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COMPUTATIONAL MEMHODS FOR THE ETGENVALUE ANATYSTS OF LARGE STRUCIURES BY COMPONENT SYNTPESIS

## By

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## ABSTRACT

The research is concerned with the normal mode approach to the dynamic analysis of undamped structural systems possessing large, but finite, degrees of freedom, and in particular with the computational application of Kron's eigenvalue method. The work originated from a study of Weinstein's method, which is shown to be equivalent to that of Kron's.

Standard eigenvalue algorithms are reviewed, with emphasis on suitability for large problems, and a unified approach to component synthesis methods is given, establishing those due to Kron and Hurty.

Kron's method is developed in detail and its computational merits discussed. New techniques for economically obtaining approximate low frequency solutions are proposed, and illustrated on beam and plate bending examples.

A computer program to implement the full and approximate Kron methods, with the minimization of core space a high priority, is described. The computational efficiency of these methods is investigated, and their applicability to repetitive structures emphasized. The approximate Kron method is compared with the component mode method, the latter being implemented in a particularly concise way.

It is concluded that where the full eigenspectrum is of interest, the full Kron method offers great economies over other methods, particularly where the number of constraints is very much less than the total problem order, or where repetition is present. The approximate Kron method enables approximate results to be obtained with great savings in computer resources. This approach may be preferable to the component mode method where highly redundant connection boundaries or awkward boundary conditions are present.

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## GLOSSARY

## PRINCIPAL MATRIX SYMBOLS (Often with various suffixes)

Relating to a fully assembled system:
$\left.\begin{array}{ll}\begin{array}{l}\mathbf{X}\end{array} & \text { Displacement vector } \\ \mathbf{X} & \text { Force vector } \\ \mathbf{K} & \text { Stiffness matrix } \\ \mathbf{M} & \text { Mass matrix } \\ \mathbf{C} & \text { Damping matrix } \\ \mathbf{\phi} & \text { Modal matrix (mass normalised eigenvectors) } \\ \mathbf{A} & \text { Diagonal matrix of system eigenvalues } \\ \mathbf{q} & \text { Modal coordinate vector } \\ \mathbf{A} & \text { Dynamical matrix of system } \\ \mathbf{Y} & \text { Displacement vector } \\ \mathbf{Y} & \text { Force vector } \\ \mathbf{\Psi} & \text { Modal matrix }\end{array}\right\} \quad$ in dynamical system coordinates

Relating to a typical component:

| $\mathrm{X}_{0}$ | Displacement vector |
| :---: | :---: |
| $\mathrm{X}_{0}$ | Force vector |
| $\mathbf{K}_{0}$ | Stiffness matrix |
| $M_{0}$ | Mass matrix |
| $\Phi_{0}$ | Modal matrix (mass normalised eigenvectors) |
| $\wedge_{0}$ | Diagonal matrix of eigenvalues |
| $S_{0}$ | Dynamic stiffness matrix |
| $\mathrm{R}_{0}$ | Receptance matrix |
| D | $\left(\Lambda_{0}-\lambda I\right)$ |
| ${ }^{\circ} \mathrm{X}$ | Connection coordinate vector |
| ${ }^{\circ} \Phi_{C}$ | Partition of modal matrix at connection coor |

Relating to a typical component
$\left.\begin{array}{ll}n_{0} & \text { typical component order } \\ b_{0} & \text { component semi-bandwidth } \\ n_{o c} & \text { number of component connection freedoms } \\ n_{n o} & \text { number of component normal modes included } \\ n_{c o} & \text { number of component constraint modes included } \\ n_{g o} & \text { total number of component generalised freedoms } \\ m_{0} & \text { number of simple constraints } \\ m_{g o} & \text { number of generalised constraints }\end{array}\right\}$ in a typical set

Relating to the base and composite systems (principally Kron's method)

| $q$ | number of components |
| :---: | :---: |
| $\mathrm{n}_{\mathrm{pc}}$ | number of primary components |
| $\mathrm{m}_{\mathrm{ps}}$ | number of partial constraint sets |
| $\mathrm{n}_{\mathrm{t}}$ | total number of displacement freedoms 7 |
| $\mathrm{n}_{\mathrm{g}}$ | total number of generalised (modal) freedoms |
| m | total number of simple constraints |
| . ${ }^{\text {g }}$ | total number of generalised constraints |
| $\mathrm{n}_{\mathrm{c}}$ | total number of connection freedoms in base system |
| 1 | number of unique connection freedoms in composite system |
| $\mathrm{d}, \mathrm{d}_{\mathrm{g}}$. | semi-bandwidth of $\mathbf{R}$ matrix |
| $g_{k l}$ | typical tern in G matrix |
| $r_{i j}$ | typical term in $\mathbf{R}$ matrix |
| $\lambda_{\mathrm{k}}$ | typical base system eigenvalue |
| $\lambda_{\min }$ | minimum of $\}$ the set of all highest component normal |
| $\lambda_{\text {max }}$ | maximum of $\int$ mode eigenvalues |
| - $\lambda_{c}$ min | minimum of all constraint mode pseudo-eigenvalues in base system |
| $\bar{\lambda}$ | value above which the constant part approximation is effective |


(A11 other symbols are defined in the text.)

