TO

CHRISTA AND JANE

UNIVERSITY OF LONDON

Imperial College of Science and Technology

Department of Physics
Applied Optics Section

THE USE OF WALSH FUNCTIONS IN THE DETERMINATION OF THREE-DIMENSIONAL REFRACTIVE INDEX DISTRIBUTIONS, AND OTHER DENSITY FIELDS, FROM EXPERIMENTAL DATA.

Michael Fernand Alexander DeRome.

Thesis submitted for the degree of Doctor of Philosophy
of the University of London, 1975.

ABSTRACT

The problem presented in this thesis is the determination of the three-dimensional distribution of refractive index, within a rectangular glass block, from experimental data. The accuracy of such a determination increases with the number of data points taken. Each data point, taken from an interferogram, represents the integrated optical path along a ray in a test light beam passing through the specimen. The extraction of the data from the interferogram is discussed briefly. It is shown that, for the small refractive index variations expected (i.e. of the order $\pm 5.10^{-6}$), the rays of light pass through the specimen in straight lines, and a suitable set of rays in any plane section of the specimen gives sufficient data for a determination of the two-dimensional refractive index field, within that plane section. The experimentally possible ray directions, through a plane rectangular section of the specimen, are discussed.

It is shown how the theory, developed for the determination of two-dimensional refractive index fields, applies to other density fields; for example the mass density within an opaque specimen, which can be determined when X-rays are passed through it. Each data point in this case, taken from a photographic plate, represents the density of matter along the path of each ray in the exploring X-ray beams.

The theory presented here, which involves representation of a two-dimensional density field by a finite series of orthogonal functions, is first described for any set of two-dimensional orthogonal functions and is then developed for the particular orthogonal set of two-dimensional Walsh functions. It is shown, from the theory, that only two different test beam directions are needed, and that these are easily achievable in practise. A simulated field is set up and is reconstructed. The theoretical reconstructio is compared with the original for different numbers of simulated input data points. The effect on the reconstruction, caused by random errors in the input data, is analysed.

A method is presented of finding orthogonal gradients of a two-dimensional field from the same input data. A way is suggested by which the resulting gradients can give some pseudo optical paths which, when used instead of the true optical paths as input data might give a better determination of the field.

TABLE OF CONTENTS.

Page.

CHAPTER 5. Continued.

APPENDIX 1. Continued. Page.

Chapter 1. The Extraction of Relative Optical Path Length Data from Experiments on a Rectangular Glass Block.

1.1 Introduction.

In recent years it has been possible to obtain interferograms from which the three-dimensional refractive index distribution within a phase object may be determined. A phase object is one which absorbs little or no light passing through it, but different parallel rays in a collimated beam of light, incident on the phase object, travel different optical path lengths through it. The relative path lengths can be recorded on interferograms.

Fig. (1.1) shows a schematic arrangement for obtaining suitable interferograms. Two mutually coherent beams of light are made to interfere after one of them, called the test beam or object beam, has passed through the object. The phase object, illustrated in Fig. (1.1), is shown enclosed by a rectangular boundary because the objects of main interest in this thesis are large rectangular sample blocks of glass, with dimensions of the order 100mm × 100mm × 100mm. These have refractive index variations of the order $\pm 5.10^{-6}$. Other phase objects include plasmas, transparent biological specimens and fluids with three-dimensional temperature variations within them.

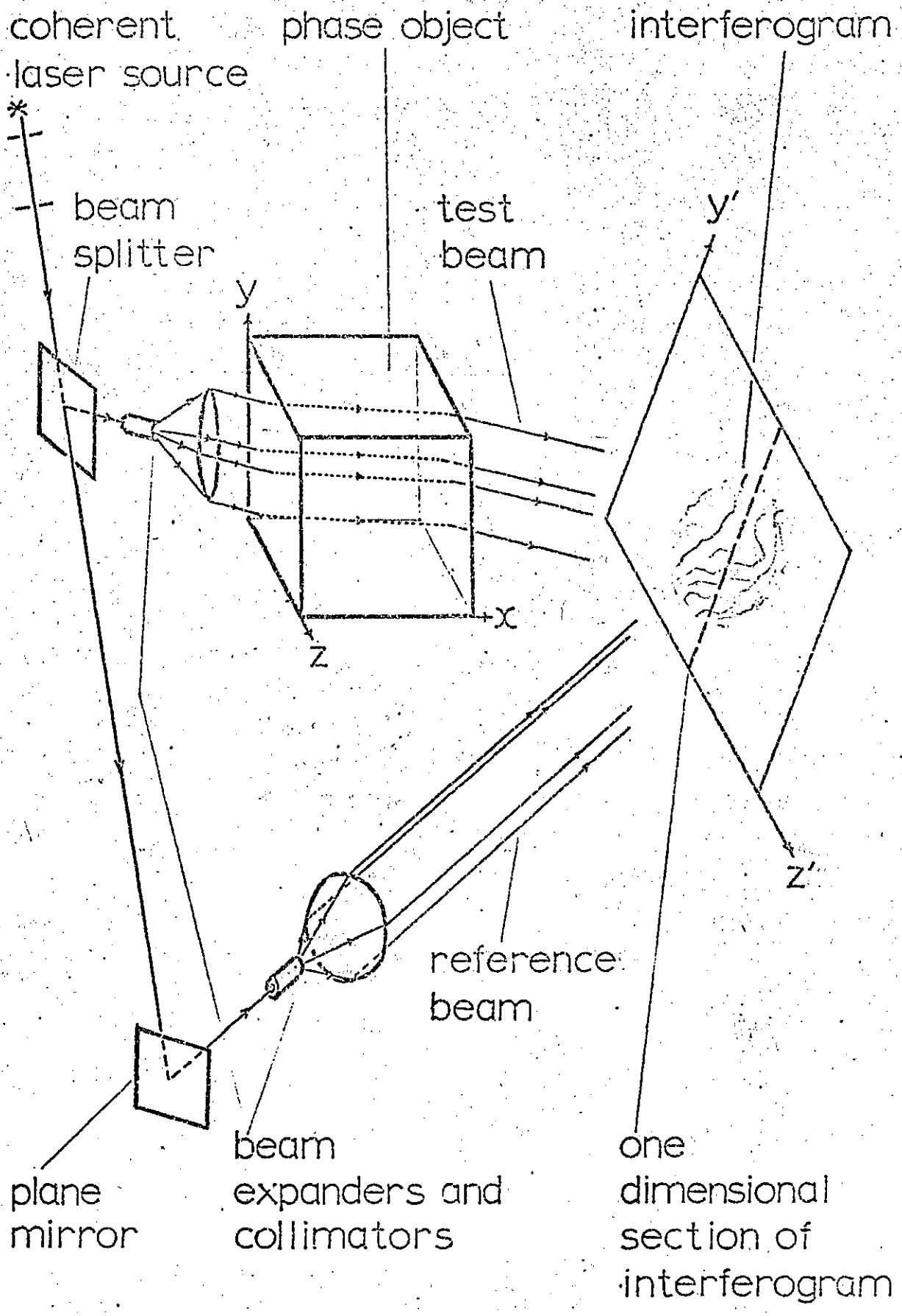
Now consider a ray, i, in the test beam which passes through the point (y_i', z_i') of the plane interferogram shown in Fig. (1.1). Its relative optical path length $p_i = p(y_i', z_i')$ can be defined as a line integral along the ray, i:

$$\int_i f(x,y,z)\,ds_i(x,y,z) = p_i = p(y_i', z_i') \quad , \qquad (1.1)$$

where $ds_i(x,y,z)$ is an element of length along the ray within the rectangular boundary, and $f(x,y,z)$ is some relative refractive index field given by

$$f(x,y,z) = n(x,y,z) - n_0 \quad .$$

n_0 is some constant refractive index, and $n(x,y,z)$ is the actual



Fig(1.1). A scheme for recording an interferogram or hologram from a phase object.

refractive index field of the phase object. If a particular relative optical path, say $p_1 = p(y_1,z_1) = 0$ at (y_1,z_1) on the interferogram, is chosen arbitrarily as zero, equation (1.1) becomes

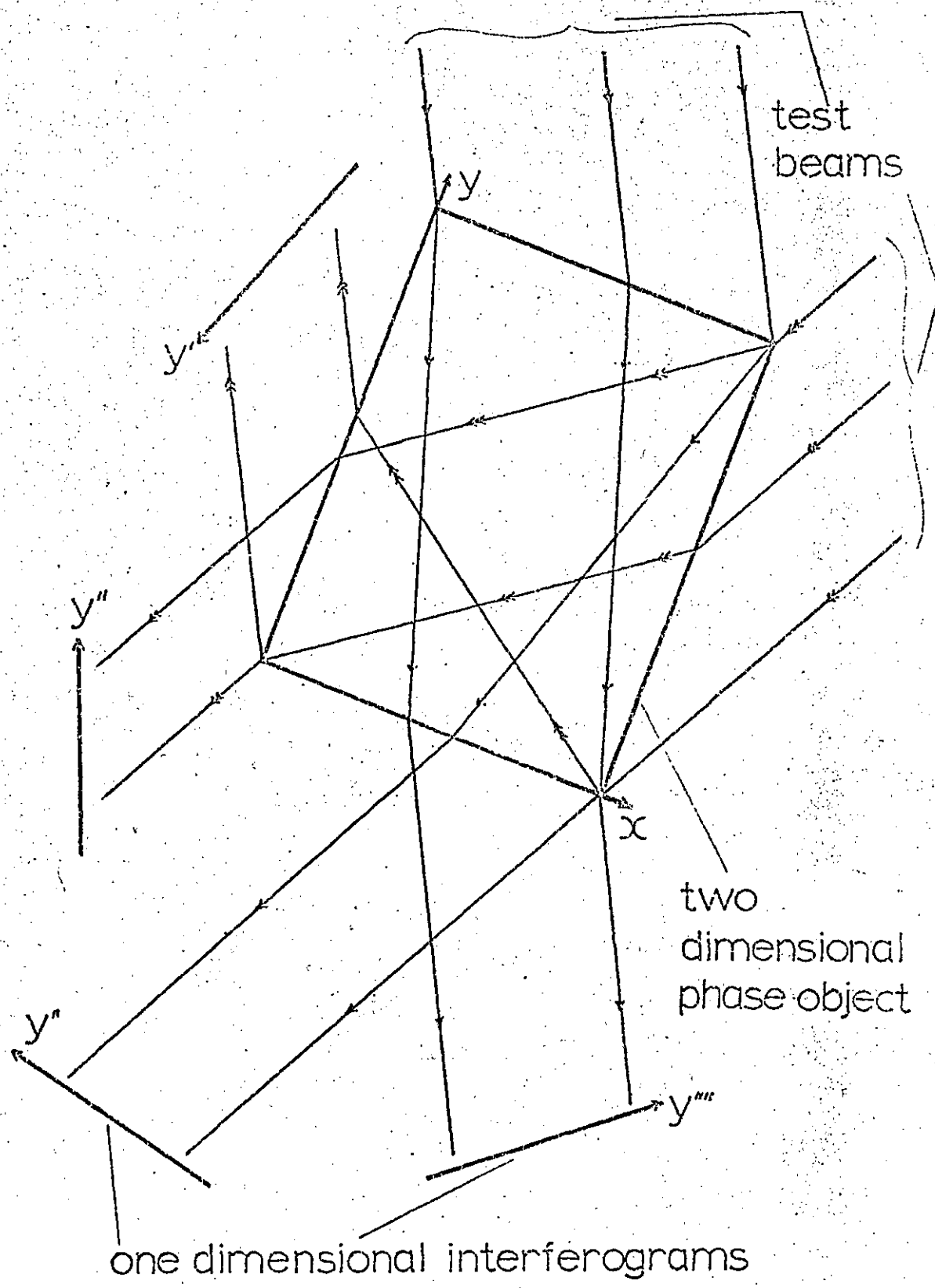
$$\int_{i=1}(n(x,y,z) - n_0)ds_1(x,y,z) = 0 \qquad (1.2)$$

and all other paths are measured relative to p_1. n_0 defined by equation (1.2) cannot be obtained from the interferogram, since only relative measurements of optical paths can be obtained from the latter. The theory in later chapters concerns itself with determining relative fields, $f(x,y,z)$, from relative optical path measurements. In general, several interferograms are required for the determination of a field $f(x,y,z)$, a different test beam being used each time. The explicit relationship between the intensity, $I(y_i',z_i')$, of a point in the interferogram and the relative path length, $p(y_i',z_i')$, at that point is discussed in section 1.2 below.

An important simplification of the theory can be made by assuming that the rays in the test beam passing through the phase object are straight lines. This assumption is justified in the case of the glass block having the small variations in refractive index already mentioned (i.e. $\pm 5.10^{-6}$). A phase object with such small refractive index variations that the straight line assumption is valid is called a "weak" phase object. By chosing the test beams, in Fig. (1.1), such that they are all perpendicular to the z axis of the weak phase object, any plane section z_c = a constant, of the object contains rays which give rise to a one-dimensional section $z' = z_c$ in the plane interferogram. Such plane sections of a weak phase object can be considered separately together with the one-dimensional section of the interferogram containing the relevant path length data. Dropping the constant value of z in equation (1.1) gives

$$\int_i f(x,y)ds_i(x,y) = p_i = p(y_i') . \qquad (1.3)$$

Fig. (1.2) shows a single "two-dimensional phase object" (i.e. a section z = constant of the three-dimensional phase object) with a



Fig(1.2) Some possible positions of one-dimensional interferograms for different test beam directions through a given two-dimensional phase object. The interferograms are in the same plane as the phase object. The reference beams for each of the test beams are not shown to avoid confusion.

relative refractive index field f(x,y) being "explored" by several test beams which ., together with a reference beam, form various "one-dimensional interferograms". (i.e. a section through a two-dimensional interferogram).

In any experiment, interferograms are not set up simulta neously, as shown, but sequentially.

From now on "phase object" will mean a "two-dimensional phase object" and "interferogram" will mean a "one-dimensional interferogram".

In general, the data from several interferograms is required for a determination of the relative two-dimensional refractive index field, f(x,y). Each point on an interferogram will be called a "data point". The relative optical paths from a number of data points each give rise to an equation like equation (1.3). From a finite number of data points from a finite number of interferograms, it is only possible to determine the relative field, f(x,y), approximately, even if no experimental errors are assumed. In general, the approximation to the field will improve as more data points are used.

1.2 An Explicit Expression for the Optical Paths at a double exposed Hologram in terms of the intensity of the reconstructed object beam.

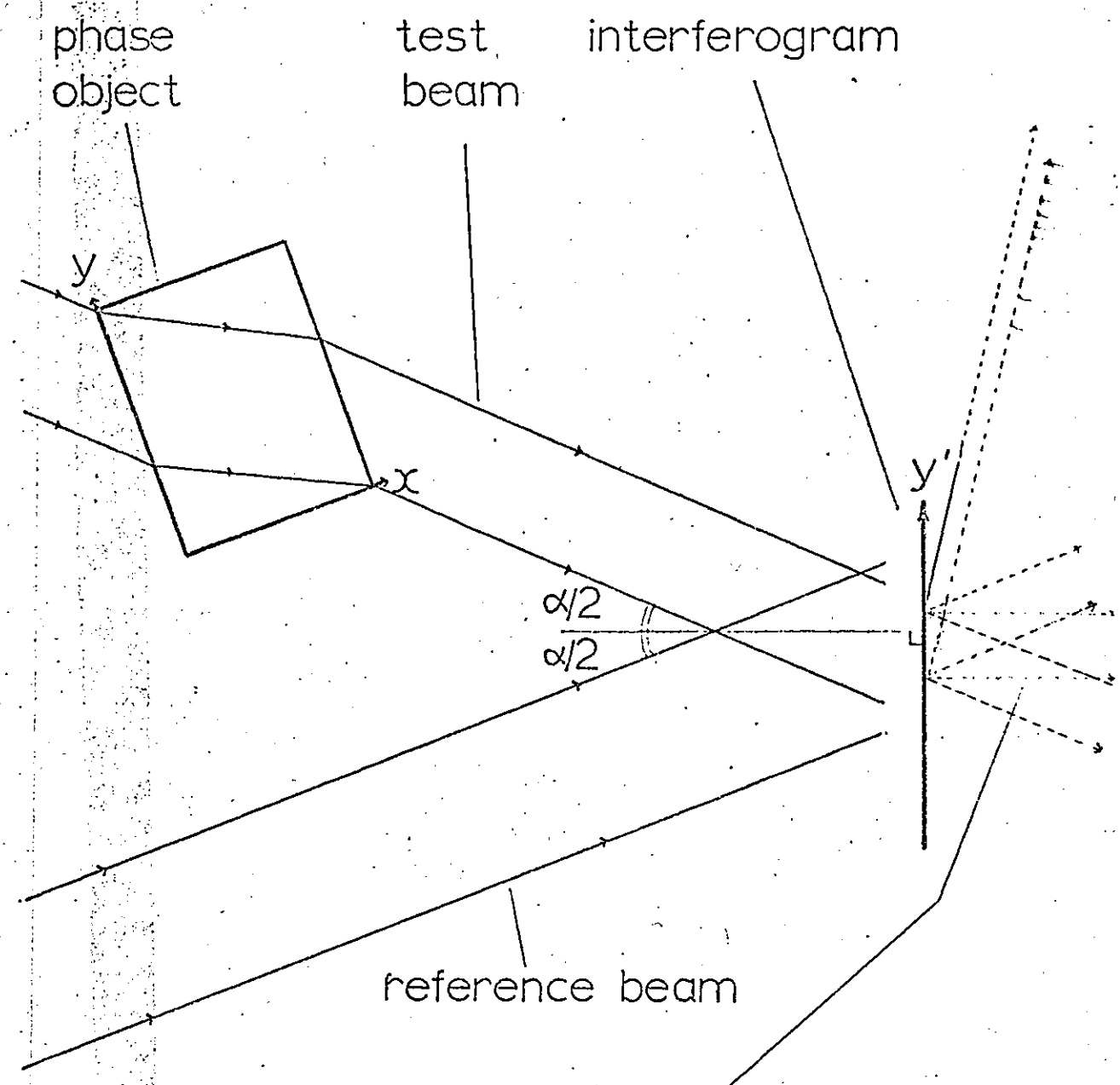
Suppose that the photographic or holographic plate used to record the interferogram is placed perpendicular to the bisector of the angle, α, between the collimated test and reference beams, as shown in Fig. (1.3). First consider the complex amplitude distribution at the hologram if the phase object is not present. The initial real amplitude of both beams is taken as $\frac{1}{2}$. λ is the wavelength of the light used. Then:

The complex amplitude distribution due to the reference beam is

$$V_r = \tfrac{1}{2} e^{2\pi i y' / (\lambda / \sin(\frac{\alpha}{2}))}$$

The complex amplitude distribution due to the test beam is

$$V_o = \tfrac{1}{2} e^{-2\pi i y' / (\lambda / \sin(\frac{\alpha}{2}))}$$



complex amplitude of object beam is

$$U(y') = e^{-\pi i y/\bar{\lambda}}(1 + e^{2\pi i p(y')/\lambda})/2, \text{ where}$$

$$\bar{\lambda} = \lambda/2\sin(\alpha/2),\ i = \sqrt{-1}.$$

Fig(13) The geometry used in the formation of the hologram is shown in bold lines. The three beams formed when the hologram is reconstructed with the same reference beam are shown dotted.

The first exposure is taken when the phase object is absent and the intensity recorded on the hologram is

$$I_o = (V_r + V_o)(V_r + V_o)^* = |V_r|^2 + |V_o|^2 + V_r V_o^* + V_r^* V_o$$
$$\simeq \tfrac{1}{2} + V_r V_o^* + V_r^* V_o \,. \qquad (1.4)$$

When the phase object is introduced into the test beam, the optical path of each ray arriving at each point, y', on the hologram is increased by p(y') so that the complex amplitude of the test ray is now

$$V_1 = \tfrac{1}{2} e^{-2\pi i y'/\lambda / \sin(\frac{\alpha}{2})} . e^{2\pi i p(y')/\lambda} \,. \qquad (1.5)$$

The second exposure is taken with the phase object present and the intensity recorded on the hologram is now

$$I_1 = (V_r + V_1)(V_r + V_1)^* = |V_r|^2 + |V_1|^2 + V_r V_1^* + V_r^* V_1$$
$$= \tfrac{1}{2} + V_r V_1^* + V_r^* V_1 \,. \qquad (1.6)$$

The total intensity recorded on the hologram is

$$I_T = I_1 + I_2 = 1 + (V_o + V_1)^* V_r + (V_o + V_1) V_r^* \,. \qquad (1.7)$$

If the ranges of exposure are chosen such that the amplitude transmittance, T = T(y'), of the hologram varies linearly with exposure we have

$$T = A - B.I_T = A - B - BV_r(V_o + V_1)^* - BV_r^*(V_o + V_1) \,. \qquad (1.8)$$

Now this condition is only possible if the reference beam has a higher intensity than the test beams. (Typically three times). The effect of including this in the analysis is that the last two terms in equation (1.7) are multiplied by a fraction. Physically this means that the modulation of the hologram fringe pattern is reduced from 1 to a fraction of 1 as they must be for the correct exposure condition already mentioned. As will be seen however the final form of the reconstructed object beam does not depend on the ratio of reconstruction beam intensity and test beam intensity. With the provisos above in mind the analysis is continued.

The sum of the two test beam amplitudes or the object wave, $V_o + V_1$, can be written as

$$U = U(y') = \tfrac{1}{2}e^{-\pi i y'/\lambda'}(1 + e^{2\pi i p(y')/\lambda}) \quad , \tag{1.9}$$

where

$$\lambda' = \lambda/2\sin(\tfrac{\alpha}{2}) \quad . \tag{1.10}$$

Now if the hologram is reconstructed with the same plane reference beam, V_r, the complex amplitude just in front of the hologram is given by

$$TV_r = CV_r - BV_r^2U^* - \tfrac{1}{2}BU \quad \text{, where } C = A - B. \tag{1.11}$$

Equation(1.11) represents three beams diffracted in three different directions. CV_r is the undeviated reference beam travelling at an angle to the normal to the hologram. $BV_r^2U^* = \tfrac{1}{4}Be^{3\pi i y'/\lambda'}(1 + e^{2\pi i p(y')/\lambda})$ is the conjugate object beam which travels at an angle $\sin^{-1}(3\sin\frac{\alpha}{2})$ to the normal(assuming $\sin\frac{\alpha}{2} \leq \frac{1}{3}$). $\tfrac{1}{2}BU$ is the required object beam which travels at $-\alpha/2$ to the hologram. The intensity of this beam, if recorded sufficiently far from the hologram for it not to interfere with the other diffracted beams, is proportional to

$$I = UU^* = 1 + \cos 2\pi p(y')/\lambda' \quad . \tag{1.12}$$

This represents a fringe pattern with maxima at points given by

$$p(y') = n\lambda' \quad , n = \text{any integer}, \tag{1.13}$$

and minima at points given by

$$p(y') = (n + \tfrac{1}{2})\lambda' \quad . \tag{1.14}$$

The intensity across the object beam can be deduced from a densitometer plot of a photograph of it.

The author worked closely with I.M.Siddiqui who gives full details of the holographic technique in his thesis,(3).

[illegible] is very [illegible]. [illegible] difference in the [illegible] refractive index [illegible] will be [illegible] than 10^{-5} so that the [illegible] path difference expected [illegible].

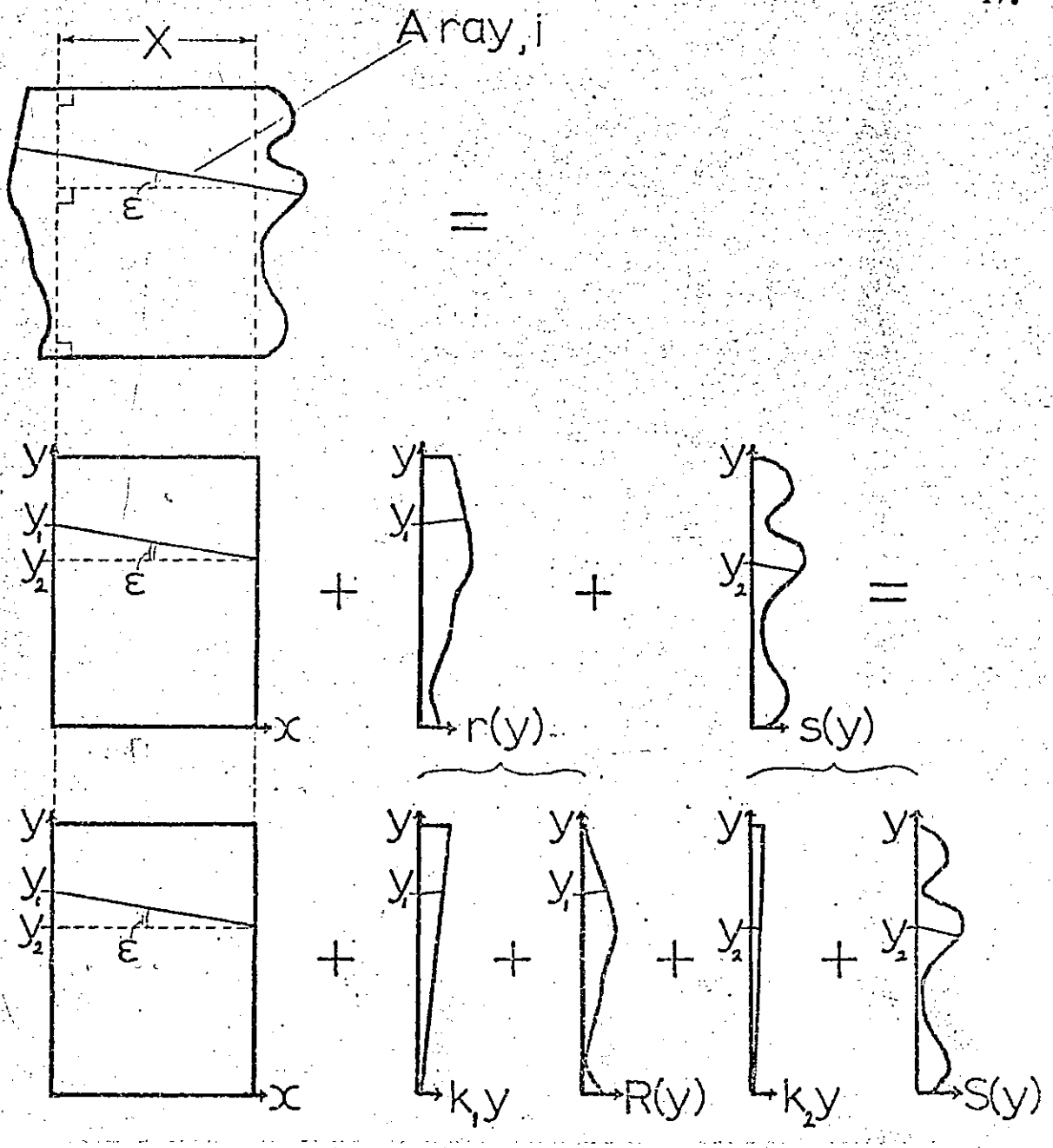
[illegible] for obtaining path difference data from interferograms is only one of many such methods. The choice of method [illegible] on the [illegible] particular experimental technique, and the [illegible] worked closely with J. [illegible] (3), who used a technique of holographic interferometry to obtain the path difference data.

1.3 "Surface Effects."

The faces of the glass block are not perfectly flat, and neither are the mean levels of opposite faces perfectly parallel. This means that a plane section as shown in Fig. (1.2) is not exactly rectangular. Now the theory developed later relies on perfectly rectangular boundaries for the phase object being considered. Thus the actual boundary can best be considered as a perfect rectangle plus a very narrow strip of variable width around it. The theory developed later uses as its input data relative optical paths of rays in a collimated beam, passing from one side of an exact rectangular boundary, through the phase object, and out of the opposite side. Thus the path difference obtained in an experiment must be "adjusted" somehow to allow for the contribution made to the relative paths by the "narrow strip of variable width" mentioned above. The contributions to relative paths introduced by the "narrow strip" can be called "surface effects".

If two opposite sides of the boundary are considered, it is possible to consider these as follows: two perfectly straight parallel lines plus a wedge plus higher order variations. This is illustrated in Fig. (1.4).

The effect of the wedge will be to change the direction of the test beam so that the angle, α, between reference and test beams at the interferogram changes. This causes the mean fringe spacing to



$$r(y)=k_1y+R(y),\ s(y)=k_2y+S(y),\ \cos\varepsilon=\frac{X}{\sqrt{X^2+(y_1-y_2)^2}}.$$

$$t_i=(r(y_1)+s(y_2))/\cos\varepsilon$$

Fig(1.4) The contributions to "surface effects".A ray,i,is defined by its intercepts,y_1 and y_2,at opposite edges of the "ideal" rectangular boundary.The two surfaces are given by the polynomials,r(y) and s(y). k_1 and k_2 are constants.R(y) and S(y) are polynomials in terms of order 2 and above.t_i is the total extra glass thickness traversed by the ray,i.The deviation of the two surfaces from being perfectly plane and parallel with one another is grossly exagerated.

change. By counting a large number of fringes in a given distance before and after the phase object has been introduced, it is possible to measure the change in α and hence the angle of the wedge, β, in Fig. (1.4).

It is possible to obtain fringes from any <u>surface</u> of a glass block by reflecting a test beam from it. From these fringes it is possible to work out the shape of the surface under test.

The surface effects due to the wedge and "higher order" variations can be calculated easily enough: If t_i is the total extra thickness of glass that the ray, i, must travel between the ideal boundary and the true surface, then the surface effect contribution to the relative path, p_i, is given, to a sufficiently good approximation, by $n_{av}t_i$, where n_{av} is the mean refractive index of the glass (known to at least four places of decimals). If the mean refractive index over the short traverse of t_i were actually $n_{av} + \delta n$ (where $|\delta n| \leqslant 10^{-5}$) the magnitude of the error in using n_{av} as the index would be $\delta n\, t_i$. Now suppose t_i were as much as 1mm. The maximum error in relative path would then be $10^{-5}.10^{6} \approx 10$nm., a figure representing a shift in fringe maxima of about $\lambda/50$, which is well below the limits of detection. (i.e. $\lambda/10$ or so.) I. Siddiqui, in his thesis,(3), describes in detail how surface effects are measured using holographic interferometry.

Chapter 2 The Determination of any Two-Dimensional Density Field from Line Integral Data, using Orthogonal Functions.

2.1 The Definition of a Density Field.

A three-dimensional scalar field describes a property of a substance at any particular point by specifying its magnitude at that point. When a specimen of a substance is "explored" with collimated beams of electromagnetic or electron radiation, it is often possible to record some property associated with the rays of the radiation. If the line integral of the scalar field along the ray can be related to this property, the field is called a density field. Refractive index is an example of a property describable by a density field.

A theory which uses relative optical path length data to make a determination of the three-dimensional refractive index field can be used to determine any three-dimensional density field from the line integrals, along rays passing through that density field.

In X-ray photography of opaque three-dimensional objects and also, to some extent, in electron microscopy of non-thin specimens, the density field of concern is simply mass density, $\rho(x,y,z)$. Since X-rays and electron beams suffer negligible refraction in most specimens of interest, the straight line approximation reduces the problem to finding plane (z = constant, say) sections of field, $\rho(x,y)$, from rays passing through the given plane.

The quantity actually measured is photographic density, which can be related to intensity by a relationship depending on the type of film used. The intensity is related to the line integral,

$$\int_i \rho(x,y)\,ds_i(x,y) = \sigma_i = \sigma(y_i'), \tag{2.1}$$

by the relation

$$I(y_i') = I_0 e^{-\sigma(y_i')}, \tag{2.2}$$

where y' in equations (2.1) and (2.2) is a coordinate passing through the photographic plate, and I_0 is the intensity of all the rays before

entering the specimen. Equations (2.1) and (1.3) are analogous. The line integral quantity, σ_i, analogous with optical path, is a measure of the total mass traversed by the ray, i, per unit area.

The rest of this chapter describes a method, using orthogonal functions, for determining any particular two-dimensional density field, f(x,y), within an exact rectangular boundary, from line integral data, corrected for "surface effects". The set of rays passing through the rectangular boundary, which leads to a successful determination of the density field, depends on which set of orthogonal functions is used. The choice of ray directions often depends on what is experimentally possible for any particular density field. The discussion on the choice of the orthogonal set of Walsh functions for the refractive index density dield is deferred until chapter 3.

2.2 The Representation of the Density Field in terms of a Series of Two-Dimensional Orthogonal Functions.

Fig. $(2.1)_{(a)}$ shows the rectangular boundary of the two-dimensional object having dimensions X by Y. Let $f(\frac{x'}{X},\frac{y'}{Y})$ represent the density field within this boundary, so that f(x,y) represents the density field in a unit sided square boundary obtained by scaling the actual boundary by factors $\frac{1}{X}$ and $\frac{1}{Y}$. This is done for mathematical convenience, many constants involving X and Y being avoided in the theory.

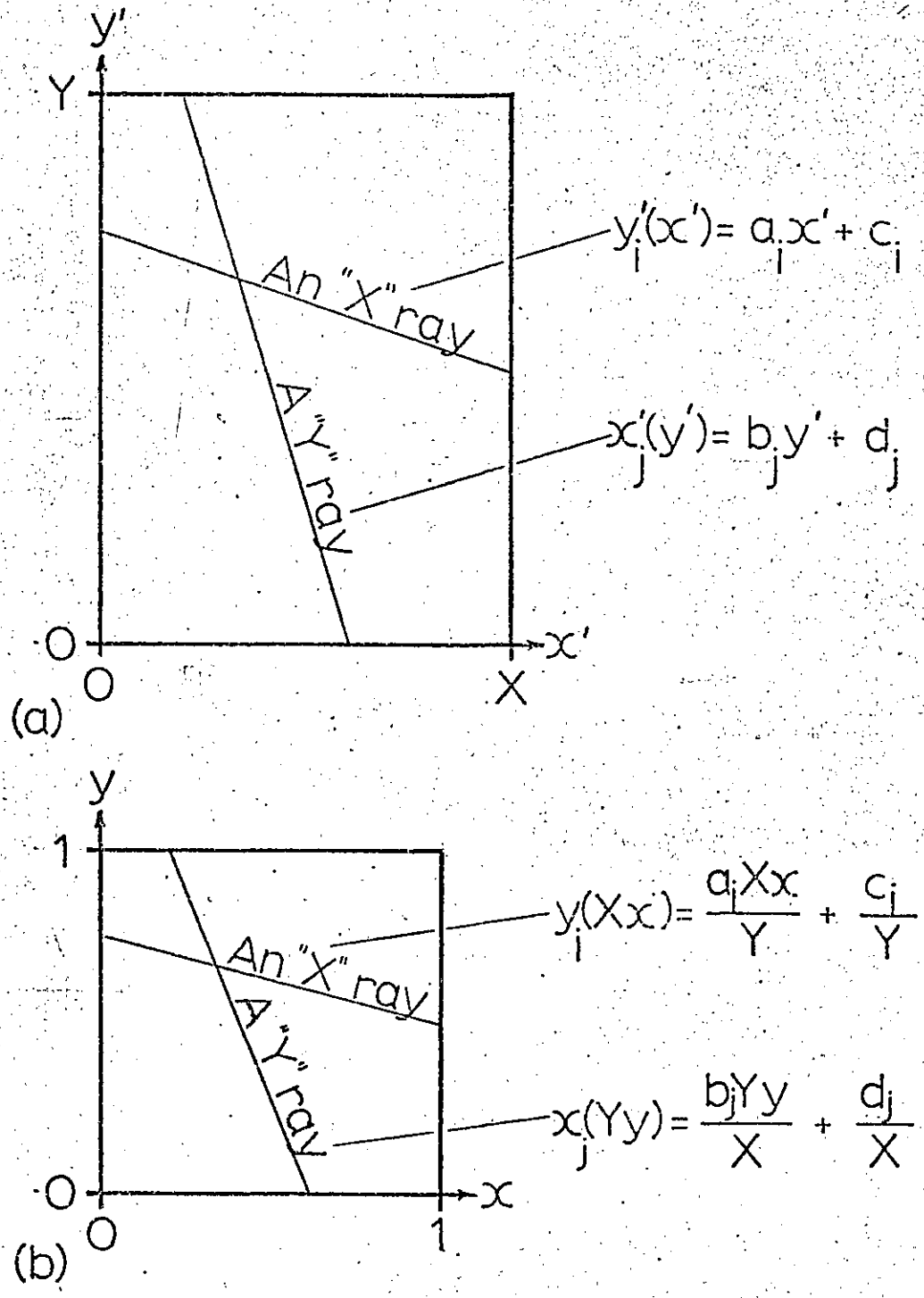
Extensive reference will be made in the rest of this chapter to results in Appendix 1, which deals with orthogonal sets of functions and the representations of one and two-dimensional functions in terms of such sets. In the formulae of Appendix 1, the values of x_2 and y_2 should be replaced by zero to be directly applicable in this chapter.

Let $\{\psi_{pq}(x,y)\}$ be an infinite two-dimensional set of orthonormal functions in the range $0 \leq x < 1$, $0 \leq y < 1$ defined by the orthogonality relation

$$\int_0^1\int_0^1 \psi_{pq}(x,y)\psi^*_{pq}(x,y)\,dxdy = \delta_{pp',qq'} = \begin{cases} 0 \text{ if } p \neq p' \text{ or } q \neq q' \\ 1 \text{ if } p = p' \text{ and } q = q' \end{cases} \qquad (2.3)$$

and

$$\psi_{00}(x,y) = 1, \text{ for all } x \text{ and } y. \qquad (2.4)$$



Fig(2.1). (a) The rectangular boundary of a phase object with an "X" ray and "Y" ray superimposed on it. (b) The corresponding unit square boundary and corresponding "X" ray and "Y" ray. In the diagrams $X=6/5, Y=8/5, a_i=-1/3, c_i=6/5, b_j=-3/10, d_j=18/5$.

p and q are consecutive integer labels which are contained in two infinite sets, $\{(p,\infty)\}$ and $\{(q,\infty)\}$. (For most sets of orthonormal functions, $\{\psi_{pq}(x,y)\}$ these label sets are given by $\{(p,\infty)\} = \{-\infty<p<\infty\}$ and $\{(q,\infty)\} = \{-\infty<q<\infty\}$ or by $\{(p,\infty)\} = \{0\leq p<\infty\}$ and $\{(q,\infty)\} = \{0\leq q<\infty\}$. See sections A1.1.1 and A1.3.1).

For a complete closed set, $\{\psi_{pq}(x,y)\}$, it is possible to express a continuous function f(x,y) exactly by the infinite series

$$f(x,y) = \sum_{p,\infty}\sum_{q,\infty} F_{pq}\psi_{pq}(x,y), \tag{2.5}$$

where the coefficients, F_{pq}, are uniquely given by the relation

$$F_{pq} = \int_0^1\int_0^1 f(x,y)\psi^*_{pq}(x,y)dxdy. \tag{2.6}$$

The notation "p,∞" and "q,∞" in equation (2.5) denotes the summation of p and q over all values in the sets $\{(p,\infty)\}$ and $\{(q,\infty)\}$.

An approximation for f(x,y) may be given by the finite series of MN terms:

$$f(x,y) \simeq f_a(x,y) = \sum_{p,M}\sum_{q,N} E_{pq}\psi_{pq}(x,y), \tag{2.7}$$

where M and N are integers, and the notation "p,M" and "q,N" denotes respectively, summation of p and q over a subset $\{(p,M)\}$ of $\{(p,\infty)\}$ containing M values of p from the latter, and a subset $\{(q,N)\}$ of $\{(q,\infty)\}$ containing N values of q from the latter. E_{pq} are coefficients as yet undetermined. It is shown in Appendix 1. that the best root mean square approximation to f(x,y) i.e. the one for which $Q = \int_0^1\int_0^1 |f(x,y) - f_a(x,y)|^2 dxdy$ is a minimum, occurs when $E_{pq} = F_{pq}$ (given by equation (2.6)), regardless of which subsets $\{(p,M)\}$ and $\{(q,N)\}$ are used for the series in equation (2.7). Thus the best series approximation for f(x,y), using MN orthonormal functions from the set $\{\psi_{pq}(x,y)\}$ is given by

$$f(x,y) \simeq f_a(x,y) = \sum_{p,M}\sum_{q,N} F_{pq}\psi_{pq}(x,y). \tag{2.8}$$

For most functions, f(x,y), the subsets $\{(p,M)\}$, $\{(q,N)\}$ which minimize Q are those containing the smallest M and N consecutive values respectively

of $|p|$ and $|q|$ in the sets $\{(p,\infty)\}$ and $\{(q,\infty)\}$. Thus if $\{(p,\infty)\} = \{0 \leq p < \infty\}$ and $\{(q,\infty)\} = \{0 \leq q < \infty\}$ then $\{(p,M)\} = \{0 \leq p < M\}$ and $\{(q,N)\} = \{0 \leq q < N\}$. (This is discussed for one-dimensional functions more fully in section A1.2.1)

From the defining relations, equation (2.3) and (2.4), it can easily be shown (see lemma 1, section A1.1.1 for the one-dimensional case) that the mean value of each function in the infinite set $\{\psi_{pq}(x,y)\}$, except for $\psi_{00}(x,y)$, is zero. From this it follows that, in the series for $f_a(x,y)$ above, F_{00} represents the mean value of $f_a(x,y)$ and all the other terms $F_{pq}\psi_{pq}(x,y)$, p and q not both zero, have a mean value of zero, so together they represent the variations about the mean of $f_a(x,y)$. It is these "a.c." terms which are of interest in most cases.

It is shown in section A1.3.2 that the two-dimensional set of functions $\{\phi_p(x)\psi_q(y)\}$ is an orthonormal set where $\{\phi_p(x)\}$ and $\{\psi_q(y)\}$ are one-dimensional orthonormal sets in the ranges $0 \leq x < 1$ and $0 \leq y < 1$ respectively. Such a set is called a separable two-dimensional set.

2.3 A Scheme for obtaining the MN Coefficients, F_{pq}, of the Finite Series for the Density Field from Line Integral Data.

Two types of rays in the test beams passing through the object will be distinguished. They are called "X" rays and "Y" rays respectively.

(a) "X" rays pass from points on the boundary line $x' = 0$ to points on the ray $x' = X$ in Fig. $(2.1)_{(a)}$. An "X" ray, i, can be given by the equation

$$y_i'(x') = a_i x' + c_i, \tag{2.9}$$

where a_i is the constant gradient $\frac{dy'}{dx'}$ of the ray, and c_i is its intercept on the y' axis.

From equation (1.3), the line integral p_i is given by

$$\int_o^X f\left(\frac{x'}{X}, \frac{y_i'(x')}{Y}\right) ds_i(x', y'(x')) = p_i. \tag{2.10}$$

But $ds_i = \sqrt{dx'^2 + dy'^2} = dx\sqrt{1 + \left(\frac{dy'}{dx'}\right)^2} = dx'\sqrt{1 + a_i^2}$. The scaling operation

$x = \frac{x'}{X}$ and $y = \frac{y'}{Y}$ transforms an X by Y rectangle in the $x'-y'$ plane into a unit sided square in the x-y plane. $dx' = Xdx$. This change of variables in equation (2.10) gives

$$X\sqrt{1 + a_i^2}\int_0^1 f(x, y_i(xX))dx = p_i, \tag{2.11}$$

where $y_i(xX)$ is given explicitly in terms of x by putting $x' = xX$ and $y' = yY$ in equation (2.9):

$$y_i(xX) = (\frac{a_i X}{Y})x + \frac{c_i}{Y}. \tag{2.12}$$

Equation (2.12) is the expression for the "X" ray, i, on the scaled unit square as shown in Fig. $(2.1)_{(b)}$.

(b) "Y" rays pass from points on the boundary $y' = 0$, to points on the ray, $y' = Y$ in Fig. $(2.1)_{(a)}$. A "Y" ray, j, can be given by the equation

$$x_j'(y') = b_j y' + d_j, \tag{2.13}$$

where b_j is the constant gradient $\frac{dx'}{dy'}$ of the rays, and d_j is its intercept on the x' axis.

From equation (1.3) the line integral p_j is given by

$$\int_0^Y f(x_j'(y')/X, y'/Y)ds_j(x'(y'), y') = p_j. \tag{2.14}$$

But $ds_j = \sqrt{dx'^2 + dy'^2} = dy\sqrt{1 + \left(\frac{dx'}{dy'}\right)^2} = dy'\sqrt{1 + b_j^2}$. The scaling operation $x = \frac{x'}{X}$ and $y = \frac{y'}{Y}$ transforms equation (2.14) to

$$Y\sqrt{1 + b_j^2}\int_0^1 f(x_j(yY), y)dy = p_j, \tag{2.15}$$

where $x_j(yY)$ is given explicitly in terms of y by

$$x_j(yY) = (\frac{b_j Y}{X})y + \frac{d_j}{X}, \tag{2.16}$$

equation (2.16) being the "Y" line, j, on the scaled unit square.

In equation (2.11) it can be seen that $X\sqrt{1 + a_i^2}$ is the actual length of the ray, i. By dividing both sides of equation (2.11) by this length, an important quantity can be defined:

$$B_i = \frac{p_i}{X\sqrt{(1 + a_i^2)}} = \int_0^1 f(x, y_i(xX))dx \qquad (2.17)$$

likewise

$$B_j = \frac{p_j}{Y\sqrt{(1 + b_j^2)}} = \int_0^1 f(x_j(yY), y)dy \qquad (2.18)$$

In both cases the quantity "B" is the line integral which the ray would have if its length were scaled to unity. This will be called the normalized line integral or, in the case of f(x,y) being a refractive index field, it is the normalized relative path. The normalized line integral is given by the line integral of the scaled density field along the corresponding scaled line divided by the actual length of the ray.

Now substituting the finite series approximation $f_a(x,y)$, given by equation (2.8), into the expression for normalized line integrals, equations (2.17) and (2.18) give:

(a) The normalized line integral for an "X" ray, i, is

$$B_i = \frac{p_i}{X\sqrt{(1 + a_i^2)}} \simeq \sum_{p=M} \sum_{q=N} F_{pq} \int_0^1 \phi_{pq}(x, y_i(xX))dx \qquad (2.19)$$

and

(b) The normalized line integral for a "Y" ray, j, is

$$B_j = \frac{p_j}{Y\sqrt{(1 + b_j^2)}} \simeq \sum_{p=M} \sum_{q=N} F_{pq} \int_0^1 \phi_{pq}(x_j(yY), y)dy. \qquad (2.20)$$

Equations (2.19) and (2.20) can be written

$$B_m \simeq \sum_{p=M} \sum_{q=N} F_{pq} \theta^m_{pq}, \qquad (2.21)$$

where for an "X" ray, m, of normalized line integral B_m:

$$\theta^m_{pq} = \int_0^1 \phi_{pq}(x, y_m(xX))dx, \qquad (2.22)$$

while for a "Y" ray , m, of normalized line integral, B_m:

$$\theta^m_{pq} = \int_0^1 \psi_{pq}(x_m(yY),\ y)dy. \qquad (2.23)$$

In equations (2.21), (2.22) and (2.23) m is a superscript, not a power.

Equation (2.21) contains MN unknowns, namely the coefficients F_{pq}. By solving a set of MN or more such equations, all independant of one another, it should be possible to solve for all MN coefficients and hence, by equation (2.8), determine $f_a(x,y)$, the approximation for f(x,y). To obtain the MN or more such equations requires choosing a "correct" set of lines.

To illustrate some possible solution schemes consider the set $\{\psi_{pq}(x,y),\ 0 \leq p < \infty,\ 0 \leq q < \infty\}$ so that $\{(p,N)\} = \{0 \leq p < N\}$ and $\{(q,N)\} = \{0 \leq q < N\}$. A set of MN equations of the type, equation (2.21), can be written in matrix form as follows:

$$\begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{MN} \end{bmatrix} = \begin{bmatrix} \theta^1_{0,0} & \theta^1_{1,0} \cdots \theta^1_{M-1,0} & \theta^1_{01} & \theta^1_{11} \cdots \theta^1_{M-1,1} & \theta^1_{02} \cdots & \theta^1_{M-1,N-1} \\ \theta^2_{0,0} & \theta^2_{1,0} \cdots \theta^2_{M-1,0} & \theta^2_{01} & \theta^2_{11} \cdots \theta^2_{M-1,1} & \theta^2_{02} \cdots & \theta^2_{M-1,N-1} \\ \vdots & & & & & \\ \theta^{MN}_{0,0} & \theta^{MN}_{1,0} \cdots \theta^{MN}_{M-1,0} & \theta^{MN}_{01} & \theta^{MN}_{11} \cdots \theta^{MN}_{M-1,1} & \theta^{MN}_{02} \cdots & \theta^{MN}_{M-1,N-1} \end{bmatrix} \begin{bmatrix} F_{0,0} \\ F_{1,0} \\ \vdots \\ F_{M-1,0} \\ F_{0,1} \\ F_{1,1} \\ \vdots \\ F_{M-1,1} \\ F_{0,2} \\ \vdots \\ F_{M-1,N-1} \end{bmatrix}. \qquad (2.24)$$

θ^m_{pq} and F_{pq} are ordered so that p changes before q does. From the defining equation (2.4) (i.e. $\psi_{0,0}(x,y) = 1$) and equations (2.22) and (2.23) it is true that for all lines, m:

$$\theta^m_{0,0} = 1\ , \qquad (2.25)$$

so that all the line integrals in equation (2.24) all contain F_{00} as the first term of the sum of MN terms. But the measurement of line integrals, p_m, and hence the normalized quantity, B_m, is nearly always relative. The path lengths in Chapter 1. were relative, not absolute. The quantity, $F_{0,0}$, found by solving the equations , equation (2.24), is arbitrary depending on which line integral all the others were measured relative to. This arbitrary constant is different for different sets of data points. For example, in the case of the refractive index determination several interferograms each constitute a different set of data points. The members of any one set are measured relative to a chosen one, whose path length is arbitrarily chosen to be zero. Thus the "$F_{0,0}$'s" from different sets will, in general, be different, and more than MN equations will be required to solve for the MN - 1 coefficients, F_{pq}, p and q not zero together.

To make this clearer, suppose that the MN data points are in three sets of I, J and K data points respectively (I + J + K = MN). Then equation (2.24) must be rewritten as follows:

$$\begin{bmatrix} B_1 \\ \vdots \\ B_I \\ B_{I+1} \\ \vdots \\ B_{I+J} \\ B_{I+J+1} \\ \vdots \\ B_{MN} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \theta'_{1,0} & \cdots & \theta'_{M-1,N-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \theta^{I}_{1,0} & \cdots & \theta^{I}_{M-1,N-1} \\ 0 & 1 & 0 & \theta^{I+1}_{1,0} & \cdots & \theta^{I+1}_{M-1,N-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 1 & 0 & \theta^{I+J}_{1,0} & \cdots & \theta^{I+J}_{M-1,N-1} \\ 0 & 0 & 1 & \theta^{I+J+1}_{1,0} & \cdots & \theta^{I+J+1}_{M-1,N-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 1 & \theta^{MN}_{1,0} & \cdots & \theta^{MN}_{M-1,N-1} \end{bmatrix} \begin{bmatrix} F'_{0,0} \\ F''_{0,0} \\ F'''_{0,0} \\ F_{1,0} \\ \vdots \\ F_{M-1,N-1} \end{bmatrix} \qquad (2.26)$$

The elements $\theta^{m}_{0,0}$ have been written as 1 in equation (2.26). The last MN-1 columns of the "θ" matrix are identical with those of the "θ" matrix in equation (2.24). The last MN-1 elements in the column matrix of coefficients F_{pq} are the same in both equations (2.24) and (2.26).

$F'_{0,0}$, $F''_{0,0}$, and $F'''_{0,0}$ are different in general, because each of the three separate sets of data points is relative to a different reference level. Equation (2.26) now represents a set of MN equations in MN+2 unknowns, and so cannot be solved. Another two data points are required for a solution to be possible. In general, for a number, L, of separate sets of data points, the minimum number of data points required for a complete solution set of coefficients, $\{F'_{0,0}, F''_{0,0}, \ldots F_{pq}$, p and q not both zero, $0 \leq p < M$, $0 \leq q < N\}$, is MN-1+L.

Under certain conditions the set of coefficients, $\{F_{pq}\}$, can be solved for by independantly solving for the subsets of it. A subset of, say, P (P < MN-1+L, where L is the number of distinct sets of line integral data points) coefficients will be the solution of P equations of the type equation (2.21), in P unknowns. To find out these conditions for the case where the sets of line integral data points can be solved independantly from one another, consider once again the example of three such sets. Suppose that the three sets contain I, J+1 and K+1 data points respectively (I+J+K = MN) then a matrix equation like equation (2.26), but with J replaced by J+1 and K replaced by K+1, would be the equation to solve. Such an equation could be written as three separate matrix equations as follows:

$$\begin{bmatrix} B_1 \\ \vdots \\ B_I \end{bmatrix} = \begin{bmatrix} 1 & \theta'_{1,0} & \cdots & \theta'_{M-1,N-1} \\ \vdots & \vdots & & \vdots \\ 1 & \theta^I_{1,0} & \cdots & \theta^I_{M-1,N-1} \end{bmatrix} \begin{bmatrix} F'_{0,0} \\ F_{1,0} \\ \vdots \\ F_{M-1,N-1} \end{bmatrix}, \qquad (2.27)$$

$$\begin{bmatrix} B_{I+1} \\ \vdots \\ B_{I+J+1} \end{bmatrix} = \begin{bmatrix} 1 & \theta^{I+1}_{1,0} & \cdots & \theta^{I+1}_{M-1,N-1} \\ \vdots & \vdots & & \vdots \\ 1 & \theta^{I+J+1}_{1,0} & \cdots & \theta^{I+J+1}_{M-1,N-1} \end{bmatrix} \begin{bmatrix} F''_{0,0} \\ F_{1,0} \\ \vdots \\ F_{M-1,N-1} \end{bmatrix} \qquad (2.28)$$

and

$$\begin{bmatrix} B_{I+J+2} \\ \vdots \\ B_{MN+2} \end{bmatrix} = \begin{bmatrix} 1 & \theta_{1,0}^{I+J+2} & \cdots\cdots & \theta_{M-1,N-1}^{I+J+2} \\ \vdots & \vdots & & \vdots \\ 1 & \theta_{1,0}^{MN+2} & \cdots\cdots & \theta_{M-1,N-1}^{MN+2} \end{bmatrix} \begin{bmatrix} F_{0,0}''' \\ F_{1,0} \\ \vdots \\ F_{M-1,N-1} \end{bmatrix} \quad (2.29)$$

A necessary condition for equation (2.27) to represent I equations in I unknowns is that all but I-1 of the last MN-1 columns of the "θ" matrix must be zero (a column of elements is zero when all the elements in the column are zero). Similarly equations (2.28) and (2.29) may have solution sets of J+1 and K+1 coefficients respectively, if all but J and K of the last MN-1 columns of their "θ" matrices are zero. If the three solution sets containing I, J+1 and K+1 coefficients are mutually exclusive (i.e. no coefficient occurs in more than one set) then the three sets together form the complete solution set of MN+2 coefficients. The condition for this to be true is that the numbers of no two non-zero columns of the last MN-1 columns of the "θ" matrices, in equations (2.27),(2.28) and (2.29) shall be the same. If there are some coefficients common to two or more of the three solution subsets, then clearly some other coefficients are not solved for, and more than MN+2 equations will be required to provide solutions including them.

To illustrate the solving of L separate subsets of coefficients from a total number MN-1+L of data points, the following example is considered:

Let L = 3, M = 4, N = 2, so that the complete solution set is of the 10 coefficients $F'_{0,0}$, $F''_{0,0}$, $F'''_{0,0}$, $F_{1,0}$, $F_{2,0}$, $F_{3,0}$, $F_{0,1}$, $F_{1,1}$, $F_{2,1}$, $F_{3,1}$. Let the three independant solution subsets of coefficients contain 4, 3 and 3 coefficients respectively, and let the 3, 2 and 2 respectively non-zero columns out of the last MN-1 = 7 columns of the "θ" matrices in the relevant matrix equations, of the types in equations (2.27), (2.28) and (2.29), be given by:

(a) For the first solution subset $\theta^m_{2,0}$, $\theta^m_{3,0}$, $\theta^m_{3,1}$, $1 \leqslant m \leqslant 4$

(b) " " second " " $\theta^m_{1,0}$, $\theta^m_{1,1}$ $5 \leqslant m \leqslant 7$

(c) " " third " " $\theta^m_{0,1}$, $\theta^m_{2,1}$ $8 \leqslant m \leqslant 10$

The first of the three matrix equations can then be written as:

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \theta^1_{2,0} & \theta^1_{3,0} & 0 & 0 & 0 & \theta^1_{3,1} \\ 1 & 0 & \theta^2_{2,0} & \theta^2_{3,0} & 0 & 0 & 0 & \theta^2_{3,1} \\ 1 & 0 & \theta^3_{2,0} & \theta^3_{3,0} & 0 & 0 & 0 & \theta^3_{3,1} \\ 1 & 0 & \theta^4_{2,0} & \theta^4_{3,0} & 0 & 0 & 0 & \theta^4_{3,1} \end{bmatrix} \begin{bmatrix} F'_{0,0} \\ F_{1,0} \\ F_{2,0} \\ F_{3,0} \\ F_{0,1} \\ F_{1,1} \\ F_{2,1} \\ F_{3,1} \end{bmatrix} = \begin{bmatrix} 1 & \theta^1_{2,0} & \theta^1_{3,0} & \theta^1_{3,1} \\ 1 & \theta^2_{2,0} & \theta^2_{3,0} & \theta^2_{3,1} \\ 1 & \theta^3_{2,0} & \theta^3_{3,0} & \theta^3_{3,1} \\ 1 & \theta^4_{2,0} & \theta^4_{3,0} & \theta^4_{3,1} \end{bmatrix} \begin{bmatrix} F'_{0,0} \\ F_{2,0} \\ F_{3,0} \\ F_{3,1} \end{bmatrix}.$$

The other two matrix equations likewise become:

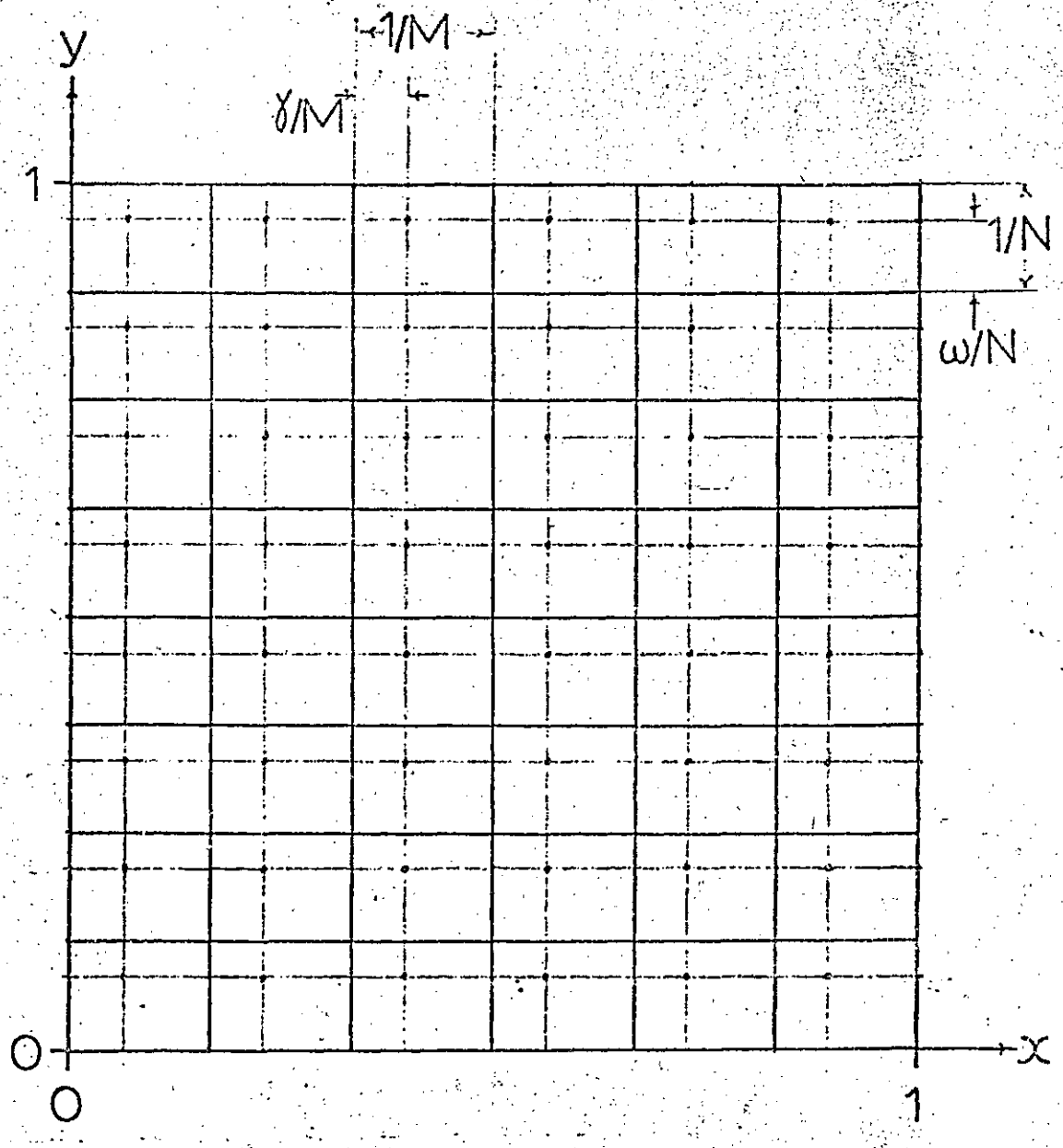
$$\begin{bmatrix} B_5 \\ B_6 \\ B_7 \end{bmatrix} = \begin{bmatrix} 1 & \theta^5_{1,0} & \theta^5_{1,1} \\ 1 & \theta^6_{1,0} & \theta^6_{1,1} \\ 1 & \theta^7_{1,0} & \theta^7_{1,1} \end{bmatrix} \begin{bmatrix} F''_{0,0} \\ F_{1,0} \\ F_{1,1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} B_8 \\ B_9 \\ B_{10} \end{bmatrix} = \begin{bmatrix} 1 & \theta^8_{0,1} & \theta^8_{2,1} \\ 1 & \theta^9_{0,1} & \theta^9_{2,1} \\ 1 & \theta^{10}_{0,1} & \theta^{10}_{2,1} \end{bmatrix} \begin{bmatrix} F'''_{0,0} \\ F_{0,1} \\ F_{2,1} \end{bmatrix}.$$

Assuming that the three matrix equations are solvable, the three subsets of coefficients are $\{F'_{0,0}, F_{2,0}, F_{3,0}, F_{3,1}\}$, $\{F''_{0,0}, F_{1,0}, F_{1,1}\}$ and $\{F'''_{0,0}, F_{0,1}, F_{2,1}\}$.

2.4 The Density Field determined at MN Sample Points expressed as a Discrete Inverse Unitary Transform of the MN Coefficients.

When a solution has been found for the set of coefficients, $\{F_{pq}$, p and q not both zero$\}$, the series given by equation (2.8) with $F_{0,0}$ set equal to zero will give the value of $f_a(x,y)$ about a mean value of zero for any chosen pair of values, (x,y), in the range $0 \leqslant x < 1$, $0 \leqslant y < 1$.

If a regular array of MN points, (x,y), given by $x = \frac{k + \gamma}{M}$ and $y = \frac{l + \omega}{N}$, where k and l are integer variables, $0 \leqslant k < M$, $0 \leqslant l < N$ and γ and ω are constants in the range 0 to 1, then some rather special results may hold. Fig. (2.2) shows this regular array of sample points, one per "cell". If x_2 and y_2 in section A1.4.2 are set equal to zero, the



sample points are: $x = \frac{k+\gamma}{M}$, $y = \frac{l+\omega}{N}$.

$k = 0,1,2,\ldots M-1.$ $\quad l = 0,1,2,\ldots N-1.$

γ is a constant, $0 \leqslant \gamma < 1.$

ω is a constant, $0 \leqslant \omega < 1.$

Fig(2.2) The MN sampling positions for $f_a(x,y)$ and the orthogonal functions, $\psi_{pq}(x,y)$, in the unit square, $0 \leqslant x < 1, 0 \leqslant y < 1$. M=6 and N=8 in the diagram.

results in that section become directly applicable here. Direct substitution of $x = \frac{k+\gamma}{M}$ and $y = \frac{l+\omega}{N}$ into equation (2.8) gives

$$\gamma\omega f_{kl} = f_a\left(\frac{k+\gamma}{M}, \frac{l+\omega}{N}\right) = \sum_{p,M}\sum_{q,N} F_{pq}\,\psi_{pq}\left(\frac{k+\gamma}{M}, \frac{l+\omega}{N}\right). \qquad (2.30)$$

For certain values of γ, ω, M and N, given by equation A 1.42, equation (2.30) may define set $\{\gamma\omega f_{kl}\}$ as a <u>discrete or finite inverse unitary transform</u> of the set of coefficients $\{F_{pq}\}$. For these values of γ, ω, M and N, equation A 1.43 gives the set $\{F_{pq}\}$ as the corresponding <u>discrete or finite unitary transform.</u>

The fact that equation (2.30) is, under certain conditions, a discrete inverse unitary transform, becomes significant when $\{\psi_{pq}(x,y)\}$ is a separable set $\{\phi_p(x)\psi_q(y)\}$ so that equation (2.30) becomes

$$\gamma\omega f_{kl} = f_a\left(\frac{k+\gamma}{M}, \frac{l+\omega}{N}\right) = \sum_{p,M}\phi_p\left(\frac{k+\gamma}{M}\right)\left(\sum_{q,N} F_{pq}\,\psi_q\left(\frac{l+\omega}{N}\right)\right). \qquad (2.31)$$

It is shown in section A 1.4.3 that equation (2.31) can be calculated as M one-dimensional discrete inverse unitary transforms of order N, based on the set of sampled functions $\{\psi_q(\frac{l+\omega}{N})\}$, and N one-dimensional discrete inverse unitary transforms of order M, based on the set of sampled functions $\{\phi_p(\frac{k+\gamma}{M})\}$.

Now, for a variety of sets of one-dimensional orthogonal functions, algorithms have been developed by which the discrete transform (and its inverse) may be implemented by far fewer operations than the straightforward summation would indicate. Cooley and Tukey, (4), first developed such an algorithm based on the complex exponential set $\{e^{2\pi ipk/M}\}$. This is the well-known "Fast Fourier Transform". Harmuth, in his book, (2), describes corresponding "Fast Walsh" and "Fast Haar" transforms. The summation, $\sum_{q,N} F_{pq}\psi_q(\frac{l+\omega}{N})$, in equation (2.31) is a one-dimensional discrete inverse unitary transform of $\{F_{pq}$, p a constant$\}$, based on the set $\{\psi_q(\frac{l+\omega}{N})\}$. If "operation" means a multiplication followed by an addition, then there are clearly N operations for each value of l. There are N values of l, so there are a total of N^2 operations in calculating the set $\{H_l = \sum_{q,N} F_{pq}\psi_q(\frac{l+\omega}{N})\}$. Now, a fast transform algorithm enables the complete transform to be calculated, in between

$N\log_2 N$ and N^2 operations, depending how "factorizable" N is. If N is a power of 2, say $N = 2^n$, then the number of operations is $N\log_2 N = nN$. Thus the ratio of number of operations by fast transform to number of operations by straightforward calculation is $n/2^n$ for $N = 2^n$. If $M = 2^m$, the number of operations for a complete two-dimensional inverse discrete transform is $(MN)(M+N)$ by straightforward calculation, and $(MN)(n+m)$ using a fast transform algorithm.

A very important result from section A1.4.2 concerning which values of γ and ω are "allowed", for equation (2.31) to be a discrete inverse unitary transform relation, is as follows:

If the set $\{\phi_p(x)\}$ is such that, for particular value(s) of M, each function, $\phi_p(x)$, is constant in the range $\frac{k}{M} \leq x < \frac{k+1}{M}$, where $k = 0,1,2,\ldots M-1$, and $\{\psi_q(y)\}$ is such that, for particular value(s) of N, each function, $\psi_q(y)$, is constant in the range $\frac{l}{N} \leq y < \frac{l+1}{N}$, where $l = 0,1,2\ldots N-1$, then equation (2.31) is a discrete transform for any pair of values of γ and ω where both of them are in the range 0 to 1. Clearly, in this case, the separable set $\{\phi_p(x)\psi_q(y)\}$ is such that the value of each function in a "sample cell" $\frac{k}{M} \leq x < \frac{k+1}{M}$, $\frac{l}{N} \leq y < \frac{l+1}{N}$ is constant. Then $f_a(\frac{k+\gamma}{M}, \frac{l+\omega}{N})$ is independant of γ and ω, and there is only one distinct two-dimensional discrete inverse unitary transform of the set of coefficients $\{F_{pq}\}$ given by equation (2.31). It gives the MN values of a two-dimensional step function $f_a(x,y)$ representing $f(x,y)$.

The set of Walsh functions $\{wal(p,x)wal(q,y), 0 \leq p < M,\ 0 \leq q < N\}$ is, for M and N powers of 2, a set for which the results in the previous paragraph apply. The first 64 two-dimensional Walsh functions, for $M = 8$, $N = 8$, are shown in Fig. (3.2).

Chapter 3. The Determination of a Two-Dimensional Density Field from Line Integral Data, using a Set of Orthogonal Walsh Functions.

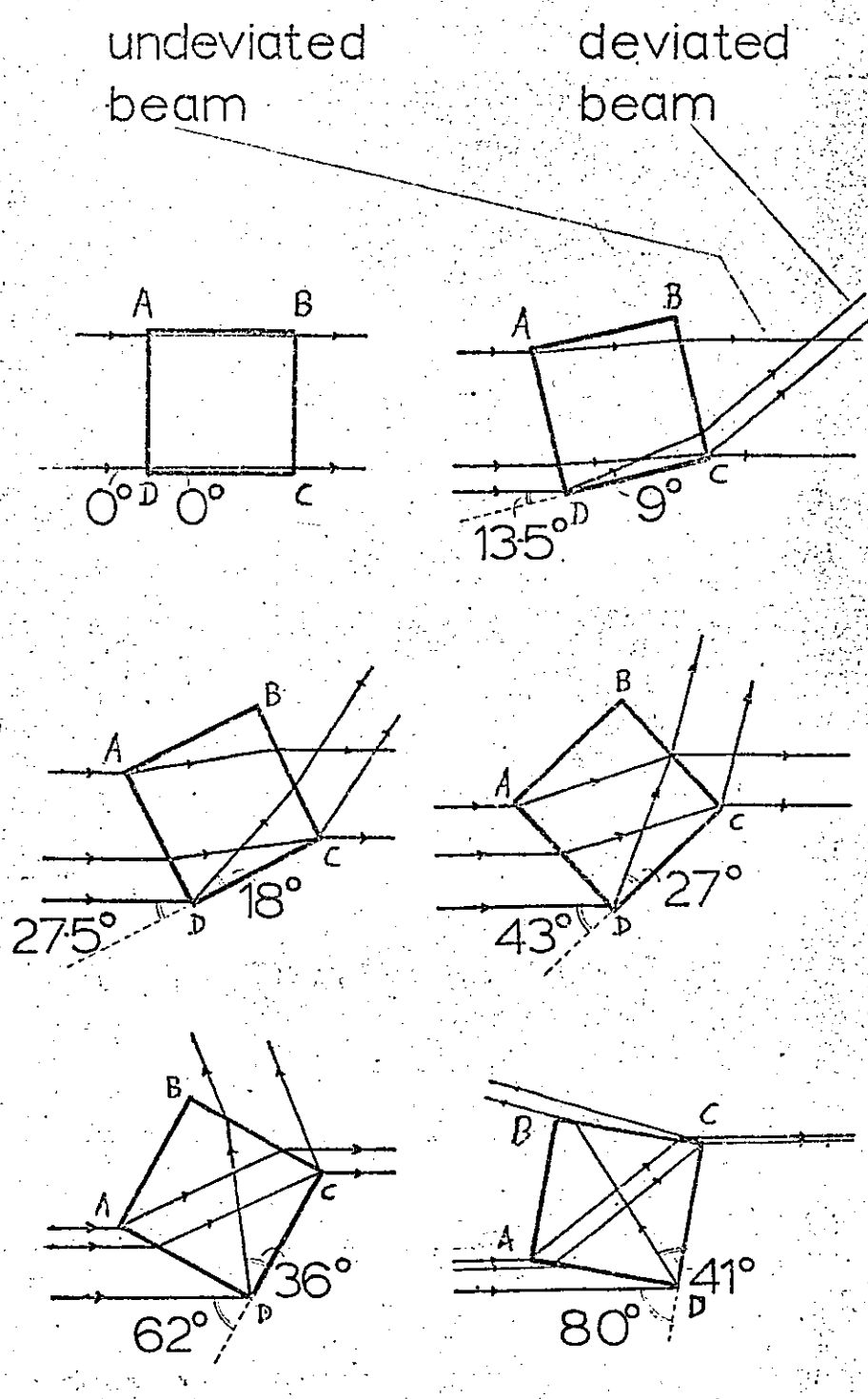
3.1 Ray Tracing through a Glass Block to establish Possible Ray Directions.

In choosing any theoretical approach to the particular problem of the determination of the two-dimensional refractive index density field in a rectangular boundary, it is important to find out those possible ray directions which can be used in an experiment.

Fig. (3.1) shows rays traced through a square cross-section of a glass block, of mean refractive index 1.5, for a collimated beam incident on a given side, AD in Fig. (3.1), of the block. The ray tracing is done for a variety of angles of incidence from 0° to 80°. It can be seen easily that, since the critical angle for this glass is 42°, no rays can emerge from this block from any side but the side opposite the one through which the rays entered. (This result applies also to a rectangular cross-section of any two relative dimensions.) The parallel rays in the block which strike the adjacent side DC are totally internally reflected, and emerge from the side BC at an angle to the incident rays equal to twice its angle of incidence on the side AD. The rest of the parallel rays in the block pass to the side BC [illegible], and emerge parallel to the incident rays.

From a practical point of view, it is easier to make use of the undeviated rays, for a variety of angles of incidence, since the direction of the test beam at the interferogram does not change (or only slightly due to surface effects) and there is only a small shift in the position of the interferogram. Also, from a theoretical point of view, it turns out that rays which pass "straight through" are much easier to deal with than those which are totally internally reflected.

Without going into any theoretical detail, it is clear that nothing can be determined about the refractive index field in a region within the boundary through which no rays at all are passing. For high angles of incidence, Fig. (3.1) shows that a high proportion of the



Fig(3.1). The ray tracing through a square phase object of refractive index, 1.5, of rays in a collimated beam incident in air, at a variety of angles from 0° to 180°.

boundary is not "explored" by the test beam, which means that several angles of incidence at the sides AD and AB may have to be used to "cover" the whole region. On the other hand, the lower the angle of incidence, the larger the bounded area that is traversed by the test beam. A single interferogram for angle of incidence near to zero is not sufficient, as any change in the field in the directions of the rays is not shown up, since the relative paths are the line integrals along the rays in the field. At least one more direction is needed for the test beam to detect changes in the direction mentioned. The simplest second direction is that at right angles to the first, i.e. a test beam incident normally on the side AB of the square boundary as shown in Fig. (3.1).

In Chapter 2. it was shown how orthogonal functions could be used in making a determination of a density field, $f(x,y)$. It was shown how data points from each interferogram led to a solution for some of the coefficients in the finite orthogonal series approximation for $f(x,y)$. Several interferograms, in general, were needed to obtain the necessary number of coefficients for this complete finite series. The particular set of orthogonal functions chosen determines how many different such sets are required, and which rays in each set lead to a solution for a particular subset of the set of all coefficients in the finite series. In the rest of this chapter it is shown that only two interferograms are required to obtain a determination of $f(x,y)$, if the two-dimensional orthogonal set of Walsh functions is chosen as the basis of the finite series. Furthermore, it emerges that, as the number of data points taken is increased (giving a better determination of $f(x,y)$), the two directions approach more and more closely to the ideal experimental ones mentioned above: Two directions at right angles. A complete solution scheme for finding $f(x,y)$, given the appropriate relative optical paths of the rays in the two beams is worked out for Walsh function series.

3.2 The Two-Dimensional set of Walsh Functions and the Two-Dimensional Walsh Transform.

In section A1.3.2. it is shown that $\{\phi_p(x)\ \Psi_q(y)\}$ is a separable two-dimensional orthonormal set in the range $x_2 \leq x < x_2+1, y_2 \leq y < y_2+1$, where $\{\phi_p(x)\}$ and $\{\Psi_q(y)\}$ are one-dimensional orthonormal sets

in the ranges $x_2 \leq x < x_2$ and $y_2 \leq y < y_2$ respectively. If $\phi_p(x)$ and $\psi_q(y)$ are both one-dimensional Walsh functions, being members of the infinite sets $\{wal(p,x),\ 0 \leq p < \infty\}$ and $\{wal(q,y),\ 0 \leq q < \infty\}$ respectively, the the separable two-dimensional orthonormal set is $\{wal(p,x)wal(q,y),\ 0 \leq p < \infty,\ 0 \leq q < \infty\}$. The 64 members of this set, $wal(p,x), wal(q,y)$, $0 \leq p < 8$, $0 \leq q < 8$ in the range $0 \leq x < 1$, $0 \leq y < 1$, are shown in Fig.(3.2). The properties of one-dimensional Walsh functions are discussed in Appendix 2. If M is a power of 2, say $M = 2^m$, then $wal(p,x)$ in the range $\frac{k}{m} \leq x < \frac{k+1}{M}$, $k = 0,1,\ldots,M-1$, and $0 \leq p < M$, is constant. Thus $wal(p,x)wal(q,y)$ is constant in each of the "sample cells" shown in Fig.(2.2), $\frac{k}{M} \leq x < \frac{k+1}{M}$, $\frac{l}{N} \leq y < \frac{l+1}{N}$, $k = 0,1,\ldots,M-1, l = 0,1,\ldots N-1$, $0 \leq p < M$, and $0 \leq q < N$, where M and N are powers of 2. Thus using the results of section A1.4.2, the function $f_a(x,y)$, which approximates the true scaled density field $f(x,y)$, is a step function of MN values given by (c.f. equation (A1.51))

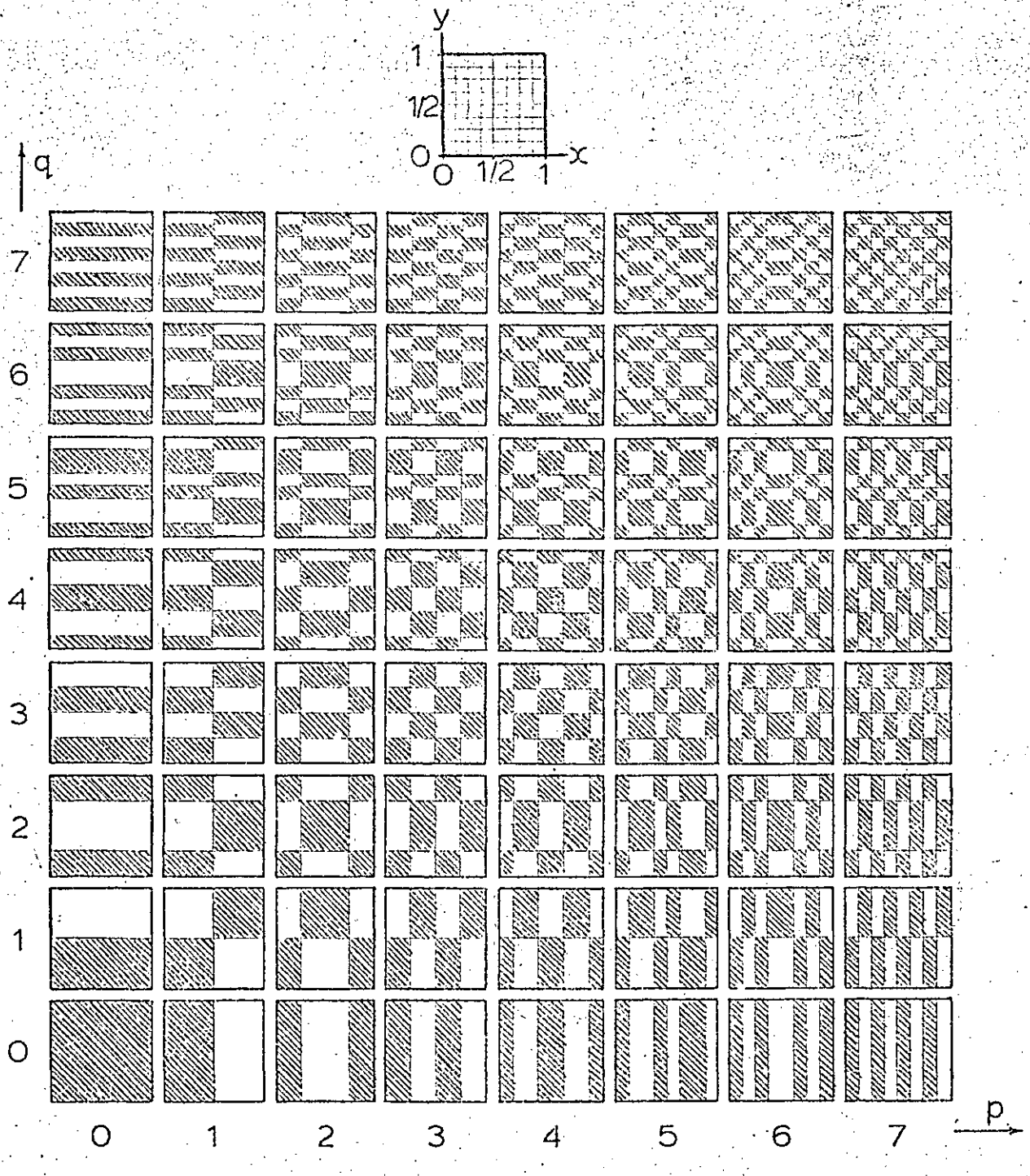
$$f_{kl} = f_a\left(\frac{k+\gamma}{2^m}, \frac{l+\omega}{2^n}\right) = \sum_{p=0}^{2^m-1} \sum_{q=0}^{2^n-1} F_{pq}\, wal\left(p, \frac{k+\gamma}{2^m}\right) wal\left(q, \frac{l+\omega}{2^n}\right),$$

$$= \sum_{p=0}^{2^m-1} wal\left(p, \frac{k+\gamma}{2^m}\right) \left(\sum_{q=0}^{2^n-1} F_{pq}\, wal\left(q, \frac{l+\omega}{2^n}\right) \right), \qquad (3.1)$$

where γ and ω are any constants in the range 0 to 1 and m and n are positive integers. The prefixes γ and ω have been dropped from $\gamma_\omega f_{kl}$ because the latter is independent of them. The set $\{f_{kl},\ 0 \leq k < M,\ 0 \leq l < N\}$ is the <u>discrete two-dimensional inverse Walsh transform</u> of the set of coefficients $\{F_{pq},\ 0 \leq p < M,\ 0 \leq q < N\}$. The corresponding discrete two-dimensional Walsh transform giving $\{F_{pq}\}$ in terms of $\{f_{kl}\}$ is (c.f. equation (A1.52))

$$F_{pq} = \frac{1}{MN} \sum_{k=0}^{2^m-1} \sum_{l=0}^{2^n-1} f_{kl}\, wal\left(p, \frac{k+\gamma}{2^m}\right) wal\left(q, \frac{l+\omega}{2^n}\right)$$

$$= \frac{1}{2^{m+n}} \sum_{k=0}^{2^m-1} wal\left(p, \frac{k+\gamma}{2^m}\right) \left(\sum_{l=0}^{2^n-1} f_{kl}\, wal\left(q, \frac{l+\omega}{2^n}\right) \right). \qquad (3.2)$$



Fig(3.2). The 64 two-dimensional Walsh functions, $wal(p,x)wal(q,y)$, $0 \leqslant p < 8, 0 \leqslant q < 8$, in the range, $0 \leqslant x < 1, 0 \leqslant y < 1$. Shaded areas represent the value, 1, and unshaded areas the value, -1.

The relevant discrete orthogonality relations are (c.f. equations (A1.46) and (A1.47)).

$$\sum_{k=0}^{2^s-1} wal(p, \frac{k+\gamma}{2^s}) wal(p', \frac{k+\gamma}{2^s}) = 2^s \delta_{pp'} , \qquad (3.3)$$

where $0 \leq p < 2^s$; $0 \leq p' < 2^s$; s = m or n and γ is a constant, $0 \leq \gamma < 1$. The discrete reciprocal orthogonality relations are (c.f. equations (A1.49) and (A1.50))

$$\sum_{p=0}^{2^s-1} wal(p, \frac{k+\gamma}{2^s}) wal(p, \frac{k'+\gamma}{2^s}) = 2^s \delta_{kk'} , \qquad (3.4)$$

where $0 \leq k < 2^s$; $0 \leq k' < 2^s$, s = m or n and γ is a constant, $0 \leq \gamma < 1$.

The results of section A1.4.3 and the end of section A2.5 can be used to express the relations above in matrix form: Define the Walsh matrix of order 2^s by the relation (c.f.equation (A2.31))

$$(W_s)_{jk} = wal(j, \frac{k+\gamma}{2^s}), \qquad (3.5)$$

where $0 \leq j < 2^s$; $0 \leq k < 2^s$, s = m or n. Walsh matrices are symmetric (c.f.equations (A2.38) to (A2.41)) so that $(\widetilde{W}_s) = (W_s)$ and both orthogonality and redprocal orthogonality relations reduce to

$$(W_s)(W_s) = 2^s (I_{2s}), \qquad (3.6)$$

where (I_{2s}) is the unit matrix of order 2^s.

The discrete Walsh transform (equation (3.2)) and inverse transform (equation (3.1)) become (c.f. equations (A1.62) and (A1.60))

$$(F) = \frac{1}{2^{m+n}} (W_m)(f)(W_n) \qquad (3.7)$$

and

$$(f) = (W_m)(F)(W_n), \qquad (3.8)$$

respectively where (f) and (F) are $2^m \times 2^n$ matrices containing respectively the elements f_{kl} and F_{pq} given by the relations (c.f. equations (A1.57) and (A1.58))

$$(f)_{kl} = f_{kl} = f_a\left(\frac{k+\gamma}{2^m}, \frac{l+\omega}{2^n}\right) \tag{3.9}$$

and

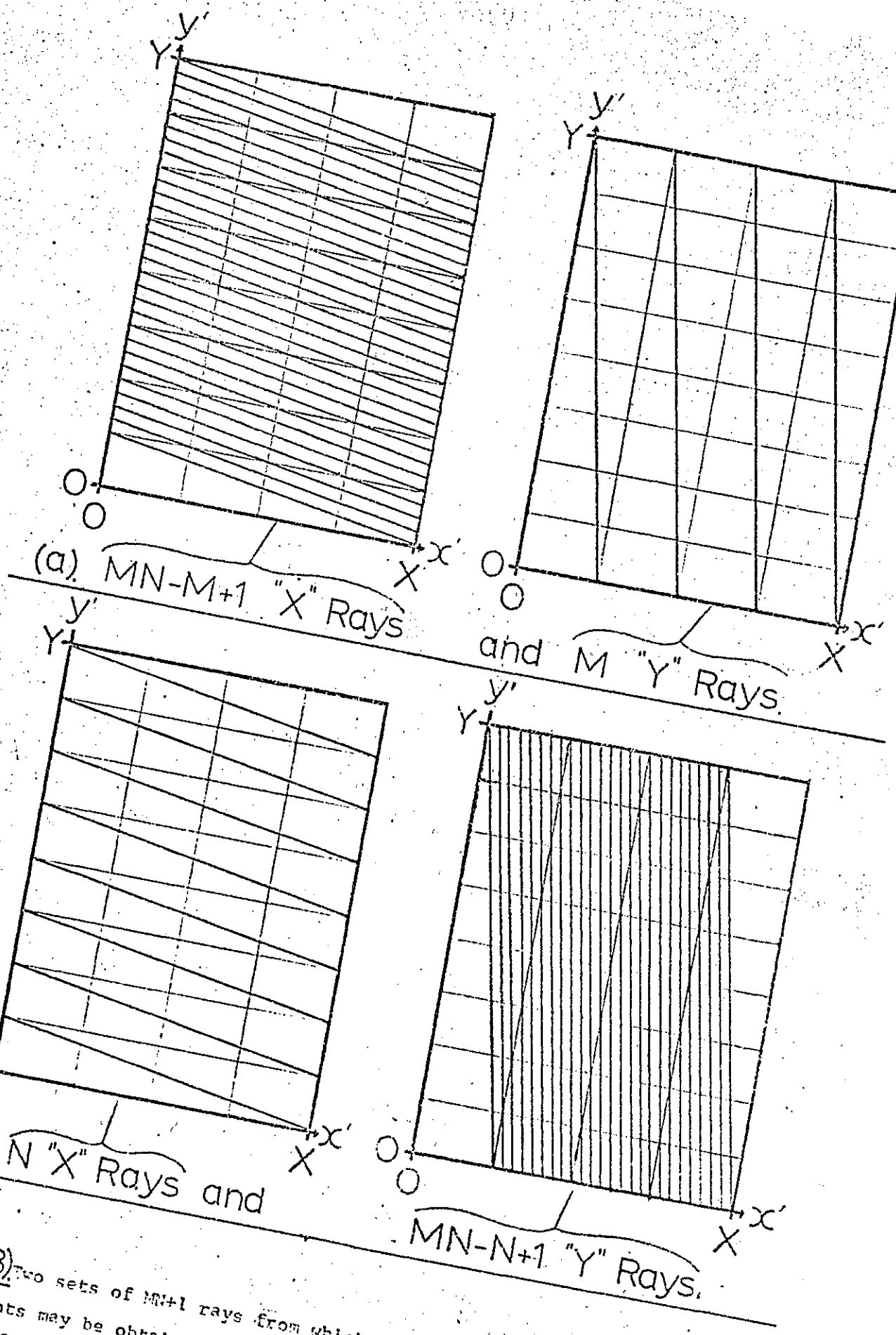
$$(F)_{pq} = F_{pq} \tag{3.10}$$

When the set of 2^{m+n} coefficients, $\left\{F_{pq},\ 0 \leq p < 2^m,\ 0 \leq q < 2^n\right\}$, has been solved for, the inverse Walsh transform, represented by the matrix equation, equation (3.8), will give the $2^m \times 2^n$ two-dimensional step function values. From section (A1.4.3) it is true that equation (3.8) can be implemented by 2^n one-dimensional inverse Walsh transforms (see equation A2.35) of order 2^m and 2^m one-dimensional inverse Walsh transforms of order 2^n. As mentioned in section (2.4) there exist "Fast" Walsh transforms for the most efficient computation of a one-dimensional transform. The author used Harmuth's, (2), Fast Walsh transform which is based on the set $\left\{wal(p,x),\ -\tfrac{1}{2} \leq x < \tfrac{1}{2}\right\}$ and adapted it to a set of sampled Walsh functions based on the set $\left\{wal(p,x),\ 0 \leq x < 1\right\}$.

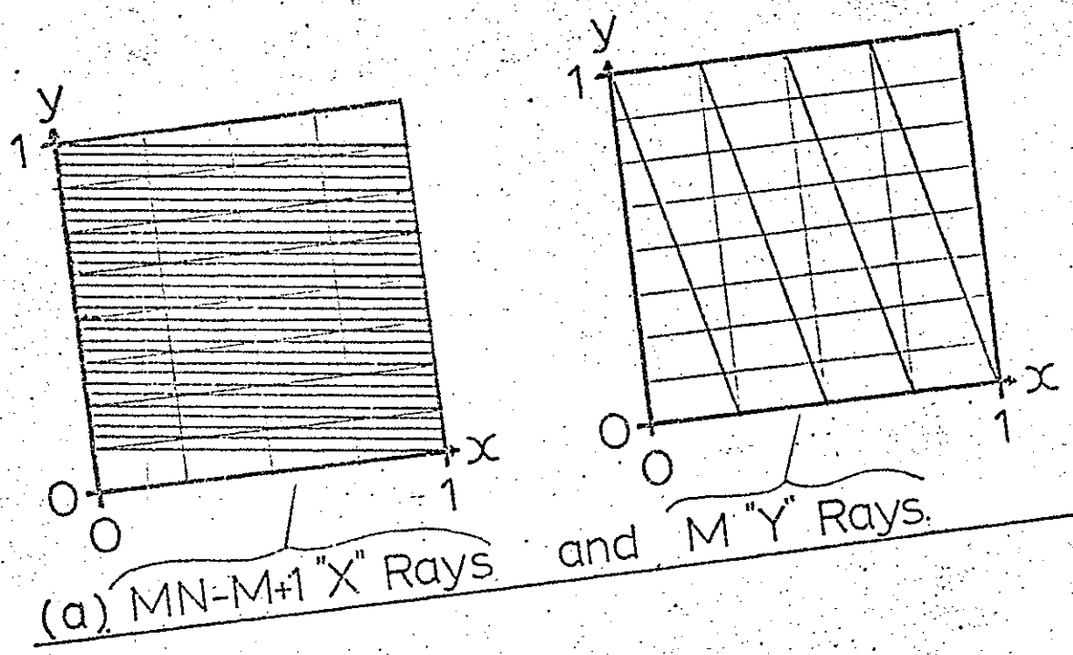
3.3 The sets of Rays Which Lead to Solutions for the set of Coefficients $\left\{F_{pq}, 0 \leq p < M,\ 0 \leq q < N\right\}$

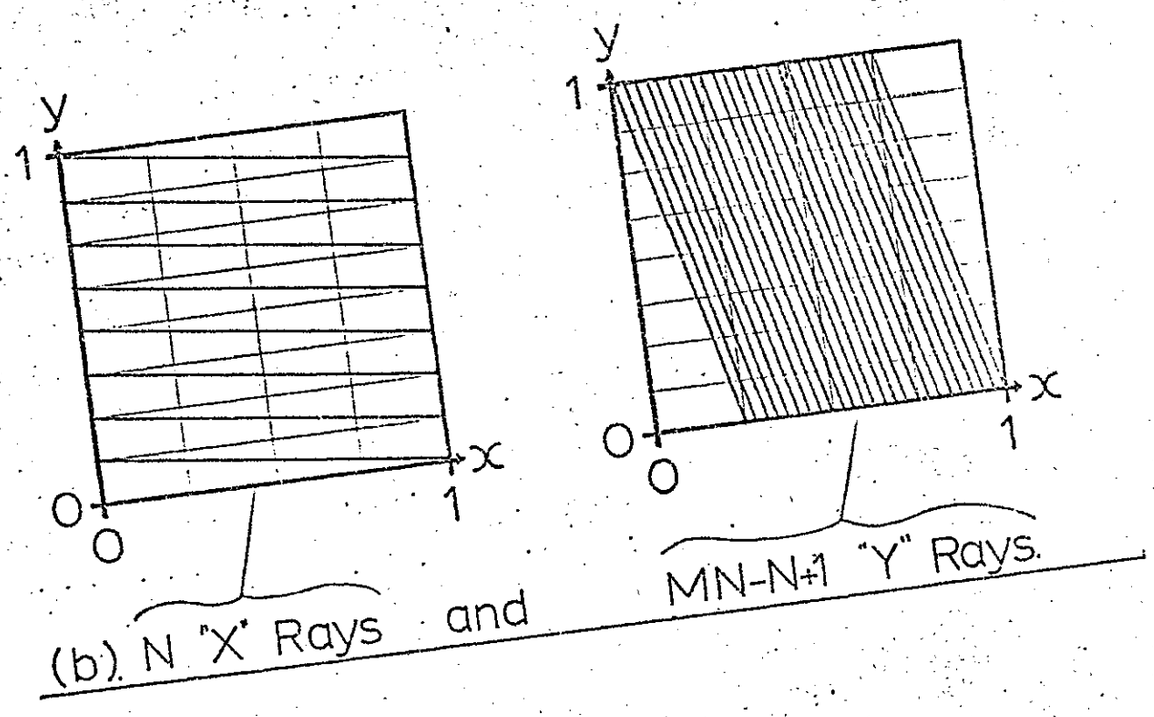
Solutions of at least $MN = 2^{m+n}$, where m and n are positive integers, equations of the type (2.21) for normalized line integrals given by equations (2.19) and (2.20) were attempted for values of M and N : 2 and 2 2 and 4, 4 and 4. In all three cases two sets of rays gave a successful solutions set, $\left\{F_{pq}\right\}$, for a given set of normalized line integrals associated with each ray. In section (2.3) it was stated that for L different sets of parallel rays there must be a minimum of $MN - L + 1$ rays chosen to obtain a solution set. For $L = 2$ this is $MN + 1$ rays. This number of rays was the number required for the values of M and chosen above. The set of lines chosen suggested some obvious generalisation to similar sets of $MN + 1$ rays for higher values of M and N.

In Fig (3.3) the rectangular area, X by Y units, is divided up into M by N rectangular "resolution" cells by $M - 1$ vertical lines and $N - 1$ horizontal lines. Let the intersection of a vertical and a horizontal line be called a "node" and any ray passing through a node be called a "nodal" ray. There are clearly $(M - 1)(N - 1)$ nodes. (The intersection of the lines with the boundary lines are not counted as nodes).



Fig(3.3) Two sets of MN+1 rays from which two solution sets of coefficients may be obtained by the use of Walsh functions. M=4 and N=8 in the diagrams. The resolution cells, each X/M by Y/N, are also shown.





Fig(3.4) The same sets of rays as in Fig.(3.3), scaled into the unit square boundary. The resolution cells, each X/M by Y/N, are also shown.

Now the "X" rays are all at an angle of $\tan^{-1}\frac{Y}{NX}$ to the x' axis. They are divided up into two types:

(a) The "nodal" "X" rays pass through nodes. There are therefore $(M - 1)(N - 1)$ of these.

(b) The "non-nodal" "X" rays do not pass through any of the $N - 1$ horizontal dividing lines. There are therefore N of these.

The formula for each of the $(M - 1)(N - 1) + N = MN - M + 1$ "X" rays is given by (c.f. equation (2.9) with $a_i = \frac{-Y}{NX}$ and $c_i = \frac{Y}{N}\left(1 + \frac{i - 1}{M}\right)$)

$$y_i'(x) = -\left(\frac{Y}{NX}\right) x' + \frac{Y}{N}\left(1 + \frac{i - 1}{M}\right), \tag{3.11}$$

where $i = 1,2,\ldots\ldots MN - M + 1$. The N "non-nodal" "X" rays are for values of i given by

$$i = qM + 1, \tag{3.12}$$

where $q = 0,1,\ldots\ldots\ldots, N - 1$. The corresponding "scaled" "X" rays, shown in Fig. (3.4) are given by (c.f. equation (2.12)).

$$y_i(Xx) = -\frac{X}{N} + \frac{1}{N}\left(1 + \frac{i - 1}{M}\right) \tag{3.13}$$

The "Y" rays are at an angle of $\tan^{-1}\frac{X}{MY}$ to the y' axis and are similarly divided into two types:

(a) There are $(M - 1)(N - 1)$ "nodal" "Y" rays which pass through the nodes.

(b) There are M "non-nodal "Y" rays which do not pass through any of the $M - 1$ vertical dividing lines.

The formula for each of the $(M - 1)(N - 1) + M = MN - N + 1$ "Y" rays is given by (c.f.equation (2.13) with $b_j = \frac{-X}{MY}$ and $d_j = \frac{X}{M}\left(1 + \frac{j - 1}{N}\right)$)

$$x_j'(y') = -\left(\frac{X}{MY}\right) y' + \frac{X}{M}\left(1 + \frac{j - 1}{N}\right), \tag{3.14}$$

where $j = 1,2,\ldots\ldots,MN - N + 1$. The N "non-nodal" "Y" rays are for values of j given by

$$j = pN + 1, \tag{3.15}$$

where $p = 0,1,\ldots,M-1$. The corresponding "scaled" "Y" rays, shown in Fig. (3.4), are given by c.f. equation (2.16))

$$x_j(Yy) = -\frac{X}{M} + \frac{1}{M}\left(1 + \frac{j-1}{N}\right) \qquad (3.16)$$

In Figs (3.3) and (3.4) are shown two sets of MN + 1 rays from which separate solution sets $\{F_{pq}\}_1$ and $\{F_{pq}\}_2$ may be obtained:

(1) The normalized set of line integrals, $\{B_m,\ 1 \leq m < MN+1\}_1$, taken from the line integral data of all MN - M + 1 "X" rays and the M "non-nodal" "Y" rays leads to the solution set $\{F_{pq}\}_1$. The normalized line integrals are ordered as follows : $B_i, i = 1,2,\ldots,MN - M + 1$ correspond to all the "X" rays (equations (3.11) or (3.13)). $B_{(MN-M+2)+p}$, $p = 0,1,\ldots,M-1$ correspond to the M "non-nodal" "Y" rays (equations (3.14) or (3.16) subject to equation (3.15)).

(2) The normalized set of line integrals $\{B_m,\ 1 \leq m < MN+1\}_2$, taken from the line integral data of all a MN - N + 1 "Y" rays and the N "non-nodal" "X" rays leads to a solution set $\{F_{pq}\}_2$. The normalized line integrals are ordered as follows : $B_j, j = 1,2,\ldots,$ MN - N + 1 correspond to all the "Y" rays (equations (3.14) and (3.16)). $B_{(MN-N+2)+q}$, $q = 0,1,\ldots,$ N - 1 correspond to the N "non-nodal" "X" rays (equations (3.11) and (3.13) subject to equation (3.12)).

