Optimal Experiment Design for Dynamic System Identification

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Abstract

The dynamic system experiment design problem is treated from a Bayesian viewpoint. It is shown how experiments may be designed using available prior information to optimise some measure of goodness of the experiment. The measure of goodness may be an information measure or, if a decision theoretic approach is adopted, a measure of the ultimate use of the experimental data. It is further shown that experiments may be designed for joint structure determination and parameter estimation.

Design algorithms for the realization of optimal input sequences are described for general linear discrete time dynamic systems. Characterisation theorems are obtained for amplitude and energy constrained inputs for general linear discrete time and continuous time dynamic systems. Stronger results are obtained for scalar output systems.

The problem of optimal sampling rate determination is also formulated and a simple design algorithm is described.

Several examples of designs carried out in the time and frequency domains are given.

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CHAPTER 1

Preliminaries

1. Introduction to the Thesis

In this thesis aspects of the problem of optimal planning of experiments for dynamic system identification are considered. The problem is approached from a control theoretic point of view and much of the notation and terminology reflects this. The results obtained are, however, applicable in many time series analysis applications where the experimenter has a degree of control over the methods of data generation and collection.

In general the experimenter must decide which variables (inputs, outputs, intermediate outputs etc.) to measure and how to measure them. It may also be possible to introduce input signals to further excite the system under test and the experimenter must decide upon the form of these test signals. These decisions will in general depend on a large number of factors (physical and economic constraints, current engineering practice, methods available for data analysis, purpose of experiment, etc.). In this thesis it is assumed that only the form of the test inputs and of the measurement system are left to be decided. It is further assumed that there are amplitude or energy (power) constraints on the test inputs, that the data collected consists of a fixed number of samples equally spaced in time, and that efficient use is made of the data. Subject to these constraints, test inputs and measurement system (sampling rate) are chosen to optimise some suitable measure of goodness of the experiment. This measure of goodness may be related to the purpose of the experiment which might,

for example, be the design of a controller or simulator for the dynamic system under test.

Obvious applications for the results obtained are in areas where the carrying out of an experiment is expensive. For example, aircraft flight tests and production line tests usually have costs proportional to experiment time and it is important to obtain the required information as quickly as possible. However, the results may also be applied advantageously in less critical applications since there will usually be savings due to smaller experiment and analysis times, or perhaps due to superior performance of a controller or simulator designed using information obtained from the experiment.

2. Originality and Significance

The results in this thesis which are believed by the author to be original are listed below, together with the chapter and section in which they appear.

1. The experiment design problem for dynamic system identification is formulated from a Bayesian decision theoretic viewpoint. The Bayesian approach answers the criticism that is often levelled at alternative formulations; viz. that, in order to design an experiment it is necessary to know the parameters that the experiment is being designed to find. The decision theoretic approach is conceptually pleasing since the experiment is designed to optimise a measure of the ultimate use to which the experimental data is to be put. (Chapter 3, section 2.)

2. A new optimality criterion is proposed. This is of the form:

$$J = \text{trace } \{\Gamma M^{-1}\} \qquad (1)$$

where Γ is a positive semi definite matrix and M is the "posterior information matrix" which is related to the expected posterior covariance. It is shown that Γ may be chosen so that J reflects the ultimate use to which the data is to be put. (Chapter 3, section 2, theorem 1.)

3. Several properties of information matrices for general innovations models are derived. These prove to be useful for experiment design purposes. In particular, they allow minimum identifiability conditions to be stated, and reduce the dimensionality of the design problem considerably. (Chapter 4, Section 4, results 1 to 7.)

4. Methods for evaluating the expected values of the posterior cost and posterior information based on first and second moments of the prior probability distribution are described. These methods offer significant reductions in computational effort over alternative approaches. (Chapter 4, section 5.)

5. Necessary conditions for optimality for both amplitude and power constrained inputs are derived for the general innovations model. (Chapter 4, section 6.)

6. A theorem on amplitude constrained optimal inputs is stated and proved. It is shown that optimal amplitude constrained inputs for linear systems are binary. This allows attention to be restricted to binary signals when performing an amplitude constrained design. (Chapter 4, section 6, theorem 2.)

7. A theorem showing that it is sufficient to consider only a subset of the parameters is stated and proved. (Chapter 4, section 6, theorem 1.)

8. A general design algorithm based on the necessary conditions for optimality is described. A number of design examples based on the algorithm are presented. (Chapter 4, section 7.)

9. A theorem on the choice of Lagrange multiplier for energy constraints in systems with disjoint system and noise modes is stated and proved. It is shown that the design can be performed with any value of the multiplier provided that the input is later scaled in amplitude. This fact leads to reductions in the computational effort. (Chapter 4, section 7, theorem 1.)

10. A detailed analysis of storage and computational requirements of the algorithm is made and methods proposed for reducing these requirements. For a typical model, it is shown that a 90% reduction is possible. (Chapter 4, section 8.)

11. It is shown that the computational requirements may be further reduced by exploiting the special structures of single output systems. (Chapter 4, section 9.)

12. A spectral characterisation theorem is stated and proved, showing that the average information matrix depends only on the spectral properties of the input. (Chapter 5, section 2, theorem 1.)

13. It is shown that the information matrix may be specified to any desired accuracy by considering only the first few shifts of the input autocovariance. (Chapter 5, section 2, theorem 2.)

14. Expressions for the information matrix are obtained in the frequency domain for general multi-input innovations models. This extends Mehra's frequency domain results which were restricted to single input discrete time systems having known disturbance characteristics. (Chapter 5, section 3.)

15. It is shown that for complete identifiability, the input spectrum

must contain at least p'/2m lines, where p' is the number of B and D parameters (B and D are the input gain matrices) and m is the number of outputs. (Chapter 5, section 3, theorem 2.)

16. It is shown that optimal inputs may be found by optimisation in a rp"(p"+1)-1 dimensional space where r is the number of inputs and p" is the number of system (A, B, C, D) parameters. (Chapter 4, section 3, theorem 4; section 4, result 8.)

17. Tests for optimality of test signals are derived using a theorem due to Whittle. The tests enable signals obtained by any means to be tested for optimality. (Chapter 5, section 4, theorems 2 and 3, results 3, 4, 5 and 6.)

18. It is shown that, for single output systems, it is sufficient to consider input spectra containing not more than $(2n+1)r^2$ lines where n is the state dimension and r the number of inputs. (Chapter 5, section 5, result 1.)

19. Frequency domain algorithms and corresponding time domain realization procedures are described. It is further demonstrated that amplitude constraints may be handled in the frequency domain. (Chapter 5, sections 6 and 7.)

20. It is shown that the results of the thesis can be applied to continuous time linear systems. (Chapter 5, section 8.)

21. A theorem is established showing that, for single input-single output linear systems, it is sufficient to consider input spectra containing not more than p" lines, where p" is the number of system parameters which may include a pure time delay for continuous time systems. (Chapter 5, section 8, theorem 1.)

22. The problem of optimal sampling rate determination for the case of a fixed number of samples is formulated. It is shown that joint

optimal design of sampling rate and input may be carried out. (Chapter 5, section 9.)

3. Conventions and Symbols

The system of numbering and cross-referencing is a standard one and is described as follows: within each section, equations, theorems, lemmas, results, definitions and so forth are given a single number. When refered to from within the same section only this number is used, but when refered to from another section the section number is also given. A similar convention applies to the numbering of sections within chapters. The chapter and section number appear at the top of each page.

The end of a proof or the end of a particular train of thought is denoted by #. The usage of other symbols is indicated in table 1.

Table 1 - Symbols

A	$n \times n$ state transition matrix
A	n^{th} order polynomial in z or z^{-1}
B	$n \times r$ matrix
B	n^{th} order polynomial in z or z^{-1}
B^{ℓ}	n^{th} order polynomial in z or z^{-1}
C	$m \times n$ output matrix
C	n^{th} order polynomial in z or z^{-1}
D	$m \times r$ matrix
D	polynomial in z^{-1}
D_j	$-\partial D/\partial \beta_j$

$\overline{D}(\xi)$ maximal rate of descent

det determinant

E total energy (power)

E $E_{\omega,\beta}$

$E_{\omega,\beta}$ expectation over $p(\omega,\beta)$; similarly E_{β}, $E_{\omega/\beta}$, $E_{\beta/\omega}$, E_{ω}

e exp

exp exponential

F sensitivity state transition matrix

G sensitivity input matrix for z'

G_k sensitivity output matrix $(\partial\omega_k/\partial\beta)^T$

g_k gradient vector - dimension r

g_k^{ℓ} $(g_k)_{\ell}$

g_k'' gradient g_k projected onto constraint surface

H_i (4.6.13); also $H_i^{(1)}$, $H_i^{(2)}$, $H_i^{(3)}$

H Hamiltonian

I unit matrix

J general cost

J_i i = 1: log det; i = 2: trace

j $\sqrt{-1}$; dummy suffix

K n×m Kalman filter gain matrix

L sensitivity input matrix for z"

ℓ_k noise sequence; ℓ - dummy suffix

log natural logarithm (base e)

M information matrix. Also M', M", $M_{\alpha\alpha}$ etc.

$\overline{M}$ average information matrix, $\overline{M}'$ - part affected by u

m output dimension (y_k,ω_k)

N experiment length

n state dimension

PRBN PRBS pseudo random binary noise (sequence)

P prior information matrix. Also P', $P_{\alpha\alpha}$ etc.

P_β prior covariance matrix for β

p number of parameters, dimension of β. (θ,α)

$p(\cdot)$ generic symbol for probability density function

Q steady state covariance of z''

Q_β posterior covariance; $\overline{Q}_\beta = E_\omega \cdot Q_\beta$

$q_\omega(\beta)$ $p(\beta/\omega)$; q, q': dummy suffix

$R_e\{\cdot\}$ real part of

r dimension of input (u_k)

S Laplace operator; $j\omega$

$\overline{S}$ complex conjugate of s

s_k A-parameter sensitivities

T as a superscript: transpose

t_k^ℓ B^ℓ parameter sensitivities; t: dummy suffix

u_k input at time k - dimension r

u $(u_1^T,\ldots u_N^T)^T$

u_k^ℓ $(u_k)_\ell$

W symmetric PD weighting matrix

x_k state vector at time k - dimension n

$\overline{y}_k$ conditional mean

y_k output vector at time k - dimension m

Z system transfer function

z_k sensitivity state vector; also z_k', z_k''

z unit shift operator; $e^{j\omega}$

α parameters in A, B, C, D and K

β full parameter vector

$\tilde{\beta}$ maximum likelihood estimate

$\bar{\beta}$	prior mean
$\hat{\beta}$	posterior mean
Γ	weighting matrix (3.2.14)
γ	initial conditions
δ_ω	a measure assigning all the power to ω
ε_k	noise sequence
η	a measure $\in \Xi$
Θ	parameters in α and γ
Θ_i	i^{th} component of Θ or generalised polar coordinate
Λ	set of all possible frequencies, ω (design space)
λ	Lagrange multiplier
υ	noise variance (scalar)
ξ	a measure
Ξ	set of all measures defined on the Borel field generated by the open sets of Λ
Σ	noise covariance
σ	parameters in Σ
Φ	noise spectral density
$\Phi(\xi,\eta)$	directional derivative
ϕ	derivative of J w.r.t. M'
χ	$\chi^T\chi = \Sigma$
Ψ	noise transfer function
Ω	space of all possible innovations sequences, $\{\omega_k\}$
ω	radian frequency; vector of innovations
ω_k	innovations at time k
$(\cdot)_i$	i^{th} component of $\cdot$; also $(\cdot)_{ij}$
$\{\cdot_k\}$	sequence
*	as a superscript - optimal path or complex conjugate transpose
#	end of proof or current discussion
$\lvert\cdot\rvert$	modulus

CHAPTER 2

Background

1. Introduction

In this chapter the problem of identification experiment design is briefly discussed. For details of various identification methods the reader is referred to the survey papers by Åström and Eykhoff, [43], [44]. A survey of applications of identification in physical and chemical processes is given by Gustavsson [42]. This latter paper and that of Bohlin [40], present general experiment design principles. This is also the topic of section 2 of this chapter.

In section 3, various techniques for optimal test signal design are discussed.

2. Design of Identification Experiments - Some General Considerations

It is generally true that efficient experiments for dynamic system identification can only be designed if the system dynamics and disturbance characteristics are fairly well known. It is also true that the experiment should be designed with the ultimate purpose in mind. The first two steps of the design should be:

(a) *Specify the purpose of the experiment.* For example to design a control strategy, to build a simulator, or just to gain knowledge about the process.

(b) *Perform a physical analysis on those parts of the process and disturbances which are relevant to the purpose.* If possible

perform simple preliminary experiments to determine gross characteristics of the process and disturbances.

Steps (a) and (b) are possibly the most important aspects of experiment design. A good knowledge of the physics of the process is not only essential for designing an experiment but also for interpretation of the results. The remaining steps of the design are:

(c) *Choose variables*. That is, choose the inputs and outputs to be measured. This choice depends on the purpose of the experiment, for example, for control purposes the inputs and outputs should be the same as those used by the controller. In general, it is advisable to measure as many variables as possible, as the purpose of the experiment may change after preliminary analysis of the data. In cases where it is uncertain whether a variable is an input or an output, it may be treated as an output and the model adjusted accordingly [42].

(d) *Choose a class of model structures*. This choice depends primarily on the purpose of the identification; for example, if the model is required to design a regulator, a linear model of a non-linear process is often adequate. If, however, a controller valid over a wide range of operating conditions is desired, then the non-linearities should be included in the model. The results of the physical analysis also affect the choice of structure. Sometimes the model structure is well defined by this preliminary analysis and it only remains to find several undetermined parameters. Often, however, the model obtained from physical considerations is far too complex, and model reduction techniques or simulations have to be

used to obtain simpler models that are adequate for the purpose. The model structure must also be suitable for the identification method chosen.

(e) *Choose identification method.* This choice depends basically on the cost involved in relation to the ultimate purpose. Sophisticated procedures which make use of prior information and estimate disturbance characteristics are generally much more expensive and difficult to use than the simpler, more restrictive methods. The choice also depends on the model structure chosen and on the form of the inputs, [42], [43], [44].

(f) *Choose sampling rate.* Most computer programs require the data to be in the form of samples equally spaced in time. Furthermore it is usually the case that the number of samples is limited due to the increased cost of analysis with large amounts of data. There is, therefore, the problem of choosing a suitable sampling rate. In practice this choice has not been found to be critical and the sampling period is usually chosen to be of the same order of magnitude as the smallest time constant of interest, [42], [46]. In order to prevent loss of accuracy due to collection of poor data ~~inefficient estimation~~ it is necessary in most cases to include an "aliasing filter" before the sampler [42]. This is a low-pass filter with steep cut-off characteristics at half the sampling frequency. The problem of optimally choosing the sampling rate has been investigated by Åström , [45], and Zarrop, [50].

A disadvantage of equi-spaced samples is that it is difficult to find process characteristics over more than a few decades. To overcome this, non-uniform sampling may be used but special identification procedures are necessary. The problem of optimally

placing non-uniform samples has been investigated by Goodwin and Zarrop [49].

(g) *Specify the input.* Whenever possible a perturbation signal should be introduced. The source of the perturbation should be independent of the process disturbances [40], [41]. The form of the input is often determined by the identification method. For example, pseudo-random binary sequences for cross-correlation methods [42]. For non-linear systems, the form of the input should approximate normal operating signals. The amplitude of the input should be as large as possible subject to constraints. Constraints may be technical (non-linearities), economic (quality of output of process) or political. The frequency spectrum of the input should be chosen with reference to the characteristics of the the process and disturbances, and to the ultimate purpose of the experiment. Minimal properties of test signals are discussed by Åström, [44], Staley and Yue, [51], Tse, [52], and Ljung, [54]. Simple and robust procedures for generating binary sequences whose properties are related to the estimation accuracy have been described by Keviczky [20], [21] and Arimoto and Kimura, [8]. The generation of random signals with prescribed amplitude probability density function and prescribed power density spectrum has been described by Veltman et al, [53], and Gujar and Kavanagh , [55]. Care must be taken when using deterministic signals with line spectra due to "confounding". For example it would be unwise to use a test signal with a component at 150 Hz in an environment containing third harmonic interference from power transformers, or to use a signal with one day period on a process that may depend on ambient temperatures. In the next section the

optimal choice of input is discussed.

Problems that arise due to poor design of experiments and other reasons are discussed by Bohlin, [40], [41].

3. Optimal Design of Inputs

The choice of optimal inputs for static systems has received a great deal of attention during the last fifteen years. (For example Kiefer and Wolfowitz, [33], Karlin and Studden, [34], Federov, [38] and Whittle, [56].) A linear (in the parameters) multiple input single output static system is described by equation (1):

$$y_j = \theta^T f(x_j) + \ell_j, \quad j = 1, \ldots, N \tag{1}$$

where y_j is the j^{th} observation and $\ell_1, \ldots, \ell_N$ have zero mean, variance v, are uncorrelated with one another and are statistically independent of the x_j. The design of an experiment consists of choosing the vectors x_j from a set of allowable inputs, X. From the corresponding observations y_j, $j = 1, \ldots, N$ a minimum variance unbiased linear estimate, $\hat{\theta}$, of the parameters, θ, may be calculated and can be shown to have covariance matrix given by:

$$\mathrm{cov}(\hat{\theta}) = v\Big[\sum_{j=1}^{N} f(x_j) f^T(x_j)\Big]^{-1} \tag{2}$$

The optimal design of the experiment thus consists of choice of $x_1, \ldots, x_N$ from X, to maximise some scalar function of $\mathrm{cov}(\hat{\theta})$. Algorithms have been devised which converge to optimal designs, [38], [57]. Typical applications of this theory are in the design of reaction

experiments in chemistry where x is a vector of reagent concentrations and y the yield, or in the design of agricultural experiments where x might represent fertilizer composition and y the dry weight of the crop.

Much less attention has been devoted to the design of optimal inputs for dynamic systems.

Turin (1957, [39]) considers the design of optimal signals for the estimation of the weighting function of a linear system in the case that the estimate is obtained as a convolution of the system output (matched filter).

Levin (1960, [1]) considers the same problem as Turin but uses a Markov estimator of the system weighting function. In the case of white output noise, the important result that an input with impulsive autocorrelation is optimal with respect to several common optimality criteria, is obtained.

Levadi (1966, [3]) considers a linear time varying system with non-stationary coloured output noise. (The time variations in both system and noise are known.) The system output is assumed to be linear in the parameters and a Markov estimator is employed. Necessary conditions for optimality are obtained but no realization procedure is given.

Aoki and Staley, [22], [23], [24], Nahi and Wallis, [12], Mehra, [9], Schmidt, [63] and Napjus, [25], [26] all use the trace of Fisher's information matrix (or slight variants) as an optimality criterion. This choice of cost function leads to a standard quadratic performance index optimal control problem to which numerical solutions may be found. However, as has been pointed out by Goodwin, [17], [37], Reid, [18] and

Tse [52], the use of the trace of the information matrix can lead to inputs with little engineering appeal. (In fact the optimal inputs obtained may not even be "persistently exciting" in the sense of [44], [7].)

Goodwin (1970, [37], [16], [17]) treats the case of a general time varying, discrete-time, non-linear system with coloured non-stationary output noise (known time variations). The trace of the inverse of Fisher's information matrix is used as an optimality criterion. Necessary conditions for optimality are derived and a realization algorithm capable of handling state inequality constraints is described. The algorithm solves the two point boundary value problem by a steepest descent method, and is able to handle moderately complex systems, but at the cost of very long computation times.

Reid (1972, [18]), obtains results similar to Goodwin's for the linear continuous-time case. Reid's realizations algorithm generates a binary signal by searching on a fixed number of switching times. The algorithm appears to be limited to rather simple systems and inputs with few switches.

Box and Jenkins, [11], and Minnich, [59] obtain interesting results for simple first order linear models. Minnich shows that an optimal input for a first order, two parameter model with least squares structure is a first order autoregressive process.

An approach which ignores the statistical aspects of the problem is the so-called "sensitivity approach". (Rault et al, [14], Inoue et al, [13], Sawaragi et al, [19], Kalaba and Spingarn, [60].) If the system can be represented by

$$y(t) = h(u(t),t,\theta) \qquad (3)$$

where $u(t)$, $y(t)$ are the input and output at time t and Θ is a vector of parameters, $\Theta^T = (\Theta_1, \ldots \Theta_p)$, then the "sensitivity equations" are defined by:

$$g^T(t) = \left.\frac{\partial h(u(t),t,\Theta)}{\partial \Theta}\right|_{\Theta=\Theta_o} \tag{4}$$

where Θ_o is some nominal value of Θ. The output sensitivity is usually measured by some scalar function of the matrix:

$$G = \int_0^T g(t)\, g^T(t)\, dt \tag{5}$$

where T is the experiment time. If, in fact, there is white gaussian observation noise with variance σ^2, G can be interpreted as $\sigma^2 M$ where M is Fisher's information matrix. Litman and Huggins, [2], follow a related approach and find the optimal probing signal, within the space matched to that spanned by the elements of $g(t)$, $t \in [0,\infty)$, for a simple two parameter system.

A problem closely related to the problem of optimal input design for identification is that of optimally selecting probing signals, from a finite set of allowable signals, to determine which of a finite number of possible systems is in fact present. A multiple hypothesis test is used for the identification and the input is selected to optimise in some sense the power of the test. Details may be found in the papers by Smith, [10], Gagliardi, [4], Mosca, [61] and Eposito and Schumer, [62].

4. Concluding Remarks

In this chapter the engineering problems associated with the design of identification experiments have been briefly indicated. The problem of optimal input design has also been discussed and a short survey of relevant material has been presented. In the next chapter a more concise statement of the problem is given.

CHAPTER 3

Bayesian Problem Statement

1. Introduction

As has been pointed out in chapter 2, efficient experiment design depends upon a knowledge of the characteristics of the system. In fact, as the problem is usually stated, optimal designs depend upon a *complete* knowledge of the system characteristics - a situation in which it is hardly necessary to perform an experiment! This point has been made by Box and Jenkins [11]. In this thesis the problem is reformulated in a Bayesian framework which is shown to resolve the above paradox, [24], [25], [26]. There has been a good deal of controversy in statistical literature regarding the use of Bayesian methods due mainly to the subjective nature of choosing prior probability distributions. (See for example chapter 1 of [64].) The methods are, however, gaining in popularity, [65], and are finding acceptance in control engineering applications where the choice of prior distributions is often uncontroversial, [66].

In section 2 it is shown how Bayesian Decision theory may be employed to design experiments which are optimal with respect to the ultimate purpose of the experiment. In many cases, however, a concise mathematical statement of the purpose is not possible or is far too complicated. In these cases it is proposed to use an information measure. This will be discussed further in section 3.

Design for structure determination is discussed in section 4 from both decision theoretic and information theoretic viewpoints.

Examples of decision theoretic design criteria are given in appendices A and B.

2. Decision Theoretic Design

Optimal experiment designs which are independent of the system and noise parameters exist for simple classes of systems, [1], [38]. In general, however, this is not true for dynamic systems. Hence it is appropriate to adopt a Bayesian viewpoint and to express the prior knowledge regarding the parameters $\beta \in B$ as a probability distribution $p(\beta)$. This prior information can be obtained from physical reasoning or a preliminary experiment.

For any experiment, the posterior distribution $p(\beta/\omega)$, summarises all the information contained in the data, ω, and the prior distribution $p(\beta)$. The posterior distribution is obtained from Bayes' Rule:

$$p(\beta/\omega) = \frac{p(\omega/\beta)p(\beta)}{p(\omega)} \tag{1}$$

where

$$p(\omega) = E_\beta[p(\omega/\beta)] = \int_B p(\omega/\beta)p(\beta)d\beta \tag{2}$$

is a scaling factor which ensures that $p(\beta/\omega)$ integrates to unity. The distribution $p(\omega/\beta)$ of the data given the parameters is completely specified by the model structure and experimental conditions, ε. When regarded as a function of β, $p(\omega/\beta)$ is called the likelihood function of β for given ω.

In order to design an optimal experiment for parameter estimation, a measure of the return from the experiment is required. In principle, the measure should reflect the use to which the model will be put,

and, in general, will be a scalar function of the posterior distribution, $p(\beta/\omega)$. Let $q_\omega(\beta) = p(\beta/\omega)$. Then the cost associated with the density function $p(\cdot/\omega)$ when the true parameter value is β is given by:

$$s = s(q_\omega(\cdot),\beta) \tag{3}$$

For example s might be the output variance of a linear optimal regulator designed using $p(\cdot/\omega)$ when the true parameter is β. (See appendix A.) Thus after the experiment, a measure of the return from the experiment may be defined as:

$$\hat{s} = E_{\beta/\omega}[s(q_\omega(\cdot),\beta)] \tag{4}$$

where $E_{\beta/\omega}$ denotes expectation over the distribution $p(\beta/\omega)$.

$$E_{\beta/\omega}[\cdot] = \int_B (\cdot)p(\beta/\omega)d\beta \tag{5}$$

$\hat{s}$ is a function of $q_\omega(\cdot)$ only.