## PRINCIPAL SCALAR SYMBOLS

## General symbols

$\left.\begin{array}{ll}\mathrm{n} & \text { order of full system } \\ \mathrm{b} & \text { semi-bandwidth of full system } \\ \mathrm{r} & \text { number of eigenvalues/eigenvectors required } \\ \mathrm{t} & \text { number of iterations } \\ \mathrm{p} & \text { multiplicity } \\ \boldsymbol{\lambda} & \text { eigenvalue } \\ \omega & \text { circulan frequency } \\ f & - \\ \text { frequency in } H z\end{array}\right\}$ various suffixes,

ABBREVIATIONS
M.D.S. Minimum data set
P.C. Primary component
P.C.S. Partial constraint set
G.C.S. Generalised constraint set
F.K.M. . Full Kron Method
A.K.M. Approximate Knon Method
C.M.M. Component mode method
S.I. Simultaneous iteration
E.J. Extended Jacobi method

## CHAPTER 1

INTRODUCTION

### 1.1. MATRIX METHODS OF STRUCTURAL DYNAMICS

The prediction of the vibrational characteristics of large complex structures is of great importance in a variety of Aeronautical, Civil and Mechanical engineering situations. It may be, for example, that the response of a structure to a known force input is required, or that natural frequencies are required for design purposes.

With the advent of the high speed digital computer, the last decade has seen a tremendous growth in the field of numerical methods applied to solve engineering problems [1]. In the field of structural analysis, the finite element method [2][3] is well established as a means of expressing continuous systems in discretised form, that is in terms of a finite number of unknowns.

Perhaps the greatest strength of the finite element method is the ease with which it may be extended to non-linear problems, general boundary value problems, and to structural dynamics [4][5].

The displacement method, in which the unknowns in the mathematical model are displacements at points within the structure, is the most successful and widely used approach, and is employed throughout this work.

Utilising a finite element displacement idealisation, the general equation of motion of a structure may be written

$$
\begin{equation*}
\mathbf{M} \ddot{\mathbf{x}}+\dot{\mathbf{C}} \dot{\mathbf{x}}+\mathbf{K} \mathbf{x}=\mathbf{X}(\mathrm{t}) \tag{1.1.1}
\end{equation*}
$$

where $\mathbf{X}$ is a vector of $n$ displacements
M, C, K are the mass, damping and stiffness matrices respectively (symmetric $\mathrm{n} \times \mathrm{n}$ matrices)
$X(t)$ is a vector of $n$ time dependent applied forces.
If the inertia distribution uses the same displacement functions as the stiffness distribution, a 'kinematically consistant' mass matrix is formed [6] which ensures that frequencies are upper bounds. Alternatively, a simpler diagonal 'lumped mass' matrix is often formed thus forfeiting the bound property. Methods for forming damping matrices are considered in references [7][8].

Restricting this discussion to deterministic problems there are
two principle approaches to the solution of equ. (1.1.1) in the time domain.
(a) Calculation of eigenvalues (natural frequencies) and eigenvectors (mode shapes), thus defining a modal transformation to uncouple the equations of motion prior to time integration.
(b) Direct integration in time of the coupled equations of motion.

The approach used depends upon the characteristics of the particular problem.

The first approach is only applicable to linear problems. If damping is neglected, or $\mathbf{C}$ is assumed to be proportional to $K$ and/or $M$, the undanped free vibration problem

$$
\begin{equation*}
M \ddot{x}+K x=0 \tag{1.1.2}
\end{equation*}
$$

is converted to an algebraic eigenvalue problem, the solution of which yields the undamped frequencies and mode shapes (Section 1.4). These mode shapes define a transformation to modal coordinates which orthogonalises equation (1.1.2). It is often convenient to introduce a modal damping term based on experimental evidence at this stage. The uncoupled equations of motion may then be integrated in time. The effort involved in this part is small compared to that involved in the eigenreduction.

Where a damping matrix cannot be assumed proportional, a complex eigenvalue problem of order $2 n$ is formed, and the damped eigenvalues and eigenvectors calculated [9].

Direct integration methods must be used where non-linearities are present, for example when $\mathbf{K}, \mathbf{M}$ or $\mathbf{C}$ vary in time [10]. The costly eigensolution step is eliminated, but the step-wise integration of the coupled equations is only advisable for relatively short time response. Inaccuracies are inevitably introduced, and stability is often a problem.

