. To do this :

Firstly , the case in which $U \neq 0$, we assumed x/t is fixed we obtain

(1) If the observer moving in the downstream direction with velocity greater than group velocity or moving in the upstream direction, he observes only a system of a transient waves with an amplitude behaves like $(1/t)^{\frac{1}{2}}$.

(2) The steady-state solution occures only when the observer moving with velocity less than group velocity in the downstream direction.

(3) In the transition case, i.e. the observer moving with velocity near from the value of the group velocity in the downstream direction. We can describe the nature of this case in terms of the Error function. Hence, we see that the transition occures by means of an amplitude modulation which behaves like an Error function.

Secondly, the case in which U = 0 and x/t is fixed, we had

(1) If the observer moves with group velocity, then the solution behaves like $(1/t)^{\frac{1}{2}}$.

(2) If he moves with velocity does not equal the group velocity,

then the solution behaves like (1/t).

Thirdly, in this case $U \neq 0$ and we are fixed our attention on one value of x then let t increasing, we obtained

(1) If x/t is positive and greater than the the group velocity, the observer will observe waves of continually decreasing wave-number , i.e. increasing wave length and the amplitude is decreasing as $(1/t)^{\frac{1}{2}}$.

(2) If x/t is negative , the observer will observe waves of continually increasing wave number, i.e. decreasing wave length, and the amplitude behaves like $(1/t)^{\frac{1}{2}}$.

(3) If x/t is positive and less than group velocity , we have a system of a steady waves.

PART II

Higher-Order Theory of Infinitesimal Waves

Cauchy - Poisson Problem

Cauchy (1827) and Poisson (1815) discussed the problem of " generation of waves " when a local disturbance is given on the surface of deep water . The wave thus created is the so called " Cauchy - Poisson " .

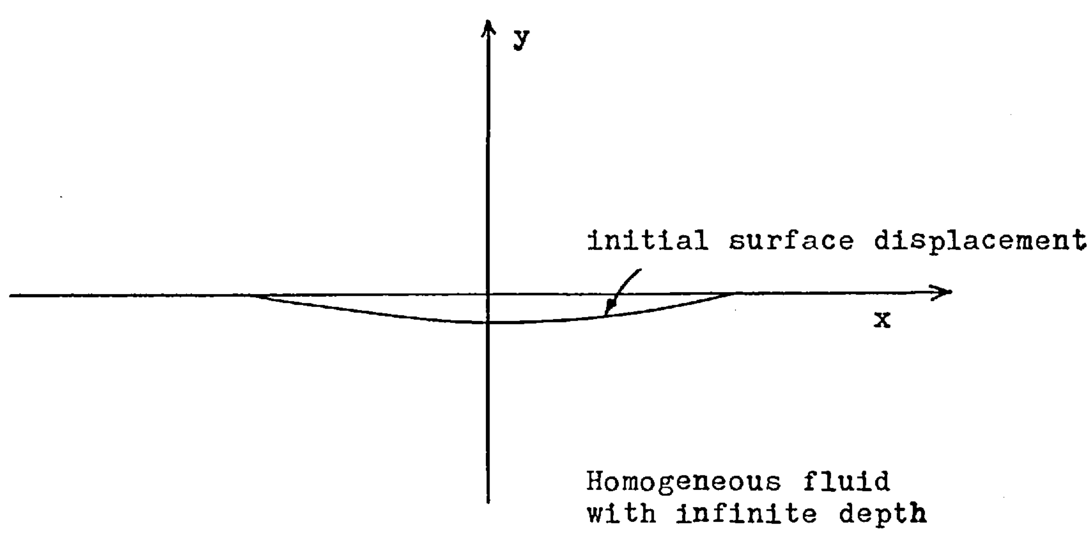
In this work we considered the two - dimensional case in which the fluid extending to infinity , horizontally and downwards , the pressure over the free surface is constant , say zero . Taking the axis of x on the undisturbed free surface ($y = 0$) and that y vertically upwards . At time $t = 0$ a disturbance is suddenly created on the free surface . It is required to find the displacement and the velocity of every particle of the fluid at any future time . The fluid being assumed incompressible and frictionless , its motion , starting primarily from rest by a disturbance applied to the free surface is essentially irrotational . The modes of irrotational motion of fluid by surface disturbance present themselves , as H . Lamb points out , in two forms : (1) by an initial displacement of the surface without initial velocity ; (2) by an initial impulse applied on the surface , without initial surface displacement . In the present problem we consider the first case , i.e. considering an initial small displacement of the free surface without initial velocity ; hence , by the infinitesmal - wave theory and using the technique of Fourier transform we can obtain the solution up the third order correction . The representation integral for the solution can be evaluated asymptotically for large x and t by the method of stationary phase (due to Lord Kelvin) .

Formulation :

For two - dimenstional motion in the x - y plane , a coordinaite system is chosen with the origin on the undisturbed free surface (y = o) , where the y - axis is positive upward and the x - axis positive to the right . Upon the usual assumptions of classical hydrodynamics , if the motion - and the fluid is inviscid with uniform density - is generated originally from rest by an initial displacement, applied on the free surface at t = o described by the relation $y = \eta(x,o)$, it will be irrotational throughout all time and we may describe the motion in terms of a velocity potential $\phi(x,y;t)$ satisfying Laplace's equation

$$\nabla^2_{x,y}\, \phi(x,y;t) = o \qquad (1)$$

where , $\nabla^2_{x,y} = \partial^2/\partial x^2 + \partial^2/\partial y^2$.



The Boundary Coditions:

(a) At the free surface we have two conditions;

(1) The dynamic condition (from Bernoulli's equation)is

$$\phi_t + \tfrac{1}{2} (\phi_x^2 + \phi_y^2) + g\eta(x,t) = F(t) ,$$

where $y = \eta(x,t)$ is the vertical displacement of the free surface about the undisturbed surface $y = o$, and $F(t)$ is an arbitrary function of t , can absorbed in ϕ_t , i.e. the condition can be written as

$$\phi_t + \tfrac{1}{2} (\phi_x^2 + \phi_y^2) + g\eta(x,t) = o \qquad (2)$$

and g is the constant of gravity .

(2) The kinematic condition (state that " no transfer of the matter across the free surface " .) Let us suppose that the free surface $S(t)$ is given by the equation $F(x,y;t) = o$, where $F(x,y;t) = y - \eta(x;t)$. Mathematically the condition can be written as $\left[D/Dt \, (y - \eta(x;t)) \right] = o$,where D/Dt is the material derivative and $=(\partial/\partial t + \phi_x \partial/\partial x + \phi_y \partial/\partial y)$, hence , the condition can be written as

$$\phi_y - (\phi_x \eta_x + \eta_t) = o \qquad (3)$$

We complete the statement of the boundary conditions by invoking the finiteness conditions

$$|\phi| < \infty \quad \text{as } |x| \to \infty \ \& \ y \to -\infty \qquad (4.a)$$

and

$$|\eta(x,t)| < \infty \quad \text{as } |x| \to \infty \qquad (4.b)$$

Prescribing the initial displacement $y = \eta(x,o)$ and invoking the assumption that the fluid is initially at rest , we obtain the initial conditions

$$\phi = o \; ,$$

and $$\eta(x,o) = f(x) \quad \text{at } t=o \qquad (5)$$

$$= \text{given, and, symetric about the origin} \; .$$

We see that the boundary conditions involve non - linear terms, for example $\frac{1}{2}(\phi_x^2 + \phi_y^2)$ in the dynamical condition and $\phi_x \eta_x$ in the kinematic condition , these lead to analytical difficults which may be overcome by expanding various functions entering into the problem into power series in ϵ - small parameter - (say in the present problem the maximamum value of the original surface displacement) . The different series are substituted into the governed equation to gether the boundary conditions at the free surface, then grouped according to powers of ϵ . The coefficients of each power yields a sequence of equations and boundary conditions, the coefficients of ϵ giving the first - order theory , these of ϵ^2 the second - order , etc.

Carrying out this program . Let us first assume that $\phi(x,y;t)$ & $\eta(x;t)$ may be expanded in a perturbation series in ϵ , as

$$\phi(x,y;t) = \epsilon \phi^{(1)}(x,y;t) + \epsilon^2 \phi^{(2)}(x,y;t) + \epsilon^3 \phi^{(3)}(x,y;t) + \ldots \; ,$$

and (6)

$$\eta(x,t) = \epsilon \eta^{(1)}(x;t) + \epsilon^2 \eta^{(2)}(x;t) + \epsilon^3 \eta^{(3)}(x;t) + \ldots \; .$$

It follows first of all that each of the functions $\phi^{(k)}(x,y;t)$ are solutions of the Laplace equation and $k = 1,2,3,\ldots$, i.e.

$$\nabla^2_{x,y}\phi^{(k)}(x,y;t) = o \qquad (7)$$

Substituting for $\phi(x,y;t)$ and $\eta(x;t)$ in (2) and (3) , remembering in addition that formal expansions of the following sort,for example , hold ;

$$\phi(x,y;t)=\phi(x,\eta(x,t),t)=\phi(x,o;t)+\eta(x,t)\phi_y(x,o;t)+ \ldots$$

$$=\epsilon\phi^{(1)}(x,o;t) + \epsilon^2\phi^{(2)}(x,o;t) + \ldots$$

$$+(\epsilon\eta^{(1)}(x;t)+ \epsilon^2\eta^{(2)}(x,t)+\ldots)(\epsilon\phi_y^{(1)}(x,o;t)+\epsilon^2\phi_y^{(2)}(x,o;t)+\ldots)$$

$$=\epsilon\phi^{(1)}(x,o;t)+\epsilon^2(\phi^{(2)}(x,o;t) +\eta^{(1)}(x;t)\phi_y^{(1)}(x,o;t)) + \ldots$$

One finds that ,

$$\phi_x(x,y;t)=\epsilon\phi_x^{(1)}(x,o;t)+\epsilon^2(\eta^{(1)}(x,t)\phi_{xy}^{(1)}(x,o;t)+\phi_x^{(2)}(x,o;t))$$

$$+\epsilon^3(\eta^{(2)}\phi_{xy}^{(1)} + \tfrac{1}{2}(\eta^{(1)})^2\phi_{xyy}^{(1)} +\eta^{(1)}\phi_{xy}^{(2)} + \phi_x^{(3)}) + \ldots ,$$

$$\phi_y(x,y;t)=\epsilon\phi_y^{(1)} +\epsilon^2(\eta^{(1)}\phi_{yy}^{(1)} + \phi_y^{(2)})$$

$$+\epsilon^3 (\eta^{(2)}\phi_{yy}^{(1)} +\tfrac{1}{2}(\eta^{(1)})^2\phi_{yyy}^{(1)} +\eta^{(1)}\phi_{yy}^{(2)} + \phi_y^{(3)}) + \ldots$$

$$\phi_t(x,y;t)=\epsilon\phi_t^{(1)} + \epsilon^2 (\eta^{(1)}\phi_{ty}^{(1)} + \phi_t^{(2)})$$

$$+ \epsilon^3(\eta^{(2)}\phi_{ty}^{(1)}+\tfrac{1}{2}(\eta^{(1)})^2\phi_{tyy}^{(1)} +\eta^{(1)}\phi_{ty}^{(2)} + \phi_t^{(3)})+ \ldots$$

Substituting these expressions in the boundary conditions at the the free surface ,and collecting the coefficients of the different powers of ϵ , one finds

(1) The first - order boundary conditions(coefficientsof ϵ)are

$$\phi_t^{(1)}(x,o;t) + g\eta^{(1)}(x;t) = o ,$$

and

$$\phi_y^{(1)}(x,o;t) - \eta_t^{(1)}(x;t) = o .$$

(2) The boundary conditions for the second - order corrections are,(coefficientssof ϵ^2)

$$\phi_t^{(2)}(x,o;t) + g\eta^{(2)}(x,t) = -\eta^{(1)}\,\phi_{ty}^{(1)} + \tfrac{1}{2}\left((\phi_x^{(1)})^2 + (\phi_y^{(1)})^2 \right),$$

$$\phi_y^{(2)}(x,o;t) - \eta_t^{(2)}(x;t) = \phi_x^{(1)}\,\eta_x^{(1)} - \eta^{(1)}\,\phi_{yy}^{(1)}$$

(3) The boundary conditions correponding the third - correction (the coefficients of ϵ^3) are

$$\phi_t^{(3)}(x,o;t) + g\eta^{(3)}(x,t) = -\left(\eta^{(2)}\phi_{ty}^{(1)} + \tfrac{1}{2}(\eta^{(1)})^2\,\phi_{tyy}^{(1)} + \eta^{(1)}\phi_{ty}^{(2)} \right)$$
$$-\phi_x^{(1)}\left(\eta^{(1)}\phi_{xy}^{(1)} + \phi_x^{(2)}\right) - \phi_y^{(1)}\left(\eta^{(1)}\phi_{yy}^{(1)} + \phi_y^{(2)}\right),$$

$$\phi_y^{(3)}(x,o;t) - \eta_t^{(3)}(x,t) = \phi_x^{(1)}\eta_x^{(2)} + \eta_x^{(1)}\left(\eta^{(1)}\,\phi_{xy}^{(1)} + \phi_x^{(2)}\right)$$
$$-\left(\eta^{(2)}\phi_{yy}^{(1)} + \tfrac{1}{2}(\eta^{(1)})^2\,\phi_{yyy}^{(1)} + \eta^{(1)}\,\phi_{yy}^{(2)} \right).$$

The First Order Problem

Due to the infinitesmal - wave theory , the first order problem is defined by

(1) The partial differential equation is

$$\nabla^2_{x,y}\,\phi^{(1)}(x,y;t) = 0 . \qquad (2.1)$$

(2) The boundary conditions at the free surface (y=o) are

$$\phi^{(1)}_y(x,0;t) - \eta^{(1)}_t(x;t) = 0 \qquad (2.2)$$

$$\phi^{(1)}_t(x,0;t) + g\,\eta^{(1)}(x;t) = 0 \qquad (2.3)$$

(3) The finiteness conditions are

$$\left|\phi^{(1)}\right| \to 0 \qquad \text{as } |x| \to \infty \text{ or } y \to -\infty \qquad (2.4a)$$

and

$$\left|\eta^{(1)}\right| < \infty \qquad \text{as } |x| \to \infty \qquad (2.4b)$$

(4) The initial condition , when t=o ,

$$\eta^{(1)}(x,0) = f(x) \qquad (2.5a)$$

$$\phi^{(1)} = 0 \qquad (2.5b)$$

Let the initial displacement be given by the function

$$f(x) = 1/(b^2 + x^2) ,$$

or

$$f(x) = ab^2/(b^2 + x^2) .$$

We attack the mathematical problem posed by (2.1 - 2.5) by invoking a Fourier transformation with respect to x . Let

$$\bar{\phi}(k,y;t) = \mathcal{F}\phi(x,y;t)$$
$$= \int_{-\infty}^{\infty} \phi(x,y;t) \exp(ikx)\, dx .$$

by integrating n times by parts ,weobtain

$$\mathcal{F}(\partial^n/\partial k^n \phi) = (-ik)^n \bar{\phi} ,$$

where $\mathcal{F}$ implies Fourier transformation with respect to x ($-\infty < x < \infty$).

Transforming (2.1) , we obtain

$$\bar{\phi}^{(1)}_{yy}(k,y;t) - k^2\bar{\phi}^{(1)}(k,y;t) = o, \qquad (2.6)$$

by taking in cosideration that,

$$\phi(x,y;t) \text{ and } \partial\phi/\partial x \longrightarrow o \text{ as } |x| \longrightarrow \infty .$$

The second order differential equation (2.6) has a solution

$$\bar{\phi}^{(1)}(k,y;t) = A(k;t)\, e^{|k|y} + B(k;t)\, e^{-|k|y} ,$$

where A(k,t) and B(k,t) are arbitrary constants .
From the finiteness condition (2.4a) , one finds that B(k;t)=o, hence , the required solution has the form

$$\bar{\phi}^{(1)}(k,y;t) = A(k;t)\, e^{|k|y} \qquad (2.7)$$

Transforming (2.2 & 2,3) , we obtain

$$\bar{\phi}^{(1)}_y(k,o;t) - \bar{\eta}^{(1)}_t(k;t) = o ,$$

and $$\bar{\phi}^{(1)}_t(k,o;t) + g\, \bar{\eta}^{(1)}(k;t) = o , \qquad \text{at } y=o .$$

Eliminating $\bar{\eta}^{(1)}(k;t)$ between the above two equations, we obtain

$$\bar{\phi}^{(1)}_{tt}(k,o;t) + g\,\bar{\phi}^{(1)}_{y}(k,o;t) = o\ , \quad \text{at } y=o \qquad (2.8)$$

Substituting (2.7) into (2.8) , we get

$$A_{tt}(k;t) + g|k|A(k;t) = o\ ,$$

has a solution

$$A(k;t)=\alpha^{(1)}(k)\ \cos(g|k|)^{\frac{1}{2}}t + \beta^{(1)}(k)\ \sin(g|k|)^{\frac{1}{2}}t, \quad (2.9)$$

where , $\alpha^{(1)}(k)$ and $\beta^{(1)}(k)$ are arbitrary functions in k .