Before the experiment, the expected value of $\hat{s}$ is given by

$$\bar{s} = E_\omega[\hat{s}] = E_\omega[E_{\beta/\omega}[s(q_\omega(\cdot),\beta)]] \tag{6}$$

where $E_\omega[\cdot]$ denotes expectation over $p(\omega)$. From Bayes' rule, (1), it follows that:

$$\bar{s} = E_{\omega,\beta}[s(q_\omega(\cdot),\beta)] = E_\beta[E_{\omega/\beta}[s(q_\omega(\cdot),\beta)]] \tag{7}$$

It can be seen that $\bar{s}$ depends upon the functional form of the likelihood function which in turn depends on the experimental conditions, ε. Thus the experiment can be designed to minimise $\bar{s}$, the expected cost before the experiment of using the data to be obtained from the experiment. In order to perform this minimisation it is preferable to have parametric forms for the likelihood function and posterior distributions. These are readily obtained by making use of the large sample properties of the likelihood function, [27], [47], viz.: subject to mild regularity conditions, the likelihood, $p(\omega/\beta)$ is asymptotically normal with mean $\tilde{\beta}$ and covariance M^{-1} where M is Fisher's information matrix ~~defined by:~~ (i.e. log $p(\omega/\beta)$ is a quadratic function of β) and :

$$M = E_{\omega/\beta}\left[\left(\frac{\partial \log p(\omega/\beta)}{\partial \beta}\right)^T \left(\frac{\partial \log p(\omega/\beta)}{\partial \beta}\right)\right] \tag{8}$$

Also:

$$E_{\omega/\beta}[\tilde{\beta}] = \beta \tag{9}$$

For a normal prior distribution it is readily shown that the mean $\hat{\beta}$ and covariance of the posterior distribution are given asymptotically by:

$$Q_\beta = (M+P_\beta^{-1})^{-1} \tag{10}$$

and

$$\hat{\beta} = Q_\beta (M\tilde{\beta}+P_\beta^{-1}\bar{\beta}) \tag{11}$$

where $\bar{\beta}$, P_β are the mean and covariance respectively of the prior distribution, $p(\beta)$. Equations (10) and (11) are true for large

samples even in the case where $p(\beta)$ is not normal, provided $p(\beta)$ is well behaved, [68]. Since a normal distribution is completely described by its mean and covariance, (7) may be written as:

$$\bar{s} = E_{\omega,\beta}[s(\hat{\beta},\Omega_{\beta},\beta)] \tag{12}$$

A theorem that allows $\bar{s}$ to be implemented as a design criterion is now stated and proved:

Theorem 1: For experiment designs based on the first and second moments of the prior and posterior distribution functions, and for large samples, the following design criteria are equivalent:

$$(i) \quad J_1 = \bar{s} = E_{\omega,\beta}[s(\hat{\beta},\Omega_{\beta},\beta)] \tag{13}$$

where s *is a function such that the first and second derivatives of* s *with respect to* β *and* $\hat{\beta}$*, and the first derivative with respect to* Ω_{β} *exist, and that:*

$$\Gamma = \frac{\partial s}{\partial \Omega_{\beta}}\bigg|_{\bar{\beta},P_{\beta}} + \frac{1}{2}\frac{\partial^2 s}{\partial \hat{\beta}^2}\bigg|_{\bar{\beta},P_{\beta}} \tag{14}$$

is positive semi-definite (PSD).

$$(ii) \quad J_2 = \text{trace}\,[\Gamma\bar{\Omega}_{\beta}] \tag{15}$$

where

$$\bar{\Omega}_{\beta} = E_{\omega}\Omega_{\beta} \tag{16}$$

Proof: Expand $s(\hat{\beta},Q_\beta,\beta)$ in a Taylor series about the prior mean $\bar{\beta}$ and covariance P_β:

$$s(\hat{\beta},Q_\beta,\beta) = s(\bar{\beta},P_\beta,\bar{\beta}) + \frac{\partial s}{\partial \hat{\beta}}(\hat{\beta}-\bar{\beta}) + \frac{\partial s}{\partial \beta}(\beta-\bar{\beta}) + \text{trace}[\frac{\partial s}{\partial Q_\beta}(Q_\beta-P_\beta)]$$

$$+ \frac{1}{2}(\hat{\beta}-\bar{\beta})^T\frac{\partial^2 s}{\partial \hat{\beta}^2}(\hat{\beta}-\bar{\beta})$$

$$+ (\hat{\beta}-\bar{\beta})^T\frac{\partial^2 s}{\partial \hat{\beta}\partial\beta}(\beta-\bar{\beta})$$

$$+ \frac{1}{2}(\beta-\bar{\beta})^T\frac{\partial^2 s}{\partial \beta^2}(\beta-\bar{\beta}) + \ldots$$

$$\therefore \quad E_{\omega,\beta}[s(\hat{\beta},Q_\beta,\beta)] = s(\bar{\beta},P_\beta,\bar{\beta}) + \text{trace}[\frac{\partial s}{\partial Q_\beta}(\bar{Q}_\beta-P_\beta)]$$

$$+ \frac{1}{2}\,\text{trace}[\frac{\partial^2 s}{\partial \hat{\beta}^2}\bar{Q}_\beta] + \frac{1}{2}\,\text{trace}[\frac{\partial^2 s}{\partial \hat{\beta}^s}P_\beta]$$

$$+ \text{trace}[\frac{\partial^2 s}{\partial \hat{\beta}\partial\beta}P_\beta]$$

$$+ \frac{1}{2}\,\text{trace}[\frac{\partial^2 s}{\partial \beta^2}P_\beta] + \ldots$$

$$= s(\bar{\beta},P_\beta,\bar{\beta}) + \text{trace}[(\frac{\partial^2 s}{\partial \hat{\beta}\partial\beta}+\frac{1}{2}\frac{\partial^2 s}{\partial \beta^2}+\frac{1}{2}\frac{\partial^2 s}{\partial \hat{\beta}^2}-\frac{\partial s}{\partial Q_\beta})P_\beta]$$

$$+ \text{trace}[(\frac{\partial s}{\partial Q_\beta}+\frac{1}{2}\frac{\partial^2 s}{\partial \hat{\beta}^2})\bar{Q}_\beta] + \ldots \qquad (17)$$

where the derivatives are evaluated at $\bar{\beta}$, P_β. The high order terms depend only on moments of third order and higher and may, therefore, be neglected since only first and second moments are of interest.

Equations (15) and (17) are therefore equivalent apart from an additive constant. The theorem is proved. #

To proceed further it is necessary to specify the model structure describing $p(\omega/\beta)$. In the next chapter, the criterion, $J_2 = \text{trace}\,[\Gamma\bar{Q}_\beta]$ is used to design optimal inputs for the class of linear time invariant dynamic systems with stationary disturbances.

3. Information Theoretic Design

The Bayesian decision theoretic approach to experiment design described in section 2 is conceptually very pleasing, but unfortunately there are many cases where it is difficult or impossible to implement. The major problem is in describing the ultimate purpose of the experiment in a suitable mathematical form. For example, if the purpose is just to gain knowledge about the process, there is no obvious mathematical statement of the ultimate purpose. Thus, there exists the need for a measure of the return from an experiment which does not depend directly upon the ultimate purpose, but which does indicate how "good" the experiment is in some well defined sense.

A suitable measure for this purpose is the *average information increment* provided by an experiment as defined by Lindley [68]. This quantity is now defined:

Definition 2: The amount of information provided by the experiment, ε, with prior knowledge $p(\beta)$, is

$$I(\varepsilon,p(\cdot)) = E_\omega[I_1(\omega)-I_o] \qquad (1)$$

where $I_1(\omega)$ *is the amount of information about* β *contained in the posterior distribution and is defined by:*

$$I_1(\omega) = \int_B p(\beta/\omega) \log p(\beta/\omega) d\beta \tag{2}$$

and I_0 *is the prior information defined as:*

$$I_0 = \int_B p(\beta) \log p(\beta) d\beta \tag{3}$$

$I(\varepsilon, p(\cdot))$ is called the average information increment for the experiment, ε, with prior distribution $p(\beta)$. It has also been called the *mutual information* between parameters and data, [8], and the *sensor channel transmittance*, [70].

An optimal experiment, ε^*, may now be defined by:

$$I(\varepsilon^*, p(\cdot)) \geq I(\varepsilon, p(\cdot)) \quad \forall \quad \varepsilon \tag{4}$$

If $p(\beta)$ and $p(\beta/\omega)$ are assumed to be normal with covariances P_β and Q_β respectively, it is possible to obtain an expression for $I(\varepsilon, p(\cdot))$ in terms of these covariances.

Result 1

$$I_0 = \frac{1}{2} \log\det[P_\beta^{-1}] - \frac{p}{2} \log(2\pi e) \tag{5}$$

where p *is the dimension of* β.

Proof: From (3)

$$I_o = \int_B [-\frac{p}{2}\log(2\pi)+\frac{1}{2}\log\det[P_\beta^{-1}]-\frac{1}{2}(\beta-\bar{\beta})^T P_\beta^{-1}(\beta-\bar{\beta})]p(\beta)\,d\beta$$

$$= \frac{1}{2}\log\det[P_\beta^{-1}]-\frac{p}{2}\log(2\pi)-\frac{1}{2}\text{trace}[P_\beta^{-1}.E_\beta[(\beta-\bar{\beta})(\beta-\bar{\beta})^T]]$$

$$= \frac{1}{2}\log\det[P_\beta^{-1}]-\frac{p}{2}\log(2\pi)-\frac{p}{2} \qquad \#$$

Result 2:

$$I_1(\omega) = \frac{1}{2}\log\det[Q_\beta^{-1}]-\frac{p}{2}\log(2\pi e) \qquad (6)$$

Proof:

Follows immediately from (2) and (5) by analogy. #

Result 3:

$$I(\varepsilon,p(\cdot)) = \frac{1}{2}E_\omega[\log\det[P_\beta Q_\beta^{-1}]] \qquad (7)$$

Proof:

From (1), (5) and (6). #

As was discussed in section 2, the assumption of normality of $p(\beta/\omega)$ is realistic for large samples. If $p(\beta/\omega)$ is not normal then the following inequality holds [70]:

$$I_1(\omega) \geq \frac{1}{2}\log\det[Q_\beta^{-1}]-\frac{p}{2}\log(2\pi e) \qquad (8)$$

so that (7) becomes:

$$I(\varepsilon,p(\cdot)) \geq \frac{1}{2}E_{\omega}[\text{logdet}[P_{\beta}Q_{\beta}^{-1}]] \tag{9}$$

Thus, use of the optimality criterion, J, defined by:

$$J = E_{\omega}[\text{logdet}[P_{\beta}Q_{\beta}^{-1}]] \tag{10}$$

is sensible even in the case of non-normal posterior distribution. Non-normal prior distributions do not usually cause any concern, since, for long experiments, $I_1(\omega)$ dominates I_0.

In the next chapter it is shown how J defined in (10) may be used as an optimality criterion for linear systems.

4. Design for Structure Determination

In this section the results of sections 2 and 3 are extended to the case where there is the need to design an experiment to discriminate between alternative model structures. For simplicity, only the case of two alternative structures, H_1 and H_2, is considered. It is assumed that one of H_1, H_2 is the true structure and that the prior probabilities of H_1, H_2 are $p(H_1)$, $p(H_2)$ respectively ($P(H_2) = 1-P(H_1)$).

Following section 2 define cost functions:

$$s_i = s_i(q_{\omega}^1(\cdot), q_{\omega}^2(\cdot), \beta_i), \quad i = 1, 2 \tag{1}$$

where s_i is the cost associated with posterior distributions:

$$q_\omega^1(\beta_1) = p(\beta_1/\omega,H_1)p(H_1/\omega) \tag{2}$$

$$q_\omega^2(\beta_2) = p(\beta_2/\omega,H_2)p(H_2/\omega) \tag{3}$$

when the true structure is H_i with parameters β_i. After the experiment, the expected cost is:

$$\hat{s} = \sum_i p(H_i/\omega) E_{\beta_i/\omega,H_i}[s_i(q_\omega^1(\cdot),q_\omega^2(\cdot),\beta_i)] \tag{4}$$

Before the experiment, the expected cost is:

$$\bar{s} = E_\omega \hat{s} = E_\omega[\sum_i p(H_i/\omega) E_{\beta_i/\omega,H_i}[s_i]] \tag{5}$$

which, from Bayes' rule, gives:

$$\bar{s} = \sum_i p(H_i) E_{\beta_i/H_i}[E_{\omega/\beta_i,H_i}[s_i(q_\omega^1(\cdot),q_\omega^2(\cdot),\beta_i]] \tag{6}$$

In general, the use of $\bar{s}$ as a design criterion would be extremely complicated. However, as in section 2, it is possible to simplify the form of $\bar{s}$ considerably by considering large samples and parametric forms of s_i. In appendix B an example illustrating a decision procedure for controller design is given. Even for the simple case considered, the problem of optimal experiment design is very complicated.

The information theoretic approach of section 3 can also be extended by redefining the information [38]:

$$I_0 = \sum_i p(H_i) \int_{B_i} p(\beta_i/H_i) \log p(\beta_i, H_i) d\beta_i \tag{7}$$

$$I_1(\omega) = \sum_i p(H_i/\omega) \int_{B_i} p(\beta_i/H_i, \omega) \log p(\beta_i, H_i/\omega) d\beta_i \tag{8}$$

In the case where β_i is known for i = 1, 2, then the *divergence*, J, as defined by Kullback, [67] is also a suitable measure:

$$J(\varepsilon, p(\cdot)) = E_{\omega/H_1}\left[\log \frac{p(H_1/\omega)}{p(H_2/\omega)}\right] + E_{\omega/H_2}\left[\log \frac{p(H_2/\omega)}{p(H_1/\omega)}\right] \tag{9}$$

The design of experiments to increase the power of commonly used experimental tests such as F-tests is also theoretically possible, although more work needs to be done in this area. A detailed discussion of the problem is given in [38].

5. Concluding Remarks

In this chapter two alternative formulations of the experimental design problem have been given. Both formulations base the design on the information contained in the prior distributions of model structure and parameters. The decision theoretic approach assumes the existence of a scalar function of the experimental data related to the ultimate purpose of the experiment, and leads to a cost function of the form:

$$J_1 = E[\text{trace}\Gamma Q_\beta] \tag{1}$$

The information theoretic approach does not make use of a knowledge of ultimate purpose (except perhaps in the initial choice of structure and parameter set) and leads to the following cost:

$$J_2 = E[\det Q_\beta] \tag{2}$$

This latter cost function has another interpretation as the expected volume of the posterior highest probability density region for β. Other cost functions such as the expected value of the maximum eigenvalue of Q_β (major diagonal of highest probability density ellipsoid) are possible but are not discussed further here.

Since Q_β is given asymptotically by $(M+P_\beta^{-1})^{-1}$, (equation (2.10)), where M is evaluated at β, the "true" parameter value, it suffices to take the expectation indicated in (1) and (2) over the prior distribution of β, $p(\beta)$.

APPENDIX A

An Example of Ultimate Model Use

A simple first order linear system under minimum variance control is considered.

System Structure:

$$y_{k+1} = ay_k + bu_k + \omega_k \tag{1}$$

where $\{u_k\}$, $\{y_k\}$ are the input and output sequences respectively and $\{\omega_k\}$ is a sequence of independent identically distributed random variables having zero mean and variance υ.

Minimum Variance Control Law:

$$u_k = -\frac{\hat{a}}{\hat{b}} y_k \tag{2}$$

where $(\hat{a},\hat{b})$ is the mean of the posterior distribution for (a,b). (The posterior covariance information has been discarded - see [48] for a general treatment.)

Measure of Ultimate Model Performance:

$$s(\hat{\beta},\beta) = s(\hat{a},\hat{b},\hat{\upsilon},a,b,\upsilon)$$

$$= E_{\omega/a,b,\upsilon}\{y_{k+1}^2\} \tag{3}$$

Finally from (1) and (2) $s(\hat{\beta},\beta)$ becomes:

$$s(\hat{\beta},\beta) = \frac{U}{1-[a-\frac{b\hat{a}}{\hat{b}}]^2} \tag{4}$$

The weighting matrix, Γ, may be readily evaluated from (4) and (2.14).

APPENDIX B

A Criterion for Structure Determination

Firstly the (non-optimal) decision function is described for a simple minimum variance control application:

Structure 1:

$$H_1 : y_k = \beta_1 u_k + \varepsilon_k \tag{1}$$

where $\{\varepsilon_k\}$ is a sequence of independent identically distributed random variables having zero mean and variance β_2.

Structure 2:

$$H_2 : y_k = \theta_1 y_k + \theta_2 u_k + \omega_k \tag{2}$$

where $\{\omega_k\}$ is a sequence of independent identically distributed random variables having zero mean and variance θ_3.

From analysis of a given finite set of data, $\{d\}$, the random variable (β_1,β_2) has posterior distribution (given H_1) with mean

$(\hat{\beta}_1,\hat{\beta}_2)$ and covariance R_β. From the same set of data the random variable $(\theta_1,\theta_2,\theta_3)$ has posterior distribution (given H_2) with mean $(\hat{\theta}_1,\hat{\theta}_2,\hat{\theta}_3)$ and covariance R_θ. It is assumed that $p(H_1/\{d\})$ and $p(H_2/\{d\})$ cannot be found with the analysis program available.

Two controller structures are proposed, the parameters of which are based only on the posterior means of the parameters in the corresponding model structure:

Controller 1:

$$u_k = 0 \tag{3}$$

This is the minimum variance control law corresponding to H_1.

Controller 2:

$$u_k = -\frac{\hat{\theta}_1}{\hat{\theta}_2} y_k \tag{4}$$

This is the minimum variance control law corresponding to H_2.

Now the costs s_{ij} associated with true structure i and controller j are calculated:

$$\begin{aligned} s_{11}(\beta,\hat{\beta}) &= E_{\varepsilon/\beta}[y_{k+1}^2] \\ &= \beta_2 \end{aligned} \tag{5}$$

$$s_{12}(\beta,\hat{\theta}) = E_{\varepsilon/\beta}[y_{k+1}^2]$$

$$= \frac{\beta_2}{[1+\beta_1(\frac{\hat{\theta}_1}{\hat{\theta}_2})]^2} \tag{6}$$

$$s_{21}(\theta,\hat{\beta}) = E_{\omega/\theta}[y_{k+1}^2]$$

$$= \frac{\theta_3}{1-\theta_1^2} \tag{7}$$

$$s_{22}(\theta,\hat{\theta}) = E_{\omega/\theta}[y_{k+1}^2]$$

$$= \frac{\theta_3}{1-[\theta_1 - \frac{\theta_2\hat{\theta}_1}{\hat{\theta}_2}]^2} \tag{8}$$

The average risk of controller i is now defined:

$$\bar{r}_i = \frac{1}{2}[E_{\beta/\{d\},H_1} s_{1i}(\beta) + E_{\theta/\{d\},H_2} s_{2i}(\theta)] \tag{9}$$

which yields (neglecting moments of third order and higher):

$$\bar{r}_1 = \frac{1}{2}[\hat{\beta}_2 + \frac{\hat{\theta}_3}{1-\hat{\theta}_1^2}\{1+\frac{R_\theta^{11}(1+3\theta_1^2)}{(1-\hat{\theta}_1^2)}\} + \frac{2R_\theta^{13}\hat{\theta}_1}{(1-\hat{\theta}_1^2)^2}] \tag{10}$$

$$\bar{r}_2 = \frac{1}{2}\frac{\hat{\beta}_2}{[1+\hat{\beta}_1(\frac{\hat{\theta}_1}{\hat{\theta}_2})]^2}\{1+\frac{3R_\beta^{11}(\frac{\hat{\theta}_1}{\hat{\theta}_2})^2}{[1+\hat{\beta}_1(\frac{\hat{\theta}_1}{\hat{\theta}_2})]^2}\}$$

$$+ \hat{\theta}_3\{1+R_\theta^{11}-2R^{12}\frac{\hat{\theta}_1}{\hat{\theta}_2}+R_\theta^{22}(\frac{\hat{\theta}_1}{\hat{\theta}_2})^2\}$$

$$- \frac{2R_\beta^{12}(\frac{\hat{\theta}_1}{\hat{\theta}_2})}{[1+\beta_1(\frac{\hat{\theta}_1}{\hat{\theta}_2})]^3} \tag{11}$$

where R_β^{ij} and R_θ^{ij} refer to elements in R_β, R_θ respectively.

The decision rule is: choose controller i such that $\bar{r}_i = \min(\bar{r}_1, \bar{r}_2)$.

The experiment may now be designed to minimise the expected average risk or some other suitable criterion. In spite of the simplifying assumptions and the very simple model structures, it can be seen that this is far from trivial!

CHAPTER 4

Time Domain Designs

1. Introduction

In this chapter the problem of optimal input sequence design for linear discrete time systems with stationary disturbances is considered. In section 2 the "innovations" model of such systems is introduced and in sections 3 and 4, the likelihood function and Fisher's information matrix are derived. Several properties of the information matrix which will prove to be useful for design purposes are stated and proved.

In section 5, a method for computing the expected return from an experiment is presented. The method is based on first and/or second moments of the prior probability density function $p(\beta)$.

In section 6, a theorem on the equivalence of optimality criteria is stated and proved and is shown to lead to simplified designs. Necessary conditions for optimality are also stated and proved via the Minimum Principle, for both energy and amplitude constraints. It is further shown that an optimal amplitude constrained input is binary.

In section 7, a general design algorithm based on the necessary conditions for optimality is described. A theorem which leads to simplification of energy constrained designs is stated and proved. Section 8 contains a discussion on the computational aspects of the algorithm and it is shown that significant simplifications can be obtained by exploiting the structure of the sensitivity equations. In section 9 it is shown that even greater simplifications are possible for single output systems. Several examples of optimal designs obtained using these algorithms are given in the appendices.

2. Model Structure

The model considered is for a system having r-dimensional control input sequence, $\{u_k\}$, and m-dimensional output sequence, $\{y_k\}$. The output (with no control input) has spectral density $\Phi(z)$, assumed to be rational, having full rank and no poles on the unit circle. Youla , [29], has shown that there exists a factorisation of $\Phi(z)$ such that

$$\Phi(z) = \Psi(z)\Sigma\Psi^T(z^{-1}) \tag{1}$$

where Ψ and Ψ^{-1} are stable. The spectral density given in (1) can be realized by the following noise model:

$$y_k = \Psi(z)\ell_k \tag{2}$$

where $\{\ell_k\}$ is a sequence of zero mean, m-dimensional independent random variables having covariance Σ. The complete model is thus:

$$y_k = Z(z)u_k + \Psi(z)\ell_k \tag{3}$$

where Z(z) is the rational transfer function from input to output. (Note: z may be interpreted either as the unit shift operator: $zx_k=x_{k+1}$, e.g. (2) and (3), or as $z=e^{j\omega}$ as in (1).) The transfer function model, (3), has a minimal n-state space representation of the form:

$$x_{k+1} = Ax_k + Bu_k + K\ell_k \tag{4}$$

$$y_k = Cx_k + Du_k + \ell_k \tag{5}$$

$$x_1 = \gamma \tag{6}$$

where both A and A-KC are stable (i.e. have eigenvalues with negative real parts). See for example [71].

This is a general model for a linear time invariant discrete dynamic system and includes the special canonical structures proposed, for example, by Mayne, [30], and Caines, [31].

3. The Likelihood Function

For the purpose of constructing the likelihood function, the noise sequence, $\{\ell_k, k=1,\ldots,N\}$, is assumed to be normally distributed. That is, the joint probability distribution of $\ell_1,\ldots,\ell_N$ is of the form:

$$p(\ell_1,\ldots,\ell_N\} = ((2\pi)^m \det\Sigma)^{-\frac{N}{2}} \exp\{-\frac{1}{2}\sum_{k=1}^{N} \ell_k^T \Sigma^{-1} \ell_k\} \tag{1}$$

Now equations (2.4), (2.5) and (2.6) may be rewritten in the form:

$$y = E\ell + Fx_1 + Gu \tag{2}$$

where $y^T = (y_1^T,\ldots,y_N^T)$, $\ell^T = (\ell_1^T,\ldots,\ell_N^T)$ and E, F, G, are $Nm\times Nm$, $Nm\times n$ and $Nm\times Nr$ matrices respectively and may be derived from (2.4)-(2.6). The important thing to note however, is that the diagonal elements of E are unity and that all elements above the diagonal of E are zero. Thus the Jacobian of the transformation from ℓ to y given by (2) is $\det E = 1$ and it can be shown that:

$$p(y/u,\beta) = ((2\pi)^m \det\Sigma)^{-\frac{N}{2}} \exp\{-\frac{1}{2}\sum_{k=1}^{N}(y_k-\bar{y}_k)^T\Sigma^{-1}(y_k-\bar{y}_k)\} \quad (3)$$

where $\bar{y}_k$, the conditional mean given $y_{k-1},\ldots,y_1$, is given by:

$$\bar{y}_k = C\bar{x}_k + Du_k \quad (4)$$

$$\bar{x}_{k+1} = A\bar{x}_k + Bu_k + K(y_k-\bar{y}_k) \quad (5)$$

$$\bar{x}_1 = \gamma \quad (6)$$

and

$$u = (u_1^T,\ldots,u_N^T)^T \quad (7)$$

Thus the likelihood function for β is given by

$$lik(\beta/u,y) = ((2\pi)^m \det\Sigma)^{-\frac{N}{2}} \exp\{-\frac{1}{2}\sum_{k=1}^{N}\omega_k^T\Sigma^{-1}\omega_k\} \quad (8)$$

where the innovations sequence $\{\omega_k\}$, [74], is given by:

$$\omega_k = y_k - \bar{y}_k \ , \quad k = 1, \ldots, N \quad (9)$$

The sequence $\{\omega_k, k=1,\ldots,N\}$ is a sequence of independent normally distributed random variables with joint distribution:

$$p(\omega/\beta) = ((2\pi)^m \det\Sigma)^{-\frac{N}{2}} \exp\{-\frac{1}{2}\sum_{k=1}^{N}\omega_k^T\Sigma^{-1}\omega_k\} \quad (10)$$

where:

$$\omega = (\omega_1^T, \ldots, \omega_n^T)^T \tag{11}$$

Comparison of (8) and (10) yields:

$$\text{lik}(\beta/u,y) = p(\omega/\beta) \tag{12}$$

This ties in with the notation used for the likelihood function throughout the rest of this thesis.

4. Fisher's Information Matrix

Fisher's information matrix, M, is defined by:

$$M = E_{\omega/\beta}\left[\left(\frac{\partial \log p(\omega/\beta)}{\partial \beta}\right)^T \left(\frac{\partial \log p(\omega/\beta)}{\partial \beta}\right)\right] \tag{1}$$

where the likelihood function for β, $p(\omega/\beta)$, is given by (3.10) and $E_{\omega/\beta}[\cdot]$ is the mathematical expectation defined by:

$$E_{\omega/\beta}[\cdot] = \int_{\Omega} (\cdot) p(\omega/\beta) d\omega \tag{2}$$

Ω is the space of all ω defined by (2.9), (2.4)-(2.6).

For the purposes of calculating the derivatives indicated in (1), the following partition of β is considered:

$$\beta^T = (\theta^T, \sigma^T) \tag{3}$$

where Θ is taken to be that part of β which contains elements of A, B, K,C,D and γ, and σ is the vector of elements of Σ. Thus M may be partitioned in an obvious manner:

$$M = \begin{bmatrix} M_{\Theta\Theta} & M_{\Theta\sigma} \\ \hline M_{\Theta\sigma}^{T} & M_{\sigma\sigma} \end{bmatrix} \tag{4}$$

To obtain expressions for the submatrices in (4), the following lemmas will prove useful:

Lemma 1:

For any random variables X, Y, and Z with Y independent of X and Z, it is true that:

$$E[A(X)B(Y)C(Z)] = E[A(X)E[B(Y)]C(Z)] \tag{5}$$

where A, B, C are matrix functions of X, Y, Z respectively.

Proof:

The ij^{th} element of $E[ABC]$ denoted d_{ij}, is given by:

$$d_{ij} = \sum_{\rho,q} E[a_{i\rho}(X)b_{\rho q}(Y)c_{qj}(Z)]$$

$$= \sum_{\rho,q} E[a_{i\rho}(X)c_{qj}(Z)]E[b_{\rho q}(Y)]$$

since Y is independent of X and Z. Thus d_{ij} is also the ij^{th} term of $E[A\ E[B]C]$. #

Lemma 2:

The random variables ω_k, k = 1, ...N, *defined by* (3.9) *are functions of* θ *and have derivatives* $\frac{\partial \omega_k}{\partial \theta_i}$ *which are independent of* ω_ℓ *for all* ℓ *greater than or equal to* k.