Where possible, the modal approach is in general preferable. The natural frequencies and mode shapes are often required in their own right, and in any case provide valuable insight into the physical behaviour of the structure.

### 1.2. FIGENVALUES OF LARGE STRUCIURES

Large or complex structural idealisations inevitably involve very large numbers of unknowns, e.g., several thousand. The mass and stiffness matrices, while of large order, are generally sparse, and special solution routines have been developed to take advantage of this in static analysis [11][12].

In structural dynamics it is usually the lower frequency spectrum of the structure that is of interest. It is well known that the number of freedoms required for a dynamic representation of a structure is less than for a static one. Coupled with the fact that the 'cost' of an eigensolution varies as $n^{3}$ (as against $n^{2}$ for a static solution) it is often desirable to reduce the problem order prior to eigenvalue analysis. This is usually achieved by static condensation (Guyan reduction) [13], however, the process induces errors in the resulting eigenspectrum which are often difficult to predict. Provided the condensed matrices are reasonably small (for example 150 freedoms) efficient in-core eigenvalue algorithms may be used.

Alternatively, the condensation step may be eliminated, and eigenvalue techniques which work directly on the sparse matrices employed.

A third alternative is to avoid using the assembled mass and stiffness matrices by a piecewise approach where the structure is considered to be formed from components. The synthesis of the normal modes of the composite structure may or may not involve some form of condensation procedure.

A further advantage of the modal technique in structural dynamics is the possibility of reducing the number of freedoms used in the response part by discarding coordinates corresponding to modes that will contribute little. Figure l.l summarises the various approaches to the spectral analysis of large structures.

### 1.3. SCOPE OF RESEARCH

The trend to more and more complex structures coupled with the relatively high 'cost' of eigenvalue analysis has ensured that the search for more efficient computational algorithms for the latter has not diminished.

This thesis investigates the component-wise approach to undamped

MODAL RESPONSE ANAIYSIS OF IARGE STRUCTURES

eigenvalue analysis, and in particular the computational implementation of Kron's eigenvalue procedure [14] in a finite element environment.

To provide necessary background, stendard eigenvalue algorithms are reviewed in Chapter 2, with particular reference to large problem applicability.

The research originated in a study of the variational characteristics of eigenvalues and the classical Weinstein method [15] which are described in Chapter 3.

Component synthesis methods are established in Chapter 4, in particular the methods of Kron and Hurty. The former involves no inherent 'condensation' procedure, while the latter relies upon one.

Kron's method is investigated in detail in Chapter 5 and is shown to be equivalent to Weinstein's method. New 'approximate Kron methods' to yield an approximate low frequency spectrum with good savings in core space and computer time requirements and introduced and illustrated in Chapter 6. Some concepts from Weinstein's method are used.

The design and implementation of computational systems for both the full and approximate Kron methods is described in Chapter 7, while a form of Hurty's method is concisely implemented for comparison purposes (Appendix 2).

Finally the computational efficiency of the Kron methods is assessed with reference to alternative algorithms. The advantages of the Kron methods in handling repetitive structures and awkward boundary conditions is illustrated.

### 1.4. THE UNDAMPFD STRUCTURAL EIGENVALUE PROBIEM

In this section, certain basic results and associated notation is established.

Assuming a harmonic response $\cdot \mathbf{x}(\mathrm{t})=\mathbf{x} \mathrm{e}^{i \omega t}$, where $\mathbf{x}$ is now taken to contain amplitudes and $\omega$ is circular frequency, equation (1.1.2), which may be formed via Lagrange's equations [5], reduces to the well-known algebraic eigenvalue problem:

$$
(\mathbf{K}-\lambda \mathbf{M}) \mathbf{x}=\mathbf{0}, \quad \lambda=\omega^{2}
$$

For non-trivial $\mathbf{x}$, the solution to this equation is given by

$$
\begin{equation*}
|K-\lambda M|=0 \tag{1.4.2}
\end{equation*}
$$

which defines a polynomial of order $n$ in $\lambda$. The $n$ roots of this polynomial are the eigenvalues of the system, equal to the squares of the natural frequencies. Corresponding to each eigenvalue $\lambda_{i}$ there is a non-trivial $X$ vector denoted by $\boldsymbol{\phi}_{i}^{\prime}$. The set $\phi^{\prime}=\left[\phi_{1}^{\prime} \phi_{2}^{\prime} \ldots \phi_{n}^{\prime}\right]$ are the eigenvectors or 'normal modes' of the problem and have the property of orthogonalising equation (1.4.1). Thus defining a transformation to modal coordinates $\mathbf{q}^{\prime}$ as

$$
\begin{equation*}
x=\Phi^{\prime} q^{\prime} \tag{1.4.3}
\end{equation*}
$$

equation (1.4.1) becomes

$$
\begin{equation*}
(k-\lambda m) q^{\prime}=0 \tag{1.4.4}
\end{equation*}
$$

where

$$
\begin{aligned}
& \mathbf{k}=\boldsymbol{\Phi}^{t} \mathbf{K} \boldsymbol{\phi}^{\prime}, \text { diagonal matrix of generalised stiffness, } \\
& \mathbf{m}=\boldsymbol{\phi}^{t} \mathbf{M} \boldsymbol{\phi}^{\prime}, \text { diagonal matrix of generalised mass. }
\end{aligned}
$$