From (2.9) and (2.7) , we obtain

$$\bar{\phi}^{(1)}(k,y;t) = (\ \alpha^{(1)}(k)\ \cos(g|k|)^{\frac{1}{2}}t + \beta^{(1)}(k)\ \sin(g|k|)^{\frac{1}{2}}t\ )e^{|k|y}.$$

But $\alpha^{(1)}(k) = o$, from the initial condition (2.5a) , hence ,

$$\bar{\phi}^{(1)}(k;y;t) = \beta^{(1)}(k)\ \sin(g|k|)^{\frac{1}{2}}t\ \exp(|k|y) \qquad (2.1o)$$

By the application of the inversion theorem , we get

$$\phi^{(1)}(x,y;t)=(1/2\pi)\int_{-\infty}^{\infty}\beta^{(1)}\ \sin[(g|k|)^{\frac{1}{2}}t]\ e^{|k|y}\ e^{-ikx}\ dk \quad (2.11)$$

The corresponding expression for $\eta^{(1)}(x;t)$ (the first correction of the vertical displacement of the fluid - level at the point (x,o) at time t relative to the undisturbed surface y=o) is given by

$$\eta^{(1)}(x;t) = -\ (1/g)\ \phi^{(1)}_{t}(x,o;t)\quad ,$$

from (2.11) , we obtain

$$\eta^{(1)}(x,t) = (1/2\pi)\int_{-\infty}^{\infty}(\ -(|k|/g)^{\frac{1}{2}}\beta^{(1)})\ \cos[(g|k|)^{\frac{1}{2}}t]\ e^{-ikx}\ dk \quad (2.12)$$

When t = o , we have an initial displacement of the free surface without initial velocity which is given by

$$\eta^{(1)}(x;o) = f(x) ,$$

it is better to consider the initial vertical displacement is symmetric about the origin (x = o) , for example

$$f(x) = 1 /(b^2 + x^2) ,$$

$$\text{or} \quad f(x) = a^2b^2 /(b^2 + x^2) ,$$

i.e. f(x) is an even function .

Then , from (2.12) , we get

$$f(x) = (1/2\pi)\int_{-\infty}^{\infty} -(|k|/g)^{\frac{1}{2}} \beta^{(1)}(k)\, e^{-ikx}\, dk \qquad (2.13)$$

i.e. , we can write ,

$$-(k /g)^{\frac{1}{2}} \beta^{(1)}(k) = \int_{-\infty}^{\infty} f^{(1)}(x)\, e^{ikx}\, dk ,$$

$$= \bar{f}^{(1)}(k) ,$$

therefore ,

$$\beta^{(1)}(k) = - (g/|k|)^{\frac{1}{2}} \bar{f}^{(1)}(k) . \qquad (2.14)$$

Then , for the first order problem we have the solution

$$\phi^{(1)}(x,y;t)=(1/2\pi)\int_{-\infty}^{\infty} \beta^{(1)}(k) \sin[(g|k|)^{\frac{1}{2}}t]\, e^{|k|y}\, e^{-ikx}\, dk ,$$

and

$$\eta^{(1)}(x;t)=(1/2\pi)\int_{-\infty}^{\infty} -(|k|/g)^{\frac{1}{2}} \beta^{(1)}(k) \cos[(g|k|)^{\frac{1}{2}}t]\, e^{-ikx}\, dk ,$$

but , f(x) is an even function , it is better to rewrite $\phi^{(1)}$ & $\eta^{(1)}$ in a suitable form ,as

$$\eta^{(1)}(x;t)=(\mathrm{Re}/\pi)\int_0^\infty \bar{f}^{(1)}(k)\ \cos((gk)^{\frac{1}{2}}t)\ e^{-ikx}\ dk\ ,$$

and, (2.15)

$$\phi^{(1)}(x,y;t)=(\mathrm{Re}/\pi)\int_0^\infty -(g/k)^{\frac{1}{2}}\ \bar{f}^{(1)}(k)\ \sin((gk)^{\frac{1}{2}}t)\ e^{-ikx}\ e^{ky}\ dk.$$

where, Re means the real part.

The Second Order Problem

This problem is defined by

(1) The second correction for the velocity potential $\phi^{(2)}(x,y;t)$ must satisfy Laplace equation, i.e.,

$$\nabla^2_{x,y}\,\phi^{(2)}(x,y;t) = 0 \qquad (3.1)$$

(2) The boundary conditions at y = o

(a) The kinematic condition is

$$\phi^{(2)}_y - \eta^{(1)}_t = \left[\eta^{(1)}_x\,\phi^{(1)}_x - \eta^{(1)}\,\phi^{(1)}_{yy}\right] \qquad (3.2)$$

(b) The dynamic-al condition is given by

$$\phi^{(2)}_t + g\,\eta^{(2)} = \left[-\eta^{(1)}\phi^{(1)}_{ty} - \tfrac{1}{2}\left((\phi^{(1)}_x)^2 + (\phi^{(1)}_y)^2\right)\right] \qquad (3.3)$$

(3) The finiteness conditions ,

$$\left|\phi^{(2)}\right| \longrightarrow 0 \qquad \text{as } |x| \longrightarrow \infty \ \&\ y \longrightarrow -\infty \qquad (3.4a)$$

and

$$\left|\eta^{(2)}\right| < \infty \qquad \text{as } |x| \longrightarrow \infty \qquad (3.4b)$$

(4) The initial condition , when t = o , we have

$$\eta^{(2)}(x;0) = 0 \qquad (3.5a)$$

and

$$\phi^{(2)} = 0 \qquad (3.5b)$$

Eliminating $\eta^{(2)}(x;t)$ between (3.2) and (3.3) ,we obtain

$$\phi^{(2)}_{tt} + g\,\phi^{(2)}_{y} = \left[g\,\eta^{(1)}_{x}\,\phi^{(1)}_{x} - g\,\eta^{(1)}\,\phi^{(1)}_{yy} - \eta^{(1)}_{t}\,\phi^{(1)}_{ty} - \eta^{(1)}\,\phi^{(1)}_{tty} \right] \quad (3.6)$$

valid at $y = o$.

Substituting for $\phi^{(1)}$, $\eta^{(1)}$ and their derivatives from (2.15) in the right hand side of (3.6),one can find that it is equal to zero, hence the single condition on $\phi^{(2)}$ at $y = o$ is

$$\phi^{(2)}_{tt} + g\,\phi^{(2)}_{y} = o \quad (3.7)$$

By Fourier transform technique , then the Laplace equation (3,1) is equivalent to

$$\frac{d^2\,\bar{\phi}^{(2)}}{dy^2} - k^2\,\bar{\phi}^{(2)} = o\ , \quad (3.8)$$

of which the solution which tends to zero as $y \longrightarrow -\infty$ is

$$\bar{\phi}^{(2)}(k,y;t) = A_2(k;t)\ \exp(\,|k|y) \quad (3.9)$$

where $A_2(k;t)$ is an arbitrary function .
Multiplying both sides of the equation (3.7) by exp(ikx) and integrating over the entire range of variation of x , we find

$$A_{tt}(k,t) + g|k|A(k;t) = o\ ,$$

whence it follows that

$$A(k;t) = \alpha^{(2)}(k)\ \cos[(g|k|)^{\frac{1}{2}}t] + \beta^{(2)}(k)\ \sin[(g|k|)^{\frac{1}{2}}t]\ ,$$

where , $\alpha^{(2)}(k)$ and $\beta^{(2)}(k)$ are constants of integration .
Initially ,at $t = o$; $\phi^{(2)} = o$ this leads to $\alpha^{(2)}(k) = o$, hence , we get

$$\phi^{(2)}(k,y;t) = \beta^{(2)}(k)\ \sin[(g|k|)^{\frac{1}{2}}t]\ \exp(|k|y).$$

By the application the inversion theorem , we obtain

$$\phi^{(2)}(x,y;t)=(1/2\pi)\int_{-\infty}^{\infty}\beta^{(2)}\sin[(g|k|)^{\frac{1}{2}}t]\ e^{|k|y}\ e^{-ikx}\ dk\ . \quad (3.1o)$$

The corresponding expression for $\eta^{(2)}(x;t)$ (the second correction of the vertical displacement of the fluid - level at the point (x,o) and at time t relative the undisturbed surface y = o) is given by the relation

$$\eta^{(2)}(x;t) =(-1/g)(\ \eta^{(1)}\ \phi^{(1)}_{ty} + \phi^{(2)}_{t}\ )_{y=o}\ ,$$

substituting for $\eta^{(1)}$, $\phi^{(1)}$, $\phi^{(2)}$ and their derivatives in the right hand side of the above relation , we obtain

$$\eta^{(2)}(x;t)=\left[(1/2\pi)\int_{-\infty}^{\infty}\bar{f}^{(1)}\cos[(g|k|)^{\frac{1}{2}}t]\ e^{-ikx}\ dk\right]x \left[(1/2\pi)\int_{-\infty}^{\infty}|k|\bar{f}^{(1)}\ \cos[(g|k|)^{\frac{1}{2}}t]\ e^{-ikx}\ dk\right] - (1/2\pi)\int_{\infty}^{\infty}(|k|/g)^{\frac{1}{2}}\beta^{(2)}\ \cos[(g|k|)^{\frac{1}{2}}t]\ e^{-ikx}\ dk \quad (3.11)$$

Initially , when t = o , we have $\eta^{(2)}(x;o) = o$, from(3.11) we get

$$\left[(1/2\pi)\int_{-\infty}^{\infty}\bar{f}^{(1)}\ e^{-ikx}\ dk\right]\ x\ \left[(1/2\pi)\int_{-\infty}^{\infty}|k|\bar{f}^{(1)}\ e^{-ikx}\ dk\right] =(1/2\pi)\int_{-\infty}^{\infty}(|k|/g)^{\frac{1}{2}}\ \beta^{(2)}\ \ e^{-ikx}\ dk\ ,$$

hence , the function $\beta^{(2)}$ be defined by the relation

$$(1/2\pi)\int_{-\infty}^{\infty}(|k|/g)^{\frac{1}{2}}\beta^{(2)}\ e^{-ikx}dk = f^{(1)}(x)\left[(1/2\pi)\int_{-\infty}^{\infty}|k|\bar{f}^{(1)}\ e^{-ikx}\ dk\right] \quad (3.12)$$

The Third Order Problem

This problem is defined by,

(1) The partial differential equation is

$$\phi^{(3)}_{xx}(x,y;t) + \phi^{(3)}_{yy}(x,y;t) = o \qquad (4.1)$$

(2) The boundary conditions when y = o ,

(a) The kinematic condition is

$$\phi^{(3)}_{y} - \eta^{(3)}_{t} = \Big[\phi^{(1)}_{x}\eta^{(2)}_{x} + \eta^{(1)}_{x}\,(\eta^{(1)}\,\phi^{(1)}_{xy} + \phi^{(2)}_{x}) - \eta^{(2)}\,\phi^{(1)}_{yy} - \tfrac{1}{2}(\eta^{(1)})^2\,\phi^{(1)}_{yyy} - \eta^{(1)}\,\phi^{(2)}_{yy}\Big] \qquad (4.2)$$

(b) The dynamical condition is

$$\phi^{(3)}_{t} + g\eta^{(3)} = \Big[-(\eta^{(2)}\phi^{(1)}_{ty} + \tfrac{1}{2}(\eta^{(1)})^2\,\phi^{(1)}_{tyy} + \eta^{(1)}\,\phi^{(2)}) - \phi^{(1)}_{x}(\eta^{(1)}\,\phi^{(1)}_{xy} + \phi^{(2)}_{x}) - \phi^{(1)}_{y}(\eta^{(1)}\,\phi^{(1)}_{yy} + \phi^{(2)}_{y})\Big] \qquad (4.3)$$

(3) The finiteness conditions are

$$|\phi^{(3)}| \longrightarrow o \quad \text{as } |x| \longrightarrow \infty \quad \& \quad y \longrightarrow -\infty \qquad (4.4a)$$

and

$$|\eta^{(3)}| < \infty \quad \text{as } |x| \longrightarrow \infty \qquad (4.4b)$$

(4) The initial condition , when t = o , given by

$$\eta^{(3)}(x;o) = o \qquad (4.5a)$$

$$\phi^{(3)} = o \qquad (4,5b)$$

Eliminating $\eta^{(3)}$ between (4.2) and (4.3) , we obtain

$$\phi^{(3)}_{tt} + g\,\phi^{(3)}_{y} = \Big[\phi^{(1)}_{yy}\eta^{(1)}\,\phi^{(1)}_{ty} + \phi^{(1)}_{yy}\,\phi^{(2)}_{t} - \tfrac{1}{2}g(\eta^{(1)})^2\,\phi^{(1)}_{yyy} - g\eta^{(1)}\,\phi^{(2)}_{yy}$$

$$+ g\,\eta^{(1)}\eta^{(1)}_{x}\phi^{(1)}_{xy} + g\eta^{(1)}_{x}\phi^{(2)}_{x} - \phi^{(1)}_{x}\eta^{(1)}\,\phi^{(1)}_{tyx} - \phi^{(1)}_{x}\eta^{(1)}_{x}\,\phi^{(1)}_{ty}$$

$$- \phi^{(1)}_{x}\,\phi^{(2)}_{tx} + 1/g\,\phi^{(1)}_{tty}\eta^{(1)}\phi^{(1)}_{ty} + 1/g\,\phi^{(1)}_{tty}\,\phi^{(2)}_{t} +$$

$$+ 1/g\,\eta^{(1)}\phi^{(1)}_{ty}\,\phi^{(1)}_{tty} + L/g\,\eta^{(1)}_{t}(\phi^{(1)}_{ty})^2 + 1/g\,\phi^{(1)}_{ty}\,\phi^{(2)}_{tt}$$

$$- \eta^{(1)}\eta^{(1)}_{t}\phi^{(1)}_{tyy} - \tfrac{1}{2}(\eta^{(1)})^2\,\phi^{(1)}_{ttyy} - \eta^{(1)}\,\phi^{(2)}_{tty} - \eta^{(1)}_{t}\phi^{(2)}_{ty}\Big] \qquad (4.6)$$

substituting for the values in the right hand side,this condition reduced to

$$\phi^{(3)}_{tt} + g\,\phi^{(3)}_{y} = o \qquad , \text{ at } y = o \qquad (4.7)$$

Applying the Fourier transform to (4.1) , we obtain

$$\bar{\phi}^{(3)}_{yy}(k,y;t) - k^2\,\bar{\phi}^{(3)}(k,y;t) = o \quad , \qquad (4.8)$$

has a solution

$$\bar{\phi}^{(3)}(k,y;t) = A(k;t)\,e^{|k|\,y} + B(k;t)\,e^{-|k|\,y} \qquad (4.9)$$

where , A(k,t) and B(k;t) are arbitrary constants. By the condition (4.4a) we find B(k;t) = o,hence , the solution is

$$\bar{\phi}^{(3)}(k,y;t) = A(k;t)\,\exp(|k|y) \quad . \qquad (4.1o)$$

The Fourier transform is next applied to the condition (4.7), hence , A(k;t),bedefined by the differential equation ,

$$A_{tt}(k;t) + g|k|A(k;t) = o \ , \qquad (4.11)$$

Finally , the initial conditions must be taken into account, and the solution of the above equation satisfying the initial conditions is

$$A(k;t) = \beta^{(3)}(k)\ \sin[(g|k|)^{\frac{1}{2}}t] \ ,$$

hence, we set from (4.1o)

$$\bar{\phi}^{(3)}(k,y;t) = \beta^{(3)}(k)\ \sin[(g|k|)^{\frac{1}{2}}t]\ e^{|k|y} \ ,$$

the inversion theórem leads immediately to the solution

$$\phi^{(3)}(x,y;t) = (1/2\pi)\int_{-\infty}^{\infty} \beta^{(3)}\ \sin[(g|k|)^{\frac{1}{2}}t]\ e^{|k|y}\ e^{ikx}\ dk \ . \qquad (4.12)$$

The corresponding expression for $\eta^{(3)}(x,t)$ (the third correction of the vertical displacement of the fluid - level atthe point (x,o) and at time t relative the undisturbed surface y=o) is given by

$$g\,\eta^{(3)}(x;t) = \Big[\eta^{(1)}.1/g(\phi^{(1)}_{ty})^2 + g^{-1}\,\phi^{(1)}_{ty}\,\phi^{(2)}_{t} - \tfrac{1}{2}(\eta^{(1)})^2\,\phi^{(1)}_{tyy} - \eta^{(1)}\,\phi^{(2)}_{ty} - \phi^{(3)}_{t}\Big]_{y=o} \ , \qquad (4.13)$$

substituting for the values in the right hand side , we obtain

$$\begin{aligned} g\,\eta^{(3)}(x;t) = &\Big[(1/2\pi)\int_{-\infty}^{\infty} -(|k|/g)^{\frac{1}{2}}\beta^{(1)}(k)\ \cos[(g|k|)^{\frac{1}{2}}t]\ e^{-ikx}\ dk\Big]\ x \\ &\Big[(1/2\pi)\int_{-\infty}^{\infty} |k|(|k|)^{\frac{1}{2}}\beta^{(1)}(k)\ \cos[(g|k|)^{\frac{1}{2}}t]\ e^{-ikx}\ dk\Big]^2 \\ &+g^{-1}\Big[(1/2\pi)\int_{-\infty}^{\infty} |k|(g|k|)^{\frac{1}{2}}\beta^{(1)}\cos[(g|k|)^{\frac{1}{2}}t]\ e^{-ikx}dk\Big]\ x \\ &\Big[(1/2\pi)\int_{-\infty}^{\infty} (g|k|)^{\frac{1}{2}}\beta^{(2)}(k)\ \cos[(g|k|)^{\frac{1}{2}}t]\ e^{-ikx}\ dk\Big] \\ &-\tfrac{1}{2}\Big[(1/2\pi)\int_{-\infty}^{\infty} (g^{-1}|k|)^{\frac{1}{2}}\beta^{(1)}\cos[(g|k|)^{\frac{1}{2}}t]\ e^{-ikx}\ dk\Big]^2 x \\ &\Big[(1/2\pi)\int_{-\infty}^{\infty} (|k|)^2(g|k|)^{\frac{1}{2}}\beta^{(1)}\ \cos[(g|k|)^{\frac{1}{2}}t]\ e^{-ikx}dk\Big] \end{aligned}$$

$$- \left[(1/2\pi) \int_{-\infty}^{\infty} -(g^{-1}|k|)^{\frac{1}{2}} \beta^{(1)}(k) \cos[(g|k|)^{\frac{1}{2}} t]\, e^{-ikx}\, dk \right] \times$$

$$\left[(1/2\pi) \int_{-\infty}^{\infty} |k|(g|k|)^{\frac{1}{2}} \beta^{(2)} \cos[(g|k|)^{\frac{1}{2}} t]\, e^{-ikx}\, dk \right]$$

$$- \left[(1/2\pi) \int_{-\infty}^{\infty} (g|k|)^{\frac{1}{2}} \beta^{(3)}(k) \cos[(g|k|)^{\frac{1}{2}} t]\, e^{-ikx}\, dk \right] \qquad (4.14)$$

Initially, when $t = o$, $\eta^{(3)}(x,o) = o$; hence, from (4.14) we obtain an expression for the function $\beta^{(3)}(k)$ which is defined by

$$(1/2\pi) \int_{-\infty}^{\infty} (g|k|)^{\frac{1}{2}} \beta^{(3)}(k)\, e^{-ikx}\, dk =$$

$$(f^{(1)}(x))/g \left[(1/2\pi) \int_{-\infty}^{\infty} -g|k| \bar{f}^{(1)}(k)\, e^{-ikx}\, dk \right]^2$$

$$+ \left[(1/2\pi) \int_{-\infty}^{\infty} -|k| g \bar{f}^{(1)}\, e^{-ikx} dk \right] \left[(1/2\pi) \int_{-\infty}^{\infty} (g|k|)^{\frac{1}{2}} \beta^{(2)}\, e^{)ikx} dk \right]$$

$$- \tfrac{1}{2} (f^{(1)}(x))^2 \left[(1/2\pi) \int_{-\infty}^{\infty} -gk^2\, \bar{f}^{(1)}(k)\, e^{-ikx}\, dk \right]$$

$$- f^{(1)}(x) \left[(1/2\pi) \int_{-\infty}^{\infty} |k|(g|k|)^{\frac{1}{2}} \beta^{(2)}(k)\, e^{-ikx}\, dk \right]. \qquad (4.15)$$

The Asymptotic Solution

For the first order problem , we have an expression for the potential which is given by

$$\phi^{(1)}(x,y;t) = (Re/2\pi)\int_0^\infty \beta^{(1)}(k)\,\sin[(g\,k)^{\frac{1}{2}}t]\,e^{-ikx+ky}\,dk\ ,$$

the corresponding expression for $\eta^{(1)}(x,y;t)$ is

$$\eta^{(1)}(x;t) = (Re/2\pi)\int_0^\infty -(k/g)^{\frac{1}{2}}\beta^{(1)}(k)\,\cos[(g\,k)^{\frac{1}{2}}t]\,e^{-ikx}\,dk\ .$$

The integral for $\eta^{(1)}(x,t)$ can be evaluated asymptotically for large x and t . To do this we assume that the ratio x/t is fixed. The resulting integral contains just one large parameter , t ,say. Then we apply Kelvin's stationary phase formula , assuming that t is large . To carry out this programe , first we put

$$x = Vt\ , \tag{5.1}$$

where , V is constant .