It can be seen that data from the "non-nodal" rays, i.e. the N "X" rays and M "Y" rays, are common to both sets $\{B_m\}_1$ and $\{B_m\}_2$ so that the solution sets $\{F_{pq}\}_1$ and $\{F_{pq}\}_2$ are not independent of one another. As M and N increase however the fraction of common data points, $(M+N)/(MN+1)$ decreases so that the two different solution sets $\{F_{pq}\}_1$ and $\{F_{pq}\}_2$ become increasingly independent of one another.

There are a further two sets of MN + 1 rays each, which lead to two solution sets $\{F_{pq}\}_3$ and $\{F_{pq}\}_4$ which are "nearly independent" of one another for large M and N, in the same way that $\{F_{pq}\}_1$ and $\{F_{pq}\}_2$ are. These rays may be generated from the rays in Fig. (3.3) by reflecting the "X" rays in a mirror placed along $x' = X/2$ and the "Y" rays in a mirror placed along $y' = Y/2$. (In the case of the scaled rays of Fig. (3.4) the mirrors are along $x = \frac{1}{2}$, $y = \frac{1}{2}$ respectively). These scaled rays are given by replacing -x and -y in equations (3.13) and

(3.16) by x = 1 and y = 1 respectively. Since none of the rays in this "reflected" set of rays are the same as any of the rays in the original set, the solution sets $\{F_{pq}\}_3$ and $\{F_{pq}\}_4$ obtained from the former set will be completely independent of $\{F_{pq}\}_1$ and $\{F_{pq}\}_2$, which are obtained from the latter set.

In the rest of this chapter a way is developed of solving for set of coefficients, $\{F_{pq}\}_1$, for the special case M = N = 2^m (m a positive integer). This requires data points from all $M^2 - M + 1$ "X" rays and the M "non-nodal" "Y" rays, shown in the top half of Fig. (3.4)

In the special case M = N, the scheme worked out to solve for $\{F_{pq}\}_1$ from $\{B_m\}_1$ can be very easily adapted to solve for $\{F_{pq}\}_2$ from $\{B_m\}_2$. This requires all $M^2 - M + 1$ "Y" rays and the M "non-nodal" "X" rays, shown in the bottom half of Fig. (3.4) : Suppose that the scaled density field, f(x,y) is rotated through 180° about y = x leaving the axes and first set of M^2 +1 lines above (i.e. those in the top half of Fig. (3.4)) as they were. This means that the function inside the unit square boundary is g(x,y) = f(y,x). But physically, in relation to the parts of the function they cross, the set of rays is now equivalent to the second set of rays (bottom half of Fig.(3.4)). The first solution scheme can be applied to solve for a coefficient set $\{G_{pq}\}$, which contains, in some as yet undetermined order, the coefficients of the set $\{F_{pq}\}_2$. Now the step function $g_a(x,y)=g(\frac{k+\gamma}{M}, \frac{l+\omega}{M}) = g_{kl} = g_{(kl)}$, which approximates g(x,y), may be obtained by the inverse Walsh transform (c.f. equation (3.8.)).

$$(g) = (W_m)(G)(W_m). \qquad (3.17)$$

But since $g(x,y) = f(y,x)$ it must be true that $g(\frac{k+\gamma}{M}, \frac{l+\omega}{M}) = f(\frac{l+\omega}{M}, \frac{k+\gamma}{M})$ or $g_{kl} = f_{lk}$ or in matrix terms $(g) = (\tilde{f})$. Now from equation (3.8) for M = N = 2^m we have $(f) = \widetilde{(W_m)(F)(W_m)} = (\widetilde{W_m})(\widetilde{F})(\widetilde{W_m})$, but $(\widetilde{W_m}) = (W_m)$ so

$$(f)=(W_m)(\widetilde{F})(W_m) \qquad (3.18)$$

By comparing equations (3.17) and (3.18) it is clear that

$$(G) = (\widetilde{F}), \qquad (3.19)$$

or in terms of elements of these matrices:

$$G_{pq} = F_{qp} \tag{3.20}$$

These results can be summarized as follows:

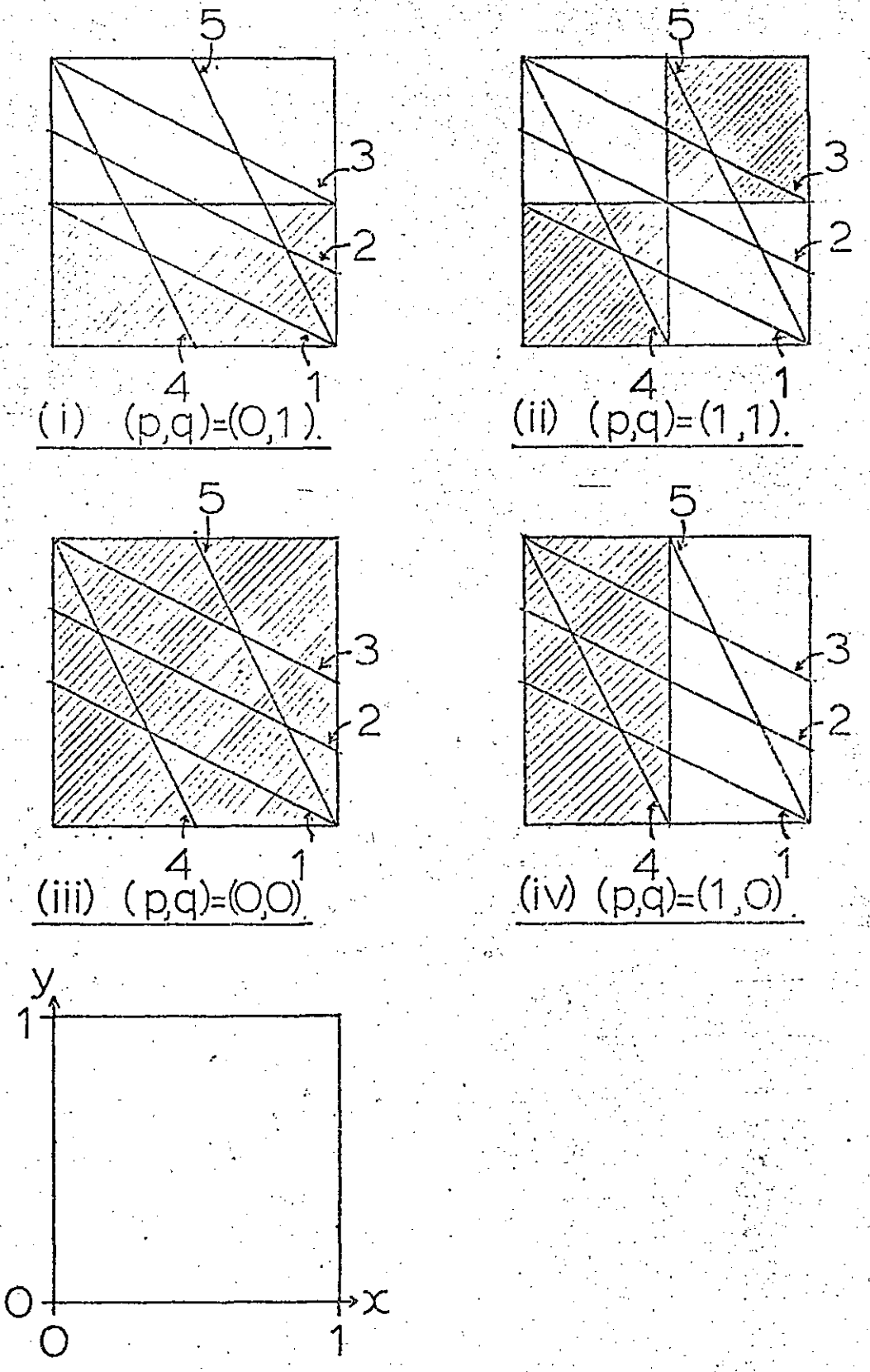
The solution scheme described later gives the set $\{F_{pq}\}_1$ for the input data set $\{B_m\}_1$. For the input set $\{B_m\}_2$ the same solution scheme gives the solution set $\{F_{qp}\}_2$, where the reversal of indices indicates that wherever a coefficient, F_{pq}, appears in the solution using input data $\{B_m\}_1$, the coefficient, F_{qp}, will appear in the solution using input data $\{B_m\}_2$.

3.4. A Scheme to solve for the set of coefficients $\{F_{pq}, 0 \leq p < 2^n, \ 0 \leq q < 2^n\}_1$, from a set of line integral data, $\{B_r, 1 \leq m \leq 2^{2n} + 1\}$.

3.4.1 M = N = 2

Fig (3.5) shows the $M^2 + 1 = 5$ rays in the unit square boundary superimposed on the four two-dimensional Walsh functions required to calculate the values of θ^m_{pq}, $0 \leq p \leq 2$, $0 \leq q < 2$, $1 \leq m \leq 5$, from equations (2.22) and (2.23). Since there is one set of "X" rays and one set of "Y" rays, from which the line integral data, P_m and hence normalized line integrals B_m are obtained, there will be two different "d.c." coefficients: F'_{00} for the "X" ray set of data, and F''_{00} for the "Y" ray set of data. The 5 equations in 5 unknowns, of the type equation (2.21), can be written in matrix form as (c.f. equation (2.26)).

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \theta^1_{10} & \theta^1_{01} & \theta^1_{11} \\ 1 & 0 & \theta^2_{10} & \theta^2_{01} & \theta^2_{11} \\ 1 & 0 & \theta^3_{10} & \theta^3_{01} & \theta^3_{11} \\ 0 & 1 & \theta^4_{10} & \theta^4_{01} & \theta^4_{11} \\ 0 & 1 & \theta^5_{10} & \theta^5_{01} & \theta^5_{11} \end{bmatrix} \begin{bmatrix} F'_{00} \\ F''_{00} \\ F_{10} \\ F_{01} \\ F_{11} \end{bmatrix}. \tag{3.21}$$



Fig(3.5). The three scaled "X" rays and four scaled "Y" rays, in the unit sided square, $0 \leqslant x < 1, 0 \leqslant y < 1$, from which the sets of coefficients, $\{F'_{00}, F_{01}, F_{11}\}$ and $\{F''_{00}, F_{10}\}$, respectively are determined. The four Walsh functions, wal(p,x)wal(q,y); p=0,1; q=0,1, required in calculating the values of θ^{m}_{pq}; p=0,1; q=0,1; $1 \leqslant m \leqslant 5$ are superimposed on the square. The numbers in the diagrams are the ray number labels, m.

The calculation of the elements in the first three rows of the 5 x 5 "θ" matrix requires equation (2.22) and the "X" rays in Fig (3.5):

Example 1 : (see Fig.(3.5)(i))

$$\theta^{1}_{0,1} = \int_0^1 wal(0,x)\,wal(1,y_1(X\ x))\,dx = \int_0^1 (1)\,dx = 1$$

Example 2 : (see Fig.(3.5)(ii))

$$\theta^{2}_{1,1} = \int_0^1 wal(1,x)\,wal(1,y_2(X\ x))\,dx = \int_0^{1/2} (-1)\,dx + \int_{1/2}^1 (-1)\,dx = -1$$

The calculation of the elements in the last two rows of the 5 x 5 "θ" matrix requires equation (2.23) and the "Y" rays in Fig. (3.5):

Example 3 : (see Fig.(3.5)(i))

$$\theta^{4}_{0,1} = \int_0^1 wal(0,x_4(Y\ y))\,wal(1,y)\,dy = \int_0^{1/2} (1)\,dy + \int_{1/2}^1 (-1)\,dy = 0$$

Example 4 : (see Fig. (3.5)(iv)

$$\theta^{5}_{1,0} = \int_0^1 wal(1,x_5(Y\ y))\,wal(0,y)\,dy = \int_0^1 (-1)\,dy = -1$$

The other coefficients are calculated in a similar way and the matrix equation, equation (3.21), becomes

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} F'_{00} \\ F''_{00} \\ F_{10} \\ F_{01} \\ F_{11} \end{bmatrix}, \qquad (3.22)$$

which can be split up into two matrix equations, one containing "X" ray data only and the other containing "Y" ray data only :

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} F'_{00} \\ F_{10} \\ F_{01} \\ F_{11} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} F'_{00} \\ F_{01} \\ F_{11} \end{bmatrix} \qquad (3.23)$$

and

$$\begin{bmatrix} B_4 \\ B_5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} F''_{00} \\ F_{10} \\ F_{01} \\ F_{11} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} F''_{00} \\ F_{10} \end{bmatrix} \tag{3.24}$$

Consider the solution of equation (3.24). Equation (A2.38), near the end of Appendix 2, states that the Walsh matrix of order 2 is given by $(W_1) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Equation (A2.34) states that, bar a power of 2, a Walsh matrix of order 2^n, is its own inverse. More precisely, equation (A2.34) was

$$(W_n)(W_n) = 2^n(I_{2^n}), \tag{3.25}$$

where n is a positive integer. (I_{2^n}) is the unit matrix of order 2^n and (W_n) is a Walsh matrix of order 2^n. The first five Walsh matrices of orders $2^0, 2^1, 2^2, 2^3$, and 2^4 are given by equations (A2.37) to(A2.41) inclusive,near the end of Appendix 2. Now pre-multiplying equation (3.24) by $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}(W_1)$ gives

$$\begin{bmatrix} F''_{00} \\ F_{10} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} B_4 \\ B_5 \end{bmatrix} = \frac{1}{2}(W_1) \begin{bmatrix} B_4 \\ B_5 \end{bmatrix}, \tag{3.26}$$

or $F''_{00} = \frac{1}{2}(B_4 + B_5)$ and $F_{10} = \frac{1}{2}(B_4 - B_5)$.

Consider now the solution of equation (3.23). It can be seen that from these three equations in three unknowns, two equations in two unknowns can be "extracted":

$$\begin{bmatrix} B_1 \\ B_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} F'_{00} \\ F_{01} \end{bmatrix}, \tag{3.27}$$

which is similar to equation (3.24) and the solution can be written down at once as

$$\begin{bmatrix} F'_{00} \\ F_{01} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} B_1 \\ B_3 \end{bmatrix} = \frac{1}{2}(W_1) \begin{bmatrix} B_1 \\ B_3 \end{bmatrix}. \tag{3.28}$$

The only remaining unsolved coefficient is F_{11}. From equation (3.23),

$$\begin{bmatrix} B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} F'_{0,0} \\ F_{0,1} \\ F_{1,1} \end{bmatrix} \tag{3.29}$$

or $B_2 = F'_{0,0} - F_{1,1}$,

so that the solution for $F_{1,1}$ is given in terms of the already solved for coefficient, $F'_{0,0}$, by

$$F_{1,1} = -B_2 + F'_{0,0}. \tag{3.30}$$

Equation (3.30) shows that the solution for $F_{1,1}$ is given in terms of a normalized line integral and a coefficient already solved for. It turns out that for $M = 2^n$, where $n \geq 2$, coefficients F_{pq}, p and q not zero, are given in terms of normalized line integrals and coefficients already solved for.

An important generalization, of equations (3.25) and (3.26), can be made at this stage which is later shown to be true for $M = N = 2^n$, n being any positive integer. This is that: "$\frac{1}{2^n}$ times the discrete Walsh transform of the normalized line integrals of the "non-nodal" rays is the set of coefficients $\{F_{0,0}, F_{0,1}, \ldots\ldots, F_{0,M-1}\}$ if the rays are "X" rays, and $\{F_{0,0}, F_{1,0}, \ldots\ldots, F_{M-1,0}\}$ if the rays are "Y" rays. In matrix form this generalisation becomes for the "non-nodal" "X" ray data:

$$\begin{bmatrix} F'_{0,0} \\ F_{0,1} \\ \vdots \\ F_{0,q} \\ \vdots \\ F_{0,M-1} \end{bmatrix} = \frac{1}{2^n}(W_n) \begin{bmatrix} B_1 \\ B_M \\ \vdots \\ B_{qM+1} \\ \vdots \\ B_{M^2-M+1} \end{bmatrix}, \tag{3.31}$$

where $M = 2^n$ and $0 \leq q < M$. The scaled "non-nodal" "X" rays are given by equation (3.13), subject to the conditions imposed by equation (3.12). For the non-nodal "Y" ray data the generalisation becomes

$$\begin{bmatrix} F''_{00} \\ F_{10} \\ \vdots \\ F_{p0} \\ \vdots \\ F_{M-1,0} \end{bmatrix} = \frac{1}{2^n}(W_n) \begin{bmatrix} B_{M^2-M+2} \\ B_{M^2-M+3} \\ \vdots \\ B_{M^2-M+2+p} \\ \vdots \\ B_{M^2+1} \end{bmatrix}, \text{ where } M=2^n \text{ and } 0 \leq p < M. \tag{3.32}$$

3.4.2 $M = N = 4$

Fig 3.6 shows the $M^2 + 1 = 17$ rays required in six groups: (a) to (f) inclusive. The rays are repeated on an acetate sheet labelled "insert 1", which is loose in a pocket at the end of this thesis. By superimposing these rays on all the 16 Walsh functions of the set $\{wal(p,x)wal(q,y),\ 0 \leq p < 4,\ 0 \leq q < 4\}$, which are drawn in the first two pages of Appendix 3, it is possible to calculate all the $17^2 = 289$ elements, θ^m_{pq}, $0 \leq p < 4$, $0 \leq q < 4$, $1 \leq m \leq 17$, required to set up the 17 equations in the 17 unknowns needed to solve for the set of coefficients $\{F'_{00}, F''_{00}, F_{pq},\ 0 \leq p < 4,\ 0 \leq q < 4,\ p \text{ and } q \text{ not both zero}\}$.

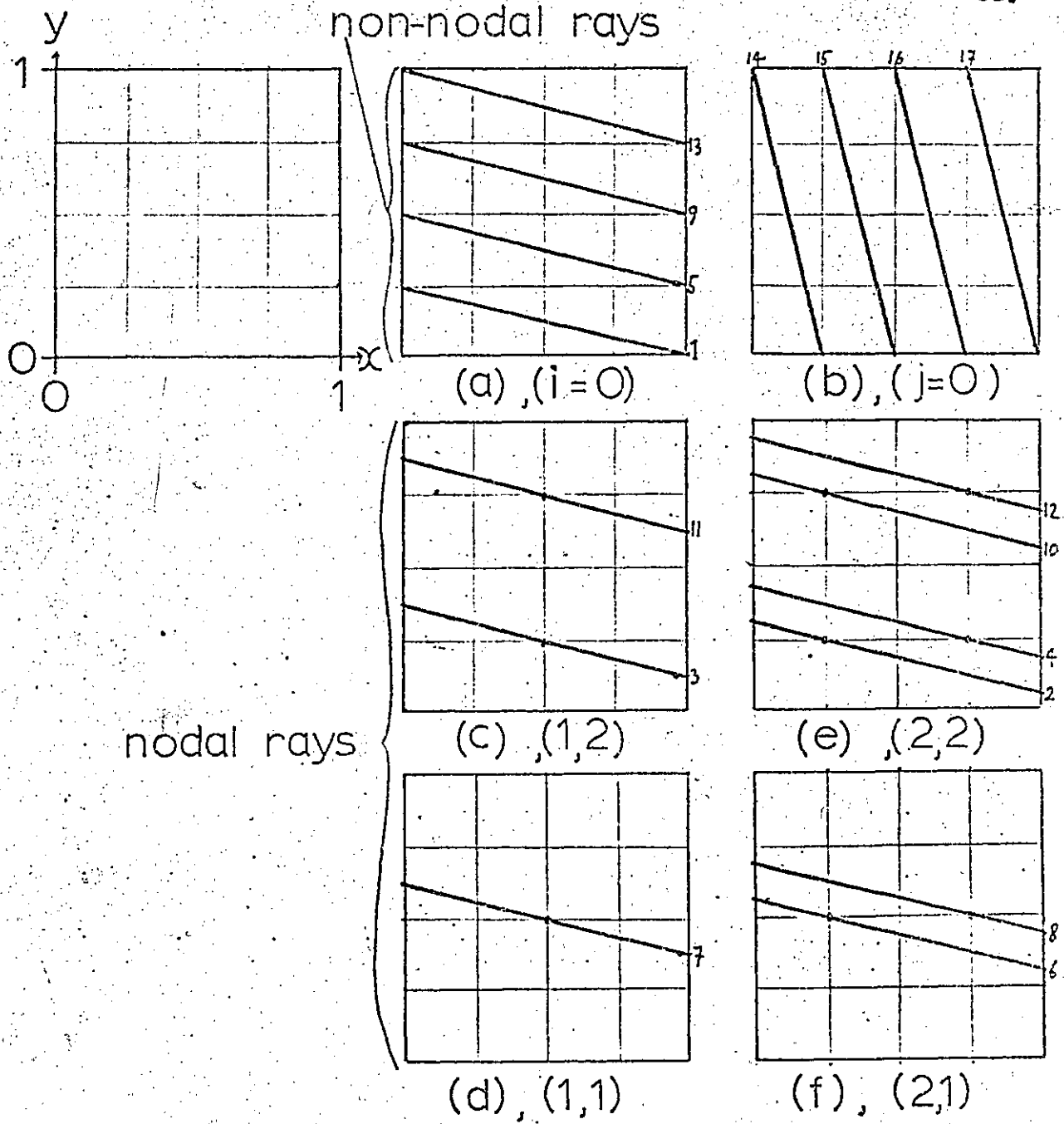
Groups (a) and (b)

Consider first the non-nodal "X" rays and "non-nodal" "Y" rays shown in Fig (3.6)(a) and (3.6)(b) respectively. When insert 1(a) and 1(b) are superimposed on the 16 Walsh functions mentioned above and equations (2.22), and (2.23) are used as in the $M = 2$ case to calculate the appropriate elements, θ^m_{pq}, it is found in each case that all but four elements for each ray are zero giving the matrix equations:

$$\begin{bmatrix} B_1 \\ B_5 \\ B_9 \\ B_{13} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} F'_{00} \\ F_{01} \\ F_{02} \\ F_{03} \end{bmatrix} \tag{3.33}$$

for the "non-nodal" "X" rays and

$$\begin{bmatrix} B_{14} \\ B_{15} \\ B_{16} \\ B_{17} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} F''_{00} \\ F_{10} \\ F_{20} \\ F_{30} \end{bmatrix} \tag{3.34}$$



Fig(3.6). The 17 scaled rays in the unit square, $0 \leqslant x < 1, 0 \leqslant y < 1$, from which the set of coefficients, $\{F'_{00}, F''_{00}, F_{pq}\ 0 \leqslant p < 4, 0 \leqslant q < 4$, p and q not zero$\}$, is determined. They are divided up into 6 groups (a) to (f) inclusive. The number pairs in brackets, (i,j), label the groups of coefficients with which each group of rays is associated (see Fig.(3.8)). These rays, which are repeated on tracing paper as insert 1, are used with Walsh functions in Appendix 3 to calculate values of θ^m_{pq}, $0 \leqslant p < 4$, $0 \leqslant q < 4$; $1 \leqslant m \leqslant 17$ (see equations (2.22) and (2.23)). The small numbers are the ray number labels, m.

for the "non-nodal" "Y" rays. The 4 x 4 matrix in both equations (3.32) and (3.33) is (W_2), the Walsh matrix of order $2^2 = 4$ (equation (A2.39)) having as its inverse (see equation (3.25) or (A2.34)), $\frac{1}{4}(W_2)$. Premultiplication of equations (3.33) and (3.34) by $\frac{1}{4}(W_2)$, therefore, gives the solution of the sets $\{F_{00}, F_{0q},\ 1 \leq q < 4\}$ and $\{F_{00}, F_{p0},\ 1 \leq p < 4\}$:

$$\begin{bmatrix} F'_{00} \\ F_{01} \\ F_{02} \\ F_{03} \end{bmatrix} = \frac{1}{4}(W_2) \begin{bmatrix} B_1 \\ B_5 \\ B_9 \\ B_{13} \end{bmatrix} \tag{3.35}$$

and

$$\begin{bmatrix} F''_{00} \\ F_{10} \\ F_{20} \\ F_{30} \end{bmatrix} = \frac{1}{4}(W_2) \begin{bmatrix} B_{14} \\ B_{15} \\ B_{16} \\ B_{17} \end{bmatrix} \tag{3.36}$$

Equations (3.35) and (3.36) are in agreement with the generalisation suggested in section 3.4.1 (equations (3.31) and (3.32)).

There are 9 unsolved coefficients F_{pq}, $1 \leq p < 4$, $1 \leq q < 4$. By superimposing the nine nodal "X" rays on the 16 Walsh functions mentioned above, the corresponding θ^m_{pq}'s were worked out and a matrix equation set up analogous with equation (3.29). The resulting matrix equation was "broken down" into four matrix equations such that solutions to groups of the coefficients could be solved in terms of the groups of line integral data associated with the rays in the groups shown in Fig.(3.6) (c),(d),(e) and (f). The matrix equations had to be solved in a particular order. This was because coefficients solved for in one group of equations are required for the solutions of coefficients in later groups. To make this clear the groups are taken in the "correct" order and the solutions for the coefficients worked out:

Group (c)

The two nodal "X" rays 3 and 11 shown in Insert 1(c) are superimposed on the Walsh functions shown on the first two pages of Appendix 3 and the appropriate elements θ^3_{pq} and θ^{11}_{pq} are calculated from equation (2.22). All but $\theta^3_{00}, \theta^3_{01}, \theta^3_{12}, \theta^3_{13}, \theta^{11}_{00}, \theta^{11}_{01}, \theta^{11}_{12}, \theta^{11}_{13}$ are zero so that, using the non-zero values of θ^3_{pq} and θ^{11}_{pq}, gives the matrix equation:

$$\begin{bmatrix} B_3 \\ B_{11} \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{vmatrix} F'_{00} \\ F_{01} \\ F_{12} \\ F_{13} \end{vmatrix} = -\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} F_{13} \\ F_{12} \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} F'_{00} \\ F_{01} \end{bmatrix}.$$

Premultiplication of the latter equation by $\frac{1}{2}(W_1) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and using equation (3.25) (i.e. $(W_n)(W_n) = 2^n(I_{2^n})$) gives, after some rearrangement of terms

$$\begin{bmatrix} F_{13} \\ F_{12} \end{bmatrix} = -\tfrac{1}{2}(W_1) \begin{bmatrix} B_3 \\ B_{11} \end{bmatrix} + \begin{bmatrix} F'_{00} \\ F_{01} \end{bmatrix}, \qquad (3.37)$$

which is the explicit solution for F_{13} and F_{12} since F'_{00} and F_{01} have already been solved for (equation (3.35)).

Group (d)

The ray,7,is the only member of this group. Proceeding as above with insert 1(d) instead of 1(c) the following matrix equation is found:

$$[B_7] = \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} F'_{00} \\ F_{02} \\ F_{11} \\ F_{13} \end{bmatrix} = [-1][F_{11}] + \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} F'_{00} \\ F_{02} \\ F_{13} \end{bmatrix},$$

so that

$$[F_{11}] = -[B_7] + [F'_{00} + F_{13} - F_{02}]. \qquad (3.38)$$

F'_{00} and F_{02} have been solved in group (a) and F_{13} has been solved for in group (c) thus all the quantities on the right hand side of equation (3.38) are known.

Group(e)

Using insert 1(e) on the Walsh functions as for the previous groups gives

$$\begin{bmatrix} B_2 \\ B_4 \\ B_{10} \\ B_{12} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 1 & -1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \left(F_{00}F_{01}F_{02}F_{03}F_{12}F_{13}F_{22}F_{23}F_{32}F_{33}\right)^T$$

$$= -\tfrac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} F_{33} \\ F_{32} \\ F_{22} \\ F_{23} \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} F'_{00} \\ F_{01} \\ F_{02} \\ F_{03} \end{bmatrix} - \tfrac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} F_{13} \\ F_{12} \\ F_{02} \\ F_{03} \end{bmatrix},$$

where the superscript "T" means "transpose of" (in this case the row matrix transposed is a column matrix of the F's). The Walsh matrix(W_2) (see equation (A2.39)) appears on the right hand side of the matrix equation above. Premultiplication of both sides of this equation by $\frac{1}{2}(W_2)$ gives after some re-arrangement of terms:

$$\begin{bmatrix} F_{3,3} \\ F_{3,2} \\ F_{2,2} \\ F_{2,3} \end{bmatrix} = -\tfrac{1}{2}(W_2) \begin{bmatrix} B_2 \\ B_4 \\ B_{10} \\ B_{12} \end{bmatrix} + 2 \begin{bmatrix} F'_{0,0} \\ F_{0,1} \\ F_{0,2} \\ F_{0,3} \end{bmatrix} - \begin{bmatrix} F_{1,3} \\ F_{1,2} \\ F_{0,2} \\ F_{0,3} \end{bmatrix}. \qquad (3.39)$$

Group (f)

Using insert 1(f) on the Walsh functions as in the previous groups gives

$$\begin{bmatrix} B_6 \\ B_8 \end{bmatrix} = \begin{bmatrix} 1 & \tfrac{1}{2} & 1 & -\tfrac{1}{2} & -\tfrac{1}{2} & \tfrac{1}{2} & -\tfrac{1}{2} & \tfrac{1}{2} & -\tfrac{1}{2} & \tfrac{1}{2} \\ 1 & -\tfrac{1}{2} & -1 & \tfrac{1}{2} & -\tfrac{1}{2} & \tfrac{1}{2} & \tfrac{1}{2} & -\tfrac{1}{2} & -\tfrac{1}{2} & \tfrac{1}{2} \end{bmatrix} \left(F'_{0,0} F_{0,1} F_{0,2} F_{0,3} F_{1,1} F_{1,3} F_{2,1} F_{2,3} F_{3,1} F_{3,3}\right)^T$$

$$= -\tfrac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} F_{3,1} \\ F_{2,1} \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} (F'_{0,0}-F_{0,2}) & +\tfrac{1}{2}(F_{3,3}+F_{1,3}) & -\tfrac{1}{2}F_{1,1} \\ & \tfrac{1}{2}(F_{2,3}+F_{0,1}) & -\tfrac{1}{2}F_{0,3} \end{bmatrix}$$

$$= -\tfrac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} F_{3,1} \\ F_{2,1} \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} (F'_{0,0}-F_{0,2}) & -\tfrac{1}{2}(F_{1,1}-F_{1,3}-F_{3,3}) \\ (F_{0,1}-F_{0,3}) & -\tfrac{1}{2}(F_{0,1}-F_{0,3}-F_{2,3}) \end{bmatrix}.$$

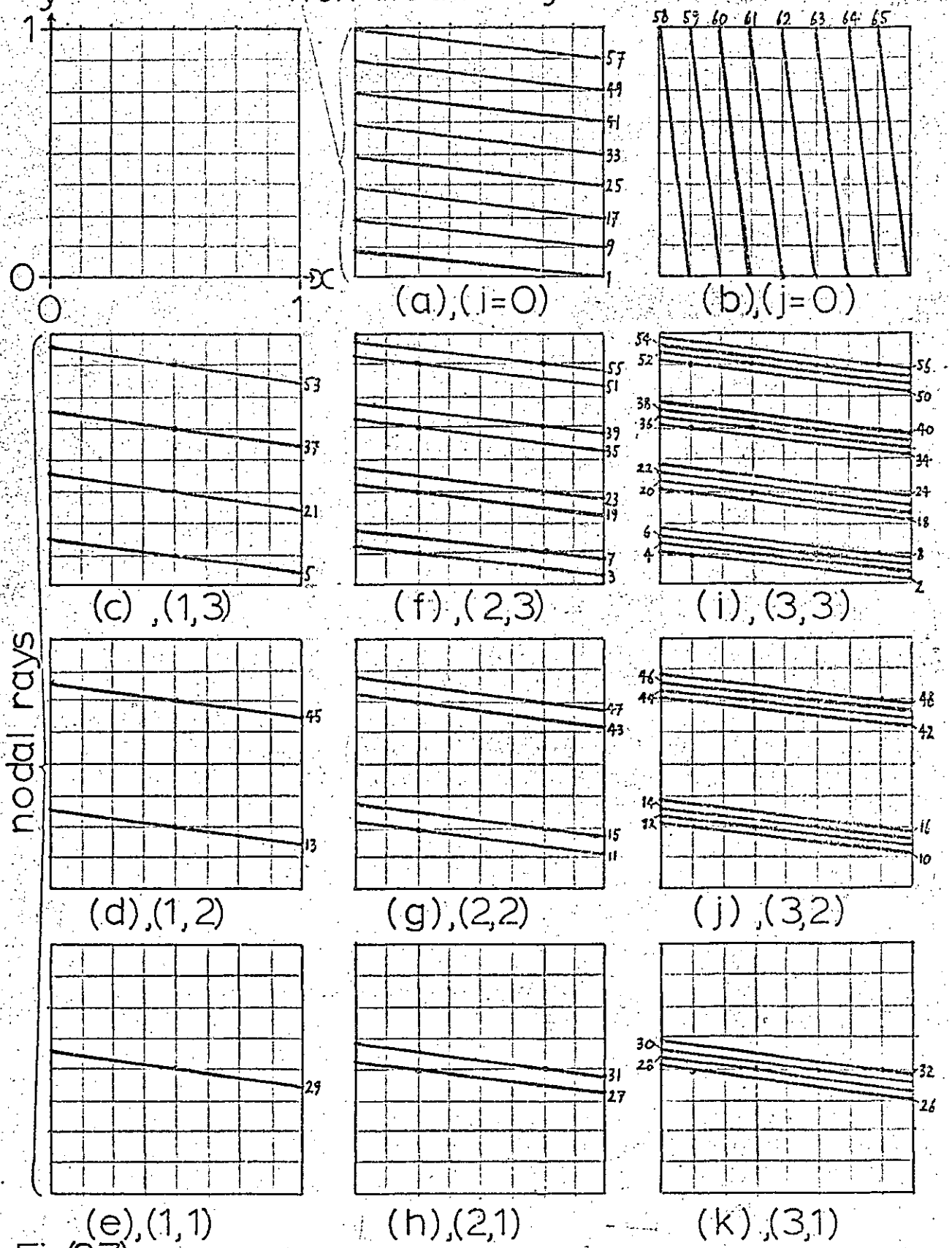
Premultiplication of both sides of the above equations by (W_1) gives, after some re-arrangement of terms

$$\begin{bmatrix} F_{3,1} \\ F_{2,1} \end{bmatrix} = -(W_1) \begin{bmatrix} B_6 \\ B_8 \end{bmatrix} + 2 \begin{bmatrix} F'_{0,0}-F_{0,2} \\ F_{0,1}-F_{0,3} \end{bmatrix} - \begin{bmatrix} F_{1,1}-F_{1,3}-F_{3,3} \\ F_{0,1}-F_{0,3}-F_{2,3} \end{bmatrix}, \qquad (3.40)$$

This completes the solutions of all the 17 coefficients.

3.4.3 M = N = 8

Fig (3.7) shows the 65 rays required in 11 groups, (a) to (k) inclusive. These rays are repeated on an acetate sheet called Insert 2 which is loose in a pocket at the end of this thesis. By superimposing these rays on all the 64 Walsh functions of the set $\{wal(p,x)wal(q,x), 0 \leq p < 8, 0 \leq q < 8\}$, in Appendix 3, it is possible to calculate all the $65^2 = 4225$ elements, θ^m_{pq}, $0 \leq p < 8$, $0 \leq q < 8$, $1 \leq m \leq 65$, required to



Fig(3.7). The 65 scaled rays in the unit square, $0 \leq x < 1$, $0 \leq y < 1$, from which the set of coefficients, $\{F'_{00}, F''_{00}, F_{pq}, 0 \leq p < 8, 0 \leq q < 8, p \text{ and } q \text{ not zero}\}$, is determined. They are divided up into 11 groups (a) to (k) inclusive. The number pairs in brackets, (i,j), label the groups of coefficients with which each group of rays is associated, (see Fig.(3.8)). These rays, which are repeated on tracing paper as insert 1, are used with Walsh functions in appendix 3 to calculate values of θ^m_{pq}, $0 \leq p < 8$, $0 \leq q < 8$, $1 \leq m \leq 65$, (see equations (2.22) and (2.23)). The small numbers are the ray number labels, m.

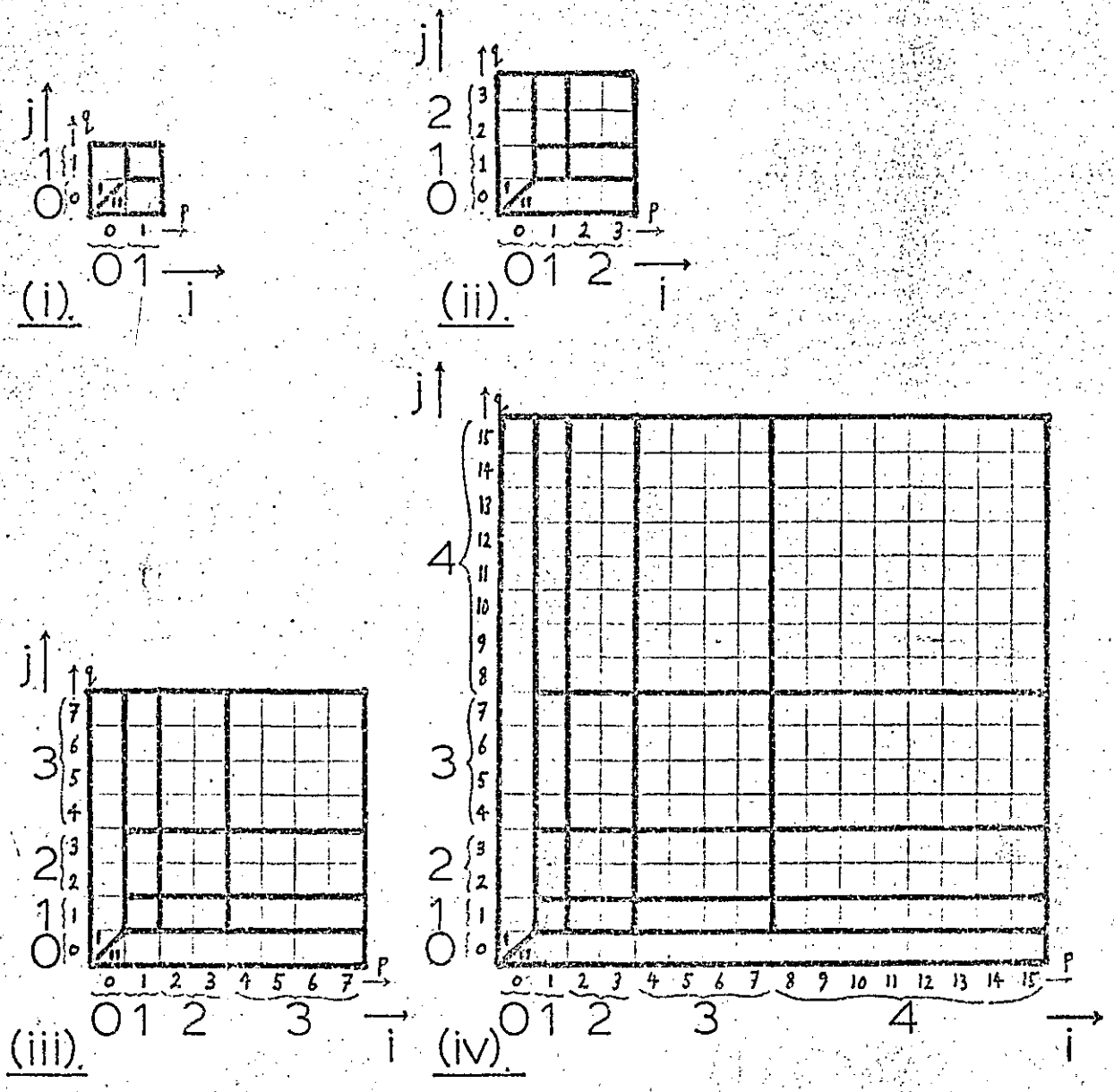
set up the 65 equations in 65 unknowns needed to solve for the set of coefficients, $\{F'_{00}, F''_{00}, F_{pq},\ 0 \leq p < 8,\ 0 \leq q < 8,\ p \text{ and } q \text{ not both zero}\}$.

It has not yet been said how the groups of rays were arrived at nor which order to attempt to solve the simultaneous equations arising from each group. The task of setting up a 65 × 65 matrix correctly and picking out suitable solvable sub matrix equations in the correct order is somewhat more difficult than in the previous case for a 17 × 17 matrix. Even assuming that the sets of coefficients $\{F'_{00}, F_{0q},\ 1 \leq q < 8\}$ and $\{F''_{00}, F_{p0},\ 1 \leq p < 8\}$ are independently solvable, according to the generalisation expressed by equations (3.31) and (3.32), there are still 49 unsolved coefficients, F_{pq}, $1 \leq p < 8$, $1 \leq q < 8$, as opposed to only 9 in the M = N = 4 case. Clearly some generalisations from the M = N = 2 and M = N = 4 cases are needed to predict :

(a) Which rays can be grouped together so that a group of coefficients (equal in number to the number of rays grouped together) can be solved from the normalized line integral data from these rays.

(b) In which order to solve the groups of simultaneous equations set up from each group of rays such that the coefficients which appear in the solutions for the group of coefficients being solved for, have themselves already been solved previously.

Fig (3.8) represents each coefficient, F_{pq}, by a small square cell, labelled by the two integers p and q, which are marked off along the conventional x-axis and y-axis directions respectively. Figs (3.8)(i) and (3.8)(ii) give the sets of coefficients for M = N = 2 and M = N = 4 respectively. The cells are divided up by bold lines into blocks of cells representing the coefficients which are solved for together from each group of simultaneous equations (see sections 3.4.1 and 3.4.2). The coefficients F'_{00} and F''_{00} share the same cell (shown "split" along diagonal), (p,q) = (0,0). The blocks of coefficients F'_{00}, F_{0q}; $1 \leq q < 2^n$ and F''_{00}, F_{p0}; $1 \leq p < 2^n$ for n = 1 and 2 (so far) are represented by the single integers i = 0 and j = 0 respectively. All the other groups of cells(i.e. for $1 \leq p < 2^n$, $1 \leq q < 2^n$) are in rectangular blocks labelled by a pair of positive integers i and j which increase in the same directions as p and q increase respectively. For n = 1 and 2 (so far) i and j are in the ranges $1 \leq i \leq n$ and $1 \leq j \leq n$ where they label coefficients in the set $\{F_{pq},\ 1 \leq p < 2^n,\ 1 \leq q < 2^n\}$.



Fig(3.8). The sets of coefficients, $C_n = \{F'_{00}, F''_{00}, F_{pq}, 0 \leq p < 2^n, 0 \leq q < 2^n,$ p and q not both zero$\}$, for: (i) n=1, (ii) n=2, (iii) n=3, (iv) n=4. Each coefficient is represented by a square cell labelled by the pair of indices p and q. The coefficient cells are divided up into rectangular groups of cells, labelled by the pair of indices i and j, and representing solution subsets of C_n which appear in successive stages in the solution of C_n. (The particular groups of rays, giving the line integral data required for the solution of each group of coefficients, (i,j), for the cases n=2 and n=3 are shown in Figs. (3.6) and (3.7) respectively.) The two coefficients, F'_{00} and F''_{00}, are shown by the cell for p=q=0 being "split".

In both the cases $M = N = 2$ and $M = N = 4$, the two blocks of coefficients represented by $i = 0$ and $j = 0$ in Fig(3.8)(i) and (3.8)(ii) can be solved independently, from the line integral data obtained from the "non-nodal" "X" and "non-nodal" "Y" rays. Equations (3.31) and (3,32) express these solutions explicitly for any value of n in $M = N = 2^n$. (This has yet to be established for any positive integer n). In the case $M = 2$ (section 3.4.1) the group of coefficients $(i,j) = (1,1)$, which contains only one coefficient, F_{11}, was solved in terms of F'_{00}, which was already solved for (equation (3.30)), and the normalized line integral, B_2. In the case $M = 4$ (section 3.4.2) the order in which the solutions of the groups of coefficients appeared was (after $i = 0$ and $j = 0$):

$$(i,j) = (1,2),(1,1),(2,2),(2,1), \text{ for } n = 2 \qquad (3.41)$$

Fig (3.6) shows the rays in each group which were used in the solution of each group of coefficients, (i,j) (see equations (3.37), (3.38), (3.39) and (3.40))

Figs (3.8)(i) and (ii) suggest, for $n = 3$ and 4, the groups of coefficients solvable together shown in Figs(3.8)(iii) and (iv). Any group of coefficients (i,j) is 2^{i-1} cells "wide" and 2^{j-1} cells "high" and contains therefore $(2^{i-1})(2^{j-1}) = 2^{i+j-2}$ cells. This latter number must also be the number of rays "associated with" each group of coefficients, where "associated with" is taken to mean those rays whose normalized line integral data is used in solving for the coefficients.

Equation (3.41) suggests that the general order for solving for the groups of coefficients (not including the $i = 0$ and $j = 0$ groups) is given by:

$$(i,j) = (1,n),(1,n-1),\ldots,(1,1),(2,n),(2,n-1),(2,1)\ldots\ldots,(n,n),(n,n-1),\ldots\ldots,(n,1). \qquad (3.42)$$

In equation (3.42) j changes before i from its highest value (n) in unit steps to 1. i changes from 1, in unit steps to n.

It now remains, for any positive integer n, to associate with each group of coefficients, (i,j), a particular set of rays containing the correct number of rays, 2^{i+j-2}. The two sets of "non-nodal" rays (one set of M "X" rays and the other a set of M "Y" rays) are associated

with the blocks of coefficients $i = 0$ and $j = 0$ respectively for any positive integer n. Equations (3.31) and (3.32) then give the solutions of the coefficients in these two blocks in terms of the normalized line integrals of the rays in the two sets.

This leaves the $(M - 1)(N - 1)$ "nodal" "X" rays to be grouped together. These groups of rays for $M = N = 4$ are shown in Fig (3.6) (c),(d),(e), and (f). From the latter four diagrams it seems that each ray, associated with the block of coefficients (1,j) "generates" two rays associated with the block of coefficients (2,j) by adding and subtracting 1 from the number of the generating ray. Now the rays "of" (i.e.associated with") the blocks $(1,j), j = 1,2$, are nodal rays which pass through nodes along the line $x = \frac{1}{2}$ and these nodes are symmetrically placed about $y = \frac{1}{2}$ for each block, (1,j).

From the information in the previous paragraph and the condition that there must be 2^{i+j-2} rays in the group of rays "of" the block (i,j) a plausible ray grouping scheme can be worked out for $M = 8$ (and $M = 2^n$, $n > 3$). The groups of rays suggested are shown in Fig (3.7). The rays "of" the blocks $(1,j), j = 1,2,3$ pass through all the nodes along $x = \frac{1}{2}$ (Fig (3.7)(c),(d)). Adding and subtracting 2 to all the labels of these rays gives the rays "of" the blocks $(2,j), j = 1,2,3$. These rays pass through all the nodes along $x = \frac{1}{4}$ and $x = \frac{3}{4}$, (Fig (3.7)(f),(g),(h)). Finally adding and subtracting 1 from the labels of the rays "of" the blocks (2,j), gives the rays "of" the blocks (3,j), $j = 1,2,3$. These rays pass through all the nodes along $x = \frac{1}{8}, \frac{3}{8}, \frac{5}{8}$ and $\frac{7}{8}$. All the nodes have been used up once each so all the nodal rays have been grouped. In general it seems that adding and subtracting 2^{n-i} to the labels of a ray "of" block (i-1,j) gives two rays of the block (i,j). Explicit formulae for the labels, m(i,j) of the different groups of rays associated with the blocks, (i,j), of coefficients must now be worked out:

Lemma 1

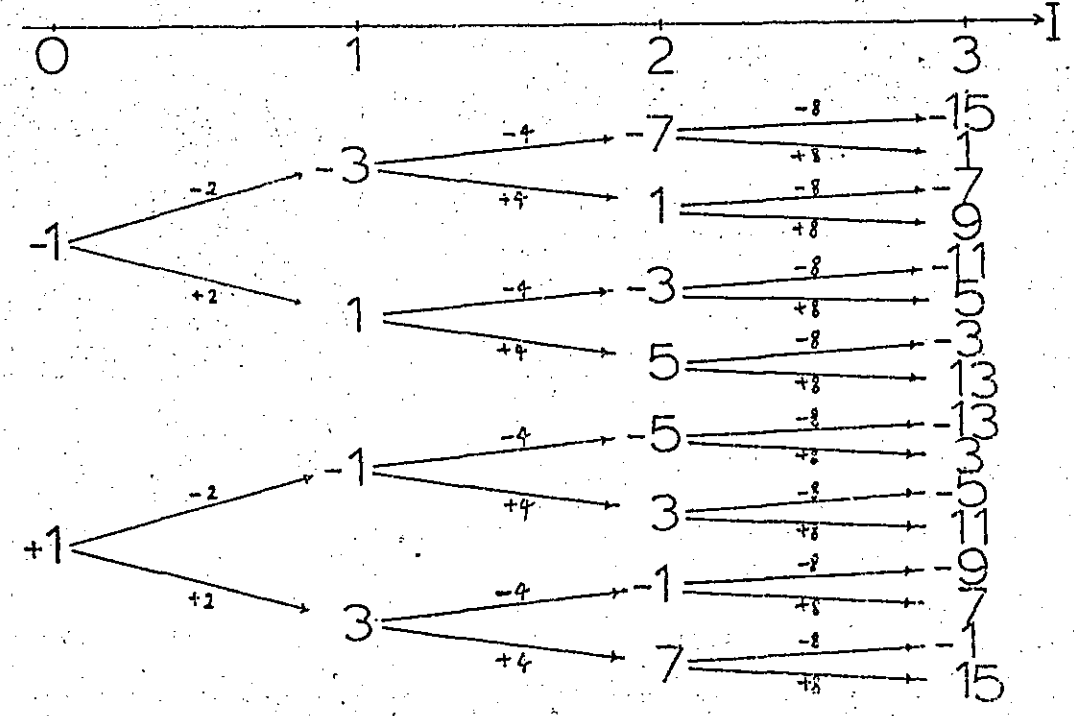
The integers, h generated by a series $\mp 1 \mp 2 \mp 2^2 \ldots \mp 2^I$, where I is an integer, are all the odd numbers between $-(2^{I+1} - 1)$ and $(2^{I+1} - 1)$ inclusive, that is the integers h are given by

$$h = 2d - 1 - 2^{I+1}, \quad d = 1,\ldots,2^{I+1}. \tag{3.43}$$

where d is an integer increasing in unit steps.

Verification.

The easiest way to show the truth of lemma 1 is to simply draw up a table of possible values of j for the first few values of I :



Groups (i,j) = (1,i), $1 \leq i \leq n$, n a positive integer.

The single ray of group (i,1) is the ray passing through the node (x,y) = $(\frac{1}{2},\frac{1}{2})$. (see Figs (3.6)(d) and (3.7)(e)). The label of this "central" ray is given by m = $\frac{1}{2}$ (M^2 - M + 1 + 1) or since M = 2^n it is

$$m(1,1) = 2^{n-1}(2^n - 1) + 1 \qquad (3.44)$$

The two rays of the next group (1,2) pass through the nodes (x,y) = $(\frac{1}{2},\frac{1}{4})$ and $(\frac{1}{2},\frac{3}{4})$ respectively. There are $t(2^n) - 1$ rays between two rays passing through two nodes, t nodes appart from one another on a vertical dividing line. ($x = \frac{l}{2^n}$, where $1 \leq l < 2^n - 1$)
Now since the closest distance between nodes on a dividing line is $\frac{1}{2^n}$, there are $t = (\frac{1}{4} \div \frac{1}{2^n}) - 1 = 2^{n-2} - 1$ nodes between the node (x,y) = $(\frac{1}{2},\frac{1}{4})$ and any of the two nodes (x,y) = $(\frac{1}{2},\frac{1}{4} \mp \frac{1}{4})$. The labels of rays of group (1,2) are therefore given by equation (3.44) $\mp 2^{n-2}$:

$$\begin{aligned} m(1,2) &= m(1,1) + (\mp 2^{n-2})2^n \\ &= 2^{n-1}(2^n - 1) + 1 + (\mp 2^{n-2})2^n \end{aligned} \qquad (3.45)$$

The true rays of the group (1,3) pass through the nodes $(x,y) = (\frac{1}{2},\frac{1}{2}\mp\frac{1}{4})$ and it is easy to see that the ray labels of this group are given by equation((3.45) $\mp 2^{n-3}$:

$$m(1,3) = m(1,2) + (\mp 2^{n-3})2^n$$
$$= 2^{n-1}(2^n - 1) + 1 + (\mp 2^{n-2} \mp 2^{n-3})2^n. \qquad (3.46)$$

The ray labels, $m(1,j)$ of the 2^{j-1} rays of the group $(1,j)$ are given in general in terms of these labels, $m(1,j-1)$ of the 2^{j-2} labels of group $(1,j-1)$ by

$$m(1,j) = m(1,j-1) + (\mp 2^{n-j})2^n \qquad (3.47)$$

or

$$m(1,j) = 2^{n-1}(2^n - 1) + 1 + (\mp 2^{n-2} \mp 2^{n-3} \ldots \ldots 2^{n-j})2^n$$
$$= 2^{n-1}(2^n - 1) + 1 + 2^{n-j}(\mp 1 \mp 2 \ldots \ldots \mp 2^{j-2})2^n,$$

which by using lemma 1 with $I = j - 2$ becomes

$$m(1,j) = 2^{n-1}(2^n - 1) + 1 + 2^{n-j}(2s - 1 - 2^{j-1})2^n$$

or

$$m(1,j) = 2^{n-1}(2^{n-j+1}(2s - 1) - 1) + 1, \qquad (3.48)$$

where $s = 1, \ldots, 2^{j-1}$.

Groups (i,j), $1 \leq i \leq n$ and $i \geq 2$, n a positive integer.

It was suggested, just before lemma 1, that adding and subtracting 2^{n-i} to all the ray number labels of the group $(i-1,j)$ gave all the ray number labels $m(i,j)$, of the group (i,j). Thus :

$$m(i,j) = m(i - 1,j) \mp 2^{n-i}. \qquad (3.49)$$

Starting with i=2,repeated use of equation (3.49) gives

$$m(i,j) = m(1,j) \mp 2^{n-i} \mp 2^{n-i-1} \ldots \mp 2^{n-2}$$
$$= m(1,j) + 2^{n-i}(\mp 1 \mp 2 \ldots \mp 2^{i-2}),$$

which by using lemma 1 with I = i - 2 becomes

$$m(i,j) = m(1,j) + 2^{n-i}(2r - 1 - 2^{i-1})$$

or

$$m(i,j) = m(1,j) + 2^{n-i}(2r - 1) - 2^{n-1}, \qquad (3.50)$$

where $r = 1,\ldots,2^{i-1}$

Substiution of m(1,j) in equation(3.50) by equation (3.48) gives

$$m(i,j) = 2^n(2^{n-j}(2s - 1) - 1) + 2^{n-i}(2r - 1) + 1,$$

It is better to write m(n,i,j,r,s) instead of m(i,j) to show the dependence of m on r and s as well as n,i and j. Thus:

$$m(n,i,j,r,s) = 2^n(2^{n-j}(2s - 1) - 1) + 2^{n-i}(2r - 1) + 1, \qquad (3.51)$$

where $r = 1,\ldots,2^{i-1}$, $s = 1,\ldots,2^{j-1}$ and i and j are integers in the ranges $1 \leqslant i \leqslant n$, $1 \leqslant j < n$. For the 2^{i+j-2} integers, m(n,i,j,r,s), to be in increasing order of value, r must increase before s does. Equation (3.51) gives all the nodal rays for n = 2 and 3 shown in Figs(3.6) and (3.7). It also correctly gives the only nodal ray for n = 1 i.e m(1,1) = 2.

The next stage is to attempt to solve for the set of 65 coefficients for n = 3 using the groups of rays i = 0, j = 0 and the groups labelled m(i,j) as predicted by equation (3.51). The 11 groups of rays, shown in Fig(3.7) and repeated on acetate sheets as Insert 2, can be used for this purpose. All the 64 Walsh functions in Appendix 3 are required. The groups are taken in the order suggested by equation (3.42). Equations (2.22) and (2.23) are used to calculate the line integrals, θ^m_{pq}, of the rays over the Walsh functions, in exactly the same way as in sections 3.4.1 and 3.4.2 for M = N = 2 and M = N = 4, repectively.

Groups (a) and (b), or j=0 and i=0. (The "non-nodal" "X" and "Y" rays).

Insert 2(a) is used on all the 64 Walsh functions in Appendix 3 to calculate the appropriate elements θ^m_{pq} from equation (2.22). For each ray all but eight elements are zero giving the matrix equation

$$\begin{bmatrix} B_1 \\ B_9 \\ B_{17} \\ B_{25} \\ B_{37} \\ B_{41} \\ B_{49} \\ B_{57} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} F'_{00} \\ F_{01} \\ F_{02} \\ F_{03} \\ F_{04} \\ F_{05} \\ F_{06} \\ F_{07} \end{bmatrix}. \tag{3.52}$$

The 8×8 matrix in equation (3.52) is the Walsh matrix (W_3) (equation (A2.40)). The inverse of (W_3) is $\frac{1}{8}(W_3)$ (equation (3.25) or(A2.34) with $n = 3$), so premultiplication of equation (3.52) by $\frac{1}{8}(W_3)$ gives

$$(F'_{00}F_{01}F_{02}F_{03}F_{04}F_{05}F_{06}F_{07})^T = \tfrac{1}{8}(W_3)(B_1B_9B_{17}B_{25}B_{33}B_{41}B_{49}B_{57})^T \tag{3.53}$$

Using insert 2(b) on the Walsh functions and using equation (2.23) gives matrix equation like equation (3.52) except that the set of "Y" ray normalized line ingegrals replace the corresponding "X" ray ones and $\{F''_{00}, F_{p0},\ 1 \leqslant p \leqslant 7\}$ replaces $\{F'_{00}, F_{0q},\ 1 \leqslant q \leqslant 7\}$. Thus the solution of the former set above can be written (c.f equation (3.53) as

$$(F''_{00}F_{10}F_{20}F_{30}F_{40}F_{50}F_{60}F_{70})^T = \tfrac{1}{8}(W_3)(B_{58}B_{59}B_{60}B_{61}B_{62}B_{63}B_{64}B_{65})^T. \tag{3.54}$$

Equation (3.54) could be also obtained directly from equation (3.53) by using equation (3.20).

Groups (c) to (k) (The nodal "X" rays).

Inserts 2(c) to 2(k) are used with the 64 Walsh functions of Appendix 3 and the appropriate matrix equations are obtained from equation (2.22):

Group (c) or $(i, j) = (1, 3)$

The matrix equation obtained is

$$\begin{bmatrix} B_5 \\ B_{21} \\ B_{37} \\ B_{53} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} \left[F'_{00}F_{01}F_{02}F_{03}F_{14}F_{15}F_{16}F_{17}\right]^T \tag{3.55}$$

The right hand side of this equation can be written as the sum of two terms involving easily invertable, Walsh, matrices :

$$\begin{bmatrix} B_5 \\ B_{21} \\ B_{37} \\ B_{53} \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} F_{1,7} \\ F_{1,6} \\ F_{1,5} \\ F_{1,4} \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} F'_{0,0} \\ F_{0,1} \\ F_{0,2} \\ F_{0,3} \end{bmatrix} . \qquad (3.56)$$

The 4×4 matrices here are $-(W_2)$ and (W_2) (equation (A2.39)). Premultiplying the equation above by $\frac{1}{4}(W_2)$ and using equation 3.25 (or(A2.34)) gives after some re-arrangement of terms:

$$\begin{bmatrix} F_{17} \\ F_{16} \\ F_{15} \\ F_{14} \end{bmatrix} = -\tfrac{1}{4}(W_2) \begin{bmatrix} B_5 \\ B_{21} \\ B_{37} \\ B_{53} \end{bmatrix} + \begin{bmatrix} F'_{0,0} \\ F_{0,1} \\ F_{0,2} \\ F_{0,3} \end{bmatrix} . \qquad (3.57)$$

From now on, wherever possible the matrix equations, like equation (3.55), obtained from each group of rays will be written immediately as the sum of terms involving Walsh matrices, like equation (3.56).

Group (d) or $(i,j) = (1,2)$.

The matrix equation obtained can be written

$$\begin{bmatrix} B_{13} \\ B_{45} \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} F_{1,3} \\ F_{1,2} \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} F'_{0,0} \\ F_{0,2} \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} F_{0,4} \\ F_{0,5} \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} F_{1,7} \\ F_{1,6} \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -F_{1,3} + F'_{0,0} - F_{0,4} + F_{1,7} \\ -F_{1,2} + F_{0,1} - F_{0,5} + F_{1,6} \end{bmatrix} .$$

The 2×2 matrices here are $\pm(W_1)$ (equation (A2.38)). Premultiplication of the equation by $\frac{1}{2}(W_1)$, using $\frac{1}{2}(W_1)(W_1) = (I_2)$ (equation (3.25) or (A2.34)) gives after some re-arrangement of terms

$$\begin{bmatrix} F_{1,3} \\ F_{1,2} \end{bmatrix} = -\tfrac{1}{2}(W_1) \begin{bmatrix} B_{13} \\ B_{45} \end{bmatrix} + \begin{bmatrix} F'_{0,0} - F_{0,4} + F_{1,7} \\ F_{0,1} - F_{0,5} + F_{1,6} \end{bmatrix} . \qquad (3.58)$$

Group (e) or $(i,j) = (1,1)$.

The matrix equation obtained is

$$\left[B_{29}\right] = \left[-1\right]\left[B_{11}\right] + \left[1\ -1\ \ 1\ -1\ \ 1\ -1\ \ 1\right]\left[F'_{00}F_{02}F_{04}F_{06}F_{13}F_{15}F_{17}\right]^T.$$

The solution for F_{11} can be directly written as

$$\left[F_{11}\right] = -(W_0)\left[B_{29}\right] + \left[F'_{00}-F_{02}+F_{04}-F_{06}+F_{13}-F_{15}+F_{17}\right], \qquad (3.59)$$

where the "trivial" Walsh matrix, $(W_0) = 1$ (equation (A2.37)) has been written to make clearer a pattern which is appearing in all these solutions namely that the first part of the solution is minus the discrete one-dimensional Walsh transform of the normalized line integrals, divided by a power of 2.

Group (f) or $(i,j) = (2,3)$.

The matrix equations obtained can be written

$$\begin{bmatrix} B_3 \\ B_7 \\ B_{19} \\ B_{23} \\ B_{35} \\ B_{39} \\ B_{51} \\ B_{55} \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} -F_{37} - F_{17} + 2F'_{00} \\ -F_{36} - F_{16} + 2F_{01} \\ -F_{35} - F_{15} + 2F_{02} \\ -F_{34} - F_{14} + 2F_{03} \\ -F_{24} + F_{04} \\ -F_{25} + F_{05} \\ -F_{26} + F_{06} \\ -F_{27} + F_{07} \end{bmatrix}.$$

The 8×8 matrix above is (W_3) (equation (A2.40)). Premultiplication of the equation above by $\frac{1}{8}(W_3)$, using $\frac{1}{8}(W_3)(W_3) = I_8$ (equation (3.25) or (A2.34)) gives, after some re-arrangement of terms:

$$\begin{bmatrix} F_{37} \\ F_{36} \\ F_{35} \\ F_{34} \\ F_{24} \\ F_{25} \\ F_{26} \\ F_{27} \end{bmatrix} = -\tfrac{1}{2}(W_3)\begin{bmatrix} B_3 \\ B_7 \\ B_{19} \\ B_{23} \\ B_{35} \\ B_{39} \\ B_{51} \\ B_{55} \end{bmatrix} + 2\begin{bmatrix} F'_{00} \\ F_{01} \\ F_{02} \\ F_{03} \\ F_{04} \\ F_{05} \\ F_{06} \\ F_{07} \end{bmatrix} - \begin{bmatrix} F_{17} \\ F_{16} \\ F_{15} \\ F_{14} \\ F_{04} \\ F_{05} \\ F_{06} \\ F_{07} \end{bmatrix}. \qquad (3.60)$$

Group (α) Or (i,i) = (2,2).

The matrix equation obtained can be written

$$\begin{bmatrix} B_{11} \\ B_{15} \\ B_{43} \\ B_{47} \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}\begin{bmatrix} -F_{3,3} + 2F'_{0,0} - 2F_{0,4} - F_{1,3} + F_{1,7} + F_{3,7} \\ -F_{3,2} + 2F_{0,1} - 2F_{0,5} - F_{1,2} + F_{1,6} + F_{3,6} \\ -F_{2,2} + F_{0,2} - F_{0,6} + F_{2,6} \\ -F_{2,3} + F_{0,3} - F_{0,7} + F_{2,7} \end{bmatrix}.$$

Premultiplying this equation by $\frac{1}{4}(W_2)$ and using $\frac{1}{4}(W_2)(W_2) = I_4$ (see equation (A2.39) and (A2.34)) gives after some re-arrangement of terms:

$$\begin{bmatrix} F_{3,3} \\ F_{3,2} \\ F_{2,2} \\ F_{2,3} \end{bmatrix} = -\tfrac{1}{2}(W_2)\begin{bmatrix} B_{11} \\ B_{15} \\ B_{43} \\ B_{47} \end{bmatrix} + 2\begin{bmatrix} F'_{0,0} - F_{0,4} \\ F_{0,1} - F_{0,5} \\ F_{0,2} - F_{0,6} \\ F_{0,3} - F_{0,7} \end{bmatrix} - \begin{bmatrix} F_{1,3} - F_{1,7} - F_{3,7} \\ F_{1,2} - F_{1,6} - F_{3,6} \\ F_{0,2} - F_{0,6} - F_{2,6} \\ F_{0,3} - F_{0,7} - F_{2,7} \end{bmatrix}. \qquad (3.61)$$

Group (h) or (i,i) = (2,1)

The matrix equations obtained can be written

$$\begin{bmatrix} B_{27} \\ B_{31} \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -F_{3,1} + 2F_{0,0} - 2F_{0,2} + 2F_{0,4} - 2F_{0,6} \\ -F_{2,1} + F_{0,1} - F_{0,3} + F_{0,5} - F_{0,7} \end{bmatrix}$$

$$\begin{bmatrix} -F_{1,1} + F_{1,3} - F_{1,5} + F_{1,7} + F_{3,3} - F_{3,5} + F_{3,7} \\ + F_{2,3} - F_{2,5} + F_{2,7} \end{bmatrix}.$$

Premultiplying this equation by (W_1) and using $\frac{1}{2}(W_1)(W_1) = I_2$ (see equation (A2.38) and (A2.34)) gives after some re-arrangement of terms:

$$\begin{bmatrix} F_{3,1} \\ F_{2,1} \end{bmatrix} = -(W_1)\begin{bmatrix} B_{27} \\ B_{31} \end{bmatrix} + 2\begin{bmatrix} F'_{0,0} - F_{0,2} + F_{0,4} - F_{0,6} \\ F_{0,1} - F_{0,3} + F_{0,5} - F_{0,7} \end{bmatrix} - \begin{bmatrix} F_{1,1} - F_{1,3} + F_{1,5} - F_{1,7} \\ F_{0,1} - F_{0,3} + F_{0,5} - F_{0,7} \end{bmatrix}$$

$$\begin{bmatrix} + F_{3,3} - F_{3,5} + F_{3,7} \\ + F_{2,3} - F_{2,5} + F_{2,7} \end{bmatrix}. \qquad (3.62)$$

Group (i) or $(i,j) = (3,3)$.

The matrix equation obtained can be written

$$
\begin{bmatrix} B_2 \\ B_4 \\ B_6 \\ B_8 \\ B_{18} \\ B_{20} \\ B_{22} \\ B_{24} \\ B_{34} \\ B_{36} \\ B_{38} \\ B_{40} \\ B_{50} \\ B_{52} \\ B_{54} \\ B_{56} \end{bmatrix} = \frac{1}{4}
\left[\begin{array}{cccc|cccc|cccc|cccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
\hline
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
\hline
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
\hline
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1
\end{array}\right]
\begin{bmatrix} -F_{77} + 4F'_{00} - F_{37} \\ -F_{76} + 4F_{01} - F_{36} \\ -F_{75} + 4F_{02} - F_{35} \\ -F_{74} + 4F_{03} - F_{34} \\ -F_{64} \quad\quad - F_{24} \\ -F_{65} \quad\quad - F_{25} \\ -F_{66} \quad\quad - F_{26} \\ -F_{67} \quad\quad - F_{27} \\ -F_{57} \\ -F_{56} \\ -F_{55} \\ -F_{54} \\ -F_{44} \\ -F_{45} \\ -F_{46} \\ -F_{47} \end{bmatrix}
$$

$$
+\frac{1}{4}\begin{bmatrix} 3 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ -3 & -3 & -3 & -3 \\ \hline -3 & -3 & 3 & 3 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 3 & 3 & -3 & -3 \\ \hline 3 & -3 & -3 & 3 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -3 & 3 & 3 & -3 \\ \hline -3 & 3 & -3 & 3 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 3 & -3 & 3 & -3 \end{bmatrix}
\begin{bmatrix} F_{04} \\ F_{05} \\ F_{06} \\ F_{07} \end{bmatrix}
+\frac{1}{4}\begin{bmatrix} -1 & -1 & -1 & -1 \\ -3 & -3 & -3 & -3 \\ -3 & -3 & -3 & -3 \\ -1 & -1 & -1 & -1 \\ \hline -1 & -1 & 1 & 1 \\ -3 & -3 & 3 & 3 \\ -3 & -3 & 3 & 3 \\ -1 & -1 & 1 & 1 \\ \hline -1 & 1 & 1 & -1 \\ -3 & 3 & 3 & -3 \\ -3 & 3 & 3 & -3 \\ -1 & 1 & 1 & -1 \\ \hline -1 & -1 & -1 & 1 \\ -3 & 3 & -3 & 3 \\ -3 & 3 & -3 & 3 \\ -1 & 1 & -1 & 1 \end{bmatrix}
\begin{bmatrix} F_{17} \\ F_{16} \\ F_{15} \\ F_{14} \end{bmatrix}
\qquad (3.63)
$$

The 16 × 16 matrix is (W_4) (equation (A2.41)). Now the 16 × 4 matrix which multiplies $\left[F_{04}\ F_{05}\ F_{06}\ F_{07}\right]^T$ is the same as twice columns 5 to 8 of (W_4) plus columns 13 to 16 of (W_4). Also the 16 × 4 matrix which multiplies $\left[F_{17}\ F_{16}\ F_{15}\ F_{14}\right]^T$ is equal to the columns 9 to 12 of (W_4) minus twice columns 1 to 4 of (W_4). This means that equation (3.63) can be rewritten as

$$\begin{bmatrix} B_2 \\ B_4 \\ B_6 \\ B_8 \\ B_{18} \\ B_{20} \\ B_{22} \\ B_{24} \\ B_{34} \\ B_{36} \\ B_{38} \\ B_{40} \\ B_{50} \\ B_{52} \\ B_{54} \\ B_{56} \end{bmatrix} = \tfrac{1}{4}(W_4) \begin{bmatrix} -F_{77}+4F_{00} & -2F_{17} & -F_{37} \\ -F_{76}+4F_{01} & -2F_{16} & -F_{36} \\ -F_{75}+4F_{02} & -2F_{15} & -F_{35} \\ -F_{74}+4F_{03} & -2F_{14} & -F_{34} \\ -F_{64}+2F_{04} & & -F_{24} \\ -F_{65}+2F_{05} & & -F_{25} \\ -F_{66}+2F_{06} & & -F_{26} \\ -F_{67}+2F_{07} & & -F_{27} \\ -F_{57} & +F_{17} & \\ -F_{56} & +F_{16} & \\ -F_{55} & +F_{15} & \\ -F_{54} & +F_{14} & \\ -F_{44}+F_{04} & & \\ -F_{45}+F_{05} & & \\ -F_{46}+F_{06} & & \\ -F_{47}+F_{07} & & \end{bmatrix}.$$

The faint boxes shows the new terms "absorbed" into the column matrix premultiplied by $\frac{1}{2}(W_4)$. Premultiplication of this equation by $\frac{1}{4}(W_4)$, using $\frac{1}{16}(W_4)(W_4) = I_{16}$ (equation (3.25) or(A2.34) with n = 4) gives after some re-arrangement of terms:

$$\begin{bmatrix} F_{77} \\ F_{76} \\ F_{75} \\ F_{74} \\ F_{64} \\ F_{65} \\ F_{66} \\ F_{67} \\ F_{57} \\ F_{56} \\ F_{55} \\ F_{54} \\ F_{44} \\ F_{45} \\ F_{46} \\ F_{47} \end{bmatrix} = -\tfrac{1}{4}(W_4) \begin{bmatrix} \beta_2 \\ \beta_4 \\ \beta_6 \\ \beta_8 \\ \beta_{18} \\ \beta_{20} \\ \beta_{22} \\ \beta_{24} \\ \beta_{34} \\ \beta_{36} \\ \beta_{38} \\ \beta_{40} \\ \beta_{50} \\ \beta_{52} \\ \beta_{54} \\ \beta_{56} \end{bmatrix} + 4 \begin{bmatrix} F'_{00} \\ F_{01} \\ F_{02} \\ F_{03} \\ F_{04} \\ F_{05} \\ F_{06} \\ F_{07} \\ F_{17} \\ F_{16} \\ F_{15} \\ F_{14} \\ F_{04} \\ F_{05} \\ F_{06} \\ F_{07} \end{bmatrix} - 2 \begin{bmatrix} F_{17} \\ F_{16} \\ F_{15} \\ F_{14} \\ F_{04} \\ F_{05} \\ F_{06} \\ F_{07} \\ F_{17} \\ F_{16} \\ F_{15} \\ F_{14} \\ F_{04} \\ F_{05} \\ F_{06} \\ F_{07} \end{bmatrix} - 1 \begin{bmatrix} F_{37} \\ F_{36} \\ F_{35} \\ F_{34} \\ F_{24} \\ F_{25} \\ F_{26} \\ F_{27} \\ F_{17} \\ F_{16} \\ F_{15} \\ F_{14} \\ F_{04} \\ F_{05} \\ F_{06} \\ F_{07} \end{bmatrix} \tag{3.64}$$

Group (i) or (i,j) = (3,2).

The matrix equation obtained can be written

$$\begin{bmatrix} B_{10} \\ B_{12} \\ B_{14} \\ B_{16} \\ B_{42} \\ B_{44} \\ B_{46} \\ B_{48} \end{bmatrix} = \tfrac{1}{4} \left[\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ \hline 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{array}\right] \begin{bmatrix} -F_{73} + 4F'_{00} - 4F_{04} - F_{33} + F_{37} + F_{77} \\ -F_{72} + 4F_{01} - 4F_{05} - F_{32} + F_{36} + F_{76} \\ -F_{62} \qquad\qquad\qquad - F_{22} + F_{26} + F_{66} \\ -F_{63} \qquad\qquad\qquad - F_{23} + F_{27} + F_{67} \\ -F_{53} \qquad\qquad\qquad\qquad\qquad + F_{57} \\ -F_{52} \qquad\qquad\qquad\qquad\qquad + F_{56} \\ -F_{42} \qquad\qquad\qquad\qquad\qquad + F_{46} \\ -F_{43} \qquad\qquad\qquad\qquad\qquad + F_{47} \end{bmatrix}$$

$$+ \tfrac{1}{4} \left[\begin{array}{cccc} 1 & 1 & -3 & -3 \\ 3 & 3 & -1 & -1 \\ 3 & 3 & 1 & 1 \\ 1 & 1 & 3 & 3 \\ \hline 1 & -1 & 3 & -3 \\ 3 & -3 & 1 & -1 \\ 3 & -3 & -1 & 1 \\ 1 & -1 & -3 & 3 \end{array}\right] \begin{bmatrix} F_{17} - F_{13} \\ F_{16} - F_{12} \\ F_{06} - F_{02} \\ F_{07} - F_{03} \end{bmatrix} \tag{3.65}$$

The 8 X 8 matrix is (W_3) (equation (A2.40)). The first two columns of the 8 X 4 matrix in the matrix equation above are equal to twice the columns 1 and 2 of (W_3) minus columns 5 and 6 of (W_3). The last two columns of the 8 X 4 matrix above are equal to minus twice columns 3 and 4 of (W_3) minus columns 7 and 8 of (W_3). This means that equation (3.65) can be rewritten as

$$\begin{bmatrix} B_{10} \\ B_{12} \\ B_{14} \\ B_{16} \\ B_{42} \\ B_{44} \\ B_{46} \\ B_{48} \end{bmatrix} = \tfrac{1}{4}(W_3) \begin{bmatrix} -F_{73} + 4F'_{00} - 4F_{04} & +2F_{17} - 2F_{13} & -F_{33} + F_{37} + F_{77} \\ -F_{72} + 4F_{01} - 4F_{05} & +2F_{16} - 2F_{12} & -F_{32} + F_{36} + F_{76} \\ -F_{62} + 2F_{02} - 2F_{06} & & -F_{22} + F_{26} + F_{66} \\ -F_{63} + 2F_{03} - 2F_{07} & & -F_{23} + F_{27} + F_{67} \\ -F_{53} & -F_{17} + F_{13} & +F_{57} \\ -F_{52} & -F_{16} + F_{12} & +F_{56} \\ -F_{42} + F_{02} - F_{06} & & +F_{46} \\ -F_{43} + F_{03} - F_{07} & & +F_{47} \end{bmatrix}$$

Premultiplication of the equation above by $\frac{1}{8}(W_3)$ and using $\frac{1}{8}(W_3)(W_3) = I_8$ (equation (3.25) or (A2.34) with n = 3) gives after some re-arrangement of terms

$$\begin{bmatrix} F_{73} \\ F_{72} \\ F_{62} \\ F_{63} \\ F_{53} \\ F_{52} \\ F_{42} \\ F_{43} \end{bmatrix} = -\tfrac{1}{2}(W_3) \begin{bmatrix} B_{10} \\ B_{12} \\ B_{14} \\ B_{16} \\ B_{42} \\ B_{44} \\ B_{46} \\ B_{48} \end{bmatrix} + 4 \begin{bmatrix} F'_{00} - F_{04} \\ F_{01} - F_{05} \\ F_{02} - F_{06} \\ F_{03} - F_{07} \\ F_{13} - F_{17} \\ F_{12} - F_{16} \\ F_{02} - F_{06} \\ F_{03} - F_{07} \end{bmatrix} - 2 \begin{bmatrix} F_{13} - F_{17} \\ F_{12} - F_{16} \\ F_{02} - F_{06} \\ F_{03} - F_{07} \\ F_{13} - F_{17} \\ F_{12} - F_{16} \\ F_{02} - F_{06} \\ F_{03} - F_{07} \end{bmatrix} - 1 \begin{bmatrix} F_{33} - F_{37} - F_{77} \\ F_{32} - F_{36} - F_{76} \\ F_{22} - F_{26} - F_{66} \\ F_{23} - F_{27} - F_{67} \\ F_{13} - F_{17} - F_{57} \\ F_{12} - F_{16} - F_{56} \\ F_{02} - F_{06} - F_{46} \\ F_{03} - F_{07} - F_{47} \end{bmatrix} \tag{3.66}$$

Group (K) Or (i,j) = (3.1).

The matrix equation can be written

$$\begin{bmatrix} B_{26} \\ B_{28} \\ B_{30} \\ B_{32} \end{bmatrix} = \tfrac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -F_{71} + F_{73} - F_{75} + F_{77} \\ -F_{61} + F_{63} - F_{65} + F_{67} \\ -F_{51} + F_{53} - F_{55} + F_{57} \\ -F_{41} + F_{43} - F_{45} + F_{47} \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 1 & -1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} F'_{00} + F_{04} \\ F_{02} + F_{06} \end{bmatrix}$$

$$+\tfrac{1}{4}\begin{bmatrix}1 & 1 & 1 & -3\\ 1 & 1 & 3 & -1\\ 1 & -1 & 3 & 1\\ 1 & -1 & 1 & 3\end{bmatrix}\begin{bmatrix}-F_{31}+F_{33}-F_{35}+F_{37}\\ -F_{21}+F_{23}-F_{25}+F_{27}\\ -F_{11}+F_{13}-F_{15}+F_{17}\\ -F_{01}+F_{03}-F_{05}+F_{07}\end{bmatrix} \qquad (3.67)$$

The first 4×4 matrix on the right hand side of equation (3.67) is (W_2) (equation (A2.39)). The solution matrix $\begin{bmatrix}F_{71} & F_{61} & F_{51} & F_{41}\end{bmatrix}^T$ is obtained by premultiplying both sides of equation (3.67) by (W_2) and re-arranging terms $\frac{1}{4}(W_2)(W_2)^2 = I_4$ (equation (3.5) or (A2.34) with $n = 2$). The other relevant matrix multiplications are

$$\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\\ 1 & -1 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & -1\\ 1 & -1\\ 1 & -1\end{bmatrix} = \begin{bmatrix}4 & -4\\ 0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix} \qquad (3.68)$$

and

$$\tfrac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1\\ 1 & -1 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & -3\\ 1 & 1 & 3 & -1\\ 1 & -1 & 3 & 1\\ 1 & -1 & 1 & 3\end{bmatrix} = \begin{bmatrix}1 & 0 & 2 & 0\\ 0 & 1 & 0 & -2\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix} \qquad (3.69)$$

so that the solution matrix is

$$\begin{bmatrix}F_{71}\\ F_{61}\\ F_{51}\\ F_{41}\end{bmatrix} = -(W_2)\begin{bmatrix}B_{26}\\ B_{28}\\ B_{30}\\ B_{32}\end{bmatrix}+\Big[4F_{00}+4F_{04}-4F_{02}-4F_{06}-2F_{11}+2F_{13}-2F_{15}+2F_{17}+2F_{01}-2F_{03}+2F_{05}-2F_{07}$$

$$\begin{matrix}-F_{31}+F_{33}-F_{35}+F_{37}+F_{73}-F_{75}+F_{77}\\ -F_{21}+F_{23}-F_{25}+F_{27}+F_{63}-F_{65}+F_{67}\\ +F_{11}-F_{13}+F_{15}-F_{17}+F_{53}-F_{55}+F_{57}\\ +F_{01}-F_{03}+F_{05}-F_{07}+F_{43}-F_{45}+F_{47}\end{matrix}\Big]$$

This can be written:

$$\begin{bmatrix} F_{71} \\ F_{61} \\ F_{51} \\ F_{41} \end{bmatrix} = -(W_2)\begin{bmatrix} B_{26} \\ B_{28} \\ B_{30} \\ B_{32} \end{bmatrix} + 4\begin{bmatrix} F'_{00} - F_{02} + F_{04} - F_{06} \\ F_{01} - F_{03} + F_{05} - F_{07} \\ F_{11} - F_{13} + F_{15} - F_{17} \\ F_{01} - F_{03} + F_{05} - F_{07} \end{bmatrix} - 2\begin{bmatrix} F_{11} - F_{13} + F_{15} - F_{17} \\ F_{01} - F_{03} + F_{05} - F_{07} \\ F_{11} - F_{13} + F_{15} - F_{17} \\ F_{01} - F_{03} + F_{05} - F_{07} \end{bmatrix}$$

$$-1\begin{bmatrix} F_{31} - F_{33} + F_{35} - F_{37} - F_{73} + F_{75} - F_{77} \\ F_{21} - F_{23} + F_{25} - F_{27} - F_{63} + F_{65} - F_{67} \\ F_{11} - F_{13} + F_{15} - F_{17} - F_{53} + F_{55} - F_{57} \\ F_{01} - F_{03} + F_{05} - F_{07} - F_{43} + F_{45} - F_{47} \end{bmatrix} \qquad (3.70)$$

This completes the solutions of all the 65 coefficients.

3.4.4 The General Solution For the Set of Coefficients $\{F'_{00}, F''_{00}, F_{pq};\ 0 \leq p < 2^n,\ 0 \leq q < 2^n\}$, where n is any positive integer.

The generalised solutions, first suggested in section (3.4.1), for the sets of coefficients, $\{F'_{00}, F_{0q};\ 1 \leq q < 2^n\}$ and $\{F''_{00}, F_{p0};\ 1 \leq p < 2^n\}$, from the normalized line integral data from the M "non-nodal" "X" rays and "Y" rays respectively are valid for M = N=2,4 and 8 and are assumed to be true for any positive value of n. These solutions, equations (3.31) and (3.32) are rewritten here as a single matrix equation:

$$\left[\begin{array}{c|c} F'_{00} & F''_{00} \\ F_{01} & F_{10} \\ \\ F_{0p} & F_{p0} \\ \\ F_{0M-1} & F_{M-1,0} \end{array}\right] = \frac{1}{2^n}(W_n)\left[\begin{array}{c|c} B_1 & B_{M^2-M+2} \\ B_{M+1} & B_{M^2-M+3} \\ \\ B_{pM+1} & B_{M^2-M+2+p} \\ \\ B_{M^2-M+1} & B_{M^2+1} \end{array}\right] \qquad (3.71)$$

where $M = 2^n$, and $0 \leq p < M$ and (W_n) is the Walsh matrix of order 2^n (see equations (A2.37) to (A2.41) inclusive for the first five Walsh matrices).

All the other $(2^n - 1)^2$ solution column matrices are given in terms of the normalized line integrals associated with the group of coefficients, (i,j), plus other column matrices containing coefficients previously solved for in other groups.

Define the column matrix, $(F^{(n,i,j)})$, to contain the 2^{i+j-2} coefficients of the group (i,j), as given by Fig(3.8), in the order in which they appear in the solutions worked out in sections 3.4.1, 3.4.2 and 3.4.2 for n = 1,2 and 3 repectively. The relavant solutions are

Section 3.4.1. $n = 1$

Equation (3.30)

Section 3.4.2. $n = 2$

Equations (3.37), (3.38), (3.39), (3.40)

Section 3.4.3. $n = 3$

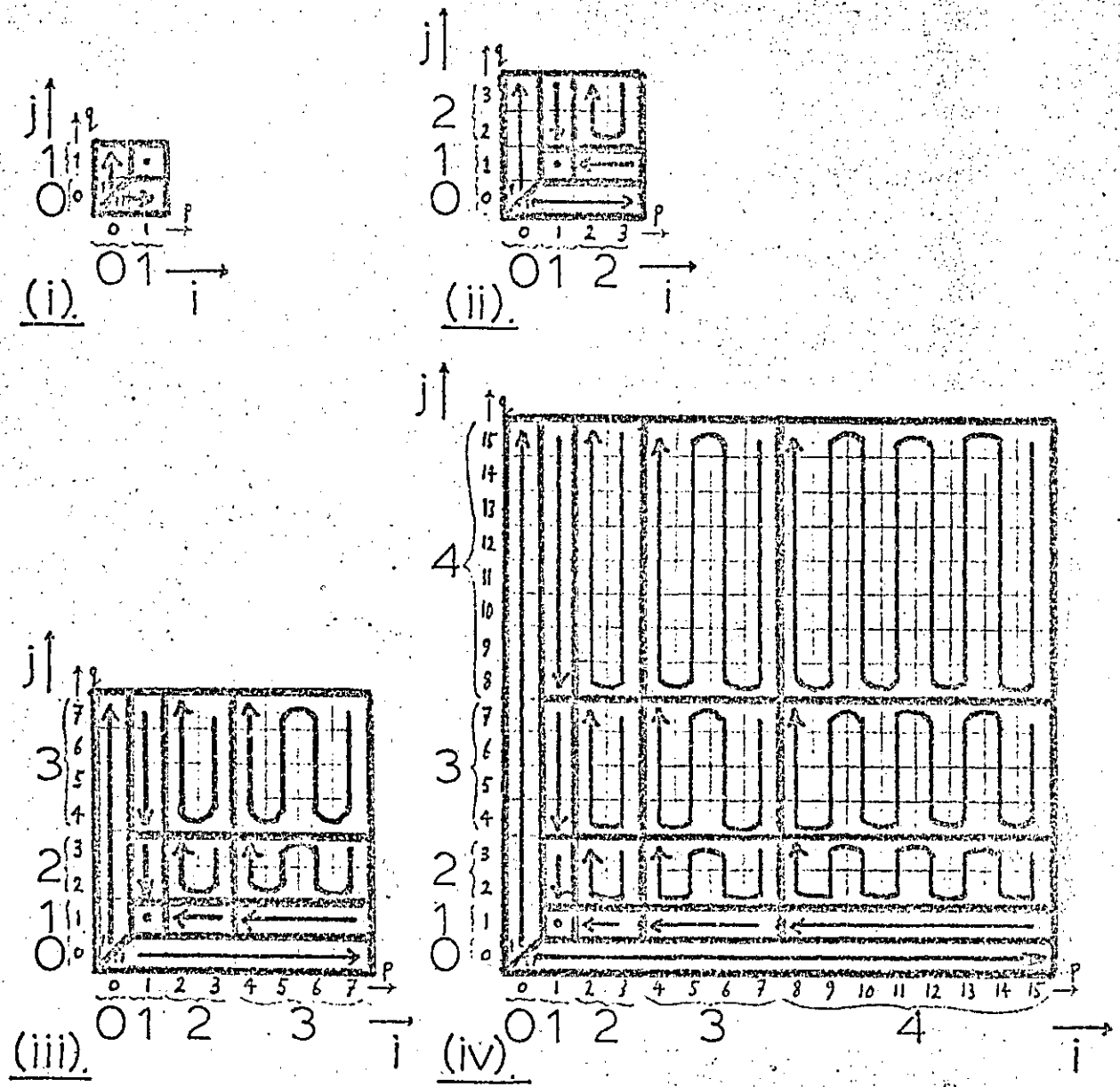
Equations (3.57), (3.58), (3.59), (3.60), (3.61), (3.62), (3.64), (3.66), and (3.70)

All the solution column matrices are of the following general form:

$$(F^{(n,i,j)}) = (\Delta^{(0)}F(n,i,j)) + (\Delta^{(1)}F(n,i,j)) + \ldots + (\Delta^{(i)}F(n,i,j)), \tag{3.72}$$

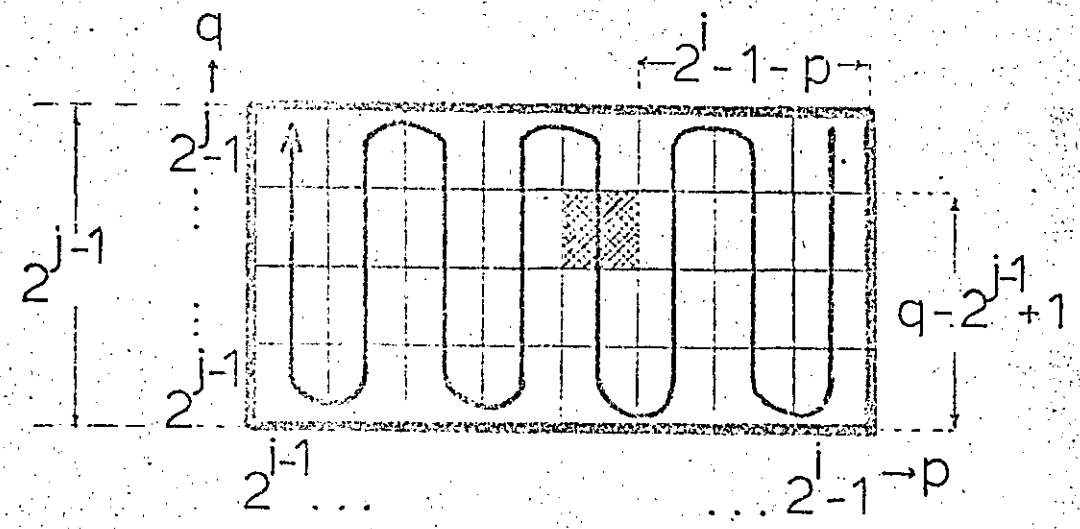
where the column matrix $(\Delta^{(0)}F(n,i,j))$, of length 2^{i+j-2}, depends on the normalized line integral data obtained from the rays associated with the group of coefficients (i,j) (see Figs (3.6) and (3.7). Each of the 2^{i+j-2} elements of the column matrices, $(\Delta^{(l)}F(n,i,j))$, $l = 1, \ldots, i$, is a sum of some coefficients, F_{pq}, which have been solved already. (The equations of the form (3.72) must be worked out in the order for (i,j) given by equation (3.42)).

Before considering the general form for the column matrices on the right hand side of equation (3.72) an explicit formula must be found equating a particular coefficient F_{pq}, in the group (i,j), to a particular element $(F^{(n,i,j)})_I$, $1 \leq I \leq 2^{i+j-2}$, of $(F^{(n,i,j)})$. This "ordering" of coefficients within a block (i,j) is not given in the coefficient diagrams of Fig(3.8). A copy of Fig(3.8) was made and by looking at the left hand side of the coefficient solution equations for n = 1,2 and 3, already referred to above, the column matrices were traced out as lines in each block,(i,j). Starting at the first coefficient, $F_{2^i-1,2^j-1}$, in each case the line passed from cell to cell in an "down and up" and "right to left" scan, finishing up at the coefficient $F_{2^{i-1},2^j-1}$ in each case. The results are shown in Fig(3.9) (i), (ii) and (iii). The lines traced out in Fig(3.9) (iv) are conjectural, based on the previous three diagrams. Fig(3.10) demonstrates, by using as an example the block (i,j) = (4,3), that the relevant formula for any block (i,j) is given by:

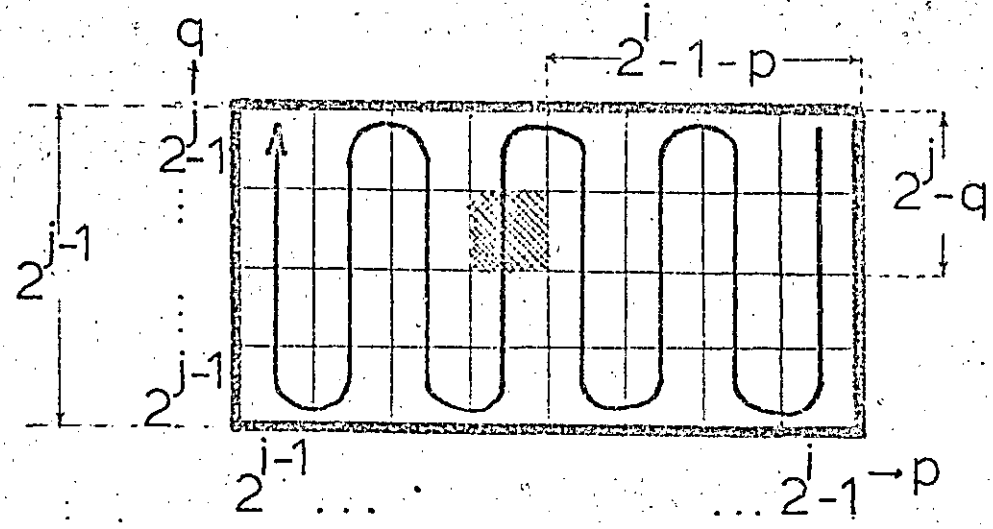


Fig(3.9). The column matrices, $(F^{(n,i,j)})$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, each made up from the 2^{i+j-2} coefficients in each of the blocks (i,j) for: (i) n=1, (ii) n=2, (iii) n=3, (iv) n=4. The thick line traced out in each block, (i,j), represents the order in which the coefficients appear in $(F^{(n,i,j)})$. The arrow head of each line is in the cell representing the last coefficient in the appropriate column matrix while the other end of the line gives the first coefficient in this matrix. $(F^{(n,1,1)}) = (F_{11})$ are shown as dots.

Also shown are the two column matrices $(F^{(n,i=0)}) = (F'_{0,0},\ldots,F_{0,2^n-1})^T$ and $(F^{(n,j=0)}) = (F''_{0,0},\ldots,F_{2^n-1,0})^T$.



(i). p is EVEN: $I = 2^{j-1}(2^{j}-1-p)+q-2^{j-1}+1.$



(ii). p is ODD: $I = 2^{j-1}(2^{j}-1-p) + 2^{j}-q.$

Fig(3.10). The relationship between a particular coefficient, F^{pq}, shown shaded, in the block of coefficients, (i,j), and the I^{th} element, $(F^{(n,i,j)})_I$, of the column matrix, $(F^{(n,i,j)})$. The examples shown are: (i) $(F^{(4,4,3)})_{15} - F_{12,6}$ for p even and (ii) $(F^{(4,4,3)})_{18} - F_{11,6}$ for p odd.

$$(F^{(n,i,j)})_I = F_{pq},$$

where

$$I = (2^{j-1})(2^i - 1 - p) + \begin{cases} q - 2^{j-1} + 1 & \text{If p is EVEN} \\ 2^j - q & \text{If p is ODD} \end{cases} \tag{3.73}$$

and $2^{i-1} \leqslant p < 2^i$, $2^{j-1} \leqslant q < 2^j$, $1 \leqslant I \leqslant 2^{i+j-2}$. For a single pair of values of p and q in equation (3.73) there is a unique value of I.

Example 1.

To which column matrix does F_{35} "belong" and which element of that column matrix is it equal to?

The range of p=3 and q = 5 is $2^{2-1} \leqslant 3 < 2^2$ and $2^{3-1} \leqslant 5 < 2^3$ (see also Fig (3.9) (iii)) so (i,j) = (2,3) and from equation (3.73)

$(F^{(n,2,3)})_I = F_{23}$ where I=4(4 - 1 3) + 8 - 5 = 3. Thus

$(F^{(n,2,3)})_3 = F_{35}$, where $n \geqslant 3$,

which from equation (3.60) is correct for n = 3.

Example 2.

To which column matrix does F_{23} "belong" and which element of that column matrix is it equal to? Proceeding as in example 1 gives (i,j) = (2,2) (see also Fig(3.9) (ii) and (iii)). Equation (3.73) then gives $(F^{(n,2,2)})_I = F_{23}$, where I = 2(4 - 1 - 2) + 3 - 2 + 1 = 4

Thus

$(F^{(n,2,2)})_4 = F_{23}$, where $n \geqslant 2$,

which from equations (3.39) and (3.61) is correct for n=2 and n=3.

The general form for $(\Delta^{(0)}F^{(n,i,j)})$.

Define a column matrix, $(B^{(n,i,j)})$, which contains the normalized line integrals in the order in which they appear on the right hand side of the coefficient equations for n=1,2 and 3. In $(B^{(n,i,j)})$, the elements $B_{m(n,i,j,r,s)}$ (see equation (3.51)) are arranged in order of increasing m(n,i,j,r,s) which means that in equation (3.51) r must change over its whole range before s is increased by 1. Thus the I^{th} element, $(B^{(n,i,j)})_I$, of $(B^{(n,i,j)})$ can be written explicitly as:

$$(B^{(n,i,j)})_I = B_{m(n,i,j,r,s)},$$

where (3.74)

$$I = (s - 1)r_{max} + r = (s - 1)2^{i-1} + r$$

and

$$m(n,i,j,r,s) = 2^n(2^{n-j}(2s - 1) - 1) + 2^{n-i}(2r - 1) + 1, \tag{3.51}$$

where $1 \leqslant r \leqslant 2^{i-1}$ and $1 \leqslant s \leqslant 2^{j-i}$.

Example 1.

Which normalized line integral, $B_{m(3,2,3,r,s)}$, is the 3rd. element, $(B^{(3,2,3)})_3$, of the column matrix, $(B^{(3,2,3)})$? From equation (3.74), $3 = (s-1)2 + r$ which, since the possible r and s values are, $r = 1,2$ and $s = 1,2,3,4$, has the unique solution $r = 1$, $s = 2$. Equation (3.51) then gives $m(n,i,j,r,s) = m(3,2,3,1,2) = 8(2^n(3) - 1) + 2(1) + 1 = 16 + 3 = 19$. Thus $(B^{(3,2,3)})_3 = B_{19}$, which from equation (3.60) is correct.

Example 2(a).

Which normalized line integral $B_{m(2,2,2,r,s)}$ is the 4th. element $(B^{(2,2,2)})_4$, of the column matrix $(B^{(2,2,2)})$ for $n = 2$, $i = 2$, $j = 2$? Equation (3.74) gives $4 = (s-1)2 + r$. The ranges of r and s are $r = 1,2$, $s = 1,2$ so $r = 2$, $s = 2$ and equation (3.51) then gives $m(2,2,2,2,2) = 4(1(3) - 1) + 1(3) + 1 = 8 + 4 = 12$. Thus

$$(B^{(2,2,2)})_4 = B_{12},$$

which from equation (3.39) is correct.

Example 2(b)

This is the same as example 2(a) except that $n = 3$ instead of 2. As before $r = 2$, $s = 2$ so equation (3.51) now becomes $m(3,2,2,2,2) = 8(2(3) - 1) + 2(3) + 1 = 40 + 6 + 1 = 47$. Thus

$$(B^{(3,2,2)})_4 = B_{47},$$

which from equation (3.61) is correct.

Now the general form for $(\Delta^{(0)}F^{(n,i,j)})$ in the coefficient solution equations of the type, equation (3.72), is $-(1/2^{i-1})$ times the discrete Walsh transform (see equation (A2.35)) of the column matrix $(B^{(n,i,j)})$:

$$(\Delta^{(0)}F^{(n,i,j)}) = -\frac{1}{2^{i-1}}(W_{i+j-2})(B^{(n,i,j)}), \qquad (3.75)$$

where (W_{i+j-2}) is the Walsh matrix of order 2^{i+j-2} (see equations (A2.37) to (A2.41) inclusive for the first five Walsh matrices)

The General Form For the l column matrices $(\Delta^{(l)}F^{(n,i,j)})$ of equation (3.72).