There is an arbitrary multiplier associated with each eigenvector, and it is often convenient to "mass normalise" the vectors by setting

$$
\phi=\phi^{\prime} m^{-1_{2}}
$$

so that

$$
\Phi^{t} \mathbf{M} \Phi=\mathbf{I}
$$

and

$$
\Phi^{t} \boldsymbol{K} \Phi=\Lambda
$$

where $\Lambda$ is the ( $n \times n$ ) diagonal matrix of eigenvalues. With a modal transformation

$$
\begin{equation*}
x=\Phi q \tag{1.4.5}
\end{equation*}
$$

equation (1.4.1) becomes

$$
\begin{equation*}
(\boldsymbol{\Lambda}-\lambda \mathbf{I}) \mathbf{q}=\mathbf{0} \tag{1.4.6}
\end{equation*}
$$

This is known as the 'canonical' or 'normal' form of the eigenvalue problem. If the mass matrix is consistent, its form will be identical to that of the stiffness matrix. Both are invariably real and symmetric. In general, $M$ is positive definite, while $K$ is positive semi-definite .
if the structure is unconstrained to the extent that rigid body modes can occur, such modes corresponding to $\lambda=\omega^{2}=0$.

The eigenvalue problem of equation (1.4.1) is in GENERAL form. Some eigenvalue algorithms require conversion to the SPECIAL eigenvalue problem form

$$
\begin{equation*}
(A-\lambda I)_{y}=0 \tag{1.4.7}
\end{equation*}
$$

To retain A symmetric, use is made of the fact that any positive definite symmetric matrix may be factorised into a lower triangular matrix multiplied by its transpose. Hence using a Choleski decomposition [16]:

$$
\mathbf{K}=\mathbf{L}_{k} \mathbf{L}_{\mathrm{k}}^{\mathrm{t}} \quad \text { or } \quad \mathbf{M}=\mathbf{L}_{m} \mathbf{L}_{m}^{t}
$$

provided the matrix to be decomposed is positive definite. $\mathbf{L}_{\mathrm{k}}$ and $\mathbf{L}_{\mathrm{m}}$ are lower triangular matrices. Utilising the transformations

$$
\begin{equation*}
\mathbf{x}=L_{k}^{-t} \mathbf{y} \tag{1.4.8}
\end{equation*}
$$

or

$$
\begin{equation*}
\mathbf{x}=\mathbf{L}_{\mathrm{m}}^{-t} \mathbf{y} \tag{1.4.9}
\end{equation*}
$$

equation (1.4.1) becomes

$$
(\mathbf{A}-\lambda I) \mathbf{y}=0, \quad \lambda=\frac{1}{\omega^{2}} \quad \text { or } \quad(\mathbf{A}-\lambda \mathbf{I}) \mathbf{y}=0, \lambda=\omega^{2}
$$

where

$$
\begin{align*}
A & =L_{k}^{-1} M L_{k}^{-t}  \tag{1.4.10}\\
\text { or } \quad A & =L_{m}^{-1} K L_{m}^{-t} \quad \text { respectively } \tag{1.4.11}
\end{align*}
$$

The mass and stiffness matrices are usually banded to some extent. The Choleski factorisation preserves banding although zeros within the band are in general destroyed. Banding is not usually carried through to the inverse and thus $\mathbf{A}$ will be fully populated whichever approach is used.

If a lumped mass idealisation is employed, the factorisation and inverse of $\mathbf{M}$ will also be diagonal, so that use of equation (1.4.9) carries banding in $\mathbf{K}$ through to $\mathbf{A}$.

### 1.5. MATRIX STORAGE METHODS

Before embarking upon the survey of eigenvalue algorithms, it is convenient to consider the common techniques of storing matrices 'in-core', as considerable emphasis on the relative core requirements is made.

A symmetric matrix $A$ of order $n$ is considered which may be thought of as a mass or stiffness matrix.

### 1.5.1. Use of Symmetry

The simplest method is to store (say) the lower triangular part of the matrix including the leading diagonal in a one-dimensional array. Termed 'triangular columnwise storage' the requirement is $\frac{n x(n+1)}{2}$ locations, and is thus limited to small n (Fig. 1.2a).