It is better to write the expression for $\eta^{(1)}$ as

$$\begin{aligned}\eta^{(1)}(x,t) &= (Re/2\pi)\int_0^\infty -(k/g)^{\frac{1}{2}}\beta^{(1)}(k)\,e^{-it(kV-(gk)^{\frac{1}{2}})}\,dk \\ &+ (Re/2\pi)\int_0^\infty -(k/g)^{\frac{1}{2}}\beta^{(1)}(k)\,e^{-ti(kV+(gk)^{\frac{1}{2}})}\,dk \qquad (5.2) \\ &= I_1 + I_2\ .\end{aligned}$$

We begin by discussing the integral I_2 , can be written as

$$I_2 = (Re/2\pi)\int_0^\infty -(k/g)^{\frac{1}{2}}\beta^{(1)}(k)\,e^{-itg_2(k)}\,dk\ ,$$

where , $g_2(k) = (kV + (gk)^{\frac{1}{2}})$, then

$$[d/dk\ (g_2(k))] = [V + \tfrac{1}{2}\ (g/k)^{\frac{1}{2}}] \neq o\ ,$$

this means that the function $g_2(k)$ is free from any stationary point within the range of k (o , ∞) . It is easy to prove by a change of variable in the integral I_2 , say $u = g_2(k)$, and integration by parts , we find the integral behaves like (1/t).

Secondely , the integral $I_1 = (Re/2\pi)\int_o^{\infty} -(k/g)^{\frac{1}{2}} \beta^{(1)}\ e^{-itg_1(k)}\ dk$,

where , $g_1(k) = [kV - (gk)^{\frac{1}{2}}]$,

then, $[d/dk\ (\ g_1(k))] = [V - \tfrac{1}{2}(g/k)^{\frac{1}{2}}]$,

therefore , the function $g_1(k)$ has a stationary point at $k=k_o$ which is satisfied the relation $[d/dk\ (kg)^{\frac{1}{2}}] = V$ and $k_o = g/4V^2$.

Due to Kelvin method , the most contribution comes from a small range of k,for which $g_1(k)$ is stationary,such that $(k_o-\delta, k_o+\delta)$, we consider the function $(-(k/g)^{\frac{1}{2}} \beta^{(1)}(k))$ over this small range of k is constant and equal $(-(k_o/g)^{\frac{1}{2}} \beta^{(1)}(k_o))$. Then by Taylor theory we expand $g_1(k)$ about k_o , we get

$$g_1(k) = g_1(k_o) + \tfrac{1}{2}\ (k - k_o)^2\ g_1''(k_o) + \dots\ ,$$

hence, the contribution of the range ($k_o-\delta$, $k_o+\delta$) to I_1 is given by

$$(Re/2\pi)(-(k_o/g)^{\frac{1}{2}}\beta^{(1)}(k_o))e^{[-ig_1(k_o)t+\frac{1}{4}i\pi sgn(g_1'')]}\left(\frac{2}{t|g_1''(k_o)|}\right)^{\frac{1}{2}}$$

provide $g_1''(k_o) \neq o$, in the present case $g_1''(k_o) = -2V^3/g$.

The contributions from the two ranges (o , $k_o-\delta$)&($k_o+\delta$, ∞), over theseranges the function $g_1(k)$ is monotonic function ,i.e. free from stationary points , then , by integration by parts giving a contribution of order O(1/t)

Hence , the asymptotic expression for $\eta^{(1)}(x;t)$ is

$$\eta^{(1)}(x;t) = (Re/2\pi)(-(k_0/g)^{\frac{1}{2}}\beta^{(1)})\, e^{[-ig_1(k_0)t+\frac{1}{4}i\pi\,\mathrm{sgn}(g_1'')]}\left(\frac{2}{t\,|g_1''(k_0)|}\right)^{\frac{1}{2}} + O(1/t)\ .$$

For the second order problem , we have

$$\phi^{(2)}(x,y;t)=(Re/\pi)\int_0^\infty \beta^{(2)}(k)\,\sin(gk)^{\frac{1}{2}}t\ \ e^{ky}\, e^{-ikx}\,dk$$

where $\beta^{(2)}(k)$ is an even function , the corresponding expression for $\eta^{(2)}(x;t)$ is

$$\eta^{(2)}(x;t) = \left[(Re/2\pi)\int_0^\infty -(k/g)^{\frac{1}{2}}\beta^{(1)}(k)\,\cos[(gk)^{\frac{1}{2}}t]\,e^{-ikx}\,dk\right]\times \left[(Re/2\pi)\int_0^\infty -k(k/g)^{\frac{1}{2}}\beta^{(1)}(k)\,\cos[(gk)^{\frac{1}{2}}t]\,e^{-ikx}\,dk\right] -(Re/\pi)\int_0^\infty (k/g)^{\frac{1}{2}}\,\beta^{(2)}(k)\,\cos[(gk)^{\frac{1}{2}}t]\,e^{-ikx}\,dk\ .$$

Like $\eta^{(1)}$, we evaluate asymptotically for large x and t the expression for $\eta^{(2)}$. Consider $x = Vt$, where V is constant , and rewrite $\eta^{(2)}$ as

$$\begin{aligned}\eta^{(2)}(x;t)= &\left[(Re/2\pi)\int_0^\infty -(k/g)^{\frac{1}{2}}\beta^{(1)}\ e^{-it(kV-(gk)^{\frac{1}{2}})}\,dk\right.\\ &\left.+(Re/2\pi)\int_0^\infty -(k/g)^{\frac{1}{2}}\beta^{(1)}\ e^{-it(kV+(gk)^{\frac{1}{2}})}\,dk\right]\times\\ &\left[(Re/2\pi)\int_0^\infty -k(k/g)^{\frac{1}{2}}\beta^{(1)}\ e^{-it(kV-(gk)^{\frac{1}{2}})}\,dk\right.\\ &\left.+(Re/2\pi)\int_0^\infty -k(k/g)^{\frac{1}{2}}\beta^{(1)}\ e^{-it(kV+(gk)^{\frac{1}{2}})}\,dk\right]\\ &-(Re/2\pi)\int_0^\infty (k/g)^{\frac{1}{2}}\beta^{(2)}\ e^{-it(kV-(gk)^{\frac{1}{2}})}\,dk\\ &+(Re/2\pi)\int_0^\infty -(k/g)^{\frac{1}{2}}\beta^{(2)}\ e^{-it(kV+(gk)^{\frac{1}{2}})}\,dk\end{aligned}$$

as t becomes large , we have

(i) integrals which are free from stationary points , it is easy to show by change of variable , and integration by parts that they behave like 1/t .

(ii) integrals with stationary points , by the principal of stationary phase we get their asymptotic expressions .

Hence , the asymptotic expression for $\eta^{(2)}(x;t)$ is

$$\eta^{(2)}(x;t) \simeq \left[(Re/2\pi)\left[-(k/g)^{\frac{1}{2}}_{k=k_o}\beta^{(1)}\right]e^{[-itg_1(k_o)+\frac{1}{4}i\pi]}\left(\frac{2\pi}{t\,|g_1''(k_o)|}\right)^{\frac{1}{2}}+O(1/t)\right] x$$
$$\left[(Re/2\pi)\left[-k_o(k_o/g)^{\frac{1}{2}}\beta^{(1)}\right]e^{[-ig_1(k_o)t+\frac{1}{4}i\pi]}\left(\frac{2\pi}{t\,|g_1''(k_o)|}\right)^{\frac{1}{2}}+O(1/t)\right]$$
$$-(Re/2\pi)\left[(k_o/g)^{\frac{1}{2}}\beta^{(2)}\right]e^{[-itg_1(k_o)+\frac{1}{4}i\pi]}\left(\frac{2\pi}{t\,|g_1''(k_o)|}\right)^{\frac{1}{2}}+O(1/t) .$$

where , $g_1(k) = [kV - (gk)^{\frac{1}{2}}]$,and , k_o is defined by the relation $[d/dk\,(g_1(k))] = o$ or $V = [d/dk(gk)^{\frac{1}{2}}]$.

The third order problem given the solution

$$\phi^{(3)}(x,y;t) = (Re/\pi)\int_o^\infty \beta^{(3)}(k)\,\sin[(gk)^{\frac{1}{2}}t]\,e^{ky-ikx}\,dk$$

where the function $\beta^{(3)}(k)$ is an even function . The corresponding expression for $\eta^{(3)}(x;t)$ is

$$g\eta^{(3)}(x;t) = \left[(Re/\pi)\int_o^\infty -(k/g)^{\frac{1}{2}}\beta^{(1)}\cos[(gk)^{\frac{1}{2}}t]\,e^{-ikx}\,dk\right] x$$
$$\left[(Re/\pi)\int_o^\infty k(k)^{\frac{1}{2}}\beta^{(1)}(k)\cos[(gk)^{\frac{1}{2}}t]\,e^{-ikx}\,dk\right]$$
$$+1/g\left[(Re/\pi)\int_o^\infty k(gk)^{\frac{1}{2}}\beta^{(1)}(k)\cos[(gk)^{\frac{1}{2}}t]\,e^{-ikx}\,dk\right] x$$
$$\left[(Re/\pi)\int_o^\infty (gk)^{\frac{1}{2}}\beta^{(2)}(k)\cos[(gk)^{\frac{1}{2}}t]\,e^{-ikx}\,dk\right]$$
$$-\tfrac{1}{2}\left[(Re/\pi)\int_o^\infty -(k/g)^{\frac{1}{2}}\beta^{(1)}(k)\cos[(gk)^{\frac{1}{2}}t]\,e^{-ikx}\,dk\right] x$$

$$\left[(\mathrm{Re}/\pi)\int_0^\infty k^2(gk)^{\frac{1}{2}}\beta^{(1)}(k)\cos[(gk)^{\frac{1}{2}}t]\,e^{-ikx}\,dk\right]$$

$$-\left[(\mathrm{Re}/\pi)\int_0^\infty -(k/g)^{\frac{1}{2}}\beta^{(1)}(k)\cos[(kg)^{\frac{1}{2}}t]\,e^{-ikx}\,dk\right] \times$$

$$\left[(\mathrm{Re}/\pi)\int_0^\infty k(gk)^{\frac{1}{2}}\beta^{(2)}(k)\cos[(gk)^{\frac{1}{2}}t]\,e^{-ikx}\,dk\right]$$

$$-(\mathrm{Re}/\pi)\int_0^\infty (gk)^{\frac{1}{2}}\beta^{(3)}(k)\cos[(gk)^{\frac{1}{2}}t]\,e^{-ikx}\,dk \;.$$

To obtain its asymptotic expression when t becomes large , it is better to consider x/t is fixed , i.e. x = Vt where V is constant . Before the evaluation the asymptotic expression for $\eta^{(3)}$, it is better to rewrite $\eta^{(3)}$ in a suitable form as

$$\begin{aligned}
g\eta^{(3)}(x;t) = &\left[(\mathrm{Re}/2\pi)\int_0^\infty -(k/g)^{\frac{1}{2}}\beta^{(1)}(k)\,e^{[-it(kV-(gk)^{\frac{1}{2}})]}\,dk\right. + \\
&\left.+(\mathrm{Re}/2\pi)\int_0^\infty -(k/g)^{\frac{1}{2}}\beta^{(1)}(k)\,e^{[(kV+(gk)^{\frac{1}{2}})(-it)]}\,dk\right] \times \\
&\left[(\mathrm{Re}/2\pi)\int_0^\infty (k)^{3/2}\beta^{(1)}(k)\,e^{[-it(kV-(gk)^{\frac{1}{2}})]}\,dk\right. + \\
&\left.+(\mathrm{Re}/2\pi)\int_0^\infty (k)^{3/2}\beta^{(1)}(k)\,e^{[-i(kV+(gk)^{\frac{1}{2}})t]}\,dk\right]^2 \\
&+1/g\left[(\mathrm{Re}/2\pi)\int_0^\infty k(gk)^{\frac{1}{2}}\beta^{(1)}(k)\,e^{[-it(kV-(gk)^{\frac{1}{2}})]}\,dk\right. + \\
&\left.+(\mathrm{Re}/2\pi)\int_0^\infty k(gk)^{\frac{1}{2}}\beta^{(1)}(k)\,e^{[-it(kV+(gk)^{\frac{1}{2}})]}\,dk\right] \times \\
&-\tfrac{1}{2}\left[(\mathrm{Re}/2\pi)\int_0^\infty -(k/g)^{\frac{1}{2}}\beta^{(1)}(k)\,e^{[-it(kV-(gk)^{\frac{1}{2}})]}\,dk\right. + \\
&\left.+(\mathrm{Re}/2\pi)\int_0^\infty -(k/g)^{\frac{1}{2}}\beta^{(1)}(k)\,e^{[-it(kV+(gk)^{\frac{1}{2}})]}\,dk\right]^2 \times \\
&\left[(\mathrm{Re}/2\pi)\int_0^\infty k^2(gk)^{\frac{1}{2}}\beta^{(1)}(k)\,e^{[-it(kV-(gk)^{\frac{1}{2}})]}\,dk\right. + \\
&\left.+(\mathrm{Re}/2\pi)\int_0^\infty k^2(gk)^{\frac{1}{2}}\beta^{(1)}(k)\,e^{[-it(kV+(gk)^{\frac{1}{2}})]}\,dk\right] \\
&-\left[(\mathrm{Re}/2\pi)\int_0^\infty -(k/g)^{\frac{1}{2}}\beta^{(1)}(k)\,e^{[-it(kV-(gk)^{\frac{1}{2}})]}\,dk\right. + \\
&\left.+(\mathrm{Re}/2\pi)\int_0^\infty -(k/g)^{\frac{1}{2}}\beta^{(1)}(k)\,e^{[it(kV+(gk)^{\frac{1}{2}})]}\,dk\right] \times
\end{aligned}$$

$$\left[(Re/2\pi)\int_0^\infty k(gk)^{\frac{1}{2}}\beta^{(2)}(k)\, e^{[-it(kV-(gk)^{\frac{1}{2}})]}\,dk\right. +$$
$$+ \left.(Re/2\pi)\int_0^\infty k(gk)^{\frac{1}{2}}\beta^{(2)}(k)\, e^{[-it(kV+(gk)^{\frac{1}{2}})]}\,dk\right]$$
$$-\left[(Re/2\pi)\int_0^\infty (gk)^{\frac{1}{2}}\beta^{(3)}(k)\, e^{[-it(kV-(gk)^{\frac{1}{2}})]}\,dk\right]+$$
$$+\left[(Re/2\pi)\int_0^\infty (gk)^{\frac{1}{2}}\beta^{(3)}(k)\, e^{[-it(kV+(gk)^{\frac{1}{2}})]}\,dk\right].$$

as t becomes large , we have

(i) All integrals without stationary points must behave like 1/t ,

(ii) All integrals with stationary points , we must use the method of stationary phase to evaluate their values.