From the coefficient solution equations of the type equation (3.72) in sections 3.4.1 to 3.4.2 inclusive for n = 1,2, and 3, an $l^{th.}$ column matrix, $(\Delta^{(l)}F^{(n,i,j)})$, can be written as the sum of a number of column matrices $(H^{(h)})$:

$$(\Delta^{(l)}F^{(n,i,j)}) = 2^{i-l}(H^{(0)} + (H)^{(1)} + \cdots ?) \qquad (3.76)$$

where $j \leq l < i$. The column matrices, $(H^{(h)})$, contain as elements, only single coefficients, F_{pq}, or $-F_{pq}$, (which have been solved in previous groups, the order for (i,j) being given by equation (3.42)). Strictly $(H^{(h)})$ should be written with further labels n,i,j,l, say as $(H^{(n,i,j,l,h)})$ but this is omitted if it is clear which n,i,j, and l are being discussed.

Example

The solution column matrix $(F^{(3,2,1)}) = [F_{31}\ F_{21}]^T$ given by equation (3.62) has two column matrices $(\Delta^{(1)}F^{(3,2,1)})$ and $(\Delta^{(2)}F^{(3,2,1)})$, the first being given by the sum of four column matrices

$$(\Delta^{(1)}F^{(3,2,1)}) = 2\left(\begin{bmatrix}F'_{00}\\F_{01}\end{bmatrix} - \begin{bmatrix}F_{02}\\F_{03}\end{bmatrix} + \begin{bmatrix}F_{04}\\F_{05}\end{bmatrix} - \begin{bmatrix}F_{06}\\F_{07}\end{bmatrix}\right)$$

In an attempt to discover some general pattern for the matrices $(H^{(h)})$ making up $(\Delta^{(l)}F^{(n,i,j)})$, the coefficients in them are traced out on "coefficient diagrams" (like Figs (3.8) and (3.9)) in the same way that $(F^{(n,i,j)})$ were in Fig(3.9). A separate coefficient diagram is used for each different value of l. The results are shown in Figs (3.11), (3.12) and (3.13). The column matrix being solved, $(F^{(n,i,j)})$, is shown (as in Fig (3.9)) by a thick line in the block (i,j).

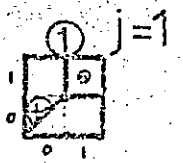
$(\Delta^{(1)}F^{(n,i,j)})$ for i = 1 (see Fig(3.11))

The column matrix $(F^{(n,i,j)})$ being solved in this case is

$$F^{(n,i,j)}) = [F_{1,2^j-1}\ F_{1,2^j-2}\ \cdots\ F_{1,2^{j-1}}]^T \qquad (3.77)$$

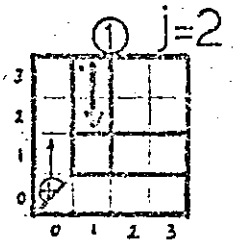
There is only one matrix $(\Delta^{(l)}F^{(n,i,j)})$ for i = 1 and in accordance with equation (3.76), it can be written

i=1

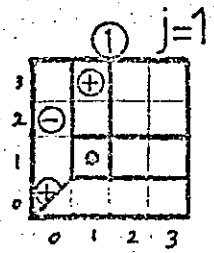


(i) (3.30)

l=1

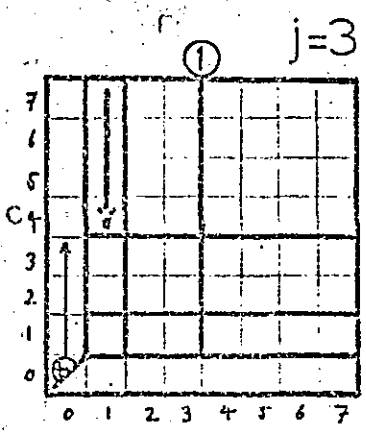


(ii) (3.37)

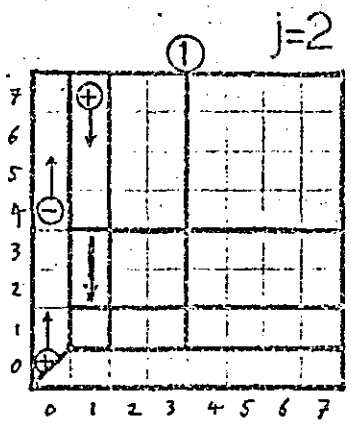


(iii) (3.38)

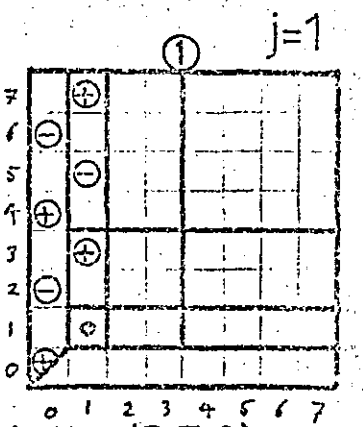
l=1



(iv) (3.57)



(v) (3.58)

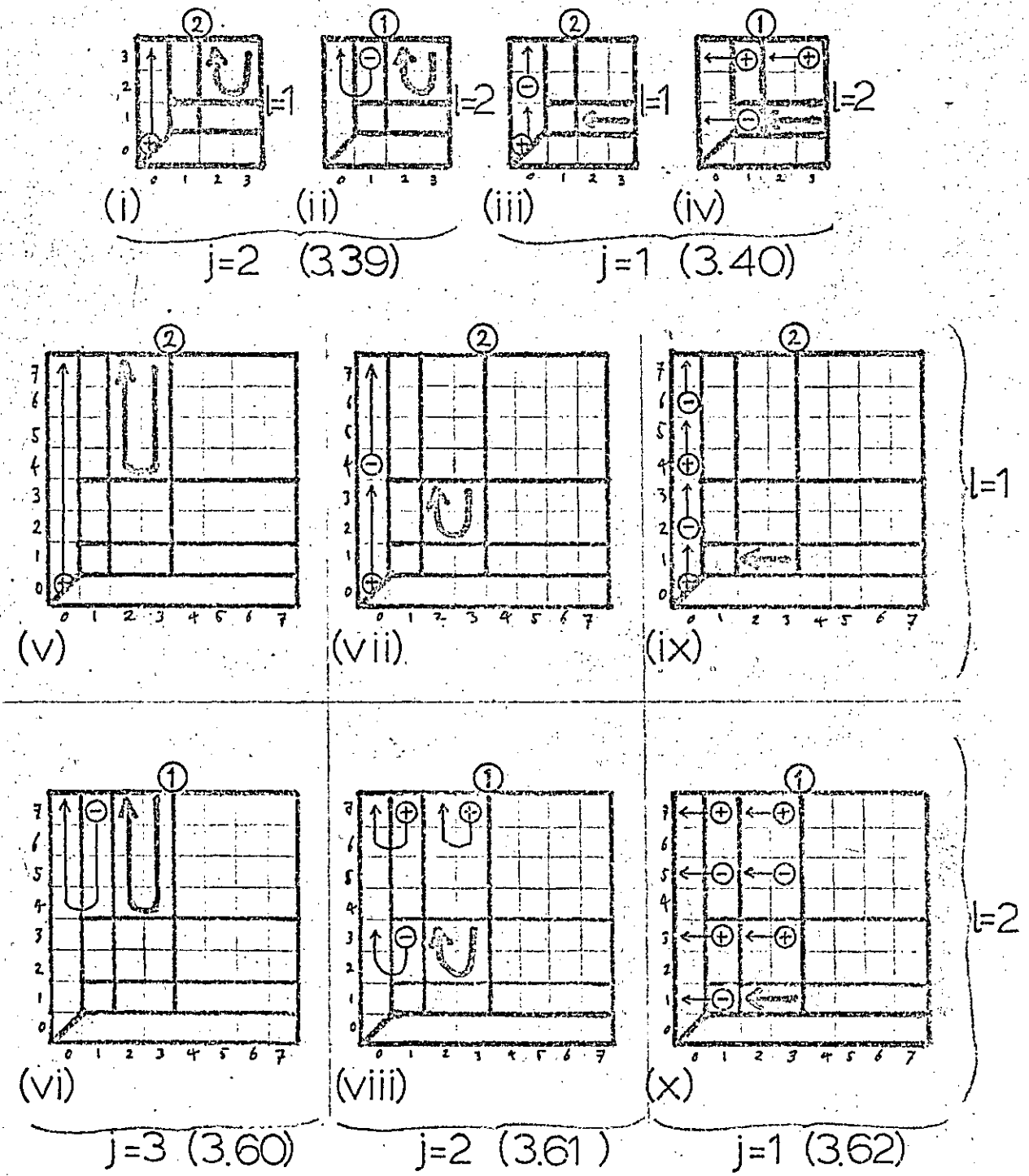


(vi) (3.59)

l=1

Fig(3.11). A representation of each of the i=1 column matrices, $(\Delta^{(l)}{}_F(n,i,j))$, $1 \leqslant i \leqslant i$, for each pair of values $(i,j)=(1,j), 1 \leqslant j \leqslant n$ where n=1,2,3. Each of the coefficient diagrams, (i) to(vi) has, traced on it, all the component column matrices, $(H^{(n,i,j,l,h)})$, of $(\Delta^{(l)}{}_F(n,i,j))$. Each of the matrices, $(H^{(n,i,j,l,h)})$, is represented by a line tracing out the coefficients in it. The factor, 2^{i-l}, which multiplies all the $(H^{(n,i,j,l,h)})$ for a given l (see equation (3.76)) is shown in a circle over each diagram.
If the elements of $(H^{(n,i,j,l,h)})$ are all of the form, $-F_{pq}$, then "⊖" represents the first element whereas "⊖" represents it if the elements are all of the form $+F_{pq}$. The arrowhead on each line trace represents the last coefficient in $(H^{(n,i,pq}{}_{j,l,h)})$. The solution column matrices, $(F^{(n,i,j)})$, are also traced out in each diagram as in Fig.(3.9). The numbers, (3.--), show the equation in which each $(\Delta^{(l)}{}_F(n,i,j))$ occurs. The integers along the axes are p and q as in Figs. (3.8) and (3.9).

j=2.



Fig(3.12). A representation of each of the j=2 column matrices, $(\Delta^{(l)}{}_F(n,i,j))$, $1 \leqslant l \leqslant i$, for $1 \leqslant j \leqslant n$ where n=2 and 3. Each coefficient diagram, (i) to (x), has, traced out on it, all the component column matrices, $(H^{(n,i,j,l,h)})$, of $(\Delta^{(l)}{}_F(n,i,j))$. The caption of Fig.(3.11), except for the first two sentences, applies here as well.

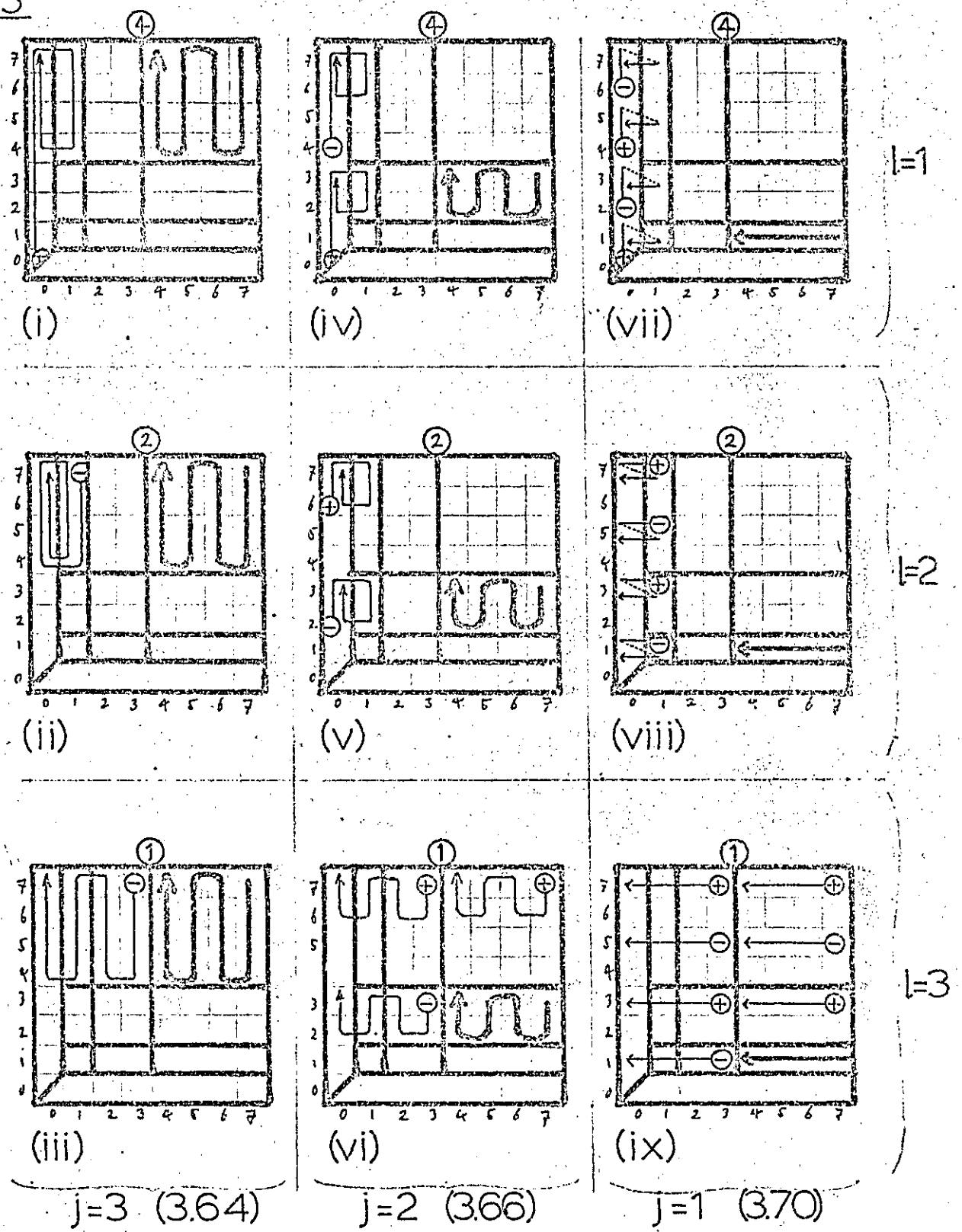


Fig.(3.13)... representation of each of the i=3 column matrices, $(\Delta^{(l)}{}_F{}^{(n,i,j)})$, $l \leqslant l \leqslant i$, for $1 \leqslant j \leqslant n$ where n=3. Each coefficient diagram, (i) to (ix), has traced out on it, all the component column matrices, $(H^{(n,i,j,l,h)})$, of $(\Delta^{(l)}{}_F{}^{(n,i,j)})$. The caption of Fig.(3.11), except for the first two sentences, applies here as well. ...

$$(\Delta^{(1)}F(n,1,j)) = (H^{(0)}) + (H^{(1)}) + \ldots\ldots + (H^{(h)}) + \ldots\ldots \qquad (3.78)$$

For $j = n$ a generalization for any n which seems consistent, (see Fig (3.11) (i), (ii) and (iv)) is given by

$$(\Delta^{(1)}F(n,1,1)) = \left[F'_{00}F_{01}\ldots\ldots F_{0,2^{j-1}-1}\right]^T = \left[F'_{00}F_{01}\ldots\ldots F_{0,2^{n-1}-1}\right]^T \qquad (3.79)$$

so that

$$(\Delta^{(1)}F(n,1,1))_I = F_{0,I-1} \qquad (3.80)$$

where $1 \leqslant I \leqslant 2^{n-1}$.

For $1 \leqslant j \leqslant n-1$ there are two types of matrices, $(H^{(h)})$, in equation (3.78) (see Fig(3.11) (iii), (v) and (vi)):

(a) These can be generated from the first one,

$$(H^{(0)}) = \left[F'_{00}F_{01}\ldots\ldots F_{0,2^{j-1}-1}\right]^T, \qquad (3.81)$$

by adding $h2^j$, where $h = 1,2,\ldots,2^{n-j}-1$, to the label $I-1$ of each element, $F_{0,I-1}$, in equation 3.81 If h is odd then all the elements $F_{0,I-1+h2^j}$ are negatived. Thus

$$(H^{(h)})_I = (-1)^h F_{0,(I-1)+h2^j}, \qquad (3.82)$$

where $h = 0,1,2,\ldots,2^{n-j}-1$, $1 \leqslant I \leqslant 2^{j-1}$ and $1 \leqslant j \leqslant n-1$, $n \geqslant 2$.

(b) These can be generated from $-(F^{(n,1,j)})$ by adding $h2^j$ where $h = 1,2,\ldots,2^{n-j}-1$, to the label $2^j - I$ of each negatived element, $-F_{1,2^j-I}$ in equation (3.77). If h is odd then all the elements $-F_{1,2^j-I+h2^j}$ are negatived. Thus

$$(H^{(h+2^{n-j})}) = -(-1)^h F_{1,(h+1)2^j-I} \qquad (3.83)$$

where $h = 1,2\ldots,2^{n-j}-1$, $1 \leqslant I \leqslant 2^{j-1}$ and $1 \leqslant j \leqslant n-1$.

Now the sum of equations (3.82) and (3.83) gives the $I^{th.}$ element of $(\Delta^{(1)}F(n,1,j))$:

$$(\Delta^{(1)}F(n,1,j))_I = F_{0,I-1} + \sum_{h=1}^{2^{n-j}-1}(-1)^h(F_{0,I-1+h2^j} - F_{1,(h+1)2^j-I}). \qquad (3.84)$$

where $1 \leqslant I \leqslant 2^{j-1}$ and $1 \leqslant j \leqslant n-1$, $n \geqslant 2$ and the coefficient F_{00} on the right hand side of equation (3.84), when $I = 1$, is, of course, F'_{00}. It is assumed that equations (3.80) and (3.84) hold for all values of $n \geqslant 1$.

$(\Delta^{(l)}F(n,i,j))$, $1 \leqslant l \leqslant i$ for $i=2$, $n \geqslant 2$. (See Fig.(3.12)).

It turns out that general formulae can be found for the i column matrices $(\Delta^{(l)}F(n,i,j))$, $1 \leqslant l \leqslant i$, for $n \geqslant 2$ but it is useful to find the formulae for the two column matrices $(\Delta^{(1)}F(n,2,j))$ and $(\Delta^{(2)}F(n,2,j))$ before attempting further generalization.

$(\Delta^{(1)}F(n,2,j))$. (See Fig(3.12),(i),(iii),(v),(vii),(ix)).

The first contribution is the column matrix $(H^{(0)})$ given by

$$(H^{(0)})_I = F_{0,I-1}, \tag{3.85}$$

where $1 \leqslant I \leqslant 2^j$.

If $j = n$ (see Fig(3.12) (i) and (v)), there are no further contributions and

$$(\Delta^{(1)}F(n,2,n))_I = 2F_{0,I-1}, \tag{3.86}$$

where $1 \leqslant I \leqslant 2^n$.

If $j < n$ the other $2^{n-j} - 1$ column matrices $(H^{(h)})$, $h = 1,\ldots,2^{n-j}-1$, are given by

$$(H^{(h)})_I = (-1)^h F_{0,I-1+h2^j}, \tag{3.87}$$

so that

$$(\Delta^{(1)}F(n,2,j))_I = 2\sum_{h=0}^{2^{n-j}-1}(H^{(h)})_I = 2\sum_{h=0}^{2^{n-j}-1}(-1)^h F_{0,I-1+h2^j} \tag{3.88}$$

where $1 \leqslant I \leqslant 2^j$.

$(\Delta^{(2)}F(n,2,j))$ (See Fig(3.12) (ii),(iv),(vi),(viii) and (x)).

The first contributing matrix, $(H^{(0)})$, is given by a block of coefficients immediately to the left of the scanned block (i,j), making up $(F^{(n,2,j)})$. The coefficients are "scanned" in an identical way to those in the block (i,j) and are all negative. Thus if $(F^{(n,2,j)})_I$ is the coefficent F_{pq} (given by equation (3.73)).

$$(H^{(0)})_I = -F_{p-2^{i-1},q} = -F_{p-2,q},$$

where I is given in terms of p and q by equation (3,73) with i = 2. Alternatively, writing p instead of p-2 above and in equation (3.73) gives

$$(H^{(0)})_I = -F_{p,q}, \tag{3.89}$$

where

$$I = 2^{j-1}(2 - 1 - p) + \begin{cases} q - 2^{j-1} + 1 & \text{If } p = 0 \text{ (EVEN)} \\ 2^j - q & \text{If } p = 1 \text{ (ODD)} \end{cases}, \tag{3.90}$$

for $2^{j-1} \leqslant q < 2^j$.

If j = n there are no further contributions and

$$(\Delta^{(2)}{}_F(n,2,n))_I = -F_{p,q}. \tag{3.91}$$

If $j < n$ there are an extra $2(2^{n-j} - 1)$ other column matrices, $(H^{(h)})$, $h = 1,\ldots,2^{n-j+1} - 2$ given as follows:

(a) $2^{n-j} - 1$ matrices are "generated" from $(H^{(0)})$ in exactly the same way as for the corresponding ones for $(\Delta^{(1)}{}_F(n,2,j))$. Thus (c.f. equation (3.87))

$$(H^{(h)})_I = -(-1)^h F_{p,q+h2^j} \tag{3.92}$$

where $0 \leqslant h < 2^{n-j}$ and I is given by equation (3.91)

(b) A further $2^{n-j} - 1$ matrices "generated" from the above ones by adding 2^{i-1} to all the first subscripts of the coefficients:

$$(H^{(h+2^{n-j}-1)})_I = -(-1)^h F_{p+2^{i-1},q+h2^j} = -(-1)^h F_{p+2,q+h2^j}, \tag{3.93}$$

where $1 \leqslant h < 2^{n-j}$ and I is given by equation (3.).

Taking equations (3,92) and (3.93) together gives

$$(\Delta^{(2)}{}_F(n,2,j))_I = -F_{p,q} - \sum_{h=1}^{2^{n-j}-1} (-1)^h (F_{p,q+h2^j} + F_{p+2,q+h2^j}), \tag{3.94}$$

where $j < n$ and I is given by equation (3.91) with i = 2 so p = 0,1 and $2^{j-1} \leqslant q < 2^j$. It is assumed that equation (3.94) is true for all $n \geqslant 2$.

$(\Delta^{(l)}{}_F(n,i,j))$, $1 \leqslant l \leqslant i$ for any $i \geqslant 2$ (see Fig.(3.13)).

Fig.(3.13) shows the column matrices $(\Delta^{(l)}{}_F(n,i,j))$ for $i=3$ with $n=3$. There are three such matrices for each block of coefficients, $(i,j) = (3,j)$.

$(\Delta^{(i)}(n,i,j))$ (see Fig(3.13) (iii),(vi) and (ix))

This is the easiest one to generalize as it follows the same pattern as the corresponding matrix for $i = 2$ already worked out, (c.f. Figs (3.12) (ii),(iv),(vi),(viii),and (x) for the contributions to $(\Delta^{(2)}{}_F(n,2,j))$:

For $j = n$ the only contribution $(H^{(0)})$, is given by the first form of the right-hand side of equation (3.89) and (3.90). Thus

$$(\Delta^{(i)}{}_F(n,i,n))_I = (H^{(0)})_I = -F_{p,q}. \tag{3.95}$$

Equation (3.95) includes equation (3.85) as $i = 2$ in the former gives the latter.

For $j < n$, the first forms of the right hand sides of equations (3.92) and (3.93) give all the contributions so that

$$(\Delta^{(i)}{}_F(n,i,j))_I = -F_{p,q} - \sum_{h=1}^{2^{n-j}-1} (-1)^h (F_{p,q+h2^j} + F_{p+2^{i-1},q+h2^j}), \tag{3.96}$$

where, in both equations (3.95) and (3.96), I is given by replacing p by $p+2^{i-1}$ in equation(3.73):

$$I = 2^{j-1}(2^{i-1} - 1 - p) + \begin{cases} q - 2^{j-1} + 1 & \text{If p is EVEN} \\ 2^j - q & \text{If p is ODD} \end{cases} \tag{3.97}$$

and

$2 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n-1$, $\quad 0 \leqslant p < 2^{i-1}$, $2^{j-1} \leqslant q < 2^j$.

Equation (3.73) with all the values of p and q in the range given will generate all the integers I in the range $1 \leqslant I \leqslant 2^{i+j-2}$. Equation (3.96) for $l = i = 2$ gives equation (3.94). It is assumed that equations (3.95) and (3.96) hold for all values of $n \geqslant 2$.

$(\Delta^{(l)}{}_F(n,i,j))$, $1 \leqslant l \leqslant i-1$. (see Fig(3.13) (i),(ii),(iv),(v),(vii) and(viii) for $i = 3$)

At first glance the picture gets much more confusing for $i = 3$, and $1 \leqslant l < i-1$. All the contributing column matrices contain some coefficients more than once so the line tracing them out crosses the same cell of the coefficient diagram more than once.

One important feature however is clear. The 2^{n-j} contributions to $(\Delta^{(l)}{}_F(n,i,j))$, $i = 3$ and $l = 1$ and $2 (= i-1)$ are "generated" from the first contribution $(H^{(n,i,j,l,h)})$, $h = 0$, $i = 3$, $l = 1$ and 2 respectively, in exactly the same way that the 2^{n-j} contributions to

$(\Delta^{(1)}{}_F(n,2,j))$ were generated from a first contribution (i.e $(H^{(0)})$ of equation (3.85)). If j = n there is only the first contribution.

Suppose then that the first contribution to $(\Delta^{(l)}{}_F(n,i,j))$ is $(H^{(n,i,j,1,0)})$ and that

$$(H^{(n,i,j,1,0)})_{I(p,q;i,j,l)} = F_{p,q} \qquad (3.98)$$

where there is a particular relation between p,q and I(p,q;i,j,l) for a particular set of values i,j and l. (The range of I is of course $1 \leqslant I \leqslant 2^{i+j-2}$).

So far the relationships between I and p,q of $F_{p,q}$ for a particular i,j, and l have been unique: i.e. for a given pair of values of p and q there is only one value of I and vice versa. There are 2^{i+j-2} different number pairs (p,q) giving the same number of I values, $1 \leqslant I \leqslant 2^{i+j-2}$ (e.g. equations (3.90) and (3.97)). From Fig. (3.13) it is clear that, for $1 \leqslant l \leqslant i-1$, there are less pairs, (p,q), than the number, 2^{i+j-3}, of different I values. I is then a several valued function of (p,q). From Fig (3.13) it is clear that a given coefficient F_{pq} may occur more than once in $(H^{(n,i,j,l,0)})$, so for a given (p,q) there may be more than one value of I.

Now the other $2^{n-j}-1$ contributions to $(\Delta^{(l)}{}_F(n,i,j))$ are given by

$$(H^{(n,i,j,l,h)})_I = (-1)^h \begin{cases} F_{p,q+h2^j} & \text{If } l = 1 \\ -F_{p,q+h2^j} & \text{If } 2 \leqslant l \leqslant i-1 \end{cases} \qquad (3.99)$$

so that

$$(\Delta^{(l)}{}_F(n,i,j))_{I(p,q,i,j,l)} = \sum_{h=0}^{2^{n-j}-1} (H^{(n,i,j,l,h)})_I \qquad (3.100)$$

For i = 1 and i = 2 the matrices, $(H^{(n,i,j,l,0)})$, from which the others, $(H^{(n,i,j,l,h)})$, h = 1,2...,$2^{n-j}-1$ or $2^{n-j+1}-2$ (see equations (3.81) and (3.82);(3.85) and (3.87);(3.90) and (3.93)), are "generated" are independent of n (as long as $n \geqslant i$ otherwise they do not exist of course), for n = 1,2 and n = 3 in the case l = i (see equation (3.95)). Compare for example Figs. (3.11) (i),(iii) and (vi) or Fig(3.12) (ii) with (viii) or Fig (3.12) (iv) with (x) and so on. On this basis it is assumed that equation (3.99) is independent of n (as long as $n \geqslant 1$, and $n \geqslant j$) so that equation (3.100) is true for all $n \geqslant i$ and $n \geqslant j$.

(just as it was assumed that equations (3.84), (3.88) and (3.85) were true for all $n \geqslant i$ because equations (3.81), giving $(H^{(n,1,1,1,0)}$, (3.85), giving $(H^{(n,2,j,1,0)})$ and (3.95) giving $(H^{(n,i,j,i,0)})$ were independent of $n \geqslant i$).

If now remains to establish the relationship between $I(p,q;i,j,l)$ and p,q,i,j and l, or in other words to find for a given (i,j) the $i - 1$ column matrices, $(H^{(n,i,j,l,0)})$, $1 \leqslant l < i - 1$.

The parts of Fig (3.13) (i),(ii),(iv),(v),(vii) and (viii) which are relevant to this end, are the bottom 2^j rows of coefficient cells (from $q = 0$ to $q = 2^j - 1$) because, for $n = 3$, these contain all the coefficients making up $(H^{(n,i,j,l,0)})$ and the column matrix being solved, $(F^{(n,i,j)})$. It is assumed that this is true for $n > 3$.

A particular formula for the two column matrices $(H^{(n,i,j,l,0)})$, $i = 3$, $1 \leqslant l \leqslant (i - 1) = 2$, could easily be worked out from the parts of Fig (3.13) referred to above but from these it is far from obvious what the form of the matrices for $i > 3$ could be.

The first 2^j elements of $(H^{(n,i,j,l,0)})$ for $l = 1$ are amenable to complete generalisation. From Fig (3.13) (i),(iv) and (vii), these can be written

$$(H^{(n,i,j,1,0)})_I = F_{0,I-1}, \qquad (3.101)$$

where $1 \leqslant I \leqslant 2^j$. It is interesting to note that if $i = 2$ (see Fig (3.12) (i),(iii),(v) and (vii) for $i = 2$, $l = 1$ cases) equation (3.101) gives <u>all</u> the 2^j elements of $(H^{(n,2,j,1,0)})$ given by equation (3.85).

It is a plausable guess that all the remaining coefficient elements of $(H^{(n,i,j,1,0)})$ are in general confined to the same range of q that F_{pq} of $(F^{(n,i,j)})$ have: i.e. $2^{j-1} \leqslant q < 2^j$. The range of p's is $0 \leqslant p < 2^{i-1}$. This also applies to all the coefficient elements of all the remaining matrices $(H^{(n,i,j,l,0)})$ for $2 \leqslant l \leqslant i$. (It is certainly true for $l = i$ if the generalization already suggested for $(H^{(n,i,j,i,0)})$, equation (3.95), is true).

It is really necessary to work out the solution schemes for the groups $(i,j) = (4,j)$ using the appropriate rays (calculuated from equation (3.51)) to "see" a pattern from which a general formula for

$(H^{(n,i,j,l,0)})$ might be developed. This is a daunting task. It can be reduced to finding the solution scheme for just one block of coefficients $(i,j) = (n,1) = (4,1)$, (see Fig.(3.9) (iv)) by an arguement based on certain observations of the lowest 2^j rows of the coefficient diagrams of Fig.(3.13) (i),(ii),(iv),(v),(vii) and (viii):

For a given value of l, 1, 2 or 3 for $i = 3$, the matrices $(H^{(n,3,j,l,0)})$, for the different values of j, 1 to n, follow a similar pattern. Compare Fig.(3.13)(i),(iv) and (vii),(ii),(v) and (viii) and (iii),(vi) and (ix). The line tracing out $(H^{(n,3,j,l,0)})$ is broken into two sections. The first line section traces out the first half of the elements of this column matrix and the second section traces out the other half of the elements.

Now appart from the first section of $(H^{(n,i,j,l,0)})$, for $l = 1$, already dealt with (equation (3.101)) all the line sections making up matrices $(H^{(n,i,j,l,0)})$ start in the cell in the row $q = 2^j - 1$ and perform a multiple "U" type scan, (one "U" for $l = 1$ and two for $l = 2$), until the column $p = 0$ is reached. The scans are of the same type as those for the column matrix $(F^{(n,i,j)})$ (see Figs.(3.9) and (3.10)). For $j = 1$ the range of rows of coefficients in $(H^{(n,i,j,l,0)})$, $2^{j-1} \leqslant q < 2^j$, is 1 so the scans reduce to straight lines drawn from right to left.

If it were known that the line sections which trace out the elements of $(H^{(n,i,j,l,0)})$ always performed multiple "U" type scans, which for a given l value start in the same column of a coefficient diagram and finish in the column $p = 0$ regardless of the value of j, then it would be sufficient to know $(H^{(n,i,j,l,0)})$ for a single value of j, say 1, to work out all the column matrices $(H^{(n,i,j,l,0)})$ having the other j values, say $2 \leqslant j \leqslant n$.

It will be assumed then, that if $(H^{(n,i,1,l,0)})$ is known for all l, $1 \leqslant l \leqslant i$, the remaining column matrices, $(H^{(n,i,j,l,0)})$, for $2 \leqslant j \leqslant n$, can be worked out according to the above arguement.

The four column matrices, $(H^{(4,4,1,l,0)})$, $l = 1,2,3,4$ will now be worked out. The range of the coefficients in this matrix, appart from the first $2^j - 2$ elements of $(H^{(4,4,1,1,0)})$ given by $n = i = 4$ and $j = 1$ in equation (3.101), are assumed, as for $i = 2$ and 3, to lie in the ranges

$$0 \leqslant p < 2^{i-1} \quad \text{and} \quad 2^{j-1} \leqslant q < 2^j, \tag{3.102}$$

which for $i = 4$ and $j = 1$ is $0 \leqslant p < 8$ and $q = 1$.

The rays associated with $(F^{(4,4,1)})$ are found by substituting $n = 4$, $i = 4$ and $j = 1$ into equation (3.51). This gives

$$m(4,4,1,r,s) = 16(8(2s-1)-1)+(2r-1)+1$$
$$= 128(2s-1)-16+2r,$$

where $1 \leqslant r \leqslant 2^{i-1}=8$ and $1 \leqslant s \leqslant 2^{j-1}=1$ so $r = 1,2,3,4,5,6,7,8$ and $s = 1$. Thus

$$m(4,4,1,r,1) = 112+2r = 114,116,118,120,122,124,126,128. \quad (3.103)$$

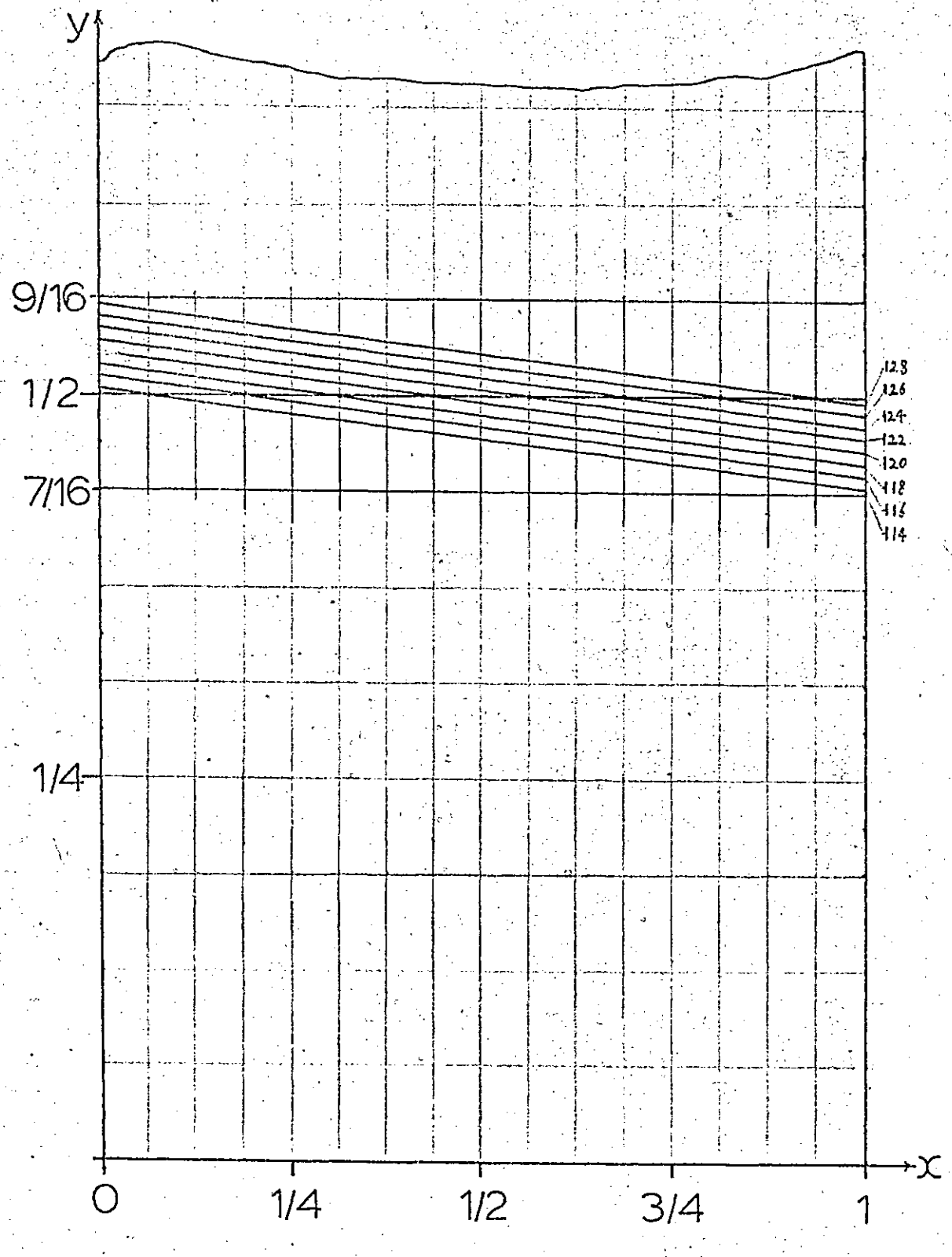
These eight rays pass through the eight odd nodes on the line $y = \frac{1}{2}$ within the scaled boundary $0 \leqslant x < 1, 0 \leqslant y < 1$. They are shown in Fig.(3.14) and are repeated on an acetate sheet, Insert 3 (on a larger scale than that used for Inserts 1 and 2 because the rays in the former are much closer together than those in the latter). The sixteen Walsh functions $wal(p,x)wal(1,y)$, $0 \leqslant p < 16$, in the range $0 \leqslant x < 1, 7/16 \leqslant y < 9/16$, are drawn in Appendix 4. By superimposing the rays given by equation(3.103) on these Walsh functions and using equation(2.22), it is possible to find the part of the matrix equation(made up of eight equations of the type equation(2.21)) for $n = i = 4, j = 1$ which include the unknown coefficients, $F_{p,1}$, $8 \leqslant p < 16$, and the coefficients $F_{p,1}$, $0 \leqslant p < 8$, making up the column matrices, $(H^{(4,4,1,l,0)})$, $1 \leqslant l \leqslant 4$. The outcome will be a "partial solution equation" of the form(see equations(3.72),(3.75),and(3.76))

$$(F^{4,4,1})_c = \left[F_{15,1},F_{14,1},\ldots,F_{0,1}\right]^T = -(W_3)(B^{(4,4,1)})$$
$$+ 8((H^{(4,4,1,1,0)}) + \ldots) + 4((H^{(4,4,4,2,0)}) + \ldots)$$
$$+ 2((H^{(4,4,4,3,0)}) + \ldots) + 1((H^{(4,4,1,4,0)}) + \ldots), \quad (3.104)$$

where the dots,...,on the right hand side of equation(3.104) indicate column matrices $(H^{(4,4,1,l,h)})$, $h \geqslant 1$, which are not worked out directly. (They can be worked out from $(H^{(4,4,1,l,0)})$; see equations(3.98) and (3.99))

To make the method of finding the equation of the form, equation(3.104), clearer, the corresponding method will be applied first to find a "partial solution equation" for $(F^{(3,3,1)})$. The complete solution equation for $(F^{(3,3,1)}) = \left[F_{7,1},F_{6,1},F_{5,1},F_{4,1}\right]^T$ has already been determined at the end of section 3.3.3.

Using only the eight Walsh functions, $wal(p,x)wal(1,y)$, $0 \leqslant p < 8$ with the rays 26,28,30,32 of the group $(n,i,j) = (3,3,1)$ and also $\theta^m_{00} = 1$, gives the "partial matrix equation"



Fig(3.14). The eight scaled rays, $m=12+2r, r=1,2,3,4,5,6,7,8$ associated with the block of coefficients, $(i,j) = (4,1)$. The vertical scale is twice the horizontal one.

$$(B^{(3,3,1)}) =$$

$$\begin{bmatrix} B_{26} \\ B_{28} \\ B_{30} \\ B_{32} \end{bmatrix} = -\tfrac{1}{4}(W_2)\begin{bmatrix} F_{7,1} \\ F_{6,1} \\ F_{5,1} \\ F_{4,1} \end{bmatrix} + \cdots + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} F'_{00} \end{bmatrix} + \cdots -\tfrac{1}{4}\begin{bmatrix} 1 & 1 & 1 & -3 \\ 1 & 1 & 3 & -1 \\ 1 & -1 & 3 & 1 \\ 1 & -1 & 1 & 3 \end{bmatrix}\begin{bmatrix} F_{3,1} \\ F_{2,1} \\ F_{1,1} \\ F_{0,1} \end{bmatrix} + \cdots$$

The dots indicate the "missing" terms not calculated. Premultiplication of the above equation by (W_2) (see equation (A2.34) gives

$$(F^{(3,3,1)}) = \begin{bmatrix} F_{7,1} \\ F_{6,1} \\ F_{5,1} \\ F_{4,1} \end{bmatrix} = -(W_2)(B^{(3,3,1)}) + \begin{bmatrix} 4F'_{00} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} -1 & 0 & -2 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} F_{3,1} \\ F_{2,1} \\ F_{1,1} \\ F_{0,1} \end{bmatrix} + \cdots$$

$$= -(W_2)(B^{(3,3,1)}) + 4\begin{bmatrix} F'_{00} + \cdots \\ F_{0,1} \\ F_{1,1} \\ F_{0,1} \end{bmatrix} - 2\begin{bmatrix} F_{1,1} + \cdots \\ F_{0,1} \\ F_{1,1} \\ F_{0,1} \end{bmatrix} - 1\begin{bmatrix} F_{3,1} + \cdots \\ F_{2,1} \\ F_{1,1} \\ F_{0,1} \end{bmatrix},$$

which is the partial solution equation containing the column matrices, $(H^{(3,3,1,1,0)})$ for 1 = 1,2 and 3. Equation(3.70) is the complete solution equation.

The "partial matrix" equation for (n,i,j) = (4,4,1) is

$$(B^{(4,4,1)}) =$$

$$\begin{bmatrix} B_{114} \\ B_{116} \\ B_{118} \\ B_{120} \\ B_{122} \\ B_{124} \\ B_{126} \\ B_{128} \end{bmatrix} = -\tfrac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} F_{15,1} \\ F_{14,1} \\ F_{13,1} \\ F_{12,1} \\ F_{11,1} \\ F_{10,1} \\ F_{9,1} \\ F_{8,1} \end{bmatrix} + \cdots + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} F'_{00} \end{bmatrix} + \cdots$$

$$-\tfrac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & -7 \\ 1 & 1 & 1 & 1 & 3 & 3 & 3 & -5 \\ 1 & 1 & -1 & -1 & 3 & 3 & 5 & -3 \\ 1 & 1 & -1 & -1 & 1 & 1 & 7 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & 7 & 1 \\ 1 & -1 & -1 & 1 & 3 & -3 & 5 & 3 \\ 1 & -1 & -1 & -1 & 3 & -3 & 3 & 5 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & 7 \end{bmatrix}\begin{bmatrix} F_{7,1} \\ F_{6,1} \\ F_{5,1} \\ F_{4,1} \\ F_{3,1} \\ F_{2,1} \\ F_{1,1} \\ F_{0,1} \end{bmatrix} + \cdots$$

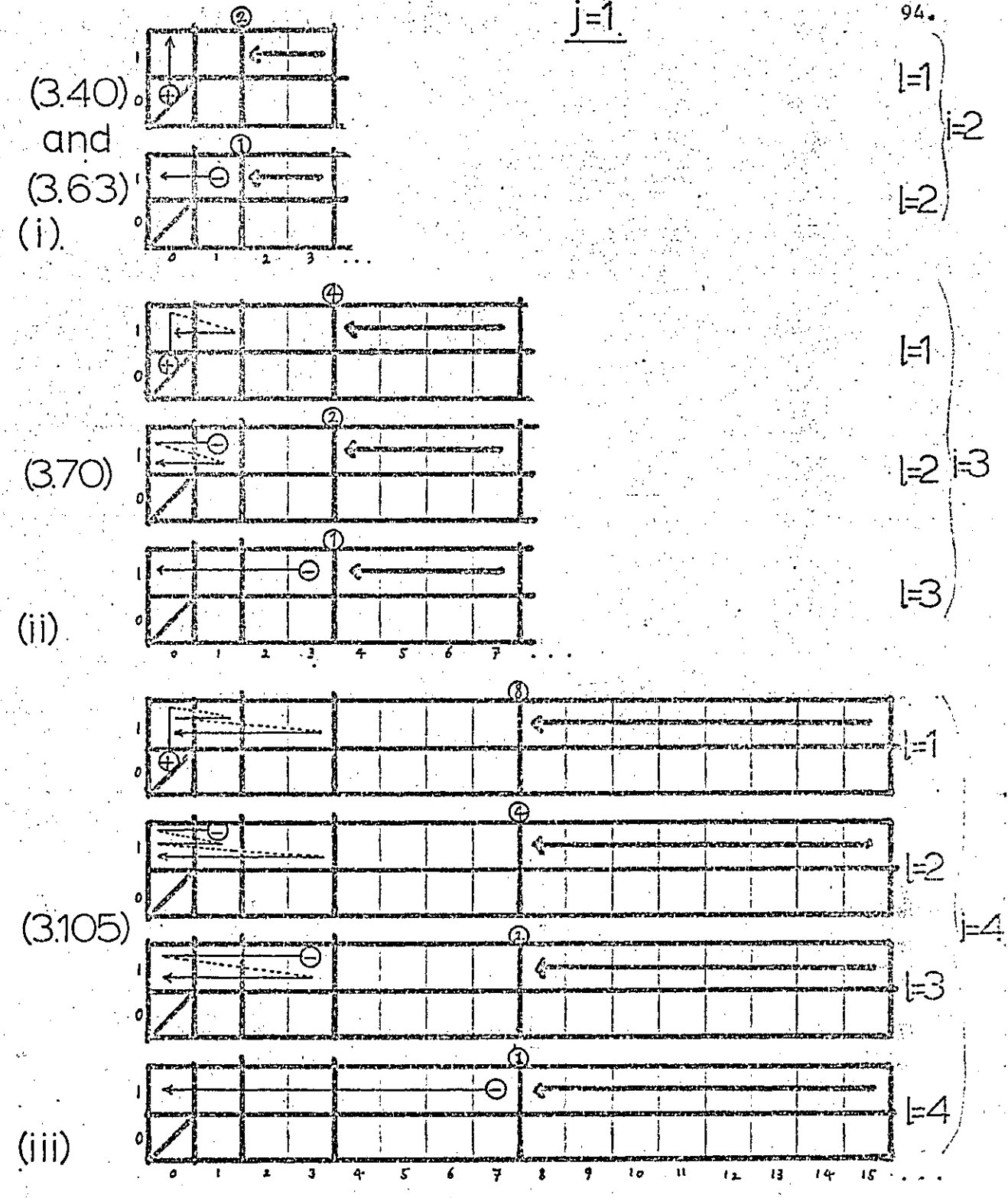
Premultiplication of this equation by (W_3) (see equation(A2.40) using $(W_3)(W_3) = 8I_8$ (equation (A2.34) with n=3) gives

$$(F^{(4,4,1)}) = \begin{bmatrix} F_{15,1} \\ F_{14,1} \\ F_{13,1} \\ F_{12,1} \\ F_{11,1} \\ F_{10,1} \\ F_{9,1} \\ F_{8,1} \end{bmatrix} - (W_3)(B^{(4,4,1)}) + \begin{bmatrix} 8F'_{00} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 & 0 & -2 & 0 & -4 & 0 \\ 0 & -1 & 0 & 0 & 0 & -2 & 0 & 4 \\ 0 & 0 & -1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_{7,1} \\ F_{6,1} \\ F_{5,1} \\ F_{4,1} \\ F_{3,1} \\ F_{2,1} \\ F_{1,1} \\ F_{0,1} \end{bmatrix}$$

$$+ \ldots$$

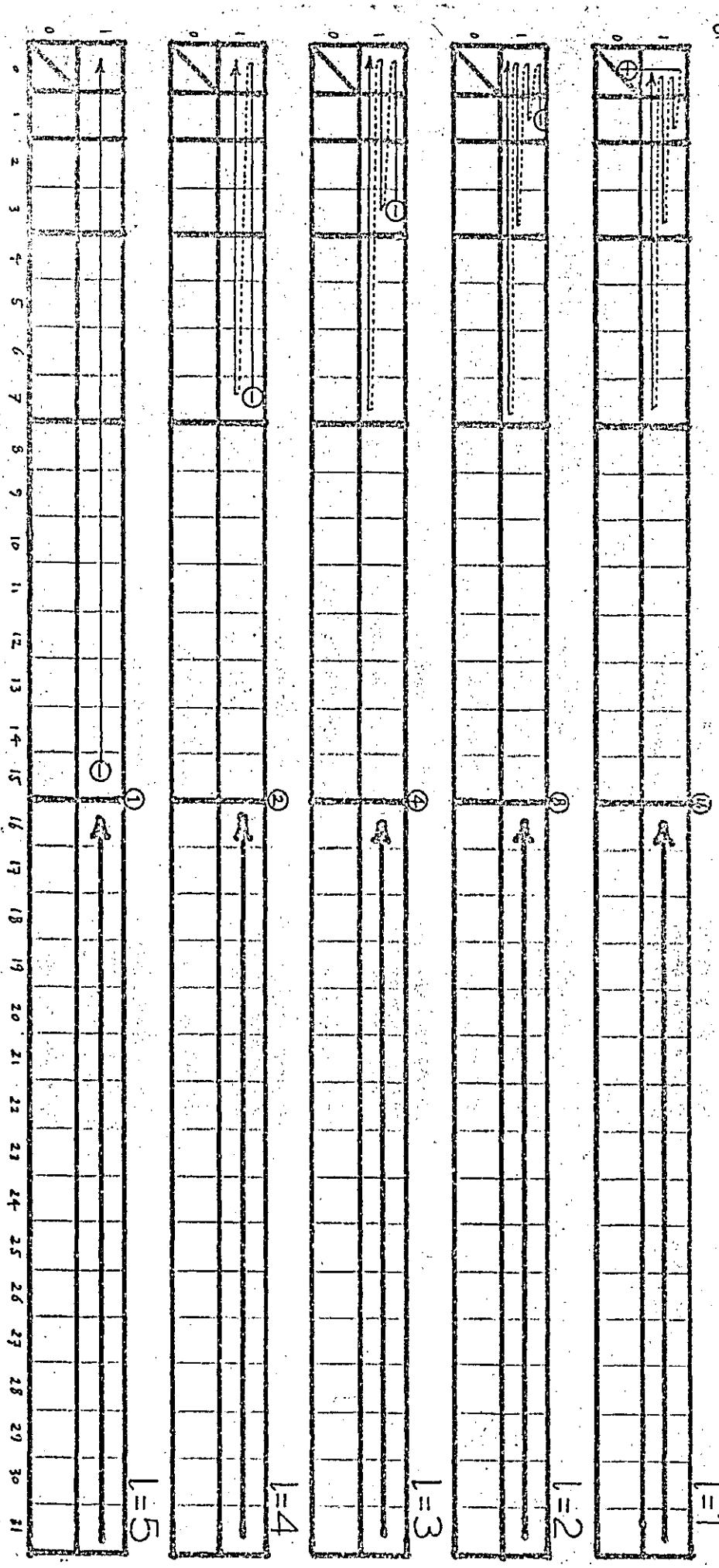
$$= -(W_3)(B^{(4,4,1)}) + 8\begin{bmatrix} F'_{0,0} + \ldots \\ F_{0,1} \\ F_{1,1} \\ F_{0,1} \\ F_{3,1} \\ F_{2,1} \\ F_{1,1} \\ F_{0,1} \end{bmatrix} - 4\begin{bmatrix} F_{1,1} + \ldots \\ F_{0,1} \\ F_{1,1} \\ F_{0,1} \\ F_{3,1} \\ F_{2,1} \\ F_{1,1} \\ F_{0,1} \end{bmatrix} - 2\begin{bmatrix} F_{3,1} + \ldots \\ F_{2,1} \\ F_{1,1} \\ F_{0,1} \\ F_{3,1} \\ F_{2,1} \\ F_{1,1} \\ F_{0,1} \end{bmatrix} - \begin{bmatrix} F_{7,1} + \ldots \\ F_{6,1} \\ F_{5,1} \\ F_{4,1} \\ F_{3,1} \\ F_{2,1} \\ F_{1,1} \\ F_{0,1} \end{bmatrix}, \qquad (3.105)$$

Which is the "partial solution equation" containing the four column matrices, $(H^{(4,4,1,l,0)})$, for $l = 1,2,3$ and 4. These matrices are traced out on the bottom two rows of coefficient diagrams in Fig.(3.15) (iii). On the basis of the observation, made after equation(3.100), that $(H^{(n,i,j,l,0)})$ is independent of n ($i \leqslant n$ of course) for $i = 1$ and 2, it is assumed that this is true for all i. Thus the matrices determined from equation(3.105) and represented in Fig.(3.15)(iii) are $(H^{(n,4,1,l,0)})$, for $l = 1,2,3$ and 4 where $n \geqslant 4$. The "pattern" of the matrices represented in Fig.(3.15) suggests the corresponding "pattern" for $(H^{(n,5,1,l,0)})$, for $l = 1,2,3,4$, and 5 where $n \geqslant 5$. These five matrices are represented in Fig.(3.16) in the same way as those in Fig.(3.15). The "patterns" in Fig.(3.16) suggest in their turn a general form for the i column matrices $(H^{(n,i,1,l,0)})$, $1 \leqslant l \leqslant i$, where i is any integer greater than or equal to 2:



Fig(3.15). representation of the first component column matrix, $(H^{(n,i,i,L,0)}), 1 \leqslant L \leqslant i$, of $(\Delta^{(L)}F^{(n,i,1)})$ for: (i) i=2(see also Fig.(3.12) (iii),(iv),(ix) and (x)),(ii) i=3(see also Fig(3.13) (vii),(viii) and (ix)) and (iii) i=4.The method of representation is the same as for the $(H^{(n,i,j,L,h)})$ shown in Figs. (3.11),(3.12) and (3.13) but only the bottom two rows of the coefficient diagrams are required for h=0.The dotted lines link different sections of the same column matrix in each diagram.For example,in (iii) for L=1, $(H^{(n,3,1,1,0)}) = \left[F'_{00}F_{0p} \quad F_{1p}F_{0p} \quad F_{3p}F_{2p}F_{1p}F_{01}\right]^T$, where the spaces indicate where the breaks occur.

$(i,j)=(5,1).$



Fig(3.16). The first component column matrix, $(H^{(n,i,l,l,0)})$, of $(\Delta^{(l)}{}_F(n,i,l))$, for $1 \leqslant l \leqslant i$, where $i=5$. The method of representation is as described for Fig(3.14)

The general form of $(H^{(n,i,1,l,0)})$ for $1\leqslant l\leqslant i$ where $2\leqslant i\leqslant n$.

(i) $2\leqslant l\leqslant i$.

$$(H^{(n,i,1,l,0)})_I = -F_{pq}. \tag{3.106}$$

The elements of $(H^{(p,i,1,l,0)})$ consist of one right to left scan of coefficients starting in the column $p = 2^{l-1}-1$ and a further $i-l$ such scans starting in the columns $p = 2^{l-1}-1, 2^{l}-1, 2^{l+1}-1, \ldots, 2^{(l-2)-(i-l)}-1 = 2^{i-2}-1$, respectively so that in equation (3.106), $I = I(p,q,i,1,l)$ is given by:

First scan.

$I = 2^{l-1} - p$, for $0\leqslant p\leqslant 2^{l-1}-1$ and $q = 1$ gives I in the range $1\leqslant I\leqslant 2^{l-1}$. (3.107)

For $l = i$ there are no further scans since $i-l=0$ (see equations (3.95) and (3.96) for the completely general form for $(\Delta^{(i)}F^{(n,i,j)})$).

The remaining $i-1$ scans.

$I = 2^{l} - p$, $0\leqslant p\leqslant 2^{l-1}-1$ and $q = 1$ gives $2^{l-1}+1\leqslant I\leqslant 2^{l}$.
$I = 2^{l+1} - p$, $0\leqslant p\leqslant 2^{l}-1$ and $q = 1$ gives $2^{l}+1\leqslant I\leqslant 2^{l+1}$.
$I = 2^{l+2} - p$, $0\leqslant p\leqslant 2^{l+1}-1$ and $q = 1$ gives $2^{l+1}+1\leqslant I\leqslant 2^{l+2}$.
and so on up to
$I = 2^{i-1} - p$, $0\leqslant p\leqslant 2^{i-2}-1$ and $q = 1$ gives $2^{i-2}+1\leqslant I\leqslant 2^{i-1}$.

These equations can be written more concisely as

$$I = 2^{k+l-1} - p, \tag{3.108}$$

for $0\leqslant p\leqslant 2^{k+l-2}-1$ and $q = 1$, where k is an integer in the range $1\leqslant k\leqslant i-l$, gives I in the range $2^{l-1}+1\leqslant I\leqslant 2^{i-1}$.

It is clear from equation (3.108) that for a given i and l there may be from one to $i-l+1$ values of I for each value of p depending on the range of values in which p falls.

(ii) $l = 1$.

$$(H^{(n,i,1,1,0)})_I = +F_{pq}. \tag{3.109}$$

The elements of $(H^{(n,i,1,1,0)})$ consist of one scan of $2^{i}-2$ elements,

$F'_{0,0}$ and $F_{0,1}$ (already given for any j by equation (3.101)) and a further i-2 scans identical to the i-2 scans in the case of l=2 above. Thus in equation(3.100) $I = I(p,q,i,l,1)$ is given by:

First scan.

$$I = q + 1, \tag{3.110}$$

for $p = 0$ and $0 \leqslant q \leqslant 1$ gives I in the range $1 \leqslant I \leqslant 2$.

The remaining i - 2 scans.

This is given by $l = 2$ in equation (3.108):

$$I = 2^{k+1} - p , \tag{3.111}$$

for $0 \leqslant p \leqslant 2^k - 1$ and $q = 1$, where k is an integer in the range $1 \leqslant k \leqslant i-2$, gives I in the range $3 \leqslant I \leqslant 2^{i-1}$.

The general form of $(H^{(n,i,j,l,0)})$ for $1 \leqslant l \leqslant i$ where $2 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$.

The elements of $(H^{(n,i,j,l,0)})$ are given by $h = 0$ in equation (3.99):

$$(H^{(n,i,j,l,0)})_I = \begin{cases} +F_{p,q} & \text{for } l = 1 \\ -F_{p,q} & \text{for } 2 \leqslant l \leqslant i \end{cases} . \tag{3.112}$$

In generalizing $(H^{(n,i,j,l,0)})$ from $j = 1$ to any $j, 1 \leqslant j \leqslant n$, the straight, right to left, line scans, shown in Figs.(3.15) and (3.16), become the multiple "U" scans like those of Figs.(3.9) and (3.10): As has been said already, the scans for $(H^{(n,i,j,l,0)}), 2 \leqslant j \leqslant n$ start in the same columns of coefficients (i.e. p values) as the straight line scans for $j = 1$ which have been given by equations (3.107), (3.108), (3.110) and (3.111).

Equation(3.73) relates I and p and q for given values of i and j where $(F^{(n,i,j)})_I = F_{p,q}$. Now by replacing $(2^i - p)$ in equation(3.73) by the functions of p in equations (3.107), (3.108), (3.110) and (3.111), the required formulae for $I = I(p,q,i,j,1)$ in equation(3.112) are obtained:-

(i) $2 \leqslant l \leqslant i$.

First scan.

$$I = 2^{j-1}(2^{l-1} - 1 - p) + \begin{cases} q - 2^{j-1} + 1 & \text{If p is EVEN} \\ 2^j - q & \text{If p is ODD} \end{cases}, \tag{3.113}$$

for $0 \leqslant p \leqslant 2^{l-1} - 1$ and $2^{j-1} \leqslant q \leqslant 2^j - 1$, gives I in the range $1 \leqslant I \leqslant 2^{l+j-2}$.

For $l = i$ there are no more scans since $i-l=0$ (see equations(3.95) and (3.96) for the completely general form for $(\Delta^{(i)}{}_F(n,i,j))$.

If $j = 1$, equation(3.113) reduces to equation(3.107)

The remaining $i - 1$ scans.

$$I = 2^{j-1}(2^{k+1-i} - 1 - p) + \begin{cases} q - 2^{j-1} + 1 & \text{If } p \text{ is EVEN} \\ 2^j - q & \text{If } p \text{ is ODD} \end{cases}, \qquad (3.114)$$

for $0 \leqslant p \leqslant 2^{k+1-2} - 1$ and $2^{j-1} \leqslant q \leqslant 2^j - 1$, where k is an integer in the range $1 \leqslant k \leqslant i - 1$, gives I in the range $2^{i+j-2} - 2^{1+j-2} + 1 \leqslant I \leqslant 2^{i+j-2}$.

(ii) $l = 1$.

First scan.

$$I = q + 1, \qquad (3.115)$$

for $p = 0$ and $0 \leqslant q \leqslant 2^j - 1$ gives I in the range $1 \leqslant I \leqslant 2^j$.

The remaining $i - 2$ scans.

$$I = 2^{j-1}(2^{k+1} - 1 - p) + \begin{cases} q - 2^{j-1} + 1 & \text{If } p \text{ is EVEN} \\ 2^j - q & \text{If } p \text{ is ODD} \end{cases}, \qquad (3.116)$$

for $0 \leqslant p \leqslant 2^k - 1$ and $2^{j-1} \leqslant q \leqslant 2^j - 1$, where k is an integer in the range $1 \leqslant k \leqslant i - 2$, gives I in the range $2^{i+j-2} - 2^{j+1} \leqslant I \leqslant 2^{i+j-2}$.

3.4.5. A Summary of The Complete Algorithm for solving the set of Coefficients $\{F'_{00}, F''_{00}, F_{pq}, 0 \leqslant p < 2^n, 0 \leqslant q < 2^n\}$ from a set of Normalized Line Integral data, $\{B_m, 1 \leqslant m \leqslant 2^{2n} + 1\}$.

The $2^{2n} + 1$ rays are those shown in Fig.(3.4) (a) for $M = N = 2^n$, where $n = 1,2,3,\ldots$. The "X" rays are labelled from $m = 1$, for the ray passing through $(x,y) = (1,0)$, to $m = M^2 - M + 1$, for the ray passing through $(x,y) = (0,1)$. The remaining M rays are the "non-nodal" "Y" rays which are labelled from $m = M^2 - M + 2$, for the ray passing through $(x,y) = (0,1)$, to $m = M^2 + 1$, for the ray passing through $(x,y) = (1,0)$. Each ray, m, yields an experimentaly determined line integral quantity, p_m, from which the corresponding normalized line integral, B_m, is obtained using equation(2.17) or (2.18).

(i) The solutions of the sets of coefficients $\{F'_{0q}, 0 \leq q < 2^n\}$ $\{F''_{p0}, 0 \leq p < 2^n\}$.

From equation(3.71)

$$(F^{(n,i=0)}) = \frac{1}{2^n}(W_n)(B^{(n,i=0)}) \qquad (3.117)$$

and

$$(F^{(n,j=0)}) = \frac{1}{2^n}(W_n)(B^{(n,j=0)}), \qquad (3.118)$$

where (W_n) is the Walsh matrix of order 2^n (see equations (A2.37) to (A2.41)) and the other matrices are column matrices of length 2^n whose I^{th} elements are given by:

$$(F^{(n,i=0)})_1 = F'_{00} \text{ and } (F^{(n,i=0)})_I = F_{0,I-1}, \qquad (3.119)$$

and

$$(F^{(n,j=0)})_1 = F'_{00} \text{ and } (F^{(n,j=0)})_I = F_{I-1,0}, \qquad (3.120)$$

where $2 \leq I \leq 2^n$.

$$(B^{(n,i=0)}) = B_{M(I-1)+1}, \qquad (3.121)$$

and

$$(B^{(n,j=0)})_I = B_{M(M-1)+1+I}, \qquad (3.122)$$

where $1 \leq I \leq 2^n$.

(ii) The solution of the remaining set of coefficients $\{F_{pq}, 1 \leq p < I^n, 1 \leq q < 2^n\}$.

The n^2 column matrices, $(F^{(n,i,j)})$, containing the 2^{i+j-2} coefficients defined by equation(3.73) are given by

$$(F^{(n,i,j)})_I = F_{pq},$$

where

$$I = 2^{j-1}(2^i - 1 - p) + \begin{cases} q - 2^{j-1} + 1 & \text{If p is EVEN} \\ 2^j - q & \text{If p is ODD} \end{cases}, \qquad (3.73)$$

for $2^{i-1} \leq p < 2^i$ and $2^{j-1} \leq q < 2^j$, i and j being in the ranges $1 \leq i \leq n$ and $1 \leq j \leq n$.

The 2^{i+j-2} rays associated with $(F^{(n,i,j)})$ are defined by equation(3.51):

$$m(n,i,j,r,s) = 2^n(2^{n-j}(2s - 1) - 1) + 2^{n-i}(2r - 1) + 1, \qquad (3.51)$$

where $1 \leq r \leq 2^{i-1}$ and $1 \leq s \leq 2^{j-1}$.

Now from equations(3.72) and (3.77) the solution of the column matrix, $(F^{(n,i,j)})$, is given by

$$(F^{(n,i,j)}) = -\frac{1}{2^{j-1}}(W_{i+j-2})(B^{(n,i,j)}) + \sum_{l=1}^{i} (\Delta^{(l)}F^{(n,i,j)}) \quad , \tag{3.123}$$

where $(\Delta^{(l)}F^{(n,i,j)})$ are column matrices of length 2^{i+j-2} containing as elements linear expressions of coefficients which have already been solved if the column matrices, $(F^{(n,i,j)})$, have been solved in the "correct" order. This "correct" order is, from equation(3.42), given by

$$O(n,i,j) = ni - j + 1 \quad , \tag{3.124}$$

where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$.

The formulae for $(\Delta^{(l)}F^{(n,i,j)}), 1 \leqslant l \leqslant i$ are given for $i = 1$ and for $i \geqslant 2$:

$(\Delta^{(l)}F^{(n,i,j)})$ for $i = l = 1$.

From equations(3.80)and (3.84)

$$(\Delta^{(1)}F^{(n,1,j)})_I = F_{0,I-1} + J.\sum_{h=1}^{2^{n-j}-1}(-1)^h(F_{0,I-1+h2^j} - F_{1,(h+1)2^j-I}) \quad ;$$

where $J = J(n,j) = \begin{cases} 0 \text{ If } j = n \\ 1 \text{ Otherwise} \end{cases}$ and $1 \leqslant I \leqslant 2^{j-1}$. (3.125)

$(\Delta^{(l)}F^{(n,i,j)})$ for $i \geqslant 2$.

From equations(3.76),(3.96),(3.99) and (3.100)

$$(\Delta^{(l)}F^{(n,i,j)})_I =$$

$$2^{i-l}.(-1)^a\left[F_{pq} + J.\sum_{h=1}^{2^{n-j}-1}(-1)^h(F_{p,q+h2^j} + L.F_{p+2^{i-l},q+h2^j})\right] \quad ; \tag{3.126}$$

where

$$J = J(n,j) = \begin{cases} 0 \text{ If } j = n \\ 1 \text{ Otherwise} \end{cases}, \quad L = L(i,l) = \begin{cases} 1 \text{ If } l = i \\ 0 \text{ Otherwise} \end{cases}$$

$$a = a(l) = \begin{cases} 0 \text{ If } l = 1 \\ 1 \text{ Otherwise} \end{cases} \text{ and where } I = I(p,q,i,j,l) \text{ is given}$$

by equations(3.115),(3.116),(3.113) and (3.114):-

For $l = 1$,

$$I = q + 1, \tag{3.115}$$

for $p = 0$ and $0 \leqslant q \leqslant 2^{j} - 1$ gives I in the range $1 \leqslant I \leqslant 2^{j}$.

Equation(3.116) can be written

$$I = 2^{j-1}(2^{k+1} - 1 - p) + Q(j,q) , \tag{3.127}$$

where

$$Q(j,q) = \begin{cases} q - 2^{j-1} + 1 & \text{If } p \text{ is EVEN} \\ 2^{j} - q & \text{If } p \text{ is ODD} \end{cases} , \tag{3.128}$$

for $0 \leqslant p \leqslant 2^{k} - 1$ and $2^{j-1} \leqslant q \leqslant 2^{j} - 1$, where k is in the range $1 \leqslant k \leqslant i - 2$, gives I in the range $2^{i+j-2} - 2^{j} + 1 \leqslant I \leqslant 2^{i+j-2}$.

For $2 \leqslant l \leqslant i$.

Using $Q(j,q)$ as defined by equation(3.128), equation(3.113) can be written

$$I = 2^{j-1}(2^{l-1} - 1 - p) + Q(j,q) \tag{3.129}$$

for $0 \leqslant p \leqslant 2^{l-1} - 1$ and $2^{j-1} \leqslant q \leqslant 2^{j} - 1$, gives I in the range $1 \leqslant I \leqslant 2^{i+j-2}$.

Equation(3.114) can be written

$$I = 2^{j-1}(2^{k+l-1} - 1 - p) + Q(j,q) , \tag{3.130}$$

for $0 \leqslant p \leqslant 2^{k+l-2} - 1$ and $2^{j-1} \leqslant q \leqslant 2^{j} - 1$, where k is in the range $1 \leqslant k \leqslant i - l$, gives I in the range $2^{i+j-2} - 2^{l+j-2} + 1 \leqslant I \leqslant 2^{i+j-2}$.

Chapter 4. The Testing of "The Walsh Function Method" using Data Generated from Simulated Density Function Distributions.

4.1 Introduction. Computer Subroutines To Implement "The Walsh Function Method".

"The Walsh Function Method" for determining density function distributions, described in the previous chapters, can be summarized in two parts as follows:

(i) The solutions, from given line intregal data, of all the coefficients, F_{pq}, $0 \leqslant p < M$, $0 \leqslant q < N$, $n \geqslant 1$ and $M = N = 2^n$ of a finite series of two-dimensional Walsh functions representing the density function, $f(x,y)$. These solutions are calculated using the algorithm developed in Chapter 3.

(ii) The two-dimensional descrete inverse Walsh transform (see equation (3.8)) of these coefficients to give the "best" approximation to $f(x,y)$ which is possible using the finite series of Walsh functions. This approximation, $f_a(x,y)$, is a two-dimensional step function whose values within each of the $M \times N$ (where $M = N = 2^n$) resolution cells is a constant. (See Figs (3.3) and (3.

Computer subroutines (written in FORTRAN IV) were developed to implement "The Walsh Function Method". The main subroutine which implements the algorithm is named "REF". The input to "REF" is line integral data from previous subroutines "PATHS1" or "PATHS2" or "PATH3" depending on the testing procedure used (see sections (4.2), (4.3.1), (4.3.2) and (4.3.3)), and the output is the step function values in a $2^n \times 2^n$ array with information about errors (see section (4.4))

In section (3.3) it was shown that two "almost independent" sets of coefficients, $\{F_{pq}\}_1$, and $\{F_{pq}\}_2$, could be obtained from the sets of rays shown in Fig (3.3)(a) and Fig(3.3)(b) respectively: The solution to the set $\{F_{pq}\}_2$ could be obtained from the set of normalized line integrals $\{B_m\}_{1 \leqslant m \leqslant MN - N + 1}$ using the algorithm for solving $\{F_{pq}\}_1$ from $\{B_m\}_1$. The coefficients in the former solution are F_{qp} where F_{pq} appear in the solution of the latter case. Also F''_{00} is solved instead of F_{00}.

The subroutines in "The Method" are designed to give the solutions of both $\{F_{pq}\}_1$ and $\{F_{pq}\}_2$ for the sets $\{B_m\}_1$ and $\{B_m\}_2$. The input to "REF", the main subroutine of "The Method" however, is not the latter two sets nor even the corresponding un-normalized sets $\{P_m\}_1$ and $\{P_m\}_2$ but is the un-normalized sets of line integrals of all the $MN - M + 1$ "X" rays and $MN - N + 1$ "Y" rays, $\left\{P_i^{(X)}, 1 \leqslant i \leqslant MN - M + 1\right\}$ and $\left\{P_j^{(Y)}, 1 \leqslant j \leqslant MN - N + 1\right\}$. The first stage of "REF" extracts $\{P_m\}_1$ and $\{P_m\}_2$, $1 \leqslant m \leqslant MN + 1$ from $\left\{P_i^{(X)}\right\}$ and $\left\{P_j^{(Y)}\right\}$:-

From section (3.3) the line integrals of the set $\{P_m\}_1$ are made up of:

(i) The line integrals of all the "X" rays given by

$$P_m = p_m^{(X)} \quad , \tag{4.1}$$

where $1 \leqslant m \leqslant MN - M + 1$, and

(ii) The line integrals of the "non-nodal" "Y" rays given by

$$P_m = p_{iN+1}^{(Y)}, \text{ where } m = (MN - M + 2) + i, \text{ for } 0 \leqslant i \leqslant M - 1. \tag{4.2}$$

This gives P_m for m in the range $MN - M + 2 \leqslant m \leqslant MN + 1$.
Likewise the line integrals of the set $\{P_m\}_2$ are made up of

(i) The line integrals of all the "Y" rays given by

$$P_m = p_m^{(Y)}, \tag{4.3}$$

where $1 \leqslant m \leqslant MN - N + 1$; and

(ii) The line integrals of the "non-nodal" "X" rays given by

$$P_m = p_{jM+1}^{(X)}, \text{ where } m = (MN - N + 2) + j \text{ for } 0 \leqslant j \leqslant N - 1. \tag{4.4}$$

This gives p_m for m in the range $MN - N + 2 \leqslant m \leqslant MN + 1$.

Various repeated operations in the implementation of the algorithm are achieved by other subroutines called for by "REF", in the appropriate places. These subroutines are named, "WALSH", "SCAN", "SCAN1", "SCAN2" and "SCAN3". Throughout the complete computer program, listed in Appendix 5, all references in comment cards and output listings are made with the special case of density function, the refractive index distribution, in mind. Thus "line integral" and "density function distribution" are referred to as "Paths" and "Ref" Index distribution".

It has already been pointed out, at the end of section (3.2), that the two-dimensional inverse Walsh transform can be determined by the one-dimensional inverse Walsh transforms of order 2^n of each of the 2^n rows of the array of $2^n \times 2^n$ coefficients followed by the one-dimensional inverse Walsh transforms of each of the 2^n columns of the resulting $2^n \times 2^n$ array to give a final $2^n \times 2^n$ array of all the step function values in the resolution cells (see section A1.4.3). The underlined words rows and columns may be interchanged with the same result.

The $n^2 + 2$ separate solution column matrices in the algorithm each include one one-dimensional inverse Walsh transform (of the normalized line integral data: see equations (3.118),(3.123) and (3.75)) times a certain power of 2. Now the Walsh transform and its inverse are the same, appart from a constant factor - a power of 2, (see equations (A2.35) and (A2.36)) so that a single subroutine, named "WALSH", which performs the inverse one-dimensional Walsh transform on any column matrix of 2^b elements, $b=1,2,..$ can be used both in working out the Walsh transforms in the algorithm and those 2^{n+1} inverse transforms required for the two-dimensional inverse transform. A subroutine named "WALSH2" implements the latter by "calling" (i.e. making use of), the subroutine named "WALSH", 2^{n+1} times. The subroutine, "WALSH" is a Fast Walsh Transform based on a program developed by the author,(5), which was developed from Harmuth's theory,(2). The latter program was a transform based on the functions $wal(p,x)wal(q,y)$ in the range $-\frac{1}{2}\leqslant x<\frac{1}{2}$, $-\frac{1}{2}\leqslant y<\frac{1}{2}$. The subroutine, "WALSH", used here involved a few changes to base the transform on the same functions but in the range $0\leqslant x<1$, $0\leqslant y<1$.

The other parts of the algorithm, summarized in section 3.4.5, not involving Walsh transforms, require much re-ordering of arrays, additions, subtractions and multiplications by powers of 2. The subroutines which implement these besides "REF" are "SCAN", "SCAN1","SCAN2" and "SCAN3", The remaining two subroutines "PLOT" and "MAXMIN" called for by "REF" are required in the output format from "REF" and error calculations respectively

If "PLOT" is called for, the output, which normally includes an array of $2^n \times 2^n$ numbers giving the two dimensional step function $f_a(x,y)$, can be represented in "A" format in which each value of the step function falling between two specified values is represented by printing a single symbol or letter. In this way a "picture" is built up which for large values of 2^n gives the general "shape" of the function at a glance.

"MAXMIN" calculates the largest and smallest numbers of a given $2^n \times 2^n$ array of numbers and calculates their difference. This is required in error calculations (see section (4.4)).

Fig (4.1) (i),(ii),(iii),(iv), shows the four different ways in which the program can be used. The box labelled "The Method " represents the eight subroutines "REF" and the other seven subroutines "WALSH" to "MAXMIN" mentioned already. The subroutines named "PATHS1", "PATHS2" and "PATHS3" are used to generate line integrals to input into the main

program. The four sections (4.2),(4.3.1), (4.3.2) and (4.3.3) describe how they are used.

The first subroutine called "INDEX" is used to select which of the four different ways the program can be used. All the input data including the dimensions X by Y, of the rectangular boundary and the value of n is read into "INDEX" from where it is transferred to either "PATHS1", "PATHS2" or "PATHS3".

4.2 Real Experimental Data.

Fig (4.1)(i) shows how the program would be used with real experimental data. The $2^{2n} - 2^n + 1$ line integrals from the same number of "X" rays and the same number from the same number of "Y" rays is read into the subroutine "PATHS3" and from there it is inputted into subroutine "REF", All the $2^{2n} - 2^n + 1$ line integrals of each of the two sets are measured relative to the first one of the set (i.e. of the "X" or "Y" ray $m = 1$) which is taken as zero.

4.3 Simulated Data From Simulated Density Functions.

The remaining three sections deal with simulated density functions and their reconstructions by "The Method". In each case the simulated density functions are f(x,y), for a unit sided "scaled" square boundary (see Fig(3.4)) but line integrals for "real" boundaries X by Y are calculated using equations (2.11) and (2.15). By choosing realistic values for "X" and "Y" the simulated line integrals will be realistic and of the same order of magnitude as true experimental values. The first stage of "The Method" in subroutine"REF" is to evaluate the normalized line integrals necessary as the input data for the algorithm.

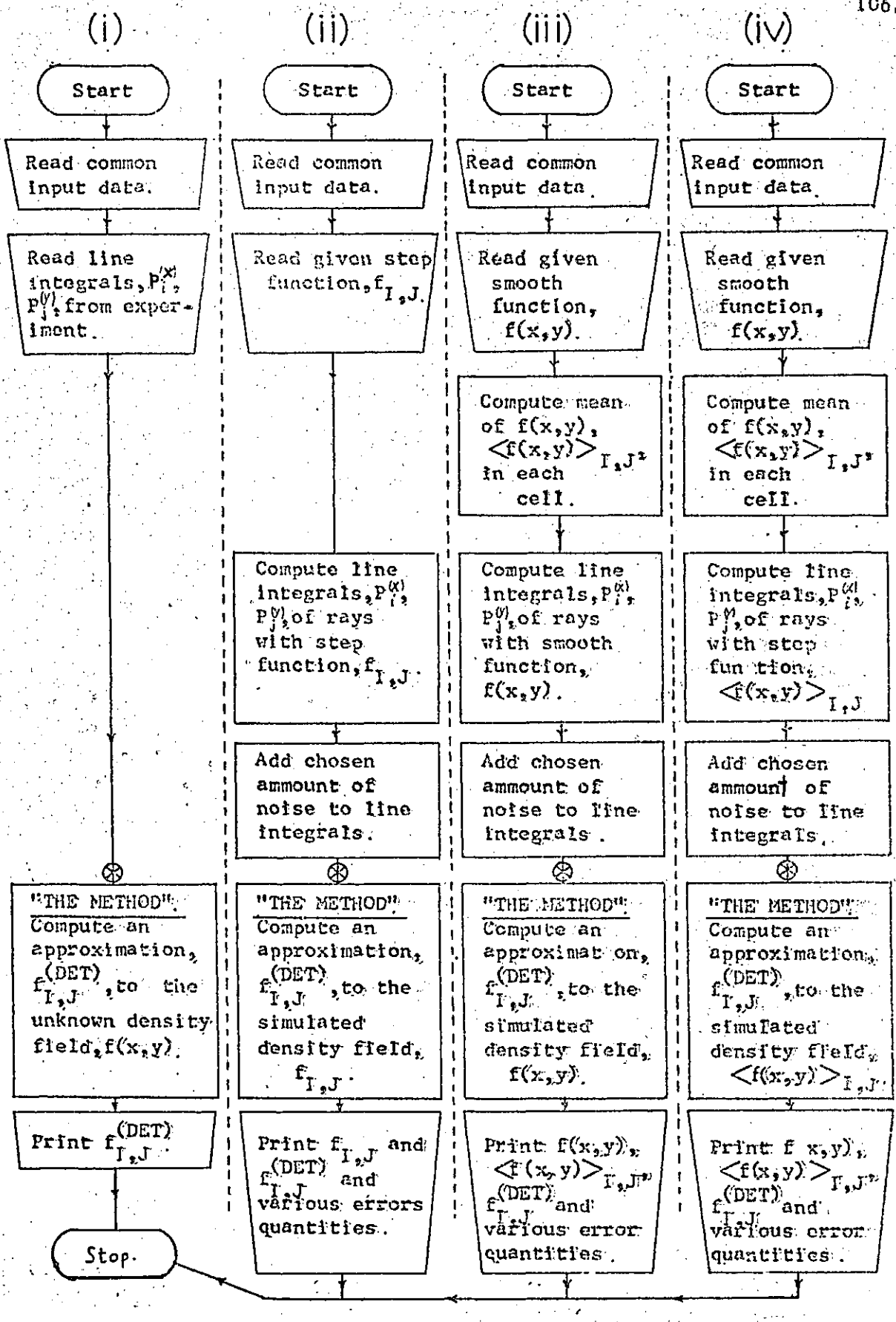
Equations (2.11) and (2.15) for the line integrals of "X" ray, i and "Y" ray, j, respectively are given by

$$P_i^{(X)} = \sqrt{X^2 + Y^2/N^2} \int_0^1 f(x, y_i(xX))dx \qquad (4.5)$$

and

$$P_j^{(Y)} = \sqrt{Y^2 + X^2/M^2} \int_0^1 f(x_j(yY), y)dy, \qquad (4.6)$$

where the appropriate values of the ray gradients "a_i" and "b_j" have



Fig(4.1) (i) to (iv). The four different ways in which the program can be used. The common input data is the first five or six groups of data shown in Fig(4.2). Further quantities may be printed depending on the value of the input variable, KO (See section 4.5). The symbol ⊛ shows where extra stages can be inserted to implement the method, described in Chapter 6, for generating "pseudo" line integrals. (See Fig(6.2)).

been substituted in from section (3.3) where "X" and "Y" rays were first defined in terms of an $M \times N$ array of identical resolution cells (see Figs (3.3) and (3.4)). $y_i(xX)$ and $x_j(yY)$ in equations (4.5) and (4.6) are the rays within the "scaled" unit sided square boundary and are given by equations (3.13) and (3.16):

$$y_i(Xx) = -\frac{x}{N} + \frac{1}{N}\left(1 + \frac{i-1}{M}\right) \qquad (3.13)$$

and

$$x_j(Yy) = \frac{y}{M} + \frac{1}{M}\left(1 + \frac{j-1}{N}\right), \qquad (3.16)$$

where $1 \leqslant i \leqslant MN - M + 1$ and $1 \leqslant j \leqslant MN - N + 1$.

The values of M and N were, later in chapter3, restricted to $M = N = 2^n$, $n \geqslant 1$ because of the limitations imposed by Walsh functions. M and N will be left in all the formulae for line integrals developed in sections (4.3.1) and (4.3.2) because in Chapter 6 the results will be useful for arrays $M \times N$ where M and N are not restricted to powers of 2. In the computer program however $M = N = 2^n$.

4.3.1. A Two-Dimensional Step Function.

Fig (4.1)(ii) shows how the computer program is used in this case. The subroutine named "PATHS1" implements all the results in this section.

A two-dimensional step function of $M \times N$ specified values, each value within a resolution cell (see Fig (3.4)) being a constant, is given by

$$f(x,y) = f\left(\frac{I+\gamma}{M}, \frac{J+\omega}{N}\right) = f_{I,J}, \qquad (4.7)$$

where I and J are integers in the ranges $0 \leqslant I \leqslant M-1$, $0 \leqslant J \leqslant N-1$ and γ and ω are constants in the ranges $0 \leqslant \gamma < 1$, $0 \leqslant \omega < 1$.

Although a step function is a physically unreasonable simulation of a density field it is vital as a test for the algorithm in "The Method". This is because any step function of the type defined by equation (4.7) with $M = N = 2^n$ can be exactly represented by the finite series of Walsh functions $wal(p,x)wal(q,y)$, $0 \leqslant p < 2^n$, $0 \leqslant q < 2^n$. This means that "The Method" should give an exact reconstruction for a step function. Many different step functions were used for various values of n up to $n = 6$, and in every case the reconstruction was exact. (see section (5.1).

Clearly the "X" rays shown in Fig (3.3)(a) are labelled by the equation

$$i = I + JM + 1 \qquad (4.8)$$

where I and J are in the following ranges : $0 \leqslant I \leqslant M - 1$, $0 \leqslant J \leqslant N - 2$, which give i in the range $1 \leqslant i \leqslant MN - M$. The last "X" ray $i = MN - M + 1$ is given by $I = 0$ and $J = N - 1$ in equation (4.8). The "X" ray, i, given by equation (4.8) for a particular pair of numbers (I,J) passes through the top left hand corner of the resolution cell $\frac{I}{M} \leqslant x < \frac{I+1}{M}$, $\frac{J}{N} \leqslant y < \frac{J+1}{N}$ which can be called the resolution cell, (I,J). Now the next "X" ray, i + 1, passes through the right hand corner of resolution cell (I,J) and it can be seen that the only resolution cells not passed through by both "X" rays i and i + 1 are (I,J) and (I,J + 1) so that the difference in the line integrals of these two $P^{(X)}_{i+1} - P^{(X)}_{i}$, is given by $\frac{1}{M}\sqrt{X^2 + Y^2/N^2}\,(f_{I,J+1} - f_{I,J})$ so that

$$P^{(X)}_{i+1} = P^{(X)}_{i} + \frac{1}{M}\sqrt{X^2 + Y^2/N^2}\,(f_{I,J+1} - f_{I,J}) \qquad (4.9)$$

By setting P_1 arbitrarily equal to zero, equation (4.9) can be used to generate all MN - M + 1 line integrals for the "X" rays.

Likewise the "Y" rays shown in Fig (3.3) (b) are labelled by

$$j = J + IN + 1, \qquad (4.10)$$

where $0 \leqslant I \leqslant N - 2$, $0 \leqslant J \leqslant N - 1$ gives j in the range $1 \leqslant j \leqslant MN - N$. The last "Y" ray $j = MN - N + 1$ is given by $I = M - 1$, $J = 0$ in equation (4.10). The difference in the line integrals of the "Y" rays j + 1 and j is given by $\frac{1}{N}\sqrt{Y^2 + X^2/M^2}\,(f_{I+1,J} - f_{I,J})$ so that

$$P^{(Y)}_{j+1} = P^{(Y)}_{j} + \frac{1}{N}\sqrt{Y^2 + X^2/M^2}\,(f_{I+1,J} - f_{I,J}) \qquad (4.11)$$

By setting P_1 arbitrarily to zero, equation (4.11) can be used to generate all the MN - N + 1 line integrals for the "Y" rays.

Before the line integrals are inputed to "The Method", noise is added to all of them in the form of random numbers in a range given by a specified fraction α of the difference of maximum and minimum line integrals. Let $\Delta P^{(X)}_{max}$ and $\Delta P^{(Y)}_{max}$ be these differences for the "X" ray data and "Y" ray data respectively. Let $p^{(X)}_i$ and $p^{(Y)}_j$ be random numbers defined such the probability of them being any number in the range 0 to 1 is equal and is zero of being outside this range. Then

$$\text{NOISY}P_i^{(X)} = P_i^{(X)} + \alpha P_{max}^{(X)} \cdot (R_i^{(X)} - \tfrac{1}{2}) \qquad (4.12)$$

and

$$\text{NOISY}P_j^{(Y)} = P_j^{(Y)} + \alpha P_{max}^{(Y)} \cdot (R_j^{(Y)} - \tfrac{1}{2}) \qquad (4.13)$$

where i and j are the ranges $1 \leqslant i \leqslant MN - M + 1$ and $1 \leqslant j \leqslant MN - N + 1$.

An exact reconstruction obviously only occurs when $\alpha = 0$. In the computer program α is called "RNOISE".

4.3.2 A Variety of Continuous Two-Dimensional Functions.

Fig (4.1)(iii) shows how the computer program is used in this case. The subroutine named "PATHS2" implements all the results in this section.

A very wide variety of continuous two-dimentional functions can be generated by

$$f(x,y) = f^{(\text{POWER})}(x,y) + f^{(\cos)}(x,y) \qquad (4.14)$$

where $f^{(\text{POWER})}(x,y)$ and $f^{(\text{COS})}(x,y)$ are a power series and a series of cosines given by:-

$$f^{(\text{POWER})}(x,y) = (c_1) + (c_2y + c_3x) + (c_4y^2 + c_5x^3y + c_6x^2) + (c_7y^3 + c_8xy^2 + c_9x^2y + c_{10}x^3) + \ldots + c_{(K+1)(K+2)/2} \cdot x^K$$

or

$$f^{(\text{POWER})}(x,y) = \sum_{k=1}^{K+1} \sum_{l=1}^{k} c_{k(k-1)/2+l} \cdot x^{l-1} y^{k-l}, \qquad (4.15)$$

K being the highest order of powers appearing in the series. k labels the order of the terms within any bracketed group in the first form of equation (4.15) and l labels the number of each term within that group. There are $(K + 1)(K + 2)/2$ terms in the series, or less if some of the numbers, c_t $1 \leqslant t \leqslant (K + 1)(K + 2)/2$, are zero. In the computer program K is named "KMAX" and the numbers $c_1, c_2, c_3, \ldots$, are stored in an array named "C(I)".

The finite series of cosines is given by

$$f^{(\text{COS})}(x,y) = \sum_{r=1}^{R} d_r \cos 2\pi (s_r x + t_r y + u_r), \qquad (4.16)$$

where d_r, s_r, t_r, and u_r are specified for each of the R terms.

In the computer program R is named "NCOS" and the numbers d_r, s_r, t_r, and

u_r are stored in arrays "CC(I)","SC(I)", "TC(I)" and "UC(I)".

The first section of "PATHS2" , for a given power series and cosine series, expresses the mean value of f(x,y) in each of the M×N resolution cells. This distribution of the mean values, $\langle f(x,y)\rangle_{I,J}$ can be compared with the step function distribution, $f^{(DET)}_{I,J}$, as determined by "The Method". If the surface f(x,y), within all the resolution cells are nearly plane then the distribution of "centre of cell" sampled values, $f(\frac{I+\frac{1}{2}}{M}, \frac{J+\frac{1}{2}}{N}) \approx \langle f(x,y)\rangle_{I,J}$ and is easier to calculate than the latter. Clearly from equation (4.14)

$$\langle f(x,y)\rangle_{I,J} = \langle f^{(POWER)}(x,y)\rangle_{I,J} + \langle f^{(COS)}(x,y)\rangle_{I,J} \qquad (4.17)$$

$\langle f^{(POWER)}(x,y)\rangle_{IJ}$ for the given function $f^{(POWER)}(X,Y)$.

By definition

$$\langle f^{(POWER)}(x,y)\rangle_{I,J} = \frac{1}{1/MN} \int_{J/N}^{(J+1)/N} \int_{I/M}^{(I+1)/M} f^{(POWER)}(x,y)\,dx\,dy \qquad (4.18)$$

From equation (4.15)

$$\langle f^{(POWER)}(x,y)\rangle_{I,J} = \sum_{k=1}^{K+1} \sum_{l=1}^{k} c_{k(k-1)/2+l} \left(MN \int_{J/N}^{(J+1)/N} \int_{I/M}^{(I+1)/N} x^{l-1} y^{k-l}\,dx\,dy \right) \qquad (4.19)$$

The double integral in the brackets becomes

$$\frac{MN}{l(k-l+1)} \left(\frac{1}{M}\right)^{l} \cdot \left(\frac{1}{N}\right)^{k-l+1} \cdot \left((I+1)^{l} - I^{l}\right) \cdot \left((J+1)^{k-l+1} - J^{k-l+1}\right)$$

$$= \left(\frac{I+\frac{1}{2}}{M}\right)^{l-1} \left(\frac{J+\frac{1}{2}}{N}\right)^{k-l} \left\{ \frac{1}{l(k-l+1)} \left(\frac{(I+1)^{l} - I^{l}}{(I+\frac{1}{2})^{l-1}} \right) \left(\frac{(J+1)^{k-l+1} - J^{k-l+1}}{(J+\frac{1}{2})^{k-l}} \right) \right\}.$$

Thus

$$\langle f^{(POWER)}(x,y)\rangle_{I,J} = \sum_{k=1}^{K+1} \sum_{l=1}^{k} c_{k(k-1)/2+l} \cdot \left(\frac{I+\frac{1}{2}}{M}\right)^{l-1} \cdot \left(\frac{J+\frac{1}{2}}{N}\right)^{k-l} \cdot \beta, \qquad (4.20)$$

where

$$\beta = \frac{1}{l(k-l+1)} \left(\frac{(I+1)^{l} - I^{l}}{(I+\frac{1}{2})^{l-1}} \right) \left(\frac{(J+1)^{k-l+1} - J^{k-l+1}}{(J+\frac{1}{2})^{k-l}} \right) \qquad (4.21)$$

If $\beta = 1$ equation (4.20) is the same as the value of f(x,y) sampled

at the centre of the resolution cell (I,J). β is closer to 1 the smaller I and K are i.e. the lower the order of the terms. For a ramp function $f(x,y) = c_1 + c_2x + c_3y$, (k,l) are (1,1),(2,1) and (2,2) all of which give $\beta = 1$ in equation (4.21) for any I and J, so as expected the mean value and the "centre of cell" sampled value are the same.

$\langle f^{(COS)}(x,y)\rangle_{I,J}$ for a given function $f^{(COS)}(x,y)$

By definition

$$\langle f^{(COS)}(x,y)\rangle_{I,J} = \frac{1}{1/MN}\int_{I/M}^{(I+1)/M}\int_{J/N}^{(J+1)/N} f^{(COS)}(x,y)\,dy\,dx \qquad (4.22)$$

From equation (4.16)

$$\langle f^{(COS)}(x,y)\rangle_{I,J} = \sum_{r=1}^{R} d_r \Re\left\{MN\int_{I/M}^{(I+1)/M}\int_{J/N}^{(J+1)/N} e^{2\pi i(s_rx+t_ry+u_r)}\,dy\,dx\right\} \qquad (4.23)$$

where $\Re$ means "the real part of" and $i = \sqrt{-1}$.

The double integral in brackets becomes

$$MN\,\Re\left\{e^{2\pi iu_r}\cdot e^{2\pi is_r(I+\frac{1}{2})/M}\cdot\left(\frac{\sin\pi s_r/M}{\pi s_r}\right)\cdot e^{2\pi it_r(J+\frac{1}{2})/N}\cdot\left(\frac{\sin\pi t_r/N}{\pi t_r}\right)\right.$$

$$= \cos 2\pi\left(s_r\left(\frac{I+\frac{1}{2}}{M}\right) + t_r\left(\frac{J+\frac{1}{2}}{N}\right) + u_r\right)\cdot\left(\frac{\sin\pi s_r/M}{\pi s_r/M}\right)\cdot\left(\frac{\sin\pi t_r/N}{\pi t_r/N}\right).$$

Thus

$$\langle f^{(COS)}(x,y)\rangle_{I,J} = \sum_{r=1}^{R} d_r \cos 2\pi\left(s_r\left(\frac{I+\frac{1}{2}}{M}\right) + t_r\left(\frac{J+\frac{1}{2}}{N}\right) + u_r\right).\varepsilon$$

where (4.24)

$$\varepsilon = \left(\frac{\sin\pi s_r/M}{\pi s_r/M}\right)\cdot\left(\frac{\sin\pi t_r/N}{\pi t_r/N}\right) \qquad (4.25)$$

If $s_r \ll M$ and $t_r \ll N$, ε tends to unity and the value of $\langle f^{(COS)}(x,y)\rangle$ is the same as the value of f(x,y) sampled at the centre of the resolution cell (I,J).

The second section of "PATH2" is the calculation of the line integrals $P_i^{(X)}$ and $P_j^{(Y)}$ of the "X" and "Y" rays given by equations (4.5) and (4.6) respectively. From these latter two equations and equation (4.14) there is a contribution to the line integrals from both the power series and the cosine series making up f(x,y);

Contribution to line integrals of the power series.

The contribution to the line integrals of the "X" rays, $(\text{POWER})_{p_i}(X)$ is given by (c.f equation 4.5)

$$(\text{POWER})_{p_i}(X) = \sqrt{X^2 + Y^2/N^2} \int_0^1 f(\text{POWER})_{(x,y_i(xX))}dx - \sqrt{X^2 + Y^2/N^2} \sum_{k=1}^{K+1} \sum_{l=1}^{k} c_{k(k-1)/2+l} \int_0^1 x^{l-1} y_i^{k-l}(xX)dx, \quad (4.26)$$

where the "X" ray, i given by equation (3.13) can be written:

$$y_i(Xx) = \frac{gy}{N}(1 - \frac{x}{g_i}), \quad (4.27)$$

where $g_i = 1 + \frac{i-1}{M}$, $1 \leqslant i \leqslant MN - M + 1$. (4.28)

The integral in equation (4.26) becomes

$$\left(\frac{g_i}{N}\right)^{k-l} \int_0^1 x^{l-1}(1 - \frac{x}{g_i})^{k-l}dx = \left(\frac{g_i}{N}\right)^{k-l} \int_0^1 x^{l-1} \sum_{q=0}^{k-l} \binom{k-l}{q} (\frac{-1}{g_i})^q x^q$$

$$= \left(\frac{g_i}{N}\right)^{k-l} \cdot \sum_{q=0}^{k-l} \binom{k-l}{q} (\frac{-1}{g_i})^q \cdot \int_0^1 x^{q+l-1}dx = \left(\frac{g_i}{N}\right)^{k-l} \sum_{q=0}^{k-l} \binom{k-l}{q} (\frac{-1}{g_i})^q (\frac{1}{q+l}), \quad (4.29)$$

so that

$$(\text{POWER})_{p_i}(X) = \sqrt{X^2 + Y^2/N^2} \sum_{k=1}^{K+1} \sum_{l=1}^{k} c_{k(k-1)/2+l} \left\{ \left(\frac{g_i}{N}\right)^{k-l} \sum_{q=0}^{k-l} \binom{k-l}{q} (\frac{-1}{g_i})^q (\frac{1}{q+l}) \right\}. \quad (4.30)$$

Likewise for the "Y" rays, swapping k-l and l-1, M and N in equation (4.29) and using equation (4.6) instead of (4.5) gives

$$(\text{POWER})_{p_j}(Y) = \sqrt{Y^2 + X^2/N^2} \sum_{k=1}^{K+1} \sum_{l=1}^{k} c_{k(k-1)/2+l} \left\{ \left(\frac{h_j}{M}\right)^{l-1} \sum_{q=0}^{l-1} \binom{l-1}{q} (\frac{-1}{h_j})^q (\frac{1}{q+k-l-1}) \right\}$$

where (4.31)

$$h_j = 1 + \frac{j-1}{N}, \quad 1 \leqslant j \leqslant MN - N + 1 \quad (4.32)$$

* $\binom{k-l}{q}$ is sometimes written as ${}^{k-l}C_q$ or ${}_{k-l}C_q$.

Contribution to line integrals of the cosine series.

From equation (4.5)

$$(COS)_{P_i}(X) = \sqrt{(X^2 + Y^2/N^2)} \int_0^1 g^{(COS)}(x, y_i(xX))dx,$$

$$= \sqrt{(X^2 + Y^2/N^2)} \sum_{r=1}^{R} d_r \int_0^1 \cos 2\pi\left(s_r x + t_r\left(\frac{g_i}{N}\right)\left(1 - \frac{x}{g_i}\right) + u_r\right)dx, \tag{4.33}$$

where g_i is given by equation (4.28). The integral in equation (4.33) can be written

$$\Re \int_0^1 e^{2\pi i(s - t/N)x}dx = \Re e^{2\pi i(g_i t_r/N + u_r)} \cdot e^{\pi i(s_r - t_r/N)} \frac{\sin\pi(s_r - t_r/N)}{\pi(s_r - t_r/N)} \tag{4.34}$$

where $\Re$ means real part of and $i = \sqrt{-1}$ in πi only. Equation (4.33) becomes

$$(COS)_{P_i}(X) = \sqrt{(X^2 + Y^2/N^2)} \sum_{r=1}^{R} d_r \cos\pi((s_r - t_r/N + 2(g_i t_r/N + u_r) \frac{\sin(s_r - t_r/N)}{(s_r - t_r/N)} \tag{4.35}$$

where g_i is given by equation (4.28)

Likewise

$$(COS)_{P_j}(Y) = \sqrt{(X^2 + X^2/M^2)} \sum_{r=1}^{R} d_r \cos\pi((t_r - s_r/M + 2(h_j s_r/M + u_r) \frac{\sin(t_r - s_r/M)}{(t_r - s_r/M)} \tag{4.36}$$

where h_j is given by equation (4.32)

The third and last section of "PATHS2" is the addition of noise to all the line integrals in a manner identical to that in "PATHS1". See equations (4.12) and (4.13).

