Scale invariance, Gauge theory and Renormalisation

by

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Abstracts

Part 1 Dynamical symmetry breaking of massless Yang-Mills theory

We investigate dynamical symmetry breaking D.S.B. of massless Yang-Mills theory in the context of SU(2). Within the approximation scheme we are using, we possibly find dimensional transmutation. More precisely, this massless theory with only one coupling constant g acquires a massive spectrum spontaneously provided the eigenequation for g^2 has a positive solution. The generated mass decouples from the theory and has become a dimensional parameter, i.e. taking place of the previous dimensionless parameter g which is now subject to constraint.

Part 2 Phenomenological applications and Renormalisation of scaling theory

We explore the scaling behaviours of inclusive and exclusive processes alike using two approaches: (1) Quark model fits in well with phenomenology and the idea of anomalous dimension which, besides its phenomenological significance, is more linked with renormalisation. (2) Operator product expansion gives generalised scaling rules

on a model-free basis. We also add a few remarks on the scaling rules of renormalised theories.

Part 3 A joint paper on Conformal invariance and Helicity conservation

By reformulating conformal invariance in terms of differential operators acting directly on helicity states we are able to establish the restrictions placed by this invariance on the helicity amplitudes for the scattering of four particles of arbitrary spins. The result is helicity conservation in the form $\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4$ except for exceptional amplitudes $T_{\lambda,\lambda;-\lambda,-\lambda}$, which survive, subject, however, to a differential constraint. It is conjectured that traces of these restrictions will survive in hadron physics at fixed angle and high energy if indeed the underlying dynamics is asymptotically free.

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Preface

The work described in this thesis was carried out under the supervision of Prof. T.W.B. Kibble between October 1972 and June 1975 in the Department of Physics, Imperial College, University of London.

Except where otherwise stated, the material contained herein is original and has not been previously presented for a degree in this or any other University.

The first part on dynamical symmetry breaking was carried out in the expert guidance of Prof. T.W.B. Kibble. The second part on the application of scale invariance and renormalisation was a continued investigation of this symmetry after my first study year with Dr. H.F. Jones, whose kind permission to include here a joint paper co-authored with him is greatly appreciated.

I wish to express my sincere appreciation to the staffs of the theoretical group, for their warmth, friendliness and their contributions to the facilities and the valuable atmosphere here in Imperial College which I have enjoyed greatly.

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I wish to express my gratitude to Prof. T.W.B. Kibble, for his great kindness, many wise advices and his critical reading of the thesis.

I am very grateful to my mother and Lai, for their understanding, encouragements and substantial assistances without which I could not have completed my study here.

F.Y.C.

Overall Introduction.

It is well known that symmetry principles, whether they are of the kinds corresponding to space-time transformations or they are of a more intrinsic nature connecting seemingly unrelated aspects, always provide us, in as much as orders and classifications, the best possible understanding of nature's laws on the deepest levels.

Of prime significance is, perhaps, gauge symmetry, a symmetry that requires invariance under 'rotations' performed independently at each point of space-time. These 'rotations', instead of connecting states of different spatial orientations, group particles of the same mass values into families. The bold attempt of Salam[1] and Weinberg[2] to put photon and W particles into the same family using a gauge symmetry group is very attractive: in this way the two kinds of interactions, the elctromagnetic and weak interactions, become unified. But how did they answer the mystery about their big difference in masses? They used Higg's mechanism:[3] the use of Higg's scalar particles in the Lagrangian in order to induce spontaneous symmetry breaking. The photon corresponds, in their models, to the unbroken gauge subgroup U(1) of electromagnetism and hence has mass zero, while the W particles, associated with broken gauge symmetries, pick up large masses from the symmetry-breaking.

With the understanding of the power of the spontaneous symmetry breaking, recent efforts have been centred around extending Higg's pioneering work on spontaneous mass generation. It is conceivable that these Higg's scalar particles, though sufficient to induce symmetry breakdown, may not be of primary significance. It should be possible that spontaneous symmetry breakdown can occur in the absence of these scalar particles, for instance, as effects of higher order processes involving virtual Goldstone bosons. In this mood, models on dynamical broken gauge symmetries flourish. We shall discuss in part one how the simplest non-abelian gauge theory consisting only of pure massless Yang-Mills fields can acquire mass via the Goldstone mechanism.[4] The result is encouraging. It supports our belief that in the near future we should be able to develope a more general formalism to deal with spontaneous symmetry breaking theories, some method that can work for general fields and give the familiar features of spontaneous symmetry breaking in scalar field theories.

There are other symmetries we like to discuss too, the scale and conformal invariances. In constrast to the previous case of breaking the gauge symmetry to generate mass, we work in the assumption of strict scale and conformal invariances, the symmetries that require zero mass. Presumably this requirement is approximately fulfilled in high energy scattering phenomena and it should be rewarding to be able to identify the underlying principle to the diverse and seemingly complex data. Again, symmetry is the answer to the corresponding power rules and 'scaling rules' in the high energy regions for the exclusive and inclusive processes. It is the scale symmetry. To our surprise, though, this invariance, giving general predictions consistent with the phenomenological data, seems to indicate that high energy interactions proceed via the basic entities which are asymptotically free.

The last part is concerned with conformal invariance, where we work with massless fields. It can be easily obtained, based on the auxiliary represenation $(s,o) + (o,s)$ of the Lorentz group for spin s , massless fields, that the representation of the Lorentz transformations on helicity states of massless particles is given in terms of the rotation angles of Wigner rotations. By considering the mathematical properties of the generalised spinors in $(s,o) + (o,s)$, it is interesting that we can extend the helicity representation of the Lorentz group to the Conformal group. With the use of this helicity formalism, conformal symmetry has its own prediction for the helicity rules in high energy scatterings. Hopefully these can be verified by the experiments in the future.

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Part 1 Dynamical symmetry breaking of massless Yang-Mills theory

Abstract

We investigate dynamical symmetry breaking D.S.B. of massless Yang-Mills theory in the context of SU(2). Within the approximation scheme we are using, we possibly find dimensional transmutation. More precisely, this massless theory with only one coupling constant g acquires a massive spectrum spontaneously provided the eigenequation for g^2 has a positive solution. The generated mass decouples from the theory and has become a dimensional parameter, i.e. taking place of the previous dimensionless parameter g which is now subject to constraint.

Introduction

Quite sometime ago it was realized[1] that a Lagrangian can admit a symmetry which is not a symmetry of its physical Hilbert space. This happens, for instance, when we have symmetry violating vacuum expectation values, or, symmetry violating n points Green's functions. Then, invariably as a result, massless excitations (Goldstone bosons) occur which, on combination with massless vector gauge fields, produce massive vector meson particles. Thus symmetry breaking theory provides us a mechanism for generating massive spectrum.

Previous investigations of spontaneous symmetry breaking theories have been centred around scalar fields. Higgs,[2] Kibble[2] and others introduced mass term of canonical scalar fields ϕ of wrong sign into the Lagrangian for ϕ to develope non-vanishing vacuum expectation value. Jona-Lasinio[3] developed effective potential method in which the minima of the effective potential give the true vacuum states of the theory. This method is especially suitable for scalar fields theories, as examplified by Coleman and Weinberg:[4] They go beyond tree approximations and show, on inclusion of one loop corrections to the effective potential, that spontaneous symmetry breakdown can occur as a consequence of radiative corrections, i.e. of a dynamical nature.

In order to progress from the scalar field theories, we investigate spontaneous symmetry breakdown due to symmetry violating n points Green's functions which specificly arise from Goldstone mechanism of a dynamical origin, i.e. in the absence of sclar fields in the Lagrangian. Approach along this line is involved with finding symmetry breaking solutions to the various integral equations in the theory and usually one thus needs to use judicious approximations. In the investigation presented in below, we mainly model the argument for this simplest non abelian case after the abelian case by Jackiw and Johnson[5]: We explicitly introduce Goldstone boson couplings to the Yang-Mills particles so that they produce a massless pole in the Yang-Mills polarisation tensor $\pi^{\alpha\beta}_{\mu\nu}$. This in turn generates $k^2 \neq 0$ pole in the Yang-Mills propagator and consequently mass for the Yang-Mills particles.

The presentation is again parallel to the abelian case. In section (1) we write down the Ward identity for the generating functional and then derive from it the Ward identities for the Yang-Mills propagator and the Yang-Mills three points vertex. The latter Ward identity is used to show the relationship between this vertex and the symmetry breaking solutions to the self energy. In section (2), and more in appendix (2), we display the various Dyson equations and Bethe-Salpeter equations. These equations essentially govern the behaviours of the coupling functions of the

Goldstone bosons to the Yang-Mills particles and to the ghosts. In section (3) we develope the criteria for decoupling of these massless excitations from the physical S matrix. It turns out to be a satisfied requirement on the residue of the massless pole in $\pi^{\alpha\beta}_{\mu\nu}$. We consider some examples showing that the number of Goldstone bosons equal to the number of Yang-Mills particles which have obtained mass through this symmetry breaking scheme. In section (4) we consider explicitly an approximation and compute for a non-trivial solution. We find that the mass value in fact decouples from the theory. In the conclusion we further comment on the mass value and extend our consideration to the mass ratios for theories which contain many fields.

Appendix (1) gives the various notations and the Feynman rules which are employed in the four sections. Appendix (2) displays the Bethe-Salpeter equations. Appendix (3) gives the coefficients in the constraint equation on δ and g^2.

(1) Ward Identities.

When we are considering massless Yang-Mills gauge theories, we have Ward identities as a consequence of gauge invariance of the Lagrangian. As result, the Ward identities one obtains usually depend on the gauge conditions, which should be so chosen as consistent with the second quantisation. In light cone gauge[6] using a light like four vectors η_μ and $\eta_\mu \cdot A^\mu = 0$, we have considerably simplified Ward identities. This is because there are no ghosts in this gauge. The drawback, however, is that loop integrations in momentum space sometimes lead to unmanageable (at present) divergences: of the kind like $\int \frac{1}{\alpha} d\alpha$, where α is Feynmann parameter. In the covariant gauge $\partial_\mu A^\mu = 0$, the Ward identities become involved with ghost entities. We adopt this gauge in the text, as we only need the Ward identities for the Yang-Mills two points and three points functions, which are still quite simple. The Ward identity for the Yang-Mills two points function is used later to derive the form for the propagator $D^{\alpha\beta}_{\mu\nu}$ and to show how the Schwinger mechanism works. The Ward identity for the Yang-Mills three points function $T^{\alpha\beta\gamma}_{\mu\nu\sigma}$ illustrates explicitly the relation between its pole structure and the isotopic symmetry of the polarisation tensor $\pi^{\alpha\beta}_{\mu\nu}$.

Let us derive the Ward identity of the generating functional a la Lee and Zinn-Justin.[7] The gauge transformation is

$$\phi_i^g = \phi_i + [\, T^\alpha_{ij} \phi_j + \Lambda^\alpha_i \,] g_\alpha .$$

where T^{α}_{ij} is reducible representation of the generators and g_α is space time dependent parameter of the Lie group. Also let

$$M_{\alpha\beta} = \frac{\partial F_\alpha}{\partial \phi_i}\frac{\partial \phi_i}{\partial g_\beta} = \frac{\partial F_\alpha}{\partial \phi_i}[T^{\beta}_{ij}\phi_j + \Lambda^{\beta}_i]$$

with $F_\alpha(\phi) = a_\alpha$ specifying the gauge condition and a_α independent of ϕ_i and g .

With these definitions we can begin with the generating functional

$$\exp i[W(J)] \equiv \int [d\phi] \det M \cdot \exp i[s(\phi) - \frac{1}{2\xi}F_\alpha^2 + \phi_i J_i] \qquad (1)$$

as defined by Lee and Zinn-Justin.[7] Here: $s(\phi) = \int d^4x \cdot L(\phi)$ and the term $-\frac{1}{2\xi}F_\alpha^2$ is a weight function. Also, $[d\phi] \det M$ can be simplified by using the technique of Faddeev and Popov i.e. by introducing ghost fields. This is best reviewed by G. 't Hooft.[8]

This generating functional is independent of a_α. Now let $\lambda_\alpha = \delta a_\alpha = \delta F_\alpha(\phi)$. Assuming $[d\phi]\det M$ is independent of a_α, which is in fact proved by Lee and Zinn-Justin,[7] we have

$$\exp i[W(J)]\frac{\partial W(J)}{\partial a_\alpha} = \frac{\partial}{\partial a_\alpha}\int [d\phi] \det M \exp i[s(\phi) - \frac{1}{2\xi}F_\alpha^2 + \phi_i J_i]$$

$$= \int [d\phi] \det M \exp i[s(\phi) - \frac{1}{2\xi}F_\alpha^2 + \phi_i J_i][-\frac{1}{\xi}F_\alpha + J_i\frac{\partial \phi_i}{\partial a_\alpha}]$$

$$= 0.$$

Also, as $\partial\phi_i/\partial a_\alpha = \partial\phi_i/\partial F_\alpha$ and

$$g_{\alpha\gamma} = M_{\alpha\beta}[M^{-1}]_{\beta\gamma} = [\frac{\partial F_\alpha}{\partial \phi_i}\frac{\partial \phi_i}{\partial g_\beta}][M^{-1}]_{\beta\gamma} ,$$

we note that

$$\frac{\partial \phi_i}{\partial F_\gamma} = \frac{\partial \phi_i}{\partial g_\beta}[M^{-1}]_{\beta\gamma} = [T^{\beta}_{ij}\phi_j + \Lambda^{\beta}_i][M^{-1}]_{\beta\gamma} \qquad (2)$$

Thus we have the Ward identity for the generating functional,

$$\int [d\phi] \det M \exp\{i[S(\phi) - \tfrac{1}{2\xi} F_\alpha^2 + J_i \phi_i]\} \{-\tfrac{1}{\xi} F_\alpha + J_i [T_{ij}^\beta \phi_j + \Lambda_i^\beta] [M^{-1}]_{\beta\alpha}\} = 0.$$

Putting it back in terms of W(J), we have

$$\{-\tfrac{1}{\xi} F_\alpha(\tfrac{1}{i}\tfrac{\partial}{\partial J}) + J_i [T_{ij}^\beta \tfrac{1}{i}\tfrac{\partial}{\partial J_j} + \Lambda_i^\beta] [M^{-1}(\tfrac{1}{i}\tfrac{\partial}{\partial J})]_{\beta\alpha}\} \exp i[W(J)] = 0 \qquad (3)$$

where we have included the arguments of F_α and M^{-1}. This equation gives rise to all the other Ward identities of the fully connected Green's functions $W^{x_1 x_2 \cdots x_n}$. In the specific gauge $F_\alpha \equiv \partial^\mu \phi_\mu^\alpha$, this equation becomes

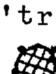
$$\{\tfrac{i}{\xi} \partial_\mu \frac{\partial}{\partial J_{\alpha\mu}(x)} + g \int d^4y\, J_\nu^\gamma(y) [\partial^\nu \delta_{\gamma\beta} - i g C_{\gamma\delta\beta} \frac{\partial}{\partial J_\delta^\nu(y)}] G^{\beta\alpha}(y, x; \tfrac{\partial}{\partial J})\} \exp i W(J) = 0. \quad (4)$$

where $G^{\beta\gamma}(x, y; \frac{\partial}{\partial J})$ is the ghost propagator satisfying

$$[-\Box_x \delta^{\alpha\beta} + i g\, \partial^\mu C_{\alpha\gamma'\beta} \frac{\partial}{\partial J_{\gamma'}^\mu(x)}]\, G^{\beta\gamma}(x, y; \tfrac{\partial}{\partial J}) = \delta^{\alpha\gamma} \delta^4(x-y) \qquad (5)$$

which is actually eqn.(2) in this gauge.

Now we can derive the Ward identities for the Yang-Mills two points and three points functions. The general rule is to use appropriate differentiations of eqn.(3) and then put the external sources equal to zero. This thus gives the Ward identities of the fully connected Green's functions $\partial^n W/\partial x_1 \partial x_2 \cdots \partial x_n$ which is usually denoted as $W^{x_1 x_2 \cdots x_n}$. To obtain the Ward identities of the proper (one particle irreducible) vertices $T^{y_1 y_2 \cdots y_n}$ one can use the expressions of $W^{x_1 x_2 \cdots x_n}$ in terms of $T^{y_1 y_2 \cdots y_n}$, i.e.

$W^{x_1 x_2 \cdots x_n} = \Sigma$ all 'trees' with n external vertices with vertices $T =$ , branches $W^{xy} =$ 

.

We can easily deduce that the longitudinal part of the vector propagator $D^{\alpha\beta}_{\mu\nu}$ is unrenormalised by applying $\partial_\tau \frac{\partial}{\partial J_\tau(3)}\Big|_{J=0}$ to eqn.(4) and using eqn.(5). Thus

$$\frac{i}{\xi} k_\mu k_\nu D^{\alpha\beta}_{\mu\nu}(k) = \delta_{\alpha\beta} .$$

That is, $D^{\alpha\beta}_{\mu\nu}$ can be put in the form

$$D^{\alpha\beta}_{\mu\nu}(k) = -i\left[\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) D^{\alpha\beta}(k^2) + \frac{k_\mu k_\nu}{k^2}\frac{\xi}{k^2}\delta^{\alpha\beta}\right].$$

This will be used later to illustrate the Schwinger mechanism.

In the abelian case using fermion field the proper vertex function $\Gamma_5^\mu(p,p+q)$ associated with J_5^μ satisfies a Ward identity

$$q_\mu \Gamma_5^\mu(p,p+q) = i[\gamma^5 \Sigma(p+q) + \Sigma(p)\gamma^5]$$

with $\Sigma(p)$ the self energy of the fermion, i.e. $\Gamma_5^\mu(p,p+q)$ has a pole at q = 0 when the self energy has a symmetry breaking part. This is obvious because

$$Lim_{q\to 0}\, q_\mu \Gamma_5^\mu(p,p+q) = i\{\gamma^5, \Sigma(p)\} .$$

It is interesting that in pure massless Yang-Mills case the same kind of relation holds between the symmetry of the self energy and the singularity structure of the three Yang-Mills vertex. This is best seen in the light cone gauge. The vertex function satisfies a Ward identity

$$k^\mu \Gamma^{\alpha\beta\gamma}_{\mu\nu\sigma}(k,p,r) = \varepsilon_{\alpha\beta\gamma}\{[D^{-1}(p)]^\beta_{\mu\nu} - [D^{-1}(r)]^\gamma_{\mu\nu}\}$$

where $D^\beta_{\mu\nu}$ is Yang-Mills two points Green's function in this gauge. As a check, one notes that this equation should be satisfied by the respective bare quantities.