Hence , the asymptotic expression for $\eta^{(3)}$ is

$$\eta^{(3)}(x;t) = -\left[(Re/2\pi)(k_0/g)^{\frac{1}{2}}\beta^{(1)}\, e^{[-itg_1(k_0)+\frac{1}{4}i\pi)]}\left(\frac{2\pi}{t|g''_1(k_0)|}\right)^{\frac{1}{2}} + O(1/t)\right] \times$$
$$\left[(Re/2\pi)(k_0)^{3/2}\beta^{(1)}\, e^{[-itg_1(k_0)+\frac{1}{4}i\pi)]}\left(\frac{2\pi}{t|g''_1(k_0)|}\right)^{\frac{1}{2}} + O(1/t)\right]^2$$
$$+1/g\left[(Re/2\pi)k_0(gk_0)^{\frac{1}{2}}\beta^{(1)}\, e^{[-itg_1(k_0)+\frac{1}{4}i\pi]}\left(\frac{2\pi}{t|g''_1(k_0)|}\right)^{\frac{1}{2}} + O(1/t)\right] \times$$
$$\left[(Re/2\pi)(gk_0)^{\frac{1}{2}}\beta^{(2)}\, e^{[itg_1(k_0)+\frac{1}{4}i\pi)]}\left(\frac{2\pi}{t|g''_1(k_0)|}\right)^{\frac{1}{2}} + O(1/t)\right]$$
$$-\frac{1}{2}\left[(Re/2\pi).-(k_0/g)^{\frac{1}{2}}\beta^{(1)}\, e^{[-itg_1(k_0)+\frac{1}{4}i\pi]}\left(\frac{2\pi}{t|g''_1(k_0)|}\right)^{\frac{1}{2}} + O(1/t)\right]^2 \times$$
$$\left[(Re/2\pi)k_0^2(gk_0)^{\frac{1}{2}}\beta^{(1)}\, e^{[-itg_1(k_0)+\frac{1}{4}i\pi)]}\left(\frac{2\pi}{t|g''_1(k_0)|}\right)^{\frac{1}{2}} + O(1/t)\right]$$
$$+\left[(Re/2\pi)k_0(gk_0)^{\frac{1}{2}}\beta^{(2)}\, e^{[-itg_1(k_0)+\frac{1}{4}i\pi]}\left(\frac{2\pi}{t|g''_1(k_0)|}\right)^{\frac{1}{2}} + O(1/t)\right] \times$$
$$\left[(Re/2\pi)\ (k_0/g)^{\frac{1}{2}}\beta^{(1)}\, e^{[-itg_1(k_0)+\frac{1}{4}i\pi)]}\left(\frac{2\pi}{t|g''_1(k_0)|}\right)^{\frac{1}{2}} + O(1/t)\right]$$
$$-\ (Re/2\pi)(gk_0)^{\frac{1}{2}}\beta^{(3)}\, e^{[-itg_1(k_0)+\frac{1}{4}i\pi]}\left(\frac{2\pi}{t|g''_1(k_0)|}\right)^{\frac{1}{2}} + O(1/t)$$

where , $g_1(k) = [kV - (gk)^{\frac{1}{2}}]$, and the stationary point $k = k_o$, be defined by $[d/dk\,(kV - (gk)^{\frac{1}{2}})] = o$.

As we known , the velocity potential and the wave profile are given by

$$\phi(x,y;t) = \epsilon\,\phi^{(1)}(x,y;t) + \epsilon^2\,\phi^{(2)}(x,y;t) + \epsilon^3\,\phi^{(3)}(x,y;t) + \ldots$$

and

$$\eta(x;t) = \epsilon\,\eta^{(1)}(x;t) + \epsilon^2\,\eta^{(2)}(x;t) + \epsilon^3\,\eta^{(3)}(x;t) + \ldots$$

Substituting for $\eta^{(1)}(x;t)$, $\eta^{(2)}(x;t)$ and $\eta^{(3)}(x;t)$ from their asymptotic expressions , then , to the third order the asymptotic expression for the wave profile can written as

$$\eta(x;t) \simeq (Re/2\pi)\left[-(k_o/g)^{\frac{1}{2}}\left[\epsilon\beta^{(1)}(k_o) + \epsilon^2\beta^{(2)}(k_o) + \epsilon^3\beta^{(3)}(k_o) + \ldots\right] e^{[-itg_1(k_o)+\frac{1}{4}i\pi]}\left(\frac{2}{t|g''_1(k_o)|}\right)^{\frac{1}{2}}\right]$$

+ higher order terms in (1/t) .

Evaluation The Values Of The Arbitrary Functions $\beta^{(1)}$, $\beta^{(2)}$ and $\beta^{(3)}$

(1) The function $\beta^{(1)}(k)$:

The function $\beta^{(1)}(k)$ is defined by the relation

$$-(k/g)\,\beta^{(1)}(k) = \int_{-\infty}^{\infty} f^{(1)}(x)\; e^{ikx}\, dx \qquad (6.1)$$

$$= \bar{f}^{(1)}(k) \quad ,$$

where the function $f^{(1)}(x)$ representing the initial elevation of the free surface & it is given by

$$f^{(1)}(x) = 1/(\, b^2 + x^2 \,) \quad ,$$

then , $$-(k/g)^{\frac{1}{2}}\,\beta^{(1)}(k) = \int_{-\infty}^{\infty} (1/(b^2 + x^2)\,)\; e^{ikx}\, dk$$

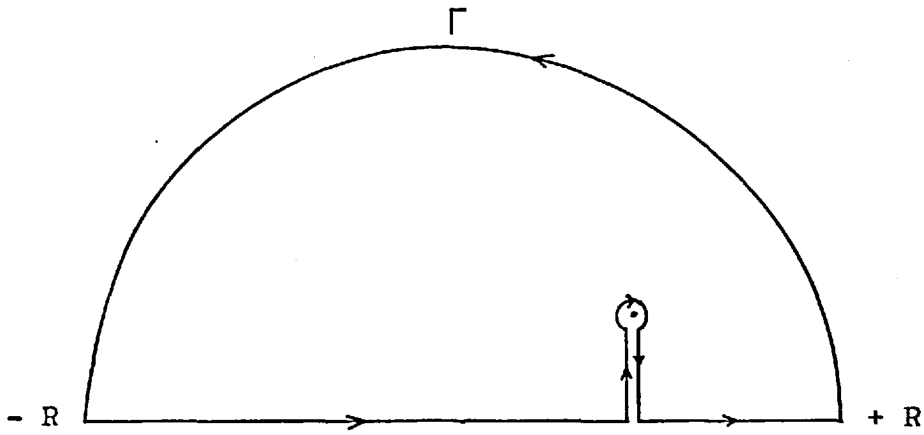
$$= \int_{-\infty}^{\infty} \left(\frac{e^{ikx}}{(x + ib)(x - ib)}\right) dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{g(x)}{(x - ib)}\right) dx \quad ,$$

to evaluate this integral , we try to evaluate

$$\int_L \frac{g(z)}{(z - ib)}\, dz \; ,$$

in the z - plane round the contour L as in the figure , where the function $g(z) = e^{ikz}/(z+ib)$.



z - plane

Then , we have

$$\int_L \cdots = \lim_{R\to\infty}\int_{-R}^{R} \cdots + \int_{\frown} \cdots + \int_{\Gamma} \cdots = 0$$

$$\therefore \tfrac{1}{2}\int_{-\infty}^{\infty} g(x)/(x-ib)\ dx = -\int_{\frown} g(z)/(z-ib)\ dz$$

$$= (\pi/b)\ e^{-|k|b}$$

$$\text{i.e.,}\quad \beta^{(1)}(k) = -\left(g/|k|\right)^{\frac{1}{2}}(\pi/b)\ \exp(|k|b) \qquad (6.2)$$

(2) <u>The function $\beta^{(2)}(k)$</u> :

The function $\beta^{(2)}$ is defined by the relation

$$(1/2\pi)\int_{-\infty}^{\infty}(g|k|)^{\frac{1}{2}}\beta^{(2)}(k)\ e^{-ikx}dk = \left[(1/2\pi)\int_{-\infty}^{\infty}(|k|/g)^{\frac{1}{2}}\beta^{(1)}\ e^{-ikx}\ dk\right] x$$

$$\left[(1/2\pi)\int_{-\infty}^{\infty}|k|\,(g|k|)^{\frac{1}{2}}\beta^{(1)}\ e^{-ikx}\ dk\right] \qquad (6.3)$$

sustituting (6.2) in (6.3) , we obtain

$$(1/2\pi)\int_{-\infty}^{\infty}(g|k|)^{\frac{1}{2}}\beta^{(2)}\ e^{-ikx}dk = (g\pi^2/b^2)\left[(1/2\pi)\int_{-\infty}^{\infty}e^{-|k|b}\ e^{-ikx}\ dk\right] x$$

$$\left[(1/2\pi)\int_{-\infty}^{\infty}|k|\ \exp(-ikx-|k|b)\ dk\right]$$

$$= -8ig\pi^2 \; (x/(b^2+x^2)^3) ,$$

hence, $$(g|k|)^{\frac{1}{2}}\beta^{(2)}(k) = -i8g\pi^2 \int_{-\infty}^{\infty} g(x)/(x-ib)^3 \; dx \qquad (6.4)$$

where , $g(x) = xe^{ikx}/(x+ib)^3$.

To evaluate the integral (6.4) , we try to integrate the integral

$$\int_L g(z)/(z-ib)^2 \; dz ,$$

round the contour L in the z - plane , we obtain

$$\int_{-\infty}^{\infty} g(x)/(x-ib)^3 \; dx =(2i\pi/3!) \; e^{-b|k|} (\; (|k|)^2/8b^2 + (|k|)/b^3 \;),$$

hence, $\beta^{(2)}(k)$ is given by the relation

$$(g|k|)^{\frac{1}{2}} \beta^{(2)}(k) =(\pi^3 g/3) \; e^{-b|k|} (\; (|k|)^2/b^2 + (|k|)/b^3 \;) . \qquad (6.5)$$

(3) The function $\beta^{(3)}(k)$:

The function $\beta^{(3)}$ is given by the relation

$$\begin{aligned}
(1/2\pi)\int_{-\infty}^{\infty} (g|k|)^{\frac{1}{2}} \beta^{(3)} \; e^{-ikx} \; dk = & \; 1/g \left[(1/2\pi)\int_{-\infty}^{\infty} -(|k|/g)^{\frac{1}{2}} \beta^{(1)} \; e^{-ikx} \; dk\right]^2 \times \\
& \left[(1/2\pi) \int_{-\infty}^{\infty} (|k|)^{3/2} g^{\frac{1}{2}} \beta^{(1)} \; e^{-ikx} \; dk\right] \\
& + 1/g \left[(1/2\pi) \int_{-\infty}^{\infty} (|k|)^{3/2} g^{\frac{1}{2}} \beta^{(1)} e^{-ikx} \; dk\right] \times \\
& \left[(1/2\pi)\int_{-\infty}^{\infty} (g|k|)^{\frac{1}{2}} \beta^{(2)} \; e^{-ikx} \; dk\right] \\
& -\tfrac{1}{2} \left[(1/2\pi)\int_{-\infty}^{\infty} -(|k|/g)^{\frac{1}{2}} \beta^{(1)} \; e^{-ikx} \; dk\right]^2 \times \\
& \left[(1/2\pi)\int_{-\infty}^{\infty} k^2 (g|k|)^{\frac{1}{2}} \beta^{(1)} \; e^{-ikx} \; dk\right] \\
& -\tfrac{1}{2} \left[(1/2\pi) \int_{-\infty}^{\infty} -(|k|/g)^{\frac{1}{2}} \beta^{(1)} \; e^{-ikx} \; dk\right] \times \\
& \left[(1/2\pi)\int_{-\infty}^{\infty} (|k|)^{3/2} g^{\frac{1}{2}} \beta^{(2)} \; e^{-ikx} \; dk\right].
\end{aligned} \qquad (6.6)$$

Substituting (6.2) and (6.5) in (6.6) , we obtain

$$(1/2\pi)\int_{-\infty}^{\infty}(g|k|)^{\frac{1}{2}}\beta^{(3)}(k)\ e^{-ikx}\ dk =$$

$$(g\pi^3/b^3)\left[(1/2\pi)\int_{-\infty}^{\infty} e^{-|k|b\ -ikx}\ dk\right]\left[(1/2\pi)\int_{-\infty}^{\infty} k\ e^{-|k|b\ -ikx}\ dk\right]$$
$$-(g\pi^4/3b^3)\left[(1/2\pi)\int_{-\infty}^{\infty} k\ e^{-|k|b-ikx}\ dk\right]\left[(1/2\pi)\int_{-\infty}^{\infty} k^2\ e^{-b|k|-ikx}\ dk\right]$$
$$-(g\pi^4/3b^4)\left[(1/2\pi)\int_{-\infty}^{\infty} k\ e^{-b|k|-ikx}\ dk\right]\left[(1/2\pi)\int_{-\infty}^{\infty} |k| e^{-b|k|-ikx}\ dk\right]$$
$$+(g\pi^3/2b^3)\left[(1/2\pi)\int_{-\infty}^{\infty} e^{-b|k|-ikx}\ dk\right]\left[(1/2\pi)\int_{-\infty}^{\infty} k^2\ e^{-b|k|-ikx}\ dk\right]$$
$$-(g\pi^4/3b^3)\left[(1/2\pi)\int_{-\infty}^{\infty} e^{-b|k|-ikx}\ dk\right]\left[(1/2\pi)\int_{-\infty}^{\infty} (|k|)^3\ e^{-b|k|-ikx}\ dk\right]$$
$$-(g\pi^4/3b^4)\left[(1/2\pi)\int_{-\infty}^{\infty} e^{-b|k|-ikx}\ dk\right]\left[(1/2\pi)\int_{-\infty}^{\infty} (|k|)^2\ e^{-b|k|-ikx}\ dk\right]$$

evaluating the different integrals in the right hand side,we get

$$(1/2\pi)\int_{-\infty}^{\infty}(g|k|)^{\frac{1}{2}}\beta^{(3)}(k)\ e^{-ikx}\ dk = \frac{A(b^2-5x^2)}{(b^2+x^2)^4} + \frac{B(b^2-7x^2)+C(-7b^2+9x^3)}{(b^2+x^2)^5} \tag{6.7}$$

where , $A = -\ 8g\pi^4/3b^2$, $B = 8g\pi^3$,

and $C = -16ig\pi^4/3b$.

hence , we have

$$(g|k|)^{\frac{1}{2}}\beta^{(3)}(k) = \int_{-\infty}^{\infty} F(x)/(b^2+x^2)^4\ dx + \int_{-\infty}^{\infty} G(x)/(b^2+x^2)^5\ dx \tag{6.8}$$

where , $F(x) = A\ (\ b^2 - 5x^2\ )\ e^{ikx}$,

and $G(x) = (\ B(\ b^2 - 7x^2\ ) + C(\ -7b^2x + 9x^3\ ))\ e^{ikx}$.

We can evaluate these integrals by integrating $\int_L F_1(z)/(z-ib)^4\,dz$ and $\int_L G_1(z)/(z-ib)^5\,dz$ round the contour L in the z - plane , where ,

$$F_1(z) = \left[\frac{A(b^2 - 5z^2)\,e^{ikz}}{(z + ib)^4}\right],$$

and
$$G_1(z) = \left[\frac{(B(b^2 - 7z^2) + C(-7b^2 z + 9z^3))e^{ikz}}{(z + ib)^5}\right],$$

hence , the expression for $\beta^{(3)}$ is

$$\begin{aligned}(g|k|)^{\frac{1}{2}}\beta^{(3)}(k) = {} & \left(\frac{A\,e^{-b|k|}}{8\,b^2}\right)\left((|k|)^3 + (|k|)^2/b\right) \\ & +\left(\frac{B\,e^{-b|k|}}{48\,b^2}\right)\left((|k|)^4 + 3(|k|)^2/b + 3(|k|)^2/b^2\right) \\ & +\left(\frac{bC\,e^{-b|k|}}{48\,b^3}\right)\left(2(|k|)^4 + 3(|k|)^3/b + 3(|k|)^2/b^2\right. \\ & \left. + 3(|k|)/b^3\right) \qquad (6.9)\end{aligned}$$

In fact , the values of the arbitrary functions $\beta^{(1)}(k)$, $\beta^{(2)}(k)$ and $\beta^{(3)}(k)$ are depend on the function which describe the the initial displacement of the free surface , for example

(1) The initial displacement is given by

$$f(x) = 1/(b^2+x^2) ,$$

the corresponding values of $\beta^{(1)}$, $\beta^{(2)}$ and $\beta^{(3)}$ are

$$(|k|/g)^{\frac{1}{2}} \beta^{(1)}(k) = - (\pi/b)\ e^{-b|k|} ,$$

$$(g|k|)^{\frac{1}{2}} \beta^{(2)}(k) = (g\pi^3/3)\ e^{-b|k|} ((|k|)^2/b^2 + (|k|)/b^3),$$

and
$$(g|k|)^{\frac{1}{2}} \beta^{(3)}(k)=(\frac{Ae^{-b|k|}}{8\ b^2})((|k|)^3 + (|k|)^2/b) + (\frac{Be^{-b|k|}}{48\ b^2})((|k|)^4+ 3(|k|)^3/b + 3(|k|)^2/b^2) + (bCe^{-b|k|})(2(|k|)^4+2(|k|)^3/b^2+3(|k|)^2/b^2+3(|k|)/b$$

where , $A = -8g\pi^4/3b^2$, $B = 8g\pi^3$ and $C = -16g\pi^4/3b$.

(2) The initial displacement is given by

$$f(x) = ab^2/(b^2+x^2) ,$$

the corresponding values of $\beta^{(1)}$, $\beta^{(2)}$ and $\beta^{(3)}$ are

$$(|k|/g)^{\frac{1}{2}} \beta^{(1)} = ab\pi e^{-b|k|} ,$$

$$(g|k|)^{\frac{1}{2}} \beta^{(2)} = 1/3\ a^2b^2g\pi^3\ e^{-b|k|} ((|k|)^2 + (|k|)/b) ,$$

and
$$(g|k|)^{\frac{1}{2}} \beta^{(3)} = (\frac{Ae^{-b|k|}}{8\ b^2})((|k|)^3 + (|k|)^2/b) + (\frac{Be^{-b|k|}}{48\ b^3})((|k|)^4+3(|k|)^3/b +3(|k|)^2/b^2) + (\frac{bCe^{-b|k|}}{48\ b^3})(2(|k|)^4+3(|k|)^3/b+3(|k|)^2/b^2+3(|k|)/b^3)$$

where , $A = (-8ab\pi)(1/3\ a^2b^3g\pi^3)$, $B = 8gb^3(ab\pi)^3$

and , $C = -16/3(ab^3\pi)(a^2b^2g\pi^3)$.