No finite series of two-dimensional Walsh functions can ever exactly represent a continous two-dimansional function so, however high 2^n is, the reconstruction will always differ from the original function even with no random errors imposed on the line integrals.

4.3.3 A two dimensional step function generated from a continuous function.

Fig (4.1)(iv) shows how the program is used in this case. The first section of "PATHS2" is used to generate the mean values, $\langle f(x,y)\rangle_{I,J}$ of a given continuous function , f(x,y), of the form given by equation (4.14). This is then inputed to "PATHS1" as a given step function , $f_{I,J}$, after which the procedure followed is the same as in section 4.3.1. Not only is it more convenient to input a step function in this way rather than unput all the MXN values as data but comparisons can be made between the line integrals of the step function distribution and those of the continuous function which gave rise to it. (See Fig (5.10)).

The line integrals which would give an exact reconstruction i.e. such that $f^{(DET)}_{I,J} = \langle f(x,y)\rangle_{I,J}$ are those given by equations (4.9) and (4.11) with $\langle f(x,y)\rangle_{I,J}$ as the given step function, $f_{I,J}$. The results (section (5.1)) confirm this. But for a continuous distribution, f(x,y), the true line integrals, given by equations (4.30),(4.31),(4.35) and (4.36) are clearly not the same as these so as has been said already in section (4.3.2), the determined distribution $f^{(DET)}_{I,J}$ will not be the same as $\langle f(x,y)\rangle_{I,J}$. Later on in Chapter 6 a way of taking the true line integrals of f(x,y) and "doctoring them" to be closer to those which a step function $f_{I,J} = \langle f(x,y)\rangle_{I,J}$, would give is explored as a method of improving the results obtained for $f^{(DET)}_{I,J}$ for a smooth "input" function f(x,y).

4.4 Errors In Reconstructed Density Fields.

In this section three types of mean appear defined below and using the notations "< >" and "———" as follows

(i) $\langle g(x,y)\rangle_{I,J}$ is the mean value of a function g(x,y) within the resolution cell I,J of area 1/MN. It is defined by the relation

$$\langle g(x,y)\rangle_{I,J} = MN \int_{J/N}^{(J+1)/N} \int_{I/M}^{(I+1)/M} g(x,y)\,dx\,dy, \qquad (4.37)$$

where $0 \leqslant I < M$, $0 \leqslant J < N$. This type has already appeared (equations (4.18) and (4.22).

(ii) $\overline{g(x,y)}$ is the mean value of the function g(x,y) over the whole domain $0 \leqslant x < 1$ defined by the relation

$$\overline{g(x,y)} = \int_0^1 \int_0^1 g(x,y)\,dx\,dy \qquad (4.38)$$

(iii) $\bar{G}$ is the mean value of some distribution of MN values, $G_{I,J}$, where $G_{I,J}$ is some quantity related to the resolution cell (I,J). G is defined

by the relation

$$\overline{G} = \frac{1}{MN} \sum_{I=0}^{M-1} \sum_{J=0}^{N-1} G_{I,J} \qquad (3.39)$$

If $G_{I,J} = \langle g(x,y) \rangle_{I,J}$ as defined by equation (4.37) then

$$\overline{G} = \sum_{I=0}^{M-1} \sum_{J=0}^{N-1} \int_{J/N}^{(J+1)/N} \int_{I/M}^{(I+1)/M} g(x,y)\,dx\,dy = \int_0^1 \int_0^1 g(x,y)\,dx\,dy$$

which from equation (4.38) is $\overline{g(x,y)}$. Thus

$$\overline{\langle g(x,y) \rangle_{I,J}} = \overline{g(x,y)}. \qquad (3.40)$$

Point Errors.

The point error distribution $F_{I,J}$ of a determined density function $f_a(x,y) = f_{I,J}^{(DET)}$ is given by

$$E_{I,J} \quad \frac{\langle f(x,y) \rangle_{I,J} - f_{I,J}^{(DET)}}{\langle f(x,y) \rangle_{MAX} - \langle f(x,y) \rangle_{MIN}} \qquad (4.41)$$

where

$$\langle f(x,y) \rangle_{I,J} = MN \int_{J/N}^{(J+1)/N} \int_{I/M}^{(I+1)/M} f(x,y)\,dx\,dy. \qquad (4.42)$$

and $\langle f(x,y) \rangle_{MAX}$ and $\langle f(x,y) \rangle_{MIN}$ are the maximum and minimum values of the point errors given by equation (4.42).

$E_{I,J}$ is a relative quantity depending on the "shape" of f(x,y) not its absolute values: $E_{I,J}$ would be the same for a distribution $Kf(x,y)$, where K is a positive constant. It varies with $M = N = 2^n$ and the amount of noise imposed on the line integrals.

The average error was defined as

$$E_{AV} = \frac{1}{MN} \sum_{I=0}^{M-1} \sum_{J=0}^{N-1} |E_{I,J}| = \overline{|E_{I,J}|}. \qquad (4.43)$$

This definition of an error measurement for the whole determined distribution is the one most commonly used in papers on density function determination. For example see (6), a recent review paper by Sweeney and Vest.

In the computer program output E_{AV} is expressed as a percentage and named "EAV". $\langle f(x,y) \rangle_{MAX}$, $\langle f(x,y) \rangle_{MIN}$ and their difference are named "REFMAX", "REFMIN" and "RANGE" respectively. $E_{I,J}$ is named "ACTUAL MINUS RECONSTRUCTED REF INDEX DISTRIBUTION OVER RANGE OR DISTRIBUTION OF POINT ERRORS".

If $f(x,y) = f_{I,J}$ a two-dimensional step function then clearly $\langle f(x,y)\rangle_{I,J} = f_{I,J}$ and, if no noise is imposed on the line integral data, $E_{I,J} = 0$ for all I and J.

Mean Square Deviation Error.

Another way of expressing an overall error in the determined density function is by the relation

$$E_{OVERALL} = \sqrt{Q/S} \tag{4.44}$$

where Q is the mean square deviation of the true function, $f(x,y)$, minus the determined function: $f_a(x,y) = f^{(DET)}_{I,J}$:

$$Q = \int_0^1 \int_0^1 \left|f(x,y) - f_a(x,y)\right|^2 dx\,dy, \tag{4.45}$$

and S is the mean square deviation of the true function about its mean, $\overline{f(x,y)}$:

$$S = \int_0^1 \int_0^1 \left|f(x,y) - \overline{f(x,y)}\right|^2 dx\,dy \tag{4.46}$$

By splitting up the integral into the sum of MN integrals over each resolution cell and substituting $f^{(DET)}_{I,J}$ for $f_a(x,y)$, equation (4.45) becomes:

$$Q = \sum_{I=0}^{M-1}\sum_{J=0}^{N-1}\left\{\int_{J/N}^{(J+1)/N}\int_{I/M}^{(I+1)/M} f^2(x,y)\,dx\,dy - 2f^{(DET)}_{I,J}\int_{J/N}^{(J+1)/N}\int_{I/M}^{(I+1)/M} f(x,y)\,dx\,dy + MN\left(f^{(DET)}_{I,J}\right)^2\right\}$$

$$= \frac{1}{MN}\sum_{I=0}^{M-1}\sum_{J=0}^{N-1}\left\{\langle f^2(x,y)\rangle_{I,J} - 2f^{(DET)}_{I,J}\langle f(x,y)\rangle_{I,J} + \left(f^{(DET)}_{I,J}\right)^2\right\} \tag{4.47}$$

which can be written

$$Q = Q_1 + Q_2, \tag{4.48}$$

where

$$Q_1 = \frac{1}{MN}\sum_{I=0}^{M-1}\sum_{J=0}^{N-1}\left(\langle f(x,y)\rangle_{I,J} - f^{(DET)}_{I,J}\right)^2 \tag{4.49}$$

and

$$Q_2 = \frac{1}{MN}\sum_{I=0}^{M-1}\sum_{J=0}^{N-1}\left(\langle f^2(x,y)\rangle_{I,J} - \langle f(x,y)\rangle^2_{I,J}\right) \tag{4.50}$$

Now from equations (4.40) $\overline{f(x,y)} = \overline{\langle f(x,y)\rangle}$ so that S is given by the right hand side of equation (4.47) with $\overline{\langle f(x,y)\rangle}$ replacing $f^{(DET)}_{I,J}$ and equation (4.46) can be written

$$S = S_1 + Q_2 \tag{4.51}$$

where

$$S_1 = \frac{1}{MN}\sum_{I=0}^{M-1}\sum_{J=0}^{N-1}\left(\langle f(x,y)\rangle_{I,J} - \overline{\langle f(x,y)\rangle}\right)^2 \tag{4.52}$$

so that $E_{OVERALL}$ is given by

$$E_{OVERALL} = \sqrt{\frac{Q_1 + Q_2}{S_1 + Q_2}} \tag{4.53}$$

$E_{OVERALL}$, Like E_{AV}, is a unitless relative quantity which is the same for $K.f(x,y)$, where K is any real constant.

Now for continuous functions which, over the area of each resolution cell, deviate only a small fraction of $\langle f(x,y)\rangle_{I,J}$ from $\langle f(x,y)\rangle_{I,J}$ itself, $\langle f^2(x,y)\rangle_{I,J} - \langle f(x,y)\rangle^2_{I,J}$ will certainly be much less than $(\langle f(x,y)\rangle_{I,J} - \overline{\langle f(x,y)\rangle})^2$ and it turns out that it is also much less than $(\langle f(x,y)\rangle_{I,J} - f^{(DET)}_{I,J})^2$ so that to a good approximation

$$E_{OVERALL} \approx \sqrt{Q_1/S_1} \tag{4.54}$$

If $f(x,y) = f_{I,J}$, a step function, then $\langle f^2(x,y)\rangle_{I,J} = \langle f(x,y)\rangle^2_{I,J} = f^2_{I,J}$ and so $Q_2 = 0$ and equation (4.54) is exact (but of course will only be non zero if noise is introduced in the line integrals since otherwise $f^{(DET)}_{I,J} = f_{I,J}$ and $Q_1 = 0$)

$E_{OVERALL}$ is a much more sensitive measure of error than E_{AV} is. It is typically five times as large as E_{AV} (see chapter 5).

In the computer program outputs, the quantities $\sqrt{S_1}$ and $\overline{\langle f(x,y)\rangle}$ (see equations (4.48) and (4.9)) are named "SIGMA" and "AV". They are calculated in "PATHS1" or "PATHS2" according to whether the input function $f(x,y)$ is a step function or a continuous function. The quantity $\sqrt{Q_1}$, calculated in subroutine "REF" is named "SIGDIF".

Unless otherwise specified the input and output distributions printed are $\langle f(x,y)\rangle_{I,J}$, $\overline{\langle f(x,y)\rangle}$ and $f^{(DET)}_{I,J}$ which both have

a mean value of zero. It is possible to add a different specified constant to each one for the purposes of displaying the distribution. These constants are named "DCIN" and"DCOUT" in the computer outputs.

4.5 Practical Details On the Use Of The Computer Program.

(a) The program is designed to deal with a number, "NDATA", of different sets of input data. A number, "IDATA", running from 1 to NDATA labels each set of data where NDATA is the first number read in on the first data card and labelled "(A)" in the computer program listing in Appendix 5.

(b) The next seven of eight groups of data cards contain the input data for the first set of data (IDATA = 1) :-

The first five groups of data cards , (consisting of one card per group) read in, required for all the four different methods of using the program are:

Card 1 reads in the three integers "NSOLN", "K2" and "K0",

If NSOLN is 2 the program determines both sets of coefficients $\{F_{pq}\}_1$ and $\{F_{pq}\}_2$ and the corresponding density functions determined from them. The value of K2 is irrelevant in this case and is usually left as zero (i.e not printed on the data card). When NSOLN is 1 the program determines $\{F_{pq}\}_1$ if K2 is 1 and $\{F_{pq}\}_2$ if K2 is 2 and the corresponding density function distribution.

"K0" is an integer which controls how much output data is printed and in what form:

K0 = 1. Input and output density function distributions, line integrals, the distribution of coefficients $\{F_{pq}\}$, the distribution of point errors and other details about errors are all printed.

K0 = 2. The same as for K0 = 1 except that the line integrals are not printed.

K0 = 3. The same as for K0 = 1 except that the distribution of coefficients is not printed.

K0 = 4. The same as for K0 = 1 except that both line integrals and the distribution of coefficients are not printed.

K0 = 5. The same as for K0 = 4 except that the distribution of point errors is not printed.

K0 = 11 to K0 = 15. The same as for K0 = 1 to K0 = 5 except that distributions are represented alphanumerically. That is each of the MN numbers is represented by a singlesymbol in the appropriate position in the M X N array. Each symbol represents a certain range of values.

K0 = 6. Only the error quantities are printed.

Card 2 reads in "K1","NSMALL".

K1 can be 1,2,3,or 4. The different numbers select the four different ways in which the program can be used as shown in Fig (4.1)(i),(ii) (iii) and (iv). NSMALL is the integer n in $M = N = 2^n$ which determines the number of resolution cells MN.

Card 3 reads in "X","Y" and "P".

X and Y are the dimensions of the real or simulated rectangular area. P is a number by which all the remaining input quantities,except for s_r, t_r,θ_r, and α,are divided before being read in.This is to allow the display of the distributions to four figures without using an exponent after each value.

Card 4 reads in "DCIN", "DCOUT" and "RNOISE".

DCIN and DCOUT are the mean values which are imposed on the input and output distributions respectively, for display purposes: for example to make all the values positive. RNOISE is α (see equation (4.12)) the measure of noise or random errors imposed on the line integrals.

The data read in on the next three or four groups of cards depends on the values of K0, K1 and NSMALL, already read in.

Card 5 reads in "NS" and SYMBOLS(I) only if K0 is greater than 10, otherwise this data card is left out.

NS is the number of different symbols used if the input, output and error distributions are displayed alphanumerically. The array SYMBOLS(I) contains the symbols themselves.

If K1 = 1 the last group of cards is:

Card(s) 5 or 6 reads "F(I,J)".

F(I,J) is the step funtion $f_{I,J}$.

If K1 = 3 the last two groups of cards are:

Card(s) 5 or 6 reads "BX(I)".

The array BX(I) contains the line integrals, $P_i^{(X)}$,of all the "X" rays from an experiment.

Card(s) 6 or 7 reads "BY(I)",

The array BY(I) contains the line integrals, $P_j^{(Y)}$, of all the "Y" rays from an experiment.

If K1 = 2 or K1 = 4 the next data card is:

Card(s) 5 or 6 reads "KMAX" and "NCOS".

KMAX is K the highest power of x or y in the power series, $f^{(POWER)}(x,y)$.

NCOS is R the number of terms in the cosine series $f^{(COS)}(x,y)$.

Card(s) 6 or 7 reads C(I).

The array C(I) contains $C_{k(k-1)/2+l}$ the coefficients of the terms

of the power series $f^{(POWER)}(x,y)$. If KMAX = 0 this group of card(s) is not read.

Card(s) 6 or 7 or 8 reads CC(I), SC(I),TC(I),UC(I). These four arrays are respectively d_r, s_r, t_r and u_r in the cosine series $f^{(COS)}(x,y)$. If NCOS = 0 this group of card(s) is not read.

These six to eight different groups of cards constitute one complete set of input data. Before the second set of data (IDATA = 2) is read a special array named SAMDAT(J) is read in. If the J^{th} group of data cards is to be left entirely unchanged SAMDAT(J) = 0 and if some data on this group is to be changed then SAMDAT(J) = 1. In the former case the group of data cards for the data to be left unchanged is left out. The order for the groups of cards for IDATA = 2 follows that above for IDATA = 1 except that the group(s) for which SAMDAT = 0 are left out.

The same procedure is followed for IDATA = 3,4 and so on. The array SAMDAT(J) is read before each new set of input data and only the groups of data cards for which SAMDAT(J) = 1 are included except for the cards for F(I,J), BX(I) and BY(I), which are read each time even if they are unchanged. (The reason for this is that these arrays are changed in the program and hence cannot act as the same input data for the next set of input data). A summary of the input data scheme is shown in Fig (4.2).

It was not thought useful to give detailed flow diagrams for each subroutine because some of them, especially the ones implementing the algorithm, follow a somewhat more clumsy older form of the theory given in Chapter 3. The program and theory were developed together and the final theory is in fact considerably "neater" than the program which implements it.

ONE SET OF INPUT DATA		CONDITIONS UNDER WHICH THE DIFFERENT GROUPS OF INPUT DATA ARE READ. ("." shows input data not read. Numbers show order of reading groups)									
		First set of I/P data					Other sets of I/P data				
NDATA		1					.				
SAMDAT(J)		.					1				
		KO greater than 10					KO less than 10				
		K1=3	K1=1	K1=2 or 4			K1=3	K1=1	K1=2 or 4		
	J				KMAX =0	NCOS =0				KMAX =0	NCOS =0
NSOLN, K2, KO	1	2	2	2	2	2	2	2	2	2	2
K1, NSMALL	2	3	3	3	3	3	3	3	3	3	3
X, Y, P	3	4	4	4	4	4	4	4	4	4	4
DCIN, DCOUT, RNOISE	4	5	5	5	5	5	5	5	5	5	5
NS, SYMBOLS(I)	5	6	6	6	6	6	.	.	.	.	.
F(I,J)		.	7	.	.	.	.	6	.	.	.
BX(I)		7	.	.	.	.	6	.	.	.	.
BY(I)		8	.	.	.	.	7	.	.	.	.
KMAX, NCOS	6	.	.	7	7	7	.	.	6	6	6
C(I)	7	.	.	8	.	8	.	.	7	.	7
CC(I), SC(I), TC(I), UC(I)	8	.	.	9	8	.	.	.	8	7	.

Fig(4.2) The scheme for reading the input data. For the second and higher sets of input data, the groups of data labelled, J, are read if SAMDAT(J)=1 and not read if SAMDAT(J)=0, In the latter case the values of the group of data, J, are the same as those given by the previous set of input data, J.

Chapter 5. Some Functions and their reconstructions.

5.1 Introduction.

The computer program in all the examples given in this chapter was used as illustrated in Fig(4.1) (iv) and (iii).

In the former case a given smooth function, $f(x,y)$, was chosen and an MxM step function, $\langle f(x,y)\rangle_{I,J}$, defined from it was computed. (see equations (4.17), (4.18) and (4.19)). The line integrals from this step function were calculated (see equations (4.9) and (4.11) and both reconstructions were computed from this line integral input data. For every function chosen and for all resolutions, $M = 2,4,8,16,32$ and 64 ($M = N = 2^n, n = 1,2,3\ldots$) the reconstructed functions were, as expected, identical with the initial step function. This showed that The Walsh function reconstruction method was working correctly. One of the two reconstructions of the first example function is shown in Fig(5.1) for $M = 4$ and in Figs(5.2) and (5.3) for $M = 64$.

In the latter case the line integrals from the smooth function itself were computed (see equations (4.30), (4.31), (4.35) and (4.36)) and the two reconstructions of the function, MxM step functions, were calculated for different values of M. To simulate a realistic block of glass of refractive index variations $\pm 5.10^{-6}$ the rectangular boundary of dimensions 10cms by 10 cms was chosen and all the example functions were confined to a range of values of $\sim 10^{-5}$. The line integrals computed then simulate realistically values to be expected from an experiment. The effect of noisy input data is analysed for example 1.

5.2 Example 1. $10^5 f(x,y) = \sin(\pi x).\sin(\pi y)$.

The function expanded in terms of a cosine series is

$$10^5 . f(x,y) = \tfrac{1}{2}\cos\pi(x-y) - \tfrac{1}{2}\cos\pi(x+y) \; . \qquad (5.1)$$

$f(x,y)$ is a "hill" function symmetric to an exchange of x and y, where $0 \leq x \leq 1$, $0 \leq y \leq 1$ so for a given resolution, M, both reconstructions are identical. Only one of the two reconstructions are then given in Figs (5.4) to (5.9) inclusive.

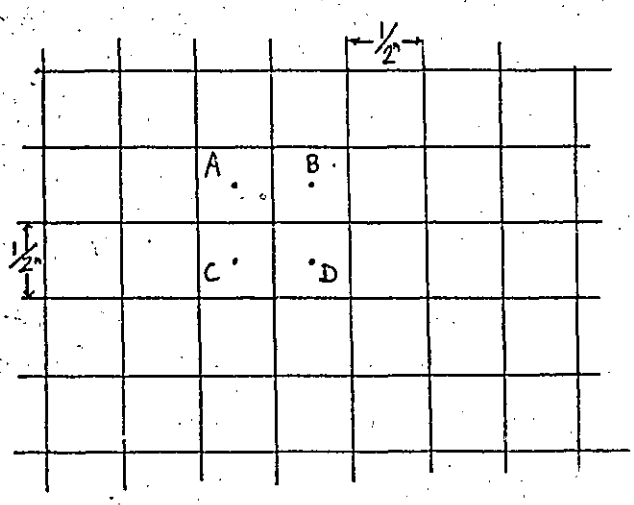
Fig(5.11) is a plot of average and overall errors versus $n = \log_2 M$ (since $M = N = 2^n$) for $n = 2,3,4,5$ and 6. A reconstruction for $n = 1$ is meaningless as the mean value of the function in each cell is the same. It could be said that the function is not resolved for $M = 2$.

Fig(5.11) shows that for $4 \leq n \leq 6$ there is only a slight increase in the error quantities E_{AV} and $E_{OVERALL}$.

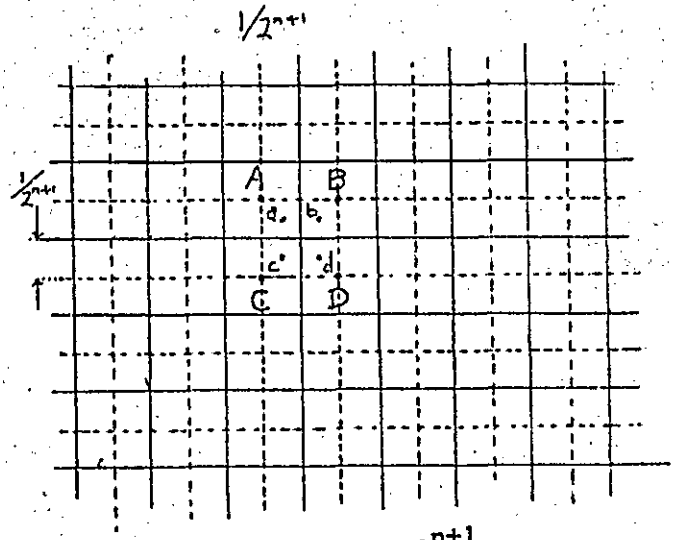
The Walsh function method effectively assumes that the function, $f(x,y)$, is a 2^n by 2^n step function whereas, however many cells the function is divided into, it will never consist of flat topped steps in each cell but will tend to planes tilted at angles $\tan^{-1}(\partial f(x,y)/\partial x)$ and $\tan^{-1}(\partial f(x,y)/\partial y)$ to the x and y axes respectively. This is developed further in Chapter 6.

In the example here little is gained by increasing the resolution beyond M = 16 especially as about four times as much input data is required for each doubling of M. Linear interpolation can generate the higher resolutions:

If the steps of the reconstructed function for $M = 2^n$ are taken as the values of the function at the centres of the corresponding cells then the linear interpolation in two dimensions of these values can give the values at the centres of the cells of the function for $M = 2^{n+1}$. To make this clearer consider the sketches below. The values of the function in the cells for $M = 2^{n+1}$ are a, b, c and d while A, B, C and D are the values of the function in the "surrounding" cells for $M = 2^n$. The former can be given, to a first approximation, in terms of the latter by the four equations (5.2) to (5.5) inclusive.



Resolution $M = 2^n$



Resolution $M = 2^{n+1}$

$$a = A + \tfrac{1}{4}(B - A) + \tfrac{1}{4}(C - A) = \tfrac{1}{2}A + \tfrac{1}{4}(B + C) \qquad (5.2)$$

$$b = B - \tfrac{1}{4}(B - A) + \tfrac{1}{4}(D - B) = \tfrac{1}{2}B + \tfrac{1}{4}(A + D) \qquad (5.3)$$

$$c = C - \tfrac{1}{4}(C - A) + \tfrac{1}{4}(D - C) = \tfrac{1}{2}C + \tfrac{1}{4}(A + D) \qquad (5.4)$$

$$d = D - \tfrac{1}{4}(D - C) - \tfrac{1}{4}(D - B) = \tfrac{1}{2}D + \tfrac{1}{4}(B + C) \qquad (5.5)$$

The equations above give quite accurate results for n = 4. (see Figs(5.5) and (5.6)).

Fig(5.10) compares the relative line integrals, for M = 16, of the smooth function, $f(x,y)$, and the step function, $\langle f(x,y)\rangle_{I,J}$, derived from it. The line integrals computed from the step function would, of course, give the "correct" reconstruction of the smooth function for a given

resolution,M.The difference in the two sets of line integrals is another measure of the errors inherent in The Walsh function method for a given resolution,M.From Fig(5.10) it is clear that the difference in the two sets of line integrals is greatest,both in absolute terms and as a fraction of the true(i.e. "smooth")line integrals,in regions near the edges. Rays giving these data points pass near the boundary of the function where the reconstruction does appear to be worse than in more central regions.(see Figs(5.5) to (5.9)).

The reconstructions of f(x,y),when a random number is added to each line integral,is examined for M = 8 and M = 16.Equations (4.12) and (4.13) show how the quantity,α,("Rnoise" in the computer program)defining the range of the random numbers controls the ammount of noise added to the input data.Fig(5.12) shows how E_{AV} changes with α for M = 8 .For each value of α,five different sets of $2(M^2-M+1)=114$ random numbers were added to $P_i^{(X)}$ and $P_j^{(Y)}$and the five resulting values of E_{AV} were plotted.The mean value of each of the five sets of E_{AV}'s was also plotted.Fig(5.13) shows the same as Fig(5.12) but for M = 16.The spread of E_{AV} for a given α is quite large but the mean values of E_{AV} generally increase with increasing αas might be expected.The increase is faster for M = 16 than for M = 8.

The separations of the exploring rays passing through the density field for a resolution $M = 2^{n+1}$ are one quarter of that for $M = 2^n$. Now the sets of line integrals,$P_i^{(X)}$and $P_j^{(Y)}$ are samples of continuous functions of line integrals,p(y') and p(y'') say,(see Chapter 1)where the sampling interval is proportional to the separation of the rays. p(y') and p(y'') will include some noise function.This ,when sampled at the sampling interval for the given resolution,M,will give a discrete set of numbers which,in general,will not be totally uncorrelated:Each sample will be better correlated with samples in its immediate neighbourhood than with more distant ones.As the sampling distance decreases the correlation between a given number of close neighbours will increase. In the noise simulation scheme used here however there is zero correlation between all the noise "samples" however close together they are.It is hoped that a more realistic simulation of noisy input data,based on sampling some given noise function as outlined above,will show that The Walsh Function Method is less sensetive to noise than Figs(5.12) and (5.13) indicate.

For the remaining example functions the inadequate simulation of noisy input data is not used.

5.3 Example 2. $10^5.f(x,y) = \sin^2(2\pi x).\sin^2(2\pi y)$.

The cosine series for this function is given by

$$10^5.f(x,y) = \tfrac{1}{4} - \tfrac{1}{4}\cos 4\pi x - \tfrac{1}{4}\cos 4\pi y + \tfrac{1}{8}\cos 4\pi(x-y) + \tfrac{1}{8}\cos 4\pi(x+y). \tag{5.6}$$

f(x,y),as in Example 1.,is unchanged by an exchange of x and y so that both reconstructions are identical.It has four peaks at (x,y) = $(\tfrac{1}{4},\tfrac{1}{4})$,$(\tfrac{1}{4},\tfrac{3}{4})$,$(\tfrac{3}{4},\tfrac{1}{4})$ and $(\tfrac{3}{4},\tfrac{3}{4})$.One of the two reconstructions is shown in Figs(5.14) to (5.19) inclusive for M = 8,16,32 and 64.The function is not resolved for M = 2 or 4.

Fig(5.20) is a plot of average and overall errors versus n ($M = N = 2^n$) for n = 3,4,5 and 6.From the plot it seems that there is very little improvement in the reconstruction for resolutions higher than M = 32.

5.4 Example 3. $10^5.f(x,y) = 30x^2(1-x)(1-y)^2y + \tfrac{1}{2}\sin^2 2\pi x.\sin^2 \pi y$.

The power and cosine series for this function is given by

$$10^5.f(x,y) = 30yx^2 - 60y^2x^2 - 30yx^3 + 30y^3x + 60y^2x^3 - 30y^3x^3 + \tfrac{1}{8} - \tfrac{1}{16}\cos 4\pi x - \tfrac{1}{16}\cos 2\pi y + \tfrac{1}{16}\cos 2\pi(2x-y) + \tfrac{1}{16}\cos 2\pi(2x+y). \tag{5.7}$$

The power series part of $10^5.f(x,y)$ is a "single hill" function, zero at the boundary,x = y = 0 and x = y = 1,with its peak of height 160/243 ≈ .658 at (x,y) = $(\tfrac{2}{3},\tfrac{1}{3})$.The cosine series is a "double hill" function,zero at the boundary,with its two peaks of equal heights,$\tfrac{1}{16}$, at (x,y) = $(\tfrac{1}{2},\tfrac{1}{4})$ and $(\tfrac{1}{2},\tfrac{3}{4})$.The sum of these two functions gives a function, zero at the boundary,with two peaks of unequal heights and a saddle point approximately in between.(See Figs(5.26) to (5.28) inclusive). The smaller peak is about .70 in height and is at (x,y) ≈ (.29,.45). The saddle point is of value about .54 and is at (x,y) ≈ (.45,.34). The taller peak is about 1.07 in height and is at (x,y) ≈ (.73,.44).

The function,f(x,y),is different from f(y,x) and,as would be expected, the two reconstructions are not identical.The reconstructions are shown in Figs(5.21) to (5.30) inclusive.As the resolution,M,increases the two reconstructions tend closer to one another.For M ⩾ 16 the two are virtually the same.

An apparent anomally is shown in Fig(5.31),the plot of average and overall errors versus n:The reconstructions for M = 4 and M = 8 are apparently "better" than those for higher values of M.In fact of course this is not true.It is recalled(See equations(4.41),(4.43),(4.49) and (4.54))

that the average and overall errors compare the reconstructed value in each resolution cell, $f_{I,J}^{(DET)}$, to the mean value of the true function in that cell, $\langle f(x,y) \rangle_{I,J}$. If the cells are so large that f(x,y) varies significantly over each cell (as is clearly the case in this example for M = 4 and to a lesser degree for M = 8) then the reconstruction is inadequate because the changes in f(x,y) within the cells cannot be represented. The average and overall errors are only comparable from one resolution, M_1, to another, M_2, when the cells in both cases are so small that the change in f(x,y) relative to the range, $f(x,y)_{max} - f(x,y)_{min}$, within all the cells is very small. The best way of comparing the reconstructions at different resolutions, M_1 and M_2, where $M_2 > M_1$, would probably be to interpolate the reconstruction for M_1 (perhaps by using equations (5.2) to (5.5) for all but the "edge" cells) to generate a reconstruction of resolution, M_2; to compute the errors, E_{AV} and $E_{overall}$, with respect to $\langle f(x,y) \rangle_{I,J}$, for M_2 and finally to compare these errors with those of the true reconstruction at resolution M_2.

The flattening off of the plot in Fig (5.31) with increasing n suggests that there is only a slight improvement in reconstruction when resolutions higher than M = 16 are used. (It is judged that f(x,y) changes very little in cells for $M \geq 16$ so that the errors at different resolutions, $M \geq 16$, can be meaningfully compared).

The two peaks of the function, f(x,y), are not resolved by the reconstructions. The "strength" of the two peaks could be described in terms of their modulation, which could be defined as their differences in values from the local minimum between them divided by the range of the function. Using this definition gives modulations of 15% and 50% respectively for the smaller peak and the larger peak. The reconstruction of the two peaks is a single peak in the region covered by them. (See Figs (5.26), (5.29) and (5.30). In Example 2 there were four peaks of 100% modulations. In the reconstructions however, the pairs of peaks aligned horizontally (in the y direction) or vertically (in the x direction) had modulations of about 50% while the pairs of peaks aligned diagonally (on the lines y=x and y=1-x) had modulations of about the correct value, 100%.

5.5 Examples 4 and 5. $10^5.f(x,y) = K_1\sin(\pi x).\sin(\pi y) + K_2\sin^2(2\pi x).\sin^2(2\pi y)$ for $K_1 = \frac{1}{2}, K_2 = \frac{1}{2}$ and $K_1 = \frac{1}{2}, K_2 = \frac{5}{8}$.

This function is the sum of K_1 times Example 1 and K_2 times Example 2. The modulation of a pair of peaks aligned in a horizontal or vertical direction is 58% for Example 4 and 41% for Example 5. The modulation of diagonally aligned peaks is 49½% and 27% in the two examples. From the reconstruction of the two examples, for M = 32, the modulation of pairs of horizontally and vertically aligned peaks was computed to be 23% and 13½% while the figures for pairs of diagonally aligned peaks are 47½% and 28%. According to the criterion of modulation the diagonally aligned peaks are "correctly" resolved for both examples, as they are also in Example 2 which is the function, f(x,y), given above with $K_1 = 0$ and $K_2 = 1$. The horizontally and vertically aligned pairs of peaks however are decreasingly well resolved as K_1/K_2 increases.

There is hardly any improvement in the reconstructions for resolutions higher than M = 16: For Example 4 E_{AV} is 8.912%, 8.559% and 8.505% for M = 16, 32 and 64 respectively. The corresponding figures for Example 5 are 8.726%, 8.479% and 8.436%.

5.6 Examples 5, 6 and 7. $10^5.f(x,y) = \sin^2(K\pi x).\sin^2(K\pi y)$, where K = 3, 4 and 5.

The cosine series for these functions is given by

$$10^5.f(x,y) = \tfrac{1}{4} - \tfrac{1}{8}\cos 2K\pi x - \tfrac{1}{8}\cos 2K\pi y + \tfrac{1}{8}\cos 2K\pi(x+y) + \tfrac{1}{8}\cos 2K\pi(x-y). \qquad (5.8)$$

The function given by K = 2 in equation (5.8) is Example 2. In general the functions have K^2 peaks of unit height and modulation placed symmetrically within the boundary at (x,y) = (p/2K, q/2K), where p and q are the K odd numbers, 1, 3, ... 2K-1 inclusive.

The reconstruction of Example 5 (K=3) for M = 16, 32 and 64 gives average errors of 11.534%, 10.519% and 10.255% respectively. M = 32 seems to give the optimum resolution, where "optimum" is taken to mean that the average error barely improves for an increase in resolution. In Example 6 the average errors for resolutions, M = 16, 32 and 64 are 16.688%, 11.705% and 10.541%. The corresponding figures for Example 7 are 13.030%, 10.979% and 10.417% respectively. Figs (5.32) and (5.33) show this example and its reconstruction for M = 64. A region of this reconstruction containing four peaks looks very similar to Fig (5.19), the reconstruction of Example 2. Examples 5 and 6 also have this property.

5.7 Conclusions.

From the very limited type and number of functions reconstructed using the Walsh function method developed in this thesis it is possible to draw some tentative conclusions.

The accuracy of the method for noiseless input data is much poorer than methods using more than two ray viewing directions, as might be expected. A recent review paper by Sweeney and West, (6), which discusses and compares these methods with one another, gives the average errors, E_{AV}, of the reconstructions of various functions. One of these functions quoted as "constant + 2.5(1 - $\cos 4\pi(x-\frac{1}{2})$)(1 - $\cos 4\pi(y-\frac{1}{2})$)" is, apart from a constant and multiplicative constant, the same as Example 2. The average errors quoted for the reconstruction for M = N = 7 using data from 16 different viewing directions each having 19 rays, are: .22%, .28%, 2.21% (M = N = 9 for this method), 3.08%, and 3.10% depending on the method. The viewing directions range over a complete field angle of $\pm 90^{\circ}$. The Walsh function method reconstruction which uses about the same ammount of input data is for M = N = 16. The average error was 11.651%.

Despite its relatively poor accuracy it must be remembered that the Walsh function method was specifically developed for situations where only two ray directions at near to 90° to one another are practicable as in the glass block problem described in Chapter 1 and the early part of Chapter 3. A fuller evaluation of the method requires a realistic noise analysis, as pointed out in section (5.2), and a systemmatic study of the reconstructions of many more functions than has been attempted here.

If, as seems reasonable, a given function gives unique reconstructions; it is conceivable that a <u>determination</u> of an unknown density function from experimental line integral data could be compared with the <u>reconstruction</u> of some guessed at function explored by the same rays. By altering the guessed at function it might be possible to obtain a reconstruction closely res embling the experimental determination. The known guessed at function could then be taken as a good approximation to the unknown density function. A library" of the reconstructions of known functions could be built up to assist in guessing the best function which has a reconstruction most like any given determination.

Chapter 6 is an attempt to improve directly the determination of density functions using the Walsh function method. It contains theory which as yet is completely untested .

PRINTOUT MODE IS KO= 1

IN ALL DISTRIBUTIONS BELOW Y AXIS IS HORIZONTAL TO THE RIGHT,X AXIS IS VERTICALLY DOWN,ORIGIN IS TOP LEFT HAND CORNER

INPUT MODE KI= 4,PATHS COMPUTED FROM A STEP FUNCTION GENERATED FROM A SMOOTH FINITE POWER AND/OR COSINE SERIES
RESOLUTION IS M= 4 BY N= 4,DIMENSIONS OF BOUNDARY ARE X= 1.000E+05 BY Y= 1.000E+05 ,MICRONS

ALL FIGS BELOW EXCEPT ERRORS,SC(I),TC(I),UC(I) AND RNOISE TO BE MULTIPLIED BY P = 1.000000E-05

COS SERIES SUM(I=1 TO 2) OF CC(I)COS(PI2)(SC(I)X+TC(I)Y+UC(I)) IS CC(I),SC(I),TC(I),UC(I)=
5.000000E-01 5.000000E-01 -5.000000E-01 0. , -5.000000E-01 5.000000E-01 5.000000E-01

$= \frac{1}{2}\sin(\pi x).\sin(\pi y)$

THE TRUE STEP FUNCTION REF INDEX DISTRIBUTION GIVEN BY VALUES IN THE 4 BY 4 RESOLUTION RECTANGLES IS

.1391	.3358	.3358	.1391
.3358	.8106	.8106	.3358
.3358	.8106	.8106	.3358
.1391	.3358	.3358	.1391

← INPUT FUNCTION

MAX VALUE OF REF INDEX DISTRIBUTION IS REFMAX= .810588E+00
MIN VALUE OF REF INDEX DISTRIBUTION IS REFMIN= .139090E+00
REFMAX MINUS REFMIN IS RANGE= .671598E+00
MEAN VALUE OF REF INDEX DISTRIBUTION IS AV= .405288E+00

THE STANDARD DEVIATION OF THE REF INDEX DISTRIBUTION ABOVE,ABOUT MEAN OF ZERO,IS SIGMA= .247385E+00
THE MEAN VALUE IMPOSED ON THE REF INDEX DISTRIBUTION ABOVE IS DCIN= .405300E+00

-|-|--8|-----16|-------------32|-------------------------64|--|--

THE RANGE OF NOISE IMPOSED ON THE RELATIVE PATHS/MAX PATH MINUS MIN PATH IS 0.000 PER CENT

THE RELATIVE PATHS OF ALL 13 X RAYS,IN MICRONS,INCLUDING IMPOSED NOISE,IS
0. 5.0683E+03 1.7304E+04 2.9540E+04 3.4608E+04 3.4608E+04 3.4608E+04 3.4608E+04 2.9541E+04
1.7304E+04 5.0683E+03 -2.4234E-03

THE RELATIVE PATHS OF ALL 13 Y RAYS,IN MICRONS,INCLUDING IMPOSED NOISE,IS
0. 5.0683E+03 1.7304E+04 2.9540E+04 3.4608E+04 3.4608E+04 3.4608E+04 3.4608E+04 2.9541E+04
1.7304E+04 5.0683E+03 -2.4234E-03

THE COEFFICIENTS GIVEN BY THE PATHS OF ALL 13 X RAYS AND THE 4 Y RAYS 4K+1 (0<K< 4) ARE
.1679 .0000 -.1679 .0000
.0000 .0000 .0000 .0000
-.1679 .0000 .1679 -.0000
.0000 .0000 -.0000 .0000

REF INDEX DISTBN GIVEN BY THE PATHS OF ALL 13 X RAYS AND THE 4 Y RAYS 4K+1 (0<K< 4) IS

.1391	.3358	.3358	.1391
.3358	.8106	.8106	.3358
.3358	.8106	.8106	.3358
.1391	.3358	.3358	.1391

← RECONSTRUCTION

ACTUAL MINUS DETERMINED REF INDEX DISTRIBUTION OVER RANGE,ABOUT ZERO MEAN FOR BOTH,OR DISTRIBUTION OF POINT ERRORS IS
.0000 .0000 .0000 .0000
.0000 .0000 .0000 .0000
.0000 .0000 .0000 .0000
.0000 .0000 .0000 .0000

THE RANGE OF NOISE IMPOSED/MAX PATH MINUS MIN PATH IS RNOISE= 0.000 PER CENT

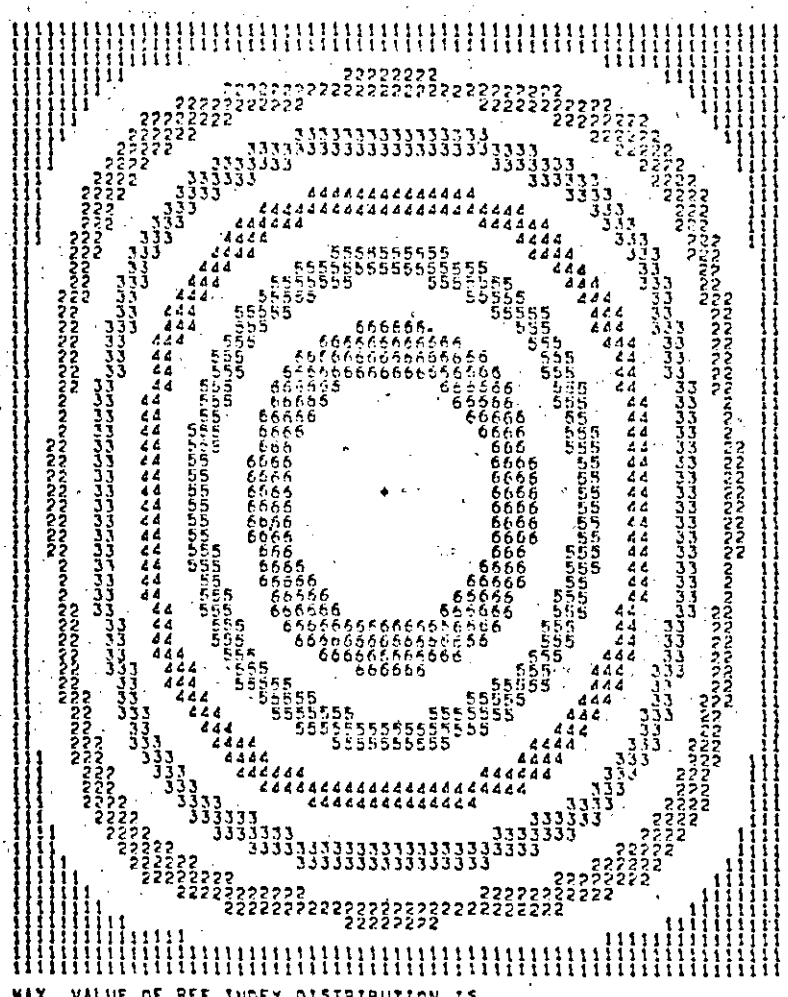
THE STANDARD DEVIATION OF ACTUAL MINUS RECONSTRUCTED REF INDEX DISTRIBUTION IS SIGDIF= .132116E-13
THE MEAN VALUE IMPOSED ON THE REF INDEX DISTRIBUTION ABOVE IS DCOUT= .405300E+00

OVERALL ERROR IS SIGDIF/SIGMA= .000 PER CENT
AVERAGE ERROR,SUM(ABS(POINT ERRORS))/MN,IS EAV= .000 PER CENT
THE MAXIMUM POINT ERRORS ARE EMAX= .000 PER CENT
EMIN= .000 PER CENT

-|-|--8|-----16|-------------32|-------------------------64|--|--

Fig(5.1). The step function, $<10^{-5}.\sin(\pi x).\sin(\pi y)>_{I,J}$, for M = N = 4, and its reconstruction.

```
PRINTOUT MODE IS NO= 15
IN ALL DISTRIBUTIONS BELOW Y AXIS IS HORIZONTAL TO THE RIGHT,X AXIS IS VERTICALLY DOWN,ORIGIN IS TOP LEFT CORNER
INPUT MODE NO= 4,PATHS COMPUTED FROM A STEP FUNCTION GENERATED FROM A SMOOTH CIRCLE FOUND AND/OR COSINE SERIES
RESOLUTION IS M= 64 BY N= 64,DIMENSIONS OF BOUNDARY ARE X= 1.000E+00 BY Y= 1.000E+00 MICRONS
ALL FIGS BELOW EXCEPT ERRORS,SC(I),TC(I),UC(I) AND RADIUS TO BE MULTIPLIED BY P = 1.000E-05
COS SERIES SUM(I=1 TO  2 ) OF CC(I)COS(PI2)(SC(I)X+TC(I)Y+UC(I)) IS CC(I),SC(I),TC(I),UC(I)=
  5.000E-01  5.000E-01  -5.000E-01  0.  0.  -5.000E-01  5.000E-01  5.000E-1  0.
THE TRUE STEP FUNCTION REF INDEX DISTRIBUTION GIVEN BY VALUES IN THE 64 BY 64 RESOLUTION RECTANGLES IS.
KEY TO THE 12 SYMBOLS USED,VALUE=SYMBOL<VALUE, IF VALUE< .617JE-03SYMBOL IS =,IF VALUE > .5992E+00:SYMBO
  .091 1  .084  .157 2  .250  .333 3  .417  .500 4  .583  .666 5  .752  .15+
  .833 6  .916  .999
```

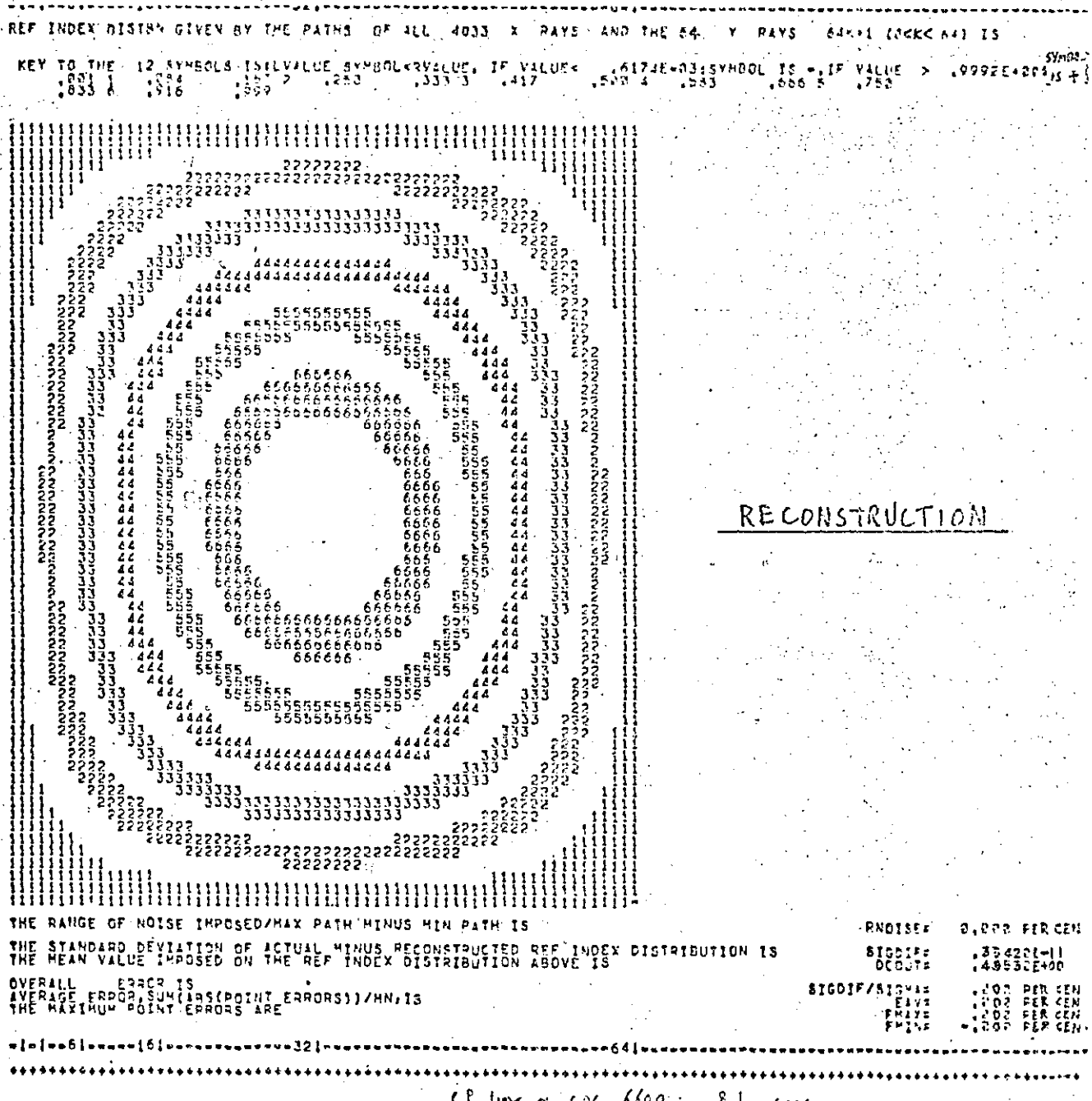


INPUT FUNCTION.

```
MAX  VALUE OF REF INDEX DISTRIBUTION IS                                   REFMAX=  .9992E+00
MIN  VALUE OF REF INDEX DISTRIBUTION IS                                   REFMIN=  .6174E-03
REFMAX MINUS REFMIN IS                                                    RANGE=   .9982E+00
MEAN VALUE OF REF INDEX DISTRIBUTION IS                                   AV=      .4053E+00

THE STANDARD DEVIATION OF THE REF INDEX DISTRIBUTION ABOVE,ABOUT MEAN OF ZERO,IS  SIGMA=  .2026E+00
THE MEAN VALUE IMPOSED ON THE REF INDEX DISTRIBUTION ABOVE IS                     DCIN=   .4053E+00
```

Fig(5.2) The step function, $\langle 10^{-5}.\sin(\pi x).\sin(\pi y)\rangle_{I,J}$, for M = N = 64.



C.P. time on cdc 6600 : 8.1 secs

Fig(5.3) The reconstruction of the step function in Fig(5.2).

PRINTOUT MODE IS K0= 1

IN ALL DISTRIBUTIONS BELOW Y AXIS IS HORIZONTAL TO THE RIGHT,X AXIS IS VERTICALLY DOWN.ORIGIN IS TOP LEFT HAND CORNER

INPUT MODE K1= 2,PATHS COMPUTED FROM A SMOOTH FUNCTION GIVEN AS A FINITE POWER AND/OR COSINE SERIES
RESOLUTION IS M= 4 BY N= 4,DIMENSIONS OF BOUNDARY ARE X= 1.000E+05 BY Y= 1.000E+05 ,MICRONS

ALL FIGS BELOW EXCEPT ERRORS,SC(I),TC(I),UC(I) AND RNOISE TO BE MULTIPLIED BY P = 1.000000E-05

COS SERIES SUM(I=1 TO 2) OF CC(I)COS(PI2)(SC(I)X+TC(I)Y+UC(I)) IS CC(I),SC(I),TC(I),UC(I)=
5.000000E-01 5.000000E-01 -5.000000E-01 0. , -5.000000E-01 5.000000E-01 5.000000E-01 0.

$= \sin(\pi x)\sin(\pi y)$

TRUE REF INDEX DISTRIBUTION OF SMOOTH FUNCTION EXPRESSED AS THE MEAN VALUES OF F(X,Y) IN THE 4 BY 4 RESOLUTION RECTANGL S IS

.1391	.3358	.3358	.1391
.3358	.8106	.8106	.3358
.3358	.8106	.8106	.3358
.1391	.3358	.3358	.1391

← INPUT FUNCTION

MAX VALUE OF REF INDEX DISTRIBUTION IS REFMAX= .9106[illegible]E+00
MIN VALUE OF REF INDEX DISTRIBUTION IS REFMIN= .1391[illegible]E+00
REFMAX MINUS REFMIN IS RANGE= .7715[illegible]E+00
MEAN VALUE OF REF INDEX DISTRIBUTION IS AV= .4[illegible]E+00

THE STANDARD DEVIATION OF THE REF INDEX DISTRIBUTION ABOVE,ABOUT MEAN OF ZERO,IS SIGMA= .2473[illegible]E+00
THE MEAN VALUE IMPOSED ON THE REF INDEX DISTRIBUTION ABOVE IS DCIN= .4[illegible]E+00

-1-1--81-----161-----------321-----------------------641---1281---

THE RANGE OF NOISE IMPOSED ON THE RELATIVE PATHS/MAX PATH MINUS MIN PATH IS 0.000 PER CENT

THE RELATIVE PATHS OF ALL 13 X RAYS,IN MICRONS,INCLUDING IMPOSED NOISE,IS
0. 1.1180E+04 2.0980E+04 2.9022E+04 3.4998E+04 3.8678E+04 3.9921E+04 3.8678E+04 3.4998E+04 2.9022E+04
2.0980E+04 1.1180E+04 -3.0634E-05

THE RELATIVE PATHS OF ALL 13 Y RAYS,IN MICRONS,INCLUDING IMPOSED NOISE,IS
0. 1.1180E+04 2.0980E+04 2.9022E+04 3.4998E+04 3.8678E+04 3.9921E+04 3.8678E+04 3.4998E+04 2.9022E+04
2.0980E+04 1.1180E+04 -3.0634E-05

THE COEFFICIENTS GIVEN BY THE PATHS OF ALL 13 X RAYS AND THE 4 Y RAYS 4K+1 (0<K< 4) ARE

.1698	.0000	-.1698	.0000
.0000	-.0815	-.0000	-.0338
.1698	-.0000	.0033	-.0000
.0000	-.9404	-.0000	-.0167

REF INDEX DISTBN GIVEN BY THE PATHS OF ALL 13 X RAYS AND THE 4 Y RAYS 4K+1 (0<K< 4) IS

-.1033	.3306	.4734	.2415
.3438	.7240	.7723	.4682
.4682	.7723	.7240	.3438
.2415	.4734	.3306	-.1033

← RECONSTRUCTION

ACTUAL MINUS DETERMINED REF INDEX DISTRIBUTION OVER RANGE,ABOUT ZERO MEAN FOR BOTH,OR DISTRIBUTION OF POINT ERRORS IS

.3609	.0077	-.2049	-.1525
-.0119	.1289	.0571	-.1053
.1053	.0571	.1289	-.0119
-.1525	-.2049	.0077	.3609

THE RANGE OF NOISE IMPOSED/MAX PATH MINUS MIN PATH IS RNOISE= 0.000 PER CENT

THE STANDARD DEVIATION OF ACTUAL MINUS RECONSTRUCTED REF INDEX DISTRIBUTION IS SIGDIF= .1[illegible]E+00
THE MEAN VALUE IMPOSED ON THE REF INDEX DISTRIBUTION ABOVE IS DCOUT= .4[illegible]E+00

OVERALL ERROR IS SIGDIF/SIGMA= 47.977 PER CENT
AVERAGE ERROR,SUM(ABS(POINT ERRORS))/MN,IS AVE= 13.865 PER CENT
THE MAXIMUM POINT ERRORS ARE EMAX= 36.094 PER CENT
EMIN= -20.493 PER CENT

-1-1--81-----161-----------321-----------------------641---1281---

****

Fig(5.4) The function, $10^{-5} \cdot \sin(\pi x) \cdot \sin(\pi y)$, and its reconstruction for M = N = 4.

M=N=8

REF INDEX DISTBN GIVEN BY THE PATHS OF ALL 57 X RAYS AND THE 8 Y RAYS 8K+1 (0≤K≤ 8) IS 10^{-5} TIMES

-.2647	-.0217	.1820	.3152	.3578	.3032	.1598	-.0506
-.0077	.2324	.4287	.5515	.5820	.5155	.3623	.1456
.1965	.4330	.6216	.7336	.7519	.6738	.5111	.2886
.3213	.5536	.7340	.8349	.8411	.7514	.5797	.3520
.3520	.5797	.7514	.8411	.8349	.7340	.5536	.3213
.2886	.5111	.6738	.7519	.7336	.6216	.4330	.1965
.1456	.3623	.5155	.5820	.5515	.4287	.2324	-.0077
-.0506	.1598	.3032	.3578	.3152	.1820	-.0217	-.2647

THE RANGE OF NOISE IMPOSED/MAX PATH MINUS MIN PATH IS RNOISE= 0.000 PER CENT

THE STANDARD DEVIATION OF ACTUAL MINUS RECONSTRUCTED REF INDEX DISTRIBUTION IS SIGDIF= .10087E+04
THE MEAN VALUE IMPOSED ON THE REF INDEX DISTRIBUTION ABOVE IS DCOUT= .40530E+00

OVERALL ERROR IS SIGDIF/SIGMA= 35.800 PER CENT
AVERAGE ERROR,SUM(ABS(POINT ERRORS))/MN,IS EAV= 8.900 PER CENT
THE MAXIMUM POINT ERRORS ARE EMAX= 33.141 PER CENT
EMIN= -18.517 PER CENT

M=N=16

REF INDEX DISTBN GIVEN BY THE PATHS OF ALL 241 X RAYS AND THE 16 Y RAYS 16K+1 (0≤K≤ 16) IS 10^{-5} TIMES

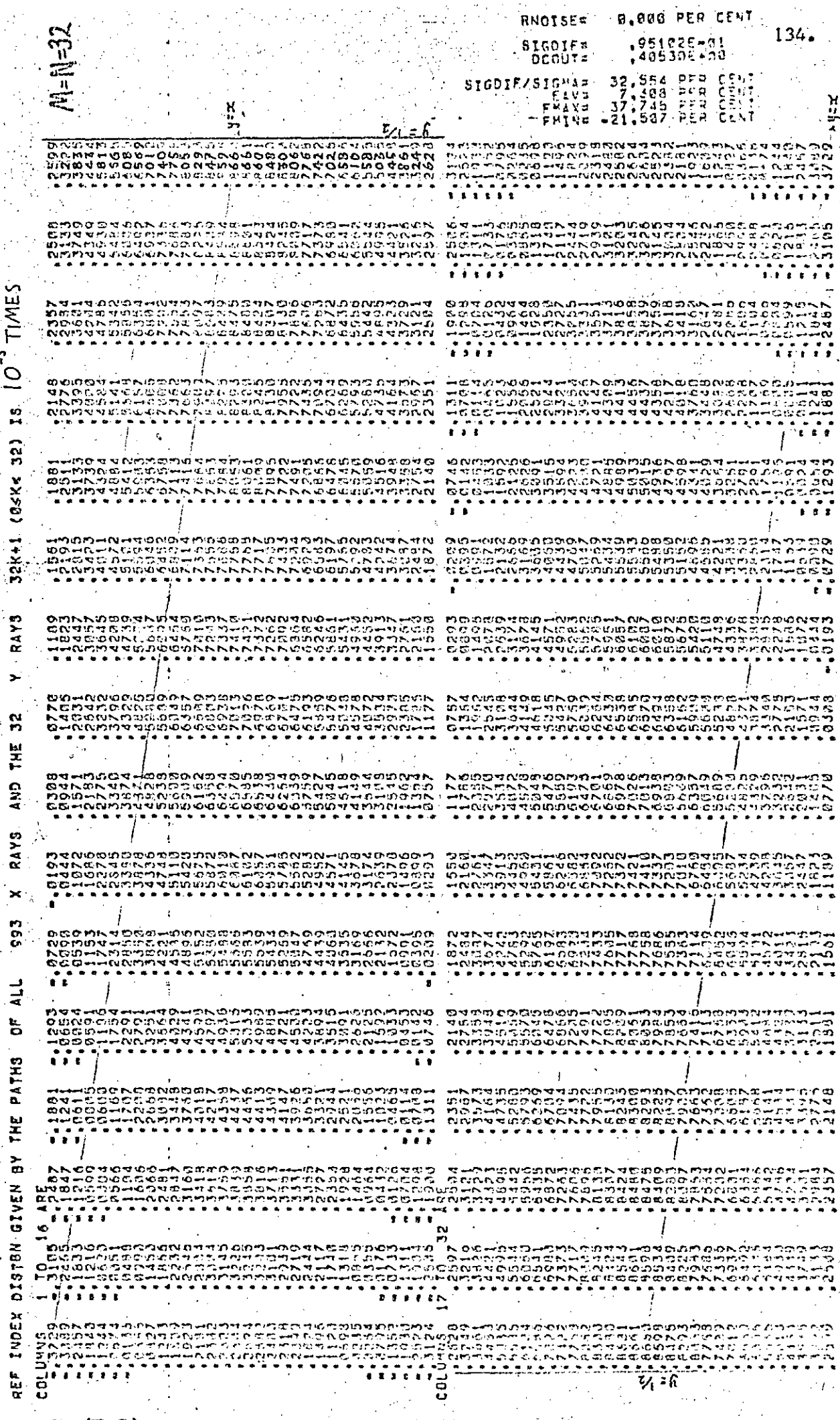
-.3385	-.2143	-.0950	.0148	.1108	.1894	.2476	.2831	.2946	.2816	.2446	.1851	.1053	.0083	-.1022	-.2220
-.2088	-.0848	.0340	.1431	.2381	.3156	.3724	.4064	.4164	.4019	.3634	.3025	.2215	.1236	.0124	-.1078
-.0862	.0377	.1560	.2643	.3583	.4345	.4980	.5226	.5310	.5150	.4751	.4129	.3307	.2318	.1199	-.0027
.0250	.1456	.2665	.3743	.4670	.5420	.5961	.6272	.6341	.6165	.5752	.5116	.4283	.3265	.2159	.0953
.1237	.2441	.3615	.4682	.5572	.6340	.6857	.7163	.7216	.7026	.6593	.5949	.5105	.4097	.2965	.1752
.1975	.3238	.4376	.5435	.6345	.7070	.7583	.7864	.7983	.7697	.7254	.6593	.5737	.4720	.3582	.2365
.2530	.3759	.4922	.5973	.6872	.7585	.8094	.8350	.8373	.8152	.7695	.7020	.6154	.5128	.3984	.2763
.2851	.4077	.5234	.6276	.7165	.7865	.8350	.8601	.8609	.8373	.7902	.7214	.6337	.5303	.4152	.2928
.2928	.4152	.5303	.6337	.7214	.7902	.8373	.8609	.8601	.8350	.7865	.7165	.6276	.5234	.4077	.2851
.2763	.3984	.5128	.6154	.7020	.7695	.8152	.8373	.8350	.8094	.7585	.6872	.5973	.4922	.3759	.2530
.2365	.3582	.4720	.5737	.6593	.7254	.7697	.7983	.7864	.7583	.7070	.6345	.5435	.4376	.3238	.1975
.1752	.2965	.4097	.5105	.5949	.6593	.7026	.7216	.7163	.6857	.6340	.5572	.4682	.3615	.2441	.1237
.0953	.2159	.3265	.4283	.5116	.5752	.6165	.6341	.6272	.5961	.5420	.4670	.3743	.2665	.1456	.0250
-.0037	.1199	.2318	.3307	.4129	.4751	.5150	.5310	.5226	.4980	.4345	.3583	.2643	.1560	.0377	-.0862
-.1078	.0124	.1236	.2215	.3025	.3634	.4019	.4164	.4064	.3724	.3156	.2381	.1431	.0340	-.0848	-.2088
-.2220	-.1022	.0083	.1053	.1851	.2446	.2816	.2946	.2831	.2476	.1894	.1108	.0148	-.0950	-.2143	-.3385

y=1/2

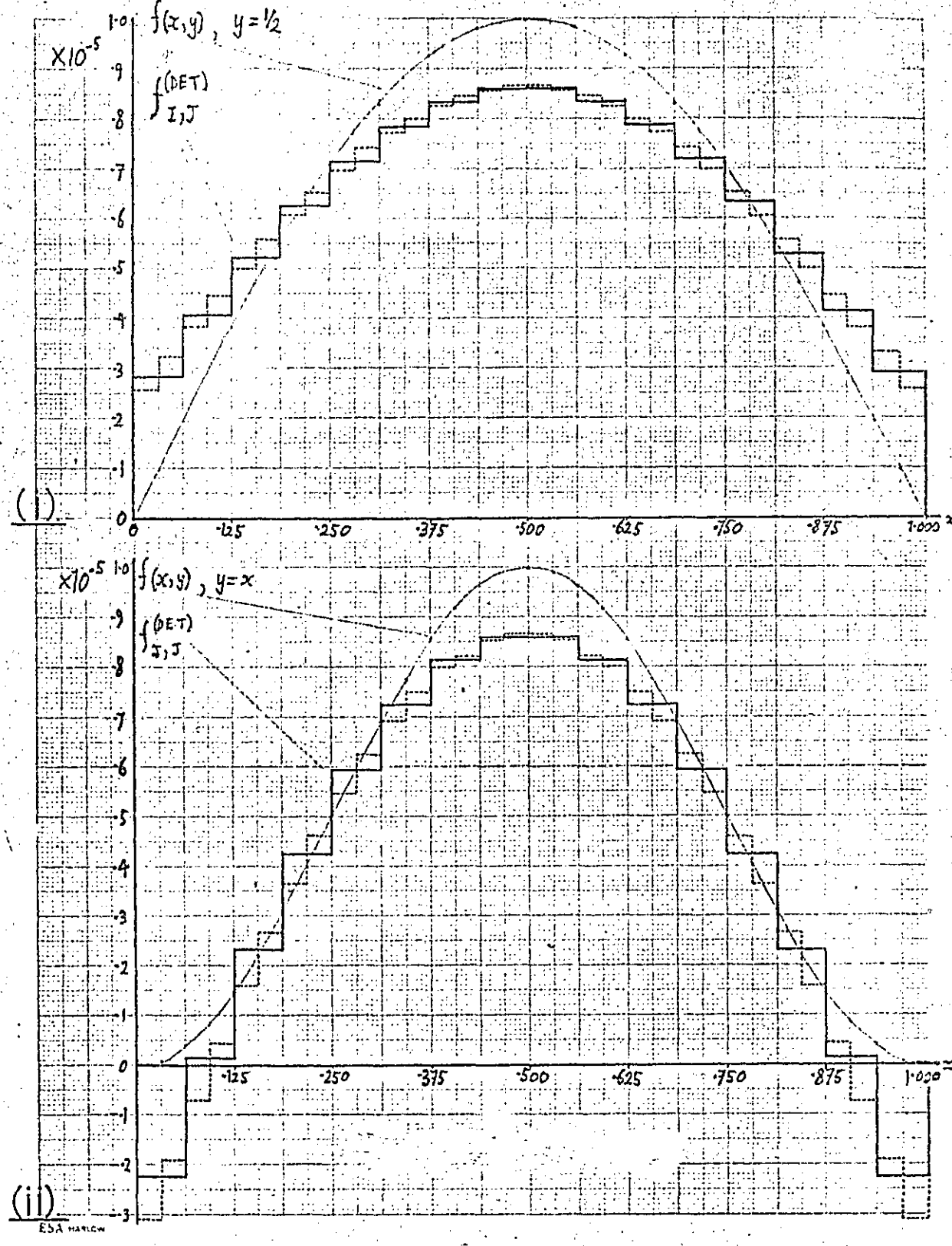
y=x

RNOISE= 0.000 PER CENT
SIGDIF= .96262E-01
DCOUT= .40530E+00
SIGDIF/SIGMA= 33.186 PER CENT
EAV= 7.646 PER CENT
EMAX= 35.666 PER CENT
EMIN= -28.189 PER CENT

Fig(5.5) The reconstruction of the function, $10^{-5}.\sin(\pi x).\sin(\pi y)$, for M = 8 and M = 16.



Fig(5.6) The reconstruction of the function, $10^{-5}.\sin(\pi x).\sin(\pi y)$, for M = 32.



Fig(5.7) Two sections of the function, $10^{-5}.\sin(\pi x).\sin(\pi y)$, and its reconstructions for M = N = 16 and M = N = 32. (The bold steps are for the former and the dotted ones the latter). The indices, I and J, of the reconstructed sections, $f^{(DET)}_{I,J}$, are: (i) $0 \leq I < M, J = M/2 - 1$, (ii) $0 \leq I < M, J = I$.

PRINTOUT MODE IS KO= 15

IN ALL DISTRIBUTIONS BELOW Y AXIS IS HORIZONTAL TO THE RIGHT,X AXIS IS VERTICALLY DOWN,ORIGIN IS TOP LEFT&OR

INPUT MODE KI= 2,PATHS COMPUTED FROM A SMOOTH FUNCTION GIVEN AS A FINITE POWER AND/OR COSINE SERIES
RESOLUTION IS MX= 64 BY MY= 64,DIMENSIONS OF BOUNDARY ARE XM= 1.0000E+00 BY YM= 1.0000E+00 ,METERS

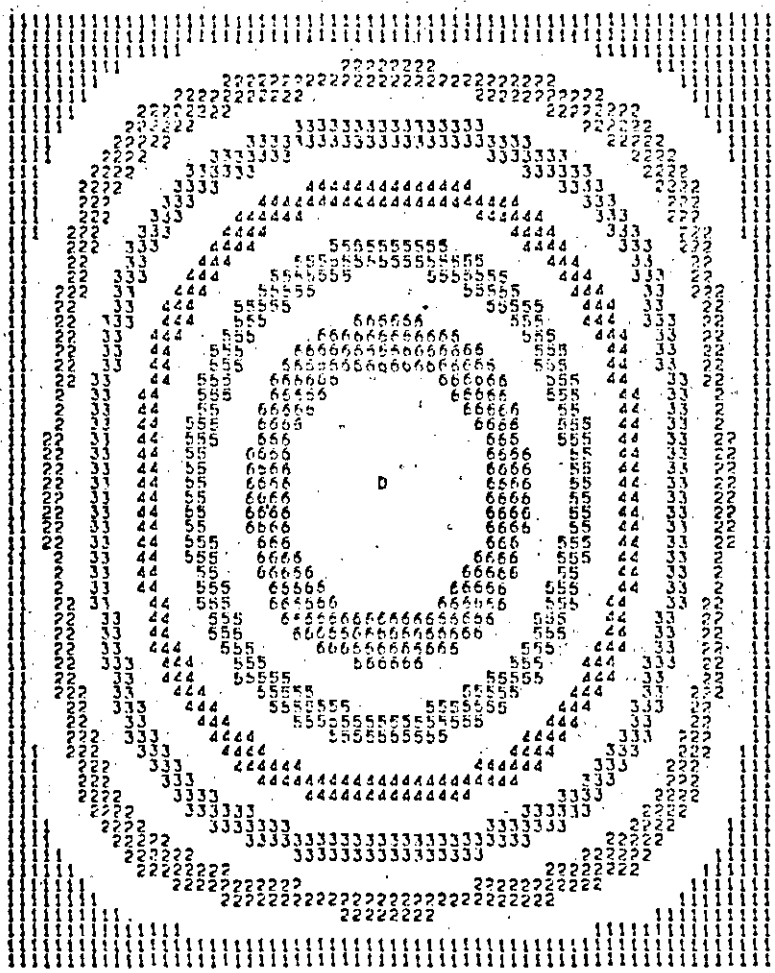
ALL FIGS BELOW EXCEPT ERRORS,SC(I),TC(I),UC(I) AND ANOISE TO BE MULTIPLIED BY P = 1.0000E-05

COS SERIES SUM(I=1 TO 2) OF CC(I)COS(PI2)(SC(I)X+TC(I)Y+UC(I)) IS CC(I),SC(I),TC(I),UC(I)=
5.0000E-01 5.0000E-01 -5.0000E-01 0. , -5.0000E-01 5.0000E-01 5.0000E-01 0.

TRUE REF INDEX DISTRIBUTION OF SMOOTH FUNCTION EXPRESSED AS THE MEAN VALUES OF F(X,Y) IN THE 64 BY 64 CELLS K=

KEY TO THE 24 SYMBOLS IS(VALUE,SYMBOL<VALUE, IF VALUE< -.4997E+00IS SYMBOL IS -,IF VALUE > .1499E+01 SYMBOL

-.499	A	-.415	-.332	B	-.249	-.166	C	-.083	.001	1	.084	.167	2	.250									
.333	3	.417	.500	4	.583	.666	5	.750	.833	6	.916	.999	D	1.082									
1.166	E	1.249	1.332	F	1.415	1.499																	

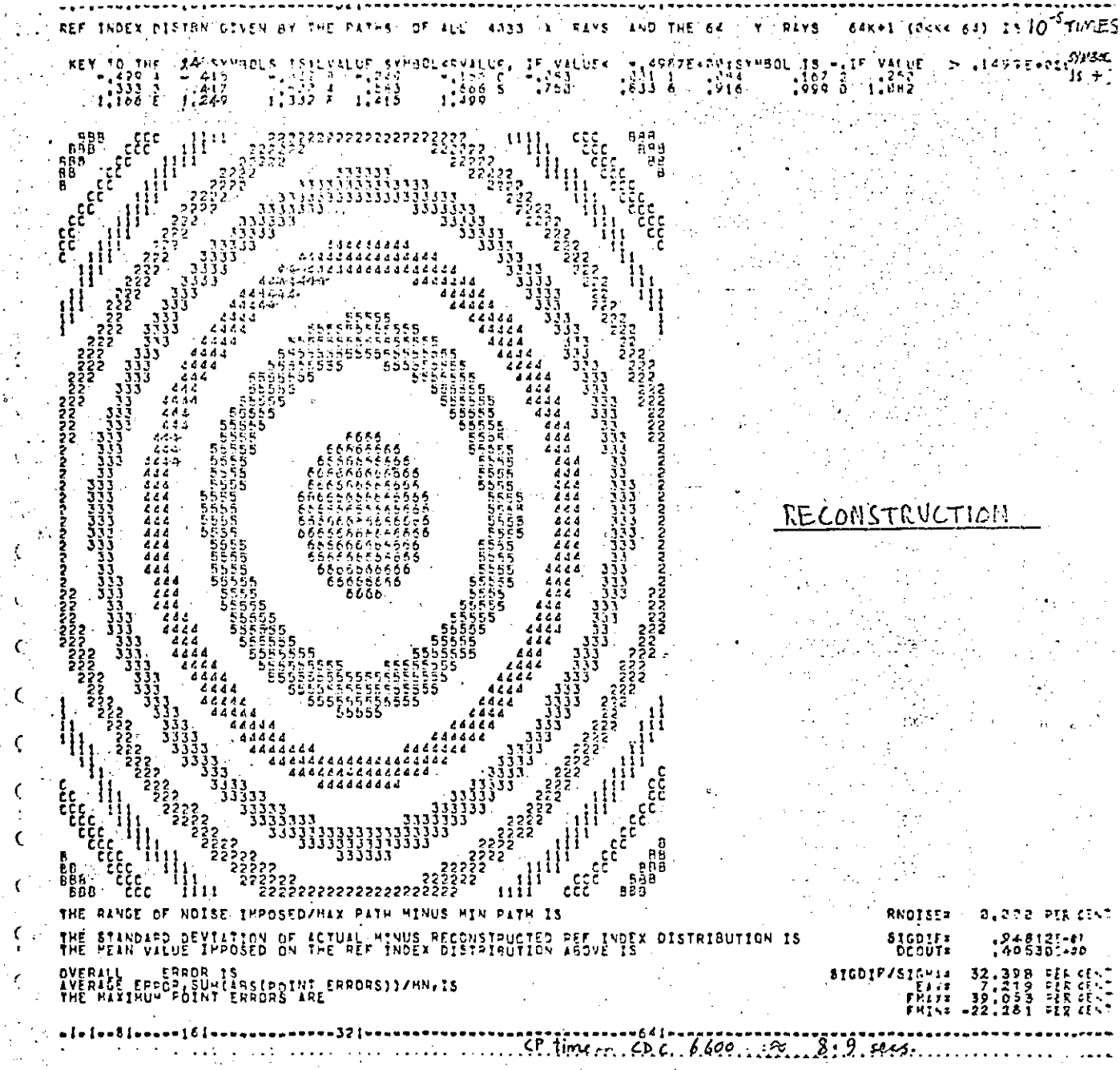


INPUT FUNCTION

MAX VALUE OF REF INDEX DISTRIBUTION IS REFMAX= .9992E+00
MIN VALUE OF REF INDEX DISTRIBUTION IS REFMIN= .6174E-03
REFMAX MINUS REFMIN IS RANGE= .9986E+00
MEAN VALUE OF REF INDEX DISTRIBUTION IS AV= .4352E+00

THE STANDARD DEVIATION OF THE REF INDEX DISTRIBUTION ABOVE,ABOUT MEAN OF ZERO,IS SIGMA= .2926E+00
THE MEAN VALUE IMPOSED ON THE REF INDEX DISTRIBUTION ABOVE IS DC1= .4053E+00

Fig(5.8) The function, $10^{-5}.\sin(\pi x).\sin(\pi y)$, displayed as its mean value in each of the 64×64 resolution cells.



Fig(59) The reconstruction of the function, $10^{-5}.\sin(\pi x).\sin(\pi y)$, for M = N = 64.

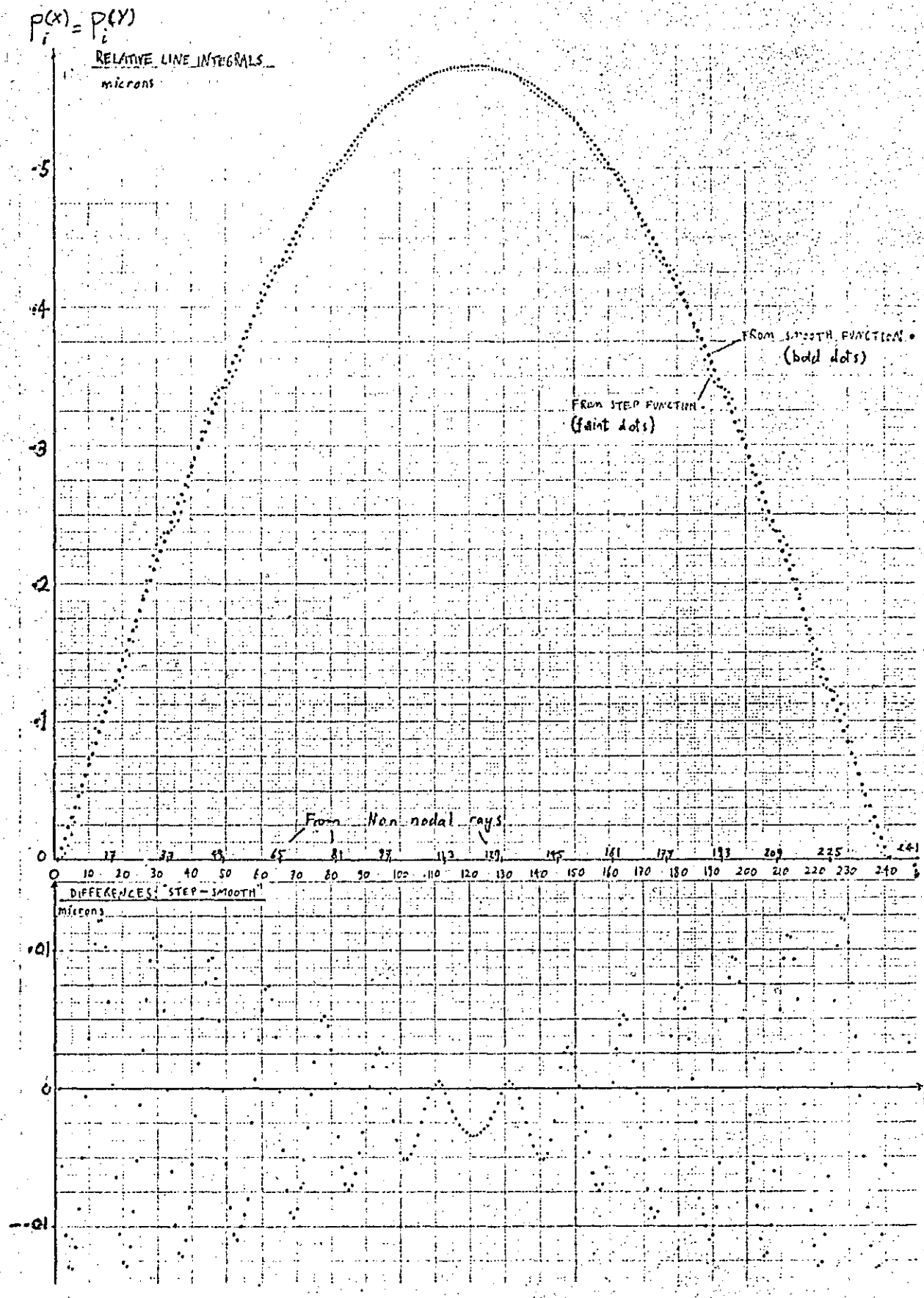
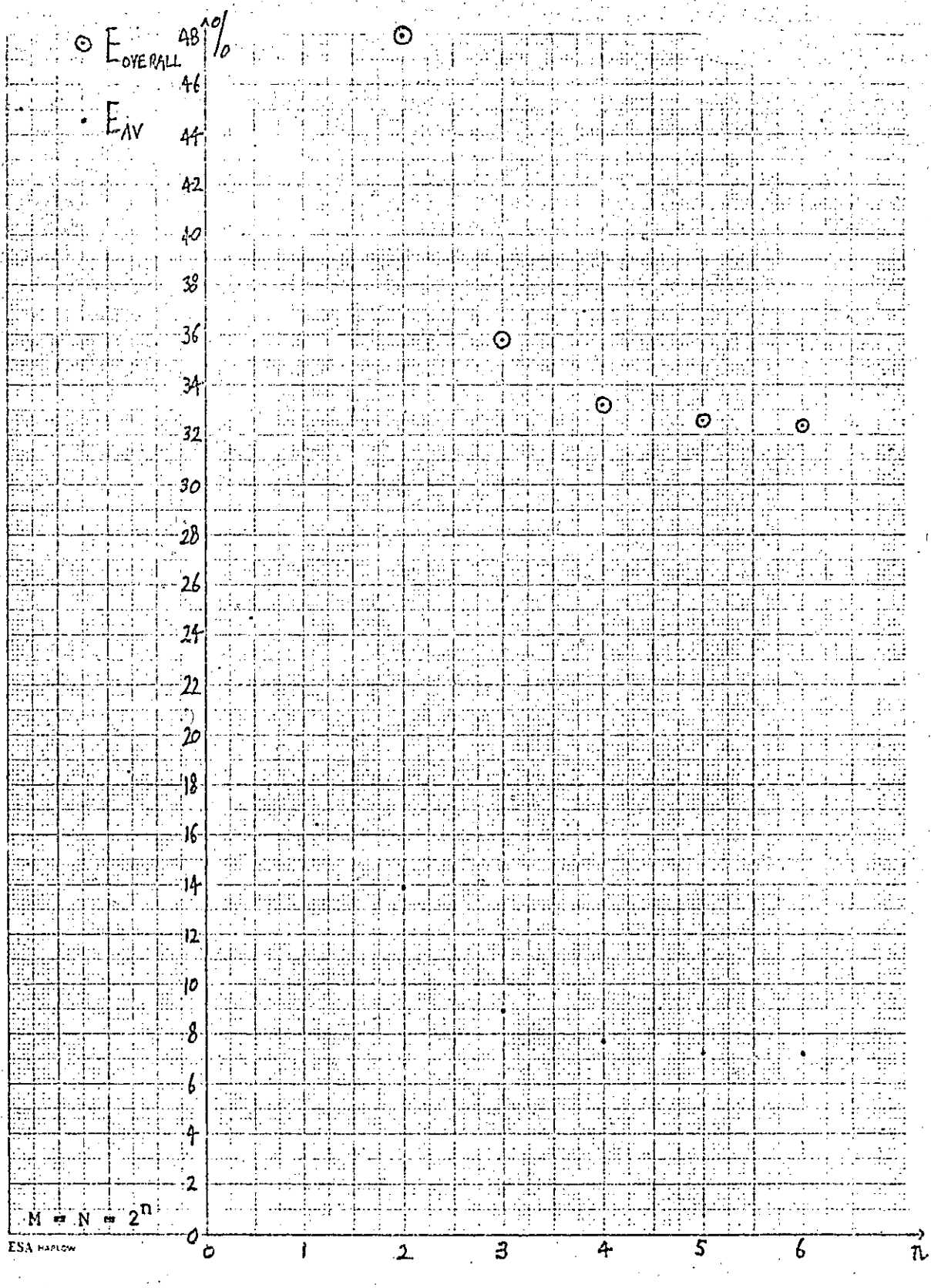
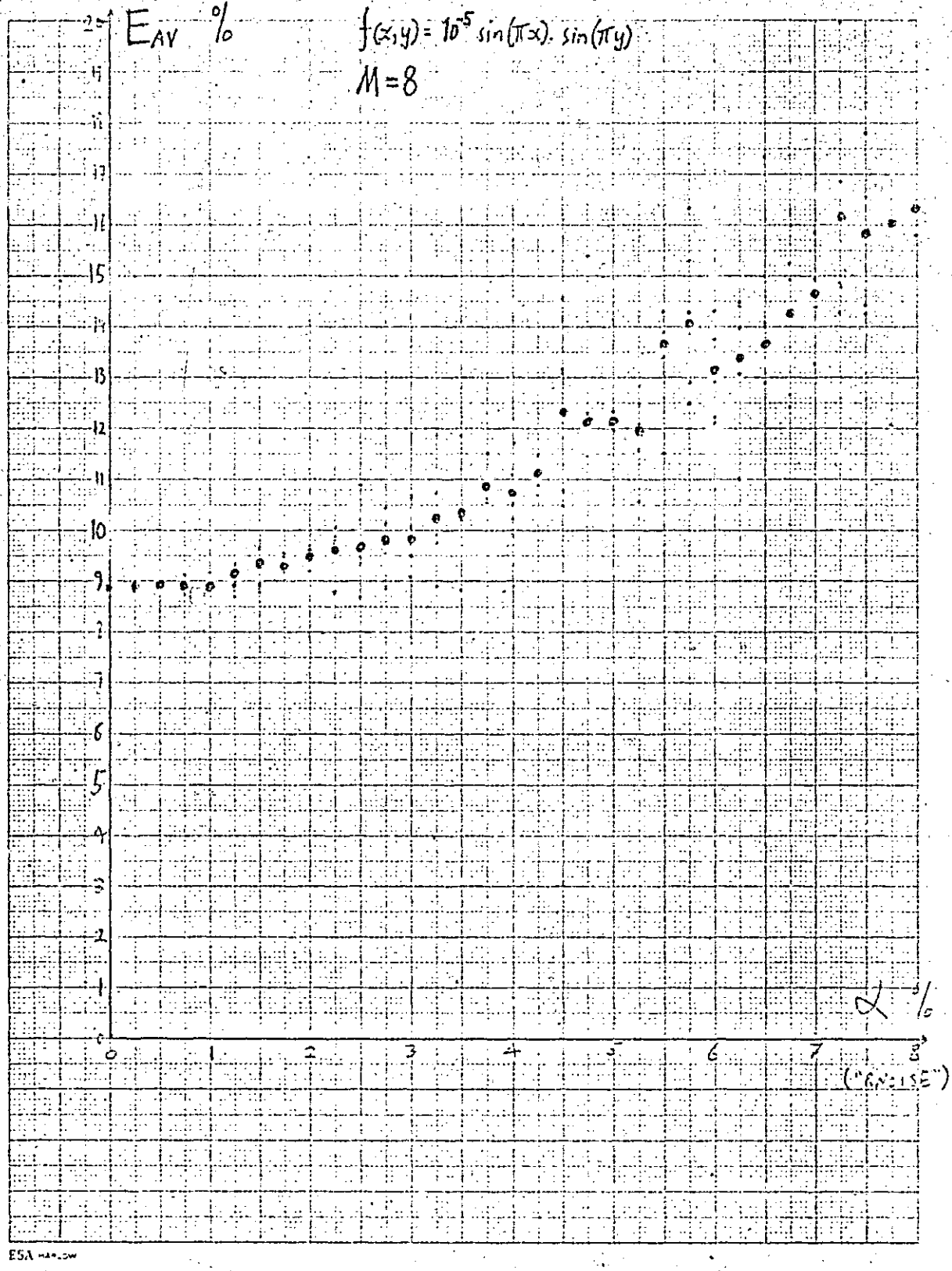


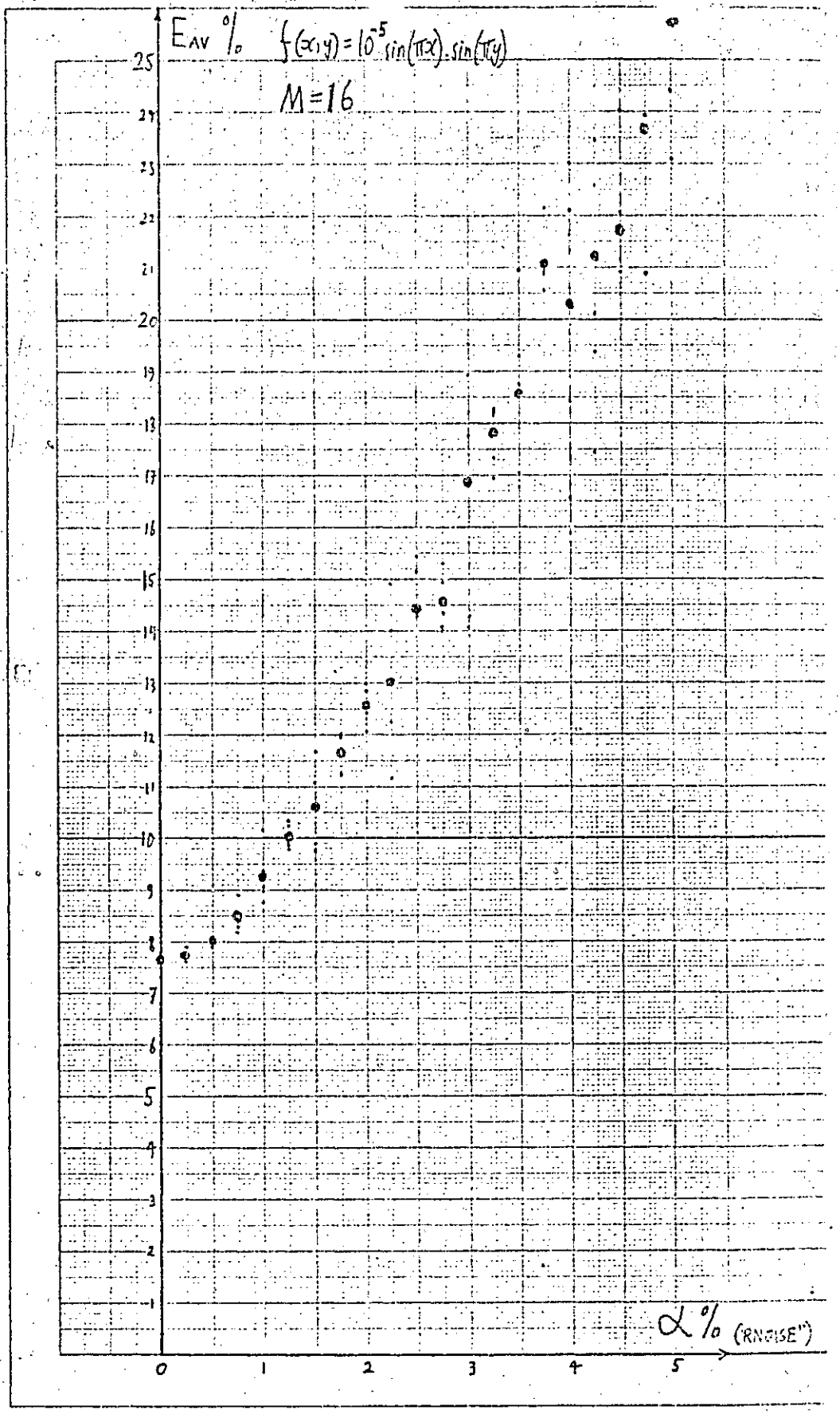
Fig.(5.10) The 241 line integrals, $P_i^{(X)} = P_i^{(Y)}$ for: (a) the function, $10^{-5} \cdot \sin(\pi x) \cdot \sin(\pi y)$, "explored" to a resolution of 16X16 cells and (b) the step function $\langle 10^{-5} \cdot \sin(\pi x) \cdot \sin(\pi y) \rangle_{I,J}$, where M = N = 16. The lower plot shows the differences in the two sets of line integrals.



Fig(511) The average and overall errors in the reconstructions of the function, $10^{-5}.\sin(\pi x).\sin(\pi y)$, for the different values of M = N , 4,8,16,32 and 64.



Fig(5.12) The average reconstruction error, E_{AV}, versus α for five different sets of random noise, where M = 8. (Faint dots). The bold dots are the mean of the five values of E_{AV} for each value of α. The values in each set of random noise are all totally uncorrellated with one another.



Fig(5.13) The average reconstruction error, E_{AV}, versus α for five different sets of random noise, where M = 16. (Faint dots). The bold dots are the mean of the five values of E_{AV} for each value of α. The values in each set of random noise are all totally uncorrellated with one another.

PRINTOUT MODE IS K0= 5

IN ALL DISTRIBUTIONS BELOW Y AXIS IS HORIZONTAL TO THE RIGHT, X AXIS IS VERTICALLY DOWN, ORIGIN IS TOP LEFT HAND CORNER

INPUT MODE K1= 2, PATHS COMPUTED FROM A SMOOTH FUNCTION GIVEN AS A FINITE POWER AND/OR COSINE SERIES
2, DIMENSIONS OF BOUNDARY ARE X= 1.000E+05 BY Y= 1.000E+05 , MICRONS

ALL FIGS BELOW EXCEPT ERRORS, SC(I), TC(I), UC(I) AND RNOISE TO BE MULTIPLIED BY P = 1.00000E-05

COS SERIES SUM(I=1 TO 4) OF CC(I)COS(PI2)(SC(I)X+TC(I)Y+UC(I)) IS CC(I), SC(I), TC(I), UC(I)=

-2.500000E-01	2.000000E+00	0.	0.	-2.500000E-01	0.	2.000000E+00	0.
1.250000E-01	2.000000E+00	-2.000000E+00	0.	1.250000E-01	2.000000E+00	2.000000E+00	0.

$= \sin^2(2\pi x) \cdot \sin^2(2\pi y)$.