The corresponding identity in covariant gauge becomes involved with ghost entities. Nonetheless the conclusion should be fairly the same. It is shown (9) in this gauge that this Ward identity can be simplified to result

$$\lim_{k\to 0} k^{\mu} \Gamma^{\alpha\beta\nu}_{\mu\nu\sigma}(k,p,r) \propto [T^{\alpha}, \pi(p)]^{\beta\nu}_{\nu\sigma}$$

with $\pi(p)$ polarisation tensor and $(T^{\alpha})_{\beta\gamma} = i\varepsilon_{\alpha\beta\gamma}$.
This explicitly shows that when $\pi^{\alpha\beta}_{\mu\nu}$ has a global symmetry breaking solution, $\Gamma^{\alpha\beta\gamma}_{\mu\nu\sigma}$ has a pole at k = 0 and vice versa. However, this vertex can still have a pole at $k^2 = 0$ with unbroken global symmetry. More precisely, massless poles with $k^2 = 0$ and $k \neq 0$ can exist in the Yang-Mills three points vertex independently of the symmetry nature of the self energy solution.

In fact, we can go beyond this. Because of the more varieties of elementary vertices in the massless Yang-Mills theory, viz.* three Yang-Mills vertex, four Yang-Mills vertex, one Yang-Mills and two ghosts vertex, it is quite conceivable that one could have symmetry breaking and dynamical generation of mass with  $= 0$. That is, massless poles need not exist in $\Gamma^{\alpha\beta\gamma}_{\mu\nu\sigma}$ at all for Schwinger mechanism to work. We shall give the formula for the pole part of $\Gamma^{\alpha\beta\gamma}_{\mu\nu\sigma}$ in the next section and this point can then become more transparent.

* Here Yang-Mills means Yang-Mills particle.

(2) Schwinger mechanism and Dyson equations.

We will now write down the explicit form for the Yang-Mills propagator $D^{\alpha\beta}_{\mu\nu}$. This is essential because the Yang-Mills particle mass is the location of the pole in $D^{\alpha\beta}_{\mu\nu}$. This choice of definition for mass is preferred, because it is gauge invariant. For instance, the other definition of mass as the value of inverse propagator at zero momentum is, in general, not gauge invariant. These two definitions coincide for first order calculations, but they differ when we go to higher order effects.

In external J, the complete Yang-Mills propagator can be written in terms of the polarisation tensor $\pi^{\alpha\beta}_{\mu\nu}$,

$$D^{\alpha\beta}_{\mu\nu}(x,y;J) = \mathring{D}^{\alpha\beta}_{\mu\nu}(x-y) + \int d^4 z\, d^4 z'\, \mathring{D}^{\alpha\alpha'}_{\mu\mu'}(x-z)\, \pi^{\alpha'\beta'}_{\mu'\nu'}(z,z';J)\, D^{\beta\beta'}_{\nu\nu'}(z',y;J)$$

where $\mathring{D}$ is the bare propagator. By applying $(\mathring{D})^{-1}$ to the left and $(D)^{-1}$ to the right of this integral equation, we have

$$\pi^{\alpha\beta}_{\mu\nu}(k) = -i\,[D^{-1}(k)]^{\alpha\beta}_{\mu\nu} + i\,[\mathring{D}^{-1}(k)]^{\alpha\beta}_{\mu\nu} . \tag{6}$$

This together with the fact that the longitudinal part of $D^{\alpha\beta}_{\mu\nu}\big|_{J=0}$ is unrenormalised imply that $\pi^{\alpha\beta}_{\mu\nu}\big|_{J=0}$ is transverse,

$$\pi^{\alpha\beta}_{\mu\nu}(k) = i\,[\,g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\,]\, k^2 \pi^{\alpha\beta}(k^2) .$$

Hence, the inverse of eqn.(6) gives

$$D^{\alpha\beta}_{\mu\nu}(k) = -i\left\{ [\,g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\,]\,[\,k^2 I + k^2 \pi(k^2)\,]^{-1}_{\alpha\beta} + \rho\,\frac{k_\mu k_\nu \delta^{\alpha\beta}}{k^4} \right\} . \tag{7}$$

Let us look at its transverse part. When $\pi^{\alpha\beta}(k^2)$ has a $k^2 = 0$ pole with non vanishing residue, say R, the previous $k^2 = 0$ pole in $D^{\alpha\beta}_{\mu\nu}$ is then evaded. Moreover, using the approximation where only the pole term of $\pi^{\alpha\beta}(k^2)$ is kept, we find that the Yang-Mills particle acquires a mass μ with

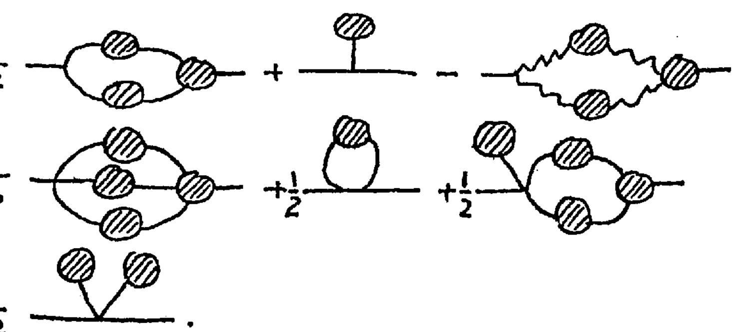
$$\mu^2 = -R.$$

This mass formula will be used very often later. This mass generating mechanism, i.e. a seemingly massless particle acquires a mass because the vacuum polarisation tensor has a pole at zero momentum transfer, is called Schwinger mechanism.

The now familiar Higgs mechanism provides a special realisation of the Schwinger mechanism. In those examples a canonical scalar field, already included in the Lagrangian, has a non-vanishing vacuum expectation value. This vacuum expectation value thus gives rise to tadpole contributions to $\pi^{\alpha\beta}$ which produce a pole. Here we aim to make such a pole occur for purely dynamical reason, i.e. in the absence of these canonical scalar fields. In other words, we are more interested in a dynamical symmetry breaking scheme: Dynamics gives rise to a zero mass bound excitation which at the end decouples from the physical S matrix and gives mass to the Yang-Mills particles.

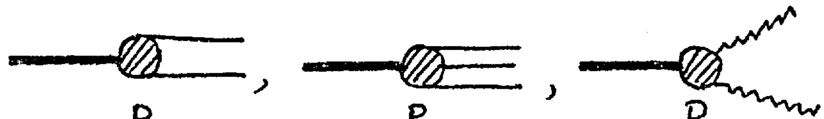
In order to explore such a possibility, we first look at the Dyson equations which relate the various

Green's functions. The Dyson equation for $\pi_{\mu\nu}^{\alpha\beta}$, also similarly for other two points Green's functions, is obtained by taking $\frac{\partial}{\partial J}\Big|_{J=0}$ of the Schwinger functional equation for the Yang-Mills field coupled to its external source J, where the Schwinger equations are derived by applying stationary action principle to the generating functional. The Dyson equation for $\pi_{\mu\nu}^{\alpha\beta}$, after taking variation δA_μ^α and $\frac{\partial}{\partial J}$ of equation (1), is diagrammatically

$\pi_{\mu\nu}^{\alpha\beta}(x,y,J) =$ 

where we have included the tadpole diagrams. At J = 0, these tadpole diagrams in fact do not contribute. This is because $\langle A_\mu(0)\rangle|_{J=0} = 0$, which can be easily seen by putting J = 0 in the Ward identity for the generating functional.

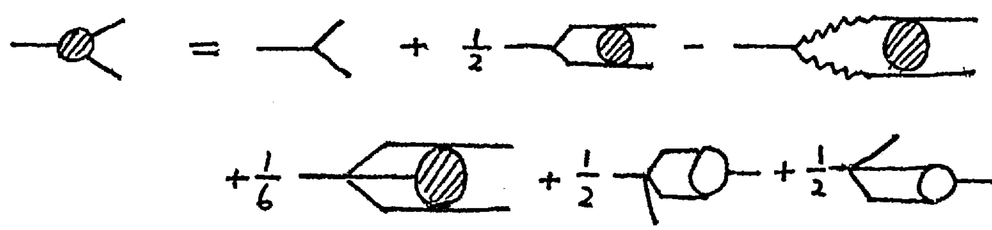
Thus from here we see that a pole in $\pi_{\mu\nu}^{\alpha\beta}$ can arise from a massless intermediate state which couples to two Yang-Mills particles, three Yang-Mills particles or to two ghosts. That is, this massless excitation (usually called Goldstone boson) must have nontrivial couplings like



P_2 , P_3 , P_g .

We denote these coupling functions with suppressed indices as P_2 , P_3 , P_g respectively.

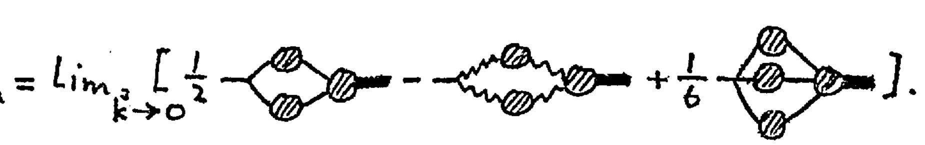
Analogously we can write down the Dyson equation for the three Yang-Mills proper vertex



This is interesting because we can read from it the pole part of this vertex.

Let us introduce $\lambda_{\alpha a}$ with the meaning

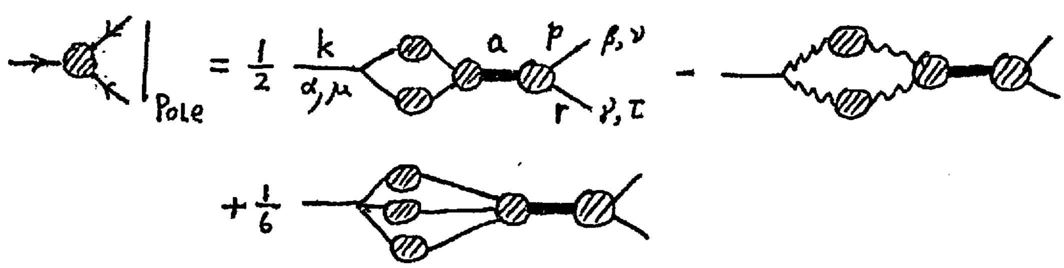
$$\lambda_{\alpha a} k_\mu = \lim_{k^2 \to 0}$$



(8)

Also let us use for the Goldstone boson propagator $D_{ab}(k) = -\frac{i d_{ab}}{k^2}$ where the factor d_{ab} can be chosen to suit our interests. For instance, for a global symmetric theory, we can simply use $d_{ab} = \delta_{ab}$. On the other hand, we can assume that there are only two Goldstone bosons and use $d_{ab} = \delta_{ab} - \delta_{a3}\delta_{b3}$. This latter thus corresponds to a broken global symmetry theory.

Thus, the pole part in this proper vertex is given by



$$= [\lambda_{\alpha a} k_\mu + O(k^2)](-i d_{ab}/k^2)\, P_{2\nu\tau}^{b\beta\gamma}(p,r) .$$

One thus observes the explicit appearance of P_2 in this residue.

Also, the pole part in $\pi_{\mu\nu}^{\alpha\beta}$ can be easily written down. It follows from using the definition equation for $\lambda_{\alpha a}$ at each end of the polarisation tensor. Thus,

$$\pi_{\mu\nu}^{\alpha\beta}\Big|_{Pole} = -i[\lambda_{\alpha a} k_\mu + O(k^2)]\, \frac{d_{ab}}{k^2} [\lambda_{\beta b} k_\nu + O(k^2)].$$

Hence, we can write, because of its transversality,

$$\Pi_{\mu\nu}^{\alpha\beta} = -i\,(k^2 g_{\mu\nu} - k_\mu k_\nu)\,\frac{\lambda_{\alpha a}\, d_{ab}\, \lambda_{\beta b}}{k^2}$$

+ higher order terms.

On substituting this formula without its higher order terms into equation (7) , we can give the mass formula for the Yang-Mills particle. It is

$$M_{\alpha\beta}^2 = \lambda_{\alpha a}\, d_{ab}\, \lambda_{\beta b}. \tag{9}$$

Thus, we see that we can have dynamical generation of mass provided $\lambda_{\alpha a} \neq 0$. However, this condition is not sufficient for a massless pole in the proper three Yang-Mills vertex. As it should be clear from above, this also requires nontrivial P_2 solution. That is, massless poles need not exist in $T_{\mu\nu\sigma}^{\alpha\beta\gamma}$ at all for Schwinger mechanism to work as we have mentioned before.

We give the other Dyson equations and the derived Bethe-Salpeter equations in appendix (3) . Any non-trivial solutions for the Goldstone boson coupling functions should be made consistent with them. Before doing the actual computation, let us look at another problem in the following section.

(3) Decoupling of Goldstone boson poles from S matrix

In this section we will develope the criteria for decoupling of these massless poles (Goldstone boson poles) from the physical S matrix. This is essential because these massless excitations, though responsible for the whole D.S.B. scheme, do not correspond to physical particles. In order to fulfil this requirement, thus we need to look for complete cancelation of pole diagrams.

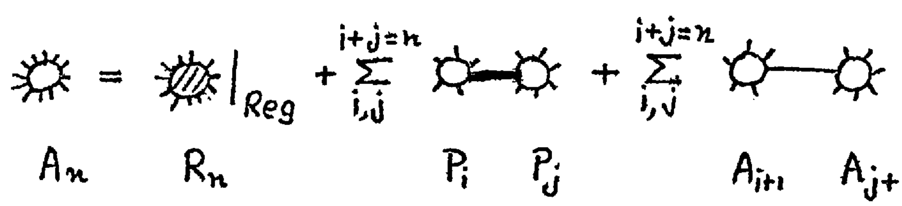
The first step is to decompose the full on mass shell amplitude A_n into three parts :

(a) part which is regular i.e. pole free, and 1 particle irreducible.

(b) part which contains those diagrams with massless poles say, at $q^2 = 0$.

(c) part which contains those reducible diagrams with $D^{\alpha\beta}_{\mu\nu}(q)$

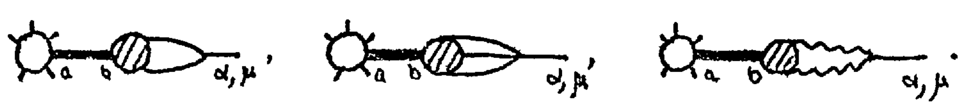
That is, in diagrams they are



The next step is to take a diagram from part (b) and choose a corresponding diagram from part (c). Two corresponding diagrams are those diagrams which become each other upon replacing a massless Goldstone boson pole with a Yang-Mills propagator, or vice versa. For instance, we can consider the following two diagrams:

The amplitude for the first diagram is easy to compute. It is $P_i^a D_{ab} P_j^b$. (10a)

The amplitude for the second diagram, however, takes more analysis. First we note that its vertex A_{i+1} and similarly for the other vertex contains a regular part and a pole part at $q^2 = o$. The pole part arises as results from such diagrams like



Thus, $A^\mu_{i+1} = A^\mu_{i+1}\big|_{Reg} + A^\mu_{i+1}\big|_{Pole}$

and we can write

$$A^\mu_{i+1}\big|_{Pole} = P_i^a D_{ab} \lambda_{\alpha b} q_\mu .$$

Now before substituting this form for A^μ_{i+1} into the amplitude which is $A^\mu_{i+1} D^{\alpha\beta}_{\mu\nu} A^\nu_{j+1}$, we note that we need only retain the $g_{\mu\nu}$ term in the Yang-Mills propagator $D^{\alpha\beta}_{\mu\nu}$. This is because the on mass shell vertices are transverse and cancel with the $k_\mu k_\nu$ term in $D^{\alpha\beta}_{\mu\nu}$. Hence, we write the amplitude for this diagram as

$$(A_{i+1})^{\mu}\left(\frac{-i}{q^2+q^2\pi(q^2)}\right)(A_{j+1})^{\mu}.$$

However, as the pole part in A^{μ}_{i+1} contracts with the rightmost vertex and cancels it, we are left with

$$\begin{aligned}
&(A_{i+1})^{\mu}_{Reg}\left[\frac{-i}{q^2+q^2\pi(q^2)}\right](A_{j+1})^{\mu}\\
&=(A_{i+1})^{\mu}_{Reg}\left[\frac{-i}{q^2+q^2\pi(q^2)}\right]\left[(A_{j+1})^{\mu}_{Reg}+(A_{j+1})^{\mu}_{Pole}\right]\\
&=(A_{i+1})^{\mu}_{Reg}\left[\frac{-i}{q^2+q^2\pi(q^2)}\right](A_{j+1})^{\mu}_{Reg}+(A_{i+1})^{\mu}_{Reg}\left[\frac{-i}{q^2+q^2\pi(q^2)}\right](A_{j+1})^{\mu}_{Pole}\\
&=(A_{i+1})^{\mu}_{Reg}\left[\frac{-i}{q^2+q^2\pi(q^2)}\right](A_{j+1})^{\mu}_{Reg}-(A_{i+1})^{\mu}_{Pole}\left[\frac{-i}{q^2+q^2\pi(q^2)}\right](A_{j+1})^{\mu}_{Pole}. \qquad (10b)
\end{aligned}$$

The last line results because

$$(A_{i+1})^{\mu}_{Reg}=(A_{i+1})^{\mu}-(A_{i+1})^{\mu}_{Pole}$$

and A_{i+1} contracts with $(A_{j+1})^{\mu}_{Pole}$ and cancels it.