Some Numerical Examples

The asymptotic expression for $\eta(x,t)$ - the wave profile-is

$$\eta(x,t)=(\mathrm{Re}/2\pi)\left[-(k_o/g)^{\frac{1}{2}}(\epsilon\beta^{(1)}+\epsilon^2\beta^{(2)}+\epsilon^3\beta^{(3)}+\dots)\left(\frac{2\pi}{t|g_1''(k_o)|}\right)^{\frac{1}{2}}\exp(-itg_1(k_o)-\tfrac{1}{4}i\pi)\right]$$

$$+\ O(1/t).$$

$$\text{or } \eta(x,t)=(-1/2\pi)\left[(k_o/g)^{\frac{1}{2}}(\epsilon\beta^{(1)}+\epsilon^2\beta^{(2)}+\epsilon^3\beta^{(3)}+\dots)\left(\frac{2\pi}{t|g_1''(k_o)|}\right)^{\frac{1}{2}}\cos(xk_o-(gk_o)^{\frac{1}{2}})t+\tfrac{1}{4}\pi)\right]$$

$$+\ O(1/t).$$

We are examining $\eta(x;t)$ from standpoint of an observer moving with group velocity appropriate to the wave number $k = k_o$,inthis case the gross outline has an amplitude which is given by

$$-(1/2\pi)\left[(k_o/g)^{\frac{1}{2}}(\epsilon\beta^{(1)}(k_o)+\epsilon^2\beta^{(2)}(k_o)+\epsilon^3\beta^{(3)}(k_o)+\dots)\left(\frac{2\pi}{t|g''_1(k_o)|}\right)^{\frac{1}{2}}\right]$$

i.e. it is $t^{-\frac{1}{2}}$ times a power series in the small parameter ϵ with a constant coefficients , this means that the power series must be exist (convergence) , this what had proved by Levi-Civita's(1925).

<u>Numerical Examples:</u>

(1) $f(x) = 1/(b^2+x^2)$, $k_o = 1$ and $b = \frac{1}{2}$

$\therefore$ $f(x) = 4/(1+4x^2)$,

$$(k_o/g)^{\frac{1}{2}} \beta^{(1)}(k_o) = -3.809$$

$$(k_o/g)^{\frac{1}{2}} \beta^{(2)}(k_o) = +75.11$$

$$(k_o/g)^{\frac{1}{2}} \beta^{(3)}(k_o) = -11671.819$$

The corresponding expression for $\eta(x;t)$ is

$$\eta(x;t)=(1/2\pi)(3.809\epsilon -75.11\epsilon^2+11671.819\epsilon^3+\ldots)(\pi/(x/t)t)^{\frac{1}{2}}\cos(x-g^{\frac{1}{2}}t+\tfrac{1}{4}\pi).$$

(2) $f(x) = 1/(b^2+x^2)$, $k_o = 1$ and $b = 1$

$\therefore$ $f(x) = 1/(1+x^2)$,

$$(k_o/g)^{\frac{1}{2}} \beta^{(1)}(k_o) = -1.155$$

$$(k_o/g)^{\frac{1}{2}} \beta^{(2)}(k_o) = +7.5928$$

$$(k_o/g)^{\frac{1}{2}} \beta^{(3)}(k_o) = -170.386$$

The corresponding expression for $\eta(x;t)$ is

$$\eta(x;t)=(1/2\pi)(1.155\epsilon -7.5928\epsilon^2+170.386\epsilon^3+\ldots)(\pi/(x/t)t)^{\frac{1}{2}}\cos(x-g^{\frac{1}{2}}t+\tfrac{1}{4}\pi).$$

(3) $f(x) = 1/(b^2+x^2)$, $k_o = 1$ and $b = 2$

$\therefore$ $f(x) = 1/(4+x^2)$,

$$(k_o/g)^{\frac{1}{2}} \beta^{(1)}(k_o) = -.21248$$

$$(k_o/g)^{\frac{1}{2}}\ \beta^{(2)}(k_o) = .52375$$

$$(k_o/g)^{\frac{1}{2}}\ \beta^{(3)}(k_o) = -3.05386$$

The corresponding expression for $\eta(x;t)$

$$\eta(x;t)=(1/2\pi)(.21248\epsilon - 0.52375\ \epsilon^2+3.05386\ \epsilon^3+\dots)(2\pi/(x/t)t)^{\frac{1}{2}} \cos(x-g^{\frac{1}{2}}t+\tfrac{1}{4}\pi)\ .$$

(4) $f(x) = ab^2/(b^2+x^2)$

$a = 1$, $b = 1$ and $k_o = 1$

$\therefore \quad f(x) = 1/(1+x^2)$

$$(k_o/g)^{\frac{1}{2}}\ \beta^{(1)}(k_o) = -1.155$$

$$(k_o/g)^{\frac{1}{2}}\ \beta^{(2)}(k_o) = 7.5928$$

$$(k_o/g)^{\frac{1}{2}}\ \beta^{(3)}(k_o) = -170.386$$

The corresponding expression for $\eta(x;t)$ is

$$\eta(x;t)=(1/2\pi)(1.155\epsilon\ \_7.5928\ \epsilon^2+170.388\ \epsilon^3+\dots)(2\pi/(x/t)t)^{\frac{1}{2}}\cos(x-g^{\frac{1}{2}}t+\tfrac{1}{4}\pi).$$

(5) $f(x) = ab^2/(b^2+x^2)$

$a = 1$, $b = 2$ and $k_o = 1$

$\therefore \quad f(x) = 4/4+x^2$

$$(k_o/g)^{\frac{1}{2}}\ \beta^{(1)}(k_o) = -0.8499$$

$$(k_o/g)^{\frac{1}{2}}\ \beta^{(2)}(k_o) = 8.38$$

$$(k_o/g)^{\frac{1}{2}}\ \beta^{(3)}(k_o) = -195.449$$

The corresponding expression for $\eta(x;t)$ is

$$\eta(x;t)=(1/2\pi)(.8499\epsilon - 8.38\epsilon^2 + 195.45\epsilon^3 + \ldots)(2\pi/(x/t)t)^{\frac{1}{2}}\cos(x - g^{\frac{1}{2}}t + \tfrac{1}{4}\pi) \ .$$

(6) $$f(x) = ab^2/(b^2+x^2)$$

$$a = 1 \quad , \quad b = \tfrac{1}{2} \quad \text{and} \quad k_o = 1$$

$$\therefore \quad f(x) = 1/(1+4x^2)$$

$$(k_o/g)^{\frac{1}{2}}\,\beta^{(1)}(k_o) = -0.9522$$

$$(k_o/g)^{\frac{1}{2}}\,\beta^{(2)}(k_o) = 4.6944$$

$$(k_o/g)^{\frac{1}{2}}\,\beta^{(3)}(k_o) = -136.0871$$

The corresponding expression for $\eta(x;t)$ is

$$\eta(x;t)=(1/2\pi)(0.952\epsilon - 4.6944\epsilon^2 + 136.087\epsilon^3 + \ldots)(2\pi/(x/t)t)^{\frac{1}{2}}\cos(x - g^{\frac{1}{2}}t + \tfrac{1}{4}\pi).$$

Historical Note :

The form of periodic waves progressing over deep water without change of type was determined by Stokes (1847) to a high degree of approximation . Later Stokes (1880) added a supplement describing a different procedure . Rayleigh turned to the problem several times (1876 , 1911 , 1915 , 1917) and introduced still another method of approximation . It should be noted that both Stokes's second method and Rayleigh's method are limited to two-dimensional irrototional progressive waves.

In all such computations there is the tacit assumption that there exists an " exact solution" which is being approximated and which can be approached more and more closely by pursuing the selected method of approximation . Unfortunately , it is seldom that one is able to prove the existance of an exact solution or of convergence of the method of approximation , and , in fact , Burnside (1916) cast doubt upon the usefulness of the Stokes-Rayleigh type of approximation of periodic progressive waves of permanent type. Burnside's objection was later met by Nekrasov's (1921 ,1922 ,1951) , Levi - Civita's (1925) and Struik's (1926) proofs of the existence of such waves for both infinite and finite depth .

PART III

SHEAR WAVES

A Model of Two Layers of inviscid fluid

We consider a two layer model in which a homogeneous, incompressible, inviscid fluid is flowing between infinite parallel plates at z=0 and $z=h_2$, in a system of parallel planes and the mean velocity is described by

$$U = U(z) \quad , \quad W = o \quad . \qquad (1)$$

We consider only two space dimensions x,z . The axis x taken toward the right coinciding with the bottom (z = o) and the axis of z being directed upwards. We note the mean velocity is every where continuous on the other hand [dU(z)/dz], the rate of shear has a discontinuity at the interface of the two layers .

The first layer(the lower layer) lies between

$$z = o \quad \text{and} \quad z = h_1 \; ,$$

through this layer the mean velocity is given by

$$U(z) = U_o + \alpha_1 z \; , \qquad (2)$$

where U_o is the mean velocity at the bottom (z = o) and $\alpha_1 = dU(z)/dz$; always positive .

The second layer (the upper layer) lies between

$$z = h_1 \quad \text{and} \quad z = h_2 \; ,$$

the mean velocity through this layer is given by

$$U_2 = U_1 + \alpha_2 (z - h_1) \tag{3}$$

where U_1 is the mean velocity at the inter-face ($z=h_1$) and $\alpha_2 = [d/dz\,(U(z))]$ is negative$(= -\alpha_3)$, where α_3 is positive.

In the present problem , we consider the case in which U(z) is a linear function in z ,i.e. α_1 (the rate of shear in the lower layer) is constant and positive and α_2 (the rate of shear in the upper layer) is constant and negative .

The discontinuity in the rate of shear shown in the figure 1

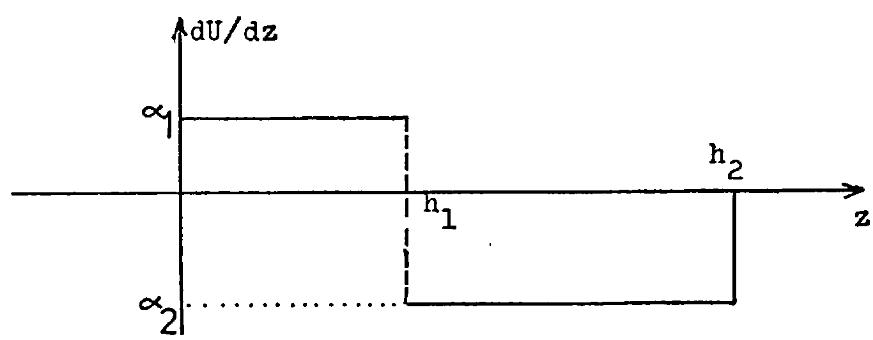


fig.1

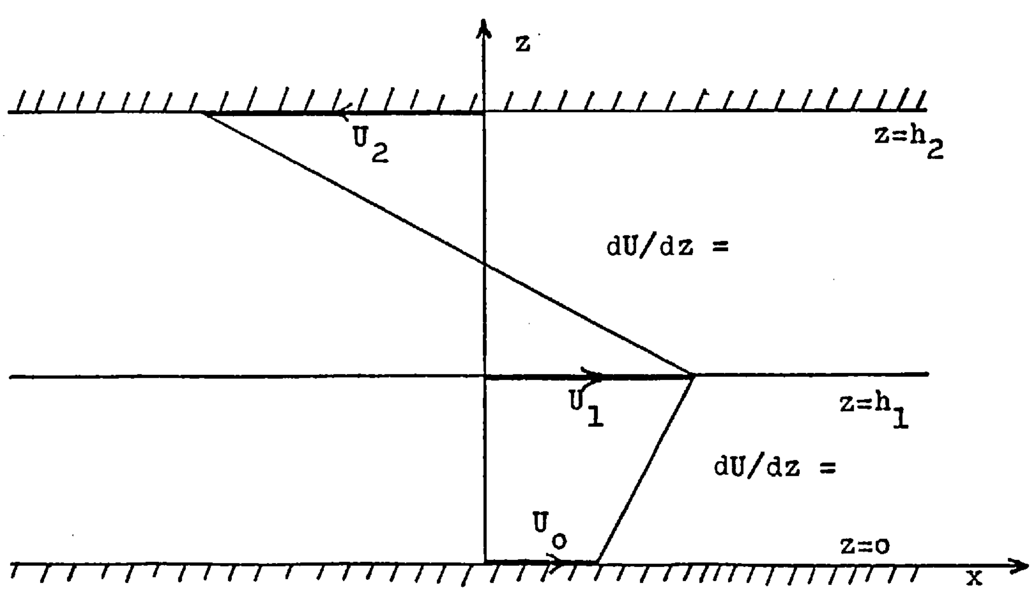


fig.2

A MODEL OF TWO LAYERS

FORMULATION :

Neglecting any external force as well as the viscosity , the equations of motion and continuity may be written as

$$\rho\left[\bar{u}_t + \bar{u}\,\bar{u}_x + \bar{w}\,\bar{u}_z\right] = -\bar{p}_x ,$$

$$\rho\left[\bar{w}_t + \bar{u}\,\bar{w}_x + \bar{w}\,\bar{w}_z\right] = -\bar{p}_z , \qquad (3)$$

$$\bar{u}_x + \bar{w}_z = 0 .$$

where ρ is the density .

The plane Couette flow is described by

$$U = U(z) \quad , \quad W = 0 \quad \& \quad P = \text{constant} .$$

At the time $t = 0$, a disturbance is created at the bottom or at the upper most level , let us write

$$\bar{u}(x,z;t) = U(z) + u(x,z;t) ,$$

$$\bar{w}(x,z;t) = w(x,z;t) , \qquad (4)$$

$$\bar{p}(x,z;t) = P + p(x,z;t) .$$

where u , w & p are the disturbance functions .

On substituting (4) in (3) and linearizing , we obtain the following equations for the disturbance :

$$-p_x = \rho\left[u_t + U(z)\,u_x + w\,U_z(z)\right] ,$$

$$-p_z = \rho\left[w_t + U(z)\,w_x\right] , \qquad (5)$$

$$u_x + w_z = 0 .$$

The continuity equation implies the existence of a stream function $\psi = \psi(x,z;t)$ such that

$$-\psi_z(x,z;t) = u(x,z;t) \ ,$$

and $$\psi_x(x,z;t) = w(x,z;t) \ .$$

Eliminating $p(x,z;t)$ between the dynamical equations , we get

$$(\partial/\partial t + U(z)\,\partial/\partial x)\,(u_z - w_x) = 0 \ .$$

In terms of stream function , this equation becomes

$$(\partial/\partial t + U(z)\,\partial/\partial x)\,\nabla^2\psi(x,z;t) = 0 \ , \qquad (14)$$

where ∇^2 is the laplacian operator, i.e.

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial z^2 \ .$$

The vertical displacement of a fluid particle $\xi(x,z;t)$ is defined by the relation

$$(\partial/\partial t + U(z)\,\partial/\partial x)\,\xi(x,z;t) = w(x,z;t) \ ,$$

i.e. $$(\partial/\partial t + U(z)\,\partial/\partial x)\,\xi(x,z;t) = \psi_x(x,z;t) \ . \qquad (15)$$

The Boundary Conditions :

(1) At the bottom which is defined by the relation

$$z = \eta(x,t) = f(x)\,g(t) \ ,$$

hence , the condition is

$$(\partial/\partial t + U(z)\,\partial/\partial x)\,f(x)g(t) = w(x,z;t) \qquad \text{at } z=0,$$

where $$U(z)\big|_{z=0} = U_0 \ .$$

in terms of stream function , the condition can be written as

$$\psi_1(x,z;t) = \psi_0(x,z;t) \qquad \text{at } z = 0 , \tag{16}$$

where , ψ_0 is given .

(2) At the top ($z = h_2$) , there is no vertical velocity

i.e. $w(x,z;t) = 0$, at $z = h_2$,

this condition can be written in terms of stream function as

$$\psi_2(x,z;t) = 0 , \qquad \text{at } z = h_2 . \tag{17}$$

(3) At the interface ($z = h_1$) , we have

(a) The continuity of the stream function , i.e.

$$\psi_1(x,z;t)\big|_{z\to h_1-0} = \psi_2(x,z;t)\big|_{z\to h_1+0} \tag{18}$$

(b) The continuity of the pressure , from the dynamical equation in the x - direction , we get

$$\Big[(\partial/\partial t + U(z)\, \partial/\partial x)\, u(x,z;t) + w(x,z;t)\, dU/dz \Big]_{h_1-0}^{h_1+0} = 0$$

in terms of stream function , the condition can be written as

$$(\partial/\partial t + U_1\, \partial/\partial x)(\psi'\big|_{h_1+0} - \psi'\big|_{h_1-0}) = -\alpha\psi_x \tag{19}$$

where , the prime denotes the differentiation with respect to z,

$U_1 = U(h_1)$, the mean velocity at the interface .

α = the jump in the rate of shear between the two layers and always positive .

ψ_1 = the stream function in the lower layer.

ψ_2 = the stream function in the upper layer.

The Solution:

We attack the mathematical problem posed by (14 - 19) by invoking Fourier transformation with respect to x and Laplace transformation with respect to t , the time . Let

$$\Psi(k,z;w) = \mathcal{L}\mathcal{F}\psi(x,z;t)$$
$$= \int_0^\infty dt\ e^{iwt} \int_{-\infty}^{\infty} dx\ e^{-ikx}\ \psi(x,z;t) ,$$

where $\mathcal{F}$ implies Fourier transformation with respect to x and $\mathcal{L}$ implies Laplace transformation with respect to t . Then we define

$$\mathcal{F}\left[f(x)\right] = \int_{-\infty}^{\infty} f(x)\ e^{-ikx}\ dx = F(k) ,$$

integrating n times by parts , we obtain

$$\mathcal{F}(d^n f(x)/dx^n) = (ik)^n\ F(k) .$$

Similarity , $\mathcal{L}(d^n f(t)/dt^n) = (-iw)^n F(w)$.