M=8

REF INDEX DISTBN GIVEN BY THE PATHS OF ALL 57 X RAYS AND THE 8 Y RAYS 8K+1 (0≤K< 8) IS 10^{-5} TIMES

-.1640	.1132	.3407	.0635	-.1640	.1132	.3407	.0635
-.1690	.4853	.6543	.3381	-.1690	.4853	.6543	.3381
-.1881	.5312	.6353	.2922	-.1881	.5312	.6353	.2922
-.1077	.2492	.2843	-.0725	-.1077	.2492	.2843	-.0725
-.0725	.2843	.2492	-.1077	-.0725	.2843	.2492	-.1077
-.2922	.6353	.5312	-.1881	-.2922	.6353	.5312	-.1881
-.3381	.6543	.4853	-.1690	-.3381	.6543	.4853	-.1690
-.0635	.3407	.1132	-.1640	-.0635	.3407	.1132	-.1640

THE RANGE OF NOISE IMPOSED/MAX PATH MINUS MIN PATH IS — RNOISE= 0.000 PER CENT

THE STANDARD DEVIATION OF ACTUAL MINUS RECONSTRUCTED REF INDEX DISTRIBUTION IS — SIGDIF= .12507E+00
THE MEAN VALUE IMPOSED ON THE REF INDEX DISTRIBUTION ABOVE IS — DCOUT= .25020E+00

OVERALL ERROR IS — SIGDIF/SIGMA= 50.670 PER CENT
AVERAGE ERROR, SUM(ABS(POINT ERRORS))/MN, IS — FAV= 16.713 PER CENT
THE MAXIMUM POINT ERRORS ARE — FMAX= 30.950 PER CENT
FMIN= -30.166 PER CENT

M=16

REF INDEX DISTBN GIVEN BY THE PATHS OF ALL 241 X RAYS AND THE 16 Y RAYS 16K+1 (0≤K< 16) IS 10^{-5} TIMES

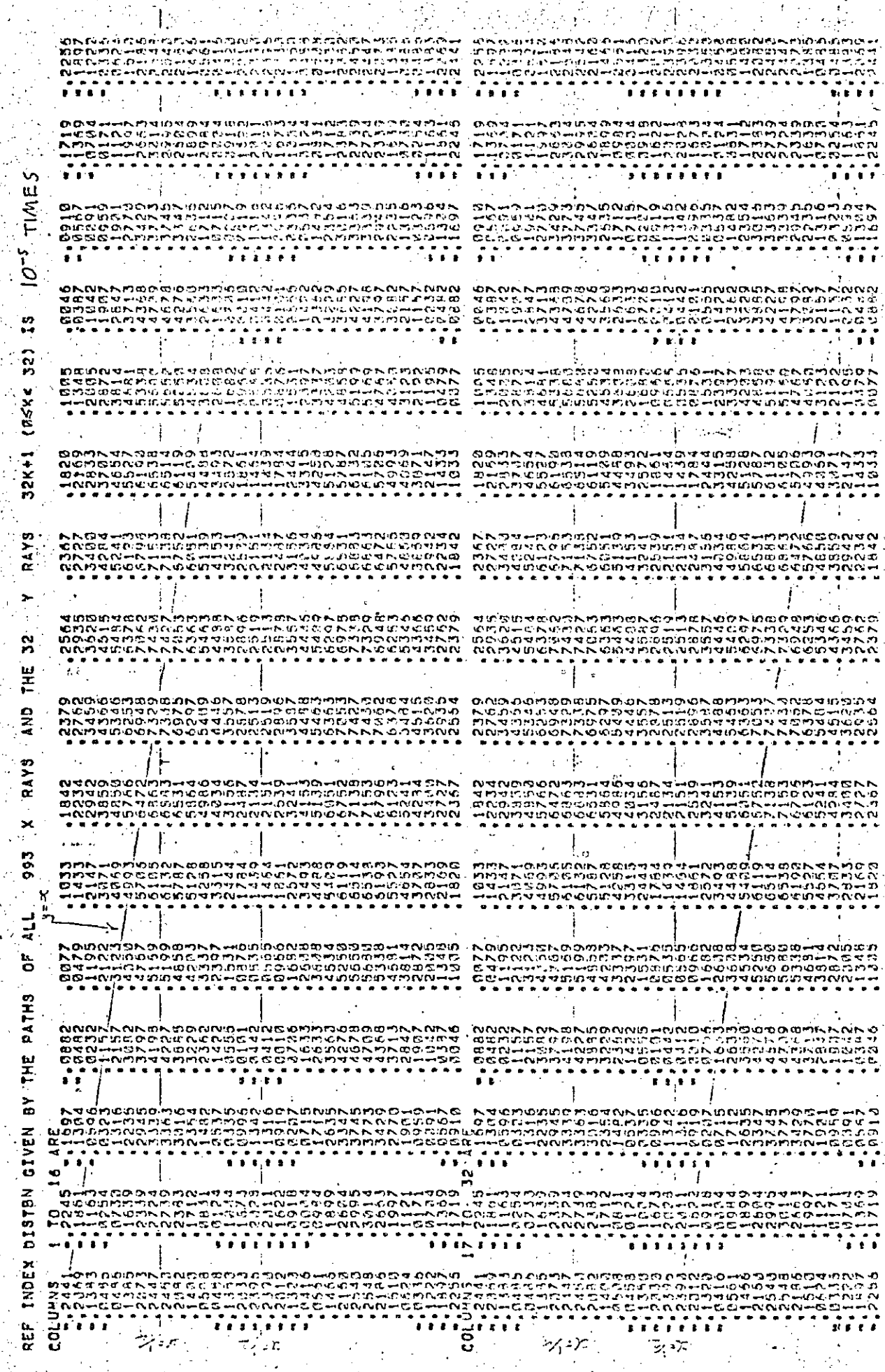
-.2268	-.1490	.0302	.2058	.2749	.1971	.0179	-.1577	-.2268	-.1490	.0302	.2058	.2749	.1971	.0179	-.1577
-.0939	-.0075	.1749	.3464	.4067	.3203	.1379	-.0336	-.0939	-.0075	.1749	.3464	.4067	.3203	.1379	-.0336
-.0945	.1892	.3743	.5414	.5927	.4988	.3129	.1458	-.0945	.1892	.3743	.5414	.5927	.4988	.3129	.1458
-.2287	.3315	.5189	.6811	.7232	.6285	.4331	.2708	-.2287	.3315	.5189	.6811	.7232	.6285	.4331	.2708
-.2311	.3417	.5309	.6880	.7208	.6102	.4210	.2639	-.2311	.3417	.5309	.6880	.7208	.6102	.4210	.2639
-.1017	.2199	.4105	.5619	.5855	.4673	.2767	.1252	-.1017	.2199	.4105	.5619	.5855	.4673	.2767	.1252
-.0819	.0436	.2351	.3626	.3947	.2692	.0777	-.0678	-.0819	.0436	.2351	.3626	.3947	.2692	.0777	-.0678
-.2054	-.0777	.1144	.2535	.2582	.1257	-.0663	-.2054	-.2054	-.0777	.1144	.2535	.2582	.1257	-.0663	-.2054
-.2054	-.0663	.1257	.2582	.2535	.1144	-.0777	-.2101	-.2054	-.0663	.1257	.2582	.2535	.1144	-.0777	-.2101
-.0678	-.0777	.2692	.3947	.3626	.2351	.0436	-.0519	-.0678	-.0777	.2692	.3947	.3626	.2351	.0436	-.0819
-.1252	-.2767	.4673	.5855	.5619	.4105	.2199	-.1017	-.1252	-.2767	.4673	.5855	.5619	.4105	.2199	-.1017
-.2639	-.4210	.6102	.7208	.6880	.5309	.3417	-.2311	-.2639	-.4210	.6102	.7208	.6880	.5309	.3417	-.2311
-.2708	-.4331	.6285	.7232	.6811	.5189	.3315	-.2287	-.2708	-.4331	.6285	.7232	.6811	.5189	.3315	-.2287
-.1458	-.3129	.4988	.5927	.5414	.3743	.1892	-.0945	-.1458	-.3129	.4988	.5927	.5414	.3743	.1892	-.0945
-.0336	-.1379	.3203	.4067	.3464	.1749	-.0075	-.0939	-.0336	-.1379	.3203	.4067	.3464	.1749	-.0075	-.0939
-.1577	-.0179	.1971	.2749	.2058	.0302	-.1490	-.2268	-.1577	-.0179	.1971	.2749	.2058	.0302	-.1490	-.2268

THE RANGE OF NOISE IMPOSED/MAX PATH MINUS MIN PATH IS — RNOISE= 0.000 PER CENT

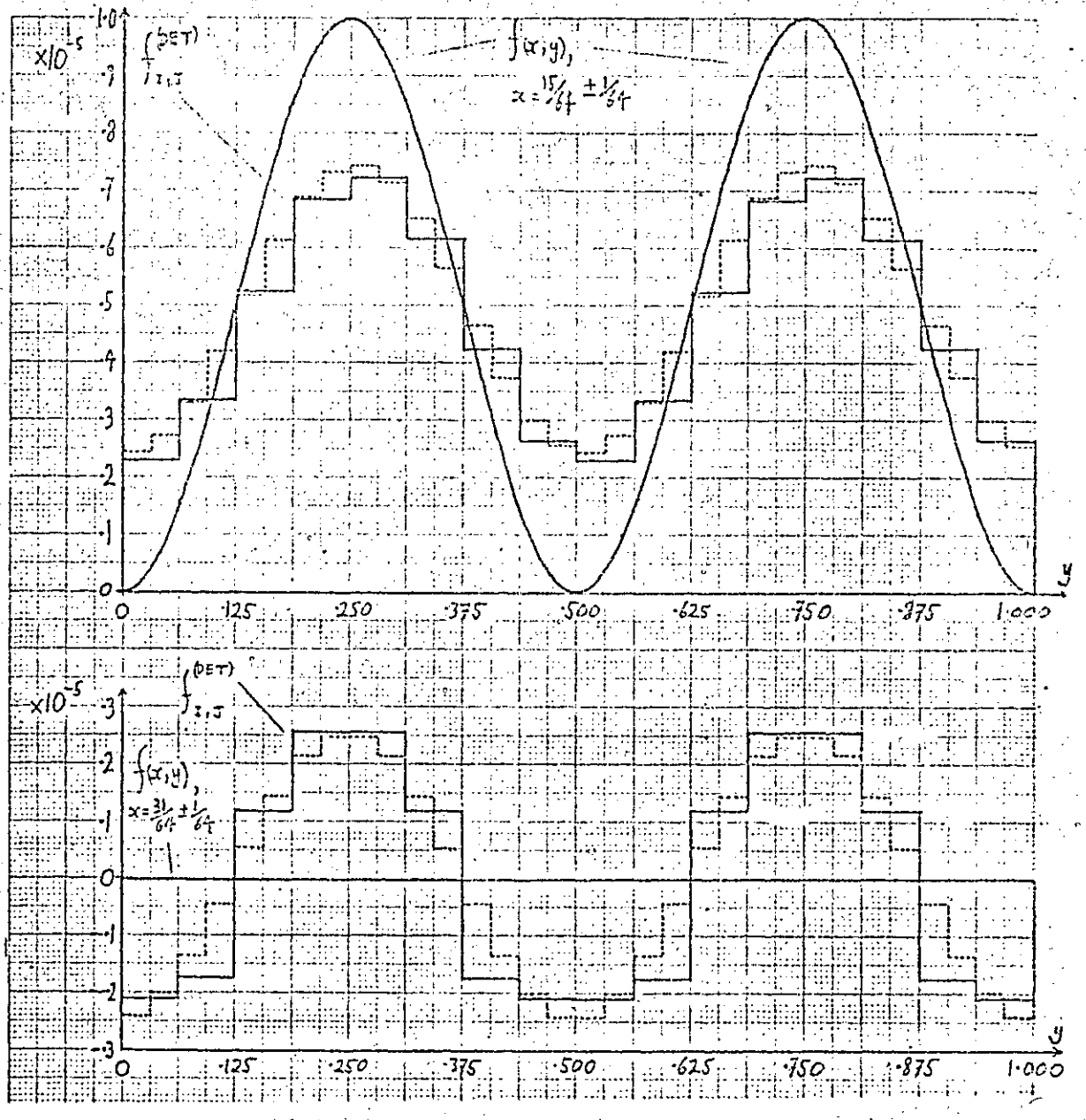
THE STANDARD DEVIATION OF ACTUAL MINUS RECONSTRUCTED REF INDEX DISTRIBUTION IS — SIGDIF= .12430E+00
THE MEAN VALUE IMPOSED ON THE REF INDEX DISTRIBUTION ABOVE IS — DCOUT= .25000E+00

OVERALL ERROR IS — SIGDIF/SIGMA= 45.866 PER CENT
AVERAGE ERROR, SUM(ABS(POINT ERRORS))/MN, IS — FAV= 11.651 PER CENT
THE MAXIMUM POINT ERRORS ARE — FMAX= 25.477 PER CENT
FMIN= -29.276 PER CENT

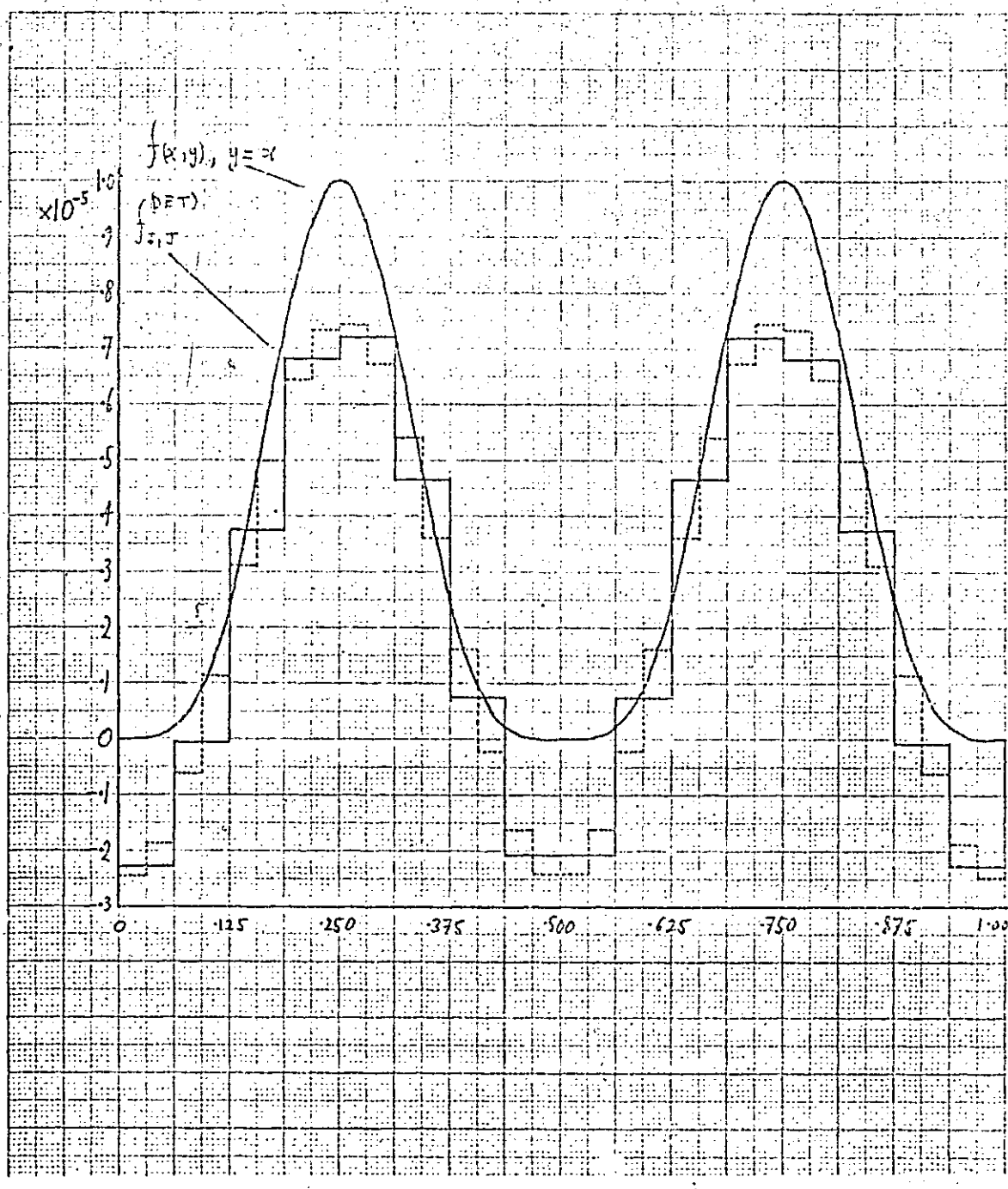
Fig(5.14) The reconstruction of the function, $10^{-5} \cdot \sin^2(2\pi x) \cdot \sin^2(2\pi y)$, for M = N = 8 and M = N = 16.



Fig(5.15) The reconstruction of the function, $10^{-5}.\sin^2(2\pi x).\sin^2(2\pi y)$, for N = N = 32.



Fig(5.16) Two sections of the function, $10^{-5}.\sin^2(2\pi x).\sin^2(2\pi y)$, and its reconstruction for M = 16 and M = 32. (The bold steps are for the former and the dotted ones the latter). The indices, I and J, of the reconstructed sections, $f^{(DET)}_{I,J}$, are: (i) $I = M/4 - 1, 0 \leq J < M$, (ii) $I = M/2 - 1, 0 \leq J < M$.

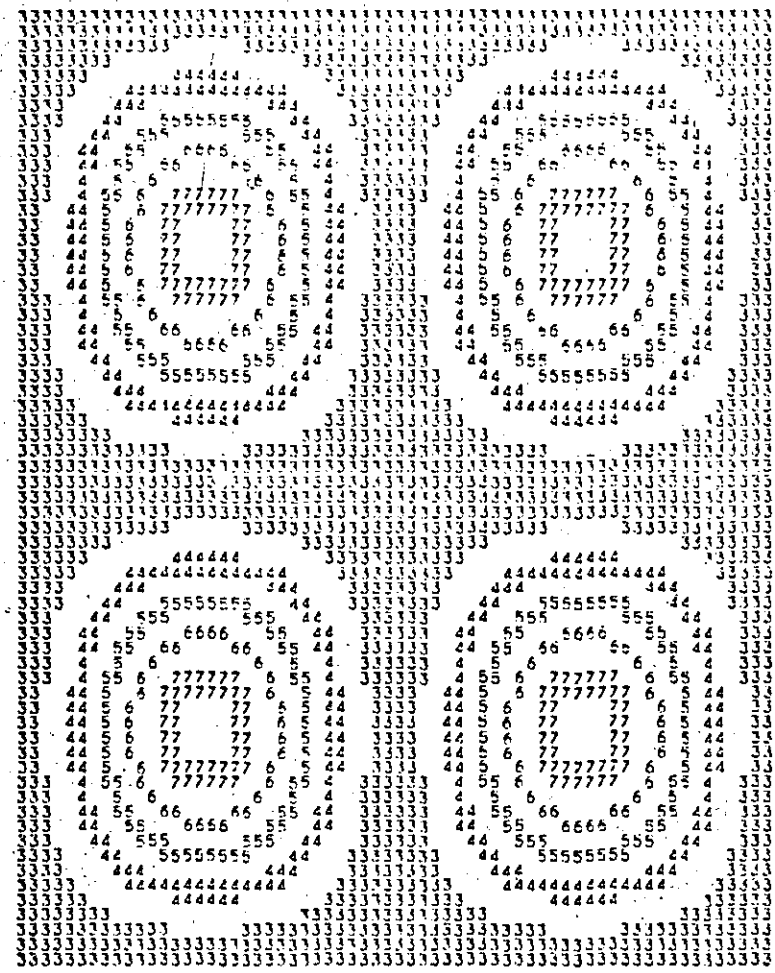


Fig(5.17) A section of the function, $10^{-5}.\sin^2(2\pi x).\sin^2(2\pi y)$, and its reconstruction for M = 16 and M = 32. (The bold steps are for the former and the dotted ones the latter). The indices, I and J, of the reconstructed section, $f^{(DET)}_{I,J}$, are: $I = J, 0 \leq J < M$.

TRUE REF INDEX DISTRIBUTION OF SMOOTH FUNCTION EXPRESSED AS THE MEAN VALUES OF F(X,Y) IN THE 64 BY 64 CELLS

KEY TO THE 10 SYMBOLS TELLS VALUES. SYMBOL 0 IF VALUE < -.1490E+01; SYMBOL 1 IF VALUE > .1490E+01; SYMBOLS 1 TO 9 IF VALUE IS IN THE RANGE -.1490E+01 TO .1490E+01

$10^{-5}\times$	-.497 1	-.386 2	-.276 3	-.166 4	-.055 5	.055 6	.166 7	.276 8	.386 9	.497
	-.607 6	-.718 7	-.828 8	-.938 9	-1.049 8	-1.159 7	-1.270 6	-1.380 5	-1.490 4	

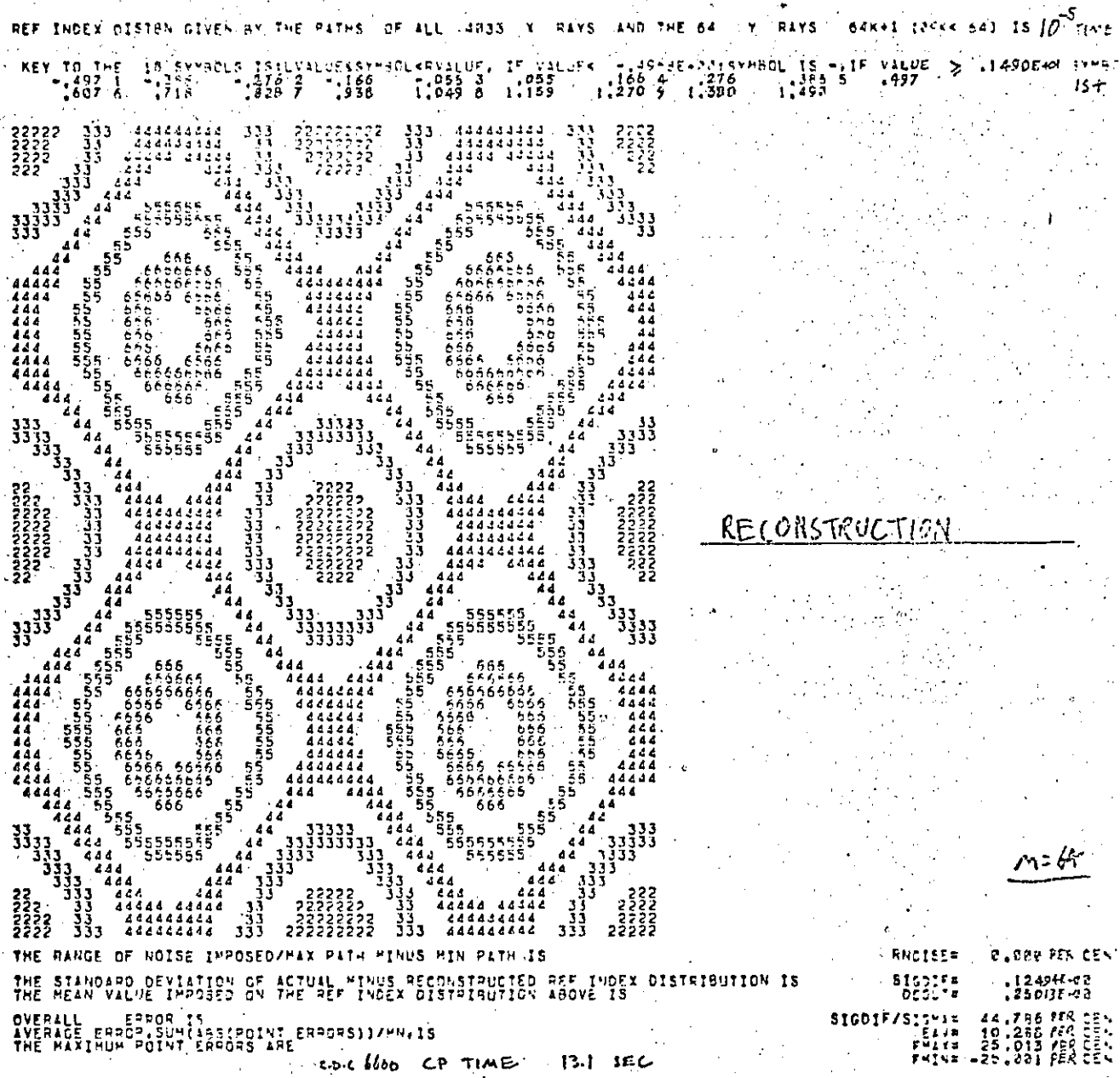


INPUT FUNCTION

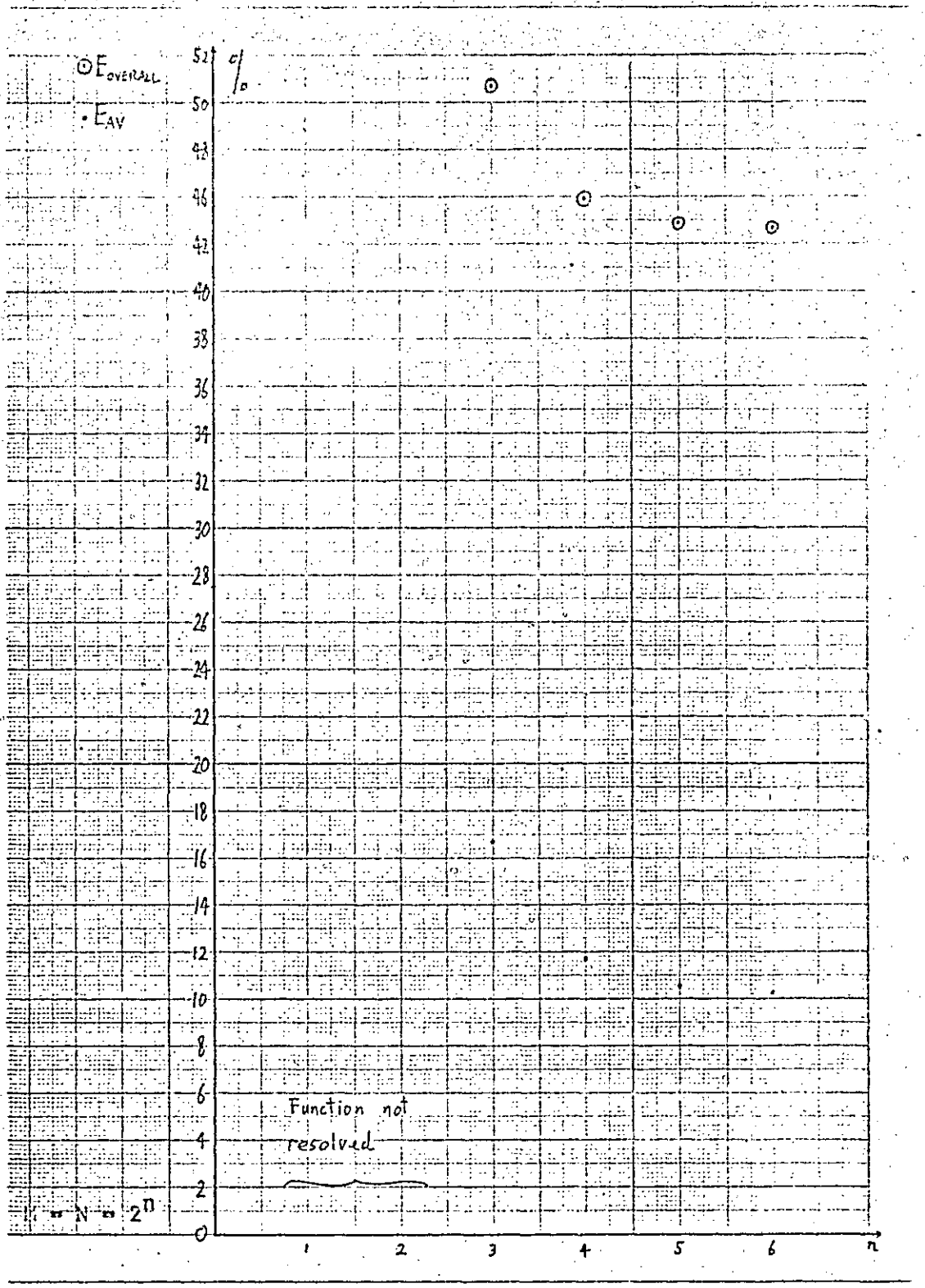
M=14

MAX VALUE OF REF INDEX DISTRIBUTION IS	REFMAX=	.9936E+01
MIN VALUE OF REF INDEX DISTRIBUTION IS	REFMIN=	.1028E-04
REFMAX MINUS REFMIN IS	RANGE=	.9935E+00
MEAN VALUE OF REF INDEX DISTRIBUTION IS	AVG=	.2500E+00
THE STANDARD DEVIATION OF THE REF INDEX DISTRIBUTION ABOVE, ABOUT MEAN OF ZERO, IS	SIGMA=	.2789E+00
THE MEAN VALUE IMPOSED ON THE REF INDEX DISTRIBUTION ABOVE IS	DCIN=	.2500E+00

Fig(5.18). The function, $10^{-5}.\sin^2(2\pi x).\sin^2(2\pi y)$, displayed as its mean value in each of the 64×64 resolution cells.



Fig(5.19) The reconstruction of the function, $10^{-5}.\sin^2(2\pi x).\sin^2(2\pi y)$, for N = N = 64.



Fig(5.20) The average and overall errors in the reconstruction of the function, $10^{-5}.\sin^2(2\pi x).\sin^2(2\pi y)$, for N = 8, 16, 32 and 64.

$30(x^3-x^2)(y^3-2y^2+y)$
$+\frac{1}{2}\sin^2(\pi x)\cdot\sin^2(\pi y)$

ALL FIGS BELOW EXCEPT ERRORS,SC(I),TC(I),UC(I) AND RNOISE TO BE MULTIPLIED BY P = 1.000000E-05

THE 28 COEFFS C(I) IN THE TAYLORS SERIES SUM(K=1 TO 7)(L=1 TO K) OF C(I)(X TO POWER L-1)(Y TO POWER K-L)
WHERE I=K(K-1)/2+L,ARE

1.2500E-01	-0.	-0.	-6.0000E+01	-3.0000E+01	-0.	-0.	-0.	-0.	-0.	-0.	-0.	3.0000E+01	6.0000E+01
-0.	-0.	-0.	-0.	-0.	-3.0000E+01	-0.	-0.	-0.	-0.	-0.	-0.		

COS SERIES SUM(I=1 TO 4) OF CC(I)COS(PI2)(SC(I)X+TC(I)Y+UC(I)) IS CC(I),SC(I),TC(I),UC(I)=

-1.2500000E-01	2.0000000E+00	0.	0.	-1.2500000E-01	0.	1.0000000E+00	0.
6.2500000E-02	2.0000000E+00	-1.0000000E+00	0.	6.2500000E-02	2.0000000E+00	1.0000000E+00	0.

TRUE REF INDEX DISTRIBUTION OF SMOOTH FUNCTION EXPRESSED AS THE MEAN VALUES OF F(X,Y) IN THE 4 BY 4 RESOLUTION RECTANGL

.0897	.2765	.2489	.0540
.2737	.5760	.4329	.0897
.4169	.8059	.5760	.1175
.2737	.5760	.4329	.0897

INPUT FUNCTION

MAX VALUE OF REF INDEX DISTRIBUTION IS — REFMAX= .82887E+00
MIN VALUE OF REF INDEX DISTRIBUTION IS — REFMIN= .54715E-01
REFMAX MINUS REFMIN IS — RANGE= .75416E+00
MEAN VALUE OF REF INDEX DISTRIBUTION IS — AV= .33333E+00

THE STANDARD DEVIATION OF THE REF INDEX DISTRIBUTION ABOVE,ABOUT MEAN OF ZERO,IS — SIGMA= .21647E+00
THE MEAN VALUE IMPOSED ON THE REF INDEX DISTRIBUTION ABOVE IS — DCIN= .33333E+00

-|--|--81-----161----------321----------641----------

REF INDEX DISTBN GIVEN BY THE PATHS OF ALL 13 X RAYS AND THE 4 Y RAYS 4K+1 (0≤K< 4) IS

.1237	.4537	.4413	.0572
.2165	.6166	.4858	.0794
.5067	.7515	.5135	.1253
.4636	.5755	.2499	-.0796

FIRST RECONSTRUCTION

THE RANGE OF NOISE IMPOSED/MAX PATH MINUS MIN PATH IS — RNOISE= 0.000 PER CENT

THE STANDARD DEVIATION OF ACTUAL MINUS RECONSTRUCTED REF INDEX DISTRIBUTION IS — SIGDIF= .12128E+00
THE MEAN VALUE IMPOSED ON THE REF INDEX DISTRIBUTION ABOVE IS — DCOUT= .33333E+00

OVERALL ERROR IS — SIGDIF/SIGMA= 56.025 PER CENT
AVERAGE ERROR,SUM(ABS(POINT ERRORS))/MN,IS — EAVE= 12.461 PER CENT
THE MAXIMUM POINT ERRORS ARE — EMAX= 28.273 PER CENT
EMIN= -25.496 PER CENT

-|--|--81-----161----------321----------641----------

REF INDEX DISTBN GIVEN BY THE PATHS OF ALL 13 Y RAYS AND THE 4 X RAYS 4K+1 (0≤K< 4) IS

.1272	.4165	.2920	-.0072
.2221	.6052	.4548	.1161
.3313	.7286	.5956	.2414
.3824	.6469	.3482	-.1681

SECOND RECONSTRUCTION

THE RANGE OF NOISE IMPOSED/MAX PATH MINUS MIN PATH IS — RNOISE= 0.000 PER CENT

THE STANDARD DEVIATION OF ACTUAL MINUS RECONSTRUCTED REF INDEX DISTRIBUTION IS — SIGDIF= .97224E-01
THE MEAN VALUE IMPOSED ON THE REF INDEX DISTRIBUTION ABOVE IS — DCOUT= .33333E+00

OVERALL ERROR IS — SIGDIF/SIGMA= 44.914 PER CENT
AVERAGE ERROR,SUM(ABS(POINT ERRORS))/MN,IS — EAVE= 10.285 PER CENT
THE MAXIMUM POINT ERRORS ARE — EMAX= 34.155 PER CENT
EMIN= -18.533 PER CENT

Fig(5.21). The function, $10^{-5}\cdot(30(x^2-x^3)(y^3-2y^2+y)+\frac{1}{2}\sin^2(2\pi x)\cdot\sin^2(\pi y))$, and its two reconstructions for M = N = 4.

REF INDEX DISTBN GIVEN BY THE PATHS OF ALL 57 X RAYS AND THE 8 Y RAYS BK+1 (0≤K≤ 8) IS 10^{-5} TIMES

-.2105	.1896	.3338	.4012	.3621	.2178	.0174	-.1586
-.0771	.2728	.4075	.4631	.4058	.2549	.0529	-.1141
.0933	.3871	.5132	.5564	.4889	.3243	.1220	-.0348
.2463	.4934	.6188	.6411	.5597	.3864	.1850	.0398
.3717	.5839	.6923	.7093	.6143	.4332	.2343	.1016
.4420	.6299	.7281	.7314	.6232	.4355	.2402	.1212
.4091	.5778	.6670	.6563	.5354	.3423	.1522	.0477
.2457	.4088	.4794	.4546	.3216	.1243	-.0595	-.1483

FIRST RECONSTRUCTION

THE RANGE OF NOISE IMPOSED/MAX PATH MINUS MIN PATH IS RNOISE= 0.000 PER CENT

THE STANDARD DEVIATION OF ACTUAL MINUS RECONSTRUCTED REF INDEX DISTRIBUTION IS SIGDIF= .13572E+00
THE MEAN VALUE IMPOSED ON THE REF INDEX DISTRIBUTION ABOVE IS DCOUT= .33333E+00

OVERALL ERROR IS SIGDIF/SIGMA= 50.882 PER CENT
AVERAGE ERROR,SUM(ABS(POINT ERRORS))/MN,IS EAV= 11.557 PER CENT
THE MAXIMUM POINT ERRORS ARE EMAX= 26.448 PER CENT
EMIN= -29.921 PER CENT

-1-1--81-----161-----------321-----------------------641----------

REF INDEX DISTBN GIVEN BY THE PATHS OF ALL 57 Y RAYS AND THE 8 X RAYS BK+1 (0≤K≤ 8) IS

.0218	.2483	.3503	.3731	.2933	.1232	-.0636	-.1584
.0282	.2854	.4095	.4445	.3699	.2088	.0263	-.0948
.1162	.3830	.5382	.5433	.4685	.3068	.1234	.0010
.2187	.4759	.5926	.6331	.5558	.3935	.2084	.0644
.3023	.5556	.6770	.7060	.6287	.4628	.2729	.1441
.3466	.6036	.7179	.7410	.6528	.4762	.2764	.1358
.3201	.5700	.6706	.6792	.5761	.3843	.1689	.0127
.1667	.4145	.4983	.4912	.3719	.1638	-.0679	-.2404

SECOND RECONSTRUCTION

THE RANGE OF NOISE IMPOSED/MAX PATH MINUS MIN PATH IS RNOISE= 0.000 PER CENT

THE STANDARD DEVIATION OF ACTUAL MINUS RECONSTRUCTED REF INDEX DISTRIBUTION IS SIGDIF= .12904E+00
THE MEAN VALUE IMPOSED ON THE REF INDEX DISTRIBUTION ABOVE IS DCOUT= .33333E+00

OVERALL ERROR IS SIGDIF/SIGMA= 48.378 PER CENT
AVERAGE ERROR,SUM(ABS(POINT ERRORS))/MN,IS EAV= 10.929 PER CENT
THE MAXIMUM POINT ERRORS ARE EMAX= 25.563 PER CENT
EMIN= -27.074 PER CENT

Fig(5.22) The two reconstructions of the function,
$10^{-5}\cdot(30(x^2-x^3)(y^3-2y^2+y)+\frac{1}{2}\sin^2(2\pi x)\cdot\sin^2(\pi y))$, for M = N = 8.

```
REF INDEX DISTBN GIVEN BY THE PATHS OF ALL   241 X RAYS AND THE 16   Y RAYS   16K+1 (0≤K≤ 16) IS .10**-5 TIMES
```



```
THE RANGE OF NOISE IMPOSED/MAX PATH MINUS MIN PATH IS                    RNOISE=    0.000 PER CENT

THE STANDARD DEVIATION OF ACTUAL MINUS RECONSTRUCTED REF INDEX DISTRIBUTION IS    SIGDIF=   .13265E+00
THE MEAN VALUE IMPOSED ON THE REF INDEX DISTRIBUTION ABOVE IS                     OCOUT=   .33333E+00

OVERALL     ERROR IS                                          SIGDIF/SIGMA=  50.369 PER CENT
AVERAGE ERROR,SUM(ABS(POINT ERRORS))/MN,IS                          EAV=  10.922 PER CENT
THE MAXIMUM POINT ERRORS ARE                                       FMAX=  31.567 PER CENT
                                                                   FMIN= -32.341 PER CENT
```

```
-1-1--8|-----16|----------32|------------------------64|------------------------------------------------------128|---
```

```
REF INDEX DISTBN GIVEN BY THE PATHS OF ALL   241 Y RAYS AND THE 16   X RAYS   16K+1 (0≤K≤ 16) IS .10**-5 TIMES
```



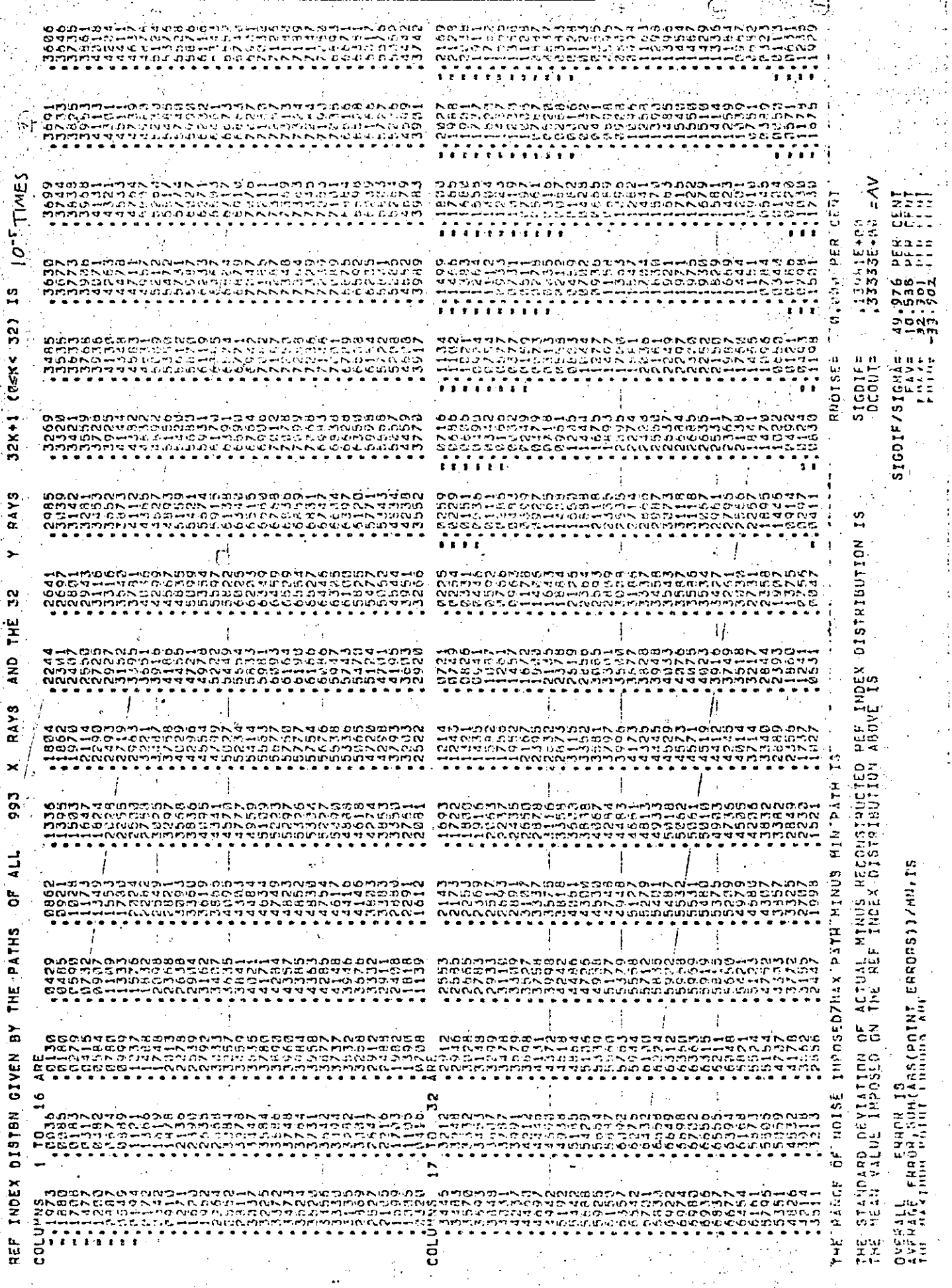
```
THE RANGE OF NOISE IMPOSED/MAX PATH MINUS MIN PATH IS                    RNOISE=    0.000 PER CENT

THE STANDARD DEVIATION OF ACTUAL MINUS RECONSTRUCTED REF INDEX DISTRIBUTION IS    SIGDIF=   .13912E+00
THE MEAN VALUE IMPOSED ON THE REF INDEX DISTRIBUTION ABOVE IS                     OCOUT=   .33333E+00

OVERALL     ERROR IS                                          SIGDIF/SIGMA=  49.841 PER CENT
AVERAGE ERROR,SUM(ABS(POINT ERRORS))/MN,IS                          EAV=  10.836 PER CENT
THE MAXIMUM POINT ERRORS ARE                                       FMAX=  30.981 PER CENT
                                                                   FMIN= -31.999 PER CENT
```

Fig(5.23). The two reconstructions of the function $10^{-5}.(30(x^2 - x^3)(y^3 - 2y^2+y) + \frac{1}{2}\sin^2(2\pi x).\sin^2(\pi y))$, for $M = N = 16$.

M=32 FIRST RECONSTRUCTION



Fig(5.24) The first reconstruction of the function, $10^{-5}.(30(x^2 - x^3)(y^3 - 2y^2 + y) + \frac{1}{2}\sin^2(2\pi x).\sin^2(\pi y))$, for M = N = 32.

Fig(5.25) The second reconstruction of the function, $10^{-5}.(30(x^2 - x)(y^3 - 2y^2 + y) + \frac{1}{2}\sin^2(2\pi x).\sin^2(\pi y))$, for M = N = 32.

M=32 SECOND RECONSTRUCTION.

REF INDEX DISTBN GIVEN BY THE PATHS OF ALL 993 Y RAYS AND THE 32 X RAYS 32K+1 (0≤K< 32) IS 10^{-6} TIMES

COLUMNS 1 TO 16 ARE

-.0643	.0090	.0263	.0632	.1085	.1556	.2009	.2427	.2798	.3112	.3363	.3544	.3653	.3685	.3640	.3518
-.0663	.0082	.0265	.0641	.1101	.1577	.2034	.2456	.2830	.3147	.3401	.3595	.3695	.3730	.3686	.3557
-.0589	.0157	.0341	.0719	.1180	.1657	.2116	.2539	.2914	.3233	.3488	.3673	.3785	.3822	.3778	.3650
-.0476	.0272	.0457	.0835	.1298	.1776	.2236	.2661	.3037	.3357	.3613	.3799	.3912	.3948	.3908	.3790
-.0327	.0421	.0607	.0987	.1451	.1930	.2391	.2816	.3194	.3514	.3771	.3958	.4072	.4110	.4078	.3953
-.0149	.0600	.0788	.1168	.1632	.2112	.2574	.3000	.3379	.3700	.3958	.4146	.4260	.4299	.4262	.4144
.0055	.0805	.0992	.1374	.1839	.2319	.2782	.3209	.3588	.3910	.4168	.4357	.4472	.4511	.4473	.4358
.0279	.1029	.1217	.1599	.2065	.2546	.3009	.3436	.3816	.4138	.4397	.4587	.4702	.4742	.4704	.4589
.0518	.1259	.1458	.1840	.2306	.2787	.3251	.3679	.4058	.4381	.4640	.4830	.4946	.4985	.4949	.4835
.0768	.1519	.1708	.2091	.2557	.3039	.3503	.3931	.4311	.4634	.4893	.5083	.5200	.5240	.5202	.5088
.1025	.1776	.1965	.2348	.2814	.3296	.3760	.4189	.4568	.4892	.5151	.5341	.5458	.5498	.5462	.5347
.1284	.2035	.2224	.2607	.3073	.3555	.4019	.4447	.4827	.5150	.5410	.5600	.5716	.5756	.5719	.5605
.1540	.2291	.2480	.2863	.3329	.3810	.4274	.4702	.5082	.5405	.5664	.5854	.5970	.6010	.5973	.5859
.1790	.2541	.2729	.3112	.3577	.4059	.4522	.4954	.5329	.5652	.5911	.6101	.6216	.6255	.6218	.6104
.2029	.2779	.2967	.3349	.3815	.4295	.4759	.5186	.5565	.5887	.6146	.6335	.6450	[illegible]	.6451	.6336
.2252	.3032	.3193	.3571	.4036	.4516	.4979	.5405	.5784	.6106	.6364	.6552	.6667	[illegible]	[illegible]	.6551
.2455	.3205	.3392	.3773	.4237	.4717	.5179	.5604	.5992	.6303	.6563	.6749	.6862	[illegible]	[illegible]	.6744
.2535	.3384	.3570	.3950	.4413	.4892	.5353	.5778	.6155	.6475	.6732	.6918	.7032	[illegible]	[illegible]	.6911
.2707	.3534	.3719	.4098	.4560	.5038	.5498	.5922	.6298	.6617	.6873	.7058	.7170	[illegible]	[illegible]	.7047
.2784	.3653	.3834	.4212	.4673	.5150	.5609	.6031	.6406	.6724	.6978	.7163	.7274	[illegible]	[illegible]	.7115
.2983	.3728	.3911	.4287	.4747	.5222	.5680	.6101	.6475	.6792	.7044	.7228	.7347	.7371	[illegible]	[illegible]
.3319	.3762	.3944	.4318	.4777	.5251	.5707	.6127	.6499	.6814	.7065	.7247	.7365	.7387	[illegible]	[illegible]
.3336	.3748	.3928	.4301	.4758	.5230	.5685	.6103	.6474	.6787	.7037	.7217	.7334	.7354	.7327	.7133
.2941	.3681	.3859	.4231	.4686	.5156	.5609	.6025	.6394	.6706	.6954	.7132	.7237	.7256	.7217	.7031
.2818	.3556	.3732	.4102	.4555	.5024	.5474	.5889	.6256	.6566	.6812	.6978	.7091	.7118	.7067	.6919
.2632	.3369	.3543	.3910	.4361	.4828	.5277	.5690	.6054	.6362	.6605	.6760	.6881	.6905	.6853	[illegible]
.2380	.3114	.3286	.3652	.4101	.4565	.5012	.5422	.5785	.6090	.6332	.6504	.6603	.6625	.6573	.6438
.2056	.2788	.2958	.3321	.3758	.4230	.4674	.5083	.5443	.5746	.5985	.6155	.6251	.6271	.6214	[illegible]
.1657	.2387	.2554	.2915	.3359	.3819	.4261	.4666	.5024	.5325	.5562	.5729	.5823	.5840	.5783	.5514
.1178	.1905	.2070	.2428	.2870	.3327	.3766	.4169	.4528	.4823	.5057	.5222	.5313	.5328	.5265	[illegible]
.0615	.1340	.1502	.1857	.2296	.2750	.3187	.3587	.3940	.4235	.4467	.4629	.4717	.4729	.4664	.4522
-.0036	.0685	.0844	.1197	.1633	.2084	.2518	.2916	.3265	.3558	.3787	.3946	.4031	.4040	.3972	.3827

COLUMNS 17 TO 32 ARE

.3323	.3057	.2729	.2345	.1916	.1452	.0967	.0472	-.0017	-.0488	-.0925	-.1317	-.1651	-.1914	-.2099	-.2196
.3373	.3109	.2783	.2400	.1973	.1511	.1026	.0533	.0045	-.0424	-.0860	-.1250	-.1582	-.1845	-.2028	-.2124
.3467	.3205	.2879	.2498	.2072	.1611	.1128	.0636	.0150	-.0318	-.0753	-.1143	-.1474	-.1735	-.1917	-.2011
.3599	.3338	.3013	.2633	.2208	.1748	.1266	.0775	.0290	-.0177	-.0611	-.0999	-.1329	-.1589	-.1770	-.1864
.3763	.3503	.3179	.2800	.2375	.1917	.1436	.0946	.0461	-.0005	-.0438	-.0825	-.1154	-.1414	-.1594	-.1686
.3954	.3695	.3372	.2994	.2570	.2112	.1632	.1143	.0659	.0194	-.0239	-.0626	-.0954	-.1212	-.1392	-.1483
.4169	.3910	.3588	.3210	.2787	.2329	.1850	.1361	.0878	.0413	-.0019	-.0404	-.0732	-.0990	-.1169	-.1260
.4401	.4143	.3821	.3444	.3021	.2564	.2085	.1597	.1114	.0650	.0218	-.0167	-.0494	-.0752	-.0930	-.1021
.4646	.4388	.4067	.3690	.3268	.2811	.2332	.1845	.1362	.0908	.0467	.0082	-.0245	-.0503	-.0681	-.0771
.4900	.4643	.4321	.3945	.3523	.3066	.2587	.2100	.1617	.1154	.0723	.0338	.0011	-.0246	-.0424	-.0515
.5159	.4901	.4580	.4203	.3781	.3325	.2845	.2359	.1876	.1412	.0981	.0596	.0270	.0012	-.0165	-.0256
.5417	.5159	.4838	.4461	.4039	.3582	.3104	.2616	.2133	.1670	.1239	.0853	.0527	.0269	.0091	[illegible]
.5670	.5412	.5091	.4714	.4292	.3835	.3356	.2868	.2385	.1921	.1490	.1104	.0777	.0520	.0342	.0251
.5915	.5657	.5335	.4958	.4535	.4078	.3599	.3110	.2627	.2163	.1731	.1345	.1018	.0766	.0581	.0498
.6137	.5888	.5566	.5188	.4765	.4307	.3827	.3339	.2855	.2390	.1958	.1572	.1244	.0955	.0805	.0714
.6351	.6102	.5779	.5401	.4977	.4518	.4038	.3549	.3064	.2599	.2166	.1779	.1450	.1191	.1011	.0919
.6554	.6293	.5970	.5591	.5166	.4707	.4226	.3736	.3250	.2784	.2350	.1963	.1633	.1373	.1192	.1099
.6719	.6458	.6134	.5754	.5328	.4869	.4386	.3895	.3409	.2942	.2507	.2118	.1788	.1527	.1345	.1231
.6854	.6592	.6266	.5885	.5459	.4998	.4514	.4022	.3535	.3067	.2631	.2241	.1910	.1648	.1465	.1359
.6953	.6689	.6363	.5980	.5553	.5090	.4606	.4112	.3624	.3154	.2717	.2328	.1994	.1730	.1546	.1453
.7011	.6740	.6418	.6034	.5605	.5141	.4655	.4161	.3671	.3200	.2762	.2369	.2035	.1770	.1585	.1487
.7023	.6757	.6427	.6042	.5611	.5146	.4659	.4162	.3671	.3198	.2759	.2365	.2029	.1762	.1575	.1476
.6985	.6717	.6385	.5999	.5566	.5099	.4610	.4112	.3619	.3145	.2703	.2308	.1970	.1702	.1513	.1412
.6891	.6621	.6288	.5899	.5465	.4997	.4506	.4006	.3511	.3035	.2591	.2194	.1855	.1585	.1394	.1291
.6737	.6460	.6130	.5740	.5304	.4833	.4340	.3838	.3342	.2864	.2418	.2019	.1677	.1405	.1213	.1108
.6519	.6245	.5918	.5515	.5077	.4604	.4109	.3605	.3106	.2625	.2179	.1777	.1434	.1160	.0965	.0857
.6232	.5956	.5616	.5221	.4781	.4306	.3809	.3302	.2801	.2319	.1869	.1465	.1119	.0843	.0646	.0536
.5871	.5593	.5251	.4854	.4411	.3933	.3434	.2925	.2421	.1937	.1484	.1078	.0730	.0451	.0252	.0140
.5432	.5152	.4837	.4408	.3962	.3482	.2983	.2469	.1963	.1476	.1021	.0612	.0261	-.0020	-.0222	-.0337
.4912	.4629	.4282	.3879	.3431	.2949	.2444	.1930	.1421	.0931	.0474	.0062	-.0291	-.0575	-.0780	-.0897
.4305	.4019	.3669	.3264	.2813	.2328	.1820	.1304	.0792	.0299	-.0161	-.0575	-.0931	-.1218	-.1426	-.1546
.3608	.3319	.2966	.2559	.2104	.1616	.1105	.0586	.0071	-.0424	-.0888	-.1305	-.1664	-.1954	-.2164	-.2287

THE RANGE OF NOISE IMPOSED/MAX PATH MINUS MIN PATH IS RNOISE= 0.000 PER CENT

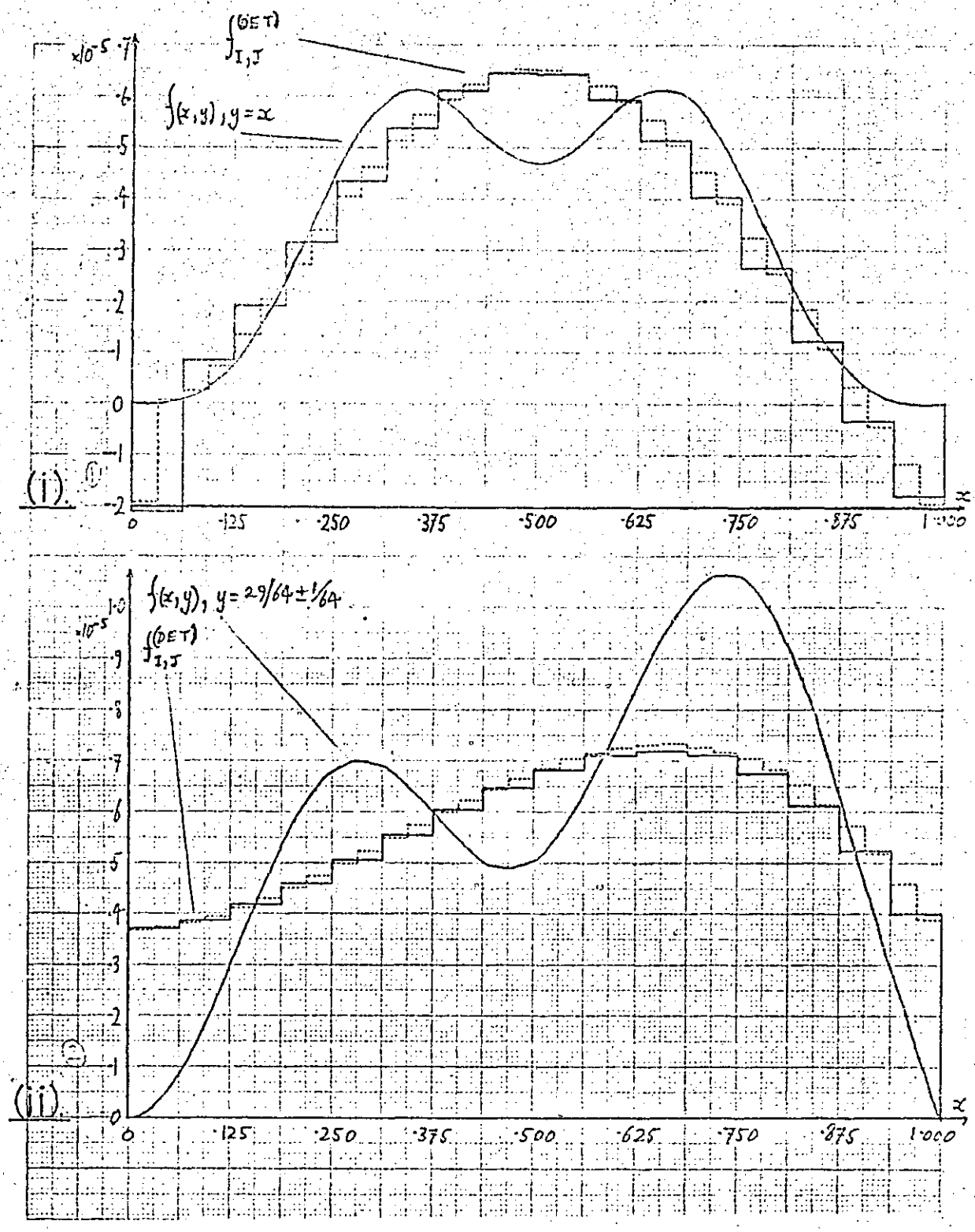
THE STANDARD DEVIATION OF ACTUAL MINUS RECONSTRUCTED REF INDEX DISTRIBUTION IS SIGDIF= .13954E+00

THE MEAN VALUE IMPOSED ON THE REF INDEX DISTRIBUTION ABOVE IS DCOUT= .33333E+00 =AV

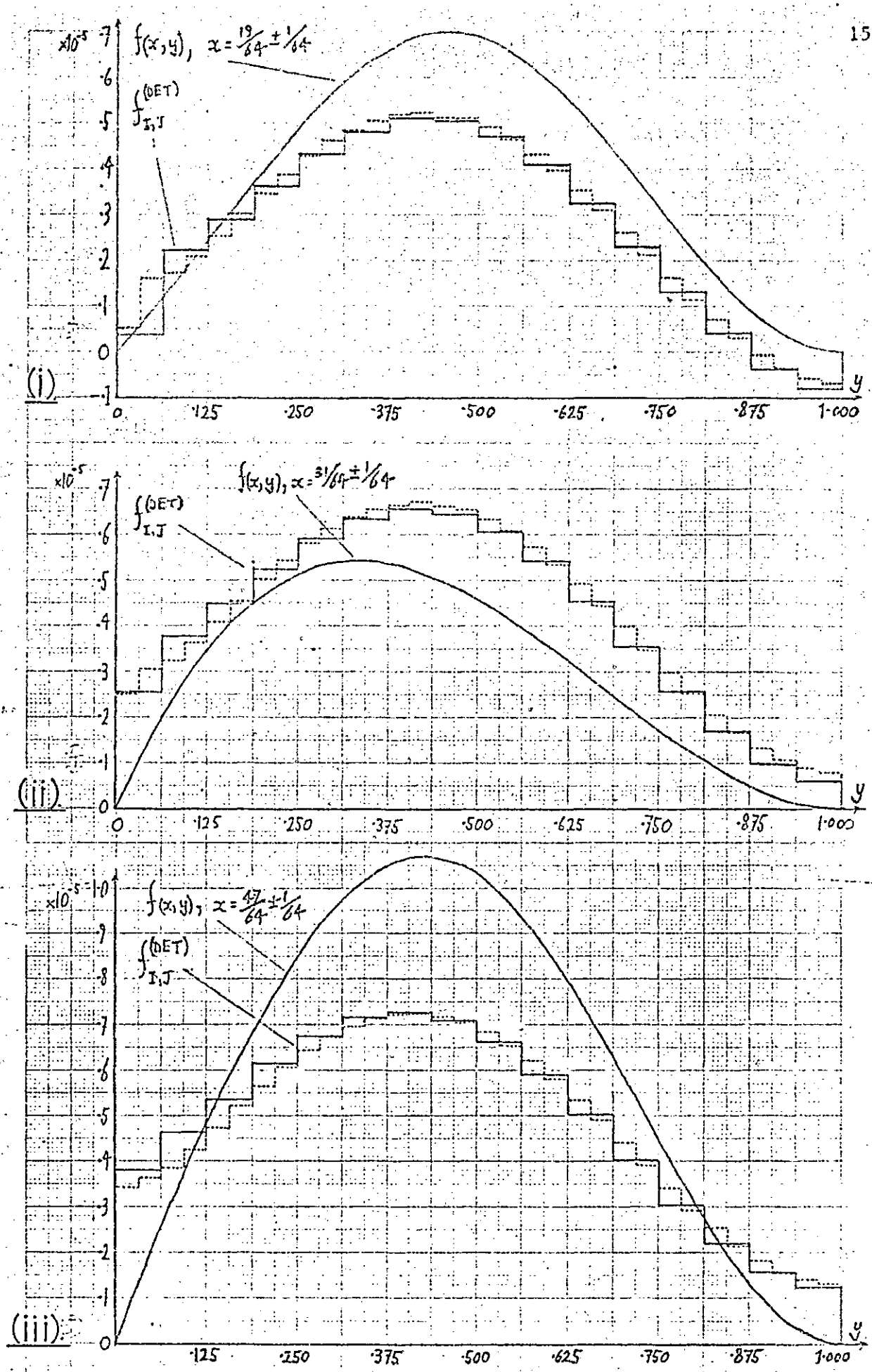
OVERALL ERROR IS SIGDIF/SIGMA= 49.880 PER CENT

AVERAGE ERROR,SUM(ABS(POINT ERROR))/MN,IS FAV= 10.544 PER CENT

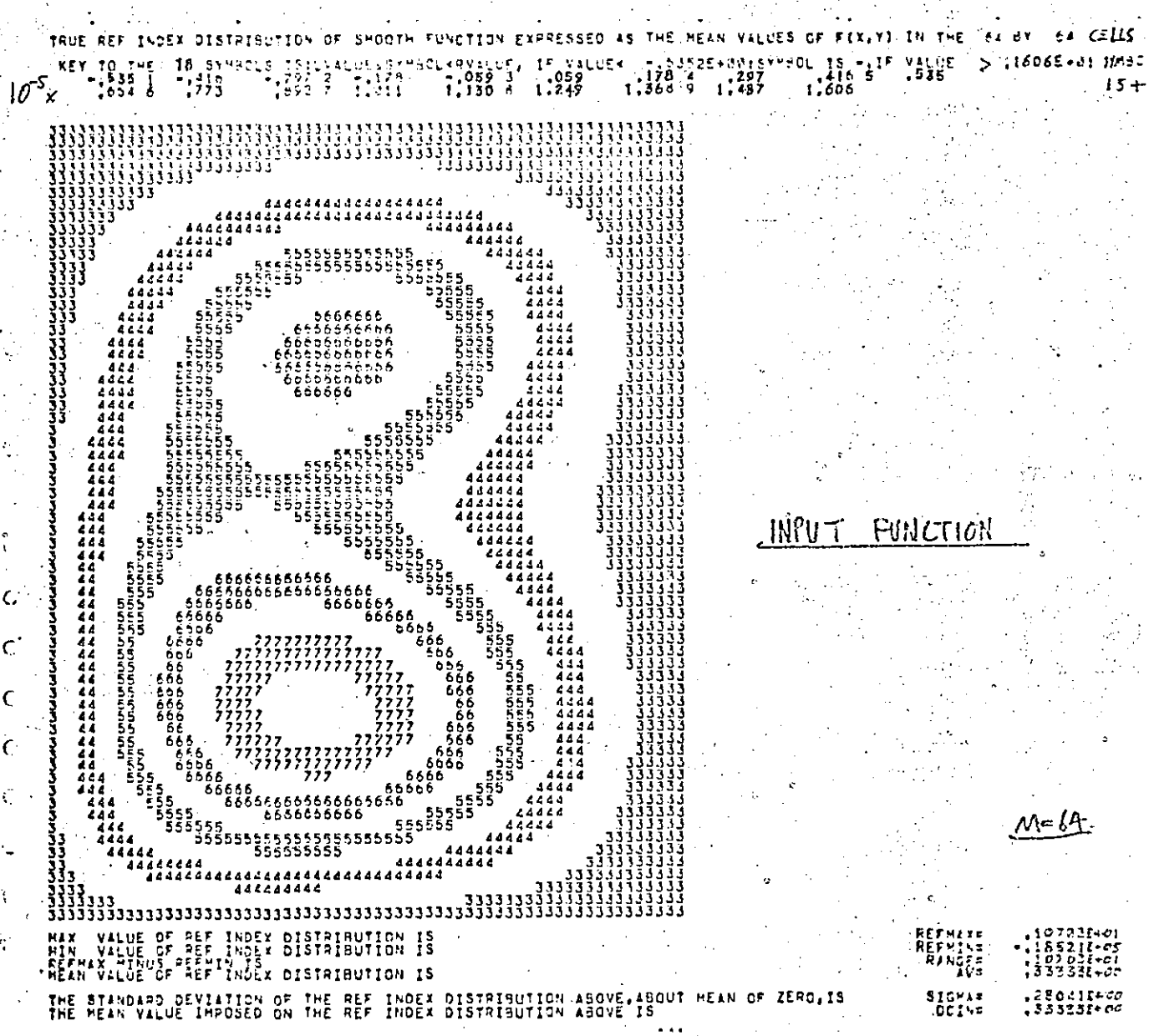
THE MAXIMUM POINT ERRORS ARE FMAX= 32.062 PER CENT FMIN= -33.771 PER CENT



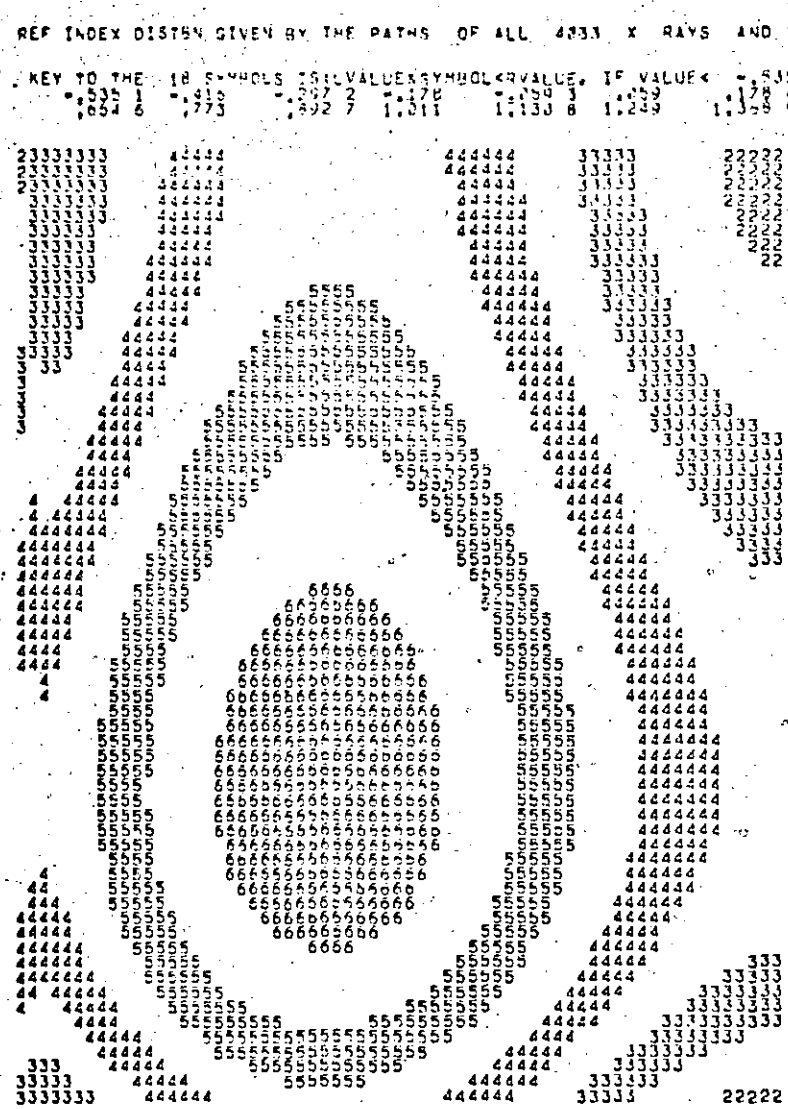
Fig(5.26). Two sections of the function, $10^{-5}(30(x^2 - x^3)(y^3 - 2y^2 + y) + \sin^2(2\pi x).\sin^2(\pi y))$, and its reconstruction for M = 16 and M = 32. (The bold steps are for the former and the dotted ones the latter). The indices, I and J, of the reconstructed sections, $f^{(DET)}_{I,J}$, are: (i) $0 \leq I < M, J = I$, (ii) $0 \leq I < M, J = M/2 - 2$.



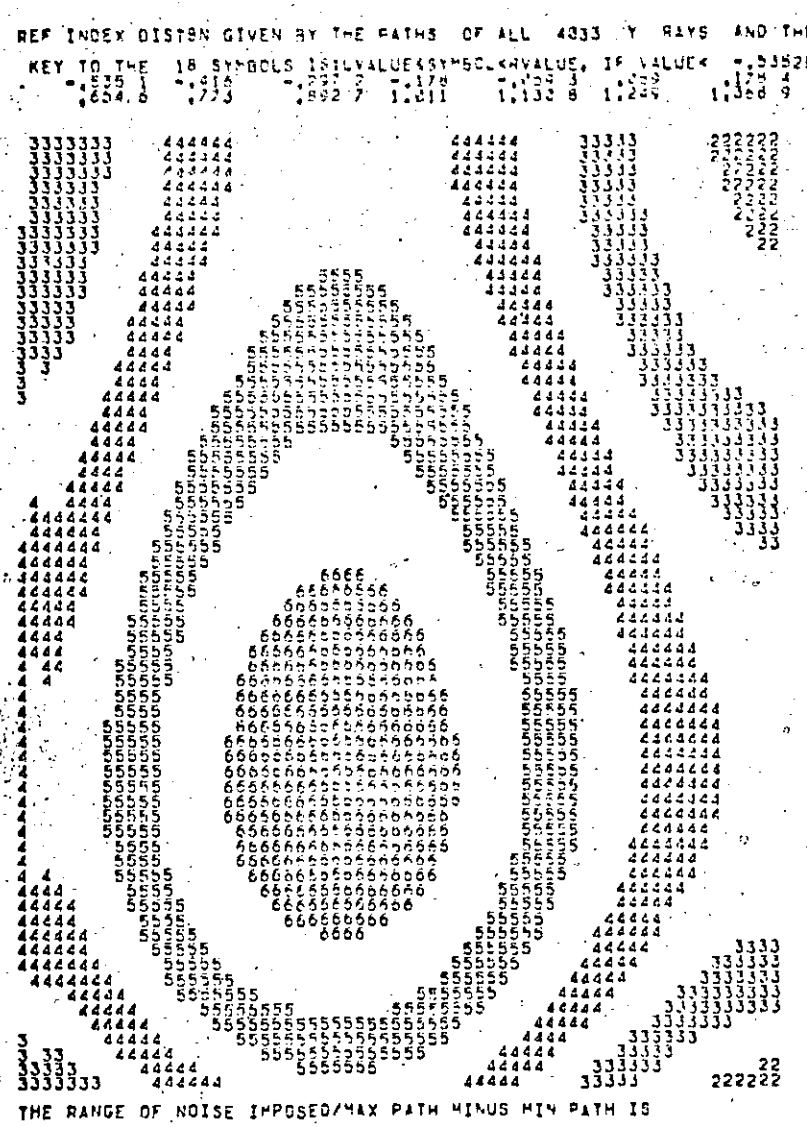
Fig(5.27) Three sections of the function, $10^{-5}(30(x^2 - x^3)(y^3 - 2y^2 + y) + \sin^2(2\pi x).\sin^2(\pi y))$, and its reconstruction for M = 16 and M = 32. (The bold steps are for the former and the dotted ones the latter). The indices, I and J, of the reconstructed sections, $f^{(DET)}_{I,J}$, are: (i) $I = (5/16)(M + 1) - 1, 0 \leq J < M$, (ii) $I = M/2 - 1, 0 \leq J < M$, (iii) $I = (3/4)(M + 1) - 1, 0 \leq J < M$.



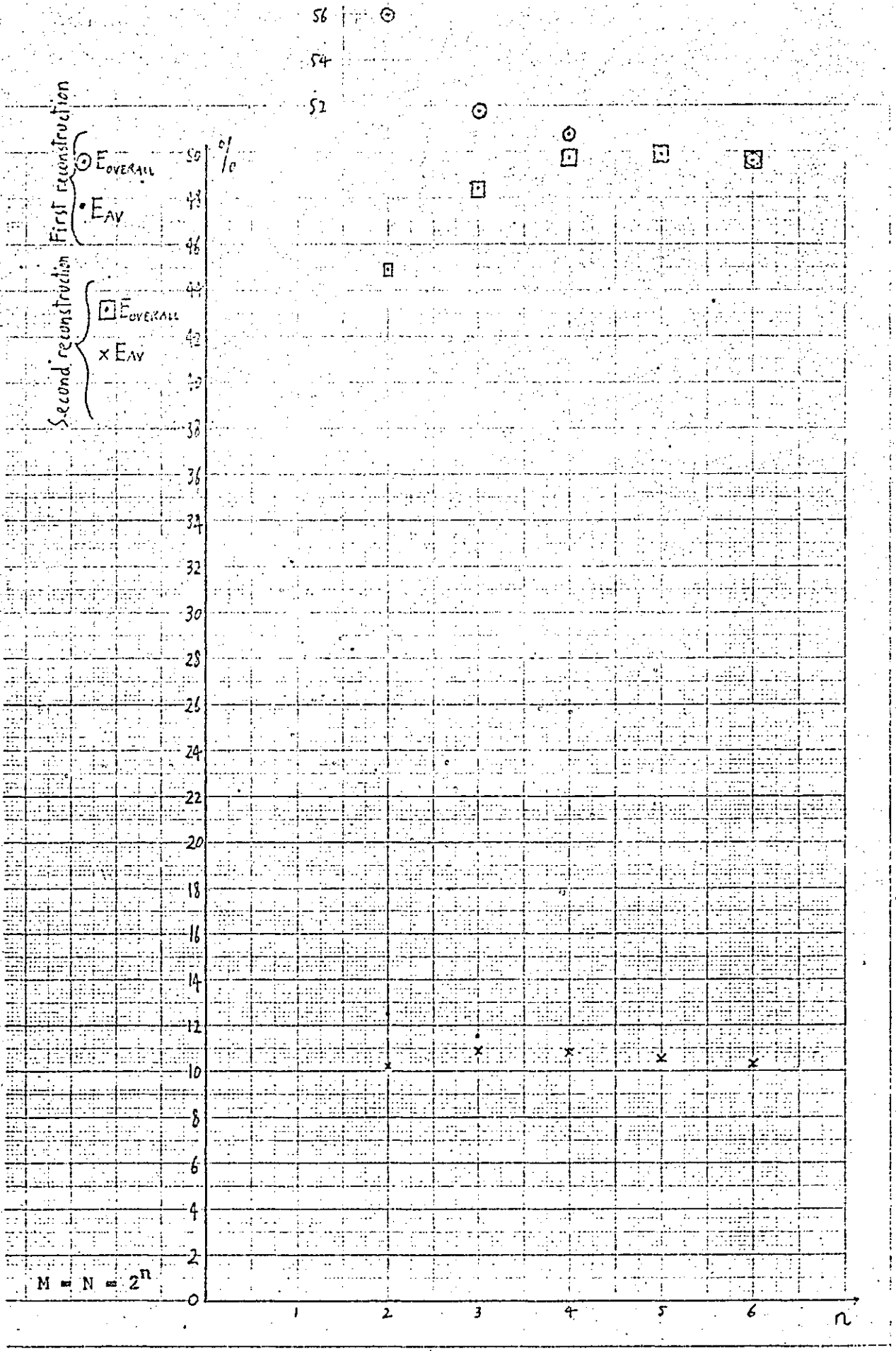
Fig(5.28) The function, $10^{-5}\cdot(30(x^2 - x^3)(y^3 - 2y^2 + y) + \frac{1}{2}\sin^2(2\pi x)\cdot\sin^2(\pi y))$, displayed as its mean in each of the 64 X 64 resolution cells.



Fig(5.29) The first reconstruction of the function,
$10^{-5}.(30(x^2 - x^3)(y^3 - 2y^2 + y) + \frac{1}{2}\sin^2(2\pi x).\sin^2(\pi y))$, for M = N = 64.



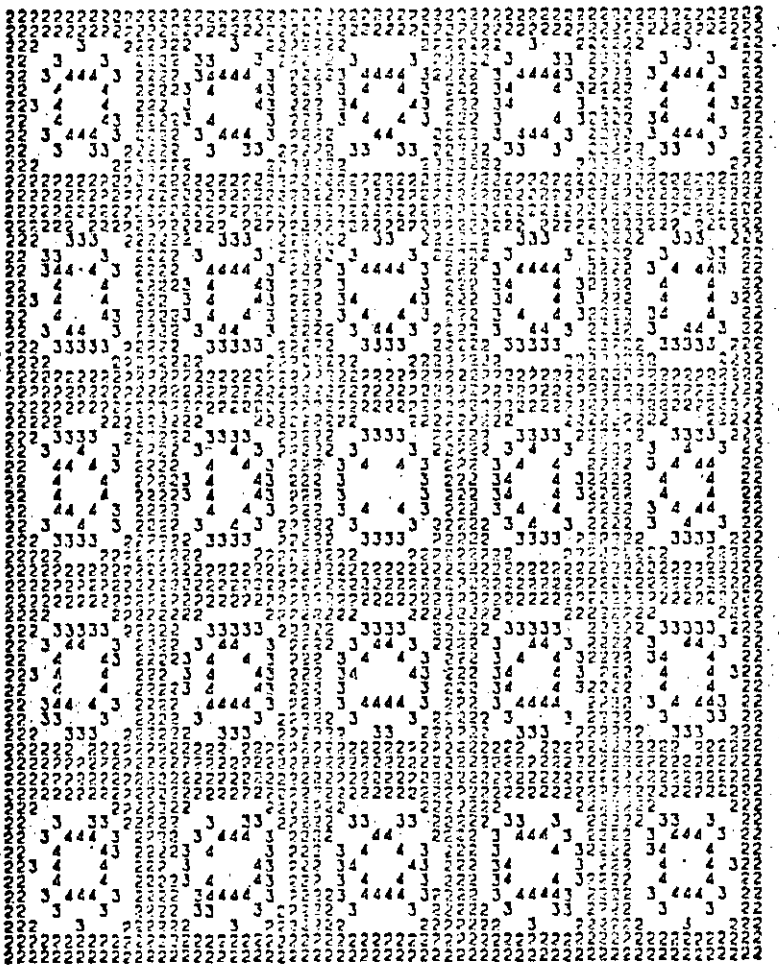
Fig(5.30) The second reconstruction of the function, $10^{-5}.(30(x^2-x^3)(y^3-2y^2+y) + \frac{1}{2}\sin^2(2\pi x).\sin^2(\pi y))$, for $N = N = 64$.



Fig(5.31). The average and overall errors in the reconstructions of the function, $10^{-5}(30(x^2 - x^3)(y^3 - 2y^2 + y) + \frac{1}{2}\sin^2(2\pi x).\sin^2(\pi y))$, for M = 2,4,8,16,32, and 64.

```
PRINTOUT MODE IS KOD 15
IN ALL DISTRIBUTIONS BELOW Y AXIS IS HORIZONTAL TO THE RIGHT,X-AXIS IS VERTICALLY DOWN.ORIGIN IS TOP LEFT@NEX
INPUT MODE KIN= 2.PATHS COMPUTED FROM A SMOOTH FUNCTION GIVEN AS A FINITE POWER AND/OR COSINE SERIES
RESOLUTION IS MA=  64 BY NA=  64,DIMENSIONS OF BOUNDARY ARE XA= 1.0000E+05 BY YA= 1.0000E+05 ,+ICRONS
ALL FIGS BELOW EXCEPT ERROR,SCII(,TC(I),UC(I) AND RNOISE TO BE MULTIPLIED BY F = 1.0000E-05
COS SERIES SUM(I=1 TO    5 ) OF CC(I)COS(PI2)(SC(I)X+TC(I)Y+UC(I)) IS CC(I),SC(I),TC(I),UC(I)=
  =2.5000E-01   5.0000E+00  -5.0000E+00   0.          ;  -2.5000E-01   5.0000E+00   5.0000E+00   0.
   1.2500E-01   5.0000E+00  -5.0000E+00   0.          ;   1.2500E-01   5.0000E+00   5.0000E+00   0.
   2.5000E-01  -5.          -5.           0.          ;
TRUE REF INDEX DISTRIBUTION OF SMOOTH FUNCTION EXPRESSED AS THE MEAN VALUES OF F(X,Y) IN THE  64 BY  64 CELLS
KEY TO THE  12 SYMBOLS IS(LVALUES&SYMBOL<RVALUE, IF VALUE<  -.49445+00;SYMBOL IS -.IF VALUE >  .14635+01;SYMBOL
 -.494    -.330 1  -.165    .000 2   .165    .330 3   .494    .659 4   .824    .989
 1.154    1.318 6  1.483
```

10^-5

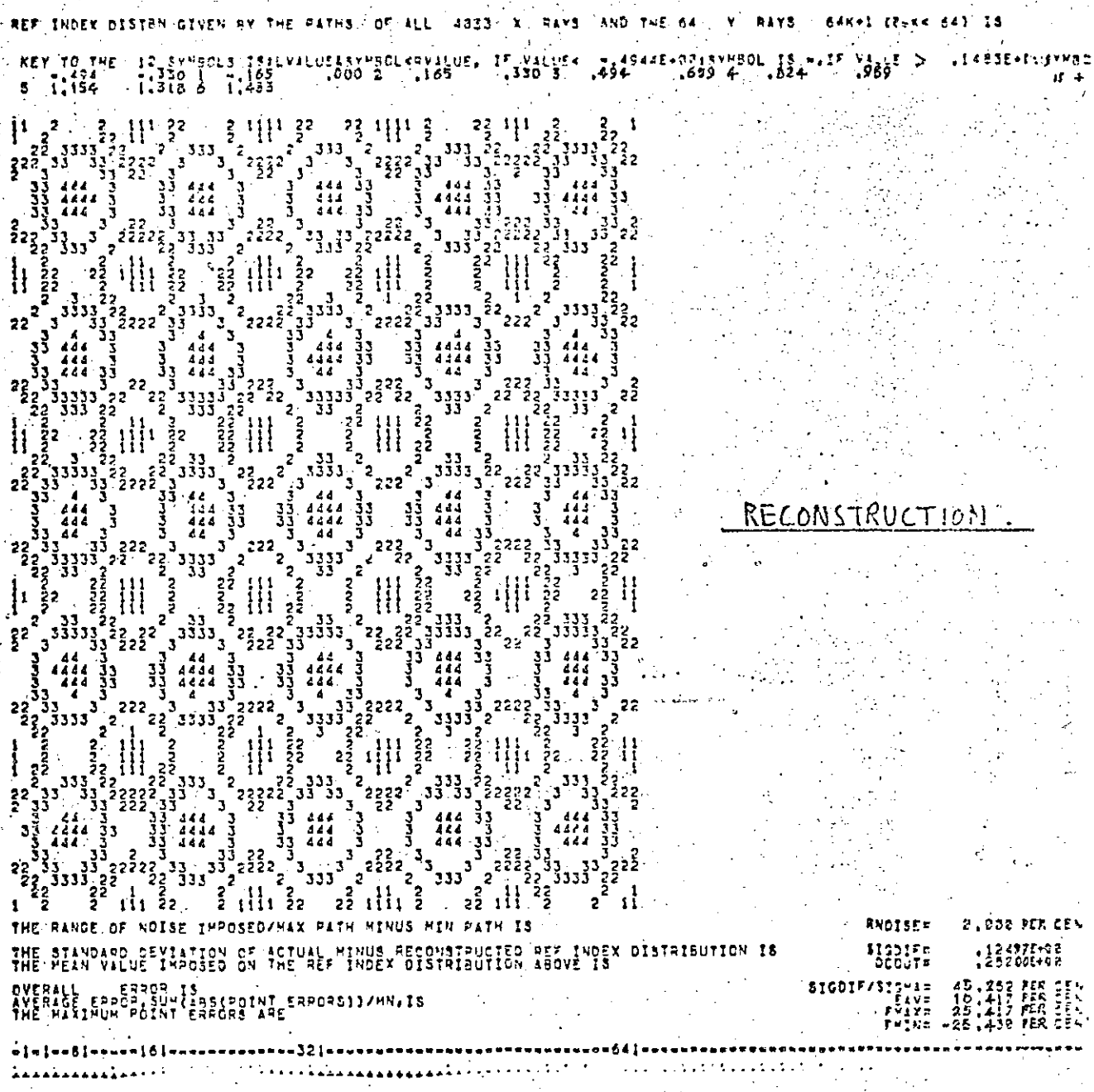


INPUT FUNCTION.

```
MAX  VALUE OF REF INDEX DISTRIBUTION IS                                  REFMAX=   .98833E+00
MIN  VALUE OF REF INDEX DISTRIBUTION IS                                  REFMIN=   .31372E-04
REFMAX MINUS REFMIN IS                                                   RANGE=    .98820E+00
MEAN VALUE OF REF INDEX DISTRIBUTION IS                                  AVR=      .25000E+00

THE STANDARD DEVIATION OF THE REF INDEX DISTRIBUTION ABOVE,ABOUT MEAN OF ZERO,IS     SIGMA=   .27616E+00
THE MEAN VALUE IMPOSED ON THE REF INDEX DISTRIBUTION ABOVE IS                        DCINH=   .25000E+00
```

Fig(5.32) The function, $10^{-5}.\sin^2(5\pi x).\sin^2(5\pi y)$, displayed as its mean value in the 64 64 resolution cells.



Fig(5.33) The reconstruction of the function, $10^{-5}.\sin^2(5\pi x).\sin^2(5\pi y)$, for M = N = 64.

Chapter 6 A Possible way of Improving The Walsh Function method by a determination of the Orthogonal Gradients of a Density Field from line integral data.

6.1 Attempts to improve the Walsh Function Method.

In Chapter 5 it was seen that, for a given number of resolution cells and a given continuous density function, the average error, E_{AV}, was much higher for the Walsh function method than for similar methods based on expanding the density function in a finite Fourier series of complex exponential or sine and cosine functions. A finite Fourier series is inherently better suited to the representation of smooth continuous density functions, which occur physically, than is a similar series of Walsh functions. In particular the orthogonal gradients $\partial f(x,y)/\partial x$ and $\partial f(x,y)/\partial y$ and higher derivatives of the density field, $f(x,y)$, can be represented to some degree of accuracy by a finite Fourier series but cannot be represented at all by a Walsh function series of any number of terms since all the derivatives of Walsh functions are zero except at the discontinuities. The Walsh function method effectively assumes that the input data, from an experiment, the line integrals $P_i^{(X)}$ from the "X" rays and $P_j^{(Y)}$ from the "Y" rays (see section (4.1)) comes from a two dimensional M×N (where $M = N = 2^n$) step function, defined by equation (4.7). In fact of course the line integrals arise from realistic continuous density functions like those defined in section 4.3.2.

In this section normalized line integrals (see section 2.3) rather than line integrals are used.

From equations (4.5) and (4.6) the normalized line integrals of all the MN - M + 1 "X" rays and MN - N + 1 "Y" rays are given in terms of the un-normalized line integrals by the two equations

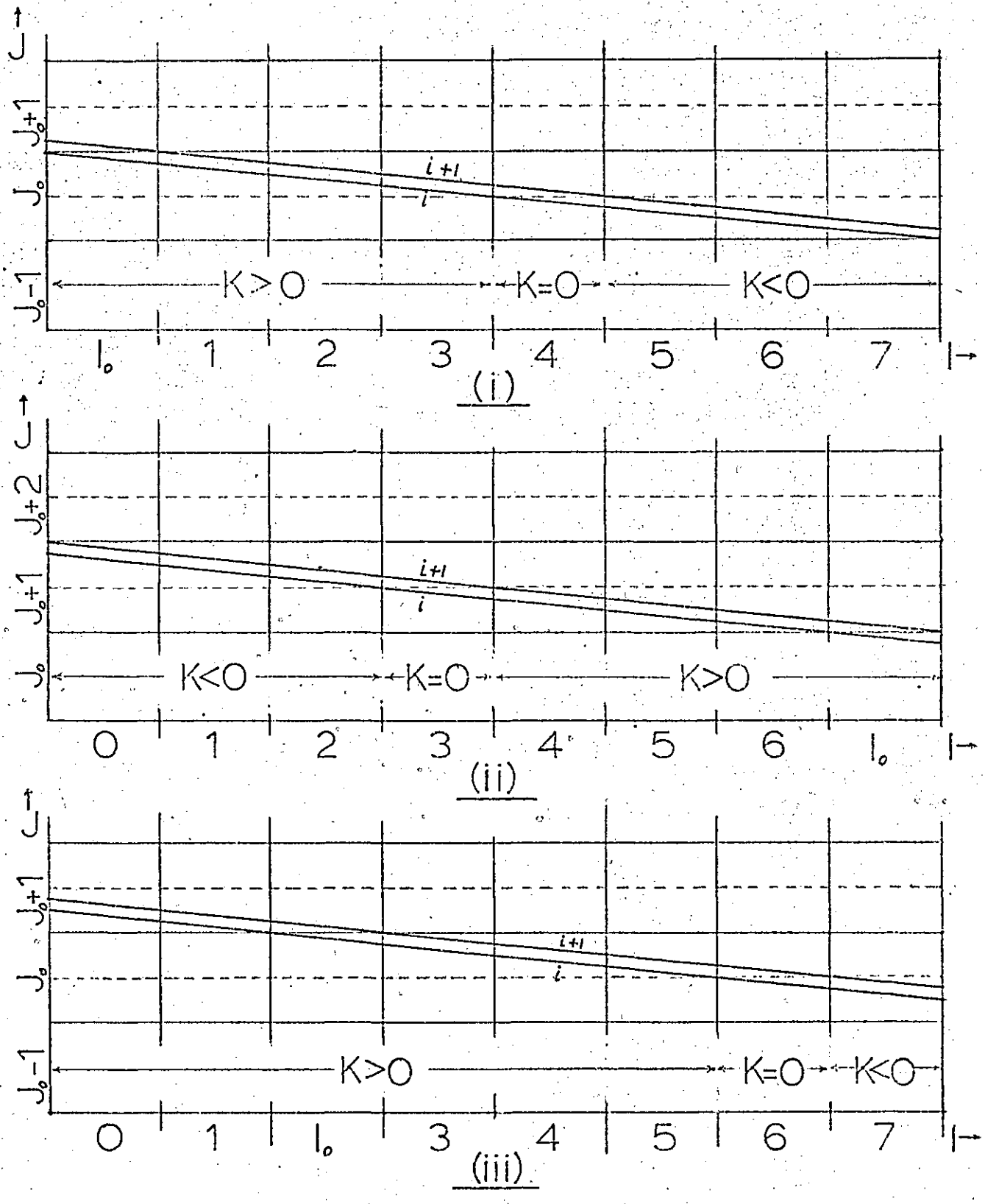
$$B_i^{(X)} = P_i^{(X)} / \sqrt{X^2 + Y^2/N^2}, \qquad (6.1)$$

where $1 \leqslant i \leqslant MN - M + 1$, and

$$B_j^{(Y)} = P_j^{(Y)} / \sqrt{Y^2 + X^2/M^2}, \qquad (6.2)$$

where $1 \leqslant j \leqslant MN - N + 1$.

Now referring to Fig (6.1) consider the two adjacent "X" rays i and



Fig(6.1). Diagrams to show the resolution cells passed through by both the "X" rays, i and i+1, for (i) $I_o=0$, (ii) $I_o=M-1$ and (iii) $1 \leqslant I_o \leqslant M-2$. The ranges of values of I for which $K=M/2-|(I_o-I)|$ is negative, positive or zero are also shown. A cell labelled by I along the x-axis and J along the y-axis is in the range $\frac{I}{M} \leqslant x \leqslant \frac{I+1}{M}$, $\frac{J}{N} \leqslant y \leqslant \frac{J+1}{N}$. The dotted lines divide each cell into two halves vertically.

i + 1. They pass through the top left hand and top right hand corners respectively of the resolution cell (I_o, J_o) where i is related to I and J by the relation (cf. equation (4.8)).

$$i = I_o + J_o M + 1, \tag{6.3}$$

the range of I_o and J_o being $0 \leqslant I_o \leqslant M - 1$ and $0 \leqslant J_o \leqslant N - 2$ giving the range of i, $1 \leqslant i \leqslant MN - M$.

Likewise two adjacent "Y" rays j and j + 1 pass through the bottom right and top right hand corners respectively of a cell (I_1, J_1) where

$$j = J_1 + I_1 N + 1, \tag{6.4}$$

the ranges of I_1 and J_1 being $0 \leqslant J_1 \leqslant N - 1$ and $0 \leqslant I_1 \leqslant M - 2$ giving the rnage of j, $1 \leqslant j \leqslant MN - N$.

The MN - M + 1 normalized line integrals of the "X" rays and MN - N + 1 normalized line integrals of the "Y" rays which, when fed into the Walsh function method would give two "almost independent" "correct" density field determinations, $f^{(DET)}_{I,J}$ for all I and J, are given by the relations (c.f. equations (4.9) and (4.11) concerning the line integrals given by a step function)

$$PSEUDO_B{}^{(X)}_{i+1} = PSEUDO_B{}^{(X)}_{i} + \frac{1}{M}\left(\langle f(x,y) \rangle_{I_o, J_o+1} - \langle f(x,y) \rangle_{I_o, J_o} \right) \tag{6.5}$$

where $B^{(X)}_1$ = an arbitrary constant (say zero),
and

$$PSEUDO_B{}^{(Y)}_{j+1} = PSEUDO_B{}^{(Y)}_{j} + \frac{1}{N}\left(\langle f(x,y) \rangle_{I_1 + 1, J_1} - \langle f(x,y) \rangle_{I_1, J_1} \right) \tag{6.6}$$

where $B^{(Y)}_1$ = an arbitrary constant (say zero). "Correct" density field means $\langle f(x,y) \rangle_{I,J}$ for all MN pairs of values, (I,J), which is the mean value of the density field f(x,y) in each resolution cell (I,J).

Now in section 6.2 it is shown that, from the true relative line integrals, $B^{(X)}_i$, it is possible to determine the gradients $b_{I,J} = \partial f(x,y)/\partial y$ at the centre of each resolution cell (I,J) for all I,J and similarly

$a_{I,J} = \partial f(x,y)/\partial x$ at the centre of each cell (I,J) for all I,J. Certain a priori assumptions about the density function, $f(x,y)$, are required for this to be possible.

If now, the density field changes slowly over regions the size of a single resolution cell, the gradient $\partial f(x,y)/\partial y$, along the common boundary between the cells, (I_o,J_o) and (I_o,J_o+1), can be approximated by the difference between the mean values of $f(x,y)$ in the cells divided by 1/N, the separation of their centres. Equating this to the mean of the two gradients within the cells gives

$$N(\langle f(x,y)\rangle_{I_o,\,J_o+1} - \langle f(x,y)\rangle_{I_o,J_o}) = \tfrac{1}{2}(b_{I_o,J_o+1} + b_{I_o,J_o+1}), \tag{6.7}$$

so that equation (6.5) can be rewritten

$$PSEUDO_B(X)_{i+1} = PSEUDO_B(X)_i + \tfrac{1}{2}(b_{I_o,J_o} + b_{I_o,J_o+1})/MN \tag{6.8}$$

where $PSEUDO_B(X)_1 = 0$ and $0 \leqslant I_o \leqslant M-1$, $0 \leqslant J_o \leqslant N-2$

Likewise

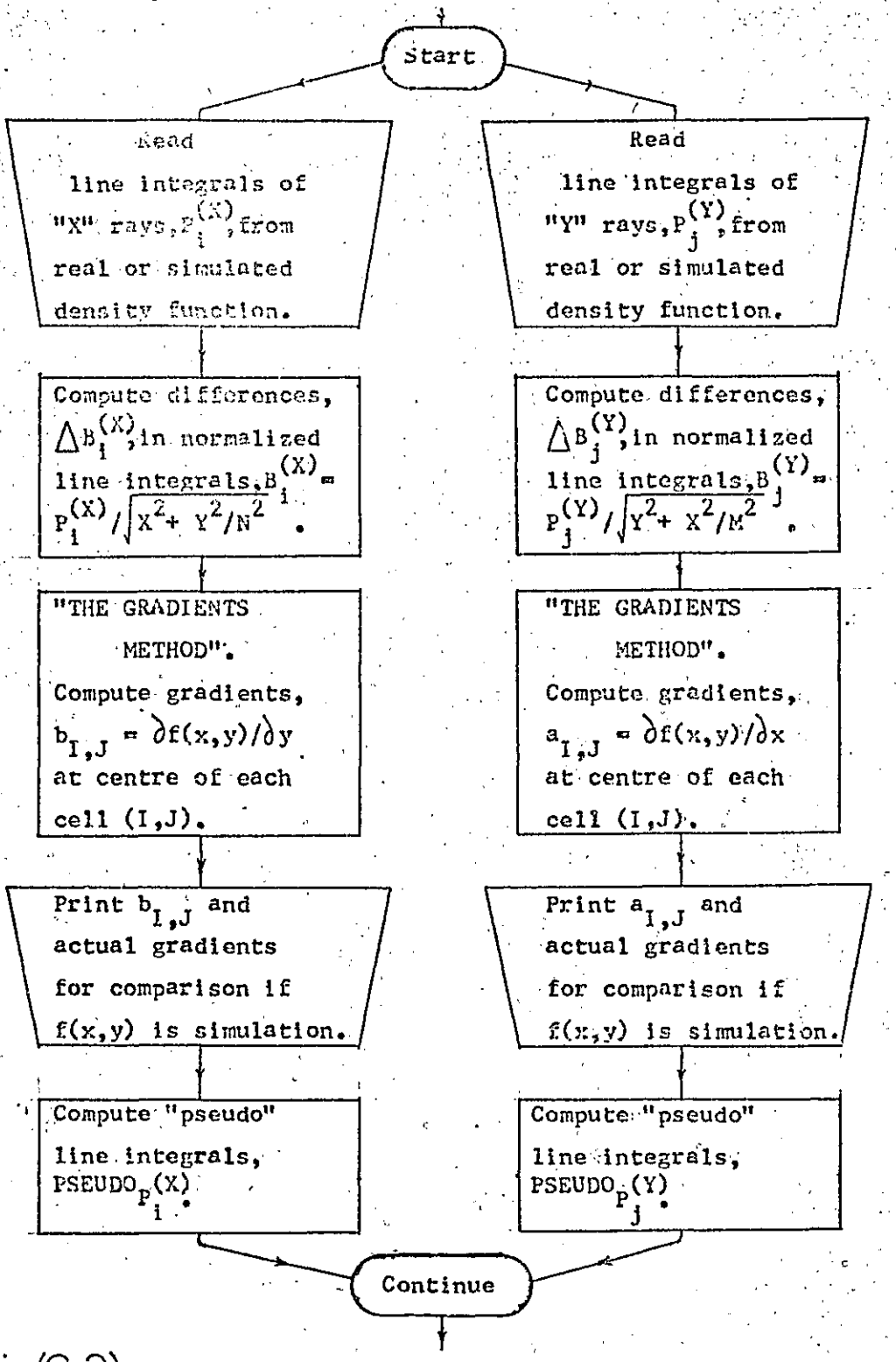
$$PSEUDO_B(Y)_{j+1} = PSEUDO_B(Y)_j + \tfrac{1}{2}(a_{I_1,J_1} + a_{I_1+1,J_1})/MN \tag{6.9}$$

where $PSEUDO_B(Y)_1 = 0$ and $0 \leqslant J_1 \leqslant N-1$, $0 \leqslant I_1 \leqslant M-2$.

Fig (6.2) is a flow diagram showing a computer program to implement the determination of orthogonal gradients of a density function for given line integral inputs. It also shows the generation of "pseudo" line integral data to act as input to the Walsh function method.

Before setting out the theory of the method for determining orthogonal gradients and generation of "pseudo" line integral data, some possible pros and cons of the method, which has not been tested, will be put forward:-

Certain reasonable a priori assumptions about the density function, which are not made in the straight-forward Walsh function method already described, are made in the orthogonal gradients method. This is the basis for the hope that better density function determination would result if "pseudo" line integrals instead of line integrals (simulated or from experiment) are used as the input data for the



Fig(6.2) A flow diagram of a proposed computer program to implement the determination of the orthogonal gradients and the generation of "pseudo" line integrals to be used as input data to The Walsh Function Method. The program between "start" and "continue" fits in the flow diagrams of Fig(4.2) in the positions marked by the symbol "⊛".

Walsh Function Method (shown as "THE METHOD" in Fig (4.1)). On the other hand, since the first stage in the orthogonal gradients method is the computation of all the MN - M differences, $\Delta B_i^{(X)}$ between the normalized line integrals of all the MN - M pairs of adjacent "X" rays i and i+1 ($1 \leqslant i \leqslant MN - M$) and likewise the MN - N differences $\Delta B_j^{(Y)}$ for the "Y" rays j and j+1 ($1 \leqslant j \leqslant MN - N$), experimental errors in relative line integrals could give rise to errors in these differences in line integrals, which are a very large fraction of those differences. This problem may perhaps be overcome as follows;

Suppose that $p(y')$ and $p(y'')$ are the two functions whose MN - M + 1 and MN - N + 1 regularly sampled values are $P_i^{(X)}$ and $P_j^{(Y)}$, the line integral data required (see section 1.1). Now if some assumptions could be made about the noise type present in $p(y')$ and $p(y'')$, these latter two could be altered in some way and re-sampled to give a better set of data points, $P_i^{(Y)}$ and $P_j^{(X)}$. For example if $p(y')$ and $p(y'')$ changed slowly compared with small "jagged" variations superimposed on this, it might be assumed that some high frequency noise was present. This could then be "smoothed" out by plotting a least squares fit curve which contains frequencies lower than any of the noise frequencies.

It is worth noting that if the samples, $P_i^{(X)}$, of $p(y')$ change slowly enough compared to the sampling space, $\Delta y'$, say then the set of differences $\Delta P_i^{(X)}$ (proportional to $\Delta B_i^{(X)}$ the normalized differences) are proportional to the first derivative, $dp(y')/dy'$, of the line integral function $p(y')$ sampled at intervals $\Delta y'$. A direct method of recording $dp(y')/dy'$ could be sampled directly to provide the input data to the orthogonal gradients method.

Finally, the orthogonal gradients may be a quantity as useful to know as the density function itself since it is usually the changes in density field over the plane which are of interest.

6.2 The Determination of the Orthogonal Gradients of a Density Function from line integral data.

6.2.1 Basic assumptions.

It is convenient to define a pair of local coordinates for each resolution cell,(I,J), within the scaled boundary (see Fig(3.4)),as follows:

$$x' = x - \frac{I+\frac{1}{2}}{M} \tag{6.10}$$

and

$$y' = y - \frac{J+\frac{1}{2}}{N} , \tag{6.11}$$

where $0 \leqslant I \leqslant M-1$ and $0 \leqslant J \leqslant N-1$. The origin of each pair of coordinates is at the centre of each cell.

The a priori assumptions made about the density function, f(x,y), are that:-

(1) f(x,y) can be fairly well represented in each cell,(I,J), by a polynomial in x' and y' up to and including terms of second order, x' and y' being in the range, $\frac{-1}{2M} \leqslant x' < \frac{1}{2M}$, $\frac{-1}{2N} \leqslant y' < \frac{1}{2N}$. Let

$$f_{I,J}(x',y') = f(x' + \frac{I+\frac{1}{2}}{M}, y' + \frac{J+\frac{1}{2}}{N}) =$$
$$f_{I,J} + b_{I,J}\cdot y' + a_{I,J}\cdot x' + \tfrac{1}{2}\beta_{I,J}\cdot y'^2 + \in_{I,J}\cdot x'y' + \tfrac{1}{2}\alpha_{I,J}\cdot x'^2 , \tag{6.12}$$

where $f_{I,J}$, $a_{I,J}$, $b_{I,J}$, and $\in_{I,J}$ are constants within the cell,(I,J). $\alpha_{I,J}$ and $\beta_{I,J}$ are both two valued constants within each cell,(I,J), and depend on the second assumption ,given below.

Clearly, from equation(6.12), $f_{I,J}$ is the value of f(x,y) at the centre, (x',y') = (0,0), of the cell,(I,J). $a_{I,J}$ and $b_{I,J}$ are respectively the gradients, $\partial f_{I,J}(x',y')/\partial x' = \partial f(x,y)/\partial x$ and $\partial f_{I,J}(x',y')/\partial y' = \partial f(x,y)/\partial y$, evaluated at the centre of the cell, (I,J).