Thus, the two corresponding amplitudes, equations (10a) and (10b), show that the pole part of the second diagram cancels with the first diagram provided

$$P_i^{a} D_{aa'} \lambda_{\alpha a'} [q^2/(q^2+q^2\pi(q^2))]_{\alpha\beta} P_j^{b'} D_{b'b} \lambda_{\beta b} = P_i^{a} D_{ab} P_j^{b}$$

for $q^2 = 0$. Here $D_{aa'}$ is the Goldstone boson propagator. We specify the region of interest : $q^2 = 0$, where these massless poles dominate the S matrix.

The L.H.S. of the above condition, seemingly containing double $q^2 = 0$ pole because $D_{aa'}$ appears twice, in fact after cancelation with q^2 in the numerator is a single $q^2 = 0$ pole. We remove the pole coupling functions and then the equivalent condition becomes

$$\mathcal{D}_{aa'} \lambda_{\alpha a'} \left[\frac{q^2}{q^2 I + q^2 \pi(q^2)}\right]_{\alpha\beta} \lambda_{\beta b'} \mathcal{D}_{b'b} = \mathcal{D}_{ab}$$

Multiplying this with $(\mathcal{D}^{-1})_{ca}$ on the left and $(\mathcal{D}^{-1})_{bd}$ on the right, we have

$$\lambda_{\alpha c} \left[\frac{q^2}{q^2 I + q^2 \pi(q^2)}\right]_{\alpha\beta} \lambda_{\beta d} = (\mathcal{D}^{-1})_{cd} .$$

It is more convenient to use matrix notation here. Its inverse, with suppressed indices, is

$$\lambda^{-1} \left[\frac{q^2 I + q^2 \pi(q^2)}{q^2}\right] \lambda^{-1} = \mathcal{D} = \frac{d}{q^2} .$$

Hence, what this requires at $q^2 = 0$ is

$$\left[q^2 I + q^2 \pi(q^2)\right]_{\alpha\beta}\Big|_{q^2=0} = \lambda_{\alpha a}\, d_{ab}\, \lambda_{b\beta}$$

which, thus, is in fact a requirement on the residue of the massless pole in $\pi(q^2)$. This is indeed what we have obtained on computing the pole part of $\pi(q^2)$ in the previous section.

Thus, we have completed our proof for the above two diagrams that the pole part of the corresponding one particle reducible diagram cancels with the pole diagram. We can perform the same argument to every pole diagram in part (b) and clearly we conclude that the on mass shell Green's functions are free from massless poles.

Though the above argument is independent of the specific values for λ and d , it is clear from the mass formula, equation (9) , that the mass value for the Yang-Mills particle depends crucially on them. Let us investigate this point more carefully. Of course, the value for $\lambda_{\alpha a}$ is explicitly dependent on the kind of the pole coupling functions and the number of the Goldstone bosons in the theory. For instance, we can consider the following two cases in SU(2) :

Case a) Suppose that we have two Goldstone bosons and thus we can choose

$$d_{ab} = \delta_{ab} - \delta_{a3}\,\delta_{b3}$$
$$\lambda_{\alpha a} \propto \varepsilon_{a\alpha 3}$$

where we use English (Greek) letters for the Goldstone boson (Yang-Mills particle) isotopic spin indices.

It is then found

$$M^2_{\alpha\beta} \propto \delta_{\alpha\beta} - \delta_{\alpha 3}\,\delta_{\beta 3}\,.$$

That is, instead of a massless Yang-Mills triplet before, we obtain two massive Yang-Mills particles and one Yang-Mills particle remains massless.

Case b) Suppose instead that we have an isotriplet of Goldstone bosons. We can accordingly assign

$$d_{ab} = \delta_{ab}$$
$$\lambda_{\alpha a} \propto \delta_{\alpha a}.$$

The mass formula will then give

$$M^2_{\alpha\beta} \propto \delta_{\alpha\beta}\,.$$

Thus, the Yang-Mills fields obtain a common mass. That is, the local isotopic symmetry is broken but the global symmetry in this case is preserved.

These two cases thus show that the number of Goldstone bosons is equal to the number of Yang-Mills particles which have obtained mass through the symmetry breaking scheme. This is in fact not peculiar to SU(2). Feinberg et al[9] showed that this is a phenomenon common to all other symmetries.

In the above two examples, we have also verified that they satisfy the lowest order Bethe-Salpeter equations so far as isotopic symmetry is concerned.

(4) Goldstone boson pole coupling functions and approximations

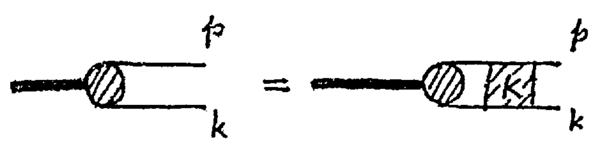
As we have noted before that the generated mass is given by the mass formula $M^2_{\alpha\beta} = \lambda_{\alpha a} d_{ab} \lambda_{\beta b}$ and

$$k_\mu(\lambda_{\alpha a} + O(k^2)) = \frac{1}{2} P^{a\alpha}_{2\mu} + \frac{1}{6} P^{a\alpha}_{3\mu} - P^{a\alpha}_{g\mu}$$

we will in this section compute for non-trivial solutions for these pole coupling functions. As solving them exactly is a formidable task, in below we will limit ourselves to consider an approximation, viz. $P_3 = 0$ and $P_g = 0$. These pole coupling functions, however, as shown in the appendix (2), are clearly self coupled. Thus our solution should be viewed as an approximation taking P_3 and P_g as higher orders in the coupling constant g and hence neglected.

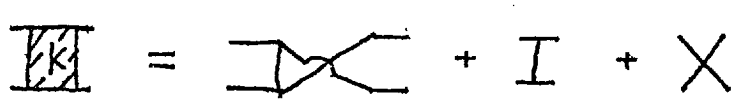
We find that taking $P_3 = 0$ is an appealing idea, because this coupling function involving three Yang-Mills particles and the Goldstone boson pole should be very complex if non-trivial. Both Feinberg et al[9] and Jan Smith[9] use this same idea to reduce the computation labour. However, subsequently Feinberg et al[9] try to break local symmetry and use a symmetry function for P_2. While Jan Smith[9] considers a global symmetry theory, similar to our work in below, his more general scheme involves more divergences than here. Though he also argues that the approximation as mentioned above is plausible, our direct approach shows, besides a consistent solution, that in this massless theory with only one coupling constant the generated mass value in fact decouples from the theory. We will consider this more in detail in below.

In this specific approximation, we have a simple integral equation of the form



$$P_2 = \int P_2 \mathcal{D}\mathcal{D}K \quad \text{with suppressed indices} \tag{11}$$

That is, similar to the previous abelian case, we also use one coupling function to characterize the theory. The kernel here, in its lowest order, is given by



where the first two diagrams on R.H.S. when substituted in equation (11) will give the same contribution because of Bose statistics.

Before solving for P_2, it is interesting to look at the isotopic symmetry of the theory. We can, for instance, choose one of the following three choices:

(1) $P_2 \propto \varepsilon_{a\alpha 3}(T^{\alpha 3})_{\beta\gamma}$, with

$$(T^{\alpha 3})_{\beta\gamma} = \delta_{3\beta}\,\delta_{\alpha\gamma} + \delta_{\beta\alpha}\,\delta_{3\gamma} - 2\,\delta_{3\alpha}\,\delta_{\gamma\beta}\,.$$

That is, there are two Goldstone bosons and the coupling function is symmetric in β, γ . It is convenient to denote coupling function of this form as $P_{2\{\beta,\gamma\}}$.

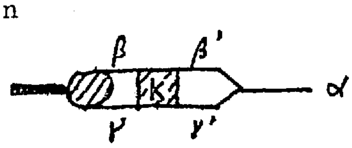
(2) $\quad P_2 \propto \varepsilon_{a\alpha 3}\, \varepsilon_{\alpha\beta\gamma}$.

This is an alternate case also with two Goldstone bosons. Only that the coupling function is antisymmetric in β, γ . Thus we denote coupling function of this kind as $P_{2[\beta,\gamma]}$.

(3) $\quad P_2 \propto \varepsilon_{a\beta\gamma}$.

This case corresponds to an isotriplet of Goldstone bosons. The coupling function is totally antisymmetric in a, β and γ . We denote this as $P_{2[a,\beta,\gamma]}$.

It is easy to verify that all three kinds of them satisfy the lowest order Bethe-Salpeter equations. But we will discard consideration of $P_{2\{\beta,\gamma\}}$ for the following reason. We are mainly interested in those solutions of P_2 which when coupled to one vector meson can give non-trivial $\lambda_{\alpha a}$. Now for a solution like $P_{2\{\beta,\gamma\}}$, every diagram in



with amplitude $P_{2\{\beta,\gamma\}}\, T_{\beta\beta'\gamma\gamma'}\, \varepsilon_{\alpha\beta'\gamma'}$ will have a corresponding diagram with amplitude $P_{2\{\beta,\gamma\}}\, T_{\gamma\gamma'\beta\beta'}\, \varepsilon_{\alpha\beta'\gamma'}$, and thus the two add up to zero. Similar argument applies to

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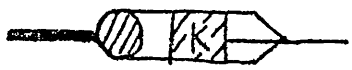
Hence in this case the main contribution will come from



which is essentially P_3, the coupling function we think of neglecting.

The other two coupling functions, $P_{2[\beta,\gamma]}$ and $P_{2[a,\beta,\gamma]}$ in lowest order, i.e. considering only

give $\lambda_{\alpha a} \propto \varepsilon_{a\alpha' 3}\,\varepsilon_{\alpha'\beta\gamma}\,\varepsilon_{\alpha\beta\gamma} \propto \varepsilon_{a\alpha 3}$.
and $\lambda_{\alpha a} \propto \varepsilon_{a\beta\gamma}\,\varepsilon_{\beta\gamma\alpha} \propto \delta_{a\alpha}$ respectively.
These are the two examples used previously to calculate the mass values. We should notice, however, that as the residue for the three Yang-Mills vertex has a factor $P_{2\mu\nu}^{a\beta\gamma}(p,-p-k)$, we cannot arrange a pole in this vertex at k = 0 while insisting an antisymmetric function for P_2. This is because Bose statistics would require

$$P_{2\mu\nu}^{a[\beta,\gamma]}(p,q) = P_{2\nu\mu}^{a[\gamma,\beta]}(q,p)$$

and this obviously cannot be satisfied for $q = -p$. Nonetheless, $P_{2\mu\nu}^{a[\beta,\gamma]}(p,-p-k)$ or $P_{2\mu\nu}^{[a,\beta,\gamma]}(p,-p-k)$ need not vanish for $k^2 = 0$, i.e. a $k^2 = 0$ pole can exist in this vertex. Thus, in below, we will furtherly use $P_{2\mu\nu}^{[a,\beta,\gamma]}$ to calculate its contribution to $\lambda_{\alpha a}$.

The solution we have in mind for equation (11) in its lowest order is a solution which asymptotically behaves as

$$P_{2\mu\nu}^{[\alpha,\beta,\gamma]}(p,k)\sim[\varepsilon_{\alpha\beta\gamma}][(p_\mu p_\nu-k_\mu k_\nu)+b g_{\mu\nu}(p^2-k^2)][(p^2)^{-\delta}+(k^2)^{-\delta}],$$

i.e. a simple power function in momentum.

It is clear that we should expect some constraint on this power δ, in order for P_2 to resurge after the operations indicated on the R.H.S. of equation (11). There we use the lowest order approximations for the kernel and the propagator which is chosen as elsewhere in Feynman gauge. The integration is then performed by continuing to Euclidean momenta. Indeed for consistency, we obtain, on matching the $p_\mu p_\nu$ and $g_{\mu\nu}$ terms on both sides, the following two conditions:

$$\begin{aligned}&[(C_{11}+E_{11})+bC_{21}]+F_{11}=1\\&[(C_{12}+E_{12})+bC_{22}]+F_{12}=b\end{aligned}\qquad(12)$$

where the coefficients C_{11}, E_{11}, C_{21}, F_{11}, C_{12}, E_{12}, C_{22}, F_{12} are polynomials in δ and g^2. They are given in the appendix (3). Eliminating b, we have a condition equation on δ as we have envisioned before

$$C_{21}\,(C_{12}+E_{12}+F_{12})=(1-C_{22})(1-F_{11}-C_{11}-E_{11})\qquad(13)$$

Now g^2, of course, should be positive. Also, in order for this solution to vanish for asymptotic momentum, we require $\delta>1$. Moreover, δ will be shown limited to a certain range of values. This, thus, makes the above equation a more stringent condition on g^2.

Naively, we would think that we can proceed now to calculate the generated mass in this approximation.

In fact this cannot be done. If we look at the mass formula again, which is

$$M^2_{\alpha\beta} = \lambda_{\alpha a} d_{ab} \lambda_{\beta b} .$$

It is clear that in our case here we can use

$\lambda_{\alpha a} \propto \delta_{a\alpha}$ and $d_{ab} = \delta_{ab}$, without any loss of generality. Thus we have

$$\lambda_{\alpha a} = \delta_{\alpha a} M$$

and then we imagine that we can calculate $\lambda_{\alpha a}$ upon taking $\left.\frac{\partial}{\partial k_\sigma}\right|_{k=0}$ of equation (8) which is, in this approximation,

$$\lambda_{\alpha a} k_\mu + O(k^2) = \int P_{2\nu\tau}^{[a,\beta,\gamma]}(k)\, D^{\nu\nu'}(k-r)\, D^{\tau\tau'}(r)\, B^{\beta\gamma\alpha}_{3\nu'\tau'\mu}\, d^4 r$$

where
$$P_{2\nu\tau}^{[a,\beta,\gamma]}(k) \equiv P_{2\nu\tau}^{[a,\beta,\gamma]}(k-r, r)$$
$$D_{\mu\nu}(k) = -i\left\{ \left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right] \frac{1}{k^2 - M^2} + \frac{k_\mu k_\nu}{k^4} \right\} .$$

Clearly, $\lambda_{\alpha a}$ is proportional to P_2 . Moreover, as the L.H.S.of this equation has dimension +2, it is convenient to write

$$P_{2\mu\nu}^{[a,\beta,\gamma]}(p,k) = cM\,\varepsilon_{a\beta\gamma}\left[\frac{p_\mu p_\nu - k_\mu k_\nu}{M^2} + b g_{\mu\nu}\frac{p^2 - k^2}{M^2}\right]\left[\left(\frac{p^2}{M^2}\right)^{-\delta} + \left(\frac{k^2}{M^2}\right)^{-\delta}\right] .$$

To our surprise, upon substituting this into the above, the value M decouples from the equation. This happens when we change the variable of integration , from r to rM . The resulting equation, after applying $\left.\frac{\partial}{\partial k_\sigma}\right|_{k=0}$ to it, is

$$g_{\mu\sigma} g_{\alpha a} = -icg^2 \varepsilon_{\beta\alpha\gamma}\varepsilon_{a\beta\gamma}\int \left[-g_{\tau\sigma} r_\nu - g_{\nu\sigma} r_\tau - 2b g_{\tau\nu} r_\sigma\right] \cdot \left[g_{\mu\tau'}(-r)_{\nu'} + g_{\tau'\nu'} 2r_\mu - g_{\mu\nu'} r_{\tau'}\right] (2(r^2)^{-\delta})\, D^{\nu\nu'}(r)\Big|_{M=1} D^{\tau\tau'}(r)\Big|_{M=1} d^4 r .$$

This integration can be performed after Wick's rotation.

The result is that, for convergence of integration, δ must be limited to the range of value $1 < \delta < 3$, and c is then given by

$$c = \frac{1}{6\pi g}\left[\sin(2-\delta)\pi\right] / \left[2b(2-\delta)+1\right] \tag{14}$$

with b previously given by equation (12) .

Thus, we have failed to calculate the generated mass. Instead we have obtained one more condition on the value of δ, which is now required to be $1 < \delta < 3$ for the existence of a solution for P_2 of the form

$$P_{2\mu\nu}^{[\alpha,\beta,\gamma]}(p,k) \propto [\varepsilon_{\alpha\beta\gamma}][(p_\mu p_\nu - k_\mu k_\nu) + b g_{\mu\nu}(p^2 - k^2)][(p^2)^{-\delta} + (k^2)^{-\delta}] .$$

We will give further comment on the mass value in the conclusion: The reason behind the decoupling of it and why it is not a sad point in this theory. Before ending this section, we like to stress that we have found the above mentioned solution. It should work, provided that the constraint equation on δ and g^2, equation (13) , can be fulfilled with all their other positivity conditions.

Conclusion

In this simplest massless non- abelian case we have found the following conclusion. For spontaneous symmetry breakdown in this model to occur as effects involving higher-order processes involving virtual Goldstone bosons, the coupling constant g becomes constraint to satisfy certain conditions and the resulting mass for the Yang-Mills particles cannot be computed.

We can view our result here in connection with that from massless scalar electrodynamics. It is found in that theory with two free parameters e and λ spontaneous symmetry breakdown can occur as effects of higher-order processes involving virtual photons. After symmetry breakdown the theory still possesses two parameters, e and $\langle\phi\rangle$, the vacuum expectation value of the scalar fields. That is, λ becomes related to e and the generated mass dependent on $\langle\phi\rangle$ in a trivial way governed by dimensional analysis is also not computable. Hence we note the similarity between these two cases: after spontaneous symmetry breakdown a dimensionless parameter is traded for a dimensional one, the phenomenon of dimensional transmutation.

There is another aspect that we like to stress on. It is found by Feinberg et al that the number of particles that have acquired mass in this dynamical symmetry breakdown scheme is equal to the number of Goldstone bosons in the theory. If we recall a general feature of spontaneously-broken gauge models where Higgs phenomenon occurs, i.e. where the driving mechanism for the instability of the theories is a non-vanishing vacuum expectation value, the number of the would-be Goldstone bosons is equal to the number of broken degrees of freedom. However, these Goldstone bosons disappear and consequently the vector mesons corresponding to the broken symmetry generators acquire mass. That is, irrespective to the driving mechanism for the symmetry breaking the number of Goldstone bosons is equal to the number of vector particles that have acquired masses.