Proof:

Equations (3.9) and (3.4) - (3.6) may be rewritten in the form:

$$\omega_k = y_k - Cx_k - Du_k \tag{6}$$

$$x_{k+1} = Ax_k + Bu_k + K\omega_k \tag{7}$$

$$x_1 = \gamma \tag{8}$$

It is obvious that ω_k depends on θ (the elements of A, B, K, C, D, γ) and the derivatives $\frac{\partial \omega_k}{\partial \theta_i}$ may be obtained by differentiating (6) - (8).

$$\frac{\partial \omega_k}{\partial \theta_i} = -C \frac{\partial x_k}{\partial \theta_i} - \frac{\partial C}{\partial \theta_i} x_k - \frac{\partial D}{\partial \theta_i} u_k \tag{9}$$

$$\frac{\partial x_{k+1}}{\partial \theta_i} = (A-KC)\frac{\partial x_k}{\partial \theta_i} + (\frac{\partial A}{\partial \theta_i} - K \frac{\partial C}{\partial \theta_i}) x_k + (\frac{\partial B}{\partial \theta_i} - K\frac{\partial D}{\partial \theta_i}) u_k + \frac{\partial K}{\partial \theta_i} \omega_k \tag{10}$$

$$\frac{\partial x_1}{\partial \theta_i} = \frac{\partial \gamma}{\partial \theta_i} \tag{11}$$

Now, from (7), it can be seen that x_k depends only on ω_ℓ for $\ell < k$ and similarly from (10) $\frac{\partial x_k}{\partial \theta_i}$ depends only on ω_ℓ for $\ell < k$. #

Result 1:

$$M_{\theta\theta} = \sum_{k=1}^{N} E_{\omega/\beta}[(\frac{\partial \omega_k}{\partial \theta})^T \Sigma^{-1} (\frac{\partial \omega_k}{\partial \theta})] \tag{12}$$

Proof:

Differentiating the log of (3.10) with respect to Θ_i yields:

$$\frac{\partial \log p(\omega/\beta)}{\partial \Theta_i} = -\sum_{k=1}^{N} \omega_k^T \Sigma^{-1} \frac{\partial \omega_k}{\partial \Theta_i} \tag{13}$$

Thus, from (1):

$$(M_{\Theta\Theta})_{ij} = E_{\omega/\beta}\Big[\sum_{k=1}^{N} \Big(\frac{\partial \omega_k}{\partial \Theta_i}\Big)^T \Sigma^{-1} \omega_k \sum_{\ell=1}^{N} \omega_\ell^T \Sigma^{-1} \Big(\frac{\partial \omega_\ell}{\partial \Theta_j}\Big)\Big].$$

$$= \sum_{k=1}^{N-1} \sum_{\ell=k+1}^{N} E_{\omega/\beta}\Big[\Big(\frac{\partial \omega_k}{\partial \Theta_i}\Big)^T \Sigma^{-1} \omega_k \omega_\ell^T \Sigma^{-1} \Big(\frac{\partial \omega_\ell}{\partial \Theta_j}\Big)\Big]$$

$$+ \sum_{\ell=1}^{N-1} \sum_{k=\ell+1}^{N} E_{\omega/\beta}\Big[\Big(\frac{\partial \omega_k}{\partial \Theta_i}\Big)^T \Sigma^{-1} \omega_k \omega_\ell^T \Sigma^{-1} \Big(\frac{\partial \omega_\ell}{\partial \Theta_j}\Big)\Big]$$

$$+ \sum_{k=1}^{N} E_{\omega/\beta}\Big[\Big(\frac{\partial \omega_k}{\partial \Theta_i}\Big)^T \Sigma^{-1} \omega_k \omega_k^T \Sigma^{-1} \Big(\frac{\partial \omega_k}{\partial \Theta_j}\Big)\Big]$$

$$= 0 + 0 + \sum_{k=1}^{N} E_{\omega/\beta}\Big[\Big(\frac{\partial \omega_k}{\partial \Theta_i}\Big)^T \Sigma^{-1} \Big(\frac{\partial \omega_k}{\partial \Theta_j}\Big)\Big]$$

where lemmas 1 and 2 together with the results:

$$E_{\omega/\beta}[\omega_k] = 0 \; ; \quad E_{\omega/\beta}[\omega_k \omega_k^T] = \Sigma \tag{14}$$

have been used. #

Result 2:

The ij^{th} element of $M_{\sigma\sigma}$ is given by:

$$(M_{\sigma\sigma})_{ij} = \frac{N}{4}\{(\Sigma^{-1})_{\rho\rho'}(\Sigma^{-1})_{qq'} + (\Sigma^{-1})_{\rho q'}(\Sigma^{-1})_{\rho' q}\} \qquad (15)$$

where σ_i corresponds to $(\Sigma)_{\rho q}$ and σ_j corresponds to $(\Sigma)_{\rho' q'}$.

Proof:

Differentiating the log of (3.10) with respect to $\sigma_i = (\Sigma)_{\rho q}$ yields:

$$\frac{\partial \log p(\omega/\beta)}{\partial \sigma_i} = -\frac{N}{2}(\Sigma^{-1})_{\rho q} + \frac{1}{2}(\Sigma^{-1}(\sum_{k=1}^{N}\omega_k\omega_k^T)\Sigma^{-1})_{\rho q} \qquad (16)$$

Taking σ_j to be the $\rho' q'^{th}$ elements of Σ it follows from (1) that:

$$(M_{\sigma\sigma})_{ij} = E_{\omega/\beta}[\{-\frac{N}{2}(\Sigma^{-1})_{\rho q} + \frac{1}{2}\sum_{s=1}^{m}\sum_{t=1}^{m}(\Sigma^{-1})_{\rho s}\sum_{k=1}^{N}(\omega_k)_s(\omega_k)_t(\Sigma^{-1})_{tq}\}$$

$$\cdot\{-\frac{N}{2}(\Sigma^{-1})_{\rho' q'} + \frac{1}{2}\sum_{s'=1}^{m}\sum_{t'=1}^{m}(\Sigma^{-1})_{\rho' s'}\sum_{\ell=1}^{N}(\omega_\ell)_{s'}(\omega_\ell)_{t'}(\Sigma^{-1})_{t' q'}\}]$$

$$= \frac{N^2}{4}(\Sigma^{-1})_{\rho q}(\Sigma^{-1})_{\rho' q'} - \frac{N^2}{2}(\Sigma^{-1})_{\rho q}(\Sigma^{-1})_{\rho' q'}$$

$$+\frac{1}{4}\sum_{s=1}^{m}\sum_{t=1}^{m}\sum_{s'=1}^{m}\sum_{t'=1}^{m}(\Sigma^{-1})_{\rho s}(\Sigma^{-1})_{tq}(\Sigma^{-1})_{\rho' s'}(\Sigma^{-1})_{t' q'}$$

$$\cdot\{\sum_{k=1}^{N-1}\sum_{\ell=k+1}^{N}E_{\omega/\beta}[(\omega_k)_s(\omega_k)_t(\omega_\ell)_{s'}(\omega_\ell)_{t'}]$$

$$+\sum_{\ell=1}^{N-1}\sum_{k=\ell+1}^{N}E_{\omega/\beta}[(\omega_k)_s(\omega_k)_t(\omega_\ell)_{s'}(\omega_\ell)_{t'}] + \sum_{k=1}^{N}E_{\omega/\beta}[(\omega_k)_s(\omega_k)_t(\omega_k)_{s'}(\omega_k)_{t'}]\}$$

$$= -\frac{N^2}{4}(\Sigma^{-1})_{\rho q}(\Sigma^{-1})_{\rho' q'} + \frac{(N^2-N)}{4}(\Sigma^{-1})_{\rho q}(\Sigma^{-1})_{\rho' q'}$$

$$+\frac{N}{4}\sum_{s=1}^{m}\sum_{t=1}^{m}\sum_{s'=1}^{m}\sum_{t'=1}^{m}(\Sigma^{-1})_{\rho s}(\Sigma^{-1})_{tq}(\Sigma^{-1})_{\rho' s'}(\Sigma^{-1})_{t'q'}\{(\Sigma)_{st}(\Sigma)_{s't'}$$

$$+(\Sigma)_{ss'}(\Sigma)_{tt'}+(\Sigma)_{st'}(\Sigma)_{ts'}\}$$

$$=-\frac{N}{4}(\Sigma^{-1})_{\rho q}(\Sigma^{-1})_{\rho' q'}+\frac{N}{4}\{(\Sigma^{-1})_{\rho q}(\Sigma^{-1})_{\rho' q'}+(\Sigma^{-1})_{\rho\rho'}(\Sigma^{-1})_{qq'}+(\Sigma^{-1})_{\rho q'}(\Sigma^{-1})_{\rho$$

$$=\frac{N}{4}\{(\Sigma^{-1})_{\rho\rho'}(\Sigma^{-1})_{qq'}+(\Sigma^{-1})_{\rho q'}(\Sigma^{-1})_{\rho' q}\}$$

where again lemmas 1 and 2 together with (14) have been used. #

Result 3:

$$M_{\theta\sigma}=0 \tag{17}$$

Proof:

From (13), (16) and (1), the ij^{th} element of $M_{\theta\sigma}$ is:

$$(M_{\theta\sigma})_{ij}=E_{\omega/\beta}\sum_{k=1}^{N}(\frac{\partial\omega_k}{\partial\theta_i})^T\Sigma^{-1}\omega_k\{-\frac{N}{2}(\Sigma^{-1})_{\rho q}$$

$$+\frac{1}{2}\sum_{s=1}^{m}\sum_{t=1}^{m}(\Sigma^{-1})_{is}\sum_{\ell=1}^{N}(\omega_\ell)_s(\omega_\ell)_t(\Sigma^{-1})_{tj}\}]$$

$$=0+\frac{1}{2}\sum_{s=1}^{m}\sum_{t=1}^{m}(\Sigma^{-1})_{is}(\Sigma^{-1})_{tj}\{\sum_{k=1}^{N-1}\sum_{\ell=k+1}^{N}E_{\omega/\beta}[(\frac{\partial\omega_k}{\partial\theta_i})^T\Sigma^{-1}\omega_k(\omega_\ell)_s(\omega_\ell$$

$$+\sum_{\ell=1}^{N-1}\sum_{k=\ell+1}^{N}E_{\omega/\beta}[(\frac{\partial\omega_k}{\partial\theta_i})^T\Sigma^{-1}\omega_k(\omega_\ell)_s(\omega_\ell)_t]$$

$$+\sum_{k=1}^{N}E_{\omega/\beta}[(\frac{\partial\omega_k}{\partial\theta_i})^T\Sigma^{-1}\omega_k(\omega_\ell)_s(\omega_\ell)_t]\}$$

$$=\frac{1}{2}\sum_{s=1}^{m}\sum_{t=1}^{m}(\Sigma^{-1})_{is}(\Sigma^{-1})_{tj}\{0+0+0\}$$

where lemmas 1 and 2 together with the additional result:

$$E_{\omega/\beta}[(\omega_k)_i(\omega_k)_j(\omega_k)_\ell] = 0 \quad \text{for all } i, j, k, \ell$$

(which is a result of the normality of ω) have been used. #

A further partition of β is now considered:

$$\beta^T = (\alpha^T, \gamma^T, \sigma^T) \tag{19}$$

where α contains elements of A, B, K, C, D; γ is the initial state vector and σ contains elements of Σ as in (3). The partition of β, (19), corresponds to a partition of θ:

$$\theta^T = (\alpha^T, \gamma^T) \tag{20}$$

and the matrix $M_{\theta\theta}$ may therefore be partitioned in an obvious way:

$$M_{\theta\theta} = \begin{bmatrix} M_{\alpha\alpha} & M_{\alpha\gamma} \\ \hline M_{\alpha\gamma}^T & M_{\gamma\gamma} \end{bmatrix} \tag{21}$$

Result 4:

The limits as N *approaches infinity of the submatrices* $M_{\gamma\gamma}$ *and* $M_{\alpha\gamma}$ *are finite.*

Proof:

The elements of $M_{\alpha\gamma}$ and $M_{\gamma\gamma}$ are readily obtained by substituting the expressions for $\frac{\partial\omega_k}{\partial\theta_i}$ obtained from (9) - (11) into (12). For this case, however, where $\theta_j = \gamma_q$, say, the equations (9) - (11) may be written as:

$$\frac{\partial \omega_k}{\partial \gamma_q} = - C \frac{\partial x_k}{\partial \gamma_q}$$

$$\frac{\partial x_{k+1}}{\partial \gamma_q} = (A-KC) \frac{\partial x_k}{\partial \gamma_q}$$

$$\frac{\partial x_1}{\partial \gamma_q} = \frac{\partial \gamma}{\partial \gamma_q}$$

Now, since (A-KC) has eigenvalues with negative real parts, [71], and since $\frac{\partial \omega_k}{\partial \theta_i}$ is bounded for bounded γ, u_k, the result follows from the convergence of the series $\sum_k e^{-bk}\cos(\nu k+\phi)$ for arbitrary ν, ϕ and positive b. #

Result 5:

The information matrix M, may be expressed as the sum of a matrix M_u depending on the input and a matrix M_c independent of the input.

Proof:

It is immediately obvious that $M_{\sigma\sigma}$ is independent of the input. Further, by superposition, equations (6) - (11) may be rewritten as:

$$\frac{\partial \omega_k}{\partial \theta_i} = \frac{\partial \omega_k'}{\partial \theta_i} + \frac{\partial \omega''}{\partial \theta_i} \tag{22}$$

where

$$\frac{\partial \omega_k'}{\partial \theta_i} = - C \frac{\partial x_k'}{\partial \theta_i} - \frac{\partial C}{\partial \theta_i} x_k' - \frac{\partial D}{\partial \theta_i} u_k \tag{23}$$

$$\frac{\partial x_{k+1}'}{\partial \theta_i} = (A-KC) \frac{\partial x_k'}{\partial \theta_i} + (\frac{\partial A}{\partial \theta_i} - K \frac{\partial C}{\partial \theta_i}) x_k' + (\frac{\partial B}{\partial \theta_i} - K \frac{\partial D}{\partial \theta_i}) u_k \tag{24}$$

$$\frac{\partial x_k'}{\partial \theta_i} = \frac{\partial \gamma}{\partial \theta_i} \tag{25}$$

$$x'_{k+1} = Ax'_k + Bu_k \tag{26}$$

$$x'_1 = \gamma \tag{27}$$

and

$$\frac{\partial \omega''_k}{\partial \theta_i} = - C \frac{\partial x''_k}{\partial \theta_i} - \frac{\partial C}{\partial \theta_i} x''_k \tag{28}$$

$$\frac{\partial x''_{k+1}}{\partial \theta_i} = (A-KC) \frac{\partial x''_k}{\partial \theta_i} + (\frac{\partial A}{\partial \theta_i} - K \frac{\partial C}{\partial \theta_i}) x''_k + \frac{\partial K}{\partial \theta_i} \omega_k \tag{29}$$

$$\frac{\partial x''_k}{\partial \theta_i} = O \tag{30}$$

$$x''_{k+1} = Ax''_k + K\omega_k \tag{31}$$

$$x''_1 = O \tag{32}$$

Thus, from (12) and (22):

$$(M_{\theta\theta})_{ij} = \sum_{k=1}^{N} E_{\omega/\beta}[(\frac{\partial \omega'_k}{\partial \theta_i}) T_\Sigma^{-1} (\frac{\partial \omega'_k}{\partial \theta_j})] + \sum_{k=1}^{N} E_{\omega/\beta}[(\frac{\partial \omega'_k}{\partial \theta_i}) T_\Sigma^{-1} (\frac{\partial \omega''_k}{\partial \theta_j})]$$

$$+ \sum_{k=1}^{N} E_{\omega/\beta}[(\frac{\partial \omega''_k}{\partial \theta_i}) T_\Sigma^{-1} (\frac{\partial \omega'_k}{\partial \theta_j})] + \sum_{k=1}^{N} E_{\omega/\beta}[(\frac{\partial \omega''_k}{\partial \theta_i}) T_\Sigma^{-1} (\frac{\partial \omega''_k}{\partial \theta_j})]$$

Since $\{u_k\}$ and $\{\omega_k\}$ are independent it follows that:

$$M = \begin{bmatrix} M' & O \\ O & O \end{bmatrix} + \begin{bmatrix} M'' & O \\ O & M_{\sigma\sigma} \end{bmatrix} \tag{33}$$

where

$$(M')_{ij} = \sum_{k=1}^{N} \left[\left(\frac{\partial \omega_k'}{\partial \theta_i}\right)^T \Sigma^{-1} \left(\frac{\partial \omega_k'}{\partial \theta_j}\right) \right] \tag{34}$$

$$(M'')_{ij} = \sum_{k=1}^{N} E_{\omega/\beta} \left[\left(\frac{\partial \omega_k''}{\partial \theta_i}\right) \Sigma^{-1} \left(\frac{\partial \omega_k''}{\partial \theta_j}\right) \right] \tag{35}$$

Since M" is independent of $\{u_k\}$, the result is proved. #

Result 6:

The matrices M_u *depending on the input and* M_c *independent of the input may be written as*

$$M_u = N \begin{bmatrix} \frac{1}{N} M_{BDCA} & 0 & \frac{1}{N} M_{\alpha\gamma} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{N} M_{\alpha\gamma}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & M_{CAK} & 0 & 0 & \\ 0 & & 0 & 0 & \\ 0 & 0 & 0 & M_{\gamma\gamma} & 0 \\ 0 & 0 & 0 & 0 & M_{\sigma\sigma} \end{bmatrix} \tag{36}$$

where the partitions correspond to the partition of β *symbolised by*

$\beta \sim [\, B, D \mid C, A \mid K \mid \gamma \mid \sigma \,]$ or $\alpha \sim [\, B, D \mid C, A \,]$

Proof:

(i) The form of M_u follows from (33), (21) and the fact that

$$\frac{\partial \omega_k'}{\partial \theta_i} = 0 \text{ for } \theta_i = \text{an element of K.}$$

(ii) The form of M_c follows from (33) and the facts that $M_{\gamma\gamma}$ is independent of $\{u_k\}$ and that $\frac{\partial \omega_k''}{\partial \theta_i} = 0$ for θ_i an element of B or D. #

Result 7:

As N approaches infinity $\frac{1}{N}M_u$ *may be written as:*

$$\lim_{N\to\infty} \frac{1}{N} M_u = \lim_{N\to\infty} \begin{bmatrix} \frac{1}{N}M_{BDCA} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{37}$$

Proof:

The result follows immediately from result 4, i.e. that $M_{\alpha\gamma}$ is finite so that $\frac{1}{N}M_{\alpha\gamma}$ tends to zero as N tends to infinity. #

5. Optimality Criterion

It was shown in chapter 3 that suitable criteria for optimality are:

(a) $$J_1 = E_\beta[\text{logdet}(M_\beta + P)^{-1}] \tag{1}$$

(b) $$J_2 = E_\beta[\text{trace}\{\Gamma(M_\beta + P)^{-1}\}] \tag{2}$$

where M_β denotes M evaluated at β, $P = P_\beta^{-1}$, and E_β denotes expectation over the prior distribution $p(\beta)$.

Definition:

A sequence $\{u_k\}$ which minimises J_i is said to be J_i-optimal, $i = 1, 2$. #

The evaluation of either J_1 or J_2 poses a problem in general due to the expectation which involves integration over a p-dimensional space. A common approach to this type of problem is to use a discrete approximation for the prior distribution so that integration is replaced by summation, [28]. This can, however, lead to a prohibitively large number of calculations, even for small systems and with coarse quantization of the prior distribution, [72]. An alternative approach using only first and second moments of the prior distribution is now proposed.

From Taylor's Formula:

$$\begin{aligned}\operatorname{logdet}(M_\beta+P)^{-1} = \operatorname{logdet}(M_{\bar\beta}+P)^{-1} &- \sum_{i=1}^{p} \operatorname{trace}\{(M_{\bar\beta}+P)^{-1}\frac{\partial M}{\partial\beta_i}\}(\beta_i-\bar\beta_i) \\ &+ \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}[\operatorname{trace}\{(M_{\bar\beta}+P)^{-1}\frac{\partial M}{\partial\beta_i}(M_{\bar\beta}+P)^{-1}\frac{\partial M}{\partial\beta_j}\} \\ &+ \operatorname{trace}\{(M_{\bar\beta}+P)^{-1}\frac{\partial^2 M}{\partial\beta_i\partial\beta_j}\}](\beta_i-\bar\beta_i)(\beta_j-\bar\beta_j) + \ldots \end{aligned} \tag{3}$$

and

$$\begin{aligned}\operatorname{Trace}\{\Gamma(M_\beta+P)^{-1}\} = \operatorname{trace}&\{\Gamma(M_{\bar\beta}+P)^{-1}\} \\ &- \sum_{i=1}^{p}\operatorname{trace}\{(M_{\bar\beta}+P)^{-1}\Gamma(M_{\bar\beta}+P)^{-1}\frac{\partial M}{\partial\beta_i}(\beta_i-\bar\beta_i) \\ &+ \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}[\operatorname{trace}\{(M_{\bar\beta}+P)^{-1}\Gamma(M_{\bar\beta}+P)^{-1}\frac{\partial M}{\partial\beta_i}(M_{\bar\beta}+P)^{-1}\frac{\partial M}{\partial\beta_j}\} \\ &+ \operatorname{trace}\{(M_{\bar\beta}+P)^{-1}\frac{\partial M}{\partial\beta_i}(M_{\bar\beta}+P)^{-1}\Gamma(M_{\bar\beta}+P)^{-1}\frac{\partial M}{\partial\beta_j}\}\end{aligned}$$

$$+ \text{trace}\{(M_{\bar{\beta}}+P)^{-1}\Gamma(M_{\bar{\beta}}+P)^{-1}\frac{\partial^2 M}{\partial\beta_i\partial\beta_j}\}](\beta_i-\bar{\beta}_j)(\beta_j-\bar{\beta}_j)$$

$$+ \ldots. \qquad (4)$$

Taking expectations as indicated in (1) and (2) and neglecting high order terms in (3) and (4) leads to:

$$J_1 = \text{logdet}(M_{\bar{\beta}}+P)^{-1} + \frac{1}{2}\text{trace}\{V_1 P_\beta\} \qquad (5)$$

$$J_2 = \text{trace}\{\Gamma(M_{\bar{\beta}}+P)^{-1}\} + \frac{1}{2}\text{trace}\{V_2 P_\beta\} \qquad (6)$$

where

$$(V_1)_{ij} = \text{trace}\{(M_{\bar{\beta}}+P)^{-1}[\frac{\partial M}{\partial\beta_i}(M_{\bar{\beta}}+P)^{-1}\frac{\partial M}{\partial\beta_j} + \frac{\partial^2 M}{\partial\beta_i\partial\beta_j}]; \qquad (7)$$

and

$$(V_2)_{ij} = \text{trace}\{(M_{\bar{\beta}}+P)^{-1}\Gamma(M_{\bar{\beta}}+P)^{-1}[\frac{\partial M}{\partial\beta_i}(M_{\bar{\beta}}+P)^{-1}\frac{\partial M}{\partial\beta_j} + \frac{\partial M}{\partial\beta_j}(M_{\bar{\beta}}+P)^{-1}\frac{\partial M}{\partial\beta_i}$$

$$+ \frac{\partial^2 M}{\partial\beta_i\partial\beta_j}]\} \qquad (8)$$

Thus, it can be seen from equations (5)-(8) that an approximation to J_1 and J_2 based on first and second moments of $p(\beta)$, can be obtained from the information matrix and its first and second derivatives with respect to β (all evaluated at the prior mean $\bar{\beta}$). Expressions for these derivatives are now obtained for the case $\beta_i \in \alpha_i$. (For simplicity of exposition γ is assumed to be zero and $\sigma(\Sigma)$ is assumed to be known.) Differentiating the expression for $(M)_{ij}$ given by (4.12) and using the fact that the third moments of $p(\omega/\beta)$ are zero for Gaussian ω leads to:

$$\frac{\partial (M)_{ij}}{\partial \beta_\ell} = \sum_{k=1}^{N} E_{\omega/\beta}\left[\left(\frac{\partial^2 \omega_k}{\partial \beta_\ell \partial \beta_i}\right)^T \Sigma^{-1} \left(\frac{\partial \omega_k}{\partial \beta_j}\right) + \left(\frac{\partial \omega_k}{\partial \beta_i}\right)^T \Sigma^{-1} \left(\frac{\partial^2 \omega_k}{\partial \beta_\ell \partial \beta_j}\right) \right] \quad (9)$$

and

$$\frac{\partial^2 (M)_{ij}}{\partial \beta_\ell \partial \beta_q} = \sum_{k=1}^{N} E_{\omega/\beta}\left[\left(\frac{\partial^3 \omega_k}{\partial \beta_\ell \partial \beta_q \partial \beta_i}\right)^T \Sigma^{-1} \left(\frac{\partial \omega_k}{\partial \beta_j}\right) + \left(\frac{\partial \omega_k}{\partial \beta_i}\right)^T \Sigma^{-1} \left(\frac{\partial^3 \omega_k}{\partial \beta_\ell \partial \beta_q \partial \beta_j}\right) \right.$$

$$\left. + \left(\frac{\partial^2 \omega_k}{\partial \beta_\ell \partial \beta_i}\right)^T \Sigma^{-1} \left(\frac{\partial^2 \omega_k}{\partial \beta_q \partial \beta_j}\right) + \left(\frac{\partial^2 \omega_k}{\partial \beta_q \partial \beta_i}\right)^T \Sigma^{-1} \left(\frac{\partial^2 \omega_k}{\partial \beta_\ell \partial \beta_j}\right) \right] \quad (10)$$

The derivatives of ω_k are obtained from the following sets of equations:

$$x_{k+1} = Ax_k + Bu_k + K\omega_k \quad (11)$$

$$x_1 = 0 \quad (12)$$

$$\frac{\partial x_{k+1}}{\partial \beta_i} = (A-KC)\frac{\partial x_k}{\partial \beta_i} + \left\{\frac{\partial A}{\partial \beta_i} - K\frac{\partial C}{\partial \beta_i}\right\} x_k$$

$$+ \left\{\frac{\partial B}{\partial \beta_i} - K\frac{\partial D}{\partial \beta_i}\right\} u_k + \frac{\partial K}{\partial \beta_i}\omega_k \quad (13)$$

$$\frac{\partial x_1}{\partial \beta_i} = 0 \quad (14)$$

$$\frac{\partial^2 x_{k+1}}{\partial \beta_i \partial \beta_j} = (A-KC)\frac{\partial^2 x_k}{\partial \beta_i \partial \beta_j} + \left\{\frac{\partial A}{\partial \beta_i} - K\frac{\partial C}{\partial \beta_i} - \frac{\partial K}{\partial \beta_i}C\right\}\frac{\partial x_k}{\partial \beta_j}$$

$$+ \left\{\frac{\partial A}{\partial \beta_j} - K\frac{\partial C}{\partial \beta_j} - \frac{\partial K}{\partial \beta_j}C\right\}\frac{\partial x_k}{\partial \beta_i} \quad (15)$$

$$\frac{\partial^2 x_1}{\partial \beta_i \partial \beta_j} = 0 \quad (16)$$

$$\frac{\partial^3 x_{k+1}}{\partial\beta_i \partial\beta_j \partial\beta_q} = (A-KC)\frac{\partial^3 x_k}{\partial\beta_i \partial\beta_j \partial\beta_q}$$

$$+ \{\frac{\partial A}{\partial\beta_i} - K\frac{\partial C}{\partial\beta_i} - \frac{\partial K}{\partial\beta_i} C\} \frac{\partial^2 x_k}{\partial\beta_j \partial\beta_q}$$

$$+ \{\frac{\partial A}{\partial\beta_j} - K\frac{\partial C}{\partial\beta_j} - \frac{\partial K}{\partial\beta_j} C\} \frac{\partial^2 x_k}{\partial\beta_i \partial\beta_q}$$

$$+ \{\frac{\partial A}{\partial\beta_q} - K\frac{\partial C}{\partial\beta_q} - \frac{\partial K}{\partial\beta_q} C\} \frac{\partial^2 x_k}{\partial\beta_i \partial\beta_j} \qquad (17)$$

$$\frac{\partial^3 x_1}{\partial\beta_i \partial\beta_j \partial\beta_q} = 0 \qquad (18)$$

$$\frac{\partial\omega_k}{\partial\beta_i} = -\frac{\partial C}{\partial\beta_i} x_k - C\frac{\partial x_k}{\partial\beta_i} - \frac{\partial D}{\partial\beta_i} u_k \qquad (19)$$

$$\frac{\partial^2\omega_k}{\partial\beta_i \partial\beta_j} = -\frac{\partial C}{\partial\beta_i}\frac{\partial x_k}{\partial\beta_j} - C\frac{\partial^2 x_k}{\partial\beta_i \partial\beta_j} \qquad (20)$$

$$\frac{\partial^3\omega_k}{\partial\beta_i \partial\beta_j \partial\beta_q} = -\frac{\partial C}{\partial\beta_i}\frac{\partial^2 x_k}{\partial\beta_j \partial\beta_q} - \frac{\partial C}{\partial\beta_j}\frac{\partial^2 x_k}{\partial\beta_i \partial\beta_q} - \frac{\partial C}{\partial\beta_q}\frac{\partial^2 x_k}{\partial\beta_i \partial\beta_j}$$

$$- C\frac{\partial^3 x_k}{\partial\beta_i \partial\beta_j \partial\beta_q} \qquad (21)$$

The "sensitivity equations", (11)-(18) can now be combined into a single state equation as follows:

$$z_{k+1} = Fz_k + Gu_k + L\omega_k \qquad (22)$$

$$z_1 = 0 \qquad (23)$$

where

$$z_k^T = [x_k^T, \ldots \frac{\partial x_k^T}{\partial\beta_i} \ldots, \ldots \frac{\partial^2 x_k^T}{\partial\beta_i \partial\beta_j} \ldots, \ldots \frac{\partial^3 x_k^T}{\partial\beta_i \partial\beta_j \partial\beta_q} \ldots] \qquad (24)$$

The form of the matrices F, G and L follow immediately from equations (13), (15) and (17). The derivatives of ω_k given in (19), (20) and (21) can now be expressed as:

$$\frac{\partial \omega_k}{\partial \beta_i} = H_i^{(1)} z_k - \frac{\partial D}{\partial \beta_i} u_k \tag{25}$$

$$\frac{\partial^2 \omega_k}{\partial \beta_i \partial \beta_j} = H_{ij}^{(2)} z_k \tag{26}$$

$$\frac{\partial^3 \omega_k}{\partial \beta_i \partial \beta_j \partial \beta_q} = H_{ijq}^{(3)} z_k \tag{27}$$

where matrices $H_i^{(1)}$, $H_{ij}^{(2)}$, $H_{ijq}^{(3)}$ follow from equations (19), (20) and (21).