(2) The orthogonal gradients, $a_{I,J}$ and $b_{I,J}$, change sufficiently slowly from cell to cell for the second derivatives in the regions between neighbouring cell centres to be expressed by the differences between the first derivatives at the centres of all pairs of such neighbouring cells, divided by the distance between their centres:

Differentiation of equation(6.12) twice with respect to x' and twice with respect to y' gives

$$\partial^2 f(x,y)/\partial x^2 = \alpha_{I,J} \qquad (6.13)$$

and

$$\partial^2 f(x,y)/\partial y^2 = \beta_{I,J} \ . \qquad (6.14)$$

Now, referring to Fig(6.3)(i), $\partial^2 f(x,y)/\partial y^2$ in the top half of cell,(I,J), and bottom half of cell,(I,J+1), is given, for all I, $0 \leqslant I \leqslant M-1$, by the relations

$$\beta^{(BOT)}_{I,J+1} = \beta^{(TOP)}_{I,J} = N(b_{I,J+1} - b_{I,J}) \ , \qquad (6.15)$$

where $0 \leqslant J \leqslant N-2$. If J=N-1 there is no cell,(I,J+1), in which case

$$\beta^{(TOP)}_{I,N-1} = 0 \qquad (6.16)$$

$\beta^{(BOT)}_{I,0}$ would be given by J+1=0 in equation(6.15), requiring the gradient, $b_{I,-1}$, which does not exist. In this case it is assumed that

$$\beta^{(BOT)}_{I,0} = 0 \qquad (6.17)$$

Fig(6.3)(ii) shows that $\partial^2 f(x,y)/\partial x^2$ in the right hand half of cell,(I,J) and left hand half of cell,(I+1,J), is given, for all J, $0 \leqslant J \leqslant N-1$, by the relations

$$\alpha^{(LEFT)}_{I+1,J} = \alpha^{(RIGHT)}_{I,J} = M(a_{I+1,J} - a_{I,J}) \ , \qquad (6.18)$$

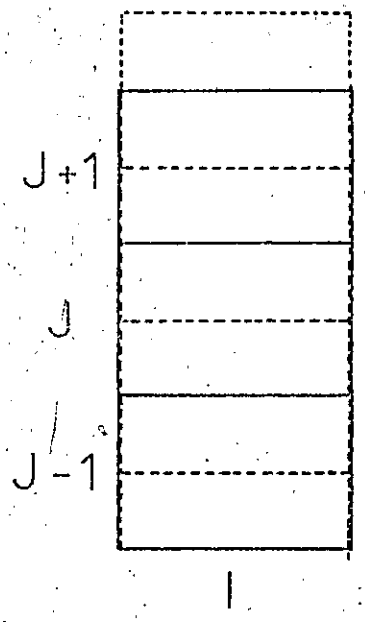
where $0 \leqslant I \leqslant M-2$. $\alpha^{(LEFT)}_{0,J}$ and $\alpha^{(RIGHT)}_{M-1,J}$ are given by

$$\alpha^{(RIGHT)}_{M-1,J} = 0 \qquad (6.19)$$

and

$$\alpha^{(LEFT)}_{0,J} = 0 \qquad (6.20)$$

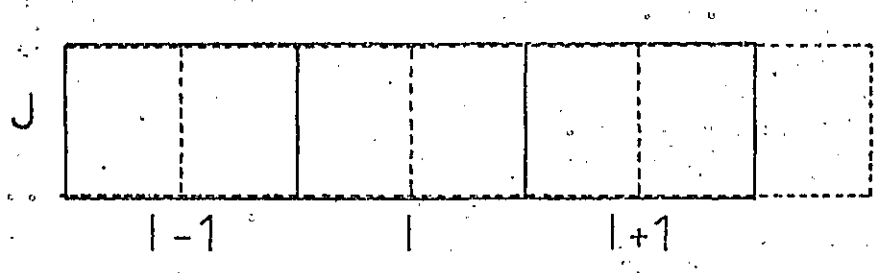
Equations (6.18) to (6.20) can be obtained from equations (6.15) to (6.17) by exchanging: (1) the order of the subscripts. (2) I and J. (3) M and N. (4) β and α. (5) a and b. The theory given below for the determination of the gradients, $b_{I,J}$, from "X" ray data, $P^{(X)}_i$, applies also to finding the gradients, $a_{I,J}$, from "Y" ray data, $P^{(Y)}_j$, if the above mentioned swaps are made. In addition $P^{(X)}_i$ and any derived quantities (e.g. $B^{(X)}_i$, the normalized line integrals) are replaced by $P^{(Y)}_j$ etc.



$$B^{(BOT)}_{I,J+1} = B^{(TOP)}_{I,J} = N(b_{I,J+1} - b_{I,J})$$

$$B^{(BOT)}_{I,J} = B^{(TOP)}_{I,J-1} = N(b_{I,J} - b_{I,J-1})$$

(i)



$$\alpha^{(LEFT)}_{I,J} = \alpha^{(RIGHT)}_{I-1,J} = M(a_{I,J} - a_{I-1,J})$$

$$\alpha^{(LEFT)}_{I+1,J} = \alpha^{(RIGHT)}_{I,J} = M(a_{I+1,J} - a_{I,J})$$

(ii)

Fig(6.3) The cell sized areas, enclosed by the dotted lines, in which the second derivatives (i) $\partial^2 f(x,y)/\partial y^2$ and (ii) $\partial^2 f(x,y)/\partial x^2$ are assumed to be constants given in terms of the first derivatives, $b_{I,J} = \partial f(x,y)/\partial y$ and $a_{I,J} = \partial f(x,y)/\partial x$ respectively, evaluated at the centres of each cell (I,J). Bold lines enclose cells.

$\partial^2 f(x,y)/\partial y^2$ in the top half of the "edge" cells, (I,N-1), and bottom half of cells, (I,0), are the same as in the rest of these cells. Likewise $\partial^2 f(x,y)/\partial x^2$ in the right hand half of the "edge" cells, (M-1,J), and left hand side of the cells, (0,J), are the same as in the rest of these cells. The range of I and J is $0 \leqslant I \leqslant M-1$ and $0 \leqslant J \leqslant N-1$.

6.2.2 The Determination of the Gradients, $b_{I,J} = \partial f(x,y)/\partial y$ at the centre of each Resolution cell, from "X" ray line integral data.

Consider a pair of "X" rays, i and i+1, which pass through the top left hand and top right hand corners, respectively, of the cell, (I,J). (See Fig(6.1)). From equation(4.5), the difference in the normalized line integrals of these two rays is given by

$$\Delta B_i^{(X)} = B_{i+1}^{(X)} - B_i^{(X)} = \int_0^1 \left\{ f(x, y_{i+1}(Xx)) - f(x, y_i(Xx)) \right\} dx \quad , \tag{6.21}$$

where $1 \leqslant i \leqslant MN-M$, and $y_i(Xx)$ is the equation of the ray, i, within the scaled boundary, and is given by (c.f. equation(3.13))

$$y_i(Xx) = -\frac{x}{N} + \frac{1}{N}\left(1 + \frac{i-1}{M}\right) \quad , \tag{6.22}$$

i being related to I_o and J_o by equation(6.3)

$$i = I_o + J_o M + 1 \quad , \tag{6.3}$$

where $0 \leqslant I_o \leqslant M-1$ and $0 \leqslant J_o \leqslant N-2$.

The integral in equation(6.21) can be rewritten as the sum of M integrals, each taken over the range I/M to (I+1)/M for $0 \leqslant I \leqslant M-1$:

$$\Delta B_i^{(X)} = \sum_{I=0}^{M-1} \delta(\Delta B_i^{(X)})_{I,J} \quad , \tag{6.23}$$

where

$$\delta(\Delta B_i^{(X)})_{I,J} = \int_{I/M}^{(I+1)/M} \left\{ f(x, y_{i+1}(Xx)) - f(x, y_i(Xx)) \right\} dx \quad . \tag{6.24}$$

The M quantities, $\delta(\Delta B_i^{(X)})_{I,J}$, will be referred to as "contributions" to the difference in the normalized line integrals of the two adjacent "X" rays, i and i+1. There are MN - M such differences, $\Delta B_i^{(X)}$, each one being the sum of M contributions, $\delta(\Delta B_i^{(X)})_{I,J}$.

From equation(6.24), each contribution, $\delta(\Delta B_i^{(X)})_{I,J}$, to $\Delta B_i^{(X)}$, involves that section of both rays which is in the range, $I/M \leqslant x < (I+1)/M$. Now, the pair of ray sections for $I_o = 1$ (i.e. the rays in the range $I_o/M \leqslant x < (I_o+1)/M$), is such that ray, i+1, is in cell, (I_o, J_o+1), whereas ray, i, is in cell, (I_o, J_o). All the other M-1 pairs of ray sections, for which $I \neq I_o$, are completely inside M-1 different cells. From Fig(6.1), (i), (ii) and (iii) respectively, the M-1 cells, (I,J), are specified in terms of I_o and J_o as follows:

$$(I,J)_{I\neq I_o} = \begin{cases} 1)\,(I,J_o), & \text{where } J_o = 0 \text{ and } 1 \leqslant I \leqslant M-1 \\ 2)\,(I,J_o+1), & \text{where } I_o = M-1 \text{ and } 0 \leqslant I \leqslant M-2 \\ 3)\begin{cases} (I,J_o+1), & \text{where } 1 \leqslant I_o \leqslant M-2 \text{ and } 0 \leqslant I \leqslant I_o-1 \\ (I,J_o), & \text{where } 1 \leqslant I_o \leqslant M-2 \text{ and } I_o+1 \leqslant I \leqslant M-1 \end{cases} \end{cases} \qquad (6.25)$$

The $M-1$ "contributions" given by equation(6.24) and(6.25) can be expressed in terms of local coordinates(equations(6.10) and (6.11)) by the relations

$$\delta\Big(\bigwedge_{I\neq I_o} B_i^{(X)}\Big)_{I,J} = \int_{-1/2M}^{1/2M} \Big\{ f_{I,J}(x', y'_{i+1}) - f_{I,J}(x', y'_i) \Big\}\, dx' \,, \qquad (6.26)$$

where y'_i and y'_{i+1} are rays,i, and,i+1,in terms of the local coordinates of cell,(I,J).Equations(6.10) and (6.11) with equation(6.22) give

$$y'_i = -\frac{x'}{N} - \frac{I+\frac{1}{2}}{MN} + \frac{1}{N} + \frac{i-1}{MN} - \frac{J+\frac{1}{2}}{N} \,,$$

which after substituting i,given by equation(6.3),becomes

$$y'_i = -\frac{x'}{N} + \frac{I_o - I}{MN} + \frac{J_o - J}{N} + \frac{1}{2N} - \frac{1}{2MN} \,. \qquad (6.27)$$

Likewise

$$y'_{i+1} = -\frac{x'}{N} + \frac{I_o - I}{MN} + \frac{J_o - J}{N} + \frac{1}{2N} + \frac{1}{2MN} \,. \qquad (6.28)$$

Of the $M-1$ ray section pairs in the cells,(I,J), $I=I_o$,given by equation(6.25),all but one are either in the top half or bottom half of a cell.Fig(6.1) shows that for $|I_o - I| = M/2$ ray,i+1, is in the top half of cell,(I,J), and ray ,i, is in the bottom half of this cell.For the $M-2$ ray pair sections in cells where $I\neq I_o$, and $|(I_o-I)| \neq M/2$,the constants $a_{I,J}, b_{I,J}, \alpha_{I,J}, \epsilon_{I,J}$ and $\beta_{I,J}$ are the same in $f_{I,J}(x', y'_i)$ as in $f_{I,J}(x', y'_{i+1})$,these being given by equation(6.12).Thus equation (6.26) becomes

$$\delta\Big(\bigwedge_{I\neq I_o} B_i^{(X)}\Big)_{I,J} = \int_{-1/2M}^{1/2M} (y'_{i+1} - y'_i)\Big\{ (b_{I,J} + \epsilon_{I,J} \cdot x') + \tfrac{1}{2}(y'_{i+1} + y'_i)\beta_{I,J} \Big\}\, dx' \,, \qquad (6.29)$$

which from equations(6.27) and 6.28) is

$$\begin{aligned} \delta\Big(\bigwedge_{I\neq I_o} B_i^{(X)}\Big)_{I,J} &= \frac{1}{MN}\int_{-1/2M}^{1/2M} (b_{I,J} + \epsilon_{I,J} \cdot x')\, dx' \\ &\quad + \frac{\beta_{I,J}}{2MN}\int_{-1/2M}^{1/2M} \left\{ -\frac{2x'}{N} + \frac{2(I_o - I)}{MN} + \frac{2(J_o - J)}{N} + \frac{1}{N} \right\} dx' \\ &= \frac{b_{I,J}}{M^2 N} + \frac{\beta_{I,J}}{M^3 N^2}\Big\{ (I_o - I) + M(J_o - J + \tfrac{1}{2}) \Big\} \,. \end{aligned} \qquad (6.30)$$

In equation(6.30), $J = J_o + 1$ if $I<I_o$ and $J = J_o$ if $I>I_o$ so that the expresion in curly brackets is $(I_o - I) - M/2$ if $I<I_o$ and $(I_o - I) + M/2$ if $I>I_o$. Thus equation(6.30) for the two values of $J, J_o + 1$ and J_o are given by the two equations

$$\underset{I<I_o}{\delta(\triangle B_i^{(X)})_{I,J_o+1}} = \frac{b_{I,J_o+1}}{M^2N} - \frac{\beta_{I,J_o+1}}{M^3N^2}\left\{\frac{M}{2} - |(I_o - I)|\right\} \qquad (6.31)$$

and

$$\underset{I>I_o}{\delta(\triangle B_i^{(X)})_{I,J_o}} = \frac{b_{I,J_o}}{M^2N} + \frac{\beta_{I,J_o}}{M^3N^2}\left\{\frac{M}{2} - |(I_o - I)|\right\} \qquad (6.32)$$

The β's in equations(6.31) and (6.32) are $\beta^{(TOP)}$ or $\beta^{(BOT)}$ depending on whether the pair of ray sections are in the top half or bottom half respectively of the cell,(I,J).(See Fig(6.1).

If $|(I_o - I)| = M/2$, the "contribution", $\delta(\triangle B_i^{(X)})_{I,J}$, could be given by equation(6.29) with $\beta_{I,J}$ replaced by the mean of the β's for the two ray sections, i, and, i+1. Thus $\beta_{I,J} = \frac{1}{2}(\beta^{(BOT)}_{I,J} + \beta^{(TOP)}_{I,J})$. The analysis then leads to equations(6.31) and (6.32) with the term in $\beta_{I,J}$ equal to zero since $M/2 - |(I_o - I)| = 0$. Thus, so far, equations(6.31) and(6.32) are valid for all I except $I = I_o$.

If $I = I_o$, the "contribution", $\delta(\triangle B_i^{(X)})_{I,J}$, due to the section of the rays, i, and, i+1, passing respectively through the cells, (I_o, J_o), and $(I_o, J_o + 1)$, will involve all the constants on the right hand side of equation(6.12) for both these cells. The result is complicated, involving many higher order quantities like $b_{I,J+1} - b_{I,J}$ compared with $b_{I,J}$ and $\beta_{I,J+1} - \beta_{I,J}$ compared with $\beta_{I,J}$ and so on. The result obtained by ignoring these is the same as that given by substituting $I = I_o$ and the appropriate $\beta_{I,J}$ in equation (6.31) or (6.32).

This $\beta_{I,J}$ is given for both equations by (c.f. equation(6.15))

$$\beta^{(BOT)}_{I_o,J_o+1} = \beta^{(TOP)}_{I_o,J_o} = N(b_{I_o,J_o+1} - b_{I_o,J_o}) \quad . \qquad (6.33)$$

Substitution of the β's given by the above equation into equations (6.31) and (6.32) gives the same result, namely

$$\delta(\triangle B_i^{(X)})_{I_o,J_o+1} = \delta(\triangle B_i^{(X)})_{I_o,J_o} = \tfrac{1}{2}(b_{I_o,J_o+1} + b_{I_o,J_o})/M^2N$$
$$= \frac{1}{M^3N}\left(\frac{M}{2}\cdot b_{I_o,J_o+1} + \frac{M}{2}\cdot b_{I_o,J_o}\right) . \qquad (6.34)$$

Equation(6.34) shows that the contribution to $\triangle B_i^{(X)}$ due to the pair of ray sections passing half-way between two cells is proportional to the mean of the gradients, b, of the two cells. ($b_{I,J}$ is $\partial f(x,y)/\partial y$

at the centre of the cell,(I,J)).Now the contribution to $\Delta B_i^{(X)}$,due to the pair of ray sections passing through the central region of a cell,is proportional to the gradient,b,in that cell since in this case $M/2 - |(I_o - 1)| = 0$ and so equations (6.31) and (6.32) become

$$\delta(\Delta B_i^{(X)})_{I,J} = \frac{1}{M^3 N}(M.b_{I,J}), \qquad (6.35)$$

where $J = J_o$ if $I > I_o$ and $J = J_o$ if $I < I_o$.

As might be expected,the contribution to $\Delta B_i^{(X)}$,due to a pair of ray sections at any general vertical position between the centres of cells,(I,J+1), and ,(I,J),turns out to be of the form

$$\delta(\Delta B_i^{(X)})_{I,J} = \frac{1}{M^3 N}(K_1 b_{I,J+1} + K_2 b_{I,J}) , \qquad (6.36)$$

if the pair of ray sections is in the cell,(I,J),or

$$\delta(\Delta B_i^{(X)})_{I,J+1} = \frac{1}{M^3 N}(K_1 b_{I,J+1} + K_2 b_{I,J}) , \qquad (6.37)$$

if the pair of ray sections is in the cell,(I,J).J is either J_o or $J_o + 1$ in equations (6.36) and (6.37) and K_1 and K_2 are weighting factors such that

$$K_1 + K_2 = M \quad . \qquad (6.38)$$

Their relative values depends on the vertical position of the pair of ray sections relative to the centres of cells,(I,J+1), and ,(I,J).K_1 and K_2 are both positive.

By substituting the appropriate β's,given by equations (6.15), (6,16) and (6.17), into equations (6.31) and (6.32) the equations of the form,equations (6.36) and (6.37) are obtained with K_1 and K_2 explicitly given for all ranges of I,for any given I_o and J_o. It is more convenient to specify the ranges of $M/2 - |(I_o - I)|$ than of I.(See Fig(6.1)):-

(1) $M/2 - |(I_o - I)| \geqslant 0$.

From Fig(6.1), the pair of ray sections in the range, $\frac{I}{M} \leqslant x < \frac{I+1}{M}$, in this case is either in the bottom half of cell, $(I, J_o + 1)$, or in the top half of cell, (I, J_o) so that the β's to substitute into equations (6.31) and (6.32) respectively are $\beta^{(BOT)}_{I,J_o+1}$ and $\beta^{(TOP)}_{I,J_o}$. These are both given by equation(6.15) with $J = J_o$ and so

$$\delta(\Delta B_i^{(X)})_{\substack{I,J_o+1 \\ I<I_o}} = \frac{1}{M^3 N}\left\{\left(\frac{M}{2} + |(I_o - I)|\right) b_{I,J_o+1} + \left(\frac{M}{2} - |(I_o - I)|\right) b_{I,J_o}\right\} \tag{6.39}$$

and

$$\delta(\Delta B_i^{(X)})_{\substack{I,J_o \\ I>I_o}} = \frac{1}{M^3 N}\left\{\left(\frac{M}{2} - |(I_o - I)|\right) b_{I,J_o+1} + \left(\frac{M}{2} + |(I_o - I)|\right) b_{I,J_o}\right\}. \tag{6.40}$$

where $0 \leqslant J_o \leqslant N-2$.

If $I = I_o$, both equations (6.39) and (6.40) reduce to equation(6.34), which is for the pair of ray sections half-way between the centres of the two cells, (I, J_o+1), and, (I, J_o).

If $M/2 - |(I_o - I)| = 0$, equations (6.39) and (6.40) reduce to the two equations contained in equation(6.35), which are for the pair of ray sections passing through the central regions of the two cells, (I, J_o+1) and, (I, J_o).

(2) $M/2 - |(I_o - I)| \leqslant 0$.

From Fig(6.1), the pair of ray sections in the range, $\frac{I}{M} \leqslant x < \frac{I+1}{M}$, in this case is either:

(i) the top half of cell, (I, J_o+1), in which case the β in equation(6.31) must be $\beta^{(TOP)}_{I,J_o+1}$, given by equation(6.15) with $J = J_o+1$ as

$$\beta^{(TOP)}_{I,J_o+1} = N(b_{I,J_o+2} - b_{I,J_o+1}), \tag{6.41}$$

where $1 \leqslant J_o+1 \leqslant N-2$. If $J_o+1 = N-1$ $\beta^{(TOP)}_{I,J_o+1}$ is given by equation(6.16) as

$$\beta^{(TOP)}_{I,N-1} = 0 \tag{6.42}$$

or (ii) the bottom half of cell, (I, J_o), in which case the β in equation(6.32) must be $\beta^{(BOT)}_{I,J_o}$, given by equation(6.15) with $J = J_o - 1$ as

$$\beta^{(BOT)}_{I,J_o} = N(b_{I,J_o} - b_{I,J_o-1}), \tag{6.43}$$

where $0 \leqslant J_o - 1 \leqslant N-2$. If $J_o = 0$ $\beta^{(BOT)}_{I,J_o}$ is given by equation(6.17) as

$$\rho^{(BOT)}_{I,0} = 0 \qquad (6.44)$$

Substituting equations (6.41) and (6.42) into equation(6.31) gives

$$\delta(\Delta B_i^{(X)})_{\substack{I,J_o+1 \\ I<I_o}} = \frac{1}{M^3N}\left\{\left(|(I_o-I)| - \frac{M}{2}\right) b_{I,J_o+2} + \left(\frac{3M}{2} - |(I_o-I)|\right) b_{I,J_o+1}\right\}, \qquad (6.45)$$

where $1 \leqslant J_o+1 \leqslant N-2$, and

$$\delta(\Delta B_i^{(X)})_{\substack{I,N-1 \\ I<I_o}} = \frac{1}{M^3N}\left\{M.b_{I,N-1}\right\}. \qquad (6.46)$$

Substituting equations (6.43) and (6.44) into equation(6.32) gives

$$\delta(\Delta B_i^{(X)})_{\substack{I,J_o \\ I>I_o}} = \frac{1}{M^3N}\left\{\left(\frac{3M}{2} - |(I_o-I)|\right) b_{I,J_o} + \left(|(I_o-I)| - \frac{M}{2}\right) b_{I,J_o-1}\right\}, \qquad (6.47)$$

where $0 \leqslant J_o-1 \leqslant N-2$, and

$$\delta(\Delta B_i^{(X)})_{\substack{I,0 \\ I>I_o}} = \frac{1}{M^3N}\left\{M.b_{I,0}\right\}. \qquad (6.48)$$

If $M/2 - |(I_o - I)| = 0$, equations (6.45) and (6.47) reduce to the two equations contained in equation (6.35), which are for the pair of ray sections passing through the central regions of the two cells, (I,J_o+1), and (I,J_o).

Between them the six equations, (6.39), (6.40), (6.45), (6.46), (6.47), and (6.48), give the contributions, $\delta(\Delta B_i^{(X)})_{I,J}$, to $\Delta B_i^{(X)}$, for all I from 0 to M-1. (Equation(6.35) are a special case of equations(6.39) and (6.40) and of equations(6.45) and (6.47). Equation(6.34) is the same special case of equations(6.39) and (6.40).)

Now depending on the value of I_o, given by equation(6.25), for the particular $i = I_o + J_o M + 1$ (equation(6.3)), $\Delta B_i^{(X)}$ given by equation(6.23) can be of three different forms as follows:

$$\Delta B_i^{(X)} = \sum_{I=0}^{M-1} \delta(\Delta B_i^{(X)})_{I,J_o}, \quad \text{if } I_o = 0, \qquad (6.49)$$

$$\Delta B_i^{(X)} = \sum_{I=0}^{M-1} \delta(\Delta B^{(X)})_{I,J_o+1}, \quad \text{if } I_o = M-1, \qquad (6.50)$$

$$\Delta B_i^{(X)} = \sum_{I=0}^{I_o-1} \delta(\Delta B_i^{(X)})_{I,J_o+1} + \sum_{I=I_o}^{M-1} \delta(\Delta B_i^{(X)})_{I,J_o} \qquad (6.51)$$

$$\text{if } I_o = 0 \text{ and } I_o = M-1.$$

(The summations in equation(6.51) could be, $\sum_{I=0}^{I_o}$ and $\sum_{I=I_o+1}^{M-1}$, since from equation(6.34), $(\triangle B_i^{(X)})_{I_o,J_o+1} = (\triangle B_i^{(X)})_{I_o,J_o}$.)

Between them these MN-M equations with $\triangle B_i^{(X)}$, i=1 to MN-M, contain the MN unknown gradients, $b_{I,J}$. In an attempt to obtain a set of MN-M equations in as many unknowns, the following initial assumptions are made:

$$b_{I,0} = 0 \quad , \tag{6.52}$$

where $0 \leqslant I \leqslant \frac{M}{2}-1$, and

$$b_{I,N-1} = 0 \quad , \tag{6.53}$$

where $\frac{M}{2} \leqslant I \leqslant M-1$. The gradients on the left hand side of the above equations are in the resolution cells least well "explored" by "X" rays. (See Fig(3.4)). They will be referred to as "corner" gradients from now on.

The method from here on is best developed by considering particular values of M and N and setting up the MN-N equations to be solved for these chosen cases.

Example 1. M=N=4.

The first stage is to set up the MN-M=12 equations in MN=16 unknowns, ignoring equations (6.52) and (6.53). The equations are of the type, equation (6.49) or (6.50) or (6.51). Each equation is the sum of M-1 "contributions" given by equations (6.39), (6.40), (6.45), (6.46), (6.47) and (6.48). To facilitate the choice of the correct equation out of these six equations for different values of I and J for each (I_o, J_o), the following table is drawn up:

I_o	0				1				2				3			
I	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
$\|(I_o-I)\|$	0	1	2	3	1	0	1	2	2	1	0	1	3	2	1	0
$K=M/2-\|(I_o-I)\|$	2	1	0	-1	1	2	1	0	0	1	2	1	-1	0	1	2
$M/2+\|(I_o-I)\|$	2	3	4	5	3	2	3	4	4	3	2	3	5	4	3	2
$3M/2-\|(I_o-I)\|$				3									3			
$I<I_o, J=J_o+1$?					yes				yes	yes			yes	yes	yes	
$I\geqslant I_o, J=J_o$?	yes	yes	yes	yes		yes	yes	yes			yes	yes				yes
$K\geqslant 0$?	yes	yes	yes		yes	yes	yes	yes	yes	yes	yes	yes		yes	yes	yes
$K<0$?				yes									yes			
Equation?	(6.40)	(6.40)	(6.40)		(6.39)	(6.40)	(6.40)	(6.40)	(6.39)	(6.39)	(6.40)	(6.40)		(6.39)	(6.39)	(6.40)

(6.47) (6.48) If $1\leqslant J\leqslant N-1$ / If $J=0$

(6.45) (6.46) If $0\leqslant J\leqslant N-2$ / If $J=N-1$

With the aid of the table above the 12 equations in 16 unknowns can be written in matrix form as follows:

$$
\begin{array}{c|c|c}
J_o & I_o & \\
\hline
0 & 0 & \Delta B_1 \\
 & 1 & \Delta B_2 \\
 & 2 & \Delta B_3 \\
 & 3 & \Delta B_4 \\
\hline
1 & 0 & \Delta B_5 \\
 & 1 & \Delta B_6 \\
 & 2 & \Delta B_7 \\
 & 3 & \Delta B_8 \\
\hline
2 & 0 & \Delta B_9 \\
 & 1 & \Delta B_{10} \\
 & 2 & \Delta B_{11} \\
 & 3 & \Delta B_{12}
\end{array}
=\frac{1}{M^3N}
\begin{array}{|cccc|cccc|cccc|cccc|}
\hline
\multicolumn{4}{|c|}{J=0} & \multicolumn{4}{c|}{1} & \multicolumn{4}{c|}{2} & \multicolumn{4}{c|}{3} \\
\hline
0&1&2&3 & 0&1&2&3 & 0&1&2&3 & 0&1&2&3 \\
\hline
2&3&4&4 & 2&1 & & & & & & & & & & \\
1&2&3&4 & 3&2&1 & & & & & & & & & \\
 &1&2&3 & 4&3&2&1 & & & & & & & & \\
 & &1&2 & 3&4&3&2 & 1 & & & & & & & \\
\hline
 & & &1 & 2&3&4&3 & 2&1 & & & & & & \\
 & & & & 1&2&3&4 & 3&2&1 & & & & & \\
 & & & & &1&2&3 & 4&3&2&1 & & & & \\
 & & & & & &1&2 & 3&4&3&2 & 1 & & & \\
\hline
 & & & & & & &1 & 2&3&4&3 & 2&1 & & \\
 & & & & & & & & 1&2&3&4 & 3&2&1 & \\
 & & & & & & & & &1&2&3 & 4&3&2&1 \\
 & & & & & & & & & &1&2 & 4&4&3&2 \\
\hline
\end{array}
\begin{bmatrix}
b_{0,0} \\ b_{1,0} \\ b_{2,0} \\ b_{3,0} \\
b_{0,1} \\ b_{1,1} \\ b_{2,1} \\ b_{3,1} \\
b_{0,2} \\ b_{1,2} \\ b_{2,2} \\ b_{3,2} \\
b_{0,3} \\ b_{1,3} \\ b_{2,3} \\ b_{3,3}
\end{bmatrix}
\qquad (6.56)
$$

The blanks in the 12X16 matrix are zeroes. The superscript, (X), in the $\Delta B_i^{(X)}$'s is left out. To make easier the filling in of the 12X16 matrix the labels I, J, I_o, J_o have been marked in:

Each pair of numbers, (I_o, J_o), corresponds to a given $\Delta B_i^{(X)}$, $(i=I_o+J_oM+1)$. Thus each row of the matrix has a particular (I_o, J_o).

Each column of the matrix which multiplies the element, $b_{I,J}$, is labelled by the appropriate I and J.

The second stage of the method is to reduce the number of unknowns in equation(6.56) from 16 to 12 by rewriting this equation with the corner gradients on the left hand side as known quantities:

$$\begin{bmatrix} M^3N\begin{bmatrix}\Delta B_1\\ \Delta B_2\\ \Delta B_3\end{bmatrix} \\ M^3N\begin{bmatrix}\Delta B_4\\ \Delta B_5\\ \Delta B_6\\ \Delta B_7\\ \Delta B_8\\ \Delta B_9\end{bmatrix} \\ M^3N\begin{bmatrix}\Delta B_{10}\\ \Delta B_{11}\\ \Delta B_{12}\end{bmatrix} \end{bmatrix} - \begin{bmatrix}2 & 3\\ 1 & 2\\ & 1\end{bmatrix}\begin{bmatrix}b_{0,0}\\ b_{1,0}\end{bmatrix} - \begin{bmatrix}1 & \\ 2 & 1\\ 3 & 2\end{bmatrix}\begin{bmatrix}b_{2,3}\\ b_{3,3}\end{bmatrix} = \left[\begin{array}{cccc|cccc|cccc|cccc} 4 & \boxed{4} & 2 & 1 & & & & & & & & & & & & \\ 3 & 4 & 3 & 2 & 1 & & & & & & & & & & & \\ 2 & 3 & 4 & 3 & 2 & 1 & & & & & & & & & & \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 & & & & & & & & & \\ \hline & 1 & 2 & 3 & 4 & 3 & 2 & 1 & & & & & & & & \\ & & 1 & 2 & 3 & 4 & 3 & 2 & 1 & & & & & & & \\ & & & 1 & 2 & 3 & 4 & 3 & 2 & 1 & & & & & & \\ & & & & 1 & 2 & 3 & 4 & 3 & 2 & 1 & & & & & \\ \hline & & & & & 1 & 2 & 3 & 4 & 3 & 2 & 1 & & & & \\ & & & & & & 1 & 2 & 3 & 4 & 3 & 2 & & & & \\ & & & & & & & 1 & 2 & 3 & 4 & 3 & & & & \\ & & & & & & & & 1 & 2 & \boxed{4} & 4 & & & & \end{array}\right]\begin{bmatrix}b_{2,0}\\ b_{3,0}\\ b_{0,1}\\ b_{1,1}\\ b_{2,1}\\ b_{3,1}\\ b_{0,2}\\ b_{1,2}\\ b_{2,2}\\ b_{3,2}\\ b_{0,3}\\ b_{1,3}\end{bmatrix}$$

"non-symmetric elements" (6.57)

Equation(6.57) is the basis for an iterative method of solving for the non-corner gradients in the column matrix on the right hand side of this matrix equation:Initially the corner gradients can be set to zero, according to equations (6.52) and (6.53).The solution to this matrix equation is then the "zeroth" solution.For the next and subsequent iterations the corner gradients on the left hand side of equation(6.57) could be equated to the nearest non-corner gradients solved in the previous iteration:

$$^{h}b_{I,0} = \left\{\begin{array}{ll} 0 & \text{If } h = 0 \\ {}^{h-1}b_{I,1} & \text{If } h \geqslant 1 \end{array}\right\} \qquad (6.58)$$

where $0 \leqslant I \leqslant M/2 - 1$ and

$$^{h}b_{I,N-1} = \left\{\begin{array}{ll} 0 & \text{If } h = 0 \\ {}^{h-1}b_{I,N-2} & \text{If } h \geqslant 1 \end{array}\right\} \qquad (6.59)$$

where $M/2 \leqslant I \leqslant M-1$.The superscript in the two equations above shows in which iteration that particular gradient occurs.If h=0 equations (6.58) and (6.59) are identical with equations (6.52) and (6.53) respectively.

The inverse of the 12X12 matrix in equation(6.57) was found by using a standard matrix inversion computer program. To three significant figures this inverse matrix is

3.23	-5.54	2.31	0.08	2.15	-3.62	1.38	0.15	1.08	-1.69	0.46	0.23
-3.00	6.00	-3.00	0.00	-2.00	4.00	-2.00	0.00	-1.00	2.00	-1.00	0.00
0.00	-1.00	2.00	-1.00	0.00	-1.00	2.00	-1.00	0.00	-1.00	2.00	-1.00
0.08	0.15	-1.23	1.69	-0.62	0.46	-1.54	1.38	-0.31	0.77	-1.85	1.08
2.15	-3.69	1.54	-0.62	2.77	-3.08	0.92	-0.23	1.38	-1.46	0.31	0.15
-2.00	4.00	-2.00	0.00	-2.00	4.00	-2.00	0.00	-1.00	2.00	-1.00	0.00
0.00	-1.00	2.00	-1.00	0.00	-2.00	4.00	-2.00	0.00	-2.00	4.00	-2.00
0.15	0.31	-1.46	1.38	-0.23	0.92	-3.08	2.77	-0.62	1.54	-3.69	2.15
1.08	-1.85	0.77	-0.31	1.38	-1.54	0.46	-0.62	1.69	-1.23	0.15	0.08
-1.00	2.00	-1.00	0.00	-1.00	2.00	-1.00	0.00	-1.00	2.00	-1.00	0.00
0.00	-1.00	2.00	-1.00	0.00	-2.00	4.00	-2.00	0.00	-3.00	6.00	-3.00
0.23	0.46	-1.69	1.08	0.15	1.38	-3.62	2.15	0.08	2.31	-5.54	3.23

Example 2 M=8, N=5.

The matrix equation analogous with equation(6.57) was worked out in the same way and found to be given by:

$$
\begin{bmatrix} M^3N \begin{bmatrix} \Delta B_1 \\ \Delta B_2 \\ \Delta B_3 \\ \Delta B_4 \\ \Delta B_5 \\ \Delta B_6 \\ \Delta B_7 \end{bmatrix} \\ M^3N \begin{bmatrix} \Delta B_8 \\ \Delta B_9 \\ \vdots \\ \Delta B_{24} \\ \Delta B_{25} \end{bmatrix} \\ M^3N \begin{bmatrix} \Delta B_{26} \\ \Delta B_{27} \\ \Delta B_{28} \\ \Delta B_{29} \\ \Delta B_{30} \\ \Delta B_{31} \\ \Delta B_{32} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \\ & 1 & 2 & 3 \\ & & 1 & 2 \\ & & & 1 \end{bmatrix} \begin{matrix} b_{0,0} \\ b_{1,0} \\ b_{2,0} \\ b_{3,0} \end{matrix} \\ \\ \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 3 & 2 & 1 & \\ 4 & 3 & 2 & 1 \\ 5 & 4 & 3 & 2 \\ 6 & 5 & 4 & 3 \\ 7 & 6 & 5 & 4 \end{bmatrix} \begin{matrix} b_{4,4} \\ b_{5,4} \\ b_{6,4} \\ b_{7,4} \end{matrix} \end{bmatrix}
$$

$$
\left[\begin{array}{cccccccc|cccccccc|cccccccc|c}
8 & \boxed{8} & 8 & \boxed{8} & 4 & 3 & 2 & 1 & & & & & & & & & & & & & & & & b_{4,0} \\
7 & 8 & \boxed{8} & \boxed{8} & 5 & 4 & 3 & 2 & 1 & & & & & & & & & & & & & & & b_{5,0} \\
6 & 7 & 8 & \boxed{8} & 6 & 5 & 4 & 3 & 2 & 1 & & & & & & & & & & & & & & b_{6,0} \\
5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & & & & & & & & & & & & & b_{7,0} \\
4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & & & & & & & & & & & & b_{0,1} \\
3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & & & & & & & & & & & b_{1,1} \\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & & & & & & & & & & b_{2,1} \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & & & & & & & & & b_{3,1} \\
\hline
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & & & & & & & & b_{4,1} \\
 & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & & & & & & & b_{5,1} \\
 & & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & & & & & & b_{6,1} \\
 & & & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & & & & & b_{7,1} \\
 & & & & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & & & & b_{0,2} \\
 & & & & & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & & & b_{1,2} \\
 & & & & & & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & & b_{2,2} \\
 & & & & & & & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & b_{3,2} \\
\hline
 & & & & & & & & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & b_{4,2} \\
 & & & & & & & & & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & b_{5,2} \\
 & & & & & & & & & & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & b_{6,2} \\
 & & & & & & & & & & & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & 4 & b_{7,2} \\
 & & & & & & & & & & & & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & 5 & b_{0,3} \\
 & & & & & & & & & & & & & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 6 & b_{1,3} \\
 & & & & & & & & & & & & & & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & b_{2,3} \\
 & & & & & & & & & & & & & & & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & b_{3,3} \\
\hline
 & & & & & & & & & & & & & & & & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & b_{4,3} \\
 & & & & & & & & & & & & & & & & & & 1 & 2 & 3 & 4 & 5 & 6 & b_{5,3} \\
 & & & & & & & & & & & & & & & & & & & 1 & 2 & 3 & 4 & 5 & b_{6,3} \\
 & & & & & & & & & & & & & & & & & & & & 1 & 2 & 3 & 4 & b_{7,3} \\
 & & & & & & & & & & & & & & & & & & & & & 1 & 2 & 3 & b_{0,4} \\
 & & & & & & & & & & & & & & & & & & & & & & 1 & 2 & b_{1,4} \\
 & & & & & & & & & & & & & & & & & & & & & & & 1 & b_{2,4} \\
 & & & & & & & & & & & & & & & & & & & & & & & & b_{3,4}
\end{array}\right]
$$

"non symmetric" elements

(6.60)

The corner gradients ^{h}b on the left hand side of equation(6.60) are given in terms of the non-corner gradients, from the h-1th solution, by equations (6.58) and (6.59).

By working out more examples it is possible to write down equations of the type, equations (6.57) and (6.60), for any M an N. Further examples will not be written down but it will be seen that the generalization given below, for M any even number and N any number greater than 3, covers both examples already considered.

The general set of equations to be solved in determining the h^{th} solution for the non-corner gradients can be written

$$({}^{h}\Delta) = (\phi)({}^{h}b) \quad . \tag{6.61}$$

$({}^{h}b)$ is the column matrix containing the MN-M non-corner gradients

$$({}^{h}b)_k = {}^{h}b_{I,J} \quad , \tag{6.62}$$

where

$$k = I + JM + 1 - M/2, \tag{6.63}$$

the ranges of I and J being (a) $J = 0$ and $M/2 \leqslant I \leqslant M - 1$, (b) $1 \leqslant J \leqslant N - 2$ and $0 \leqslant I \leqslant M - 1$ and (c) $J = N - 1$ and $0 \leqslant I \leqslant M/2 - 1$. This gives k in the range $1 \leqslant k \leqslant MN - M$ as it must of course.

$({}^{h}\Delta)$ is the column matrix of known quantities defined as follows:

(a) The "middle" (N-3)M+2 elements of $({}^{h}\Delta)$ are independent of h, being normalized line integrals times M^3N:

$$({}^{h}\Delta)_i = M^3N \cdot \Delta B_i^{(X)} \quad , \tag{6.64}$$

where $M \leqslant i \leqslant MN-2M+1$.

(b) The "top" M-1 elements are given by

$$({}^{h}\Delta)_i = M^3N \cdot \Delta B_i^{(X)} - \sum_{I=0}^{\frac{M}{2}-1} (U)_{i,I} \cdot {}^{h}b_{I,0} \quad , \tag{6.65}$$

where (U) is the $M-1 \times M/2$ matrix whose elements are given by

$$(U)_{i,j} = \begin{cases} (j - i) + M/2 & \text{If } (j - i) \geqslant -(M/2-1) \\ 0 & \text{Otherwise} \end{cases} \tag{6.66}$$

The elements of (U) are constant along the different diagonals

for which j - i is constant. In equation(6.60), for example, where M=8, all the elements in the diagonals below the diagonal, j - i = -4, are zero while those in the diagonals j - i = -4, -3, -2, -1, 0, 1, 2, 3, are M/2=4 plus these figures; i.e. 0, 1, 2, 3, 4, 5, 6, 7.

The range of i in equations (6.65) and (6.66) is $1 \leqslant i \leqslant M-1$. $h_{b_{I,0}}$ is given by equation(6.58).

c) The "bottom" M-1 elements are given by

$$({}^{h}\Delta)_{i+MN-2M+1} = M^3 N \cdot \Delta B^{(X)}_{i+MN-2M+1} - \sum_{I=0}^{\frac{M}{2}-1} (V)_{i,I} \cdot h_{b_{I+M/2,N-1}} , \tag{6.67}$$

where (V) is the M-1 X M/2 matrix whose elements are given by

$$(V)_{i,j} = \left\{ \begin{array}{ll} (i-j)+1 & \text{If } (i-j) \geqslant 0 \\ 0 & \text{Otherwise} \end{array} \right\} \tag{6.68}$$

From equation(6.60) it can be seen that The matrix (V) is (U) rotated through 180° in the plane of the paper.

The range of i in equations (6.67) and (6.68) is $1 \leqslant i \leqslant M-1$. $h_{b_{I+M/2,N-1}}$ is given by equation(6.59)

(ϕ) is the MN-M X MN-M almost symmetric matrix in equation (6.61). It is best defined by by redefining some of the elements of a symmetric matrix:

(a) The "underlying" symmetric matrix.

Elements above and on the principal diagonal, j - i = 0, are given by

$$(\phi)_{i,j} = \left\{ \begin{array}{ll} 0 & \text{If } M \leqslant j-i \leqslant MN-M-1 \\ M-(j-i) & \text{If } 0 \leqslant j-i \leqslant M \end{array} \right\}, \tag{6.69}$$

and those below the principal diagonal are given in terms of these by

$$(\phi)_{j,i} = (\phi)_{i,j} \quad . \tag{6.70}$$

(b) The non symmetric elements of (ϕ).

The following (M/2-1)(M/4) elements above the principal diagonal are redefined as follows:

$$(\phi)_{i,j} = M \text{ , for } 2 \leqslant j \leqslant M/2 \text{ and } 1 \leqslant i \leqslant j \text{ ,} \tag{6.71}$$

The remaining (M/2-1)(M/4) elements below the principal diagonal are given in terms of these as follows:

$$(\phi)_{MN-M+1-i,MN-M+1-j} = (\phi)_{i,j} \quad . \tag{6.72}$$

In the trivial case M=2 there are no non symmetric elements. The explicit solution to the non corner gradients can be written by inverting equation(6.61):

$$({}^{h}b) = (\phi)^{-1}({}^{h}\Delta) \ , \qquad (6.73)$$

where $(\phi)^{-1}$ is the inverse of the matrix,(ϕ).

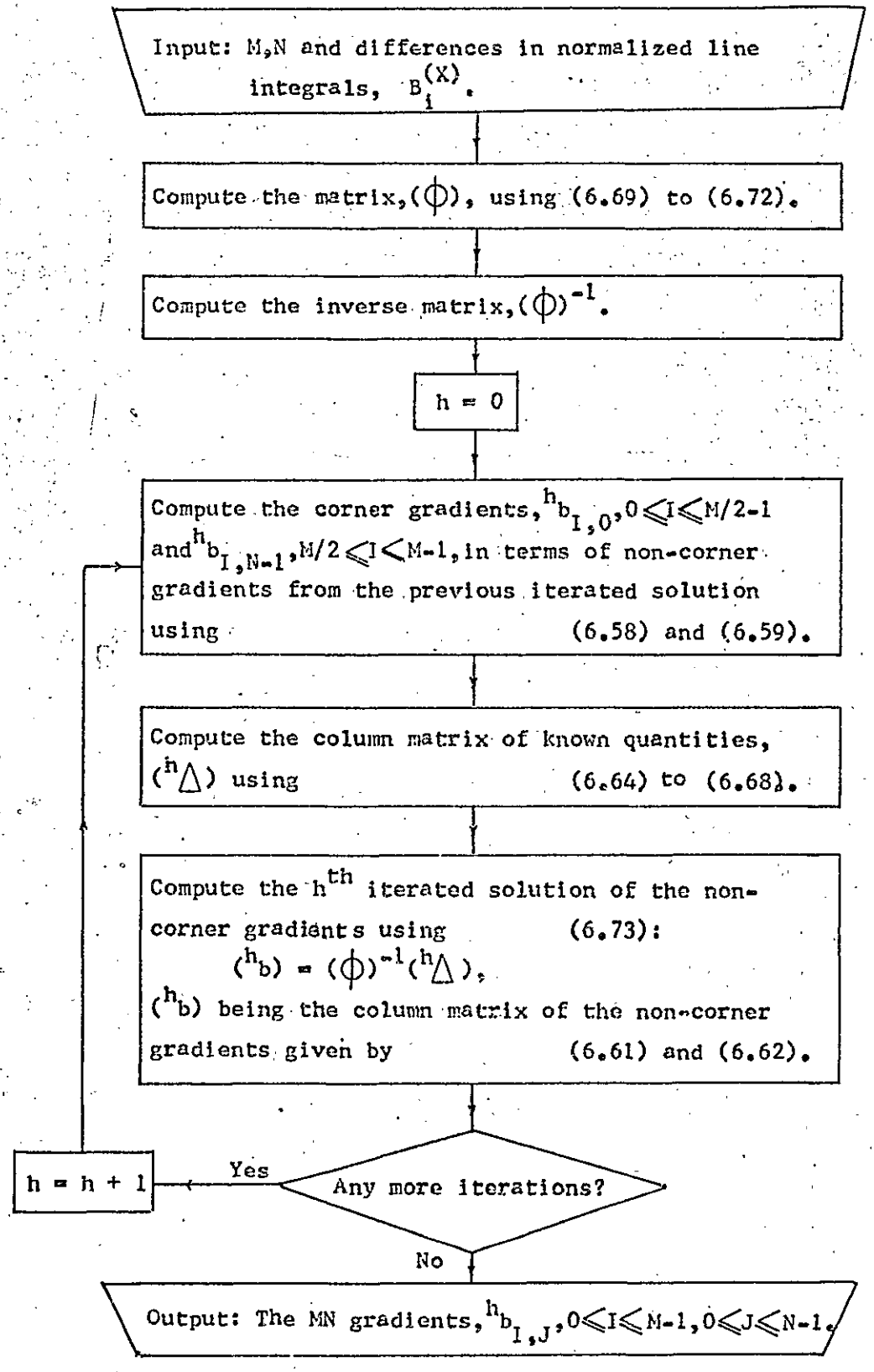
The inverses of all the matrices,(ϕ), of orders up to and including 32 ,for even M and $N \geqslant 3$,were found using the matrix inversion program mentioned before.M and N for each these matrices together with their order L = MN - M is given in the table below.

M	2														
N	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
L	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32

M	4						6				8			10	
N	3	4	5	6	7	8	3	4	5	6	3	4	5	3	4
L	8	12	16	20	24	28	12	18	24	30	16	24	32	20	30

Each matrix,(ϕ),and its inverse,$(\phi)^{-1}$, possess a symmetry which leaves them unchanged when they are rotated through an angle of 180° in the plane of the paper.This is the same symmetry exhibited by the "X" rays within their rectangular boundary.(See Fig(3.4)).

Fig(6.4) is a flow diagram summarizing the stages in the determination of the gradients.



Fig(6.4) Flow diagram summarizing the stages in the determination of the gradients, $b_{I,J}$, The numbers in brackets refer to relavant equations.

Appendix 1. The Representation of a Function in a Given Range by a Set of Functions Orthogonal in that Range.

A 1.1 The Exact Representation of a One-Dimensional Function, in any Range of Unit Width, by an Infinite Series of One-Dimensional Orthogonal Functions.

A 1.1.1 Definition and Basic Properties of the Orthogonal Set of Functions.

Define an infinite set of functions of a real variable x, $\{\phi_p(x)\}$, orthogonal in the range $x_0 \leq x < x_0+1$ by the orthogonality relation

$$\int_{x_0}^{x_0+1} \phi_p(x)\, \phi^*_{p'}(x)\, dx = c\delta_{pp'} = \begin{cases} 0 & \text{if } p \neq p' \\ 1 & \text{if } p = p' \end{cases}, \qquad (A1.1)$$

where p are integers which label the functions, and c is a real constant. The functions may take on complex values. It is convenient to define some sets of orthogonal functions such that p runs from $-\infty$ to $+\infty$, while for other sets only $p \geq 0$ is the best choice. Examples of the former are $\{e^{2\pi ipx},\ -\infty<p<\infty\}$ and $\{\sin 2\pi px + \cos 2\pi px,\ -\infty<p<\infty\}$ while examples of the latter are:-

$\{\phi_0(x) = 1,\ \phi_p(x) = \sqrt{2}\sin\pi(p+1)x,$ if p is odd, $\phi_p(x) = \sqrt{2}\cos\pi px,$ if p is even, $1 \leq p < \infty\}$

and the set of Walsh functions $\{wal(p,x),\ 0 \leq p < \infty\}$.

If c = 1 in equation (A1.1) the set $\{\phi_p(x)\}$ is said to be orthonormal. Any orthonormal set may always be obtained from an orthogonal one for which $c \neq 1$, by normalizing the functions of the latter: The set $\left\{\frac{1}{\sqrt{c}}\phi_p(x)\right\}$ is an orthonormal one, as can be seen by substituting in equation (A1.1).

The orthonormal sets of functions considered here are restricted by a further defining relation, namely

$$\phi_0(x) = 1 \ , \qquad (A1.2)$$

for all x.

Lemma 1

The mean values of all the functions of the set $\{\phi_p(x),\ p \neq 0\}$ are zero in the range of orthonormality, where $\{\phi_p(x),$ all $p\}$ is an orthonormal set of functions.

Proof.

Consider the orthogonality of $\phi_{p'}(x) = \phi_0(x)$ with all the other functions $\phi_p(x)$, $p \neq 0$ and with itself.

If $p = p' = 0$, equation (A1.1) gives $\int_{x_0}^{x_0+1}(1)(1)dx = 1$.

If $p \neq p'$, equation (A1.1) gives $\int_{x_0}^{x_0+1}\phi_p(x)(1)dx = 0$, which is the mean of $\phi_p(x)$, $p \neq 0$ in the range of orthogonality. Q.E.D.

Now let the definition of the set $\{\phi_p(x)\}$ be extended to include all values of x by periodically repeating each function:

$$\phi_p(x+n) = \phi_p(x) \qquad x_0 \leq x < x_0+1, \quad n = \pm 1, \pm 2, \ldots \tag{A1.3}$$

Lemma 2

The infinite set of functions $\{\phi_p(x)\}$ defined completely by equations (A1.1), (A1.2) and (A1.3) above is orthogonal in any range of x of unit width, $x_2 \leq x < x_2+1$, where $-\infty < x_2 < \infty$.

Proof

Equation (A1.1), the orthogonality relation, can be written

$$\int_{x_0}^{x_1}\phi_p(x)\,.\,\phi^*_{p'}(x)dx + \int_{x_1}^{x_0+1}\phi_p(x)\,.\,\phi^*_{p'}(x)dx = c\delta_{pp'}, \text{ where } x_0 < x_1 < x_0+1.$$

Now let $x' = x + n$ in the first integral, and $x'' = x + n - 1$ in the second integral, where n is an integer $\pm 1, \pm 2, \ldots$. This gives

$$\int_{x_0+n}^{x_1+n}\phi_p(x' - n)\,.\,\phi^*_{p'}(x' - n)dx' + \int_{x_1+n}^{(x_0+1)+(n-1)}\phi_p(x''-(n-1))\,.\,\phi^*_{p'}(x''-(n-1))dx'' = c\delta_{pp'}.$$

Now using equation (A1.3) and replacing the dummy variables x' and x'' by x gives

$$\int_{x_0+n}^{x_1+n}\phi_p(x)\,.\,\phi^*_{p'}(x)dx + \int_{x_1+n-1}^{x_0+n}\phi_p(x)\,.\,\phi^*_{p'}(x)dx = c\delta_{pp'}.$$

Now since $x_0 < x_1$ it follows that $x_0 + n < x_1 + n$. Also, since $x_1 < x_0 + 1$, it follows that $x_1 + n - 1 < x_0 + n$. Thus the two integrals above, written in reverse order, can be replaced by the single integral

$$\int_{x_1+n-1}^{x_1+n} \phi_p(x) \cdot \phi_{p'}^*(x)\,dx = \delta_{pp'} \qquad \text{or writing } x_2 = x_1 + n - 1$$

$$\int_{x_2}^{x_2+1} \phi_p(x) \cdot \phi_{p'}^*(x) = \delta_{pp'}, \qquad \text{(A1.4)}$$

where clearly x_2 can take on any real value from $-\infty$ to ∞ Q.E.D.

Lemma 3

The infinite set of functions $\left\{ \phi_p\left(\frac{x}{X}\right)\right\}$ is orthogonal in an interval of width X, $x_3 \leq x < x_3 + X$, given that $\{\phi_p(x)\}$ is a set of functions orthonormal in unit range of x.

Proof

The orthogonality integral for functions in a range $x_3 \leq x < x_3 + X$ is

$$I = \int_{x_3}^{x_3+X} \phi_p\left(\frac{x'}{X}\right)\phi_{p'}^*\left(\frac{x'}{X}\right)dx', \qquad x = \frac{x'}{X} \text{ gives}$$

$$I = X \int_{x_3/X}^{(x_3+X)/X} \phi_p(x)\phi_{p'}^*(x)\,dx = X \int_{x_3/X}^{(x_3/X)+1} \phi_p(x)\phi_{p'}^*(x)\,dx = X\delta_{pp'}.$$

(by equation (A1.4)) Q.E.D.

Clearly the set, $\left\{ \frac{1}{\sqrt{X}}\phi_p\left(\frac{x}{X}\right)\right\}$, would be one orthonormal in the range $x_3 \leq x < x_3 + X$.

A set of functions, orthogonal in a range of width X, can always be reduced to a set orthonormal in unit range. This is often convenient to avoid constants in the formulae.

A1.1.2 Expansion of a Function by an Infinite Series of One-Dimensional Orthogonal Functions.

Let a continuous function, f(x), be represented exactly in the range of interest $x_2 \leq x < x_2 + 1$ by the infinite series of orthonormal functions:

$$f(x) = \sum_{p,\infty} F_p \phi_p(x), \qquad \text{(A1.5)}$$

where the notation $\sum_{p,\infty}$ means $\sum_{p=-\infty}^{\infty}$ for a set $\{\phi_p(x),\ -\infty<p<\infty\}$ and $\sum_{p=0}^{\infty}$ for a set $\{\phi_p(x),\ 0\leq p<\infty\}$.

It follows immediately from the periodicity of the functions $\phi_p(x)$ that equation (A1.4) represents, for all values of x, a periodic function $f(x+n)=f(x)$, $x_0<x<x_0+1$, $n=0,\pm1,\pm2,\ldots$

Now multiplying equation (A1.5) by $\phi_p^*(x)$ and integrating from $x=x_2$ to $x=x_2+1$ gives

$$\int_{x_2}^{x_2+1} f(x)\phi_{p'}^*(x)dx = \sum_{p,\infty} F_p \int_{x_2}^{x_2+1} \phi_p(x)\phi_{p'}^*(x)dx = \sum_{p,\infty} F_p \delta_{pp'} = F_{p'}.$$

Thus the coefficients F_p form a unique infinite set given by

$$F_p = \int_{x_2}^{x_2+1} f(x)\phi_p^*(x)dx \quad . \tag{A1.6}$$

Now if it were possible to find a function f(x) (excluding $f(x)=0$ for all x) such that F_p, calculated from equation (A1.6), was zero for all values of p, then clearly the infinite series, equation (A1.5), is unable to represent f(x). But if this were so, f(x) would by definition (by equation (A1.4)) be orthogonal to all $\phi_p(x)$, and so $\{\phi_p(x)\}$ would not be a complete or <u>closed</u> set. More formally, the orthogonal set $\{\phi_p(x)\}$ is said to be <u>closed</u> if no <u>quadratically integrable</u> function, f(x) (i.e. one for which $0\leq\int_{x_2}^{x_2+1}|f(x)|^2\,dx<$ a finite positive real number) exists such that $\int_{x_2}^{x_2+1} f(x)\phi_p^*(x)dx=0$ for all p. By expressing the infinite series for f(x), equation (A1.5) as $f(x)=\lim_{N\to\infty}\sum_{p,N} F_p\phi_p(x)$ (where a certain set "$\{(p,N)\}$" of N values of p is summed) the finite quantity $\int_{x_2}^{x_2+1}|f(x)|^2dx$ is given by

$$\begin{aligned}\int_{x_2}^{x_2+1}|f(x)|^2dx &= \lim_{N\to\infty}\int_{x_2}^{x_2+1}\Big(\sum_{p,N}F_p\phi_p(x)\Big)\Big(\sum_{p',N}F_{p'}^*\phi_{p'}^*(x)\Big)dx\\ &= \lim_{N\to\infty}\sum_{p,N}\sum_{p',N}F_pF_{p'}^*\int_{x_2}^{x_2+1}\phi_p(x)\phi_{p'}^*(x)dx\\ &= \lim_{N\to\infty}\sum_{p,N}\sum_{p',N}F_pF_{p'}^*\delta_{pp'}\\ &= \lim_{N\to\infty}\sum_{p,N}|F_p|^2 \quad .\end{aligned}$$

Since $|F_p|^2$ is always positive, it is clear that for finite N

$$\int_{x_2}^{x_2+1}|f(x)|^2dx \geq \sum_{p,N}|F_p|^2 \quad \text{or} \quad \int_{x_2}^{x_2+1}|f(x)|^2dx - \sum_{p,N}|F_p|^2 \geq 0 \quad . \tag{A1.7}$$

This is the Bessel inequality which, for $N \to \infty$, becomes the equality

$$\int_{x_2}^{x_2+1} |f(x)|^2 dx = \sum_{p,\infty} |F_p|^2 . \qquad (A1.8)$$

This is the completeness relation or Parseval's theorem. It is a necessary condition for the exact representation of f(x) by the closed orthonormal set $\{\phi_p(x)\}$.

A1.2 The Approximate Representation of a One-Dimensional Function, in any Range of Unit Width, by a Finite Series of One-Dimensional Orthogonal Functions, $\{\phi_p(x)\}$.

A1.2.1 The Criteria for the "best" Finite Series Representation.

Let the function, f(x), in the range $x_2 \leq x < x_2 + 1$ be approximately represented by a finite sum involving a number, M, of the functions in the closed orthonormal set $\{\phi_p(x)\}$:

$$f(x) \simeq f_a(x) = \sum_{p,M} E_p \phi_p(x) , \qquad (A1.9)$$

where E_p are M coefficients as yet undetermined. The notation "$\sum_{p,M}$" means that M different values of p occur in the summation, not necessarily consecutive values n, n + 1,....n + M - 1, where n is some integer. The values of p are taken from the set $\{-\infty < p < \infty\}$ or $\{0 \leq p < \infty\}$ depending on the range of p values for the set $\{\phi_p(x)\}$ in question. Denote this set of M values by $\{(p,M)\}$.

A criterion that equation (A1.9) is a "good" representation of f(x) is that the mean square deviation of $f_a(x)$ with respect to f(x), given by the exact series, equation (A1.5), shall be a minimum. This mean square deviation, Q, is given by

$$\begin{aligned} Q &= \int_{x_2}^{x_2+1} |f(x) - f_a(x)|^2 dx = \int_{x_2}^{x_2+1} [f(x) - f_a(x)][f(x) - f_a(x)]^* dx \\ &= \int_{x_2}^{x_2+1} f(x) f^*(x) dx - \int_{x_2}^{x_2+1} f(x) f_a^*(x) dx - \int_{x_2}^{x_2+1} f_a(x) f^*(x) dx + \int_{x_2}^{x_2+1} f_a(x) f_a^*(x) dx \\ &= I_1 + I_2 + I_2^* + I_3 , \qquad (A1.10) \end{aligned}$$

where I_1, I_2, I_2^* and I_3 are the four integrals on the r.h.s. of equation (A1.10).

Substituting the exact series representation, equation (A1.5), and the approximate series representation, equation (A1.9) for $f(x)$ and $f_a(x)$ respectively in the integral I_2 gives

$$I_2 = -\int_{x_1}^{x_{1+1}} f(x) f_a^*(x)\,dx = -\int_{x_1}^{x_{1+1}} \left(\sum_{p,\infty} F_p \phi_p(x)\right)\left(\sum_{p',M} E_{p'} \phi_{p'}(x)\right)^* = -\sum_{p,\infty}\sum_{p',M} F_p E_{p'}^* \int_{x_1}^{x_{2+1}} \phi_p(x)\phi_{p'}^*(x)\,dx$$

$$= -\sum_{p,\infty}\sum_{p',M} F_p E_{p'}^* \delta_{pp'} = -\sum_{p,M} F_p E_p^* .$$

Likewise the integral I_3 becomes

$$I_3 = +\int_{x_1}^{x_{1+1}} f_a(x) f_a^*(x)\,dx = +\sum_{p,M} E_p E_p^* .$$

Equation (A1.10) can now be written

$$Q = \int_{x_1}^{x_{1+1}} |f(x)|^2 dx + \sum (-F_p E_p^* - F_p^* E_p + E_p E_p^*)$$

$$= \left(\int_{x_1}^{x_{1+1}} |f(x)|^2 dx - \sum_{p,M} F_p F_p^*\right) + \sum_{p,M} (F_p - E_p)(F_p - E_p)^*$$

$$= \left(\int_{x_1}^{x_{1+1}} |f(x)|^2 dx - \sum_{p,M} |F_p|^2\right) + \sum_{p,M} |(F_p - E_p)|^2 .$$

The quantity in brackets above is positive by the Bessel inequality, equation (A1.7). The second term above is positive if $E_p \neq F_p$, but zero if $E_p = F_p$, so that Q is a minimum, regardless of what $\{(p,M)\}$ is, if $E_p = F_p$. Thus, for a given set $\{(p,M)\}$

$$Q_{min} = \sum_{p,\infty} |F_p|^2 - \sum_{p,M} |F_p|^2 , \qquad \text{(A1.11)}$$

which is clearly a minimum if the set $\{(p,M)\}$ is chosen for which the coefficients, F_p, are the M largest out of the infinite set of all F_p. ("largest" for a complex coefficient refers to it s modulus). Many sets $\{\phi_p(x)\}$ are so ordered that the higher the absolute value of p, the more changes positive to negative, negative to positive occur in $\phi_p(x)$ in the range of orthonormality. A consequence of this is that if $f(x)$ is a smooth well-behaved function, the coefficients in it s series representation get smaller as $|p|$ increases. The set $\{(p,M)\}$ chosen then is nearly always $\{0 \leq p \leq M-1\}$ for an infinite orthonormal set $\{\phi_p(x),\ 0 \leq p < \infty\}$, and $\{-\frac{M}{2} \leq p < \frac{M}{2},\ M \text{ odd}\}$ or

$\left\{-\frac{1}{2}(M-1) \leq p < \frac{1}{2}(M-1),\ M \text{ even}\right\}$ for a set $\left\{\phi_p(x),\ -\infty<p<\infty\right\}$. For the appropriate set, $\left\{(p,M)\right\}$, above the "best" finite series approximation for f(x), equation (A1.9), is then

$$f_a(x) = \sum_{p,M} F_p \phi_p(x), \qquad (A1.12)$$

where each coefficient, F_p, is given by equation (A1.6).

A1.2.2 The Approximate Representation of a Function at M Discrete Values of x by a Discrete Inverse Unitary Transform of the M Coefficients of its Finite Series Representation.

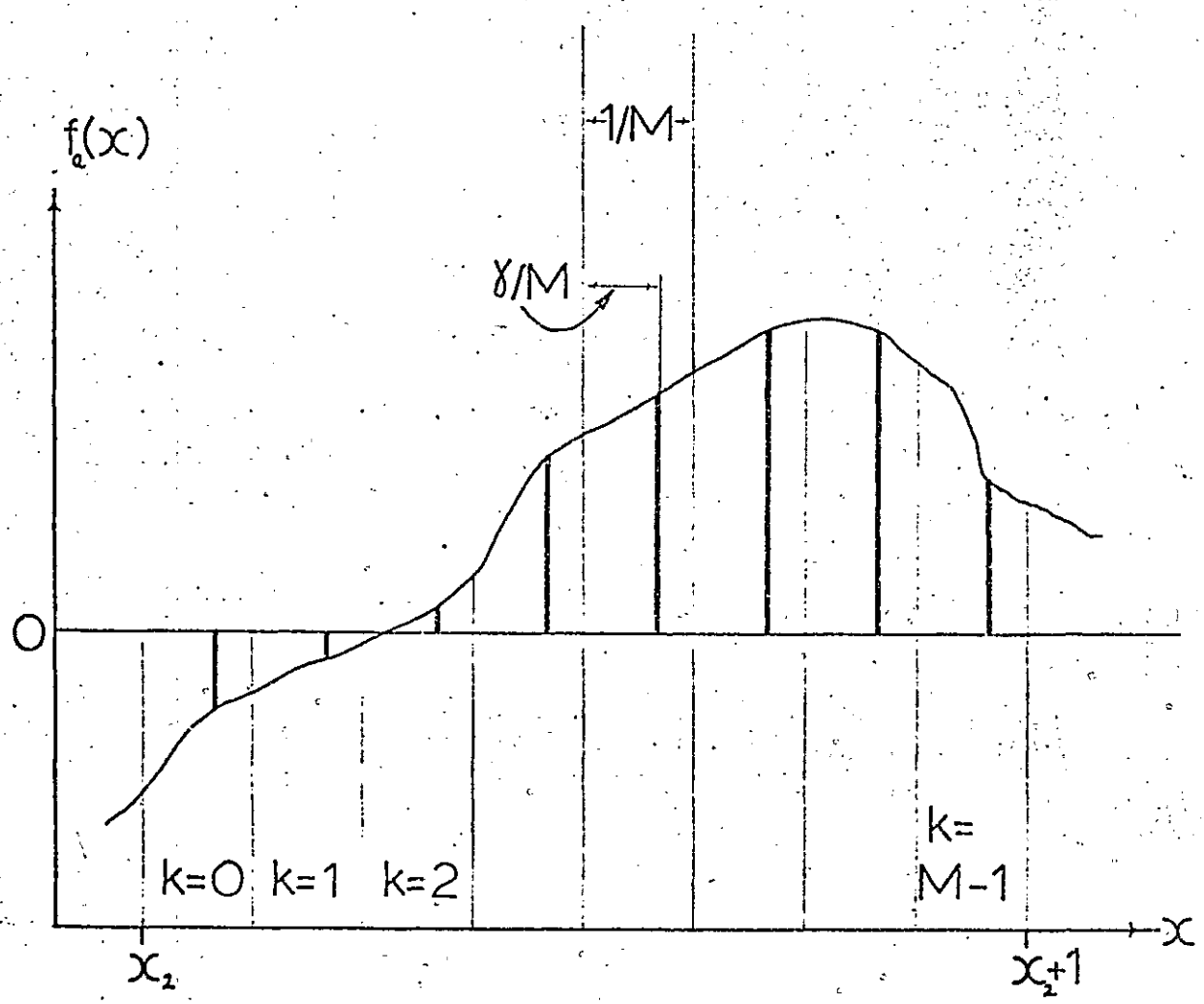
By considering $f_a(x)$, sampled at M equal intervals at distance $\frac{1}{M}$ appart, in the range $x_2 \leq x < x_2 + 1$, it is possible to express the M coefficients, F_p, $p \in \left\{(p,M)\right\}$, as finite summations rather than in the form of integrals, equation (A1.6). It will be shown that an extra "summation type" of orthogonality relation, analogous with equation (A1.4) and closely connected with it, must be satisfied for this to be possible.

Fig. (A1.1) shows $f_a(x)$ divided up into M equal sections or bins, each having a width of $\frac{1}{M}$. $f_a(x)$ is sampled once in the same relative position within each bin, so that the samples are $\frac{1}{M}$ appart. Substituting for the M discrete values $x = x_2 + \frac{k+\gamma}{M}$, where k is an integer variable and γ is a constant, in equation (A1.12) gives

$$\gamma f_k = f_a\left(x_2 + \frac{k+\gamma}{M}\right) = \sum_{p,M} F_p \phi_p\left(x_2 + \frac{k+\gamma}{M}\right);\quad 0 \leq k < M;\ 0 \leq \gamma < 1, \qquad (A1.13)$$

where each coefficient F_p was given by equation (A1.6), an integral of the function $f(x)\phi_p^*(x)$. (The set of labels $\left\{(p,M)\right\}$ in the summation $\sum_{p,M}$ is chosen as appropriate from $\left\{-\frac{M}{2} \leq p < \frac{M}{2}\right\}$, $\left\{-\frac{1}{2}(M-1) \leq p \leq \frac{1}{2}(M-1)\right\}$ or $\left\{0 \leq p < M\right\}$. Now, as a first step in finding an expression for F_p in terms of discrete sampled functions, equation (A1.13) is multiplied by $\phi_{p'}^*(x_2 + \frac{k+\gamma}{M})$ and summed over all k, $0 \leq k < M$. This gives

$$\sum_{k=0}^{M-1} f_a(x_2 + \frac{k+\gamma}{M})\phi_{p'}^*(x_2 + \frac{k+\gamma}{M}) = \sum_{p,M} F_p \left(\sum_{k=0}^{M-1} \phi_p(x_2 + \frac{k+\gamma}{M})\phi_{p'}^*(x_2 + \frac{k+\gamma}{M})\right). \qquad (A1.14)$$



$$_{\gamma}f_k = f_a(x_2 + \frac{k+\gamma}{M}) \; ; 0 \leq k < M,$$

γ is a constant, $0 \leq \gamma < 1$

Fig(A1.1). The function, $f_a(x)$, approximating $f(x)$, sampled at M equal intervals. Faint vertical lines represent bin boundaries and bold vertical lines represent the function samples. A real continuous function, $f_a(x)$, is illustrated and M=8.

Now if, for one or more values of γ, $0 \leq \gamma < 1$, the discrete orthogonality relation

$$G_{pp'}(\gamma) = \sum_{k=0}^{M-1} \phi_p(x_2 + \frac{k+\gamma}{M}) \phi^*_{p'}(x_2 + \frac{k+\gamma}{M}) = C(\gamma,M)\delta_{pp'}, \qquad (A1.15)$$

is true, where $C(\gamma,M)$ is a constant, for any particular values of M and γ, then equation (A1.14) becomes

$$F_{p'} = \frac{1}{C(\gamma,M)} \sum_{k=0}^{M-1} f(x_2 + \frac{k+\gamma}{M}) \phi^*_{p'}(x_2 + \frac{k+\gamma}{M}), \qquad (A1.16)$$

for those one or more values of γ and M for which (A1.15) is true. For $p \neq p'$ equation (A1.15) represents the $\frac{M}{2}(M-1)$ equations

$$G_{pp'}(\gamma) = 0, \qquad (A1.17)$$

while for $p = p'$ it represents the M equations

$$G_{pp}(\gamma) = C(\gamma,M). \qquad (A1.18)$$

If there are values of γ common to the solutions of equations (A1.17) and (A1.18) then the M equations represented by equation (A1.16) will be true for those values of γ. When all $\frac{M}{2}(M+1)$ equations represented by equation (A1.15) are represented as a single matrix equation (see section A1.2.3) it is particularly easy to "test" any practical cases and find values of γ for which equation (A1.16) is true. Examples from the orthonormal set $\{\phi_0 = 1, \phi_{2q-1}(x) = \sqrt{2}\sin 2\pi qx, \phi_{2q}(x) = \sqrt{2}\cos 2\pi qx, 1 \leq q < \infty\}$ are for "allowed" values of γ and the corresponding value $C(\gamma,M)$: (a) If $M = 2$, $\gamma = \frac{1}{4}, \frac{3}{4}$; $C(\gamma,M) = M = 2$, (b) If $M = 3$ all values of γ are "allowed", $0 \leq \gamma < 1$; $C(\gamma,M) = M = 3$ and (c) If $M = 4$, γ values are the two solutions of $\sin \pi\gamma = \frac{1}{4}$ for $0 \leq \gamma < 1$; $C(\gamma,M) = M = 4$. For the exponential set $\{e^{2\pi i px}, -\infty < p < \infty, i = \sqrt{-1}\}$ it is easy to show that all values of γ, $0 \leq \gamma < 1$ and all values of M are allowed. Also $C(\gamma,M) = M$ for all γ and M. For $\gamma = 0$ the set $\{e^{2\pi i \frac{pk}{M}}\}$ is the basis of the discrete complex Fourier transform as conventionally used.

Lemma 4.

It is true that

$$\int_0^1 G_{pp'}(\gamma)\,d\gamma = M\delta_{pp'}, \qquad (A1.19)$$

where $G_{pp'}(\gamma) = \sum_{k=0}^{M-1} \phi_p(x_2 + \frac{k+\gamma}{M}) \phi^*_{p'}(x_2 + \frac{k+\gamma}{M})$, whether or not value(s) of γ can be found for $G_{pp'}(\gamma) = C(\gamma,M)\ pp'$ (equations (A1.17) and (A1.18)).

Proof

The integral in equation (A1.4), the relation defining the orthogonality of the set $\{\phi_p(x)\}$ in the range $x_2 \leq x < x_2 + 1$, can be split up into M integrals each covering the range of integration of a single bin:

$$\int_{x_2}^{x_2+1} \phi_p(x)\phi^*_{p'}(x)dx = \sum_{k=0}^{M-1} \int_{x_2+\frac{k}{M}}^{x_2+\frac{k+1}{M}} \phi_p(x)\phi^*_{p'}(x)dx = \delta_{pp'}.$$

Now let $x = x_2 + \frac{k+\gamma}{M}$, where γ is a variable, so $dx = \frac{1}{M}d\gamma$ and the orthogonality relation becomes

$$\sum_{k=0}^{M-1} \int_0^1 \phi_p(x_2 + \frac{k+\gamma}{M}) \phi^*_{p'}(x_2 + \frac{k+\gamma}{M}) d\gamma = M\delta_{pp'}.$$

Now changing the order of the summation and integration:

$$\int_0^1 \underbrace{\sum_{k=0}^{M-1} \phi_p(x_2 + \frac{k+\gamma}{M}) \phi^*_{p'}(x_2 + \frac{k+\gamma}{M})}_{G_{pp'}(\gamma)} d\gamma = M\delta_{pp'}.$$

Q.E.D.

Lemma 5.

If the set of M orthogonal functions $\left\{\phi_p(x_2 + \frac{k+\gamma}{M})\right\}$ is independent of γ for all M values of p and M values of k, (i.e. if the value of each of the functions is constant within all of the sampling bins), then the discrete orthogonality relation, equation (A1.15), is true for all γ, $0 \leq \gamma < 1$ and $C(\gamma,M) = M$.

Proof

If, for all values of p, $\phi_p(x_2 + \frac{k+\gamma}{M})$ is independent of γ for all M values of k, then for all pairs of values of p and p', $G_{pp'}(\gamma) = \sum_{k=0}^{M-1} \phi_p(x_2 + \frac{k+\gamma}{M}) \phi^*_{p'}(x_2 + \frac{k+\gamma}{M})$ is also independent of γ, where $G_{pp'}(\gamma) = G_{pp'}$, say. Thus in equation (A1.19), lemma 4, $G_{pp'}$ comes outside the integral:

$$M\delta_{pp'} = \int_0^1 G_{pp'}(\gamma)d\gamma = G_{pp'} \int_0^1 d\gamma = G_{pp'}.$$

Substituting for $G_{pp'}$, this is:

$$\sum_{k=0}^{M-1} \phi_p(x_2 + \frac{k+\gamma}{M}) \phi^*_{p'}(x_2 + \frac{k+\gamma}{M}) = M\delta_{pp'}. \qquad \text{(A1.20)}$$

$G_{pp'}$

Q.E.D.

It is emphasized that for a given infinite set of orthonormal functions, $\{\phi_p(x)\}$, the result above may be true for some values of M but not for others. As will be seen in Appendix 2, this is the case with Walsh functions where the result is confined to values of $M = 2^n$, $n = 0,1,2....$

The finite set of M coefficients, $\{F_p\}$, given by equation (A1.16), is an M^{th} order discrete unitary transform of the finite set of M samples $\{{}_\gamma f_k\} = \{f_a(x_2 + \frac{k+\gamma}{M})\}$. Conversely, the set $\{{}_\gamma f_k\}$, given by equation (A1.13) is an M^{th} order inverse discrete unitary transform of the set $\{F_p\}$. $\{{}_\gamma f_k\}$ and $\{F_p\}$ are said to form a discrete unitary transform pair of sets. If the set of numbers, $\{\phi_p(x_2 + \frac{k+\gamma}{M}), {}^{p\in\{(p,M)\}}_{0\leq k<M}\}$ upon which the transform is based is real, the transform (and it's inverse) is called "orthogonal" rather than "unitary". Equation (A1.15) is called a discrete "orthogonality" relation for both real and complex sets of numbers $\{\phi_p(x_2 + \frac{k+\gamma}{M})\}$.

Equation (A1.16) was derived from equation (A1.13) by using the discrete orthogonality condition, equation (A1.15). Now clearly equation (A1.13) must be obtainable [illegible] from equation (A1.16). Multiplying equation (A1.16) by $\phi_p(x_2 + \frac{k+\gamma}{M})$ and summing over p, $p\in\{(p,M)\}$ gives

$$\sum_{p,M} F_p \phi_p(x_2 + \frac{k'+\gamma}{M}) = \frac{1}{C(\gamma,M)} \sum_{k=0}^{M-1} f(x_2 + \frac{k+\gamma}{M}) \left(\sum_{p,M} \phi^*_p(x_2 + \frac{k+\gamma}{M}) \phi_p(x_2 + \frac{k'+\gamma}{M}) \right). \qquad \text{(A1.21)}$$

By comparing equations (A1.13) and (A1.21) it must be true that

$$\sum_{p,M} \phi^*_p(x_2 + \frac{k+\gamma}{M}) \phi_p(x_2 + \frac{k'+\gamma}{M}) = C(\gamma,M)\delta_{kk'}; \qquad \text{(A1.22)}$$

where $0\leq k'<M$; $0\leq k<M$.

Equation (A1.22) may be called the "reciprocal discrete orthogonality relation".

Lemma 6.

The completeness relation or Parseval's theorem for the discrete unitary transform pair $\left\{{}_{\gamma}f_k\right\}$ and $\left\{F_p\right\}$ is

$$\frac{1}{C(\gamma,M)}\sum_{k=0}^{M-1}\left|{}_{\gamma}f_k\right|^2 = \sum_{p,M}\left|F_p\right|^2 . \tag{A1.23}$$

Proof

From equation (A1.14)

$$\sum_{k=0}^{M-1}{}_{\gamma}f_k\,{}_{\gamma}f_k^* = \sum_{k=0}^{M-1}\left(\sum_{p,M}F_p\phi_p(x_2+\frac{k+\gamma}{M})\right)\left(\sum_{p',M}F_{p'}^*\phi_{p'}^*(x_2+\frac{k+\gamma}{M})\right)$$

$$= \sum_{p,M}\sum_{p',M}F_pF_{p'}^*\sum_{k=0}^{M-1}\phi_p(x_2+\frac{k+\gamma}{M})\phi_{p'}^*(x_2+\frac{k+\gamma}{M})$$

$$= \sum_{p,M}\sum_{p',M}F_pF_{p'}^*\delta_{pp'} = \sum_{p,M}F_pF_p^* . \qquad \text{Q.E.D.}$$

A1.2.3 The One-Dimensional Discrete Unitary Transform Relationships Expressed in Matrix Form.

In the matrix notation used below, the N × M transpose matrix of an M × N matrix, (A), is denoted by $\widetilde{(A)}$ where $\widetilde{(A)}_{nm} = (A)_{mn}$. The Hermitian conjugate of a matrix, (A), which is the transpose matrix with all the elements replaced by their complex conjugates, is denoted by $(A)^{\dagger} = \widetilde{(A)}^* = \widetilde{(A)^*}$. A 1 × M, or row, matrix of a finite number, M, of elements is implied when square brackets replace rounded ones: Thus $[B]$ is a row matrix, and it's transpose $\widetilde{[B]}$ is a column matrix containing the same elements. (I_M) is the M × M, or M^{th} order, unit matrix, $(I_M)_{mn} = \delta_{mn}$. Finally, use is made of the relation $\widetilde{(A)(B)} = \widetilde{(B)}\widetilde{(A)}$.

Define an M^{th} order (i.e. M × M) unitary transform matrix, based on the first M orthogonal functions, each sampled at M points, $\left\{\phi_p(x_2+\frac{k+\gamma}{M}),\ 0\le k<M\right\}$, where γ is a constant, $0\le\gamma<1$, by the relation:

$$({}_{\gamma}\Phi_M)_{pk} = \phi_p^*(x_2+\frac{k+\gamma}{M}) . \tag{A1.24}$$

Each of the orthogonality relations, equation (A1.15), and each of the reciprocal orthogonality relations, equation (A1.22),

respectively, can be rewritten as $\sum_{k=0}^{M-1} ({}_{\gamma}\tilde{\Phi}_M)_{p'k} (\widetilde{{}_{\gamma}\Phi_M})^*_{kp} = C(\gamma,M)\delta_{pp'}$ and $\sum_{p=M} (\widetilde{{}_{\gamma}\Phi_M})^*_{k'p} ({}_{\gamma}\tilde{\Phi}_M)_{pk} = C(\gamma,M)\delta_{kk'}$. The matrix equations representing these is then:

$$({}_{\gamma}\Phi_M)({}_{\gamma}\Phi_M)^\dagger = ({}_{\gamma}\Phi_M)^\dagger ({}_{\gamma}\Phi_M) = C(\gamma,M)(I_M) . \tag{A1.25}$$

Equation (A1.25) defines the matrix $\frac{1}{\sqrt{C(\gamma,M)}}$ $({}_{\gamma}\Phi_M)$ to be a unitary matrix.

Define a row matrix $[f]$, and the corresponding column matrix $[\tilde{f}]$, which contain the M elements $f_a(x_2 + \frac{k+\gamma}{M}) = {}_{\gamma}f_k$, $0 \le k < M$. Define a row matrix $[F]$, and the corresponding column matrix $[\tilde{F}]$, which contain the M elements of the discrete unitary transform set $\{F_p\}$. If the range of p in the original infinite orthogonal set of functions, $\{\phi_p(x)\}$ is $-\infty < p < \infty$ the finite set of M values of p is : $\{(p,M)\} = \{-\frac{M}{2} \le p < \frac{M}{2}, M \text{ even}\}$ or $\{-\frac{(M-1)}{2} \le p < \frac{(M-1)}{2}, M \text{ odd}\}$. If the range of p in $\{\phi_p(x)\}$ is $0 \le p < \infty$ then $\{(p,M)\} = 0 \le p < M$. The above matrices are formally defined by the relations:

$$[f]_k = [\tilde{f}]_k = {}_{\gamma}f_k = f_a(x_2 + \frac{k+\gamma}{M}) \tag{A1.26}$$

and

$$[F]_p = [\tilde{F}]_p = F_p . \tag{A1.27}$$

The set of function samples $\{{}_{\gamma}f_k, 0 \le k < M\}$, expressed as a discrete inverse unitary transform of the set of coefficients $\{F_p\}$ by the M equations in (A1.13), can be written in row matrix and column matrix form as:

$$[f] = [F]({}_{\gamma}\Phi_M)^* \tag{A1.28}$$

and

$$[\tilde{f}] = ({}_{\gamma}\Phi_M)^\dagger [F] , \tag{A1.29}$$

while the set $\{F_p\}$, expressed as a discrete unitary transform of the set $\{{}_{\gamma}f_k\}$ by the M equations, (A1.20), can be written in row matrix and column matrix form as:

$$[F] = \frac{1}{C(\gamma,M)} [f] (\widetilde{{}_{\gamma}\Phi_M}) \tag{A1.30}$$

and

$$[\tilde{F}] = \frac{1}{C(\gamma,M)} ({}_{\gamma}\tilde{\Phi}_M) [\tilde{f}] . \tag{A1.31}$$

Parseval's theorem, equation (A1.23), for the two discrete sets $\{\gamma f_k\}$ and $\{F_p\}$ is :

$$\frac{1}{C(\gamma,M)}[f][f]^\dagger = [F][F]^\dagger \quad . \tag{A1.32}$$

It very frequently is true that $C(\gamma,M) = M$ and it has already been proved, in lemma 5, that this is true when the first M functions of $\{\phi_p(x)\}$ are such that the values of $\phi_p(x)$ are constant in each of the M bins, $\frac{k}{M} \leq x < \frac{k+1}{M}$, $0 \leq k < M$.

A1.3 The Exact Representation of a Two-Dimensional Function, within a Unit Square, by an Infinite Series of Two-Dimensional Orthogonal Functions.

A1.3.1 Definition and Simple Properties of Two-Dimensional Orthogonal Functions. The Exact Representation of a Two-Dimensional Function using an Infinite Series of them.

The extension of one-dimensional orthogonal functions to two-dimensional ones is straightforward. All the results and proofs for the latter have analogues in the former.

Define an infinite two-dimensional set of functions $\{\psi_{pq}(x,y)\}$, orthogonal in the square $y_0 \leq y < y_0 + 1$, $x_0 \leq x < x_0 + 1$, by the relations:

$$\int_{y_0}^{y_0+1}\int_{x_0}^{x_0+1} \psi_{pq}(x,y)\psi^*_{p'q'}(x,y)\,dxdy = c\delta_{pp',qq'} = \left\{\begin{array}{l} 0 \text{ if } p \neq p' \text{ or } q \neq q' \\ 1 \text{ if } p = p' \text{ and } q = q' \end{array}\right\}, \tag{A1.33}$$

$$\psi_{00}(x,y) = 1 \tag{A1.34}$$

and

$$\psi_{pq}(x+n,\, y+m) = \psi_{pq}(x,y)\,, \tag{A1.35}$$

where $x_0 \leq x < x_0 + 1$; $y_0 \leq y < y_0 + 1$, and m and n are integers.

When $c = 1$ the orthogonal set $\{\psi_{pq}(x,y)\}$ is said to be orthonormal. There are two sets $\{(p,\infty)\}$ and $\{(q,\infty)\}$ each containing an infinite number of consecutive integer labels p and q. Usually $\{(p,\infty)\} = \{-\infty < p < \infty\}$ or $\{(p,\infty)\} = \{0 \leq p < \infty\}$ and $\{(q,\infty)\} = \{-\infty < q < \infty\}$ or $\{(q,\infty)\} = \{0 \leq q < \infty\}$.

It can be shown by a proof analogous with that in lemma 1, section A1.1.1, that the mean value of all these functions in the square $x_0 \leq x < x_0 + 1$,

$y_0 \leq y < y_0 + 1$, except for the function $\psi_{00}(x,y) = 1$, is zero.

It can be shown by a proof analogous with that in lemma 2, section A1.1.1, that the set $\{\psi_{pq}(x,y)\}$ originally defined to be orthogonal in the square $x_0 \leq x < x_0 + 1$, $y_0 \leq y < y_0 + 1$ is orthogonal in any square $x_2 \leq x < x_2 + 1$, $y_2 \leq y < y_2 + 1$. Thus :

$$\int_{y_2}^{y_2+1}\int_{x_2}^{x_2+1} \psi_{pq}(x,y)\psi^*_{p'q'}(x,y)\,dxdy = c\delta_{pp',qq'} \qquad \text{(A1.36)}$$

and hence it can be shown that a smooth function, $f(x,y)$, within the square $x_2 \leq x < x_2 + 1$, $y_2 \leq y < y_2 + 1$ is given by the infinite series

$$f(x,y) = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} F_{pq}\psi_{pq}(x,y) \qquad \text{(A1.37)}$$

where the coefficients, F_{pq}, are given by

$$F_{pq} = \int_{y_2}^{y_2+1}\int_{x_2}^{x_2+1} f(x,y)\psi^*_{pq}(x,y)\,dydx \qquad \text{(A1.38)}$$

The mean value of $f(x,y)$ is F_{00} since the mean value of all coefficients, but $\psi_{00}(x,y) = 1$, is zero. The set $\{F_{pq}\psi_{pq}(x,y)$, excluding $F_{00}\}$ represents the variations of $f(x,y)$ about the mean value F_{00}.

The conditions for a set $\{\psi_{pq}(x,y)\}$ to be <u>closed</u> and <u>complete</u> are exactly analogous to those for the one-dimensional case:-

A set $\{\psi_{pq},(x,y)\}$ is closed if there exists no quadratically integrable function $g(x,y)$ (i.e. one for which $0 \leq \int_{y_2}^{y_2+1}\int_{x_2}^{x_2+1}|g(x,y)|^2 dxdy <$ a finite positive number) such that $\int_{y_2}^{y_2+1}\int_{x_2}^{x_2+1} g(x,y)\psi^*_{pq}(x,y)\,dxdy = 0$ for all p and q. (The case $g(x,y) = 0$ for all x and y is not counted.)

The completeness relation or Parseval's theorem for the set $\{\psi_{pq}(x,y)\}$ is

$$\int_{y_2}^{y_2+1}\int_{x_2}^{x_2+1}|f(x,y)|^2 dxdy = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}|F_{pq}|^2 . \qquad \text{(A1.39)}$$

A1.3.2 Separable Two-Dimensional Sets of Orthogonal Functions.

Consider the two-dimensional function formed by the product of two functions which are members respectively of the one-dimensional orthonormal sets $\{\phi_p(x)\}$ and $\{\psi_q(y)\}$. The orthogonality integral for

the set of two-dimensional functions $\{\phi_p(x)\psi_q(y)\}$ (the left hand side of equation (A1.36)) is

$$\int_{y_2}^{y_{2+1}}\int_{x_2}^{x_{2+1}}\left(\phi_p(x)\psi_q(y)\right)\left(\phi^*_{p'}(x)\psi^*_{q'}(y)\right)dxdy = \left(\int_{x_2}^{x_{2+1}}\phi_p(x)\phi^*_{p'}(x)dx\right)\left(\int_{y_2}^{y_{2+1}}\psi_q(y)\psi^*_{q'}(y)dy\right)$$
$$= \delta_{pp'}.\delta_{qq'} .$$

Now $\delta_{pp'}\delta_{qq'} = \delta_{pp',qq'}$ thus $\{\phi_p(x)\psi_q(y)\}$ is a two-dimensional orthonormal set.

Example.

If $\{\phi_p(x)\} = \{e^{2\pi ipx}\}$ and $\{\psi_q(y)\} = \{e^{2\pi iqy}\}$ the series expansion for f(x,y) given by equation (A1.37) is the complex Fourier series $f(x,y) = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}F_{pq}e^{2\pi i(px + qy)}$ where the coefficients are

$$F_{pq} = \int_{y_2}^{y_{2+1}}\int_{x_2}^{x_{2+1}}f(x,y)e^{-2\pi i(px + qy)}dxdy$$

A1.4 The Approximate Representation of a Two-Dimensional Function within a Unit Square, by a Finite Series of Two-Dimensional Orthogonal Functions.

A1.4.1 The "Best" Finite Series Representation.

It can be shown, as in section A1.2.1 for the one-dimensional case, that the "best" least squares approximation to a function, f(x,y), obtainable by using MN functions of the two-dimensional orthogonal set $\{\eta_{pq}(x,y)\}$ is

$$f(x,y) \simeq f_a(x,y) = \sum_{p}^{M}\sum_{q}^{N}F_{pq}\eta_{pq}(x,y) \quad , \qquad (A1.40)$$

where M and N are integers. The coefficients, F_{pq}, are given by equation (A1.38). The summations in equation (A1.40) are over the M smallest consecutive absolute values of p, and N of q. (e.g. if p and q in $\{\eta_{pq}(x,y)\}$ are in the ranges $0 \leq p < \infty$ and $0 \leq q < \infty$ then the summations are $\sum_{p=0}^{M-1}$ and $\sum_{q=0}^{N-1}$.)

A1.4.2 The Finite Series Representation at a Finite Number of Discrete Sample Points.

Suppose that the bounding square $x_2 \leq x < x_2 + 1$, $y_2 \leq y < y_2 + 1$ is divided up into M by N two-dimensional bins or cells as shown in Fig. (A1.2). Each cell is rectangular, of dimensions $\frac{1}{M}$ by $\frac{1}{N}$. Now suppose M x N sampling points are chosen, one in the same relative point in each cell as shown in Fig. (A1.2). Then the finite set of sampled values of $f_a(x)$ are given by substituting $x = x_2 + \frac{k+\gamma}{M}$ and $y = y_2 + \frac{l+\omega}{N}$ in equation (A1.40), to give

$$\gamma\omega f_{kl} = f_a(x_2 + \frac{k+\gamma}{M}, y_2 + \frac{l+\omega}{N}) = \sum_{p,M}\sum_{q,N} F_{pq}\psi_{pq}(x_2 + \frac{k+\gamma}{M}, y_2 + \frac{l+\omega}{N}), \quad (A1.41)$$

where $0 \leq k < M$, $0 \leq l < N$ and γ and ω are constants in the range 0 to 1.

It can be shown, in a way exactly analogous with the one-dimensional case, section A1.2.2, that if there are values of γ, ω, M and N for which all the $\frac{M}{2}(M+1)\frac{N}{2}(N+1)$ equations,

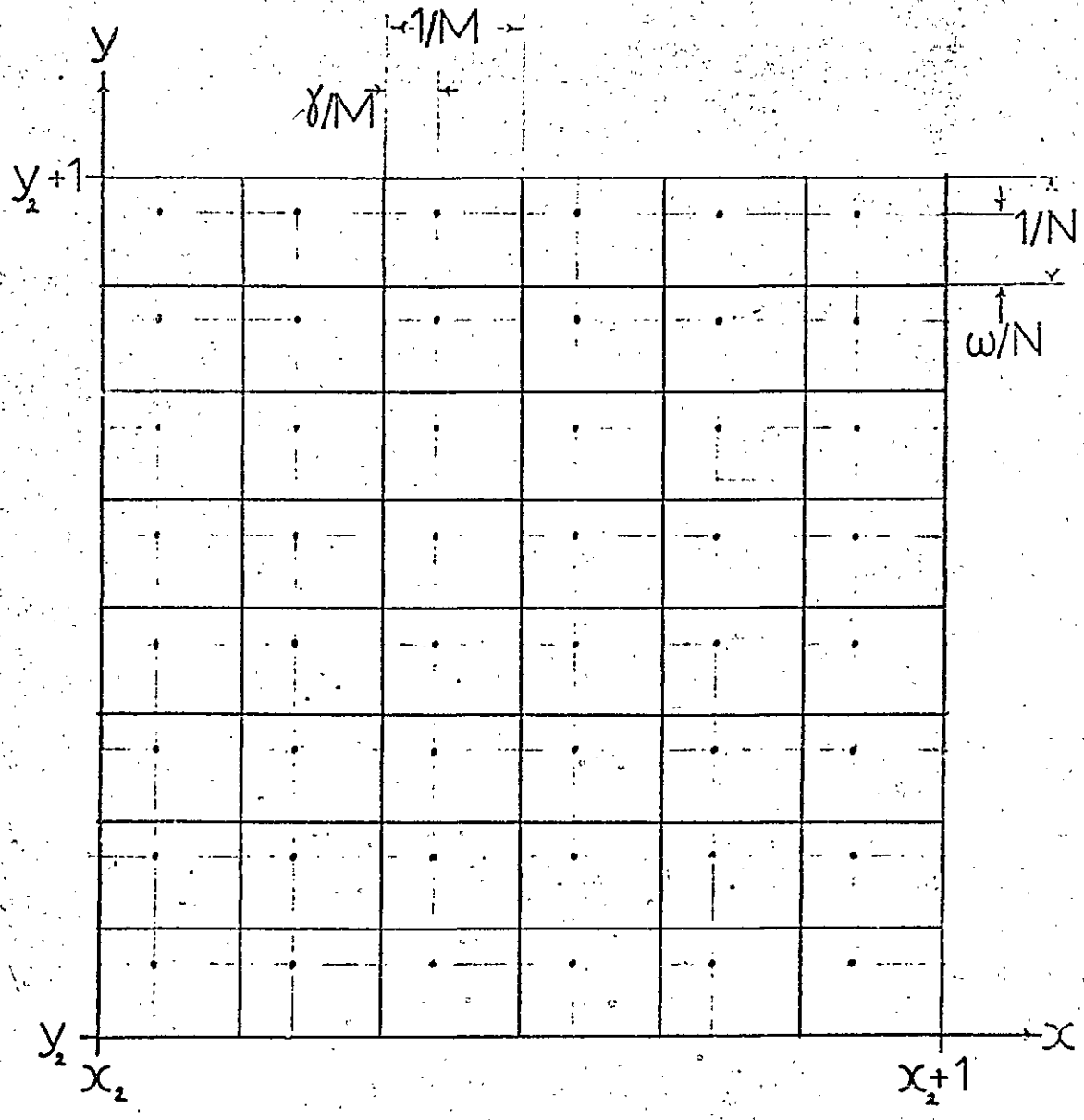
$$G_{pp',qq'}(\gamma,\omega) = \sum_{k=0}^{M-1}\sum_{l=0}^{N-1}\psi_{pq}(x_2 + \frac{k+\gamma}{M}, y_2 + \frac{l+\omega}{N})\psi^*_{p'q'}(x_2 + \frac{k+\gamma}{M}, y_2 + \frac{l+\omega}{N})$$
$$= C(\gamma,M;\omega,N)\delta_{pp',qq'}, \quad (A1.42)$$

are true, then the MN coefficients F_{pq} can be expressed by the sums:

$$F_{pq} = \frac{1}{C(\gamma,M;\omega,N)}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} f(x_2 + \frac{k+\gamma}{M}, y_2 + \frac{l+\omega}{N})\psi^*_{pq}(x_2 + \frac{k+\gamma}{M}, y_2 + \frac{l+\omega}{N}), \quad (A1.43)$$

in preference to integrals, as in equation (A1.38).

The sets $\{\gamma\omega f_{kl}\}$ and $\{F_{pq}\}$ form a finite or discrete unitary transform pair of sets in two dimensions, expressed by equations (A1.41) and (A1.43), and the orthogonality relation, equation (A1.42). A discrete "reciprocal" orthogonality relation is derivable from the assumption that equation (A1.41) is obtainable from equation (A1.43), which was derived from equation (A1.41) by use of the orthogonality relation,



sample points are: $x = x_2 + \frac{k+\gamma}{M}$, $y = y_2 + \frac{l+\omega}{N}$.

$k = 0, 1, 2, \ldots M-1.$ $\quad l = 0, 1, 2, \ldots N-1.$

γ is a constant, $0 \leqslant \gamma < 1.$

ω is a constant, $0 \leqslant \omega < 1.$

Fig(A1.2) The MN sampling positions for $f_a(x,y)$ and the orthogonal functions, $\Psi_{pq}(x,y)$, in the unit square, $x_2 \leqslant x < x_2+1, y_2 \leqslant y < y_2+1$. M=6 and N=8 in the diagram.

equation (A1.42). This "reciprocal" orthogonality relation is

$$\sum_{p}^{M}\sum_{q}^{N} \psi^{*}_{pq}\left(x_2 + \frac{k+\gamma}{M}, y_2 + \frac{l+\omega}{N}\right)\psi_{pq}\left(x_2 + \frac{k'+\gamma}{M}, y_2 + \frac{l'+\omega}{N}\right)$$

$$= C(\gamma,M;\omega,N)\delta_{kk',ll'} \quad . \qquad \text{(A1.44)}$$

Parseval's theorem relating the discrete unitary transform pair of sets $\{{}_{\gamma\omega}f_{kl}\}$ and $\{F_{pq}\}$ is

$$\frac{1}{C(\gamma,M;\omega,N)}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}\left|{}_{\gamma\omega}f_{kl}\right|^2 = \sum_{p}^{M}\sum_{q}^{N}\left|F_{pq}\right|^2 \quad . \qquad \text{(A1.45)}$$

It can be shown, by a proof analogous with that in lemma 5, section A1.2.2, that if $\left\{\psi_{pq}\left(x_2 + \frac{k+\gamma}{M}, y_2 + \frac{l+\omega}{N}\right)\right\}$ is independent of γ and ω (i.e. each function $\psi_{pq}(x,y)$ is constant within the MN sampling cells) then <u>all</u> values of γ and ω between 0 and 1 give rise to orthogonality relations, equations (A1.42) and (A1.44). Also $C(\gamma,M;\omega,N) = MN$ in these cases.