### 1.5.2. Use of Banding

If the structural idealisation is numbered such that the matrices have a reasonably constent bandwidth, $2 \mathrm{~b}+1$ where b is termed the semi-bandwidth, the leading diagonal and upper band may be stored in a rectangular array of dimension $n \times(b+1)$ (Fig. 1.2b).

### 1.5.3. Variable Bandwidth

A more general approach is to store the off-diagonal terms in each row in the lower triangle starting with the first non-zero term encountered. A variable bandwidth may thus be made use of. The off -diagonal terms are stacked in an array (say) A while the number included from each row is recorded in the array IR. Diagonal terms are held separately in the array $A B$. The total requirement is $(2 n+2)$ locations, where there are $\ell$ entries in A (Fig. 1.2c). Zeros within the variable band are of course stored. This is not a disadvantage in that the very common matrix operation of factorisation, for example by the Choleski method, in general disturbs included zeros. This scheme is used extensively in the computer programs developed in this work.

### 1.5.4. Sparse Matrix Storage [12]

For very sparse matrices, it is clearly advantageous to merely store off-diagonal terms (or blocks corresponding to a node) in an array with an integer array of locations. Thus out-of-band and in-band zeros would be omitted.

Often, the actual non-zero terms will be held on a disc file with only the addresses of the non-zero blocks in core. This technique is suitable where the matrix in question is only involved in multiplication or addition processes.

## FIGURE 1.2

## MATRIX STORAGE METHODS

(a) Triangular Columnwise Storage

$$
\left[\begin{array}{lll}
a_{11} & \text { symm. } \\
a_{21} & a_{22} & \\
a_{31} & a_{32} & a_{33}
\end{array}\right] \rightarrow \quad\left\{\begin{array}{lllll}
a_{11} & a_{21} & a_{31} & a_{22} & a_{32}
\end{array} a_{33}\right\}
$$

(b) Constant Bandwidth Storage

$$
\text { ( } 2-D \text { storage array })
$$

(c) Variable Bandwidth Storage

$$
\begin{aligned}
& \text { AB }
\end{aligned}
$$

## A REVIEN OF EIGENVALUE AIGORITHMS

### 2.1. INPRODUCTION

The central objective of this review is the presentation of algorithms for the solution of the undamped structural eigenvalue problem in a unified manner with emphasis on suitability for large problems. The component methods discussed in subsequent chapters in general utilise algorithms described here at some stage.

Large problems may involve from 200 to several thousand unknowns. To permit comparison, general criteria must be borne in mind, for example:
(a) central memory requirement,
(b) central processor time requirements,
(c) peripheral processor time requirements,
(d) suitability for few/many eigenvalues,
(e) accuracy attainable and reliability,
(f) ease of implementation.

Throughout this thesis it is assumed that principal matrix operations (e.g. factorisation) take place fully "in-core". Central memory requirements are thus formulated on this basis.

The choice of algorithm invariably depends upon the characteristics of the problem to be solved. Typical parameters affecting this choice are
(i) problem size and bandwidth (structure topology),
(ii) location and extent of eigenspectrum required,
(iii) are all corresponding eigenvectors required?

Any eigenvalue algorithm is by necessity iterative in that the eigenvalues are zeros of a polynomial function. However, it is convenient to group algorithms under 4 headings according to the mathematical property used:
(1) Transformation methods
(Section 2.2)
(2) Determinant methods (Section 2.3)
(3) Iteration methods (Section 2.4)
(4) Rayleigh quotient methods
(Section 2.5)

The process of static condensation prior to eigensolution is discussed in Section 2.5.

In this Chapter, it is assumed that the assembled mass and stiffness matrices are of order $n$, have semi-bandwidth $b$, and that $r$ éigenvalues/eigenvectors are required.

### 2.2. TRANSFORMAIION METHODS -

2.2.1. General Considerations

The title of transformation methods covers a well-established group of techniques including the methods of Jacobi [17, p.266], Givens [17, p.282], and Householder [17, p.290] and the IR [17, p.487] and $Q R$ [17, p.515] algorithms.

Most techniques require the structural eigenvalue problem to be converted to the special form: (Section 1.4)

$$
\begin{equation*}
(\mathbf{A}-\lambda \mathbf{I}) \mathbf{y}=0, \quad \lambda=\omega^{2} \tag{2.2.1}
\end{equation*}
$$

With a non-diagonal mass matrix, $\mathbf{A}$ is in general fully populated. The basis of all transformation methods is the use of a series of similarity transformations satisfying

$$
\begin{align*}
\mathbf{A}_{r+1}= & \mathbf{R}_{r+1}^{t} \mathbf{A}_{r} \mathbf{R}_{r+1} \\
& \mathbf{R}_{r+1}^{t} \mathbf{R}_{r+I}=\mathbf{I} \tag{2.2.2}
\end{align*}
$$

where $\mathbf{R}_{r+1}$ is the $(r+1)^{\text {th }}$ transformation matrix. These similarity transformations convert the problem to diagonal or tri-diagonal form without affecting the eigenvalues of the problem. In the former event, the eigenvalues are on the leading diagonal of the diagonalised $\mathbf{A}$ matrix, while in the latter the tri-diagonal eigenproblem may be readily solved by the Sturm sequence method (Section 2.3).