These two aspects, dimensional transmutation and the one above, suggest a strong possibility: in the future we should be able to develope a more general formalism to deal with spontaneous symmetry breaking theories, some method that can work for general fields and give the above and other familiar features of spontaneous symmetry breaking in scalar field theories.

Before ending it may be interesting to point out that in theories consisting of many massless fields but with only one coupling constant the mass ratios are necessarily independent of the coupling strength g. This follows from dimensional analysis and renormalisation: The generated mass

M for each particle via spontaneous symmetry breaking should be independent of μ, the point of renormalisation, while from dimension analysis it can be put in the form $M = \mu f(g)$. Hence we have $\partial M/\partial\mu = 0$ and this consequently leads to $f = c\exp[-\int\frac{1}{\beta(g)}dg]$, where we have introduced $\beta(g) \equiv \mu\frac{\partial g}{\partial\mu}$ and c is constant of integration. Hence the mass ratios are just constant ratios and are independent of the coupling strength.

Appendix (1)

Notations and Feynman rules for massless Yang-Mills fields

(1) Notations

 solid line representing Yang-Mills particle

 wavy line representing ghost particle

 thick solid line representing massless bound state (Goldstone boson)

K, proper kernel of appropriate amplitudes

 P, proper Goldstone amplitude

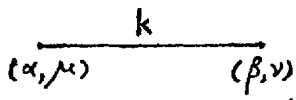
 A, connected amplitude

 A', proper connected amplitude

(2) Feynman rules for massless Yang-Mills fields

(a) Propagators: each propagator has an additional factor $1/(2\pi)^4 i$.

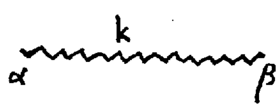
(i) Yang-Mills propagator



$$\overset{\circ}{D}{}^{\alpha\beta}_{\mu\nu}(k) = \frac{\delta^{\alpha\beta}}{k^2 - i\varepsilon}\left[g_{\mu\nu} - \lambda\, \frac{k_\mu k_\nu}{k^2 - i\varepsilon} \right]$$

where λ = 1 , 0 for Landau, Feynman gauge respectively.

(ii) Ghost propagator



$$\overset{\circ}{G}_{\alpha\beta} = \frac{\delta_{\alpha\beta}}{k^2 - i\varepsilon} .$$

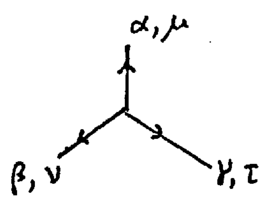
(iii) Goldstone boson propagator

$$D_{ab} = \frac{d_{ab}}{k^2 - i\varepsilon}$$

where the choice of $d_{ab} = \delta_{ab}$ or $= \delta_{ab} - \delta_{a3}\,\delta_{b3}$ depends if one is interested to conserve or break the global symmetry.

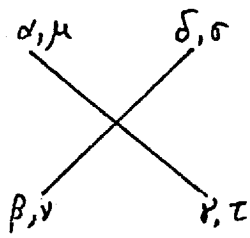
(b) Elementary vertices: each vertex has an additional factor $(2\pi)^4 i$

(i) Three Yang-Mills particles vertex



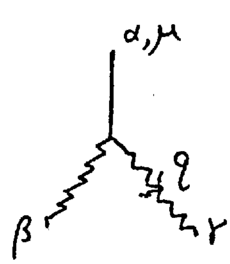
$$B_3 = -ig\,\varepsilon_{\alpha\beta\gamma}\left[g_{\nu\tau}(q-p)_\mu + g_{\mu\tau}(k-q)_\nu + g_{\nu\mu}(p-k)_\tau\right]$$

(ii) Four Yang-Mills particles vertex



$$\begin{aligned} B_4 = &-g^2 \varepsilon_{h\alpha\beta}\varepsilon_{h\gamma\delta}\left[g_{\mu\tau}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\tau}\right] \\ &-g^2 \varepsilon_{h\alpha\delta}\varepsilon_{h\gamma\beta}\left[g_{\mu\tau}g_{\nu\sigma} - g_{\mu\nu}g_{\sigma\tau}\right] \\ &-g^2 \varepsilon_{h\alpha\gamma}\varepsilon_{h\beta\delta}\left[g_{\mu\nu}g_{\tau\sigma} - g_{\mu\sigma}g_{\nu\tau}\right] \end{aligned}$$

(iii) Two ghosts and Yang-Mills particle vertex



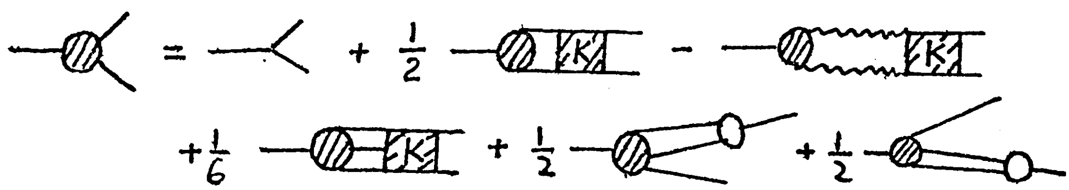
$$B_{1,2} = -ig\,\varepsilon_{\alpha\beta\gamma}\,q_\mu .$$

Appendix (2)

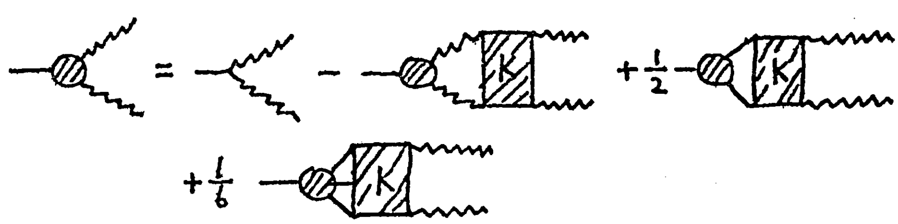
Dyson equations and Bethe-Salpeter equations

(i) We give the Dyson equations diagrammatically for the following vertices

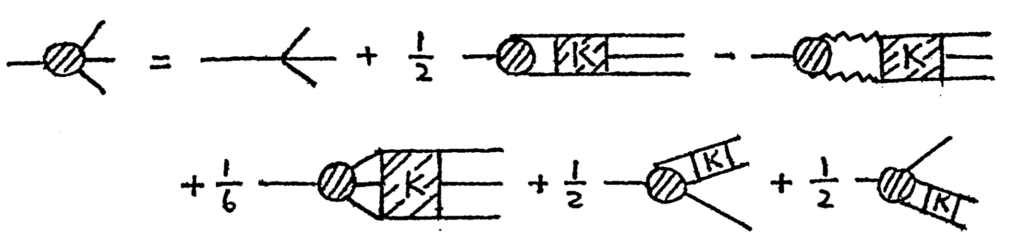
(a) Three Yang-Mills particles vertex



(b) Yang-Mills particle and two ghosts vertex

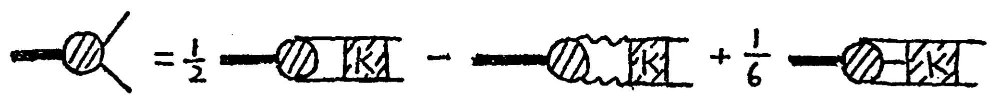


(c) Four Yang-Mills particles vertex

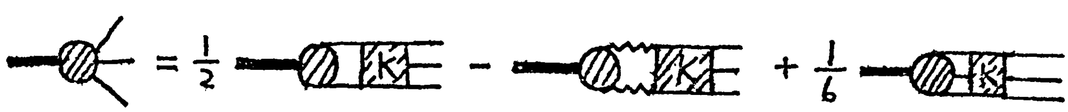


(ii) We give the derived Bethe-Salpeter equations for the following Goldstone boson coupling functions. Diagrammatically, they are

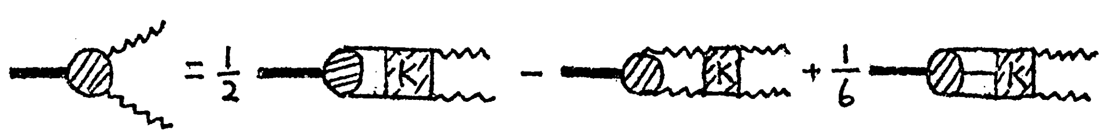
(a) Goldstone boson and two Yang-Mills particles coupling function



(b) Goldstone boson and three Yang-Mills particles coupling function



(c) Goldstone boson and two ghosts coupling function



Appendix (3)

The coefficients for the constraint equation on δ and g^2

In the self consistency computation of $P_{2\mu\nu}^{[\alpha,\beta,\gamma]}$, equation (11), the integration is performed by continuing to Euclidean space and using addition of denominators method, namely,

$$\prod_i A_i^{-r_i} = \int \left[\prod_i d\beta_i \frac{\beta_i^{r_i-1}}{(r_i-1)} \right] \frac{\delta(1-\sum_i \beta_i)(R-1)!}{[\sum_i \beta_i A_i]^R}$$

with $R = \sum_i r_i$.

In order to present the results from this integration, it is convenient to introduce the following definitions:

$$I_\alpha(d) \equiv \int \frac{d^4 r\,(d-1)!}{[r^2+\alpha(1-\alpha)p^2]^d}$$

$$C(\delta,n)\ \frac{1}{(p^2)^{\delta+1}} \equiv \int d\alpha\ \frac{\alpha(1-\alpha)^\delta(1-\alpha)^n}{\delta!} I_\alpha(\delta+3)$$

$$D(\delta,n)\ \frac{1}{(p^2)^\delta} \equiv \int d\alpha\ \frac{(1-\alpha)^\delta(1-\alpha)^n}{\delta!} I_\alpha(\delta+2)$$

and similarly,

$$I_\alpha(d,\mu\nu) \equiv \int \frac{d^4 r\,(d-1)!\,r_\mu r_\nu}{[r^2+\alpha(1-\alpha)p^2]^d}$$

$$\tilde{C}(\delta,n)\frac{g_{\mu\nu}}{(p^2)^\delta} \equiv \int d\alpha\ \frac{\alpha(1-\alpha)^\delta(1-\alpha)^n}{\delta!} I_\alpha(\delta+3,\mu\nu)$$

$$\tilde{D}(\delta,n)\frac{g_{\mu\nu}}{(p^2)^{\delta-1}} \equiv \int d\alpha\ \frac{(1-\alpha)^\delta(1-\alpha)^n}{\delta!} I_\alpha(\delta+2,\mu\nu)$$

where n, d, and δ are real numbers.

The above definitions are used to give the results from the first two diagrams in the kernel. For the contribution from the last diagram, it helps to make the following similar definitions:

$$E(\delta,n)\frac{1}{(p^2)^{\delta+1}} \equiv \int d\alpha \, \frac{\alpha^{1+\delta}}{(1+\delta)!}(1-\alpha)^n I_\alpha(\delta+3)$$

$$\tilde{E}(\delta,n)\frac{g_{\mu\nu}}{(p^2)^{\delta}} \equiv \int d\alpha \, \frac{\alpha^{1+\delta}}{(1+\delta)!}(1-\alpha)^n I_\alpha(\delta+3,\mu\nu)$$

$$F(d,n) \equiv E(d-1,n)$$

$$\tilde{F}(d,n) \equiv \tilde{E}(d-1,n) .$$

As said in the text, for P_2 to resurge after the integration operation, we arrive at two condition equations. Now, with the help of the above definitions, we can give the coefficients of these equations. They are

$$C_{11} = g^2[C(d,1) + 2C(d,3) + \tilde{C}(d) + 4\tilde{C}(d,1) - D(d) + 5D(d,1)]$$

$$C_{12} = g^2[-C(d,1) - \tilde{C}(d) + 2\tilde{C}(d,1) + D(d) + 2D(d,1)]$$

$$E_{11} = g^2[E(d,1) + 2E(d,3) + \tilde{E}(d) + 4\tilde{E}(d,1) - F(d) - 5F(d,1)]$$

$$E_{12} = g^2[-E(d,1) - \tilde{E}(d) + 2\tilde{E}(d,1) + F(d) + 2F(d,1)]$$

$$C_{21} = g^2 [5C(d,1) - 20C(d,2) + 20C(d,3) - 10\tilde{C}(d) + 40\tilde{C}(d,1)]$$

$$C_{22} = g^2 [-10\tilde{C}(d) + 20\tilde{C}(d,1) - 2D(d) + 4D(d,1) + C(d,1) - 2C(d,2) - 2\tilde{C}(d)]$$

$$F_{11} = -2g^2 [3F(d) - 2\tilde{F}(d,1)]$$

$$F_{12} = 8g^2 F(d) .$$

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Part 11 Phenomenological applications and renormalisation of scaling theory

Abstract

We explore the scaling behaviours of inclusive and exclusive processes alike using two approaches : (1) Quark model fits in well with phenomenology and the idea of anomalous dimension, which, besides its phenomenological significance, is more linked with renormalisation. (2) Operator product expansion gives generalised scaling rules on a model free basis. We also add a few remarks on the scaling rules of renormalised theories.

Introduction

Scale invariance has a rich relation to many theoretical investigations. As an asymptotic high energy symmetry of scattering amplitudes, it leads to the phenomenon of helicity conservation.[1] Group theorists can connect its algebra with improved stress energy tensor to construct stress energy tensor algebra.[2] Or, using ideas similar to P.C.A.C. , we can concern ourselves with the soft theorems of dilatons.[2] The phenomenological use of it, in fact, has been known for sometime. We often use simple power rules to describe form factors. The most well known example is, perhaps, the dipole fitting for nucleon form factors.

With the recent discovery of a simple phenomenological scaling rule for the more complex inclusive processes,[3] it is thus desirable, and it is also our aim here, to describe the scaling rules for exclusive and inclusive processes in the same theory, or using the same tools. Naturally, we begin with the principle of scale invariance. We find that its predictions for exclusive processes, implemented with the idea of anaomalous dimensions,[4] are quite successful. It shows that we can reach a consistent assignment of anomalous dimensions to the hadron fields. This assignment is in fact very agreeable with quark assignments. Thus motivated, we try for a quark description of the inclusive processes. The simplest approach is to use quark canonical anticommutator and quark descriptions of currents. Surprisingly this suffices to give the correct scaling behaviour for the inclusive processes, and, moreover, the two structure functions become related in quark models.

In order to do without any specific model, we employ Operator Product expansions[5] to the same aims, using the minimum amount of assumptions except its built-in scale invariance concept. We find that this approach is able to give the correct generalised scaling rules for both kinds of processes. It also has the advantage to identify the powers to the dimensions of the various operators. We can compare this result with the other results using the additional assumption of conformal invariance.

We give at the end the renormalisation group equation,[6] which is the replacement of the naive scaling rule in renormalised theories. We also touch on the relation between the dimensions of the coupling constants and the renormalisability of the theories.

(1) Scaling rule and exclusive processes

Before we formulate from the n-points Green's function the scaling rule for exclusive processes in which the numbers of incoming and outgoing particles are aforehand defined, let us mention briefly that some results from naive scale invariance can naturally arise from dimensional arguments using the fact that in a massless theory the dimensionless dynamical quantities become functions of dimensionless ratios of the available kinematic variables. In accordance with this idea we can deduce that the annihilation process of elctron gives σ = constant, as q is the only available kinematic variable, while inelastic scattering of electron with its two independent kinematic variables ν and q gives $\sigma = \frac{1}{q^2} f(\nu/q^2)$. That is, we can find out in this simple way the rate of decrease of the scattering cross section with energy. However, unfortunately, not all of these predictions are fulfilled, i.e. somehow naive scale invariance is broken. There are many ways to treat broken scale invariance theories. In the below we will consider mainly how anomalous dimension[4], the difference of the scale dimension from its canonical value, can implement the naive scaling rule and fit in well with the phenomenological world.

Let us thus begin with the scaling rule for the n-points Green's function. This is easily obtained from using an operator of $d + p^{\mu}\cdot\partial_{\mu}$ for each participating particle in the Green's function and assuming scale invariance. The scaling equation is

$$[n(d-4)+4-\Sigma p_{(i)}\frac{\partial}{\partial p_{(i)}}]\,G_n(p_{(1)}\cdots p_{(n-1)})=0$$

where for convenience we consider n similar particles each of which has a scale dimension d and where

$$(2\pi)^4\delta(\sum_{i=1}^{n} p_{(i)})\,G_n(p_{(1)}\cdots p_{(n-1)})\equiv\int dx_{(1)}\cdots dx_{(n)}\,e^{i\Sigma p_{(i)}x_{(i)}}\langle 0|T(\phi(x_{(1)})\cdots\phi(x_{(n)}))|0\rangle .$$

The extra 4 in this equation comes from commuting the dilation operator $\Sigma(d + p^{\mu}\cdot\partial_{\mu})$ with the momentum conserving delta-function.

We can obtain the corresponding equation for one particle irreducible Green's function after amputation of external legs,

$$[4-nd-\Sigma p_{(i)}\frac{\partial}{\partial p_{(i)}}]\,\bar{G}_n(p_{(1)}\cdots p_{(n-1)})=0 .$$

Hence in accordance with this equation, $\bar{G}_n$ is required to be a homogeneous function in momentum of order $4-nd$.

Thus, according to scale invariance M function should behave like

$$M \sim p^{4-\Sigma d_{(i)}}$$

where d_i is the scale dimension of the i participating particle. Taking into account of the contribution from the final state phase space which is $\int \prod_f \frac{d^3 p_f}{2E_f} \delta^4(\Sigma p_f - \Sigma p_{in})$ we have

$$d\sigma/d\Omega \sim (1/s)\, s^{4-\Sigma d_{(i)}}\, s^{N_f-2}$$

Hence,

$$d\sigma/dt \sim s^{2-\Sigma d_{(i)}}\, s^{N_f-2}.$$

This is the power rule predicted from scale invariance.