The expressions for the first and second derivatives of M given in (9) and (10) may now be written as:

$$\frac{\partial (M)_{ij}}{\partial \beta_\ell} = E_{\omega/\bar{\beta}}\left[\sum_{k=1}^{N} z_k^T [R_{ij\ell}^{(1)} z_k - S_{ij\ell}^{(1)} u_k]\right] \tag{28}$$

$$\frac{\partial^2 (M)_{ij}}{\partial \beta_\ell \partial \beta_q} = E_{\omega/\bar{\beta}}\left[\sum_{k=1}^{N} z_k^T [R_{ij\ell q}^{(2)} z_k - S_{ij\ell q}^{(2)} u_k]\right] \tag{29}$$

where

$$R_{ij\ell}^{(1)} = H_{\ell i}^{(2)} T_\Sigma^{-1} H_j^{(1)} T_\Sigma^{-1} H_{\ell j}^{(2)} \tag{30}$$

$$S_{ij\ell}^{(1)} = H_{\ell i}^{(2)} T_\Sigma^{-1} \frac{\partial D}{\partial \beta_j} + H_{\ell j}^{(2)} T_\Sigma^{-1} \frac{\partial D}{\partial \beta_i} \tag{31}$$

$$R_{ij\ell q}^{(2)} = H_{\ell qi}^{(3)} T_\Sigma^{-1} H_j^{(1)} + H_i^{(1)} T_\Sigma^{-1} H_{\ell qj}^{(3)} + H_{\ell i}^{(2)} T_\Sigma^{-1} H_{qj}^{(2)} + H_{qi}^{(2)} T_\Sigma^{-1} H_{\ell j}^{(2)} \tag{32}$$

$$S^{(2)}_{ij\ell q} = H^{(3)}_{\ell qi} T_{\Sigma}^{-1} \frac{\partial D}{\partial \beta_j} + H^{(3)}_{\ell qj} T_{\Sigma}^{-1} \frac{\partial D}{\partial \beta_i} \tag{33}$$

Note that, from equation (4.12) for $M_{\overline{\beta}}$:

$$(M_{\overline{\beta}})_{ij} = E_{\omega/\overline{\beta}}[\sum_{k=1}^{N} [H^{(1)}_i z_k - \frac{\partial D}{\partial \beta_i} u_k]^T \Sigma^{-1} [H^{(1)}_j z_k - \frac{\partial D}{\partial \beta_j} u_k]] \tag{34}$$

To perform the expectation operations indicated in (28), (29) and (34) it is convenient to split z_k into deterministic and stochastic components:

$$z_k = z'_k + z''_k \tag{35}$$

where, from (22) and (23):

$$z'_{k+1} = Fz'_k + Gu_k \tag{36}$$

$$z'_1 = 0 \tag{37}$$

$$z''_{k+1} = Fz''_k + L\omega_k \tag{38}$$

$$z''_1 = 0 \tag{39}$$

Equations (28) and (29) for the derivatives of M_{β} and equation (34) for $M_{\overline{\beta}}$ now reduce to:

$$(M_{\overline{\beta}})_{ij} = \sum_{k=1}^{N} \{H^{(1)}_i z'_k - \frac{\partial D}{\partial \beta_i} u_k\}^T \Sigma^{-1} \{H^{(1)}_j z'_k - \frac{\partial D}{\partial \beta_j} u_k\}$$

$$+ N \text{ trace}\{H^{(1)T}_i \Sigma^{-1} H^{(1)}_j Q\} \tag{40}$$

$$\frac{\partial (M)_{ij}}{\partial \beta_\ell} = \sum_{k=1}^{N} z_k'^T [R^{(1)}_{ij\ell} z_k' - S^{(1)}_{ij\ell} u_k] + N \, \text{trace}\{R^{(1)}_{ij\ell} Q\} \tag{41}$$

$$\frac{\partial^2 (M)_{ij}}{\partial \beta_\ell \partial \beta_q} = \sum_{k=1}^{N} z_k'^T [R^{(2)}_{ij\ell q} z_k' - S^{(2)}_{ij\ell q} u_k] + N \, \text{trace}\{R^{(2)}_{ij\ell q} Q\} \tag{42}$$

where Q is the steady state covariance of z_k'' and satisfies a simple linear equation:

$$Q - FQF^T - L\Sigma L^T = 0 \tag{43}$$

#

Remark 1: Equation (40) for $M_{\underline{\beta}}$ corresponds to expressing $M_{\underline{\beta}}$ as the sum of a constant matrix, M_c, and a matrix depending on the input, M_u. (c.f. result 4.5.) #

Now substituting (40), (41) and (42) back into (5), (6), gives a computational procedure for evaluating the costs, J_1 and J_2. In principle the sequence $\{u_k\}$ which minimises J_1 or J_2 could now be obtained by applying a standard optimisation technique such as differential dynamic programming, [36]. In practice, however, the complexity of the resulting algorithm would probably restrict its use to critical situations where the economics demand the maximum possible return from the experiment. Usually, however, the added complexity incurred by inclusion of the second order terms cannot be economically justified and it is sufficient to base designs on the mean of the prior distribution. For this case, the cost functions J_1 and J_2 reduce to:

$$J_1 = \log \det (M_{\bar{\beta}} + P)^{-1} \tag{44}$$

$$J_2 = \text{trace } \{\Gamma (M_{\bar{\beta}} + P)^{-1}\} \tag{45}$$

where $M_{\bar{\beta}}$ is given by (40).

In the next section, necessary conditions that must be satisfied by any $\{u_k^*\}$ which minimises either (44) or (45), are given. Section 8 contains a discussion of the computational requirements of the various methods for evaluating J which have been described in this section.

6. Necessary Conditions for Optimality

Following the discussion in the last section, the following two cost functions are defined:

$$J_1 = \log \det (M_{\bar{\beta}} + P)^{-1} \tag{1}$$

$$J_2 = \text{trace } \{\Gamma (M_{\bar{\beta}} + P)^{-1}\} \tag{2}$$

Throughout this section attention will be restricted to the α parameters, that is, the parameters in A, B, C, D and K. The justification for this comes from the following theorem:

Theorem 1:

Provided P *has the same "structure" as* $M_{\bar{\beta}}$, *the cost functions* J_1, J_2 *defined in* (1) *and* (2) *lead to the same optimal designs as the cost functions:*

$$J_1' = \log \det (M_{\alpha\alpha}+P_{\alpha\alpha})^{-1} \tag{3}$$

$$J_2' = \text{trace } \{\Gamma'(M_{\alpha\alpha}+P_{\alpha\alpha})^{-1}\} \tag{4}$$

respectively, where Γ' *is the principle submatrix of* Γ *corresponding to* α.

Proof:

It was shown in results (4.6) and (4.7) that, for large N, $M_{\overline{\beta}}$ has the structure:

$$M_{\overline{\beta}} = \begin{bmatrix} M_{\alpha\alpha} & O & O \\ O & M_{\gamma\gamma} & O \\ O & O & M_{\sigma\sigma} \end{bmatrix} \tag{5}$$

Assuming that P has the same 'structure', that is,

$$P = \begin{bmatrix} P_{\alpha\alpha} & O & O \\ O & P_{\gamma\gamma} & O \\ O & O & P_{\sigma\sigma} \end{bmatrix} \tag{6}$$

enables J_1 and J_2 given by (1) and (2) to be written

$$J_1 = \log \det (M_{\alpha\alpha}+P_{\alpha\alpha})^{-1} + \log \det (M_{\gamma\gamma}+P_{\gamma\gamma})^{-1} + \log \det (M_{\sigma\sigma}+P_{\sigma\sigma})^{-1} \tag{7}$$

and

$$J_2 = \text{trace } \{\Gamma'(M_{\alpha\alpha}+P_{\alpha\alpha})^{-1}\} + \text{trace } \{\Gamma_\gamma(M_{\gamma\gamma}+P_{\gamma\gamma})^{-1}\}$$

$$+ \text{trace } \{\Gamma_\sigma(M_{\sigma\sigma}+P_{\sigma\sigma})^{-1}\} \qquad (8)$$

where Γ_γ, Γ_σ are the principle submatrices of Γ corresponding to γ, σ respectively. Now, since $M_{\gamma\gamma}$ and $M_{\alpha\alpha}$ are independent of the input, the result follows. #

Remark 1: The restriction on the structure of P in the above theorem is unnecessary if there is little prior information, that is P is small compared with $M_{\bar{\beta}}$.

Remark 2: If $P=P_\beta^{-1}$ is obtained from a previous long experiment it will have the same "structure" as $M_{\bar{\beta}}$ (since P_β approaches M_β^{-1} for long experiments). #

Now making use of result (4.5) and equation (5.41), equations (7) and (8) may be rewritten as:

$$J_1 = \log \det (M'+P')^{-1} \qquad (9)$$

$$J_2 = \text{trace } \{\Gamma'(M'+P')^{-1}\} \qquad (10)$$

where M' is given by:

$$(M')_{ij} = \sum_{k=1}^{N} (H_i z_k + D_i u_k)^T \Sigma^{-1} (H_j z_k + D_j u_k) \qquad (11)$$

where

$$D_i = \frac{\partial D}{\partial \alpha_i} \qquad (12)$$

$$H_i = \left[-\frac{\partial C}{\partial \alpha_i} \;\middle|\; \underset{1 \cdots i \cdots p}{0 \cdots 0} \;\middle|\; -C \;\middle|\; 0 \cdots 0 \right] \tag{13}$$

and

$$z_{k+1} - z_k = A_1 z_k + B_1 u_k; \quad z_1 = 0 \tag{14}$$

where

$$A_1 = F - I$$

and

$$F = \left[\begin{array}{c|c|c|c} A & 0 & 0 \cdots 0 & 0 \\ \hline \frac{\partial A}{\partial \alpha_1} - K\frac{\partial C}{\partial \alpha_i} & A-KC & 0 \cdots 0 & 0 \\ \hline \vdots & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \ddots & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \frac{\partial A}{\partial \alpha_p} - K\frac{\partial C}{\partial \alpha_p} & 0 & 0 \cdots 0 & A-KC \end{array}\right] \tag{15}$$

$$B_1 = \left[\begin{array}{c} B \\ \hline \frac{\partial B}{\partial \alpha_1} - K\frac{\partial D}{\partial \alpha_1} \\ \hline \vdots \\ \hline \frac{\partial B}{\partial \alpha_p} - K\frac{\partial D}{\partial \alpha_p} \end{array}\right] \tag{16}$$

The matrix P' is given by:

$$P' = P_{\alpha\alpha} + M'' \tag{17}$$

and

$$(M'')_{ij} = N \text{ trace } \{H_i^T \Sigma^{-1} H_j Q\} \tag{18}$$

where Q satisifes:

$$Q - (I+A_1)Q(I+A_1)^T - K'\Sigma K'^T = 0 \tag{19}$$

where

$$K' = \begin{bmatrix} K \\ \hline \frac{\partial K}{\partial \alpha_1} \\ \hline \vdots \\ \hline \frac{\partial K}{\partial \alpha_p} \end{bmatrix} \tag{20}$$

(c.f. equations (5.41) and (5.44).)

Remark 1: All functions which depend upon β are evaluated at $\beta = \bar{\beta}$. #

The discrete minimum principle, [73], will now be used to obtain necessary conditions for optimality of $\{u_k, k=1,\ldots N\}$ in the case of generalised power or energy constraints:

Result 1:

The necessary conditions for J-optimality of $\{u_k^*, k=1,\ldots N\}$ *subject to the total energy constraint:*

$$\sum_{k=1}^{N} u_k^T W u_k = E \tag{21}$$

where W is symmetric positive definite and E is positive, are given by:

$$\left.\frac{\partial H(z_k^*, \gamma_{k+1}^*, \phi^*, \lambda, u_k)}{\partial u_k}\right|_{u_k=u_k^*} = 0 \tag{22}$$

where:

$$H(z_k,\gamma_{k+1},\phi,\lambda,u_k) = \gamma_{k+1}^T[A_1 z_k+B_1 u_k] + \lambda u_k^T W u_k + \sum_{i=1}^{p}\sum_{j=1}^{p}(H_i z_k+D_i u_k)^T\Sigma^{-1}(H_j z_k+D_j u_k)\phi_{ij} \tag{23}$$

$$z_{k+1}^* - z_k^* = A_1 z_k^* + B_1 u_k^* \tag{24}$$

$$z_1^* = 0 \tag{25}$$

$$\gamma_{k+1}^* - \gamma_k^* = -A_1^T\gamma_{k+1}^* + 2\sum_{i=1}^{p}\sum_{j=1}^{p} H_i^T\Sigma^{-1}(H_j z_k^*+D_j u_k^*)\phi_{ij}^* \tag{26}$$

$$\gamma_{N+1}^* = 0 \tag{27}$$

$$\phi^* = -(M'^*+P)^{-1} \quad \text{if } J = J_1$$

$$= -(M'^*+P)^{-1}\Gamma(M'^*+P)^{-1} \quad \text{if } J = J_2 \tag{28}$$

$$\lambda = -\frac{1}{2E}\sum_{k=1}^{N}\{\gamma_{k+1}^{*T}B_1 M_k^* + 2\sum_{i=1}^{p}\sum_{j=1}^{p} u_k^{*T}D_i^T\Sigma^{-1}(H_j z_k^*+D_j u_k^*)\phi_{ij}\} \tag{29}$$

$$M'^* = \sum_{k=1}^{N}(H_i z_k^*+D_i u_k^*)^T\Sigma^{-1}(H_j z_k^*+D_j u_k^*) \tag{30}$$

Proof:

The energy constraint, (21) is first adjoined to the cost function via a Lagrangian multiplier:

$$J_c = J + \lambda\{\sum_{k=1}^{N} u_k^T W u_k - E\} \tag{31}$$

Introducing the following state equations:

$$z_{k+1} - z_k = A_1 z_k + B_1 u_k \tag{32}$$

$$(M_{k+1})_{ij} - (M_k)_{ij} = (H_i z_k + D_i u_k)^T \Sigma^{-1} (H_j z_k + D_j u_k) \tag{33}$$

$$\mu_{k+1} - \mu_k = u_k^T W u_k \tag{34}$$

equation (31) can be expressed in the form:

$$J_c = J + \lambda\{\mu_{N+1} - E\} \tag{35}$$

and

$$J = \log \det (M_{N+1} + P')^{-1} \quad \text{if } J = J_1$$

$$= \text{trace } \{\Gamma (M_{N+1} + P')^{-1}\} \quad \text{if } J = J_2 \tag{36}$$

The following Hamiltonian function may now be defined:

$$H(z_k, M_k, \mu_k, \gamma_{k+1}, H_{k+1}, \lambda_{k+1}, u_k) = \lambda_{k+1} u_k^T W u_k$$

$$+ \gamma_{k+1}^T [A_1 z_k + B_1 u_k] + \sum_{i=1}^{p} \sum_{j=1}^{p} (H_i z_k + D_i u_k)^T \Sigma^{-1} (H_j z_k + D_j u_k) (H_{k+1})_{ij} \tag{37}$$

If u_k^*, $k = 1, \ldots N$ is an optimal input and z_k^*, M_k^*, μ_k^*, $k = 1, \ldots N+1$ are the corresponding states, then the discrete minimum principle states that there exist costate variables, γ_{k+1}^*, H_{k+1}^*, λ_{k+1}^* such that the following relations hold:

$$z^*_{k+1} - z^*_k = A_1 z^*_k + B_1 u^*_k \tag{38}$$

$$(M^*_{k+1})_{ij} - (M^*)_{ij} = (H_i z^*_k + D_i u^*_k)^T \Sigma^{-1} (H_j z^*_k + D_j u^*_k) \tag{39}$$

$$\mu^*_{k+1} - \mu^*_k = u^{*T}_k W u^*_k \tag{40}$$

$$\gamma^*_{k+1} - \gamma^*_k = -A_1^T \gamma^*_{k+1} + 2 \sum_{i=1}^{p} \sum_{j=1}^{p} H_i^T \Sigma^{-1} (H_j z^*_k + D_j u^*_k)(\Pi_{k+1})_{ij} \tag{41}$$

$$\Pi^*_{k+1} - \Pi^*_k = 0 \tag{42}$$

$$\lambda^*_{k+1} - \lambda^*_k = 0 \tag{43}$$

$$z^*_1 = 0 \tag{44}$$

$$\gamma^*_{N+1} = \frac{\partial J_c}{\partial z^*_{N+1}} = 0 \tag{45}$$

$$\Pi^*_{N+1} = \frac{\partial J_c}{\partial M^*_{N+1}} = \phi^* \tag{46}$$

$$\lambda^*_{N+1} = \frac{\partial J_c}{\partial \mu^*_{N+1}} = \lambda \tag{47}$$

For every $\{u_k, k=1,\ldots N\}$ and each $k = 1, \ldots, N$

$$H(z^*_k, M^*_k, \mu^*_k, \gamma^*_{k+1}, \Pi^*_{k+1}, \lambda^*_{k+1}, u^*_k) \leq H(z^*_k, M^*_k, \mu^*_k, \gamma^*_{k+1}, \Pi^*_{k+1}, \lambda^*_{k+1}, u_k) \tag{48}$$

That is, an optimal input $\{u^*_k, k=1,\ldots N\}$ minimises the Hamiltonian at each time instant. If $\{u_k\}$ is unconstrained as it is in this case, the following equivalent necessary condition for optimality holds:

$$\frac{\partial}{\partial u_k}\{H(z_k^*, M_k^*, \mu_k^*, \gamma_{k+1}^*, H_{k+1}^*, \lambda_{k+1}^*, u_k)\}\Big|_{u_k=u_k^*} = 0 \qquad (49)$$

From equations (42), (43), (46) and (47) it follows that:

$$\lambda_k^* = \lambda \quad \text{for all } k \qquad (50)$$

$$H_k^* = \phi^* \text{ for all } k \qquad (51)$$

Equation (49) may now be written as:

$$2\lambda u_k^{*T} W + \gamma_{k+1}^{*T} B_1 + 2\sum_{i=1}^{p}\sum_{j=1}^{p}(H_i z_k^* + D_i u_k^*)^T \Sigma^{-1} D_j \phi_{ij}^* = 0 \qquad (52)$$

Postmultiplying by u_k^* and summing from 1 to N yields:

$$2\lambda E + \sum_{k=1}^{N}\{\gamma_{k+1}^* B_1 u_k^* + 2\sum_{i=1}^{p}\sum_{j=1}^{p}(H_i z_k^* + D_i u_k^*)^T \Sigma^{-1} D_j u_k^* \phi_{ij}^*\} = 0 \qquad (53)$$

which may be rearranged to give (29). The remaining equations follow immediately. The result is proved. #

Result 2:

For constrained inputs the necessary conditions for optimality are:

$$H(z_k^*, \gamma_{k+1}^*, \phi_k^*, u_k^*) \leq H(z_k^*, \gamma_{k+1}^*, \phi^*, u_k) \qquad (54)$$

for every $\{u_k\}$ *and each time instant* k = 1, ... N, *where* z_k^*, γ_k^*, ϕ^* *satisfy equations* (26) *to* (28).

Proof:

From (48) with $\lambda \equiv 0$. #

Theorem 2:

For amplitude constrained inputs, that is

$$\delta_1 \leq u_k \leq \delta_2 \quad \textit{for each } k \tag{55}$$

optimal input sequences $\{u_k^*\}$ *have the property that:*

$$\begin{aligned} &\textit{either} \quad (u_k^*)_i = (\delta_1)_i \\ &\textit{or} \quad\ (u_k^*)_i = (\delta_2)_i \end{aligned} \tag{56}$$

for each i *and* k. $[(\cdot)_i$ *denotes* i^{th} *component of* $\cdot]$. *That is, each of the* m *components of the optimal input forms a binary sequence with values on the constraint.*

Proof:

From (54), u_k^* is the u_k which minimises:

$$\begin{aligned} H(u_k) &= \gamma_{k+1}^{*T}(A_1 z_k^* + B_1 u_k) + \sum_{i=1}^{p}\sum_{j=1}^{p} (H_i z_k^* + D_i u_k)^T \Sigma^{-1} (H_j z_k^* + D_j u_k)\phi_{ij}^* \\ &= C_k + d_k^T u_k + u_k^T \Omega' u_k \end{aligned} \tag{57}$$

where

$$C_k = \gamma_{k+1}^{*T} A_1 z_k^* + \sum_{i=1}^{p}\sum_{j=1}^{p} z_k^{*T} H_i^T \Sigma^{-1} H_j z_k^* \phi_{ij}^*$$

$$d_k^T = \gamma_{k+1}^{*T} B_1 + 2\sum_{i=1}^{p}\sum_{j=1}^{p} z_k^{*T} H_i^T \Sigma^{-1} D_j \phi_{ij}^*$$

and

$$\Omega' = \sum_{i=1}^{p}\sum_{j=1}^{p} D_i^T \Sigma^{-1} D_j \phi^*_{ij} \tag{58}$$

Ω' is negative semi definite as is readily shown:

Let x be an arbitrary vector. Then:

$$x^T\Omega' x = \sum_{i=1}^{p}\sum_{j=1}^{p} x^T D_i^T \Sigma^{-1} D_j x \phi^*_{ij} \tag{59}$$

Let $x^{(i)} = \chi D_i x$ where $\chi^T\chi = \Sigma^{-1}$. Then (59) becomes:

$$\begin{aligned} x^T\Omega' x &= \sum_{i=1}^{p}\sum_{j=1}^{p} x^{(i)T} x^{(j)} \phi^*_{ij} \\ &= \sum_{\ell=1}^{m}\sum_{i=1}^{p}\sum_{j=1}^{p} (x^{(i)})_\ell (x^{(j)})_\ell \phi^*_{ij} \\ &= \sum_{\ell=1}^{m} x^T_{(\ell)} \phi^* x_{(\ell)} \end{aligned} \tag{60}$$

where $x_{(\ell)}$ is a vector with i^{th} component $(x^{(i)})_\ell$. But from (28), ϕ^* is negative definite so (60) implies that

$$x^T\Omega' x \leq 0$$

so that Ω' is negative semi definite. Now it is an elementary property of quadratic forms with negative definite weighting matrices, that the minimum is achieved on the boundary. Theorem 2 follows immediately from the form of (57). #

In sections 7 and 9, algorithms are proposed for finding $\{u_k^*, k=1,\ldots N\}$ which satisfy the necessary conditions for optimality.

There is, of course, no guarantee that the sequences satisfying these conditions are in fact optimal. In chapter 5, tests for global optimality are derived, but these require a Fourier analysis of the input sequence.

7. General Design Algorithm

In section 6 it was shown that a necessary condition for optimality of $\{u_k^*, k=1,\ldots N\}$ is that u_k^* minimise the Hamiltonian $H(u_k^*)$ for every k. A simple steepest descent algorithm based on this principle is now described:

i. Evaluate the matrix M" from (6.18) and hence P' from (6.17).

ii. Choose a suitable value for λ. (=0 if no energy constraint.)

iii. Choose any non-zero input sequence $\{u_k^{(0)}\}$; set $\ell = 0$.

iv. Evaluate $\{z_k\}$:

$$z_{k+1} - z_k = A_1 z_k + B_1 u_k^{(\ell)} \tag{1}$$

$$z_1 = 0 \tag{2}$$

v. Evaluate M' and J:

$$(M')_{ij} = \sum_{k=1}^{N} (H_i z_k + D_i u_k^{(\ell)})^T \Sigma^{-1} (H_j z_k + D_j u_k^{(\ell)}) \tag{3}$$

$$J = J_1 = \log \det (M'+P')^{-1} \tag{4}$$

or

$$J = J_2 = \text{trace } \{\Gamma(M'+P')^{-1}\} \tag{5}$$

vi. Evaluate ϕ:

$$\phi = -(M'+P')^{-1} \text{ if } J = J_1 \tag{6}$$

$$\phi = -(M'+P')^{-1}\Gamma(M'+P')^{-1} \text{ if } J = J_2 \tag{7}$$

vii. Evaluate $\{\gamma_k\}$:

$$\gamma_{k+1}-\gamma_k = -A_1^T\gamma_{k+1} + 2\sum_{i=1}^{p}\sum_{j=1}^{p} H_i^T\Sigma^{-1}(H_j z_k + D_j u_k^{(\ell)})\phi_{ij} \tag{8}$$

$$\gamma_{N+1} = 0 \tag{9}$$

viii. Evaluate gradient:

$$g_k^T = \frac{\partial H(u_k)}{\partial u_k} = 2\lambda u_k^T W + \gamma_{k+1}^T B_1 + 2\sum_{i=1}^{p}\sum_{j=1}^{p}(H_i z_k + D_i u_k)^T\Sigma^{-1}D_j\phi_{ij} \tag{10}$$

ix. Move in negative gradient direction:

$$u_k^{(\ell+1)} = u_k^{(\ell)} - \delta_\ell g_k'' \tag{11}$$

where g_k'' is the gradient g_k suitably projected onto the constraint surface. δ_ℓ is a positive scalar chosen so that a decrease in J occurs.

x. Stopping condition:

$$g_k'' = 0 \text{ for all } k. \tag{12}$$

If (12) is satisfied, go to xi; else go to iv.

xi. If there are energy constraints check whether these are satisfied; if not choose a new value of λ and repeat from iii. If the constraint is satisfied, stop. #

There are many possible strategies for choosing the scalar δ_ℓ in (11) but the following has been found to work well in practice with amplitude constraints:

i. Try $\delta_\ell = \infty$ (i.e. saturate in negative gradient direction);

ii. If there is no decrease in J with $\delta_\ell = \infty$, perform a linear search until a decrease in J occurs. #

In the energy constrained case, the Lagrange multiplier, λ, must be chosen so that the constraint:

$$\sum_{k=1}^{N} u_k^{*T} W u_k^* = E \tag{13}$$

is satisfied. Often it suffices to choose an arbitrary value for λ. This is justified as follows.