Transforming (14) , we obtain

$$-i(\ w - kU(z))(\Psi_{zz} - k^2\Psi) = 0 \qquad (20)$$

if $(\ w - kU(z)) \neq 0$, we get

$$\bar{\psi}_{zz} - k^2\bar{\psi} = 0 \,, \tag{21}$$

(21) has a solution in first layer (the lower layer) is given by

$$\bar{\psi}_1(k,z;w) = a_o \sinh(kz) + b_o \sinh(k(z-h_1)) \tag{22.a}$$

in the second layer , the solution is

$$\bar{\psi}_2(k,z;w) = a_o \sinh(kz) + b_1 \sinh(k(z-h_1)) \tag{22.b}$$

where a_o , b_o and b_1 are arbitrary functions .

The condition at the bottom is

$$(\partial/\partial t + U_o \partial/\partial x)\,\eta(x,t) = w(x,z;t)$$

$$= \psi_x(x;z;t) \,.$$

Applying Fourier - Laplace transform on this condition , the equivalent relation is

$$-(w - kU_o)\,\bar{f}(k)\bar{g}(w) = k\,\bar{\psi}_o \,,$$

where $\psi_o = \psi(x,0;t)$, the stream function at the bottom .

$$\therefore\ \bar{\psi}_o(k,0;w) = \left[(kU_o - w)/k\right]\bar{f}(k)\bar{g}(w) \,. \tag{23}$$

From (16) and (23) , we get

$$\bar{\psi}_1(k,0;w) = \bar{\psi}_o(k,0;w) \,, \tag{24}$$

i.e.

$$b_o \sinh(k(-h_1)) = \left[(kU_o - w)/k\right]\bar{f}(k)\bar{g}(w) \,. \tag{25}$$

$$b_o = \left[\left(\frac{kU_o - w}{k} \right) \bar{g}(w) \left(\frac{\bar{f}(k)}{\sinh(-kh_1)} \right) \right] \tag{26}$$

At the top ($z = h_2$) , we have $\bar{\psi}_2(k,-h_2,w) = o$ (27)

From (22.b) and (27) , we get the relation

$$a_o \sinh kh_2 + b_1 \sinh k(h_2-h_1) = o \tag{28}$$

Transforming (19) , we get

$$(kU_1 - w)(b_1-b_o) = (-\alpha \sinh kh_1)\, a_o$$

hence ,
$$b_1 = -\left(\frac{\alpha \sinh kh_1}{kU_1 - w}\right) a_o + b_o \tag{29}$$

Substituting (29) into (28) , we obtain

$$a_o \left[\left(kU_1 - \frac{\alpha \sinh kh_1 \sinh k(h_2-h_1)}{\sinh kh_2} \right) - w \right] = \left(\frac{-\sinh k(h_2-h_1)}{\sinh kh_2} \right) (kU_1-w) b_o$$

By putting ,
$$\beta(k) = \left(kU_1 - \frac{\alpha \sinh kh_1 \sinh k(h_2-h_1)}{\sinh kh_2}\right) \tag{30}$$

we get ,
$$a_o = \left(\frac{(kU_1 - w)}{(\beta(k) - w)} \right) \left(-\sinh k(h_2-h_1) \,/\, \sinh kh_2 \right) b_o \tag{31}$$

and
$$b_1 = \left[(kU_1 - w)/(\beta(k) - w) \right] b_o \tag{32}$$

Substituting for b_o from (26) , a_o and b_1 can be rewritten as

$$a_o = \left[\left(\frac{(kU_1-w)(kU_o-w)}{k(\beta(k)-w)} \right) \bar{g}(w) \left(\frac{-\sinh(k(h_2-h_1))}{\sinh(-kh_1)\sinh(kh_2)} \right) \bar{f}(k) \right] \quad (31)'$$

and

$$b_1 = \left[\left(\frac{(kU_1-w)(kU_o-w)}{k(\beta(k)-w)} \right) \bar{g}(w) \left(\bar{f}(k)/\sinh(-kh_1) \right) \right] , \quad (32)'$$

Substituting the values of a_o , b_o and b_1 in (22.a) and (22.b) , we obtain

$$\bar{\Psi}_1(k,z;w) = \left[\left(\frac{-\sinh(k(h_2-h_1))}{\sinh(-kh_1)\sinh(kh_2)} \right) \bar{f}(k) \sinh(kz) \left(\frac{(kU_1-w)(kU_o-w)}{k(\beta(k)-w)} \right) \bar{g}(w) \right.$$

$$\left. + \left(\frac{\bar{f}(k)}{\sinh(-kh_1)} \right) \sinh(k(z-h_1)) \left(\frac{(kU_o-w)}{k} \right) \bar{g}(w) \right] \quad (22.a)'$$

$$\bar{\Psi}_2(k,z;W) = \left[\left(\frac{-\sinh(k(h_2-h_1))}{\sinh(-kh_1)\sinh(kh_2)} \right) \bar{f}(k) \sinh(kz) \left(\frac{(kU_1-w)(kU_o-w)}{K(\beta(k)-w)} \right) \bar{g}(w) \right.$$

$$\left. + \left(\frac{\bar{f}(k)}{\sinh(-kh_1)} \right) \sinh(k(z-h_1)) \left(\frac{(kU_1-w)(kU_o-w)}{k(\beta(k)-w)} \right) \bar{g}(w) \right] \quad (22.b)'$$

Transforming (15) , we obtain

$$\bar{\xi}(k,z;w) = (k/(kU(z)-w))\bar{\Psi}(k,z;w) \quad (33)$$

Substituting from (22.a)' & (22.b)' in (33) , we get

$$\bar{\xi}_1(k,z;t) = \left[G(k) \sinh(kz) \left(\frac{(kU_1-w)(kU_o-w)}{(\beta(k)-w)(kU(z)-w)} \right) \bar{g}(w) \right.$$

$$\left. + F(k) \sinh(k(z-h_1)) \left(\frac{(kU_o-w)}{(kU(z)-w)} \right) \bar{g}(w) \right] \quad (34.a)$$

$$\bar{\xi}_2(k,z;w) = \left[\left[G(k)\ \sinh(kz) + F(k)\ \sinh(k(z-h_1))\right] \times \left[\left(\frac{(kU_1-w)(kU_o\ -w)}{(kU(z)-w)(\beta(k)-w)}\right)\ \bar{g}(w)\right]\right] \qquad (34.b)$$

where,

$$G(k) = \left(\frac{-\sinh(k(h_2-h_1)}{\sinh(-kh_1)\sinh(kh_2)}\right)\ \bar{f}(k) \quad ,$$

and (35)

$$F(k) = \bar{f}(k)/\sinh(-kh_1) \quad .$$

ξ_1 = the vertical displacement of a fluid particle in the first layer (i.e. the lower layer)

ξ_2 = the vertical displacement of a fluid particle in the second layer (i.e. the upper layer) .

Inverting , we get

$$\xi_1 = (1/4\pi^2)\int_{-\infty}^{\infty} dk \int_L dw\, e^{i(kx-wt)}\, \bar{f}(k)\bar{g}(w)\left[\frac{\sinh(k(z-h_1))}{\sinh(-kh_1)}\,\frac{(kU_o-w)}{(kU-w)} + \frac{-\sinh(k(h_2-h_1))\,\sinh(kz)}{\sinh(-kh_1)\,\sinh(kh_2)}\,\frac{(kU_1-w)(kU_o-w)}{(\beta-w)(kU-w)}\right]$$

and

$$\xi_2 = (1/4\pi^2)\int_{-\infty}^{\infty} dk \int_L dw\, e^{i(kx-wt)}\, \bar{f}(k)\bar{g}(w)\left[\frac{-\sinh(k(h_2-h_1))\,\sinh(kz)}{\sinh(-kh_1)\,\sinh(kh_2)} + \frac{\sinh(k(z-h_1))}{\sinh(-kh_1)}\,\frac{(kU_1-w)(kU_o-w)}{(kU-w)\,(\beta-w)}\right]$$

where the path L lies in the complex w - plane .

The original equation is

$$(\partial/\partial t + U(z)\,\partial/\partial x)\,\nabla^2\psi = 0\ ,$$

or

$$(\partial/\partial t + U(z)\,\partial/\partial x)\,\nabla^2\psi_x = 0\ .$$

The vertical displacement ξ is given by

$$(\partial/\partial t + U(z)\,\partial/\partial x)\,\xi = \psi_x\ ,$$

$$\therefore\quad (\partial/\partial t + U(z)\,\partial/\partial x)\,\nabla^2\,(\partial/\partial t + U(z)\,\partial/\partial x)\,\xi = 0\ .$$

It is clear that the expression for ξ_1 & ξ_2 satisfy the above equation and

$\xi_1 = f(x)g(t)$ at $z = 0$,

$\xi_2 = 0$ at $z = h_2$,

also $\xi_1 = \xi_2$ at $z = h_1$ (the inter-face)

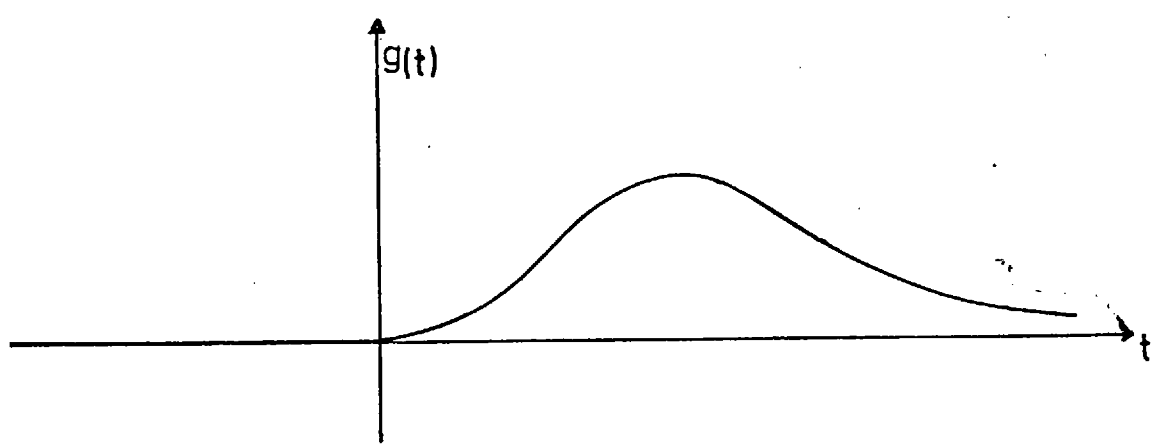
The function $g(t)$:

Let the function $g(t)$ be defined by

$$g(t) = 0 , \qquad t < 0 ,$$
$$= t^2 e^{-t\lambda} \qquad t > 0 .$$

where λ is a positive constant .

This definition means , at the time $t = 0$, a disturbance creating smoothly at the bottom , then decaying gradually with time . The function $g(t)$ behaves as in the figure



$$\therefore \mathcal{L} g(t) = \int_0^\infty t^2 e^{-t\lambda} e^{iwt} dt = \bar{g}(w) ,$$

i.e. $\bar{g}(w) = -2i/(w+i\lambda)^3$.

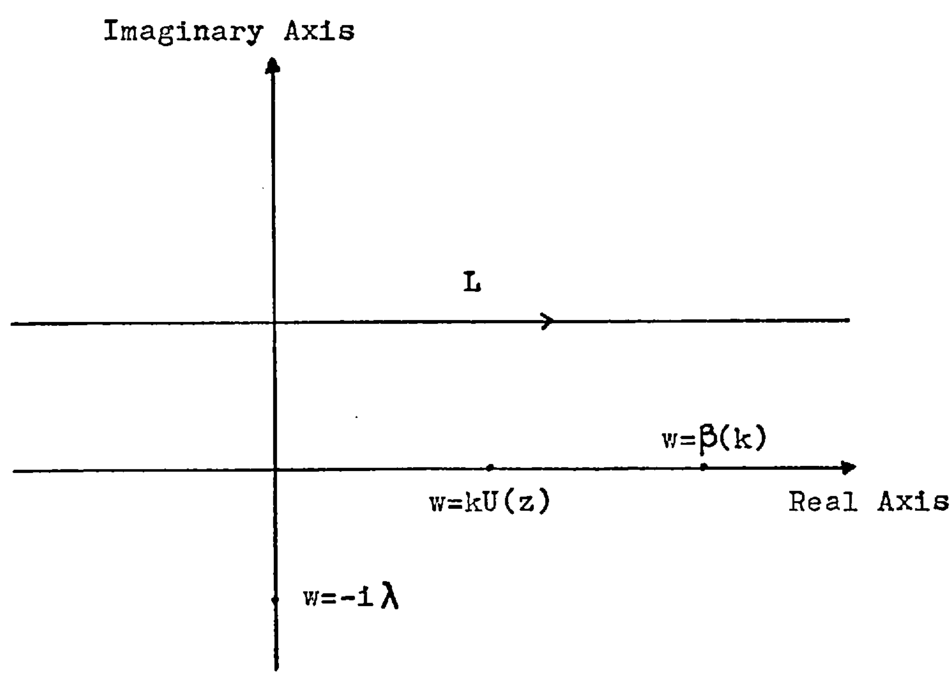
The Disturbance Functions

(1) The vertical displacement in each layer :

Substituting the value of $\bar{g}(w)$ in the expression for ξ_1 , we obtain

$$\xi_1 = (1/4\pi^2) \int_{-\infty}^{\infty} dk \int_L dw \; e^{i(kx-wt)} \; \bar{f}(k) \left[\frac{\sinh(k(z-h_1))}{\sinh(-kh_1)} \; \frac{(-kU_0+w)}{(w-kU(z))(w+i\lambda)^3} + \frac{-\sinh(k(h_2-h_1)) \; \sinh(kz)}{\sinh(-kh_1) \; \sinh(kh_2)} \; \frac{(kU_0-w)(kU_1-w)}{(w-\beta(k))(w-kU(z))(w+i\lambda)^3} \right]$$

We take the path L (as in the figure) , in the complex w-plane, above and parallel to the real axis to avoid any singularity on the real axis and on the imaginary axis in the negative half.



w - plane

Manipulating the integrals , we find

$$\zeta_1=\int_{-\infty}^{\infty} G_1(k,z)\ \frac{e^{i(kx-\beta(k)t)}}{(\beta(k)\ -\ kU(z))}\ dk\ -\int_{-\infty}^{\infty} G_2(k,z)\ \frac{e^{i(kx-kU(z)t)}}{(\beta(k)\ -\ kU(z))}\ dk$$

$$+\int_{-\infty}^{\infty} G_3\ e^{i(kx-kU(z)t)}\ dk\ +\ I_1\ ,$$

where ,

$$G_1(k,z)=(-1/\pi)\bar{f}(k)\left[\frac{-\sinh(k(h_2-h_1))\sinh(kz)}{\sinh(-kh_1)\sinh(kh_2)}\right]\left[\frac{(kU_o-\beta)(kU_1-\beta)}{(\beta(k)+i\lambda)^3}\right]\ ,$$

$$G_2(k,z)=(-1/\pi)\bar{f}(k)\left[\frac{-\sinh(k(h_2-h_1))\sinh(kz)}{\sinh(-kh_1)\sinh(kh_2)}\right]\left[\frac{(kU_o-kU(z))(kU_1-kU(z))}{(kU(z)+i\lambda)^3}\right],$$

$$G_3(k,z)=(-1/\pi)\bar{f}(k)\left[\frac{\sinh(k(z-h_1))}{\sinh(-kh_1)}\right]\left[\frac{(kU(z)-kU_o)}{(kU(z)+i\lambda)^3}\right]\ ,$$

and $$I_1=(e^{-t\lambda}/2\pi)\int_{-\infty}^{\infty}(-2)dk\ \bar{f}(k)\ e^{ikx}\left[\frac{-\sinh(k(h_2-h_1))\sinh(kz)}{\sinh(-kh_1)\sinh(kh_2)}\right]\ X$$

$$\Bigg[(-t^2/2)\left[\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta(k)+i\lambda)(kU(z)+i\lambda)}\right]-it\Bigg[\frac{-(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)}$$

$$+\frac{-(kU_o+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)}+\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)^2(kU+i\lambda)}+\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)^2}\Bigg]$$

$$+\Bigg[\frac{1}{(\beta+i\lambda)(kU+i\lambda)}+\frac{-(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)^2}+\frac{-(kU_1+i\lambda)}{(\beta+i\lambda)^2(kU+i\lambda)}$$

$$+\frac{-(kU_o+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)^2}+\frac{-(kU_o+i\lambda)}{(\beta+i\lambda)^2(kU+i\lambda)}+\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)^3}$$

$$+\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)^3(kU+i\lambda)}+\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)^2(kU+i\lambda)^2}\Bigg]\Bigg]$$

$$+(e^{-t\lambda}/2\pi)\int_{-\infty}^{\infty}(-2)dk\,\bar{f}(k)\,e^{ikx}\left(\frac{\sinh(k(z-h_1))}{\sinh(-kh_1)}\right)\left[(-\tfrac{1}{2}t^2)\left(\frac{(kU_o+i\lambda)}{(kU+I\lambda)}\right)\right.$$

$$\left.+(-it)\left[\frac{-1}{(kU+i\lambda)}+\frac{(kU_o+i\lambda)}{(kU+i\lambda)^2}+\frac{-1}{(kU+i\lambda)^2}+\frac{(kU_o+i\lambda)}{(kU+i\lambda)^3}\right]\right]$$