We can now compare this power rule with the phenomenological world, bearing in mind the the scale dimension can assume a value different from its canonical dimension. The data which can be used for this purpose are compiled by Brodsky et al[7] for some 4 particles exclusive processes and parametrised at fixed c. m. angle as $d\sigma/dt \propto 1/s^m$. The values of m for these processes are

Photon + Baryon $\longrightarrow$ Meson + Baryon :

$$m(\gamma N \longrightarrow \pi^+ N) = 7.3 \pm 0.4$$

Meson + Baryon $\longrightarrow$ Meson + Baryon :

$$m(K_L^0 P \longrightarrow K_S^0 P) = 8.5 \pm 1.4$$

$$m(\bar{K}^0 P \longrightarrow \pi^+ \Lambda^0) = 7.4 \pm 1.4$$

$$m(\bar{K}^0 P \longrightarrow \pi^+ \Sigma^0) = 8.1 \pm 1.4$$

Baryon + Baryon $\longrightarrow$ Baryon + Baryon :

$$m(pp \longrightarrow pp) = 10 \pm 2.0 \ .$$

These are based on the experiment done by R. Anderson et al[8] on large angle high energy photoproduction of single pion from liquid hydrogen at an energy range 4.0 - 7.5 Gev., and the experiment done by G. Brandenberg et al[9] on $K_L^0 p \rightarrow p K_s^0$ backward scattering in the momentum interval 1.0 - 7.5 Gev/c . A number of experimenters has contributed to the measurements of the energy power rule of the proton-proton scattering. It is best to read the paper by R. Blankenbecler et al[10] for references to these proton-proton experiments.

When we take these powers to be their closest integral values, 7, 8, 8, 8, 10 respectively, we find out that they correspond to an assignment of scale dimensions of 3 to the hadrons and of 2 to the mesons. Surely, we can turn the logics backwards and say that we have a consistent assignment of scale dimensions and it gives quite good experimental predictions. Whether it is actually very good we have to wait for more data to tell. But this assignment is interesting in a peculiar way. These numbers 3 and 2 are exactly the minimal numbers of quark components in the baryons and mesons. It seems to suggest that high energy interactions proceed via the basic entities. We shall see in fact in the next section that quark description also works for inclusive processes.

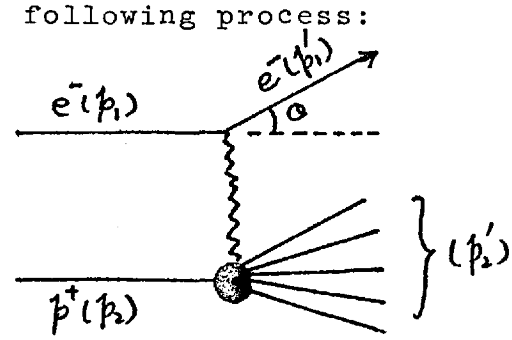
(2) Scaling rule and inclusive processes

It is very interesting that the recent data for inclusive processes[3] exhibit one very astounding feature. It is, to a good approximation, a large fraction of the data can be expressed as two functions of only one variable: the ratio of energy loss to the square of the momentum variable. This simple behaviour is usually called Scaling. Furthermore, present measurements do not exclude the possibility that we might only need one function of one variable to describe the complex inclusive processes. In this section we will find out how the quark model accounts for this amazing phenomenon.

Let us begin with the kinematics and the matrix element. For instance, let us consider the following process:

Lepton + Hadron

$\longrightarrow$ Lepton + anything



The kinematics variables are

$p_1(p_1'), p_2(p_2')$ = initial (final) momenta of lepton, hadron. The invariant variables are

$$\nu \equiv q\cdot p_2\,,\quad Q^2 \equiv -q^2 \qquad \text{with} \quad q = p_1 - p_1'.$$

As it is familiar in one photon exchange picture, the matrix element is given by

$$M = \bar{u}(p_1',\lambda')\,\gamma_\mu\, u(p,\lambda)\,(e^2/Q^2)\langle n|J^\mu(0)|p_2,\sigma\rangle$$

where λ', λ, σ, $\langle n|$, $J^\mu(0)$ are helicity indices, final state, and local current operator respectively.

Hence, the differential cross section is, after summing over all final hadron states and averaging initial spins,

$$\propto L_{\mu\nu} W_{\mu\nu}$$

where $L_{\mu\nu}$ corresponds to the lepton part, and

$$W_{\mu\nu}(p_2,q) = \int \frac{d^4x}{2\pi}\, e^{-iq\cdot x}\langle p_2|J_\mu(x)\,J_\nu(0)|p_2\rangle.$$

We can, in fact, write

$$W_{\mu\nu}(p_2,q) = \int \frac{d^4x}{2\pi}\, e^{-iq\cdot x}\langle p_2|\,[J_\mu(x), J_\nu(0)]\,|p_2\rangle. \qquad (3)$$

This is because

$$\int \frac{d^4x}{2\pi}\, e^{-iq\cdot x}\langle p_2|J_\nu(0)\,J_\mu(x)|p_2\rangle = \sum_n \langle p_2|J_\nu(0)|n\rangle\cdot$$
$$\cdot\langle n|J_\mu(0)|p_2\rangle(2\pi)^3\,\delta(p_2 - p_n - q)$$

and vanishes as in laboratory frame $E_n = M_N - q_0 < M_N$ and no baryon state exists with mass less than the nucleon mass.

Also, we can put it in terms of two invariant functions $W_1(q^2,\nu)$ and $W_2(q^2,\nu)$, the other invariant functions being eliminated because of current conservation condition and the hermiticy of the current operators,

$$W_{\mu\nu}(p_2,q) = \frac{1}{M_N^2}(p_{2\mu} - \frac{\nu}{q^2}q_\mu)(p_{2\nu} - \frac{\nu}{q^2}q_\nu)W_2(q^2,\nu) - (g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2})W_1(q^2,\nu) \quad . \tag{4}$$

The differential cross section is

$$\frac{d^2\sigma}{d\Omega' dE'} = (\frac{d^2\sigma}{d\Omega' dE'})_{Mott}\,[W_2(q^2,\nu) + 2\tan^2\frac{\theta}{2}W_1(q^2,\nu)]$$

where $\alpha = e^2/c\hbar = 1/137$ = fine structure constant,

$$(\frac{d^2\sigma}{d\Omega' dE'})_{Mott} = [\alpha^2/(4E^2\sin^4\frac{\theta}{2})]\,[\cos^2\frac{\theta}{2}(1+2\frac{E}{M_N}\sin^2\frac{\theta}{2})],$$

E, θ are the energy and scattering angle of the lepton in the laboratory frame.

The scaling behaviour of inclusive processes are usually encompassed in the statement that

$$W_1(q^2,\nu) = W_1(\omega), \quad \nu W_2(q^2,\nu) = W_2(\omega) \tag{5}$$

where $\omega \equiv Q^2/2\nu$ is a dimensionless quantity. That is, this single variable suffices to describe the structure functions which would be expected to depend on both the variables q^2 and ν . Though this was hard to imagine before it was borne out by the recent data, there is a very naive argument to account for it using scale transformation of the current operators. If we recall that

$$U(\Lambda)|p> = \Lambda|\Lambda p>$$

$$U^{-1}(\Lambda)J_\mu(x)U(\Lambda) = J_\mu'(x) = \Lambda^3 J_\mu(\Lambda x) .$$

Then

$$W_{\mu\nu}(p_2,q) = \int \frac{d^4x}{2\pi} e^{-iq\cdot x} \langle p_2 | J_\mu(x) J_\nu(0) | p_2 \rangle$$
$$= \int \frac{d^4x}{2\pi} e^{-iq\cdot x} \langle p_2 | U U^{-1} J_\mu(x) U U^{-1} J_\nu(0) U U^{-1} | p_2 \rangle$$
$$= \int \frac{d^4x}{2\pi} e^{-i\Lambda^{-1} q\cdot x} \langle \Lambda^{-1} p_2 | J_\mu(x) J_\nu(0) | \Lambda^{-1} p_2 \rangle$$
$$= W_{\mu\nu}(\Lambda^{-1} p_2, \Lambda^{-1} q).$$

Thus, $W_{\mu\nu}(p_2,q)$ should be dimensionless, and this again leads to the fact that the structure functions can be put in terms of a dimensionless quantity, $\omega = Q^2/2\nu$, in the form

$$W_1(q^2,\nu) = W_1(\omega) \ , \ \nu W_2(q^2,\nu) = W_2(\omega).$$

Now, in order to see how quark model accounts for the scaling behaviour for inclusive processes, we first of all will derive the current commutation relation from the quark anticommutation relation and the representation of the currents in terms of quark fields. They are

$$J^\mu_{\tau'}(x) = \bar{q}_\alpha(x)(\gamma^\mu \lambda_{\tau'})_{\alpha\beta}\, q_\beta(x) \equiv J^\mu_{E.M.}(x)$$
$$J^\nu_{\tau'}(x') = \bar{q}_\sigma(x')(\gamma^\nu \lambda_{\tau'})_{\sigma\sigma'}\, q_{\sigma'}(x') \equiv J^\nu_{E.M.}(x')$$
$$\{q_\tau(x), \bar{q}_\sigma(y)\} = i\, S_{\tau\sigma}(x-y) = -2\pi\, \gamma^\rho_{\tau\sigma}\, \partial_\rho [\delta((x-y)^2)\, \varepsilon(x-y)_0]$$

where

$$\lambda_{\tau'} \equiv \frac{\lambda_3}{2} + \frac{1}{\sqrt{3}}\,\frac{\lambda_8}{2}\ .$$

On using

$$\gamma^\mu \gamma^\rho \gamma^\nu = \varepsilon^{\mu\rho\nu\delta}\,\gamma_5\,\gamma_\delta + g^{\mu\rho}\gamma^\nu - g^{\mu\nu}\gamma^\rho + g^{\rho\nu}\gamma^\mu$$

we can obtain

$$[\,J^\mu_{E.M.}(x),\, J^\nu_{E.M.}(x')\,] = -2\pi\,\partial_\rho\,[\,\delta(\zeta^2)\,\varepsilon(\zeta_0)\,]\,\{\qquad\}$$

where

$$s^{\mu\nu\rho\sigma} \equiv g^{\mu\rho} g^{\nu\delta} - g^{\mu\nu} g^{\rho\delta} + g^{\rho\nu} g^{\mu\delta}$$

$$J_{5\delta,E.M.}(x,x') \equiv \bar{q}_\alpha(x)\,[\gamma_5\gamma_\delta(\frac{\lambda_3}{2} + \frac{1}{\sqrt{3}}\,\frac{\lambda_8}{2})]_{\alpha\beta}\, q_\beta(x')$$

and similarly for $J_{\delta,E.M.}(x,x')$.

Here

$$\{\qquad\} \equiv \{\,\frac{1}{3}\,\varepsilon^{\mu\nu\rho\delta}\,[\,J_{5\delta,E.M.}(x,x') + J_{5\delta,E.M.}(x',x)\,]$$

$$+ \frac{1}{3}\, s^{\mu\nu\rho\delta}\,[\,J_{\delta,E.M.}(x,x') - J_{\delta,E.M.}(x',x)\,]$$

$$+ \frac{2}{9}\,[\,\bar{q}(x)\,\gamma^\mu\gamma^\rho\gamma^\nu q(x') - \bar{q}(x')\,\gamma^\mu\gamma^\rho\gamma^\nu\, q(x)\,]\,\}\ .$$

This formula is also derived by Gellmann and Fritsh[11] for the general SU(3) currents.

We can now sandwich this commutator between equal momenta proton states. We have

$$\langle p | [J^{\mu}_{E.M.}(x), J^{\nu}_{E.M.}(x')] | p \rangle$$
$$= \frac{1}{4\pi} \partial_\rho (\varepsilon(z_0) \delta(z^2)) \, s^{\mu\nu\rho\sigma} \langle p | O_\sigma(x,x') | p \rangle$$
$$= \frac{1}{4\pi} \partial_\rho (\varepsilon(z_0), \delta(z^2)) \, s^{\mu\nu\rho\sigma} \langle p | O_\sigma(z,0) | p \rangle \ , \ z \equiv x - x'.$$

The last line is because of translation invariance. The bilocal operator $O_\sigma(x,x')$, introduced above, is proportional to $J_{\sigma,E.M.}(x,x') - J_{\sigma,E.M.}(x',x)$ and hence is analytic at $z = 0$. That is, the expression $\partial_\rho(\varepsilon(z_0)\delta(z^2))$ gives the leading singularity of the current commutator between equal momenta states. We can thus write

$$\langle p | [J^{\mu}_{E.M.}(x), J^{\nu}_{E.M.}(x')] | p \rangle$$
$$= \frac{1}{4\pi} \partial_\rho (\varepsilon(z_0) \delta(z^2)) s^{\mu\nu\rho\sigma} a(pz) p_\sigma + \text{ less singular terms.}$$

Now we should compute for its fourier transform. In its computation we need to make use of a lemma:[1]

$$\varepsilon((q+\xi p)_0)\, \delta((q+\xi p)^2) = \varepsilon(\nu)\, \delta(q^2 + 2\xi\nu) \quad \text{for } +\text{ve } p_0 .$$

This lemma is true, because

$$\text{L.H.S.} = \varepsilon((q+\xi p)_0 p_0) \delta((q+\xi p)^2) \text{ for positive } p_0$$

$$= \varepsilon(q_0 p_0 + \xi p_0^2)\,\delta((q+\xi p)^2)$$

$$= \varepsilon(q\cdot p + \xi p^2)\,\delta(q^2 + 2\xi q\cdot p) \quad \text{on neglecting } \xi^2 \text{ term}$$

$$= \varepsilon(q\cdot p - \frac{q^2 p^2}{2q\cdot p})\,\delta(q^2 + 2\xi p\cdot q)$$

$$= \varepsilon(\nu)\,\delta(q^2 + 2\xi\nu) \qquad \text{on recalling } \nu \equiv q\cdot p\,.$$

On introducing the fourier transform of $a(p\cdot z)$,

$$A(\xi) \equiv \int e^{i\xi(p\cdot z)} a(p\cdot z)\, d(p\cdot z),$$

we can now give the fourier transform of the matrix element of the current commutator. It is

$$-\frac{\pi}{2\nu} s^{\mu\nu\rho\sigma}(q_\rho p_\sigma A(\omega) + p_\rho p_\sigma \omega A(\omega)).$$

This expression, compared with equation (4) which defines the structure functions $W_1(\nu, q^2)$ and $W_2(\nu, q^2)$, gives

$$W_1(\nu, q^2) + \frac{\nu^2}{q^2}\frac{W_2(\nu, q^2)}{M_N^2} = 0\,, \quad \nu W_2(\nu, q^2) \propto \omega A(\omega)\,.$$

Thus, obviously we have the scaling phenomenon

$$W_1(q^2, \nu) = W_1(\omega)\,, \quad \nu W_2(q^2, \nu) = W_2(\omega)\,.$$

Moreover, surprisingly the two structure functions become correlated in quark model. This extra prediction of the quark model can be interpreted in the following way.

From above it should be noticed that $W_{\mu\nu}$ is proportional to the imaginary part of forward (virtual) photon-nucleon scattering, which, by the optical theorem, is proportional to the total cross section for photon on nucleon. Hence, we can view the inelastic electron scattering in terms of the processes $'\gamma' + N \longrightarrow \text{Hadrons}$. Now as the incident photon is a virtual particle, it can have any energy, mass and polarisation. In contrast to real photon processes which are characterized by σ_T, the virtual photon processes have σ_T and σ_S, the cross sections corresponding to transversely and longitudinally polarised photons. The relations[(3)] between the structure functions W_1, W_2 and these cross sections can be easily worked out. They are

$$W_1 = K \sigma_T$$

$$W_2 = \frac{K q^2}{q^2 + \nu^2} (\sigma_T + \sigma_S)$$

where K is an unimportant constant factor.

Thus, the relation given by Quark model

$$W_1(\nu, q^2) + \frac{\nu^2}{q^2} \frac{W_2(\nu, q^2)}{M_N^2} = 0$$

amounts to predicting that $\sigma_S = 0$ as $-q^2 \to \infty$. Recent experimental data indicate a reasonably low value $R = \sigma_S / \sigma_T \cong 0.18$ when the invariant mass W for the final hadron states ≥ 2 Gev . This result is also given by the parton model.[11] It is interesting that it comes out simply from the quark anticommutation relation. When we work out the generalised scaling rules in the operator product expansion method, it seems that this result $\sigma_S = 0$ as $-q^2 \to \infty$ is lost.

(3) Operator product expansions

The basic assumption of operator product expansion[5] at short distance is that, to a certain accuracy, we can represent an operator product A(x)B(y) in an expansion of the form

$$A(x)B(y) = \sum_n c_n(x-y)\, O_n(y)$$

with the c functions $c_n(x-y)$ containing all the singularities as $x \to y$. On applying scale transformation to this expansion and using linear independence of the local operators $O_n(y)$, we can easily deduce that

$$c_n(ax-ay) = a^{-d_A-d_B+d_n}\, c_n(x-y).$$

This is on the assumption of scale invariance and the fact that each operator has a scale dimension d .

The application of operator product expansion to inclusive processes was well worked out by Frishman.[11] Let us reproduce it here to illustrate the principles. We will later apply the same techniques to derive the power rules for exclusive processes.