Theorem 1:

If the matrix P' *is the null matrix, and if the input* $\{u_k^*, k=1,\ldots N\}$ *is optimal subject to power constraint* E, *then the input* $\{au_k^*, k=1,\ldots N\}$ *is optimal subject to power constraint* a^2E.

Proof:

Suppose $\{u_k^*\}$ has corresponding information matrix M'* where:

$$M'^* = \sum_{k=1}^{N} G_k^{*T} \Sigma^{-1} G_k^* \tag{14}$$

G_k^* is a matrix with elements that are the outputs of linear equations forced by $\{u_k^*\}$.

Suppose now that an input $\{u_k\}$ where $u_k = au_k^*$ for all k is used. Then it follows that the corresponding M' is given by

$$M' = \sum_{k=1}^{N} aG_k^{*T}\Sigma^{-1}G_k^*a = a^2M'^* \tag{15}$$

Suppose that $\{u_k^*\}$ has energy E:

$$\sum_{k=1}^{N} u_k^{*T}Wu_k^* = E \tag{16}$$

Then

$$\sum_{k=1}^{N} u_k^{T}Wu_k = a^2E \tag{17}$$

Consider now the costs associated with $\{u_k\}$:

$$J_1(u) = \log \det (M')^{-1}$$

$$= \log \det (M'^*)^{-1} - 2p \log a$$

$$= J_1(u^*) - 2p \log a \tag{18}$$

$$J_2(u) = \text{trace } (M')^{-1}$$

$$= \frac{1}{a^2} \text{trace } (M'^*)^{-1}$$

$$= \frac{1}{a^2} J_2(u^*) \tag{19}$$

Clearly if $\{u_k^*\}$ minimises $J_i(u^*)$ subject to (16), then $\{u_k\}$ minimises $J_i(u)$ subject to (17). #

Corollary:

If the matrix P' is the null matrix, then to find an optimal energy constrained input with specified energy, it is sufficient to find an optimal energy constrained input with arbitrary energy and then scale it by the square root of the ratio of the energies. #

Thus when the conditions of theorem 1 are met, that is P' = O, any non-zero λ will suffice. There are many cases when P' is small compared with M' and the theorem may be applied with little error. In particular, the result applies to the important class of models with disjoint system and noise modes. (These models have been used very extensively in the literature, see for example [3], [7], [9], [12], [15], [16], [25].) The result is also valid in all cases where high input energy is used since M' then dominates P'. #

If for some reason the conditions of the theorem are not met then a sequence of λ's must be chosen to satisfy the constraint. An alternative approach is to use generalised polar coordinates, for example, for a scalar input:

$$u_N = E.\sin\theta_{N-1}$$

$$u_{N-1} = E.\sin\theta_{N-2}\cos\theta_{N-1}$$

$$\vdots$$

$$u_2 = E.\sin\theta_1\cos\theta_2 \ldots \cos\theta_{N-1}$$

$$u_1 = E.\cos\theta_1\cos\theta_2 \ldots \cos\theta_{N-1} \qquad (20)$$

$$g_{\ell}^{(\theta)} = \sum_{k=1}^{N} \left(\frac{\partial u_k}{\partial \theta_{\ell}}\right)^T g_k \qquad (21)$$

g_k is given by (10) and $g_{\ell}^{(\theta)}$ is the gradient in polar coordinates. The input is thus constrained to lie on the surface of an N-dimensional hypersphere of radius E, the total energy.

In the next section, computational aspects of the general algorithm described above are discussed.

8. Computational Aspects

For all but very short sequence lengths, the most time consuming parts of the general algorithm are steps iv. and vii. viz. the calculation of the state z_k and the costate γ_k. It is also usual to store z_k for *every* k as these are needed for the calculation of γ_k. It is possible, however, to recalculate the z_k in reverse time concurrently with γ_k but this is very time consuming and requires storage of many intermediate values of z_k to prevent divergence of errors due to the reverse time instability of the z_k equation. A fairly accurate indication of the computational effort and storage requirements is given by the dimension of the sensitivity state vector, z_k. Reference to the previous section indicates that the requirements of the general algorithm are considerable. (A simple two input-two output four state system in Caines' canonical form, [31], requires the solution of a 116 dimensional state equation and for N = 100 requires about 12,000 storage locations.) However, these requirements can be vastly reduced by exploiting the structure of the sensitivity equations. The following theorem will prove useful in establishing these simplifications.

Theorem (Denery) [32]:

If x_k is the solution to a controllable single input set:

$$x_{k+1} = Fx_k + gu_k; \quad x_1 = 0 \tag{1}$$

then the solution to the set:

$$z_{k+1} = Fz_k + g'u_k, \quad z_1 = 0 \tag{2}$$

(where g' is arbitrary) is related to x_k *by the linear transformation:*

$$z_k = \sum_{\ell=0}^{n-1} a_\ell F^\ell x_k \tag{3}$$

where:

$$[a_{n-1},\ldots a_o]^T = [F^{n-1}g| \ \ldots \ |g]^{-1}g' \tag{4}$$

Proof:

Follows from superposition and the fact that g' may be expressed as a linear combination of the columns of the controllability matrix for (1). #

Now, in section 6, it was shown that the sensitivity equations could be written in the form:

$$z_{k+1} = Fz_k + B_1u_k; \quad z_1 = 0 \tag{5}$$

where F and B_1 are given by (6.15) and (6.16). The important thing to notice is that there are only two types of dynamics, viz.:

$$\bar{z}_{k+1} = A\bar{z}_k + Bu_k \tag{6}$$

and

$$\tilde{z}_{k+1} = (A-KC)\tilde{z}_k + \bar{B}\bar{z}_k + \tilde{B}u_k \tag{7}$$

Applying Denery's result to (6) and (7) it is obvious that the whole sensitivity state vector may be obtained from linear combinations of the states of the following (r+n+1) n^{th} order equations:

$$\bar{z}_{k+1} = A\bar{z}_k + Bu_k \tag{8}$$

$$\tilde{z}^{(i)}_{k+1} = (A-KC)\tilde{z}^{(i)}_{k+1} + e_n(u_k)_i \; ; \quad i = 1,\ldots,r \tag{9}$$

$$\hat{z}^{(i)}_{k+1} = (A-KC)\hat{z}^{(i)}_{k+1} + e_n(\bar{z}_k)_i \; ; \quad i = 1,\ldots,n \tag{10}$$

where

$$e_n^T = [0,\ldots 0,1] \tag{11}$$

Thus the storage requirements will be considerably reduced. The computational requirement will also be reduced since the linear combinations required can be absorbed into H_i (6.13).

Using the above results, the storage requirements of the simple two input-two output four state system drop to about 3,200 storage locations for N = 100.

Still further reductions may be obtained by transforming A and A - KC to companion form:

$$T_1 A T_1^{-1} = \left[\begin{array}{c|c} 0 & I \\ \hline -a'_n & \cdots, -a'_1 \end{array}\right] = A_1 \tag{12}$$

$$T_2(A-KC)T_2^{-1} = \left[\begin{array}{c|c} 0 & \begin{array}{ccc} & I & \\ & & \end{array} \\ \hline -a''_m & \cdots & -a''_1 \end{array}\right] = A_2 \qquad (13)$$

Equations (8) to (10) may now be replaced by:

$$\bar{z}^{(i)}_{k+1} = A_1 \bar{z}^{(i)}_k + e_n(u_k)_i \ , \ i = 1, \ \ldots \ r \qquad (14)$$

$$\tilde{z}^{(i)}_{k+1} = A_2 \tilde{z}^{(i)}_{k+1} + e_n(u_n)_i \ , \ i = 1, \ \ldots \ r \qquad (15)$$

$$\hat{z}^{(i)}_{k+1} = A_2 \hat{z}^{(i)}_{k+1} + e_n(\bar{z}_k)_i \ , \ i = 1, \ \ldots \ n \qquad (16)$$

It is still true that the whole sensitivity state vector may be obtained from linear combinations of the states of the (2r+n) n^{th} order equations, (14) - (16). A fundamental property of equations (14) - (16) is that the state vectors are such that:

$$(z_{k+1})_i = (z_k)_{i+1} \ , \ i = 1, \ \ldots, \ n - 1 \qquad (17)$$

so that in order to store the complete state vector z_k for k = 1, N it is necessary to store only $(z_k)_n$ for k = 2, ...N together with z_1. Thus only (2r+n) vectors of length (N-1) are needed to store the complete state information. The storage requirements for the simple example previously considered are thus approximately 1,200 (allowing for inputs and gradients).

Since the transformation matrices need be calculated only once and the number of iterations of the algorithm is large, the number of computations will also be reduced. In practice, the full sensitivity

state and costate are never calculated; the only effect on the algorithm as stated in section 7 is to reduce the dimension of the matrices H_i which will include the transformation matrices described above.

In particular cases, the transformation matrices may be particularly simple. This is demonstrated in the next section where a design algorithm for multiple input-single output systems in Caines' canonical form is described. It is shown that, for this case, only (r+1) n^{th} order difference equations need be solved and that only (r+1) vectors of length N are needed for the sensitivity states.

In table 1 the state dimension and storage requirements are given for the various methods discussed in section 6 and in this section. Figures are also given for the two input-two output four state system discussed earlier.

Method	Sensitivity State Dimension n_z	Storage Required	Example: n=4, m=r=2, d=2	
			Dimension	Storage
discrete prior distribution d levels/parameter	$(n^2+nr+1)d^p$	$(n_z+2r)N$	$7x10^9$	$7x10^{11}$
straightforward 2nd moment method	$n(1+p+p^2+p^3)$	$(n_z+2r)N$	91,060	$9x10^6$
2nd moment exploiting symmetry	$n(1+p+\frac{p(p+1)}{2}+\frac{p(p^2+5)}{6})$	$(n_z+2r)N$	16,468	$1.6x10^6$
2nd moment exploiting structure	$n(n^3+n^2+n+1)+nr(1+n+n^2)$	$(n_z+2r)N$	508	$5.1x10^4$
prior mean exploiting structure	$n(n+r+1)$	$(n_z+2r)N$	28	3,200
prior mean with companion form	$n(n+2r)$	$(n+4r)N$	32	1,200

Table 1

9. Design Algorithm for Single Output Systems

A canonical model for multiple input-single output systems is conveniently represented in pulse transfer function form, [44]:

$$A(z^{-1})y_k = \sum_{\ell=1}^{r} B^{\ell}(z^{-1})u_k^{\ell} + C(z^{-1})\varepsilon_k \tag{1}$$

where

$$A(z^{-1}) = a_0 + a_1 z^{-1} + \ldots + a_n z^{-n}; \quad a_n = 1 \tag{2}$$

$$B^{\ell}(z^{-1}) = b_0^{\ell} + b_1^{\ell} z^{-1} + \ldots + b_n^{\ell} z^{-n} \tag{3}$$

$$C(z^{-1}) = C_0 + C_1 z^{-1} + \ldots + C_n z^{-n}, \quad C_0 = 1 \tag{4}$$

u_k^{ℓ} is the ℓ^{th} component of u_k (i.e. the ℓ^{th} input), $\{y_k\}$ is the output sequence and $\{\varepsilon_k\}$ is a sequence of independent normally distributed random variables with zero mean and variance σ^2. The model (1) is equivalent to the innovation model described by (2.4) - (2.5), in which $C = [1, 0, \ldots, 0]$ and A is in companion form, [44].

It is readily shown that the matrix (M') defined by (6.11) may be written as:

$$M' = \frac{1}{\sigma^2} \sum_{k=1}^{N} g_k g_k^T \tag{5}$$

where

$$g_k^T = [t_k^1, \ldots t_{k-n}^1, \ldots t_{k-n}^2, \ldots t_{k-n}^r, s_k, \ldots s_{k-n+1}] \tag{6}$$

and

$$C(z^{-1})t_k^j = -u_k^j; \quad j = 1, \ldots r; \quad t_k^j = 0 \ \forall \ k \leq 0 \tag{7}$$

$$A(z^{-1})s_k = \sum_{j=1}^{r} B^j(z^{-1})t_k^j; \quad s_k = 0 \ \forall \ k \leq 0 \tag{8}$$

It is also not difficult to show that the corresponding costate equations are:

$$C(z)\lambda_k^j = -B^j(z)\gamma_k - \sum_{i=1}^{n} \phi_{i+n(j-1)} g_{k+i}; \quad j = 1,\ldots r;$$

$$\lambda_k^j = 0 \ \forall \ k > N \tag{9}$$

$$A(z)\gamma_k = -\sum_{i=1}^{n-1} \phi_{i+nr} g_{k+i}; \quad \gamma_k = 0 \ \forall \ k > N \tag{10}$$

together with the additional boundary condition:

$$g_k = 0 \ \forall \ k > N \tag{11}$$

The gradient of J with respect to u_k^j is given by:

$$g_k^j = \lambda_k^j \tag{12}$$

The row vector ϕ_i is the i^{th} row of the matrix ϕ defined by(7.6) and (7.7).

The equations (7), (8), (9) and (10) are shown diagrammatically in figure 1, and can be seen to have a particularly simple form:

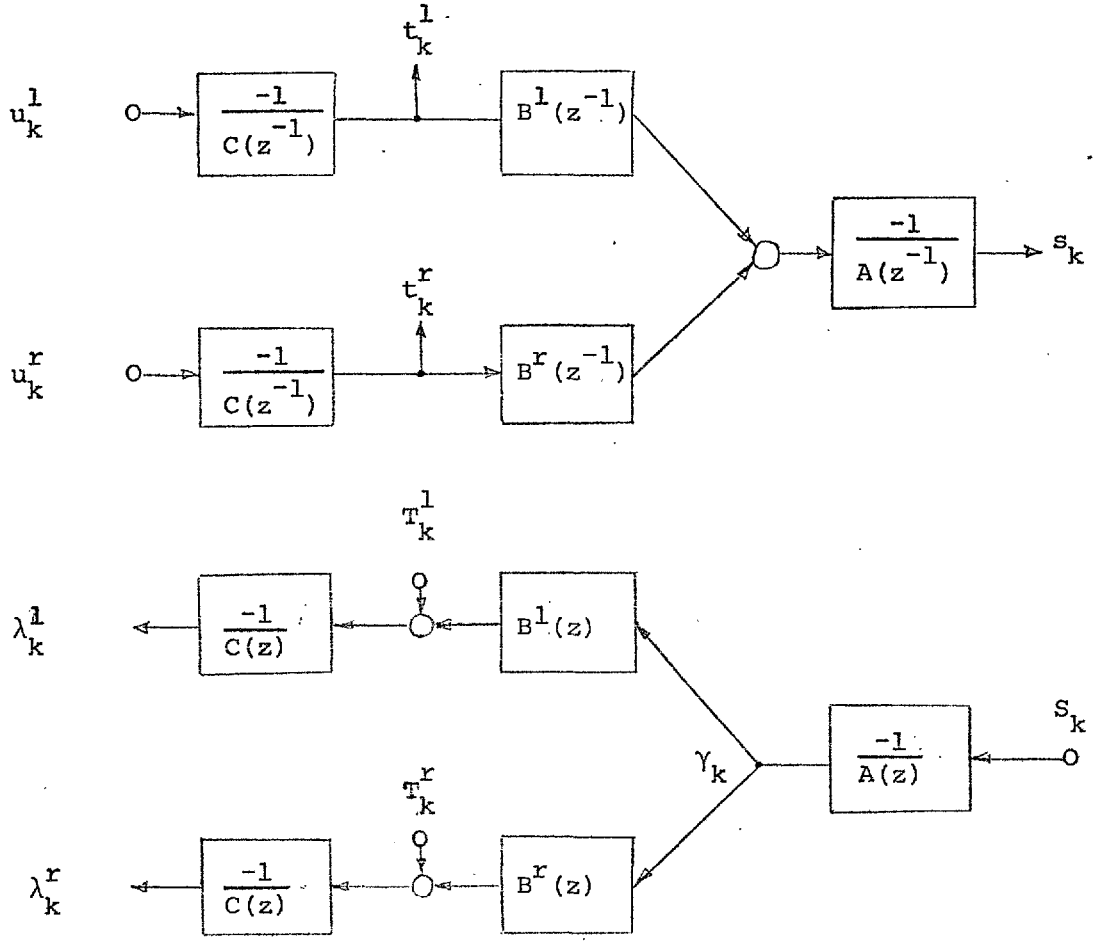

Figure 1

The quantities T_k^j and S_k are obtained from:

$$T_k^j = \sum_{i=1}^{n} \phi_{i+n(j-1)} g_{k+i} \tag{13}$$

$$S_k = \sum_{i=1}^{n-1} \phi_{i+nr} g_{k+i} \tag{14}$$

which are functions of t_k^j, $j = 1, \ldots r$, s_k, $k = 1, \ldots N$. These may be stored in $(r+1)N$ storage locations. Allowing rN more locations

for the input sequence yields a total requirement of (2r+1)N storage locations. (C.f. (4r+n)N required by the general algorithm.)

10. Concluding Remarks

In this chapter necessary conditions for the optimality of inputs have been developed. Algorithms have been given and several theorems and results which lead to simplifications have been stated and proved. The computational requirements of the algorithm have been discussed and methods for vastly reducing these described. It has also been proved that optimal amplitude constrained inputs are binary. Examples indicating the viability of the algorithms and the improvements that can typically be obtained have been given.

APPENDIX A

Amplitude Constrained Examples

Three examples of amplitude constrained designs for models with the following structure are given:

$$y_k = \frac{B(z^{-1})}{A(z^{-1})}u_k + \frac{D(z^{-1})}{C(z^{-1})}\varepsilon_k \qquad (1)$$

where $\{u_k\}$, $\{y_k\}$ are the input and output sequences, respectively and $\{\varepsilon_k\}$ is a sequence of independent normally distributed random variables with zero mean and variance σ^2. A, B, C and D are polynomials in z^{-1} and the parameter vector is defined by:

$$\beta^T = \{a_1,\ldots a_n, b_o, b_1,\ldots b_n, c_1,\ldots c_n, d_1,\ldots d_n, \sigma\} \qquad (2)$$

The design algorithm used was a single input version of that described in section 9 suitably modified to treat the output noise case given by (1). The identification procedure used to obtain the parameter estimates was Clarke's generalised-least-squares algorithm, [75].

The criterion for optimality of a test signal $\{u_k^*\}$ was that $\{u_k^*\}$ should minimise J defined by:

$$J = \text{trace}\ \{(M')^{-1}\} \qquad (3)$$

where M' is defined by (6.11). (Note that this corresponds to little prior information: $P \simeq 0$.)

Example 1:

This example (table 1) was suggested by Clarke, [75]. The number of data points was 100 and σ was 0.1. The optimal test signal design procedure gave a predicted improvement of 1.4 to 1 in the sum of the variances of the A and B parameters compared with the estimates obtained using a PRBN test signal (63 bit, with clock rate equal to sampling rate - see fig. 1). (The optimal test signal for this example is shown in fig. 2.) Table 1 shows the true parameter values and the estimates obtained from both PRBN and optimal test signals.

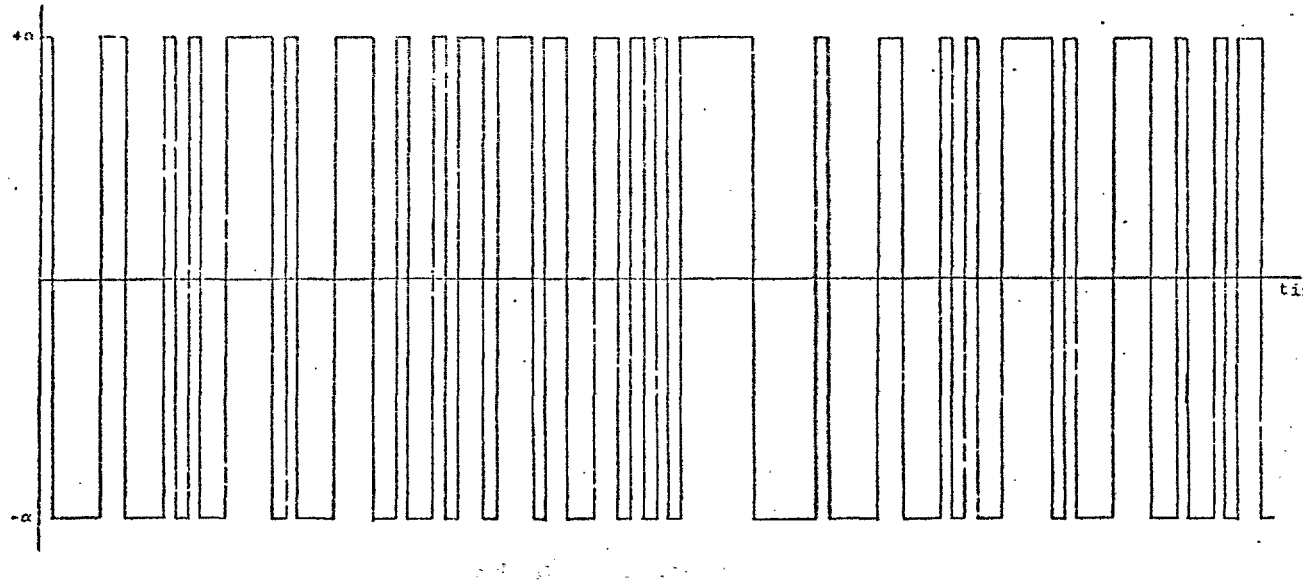

Figure 1, PRBN for Example 1

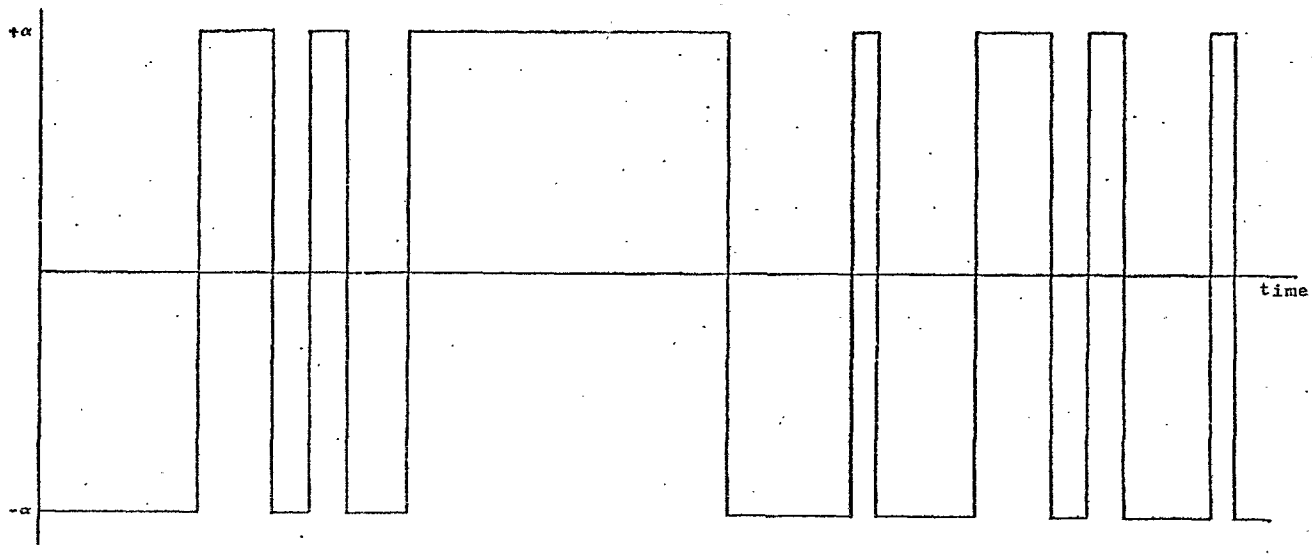

Figure 2, Optimal Test Signal for Example 1

	True Parameters	Parameters Estimated using PRBN ± Standard Deviations	Parameters Estimated using Optimal Test Signal ± Standard Deviations
a_1	-1.3	-1.2241 ± 0.0645	-1.2935 ± 0.0541
a_2	0.6	0.5446 ± 0.0634	0.5904 ± 0.0544
b_o	0.15	0.1207 ± 0.0325	0.1275 ± 0.0300
b_1	0.15	0.2070 ± 0.0347	0.1799 ± 0.0306
c_1	-0.95	-0.7145 ± 0.0836	-0.7272 ± 0.0858
d_1	0.0	0.1208 ± 0.1310	0.0670 ± 0.1319

Table 1

Estimated Parameters and Standard Deviations

for Example 1

Example 2:

For this example the number of data points was again 100 with $\sigma = 0.1$. The design procedure gave a predicted improvement of 4.5 to 1 in the sum of the variances of the A and B parameters compared with those obtained using the PRBN of example 1. Table 2 shows the true parameter values and the estimates obtained from both PRBN and optimal test signals.

	True Parameters	Parameters Estimated using PRBN ± Standard Deviations	Parameters Estimated using Optimal Test Signal ± Standard Deviations
a_1	1.0	1.00856 ± 0.01670	0.99060 ± 0.00940
a_2	0.85	0.85434 ± 0.01643	0.84525 ± 0.00874
b_o	0.15	0.13785 ± 0.00784	0.15283 ± 0.00668
b_1	0.15	0.14210 ± 0.00823	0.14550 ± 0.00714
c_1	-0.95	-0.70463 ± 0.07835	-0.72011 ± 0.07706
d_1	0.0	0.13838 ± 0.11954	0.12197 ± 0.11583

Table 2

Estimated Parameters and Standard Deviations

for Example 2

Example 3:

In this case 200 points of a 63-bit PRBN (fig. 3) were used as a starting input for a simple two parameter system:

$$y_k = \frac{bz^{-1}}{1-az^{-1}} + \varepsilon_k \qquad (4)$$

where the prior mean of a was 0.9. The predicted improvement was about 10 : 1 (see fig. 6 which shows cost versus iteration of the linear search in the first gradient direction, where almost all the decrease occurred). figure 4 shows the optimal test signal and figure 5 the gradient. It is readily seen that the necessary conditions for optimality are satisfied.