Also , the expression for ξ_2 is

$$\xi_2=\int_{-\infty}^{\infty}G_4(k,z)\,\frac{e^{i(kx-\ (k)t)}}{(\beta(k)-kU(z))}\,dk-\int_{-\infty}^{\infty}G_5(k,z)\,\frac{e^{i(kx-kU(z)t)}}{(\beta(k)-kU(z))}\,dk+I_2\ .$$

where,

$$G_4(k,z)=(-1/\pi)\bar{f}(k)\left[\frac{-\sinh(k(h_2-h_1))\sinh(kz)}{\sinh(-kh_1)\sinh(kh_2)}+\frac{\sinh k(z-h_1)}{\sinh(-kh_1)}\right]\frac{(kU_1-\beta)(kU_o-\beta)}{(\beta(k)+i\lambda)^3}$$

$$G_5(k,z)=(-1/\pi)\bar{f}(k)\left[\frac{-\sinh k(h_2-h_1)}{\sinh(-kh_1)}\frac{\sinh(kz)}{\sinh(kh_2)}+\frac{\sinh k(z-h_1)}{\sinh(-kh_1)}\right]\frac{(kU_1-kU)(kU_o-kU)}{(kU(z)+i\lambda)^3}$$

and

$$I_2=\left(\frac{e^{-t\lambda}}{2\pi}\right)\int_{-\infty}^{\infty}(-2)dk\,\bar{f}(k)\,e^{ikx}\left[\frac{-\sinh k(h_2-h_1)\ \sinh(kz)}{\sinh(-kh_1)\ \sinh(kh_2)}+\frac{\sinh(k(z-h_1))}{\sinh(-kh_1)}\right]$$

$$\left[\left(\frac{-t^2}{2}\right)\left[\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)}\right]+(-it)\left[\frac{-(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)}+\frac{-(kU_o+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)}\right.\right.$$

$$\left.+\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)^2(kU+i\lambda)}+\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)^2}\right]+\left[\frac{1}{(\beta+i\lambda)(kU+i\lambda)}\right.$$

$$+\frac{-(kU_1+i\lambda)}{(\beta+i\lambda)^2(kU+i\lambda)}+\frac{-(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)^2}+\frac{-(kU_o+i\lambda)}{(\beta+i\lambda)^2(kU+i\lambda)}+\frac{-(kU_o+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)^2}$$

$$\left.\left.+\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)^3(kU+i\lambda)}+\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)^3}+\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)^2(kU+i\lambda)^2}\right]\right]$$

(2) The velocity components :

We can obtain the vertical component of the disturbance velocity from the relation

$$(\partial/\partial t + U(z)\,\partial/\partial x)\,\xi = w ,$$

then , the horizontal component is given by using the continuity equation

$$u_x + w_z = 0 .$$

In the first layer , the expression for w_1 is

$$w_1(x,z;t) = \int_{-\infty}^{\infty} F_1(k,z)\ \exp(ikx - it\beta(k))\ dk + I_3 ,$$

where,

$$F_1(k,z) = (\tfrac{1}{2}i)\ \bar{f}(k)\ \left(\frac{-\sinh(k(h_2-h_1))\sinh(kz)}{\sinh(-kh_1)\sinh(kh_2)}\right)\left(\frac{(kU_1-\beta)(kU_0-\beta)}{(\beta+i\ )^3}\right) ,$$

and

$$I_3 = (e^{-t\lambda}/2\pi)\int_{\infty}^{\infty}(-2i)\ dk\ \bar{f}(k)\ e^{ikx}\left(\frac{-\sinh(k(h_2-h_1))\sinh(kz)}{\sinh(-kh_1)\sinh(kh_2)}\right)\ X$$

$$\Bigg[(-\tfrac{1}{2}t^2)\left(\frac{(kU_0+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)}\right) + (-it)\Bigg[\frac{-(kU_0+i\lambda)}{(\beta+i\lambda)} + \frac{-(kU_1+i\lambda)}{(\beta+i\lambda)}$$

$$+\frac{(kU_0+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)^2} + \frac{(kU_0+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)}\Bigg] + \Bigg[\frac{1}{(\beta+i\lambda)} + \frac{-(kU_1+i\lambda)}{(\beta+i\lambda)^2}$$

$$+\frac{-(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)} + \frac{-(kU_0+i\lambda)}{(\beta+i\lambda)^2} + \frac{-(kU_0+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)} + \frac{(kU_0+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)^3}$$

$$+\frac{(kU_0+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)^2} + \frac{(kU_0+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)^2(kU+i\lambda)}\Bigg]\Bigg]$$

$$+(e^{-t\lambda}/2\pi)\int_{-\infty}^{\infty}(-2)\ dk\ \bar{f}(k)\ e^{ikx}\left(\frac{-\sinh(k(h_2-h_1))\ \sinh(kz)}{\sinh(-kh_1)\ \sinh(kh_2)}\right)$$

$$\Big[(-t)\Big(\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)}\Big)+(-i)\Big[\frac{-(kU_o+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)}+\frac{-(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)}$$

$$+\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)^2(kU+i\lambda)}+\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)^2}\Big]\Big]$$

$$+(e^{-t}/2)\int_{-\infty}^{\infty}(-2)\,dk\,\bar{f}(k)\,(i)\,e^{ikx}\Big(\frac{\sinh(k(z-h_1))}{\sinh(kh_1)}\Big)\Big[(-\tfrac{1}{2}t^2)\,(kU_o+i\lambda)$$

$$+(it)\Big(-1+\frac{(kU_o+i\lambda)}{(kU+i\lambda)}\Big)+\Big(\frac{-1}{(kU+i\lambda)}+\frac{(kU_o+i\lambda)}{(kU+i\lambda)^2}\Big)\Big]$$

$$+(e^{-t}/2)\int_{-\infty}^{\infty}(-2)\,dk\,\bar{f}(k)\,e^{ikx}\Big(\frac{\sinh(k(z-h_1))}{\sinh(kh_1)}\Big)\Big[(-t)\Big(\frac{(kU_o+i\lambda)}{(kU+i\lambda)}\Big)$$

$$+(-i)\Big(\frac{-1}{(kU+i\lambda)}+\frac{(kU_o+i\lambda)}{(kU+i\lambda)}\Big]$$

and the expression for U_1 is given by

$$u_1(x,z;t)=\int_{-\infty}^{\infty}F_2(k,z)\,\exp(ikx-it\beta(k))\,dk+I_4,$$

where,

$$F_2(k,z)=(1/\pi)\,\bar{f}(k)\,\Big(\frac{-\sinh(k(h_2-h_1))\cosh(kz)}{\sinh(-kh_1)\sinh(kh_2)}\Big)\Big(\frac{(kU_o-\beta)(kU_1-\beta)}{(\beta+i\ \ )}\Big)$$

and

$$I_4=(e^{-t}/2)\quad(-2)\,dk\,\bar{f}(k)\,e^{ikx}\Big(\frac{-\sinh(k(h_2-h_1))\cosh(kz)}{\sinh(-kh_1)\sinh(kh_2)}\Big)\ X$$

$$\Big[(-\tfrac{1}{2}t^2)\Big(\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)}\Big)+(-it)\Big[\frac{-(kU_o+i\lambda)}{(\beta+i\lambda)}+\frac{-(kU_1+i\lambda)}{(\beta+i\lambda)}$$

$$+\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)}\Big]+\Big[\frac{1}{(\beta+i\lambda)}+\frac{-(kU_1+i\lambda)}{(\beta+i\lambda)^2}+\frac{-(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)}$$

$$+ \frac{-(kU_o+i\lambda)}{(\ +i\lambda)^2} + \frac{-(kU_o+i\lambda)}{(\ +i\lambda)(kU+i\lambda)} + \frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\ +i\lambda)^3} + \frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\ +i\lambda)(kU+i\lambda)^2}$$

$$+ \frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\ +i\lambda)^2(kU+i\lambda)}$$

$$+(e^{-t\lambda}/2\pi)\int_{-\infty}^{\infty}(2i)\ dk\ \bar{f}(k)\ e^{ikx}\left(\frac{-\sinh(k(h_2-h_1))\cosh(kz)}{\sinh(-kh_1)\sinh(kh_2)}\right)\ X$$

$$\left[(-t)\left(\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\ +i\lambda)(kU+i\lambda)}\right) + (-i)\left[\frac{-(kU_1+i\lambda)}{(\ +i\lambda)(kU+i\lambda)} + \frac{-(kU_o+i\lambda)}{(\ +i\lambda)(kU+i\lambda)}\right.\right.$$

$$\left.\left. + \frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\ +i\lambda)^2(kU+i\lambda)} + \frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\ +i\lambda)(kU+i\lambda)^2}\right]\right]$$

$$+(e^{-t\lambda}/2\pi)\int_{-\infty}^{\infty}(-2)\ dk\ \bar{f}(k)\ e^{ikx}\left(\frac{\sinh(k(z-h_1))}{\sinh(kh_1)}\right)\left[(-\tfrac{1}{2}t^2)\ (kU_o+i\lambda)\right.$$

$$\left. +(it)\left(-1 + \frac{(kU_o+i\lambda)}{(kU+i\lambda)}\right) + \left(\frac{-1}{(kU+i\lambda)} + \frac{(kU_o+i\lambda)}{(kU+i\lambda)^2}\right)\right]$$

$$+(e^{-t\lambda}/2\pi)\int_{-\infty}^{\infty}(2i)\ dk\ \bar{f}(k)\ e^{ikx}\left(\frac{\sinh(k(z-h_1))}{\sinh(kh_1)}\right)\left[(-t)\left(\frac{(kU_o+i\lambda)}{(kU+i\lambda)}\right)\right.$$

$$\left. +(-i)\left(\frac{-1}{(kU+i\lambda\ )} + \frac{(kU_o+i\lambda)}{(kU+i\lambda)^2}\right)\right]$$

In the second layer, the expression for w_2 is given by

$$w_2(x,z;t) = \int_{\infty}^{\infty} F_3(k,z)\exp(ikx - it\beta(k))\,dk + I_5 \;;$$

where

$$F_3(k,z) = (i/\pi)\,\bar{f}(k)\left(\frac{-\sinh k(h_2-h_1)\sinh kz}{\sinh(-kh_1)\sinh kh_2} + \frac{\sinh k(z-h_1)}{\sinh(-kh_1)}\right)\left(\frac{(kU_o-\beta)(kU_1-\beta)}{(\beta+i\lambda)^3}\right)$$

and

$$I_5 = (e^{-t\lambda}/2\pi)\int_{-\infty}^{\infty}(-2i)\,dk\,\bar{f}(k)\,e^{ikx}\left(\frac{-\sinh k(h_2-h_1)\sinh kz}{\sinh(-kh_1)\sinh(kh_2)} + \frac{\sinh k(z-h_1)}{\sinh(-kh_1)}\right)$$

$$\left[(-\tfrac{1}{2}t^2)\left(\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\ +i\lambda)}\right) + (-it)\left[\frac{-(kU_o+i\lambda)}{(\beta+i\lambda)} + \frac{-(kU_1+i\lambda)}{(\beta+i\lambda)}\right.\right.$$

$$\left.+\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)^2} + \frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)}\right] + \left[\frac{1}{(\beta+i\lambda)} + \frac{-(kU_1+i\lambda)}{(\beta+i\lambda)^2}\right.$$

$$+\frac{-(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)} + \frac{-(kU_o+i\lambda)}{(\beta+i\lambda)^2} + \frac{-(kU_o+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)} + \frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)^3}$$

$$\left.\left.+\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)^2} + \frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)^2(kU+i\lambda)}\right]\right]$$

$$+(e^{-t\lambda}/2\pi)\int_{-\infty}^{\infty}(-2)\,dk\,\bar{f}(k)\,e^{ikx}\left(\frac{-\sinh k(h_2-h_1)\sin(kz)}{\sinh(-kh_1)\sinh kh_2} + \frac{\sinh k(z-h_1)}{\sinh(-kh_1)}\right)$$

$$\left[(-t)\left(\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\ +i\lambda)(kU+i\lambda)}\right) + (-i)\left[\frac{-(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)} + \frac{-(kU_o+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)}\right.\right.$$

$$\left.\left.+\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)^2(kU+i\lambda)} + \frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)^2}\right]\right]$$

and the expression for u_2 is given by

$$u_2(x,z;t) = \int_{-\infty}^{\infty} F_4(k,z)\, \exp(ikx - it\beta(k))\, dk + I_6 ,$$

where

$$F_4(k,z) = (i/\pi)\, \bar{f}(k) \left(\frac{-\sinh k(h_2-h_1)\cosh kz}{\sinh(-kh_1)\sinh kh_2} + \frac{\cosh k(z-h_1)}{\sinh(-kh_1)}\right)\left(\frac{(kU_o-\beta)(kU_1-\beta)}{(\beta+i\lambda)^3}\right)$$

and

$$I_6 = (e^{-t\lambda}/2\pi) \int_{-\infty}^{\infty} (-2)\, dk\, \bar{f}(k)\, e^{ikx} \left(\frac{-\sinh k(h_2-h_1)\cosh kz}{\sinh(-kh_1)\sinh kh_2} + \frac{\cosh k(z-h_1)}{\sinh(-kh_1)}\right) X$$

$$\left[(-t^2/2)\left(\frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)}\right) + (-it)\left[\frac{-(kU_o+i\lambda)}{(\beta+i\lambda)} + \frac{-(kU_1+i\lambda)}{(\beta+i\lambda)}\right.\right.$$

$$\left. + \frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)^2} + \frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)}\right] + \left[\frac{1}{(\beta+i\lambda)} + \frac{-(kU_1+i\lambda)}{(\beta+i\lambda)^2}\right.$$

$$+ \frac{-(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)} + \frac{-(kU_o+i\lambda)}{(\beta+i\lambda)^2} + \frac{-(kU_o+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)} + \frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)^3}$$

$$\left.\left. + \frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)^2} + \frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)^2(kU+i\lambda)}\right]\right]$$

$$+ (e^{-t\lambda}/2\pi) \int_{-\infty}^{\infty} (2i)\, dk\, \bar{f}(k)\, e^{ikx} \left(\frac{-\sinh k(h_2-h_1)\cosh kz}{\sinh(-kh_1)\sinh kh_2} + \frac{\cosh k(z-h_1)}{\sinh(-kh_1)}\right) X$$

$$\left[(-t)\left(\frac{(kU_o+i\lambda)(ku_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)}\right) + (-i)\left[\frac{-(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)} + \frac{-(kU_0+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)}\right.\right.$$

$$\left.\left. + \frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)^2\,(kU+i\lambda)} + \frac{(kU_o+i\lambda)(kU_1+i\lambda)}{(\beta+i\lambda)(kU+i\lambda)^2}\right]\right]$$

Evaluation the different integrals when t becomes large :

The integral expression for ξ_1 (the vertical displacement of a fluid particle in the first layer) is

$$\xi_1(x,z;t) = \int_{-\infty}^{\infty} G_1(k,z)\, \frac{e^{ikx}\, e^{-it\beta(k)}}{(\beta(k)-kU(z))}\, dk - \int_{-\infty}^{\infty} G_2(k;z)\, \frac{e^{ikx}\, e^{-ikU(z)t}}{(\beta(k) - kU(z))}\, dk$$

$$+ \int_{-\infty}^{\infty} G_3(k,z)\; e^{ikx}\, e^{-ikU(z)t}\, dk + I_1 \;,$$

The function $(\beta(k)-kU(z))$ has a simple zero at $k = k^*$, where $G_1(k,z)$, $G_2(k,z)$ and $G_3(k,z)$ are analytic and not zero at $k = k^*$, therefore , we can write

$$\frac{G_1(k,z)\; e^{ikx}}{(\beta(k)-kU(z))} = \frac{G_1(k^*,z)\; e^{ik^*x}}{(k-k^*)\; d/dk(\beta(k)-kU(z))_{k=k^*}} + \phi_1(k,z)\; e^{ikx}$$

and

$$\frac{G_2(k,z)\; e^{ikx}}{(\beta(k)-kU(z))} = \frac{G_2(k^*,z)\; e^{ik^*x}}{(k-k^*)\; d/dk(\beta(k)-kU(z))_{k=k^*}} + \phi_2(k,z)\; e^{ikx}.$$

where $\phi_1(k,z)$ and $\phi_2(k,z)$ are analytic at $k = k^*$.

Since $G_1(k^*,z) = G_2(k^*,z)$, the expression for ξ_1 can be written as

$$\xi_1(x,z;t) = \frac{G_1(k^*,z)\; e^{ik^*x}}{d/dk(\beta(k)-kU(z))_{k=k^*}} \left[\int_{-\infty}^{\infty} \frac{e^{-it\beta(k)}}{(k-k^*)}\, dk - \int_{-\infty}^{\infty} \frac{e^{-itkU(z)}}{(k-k^*)}\, dk \right]$$

$$+ \int_{-\infty}^{\infty} \phi_1(k,z)\; e^{ikx-it\beta(k)}\, dk - \int_{-\infty}^{\infty} \phi_2(k,z)\; e^{ikx-ikU(z)t}\, dk$$

$$+ \int_{-\infty}^{\infty} G_3(k,z)\; e^{ikx-ikU(z)t}\, dk + I_1 \;.$$

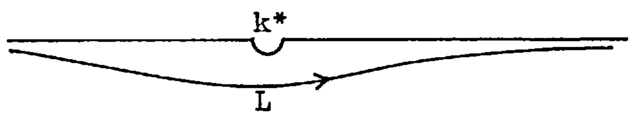
We consider first the integral $\displaystyle ⨍_{-\infty}^{\infty} \frac{e^{-itkU(z)}}{(k-k^*)}\,dk$, has a simple pole at $k = k^*$ (real) .

Put $k = k^* + \kappa$, where $\kappa = \kappa_r + i\,\kappa_i = \rho \exp(i\theta)$,

and $\rho = |\kappa| < 1$.