For the inclusive processes, let us write down the leading term, the term with the leading singularity as $x \to 0$, in the current commutator

$$[J_\mu(x), J_\nu(0)] = C(x) F_{\mu\nu}(x,0) + \cdots$$

where $F_{\mu\nu}(x,0)$ is a bilocal operator which is analytic at $x=0$. The form of it is irrelevant. However, for easy comparison with the previous definitions of the structure functions, we write

$$\langle p | [J_\mu(x), J_\nu(0)] | p \rangle = c(x) \langle p | F_{\mu\nu}(x,0) | p \rangle$$

$$= t_{\mu\nu\rho\sigma}\, c_2(x)\, p^\rho p^\sigma f_2(p\cdot x) + [(\partial_\mu \partial_\nu - \partial_{\mu\nu} \partial^2)/\partial^2]\, c_1(x) f_1(p\cdot x)$$

with

$$c_i(x) \equiv (-x^2 + i\epsilon x_0)^{d_i} - (-x^2 - i\epsilon x_0)^{d_i}$$

$$t_{\mu\nu\rho\sigma} \equiv \frac{1}{\pi_i}[2 g_{\mu\nu} \partial_\rho \partial_\sigma - g_{\rho\mu} \partial_\nu \partial_\sigma - g_{\sigma\mu} \partial_\nu \partial_\rho - g_{\sigma\nu} \partial_\mu \partial_\rho - g_{\mu\sigma} g_{\nu\rho} \partial^2 - g_{\nu\sigma} g_{\mu\rho} \partial^2] \frac{1}{\partial^2}$$

and it can be easily seen from dimensional arguments that

$$d_2 = d_1 + 1 \; ,$$

Of course, we need the fourier transform of this expression in order to obtain the structure functions. Thus, let us introduce

$$f(p\cdot x) = \int d\lambda\, g(\lambda)\, e^{i\lambda p\cdot x}$$

and hence we have

$$W_i(q^2,\nu)=\frac{1}{2\pi^2 i}\int d^4x\; e^{-iq\cdot x}\left[(-x^2+i\epsilon x_0)^{d_i}\right.$$
$$\left.-(-x^2-i\epsilon x_0)^{d_i}\right] f_i(p\cdot x). \qquad (6)$$

Now we will make use of some mathematical properties:

$$\int d^4x\; e^{ik\cdot x}\left[(-x^2+i\epsilon x_0)^d-(-x^2-i\epsilon x_0)^d\right]$$
$$=\frac{\pi^2}{i}2^{2d+4}\frac{\Gamma(d+2)}{\Gamma(-d)}\left[(-k^2+i\epsilon k_0)^{-d-2}-(-k^2-i\epsilon k_0)^{-d-2}\right]$$
$$=\frac{-2^{2d+5}\pi^3}{\Gamma(-d)\Gamma(-d-1)}\,\varepsilon(k_0)\,\theta(k^2)\,(k^2)^{-d-2}.$$

and also a lemma:

$$\varepsilon((q+\xi p)_0)\,\delta((q+\xi p)^2)=\varepsilon(\nu)\,\delta(q^2+2\xi\nu) \quad \text{for } +\text{ve } p_0 .$$

(This lemma is proved in the previous section). Then we find out that

$$W_i(q^2,\nu)\sim\int d\lambda\, g(\lambda)\,\varepsilon(\nu+\lambda M_N^2)\,\theta\left[\varepsilon(\nu)(\lambda-\omega)\right]\left[\lambda-\omega\right]^{-d_i-2}(2\nu)^{-d_i-2}, \qquad (7)$$

where $\omega\equiv Q^2/2\nu$ as before. From here it becomes apparent that $W_i(q^2,\nu)\cdot\nu^{d_i+2}$ is a dimensionless function which is dependent only on the ratio of energy loss to momentum transfer. Thus, we have obtained some generalised scaling rules for the inclusive processes. The advantage of this approach is that we did not make any assumptions about the nature of the bilocal operator $F_{\mu\nu}(x,0)$. Consequently the results here are model independent. If we wish to relate these to the Bjorken scaling, i.e. $W_1(q^2,\nu)$ and $\nu W_2(q^2,\nu)$ are dimensionless functions of ω, we have to assign $d_1=-2$ and $d_2=-1$. This assignment in turn indicates that the currents maintain their natural dimensions +3.

If we look at the current algebra commutation relations, we also see that the currents should have dimensions +3 even in presence of interactions.

Let us now use the same approach to deal with exclusive processes. To be specific, let us use the same techniques to derive in below the decreasing power rules of the electromagnetic form factors of nucleons.

Let us first consider the vertex $N+\gamma \rightarrow N_{vir}$ This semi-amplitude is described by

$$I_m \langle p| \int d^4y \; e^{iq\cdot y} T(J_\mu(y)\psi(0))|0\rangle = p_\mu W(\nu,\omega) + q_\mu \cdots$$

Now as before let us write

$$\langle p| T(J_\mu(y)\psi(0))|0\rangle \sim p_\mu c_1(y) f_1(p\cdot y) + \partial_\mu c_2(y) f_2(p\cdot y) \qquad (8)$$

where

$$c_i(y) \equiv (-y^2 + i\epsilon(y_0))^{d_i}, \quad i = 1,2 .$$

If we take the fourier transform of this expression, we can easily obtain that

$$\begin{aligned} W(\nu,\omega) &= I_m \int d^4y \; e^{iq\cdot y} c_1(y) f_1(p\cdot y) \\ &= I_m \iint d^4y \; d\lambda \; e^{iq\cdot y} g(\lambda) e^{i\lambda(p\cdot y)} c_1(y) \\ &= I_m \iint d^4y \, d\lambda \; e^{i(q+\lambda p)\cdot y} g(\lambda)\, c_1(y) \end{aligned}$$

where $g(\lambda)$ is the fourier transform of $f_1(p\cdot y)$.

We can use the mathematical property:

$$\mathrm{disc}\,(p^2+i0)^d \propto (p^2+i0)^d-(p^2-i0)^d$$

and hence we obtain, aside some unimportant constant factor,

$$W(\nu,\omega)=\mathrm{Im}\int d\lambda\, g(\lambda)\left(-(q+\lambda p)^2-i\varepsilon(q+\lambda p)_0\right)^{-d_1-2}$$

$$\propto\int d\lambda\, g(\lambda)\,\varepsilon(q_0+\lambda p_0)\,\theta(q^2+2\lambda p\cdot q)\,[q^2+2\lambda p\cdot q]^{-d_1-2}$$

On introducing ω and using the same lemma as before, we have

$$\nu^{d_1+2}W(\nu,\omega)\propto\int_{-\infty}^{\infty}d\lambda\, g(\lambda)\,\theta(\varepsilon(\nu)(\lambda-\omega))\,\varepsilon(\nu+\lambda M_N^2)[\lambda-\omega]^{-d_1-2} \quad (9)$$

with $1\leq\omega\leq\infty$, $\nu>0$ in physical region.

This does not give us the power rule yet, but we are not very far from it. Now we assume that $g(\lambda)$ is a regular function and the integral vanishes at infinity. Then we can write, on changing variable $\lambda'=\lambda-\omega$,

$$\nu^{d_1+2}W(\nu,\omega)\propto\int_{1-\omega}^{\infty}d\lambda'\, g(\lambda'+\omega)\,\theta(\lambda')(\lambda')^{-d_1-2}d\lambda'$$

and, on expanding $g(\lambda)$,

$$\nu^{d_1+2}W(\nu,\omega)\propto(1-\omega)^{-d_1+k-2}$$

for some power k and for ω close to 1. This is true when we neglect higher powers in $(1-\omega)$.

In order to obtain the power rule for form factors from $W(\nu,\omega)$, it is necessary to put it back in terms of the energy variable s and the four momentum transfer q. This can be done by observing that

$$\begin{aligned} 1-\omega &= 1 - Q^2/(2\nu) \\ &= (2\nu+q^2)/(2\nu) \\ &\cong (M_N^2+2\nu+q^2)/(2\nu) \\ &= s/(2\nu) \qquad \text{for large } -q^2. \end{aligned}$$

Hence,

$$W(\nu,\omega) \sim (q^2)^{-k}(s)^{-d_1+k-2}$$

for large $-q^2$ and ω close to 1.

But

$$W = I_m\, F_N(q^2)\, G(s)$$

where $F_N(q^2)$ is the nucleon form factor and $G(s)$ is the two points Green's function. Also, we can assume that the formula $G(s) \sim (s)^{d_N-2}$ for large s is valid up to s close to M . Then we are able to eliminate the unknown value k in equation (9) and obtain the power rule for the form factor:

$$F_N(q^2) \sim (q^2)^{-(d_N+d_1)}. \tag{10}$$

We can observe from here that the power of decrease is not determined by the dimension of the nucleon field alone. It is also determined by d_1 , which can be computed by counting dimensions in the expansion, eqn.(8). Upon taking $\dim\langle p|\psi(0)|0\rangle = 1$, we have

$$d_1 = -\frac{d_J}{2} .$$

Hence, we conclude that the decreasing power law of the nucleon form factor is given as

$$F_N(q^2) \sim (q^2)^{-(d_N - \frac{d_J}{2})} .$$

We can compare this result with that obtained by Migdal[12] using conformal invariance,

$$F_N(q^2) \sim (q^2)^{-(d_N - \frac{3}{2})} .$$

That is, the two results are the same if we make an additional assumption that $d_J = 3$, which was in fact used by Migdal in his conformal invariant expansion to $\langle p|\psi(0)|0\rangle$. It is interesting that we have obtained the above result from using scale invariance alone. For a discussion about incorporating conformal invariance into operator product expansions one should refer to the work of Ferrara et al.[13] We shall now turn our attention to a description of the scaling rules in the renormalised theories.

(4) Renormalisation

In order to complete our study of scale invariance, we shall rederive the renormalisation group equation[6] in below, which, replacing the naive scaling rule, describes the behaviour of the renormalised Green's functions upon scaling the momenta. Before doing this, we will first show how the parameters change with μ, the point of renormalisation. The approach that we will employ is that of G. 't Hooft,[14] only slightly modified with effective use of $\mu\frac{\partial}{\partial\mu}$ on the various equations. In this method, namely, dimensional regularisation method, the bare quantities (g_B, ϕ_B, M_B) are expanded in terms of the n-4 poles and the renormalised quantities (g_R, ϕ_R, M_R) which are chosen to be dimensionless and analytic in the dimension of space and time n . Their expansions take the forms

$$g_B\mu^{n-4} = g_R + \sum_{\nu=1}^{\infty}\frac{\alpha_\nu(M_R, g_R)}{(n-4)^\nu}, \tag{11}$$

$$M_B\mu^{-1} = M_R + \sum_{\nu=1}^{\infty}\frac{\beta_\nu(M_R, g_R)}{(n-4)^\nu}. \tag{12}$$

From these expansions we can consider the following scaling behaviours.

(a) Scaling behaviours of parameters

In order to find out the scaling behaviour of g_R, let us differentiate equation (11) with $\mu\frac{\partial}{\partial\mu}$. Introducing

$\mathcal{A}_{\nu,x} \equiv \dfrac{\partial \mathcal{A}_\nu}{\partial x}$,we have

$$g_R(n-4) + \sum_{\nu=1}^{\infty} \frac{a_\nu(M_R, g_R)}{(n-1)^{\nu-1}} = \mu\frac{\partial}{\partial\mu} g_R + \sum_{\nu=1}^{\infty} \frac{1}{(n-4)^\nu}\left(a_{\nu,M_R}\mu\frac{\partial M_R}{\partial\mu} + a_{\nu,g_R}\mu\frac{\partial g_R}{\partial\mu}\right) \tag{13}$$

Now as g_R, M_R are analytic in n, so are $\mu\frac{\partial M_R}{\partial\mu}$, $\mu\frac{\partial g_R}{\partial\mu}$. Thus before matching poles in equation (13), it is convenient to write

$$\mu\frac{\partial}{\partial\mu} g_R = g_R(n-4) + a$$

for some quantity a. This quantity a is, in fact, determined upon matching the first order pole. It is

$$a = a_1 + g_R\, a_{1,g_R} \tag{14}$$

Hence, we obtain

$$\mu\frac{\partial}{\partial\mu} g_R = \left(1 - g_R\frac{\partial}{\partial g_R}\right) a_1 \tag{15}$$

which describes the variation of g_R with μ .
Also, we have a recurrence relation between the residue a_ν and its next residue $a_{\nu+1}$,

$$a_{\nu+1} = a_{\nu,M_R}\left(\mu\frac{\partial M_R}{\partial\mu}\right) + a_{\nu,g_R}\left(-g_R a_{1,g_R} + a_1\right) + g_R a_{\nu+1,g_R} \tag{16}$$

Similarly, we can derive the scaling behaviour of M_R on differentiating equation (12) with $\mu\frac{\partial}{\partial\mu}$ and using equation (14). The result is

$$\mu\frac{\partial}{\partial\mu} M_R = -\left(M_R + g_R\frac{\partial}{\partial g_R}\mathcal{B}_1\right) \tag{17}$$

for the scaling behaviour of M_R. The recurrence relation among $\mathcal{B}_\nu$ and its next residue $\mathcal{B}_{\nu+1}$

$$g_R \mathcal{B}_{\nu+1, g_R} = \mathcal{B}_{\nu, M_R}(M_R + g_R \mathcal{B}_{1, g_R}) + \mathcal{B}_{\nu, g_R}(g_R \mathcal{a}_{1, g_R} - \mathcal{a}_1) - \mathcal{B}_\nu \,. \tag{18}$$

The difference in forms of equation (15) and (17) arises from the different dimensions of their bare parameters. We can see this more clearly in a theory with many parameters λ^k. Suppose that each of them has a dimension

$$D^k = \alpha^k (n-4) + \beta^k$$

and an expansion

$$\lambda^k_B \mu^{-D^k} = \lambda^k + \Sigma\, C^k_\nu(\lambda)/(n-4)^\nu.$$

We can do similar work as above and obtain for the scaling behaviour of λ^k,

$$\mu \frac{\partial}{\partial \mu} \lambda^k = -\beta^k \lambda^k - \alpha^k C^k_1 + \sum_\ell \alpha^\ell \lambda^\ell C^k_{1,\lambda^\ell} \tag{19}$$

and the recurrence relation between the successive residues

$$-\alpha^k C^k_{\nu+1} - \beta^k C^k_\nu = \sum_\ell \left[-\alpha^\ell \lambda^\ell C^k_{\nu+1,\lambda^\ell} + (\mu \frac{\partial}{\partial \mu} \lambda^\ell) C^k_{\nu,\lambda^\ell} \right] \tag{20}$$

Obviously, we can recover the equations for M_R by putting $\alpha = 0$, $\beta = 1$. The case for g_R corresponds to $\alpha = -1$, $\beta = 0$.

From these equations, we conclude that the scaling behaviour of a renormalised parameter is given completely by the residue of its first order pole. In fact, because of the recurrence relations, the residues of the higher order poles are determined by the lowers, i.e. we have a pole algorithm.[14]

It is also interesting to make the following observations.

(a)′ We can employ equation (15) to describe g_R in ϕ^4 theory. Suppose that we expand a_1 in power series in g_R, i.e.

$$a_1 = a_{11} g_R + a_{12} g_R^2 + a_{13} g_R^3 + \cdots .$$

We then have

$$\mu \frac{\partial}{\partial \mu} g_R = - a_{12} g_R^2 + O(g_R^3).$$

The value of a_{12} is given by G.'t Hooft.[15] It is

$$a_{12} = \frac{-3}{16\pi^2} .$$

If we write $\mu' = \mu(1+\varepsilon)$ and $\frac{1}{\mu}\delta\mu = \varepsilon$, then the scaling behaviour of g is given by

$$\frac{\partial g_R}{\partial \varepsilon} = \frac{3}{16\pi^2} g_R^2 + O(g_R^3). \tag{21}$$

It is also well known that we can do renormalisation by the momentum cut off method. The coupling parameter g_k in that method is shown[4] to depend on the cut-off momenta k as

$$g_{k'} = g_k + [g_k^2/(4\pi^2)][\ln(k'^2/k^2)] + O(g_k^3)$$

Again, introducing $k'^2 = (1+2\varepsilon)k^2$, we have

$$dg_k/d\varepsilon = [9/(2\pi^2)] g_k^2 + O(g_k^3). \tag{22}$$

Hence, the two renormalisation methods give the same scaling behaviour, up to and including second order in g_R, for g_R in ϕ^4 theory, (noting that $g_R = g_k \cdot 4!$ from their definitions) .

(b)$'$ We can also consider the simplest solutions for the residues C_ν^k of the parameter λ^k . For instance, we can have, from solving the recurrence relations,

(i) constant solutions where

$$\begin{aligned} C_{\nu+1}^k &= 0 \qquad , \text{ when } \alpha_k = 0 \text{ or } \beta_k = 0. \\ &= (-\beta^k/\alpha^k)^\nu C_1 \qquad , \text{ when } \alpha_k \neq 0 \text{ and } \beta_k \neq 0. \end{aligned}$$

(ii) linear solutions where

$$C_\nu^k = \sum_\ell d_{\nu;\ell}^k \lambda_\ell \, .$$

Here $d_{\nu;\ell}^k$ are constant coefficients which must vanish when $\alpha^\ell \neq \alpha^k$.

(iii) solutions which involve the minimum number of parameters when permissible. For instance, we can choose

$$\begin{aligned} C_\nu^k &= f_\nu^k(\lambda_1) \qquad , \text{ when } \beta^k = 0 . \\ &= f_\nu^k(\lambda_1)\lambda^k \qquad , \text{ when } \beta^k \neq 0 . \end{aligned}$$

For example, it is possible to choose the renormalised coupling constant g_R , the renormalised M_R , and the renormalised wave function in the following way:

$$\mu \frac{\partial g_R}{\partial \mu} \equiv \beta g(R)$$

$$\mu\frac{\partial}{\partial\mu}\ln\frac{M_B}{M_R} \equiv \mu\frac{\partial}{\partial\mu}\ln Z_M \equiv \gamma_M(g_R)$$

$$\mu\frac{\partial}{\partial\mu}\ln(\phi_B/\phi_R)^{\mathbb{L}} \equiv \mu\frac{\partial}{\partial\mu}\ln Z_\gamma \equiv \gamma_\gamma(g_R), \quad \mathbb{L}=\text{integer}.$$

That is, they are chosen to depend only on g_R. This choice is called mass independent renormalisation and is very convenient to illustrate the scaling behaviour of the Green's function.

(b) Scaling behaviour of Green's functions.