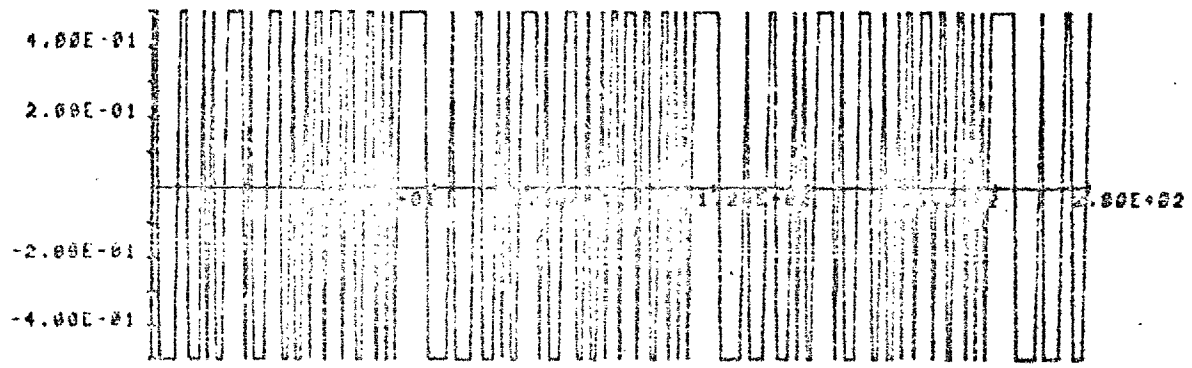

Figure 3, PRBN for Example 3

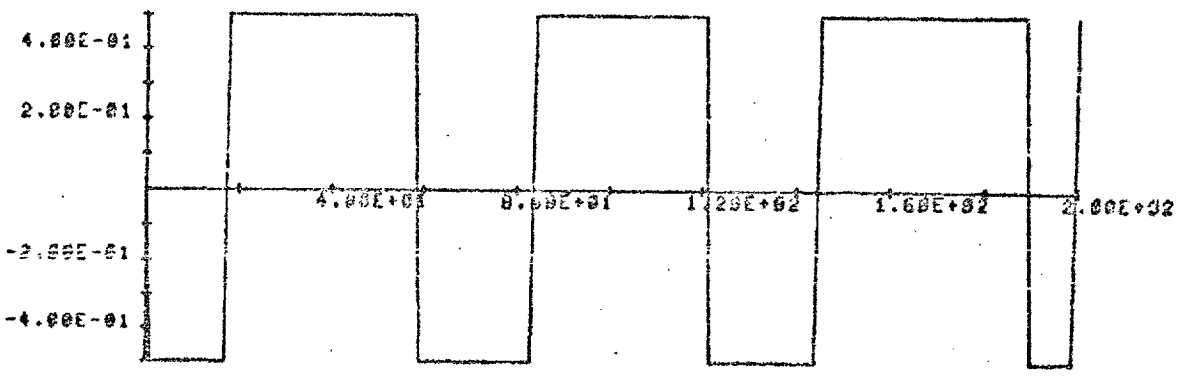

Figure 4, Optimal Test Signal for Example 3

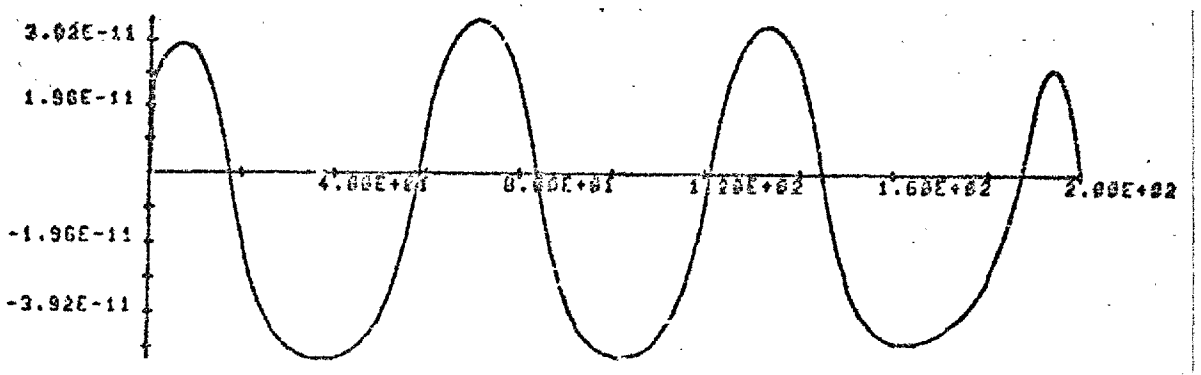

Figure 5, Gradient for Example 3

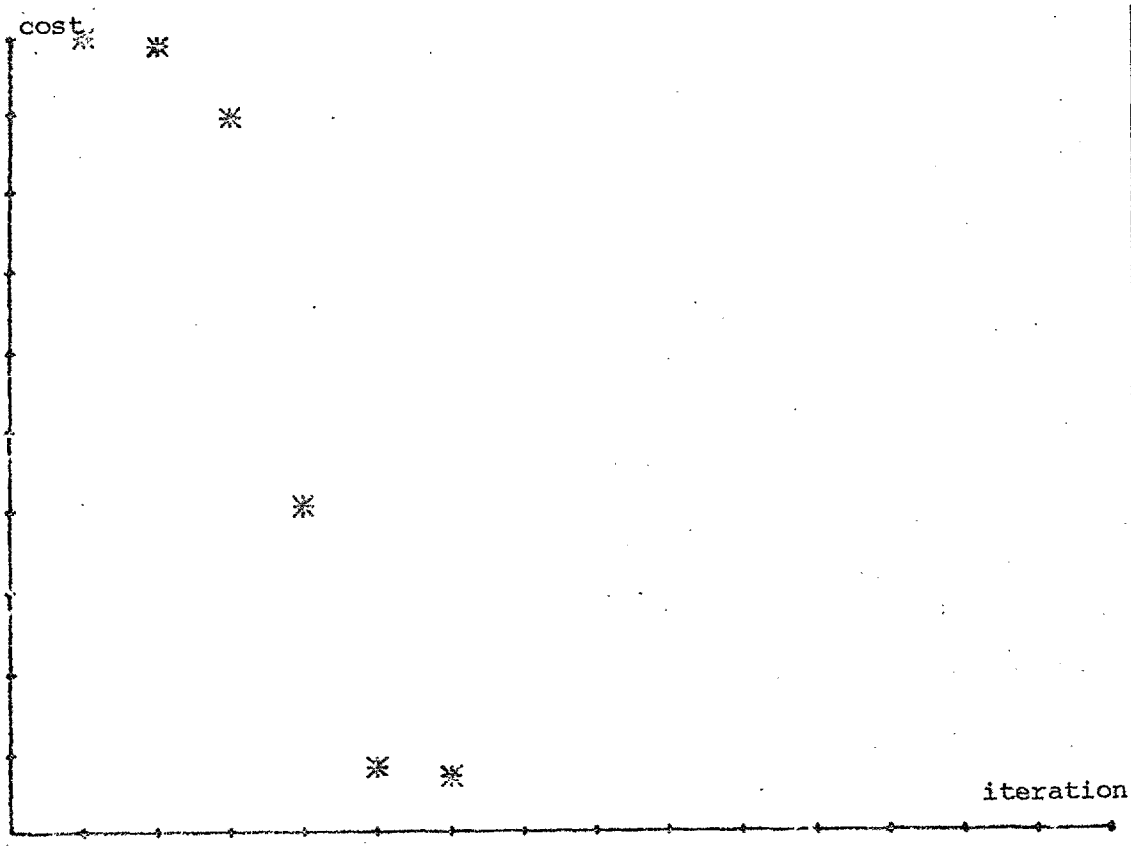

Figure 6, Cost v. Iteration for Example 3

Note: The cost in this case was taken to be var $(\hat{a})$.

APPENDIX B

Energy Constrained Examples

Three examples of energy constrained designs for models having the same structure as in appendix A are presented. The cost function used is again the trace of the posterior covariance matrix. The number of data points in all three cases was 50.

Example 1:

Fig. 1 shows the optimal energy constrained input for a simple first order model $1/(1-0.95z^{-1})$ with white output noise. The predicted improvement in the trace of the parameter covariance compared with the use of a test signal having impulsive autocorrelation is 11 to 1. It is observed from fig. 1 that the input energy is primarily low frequency which is consistent with the slow response of the system.

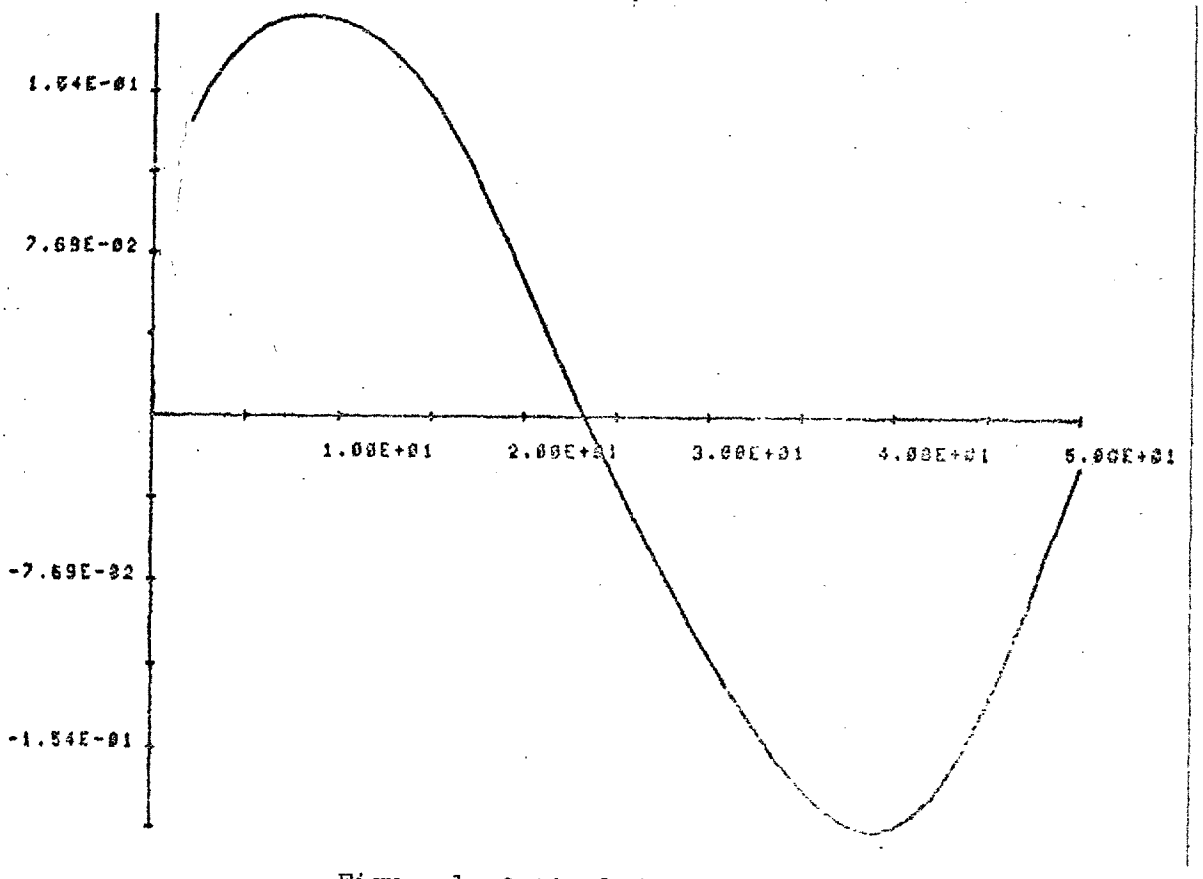

Figure 1, Optimal Input for Example 1

Example 2:

Fig. 2 shows the optimal input for the model $1/(1-0.5z^{-1})$ with white noise. The predicted improvement in the trace of the covariance matrix compared with the use of a PRBN having impulsive autocorrelation is 1.7 to 1. It is observed that the optimal input has high frequency components which is consistent with the fast response of the system.

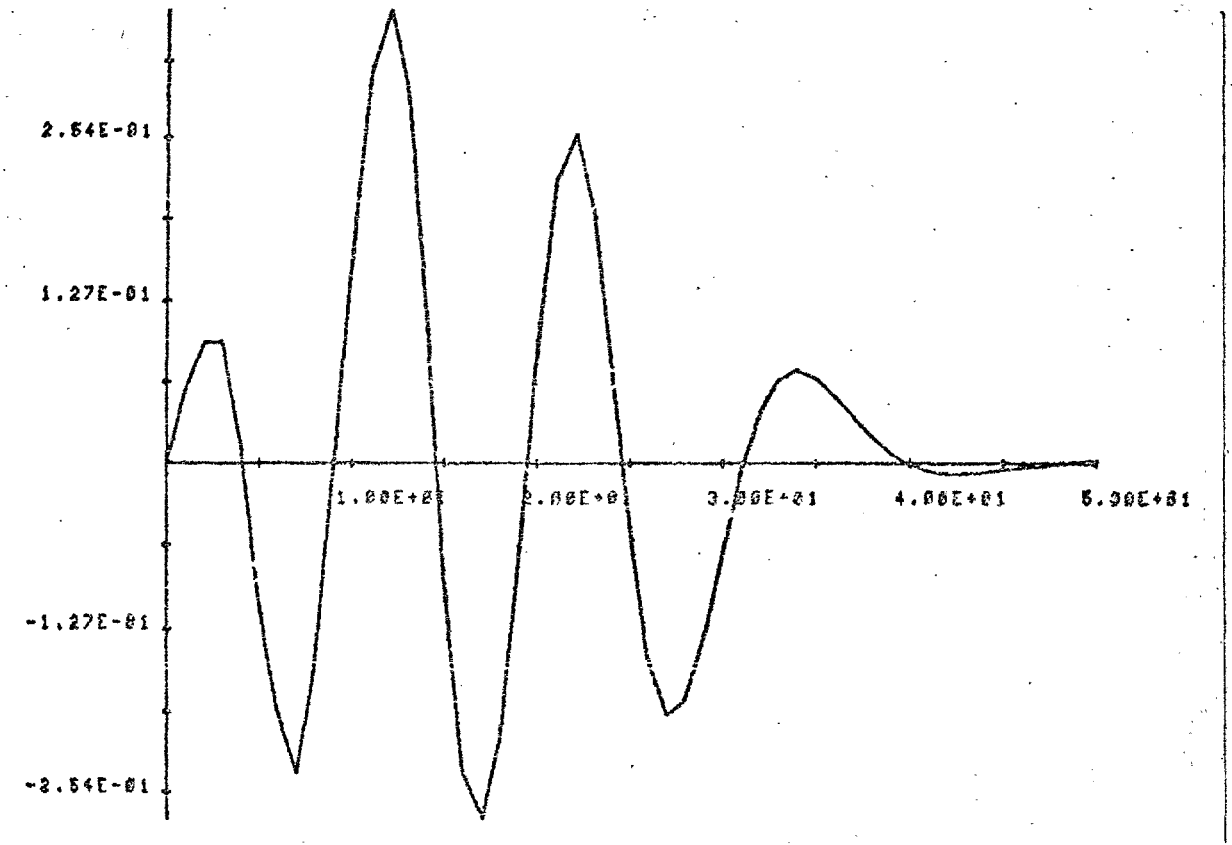

Figure 2, Optimal Input for Example 2

Example 3:

To show the effect of the noise model on the optimal test signal, a weighting sequence system model was chosen with noise model $1/(1-0.95z^{-1})$. The optimal test signal is shown in fig. 3. It is also to be noted that the optimal test signal has an autocorrelation function which is significantly different

from the well known impulsive autocorrelation ~~result~~ for the weighting sequence model with white output noise. (Levin, 1960, [1].)

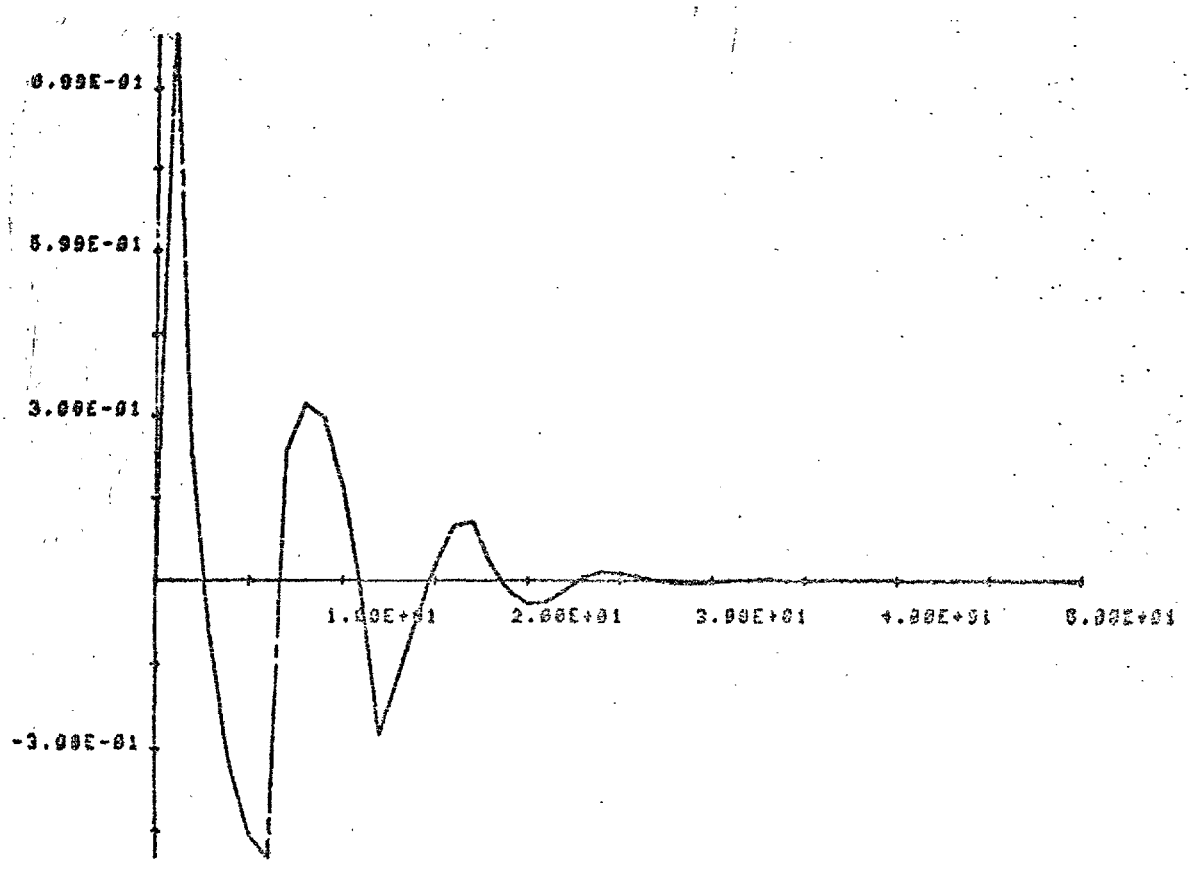

Figure 3, Optimal Input for Example 3

CHAPTER 5

Frequency Domain Designs

1. Introduction

In the last chapter, design procedures for determining test signals as functions of time were described. In section 2 of this chapter, it is shown that it is only the spectral properties of the input that are important. It is shown, in fact, that it is only ~~the first few values~~ a finite number of shifts of the input autocorrelation which have a significant effect on the cost function. In section 3 the frequency domain properties of test signals for multiple input-multiple output innovations models are derived, and identifiability conditions are described.

In section 4, Whittle's general equivalence theorem is shown to lead to tests for optimality of input signals. It is further shown that optimal spectra exist having not more than $p''(p''+1)/2$ lines where p'' is the number of parameters in the A, B, C and D matrices. In section 5 it is shown that the maximum number of lines necessary can be further reduced by exploiting the special structure of single output systems.

Sections 6 and 7 discuss various design algorithms and methods for realizing both power and amplitude constrained test signals.

In section 8, the extension of the results contained in both chapters 4 and 5 of this thesis to continuous time systems is indicated. It is shown in section 9 how the results of section 8 can be modified to allow joint optimal design of test signal and sampling rates. Several examples of designs in the frequency domain are given in the appendix.

2. Characterisation of Test Signals

In this section, large sample properties of test signals are investigated. Since the information will in general grow without bound as N increases, it is natural to consider the *average information matrix*, $\bar{M}$, defined by:

$$\bar{M} = \lim_{N\to\infty} \frac{1}{N} M \tag{1}$$

where M is Fisher's information matrix, (4.4.1). Some properties of $\bar{M}$ are now stated and proved:

Result 1:

The average information matrix, $\bar{M}$, is independent of the initial conditions, γ, provided γ is finite.

Proof:

From equation (4.4.33):

$$\bar{M} = \left[\begin{array}{c|c} \bar{M}' & 0 \\ \hline 0 & 0 \end{array}\right] + \left[\begin{array}{c|c} \bar{M}'' & 0 \\ \hline 0 & \bar{M}_{\sigma\sigma} \end{array}\right] \tag{2}$$

where

$$\bar{M}' = \lim_{N\to\infty} \frac{1}{N} M \tag{3}$$

$$\bar{M}'' = \lim_{N\to\infty} \frac{1}{N} M'' \tag{4}$$

$$\bar{M}_{\sigma\sigma} = \lim_{N\to\infty} M_{\sigma\sigma} \tag{5}$$

where M', M'' and $M_{\sigma\sigma}$ are given by (4.4.34), (4.4.35) and (4.4.15) respectively.

It was shown in chapter 4, result (4.4.7), that the submatrices of $\frac{1}{N}M'$ corresponding to γ tended to zero as $N \to \infty$. Now the submatrices of $\frac{1}{N}M'$ corresponding to the parameters in A and C also depend on γ, but similar reasoning to that of result (4.4.7) shows that these also tend to zero as $N \to \infty$. The result follows. #

It is now possible to prove the following theorem which shows that it is only the spectral properties of the test signals which affect the estimation accuracy:

Theorem 1:

The average information matrix, $\bar{M}$, depends only on the auto-covariance function, $R(\tau)$, and the mean, $\bar{u}$, of the input sequence.

Proof:

From (4.4.34), an expression for the ij^{th} element of $\bar{M}'$ is given by:

$$(\bar{M}')_{ij} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \left(\frac{\partial \omega'_k}{\partial \theta_i}\right)^T \Sigma^{-1} \left(\frac{\partial \omega'_k}{\partial \theta_j}\right) \qquad (6)$$

where $\{\frac{\partial \omega'_k}{\partial \theta_i}\}$ is the output of a linear equation driven by $\{u_k\}$. (Equations (4.4.23)-(4.4.27).) Since the choice the initial conditions, γ, does not affect $\bar{M}'$ (result 1), it is permissible to choose γ such that the output $(\frac{\partial \omega'_k}{\partial \theta_i})$, $k = 1, \ldots, N$ is the same as if the input were periodic with period N. Thus:

$$(\bar{M}')_{ij} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} u_{k-s}^T (h_s^i)^T \Sigma^{-1} (h_t^j) u_{k-t} \qquad (7)$$

where $u_k = u_{k-N}$ $\forall$ $k \leq N$ and where $\{h_t^i, t = 0, 1, \ldots\}$ is the weighting sequence of the sensitivity equations for θ_i. ~~From (7):~~ Assuming that the input is stationary, it follows from (7) that:

$$(\bar{M}')_{ij} = \sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \text{trace } \{[R(s-t)+\bar{u}\bar{u}^T][(h_s^i)^T\Sigma^{-1}(h_t^j)]\} \qquad (8)$$

where $R(\tau)$ is the input autocovariance defined by:

$$R(\tau) = \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} (u_k-\bar{u})(u_{k-\tau}-\bar{u})^T \qquad (9)$$

and

$$\bar{u} = \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} u_k$$

The result follows from (8). #

It will now be shown that it is only the first few values of the autocovariance function that are important for experiment design:

Theorem 2:

Given any autocovariance function $R_1(\tau)$, $\tau = 0, \pm 1, \ldots$ *and for any* $\varepsilon > 0$ *there exists a finite* $\Delta(\varepsilon)$ *such that if:*

$$R_2(\tau) = R_1(\tau), \quad -\Delta(\varepsilon) \leq \tau \leq \Delta(\varepsilon) \qquad (10)$$

and $R_2(\tau)$ *is arbitrary otherwise, then*

$$|(\bar{M}_1)_{ij} - (\bar{M}_2)_{ij}| < \varepsilon \quad \forall\ i, j \qquad (11)$$

where $\bar{M}_1$, $\bar{M}_2$ *correspond to* R_1, R_2 *respectively.*

Proof: From (8):

$$(\bar{M}_1)_{ij} - (\bar{M}_2)_{ij} = \sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \text{trace } \{[R_1(s-t)-R_2(s-t)][(h_s^i)^T\Sigma^{-1}(h_t^j)]\} \qquad (12)$$

Let λ be the maximum modulus of the eigenvalues of the sensitivity

equations (4.4.23)-(4.4.27). ($\lambda<1$ since the equations are stable.)

Then there exists a $b > 0$ such that:

$$|(h_t^i)_{\ell k}| \leq b\lambda^t \quad \forall\ i, t, \ell, k \tag{13}$$

Let $r_{\rho q}(\tau)$ denote the ρq^{th} element of $R_1(\tau) - R_2(\tau)$, and $\Lambda_{\rho q}(s,t)$ the ρq^{th} element of $(h_s^i)^T \Sigma^{-1} (h_t^j)$. Then:

$$|(\bar{M}_1)_{ij} - (\bar{M}_2)_{ij}| = \left| \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \sum_{\rho=1}^{r} \sum_{p=1}^{r} r_{\rho q}(s-t) \Lambda_{\rho q}(s,t) \right|$$

$$\leq \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \sum_{\rho=1}^{r} \sum_{q=1}^{r} |r_{\rho q}(s-t)| |\Lambda_{\rho q}(s,t)| \tag{14}$$

Now

$$|\Lambda_{\rho q}(s,t)| = \left| \sum_{\ell k} \sum (h_s^i)_{\ell \rho} (\Sigma^{-1})_{\ell k} (h_t^j)_{kq} \right|$$

$$\leq \sum_{\ell k} \sum b^2 \lambda^{(s+t)} |(\Sigma^{-1})_{\ell k}|$$

$$\leq \sum_{\ell k} \sum b_1^2 \lambda^{(s+t)} \tag{15}$$

(15) following from boundedness of Σ^{-1}.

If $R_2(\tau) = R_1(\tau)$, $-\Delta \leq \tau \leq \Delta$

$$|r_{\rho q}(\tau)| = 0, \quad -\Delta \leq \tau \leq \Delta \tag{16}$$

$$|r_{\rho q}(\tau)| \leq \text{trace}\ \{R_1(0)\}, \quad |\tau| > \Delta \tag{17}$$

(17) follows from the properties of autocovariance functions.

Substituting (15), (16) and (17) into (14) yields:

$$|(\bar{M}_1)_{ij} - (\bar{M}_2)_{ij}| \leq 2 \sum_{\tau=\Delta+1}^{\infty} r^2 \, \text{trace}\{R_1(0)\}.b_1^2 \lambda^{(\tau+2t)}$$

$$\leq \sum_{\tau=\Delta+1}^{\infty} b_2 \lambda^{\tau} \qquad (18)$$

$$\leq \varepsilon \; \forall \; \Delta \geq \Delta(\varepsilon)$$

since (18) is the remainder of an absolutely convergent series.

The implications of this theorem are discussed further in section 7.