The major computational effort is in the transformation stage. Little additional work is involved in obtaining all, rather than just a few, eigenvalues, the technique is thus.particularly favoured when complete eigenreductions are required.

Unfortunately, even if banding in $K$ is carried through to $\mathbf{A}$, subsequent transformation tends to produce a full matrix, hence storage
for the full lower triangle is required involving ${ }_{2} n^{2}$ words. It may also be shown that the number of multiplications involved is proportional to $n^{3}\left({ }_{3} n^{3}\right.$ for Householder). These techniques are thus limited in practice to small problems, e.g., $\leqslant 150$ freedoms. As an illustration, with $n=100$, storage of $\mathbf{A}$ requires approximately 5 K words of central memory, while at $n=200$ this has risen to 20 K .

For large problems it is imperative to employ a condensation step to reduce the number of active freedoms to around 100. However, as computers develop, backing store transfer times will progressively decrease and the application of transformation methods to "out-of-core" matrices may become a more attractive proposition.

### 2.2.2. The Extended Jacobi Method

To avoid the conversion to special form, an extended Jacobi method which simultaneously diagonalises the mass and stiffness matrices has been developed by the author. The method has been used extensively to obtain complete eigenreductions of problems up to 100 freedoms, and is employed in the computer systems developed in this research. A brief description is included in APPENDIX 4.

While the method has proved convenient and reliable, storage for both mass and stiffness lower triangles is required. Transformation methods are often more involved programming-wise and care must be taken to cater for certain numerical ill-conditioning.

### 2.3. DETERMINANT METHODS

### 2.3.1. General Considerations

The basis of determinant methods is the location of the zeros of the determinental equation

$$
\begin{equation*}
|\mathbf{K}-\lambda \mathbf{M}|=0 \tag{2.3.1}
\end{equation*}
$$

or

$$
F(\lambda) \mid=0
$$

The approach is extremely general in that terms in $F(\lambda)$ may contain non-linear functions of $\lambda$. The determinant of $F(\lambda)$ is evaluated for a series of trial $\lambda^{\prime}$ s to permit convergence to the roots of equation (2.3.1).

As described in a subsequent section, the evaluation of a determinant.
of a matrix is best achieved via a triangular factorisation. Thus, banding in $K$ and $M$ is preserved. However, each root location involves several such factorisations. The approach is thus only suitable for large problems where the bandwidth is reasonably small, end only a few eigenvalues/eigenvectors are required.

A distinct advantage of the approach is that a specific range of interest in $\lambda$ may be investigated without the need to locate eigenvalues starting at one end of the spectrum. Applied directly to the fully assembled matrices, accurate results may be obtained over a wide spectrum.

### 2.3.2. The Sturm Sequence Property

The scanning of $|F(\lambda)|$ to locate roots is not a simple task in that no a priori knowledge of the spacing and distribution of roots is in general available, and a zero of multiplicity two (for example) may easily be missed. Fortunately, the Sturm sequence method provides a powerful means of overcoming these difficulties. Originally used by Givens in 1954 for the solution of the tri-diagonal eigenproblem (Section 2.2.1), it is established here in some detail for the special algebraic eigenproblem:

$$
\begin{equation*}
(A-\lambda I) y=0 \tag{2.3.2}
\end{equation*}
$$

The property relies upon the fact that the roots of a principal minor of $\mathbf{A}$ of order $j$ separate the roots of the minor of order ( $j+1$ ). Considering for convenience the case where $j+1=n$, equation (2.3.2) may be partitioned:

$$
\left[\begin{array}{ll}
A_{n-1} & \mathbf{a}  \tag{2.3.3}\\
\mathbf{a}^{t} & a_{n}
\end{array}\right]\left[\begin{array}{l}
\bar{\psi} \\
\psi
\end{array}\right]=\lambda\left[\begin{array}{l}
\bar{\psi} \\
\psi
\end{array}\right]
$$

where $\lambda$ and $\{\bar{\psi} \psi\}$ are an eigenvalue and corresponding eigenvector of equation (2.3.2). Obtaining an expression for $\overline{\boldsymbol{\psi}}$ from the first set, and inserting in the second results in the scalar equation

$$
\begin{equation*}
\left[a^{t}\left(\mathbf{A}_{n-1}-\lambda I\right)^{-1} \mathbf{a}+\lambda-a_{n}\right] \psi=0 \tag{2.3.4}
\end{equation*}
$$