We can easily derive this with the above mentioned renormalisation method, mass independent renormalisation method.[16] First, let the $\mathbb{L}$ points renormalised Green's function be defined as

$$T_R(p, g_R, M_R, \mu) \equiv \lim_{n\to 4} \tilde{T}_R(p, g_R(n), M_R(n), \mu, n)$$

where $\tilde{T}_R(p, g_R(n), M_R(n), \mu, n)$ can be obtained from the unrenormalised Green's function $T_u(p, g_B(n), M_B(n), n)$ by a suitable factor. That is

$$\tilde{T}_R(p, g_R(n), M_R(n), \mu, n) = Z_\gamma(g_R, n)\, T_u(p, g_B(n), M_B(n), n)$$

with $Z_\gamma = (\phi_B/\phi_R)^{\mathbb{L}}$. If we differentiate this equation with respect to μ, and then put n=4, we have

$$[\mu\frac{\partial}{\partial\mu}\Big|_{\text{fixed } g_R, M_R} + \beta(g_R)\frac{\partial}{\partial g_R} - \gamma_M(g_R)M_R\frac{\partial}{\partial M_R} - \gamma_\gamma(g_R)]\, T_R = 0 .$$

Also, from dimensional argument, we have another equation for T_R ,

$$(K\frac{\partial}{\partial K}+\mu\frac{\partial}{\partial\mu}+M_R\frac{\partial}{\partial M_R}-\delta)\,T_R=0\,.$$

where δ gives the appropriate dimension of T_R in mass. Eliminating $\mu\frac{\partial T_R}{\partial\mu}$ from these two equations, we have

$$[K\frac{\partial}{\partial K}-\beta(g_R)\frac{\partial}{\partial g_R}+(1+\gamma_M(g_R))M_R\frac{\partial}{\partial M_R}-\delta+\gamma_r(g_R)]\,T_R(Kp,g_R,M_R,\mu)=0\,.$$

This equation thus describes the behaviour of $T_R(Kp,\cdots)$ upon rescaling the momenta by a factor K .

In this equation, well known as the renormalisation group equation, we can see the following points:

(i) The explicit appearance of the $\partial/\partial g_R$ term indicates clearly that the Green's functions should have correlated asymptotic dependence on momentum and coupling parameter,

(ii) If we look at the part other than the linear differential operators in this equation, there is a term $\gamma_r(g_R)$ in addition to the usual dimension δ . Thus, here we have an explicit reason for anomalous dimension. It arises from renormalisation of fields.

(iii) In gauge theories, we should include a gauge dependent term in the above equation. This is necessary because, in gauge theory, the two points Green's function is given, in terms of the gauge parameter $\tilde{\alpha}$ and the self energy $\pi(k^2)$, as

$$D_{\mu\nu} = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \frac{1}{\pi(k^2) + k^2} + \tilde{\alpha} \frac{k_\mu k_\nu}{k^4} .$$

As the $\tilde{\alpha}$ term has no explicit dependence on the coupling parameter g_R, we must include variation of $\tilde{\alpha}$ to accompany scaling of momenta. The only exception is when $\tilde{\alpha} = 0$, that is, except unitary gauge.

There are many uses of the renormalisation group equation. The most widely used application is to improve calculations from perturbation theories. However, as this will take us away from the scope of scale invariance, we will consider in below another connected aspect: the relation of the dimension and the renormalizability of a theory.

It is usually assumed that the renormalizability of a theory depends on the dimension of the coupling constant. It is said that when the dimension of the coupling constant is greater than, equal, less than, zero the theory is correspondingly super-renormalisable, renormalisable, and non-renormalisable. However, this is not necessarily true. The first counter example we know of is scalar Q.E.D.[17], where, to renormalise the theory a term Φ^4 has to be introduced into the Lagrangian. Also, recently we find that upon taking out the anomalous magnetic moment term from the Lagrangian of pure massless Yang-Mills theory the theory then becomes unrenormalisable. Thus it appears that, besides the dimension requirements, sometimes we need to add some new interaction terms to the Lagrangian in order to make it renormalisable.

Conclusion

We shall now summarize the various conclusions from these sections.

When allowing the scale dimensions to assume values different from their canonical values, we are able to arrive at a consistent assignment of dimensions to the various hadronic fields and give the correct power rules for those exclusive fixed angle high energy processes with available data. The scale dimensions for baryons and mesons are found to be 3 and 2 respectively, which are exactly the minimal numbers of quark components in these hadrons. This has the interesting suggestion that in high energy region where scale symmetry becomes an exact symmetry the hadrons decompose into their basic entities, quarks. The quark description of inclusive processes is also pleasing. It gives 'scaling': as $\nu, -q^2 \to \infty$, $W_1(q^2,\nu), \nu W_2(q^2,\nu)$ become non-trivial functions of the dimensionless ratio $\omega = -q^2/2\nu$ only, rather than functions of both ν and q^2 as 'would-be' the case. Moreover, it gives a relation between W_1 and W_2, which corresponds to $\sigma_S = 0$ as $-q^2 \to \infty$. These results are also obtained by others using complex parton models. It is surprising that they come out simply from the quark anticommutation relation.

When we describe these scaling phenomena using operator product expansions, we use the same simple form of singular function $C_i(y)$ for both kinds of processes. This approach involves the minimum amount of assumption: the assumption of scale invariance. It has the advantage to identify the powers to the dimensions of the operators. It shows that Bjorken scaling corresponds to requiring the dimension of the current equal to 3. Interestingly enough, this is also the only additional assumption we need to make in order to give the same result for the nucleon form factor power rule as from conformal invariance where this assumption is used.

The last section on renormalisation reveals that the existence of anomalous dimension is connected with renormalisation of fields. As the latter depends, in general, on the coupling strength, we have reasons to believe that scale dimension becomes a dynamical entity in renormalised field theories.

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Part 111 A joint paper on Conformal invariance and helicity conservation

Abstract

By reformulating conformal invariance in terms of differential operators acting directly on helicity states we are able to establish the restrictions placed by this invariance on the helicity amplitudes for the scattering of four particles of arbitrary spins. The result is helicity conservation in the form $\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4$ except for exceptional amplitudes $T_{\lambda,\lambda;-\lambda,-\lambda}$, which survive, subject, however, to a differential constraint. It is conjectured that traces of these restrictions will survive in hadron physics at fixed angle and high energy if indeed the underlying dynamics is asymptotically free.

I. Introduction.

The phenomenon of (s-channel) helicity conservation, which seems to be experimentally verified[1] in πN scattering and ρ^0 photoproduction, can be connected up with a number of different theoretical considerations. At the NN vertex the Dirac-type coupling of the Pomeron necessary for helicity conservation can be related by a chain of ideas including f-dominance of the Pomeron[2] and exchange degeneracy[3] to the approximate vanishing of the nucleon isoscalar magnetic moment. Alternatively the required minimal coupling of tensor mesons can be derived from tensor dominance of the matrix elements of the stress-tensor[4].

The above considerations apply to the Regge region of large s and fixed t. In the fixed angle regime, where, however, the helicity structure has not yet been experimentally explored, different theoretical considerations turn out again to be connected with helicity conservation.

These considerations arise from conformal invariance, which Gross and and Wess[5] have shown to imply (in the massless limit) helicity conservation for scalar-spinor and scalar-vector scattering. The relationship between the two was probed a little further by the present authors[6] using the method of Gross and Wess to investigate spinor-spinor scattering, where it was found that the double-helicity-flip amplitude $T_{\frac{1}{2},\frac{1}{2};-\frac{1}{2},-\frac{1}{2}}$ was not constrained to vanish by conformal invariance.

The purpose of the present paper is to identify the precise connection between helicity conservation and conformal invariance hinted at in these special cases. To do this we recast the formalism so that the infinitesimal conformal generators are represented by differential operators acting directly on helicity states (Sec. 2). In the following Section these are applied to the four-particle scattering amplitude. The spinless case is treated first, and it is shown (c.f. ref. 5) that Lorentz and scale invariance are sufficient to guarantee conformal invariance. For particles with spin we model the treatment on the spinless case, and find that extra restrictions on the helicity amplitudes are now required to satisfy conformal invariance. The final result is helicity conservation in the form $\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4$, with the exception of the possibility of double helicity flip: $T_{\lambda,\lambda;-\lambda,-\lambda} \neq 0$.

These results are derived under the assumption of strict conformal invariance for massless particles with canonical dimensions. In the last Section we touch on the modifications which might be expected when the high-energy behaviour is governed by equations of the Callan-Symanzik type rather than by strict conformal invariance.

The Appendix sets out the (unresolved) problem of deducing the spin 1 Fock-space conformal operators from the auxiliary operators in the usual A_μ rather than $F_{\mu\nu}$ basis.

II. Helicity Formalism for Conformal Operators

Differential operators which represent the infinitesimal generators of the conformal group and act on the auxiliary space[7] of fields transforming according to simple representations of the Lorentz group have been written down by many authors[5,8,9]. The momentum-space version of these operators has been used by Gross and Wess[5] to obtain the restrictions of conformal invariance on the form of the M-function in some simple cases, where the results turned out to imply helicity conservation (See, however, ref. 6).

In order to investigate the connection in more generality it would clearly be advantageous to recast the formalism so that the conformal generators were represented instead by operators acting on the physical space of helicity states or creation and annihilation operators: this is the purpose of the present Section.

In x-space the auxiliary operators referred to above take the form[9]

$$m_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \Sigma_{\mu\nu} \qquad (1)$$

$$d = i(d + x.\partial) \qquad (2)$$

$$k_\mu = i(2x_\mu d + 2x_\mu x.\partial - x^2\, \partial_\mu) + 2x^\nu\, \Sigma_{\mu\nu} \qquad (3)$$

where $\Sigma_{\mu\nu}$ is the spin matrix of the particular representation chosen (e.g. $\frac{1}{2}\, \sigma_{\mu\nu}$ for a Dirac field), d is the (canonical) scale dimension of the field and $\partial_\mu \equiv \partial/\partial x^\mu$. The above operators represent the infinitesimal generators of Lorentz transformations, dilations and special conformal transformations, to which they respectively correspond, in the sense that

$$[\psi(x),\, M_{\mu\nu}] = m_{\mu\nu}\, \psi(x) \text{ etc.}$$

The form of the operators which act on creation or annihilation operators may be found from these by applying the auxiliary operators to free fields, which are expanded in the usual way in terms of (massless) momentum eigenfunctions:

$$\psi_\alpha(x) = \int \frac{(d\underline{p})}{2E} \sum_\lambda \left(a^{(\lambda)}(\underline{p})\, u_\alpha^{(\lambda)}\, (\underline{p})\, e^{-ip.x} + b^{+(\lambda)}(\underline{p})\, u_\alpha^{(\lambda)}\, (\underline{p})\, e^{ip.x}\right) \qquad (4)$$

Here $(d\underline{p}) \equiv d^3\underline{p}/(2\pi)^3$, and we have used a generalized Dirac notation, the suffix α representing the collection of auxiliary group labels. Because of the mass-shell constraint, the energy E is not an independent variable: ambiguity can be avoided by working at $x_o = 0$. It is then also convenient to deal with the space and time components of the above operators separately.

Working, for example, with rotations, one finds that

$$m_{ij}\, \psi\,(x) = -\, i \int \frac{(d\underline{p})}{2E} \sum_\lambda e^{-ip.x} \left[(p_i\partial_j - p_j\partial_i) + i\, \Sigma_{ij}\right] (u^{(\lambda)}(\underline{p}) a^{(\lambda)}(\underline{p}))$$

$$+ \ldots .$$

where now $\partial_j \equiv \partial/\partial p_j$.

The generalized spinor, however, transforms according to the Wigner rotation. That is

$$\left[-\, i(p_i\partial_j - p_j\partial_i) + \Sigma_{ij}\right]\, u^{(\lambda)}(\underline{p}) = W_{ij}\, u^{(\lambda)}\, (\underline{p}) \qquad (5)$$

where

$$W_{ij} = \lambda \frac{\varepsilon_{ij3} + \varepsilon_{ij}(\hat{p})}{1 + \hat{p}_3} \tag{6}$$

for states defined by rotation from a canonical state with momentum aligned along the third axis. Here $\hat{p}_3$ is the third component of the unit three-vector $\hat{\underline{p}}$ and $\varepsilon_{ij}(\hat{p}) \equiv \varepsilon_{ijk}\,\hat{p}_k$. Equation (5) can be derived by finding the Wigner rotation corresponding to a rotation about each of the three axes in turn and can be verified for the spin ½ case by explicit differentiation of the Dirac spinor

$$u^{(\lambda)}(\underline{p}) = \sqrt{E}(1 + 2i\gamma_5\lambda)(\tfrac{1}{2} + \lambda\,\underline{\sigma}\cdot\hat{\underline{p}})\,\chi^{(\lambda)}\,(\tfrac{1}{2}(1 + \hat{p}_3))^{-\frac{1}{2}} \tag{7}$$

The result is that the transformed field ψ can again be cast in the form (4), with a transformed annihilation operator

$$[a(p), M_{ij}] = \tilde{m}_{ij}\; a(\underline{p}),$$

where

$$\tilde{m}_{ij} = -\,i(p_i\partial_j - p_j\partial_i) + W_{ij} \tag{8}$$

Similarly the boost operator $\tilde{m}_{oi}$ turns out to be

$$\tilde{m}_{oi} = -\,i\,E\partial_i - W_i, \tag{9}$$

where

$$W_i = \frac{\lambda\;\varepsilon_{3i}(\hat{p})}{1 + \hat{p}_3}, \tag{10}$$

by virtue of the relation

$$[-\,i\,E\,\partial_j + \Sigma_{oj}]u^{(\lambda)}(\underline{p}) = -\,W_j\;u^{(\lambda)}(\underline{p}) \tag{11}$$

For the dilation operator, now specifying the auxiliary representation for spin s to be (0,s) + (s,0), we have

$$\tilde{d} = -\,i\,(1 + \underline{p}\cdot\underline{\partial}) \tag{12}$$

by virtue of the relation

$$\underline{p}\cdot\underline{\partial}\; u^{(\lambda)}\;(\underline{p}) = s\; u^{(\lambda)}\;(\underline{p}) = (d - 1)\; u^{(\lambda)}(\underline{p}) \qquad (13)$$

The form of the operators $\tilde{k}_o$ and $\tilde{k}_i$ representing the special conformal transformations then follows from (3) after some algebra involving repeated use of eqns. (9), (11) and (13) and the property

$$s\;\Sigma_{\ell k}\; u^{(\lambda)}(\underline{p}) = - i\;\lambda\;\epsilon_{k\ell m}\;\Sigma_{om}\; u^{(\lambda)}(\underline{p}) \qquad (14)$$

characteristic of the representation (0,s) + (s,0).

Displaying these together with the operators already found we have

$$\tilde{m}_{ij} = - i(p_i\partial_j - p_j\partial_i) + W_{ij} \qquad (15)$$

$$\tilde{m}_{oi} = - iE\partial_i - W_i \qquad (16)$$

$$\tilde{d} = - i(1 + \underline{p}\cdot\underline{\partial}) \qquad (17)$$

$$\tilde{k}_o = - E\;\underline{\partial}^2 + 2i\;\underline{W}\cdot\underline{\partial} + \frac{2s^2}{E(1+\hat{p}_3)} \qquad (18)$$

$$\tilde{k}_i = 2\partial_i\;\underline{p}\cdot\underline{\partial} - p_i\underline{\partial}^2 - 2i\;W_{ik}\partial_k - \frac{2s^2\;\delta_{i3}}{E(1+\hat{p}_3)} \qquad (19)$$

As may easily be verified, these differential operators obey the commutation relations of their corresponding generators[9]. The operators which act on the creation operators are the complex conjugates of these: i.e.

$$[G,\; a^+] = \tilde{g}^*\; a^+,$$

where G is any of the conformal generators and $\tilde{g}$ the corresponding differential operator.

III. Restrictions on T-Matrix

Under the assumption of strict invariance under conformal transformations we can evaluate the quantity ${}_{out}\langle p_3p_4|G|p_1p_2\rangle_{in}$ by commuting the conformal generator G either through the creation operators $a^+(\underline{p}_1)a^+(\underline{p}_2)$ or through the annihilation operators $a(p_3)a(p_4)$ to act on the vacuum, which it annihilates.

Thus $\;{}_{out}\langle p_3p_4|G|p_1p_2\rangle_{in} = (\tilde{g}_1^* + \tilde{g}_2^*)\,{}_{out}\langle p_3p_4|p_1p_2\rangle_{in}$

$$= (\tilde{g}_3 + \tilde{g}_4)\,{}_{out}\langle p_3p_4|p_1p_2\rangle_{in},$$

where $\tilde{g}_r$ is the appropriate differential operator for particle "r".

The restriction on the S-matrix is thus

$$(\sum_{in} \tilde{g}_r^* - \sum_{out} \tilde{g}_r)\, S_{\{\lambda\}} = 0 \qquad (20)$$

To find the restriction on the T-matrix we still have to commute the conformal operators through the momentum-conserving δ-function: as in ref. 5 all the operators commute except for the dilation operator $\tilde{d}$, which picks up an extra $4i$. The structure of the resulting equations is best explored by first of all considering the spinless case and then taking into account the additional parts of the operators which occur for particles with spin.

(a) Spinless Case.