Remark 1:

If $\{u_k\}$ is a stationary stochastic process satisfying certain regularity conditions, [35], then:

$$\bar{u} = E_u[u_k] \qquad (19)$$

$$R(\tau) = E_u[(u_k - \bar{u})(u_{k-\tau} - \bar{u})^T] \qquad (20)$$

where E_u denotes expectation over the distribution of u.

Remark 2:

There is a one to one correspondence via the Fourier transform between $R(\tau)$ and the power density spectrum of the input. Thus it is only the spectral properties of the input which are important. This fact is exploited in the next section.

3. Frequency Domain Representation

From equation (2.6), the ℓq^{th} element of $\bar{M}'$ (the part of the average information matrix, $\bar{M}$, that depends on the input) is given by:

$$(\bar{M}')_{\ell q} = \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \left(\frac{\partial \omega'_k}{\partial \theta_\ell}\right)^T \Sigma^{-1} \left(\frac{\partial \omega'_k}{\partial \theta_q}\right) \qquad (1)$$

~~$$= \sum_{k=1}^{\infty} (g_k^{\ell})^T \Sigma^{-1} (g_k^{q}) \qquad (2)$$~~

~~where $\{g_k^{\ell}\}$ is the output of the sensitivity equations corresponding to a hypothetical input, u'_k, with a total energy constraint. Application of Parseval's theorem, [77], leads to:~~

~~$$(\bar{M}')_{\ell q} = \int_{-\pi}^{\pi} \tilde{g}^{\ell}(e^{-j\omega})^T \Sigma^{-1} \tilde{g}^{q}(e^{j\omega}) d\omega \qquad (3)$$~~

~~where $\tilde{g}^{\ell}(z)$ is the z transform of $\{g_k^{\ell}\}$ and is given by:~~

~~$$\tilde{g}^{\ell}(z) = H^{\ell}(z) u'(z) \qquad (4)$$~~

Definition:

$H^{\ell}(z)$ is the transfer function of the sensitivity equation corresponding to θ_ℓ ~~and $u'(z)$ is the z transform of $\{u'_k\}$. Substituting (4) into (3) yields:~~

Application of Parseval's theorem, [77], to (1) yields:

$$(\bar{M}')_{\ell q} = \int_{-\pi}^{\pi} \text{trace}\, \{[H^{\ell}(e^{-j\omega})^T \Sigma^{-1} H^{q}(e^{j\omega})] . dG(\omega)\} \qquad (5)$$

where $G(\omega)$ is the ~~cumulative energy distribution function of the input, u'_k, that is, the~~ cumulative power distribution function of u_k.

The following generalised power constraint is considered:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} u_k^T u_k = E \qquad (6)$$

Hence, introducing $\frac{1}{2}\mathbf{B}(\omega) d\xi(\omega)$ for $dG(\omega)$, equation (5) can be rewritten as:

$$(\bar{M}')_{\ell q} = \frac{1}{2}\int_{-\pi}^{\pi} \text{trace}\,\{[H^{\ell}(e^{-j\omega})\Sigma^{-1}H^{q}(e^{j\omega})]\mathbf{B}(j\omega)\}d\xi(\omega) \qquad (7)$$

where $\xi(\omega)$ satisfies:

$$\frac{1}{2}\int_{-\pi}^{\pi} d\xi(\omega) = 1 \qquad (8)$$

or, by exploiting the symmetry of $\xi(\omega)$ about $\omega = 0$:

$$\bar{M}' = \int_{0}^{\pi} M(\omega)d\xi(\omega) \qquad (9)$$

where

$$(M(\omega))_{\ell q} = R_e\{\text{trace}\,\{H^{\ell}(e^{-j\omega})\Sigma^{-1}H^{q}(e^{j\omega})\mathbf{B}(\omega)\}\} \qquad (10)$$

and $\xi(\omega)$ satisfies:

$$\int_{0}^{\pi} d\xi(\omega) = 1 \qquad (11)$$

Notation:

Denote the set of all $\xi(\omega)$ satisfying (11) by Ξ. #

Remark 1:

The fraction of the total power in the range ω to $\omega + d\omega$ is given by $d\xi(\omega)$. ~~The manner in which the power is shared between the various inputs at the frequency ω is given by the complex matrix, $H(\omega)$. The properties of $H(\omega)$ are:~~ For a general stationary input process, the matrix $B(\omega)$ can have rank 1 to r. Here attention is restricted to the case where rank $B(\omega) = 1$. This includes all deterministic inputs and stochastic inputs with multiplicity 1. [82] The properties of $B(\omega)$ for this case are:

(i) The diagonal elements of $\lambda_i(\omega) = (\mathbf{B}(\omega))_{ii}$ are real and positive. $\lambda_i(\omega)d\xi(\omega)$ is the power of input i in the frequency range ω to $\omega + d\omega$.

(ii) trace $\{\mathbf{B}(\omega)\} = \sum_{i=1}^{r} \lambda_i(\omega) = E$. That is, the total power in the range ω to $\omega + d\omega$ is $Ed\xi(\omega)$.

(iii)

$$(\mathbf{B}(\omega))_{\ell q} = (\lambda_\ell(\omega)\lambda_q(\omega))^{\frac{1}{2}} \exp\{j(\phi_\ell(\omega)-\phi_q(\omega))\} \qquad (12)$$

where $\phi_\ell(\omega), \phi_q(\omega)$ are the phases of inputs ℓ and q with respect to some reference phase (taken for convenience to be $\phi_1 = 0$). #

Remark 2:

The matrix $M(\omega)$ given by (10) is the matrix $\overline{M}'$ corresponding to an input satisfying (6) and containing only one frequency component, that is an input having a (one sided) spectrum with a single line. #

It is now possible to state and prove theorems on the properties of the information matrix and input power density spectrum. Analogues of these theorems for the case of static regression experiment design are well known, (for example [38]), and have recently been extended by Mehra, [78], to the case of single input dynamic systems with measurement noise of known characteristics. Here they are extended to the general multiple input innovations model:

Theorem 1:

The submatrix $\overline{M}_{BDCA}$ *of* $\overline{M}'$ *corresponding to the* B, D, C *and* A *matrix parameters, is singular if the input spectrum contains less than* p"/2m *lines, where* p" *is the number of* B, D, C *and* A *parameters and* m *is the number of outputs.*

Proof:

From results (4.4.6) and (4.4.7):

$$\overline{M} = \overline{M}' + \overline{M}'' \qquad (13)$$

where:

$$\bar{M}' = \begin{bmatrix} & & \vdots & 0 \\ & \bar{M}_{BDCA} & \vdots & \vdots \\ & & \vdots & 0 \\ \hline 0 & \cdots & 0 \vdots & 0 \end{bmatrix} \tag{14}$$

$$\bar{M}'' = \begin{bmatrix} 0 & \vdots & 0 & \vdots & 0 \\ \hline 0 & \vdots & \tilde{M} & & \\ \hline 0 & \vdots & & & \end{bmatrix} \tag{15}$$

where $\tilde{M}$ is a constant matrix and the partitions indicated correspond to the partition of β symbolised by:

$$\beta \sim [B, D \mid C, A \mid K, \gamma, \sigma] \tag{16}$$

From equation (3):

$$\bar{M}_{BDCA} = 2R_e\{\int_0^{\pi} \tilde{G}(e^{-j\omega})^T \Sigma^{-1} \tilde{G}(e^{j\omega})\, d\omega\} \tag{17}$$

where $\tilde{G}$ is a $(m \times p'')$ complex matrix with q^{th} column $\tilde{g}^q$ as defined by (4).

For a spectrum with ℓ lines, (17) reduces to:

$$\bar{M}_{BDCA} = 2 \sum_{i=1}^{\ell} R_e\{\tilde{G}(e^{-j\omega_i})^T \Sigma^{-1} \tilde{G}(e^{j\omega_i})\} \tag{18}$$

From (18) the maximum possible rank of $\bar{M}_{BDCA}$ is $2\ell.\min(m,p'')$. Thus $\bar{M}_{BDCA}$ is singular if $2\ell.\min(m,p'') < p''$. The result follows. #

Theorem 2:

The average information matrix, $\bar{M}$, is singular if the input spectrum contains less than $p'/2m$ lines, where p' is the number of B and D parameters and m is the number of outputs.

Proof:

From (13), (14) and (15) it can be seen that $\overline{M}$ will be singular if $\overline{M}_{BDCA}$ has rank < p'. That is, if $2\ell.\min(m,p'') < p'$. The result follows. #

Remark 2:

Theorem 2 gives a minimum condition that must be satisfied by the input for identifiability of all the parameters. #

Theorem 3:

The set of all average information matrices corresponding to the power constraint, (6), *and fixed* $\mathbf{S}(\omega)$, $\omega \in [-\pi,\pi)$ *is convex. Further, if* $\mathbf{S}(\omega)$ *is continuous and* $\omega \in [-\omega_M,\omega_M]$ where ω_M is any frequency in the range $(-\pi,\pi)$, *then the set of average information matrices is closed.*

Proof:

Let $\xi_1(\omega)$, $\xi_2(\omega) \in \Xi$. Then $\xi(\omega) \in \Xi$ where

$$\xi(\omega) = \alpha\xi_1(\omega) + (1-\alpha)\xi_2(\omega) \; ; \quad \alpha \in [0,1]$$

The corresponding matrices $\overline{M}'$ are related by:

$$\overline{M}'(\xi) = \alpha\overline{M}'(\xi_1) + (1-\alpha)\overline{M}'(\xi_2) \tag{19}$$

as is readily verified.

From (19) and (2.2) it follows that:

$$\overline{M}(\xi) = \alpha\,\overline{M}(\xi_1) + (1-\alpha)\overline{M}(\xi_2) \tag{20}$$

Thus convexity is proved. Closure follows from the continuity of H^i and $\mathbf{B}$ and the closure of $[-\omega_M, \omega_M]$. #

Theorem 4:

For any $\mathbf{B}(\omega)$, $\omega \in [-\pi,\pi)$ *and* $\xi_1(\omega) \in \Xi$ *with corresponding average information matrix,* $\bar{M}_1$, *there is always a* $\xi_2(\omega) \in \Xi$, *the spectrum of which contains at most* $p''(p''+1)/2 + 1$ *lines and* $\bar{M}_1 = \bar{M}_2$, *where* p'' *is the number of* B, D, C, *and* A *parameters and* $\bar{M}_2$ *corresponds to* $\xi_2(\omega)$.

Proof:

Equations (9) and (11) define the convex hull, C, of all matrices $M(\omega)$ corresponding to the spectrum with a single line at ω. Since $M(\omega)$ is symmetric, it can be represented as a point in a $p''(p''+1)/2$ dimensional space. It follows from a classical theorem of Caratheodory, [34], that $\bar{M}'$ may be represented as:

$$\bar{M}' = \sum_{i=1}^{\ell} M(\omega_i)\xi_i \tag{21}$$

where

$$\sum_{i=1}^{\ell} \xi_i = 1 \tag{22}$$

and

$$\ell = p''(p''+1)/2 + 1 \tag{23}$$

Defining $\xi_2(\omega)$ as a measure which assigns a fraction ξ_i of the total power at frequency ω_i for $i = 1, \ldots. \ell$, leads to the required result. #

Remark 3:

From theorem 4, *any* average information matrix, $\bar{M}$, may be achieved with an input having a line spectrum with no more than $p''(p''+1)/2 + 1$ lines. In particular, there exists an optimal spectrum

with no more than this number of lines. The search for an optimal spectrum thus reduces to a search for $2r(p"(p"+1)/2+1)-1$ quantities namely:

$$\omega_1, \omega_i, \xi_i, \lambda_j(\omega_i), \phi_j(\omega_i); \quad j = 2,\ldots r; \quad i = 2,\ldots p"(p"+1)/2+1 \tag{24}$$

NOTE:

$$\phi_1(\omega_i) = 0; \quad \lambda_1(\omega_i) = E - \sum_{j=2}^{r} \lambda_j(\omega_i), \quad \xi_1 = 1 - \sum_{i=2}^{\ell} \xi_i \tag{25}$$ #

4. Tests for Optimality

Whittle, [56], has stated a general equivalence theorem for concave optimality criteria. Special cases of this theorem for $J = \det M$ and $J = \text{trace } M^{-1}$ where M is Fisher's information matrix, have been discussed at length by Kiefer and Wolfowitz, [33], Karlin and Studden, [34], Fedorov, [38], and others. Here Whittle's general theorem is stated for convex differentiable optimality criteria and is shown to apply to the criteria J_1 and J_2 previously defined. First some definitions needed for the statement of the theorem are given:

Definition 1:

The cost function $J(\xi)$ *corresponding to any* $\xi(\omega) \in \Xi$ *is convex if:*

$$J\{(1-\alpha)\xi+\alpha\eta\} \leq (1-\alpha)J(\xi) + \alpha J(\eta) \quad \forall\ \xi, \eta \in \Xi \tag{1}$$

Definition 2:

The directional derivative $\Phi(\xi,\eta)$ *of* J *at* ξ *in the direction of* η *for all* $\xi, \eta \in \Xi$ *is given by:*

$$\Phi(\xi,\eta) = \frac{\partial}{\partial\alpha}\left[J\{(1-\alpha)\xi+\alpha\eta\}\right]\Big|_{\alpha=0} \tag{2}$$

Definition 3:

The maximal rate of descent of J from ξ is defined as:

$$\overline{D}(\xi) = \inf_{\omega\in\Lambda} \Phi(\xi,\delta_\omega) \tag{3}$$

where $\delta_\omega \in \Xi$ is a design which contains only one frequency component at ω.

Theorem 1: (Whittle):

If J is convex, then a J-optimal design, ξ^*, can be equivalently characterised by any of the three conditions:

(i) ξ^* *minimises* J (4)

(ii) ξ^* *minimises* $\overline{D}(\xi)$ (5)

(iii) $\overline{D}(\xi^*) = 0$ (6) #

The applicability of this theorem to J_1 and J_2 is now investigated:

Result 1:

$J_1 = \log\det(\overline{M}'+\overline{P}')^{-1}$ *is a convex cost function, where* $\overline{P}'$ *is a constant positive semi-definite matrix.*

Proof:

Let $\xi_1(\omega)$, $\xi_2(\omega) \in \Xi$ and let $\xi(\omega) = \alpha\xi_1(\omega) + (1-\alpha)\xi_2(\omega)$. Then

$$\overline{M}'(\xi) = \alpha\overline{M}'(\xi_1) + (1-\alpha)\overline{M}'(\xi_2) \tag{7}$$

therefore

$$J_1(\xi) = -\log\det(\alpha(\overline{M}'(\xi_1)+\overline{P}') + (1-\alpha)(\overline{M}'(\xi_2)+\overline{P}'))$$

But $\det(\alpha A+(1-\alpha)B) \geq (\det A)^{\alpha}(\det B)^{1-\alpha}$ for A, B any positive definite matrices, [38]. Thus:

$$J_1(\xi) \leq -\alpha \log \det(\bar{M}'(\xi_1)+\bar{P}') - (1-\alpha)\log\det(\bar{M}'(\xi_2)+\bar{P}')$$

$$= \alpha J_1(\xi_1) + (1-\alpha)J_1(\xi_2) \qquad \#$$

Result 2:

$J_2 = \text{trace}\{\Gamma(\bar{M}'+\bar{P}')^{-1}\}$ *is a convex cost function where* Γ, $\bar{P}'$ *are positive semi-definite matrices.*

Proof:

From (7):

$$J_2(\xi) = \text{trace}\{\Gamma(\alpha(\bar{M}'(\xi_1)+\bar{P}') + (1-\alpha)(\bar{M}'(\xi_2)+\bar{P}'))^{-1}\}$$

But

$$(\alpha A+(1-\alpha)B)^{-1} \leq \alpha A^{-1} + (1-\alpha)B^{-1}$$

for any A, B positive definite matrices, [38]. So since Γ is positive semi-definite, it follows that:

$$J_2(\xi) \leq \alpha\,\text{trace}\{\Gamma(\bar{M}'(\xi_1)+\bar{P}')^{-1}\} + (1-\alpha)\,\text{trace}\{\Gamma(\bar{M}'(\xi_2)+\bar{P}')^{-1}\}$$

$$= \alpha J_2(\xi_1) + (1-\alpha)J_2(\xi_2) \qquad \#$$

Since J_1 and J_2 satisfy the conditions for theorem 1 the following equivalence theorems may be stated:

Theorem 2:

A J_1-optimal design, ξ^*, can be equivalently characterised by any of the three conditions:

(i) ξ^* *minimises* $\log \det (\bar{M}'(\xi)+\bar{P}')^{-1}$ (8)

(ii) ξ^* *minimises* $\sup_{\omega\in\Lambda} \text{trace}\ \{(\bar{M}'(\xi)+\bar{P}')^{-1}(\bar{M}(\omega)+\bar{P}')\}$ (9)

(iii) $\sup_{\omega\in\Lambda} \text{trace}\ \{(\bar{M}'(\xi^*)+\bar{P}')^{-1}(\bar{M}(\omega)+\bar{P}')\} = p$ (10)

where p *is the total number of parameters and* $\bar{M}(\omega)$ *is the matrix* $\bar{M}'(\delta_\omega)$. *(Equation* (3.10).*)*

Proof:

J_1 is convex from result 1. Also, from (2):

$$\Phi(\xi,\eta) = \text{trace}\ \{(\bar{M}'(\xi)+\bar{P}')^{-1}\ (\bar{M}'(\xi)-\bar{M}'(\eta))\} \tag{11}$$

$$= p - \text{trace}\ \{(\bar{M}'(\xi)+\bar{P}')^{-1}(\bar{M}'(\eta)+\bar{P}')\} \tag{12}$$

Putting $\eta = \delta_\omega$ and substituting (12) into (5) and (6) leads to (9) and (10). #

Theorem 3:

A J_2-optimal design, ξ^*, can be equivalently characterised by any of the three conditions:

(i) ξ^* *minimises* $J_2(\xi) = \text{trace}\ \{\Gamma(\bar{M}'(\xi)+\bar{P}')^{-1}\}$ (13)

(ii) ξ^* *minimises* $\sup_{\omega\in\Lambda} \text{trace}\ \{(\bar{M}'(\xi)+\bar{P}')^{-1}\Gamma(\bar{M}'(\xi)+\bar{P}')(\bar{M}(\omega)+\bar{P}')\}$ (14)

(iii) $\sup_{\omega\in\Lambda} \text{trace}\ \{(\bar{M}(\xi^*)+\bar{P}')^{-1}\Gamma(\bar{M}'(\xi^*)+\bar{P}')^{-1}(\bar{M}(\omega)+\bar{P}')\} = J_2(\xi^*)$ (15)

Proof:

J_2 is convex from result 2. Also, from (2):

$$\Phi(\xi,\eta) = \text{trace}\ \{(\bar{M}'(\xi)+\bar{P}')^{-1}\Gamma(\bar{M}'(\xi)+\bar{P}')^{-1}(\bar{M}'(\xi)-\bar{M}'(\eta)) \tag{16}$$

$$= \text{trace}\ \{\Gamma(\bar{M}'(\xi)+\bar{P}')^{-1}\}$$

$$- \text{trace}\ \{(\bar{M}'(\xi)+\bar{P}')^{-1}\Gamma(\bar{M}'(\xi)+\bar{P}')^{-1}(\bar{M}'(\eta)+\bar{P}')\} \quad (17)$$

Putting $\eta = \delta_\omega$ and substituting (17) into (5) and (6) yields (14) and (15). #

Notation:

Let $V_1(\xi,\omega)$ and $V_2(\xi,\omega)$ be defined as follows:

$$V_1(\xi,\omega) = \text{trace}\ \{(\bar{M}'(\xi)+\bar{P}')^{-1}(\bar{M}(\omega)+\bar{P}')\} \quad (18)$$

$$V_2(\xi,\omega) = \text{trace}\ \{(\bar{M}'(\xi)+\bar{P}')^{-1}\Gamma(\bar{M}'(\xi)+\bar{P}')^{-1}(\bar{M}(\omega)+\bar{P}')\} \quad (19)$$

#

Tests for optimality and non-optimality of test signals based on theorems 2 and 3 are now given:

Result 3 - Test for J_1-Optimality:

A design, ξ, is J_1-optimal if $V_1(\xi,\omega)$ defined by (18) is less than or equal to p for every $\omega \in \Lambda$.

Proof:

The result follows from part (iii) of theorem 2. #

Result 4 - Test for J_2-Optimality:

A design, ξ, is J_2-optimal if $V_2(\xi,\omega)$ defined by (19) is less than or equal to $J_2(\xi)$ for every $\omega \in \Lambda$.

Proof:

The result follows from part (iii) of theorem 3. #

Result 5 - Necessary Condition for J_1-Optimality:

If ξ^* is a J_1-optimal design, then $V_1(\xi^*,\omega) = p$ for every ω to

which ξ^* assigns non zero measure.

Proof:

Suppose that $V_1(\xi^*,\omega) < p$ for some ω with non zero measure. Then since $V_1(\xi^*,\omega) \leq p \ \forall \ \omega$ by (10), it follows that:

$$\int_0^\pi V_1(\xi^*,\omega)\, d\xi^*(\omega) < p$$

But by integrating (18) it is readily shown that:

$$\int_0^\pi V_1(\xi,\omega)\, d\xi(\omega) = p \tag{20}$$

for *any* ξ; Contradiction. #

Result 6 - Necessary Condition for J_2-Optimality:

If ξ^* is a J_2-optimal design, then $V_2(\xi^*,\omega) = J_2(\xi^*)$ for every ω to which ξ^* assigns non zero measure.

Proof:

Suppose that $V_2(\xi^*,\omega) < J_2(\xi^*)$ for some ω with non zero measure. Then since $V_2(\xi^*,\omega) \leq J_2(\xi^*) \ \forall \ \omega$ by (15), it follows that:

$$\int_0^\pi V_2(\xi^*,\omega)\, d\xi^*(\omega) < J_2(\xi^*)$$

But, integrating (19) it is readily shown that:

$$\int_0^\pi V_2(\xi,\omega)\, d\xi(\omega) = J_2(\xi) \tag{21}$$

for *any* ξ; Contradiction. #

Remark 1:

Results 5 and 6 allow non-optimal designs to be detected and discarded. The test is very easy to perform when the input has a line spectrum containing few lines. #

Remark 2:

The error in the frequency response estimate due to estimation errors is given to first order by:

$$\Delta y(e^{j\omega}) = \frac{\partial y(e^{j\omega})}{\partial \beta} (\beta - \hat{\beta})$$

where $\hat{\beta}$ the posterior mean and $y(z)$ is the z-transform of the output, $\{y_k\}$, given by:

$$y_k = Cx_k + Du_k$$

$$x_{k+1} = Ax_k + Bu_k$$

$$x_1 = \gamma$$

Thus the posterior covariance of this error at ω is:

$$V(\xi,\omega) = 2R_e\{(\frac{\partial y(e^{j\omega})}{\partial \beta}) Q_\beta (\frac{\partial y(e^{-j\omega})}{\partial \beta})\} \qquad (22)$$

where $Q_\beta \simeq (M'(\xi)+P')^{-1}$ is the posterior covariance. Taking as a measure of the error covariance the quantity:

$$v(\xi,\omega) = \text{trace}\{\Sigma^{-1} V(\xi,\omega)\}$$

it is not hard to see that:

$$\upsilon(\xi,\omega) \propto \text{trace } \{(\overline{M}'(\xi)+\overline{P}')^{-1}\overline{M}(\omega)\} \tag{23}$$

Defining $w(\xi)$ as:

$$w(\xi) = \text{trace } \{(\overline{M}'(\xi)+\overline{P}')^{-1}\overline{P}'\} \tag{24}$$

it can be seen from theorem 2 that:

(i) ξ^* is any design ξ that minimises $\sup_{\omega\in\Lambda} \{\upsilon(\xi,\omega)\} + w(\xi)$ (25)

(ii) ξ^* is optimal if $\sup_{\omega\in\Lambda} \{\upsilon(\xi^*,\omega)\} + w(\xi^*) = p$ (26)

#

In section 3 it was shown that any information matrix could be achieved with an input having a line spectrum no more than $p"(p"+1)/2+1$ lines. The next result enables this to be reduced to $p"(p"+1)/2$.

Result 7:

The information matrices corresponding to either J_1-optimal designs or J_2-optimal designs are boundary points of the set of all possible information matrices.

Proof:

Let M_o be the matrix $\overline{M}'(\xi^*)$ corresponding to the optimal design ξ^* and assume that M_o is an interior point of the set M of all possible matrices $\overline{M}'$. Then there exists an $\alpha > 0$ such that $M_1 = (1+\alpha)M_o$ also belongs to the set M.

J_1-optimality:

$$J_1 = \log \det ((1+\alpha)M_o+\bar{P}')$$

$$\left.\frac{\partial J_1}{\partial \alpha}\right|_{\alpha=0} = - \text{trace } \{(M_o+\bar{P})^{-1}M_o\}$$

$$< 0$$

J_2-optimality:

$$J_2 = \text{trace } \{\Gamma((1+\alpha)M_o+\bar{P}')^{-1}\}$$

$$\left.\frac{\partial J_2}{\partial \alpha}\right|_{\alpha=0} = - \text{trace } \{(M_o+\bar{P}')^{-1}\Gamma(M_o+\bar{P}')^{-1}M_o\}$$

$$< 0 \text{ (provided } \Gamma > 0)$$

Thus in both cases the derivative of the cost function is strictly negative so an $\alpha > 0$ exists such that a decrease in cost occurs - but M_o corresponds to an optimal design - contradiction. #

Result 8:

J_1-optimal and J_2-optimal designs exist having an input spectrum containing no more than $p"(p"+1)/2$ *lines.*

Proof:

Follows from Caratheodory's theorem, [34]. #

In the next section it is shown that this number may be still further reduced by exploiting the structure of single output systems.