If the two-dimensional orthonormal set is $\{\phi_p(x)\psi_q(y)\}$, where $\{\phi_p(x)\}$ and $\{\psi_q(y)\}$ are two one-dimensional orthonormal sets, the orthogonality relation, equation (A1.42) is separable:

$$\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}\phi_p\left(x_2 + \frac{k+\gamma}{M}\right)\psi_q\left(y_2 + \frac{l+\omega}{N}\right).\phi^{*}_{p'}\left(x_2 + \frac{k+\gamma}{M}\right)\psi^{*}_{q'}\left(y_2 + \frac{l+\omega}{N}\right)$$

$$= C(\gamma,M;\omega,N)\delta_{pp',qq'} \quad ,$$

$$\text{or}\left(\sum_{k=0}^{M-1}\phi_p\left(x_2 + \frac{k+\gamma}{M}\right)\phi^{*}_{p'}\left(x_2 + \frac{k'+\gamma}{M}\right)\right).\left(\sum_{l=0}^{N-1}\psi_q\left(y_2 + \frac{l+\omega}{N}\right)\psi^{*}_{q'}\left(y_2 + \frac{l+\omega}{N}\right)\right)$$

$$= C(\gamma,M;\omega,N)\delta_{pp',qq'}.$$

The above equation is expressable as two one-dimensional orthogonality relations, namely

$$\sum_{k=0}^{M-1}\phi^{*}_{p'}\left(x_2 + \frac{k+\gamma}{M}\right)\phi_p\left(x_2 + \frac{k+\gamma}{M}\right) = C_1(\gamma,M)\delta_{pp'} \qquad \text{(A1.46)}$$

and

$$\sum_{l=0}^{N-1}\psi^{*}_{q'}\left(y_2 + \frac{l+\omega}{N}\right)\psi_q\left(y_2 + \frac{l+\omega}{N}\right) = C_2(\omega,N)\delta_{qq'} \quad , \qquad \text{(A1.47)}$$

where $C(\gamma,M;\omega,N) = C_1(\gamma,M).C_2(\omega,N)$. $\qquad$ (A1.48)

The "allowed" values of γ are those for which all $\frac{M}{2}(M+1)$ equations in equation (A1.46) are true and, likewise, the "allowed" values of ω require all $\frac{N}{2}(N+1)$ equations in equation (A1.47) to be true.

The two-dimensional "reciprocal" orthogonality relation, equation (A1.44), is similarly separable into the two one-dimensional relations:

$$\sum_{p,M} \phi_p(x_2 + \frac{k+\gamma}{M}) \phi_p^*(x_2 + \frac{k'+\gamma}{M}) = C_1(\gamma,M)\delta_{kk'} \qquad \text{(A1.49)}$$

and

$$\sum_{q,N} \psi_q(y_2 + \frac{l+\omega}{N}) \psi_q^*(y_2 + \frac{l'+\omega}{N}) = C_2(\omega,N)\delta_{ll'} \qquad \text{(A1.50)}$$

The discrete unitary transform (and inverse) relations, equations (A1.41) and (A1.43), can be written

$$\gamma\omega f_{kl} = f_a(x_2 + \frac{k+\gamma}{M}, y_2 + \frac{l+\omega}{N})$$

$$= \sum_{p,M} \phi_p(x_2 + \frac{k+\gamma}{M}) \left(\sum_{q,N} F_{pq} \psi_q(y_2 + \frac{l+\omega}{N}) \right) \qquad \text{(A1.51)}$$

and

$$F_{pq} = \frac{1}{C_1(\gamma,M)C_2(\omega,N)} \sum_{k,M} \phi_p^*(x_2 + \frac{k+\gamma}{M}) \left(\sum_{l,N} \gamma\omega f_{kl} \psi_q^*(y_2 + \frac{l+\omega}{N}) \right). \qquad \text{(A1.52)}$$

If the set $\{\phi_p(x)\}$ is such that for particular value(s) of M each function $\phi_p(x)$ is constant in the range $\frac{k}{M} \leq x < \frac{k+1}{M}$, where $k = 0,1,2 \ldots M-1$, then, from lemma 5, section A1.2.2, all values of γ, $0 \leq \gamma < 1$, are "allowed" and $C_1(\gamma,M) = M$. Likewise, if $\psi_q(y)$, for particular value(s) of N, are all constant in the range $\frac{l}{N} \leq y < \frac{l+1}{N}$, where $l = 0,1,2 \ldots N-1$, then all values of ω, $0 \leq \omega < 1$ are "allowed" and $C_2(\omega,N) = N$. In these cases each function $\phi_p(x)\psi_q(y)$ in the separable two-dimensional set will have a constant value in each rectangular sampling cell, $\frac{k}{M} \leq x < \frac{k+1}{M}$, $\frac{l}{N} \leq y < \frac{l+1}{N}$, $k = 0,1,2 \ldots M-1$, $l = 0,1,2 \ldots N-1$. From this statement and equation (A1.51) the function $f_a(x,y)$ must be constant in each cell, and hence a sample set $f_a(x_2 + \frac{k+\gamma}{M}, y_2 + \frac{l+\omega}{N})$ for any single pair of values γ,ω, $0 \leq \gamma < 1$, $0 \leq \omega < 1$ is sufficient to completely specify $f_a(x,y)$ when $\phi_p(x)\psi_q(y)$ are constant in each sampling cell.

An example of a separable two-dimensional orthonormal set for which the above statement is true is the set of two-dimensional Walsh functions $\{wal(p,x)wal(q,y),\ 0 \leq p < \infty,\ 0 \leq q < \infty\}$. The one-dimensional set $\{wal(p,x),\ 0 \leq p < \infty\}$ is discussed in Appendix 2. The first 64 two-

dimensional Walsh functions in the range $0 \leq x < 1$, $0 \leq y < 1$ are shown in Fig. (3.2).

A1.4.3 The Discrete Unitary Transform Relationships. Derived from a Separable Two-Dimensional Orthonormal Set. Expressed in Matrix Form.

Define the unitary transform matrices, $({}_{\gamma}\Phi_M)$ and $({}_{\omega}\Psi_N)$, for the two one-dimensional unitary transforms based on the one-dimensional orthonormal sets $\{\phi_p(x)\}$ and $\{{}_{\omega}\Psi_q(y)\}$ respectively, by the relations

$$({}_{\gamma}\Phi_M)_{pk} = \phi_p^*(x_2 + \frac{k+\gamma}{M}); \qquad k = 0,1,2\ldots M-1, \quad p \in \{(p,M)\} \tag{A1.53}$$

and

$$({}_{\omega}\Psi_N)_{ql} = \Psi_q^*(y_2 + \frac{l+\omega}{N}); \qquad l = 0,1,2\ldots N-1, \quad q \in \{(q,N)\} \tag{A1.54}$$

The two matrices are the M^{th} order and N^{th} order respectively.

The orthogonality relations, equation (A1.46), and "reciprocal" orthogonality relations, equation (A1.49), for the set $\{\phi_p(x_2 + \frac{k+\gamma}{M})\}$ can be written in matrix form as

$$({}_{\gamma}\Phi_M)({}_{\gamma}\Phi_M)^{\dagger} = ({}_{\gamma}\Phi_M)^{\dagger}({}_{\gamma}\Phi_M) = C_1(\gamma,M)(I_M) \tag{A1.55}$$

Likewise equations (A1.47) and (A1.50), the corresponding relations for the set $\{\Psi_q(y_2 + \frac{l+\omega}{N})\}$ are

$$({}_{\omega}\Psi_N)({}_{\omega}\Psi_N)^{\dagger} = ({}_{\omega}\Psi_N)^{\dagger}({}_{\omega}\Psi_N) = C_2(\omega,N)(I_N) . \tag{A1.56}$$

Define the $M \times N$ matrix (f) and the $M \times N$ matrix (F) as follows:

$$(f)_{kl} = {}_{\gamma\omega}f_{kl} = f_a(x_2 + \frac{k+\gamma}{M}, y_2 + \frac{l+\omega}{N}) \tag{A1.57}$$

and

$$(F)_{pq} = F_{pq} . \tag{A1.58}$$

The expression, equation (A1.51), giving one element ${}_{\gamma\omega}f_{kl}$ of the complete discrete inverse unitary transform set $\{{}_{\gamma\omega}f_{kl}\}$, can be written

$${}_{\gamma\omega}f_{kl} = \sum_{p,M} ({}_{\gamma}\Phi_M)^{\dagger}_{kp}\left(\sum_{q,M} F_{pq}({}_{\omega}\Psi_N)^*_{ql}\right), \tag{A1.59}$$

so that the complete inverse transform can be written by the matrix product:

$$(f) = ({}_{\gamma}\Phi_M)^{\dagger}(F)({}_{\omega}\Psi_N)^{*}. \tag{A1.60}$$

Likewise, equation (A1.52), giving one element F_{pq} of the complete discrete unitary transform set F_{pq}, can be written:

$$F_{pq} = \frac{1}{C_1(\gamma,M)C_2(\omega,N)}\sum_{k=0}^{M-1}({}_{\gamma}\Phi_M)_{pk}\left(\sum_{l=0}^{N-1}\gamma\omega^{f}{}_{kl}(\widetilde{{}_{\omega}\Psi_N})_{lq}\right), \tag{A1.61}$$

so that the complete transform can be written by the matrix product:

$$(F) = \frac{1}{C_1(\gamma,M)C_2(\omega,N)}({}_{\gamma}\Phi_M)(f)(\widetilde{{}_{\omega}\Psi_N}). \tag{A1.62}$$

Both the two-dimensional discrete transform above, and its inverse can be expressed completely in terms of one-dimensional unitary transforms. Consider the inverse transform given by equation (A1.60): The i^{th} row of the M X N matrix $(F)({}_{\omega}\Psi_N)^{*}$ is formed by the matrix product of the i^{th} row of F with the N x N transform matrix $({}_{\omega}\Psi_N)^{*}$. But, by comparison with equation (A1.28), this is a one-dimensional inverse discrete unitary transform, of order N, expressed as a row matrix. There are M rows in (F) so $(F)({}_{\omega}\Psi_N)^{*}$ is given by M of these one-dimensional transforms, one for each row of (F).

Now the complete inverse transform is obtained when $(F)({}_{\omega}\Psi_N)^{*}$ is premultiplied by the M × M transform matrix $({}_{\gamma}\Phi_M)^{\dagger}$. The j^{th} column of (f) is formed by the matrix multiplication of $({}_{\gamma}\Phi_M)^{\dagger}$ with the j^{th} row of $(F)({}_{\omega}\Psi_N)^{*}$. But, by comparison with equation (A1.29), this is a one-dimensional inverse discrete unitary transform, of order M, expressed as a column matrix. There are N columns in $(F)({}_{\omega}\Phi_N)^{*}$ so (f) is given by N of these one-dimensional transforms, one for each column of $(F)({}_{\omega}\Phi_N)^{*}$.

The inverse transform (f) is also given by the matrix product $({}_{\gamma}\Phi_M)(F)$ followed by the post-multiplication of this by $({}_{\omega}\Psi_N)$. In this case it is easily shown that the first matrix $({}_{\gamma}\Phi_M)(F)$ is given by the M,N^{th} order one-dimensional inverse transforms of the columns of (F), with respect to the transform matrix $({}_{\gamma}\Phi_M)$. Then (f) is given by N,M^{th} order one-dimensional inverse transforms of the rows of $({}_{\gamma}\Phi_M)(F)$, with respect to the transform matrix $({}_{\omega}\Psi_N)$.

In both cases (f) is completely calculated by N one-dimensional inverse transforms of order N, with respect to the transform matrix, $({}_{c}\Psi_{N})$, and N one-dimensional inverse transforms of order M, with respect to the transform matrix $({}_{r}\Psi_{M})$.

If "inverse" is deleted from the above sentence, and (f) replaced by (F), the resulting statement applies to equation (A1.62), the two-dimensional discrete unitary transform.

Appendix 2. The Infinite Orthonormal Set of One-Dimensional Walsh Functions.

A 2.1 Introduction.

J. L. Walsh in his paper,(1), first defined a set of function elements, orthonormal in the range $0 \leqslant x < 1$, and having only two values, $+1$ or -1. A function element is taken to mean a function which is identically zero outside of a given range. It has been shown in Lemma 2 Appendix 1 that if an infinite set of functions is defined to be orthogonal in any unit range, $x_0 \leqslant x < x_0 + 1$, then the periodic continuation of each function in the set means that the resulting set of functions is also orthogonal, but in any other chosen unit range, $x_1 \leqslant x < x_1 + 1$, say. The definitions of Walsh functions given here are taken from H.F. Harmuth's recent book,(2), in which a set of Walsh function elements are defined in the unit interval, $-\frac{1}{2} \leqslant x < \frac{1}{2}$, outside of which they are zero. This infinite set of Walsh function elements is shown to be orthonormal in the unit range $-\frac{1}{2} \leqslant x < \frac{1}{2}$. Periodic continuation of these Walsh function elements gives the Walsh functions which, by Lemma 2 Appendix 1, form an infinite set orthonormal in any unit range $x_1 \leqslant x < x_1 + 1$. * Harmuth's convenient notation, $\mathrm{wal}(j,x)$, $j = 0,1,2,\ldots,\infty$, in honour of Walsh, is used for both Walsh function elements and Walsh functions.

A 2.2 The Definition of A Set of Walsh Function Elements.

A Set of Walsh function elements can be defined by the relations

$$\mathrm{wal}(j,x) = 0, \quad x < -\tfrac{1}{2} \text{ or } x \geqslant \tfrac{1}{2} \text{ and } j = 1,2,3,\ldots,\infty, \tag{A2.1}$$

$$\mathrm{wal}(0,x) = 1, \quad -\tfrac{1}{2} \leqslant x < \tfrac{1}{2}, \tag{A2.2}$$

and

*In the "range of interest" in this thesis, $0 \leqslant x < 1$, they are identical with the function elements originally defined by J.L. Walsh in his paper,(1).

$$wal(2j+r,x) = \begin{cases} (-1)^{[j/2]+r} wal(j,2(x+\tfrac{1}{2})), & -\tfrac{1}{2} \leqslant x < 0 \\ \\ (-1)^{[j/2]+j} wal(j,2(x-\tfrac{1}{2})), & 0 \leqslant x < \tfrac{1}{2} \end{cases} \qquad (A2.3)$$

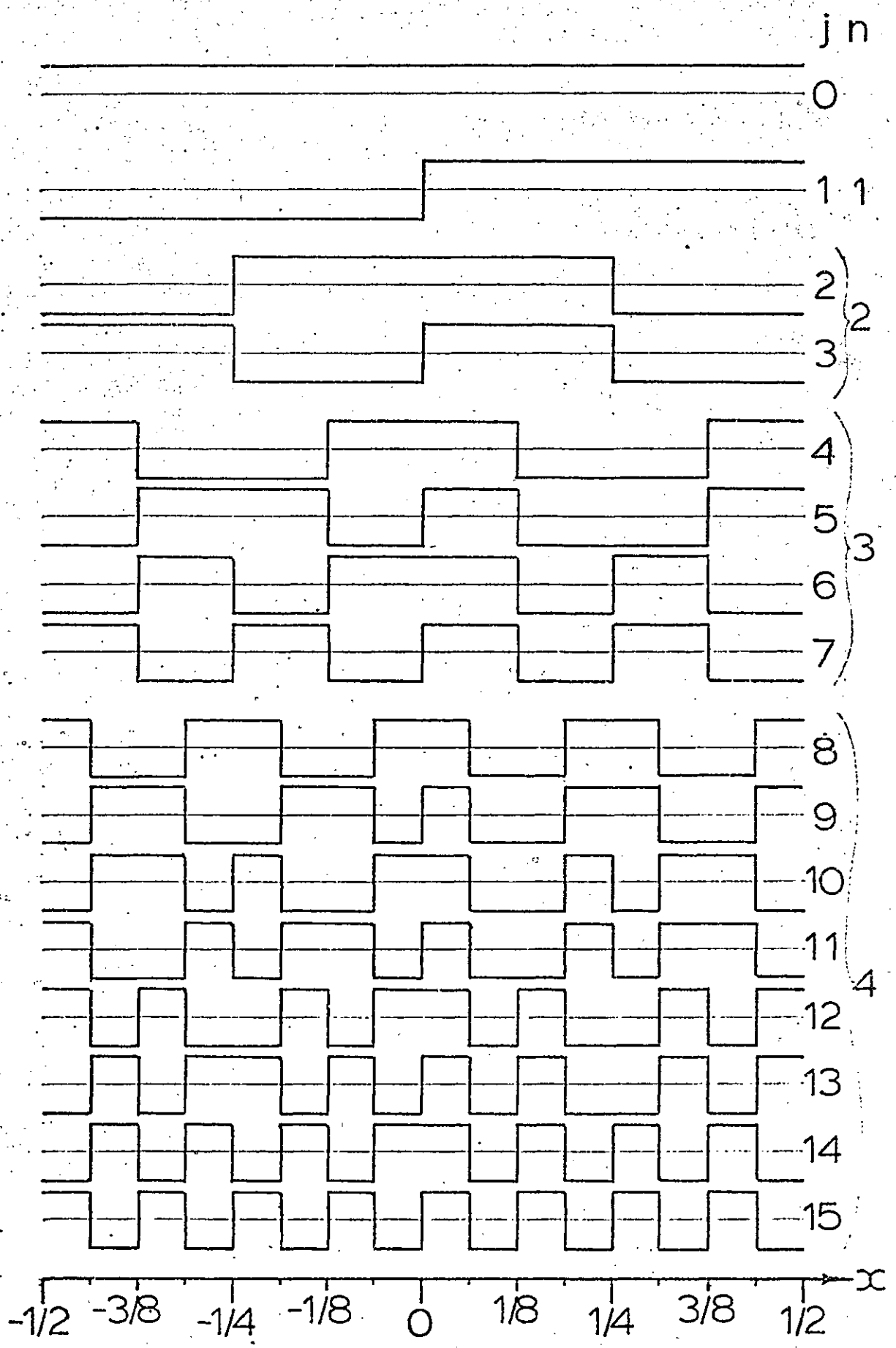
where r can take on the two values 0 or 1, the labels of the function elements, j, are $0,1,2,3,\ldots\ldots\infty$ and $[\frac{j}{2}]$ is the largest integer smaller than or equal to $\frac{j}{2}$.

The first sixteen Walsh function elements are shown in Fig. (A2.1). All the Walsh function elements have only two values $+1$ or -1, for any given value of x in the range $-\frac{1}{2} \leqslant x < \frac{1}{2}$ (except at each discontinuity, where the value infintesimally to the right of the discontinuity is taken). It is useful, to describe how the recursive relation, equation (A2.3), leads to the generation of new Walsh functions in sucessive steps from the first Walsh function, $wal(0,x)$, which is an even function.

The function element, $wal(j,2x)$, is $wal(j,x)$ "squeezed" along the x-axis to be non-zero in the range $-\frac{1}{2} \leqslant x < \frac{1}{2}$ and zero outside this range. Then $wal(j,2(x+\frac{1}{2}))$ is $wal(j,2x)$ shifted to the left along the x-axis by $\frac{1}{2}$ so that it is non-zero for $-\frac{1}{2} \leqslant x < 0$ and zero for $0 \leqslant x < \frac{1}{2}$. Similarly $wal(j,2(x-\frac{1}{2}))$ is $wal(j,2x)$ shifted to the right along the x-axis by $\frac{1}{2}$ so that it is non-zero for $0 \leqslant x < \frac{1}{2}$ and zero for $-\frac{1}{2} \leqslant x < 0$. This means that the recursive relation, equation (A2.3), can be written as the single equation:

$$wal(2j+r,x) = (-1)^{[j/2]} \cdot \left[(-1)^{r} wal(j,2(x+\tfrac{1}{2})) + (-1)^{j} wal(j,2(x-\tfrac{1}{2}))\right] \qquad (A2.4)$$

The two halves of the complete function, $wal(2j+r,x)$, consist of



Fig(A2.1). The first sixteen Walsh functions, wal (j,x), in the range $-\frac{1}{2} \leq x < \frac{1}{2}$. n is the "generation" number. Positive values are +1 and negative ones are -1.

wal $(j,2(x+\frac{1}{4}))$ and wal $(j,2(x-\frac{1}{4}))$ separately multipled by -1 or +1 according to the parity (oddness or evenness) of r and j. From each Walsh function, wal(j,x), two new Walsh functions, wal(2j,x) and wal (2j+1,x), can be generated (except for the case j=0 where wal (0,x) generates itself for r=0 and wal (1,x) for r=1). Fig. (A2.2) shows the labels of the functions generated, starting from the function wal (o,x), represented by "0". One function, wal (1,x), occurs in the first "generation", two in the second and so on. There are 2^{n-1} Walsh functions in the nth "generation" of Walsh functions. (In J.L. Walsh's original notation the kth. Walsh function in the nth. "generation" is denoted by $\varphi_n^{(k)}(x)$, where $1 \leqslant k \leqslant 2^{n-1}$. Then $\varphi_{n+1}^{(2k-1)}(x)$ and $\varphi_{n+1}^{(2k)}(x)$ are the two functions in the (n+1)th. "generation" derived from $\varphi_n^{(k)}(x)$.)

A2.3 The Classification Of Walsh Functions Into Odd and Even Functions.

It is now shown that the two statements below are true:

(1) If any given Walsh function in the nth. "generation" is an odd or an even function, then the two Walsh functions generated from it, in the n+1 th. "generation", can only be odd and even themselves.

(2) The Walsh function wal (j,x) is odd when j is odd and even when j is even.

Proof of Statement (1).

Suppose that wal (j,x) is an even function in the interval $-\frac{1}{2} \leqslant x < \frac{1}{2}$ (i.e. the function is said to have even parity), so that wal (j,x) = wal (j,-x) and so :

$$\text{wal}(j,2(x+\tfrac{1}{4})) = \text{wal}(j,-2(x+\tfrac{1}{4})) = \text{wal}(j,2(-x-\tfrac{1}{4})) \qquad \text{(A2.5)}$$

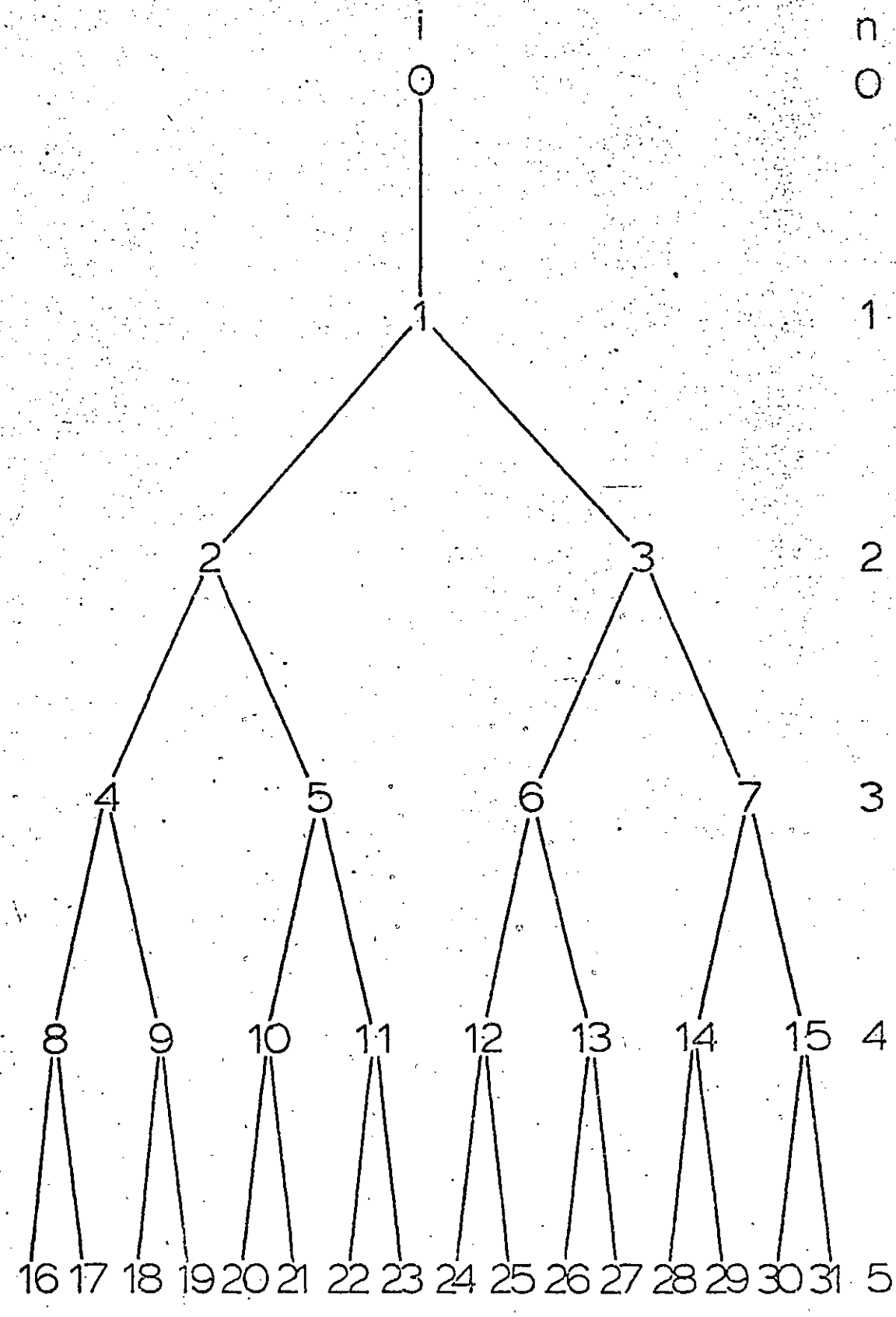
and

$$\text{wal}(j,2(x-\tfrac{1}{4})) = \text{wal}(j,-2(x-\tfrac{1}{4})) = \text{wal}(j,2(-x+\tfrac{1}{4})) \qquad \text{(A2.6)}$$

The parity of r in wal (2j+r,x), generated from wal (j,x), may be the same as that of j or different. The two cases give different results:

(a) r has the same parity as that of j so $(-1)^r = (-1)^j$

In this case the generated function given by equation (A2.4) becomes



Fig(A2.2) The "genealogy" of the first five "generations" of Walsh functions. Two Walsh functions wal (2j+r,x), where r=0,1 are generated from wal (j,x). Each Walsh function wal (i,x) is represented in the diagram by its label,i. i and j are integers greater than or equal to zero. n is the "generation" number.

$$wal(2j+r,x)=(-1)^{[j/2]+j}\left[wal(j,2(x+\tfrac{1}{2}))+wal(j,2(x-\tfrac{1}{2}))\right] \qquad (A2.7)$$

Substituting the right hand side of equation (A2.7) for that of equations (A2.5) and (A2.6) gives

$$wal(2j+r,x)=(-1)^{[j/2]+j}\left[wal(j,2(-x-\tfrac{1}{2}))+wal(j,2(-x+\tfrac{1}{2}))\right]$$

But from equation (A2.7)

$$wal(2j+r,-x)=(-1)^{[j/2]+j}\left[wal(j,2(-x+\tfrac{1}{2}))+wal(j,2(-x-\tfrac{1}{2}))\right]$$

$$\therefore wal(2j+r,x)=wal(2j+r,-x)$$

(b) r has the opposite parity to that of j so that $(-1)^r=-(-1)^j$

wal(2j+r,x) given by equation (A2.4) becomes

$$\begin{aligned} wal(2j+r,x) &= (-1)^{[j/2]+j}\left[-wal(j,2(x+\tfrac{1}{2}))+wal(j,2(x-\tfrac{1}{2}))\right] \\ &= (-1)^{[j/2]+j}\left[-wal(j,2(-x-\tfrac{1}{2}))+wal(j,2(-x+\tfrac{1}{2}))\right] \\ &= -wal(2j+r,x) \end{aligned}$$

From cases (a) and (b) it can therefore be concluded that:
For a given parity of j, a Walsh function wal(j,x), assumed to be even, leads to the generation of an odd and an even function, since r can have two values 0 and 1 of opposite parities.

Now suppose wal(j,x) is an odd function in the interval $-\tfrac{1}{2}\leqslant x<\tfrac{1}{2}$ (i.e. the function is said to have odd parity), so that wal(j,x)=-wal(j,-x) and so

$$wal(j,2(x+\tfrac{1}{2}))=-wal(j,-2(x+\tfrac{1}{2}))=-wal(j,2(-x-\tfrac{1}{2})) \qquad (A2.8)$$

and

$$wal(j,2(x-\tfrac{1}{2}))=-wal(j,-2(x-\tfrac{1}{2}))=-wal(j,2(-x+\tfrac{1}{2})) \qquad (A2.9)$$

The parity of r in wal (2j+r,x), generated from wal(j,x), may be the same as that of j or different. The two cases give different results:

(a) If r has the same parity as that of j so that $(-1)^r+(-1)^j$

Then wal (2j+r,x),given by equation (A2.4), is

$$\begin{aligned} wal(2j+r,x) &= (-1)^{[j/2]} + j\left[wal(j,2(x+\tfrac{1}{2}))+wal(j,2(x-\tfrac{1}{2}))\right] \\ &= (-1)^{[j/2]} + j\left[-wal(j,2(-x-\tfrac{1}{2}))-wal(j,2(-x+\tfrac{1}{2}))\right] \\ &= -wal(2j+r,-x) \end{aligned}$$

(b) If r has the opposite parity to that of j so that $(-1)^r+-(-1)^j$

Then wal (2j+r,x), given by equation (A2.4), is

$$\begin{aligned} wal(2j+r,x) &= (-1)^{[j/2]} + j\left[-wal(j,2(x+\tfrac{1}{2}))+wal(j,2(x-\tfrac{1}{2}))\right] \\ &= (-1)^{[j/2]} + j\left[wal(j,2(-x-\tfrac{1}{2}))-wal(j,2(-x+\tfrac{1}{2}))\right] \\ &= wal(2j+r,-x) \end{aligned}$$

From cases (a) and (b) above it can therefore be concluded that:

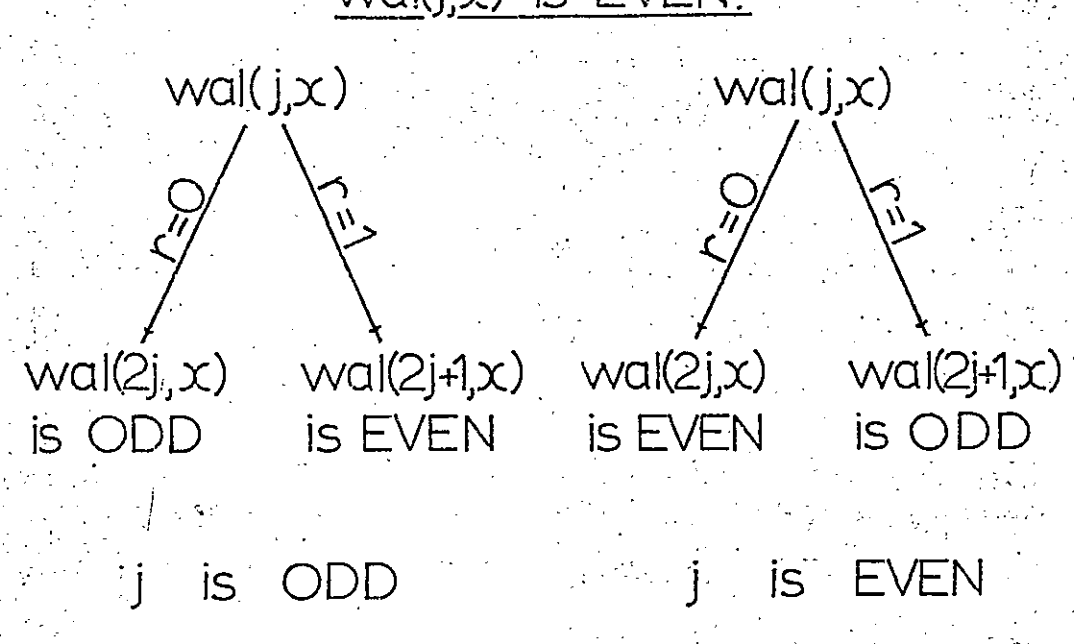
For a given parity of j, a Walsh function wal(j,x), assumed to be odd, leads to the generation of an odd and an even function, since r can have two values 0 and 1 of opposite parities.

Thus an odd and an even function wal(2j+r,x) and wal(2j+r′,x), where

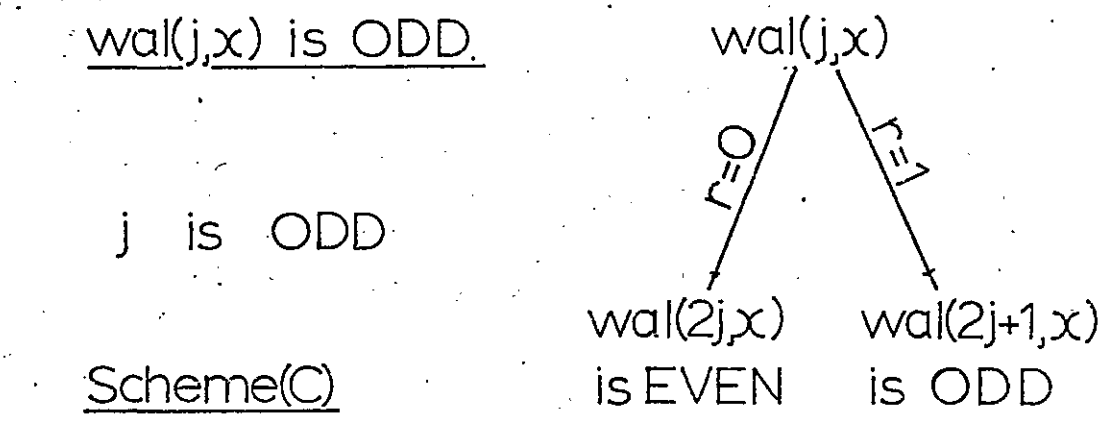
$r' = \begin{cases} 0 \text{ if } r=1 \\ 1 \text{ if } r=0 \end{cases}$, are generated from an odd or an even Walsh function, wal(j,x).

Proof of Statement (2).

The first part of the proved statement (1) above, concerning the generation of an even and an odd Walsh function from an even Walsh function leaves two possibilities: one for j odd and the other for j even:



In scheme (A), the even function generated has an odd label and vice versa, while in scheme(B) the even function generated has an even label and the odd function generated has an odd label. Now the first Walsh function, from which all the others are generated, is wal(0,x)=1, in the range $-\frac{1}{2} \leqslant x < \frac{1}{2}$, which is an even function with an even label. Thus scheme (B), applies, in this case in which the odd function wal(1,x) is generated from wal(0,x) when r=1 and the even function wal(0,x) is generated again for r=0. Now applying the second part of proved statement (1) above, concerning the generation of an even and an odd Walsh function from an odd Walsh function to this first odd Walsh function, wal(1,x), only one possible scheme is possible:



In scheme (C) an odd Walsh function with an odd label generates, as in scheme (B) an even, even labelled Walsh function and an odd, odd labelled Walsh function. Thus schemes (B) and (C) mean that all odd Walsh functions have odd labels and all even Walsh functions have even labels. Harmuth, (2), uses a convenient notation "cal" for even Walsh functions (like "cosine", an even function) and "sal" (like "sine", an odd

function), defining the even and odd sets by the relations

$$\begin{aligned} &cal(0,x) = wal(0,x) \\ &cal(k,x) = wal(2k,x) \\ &sal(k,x) = wal(2k-1,x) \end{aligned} \qquad (A2.10)$$

where $k = 1,2,3,\ldots$

A2.4 Proof of the Orthonormality of the Infinite Set of One-Dimensional Walsh Function elements $wal(j,x)$, $0 \leq j < \infty$ In the Range $-\frac{1}{2} \leq x < \frac{1}{2}$

The orthonormality condition to be proved is

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} wal(i,x)\,wal(j,x)\,dx = \delta_{ij} = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i \neq j \end{cases} \qquad (A2.11)$$

For all non-negative integers i and j.

A2.4.1 Proof that Equation (A2.11) is True for i=j.

All Walsh functions are +1 or -1 (except at infintesimally narrow discontinuities) so that $wal(i,x)\,wal(i,x) = 1$ and so

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} wal(i,x)\,wal(i,x)\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} 1.\,dx = 1$$

Q.E.D.

A2.4.2 Proof That Equation (A2.11) is True for i an odd number and j an even number.

Wal (i,x) is an odd function if i is odd and wal(j,x) is an even function for j even so the product is clearly an odd function, $G(x)$ say : $wal(i,x) \cdot wal(j,x) = G(x) = -G(-x)$. The integral in equation (A2.1) will be zero for any odd function:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} G(x)dx = \int_{-\frac{1}{2}}^{0} G(x)dx + \int_{0}^{\frac{1}{2}} G(x)dx = \int_{-\frac{1}{2}}^{0} -G(-x)dx + \int_{0}^{\frac{1}{2}} G(x)dx$$

$$= \int_{\frac{1}{2}}^{0} G(y)dy + \int_{0}^{\frac{1}{2}} G(x)dx = \int_{0}^{\frac{1}{2}} -G(y)dy + \int_{0}^{\frac{1}{2}} G(x)dx = 0 \qquad \text{Q.E.D.}$$

A2.4.3. Proof That Equation (A2.9) is True for i and j Both Even or Both Odd, i≠j.

Suppose that wal(k,x) and wal(l,x), k≠l, are orthogonal:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \text{wal}(k,x)\,\text{wal}(l,x)dx = 0 \qquad k \neq l \qquad \text{(A2.12)}$$

for a pair of integers l and k where l≠k.

(1) It is first proved that if equation (A2.12) is generally true then (a) the even functions, wal(2k,x) and wal(2l,x), generated from wal(k,x) and wal(l,x), respectively are orthogonal and (b) the odd functions wal(2k+1,x) and wal(2l+1,x), generated from wal(k,x) and wal(l,x) respectively are orthogonal also.

(2) It is then proved by induction that, since equation (A2.12) is true for k even and l odd (or vice versa) from section A2.4.2. above, wal(i,x) and wal(j,x) are orthogonal where i and j are two odd or two even numbers: $i=2^n k+2^{n-1}r_1.....+r_n$ and $j=2^n l+2^{n-1}r_1+2^{n-2}r_2.....r_n$ where $r_1, r_2, r_3, ...r_n$ are 1 or 0 and n is a positive integer.

Any pair of even or odd numbers i,j can be given by the expressions above so the two proofs of (1) and (2) will complete the proof of the orthogonality of the Walsh function elements.

Proof (1)

From equation (A2.4), the two even functions or two odd functions, generated from wal(k,x) and wal(l,x) respectively, are

$$\text{wal}(2k+r_1,x) = (-1)^{[k/2]}\left[(-1)^{r_1}\text{wal}(k,2(x+\tfrac{1}{2}))+(-1)^{k}\text{wal}(k,2(x-\tfrac{1}{2}))\right]$$

and (A2.13)

$$\text{wal}(2l+r_1,x) = (-1)^{[l/2]}\left[(-1)^{r_1}\text{wal}(l,2(x+\tfrac{1}{2}))+(-1)^{l}\text{wal}(l,2(x-\tfrac{1}{2}))\right]$$

If $r_1 = 0$ the functions are even and if $r_1 = 1$ they are odd.
The orthogonality integral for these two functions is then

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\text{wal}(2k+r_1,x)\,\text{wal}(2l+r_1,x)\,dx$$

$$= (-1)^{[k/2]+[l/2]}\int_{-\frac{1}{2}}^{\frac{1}{2}}\left[(-1)^{r_1}\text{wal}(k,2(x+\tfrac{1}{2}))+(-1)^{k}\text{wal}(k,2(x-\tfrac{1}{2}))\right] \qquad \text{(A2.14)}$$

$$\cdot\left[(-1)^{r_1}\text{wal}(l,2(x+\tfrac{1}{2}))+(-1)^{l}\text{wal}(l,2(x-\tfrac{1}{2}))\right]\cdot dx$$

Now cross terms $\text{wal}(k,2(x+\tfrac{1}{2}))\ \text{wal}(l,2(x-\tfrac{1}{2}))$ and $\text{wal}(k,2(x-\tfrac{1}{2}))\ \text{wal}(l,2(x+\tfrac{1}{2}))$ of the integral on the r.h.s are zero since the Walsh function element, $\text{wal}(i,2(x+\tfrac{1}{2}))=0$ when $0\leqslant x<\tfrac{1}{2}$ while $\text{wal}(i,2(x-\tfrac{1}{2}))=0$ when $-\tfrac{1}{2}\leqslant x<0$ for all i, $0\leqslant i<\infty$. Equation (A2.14) then becomes:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}\text{wal}(2k+r,x)\,\text{wal}(2l+r,x)\,dx = (-1)^{[k/2]+[l/2]}\left\{\int_{-\frac{1}{2}}^{\frac{1}{2}}\text{wal}(k,2(x+\tfrac{1}{2}))\,\text{wal}(l,2(x+\tfrac{1}{2}))\,dx\right.$$

$$\left.+(-1)^{k+l}\int_{-\frac{1}{2}}^{\frac{1}{2}}\text{wal}(k,2(x-\tfrac{1}{2}))\,\text{wal}(l,2(x-\tfrac{1}{2}))\,dx\right\} = I_1+I_2 \qquad \text{(A2.15)}$$

where I_1 and I_2 are the two integrals on the r.h.s. of equation (A2.15).

The integrand of I_1 on the r.h.s. is zero in the range $0 \leqslant x < \frac{1}{2}$ while that of I_2 is zero in the range $-\frac{1}{2} \leqslant x < 0$ so that these can be rewritten

$$I_1 = \int_{-\frac{1}{2}}^{0} wal(k,2(x+\tfrac{1}{2}))wal(l,2(x+\tfrac{1}{2}))dx \qquad (A2.16)$$

$$I_2 = \int_{0}^{\frac{1}{2}} wal(k,2(x-\tfrac{1}{2}))wal(l,2(x-\tfrac{1}{2}))dx \qquad (A2.17)$$

$y=2(x+\frac{1}{2})$ in equation (A2.16) and $y = 2(x-\frac{1}{2})$ in equation (A2.17) gives for <u>both</u> equations

$$I_1 = I_2 = \tfrac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} wal(k,y)wal(l,y)dy \qquad (A2.18)$$

This is identical with the definite integral of equation (A2.12) which was assumed to be identically zero. Thus it is true that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} wal(2k+r_1,x)wal(2l+r_1,x)dx = 0 \qquad (A2.19)$$

$k \neq l$

and

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} wal(2k,x)wal(l,x)dx = 0 \qquad (A2.12)$$

$k \neq l$, $r_1 = 0$ or 1

Q.E.D.

Proof (2)

Proof (1) allows the statement that the pair of functions wal(2k+r_1 ,x) and wal(2l+r_1 ,x) are orthogonal if wal(k,x) and wal(l,x) are orthogonal for any pair of non equal integers k and l. Proof(1) also allows the statement that wal(2(2k+r_1)+r_2 ,x) and wal(2(2l+r_1)+r_2 ,x) are orthogonal if wal(2k+r_1 ,x) and wal(2l+r_1 ,x) are (r_1 , r_2 =0 or 1). Clearly (1) can be extended to a more general statement that:

If wal(k,x) and wal(l,x) are orthogonal then wal(2^nk+$2^{n-1}r_1$ +$2^{n-2}r_2$ ..+r_n,x) and wal(2^nl+$2^{n-1}r_1$ +$2^{n-2}r_2$ .....+r_n,x)are orthogonal where r_1 ,r_2 ,r_3 .....r_n can each be 0 or 1 and n is a positive integer.

But it has already been shown in section(A2.4.2) that wal(k,x) and wal(l,x) are orthogonal if k and l are even and odd respectively (or odd and even). Thus it has been shown that "if k and l are even and odd numbers respectively (or odd and even respectively) the pair of odd functions or pair of even functions wal(i,x) and wal(j,x) are orthogonal where i and j are given by

$$i = 2^n k = 2^{n-1} r_1 + 2^{n-2} r_2 \cdots\cdots r_n \quad \text{(A2.20)}$$

and

$$j = 2^n l = 2^{n-1} r_1 + 2^{n-2} r_2 \cdots\cdots r_n . \quad \text{(A2.21)}$$

where r_1 ,r_2 ,r_3 .....r_n can each be 1 or 0. If r_n is 0 i and j are even. If r_n is 1, i and j are odd. n is a positive integer.

Any pair of odd or even numbers can be expressed in terms of an even number, k, and an odd number, l, as in equations (A2.20) and (A2.21)

This completes the proof that equation (A2.11) is true for all possible pairs of non-negative integers, i and j, i.e. that the infinite set of Walsh function elements $\{ \text{wal}(j,x) \; 0 \leqslant j < \infty \}$ is an orthonormal set in the range $-\frac{1}{2} \leqslant x < \frac{1}{2}$.

A2.5 The Walsh Functions Orthonormal In The Range $0 \leq x < 1$.

If all the Walsh function elements defined by equations (A2.1), (A2.2) and (A2.3) are periodically extended such that

$$wal\ (j,x+m) = wal_{el}\ (j,x), \qquad -\tfrac{1}{2} \leq x < \tfrac{1}{2},\quad m = \pm 1, \pm 2, \ldots\ldots \tag{A2.22}$$

where the subscript "el" means Walsh function element, then, by Lemma2, Appendix 1, the resulting set of Walsh functions will form an infinite orthonormal set in any unit range, $x_o \leq x < x_o + 1$, for instance. Of interest here is the range $0 \leq x < 1$ so that from now on "wal(j,x)" will mean the Walsh function $wal_{el}\ (j,x)$ for $0 \leq x < \tfrac{1}{2}$ and $wal_{el}\ (j,x-1)$ for $+\tfrac{1}{2} \leq x < 1$. The first sixteen Walsh functions in the range $0 \leq x < 1$ are shown in Fig.(A2.3). The orthonormality relation for these Walsh functions is

$$\int_0^1 wal(i,x)\ .\ wal(j,x)dx = \delta_{ij}. \tag{A2.23}$$

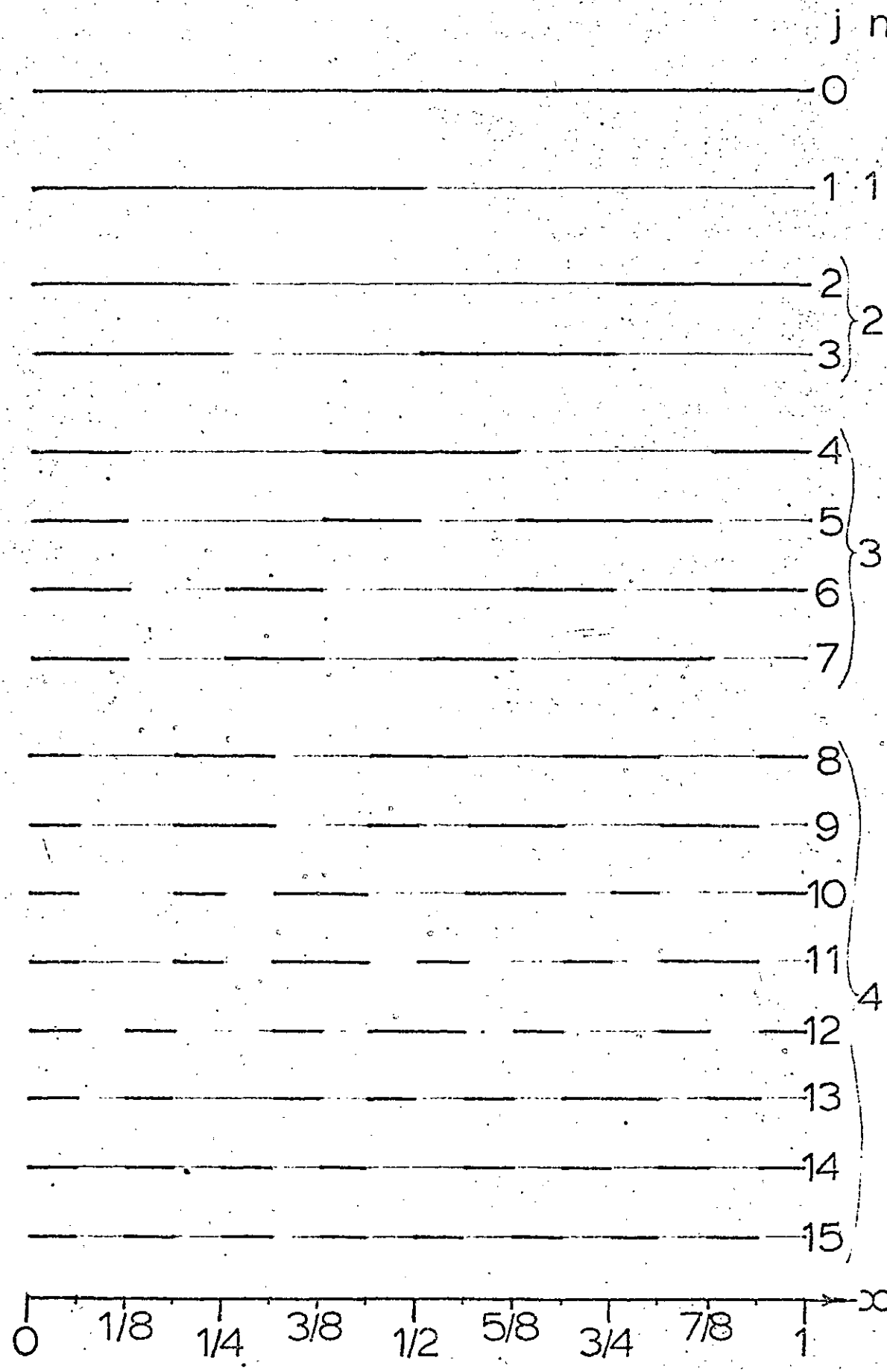
A2.6 Expansion of A Function, In the Range $0 \leq x < 1$, By Infinite and Finite Series of Walsh Functions.

The results, of sections A1.1 and A1.2 in Appendix 1, for expressing bounded functions exactly and also approximately by infinite and finite series of orthogonal functions apply in the special case of Walsh functions. Thus a function f(x), $0 \leq x < 1$ can be given exactly:

$$f(x) = \sum_{i=0}^{\infty} F_i\ \ wal(i,x), \tag{A2.24}$$

where

$$F_i = \int_0^1 f(x)\ wal(i,x)dx; \qquad 0 \leq i < M. \tag{A2.25}$$



Fig(A2.3). The first sixteen Walsh functions, wal (j,x), in the range $0 \leq x < 1$. n is the "generation" number. Bold and faint line segments represent the values +1 and -1 respectively.

The "best" (see section A1.2.1 for discussion on "best" Finite Series) finite series $\left\{\begin{matrix}\text{obtainable}\\ \text{for } f(x)\end{matrix}\right.$ using M Walsh functions from $\{wal(j,x)\}$ is given by

$$f(x) \simeq f_a(x) = \sum_{i=0}^{M-1} F_i \, wal(i,x), \qquad (A2.26)$$

where the coefficients, F_i, are given by equation (A2.25). A set of M values of $f_a(x)$, at the values of x given by $x = \frac{k+\gamma}{M}$, $0 \le k < M$ and γ is a constant in the range $0 \le \gamma < 1$, is given by

$$f_a\left(\frac{k+\gamma}{M}\right) = {}_{\gamma}f_k = \sum_{i=0}^{-1} F_i \, wal\left(i, \frac{k+\gamma}{M}\right); \qquad (A2.27)$$

γ, a constant, $0 \le \gamma < 1$

Now in Lemma 5, section A1.2.2, it was shown that if M can be chosen for a particular orthonormal set, $\{\phi_p(x)\}$, where the M orthomormal function samples, $\phi_p\left(\frac{k+\gamma}{M}\right)$, $0 \le k < M$, are independent of γ (i.e. $\phi_p(x)$ is constant within each sampling bin), then discrete orthogonality relations, equation (A1.20), are true for all values of γ, $0 \le \gamma < 1$. This relation enables the set of M coefficients, $\{F_i,\ 0 \le i < M\}$, to be expressed as summations in terms of the set of function samples $\left\{f_a\left(\frac{k+\gamma}{M}\right),\ 0 \le k < M\right\}$. In fact the set $\{F_i\}$ is a discrete or finite unitary transform of the set $\left\{f_a\left(\frac{k+\gamma}{M}\right)\right\}$. Now Walsh functions have values +1 or -1 only and their discontinuities in the finite set of Walsh functions up to and including some of the nth "generation" i.e.

$\left\{wal(j,x),\ \begin{matrix}0 \le x < 1\\ 0 \le j < M\end{matrix}\right\}$ $2^{n-1} \le M < 2^n$ may only occur at points $x = k/2^n$, $0 \le k < 2^n$,

so that, to obtain a discrete orthogonality relation for $wal(j, \frac{k+\gamma}{M})$, the bins must be of width $1/2^n$ which implies 2^n of them. Thus if and only if $M = 2^n$ $(n = 1,2.....)$ the general discrete orthogonality relations, equation (A1.20), for Walsh functions become

$$\sum_{k=0}^{2^n-1} wal\left(i, \frac{k+\gamma}{2^n}\right) wal\left(j, \frac{k+\gamma}{2^n}\right) = 2^n \delta_{ij}; \qquad (A2.28)$$

$0 \le i < 2^n$; $0 \le j < 2^n$; $n = 0,1,2...$; γ, a constant, $0 \le \gamma < 1$.

Now multiplying equation (A2.27) by $\mathrm{wal}(j, \frac{k+\gamma}{2^n})$, summing over k and using equation (A2.28) gives

$$F_j = \frac{1}{2^n} \sum_{k=0}^{2^n-1} f_a\left(\frac{k+\gamma}{2^n}\right) \mathrm{wal}\left(j, \frac{k+\gamma}{2^n}\right). \qquad (A2.29)$$

The finite set of M coefficients $\{F_j,\ 0 \le j < 2^n\}$ given by equation (A2.29) is the one-dimensional discrete Mth order Walsh transform of the finite set of M sample values $\left\{f\left(\frac{k+\gamma}{2^n}\right), 0 \le k < 2^n\right\}$. Conversely $\left\{f\left(\frac{k+\gamma}{2^n}\right)\right\}$ given by equation (A2.27) is the discrete inverse Walsh transform of the set $\{F_i,\ 0 \le i < 2^n\}$. To complete the theory requires a reciprocal discrete orthogonality relation by which equation (A2.27) can be obtained from equation (A2.29) (c.f. equation (A1.22) in the general case) This must clearly be

$$\sum_{i=1}^{2^n-1} \mathrm{wal}\left(i, \frac{k+\gamma}{2^n}\right) \mathrm{wal}\left(i, \frac{l+\gamma}{2^n}\right) = 2^n \delta_{kl}; \qquad (A2.30)$$

$$0 \le k < 2^n; \quad 0 \le l < 2^n.$$

The discrete relationships, equations (A2.27) to (A2.30), can be conveniently expressed in matrix form:

Defining the Mth order Walsh transform matrix, in accordance with the definition of equation (A1.24), for any unitary transform matrix, we have that

$$(W_n)_{jk} = \mathrm{wal}\left(j, \frac{k+\gamma}{2^n}\right); \qquad (A2.31)$$

$$0 \le k < 2^n; \quad 0 \le k < 2^n; \quad n = 0, 1, 2 \ldots$$

An important fact to emphasize is that $\mathrm{wal}\left(j, \frac{k+\gamma}{2^n}\right)$ for a given j, k, and n is constant (i.e. +1 or -1) within a bin for any value of γ in the range $0 \le \gamma < 1$ (since there are no discontinuities within any bin).

Thus the Walsh transform matrix of a given order $M=2^n$ is constant. It is often convenient to choose $\gamma = \frac{1}{2}$ when the jth row of the matrix is given by the value of wal(j,x) sampled at the centre of each of the 2^n bins. Using $\gamma = \frac{1}{2}$ in this way it is easy, from Fig. (A2.3), to work out the first five Walsh matrices. These are given at the end of Appendix 2.

The orthogonality relation equations (A2.28) and (A2.30) are given by

$$(W_n)(\widetilde{W_n}) = (\widetilde{W_n})(W_n) = 2^n(I_M); \qquad (A2.32)$$

$$M = 2^n, \; n = 0,1,2\ldots$$

I_M being the Mth order unit matrix $(I_M)ij = \delta ij$. In addition to being an orthogonal matrix, defined by equation (A2.32), (W_n) is a symmetric matrix given by $(W_n)ij = (W_n)ji$ so that

$$(\widetilde{W_n}) = (W_n) \qquad (A2.33)$$

This can be most easily demonstrated by working out a number of Walsh matrices as described above but could be shown analytically from the Walsh function definitions. The orthogonality relations thus simplify to

$$(W_n)(W_n) = 2^n(I_{2^n}) \qquad (A2.34)$$

Define a column matrix $[\widetilde{f}]$ and row matrix $[f]$ containing the same 2^n elements, $f\left(\frac{k+\gamma}{2^n}\right)$, so that $[\widetilde{f}]_k = [f]_k$. Also define a column matrix $[F]$ and row matrix $[F]$ containing the same M elements, F_i, so that $[\widetilde{F}]_i = [F]_i$. Then the discrete inverse Walsh transform, equation (A2.27), can be expressed in row matrix and column matrix form repectively as

$$[f] = [F](W_n) \quad \text{and} \quad [\widetilde{f}] = (\widetilde{W_n})[\widetilde{F}] = (W_n)[\widetilde{F}]. \qquad (A2.35)$$

Likewise the discrete Mth order Walsh transform, equation (A2.29) is

$$[F] = \frac{1}{2^n}\cdot[f]\,(\widetilde{W_n}) = \frac{1}{2^n}\cdot[f]\,(W_n) \quad \text{and} \quad [\widetilde{F}] = \frac{1}{2^n}\cdot(W_n)\,[\widetilde{f}] \tag{A2.36}$$

The first five Walsh matrices, (W_n), n = 0,1,2,3,4, found from Fig. (A2.3) by sampling the centres of the $M = 2^n$ bins of the first M = 1,2,4,8 and 16 Walsh functions respectively, are

$$(W_0) = [+], \tag{A2.37}$$

$$(W_1) = \begin{bmatrix} + & + \\ + & - \end{bmatrix}, \tag{A2.38}$$

$$(W_2) = \begin{bmatrix} + & + & + & + \\ + & + & - & - \\ + & - & - & + \\ + & - & + & - \end{bmatrix}, \tag{A2.39}$$

$$(W_3) = \begin{bmatrix} + & + & + & + & + & + & + & + \\ + & + & + & + & - & - & - & - \\ + & + & - & - & - & - & + & + \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & - & - & + & - & + & + & - \\ + & - & + & - & - & + & - & + \\ + & - & + & - & + & - & + & - \end{bmatrix}, \tag{A2.40}$$

$$(W_4) = \begin{bmatrix} + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\ + & + & + & + & + & + & + & + & - & - & - & - & - & - & - & - \\ + & + & + & + & - & - & - & - & - & - & - & - & + & + & + & + \\ + & + & + & + & - & - & - & - & + & + & + & + & - & - & - & - \\ + & + & - & - & - & - & + & + & + & + & - & - & - & - & + & + \\ + & + & - & - & - & - & + & + & - & - & + & + & + & + & - & - \\ + & + & - & - & + & + & - & - & - & - & + & + & - & - & + & + \\ + & + & - & - & + & + & - & - & + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + & + & - & - & + & + & - & - & + \\ + & - & - & + & + & - & - & + & - & + & + & - & - & + & + & - \\ + & - & - & + & - & + & + & - & + & - & - & + & - & + & + & - \\ + & - & - & + & - & + & + & - & - & + & + & - & + & - & - & + \\ + & - & + & - & - & + & - & + & + & - & + & - & - & + & - & + \\ + & - & + & - & - & + & - & + & - & + & - & + & + & - & + & - \\ + & - & + & - & + & - & + & - & - & + & - & + & - & + & - & + \\ + & - & + & - & + & - & + & - & + & - & + & - & + & - & + & - \end{bmatrix}, \tag{A2.41}$$

where "+" represents 1 and "−" represents −1.

From the form of the first five Walsh matrices it can be seen that if $\left[{}^{n}R_{j}\right]$ is the row matrix containing all 2^n elements of the j^{th} row of the Walsh matrix, (W_n), then the $(2j-1)^{th}$ row, $\left[{}^{n+1}R_{2j-1}\right]$, and the $2j^{th}$ row, $\left[{}^{n+1}R_{2j}\right]$, of (W_{n+1}) can be generated from $\left[{}^{n}R_{j}\right]$ as follows:

If j is odd:

$$\left[{}^{n+1}R_{2j-1}\right] = \left[{}^{n}R_{j}\right] ; \left[{}^{n}R_{j}\right]$$

$$\text{and } \left[{}^{n+1}R_{2j}\right] = \left[{}^{n}R_{j}\right] ; -\left[{}^{n}R_{j}\right] . \qquad \text{(A2.42)}$$

If j is even:

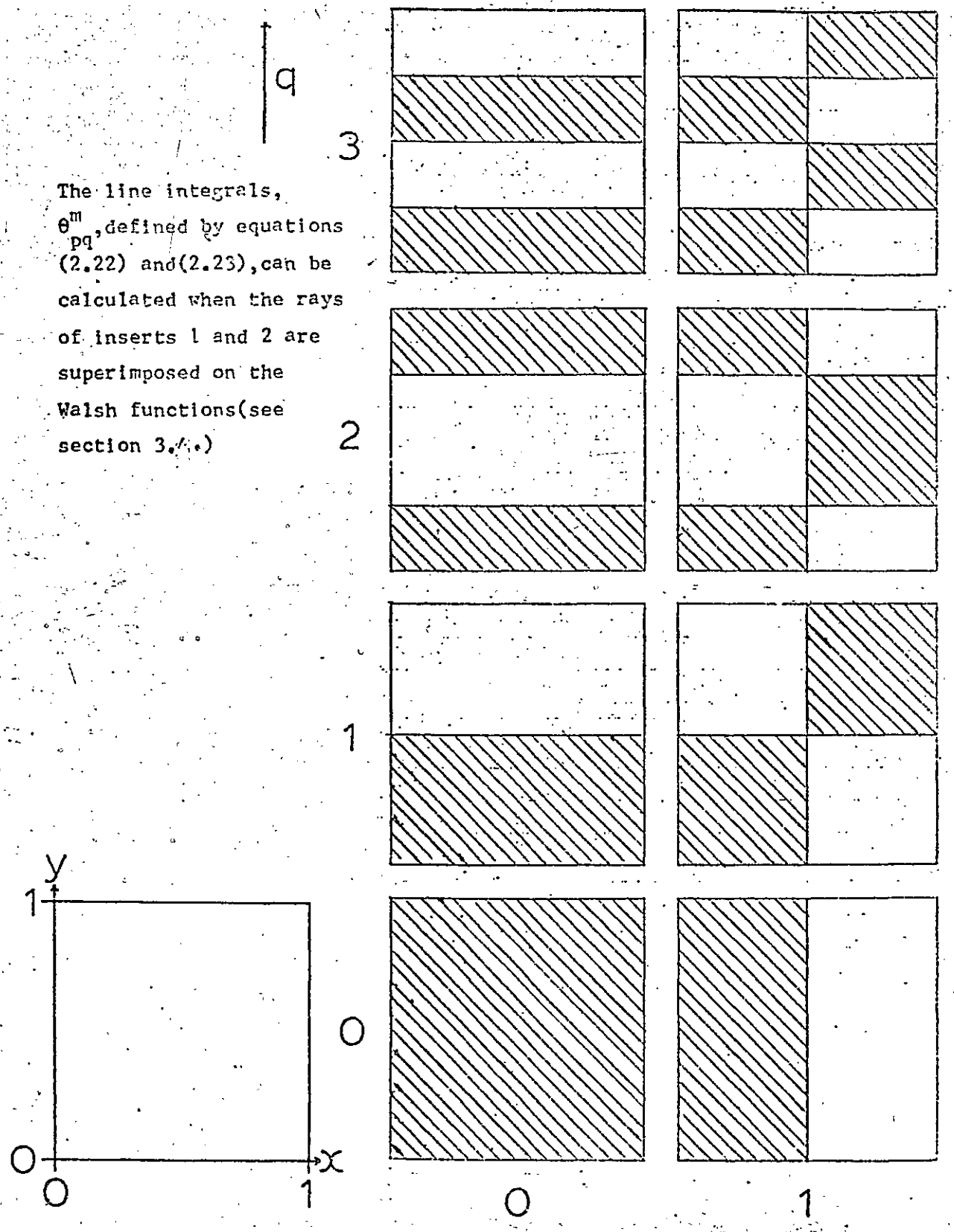
$$\left[{}^{n+1}R_{2j-1}\right] = \left[{}^{n}R_{j}\right] ; -\left[{}^{n}R_{j}\right]$$

$$\text{and } \left[{}^{n+1}R_{2j}\right] = \left[{}^{n}R_{j}\right] ; \left[{}^{n}R_{j}\right] . \qquad \text{(A2.43)}$$

where the ";" between row matrices means simply the juxtaposition of two row matrices to make a single longer one. "-" written in front of $\left[{}^{n}R_{j}\right]$ means all the elements of $\left[{}^{n}R_{j}\right]$ are multiplied by -1.

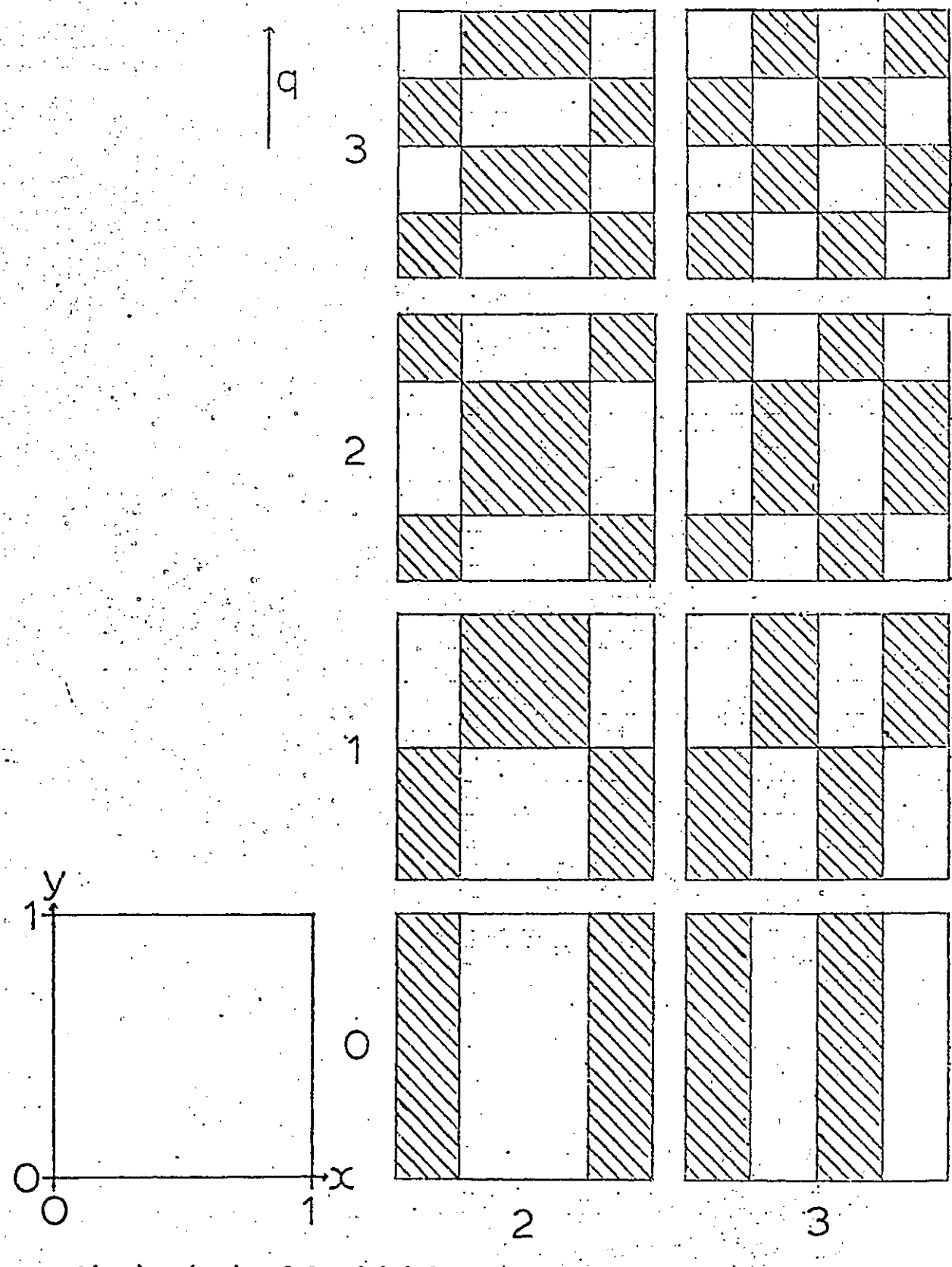
Appendix 3. The 64 two-dimensional Walsh functions, $wal(p,x)wal(q,v)$, $0 \leq x < 1, 0 \leq v < 1, 0 \leq p < 8, 0 \leq q < 8$.

The line integrals, θ^{m}_{pq}, defined by equations (2.22) and (2.23), can be calculated when the rays of inserts 1 and 2 are superimposed on the Walsh functions (see section 3.4.)



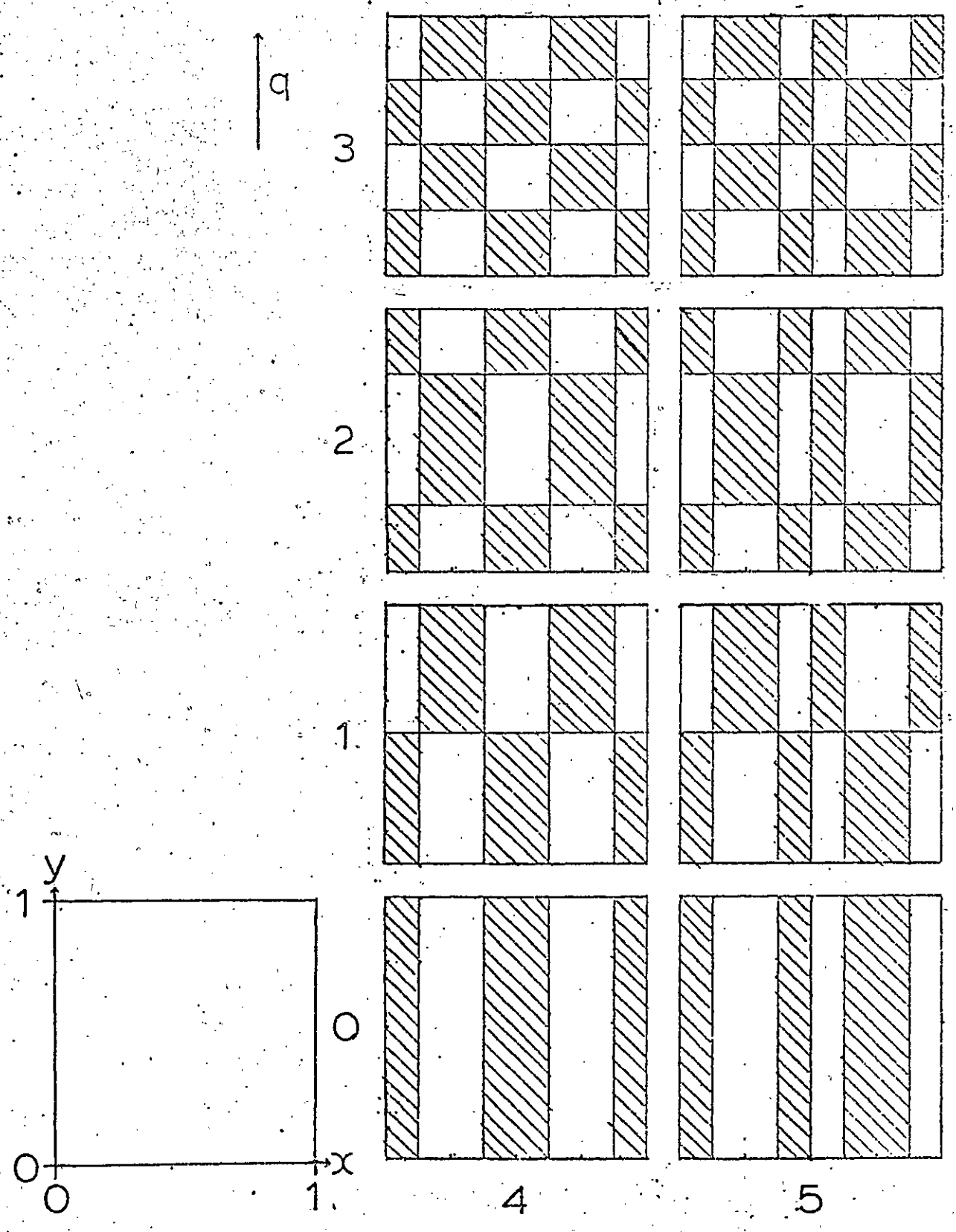
wal(p,x)wal(q,y); p=0,1; q=0,1,2,3.

Shaded areas represent +1 and unshaded areas represent -1.



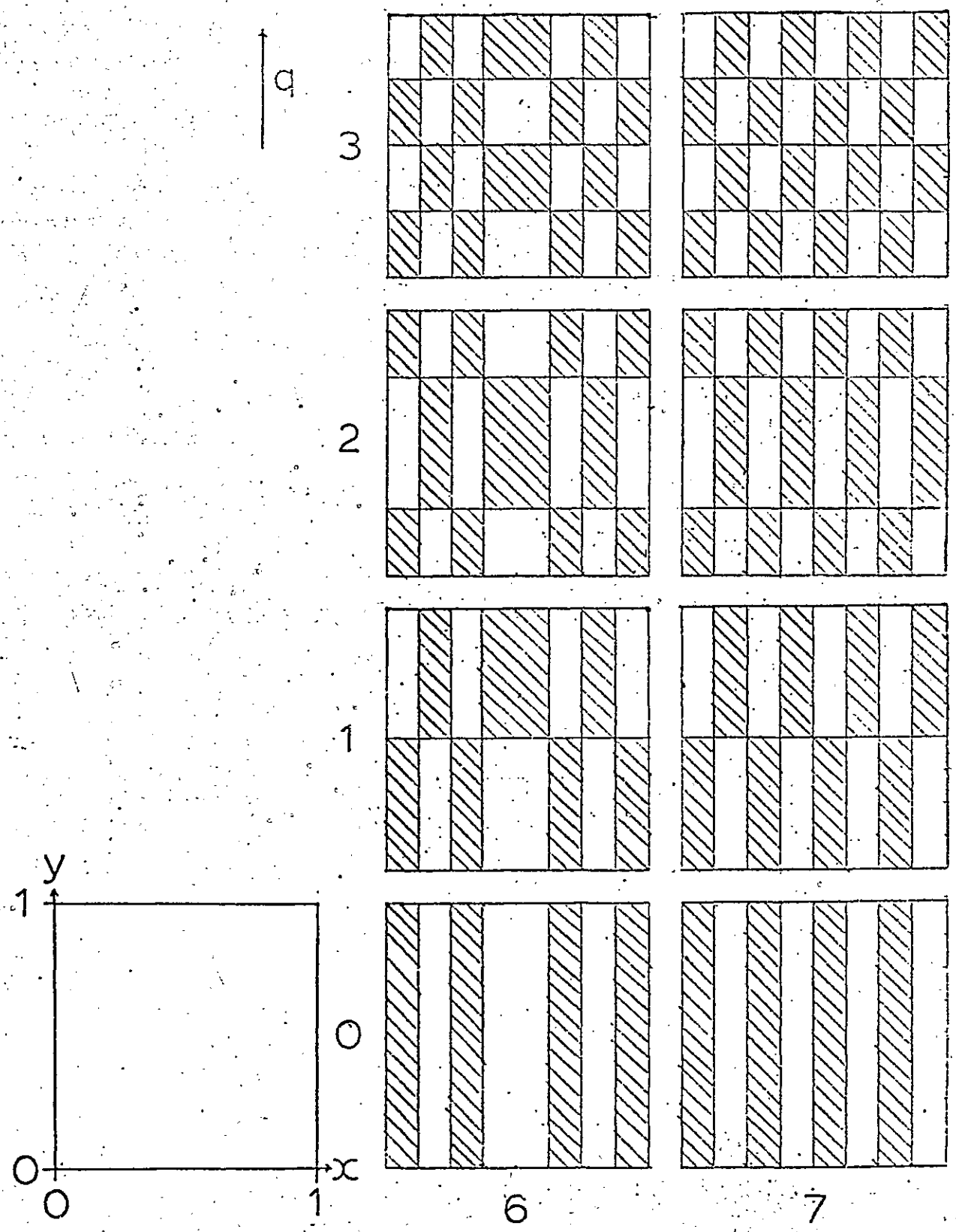
wal(p,x)wal(q,y);p=2,3;q=0,1,2,3.

Shaded areas represent +1 and unshaded areas represent -1.



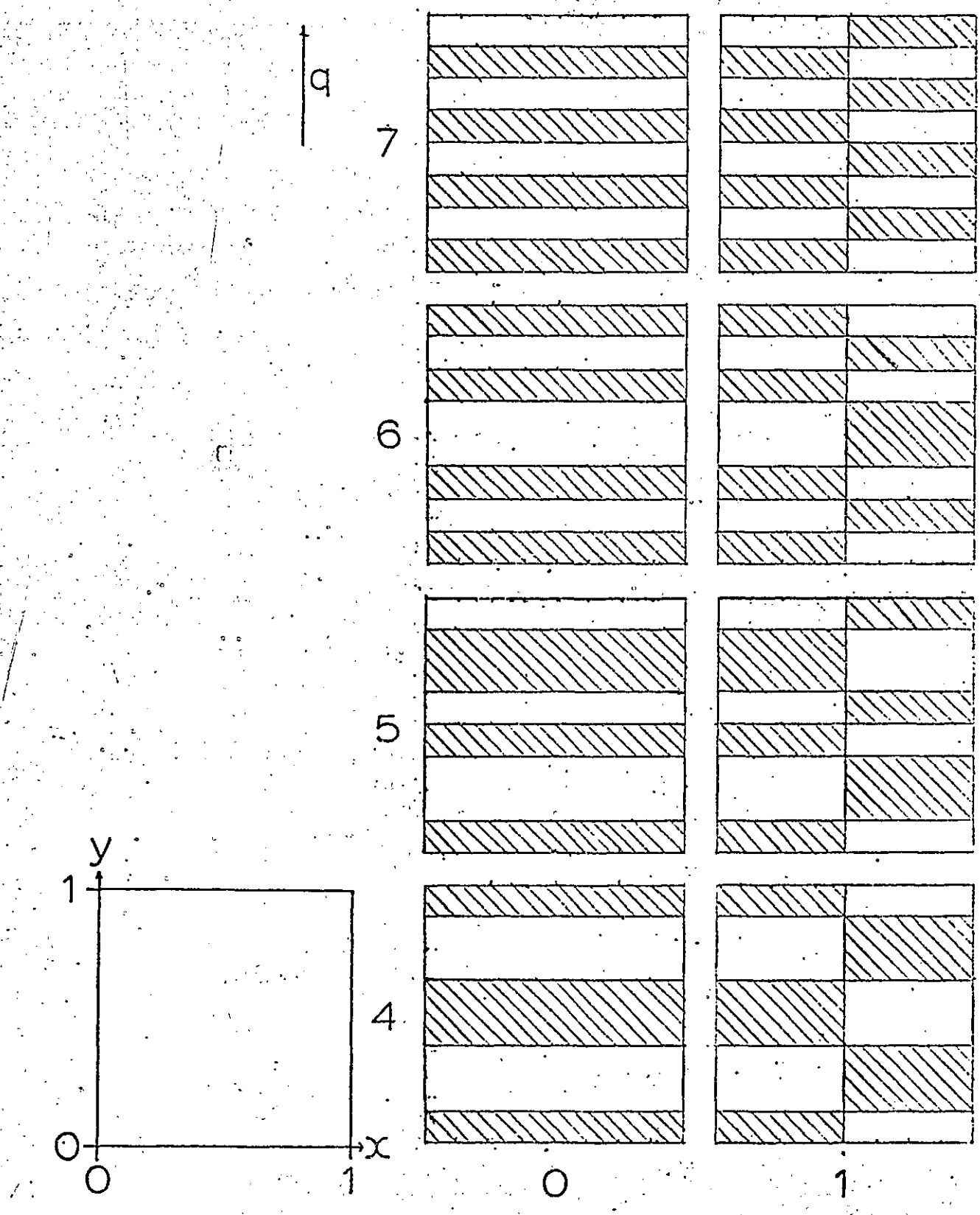
wal(p,x)wal(q,y);p=4,5;q=0,1,2,3.

Shaded areas represent +1 and unshaded areas represent -1.



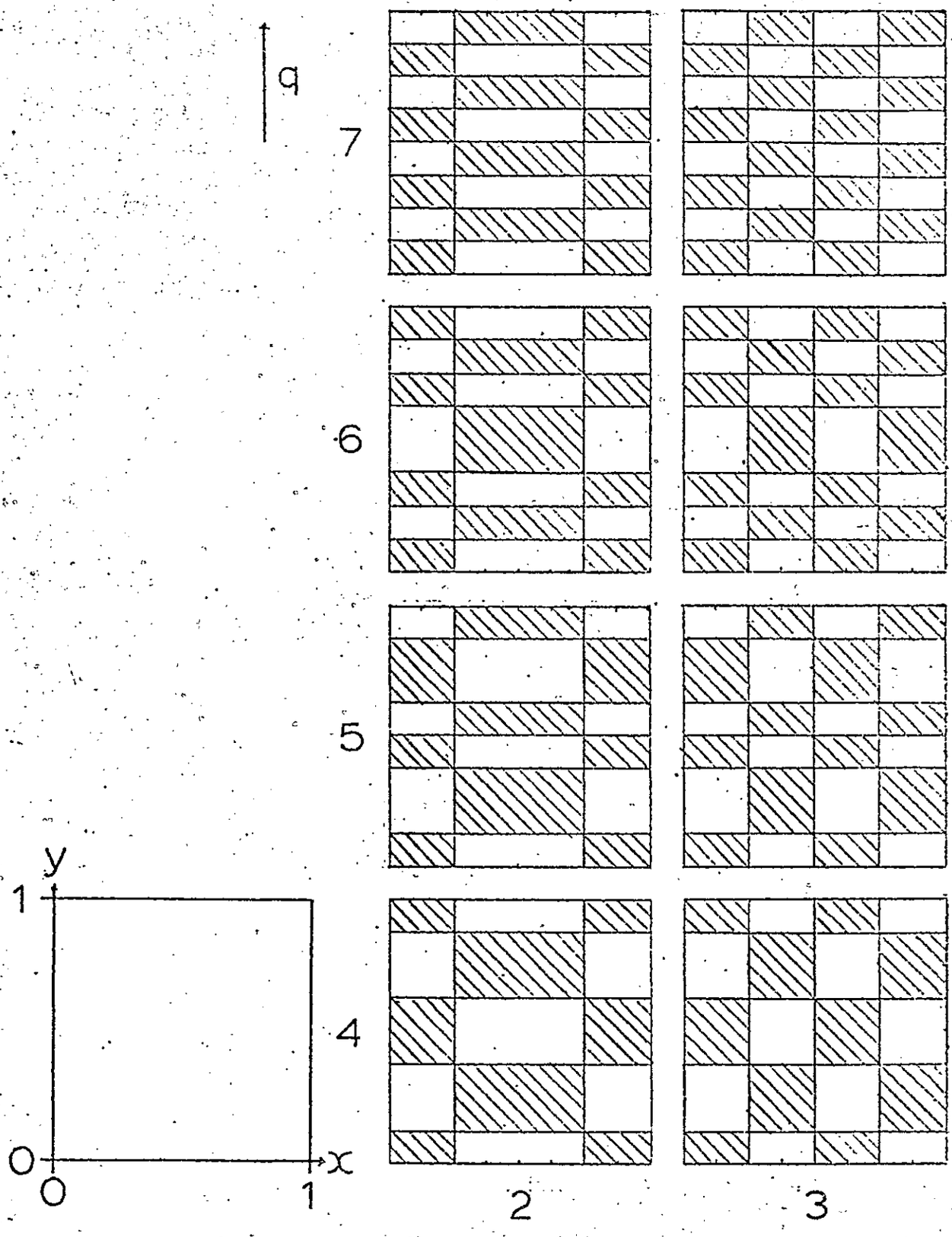
wal(p,x)wal(q,y);p=6,7;q=0,1,2,3.

Shaded areas represent +1 and unshaded areas represent -1.



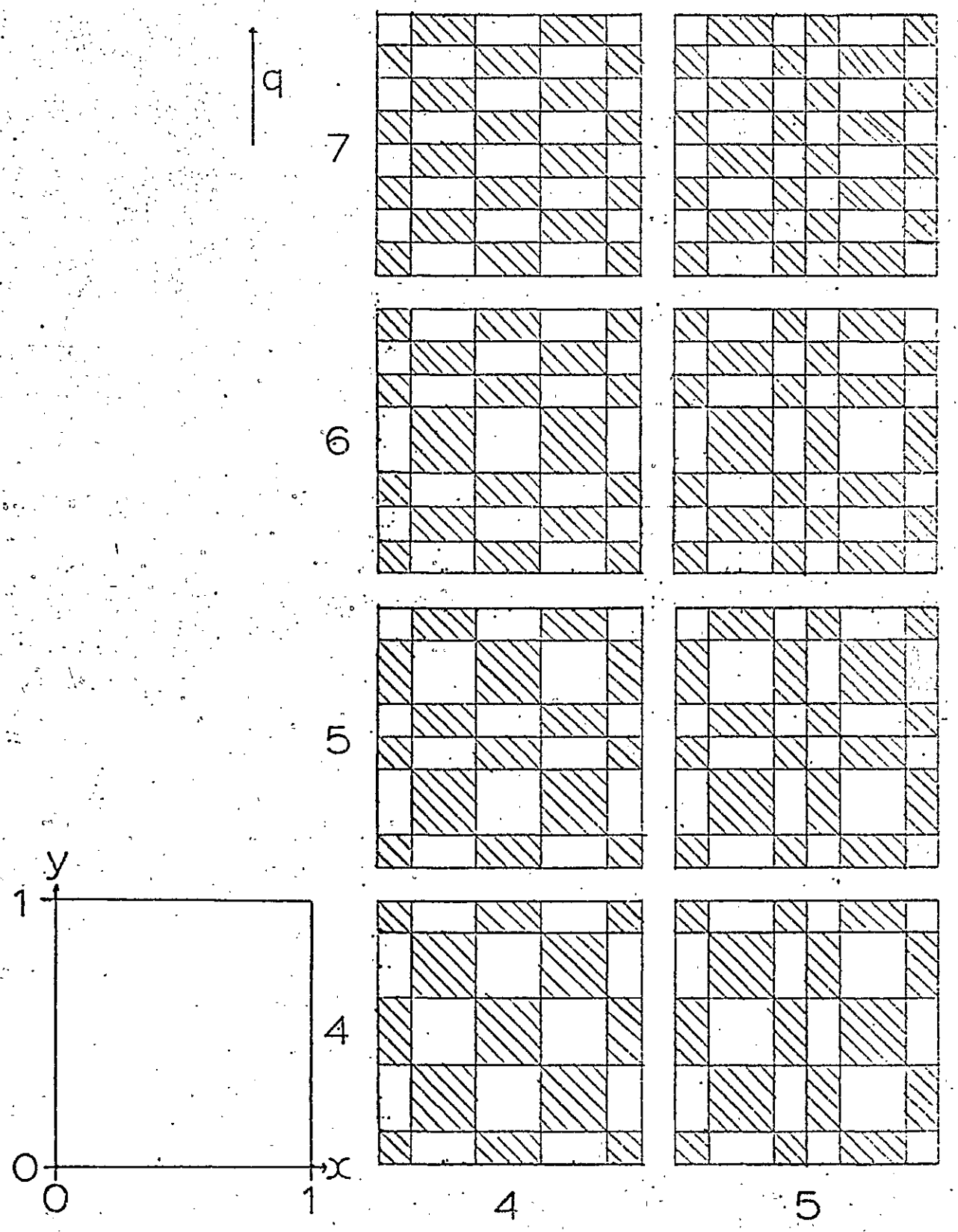
wal(p,x)wal(q,y);p=0,1;q=4,5,6,7.

Shaded areas represent +1 and unshaded areas represent -1.



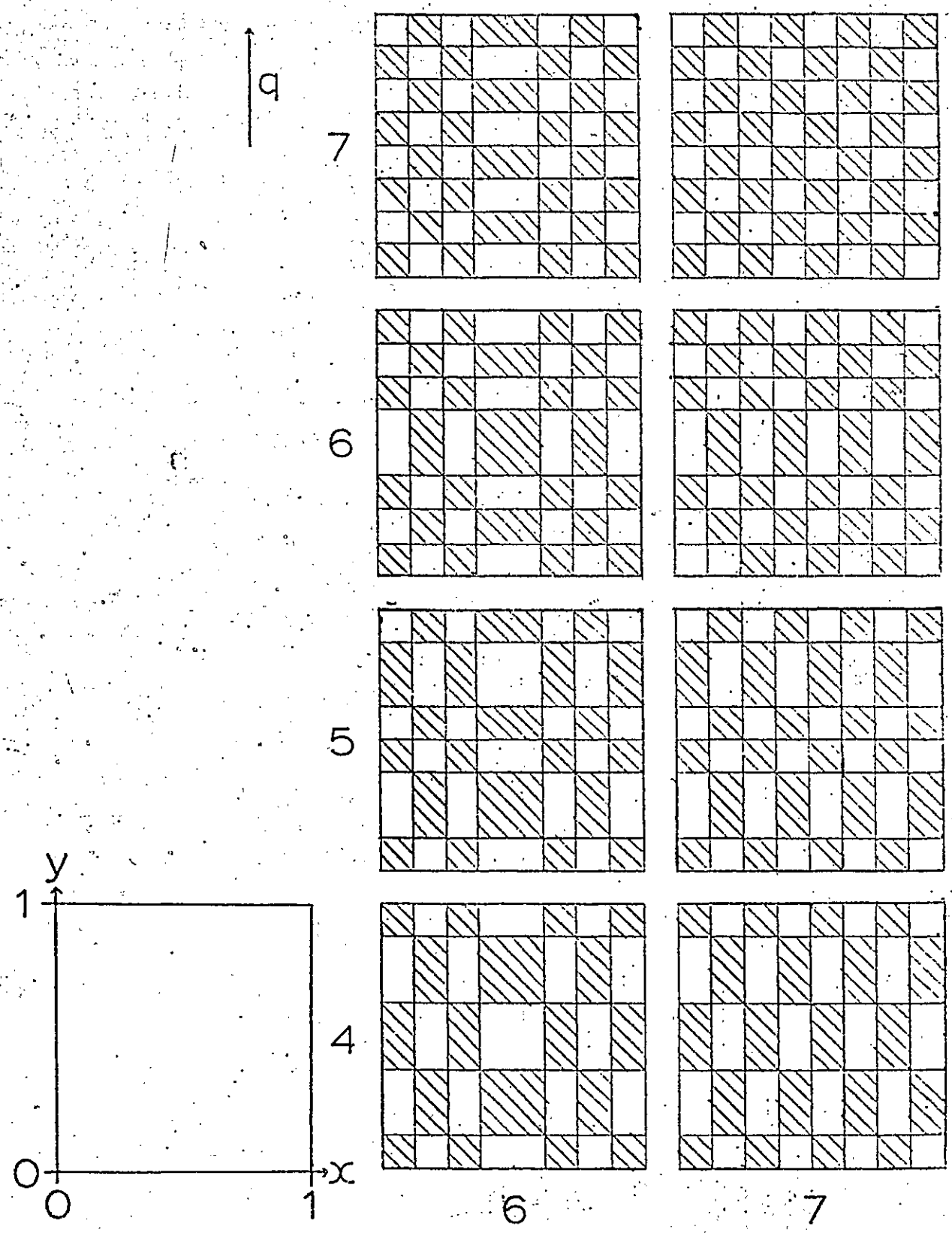
wal(p,x)wal(q,y); p=2,3; q=4,5,6,7.

Shaded areas represent +1 and unshaded areas represent -1.



wal(p,x)wal(q,y); p=4,5; q=4,5,6,7.

Shaded areas represent +1 and unshaded areas represent -1.

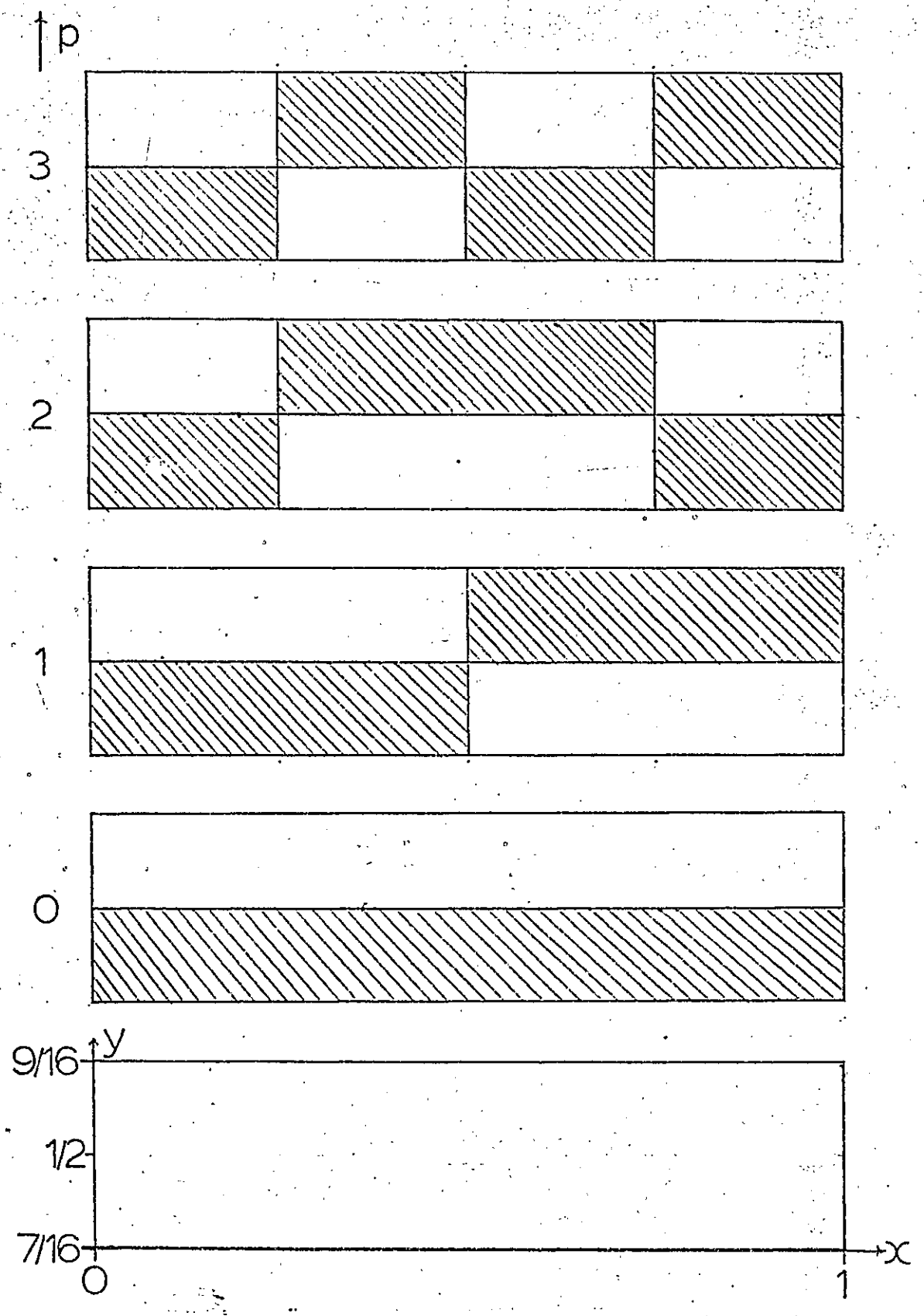


wal(p,x)wal(q,y); p=6,7; q=4,5,6,7.

Shaded areas represent +1 and unshaded areas represent -1.

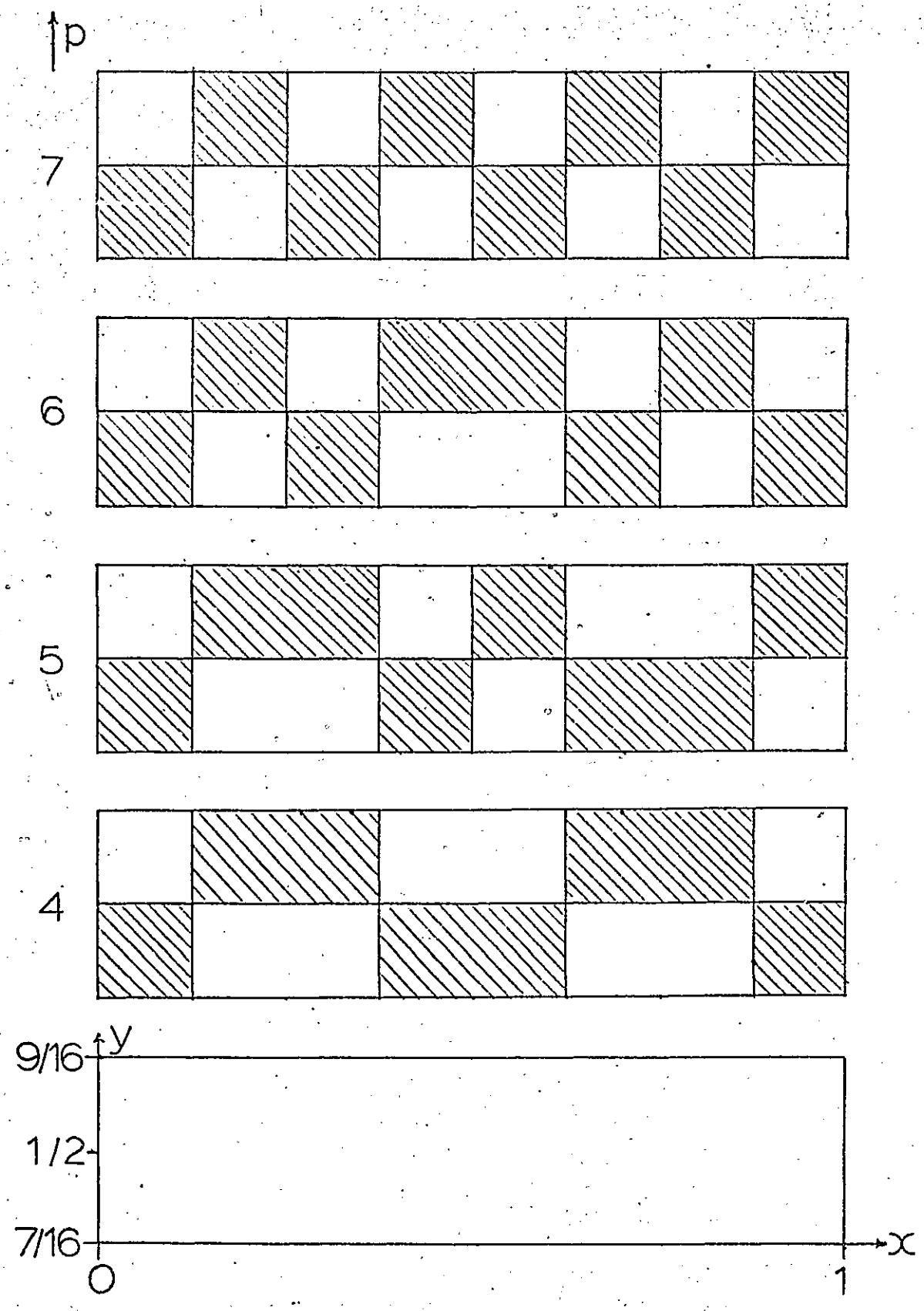
Appendix 4. The two-dimensional Walsh functions wal(p,x)wal(1,y), $0 \leqslant x < 1$, $7/16 \leqslant y < 9/16$, $0 \leqslant p < 16$.

The line integrals, θ^{m}_{pl}, $0 \leqslant p < 16$, defined by equation (2.22), can be calculated when the rays of insert 3 are superimposed on the Walsh functions.



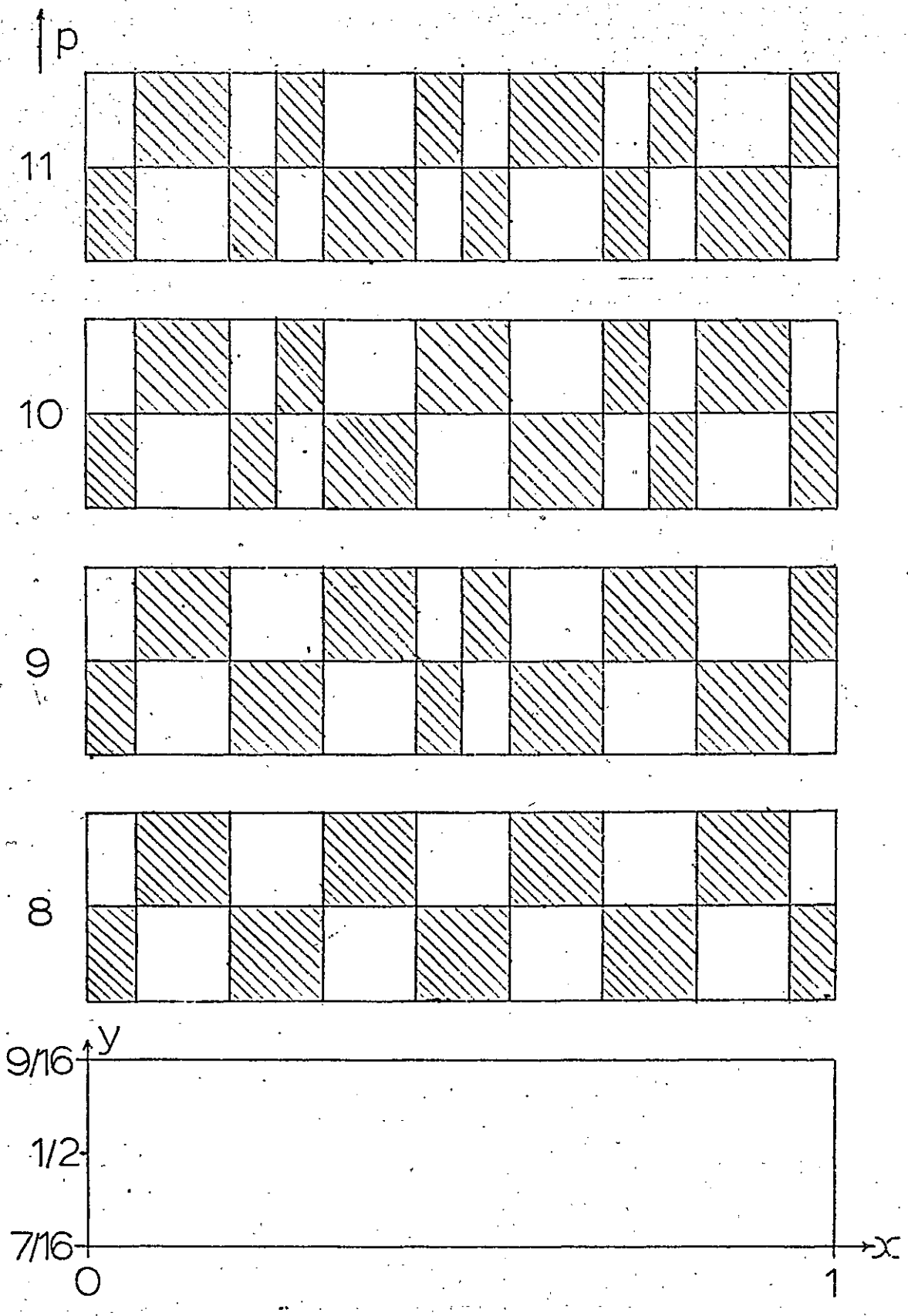
wal(p,x)wal(1,y), p=0,1,2,3.