The equations to be satisfied by the T-matrix are

$$(M_{ij} + M'_{ij})T = 0 \qquad \text{(Rotation eqn.)} \qquad (21)$$

$$(B_i + B'_i)T = 0 \qquad \text{(Boost eqn.)} \qquad (22)$$

$$(D + D')T = 0 \qquad \text{(Dilation eqn.)} \qquad (23)$$

$$K_oT = K'_o\,T \qquad \text{(Conformal } K_o\text{)} \qquad (24)$$

$$K_iT = K'_i\,T \qquad \text{(Conformal } K_i\text{)} \qquad (25)$$

where, with $\underline{P} = \underline{p}_1 + \underline{p}_2$, $\underline{q} = \frac{1}{2}(\underline{p}_1 - \underline{p}_2)$, the initial operators M_{ij} etc. are

$$M_{ij} = (P_i \frac{\partial}{\partial P_j} - P_j \frac{\partial}{\partial P_i}) + (q_i \frac{\partial}{\partial q_j} - q_j \frac{\partial}{\partial q_i}) \qquad (26)$$

$$B_i = (E_1 + E_2)\frac{\partial}{\partial P_i} + \tfrac{1}{2}(E_1 - E_2)\frac{\partial}{\partial q_i} \qquad (27)$$

$$D = \underline{P}.\frac{\partial}{\partial \underline{P}} + \underline{q}.\frac{\partial}{\partial \underline{q}} \qquad (28)$$

$$K_o = -2E\left(\frac{\partial^2}{\partial \underline{P}^2} + \frac{1}{4}\frac{\partial^2}{\partial \underline{q}^2}\right) \qquad (29)$$

$$K_i = 2\frac{\partial}{\partial P_i}\left(\underline{P}.\frac{\partial}{\partial \underline{P}} + \underline{q}.\frac{\partial}{\partial \underline{q}}\right) + 2\frac{\partial}{\partial q_i}\underline{q}.\frac{\partial}{\partial \underline{P}} - 2q_i\frac{\partial}{\partial \underline{P}}.\frac{\partial}{\partial \underline{q}} \qquad (30)$$

and similarly for the final operators M'_{ij} etc., with $\underline{q} \rightarrow \underline{q}' = \frac{1}{2}(\underline{p}_3 - \underline{p}_4)$.

What we aim to show, c.f. ref. 5 , is that once Lorentz and dilation invariance are satisfied (eqns. (21) - (23)), conformal invariance (eqns. (24) - (25)) follows automatically.

For K_o we multiply eqns. (21), (22) and (23) by $\frac{1}{2}(M_{ij} - M'_{ij})$, $(B_i - B'_i)$ and $(D - D')$ respectively, and add, resulting in the equation

$$\left[\tfrac{1}{2}(M_{ij})^2 + \underline{B}^2 + D^2\right]T = \left[\qquad\right]' T \qquad (31)$$

This is in fact the K_o equation, since by explicit calculation

$$\tfrac{1}{2}(M_{ij})^2 + \underline{B}^2 + D^2 = -2EK_o, \qquad (32)$$

and similarly for K'_o.

Again, from eqns. (21) - (23) we can derive the equation

$$\left[\{B_j, M_{ji}\} + \{B_i, D\}\right] T = \left[\qquad\right]' T, \qquad (33)$$

which is just the K_i equation, by virtue of the relation

$$\{B_j, M_{ji}\} + \{B_i, D\} = 2EK_i, \qquad (34)$$

and a similar relation for K'_i

Hence, in the spinless case, conformal invariance imposes no further restrictions once Lorentz and scale invariance are satisfied.

(b) Particles with Spin

The dilation operator is unchanged, but the Lorentz operators are augmented by their spin parts, according to

$$M_{ij} \to M_{ij} - iW_{ij}^{(1)} - iW_{ij}^{(2)} \tag{35}$$

$$B_i \to B_i + iW_i^{(1)} + iW_i^{(2)}, \tag{36}$$

and the conformal operators are similarly augmented, to give the full operators, which we will denote by $\mathcal{D}, \mathcal{M}_{ij}, \mathcal{B}_i, \mathcal{K}_o, \mathcal{K}_i$ respectively. For simplicity we take the third axis normal to the scattering plane.

The analogue of eqn. (31) is then

$$\left[\tfrac{1}{2}(\mathcal{M}_{ij})^2 + \underline{\mathcal{B}}^2 + \mathcal{D}^2\right] T_{\{\lambda\}} = [\qquad]' \; T_{\{\lambda\}} \tag{37}$$

where now, however,

$$\tfrac{1}{2}(\mathcal{M}_{ij})^2 + \underline{\mathcal{B}}^2 + \mathcal{D}^2 = -\,2E\mathcal{K}_o + (\lambda_1+\lambda_2)^2 \tag{38}$$

Thus the conformal $\mathcal{K}_o$ equation is satisfied provided that

$$(\lambda_1 + \lambda_2)^2 = (\lambda_3 + \lambda_4)^2, \tag{39}$$

which is the first restriction of conformal invariance. This equality is equivalent to conservation of the first Casimir operator of the conformal group, as can be seen by comparison with the work of Castell[10]. However, note that this condition is automatically satisfied for the elastic scattering of a spin s particle off a spinless particle, so that we must look to the conformal $\mathcal{K}_i$ equation to provide the constraint of helicity conservation in this case.

The analogue of eqn. (33) is

$$\left[\{\mathcal{B}_j, \mathcal{M}_{ji}\} + \{\mathcal{B}_i, \mathcal{D}\}\right] T_{\{\lambda\}} = [\qquad]' \; T_{\{\lambda\}} \tag{40}$$

but now

$$\{\mathcal{B}_j, \mathcal{M}_{ji}\} + \{\mathcal{B}_i, \mathcal{D}\} = 2E\mathcal{K}_i + 2(\lambda_1+\lambda_2)\left[iE\,\varepsilon_{ji}(\hat{q})\,\frac{\partial}{\partial q_j} + (\lambda_1+\lambda_2)\delta_{3i} + (\lambda_1-\lambda_2)\hat{q}_i\right] \quad (41$$

Thus conformal $\mathcal{K}_i$ invariance holds provided that

$$(\lambda_1+\lambda_2)\left[iE\varepsilon_{ji}(\hat{q})\frac{\partial}{\partial q_j}+(\lambda_1+\lambda_2)\delta_{3i}+(\lambda_1-\lambda_2)\hat{q}_i\right]T_{\{\lambda\}}=-(\lambda_3+\lambda_4)\left[\qquad\right]'T_{\{\lambda\}} \tag{42}$$

However, the angular momentum equation can be cast in the form

$$\left[iE\varepsilon_{ji}(\hat{q})\ \frac{\partial}{\partial q_j}+(\lambda_1+\lambda_2)\delta_{3i}+(\lambda_1-\lambda_2)\hat{q}_i\right]T_{\{\lambda\}}=-\left[\qquad\right]'\ T_{\{\lambda\}} \tag{43}$$

and hence the necessary condition is

$$(\lambda_1+\lambda_2-\lambda_3-\lambda_4)\left[iE\varepsilon_{ji}(\hat{q})\frac{\partial}{\partial q_j}+(\lambda_1+\lambda_2)\delta_{3i}+(\lambda_1-\lambda_2)\hat{q}_i\right]T_{\{\lambda\}}=0 \tag{44}$$

We must distinguish between several possible cases.

(i) If $\lambda_1+\lambda_2=\lambda_3+\lambda_4$ the equation is identically satisfied.

(ii) If $\lambda_1+\lambda_2\neq\lambda_3+\lambda_4$ and $\lambda_1\neq\lambda_2$ or $\lambda_3\neq\lambda_4$, we can multiply eqn. (44) by q_i, or its equivalent by q_i', to give $T_{\{\lambda\}}=0$.

(iii) If $\lambda_1+\lambda_2\neq\lambda_3+\lambda_4$ but $\lambda_1=\lambda_2$, $\lambda_3=\lambda_4$, we are left with a differential condition on $T_{\{\lambda\}}$, which does not constrain it to vanish.

Thus the restriction of conformal invariance is helicity conservation in the form $\lambda_1+\lambda_2=\lambda_3+\lambda_4$ except for the special case $T_{\lambda,\lambda;-\lambda,-\lambda}$, where the amplitude survives subject to a differential constraint. A particular example of this phenomenon was found previously using the M-function formalism for the case of spinor-spinor scattering[6].

IV. Outlook.

The above results were all derived under the assumption of strict conformal invariance for massless particles with canonical dimensions. The high-energy behaviour of, for example, fixed angle pp scattering, where it might have been thought that the masses could have been neglected and the invariance relevant, shows that, at the very least, anomalous dimensions[11] are required for hadron scattering. The problem can in fact be attacked on two levels, with the tools of equations of the Callan-Symanzik type[12]. At the more fundamental level it appears likely that the interactions of the basic entities are asymptotically free[13], in which case the above equations do reduce to conformal invariance with canonical dimensions. At the more phenomenological level of the interactions of (composite) hadrons their behaviour may then be deduced from the interactions of the basic constituents. This programme has been developed for scale invariance by Brodsky and Farrar[14] and in fact corresponds to the introduction of an anomalous dimension for hadrons equal to the number of quarks minus one, in agreement with the phenomenological observations of Theis[11]. It is our hope that the extension of this programme to conformal invariant interactions of the constituents will lead to comparably simple modifications for hadrons of the helicity rules we have established above.

Appendix

In the main body of the paper we have used fields transforming according to the (0,s) + (s,0) representation of the Lorentz group to deduce the form of the conformal operators that act on helicity states. However, spin 1 particles are commonly described by vector fields transforming according to the $(\frac{1}{2},\frac{1}{2})$ representation, which generally leads to no inconsistency provided that gauge invariance is imposed on the amplitudes. It is therefore of some interest to see how the above arguments have to be modified when the vector representation is used: in fact it turns out that the conformal operator acting in auxiliary space has to be modified by the addition of a non-gauge term.

Explicit expressions for the vector spin-matrix and radiation-gauge polarization vector are

$$(\Sigma_{\rho\sigma})_{\lambda\mu} = i(g_{\rho\lambda}g_{\sigma\mu} - g_{\rho\mu}g_{\sigma\lambda}) \qquad (A1)$$

and

$$\xi_i^{(h)}\ (\underline{p}) = \frac{1}{1+\hat{p}_3}\,(\delta_{ij} - \hat{p}_i\hat{p}_j - ih\ \varepsilon_{ij}(\hat{p}))\xi_j^{(h)} \qquad (A2)$$

where we are now using h for the helicity label and $\xi_j^{(h)}$ is a standard third-axis polarization vector.

Under rotations the polarization vector transforms in the standard way (c.f. eq. (5)):

$$\left[-\ i(p_i\partial_j - p_j\partial_i)\delta_{\ell m} + (\Sigma_{ij})_{\ell m}\right]\xi_m^{(h)}\ (\underline{p}) = W_{ij}\ \xi_\ell^{(h)}(\underline{p}) \qquad (A3)$$

However, under boosts there is an extra gauge term compared with the corresponding eq. (11), viz.

$$\left[-\ iE\partial_j g_\lambda^\mu + (\Sigma_{oj})_\lambda^\mu\right]\xi_\mu^{(h)}\ (\underline{p}) = -\ W_j\ \xi_\lambda^{(h)}\ (\underline{p}) + i\frac{p_\lambda}{E}\ \xi_j^{(h)}\ (\underline{p}) \quad (A4)$$

This latter term may, however, be ignored, because of gauge invariance.

The relation given in eq. (13) between the spin s and the canonical dimension d is no longer true, since in the vector representation we have d = s = 1. Eq. (13) is replaced by

$$\underline{p}\cdot\underline{\partial}\ \xi_m^{(h)}\ (\underline{p}) = 0 \tag{A5}$$

Working out the conformal $\tilde{k}_o$ operator in a similar manner to before we find an extra gauge term, which again can be ignored, compared with eq. (18):

$$-\tfrac{1}{2}\ \underline{\partial}^2(\xi_\lambda(\underline{p})a(\underline{p})) - 2i(\Sigma_{ok})^\mu_\lambda\quad \partial_k\left(\frac{\xi_\mu(\underline{p})a(\underline{p})}{2E}\right)$$

$$= \frac{\xi_\lambda(\underline{p})}{2E}\left[-E\ \underline{\partial}^2 + 2i\underline{W}\cdot\underline{\partial} + \frac{2}{E(1+\hat{p}_3)}\right]a(\underline{p}) + \frac{p_\lambda}{E^2}\ \underline{\partial}\cdot(\underline{\xi}(\underline{p})a(\underline{p})) \tag{A6}$$

Working out the conformal $\tilde{k}_i$ operator, however, we find an extra <u>non-gauge</u> term compared with eq. (19):

$$\left[(4\partial_i + 2\underline{p}\cdot\underline{\partial}\ \partial_i - p_i\ \underline{\partial}^2)\ \delta_{\ell m} - 2i(\Sigma_{ik})^m_\ell\ \partial_k\right]\left(\frac{\xi_m(\underline{p})a(\underline{p})}{2E}\right)$$

$$= \frac{\xi_\ell(\underline{p})}{2E}\left[2\ \partial_i\underline{p}\cdot\underline{\partial} - p_i\underline{\partial}^2 - 2iW_{ik}\partial_k - \frac{2\delta_{i3}}{E(1+\hat{p}_3)}\right]a(\underline{p}) - \xi_i(\underline{p})\frac{p_\ell}{E^3}\ a(\underline{p}) \tag{A7}$$

The difference from the previous cases is that this is not the spatial part of a four-vector equation: we cannot make the replacement $\ell \to \lambda$. The last term must therefore either be removed or supplemented with a time-like part by a modification for the vector case of the original auxiliary group conformal operator. That such a modification is necessary is also clear from the form of this operator, which does not preserve the Lorentz gauge condition.

That is, for a vector field $A_\mu(x)$, satisfying $\partial . A = 0$, we have

$$\partial_\rho\left[(2ix_\mu + i(2x_\mu x.\partial - x^2\partial_\mu))g^{\rho\sigma} + 2ix^\nu(g^\rho_\mu g^\sigma_\nu - g^\sigma_\mu g^\rho_\nu)\right]A_\sigma(x)$$

$$= -\ 4i\ A_\mu(x) \neq 0, \tag{A8}$$

in contrast to the effect of the Lorentz operator, which, as eqs. (A3), (A4) in fact show, does preserve the gauge condition:

$$\partial_\rho \left[i(x_\mu \partial_\nu - x_\nu \partial_\mu) g^{\rho\sigma} + i(g_\mu^\rho g_\nu^\sigma - g_\mu^\rho g_\nu^\sigma) \right] A_\sigma(x) = 0 \tag{A9}$$

The problem must be regarded as unresolved, however, as long as we do not have a deeper understanding of the form the modifications must take which goes beyond the fact that such modifications must be made in order to preserve gauge invariance.

Addendum

It is interesting that the helicity formulations of the Lorentz operators for massless particles find another application, namely, for investigating these particles in the vacuum. It is clear from four momentum conservation that each of the particles in the vacuum must be massless and have vanishing momentum. We hope, from this formulation, to find out if they can be spin particles.

Upon recalling the rotation operators

$$M_{ij} = -i(p_i\partial_j - p_j\partial_i) + W_{ij} \tag{15}$$

and the boost operators

$$M_{0i} = -iE\partial_i - W_i \tag{16}$$

where

$$W_i = \frac{\lambda\,\varepsilon_{3i}(\hat{p})}{1+\hat{p}_3} \tag{10}$$

$$W_{ij} = \frac{\lambda[\varepsilon_{ij3} + \varepsilon_{ij}(\hat{p})]}{1+\hat{p}_3}, \tag{6}$$

we have translation invariant equation

$$\Big[\sum_{(i)} p_\mu^{(i)}\Big]\langle a_{\lambda_1}(p_1)\cdots a_{\lambda_n}(p_n)\rangle = 0,$$

rotation invariant equation

$$\Big[\sum_{(i)} M_{ij}^{(i)}\Big]\langle a_{\lambda_1}(p_1)\cdots a_{\lambda_n}(p_n)\rangle = 0,$$

and the boost equation

$$\Big[\sum_{(i)} M_{0i}^{(i)}\Big]\langle a_{\lambda_1}(p_1)\cdots a_{\lambda_n}(p_n)\rangle = 0$$

where $\langle a_{\lambda_1}(p_1)\cdots a_{\lambda_n}(p_n)\rangle$ is vacuum expectation value of n annihilation operators of massless particles of helicity λ_i and momentum p_i.

To find out their implications, let us consider the following cases separately.

(a) When n = 1 , from translation equation we see immediately that

$$p_\mu \langle a_\lambda(p) \rangle = 0 .$$

That is, $\langle a_\lambda(p) \rangle = 0$, unless $p_\mu = 0$.

Also, from rotation equation (M_{12}) , we have

$$\lambda \langle a_\lambda(p) \rangle = 0 .$$

That is, $\langle a_\lambda(p) \rangle = 0$, unless $\lambda = 0$.

Hence we conclude that there can be no single massless particle in the vacuum unless it is a scalar particle and its four momenta are zero.

(b) When n = 2 , it is convenient to choose

$$p_i^{(1)} = (0, |p_2|, 0), \quad p_i^{(2)} = (0, -|p_2|, 0).$$

We do not put $p_\mu = 0$ until the end because helicity states for $p_\mu = 0$ are not well defined.

From M_{13} equation we have

$$(\lambda_1 - \lambda_2) \langle a_{\lambda_1}(p_1) \, a_{\lambda_2}(p_2) \rangle = 0 .$$

From M_{12} equation we have

$$[-i(p_2^{(1)} \partial_1^{(1)} + p_2^{(2)} \partial_1^{(2)} + \lambda_1 + \lambda_2] \langle a_{\lambda_1}(p_1) \, a_{\lambda_2}(p_2) \rangle = 0 .$$

From M_{01} equation we have

$$[i(E^{(1)} \partial_1^{(1)} + E^{(2)} \partial_1^{(2)}) - \lambda_1 + \lambda_2] \langle a_{\lambda_1}(p_1) \, a_{\lambda_2}(p_2) \rangle = 0 .$$

But $E^{(1)} = E^{(2)} = |p_2|$, these equations imply

$$(-i E^{(1)} \partial_1^{(1)} + \lambda_1) \langle a_{\lambda_1}(p_1) \, a_{\lambda_2}(p_2) \rangle = 0$$

and

$$\lambda_1 = \lambda_2 .$$

Putting E = 0 now, we deduce that

$$\langle a_{\lambda_1}(p_1),\ a_{\lambda_2}(p_2) \rangle = 0$$

unless $p_\mu = 0$ and $\lambda_1 = \lambda_2 = 0$.
That is, there cannot be two massless particles in the vacuum unless they are scalar particles and have zero four momenta.
(c) When n = 3 or more, we cannot reach similar conclusions as above. That is, we cannot exclude the possibility of having three or more spin particles of $p_\mu = 0$ in the vacuum.

Altogether, we should have expected the above results from angular momentum conservation principle. The above formulation, however, helps us to see these results from the first principle, i.e. directly from Lorentz invariance.

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