5. Optimal Spectra for Single Output Systems

In section 4.9 it was shown that the matrix M' was given by:

$$M' = \frac{1}{\sigma^2} \sum_{k=1}^{N} g_k g_k^T \tag{1}$$

where

$$g_k^T = [t_k^1, \ldots t_{k-n}^1, t_k^2, \ldots t_{k-n}^2, \ldots t_{k-n}^r, s_k, \ldots s_{k-n+1}] \tag{2}$$

and

$$C(z^{-1}) t_k^j = - u_k^j \; ; \quad j = 1, \ldots r \tag{3}$$

$$A(z^{-1}) s_k = \sum_{\ell=1}^{r} B^j(z^{-1}) t_k^j \tag{4}$$

Equations (3) and (4) may be rewritten in the form:

$$A(z^{-1}) C(z^{-1}) v_k^j = -u_k^j \tag{5}$$

$$t_k^j = A(z^{-1}) v_k^j \tag{6}$$

$$s_k = \sum_{j=1}^{r} B^j(z^{-1}) v_k^j \tag{7}$$

Now, from (1) and assuming stationary inputs :

$$\overline{M'} = \lim_{N \to \infty} \frac{1}{N\sigma^2} \sum_{k=1}^{N} g_k g_k^T \tag{8}$$

$$= \begin{bmatrix} \Gamma_{TT}^{11} & \cdots & \Gamma_{TT}^{1r} & \Gamma_{TS}^{1} \\ \vdots & & \vdots & \vdots \\ \Gamma_{TT}^{r1} & \cdots & \Gamma_{TT}^{rr} & \Gamma_{TS}^{r} \\ \Gamma_{ST}^{1} & \cdots & \Gamma_{ST}^{r} & \Gamma_{SS} \end{bmatrix} \tag{9}$$

where Γ_{TT}^{ij}, Γ_{TS}^{i}, Γ_{SS} are matrices with ℓq^{th} elements $\gamma_{TT}^{ij}(q-\ell)$, $\gamma_{TS}^{i}(q-\ell)$, $\gamma_{SS}(q-\ell)$ respectively. The elements are given by:

$$\gamma_{TT}^{ij}(\tau) = A(z^{-1})A(z)\gamma_{vv}^{ij}(\tau); \quad i,j=1,\ldots r; \ \tau=0,\pm1,\ldots\pm n. \tag{10}$$

$$\gamma_{SS}(\tau) = \sum_{i=1}^{r}\sum_{j=1}^{r} B^{i}(z^{-1})B^{j}(z)\gamma_{vv}^{ij}(\tau); \ \tau=0,\pm1,\ldots\pm(n-1) \tag{11}$$

$$\gamma_{TS}^{i}(\tau) = \sum_{j=1}^{r} A(z^{-1})B^{j}(z)\gamma_{vv}^{ij}(\tau); \ i=1,\ldots r; \ \tau=0,\pm1,\ldots\pm(n-1),-n \tag{12}$$

Remark 1:

$\gamma_{TT}^{ij}(\tau)$, $\tau=0,1,2\ldots$ is the cross-covariance of t_k^i and t_k^j; $\gamma_{SS}(\tau)$ is the auto-covariance of s_k; γ_{TS}^{i} is the cross-covariance between t_k^i and s_k; γ_{vv}^{ij} is the cross-covariance between v_k^i and v_k^j. #

Result 1:

For single output systems in Caines' canonical form, the average information matrix $\bar{M}$ lies in a $(2n+1)r^2$-dimensional subspace of the space of all average information matrices.

Proof:

From (10), (11) and (12) it can be seen that only $\gamma_{vv}^{ij}(\tau)$, $\tau = 0, \pm1, \pm2, \ldots \pm2n$; $i, j = 1, \ldots r$ are needed to specify $\bar{M}'$ completely. Making use of the fact that $\gamma_{vv}^{ij}(\tau) = \gamma_{vv}^{ji}(-\tau)$ leads to the conclusion that $\bar{M}'$ is completely specified by the $(2n+1)r^2$ quantities:

$$\gamma_{vv}^{(ij)}(\tau); \ \tau = 0, 1, \ldots 2n; \quad i, j = 1, \ldots r \tag{13}$$

The result follows. #

Corollary to Result 1:

For single output systems, J_1-optimal and J_2-optimal designs exist

having an input spectrum containing no more than $(2n+1)r^2$ *lines.*

Proof:

The result follows immediately from Caratheodory's theorem. #

Remark 2:

In the last section it was shown that not more than $p''(p''+1)/2$ lines were needed in general. For the single output model considered here, $p'' = r(n+1) + n$. To see under what conditions $(2n+1)r^2$ is less than $p''(p''+1)/2$ define Δ as:

$$\Delta = p''(p''+1)/2 - (2n+1)r^2 \tag{14}$$

$$= \tfrac{1}{2}[r^2(n^2-2n-1)+r(2n^2+3n+1)+(n^2+n)]$$

Δ is positive if:

(i) $n = 1$, $r \leq 3$

(ii) $n = 2$, $r \leq 15$

(iii) $n \geq 3$, r any finite integer.

Thus result 1 represents a decrease in the number of lines necessary for all but pathological cases. In particular, for $r = 1$, the decrease is very significant. This is brought out in result 2. #

Result 2:

For single input-single output systems, J_1*-optimal and* J_2*-optimal designs exist having an input spectrum containing no more than* p'' *lines.*

Proof:

From the corollary to result 1 it can be seen that only $(2n+1)$ lines are needed, Since $p'' = 2n + 1$ the result is proved. #

Remark 3:

Although result 2 was proved for single input-single output systems in Caines' canonical form, it is in fact true for systems with A and B polynomials of different order. Also, for continuous time models the time delay may be included in the parameter list without affecting the result. Both these points will be taken up again in section 8.

6. Design Algorithms

In section 4, Whittle's general equivalence theorem was given. This theorem states that a J-optimal design, ξ, is one which minimises the maximum rate of descent defined by:

$$\overline{D}(\xi) = \inf_{\omega \in \Lambda} \Phi(\xi, \delta_\omega) \tag{1}$$

where

$$\Phi(\xi, \delta_\omega) = \frac{\partial}{\partial \alpha} \left[J\{(1-\alpha)\xi + \alpha\delta_\omega\} \right] \Big|_{\alpha = 0} \tag{2}$$

It further states that any design, ξ, for which the maximal rate of descent is zero, is J-optimal.

From this theorem the following algorithm naturally suggests itself, (see also [38], [56], [57]):

1. Start with any design, ξ_0. Set $\ell = 0$.
2. Calculate $\Phi(\xi_\ell, \delta_\omega)$ for all $\omega \in \Lambda$.
3. Find $\overline{D}(\xi_\ell)$ and the corresponding minimising $\omega = \omega_\ell$.
4. Let $\xi_{\ell+1} = (1-\alpha_\ell)\xi_\ell + \alpha_\ell \delta_{\omega_\ell}$ where α_ℓ is chosen so that a decrease in J occurs.

5. $\ell = \ell + 1$; if $\overline{D}(\xi_\ell) = 0$ stop; else go to 2.

The convergence of this algorithm is evident, [56], but unfortunately its applicability to the problem at hand is limited for the following reasons:

(i) The number of points in the optimal design is in general large even if "rounded-off" designs, [38], are used. This can pose realization problems.

(ii) It requires a knowledge of the matrix $\mathbf{B}(\omega)$, defined in section 3, for all ω. This is not a restriction for single input systems since $\mathbf{B}(\omega)$ is simply E. However, for multi-input systems the algorithms cannot be directly applied.

An alternative approach is to use a steepest descent algorithm in the $(2r\ell-1)$-dimensional space of the design variables:

$$\omega_i,\ \lambda_j(\omega_i),\ \phi_j(\omega_i);\ i = 1, \ldots \ell;\ j = 2, \ldots r,\ \xi_i,\ i = 2, \ldots \ell \tag{3}$$

where ℓ is the number of lines in the input spectrum. $\lambda_j(\omega_i)$, $\phi_j(\omega_i)$ define the elements of $\mathbf{B}(\omega_i)$, $i = 1, \ldots \ell$.

The essential point is that the dimension of the space of design variables is small and hence the above algorithms should present ~~no~~ little computational difficulty. In fact, for some systems analytic designs are possible as will be shown in appendix A.

7. Realization Procedures

For power constrained inputs there is little problem in realizing a signal with the required power density spectrum. For spectra consisting of a finite number of lines a suitable realization

is obtained by simply adding sinusoids of the required frequencies, although this could be tedious for large numbers of lines. For continuous rational spectra a suitable realization procedure is to pass Gaussian white noise through a filter with rational transfer function H(z) where:

$$\Phi^*(z) = H(z)H(z^{-1}) \tag{1}$$

and $\Phi^*(e^{j\omega})$ is the required power density spectrum.

In general, the theory presented in this chapter for power constrained inputs is not strictly applicable to other types of constraints. However, as will be brought out presently, amplitude (and other) constrained inputs can often be found with spectra closely approximating the spectra of optimal power constrained inputs of the same power.

Result 1:

If, for a single input system with amplitude constraints, $\pm\delta$, on the input, a $\pm\delta$ binary signal exists with power density spectrum equal to an optimal δ^2-power constrained input, then the binary signal is an optimal amplitude constrained input.

Proof:

It was shown in theorem (4.6.2) that with amplitude constraints the optimal input must be binary. Further in theorem (2.1) it was shown that it is only the spectral properties of the input which matter. The result follows. #

Under fairly general conditions, binary signals do exist with arbitrary continuous spectra and are readily generated by means of a linear filter driven by Gaussian white noise followed by an infinite gain amplifier and clipper. The autocorrelation of the output of the

clipper is related to the input of the amplifier via the arcsine law, [35]. In fact, it is possible to generate signals with almost any continuous power density spectrum and amplitude probability distribution, [53], [55].

For line spectra, however, binary signals do not in general exist having only the required lines, [79]. In section 2, it was shown that it was possible to achieve any desired accuracy by considering only a finite portion of the input autocorrelation, $R(\tau)$, $-T \leq \tau \leq T$, say. The implication is that periodic inputs may be used with period $> T$, or, in the frequency domain, it is permissable to consider spectra with lines only at multiples of $\frac{1}{T}$. The optimal spectrum may thus be approximated by such a spectrum. Van den Bos, [80], has described methods for generating periodic binary signals with specified line spectra. It is also possible to approximate line spectra with continuous spectra. This is achieved by matching autocorrelation functions of the two spectra over an interval $[-T,T]$ where $\frac{1}{T}$ is the highest significant frequency in the line spectrum. The remainder of the autocorrelation function may then be arbitrary, (for example, maximum entropy extrapolation, [53]). The input may then be realized as described above for continuous spectra.

A point in favour of line spectra is that subsequent data analysis is greatly simplified, [15].

8. Extension to Continuous Time Systems

An appropriate model for a continuous model is the continuous analogue of the innovations model described in chapter 4:

$$dx_t = Ax_t dt + Bu_t dt + K d\varepsilon_t \tag{1}$$

$$dy_t = Cx_t dt + Du_t dt + d\varepsilon_t \tag{2}$$

where ε_t is a Wiener Process. Care must be taken with this representation, especially for sampling problems, [81], but most of the results presented in this thesis have continuous time analogues which may be obtained formally by:

(i) replacing summations over time by integrations,

(ii) replacing integrations over $[-\pi,\pi)$ by integrations over $(-\infty,\infty)$, (or $[0,\pi)$ by $[0,\infty)$),

(iii) replacing $z = e^{j\omega}$ by $s = j\omega$.

As an illustration, result 5.2 on the maximum number of spectral lines necessary for single input-single output systems is rederived for continuous time systems with unequal numerator and denominator orders and with the time delay in the parameter list.

The system may be modelled as follows:

$$\tilde{y}(s) = \frac{B(s)e^{s\tau}}{A(s)}\tilde{u}(s) + \tilde{n}(s) \tag{3}$$

where $\tilde{u}(s)$, $\tilde{y}(s)$ are the Laplace transforms of the input, u_t and the output, Y_t, respectively and $\tilde{n}(s)$ is the Laplace transform of the noise which is assumed to be Gaussian and to have spectral density expressible as:

$$\Phi(s) = \frac{D(s)D(\bar{s})}{C(s)C(\bar{s})} \tag{4}$$

where $\bar{s}$ denotes complex conjugate, A(s), B(s), C(s), D(s) are finite polynomials in s. The orders of A and B are n and m respectively.

Fisher's information matrix, M, for this model is given by ([5], [15]):

$$M = \begin{bmatrix} M_1 & O \\ O & M_2 \end{bmatrix} \tag{5}$$

where M_2 is a constant matrix and M_1 is given by:

$$M_1 = \int_0^{t_f} \left(\frac{\partial e(t)}{\partial \theta}\right)^T \left(\frac{\partial e(t)}{\partial \theta}\right) dt \tag{6}$$

where θ is a p" vector of coefficients:

$$\theta^T = [a_o, \ldots a_{n-1}, b_o, \ldots b_m, \tau] \tag{7}$$

t_f is the experiment time and is assumed to be large, and the functions $\frac{\partial e(t)}{\partial \theta}$ are obtained as the outputs of the following linear equations:

$$\frac{\partial \tilde{e}(s)}{\partial a_i} = s^i X(s) B(s) \tilde{u}(s); \quad i = O, \ldots, n-1 \tag{8}$$

$$\frac{\partial \tilde{e}(s)}{\partial b_i} = -s^i X(s) A(s) \tilde{u}(s); \quad i = O, \ldots, m \tag{9}$$

$$\frac{\partial e(s)}{\partial \tau} = sX(s)A(s)B(s)\tilde{u}(s) \tag{10}$$

where

$$X(s) = \frac{C(s)}{D(s)A^2(s)} \tag{11}$$

A direct application of Parseval's theorem, [77], to (6) leads

to the result:

$$M_1 = \int_{-\infty}^{\infty} \left(\frac{\partial e(-j\omega)}{\partial \Theta}\right)^T \left(\frac{\partial e(j\omega)}{\partial \Theta}\right) d\mu(\omega) \tag{12}$$

where $\mu(\omega)$ is the appropriate measure on the frequency space. Substituting (8)-(11) into (12) leads to:

$$M_1(\xi) = t_f \int_{-\infty}^{\infty} F(-j\omega) X(-j\omega) X(j\omega) F^T(j\omega) d\xi(\omega) \tag{13}$$

where $\xi(\omega)$ is the cumulative power distribution of the input and where $F(s)$ is a p" vector having k^{th} element:

$$\begin{aligned} F_k(s) &= s^{k-1} B(s); \quad k = 1,\ldots n \\ &= -s^{k-n-1} A(s); \quad k = n+1,\ldots n+m+1 \\ &= sA(s)B(s); \quad k = p" = n+m+2 \end{aligned} \tag{14}$$

The total input power is constrained, that is:

$$\int_0^{\infty} d\xi(\omega) = 1 \tag{15}$$

Theorem 1:

For the single input-single output continuous time system, (3), J_1*-optimal and* J_2*-optimal power constrained designs exist having no more than* p" *lines where* p" *is the number of system parameters.*

Proof:

From equations (13) and (14), M_1 can be expressed as:

$$M_1(\xi) = 2t_f \int_0^\infty R_e\{V(j\omega)\}X(-j\omega)X(j\omega)\,d\xi(\omega) \tag{16}$$

where $V(j\omega)$ is a $p'' \times p''$ matrix having ik^{th} element:

$$V_{ik}(j\omega) = (-1)^{i-1}(j\omega)^{i+k-2}B(-j\omega)B(j\omega) \quad ; \; i=1\ldots n;\; k=1\ldots n$$

$$= (-1)^{i}(j\omega)^{i+k-n-2}B(-j\omega)A(j\omega) \quad ; \; i=1\ldots n;\; k=n+1\ldots n+m+1$$

$$= (-1)^{i-1}(j\omega)^{i}B(-j\omega)B(j\omega)A(j\omega) \quad ; \; i=1\ldots n;\; k=n+m+2$$

$$= (-1)^{i-n}(j\omega)^{i+k-2n-2}A(-j\omega)A(j\omega) \quad ; \; i=n+1\ldots n+m+1,\; k=n+1\ldots n+m+1$$

$$= (-1)^{i-n}(j\omega)^{i-n}A(-j\omega)A(j\omega)B(j\omega) \quad ; \; i=n+1\ldots n+m+1,\; k=n+m+2$$

$$= \omega^2 A(-j\omega)A(j\omega)B(-j\omega)B(j\omega) \quad ; \; i = n+m+2,\; k=n+m+2 \tag{17}$$

Careful inspection of equations (16) and (17) shows that M_1 may be expressed as:

$$M_1 = t_f \sum_{i=1}^{p''} \alpha_i L_i \tag{18}$$

where $L_1, \ldots L_{p''}$ are constant matrices depending only on the coefficients of the polynomials $A(s)$, $B(s)$ and $\alpha_1, \ldots \alpha_{p''}$ are scalars which depend upon the input spectrum and are given by

$$\alpha_i = \int_0^\infty X(-j\omega)X(j\omega)\omega^{2(i-1)}\,d\xi(\omega) \tag{19}$$

Since M_1 is completely specified by the p'' quantities $\alpha_1, \ldots \alpha_{p''}$ and

since information matrices corresponding to J_1-optimal and J_2-optimal designs are boundary points of the set of all information matrices, the result follows from Caratheodory's theorem, [34]. #

9. Optimal Sampling Rate Determination

In the last section it was seen that Fisher's information matrix, M, for single input-single output systems given by equation (8.3), was of the form:

$$M = \left[\begin{array}{c|c} M_1 & 0 \\ \hline 0 & M_2 \end{array}\right] \tag{1}$$

where M_1 is the information matrix for the A and B parameters and depends on the input spectrum and M_2 is the information matrix for the C and D parameters and is independent of the input spectrum.

Suppose that the cost function is a function of M_1 only. For example:

$$J_1 = \log \det (M_1+P_1)^{-1} \tag{2}$$

or

$$J_2 = \text{trace } \{\Gamma(M_1+P_1)^{-1}\} \tag{3}$$

where P_1 is the prior information matrix for the A and B parameters. Suppose further that the measurements of the system output are of the form of N uniformly spaced samples, with sampling rate $\omega_s/2\pi$. The total experiment time t_f is therefore given by:

$$t_f = \frac{2\pi N}{\omega_s} \tag{4}$$

Now it is well known that a signal with no frequency component above ω_h, say, can be sampled without loss of information at any frequency greater than $2\omega_h$ (Nyquist rate, [42]). Furthermore, M_1 is unaffected by any frequency components in the output which are above the highest frequency in the input spectrum. Thus all the information about the system parameters is contained in the sample values of the filtered output y'_t provided:

$$\omega_s > 2\omega_c > 2\omega_h \tag{5}$$

where ω_c is the cut-off frequency of a filter with transfer function H(s) such that:

$$|H(j\omega)| = 1, \quad \omega \leq \omega_c$$

$$= 0, \quad \omega > \omega_c \tag{6}$$

The inclusion of this filter is necessary to prevent aliasing, [42]. Letting $\omega_s \to 2\omega_c \to 2\omega_h$, (4) becomes

$$t_f = \frac{\pi N}{\omega_h} \tag{7}$$

and hence, from (8.18):

$$M_1 = \frac{\pi N}{\omega_h} \overline{M} \tag{8}$$

where

$$\bar{M} = \sum_{i=1}^{p''} \alpha_i L_i \tag{9}$$

The joint problem of optimal sampling rate and input spectrum determination may now be stated as:

$$\min_{\omega_h \in [0,\infty)} \{ \min_{\substack{\xi \in \Xi \\ \Lambda = [0,\omega_h]}} J(\frac{\pi N}{\omega_h} \bar{M}(\xi)) \} \tag{10}$$

Since the optimal $\bar{M}$ is specified by p" lines the optimisation indicated in (10) may be carried out in a 2p"-1 dimensional space.

Remark 1:

Conditions under which it is meaningful to use cost functions which are functions of M_1 only are:

(a) The purpose of the experiment is to determine only the system parameters, the noise parameters being of little or no interest.

(b) The noise spectrum has predominantly low frequency components so that there is little information loss due to filtering and sampling. #

Remark 2:

Constraints on the number of samples are common when a digital computer is used for analysis. #

Remark 3:

The more general problem in which the noise parameters are also required to be estimated could be solved if suitable expressions for the information loss due to filtering and sampling were considered. #

10 Concluding Remarks

In this chapter it has been demonstrated that it is only the spectral properties of the input which affect the identification accuracy. Furthermore, it has been shown that it is sufficient to

consider only line spectra of low dimensionality. Realization procedures which are simple and robust have been described for both power and amplitude constraints. Tests for optimality of inputs have been described. It has also been demonstrated that the results of this thesis may be readily extended to include continuous time systems. For these systems, the additional problem of optimal sampling rate determination also arises. A suitable method for joint optimal design of input and sampling rate has been indicated.

APPENDIX A

Examples

1. Single Time Constant with Coloured Noise

Model:

$$y(s) = \frac{1}{\tau s+1} u(s) + e(s) \tag{1}$$

where e(s) is coloured noise with spectral density:

$$\Phi(\omega) = \frac{1}{a^2\omega^2+1} \tag{2}$$

Sensitivity equation:

$$g_T(s) = - \frac{s(as+1)}{(\tau s+1)^2} u(s) \tag{3}$$

Average information matrix corresponding to a line at ω:

$$\bar{M}(\omega) = \frac{(a^2\omega^2+1)\omega^2}{(\tau^2\omega^2+1)^2} \tag{4}$$

From theorem 8.1 it is not necessary to consider spectra containing more than one line. Therefore let $\bar{M} = \bar{M}(\omega)$.

For the single parameter case J_1-optimality is equivalent to J_2-optimality and both correspond to maximising $\bar{M}$. To find the maximum value of $\bar{M}(\omega)$ and the corresponding ω, differentiate (4) with respect to ω and equate to zero:

$$\frac{\partial \overline{M}}{\partial \omega} = \left[\frac{(4a^2\omega^3+2\omega)(\tau^2\omega^2+1)^2 - 4\tau^2\omega^3(a^2\omega^2+1)(\tau^2\omega^2+1)}{(\tau^2\omega^2+1)^4} \right] = 0$$

i.e.

$$\omega = 0$$

or

$$(a^2\omega^2+1)(\tau^2\omega^2+1) - 2\tau^2\omega^2(a^2\omega^2+1) = 0$$

i.e.

$$(2a^2-\tau^2)\omega^2 + 1 = 0$$

$$\omega^2 = \frac{1}{\tau^2-2a^2}$$

Thus the optimal input frequency is:

$$\omega^* = (\tau^2-2a^2)^{-\frac{1}{2}} \text{ if } 2a^2 < \tau^2 \tag{5}$$

$$\omega^* = \infty \text{ if } 2a^2 \geq \tau^2 \tag{6}$$

Remark:

As $a \to 0$ (i.e the noise becomes "white") the intuitively pleasing result $\omega^* \to \frac{1}{\tau}$, the 3dB frequency, is obtained. In practice, (6) # implies that a high frequency should be used.

2. Simple Gain in Coloured Noise

Model:

$$y(s) = Ku(s) + e(s) \tag{7}$$

where e(s) is coloured with spectral density given by:

$$\Phi(\omega) = H(j\omega)H(-j\omega) = |H(j\omega)|^2 \tag{8}$$

Sensitivity equation:

$$g_K(s) = \frac{1}{H(s)} u(s) \tag{9}$$

Average information matrix corresponding to a line at ω:

$$\bar{M}(\omega) = \frac{1}{|H(j\omega)|^2}$$

Thus $\bar{M}(\omega)$ is maximised for ω^* such that $|H(j\omega^*)|^2$ is a minimum. Thus the optimal frequency is at the point where the noise power density is a minimum. #

3. Simple Two Input System

Model:

$$y(s) = b_1 u^1(s) + b_2 u^2(s) + e(s) \tag{10}$$

where e(s) has spectral density given by (8).

Sensitivity equations:

$$g_1(s) = \frac{1}{H(s)} u^1(s) \tag{11}$$

$$g_2(s) = \frac{1}{H(s)} u^2(s) \tag{12}$$

Average information matrix corresponding to a line at ω:

$$\bar{M}(\omega) = \frac{1}{|H(j\omega)|^2} \begin{bmatrix} \lambda_1 & (\lambda_1\lambda_2)^{\frac{1}{2}}\cos\phi \\ (\lambda_1\lambda_2)^{\frac{1}{2}}\cos\phi & \lambda_2 \end{bmatrix} \tag{13}$$

From theorem 8.1 it is not necessary to consider spectra containing more than two lines. It is possible, however, from theorem 3.1, that an optimal input spectrum with just one line exists. This possibility is now investigated for the case of $J_1 = \log\det \bar{M}$: Now for fixed λ_1, λ_2 and ϕ, $\det \bar{M}(\omega)$ is obviously maximised for $\omega = \omega^*$ where ω^* minimises $|H(j\omega)|^2$. Introduce the test quantity $V_1(\omega^*,\omega)$ defined by:

$$V_1(\omega^*,\omega) = \text{trace}\,\{\bar{M}(\omega^*)^{-1}\bar{M}(\omega)\} \tag{14}$$

Again it is obvious that $\sup_{\omega} V_1(\omega^*,\omega)$ occurs for $\omega = \omega^*$ and when $\omega = \omega^*$, $V_1(\omega^*,\omega) = 2$. Since 2 is the number of parameters, it follows from theorem 4.2 that $\omega = \omega^*$ is the optimal spectrum for all λ_1, λ_2, ϕ. It thus only remains to maximise $\det \bar{M}(\omega^*)$ with respect to λ_1 and ϕ. $(\lambda_2 = 1-\lambda_1)$:

$$\det \bar{M}(\omega^*) = \frac{1}{|H(j\omega^*)|^4} \cdot \lambda_1(1-\lambda_1)(1-\cos^2\phi)$$

which is maximised for $\lambda_1 = \frac{1}{2}$ and $\phi = \pm\frac{\pi}{2}$. Hence the optimal inputs $u^1(s)$ and $u^2(s)$ have the same frequency ω^* which minimises the noise power density, they are of equal power and are 90° out of phase. #

4. Optimal Sampling Rate Determination

Model:

$$y(s) = \frac{1}{\tau s+1} u(s) + e(s) \tag{15}$$

where e(s) has spectral density

$$\Phi(\omega) = \frac{1}{a^2\omega^2+1} \tag{16}$$

This is the same model as for example 1. The optimal input spectrum contains only one line and the corresponding information matrix is $\overline{M}(\omega)$:

$$\overline{M}(\omega) = \frac{(a^2\omega^2+1)\omega^2}{(\tau^2\omega^2+1)^2} \tag{17}$$

For a fixed number of samples, at the Nyquist rate, 2ω, the information is proportional to M where $M = \frac{1}{\omega}\overline{M}(\omega)$. Maximisation of M with respect to ω is carried out as in example 1:

$$\frac{\partial M}{\partial \omega} = \frac{[(\tau^2\omega^2+1)^2[(a^2\omega^2+1)+2a^2\omega^2]-\omega(a^2\omega^2+1)(\tau^2\omega^2+1).4\tau^2\omega]}{(\tau^2\omega^2+1)^4} = 0$$

i.e.

$$(\tau^2\omega^2+1)(3a^2\omega^2+1) - 4\tau^2\omega^2(a^2\omega^2+1) = 0$$

i.e.

$$3a^2\tau^2\omega^4 + \tau^2\omega^2 + 3a^2\omega^2 + 1 - 4a^2\tau^2\omega^4 - 4\tau^2\omega^2 = 0$$

i.e.

$$-a^2\tau^2\omega^4 + 3(a^2-\tau^2)\omega^2 + 1 = 0$$

i.e.

$$\omega^2 = \frac{1}{2a^2\tau^2}\left[3(a^2-\tau^2) + (9(a^2-\tau^2)^2 + 4a^2\tau^2)^{\frac{1}{2}}\right] \qquad (18)$$

For $a \to 0$, $\omega \to \frac{1}{\sqrt{3}\tau}$, i.e. somewhat less than in the first example. This is to be expected since, for lower sampling rates, the experiment time, and hence the input energy, increases.

CHAPTER 6

Suggestions for Further Research

In this thesis the experiment design problem has been tackled in both the time domain and the frequency domain. The time domain approach, although leading to viable design algorithms, offers little insight into the problem. On the other hand, the frequency domain approach leads to a number of elegant mathematical results giving considerable insight. It is felt by the author that further research in this area could be very rewarding. Also, because of the success of the frequency domain techniques in dealing with the experiment design problem, it seems highly likely that further interesting results may be possible for the related problems of identification and control. In both cases this would entail a new look at some of the classical methods.

Another avenue of further research is the problem of optimal sampling rate determination in the case where system and noise modes are not distinct. Also the problems of optimal design of non-uniform sampling systems and the optimal choice of presampling filter are yet to be studied in depth.

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