Let the eigenproblem of order ( $n-1$ ) have eigenvalues $\mu_{i}$ and associated
eigenvectors $\mathbf{v}_{\mathbf{i}}$, then we may write (cf. Section 3.3.2)

$$
\begin{equation*}
\left(A_{n-I}-\lambda I\right)^{-1}=\sum_{i=I}^{(n-I)} \frac{v_{i} v_{i}^{t}}{\mu_{i}-\lambda} \tag{2.3.5}
\end{equation*}
$$

hence the L.H.S. of equation (2.3.4) may be written:

$$
f(\lambda)=\left[\Sigma \frac{\left(a^{t} \mathbf{v}_{i}\right)^{2}}{\left(\mu_{i}-\lambda\right)}+\lambda-a_{n}\right]
$$

Thus, $f(\lambda)=0$ defines $\lambda$ as an eigenvalue of equation (2.3.2) while $f(\lambda)=\infty$ when $\lambda=\mu_{i}$. The numerators in the summation term above are all positive, hence $f(\lambda)$ changes from $+\infty$ to $-\infty$ as $\lambda$ goes from $\left(\mu_{i}-\varepsilon\right) \rightarrow\left(\mu_{i}+\varepsilon\right)$, where $\varepsilon$ is a small quantity. In addition, the slope of $f^{\prime}(\lambda)$ is always positive, so it is clear that the zeros of $f(\lambda)$ interlace the poles $\lambda=\mu_{i}$. Further inspection shows that there is a root of $f(\lambda)<\mu_{1}$ and the root $>\mu_{n-1}$.

This interlacing effect extends for each pair of adjacent principal monors of ( $A-\lambda I$ ). If, for a given trial value of $\lambda$, the principal minors are evaluated and set in sequence with $P_{0}=1$, i.e., $-P_{0} P_{1} P_{2} \ldots P_{n-1} P_{n}$, a sign change between adjacent members in the sequence indicates the presence of an additional root below $\lambda$ (cf. Figure 2.1). The total number of sign changes between consecutive members in the complete sequence thus gives the number of roots of equation (2.3.2) exceeded by $\lambda$. The sequence is said to exhibit the STURM SEQUENCE PROPERTY.

The location of eigenvalues of the tri-diagonal matrix $\mathbf{C}$ (Section 2.2.1) resulting from the Givens/Householder methods is readily achieved by the Sturm sequence method in that the principal minors of $|C-\lambda I|$ may be calculated from a simple recurrance formula [18].

### 2.3.3. Sturm Sequence by Matrix Factorisation. Sign Count

The number of sign changes between adjacent principal minors may be evaluated for the matrix ( $\mathbf{A}-\lambda \mathbf{I}$ ) in a particularly simple way. For considering that the triangular decomposition of this matrix is termed
$L(\lambda)$, with leading diagonal terms $l_{i i}$, then the principal minors are given by


$$
P_{0}=1, P_{1}=\ell_{11}, P_{2}=\ell_{11} \ell_{22}, P_{j}=\prod_{i=1}^{j} \ell_{i i}
$$

etc. Hence,

$$
\ell_{11}=\frac{P_{1}}{P_{0}}, \quad \ell_{22}=\frac{P_{2}}{P_{1}}, \ldots \ldots \ldots \ell_{i i}=\frac{P_{i}}{P_{i-1}}
$$

etc., that is, a negative diagonal term on the factored form corresponds to a sign change in the Sturm sequence. The total number of negative diagonal terms on the factorised form of ( $\mathbf{A}-\lambda \mathbf{I}$ ) thus equals the number of eigenvalues exceeded by $\lambda$. If a Choleski decomposition is used instead of the Gaussian elimination implied here, the number of imaginary row/cols should be used in lieu of the number of negative diagonal terms.

Wittrick and Williams [19] extended the use of the Sturm sequence property to the general determinant method, i.e., the solution of $|F(\lambda)|=0$ where $F$ involves continuous functions of $\lambda$, (cf. Section 4.5.2) and gave the name SIGN COUNT to the number of negative leading diagonal terms on the factorised form. This term is used throughout this work, and indicated as follows for the matrix F ( $\lambda$ )

$$
S[F(\lambda)]
$$

### 2.3.4. Sturm Sequence Method for ( $K-\lambda M) X=0$. Gupta's Method

The sturm sequence method is directly applicable to the structural eigenvalue problem, involving the factorisation of ( $\mathbf{K}-\lambda \mathbf{M}$ ) for a succession of trial $\lambda$ values. Evaluation of the sign count at each stage means that it is impossible to miss any roots (as could be easily done with scanning algorithms based on determinant values alone).