Shaded areas represent +1 and unshaded areas represent -1.



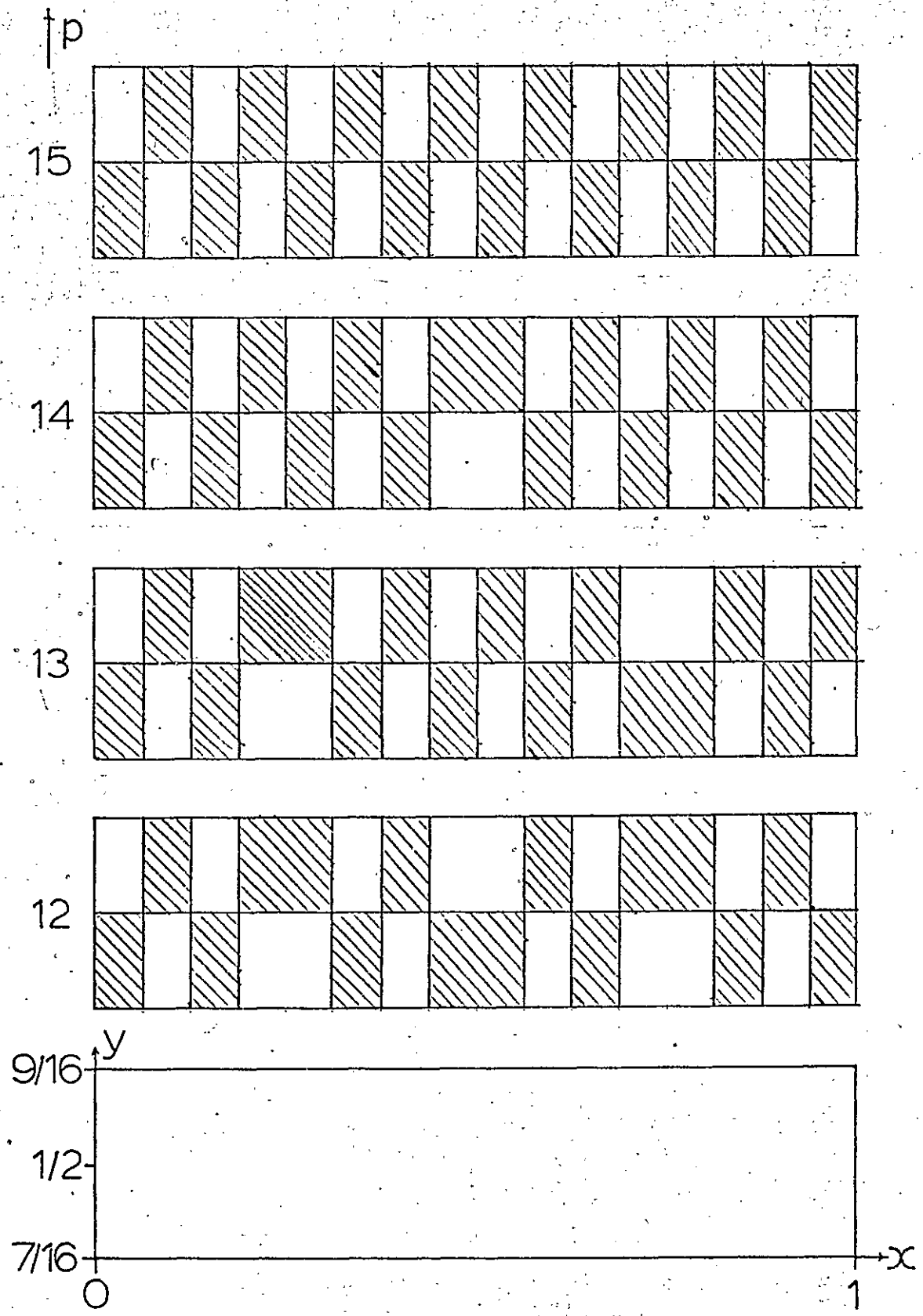
wal(p,x)wal(! ,y), p=4,5,6,7.

Shaded areas represent +1 and unshaded areas represent -1.



wal(p,x)wal(!,y),p=8,9,10,11.

Shaded areas represent +1 and unshaded areas represent -1.



wal(p,x)wal(1,y), p=12,13,14,15.

Shaded areas represent +1 and unshaded areas represent -1.

Appendix 5. The listing of the computer program which implements all four of the flow diagrams shown in Fig(4.1).

```
                PROGRAM INDEX (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT)                    SUBR  1
000002          COMMON MSMALL,NSMALL,ISMALL,KSMALL,LSMALL,MN,K2,IM1NMK,JKID2,JAJB
000002          COMMON JNMK,JIM1,JI,JNMKP1,JNMI,JNPK,IM2,JIM2,NMK,JKI,JKM2,JMI,JB
000002          COMMON ND2,NP1,MD2,MP1,X,Y,IM1,P,RANGE,REFMIN,KO,K1,KMAX,KMAXP1
000002          COMMON SIGMA,SIGDIF,AV,DCIN,DCOUT,FMAX,FMIN,NSOLN,ND16,F1,RNOISE
000002          INTEGER SAMDAT(6)
000002          DIMENSION FI(1024),GI(1024),F(64,64),G(64,64),BX(4033),BY(4033)
000002          DIMENSION FY(64),FX(64)
000002          DIMENSION FA(101),SYMBOLS(30),A(64),C(231)
000002          DIMENSION CC(15),SC(15),TC(15),UC(15)
000002          WRITE(6,9001)
000006    9001  FORMAT(14 ,'
                *                INDEX DEFINES THE RESOLUTION USED,M BY N,BY INPUTING*/
                *' LOG2(M) AND LOG2(N) WHICH ARE MSMALL AND NSMALL.OTHER CONSTANTS
                *ARE DEFINED IN TERMS OF THESE.X AND Y ARE THE BLOCK DIMENSIONS.
                *'/*P.F(I,J)IS THE REF INDEX DISTRIBUTION.K1=1 TO 4...(1)I/P,F(I,J)
                *,IS A GIVEN STEP FUNCTION OF M BY N VALUES.EACH VALUE IS READ IN..
                *'/*.(4)I/P IS A SMOOTH FUNCTION FROM WHICH A STEP FUNCTION IS GEN
                *RATED AS MEAN VALUE OF FUNCTION IN EACH RESOLUTION RECTANGLE.FROM
                *'/*STRUCTION FROM PATH DIFF DATA FROM THESE TWO SHOULD BE EXACT...
                *(2)I/P IS A SMOOTH FUNCTION FROM WHICH PATH DIFF DATA WILL BE
                *'/*BEST RECONSTRUCTION FOR THE RESOLUTION USED...(3)I/P IS PATH DI
                *FF DATA FROM EXPERIMENT.RECONSTRUCTION ACCURACY WILL DEPEND
                *'/*ON RESOLUTION USED AND EXPERIMENTAL ERRORS IN PATH DIFF DATA.
                *CORE RECORD...'OUT (9/2) M,N, KO=1 TO 5..(1)I/P AND O/P REF INDEX,
                *'/*PATH DIFFS,COEFFS AND POINT ERRORS PRINTED...(2)PATH DIFFS NOT
                *PRINTED...(3)COEFFS NOT PRINTED...(4)PATH DIFFS AND COEFFS N..
                *'/*PRINTED...(5)REF INDEX I/P AND O/P ONLY PRINTED.KO=11 TO 15 SA
                *ME AS FOR KO=1 TO 5 BUT REF INDEX I/P AND O/P IS ALPHANUMERIC.')
000006          WRITE(6,9003)
000012    9003  FORMAT(14 ,'NDATA IS NUMBER OF SETS OF DATA .IF NSOLN=1...K2=1 IS
                *FOR SOLN FROM NDX X LINES AND N Y LINES.K2=2 IS FOR NDY Y LINES AN
                *D N X LINES.IF NSOLN=2 BOTH SOLNS ARE CALCULATED.IN THIS CASE SE
                *AVE K2 BLANK(I.E. ZERO)')
```

KO = 6: ONLY ERRORS O/P

DATA CARDS

```
000012          READ(5,1001) NDATA                                                  (A)
000021     1001 FORMAT(I10)
000020          PI=3.1415926534
000022          DO 1 IDATA=1,NDATA                                                  DO  1
                C  IF NO CHANGE IN J TH I/P DATA CARD IN 1 TO IN 6,7 OR 8   ,SAMDAT(J)=0
                C  AND THIS CARD LEFT OUT.
000023          IF(IDATA.GT.1)READ(5,1010) (SAMDAT(J),J=1,8)                        (B)
000032     101) FORMAT(8I2)
000032          WRITE(6,9002)
000036     9002 FORMAT(1H ,'++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++IN 1
                *++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++ TO
                *+++++++++++++++')                                                  IN 8
000036          IF(IDATA.EQ.1) WRITE(6,9003) IDATA
000046          IF(IDATA.EQ.2) WRITE(6,9004) IDATA
000056          IF(IDATA.EQ.5) WRITE(6,9005) IDATA
000066          IF(IDATA.GE.3) WRITE(6,9006) IDATA
000076     9003 FORMAT(1H ,'+ THE *,I4,* ST SET OF DATA IS **/1H ,*++++++++++++++++
                *+++++++++++++++')
000076     9004 FORMAT(1H ,'+ THE *,I4,* ND SET OF DATA IS **/1H ,*++++++++++++++++
                *+++++++++++++++')
000076     9005 FORMAT(1H ,'+ THE *,I4,* RD SET OF DATA IS **/1H ,*++++++++++++++++
                *+++++++++++++++')
000076     9006 FORMAT(1H ,'+ THE *,I4,* TH SET OF DATA IS **/1H ,*++++++++++++++++
                *+++++++++++++++')
000076          IF(IDATA.EQ.1.OR.SAMDAT(1).EQ.1) READ(5,1009) NSOLN,K2,K0           1
000116     1009 FORMAT(3I10)
000116          IF(IDATA.EQ.1.OR.SAMDAT(2).EQ.1) READ(5,1002) K1,NSMALL             2
000134     1002 FORMAT(2I10)
000134          IF(IDATA.EQ.1.OR.SAMDAT(3).EQ.1) READ(5,1003) X,Y,P                 3
000154          IF(IDATA.EQ.1.OR.SAMDAT(4).EQ.1) READ(5,1003) DCIN,DCOUT,RNOISE     4
000174     1003 FORMAT(3E10.3)
000174          IF((K0.GT.1).AND.(IDATA.EQ.1.OR.SAMDAT(5).EQ.1))READ(5,1004) NS,    5
                *(SYMBOLS(I),I=1,NS)
000220     1004 FORMAT(I10,30A1)
000220          WRITE(6,9009) K0
000226     9009 FORMAT(1H0,'PRINTOUT MODE IS K0=*,I3)
000226          IF(K1.EQ.1.OR.K1.EQ.3) GO TO 101
                C KMAX IS HIGHEST POWER OF TAYLORS SERIES PART OF F(X,Y).
                C NCOS IS NUMBER OF TERMS IN COS SERIES PART OF F(X,Y).
000236          IF(IDATA.EQ.1.OR.SAMDAT(6).EQ.1) READ(5,1005) KMAX,NCOS             5,6
000253     1005 FORMAT(2I10)
000254          KMAXP1=KMAX+1
000255          KT=KMAXP1*(KMAXP1+1)/2
000260          IF(KMAX.EQ.0) GO TO 102
                C    TERMS UP TO ORDER 20.
                C THE ITH TERM OF THE TAYLORS SERIES OF KT TERMS IS C(I)(X TO POWER L-1)
                C (Y TO POWER (-L),WHERE 1≤L≤K AND 1≤K≤KMAXP1(L CHANGES FIRST.)
000261          IF(IDATA.EQ.1.OR.SAMDAT(7).EQ.1) READ(5,1006)(C(I),I=1,KT)          6,7


000275     1006 FORMAT(8E11.4)
000275      102 IF(NCOS.EQ.0) GO TO 101
                C THE ITH TERM OF THE COS SERIES IS CC(I)COS(PI2)(SC(I)X+TC(I)Y+UC(I)),
                C WHERE 1≤I≤NCOS.
000276          IF(IDATA.EQ.1.OR.SAMDAT(8).EQ.1) READ(5,1006)(CC(I),SC(I),TC(I),    6,7,8
                *UC(I),I=1,NCOS)
000325      101 NSMALL=NSMALL
000326          N=2**NSMALL
000332          M=N
000332          MD2=M/2
000334          MD16=M/16
000337          MP1=M+1
000340          MD2=M/2
000342          MP1=M+1
000343          NOX=M*N-M+1
000345          NOY=M*N-N+1
000347          MN=M*N
000350          MND4=MN/4
000352          IF(K1.EQ.2.OR.K1.EQ.4) CALL PATHS2(BX,BY,F1,G1,F,G,FA,SYMBOLS,A,
                *NOX,NOY,MND4,M,N,FX,FY,NS,C,KT,NCOS,CC,SC,TC,UC)
000411          IF(K1.EQ.1) READ(5,1006)((F(I,J),J=1,N),I=1,M)                      5,6
000432          IF(K1.EQ.1) CALL PATHS1(BX,BY,F1,G1,F,G,FA,SYMBOLS,A,NOX,NOY,MND4,
                *M,N,FX,FY,NS,C,KT,NCOS,CC,SC,TC,UC)
000464          IF(K1.EQ.3) READ(5,1006)(BX(I),I=1,NOX)                             5,6
000475          IF(K1.EQ.3) READ(5,1006)(BY(I),I=1,NOY)                             6,7
000506          IF(K1.EQ.3) CALL PATHS3(BX,BY,F1,G1,F,FA,SYMBOLS,A,NOX,NOY,MND4,
                *M,N,FX,FY,NS)
000530          WRITE(6,9007)
000534     9007 FORMAT(1H0,'++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
                *++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
                *+++++++++++++++')
000534        1 CONTINUE                                                            1
000537          STOP 1001                                                         END OF
000541          END                                                               INDEX

UNUSED COMPILER SPACE
014200
```

```
                      SUBROUTINE PATHS1(BX,BY,F1,G1,F,G,FA,SYMBOLS,A,NOX,NOY,MND4,M,N,    SUBR  1
                     *FX,FY,NS,C,CI,NCOS,CC,SC,TC,UC)
                     C PATHS1INPUTS A 2D STEP FUNCTION AND CALCULATES THE PATH DIFFERENCES
                     C OF THE NOX X LINES AND NOY Y LINES.
000032               COMMON NSMALL,NSMALL,ISMALL,KSMALL,LSMALL,MN,K2,IM1NMK,JKIO2,JAJB
000032               COMMON JJMK,JIM1,JI,JNMKP1,JNMI,JNPK,IM2,JIL2,NMK,JKI,JKM2,JKM1,JS
000032               COMMON ND2,NP1,MD2,MP1,X,Y,IM1,P,RANGE,REFMIN,K0,K1,KMAX,KMAXP1
000032               COMMON SIGM1,SIGDIF,AV,DCIN,DCOUT,FMAX,FMIN,NSOLN,ND16,PI,RNOISE
000032               DIMENSION BX(NOX),BY(NOY),F(M,N),G(M,N)
                     C 1/P OF ACTUAL VALUE MUST BE READ IN.
000032               RNSEP=RNOISE*100.0
000033               IF(K1.EQ.4) GO TO 101
000036               AV=0.0                                                              DO  1
000037               DO 1 J=1,M                                                          DO  1
000040               DO 1 I=1,N                                                              1
000047             1 AV=AV+F(I,J)
000060               AV=AV/FLOAT(MN)                                                     DO  2
000062               DO 2 J=1,N                                                          DO  2
000064               DO 2 I=1,M                                                              2
000074             2 F(I,J)=F(I,J)-AV+DCIN
000105               CALL MAXMIN(F,M,N)
000110               REFMIN=FMIN
000111               RANGE=FMAX-FMIN
                     C F(I,J) IS I/P DISTBN WITH IMPOSED MEAN OF DCIN.
000113               WRITE(6,902)
000122             902) FORMAT(1H0,'IN ALL DISTRIBUTIONS BELOW Y AXIS IS HORIZONTAL TO THE
                    * RIGHT,X AXIS IS VERTICALLY DOWN.ORIGIN IS TOP LEFT HAND CORNER*)
000126               IF(K1.EQ.1) WRITE(6,9031) K1,M,N,X,Y
000151             9031 FORMAT(1H0,'INPUT MODE K1=',I3,'.PATHS COMPUTED FROM A GIVEN ',I5,
                    *' BY ',I5,' STEP FUNCTION*/1H ,'DIMENSIONS OF BOUNDARY ARE X=',
                    *1PE10.3,' BY Y=',1PE10.3,' MICRONS')
000151               IF(K1.EQ.4) WRITE(6,9032) K1,M,N,X,Y
000175             9032 FORMAT(1H0,'INPUT MODE K1=',I3,'.PATHS COMPUTED FROM A ',I5,' BY '
                    *,I5,' STEP FUNCTION GENERATED FROM A GIVEN SMOOTH FUNCTION*/1H ,
Correction:         *'DIMENSIONS OF BOUNDARY ARE X=',1PE10.3,' BY Y=',1PE10.3,' MICRONS
                    *')
000175               WRITE(6,9001) P
000203             9001 FORMAT(1H0,'REAL NOS BELOW EXCEPT ERRORS,SC(I),TC(I),UC(I) AND RNO
                    *ISE TO BE MULTIPLIED BY P =',1PE13.5)
000207               CX=(SQRT(X*X+Y*Y/FLOAT(N**2)))/FLOAT(M)
000221               CY=(SQRT(Y*Y+X*X/FLOAT(M**2)))/FLOAT(N)
000233               IF(K0.EQ.6) GO TO 104
000241               WRITE(6,9004) M,N
000251             9004 FORMAT(1H0,'THE TRUE STEP FUNCTION REF INDEX DISTRIBUTION GIVEN BY
                    * VALUES IN THEF',I4,' BY',I4,' RESOLUTION RECTANGLES IS')
000255               IF(K0.GE.11) CALL PLOT(F,FA,SYMBOLS,A,M,N,NS)
000267               IF(K0.GE.11) GO TO 104
000276               IF(N.GE.32) GO TO 106
000300               DO 5    I=1,M                                                       DO  5
000302               WRITE(6,9005)(F(I,J),J=1,N)                                             *
000324             9005 FORMAT(1H ,16F8.4)                                                   *
000330               5 CONTINUE                                                              5
000333               GO TO 104                                                               *
000333             105 DO  7 I1=1,ND16                                                   DO  7
000335               I2=I1*16                                                                *
000336               I3=I2-15                                                                *
000340               WRITE(6,9011) I3,I2                                                     *
000347             9011 FORMAT(1H ,'COLUMNS',I4,' TO',I4,' ARE')                             *
000353               DO  7 I=1,M                                                       DO  7
000354               WRITE(6,9005)(F(I,J),J=I3,I2)                                           *
000401               7 CONTINUE                                                              7
000406             104 SIGSQ=0.0
000407               DO 6    J=1,N                                                       DO  6
000411               DO 6    I=1,M                                                       DO  6
000421               G(I,J)=F(I,J)-DCIN                                                      *
000422               6 SIGSQ=SIGSQ+G(I,J)*G(I,J)                                             6
                     C G(I,J) IS I/P DISTBN ABOUT ZERO MEAN(I.E. A.C. COMPONENTS OF F(I,J)).
000433               SIGMA=SQRT(SIGSQ/FLOAT(MN))
000440               WRITE(6,9030) FMAX,FMIN,RANGE,AV
000457             9030 FORMAT(1H0,'FMAX  VALUE OF REF INDEX DISTRIBUTION IS',48X,'REFMAX='
                    *,E14.5/1H ,'FMIN  VALUE OF REF INDEX DISTRIBUTION IS',48X,'REFMIN='
                    *,E14.5/1H ,'REFMAX MINUS REFMIN IS',66X,'RANGE=',E14.5/1H ,'MEAN V
                    *ALUE OF REF INDEX DISTRIBUTION IS',52X,'AV=',E14.5)
000463               WRITE(6,9005)  SIGMA,DCIN
000472             9005 FORMAT(1H0,'THE STANDARD DEVIATION OF THE REF INDEX DISTRIBUTION A
                    *BOVE,ABOUT MEAN OF ZERO,IS',8X,'SIGMA=',E14.5/1H ,'THE MEAN VALUE
                    *IMPOSED ON THE REF INDEX DISTRIBUTION ABOVE IS',28X,'DCIN=',E14.5)
000476               WRITE(6,9007)
000501             9007 FORMAT(1H0,'-+---+---5+-----16+----------32+--------------------
                    *--------64+-------------------------------------------------------
                    *----128+---')
```

```
000506          NM1=N-1
000507          MM1=M-1
000511          BX(1)=0.0
000512          BY(1)=0.0
000513          L=0
000513          BMAX=BX(1)
000515          BMIN=BX(1)
000516          DO 3 J=1,NML                                                    DO
000520          JP1=J+1
000521          DO 3 I=1,M                                                      DO
000541          L=L+1
000542          LP1=L+1
000543          BX(LP1)=BX(L)+(G(I,JP1)-G(I,J))*CX
000546          IF(RNOISE.EQ.0.0) GO TO 3
000550          IF(BX(LP1).GE.BMAX) BMAX=BX(LP1)
000553          IF(BX(LP1).LE.BMIN) BMIN=BX(LP1)
000557        3 CONTINUE
000574          BXMBN=BMAX-BMIN
000575          IF(RNOISE.EQ.0.0) GO TO 102
000577          DO 8 I=1,NOX
000600          RNSX=RNOISE*BXMBN*(RANF(0.0)-0.5)
000610        8 BX(I)=BX(I)+RNSX
000617      102 L=0
000617          BMIN=BY(1)
000621          BMAX=BY(1)
000622          DO 4 I=1,MML                                                    DO
000624          IP1=I+1
000625          DO 4 J=1,N                                                      DO
000644          L=L+1
000645          LP1=L+1
000646          BY(LP1)=BY(L)+(G(IP1,J)-G(I,J))*CY
000651          IF(RNOISE.EQ.0.0) GO TO 4
000653          IF(BY(LP1).GE.BMAX) BMAX=BY(LP1)
000656          IF(BY(LP1).LE.BMIN) BMIN=BY(LP1)
000662        4 CONTINUE
000700          BYMBN=BMAX-BMIN
000701          IF(RNOISE.EQ.0.0) GO TO 103
000703          DO 9 I=1,NOY
000704          RNSX=RNOISE*BYMBN*(RANF(0.0)-0.5)
000714        9 BY(I)=BY(I)+RNSX
000722          IF(K0.EQ.6) GO TO 105
000724      103 IF(K0.EQ.2.OR.K0.EQ.4.OR.K0.EQ.5.OR.K0.EQ.12.OR.K0.EQ.14.OR.K0.EQ.
                *15) GO TO 105
000753          WRITE(6,9014) RNSEP
000761     9014 FORMAT(1H0,*THE RANGE OF NOISE IMPOSED ON THE RELATIVE PATHS/MAX P
                *ATH MINUS MIN PATH IS*,F7.3,* PER CENT*)
000765          WRITE(6,9008) NOX
000772     9008 FORMAT(1H0,*THE RELATIVE PATHS OF ALL*,I5,* X RAYS,IN MICRONS,INCL
                *UDING IMPOSED NOISE,IS*)
000776          WRITE(6,9009)(BX(I),I=1,NOX)
001007          WRITE(6,9010) NOY
001020     9010 FORMAT(1H0,*THE RELATIVE PATHS OF ALL*,I5,* Y RAYS,IN MICRONS,INCL
                *UDING IMPOSED NOISE,IS*)
001024          WRITE(6,9009)(BY(J),J=1,NOX)
001035     9009 FORMAT(1H ,1P10E13.4)
001041      105 CALL REF(BX,BY,F1,G1,F,G,FA,SYMBOLS,A,NOX,NOY,MND4,M,N,FX,FY,NS)
001064          RETURN                                                          END OF
001065          END                                                             PATHS

UNUSED COMPILER SPACE
013200
```

```
                SUBROUTINE PATHS2(BX,BY,F1,G1,F,G,FA,SYMBOLS,A,NOX,NOY,MND4,M,N,   $UBR  3
               *FX,FY,NS,C,KT,NCOS,CC,SC,TC,UC)
      C PATHS2 INPUTS A SMOOTH 2D FUNCTION AND CALCULATES THE PATH DIFFS OF
      C NOX X LINES AND NOY Y LINES.
 000032       COMMON MS14,LSMALL,NSMALL,ISMALL,KSMALL,LSMALL,MN,K2,IM1NMK,JKID2,JAJJ
 000032       COMMON JNMK,JIM1,JI,JNMKP1,JNM1,JNPK,IM2,JI1,2,NNK,JKI,JKM2,JKM1,J3
 000032       COMMON ND2,NP1,MD2,MP1,X,Y,IM1,P,RANGE,REFMIN,K0,K1,KMAX,KMAXP1
 000032       COMMON SIGM1,SIGDIF,AV,DCIN,DCOUT,FMAX,FMIN,NSOLN,ND16,PI,RNOISE
 000032       DIMENSION BX(NOX),BY(NOY),F(M,N),G(M,N),C(KT)
 000032       DIMENSION CC(NCOS),SC(NCOS),TC(NCOS),UC(NCOS)
 000032       RNSEP=RNOISE*100.0
 000033       CX=SQRT(X*X+Y*Y/FLOAT(M*N))
 000042       CY=SQRT(Y*Y+X*X/FLOAT(M*N))
 000052       PI2=2.*PI
      C       1ST PART GIVES MEAN VALUE OF FUNCTION F(X,Y) IN EACH RESOLUTION
      C       RECTANGLE. F(X,Y) FOR A SQUARE BLOCK X=Y=1 IS INPUTED AND SCALED FOR
      C       GENERAL X AND Y.
 000053       RM=1./FLOAT(M)
 000056       RN=1./FLOAT(N)
 000061       RMD2=RM/2.
 000063       RND2=RN/2.

 000064       PIDM=PI*RM
 000065       PIDN=PI*RN
 000066       BV=0.0
 000067       DO 1 J=1,N                                                      DO  1.
 000074       JMRJ=1.-1./FLOAT(J)                                                 .
 000076       RJN=FLOAT(J)/FLOAT(N)                                               .
 000100       RJN1=(RJN-RND2)*PI2                                                 .
 000103       DO 1 I=1,M                                                      DO  1.
 000105       IMRI=1.-1./FLOAT(I)                                                 .
 000110       RIM=FLOAT(I)/FLOAT(M)                                               .
 000112       RIM1=(RIM-RMD2)*PI2                                                 .
 000114       KS=0                                                                .
 000115       R=0.0                                                               .
 000115       RR=0.                                                               .
 000116       IF(KMAX.EQ.3) GO TO 110                                             .
      C THE MEAN VALUE OF F IN THE (I,J) RESOLUTION RECTANGLE DUE TO TAYLORS
      C SERIES OF KT TERMS OF HIGHEST ORDER KMAX.                                .
 000120       DO 2 K=1,KMAXP1                                                 DO  2.
 000121       DO 2 L=1,K                                                      DO  2.
 000126       KMLP1=K-L+1                                                         **
 000130       IU=L*KMLP1                                                          **
 000131       U=1./FLOAT(IU)                                                      **
 000133       V=1.-IMRI**L                                                        **
 000140       W=1.-IMRJ**KMLP1                                                    **
 000145       KS=KS+1                                                             **
 000145     2 R=R+C(KS)*(RJN**KMLP1)*(RIM**L)*U*V*W                               2.
 000171   110 IF(NCOS.EQ.0) GO TO 101                                             .
      C THE MEAN VALUE OF F IN THE (I,J) RESOLUTION RECTANGLE DUE TO COSINE      .
      C SERIES OF NCOS TERMS.                                                    .
 000173       DO 7 IC=1,NCOS                                                  DO  7.
 000174       PISOM=PIDM*SC(IC)                                                   **
 000176       IF(PISOM.EQ.0) U=1.                                                 **
 000201       IF(PISOM.NE.0) U=SIN(PISOM)/PISOM                                   **
 000206       PITON=PIDN*TC(IC)                                                   **
 000211       IF(PITON.EQ.0) V=1.                                                 **
 000217       IF(PITON.NE.0) V=SIN(PITON)/PITON                                   **
 000224       SOI=SC(IC)*RIM1                                                     **
 000227       TOJ=TC(IC)*RJN1                                                     **
 000231       UO=UC(IC)*PI2                                                       **
 000237     7 RR=RR+CC(IC)*U*V*COS(SOI+TOJ+UO)                                    7.
 000257   101 F(I,J)=R*FLOAT(MN)+RR                                               .
 000267     1 BV=BV+F(I,J)                                                        1.
 000277       AV=BV/FLOAT(MN)                                                     .
 000301       DO 8  J=1,N                                                     DO  8
 000302       DO 8  I=1,M                                                     DO  8
 000312     8 F(I,J)=F(I,J)-AV+DCIN                                               8
 000323       CALL MAXMIN(F,M,N)                                                  .
 000326       REFMIN=FMIN                                                         .
 000327       RANGE=FMAX-FMIN                                                     .
      C F(I,J) IS I/P DISTRN WITH IMPOSED MEAN OF DCIN.                          .
 000331       WRITE(6,9020)                                                       .
 000340  9020 FORMAT(1H0,*IN ALL DISTRIBUTIONS BELOW Y AXIS IS HORIZONTAL TO THE
               * RIGHT, X AXIS IS VERTICALLY DOWN. ORIGIN IS TOP LEFT HAND CORNER*)
 000344       IF(K.EQ.2)WRITE(6,9031) M,N,X,Y                                     .
 000351  9031 FORMAT(1H0,*INPUT MODE K1=2 ,*,*PATHS COMPUTED FROM A SMOOTH FUNC
               *TION GIVEN AS A FINITE POWER AND/OR COSINE SERIES*/1H ,*RESOLUTION
               * IS N=*,I5,* BY M=*,I5,*. DIMENSIONS OF BOUNDARY ARE X=*,1PE10.3,*
               * Y=*,1PE10.3,* ,MICRONS*)
 000365       WRITE(6,9001) P                                                     .
 000372  9001 FORMAT(1H0,CALL FIGS BELOW EXCEPT ERRORS,SC(I),TC(I),UC(I) AND RNO
               *ISE TO BE MULTIPLIED BY P =*,1PE13.5)
 000376       IF(KT.LE.2) GO TO 102                                               .
 000400       WRITE(6,9012) KT,KMAXP1                                             .
 000410  9012 FORMAT(1H0,*THERE,I6,* COEFFS C(I) IN THE TAYLORS SERIES SUM(K=1 TO
               **,I5,* )(L=1 TO K) OF C(I)(X TO POWER L-1)(Y TO POWER K-L)*/1H ,*
               *WHERE I=K(K-1)/2+L,ARE*)
 000414       WRITE(6,9002) (C(I),I=1,KT)                                         .
 000422  9002 FORMAT(1H ,1P10E13.4)                                              .
 000426   102 IF(NCOS.EQ.0) GO TO 103                                            .
 000430       WRITE(6,9011) NCOS                                                  .
 000435  9011 FORMAT(1H0,'COS SERIES SUM(I=1 TO*,I5,* ) OF CC(I)COS(PI2)(SC(I)X+
               *TC(I)Y+UC(I)) IS SC(I),TC(I),UC(I)=*)
 000441       WRITE(6,9007) (CC(JC),SC(JC),TC(JC),UC(JC),JC=1,NCOS)               .
 000471  9003 FORMAT(1H ,1P4E15.6,*,*,1P4E15.6)                                  .
 000475   103 IF(K1.EQ.4) CALL PATHS1(BX,BY,F1,G1,F,G,FA,SYMBOLS,A,NOX,NOY,MND4,
               *M,N,FX,FY,NS)
 000522       IF(K1.EQ.4) RETURN                                                  .
 000531       IF(K0.EQ.6) GO TO 190                                               .
 000533       WRITE(6,9004) M,N                                                   .
 000543  9004 FORMAT(1H0,'TRUE REF INDEX DISTRIBUTION OF SMOOTH FUNCTION EXPRESS
               *ED AS THE MEAN VALUES OF F(X,Y) IN THE*,I4,* BY*,I4,* RESOLUTION R
               *ECTANGLES IS*)
 000547       IF(K0.GE.11) GO TO 104                                              .
 000551       IF(N.GE.32) GO TO 109                                               .
```

```
000554          DO 11  I=1,M                                                    DO 11
000555          WRITE(6,9005)(F(I,J),J=1,N)                                       .
000577     9005 FORMAT(1H ,15F8.4)                                                .
000603       11 CONTINUE                                                         11
000606          GO TO 104
000606      103 DO 13 I1=1,NDIV6                                                DO 13
000610          I2=I1*16                                                          .
000611          I3=I2-15                                                          .
000613          WRITE(6,9013) I3,I2                                               .
000622     9013 FORMAT(1H ,*COLUMNS*,I4,* TO*,I4,* ARE*)                          .
000626          DO 13 I=1,M                                                    DO 13
000627          WRITE(6,9005)(F(I,J),J=I3,I2)                                     .
000654       13 CONTINUE                                                         13
000661      104 IF(KO.GE.11) CALL PLOT(F,FA,SYMBOLS,A,M,N,NS)
000700      190 SIGSQ=0.0
000701          DO 12  J=1,N                                                   DO 12
000703          DO 12  I=1,M                                                   DO 12
000713          G(I,J)=F(I,J)-DCIN                                                .
000714       12 SIGSQ=SIGSQ+G(I,J)*G(I,J)                                         12
      C G(I,J) IS I/P DISTBN ABOUT ZERO MEAN(I.E. A.C. COMPONENTS OF F(I,J)).
000725          SIGMA=SQRT(SIGSQ/FLOAT(MN))
000732          WRITE(6,9030) FMAX,FMIN,RANGE,AV
000751     9030 FORMAT(1H0,*MAX  VALUE OF REF INDEX DISTRIBUTION IS*,48X,*REFMAX=*
                *,E14.5/1H ,*MIN  VALUE OF REF INDEX DISTRIBUTION IS*,48X,*REFMIN=*
                *,E14.5/1H ,*REFMAX MINUS REFMIN IS*,56X,*RANGE=*,E14.5/1H ,*MEAN V
                *ALUE OF REF INDEX DISTRIBUTION IS*,52X,*AV=*,E14.5)
000755          WRITE(6,9006)  SIGMA,DCIN
000764     9006 FORMAT(1H0,*THE STANDARD DEVIATION OF THE REF INDEX DISTRIBUTION A
                *BOVE,ABOUT MEAN OF ZERO,IS*,8X,*SIGMA=*,E14.5/1H ,*THE MEAN VALUE
                *IMPOSED ON THE REF INDEX DISTRIBUTION ABOVE IS*,28X,*DCIN=*,E14.5)
000770          WRITE(6,9007)
000773     9007 FORMAT(1H0,*--+---+---8+-----16+----------32+--------------------
                *-------64+-------------------------------------------------------
                *-----128+----*)
      C      3RD PART GIVES PATH DIFFS OF X AND Y LINES RLL TO DX(1)=BY(1)=0.0.
000777          GX=-RN
001000          GY=-RM
001002          DO 3 KK=1,2                                                    DO  3
001003          IF(KK.EQ.1) ND=NDX                                                .
001006          IF(KK.EQ.1) GRAD=GX                                               .
001011          IF(KK.EQ.1) MORN=M                                                .
001014          IF(KK.EQ.2) ND=NDY                                                .
001017          IF(KK.EQ.2) GRAD=GY                                               .
001022          IF(KK.EQ.2) MORN=N                                                .
001025          EMIN=0.0                                                          .
001026          BMAX=0.0                                                          .
001027          DO 4 I=1,ND                                                    DO  4
001030          Q=-1./(1.0E-FLOAT(I-1)/FLOAT(MORN))                               .
001035          AORB=GRAD/Q                                                       .
001036          AORB2=AORB**2.                                                    .
001040          KS=0                                                              .
001041          S2=0.0                                                            .
001042          DO 5 K=1,KMAXP1                                                DO  5
001043          DO 5 L=1,K                                                     DO  5
001044          LM1=L-1                                                           .
001046          KML=K-L                                                           .
001047          KMLP1=KML+1                                                       .
001047          IF(KK.EQ.1) KL=KML                                                .
001052          IF(KK.EQ.1) LK=LM1                                                .
001055          IF(KK.EQ.2) KL=LM1                                                .
001060          IF(KK.EQ.2) LK=KML                                                .
001063          KLP1=KL+1                                                         .
001065          LKP1=LK+1                                                         .
001066          KS=KS+1                                                           .
001070          IF(KML.GE.1.AND.LM1.GE.1) GO TO 105                               .
001077          IF(KS.EQ.1) R=C(1)                                                .
001103          IF(KL.EQ.0.AND.LK.NE.0) S2=S2+C(KS)/FLOAT(K)                      .
001115          IF(LK.EQ.0.AND.KL.NE.0) S2=S2+C(KS)*(AORB**K)*((1.+Q)**K-1.)/(GRAD
                **FLOAT(K))                                                       .
001144          IF(KS.NE.KT) S1=0.0                                               .
001147          IF(KS.EQ.KT) S1=S2                                                .
001152          GO TO 5                                                           .
001153      105 S=1./FLOAT(LKP1)                                                  .
001155          SC=1.0                                                            .
001156          IM=KLP1                                                           .
001157          LP=LKP1                                                           .
001161          DO 6 II=1,KL                                                   DO  6
001166          IM=IM-1                                                           .
001167          LP=LP+1                                                           .
001170          SC=(FLOAT(IM)/FLOAT(II))*Q*SC                                     .
001173        6 S=S+SC/FLOAT(LP)                                                  6
001177          S1=C(KS)*(1.0+R)**KL)*S                                           .
001211        5 R=R+S1                                                            5
001220          IF (NCOS.EQ.0)GO TO 106                                           .
```

```
001222          DO 9 IC=1,NCOS                                          DO  9...
001235          IF(KK.EQ.1) SO=SO(IC)                                       ...
001240          IF(KK.EQ.1) TO=TO(IC)                                       ...
001243          IF(KK.EQ.2) SO=TO(IC)                                       ...
001246          IF(KK.EQ.2) TO=SO(IC)                                       ...
001251          R1=(SO+TO*GRID)*PI                                          ...
001254          R2=R1+PI2*(TO*AORB+UO(IC))                                  ...
001261          IF(R1.LQ.0) SINK=1.                                         ...
001264          IF(R1.NE.0) SINK=SIN(R1)/R1                                 ...
001274        9 R=R+UO(IC)*COS(R2)*SINK                                     9...
001311      105 IF(I.GT.1) GO TO 107                                          ..
001315          IF(I.EQ.1.AND.KK.EQ.1) BX1=CX*R                             ..
001324          IF(I.EQ.1.AND.KK.EQ.2) BY1=CY*R                             ..
001335          GO TO 4                                                     ..
001336      107 IF(KK.EQ.1) BX(I)=CX*R-BX1                                  ..
001344          IF(KK.EQ.2) BY(I)=CY*R-BY1                                  ..
              C NOISE IMPOSED ON RELATIVE PATH DATA.                        ..
001352          IF(RNOISE.EQ.0.0) GO TO 4                                   ..
001353          IF(KK.EQ.1.AND.BX(I).GE.BMAX) BMAX=BX(I)                    ..
001365          IF(KK.EQ.2.AND.BY(I).GE.BMAX) BMAX=BY(I)                    ..
001376          IF(KK.EQ.1.AND.BX(I).LE.BMIN) BMIN=BX(I)                    ..
001407          IF(KK.EQ.2.AND.BY(I).LE.BMIN) BMIN=BY(I)                    ..
001420        4 CONTINUE                                                    4...
001423          BXMBN=BMAX-BMIN                                             .
001424          IF(RNOISE.EQ.0.0) GO TO 3                                   .
001426          DO 10 I=1,NO                                                .
001427          RNSX=RNOISE*BXMBN*(RANF(0.0)-0.5)                           .
001433          IF(KK.EQ.1) BX(I)=BX(I)+RNSX                                .
001444       10 IF(KK.EQ.2) BY(I)=BY(I)+RNSX                                .
001454        3 CONTINUE                                                    3...
001456          BX(1)=0.0                                                   .
001457          BY(1)=0.0                                                   .
001460          IF(KO.EQ.6) GO TO 108                                       .
001462          IF(KO.EQ.2.OR.KO.EQ.4.OR.KO.EQ.5.OR.KO.EQ.12.OR.KO.EQ.14.OR.KO.LQ.
               *15) GO TO 108
001510          WRITE(6,9014) RNSEP
001516     9014 FORMAT(1HO,'THE RANGE OF NOISE IMPOSED ON THE RELATIVE PATHS/MAX P
               *ATH MINUS MIN PATH IS*,F7.3,* PER CENT*)
001522          WRITE(6,9003) NOX
001527     9008 FORMAT(1HO,'THE RELATIVE PATHS OF ALL*,I5,* X RAYS,IN MICRONS,INCL
               *UDING IMPOSED NOISE,IS*)
001533          WRITE(6,9009)(BX(I),I=1,NOX)
001544          WRITE(6,9010) NOY
001555     9010 FORMAT(1HO,'THE RELATIVE PATHS OF ALL*,I5,* Y RAYS,IN MICRONS,INCL
               *UDING IMPOSED NOISE,IS*)
001561          WRITE(6,9009)(BY(J),J=1,NOY)
001572     9009 FORMAT(1H ,1P10E13.4)
001576      108 CALL REF(BX,BY,F1,G1,F,G,FA,SYMBOLS,A,NOX,NOY,MND4,M,N,FX,FY,NS)
001621          RETURN                                                  END OF
001622          END                                                     PATHS2

UNUSED COMPILER SPACE
010600
```

```
                      SUBROUTINE PATHS3(BX,BY,F1,G1,F,FA,SYMBOLS,A,NOX,NOY,MND4,M,N,      SUBR 4
                     *FX,FY,NS)
                C PATHS3 INPUTS EXPERIMENTAL OPTICAL PATH LENGTH DATA.
 000022               COMMON MSMALL,NSMALL,ISMALL,KSMALL,LSMALL,MN,K2,IM1NMK,JKID2,JAJB
 000022               COMMON JNMK,JIM1,JI,JNMKP1,JNMI,JNPK,IM2,JIL2,NMK,JKI,JKP2,JM1,JB
 000022               COMMON MD2,MP1,MD2,MP1,X,Y,IM1,P,RANGE,REFMIN,K0,K1,KMAX,KMAXP1
 000022               COMMON SIGMA,SIGDIF,AV,DDIN,LCOUT,FMAX,FMIN,NSOLN,ND16,PI,RNOISE
 000022               DIMENSION BX(NOX),BY(NOY)
                C 1/P OF ACTUAL VALUE MUST BE READ IN.
 000022               WRITE(6,3020)
 000025                9020 FORMAT(140,*IN ALL DISTRIBUTIONS BELOW Y AXIS IS HORIZONTAL TO THE
                     *  RIGHT,X AXIS IS VERTICALLY DOWN.ORIGIN IS TOP LEFT HAND CORNER*)
 000031               WRITE(6,9001) P
 000036                9001 FORMAT(140,*ALL FIGS BELOW TO BE MULTIPLIED BY P =*,1PE13.5)
 000042               WRITE(6,9031) K1,M,N
 000053                9031 FORMAT(140,*INPUT MODE K1=*,I3,*.PATHS ARE FROM EXPERIMENT*)
 000057               IF(K0.EQ.2.OR.K0.EQ.4.OR.K0.EQ.5.OR.K0.EQ.12.OR.K0.EQ.14.OR.K0.EQ.
                     *15) GO TO 111
 000105               WRITE(6,9002) NOX
 000113                9002 FORMAT(140,*THE RELETIVE PATHS OF ALL*,I5,* X RAYS,IN MICRONS,IS*)
 000117               WRITE(6,9003) (BX(I),I=1,NOX)
 000130               WRITE(6,9004) NOY
 000141                9004 FORMAT(140,*THE RELATIVE PATHS OF ALL*,I5,* Y RAYS,IN MICRONS,IS*)
 000145               WRITE(6,9003) (BY(J),J=1,NOY)
 000156                9003 FORMAT(14 ,1P10E13.4)
 000162                 101 CALL REF(BX,BY,F1,G1,F,G,FA,SYMBOLS,A,NOX,NOY,MND4,M,N,FX,FY,NS)
 000205               RETURN
 000206               END                                                                 END OF
                                                                                          PATHS3

UNUSED COMPILER SPACE
016400
```

```
                      SUBROUTINE REF(BX,BY,F1,G1,F,G,FA,SYMBOLS,A,NOX,NOY,MND4,M,N,FX,FY SUBR  5
                     *,NS)
                C REF USES REL PATHS AS INPUT AND COMPUTES THE A.C. PART OF
                C THE REF INDEX DISTRIBUTION.THE OTHER SUBROUTINES ARE NEEDED FOR THIS.
000023                COMMON MSMALL,NSMALL,ISMALL,KSMALL,LSMALL,MN,K2,IM1NMK,JKID2,JJJJ
000023                COMMON JMK,JIM1,JJ,JNMKP1,JNMI,JNPK,IM2,JIM2,NMK,JKI,JKM2,JKM1,JB
000023                COMMON ND2,NP1,ND2,MP1,X,Y,IM1,P,RANGE,REFMIN,KO,K1,KMAX,KMAXP1
000023                COMMON SIGMA,SIGDIF,AJ,DCIN,DCOUT,FMAX,FMIN,NSOLN,ND16,PI,RNDISE
000023                DIMENSION BX(NOX),BY(NOY),F1(MND4),G1(MND4),F(M,N),FY(M),FX(N)
000023                DIMENSION G(M,N)
000023                RNSEP=RNDISE*100.0
                C     PATH DIFFS ARE NORMALIZED.
000024                SX=SQRT(X*X+Y*Y/FLOAT(M**2))
000034                SY=SQRT(Y*Y+X*X/FLOAT(M**2))
000045                DO 19  I=1,NOX                                                  DO 19
000060                B1=BX(I)                                                          .
000061             19 BX(I)=B1/SX                                                       19
000062                DO 20  I=1,NOY                                                  DO 20
000075                B2=BY(I)                                                          .
000076             20 BY(I)=B2/SY                                                       20
                C     LOGE COEFFICIENTS.
                C       X LINES,
000077                DO 1  I=1,N                                                     DO  1
000112                IM1=I-1                                                           .
000113              1 F1(I)=BX(1+IM1*N)/FLOAT(M)                                        1
000121                JKI=M
000122                JKID2=ND2
000124                IM1NMK=NSMALL
000126                CALL WALSH(F1,G1,MND4)
000134                DO 2 I=1,N                                                      DO  2
000144              2 FX(I)=F1(I)                                                       2
                C       Y LINES,
000146                DO 3 I=1,M                                                      DO  3
000161                IM1=I-1                                                           .
000162              3 F1(I)=BY(1+IM1*M)/FLOAT(N)                                        3
000170                JKI=N
000171                JKID2=ND2
000173                IM1NMK=NSMALL
000175                CALL WALSH(F1,G1,MND4)
000203                DO 4 I=1,M                                                      DO  4
000213              4 FY(I)=F1(I)                                                       4
000215                DO 22  K=1,NSOLN                                                DO 22
000222                IF(NSOLN.EQ.2) K2=KK                                              .
                C     FX,FY ARE EDGE COEFFICIENTS.
000225                IF(K2.EQ.1) F(1,1)=FX(1)                                          .


000241                IF(K2.EQ.2) F(1,1)=FY(1)                                          .
000234                DO 21 I=2,N                                                     DO 21
000245             21 F(1,I)=FX(I)                                                     21
000250                DO 18 I=2,M                                                     DO 18
000263             18 F(I,1)=FY(I)                                                     18
                C     SET REST OF F TO ZERO.
000266                DO 5 J=2,N                                                      DO  5
000273                DO 5 I=2,M                                                      DO  5
000302              5 F(I,J)=0.0
                C FROM HERE ON THE SPECIAL CASE M=N ONLY IS ASSUMED.
                C     SOLN FOR BLOCK ISMALL=KSMALL=1.
                C       WALSH TRANSFORM PART OF SOLN.
000312                JNMK=ND2
000313                JIM1=1
000314                IM1NMK=NSMALL-1
000316                JKI=2**(IM1NMK)
000322                JKID2=JKI/2
000323                L=0
000324                DO 6 IS=1,ND2                                                   DO  6
000344                L=L+1                                                             .
000345                LR=ND2*(2*IS-3)+1                                                 .
000350                IF(K2.EQ.1) F1(L)=-BX(LR)/FLOAT(ND2)                              .
000354              6 IF(K2.EQ.2) F1(L)=-BY(LR)/FLOAT(ND2)                              6
000365                IF(IM1NMK.EQ.0) GO TO 101                                         .
000371                CALL WALSH(F1,G1,MND4)                                            .
000400            101 CALL SCAN(F1,F,MND4,M,N)                                          .
                C       EXTRA TERMS PART OF SOLN.
000404                DO 7 I=1,ND2                                                    DO  7
000431                IF(K2.EQ.1) F1(I)=F(1,I)                                          .
000434              7 IF(K2.EQ.2) F1(I)=F(I,1)                                          7
000441                CALL SCAN(F1,F,MND4,M,N)                                          .
                C     SOLN FOR THE NSMALL-1 BLOCKS ISMALL=1,KSMALL.GE.2
000450                IF(NSMALL.EQ.1) GO TO 102                                         .
                C       WALSH TRANSFORM PART OF SOLN.
000456                DO 8 KSMALL=2,NSMALL                                            DO  8.
000457                IM1NMK=NSMALL-KSMALL                                              .
000460                JNMK=2**IM1NMK                                                    .
000464                JNMKP1=2*JNMK                                                     .
000466                JKM2M1=2**((KSMALL-2)-1                                           .
000473                JKM1=2*(JKM2M1+1)                                                 .
000474                JK=2*JKM1                                                         .
000476                L=0                                                               .
```

```
000500          DO 9 IS=1,JNMK                                                  DO  9..
000517          L=L+1                                                             9..
000520          LR=NOX+((2*IS-1)*JK-1)+1                                          9..
000524          IF(K2.EQ.1) F1(L)=-BX(LR)/FLOAT(JNMK)                             9..
000530        9 IF(K2.EQ.2) F1(L)=-BY(LR)/FLOAT(JNMK)                             9..
000541          IF(IM1NMK.EQ.0) GO TO 103                                         ..
000545          CALL WALSH(F1,G1,MND4)                                            ..
000554      103 CALL SCAN(F1,F,MND4,M,N)                                          ..
            C       EXTRA TERMS PART OF SOLN.                                     ..
000560          DO 11 I=1,JNMK                                                  DO 11..
000570       11 F1(I)=0.0                                                         11..
000572          DO 10 I=1,JNMK                                                  DO 10..
000630          IF(K2.EQ.1) F1(I)=-F(1,N-JNMKP1+I)+F(1,I)+F(2,NP1-I)              10..
000640       10 IF(K2.EQ.2) F1(I)=-F(N-JNMKP1+I,1)+F(I,1)+F(NP1-I,2)              10..
000656          IF(KSMALL.EQ.2) GO TO 8                                           ..
000663          DO 12 I=1,JNMK                                                  DO 12..
000665          DO 12 IH=1,IKM2M1                                               DO 12..
000703          IHE=IH+IH                                                         ...
000704          IHO=IHE-1                                                         ...
000705          IF(K2.EQ.1) F1(I)=-F(1,IHO*JNMKP1+I)-F(2,NP1-I-IHO*JNMKP1)        ...
                *                 +F(1,IHE*JNMKP1+I)+F(2,NP1-I-IHE*JNMKP1)+F1(I)  ...
000740          IF(K2.EQ.2) F1(I)=-F(IHO*JNMKP1+I,1)-F(NP1-I-IHO*JNMKP1,2)        ...
                *                 +F(IHE*JNMKP1+I,1)+F(NP1-I-IHE*JNMKP1,2)+F1(I)  ...
000776       12 CONTINUE                                                          ...
001006        8 CALL SCAN(F1,F,MND4,M,N)                                        12..
            C       SOLN FOR SOLNS ISMALL.GE.2                                    6..
            C       WALSH TRANSFORM PART OF SOLN.                                 ..
001020          DO 13 ISMALL=2,NSMALL                                           DO 13..
001022          IM1=ISMALL-1                                                      ..
001023          IM2=IM1-1                                                         ..
001024          JIM2=2**IM2                                                       ..
001030          JIM1=JIM2+JIM2                                                    ..
001030          JI=JIM1+JIM1                                                      ..
001031          JNMI=2**(NSMALL-ISMALL)                                           ..
001037          DO 13 KSMALL=1,NSMALL                                             ..
001041          NMK=NSMALL-KSMALL                                                 ..
001042          NPK=NSMALL+KSMALL                                                 ..
001043          JNMK=2**NMK                                                       ..
001047          JNPK=2**NPK                                                       ..
001053          JKI=JNMK*JIM1                                                     ..
001055          JKID2=JKI/2                                                       ..
001057          JNMKP1=JNMK+JNMK                                                  ..
001057          JKM2=2**(KSMALL-2)                                                ..
001065          JKM1=JKM2*JKM2                                                    ..
001065          IF(KSMALL.EQ.1) JKM1=1                                            ..
001071          NTKM11=(JKM1-1)*N+1                                               ..
001074          IM1NMK=IM1+NMK                                                    ..
001077          DO 15 I=1,JKI                                                     ..
001103       15 F1(I)=0.0                                                       DO 15..
001105          L=0                                                               15..
001106          DO 14 IS=1,JNMK                                                   ...
001113          ISM1=IS-1                                                       DO 14..
001114          DO 14 IR=1,JIM1                                                 DO 14..
001140          L=L+1                                                             ...
001141          IR2=IR+IR-1                                                       ...
001141          LR=NTKM11+JNPK*ISM1+JNMI*IR2                                      ...
001145          IF(K2.EQ.1) F1(L)=-BX(LR)/FLOAT(JNMK)                             ...
001150       14 IF(K2.EQ.2) F1(L)=-BY(LR)/FLOAT(JNMK)                             ...
001167          CALL WALSH(F1,G1,MND4)                                            14..
            C       EXTRA TERMS PART OF SOLN.                                     ..
001171          CALL SCAN1(F1,G1,F,MND4,M,N)                                      ..
001201          CALL SCAN2(F1,G1,F,MND4,M,N)                                      ..
001211          CALL SCAN(F1,F,MND4,M,N)                                          ..
001225       13 CONTINUE                                                          ..
001231      102 IF(K0.EQ.3.OR.K0.EQ.4.OR.K0.EQ.5.OR.K0.EQ.13.OR.K0.EQ.14.OR.K0.EQ.  ..
                *15) GO TO 114                                                    ..
001260          IF(K0.EQ.6) GO TO 104                                             ..
001262          IF(K2.EQ.1) WRITE(6,9001) NOX,M,M,M                               ..
001304          IF(K2.EQ.2) WRITE(6,9002) NOY,N,N,N                               ..
001326     9001 FORMAT(1H0,'THE COEFFICIENTS GIVEN BY THE PATHS  OF ALL',I6,'  X  ..
                *RAYS  AND THE',I3,'   Y  RAYS  ',I3,'K+1 (0<K<',I3,') ARE')      ..
001326     9002 FORMAT(1H0,'THE COEFFICIENTS GIVEN BY THE PATHS  OF ALL',I6,'  Y  ..
                *RAYS  AND THE',I3,'   X  RAYS  ',I3,'K+1 (0<K<',I3,') ARE')      ..
001326          IF(N.GE.32) GO TO 107                                             ..
001331          DO 23  I=1,1                                                      ..
001332          WRITE(6,9003)(F(I,J),J=1,N)                                     DO 23..
001354     9003 FORMAT(1H ,16E8.4)                                                ...
001360       23 CONTINUE                                                          23..
```

```
001363          GO TO 104
001373      107 DO 28 I1=1,ND16                                          DO 28
001375          I2=I1*16
001376          I3=I2-15
001370          WRITE(6,9011) I3,I2
001377     9011 FORMAT(1H ,*COLUMNS*,I4,* TO*,I4,* ARE*)
001403          DO 28 I=1,M                                              DO 28
001404          WRITE(6,9006)(F(I,J),J=I3,I2)
001431       28 CONTINUE                                                    28
001436      104 F(1,1)=DCOUT
001440          CALL WALSH2(F1,G1,F,MWD4,M,N)
      C F(I,J)  HERE IS RECONSTRUCTED DISTBN WITH MEAN OF DCOUT.
001444          IF(K1.EQ.3) CALL MAXMIN(F,M,N)
001455          IF(K0.EQ.6) GO TO 191
001463          IF(K2.EQ.1) WRITE(6,9004) NGX,M,M,M
001505          IF(K2.EQ.2) WRITE(6,9005) NGY,N,N,N
001527     9004 FORMAT(1H0,*REF INDEX DISTBN GIVEN BY THE PATHS  OF ALL*,I6,*   X
                *RAYS   AND THE*,I3,*   Y  RAYS  *,I3,*K+1 (0≤K<*,I3,*) IS*/)
001527     9005 FORMAT(1H1,*REF INDEX DISTBN GIVEN BY THE PATHS  OF ALL*,I6,*   Y
                *RAYS   AND THE*,I3,*   X  RAYS  *,I3,*K+1 (0≤K<*,I3,*) IS*)
001527          IF(K0.GE.11) CALL PLOT(F,FA,SYMBOLS,A,M,N,NS)
001542          IF(K0.GT.11) GO TO 105
001551          IF(N.GT.32) GO TO 108
001553          DO 24  I=1,M                                             DO 24
001555          WRITE(6,9003)(F(I,J),J=1,N)
001603       24 CONTINUE                                                    24
001606          GO TO 105
001606      108 DO 29 I1=1,ND16                                          DO 29
001610          I2=I1*16
001611          I3=I2-15
001613          WRITE(6,9011) I3,I2
001622          DO 29 I=1,M                                              DO 29
001627          WRITE(6,9003)(F(I,J),J=I3,I2)
001654       29 CONTINUE                                                    29
001661      105 IF(K1.EQ.3) GO TO 22
      C POINT ERRORS CALCULATED.F(I,J) IS POINT ERROR DISTBN.
001663      191 SIGSQ=0.0
001664          DO 25  J=1,N                                             DO 25
001666          DO 25  I=1,M                                             DO 25
001676          FIJ=G(I,J)-F(I,J)+DCOUT
001700          F(I,J)=FIJ
001700       25 SIGSQ=SIGSQ+FIJ*FIJ                                         25
001712          SIGDIF=SQRT(SIGSQ/FLOAT(MN))
001716          IF(SIGMA.GT.0) ERROR=100.0*(SIGDIF/SIGMA)
001726          IF(SIGMA.EQ.0) ERROR=0.0
001730          BV=0.0
001731          DO 26  J=1,N                                             DO 26
001733          DO 26  I=1,M                                             DO 26
001746          FIJ=F(I,J)/RANGE
001747          F(I,J)=FIJ
001750       26 BV=BV+ABS(F(I,J))                                           26
001761          EAV=100.0*(BV/FLOAT(MN))
001763          IF(K0.EQ.5.OR.K0.EQ.15.OR.K0.EQ.6) GO TO 106
001776          WRITE(6,9007)
002002     9007 FORMAT(1H0,*ACTUAL MINUS DETERMINED    REF INDEX DISTRIBUTION OVER
               * RANGE, ABOUT ZERO MEAN FOR SCTH,OR DISTRIBUTION OF POINT ERRORS IS
               **)
002006          IF(N.GE.32) GO TO 109
002010          DO 27  I=1,M                                             DO 27
002012          WRITE(6,9003)(F(I,J),J=1,N)
002040       27 CONTINUE                                                    27
002043          GO TO 106
002043      109 DO 30 I1=1,ND16                                          DO 30
002045          I2=I1*16
002046          I3=I2-15
002050          WRITE(6,9011) I3,I2
002057          DO 30 I=1,M                                              DO 30
002064          WRITE(6,9003)(F(I,J),J=I3,I2)
002111       30 CONTINUE                                                    30
002116      106 CALL MAXMIN(F,M,N)
002121          FMX=FMAX*100.
002123          FMN=FMIN*100.
002124          FMAX=FMX
002125          FMIN=FMN
002127          WRITE(6,9005) ANSEP
002140     9005 FORMAT(1H0,*THE RANGE OF NOISE IMPOSED/MAX PATH MINUS MIN PATH IS*
               *,29X,*  =NOISE=*,F8.3,* PER CENT*)
002144          WRITE(6,9009) SIGDIF,DCOUT,ERROR,EAV,FMAX,FMIN
002163     9009 FORMAT(1H0,*THE STANDARD DEVIATION OF ACTUAL MINUS RECONSTRUCTED R
               *EF INDEX DISTRIBUTION IS*,9X,*SIGDIF=*,  E14.5/1H ,*THE MEAN VALUE
               * IMPOSED ON THE REF INDEX DISTRIBUTION ABOVE IS*,27X,*DCOUT=*,  E1
               *4.5/1H0,*OVERALL    ERROR IS*,62X,*SIGDIF/SIGMA=*,F8.3,* PER CENT*
               */1H ,*AVERAGE ERROR,SUM(ABS(POINT ERRORS))/MN,IS*,48X,*EAV=*,F8.3,
               ** PER CENT*/1H ,*THE MAXIMUM POINT ERRORS ARE*,61X,*FMAX=*,F8.3,*
               *PER CENT*/1H ,  89X,*FMIN=*,F8.3,* PER CENT*)
002167          WRITE(6,9010)
002172     9010 FORMAT(1H0,'-+-+---8+-----16+----------32+----------
               *----------64+----------------------------------------------------
               *----128+---')
002176       22 CONTINUE                                                    22
002201          DO 16 I=1,NGX                                            DO 16
002205       16 EX(I)=0.0                                                   16
002207          DO 17 I=1,NGY                                            DO 17
002217       17 EY(I)=0.0                                                   17
002221          RETURN                                                   END OF
002221          END                                                      REF

UNUSED COMPILER SPACE
007600
```

```
                SUBROUTINE WALSH2(F1,G1,F,MND4,M,N)                                    SUBR  6
          C     EVALUATION OF EDGE COEFFICIENTS.
000010          COMMON NSMALL,MSMALL,ISMALL,KSMALL,LSMALL,MN,K2,IM1NMK,JKID2,JAJB
000010          COMMON JNMK,JIM1,JI,JNMKP1,JNMI,JNPK,IM2,JIK2,NMK,JKI,JKM2,JKM1,JB
000010          COMMON ND2,NP1,MD2,MP1,X,Y,IM1,P,RANGE,REFMIN,K0,K1,KMAX,KMAXP1
000010          COMMON SIGM1,STGDIF,AV,DCIN,DCOUT,FMAX,FMIN,NSOLN,ND16,PI,RNOISE
000010          DIMENSION F1(MND4),G1(MND4),F(M,N)
000010          DO 1 J=1,N                                                             DO   1..
000011          DO 2 I=1,M                                                             DO   2.
000020        2 F1(I)=F(I,J)                                                              2.
000026          JKI=M                                                                     .
000026          JKID2=ND2                                                                 .
000030          IM1NMK=MSMALL                                                             .
000032          CALL WALSH(F1,G1,MND4)                                                    .
000033          DO 1 I=1,M                                                             DO   1.
000046        1 F(I,J)=F1(I)                                                              1.
000056          DO 3 I=1,M                                                             DO   3.
000057          DO 4 J=1,N                                                             DO   4.
000067        4 F1(J)=F(I,J)                                                              4.
000075          JKI=N                                                                     .
000075          JKID2=ND2                                                                 .
000077          IM1NMK=NSMALL                                                             .
000101          CALL WALSH(F1,G1,MND4)                                                    .
000102          DO 3 J=1,N                                                             DO   3.
000116        3 F(I,J)=F1(J)                                                              3.
000126          RETURN                                                                 END OF
000126          END                                                                    WALSH2

UNUSED COMPILER SPACE
016700


                SUBROUTINE WALSH(F1,G1,MND4)                                           SUBR  7
          C     JKI,JKID2 AND IM1NMK MUST BE SPECIFIED BEFORE CALLING WALSH.
000005          COMMON MSMALL,NSMALL,ISMALL,KSMALL,LSMALL,MN,K2,IM1NMK,JKID2,JAJB
000005          COMMON JNMK,JIM1,JI,JNMKP1,JNMI,JNPK,IM2,JIK2,NMK,JKI,JKM2,JKM1,JB
000005          COMMON ND2,NP1,MD2,MP1,X,Y,IM1,P,RANGE,REFMIN,K0,K1,KMAX,KMAXP1
000005          COMMON SIGM1,STGDIF,AV,DCIN,DCOUT,FMAX,FMIN,NSOLN,ND16,PI,RNOISE
000005          DIMENSION F1(MND4),G1(MND4)
000005          IF(IM1NMK.EQ.0) RETURN
000006          DO 1 K=1,IM1NMK                                                        DO   1.
000010          KM1=K-1                                                                   .
000011          IIKM1=2**KM1                                                              .
000015          DO 2 I=1,JKID2                                                         DO   2.
000030          IYM1=I-1                                                                  .
000031          J=MOD(IYM1,IIKM1)                                                         .
000036          RJD2=FLOAT(J)/2.                                                          .
000040          JD2I=J/2                                                                  .
000041          JD2R=MOD(J,2)                                                             .
000043          JD4R=MOD(JD2I,2)                                                          .
000045          IE=I+I                                                                    .
000047          IO=IE-1                                                                   .
000050          IOMJ=IO-J                                                                 .
000051          IOMJM=IOMJ+IIKM1                                                          .
000052          IF(JD2R.EQ.0.AND.JD4R.EQ.0)G1(IO)=+F1(IOMJ)+F1(IOMJM)                     .
000063          IF(JD2R.EQ.1.AND.JD4R.EQ.1)G1(IO)=-F1(IOMJ)-F1(IOMJM)                     .
000075          IF(JD2R.EQ.1.AND.JD4R.EQ.0)G1(IO)=+F1(IOMJ)-F1(IOMJM)                     .
000106          IF(JD2R.EQ.1.AND.JD4R.EQ.1)G1(IO)=-F1(IOMJ)+F1(IOMJM)                     .
000117          IF(JD2R.EQ.0.AND.JD4R.EQ.0)G1(IE)=-F1(IOMJ)+F1(IOMJM)                     .
000130          IF(JD2R.EQ.0.AND.JD4R.EQ.1)G1(IE)=+F1(IOMJ)-F1(IOMJM)                     .
000142          IF(JD2R.EQ.1.AND.JD4R.EQ.0)G1(IE)=-F1(IOMJ)-F1(IOMJM)                     .
000153          IF(JD2R.EQ.1.AND.JD4R.EQ.1)G1(IE)=+F1(IOMJ)+F1(IOMJM)                     .
000164          IF(K.NE.IM1NMK) GO TO 2                                                   .
000166          JJ=MOD(IYM1,2)                                                            .
000170          IF(JJ.EQ.0) G1(IE)=-G1(IE)                                                .
000172          IF(JJ.EQ.1) G1(IO)=-G1(IO)                                                .
000175        2 CONTINUE                                                                  2.
000203          DO 1 I=1,JKI                                                           DO   1.
000211        1 F1(I)=G1(I)                                                               1.
000215          RETURN                                                                 END OF
000216          END                                                                    WALSH

UNUSED COMPILER SPACE
016400
```

```
                      SUBROUTINE SCAN(F1,F,MND4,M,N)                                    SUBR 8
              C       JNMK AND JIM1          MUST BE SPECIFIED BEFORE CALLING SCAN.
000007                COMMON MSMALL,NSMALL,ISMALL,KSMALL,LSMALL,MN,K2,IM1NMK,JKID2,JAJB
000007                COMMON JNMK,JIM1,JI,JNMKP1,JNMI,JNPK,IM2,JIM2,NMK,JKI,JKN2,JKN1,JB
000007                COMMON ND2,NP1,MD2,MP1,X,Y,IM1,P,RANGE,REFMIN,K0,K1,KMAX,KMAXP1
000007                COMMON SIGMA,SIGDIF,AV,DCIN,DCOUT,FMAX,FMIN,NSOLN,ND16,PI,RNOISE
000007                DIMENSION F1(MND4),F(M,N)
              C       JIM1 MUST BE EVEN EXCEPT FOR CASE JIM1=1
000007                JI=2*JIM1
000010                JNMKP1=2*JNMK
000011                IF(JIM1.EQ.1) GO TO 101
000013                JBD2=JIM1/2
000014                DO 1 IB=1,JBD2                                                    DO  1
000016                IBE=IB+IB                                                           .
000017                IBO=IBE-1                                                           .
000020                JCMBO1=JI-IBO+1                                                     .
000021                JCMBE1=JI-IBE+1                                                     .
000023                JABO=JNMK*IBO                                                       .
000025                JABOM1=JNMK*(IBO-1)                                                 .
000030                DO 1 IA=1,JNMK                                                      .
000055                IAM1=IA-1                                                           .
000056                IF(K2.EQ.1) F(JCMBO1,JNMKP1-IAM1)=F(JCMBO1,JNMKP1-IAM1)             .
                     *+F1(JABOM1+IA)                                                    .
000066                IF(K2.EQ.1) F(JCMBE1,JNMK+IA)=F(JCMBE1,JNMK+IA)+F1(JABO+IA)         .
000072                IF(K2.EQ.2) F(JNMKP1-IAM1,JCMBO1)=F(JNMKP1-IAM1,JCMBO1)             .
                     *+F1(JABOM1+IA)                                                    .
000162              1 IF(K2.EQ.2) F(JNMK+IA,JCMBE1)=F(JNMK+IA,JCMBE1)+F1(JABO+IA)       1
000120                RETURN
000120            101 DO 2 IA=1,JNMK                                                    DO  2
000134                IAM1=IA-1                                                           .
000135                IF(K2.EQ.1) F(JI,JNMKP1-IAM1)=F1(IA)+F(JI,JNMKP1-IAM1)              .
000145              2 IF(K2.EQ.2) F(JNMKP1-IAM1,JI)=F1(IA)+F(JNMKP1-IAM1,JI)            2
000156                RETURN                                                          END OF
000157                END                                                             SCAN

      UNUSED COMPILER SPACE
      016600


                      SUBROUTINE SCAN1(F1,G1,F,MND4,M,N)                                SUBR 9
              C       SCAN1 CALCULATES 1ST COL MATRIX OF SUM OF ISMALL COL MATRICES.
000010                COMMON MSMALL,NSMALL,ISMALL,KSMALL,LSMALL,MN,K2,IM1NMK,JKID2,JAJB
000010                COMMON JNMK,JIM1,JI,JNMKP1,JNMI,JNPK,IM2,JIM2,NMK,JKI,JKN2,JKN1,JB
000010                COMMON ND2,NP1,MD2,MP1,X,Y,IM1,P,RANGE,REFMIN,K0,K1,KMAX,KMAXP1
000010                COMMON SIGMA,SIGDIF,AV,DCIN,DCOUT,FMAX,FMIN,NSOLN,ND16,PI,RNOISE
000010                DIMENSION F1(MND4),G1(MND4),F(M,N)
000010                LSMALL=1
              C       1ST JNMKP1 ELEMENTS OF THE JKI=JNMK*JIM1 TOTAL IN THIS 1ST COL MATRIX.
000011                DO 1 I=1,JNMKP1                                                   DO  1.
000012                IF(KSMALL.EQ.1) GO TO 101                                           .
000014                DO 2 IH=1,JKM2                                                    DO  2.
000033                IHE=IH+IH-2                                                         ..
000035                LE=IHE*JNMKP1                                                       ..
000036                LO=LE+JNMKPL                                                        ..
000037                IF(K2.EQ.1) F1(I)=F1(I)+FLOAT(JIM1)*(F(1,I+LE)-F(1,I+LO))           ..
000053              2 IF(K2.EQ.2) F1(I)=F1(I)+FLOAT(JIM1)*(F(I+LE,1)-F(I+LO,1))         2.
000073                GO TO 1                                                             .
000077            101 IF(K2.EQ.1) F1(I)=F1(I)+FLOAT(JIM1)*F(1,I)                        .
000107                IF(K2.EQ.2) F1(I)=F1(I)+FLOAT(JIM1)*F(I,1)                        .
000120              1 CONTINUE                                                          1.
000123                IF(ISMALL.EQ.2) RETURN
              C       THE OTHER ISL SETS OF TERMS IN THE 1ST SUB COL MATRIX.
000125                JB=1
000126                DO 4 I=1,IM2                                                      DO  4
000130                JB=JB+JB                                                            .
000131                JAJB=JNMK*JB                                                        .
000132                IP=JNMK*(2**I)                                                      .
000140                CALL SCAN3(G1,F,MND4,M,N)                                           .
000142                DO 4 J=1,IP                                                       DO  4
000156              4 F1(IP+J)=F1(IP+J)-FLOAT(JIM1)*G1(J)                               4.
000167                RETURN                                                          END OF
000170                END                                                             SCAN1

      UNUSED COMPILER SPACE
      016500


                      SUBROUTINE SCAN2(F1,G1,F,MND4,M,N)                                SUBR 10
000010                COMMON MSMALL,NSMALL,ISMALL,KSMALL,LSMALL,MN,K2,IM1NMK,JKID2,JAJB
000010                COMMON JNMK,JIM1,JI,JNMKP1,JNMI,JNPK,IM2,JIM2,NMK,JKI,JKN2,JKN1,JB
000010                COMMON ND2,NP1,MD2,MP1,X,Y,IM1,P,RANGE,REFMIN,K0,K1,KMAX,KMAXP1
000010                COMMON SIGMA,SIGDIF,AV,DCIN,DCOUT,FMAX,FMIN,NSOLN,ND16,PI,RNOISE
000010                DIMENSION F1(MND4),G1(MND4),F(M,N)
              C       1ST SCAN OF JNMK*JIM1 TERMS.
000010                DO 1 LSMALL=2,ISMALL                                              DO  1.
000011                LM2=LSMALL-2                                                        .
000012                JB=2**(LSMALL-1)                                                    .
000017                JAJB=JNMK*JB                                                        .
000021                CALL SCAN3(G1,F,MND4,M,N)                                           .
000024                IML=ISMALL-LSMALL                                                   .
000026                JIML=2**IML                                                         .
000032                DO 2 I=1,JAJB                                                     DO  2.
000044              2 F1(I)=F1(I)+FLOAT(JIML)*G1(I)                                     2.
000047                IF(IML.EQ.0) GO TO 1                                                .
              C       IML OTHER IML SETS OF TERMS IN THE LSMALLTH COL MATRIX.
000053                JB=JB/2                                                             .
000055                DO 3 I=1,IML                                                      DO  3.
000056                JB=JB+JB                                                            ..
000057                JAJB=JNMK*JB                                                        ..
000060                IPLM2=I+LM2                                                         ..
000062                IP=JNMK*(2**IPLM2)                                                  ..
000076                CALL SCAN3(G1,F,MND4,M,N)                                           ..
000072                DO 3 J=1,IP                                                       DO  3.
000106              3 F1(IP+J)=F1(IP+J)+FLOAT(JIML)*G1(J)                               3.
000117              1 CONTINUE                                                          1.
000122                RETURN                                                          END OF
000122                END                                                             SCAN2

      UNUSED COMPILER SPACE
      016700
```

```
                      SUBROUTINE SCAN3(G1,F,MND4,M,N)                                    SUBR 1
C                     JNMK,JNMKP1,JI,JB,JKM2,JAJB (SMALL TO BE SPECIFIED BEFORE CALLING SCAN3.
000007                COMMON MSMALL,NSMALL,ISMALL,KSMALL,LSMALL,MN,K2,IM1NMK,JKID2,JAJB
000007                COMMON JNMK,JIM1,JI,JNMKP1,JNMI,JNPK,IM2,JIM2,NMK,JKI,JKM2,JKM1,JB
000007                COMMON ND2,NP1,MD2,MP1,X,Y,IM1,P,RANGE,REFNIN,KD,K1,KMAX,KMAXP1
000007                COMMON SIGN1,SIGDIF,A7,DCIN,DCOUT,FMAX,FMIN,NSOLN,ND16,PI,RNOISE
000007                DIMENSION G1(MND4),F(M,N)
000007                JBD2=JB/2
000010                DO 1 I=1,JAJB                                                      DO  1
000015              1 G1(I)=0.0                                                            1
000017                DO 2 IB=1,JBD2                                                     DO  2.
000023                IBE=IB+IB                                                             .
000024                JBMB01=JB-IBE+2                                                       .
000026                JBMBE1=JBMB01-1                                                       .
000027                JO=JBMB01+JIM1                                                        .
000031                JE=JBMBE1+JIM1                                                        .
000032                JABO=JNMK*(IBE-1)                                                     .
000035                JABOM1=JABO-JNMK                                                      .
000036                DO 2 IA=1,JNMK                                                     DO  2.
000040                L1=JABOM1+IA                                                          .
000042                L2=JABO+IA                                                            .
000043                L3=1-IA                                                               .
000044                L4=IA-JNMK                                                            .
000046                L5=L3-JNMKP1                                                          .
000047                L6=L4-JNMKP1                                                          .
000050                L7=L3+JNMKP1                                                          .
000051                L8=L4+JNMKP1                                                          .
000052                L9=L7+JNMKP1                                                          .
000053                L10=L8+JNMKP1                                                         .
000054                IF(KSMALL.EQ.1) GO TO 101                                             .
000057                DO 3 IH=1,JKM2                                                     DO  3.
000075                IHE=IH+IH                                                             ..
000076                LE=IHE*JNMKP1                                                         ..
000100                IF(K2.EQ.1)G1(L1)=G1(L1)+F(JBMB01,L3+LE)-F(JBMB01,L5+LE)              ..
000115                IF(K2.EQ.1)G1(L2)=G1(L2)+F(JBMBE1,L4+LE)-F(JBMBE1,L6+LE)              ..
000132                IF(K2.EQ.2)G1(L1)=G1(L1)+F(L3+LE,JBMB01)-F(L5+LE,JBMB01)              ..
000150              3 IF(K2.EQ.2)G1(L2)=G1(L2)+F(L4+LE,JBMBE1)-F(L6+LE,JBMBE1)              3.
000167                IF(LSMALL.EQ.ISMALL) GO TO 102                                        .
000174                GO TO 2                                                               .
000174            101 IF(K2.EQ.1)G1(L1)=G1(L1)-F(JBMB01,L7)                                 .
000204                IF(K2.EQ.1)G1(L2)=G1(L2)-F(JBMBE1,L8)                                 .
000214                IF(K2.EQ.2)G1(L1)=G1(L1)-F(L7,JBMB01)                                 .
000224                IF(K2.EQ.2)G1(L2)=G1(L2)-F(L8,JBMBE1)                                 .
000234                GO TO 2                                                               .
000235            102 IF(KSMALL.EQ.2) GO TO 103                                             .
000237                DO 4 IH=2,JKM2                                                     DO  4.
000256                IHE=IH+IH                                                             ..
000257                LE=IHE*JNMKP1                                                         ..
000261                IF(K2.EQ.1)G1(L1)=G1(L1)+F(JO,L3+LE)-F(JO,L5+LE)                      ..
000276                IF(K2.EQ.1)G1(L2)=G1(L2)+F(JE,L4+LE)-F(JE,L6+LE)                      ..
000313                IF(K2.EQ.2)G1(L1)=G1(L1)+F(L3+LE,JO)-F(L5+LE,JO)                      ..
000331              4 IF(K2.EQ.2)G1(L2)=G1(L2)+F(L4+LE,JE)-F(L6+LE,JE)                      4.
000353            103 IF(K2.EQ.1)G1(L1)=G1(L1)+F(JO,L9)                                     .
000363                IF(K2.EQ.1)G1(L2)=G1(L2)+F(JE,L10)                                    .
000373                IF(K2.EQ.2)G1(L1)=G1(L1)+F(L9,JO)                                     .
000403                IF(K2.EQ.2)G1(L2)=G1(L2)+F(L10,JE)                                    .
000413              2 CONTINUE                                                              2.
000420                RETURN                                                             END OF
000421                END                                                                SCAN3

UNUSED COMPILER SPACE
015600
```

```
                      SUBROUTINE MAXMIN(F,M,N)                                          SUBR 12
000005                COMMON MSMALL,NSMALL,ISMALL,KSMALL,LSMALL,MN,K2,IM1NMK,JKID2,JAJB
000005                COMMON JNMK,JIM1,JI,JNMKP1,JNM1,JNPK,IM2,JIM2,NMK,JKI,JKM2,JKM1,JJ
000005                COMMON MD2,MP1,MD2,MP1,X,Y,IM1,P,RANGE,REFMIN,KJ,K1,KMAX,KMAXP1
000005                COMMON SIGMA,SIGDIF,AV,DCIN,DCOUT,FMAX,FMIN,NSOLN,NU16,PI,RNOISE
000005                DIMENSION F(1,M)
000005                FMAX=F(1,1)
000005                FMIN=F(1,1)
000006                DO 1 J=1,N                                                        DO  1
000010                DO 1 I=1,M                                                        DO  1
000017                IF(F(I,J).GE.FMAX) FMAX=F(I,J)                                        *
000022              1 IF(F(I,J).LE.FMIN) FMIN=F(I,J)                                        1
000033                IF(K1.EQ.3) REFMIN=FMIN
000036                IF(K1.EQ.3) RANGE=FMAX-FMIN
000041                RETURN                                                            END OF
000042                END                                                               MAXMIN

UNUSED COMPILER SPACE
017200


                      SUBROUTINE PLOT(F,FA,SYMBOLS,A,M,N,NS)                            SUBR 13
                C FOR K2=2 PLOT COVERS RANGE + AND - RANGE/2.USE TWICE THE NUMBER OF
                C SYMBOLS IN THIS CASE.
000011                COMMON MSMALL,NSMALL,ISMALL,KSMALL,LSMALL,MN,K2,IM1NMK,JKID2,JAJB
000011                COMMON JNMK,JIM1,JI,JNMKP1,JNM1,JNPK,IM2,JIM2,NMK,JKI,JKM2,JKM1,JJ
000011                COMMON MD2,MP1,MD2,MP1,X,Y,IM1,P,RANGE,REFMIN,KJ,K1,KMAX,KMAXP1
000011                COMMON SIGMA,SIGDIF,AV,DCIN,DCOUT,FMAX,FMIN,NSOLN,NU16,PI,RNOISE
000011                DIMENSION F(1,N),FA(NS),SYMBOLS(NS),A(N)
000011                DATA UNDER,OVER/1H-,1H+/
000011                REFMAX=REFMIN+RANGE
000012                HRANGE=RANGE/2.
000014                REFMIC=REFMIN-HRANGE
000015                REFMAC=REFMAX+HRANGE
000016                GAP=RANGE/FLOAT(NS)
000021                FAI=REFMIN
000023                GAP1=GAP+GAP
000024                IF(K1.EQ.2) GAP=GAP1
000030                IF(K1.EQ.2) FAI=REFMIC
000033                DO 1 I=1,NS                                                       DO  1
000041                FAI=FAI+GAP                                                           *
000042              1 FA(I)=FAI                                                             1
000043                IF(K1.NE.2) WRITE(6,9001) NS,REFMIN,REFMAX
000066                IF(K1.EQ.2) WRITE(6,9001) NS,REFMIC,REFMAC
000106     9001 FORMAT(1H0,' KEY TO THE ',I4,' SYMBOLS IS',/VALUE<SYMBOL<RVALUE, IF
               *VALUE<',E12.4,' SYMBOL IS -',/IF VALUE >',E12.4,' SYMBOL IS +')
000106                IF(K1.NE.2) REFMIC=REFMIN
000112                IF(NS.LT.10) WRITE(6,9002) REFMIC,(SYMBOLS(I),FA(I),I=1,NS)
000150                IF(NS.GE.10) WRITE(6,9002) REFMIC,(SYMBOLS(I),FA(I),I=1,9)
000205                IF(NS.GE.10) WRITE(6,9003) (SYMBOLS(I),FA(I),I=10,NS)
000242     9002 FORMAT(1H ,2X,F7.3, 9(X,A1,F7.3))
000242     9003 FORMAT((1H ,10(X,A1,F7.3)))
000242                WRITE(6,9004)
000246     9004 FORMAT(1H0)
000252                DO 2 I=1,M                                                        DO  2
000253                DO 3 J=1,N                                                        DO  3
000271                Q=F(I,J)-REFMIN                                                       *
000273                IF(K1.EQ.2) Q=Q+HRANGE                                                *
000277                K=Q/GAP+1.0                                                           *
000302                IF(K.LE.0) A(J)=UNDER                                                 *
000306                IF(K.GT.NS) A(J)=OVER                                                 *
000312              3 IF(K.GE.1.AND.K.LE.NS) A(J)=SYMBOLS(K)                                3
000332                WRITE(6,9005) A                                                       *
000347     9005 FORMAT(1H ,120A1)                                                           *
000353              2 CONTINUE                                                            2
000355                RETURN                                                            END OF
000355                END                                                               PLOT

UNUSED COMPILER SPACE
016000
```

References.

(1) Walsh,J.H.(1923),Amer.Jour. of Math.,Vol.55.pp5-24."A closed Set Of Orthogonal Functions".

(2) Harmuth,H.F.(Feb1972),Springer-Verlag,2nd Edit."Transmission Of Information By Orthogonal Functions".

(3) Siddiqui,I.M.(1974),Ph.D.,thesis,Univ. of London,Imperial College, "Design and construction of a large aperture holographic interferometer for studying the inhomogeneity of optical glass".

(4) Cooley,J.W., and Tukey,J.W.,Math.Computation(April1965),Vol.19, pp.297-301,"An algorithm for machine calculation of complex Fourier series".

(5) DeRome,M.F.A.(1972),M.Sc.,report,Univ of London,Imperial Colege, "Non-Sinusoidal orthogonal Transforms for Picture Processing".

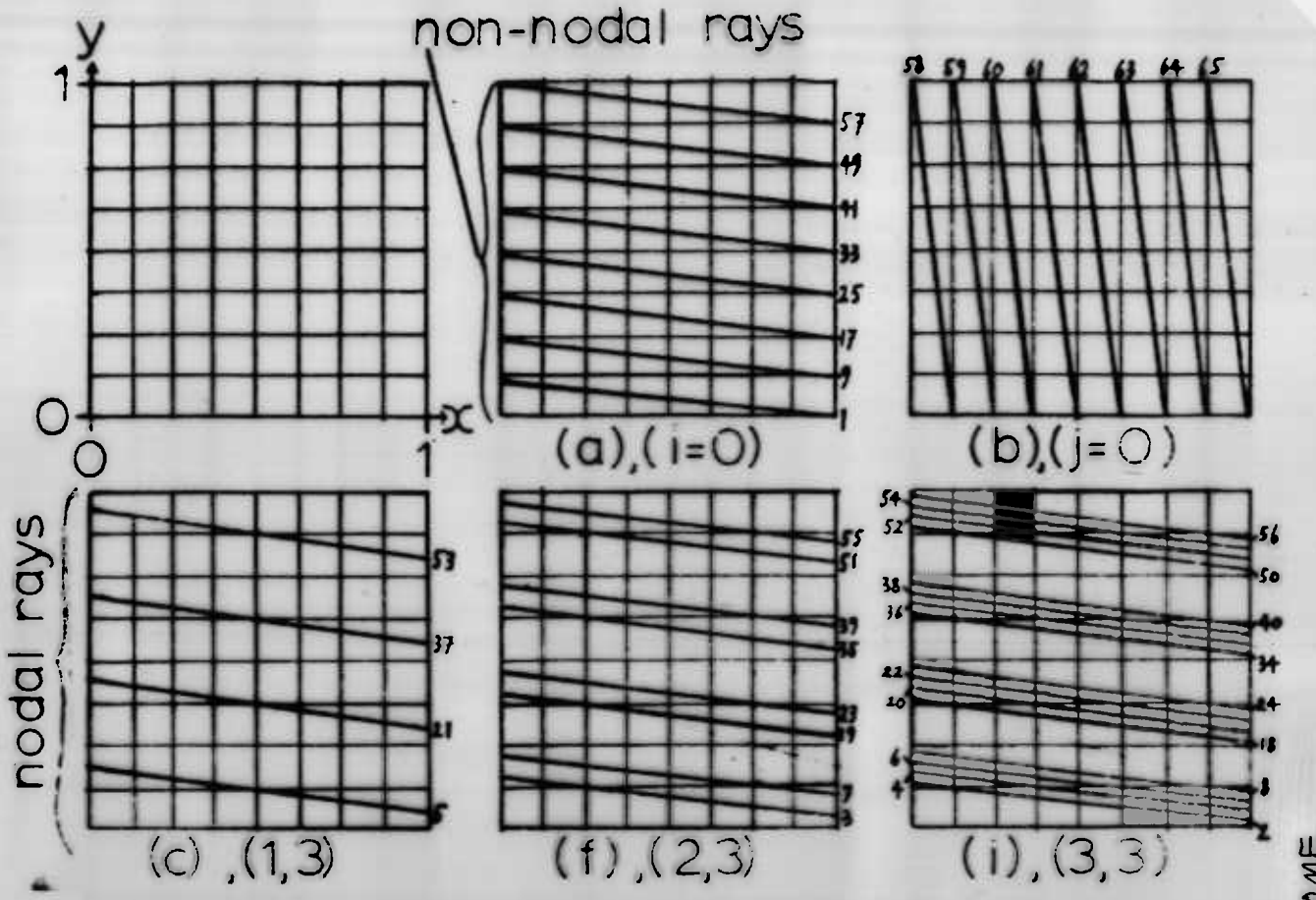
(6) Sweeney,D.W. and Vest,M.(Nov1973),Applied Optics,Vol.12.No.11. pp2649-2664."Reconstruction of Three-Dimensional Refractive Index Fields from Multidirectional Interferometric Data".

Acknowledgements.

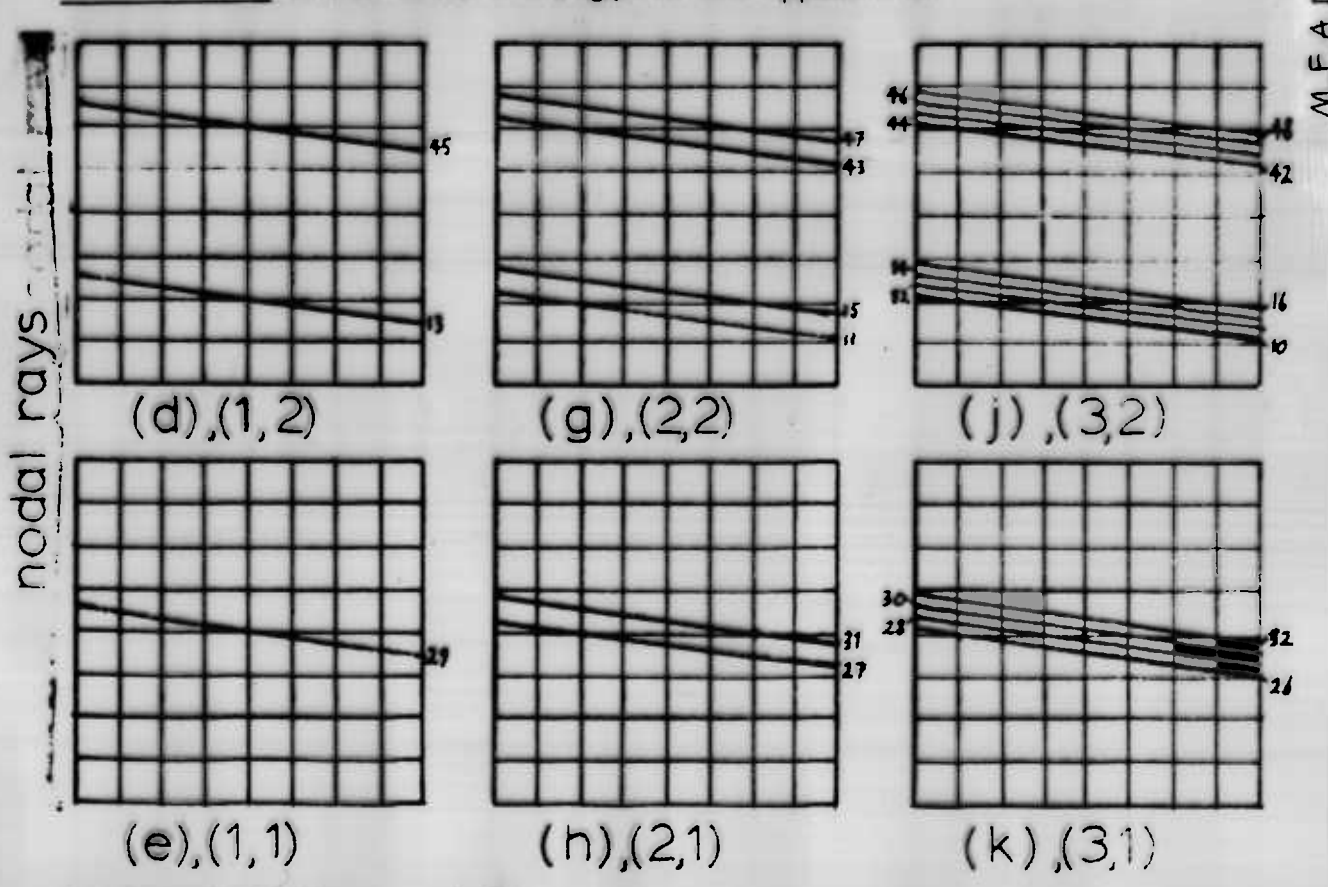
The author is very grateful to Dr. M. E. Barnett for suggesting the subject of his M.Sc. report which prepared him so well in tackling the subject of this thesis that he was able to complete the latter within two years.The author thanks Dr. R. W. Smith whose initial suggestion that Walsh functions might replace complex exponentials in the glass block problem led to the choice of subject of this thesis. Dr. Smith provided much assistance during the work and in planning the form of the thesis when the work had been completed.

The author is indebted to Jane DeRome,his sister, and to Christa Gausden for their patience in typing the thesis between them.

Finally,The Science Research Council are thanked for the financial assistance given over the past two years.

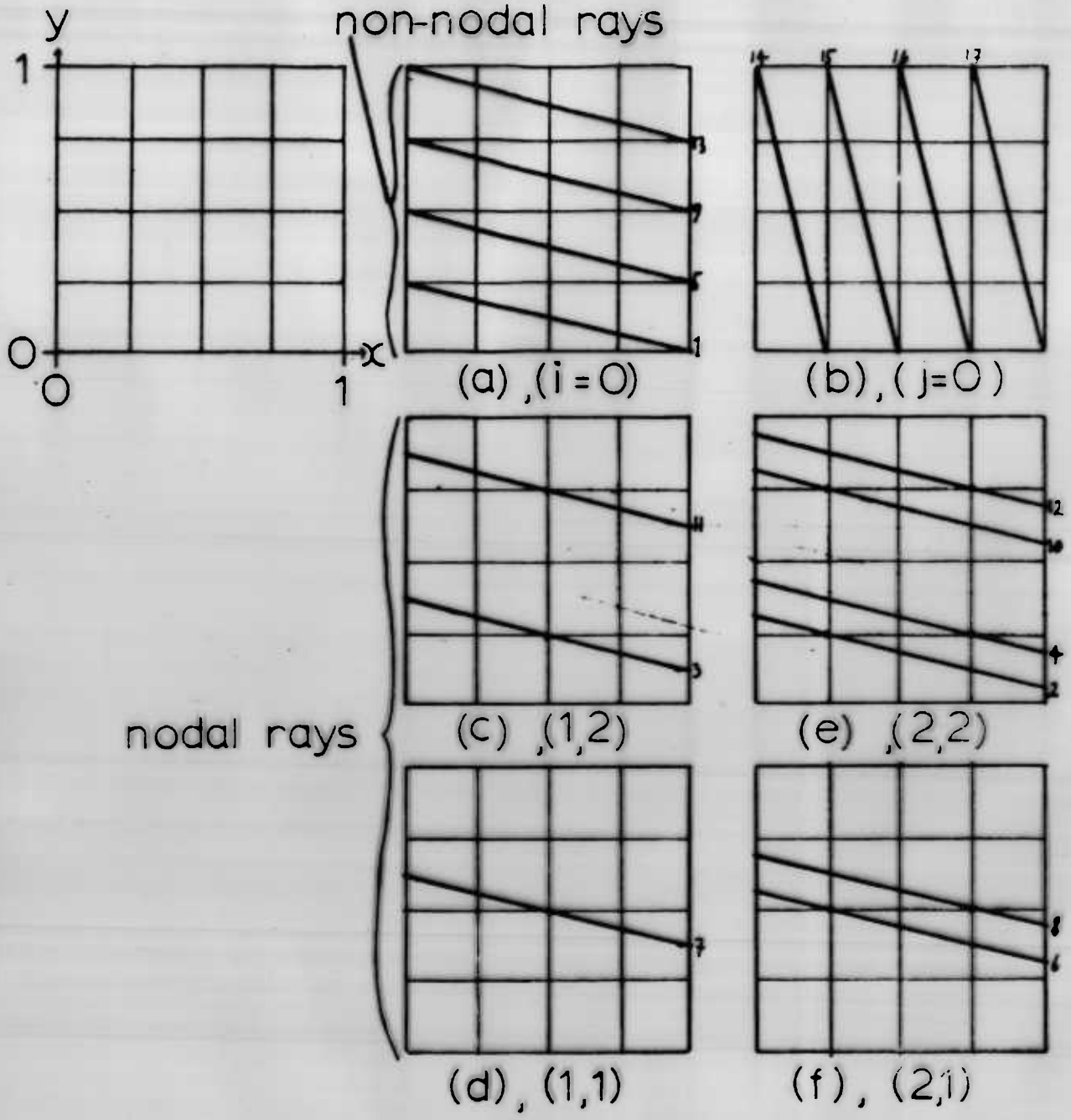


Insert(2) (first half) see Fig(3.7) and Appendix 3.



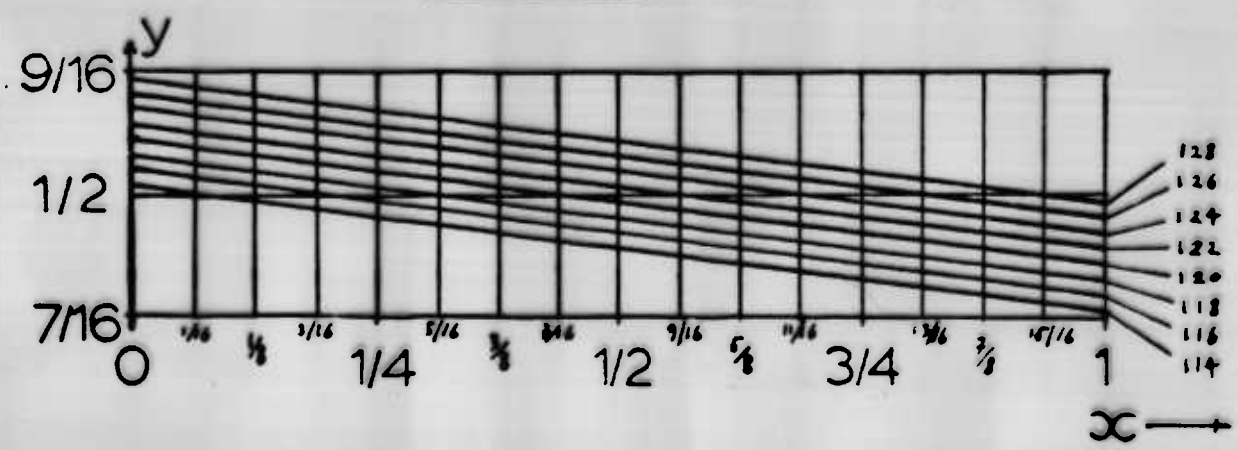
Insert(2) (second half) see Fig(3.7) and Appendix 3.

M.F.A.DEROME.



M.F.A.DEROME.

Insert(1) see Fig(3.6) and Appendix 3.



Insert(3) see Fig.(3.14) and Appendix 4.