Then , we have ;

(1) When $U(z) > 0$, then for convergence $\kappa_i < 0$, i.e. the deformed path is

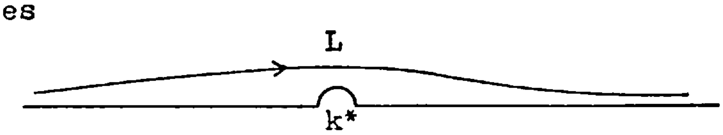


$$\therefore \int_L = ⨍ + \int_\gamma ,$$

$$\text{i.e. } ⨍ = -\int_\gamma + \int_L$$

$$\simeq -i\pi \; \exp(-itk^*U(z))$$

(2) When $U(z) < 0$, then for convergence $\kappa_i > 0$, i.e. the deformed path becomes



$$\therefore ⨍ = -\int_{\frown} + \int_L$$

$$\simeq i\pi \exp(-itk^*U(z)) .$$

$$\therefore ⨍ \frac{\exp(-itkU(z))}{(k - k^*)}\,dk = -i\pi \;\text{sgn}(U(z)) \;\exp(-itk^*U(z))$$

Now , the second integral $\int_{-\infty}^{\infty} dk \frac{\exp(-it\beta(k))}{(k - k^*)}$, where the function $\beta(k)$ is analytic and well behaved in a domain containing the real axis . The simple pole k^* is real and the principal value of the integral is implied . The function $\beta(k)$ has a saddle point at $k = k_o$ which is defined by the relation $d/dk(\beta(k)) = 0$. Then we deform the path in the manner of steepest descent .

Let $\beta(k) = P + iq$, then

$\exp(-it\beta(k)) = \exp(itp)\exp(tq)$, hence the deformed path through the saddle point is defined by :

$$p = p_o = \text{constant} , \text{ and}$$

$$q < q_o \quad , \text{ where}$$

$\beta(k_o) = p_o + iq_o$.

Since $\beta''(k_o) \neq 0$, then $\beta(k)$ near the saddle point can be expanded in the form

$$\beta(k) = \beta(k_o) + \tfrac{1}{2}(k-k_o)\,\beta''(k_o) \quad ,$$

but if we write for values of k on the path , with r real and small

$$\therefore \quad k - k_o = re^{i\alpha} \quad ,$$

and ,
$$\beta''(k_o) = |\beta''(K_o)|\, e^{i\theta} \quad .$$

Hence , we have

$$p + iq = p_o + iq_o + \tfrac{1}{2} r^2 |\beta''(k_o)|\, e^{i(2\alpha+\theta)} \quad ,$$

then ,
$$\tfrac{1}{2}r^2|\beta''| \cos(2\alpha+\theta) = o \ ,$$

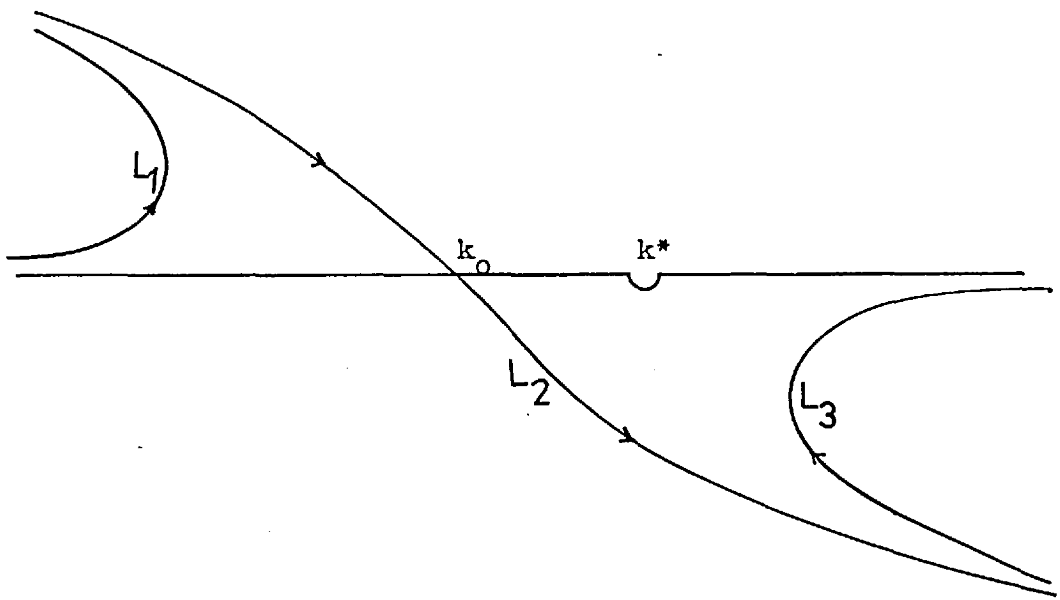
and
$$\tfrac{1}{2}r^2|\beta''| \sin(2\alpha+\theta) \text{ is negative} ,$$

$$\therefore \quad \alpha = -\tfrac{1}{4}\pi - \tfrac{1}{2}\theta \quad .$$

Here , $\beta''(k_o)$ is real and positive , i.e. $\theta = o$,

$$\therefore \quad \alpha = -\tfrac{1}{4}\pi .$$

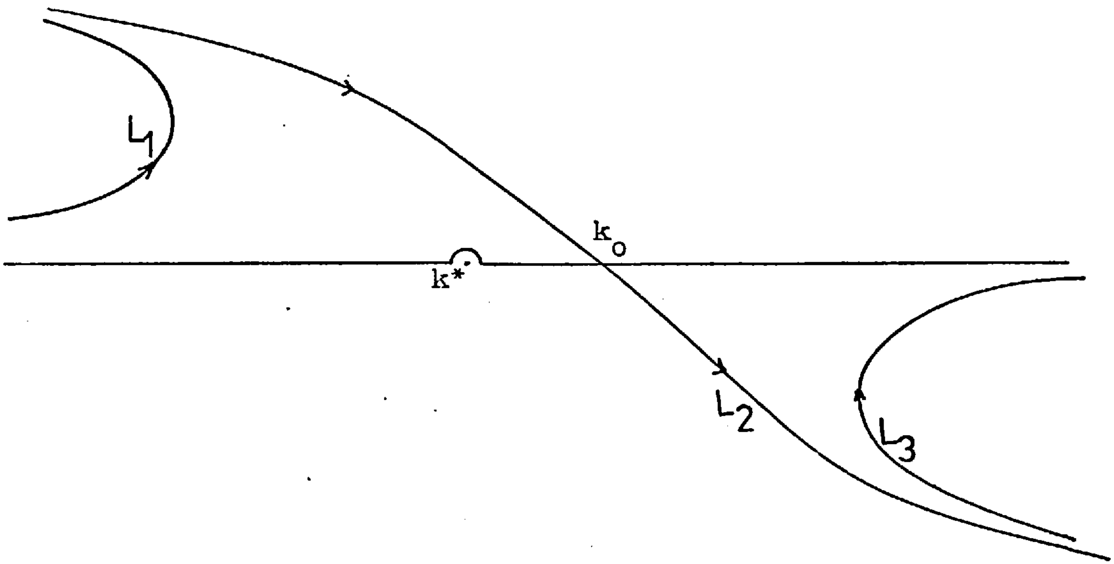
Now , if $k^* > k_o$ i.e. $\beta'(k^*) > 0$, hence , for convergence the semicircle round the simple pole lies in the negative half , and the deformed path becomes as in the figure,



$$\therefore \quad \int = -i\pi \exp(-it\beta(k^*)) + \int_{L_2} + O(1/t)$$

since the contributions from L_1 and L_3 are $O(1/t)$.

If $k^* < k_o$ i.e. $\beta'(k^*) < 0$, then , for the convergence, the semicircle round the simple pole lies in the positive half , and the deformed path becomes as in the figure,



$$\therefore \ \int\!\!\!\!\!\!- \ = i\pi \exp(-it\beta(k^*)) + \int_{L_2} + O(1/t)$$

since the contributions from L_1 and L_3 are $O(1/t)$.

Hence , in general case we have

$$\int\!\!\!\!\!\!- \frac{\exp(-it\beta(k))}{(k - k^*)} dk = -i\pi \,\mathrm{sgn}(\beta'(k^*)) \exp(-it\beta(k^*))$$

$$+ \int_{L_2} \frac{\exp(-it\beta(k))}{(k - k^*)} dk + O(1/t) .$$

We therefore consider the integral $\int_{L_2} \frac{\exp(-it\beta(k))}{(k - k^*)} dk$. By expanding $\beta(k)$ into a power series in $(k-k_o)$ and take only the first two terms :

$$\beta(k) = \beta(k_o) + \tfrac{1}{2}\beta''(k_o) \ (k-k_o)^2 .$$

Introducing a new integration variable

$$\sigma = (\tfrac{1}{2}t\beta''(k_o))^{\frac{1}{2}} \ (k-k_o)\, e^{i\pi/4}$$

and assuming $\sigma^* = (\frac{1}{2}t\ ''(k_o))^{\frac{1}{2}}\ (k^*-k_o)\ e^{\frac{1}{4}i\pi}$

Then , we obtain for the integral $\int_{-\infty}^{\infty} \frac{\exp(-it\beta(k))}{(k-k^*)}\, dk$ the

approximate expression $\simeq \exp(-it\beta(k_o)) \int \frac{\exp(-\sigma^2)}{(\sigma-\sigma^*)}\, d\sigma$

$$\simeq \exp(-it\ (k_o))\quad (2\pi)\ W(\sigma^*)$$

where $W(\sigma^*)$ is tabulated for complex values of the argument σ^* .

If k^* is not near k_o then $(\frac{1}{2}t\beta''(k_o))^{\frac{1}{2}}(k^*-k_o)$ is large , in this case $W(\sigma^*)$ can be replaced by the asymptotic expression which is the same result as is obtained by the standard method of steepest descent.

If the difference (k^*-k_o) is sufficiently small that $(\frac{1}{2}t\beta''(k_o))^{\frac{1}{2}}(k^*-k_o)$ is small , then the integral can be expressed , in terms of the function $W(\sigma^*)$.

Hence,the solution in the first layer is

$$\begin{aligned}\xi_1(x,z;t) = &-i\pi\left[\mathrm{sgn}(\beta'(k_o)) - \mathrm{sgn}(U(z))\right] \frac{G_1(k^*,z)\ e^{ik^*x-it\beta(k^*)}}{d/dk(\beta(k)-kU(z))_{k=k^*}}\\ &+ 2\pi\, W(\sigma^*)\ \frac{G_1(k^*,z)\ e^{ik^*x-it\beta(k_o)}}{d/dk(\beta(k)-kU(z))_{k=k^*}}\\ &+ \int_{-\infty}^{\infty} \Phi_1(k,z)\ e^{ikx-it\beta(k)}\, dk - \int_{-\infty}^{\infty} \Phi_2(k,z)\ e^{ikx-itkU(z)}\, dk\\ &+ \int_{-\infty}^{\infty} G_3(k,z)\ e^{ikx-itkU(z)}\, dk\\ &+ O(\ e^{-t\lambda}\ )\ .\end{aligned}$$

The integral expression for $\xi_2(x,z;t)$ (the vertical displacement of a fluid partical in the second layer) can be written as

$$\xi_2(x,z;t) = \frac{G_4(k^*,z)\ e^{ik^*x}}{d/dk(\beta(k)-kU(z))_{k=k^*}} \left[\int_{-\infty}^{\infty} \frac{e^{-it\ (k)}}{(k-k^*)}\, dk - \int_{-\infty}^{\infty} \frac{e^{-itkU(z)}}{(k-k^*)}\, dk \right]$$

$$+ \int_{-\infty}^{\infty} \phi_3(k,z)\ e^{ikx-it\beta(k)}\, dk - \int_{-\infty}^{\infty} \phi_4(k,z)\ e^{ikx-itkU(z)}\, dk$$

$$+ O(\ e^{-t\lambda}\ ) .$$

where , $\phi_3(k,z)$ and $\phi_4(k,z)$ are analytic functions , also $G_4(k^*,z) = G_5(k^*,z)$.

Then , the solution in the second layer is

$$\xi_2(x,z;t) = -i\pi \left[\mathrm{sgn}(\beta(k_o))-\mathrm{sgn}(U(z)) \right] \frac{G_4(k^*,z)\ e^{ik^*x-it\beta(k^*)}}{d/dk(\beta(k)-kU(z))_{k=k^*}}$$

$$+ 2\pi\ W(\sigma^*) \frac{G_4(k^*,z)\ e^{ik^*x-it\beta(k^*)}}{d/dk(\beta(k)-kU(z))_{k=k^*}}$$

$$+ \int_{-\infty}^{\infty} \phi_3(k,z)\ e^{ikx-it\beta(k)}\, dk - \int_{-\infty}^{\infty} \phi_4(k,z)\ e^{ikx-itkU(z)}\, dk$$

$$+ O(\ e^{-t\lambda}\ ) .$$

Now , the disturbance functions , in the first layer , are

$$w_1(x,z;t) = \int_{\infty}^{\infty} F_1(k,z)\, e^{ikx-it\beta(k)}\, dk + O(e^{-t\lambda}) ,$$

$$u_1(x,z;t) = \int_{\infty}^{\infty} F_2(k,z)\, e^{ikx-it\beta(k)}\, dk + O(e^{-t\lambda}) ,$$

$$\xi_1(x,z;t) = -i\pi\Big[sgn(\beta'(k^*))-sgn(U(z))\Big] \frac{G_1(k^*,z)\, e^{ik^*x-it\ (k^*)}}{d/dk(\beta(k)-kU(z))_{k=k^*}}$$

$$+ 2\pi\, W(\sigma^*) \frac{G_1(k^*,z)\, e^{ik^*x-it\beta(k^*)}}{d/dk(\beta(k)-kU(z))_{k=k^*}}$$

$$+ \int_{-\infty}^{\infty} \phi_1(k,z)\, e^{ikx-it\beta(k)}\, dk - \int_{-\infty}^{\infty} \phi_2(k,z)\, e^{ikx-itkU(z)}\, dk$$

$$+ \int_{-\infty}^{\infty} G_3(k,z)\, e^{ikx-itkU(z)}\, dk + O(e^{-t\lambda}) .$$

and , in the second layer , are

$$w_2(x,z;t) = \int_{\infty}^{\infty} F_3(k,z)\, e^{ikx-it\beta(k)}\, dk + O(e^{-t\lambda}) ,$$

$$u_2(x,z;t) = \int_{\infty}^{\infty} F_4(k,z)\, e^{ikx-it\beta(k)}\, dk + O(e^{-t\lambda}) ,$$

$$\xi_2(x,z;t) = -i\pi\Big[sgn(\beta'(k^*)-sgn(U(z))\Big] \frac{G_4(k^*,z)\, e^{ik^*x-it\beta(k^*)}}{d/dk(\beta(k)-kU(z))_{k=k^*}}$$

$$+ 2\pi\, W(\sigma^*) \frac{G_4(k^*,z)\, e^{ik^*x-it\beta(k^*)}}{d/dk(\beta(k)-kU(z))_{k=k^*}}$$

$$+ \int_{\infty}^{\infty} \phi_3(k,z)\, e^{ikx-it\beta(k)}\, dk - \int_{\infty}^{\infty} \phi_4(k,z)\, e^{ikx-itkU(z)}\, dk$$

$$+ O(e^{-t\lambda}) .$$

The functions $F_1(k,z)$, $F_2(k,z)$, $F_3(k,z)$ and $F_4(k,z)$ are analytic and well behaved in a domain containing the real axis . The function $\beta(k)$ has a saddle point at $k = k_o$. As t becomes large , the different components of perturbation velocity u_1, u_2, w_1 and w_2 tend to zero like ($1/t^{\frac{1}{2}}$) .

The integrals

$$\int_{-\infty}^{\infty} \phi_1(k,z)\, e^{ikx-it\beta(k)}\, dk \quad \& \quad \int_{-\infty}^{\infty} \phi_3(k,z)\, e^{ikx-it\beta(k)}\, dk$$

have saddle points , then they tend to zero like ($1/t^{\frac{1}{2}}$), since $\phi_1(k,z)$ and $\phi_3(k,z)$ are analytic and well behaved in a domain containing the real axis . But the integrals

$$\int_{-\infty}^{\infty} \phi_2(k,z)\, e^{ikx-itkU(z)}\, dk \quad , \quad \int_{-\infty}^{\infty} G_3(k,z)\, e^{ikx-itkU(z)}\, dk$$

and $\int_{-\infty}^{\infty} \phi_4(k,z)\, e^{ikx-itkU(z)}\, dk$,

where the functions $\phi_2(k,z)$, $\phi_4(k,z)$ and $G_3(k,z)$ are analytic and well behaved , they tend to zero like ($1/t$) as t becomes large, when $U(z) \neq 0$. Hence , when the model is free from any level which its mean velocity is zero,i.e. $U(z) = 0$, the vertical displacement of a fluid particle $\xi_1(x,z;t)$ and $\xi_2(x,z;t)$ tends to zero , provided $(\mathrm{sgn}(k_o-k^*)-\mathrm{sgn}(-tU(z))) = 0$, i.e. $U(z)$ and $d/dk(\beta(k))_{k=k^*}$ have the same sign . Therefore , if there is a level with zero velocity ($U(z)=o$) or $(\mathrm{sgn}(k-k^*)-\mathrm{sgn}(-tU(z))) \neq 0$, i.e. there is a width in which $U(z)$ and $d/dk(\beta(k))_{k=k^*}$ have a different sign , $\xi_1(x,z;t)$ and $\xi_2(x,z;t)$ $\not\rightarrow 0$, as $t \longrightarrow \infty$, here the linearized theory fails .

Conclusion :

At the time $t = 0$, some perturbation such that the velocity components and the vertical displacement are introduced. We use the linearized theory to see what happens for the disturbance functions as functions of time when t becomes large . From our expressions for the components of perturbation velocity ,they tend to zero as t becomes large like

(i) $1/t^{\frac{1}{2}}$, if $d/dk(\beta(k)) = 0$, i.e. there is a saddle point ,

(ii) $1/t$, if $d/dk(\beta(k)) \neq 0$,i.e. the function $\beta(k)$ is monotonic .

When the model is free from any level with $U(z) = 0$ and $(sgn(k_0-k^*)-sgn(-tU(z))) = 0$, i.e. the mean velocity $U(z)$ and $d/dk(\beta(k))_{k=k^*}$ have the same sign , hence , the vertical displacement tends to zero as t becomes large . But the linearized theory fails and the reason is that the vertical displacement of a fluid particle does not tend to zero as $t \longrightarrow \infty$, when the model contains a level with zero velocity ($U(z) = 0$) , or the model has a width in which the mean velocity $U(z)$ and $d/dk(\beta(k))_{k=k^*}$ has a different sign .

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