As proposed by Peters and Wilkinson [20], simple bisections are carried out until one root is isolated. A modified successive linear interpolation scheme is then employed to provide 'super-linear' convergence on the simple root.

Application to the large structural eigenvalue problem has been proposed by Gupta [21][22]. A feature of the program developed in the latter reference is the use of variant of Gaussian elimination for the factorisation of ( $K-\lambda M$ ). It is carried out in a working array of dimensions ( $b+1$ ) $2 b+1$ ) which 'slides' down the band of ( $K-\lambda M$ ).

The process of inverse iteration is used to obtain the eigenvectors once accurate eigenvalues have been obtained.

The total core requirements of the method assuming that $\mathbf{K}$ and M are held in constant bandwidth form is given by

$$
c \simeq 4 n b+2 b^{2}
$$

Gupta gives the number of multiplications per factorisation as $2 \mathrm{mb}^{2}$, and indicates that on average 12 evaluations per root are required, hence the number of multiplications for $r$ eigenvalues is given by

$$
M \simeq 24 n b^{2} r
$$

The efficiency of the method is clearly highly dependent upon the semi-bandwidth, b. The method is compared with ones developed in this thesis in Chapter 8.

### 2.4. ITERATIVE METHODS

### 2.4.1. The Power Method

The well known power method uses the special eigenproblem form of equation (1.4.10) with $\lambda=1 / \omega^{2}$ such that in the iterative scheme

$$
\begin{equation*}
\mathbf{y}_{\mathrm{r}+1}^{\prime}=\mathbf{A} \mathbf{y}_{\mathrm{r}} \tag{2.4.1}
\end{equation*}
$$

$\mathbf{y}_{r+1}$ converges to the eigenvector corresponding to the largest eigenvalue, [17, p.570]. $\mathbf{y}_{\mathrm{O}}$ is an arbitarry starting vector, while $\mathbf{y}_{\mathrm{r}+\mathrm{l}}^{\prime}$ is normalised w.r.t. its largest element to obtain $\mathbf{y}_{\mathrm{r}+\mathrm{I}}$. The normalisation factor at convergence is equal to the eigenvalue.

Convergence to higher frequency eigenvectors is possible provided the trial vector is orthogonalised after each iteration w.r.t. previously found eigenvectors. However, this process is slow and the method is usually only employed for locating a fundamental mode and frequency. Further disadvantages are the need to form A, i.e. a loss of banding, and the problems associated with a semi-definite stiffness matrix.

### 2.4.2. Inverse Iteration

Inverse iteration is a well established technique for obtaining the eigenvector and accurate eigenvalue $\lambda_{i}$ when an approximate eigenvalue $q$ is available. In the iterative relation

$$
\begin{equation*}
(\mathbf{A}-q \mathbf{I}) \boldsymbol{y}_{\mathrm{r}+\mathrm{l}}^{\prime}=\mathbf{y}_{\mathrm{r}} \tag{2.4.2}
\end{equation*}
$$

provided $q$ is a good estimate, $\mathbf{y}_{\mathrm{r}+\boldsymbol{l}}$ converges to the appropriate eigenvector extremely quickly. The normalisation factor at convergence is here ${ }^{I} /\left(\lambda_{i}-q\right)$, hence $\lambda_{i}$ may also be obtained. The process involves the factorisation of ( $\mathbf{A}-\mathrm{q} \mathbf{I}$ ) and forward and backward substitution steps for each iteration.

Conversion to the special eigenproblem form may be avoided by using the iterative relation

$$
\begin{equation*}
(K-q M) x_{r+1}^{i}=M x_{r} \tag{2.4.3}
\end{equation*}
$$

Banding is thus preserved. The principal drawback of inverse iteration is the slow convergence obtained where eigenvalues are closely grouped [17, p.619].

### 2.4.3. Simultaneous Iteration (S.I.)

Returning to direct iteration; the technique of simultaneous iteration, which is essentially the power method with several trial vectors, has received considerable recent attention, particularly w.r.t. large-scale problems.

Jennings (1967) [23], and with Clint (1970) [24] presented the technique applied to the special eigenproblem, i.e.,

$$
\begin{equation*}
\mathbf{u}_{\mathrm{r}+1}^{\prime}=\mathbf{A} \mathbf{u}_{\mathrm{r}} \tag{2.4.4}
\end{equation*}
$$

where $\quad \mathbf{u}_{0}$ is a set of $t$ initial trial vectors which must be orthogonal. A feature of the method was the incorporation of an interaction analysis, to define an improved set of trial vectors for use, after an orthogonalisation step, as input to the next iteration. The off-diagonal terms in the symmetric ( $t \mathrm{x} t$ ) interaction matrix, defined by

$$
\begin{equation*}
\mathbf{B}=\mathbf{u}_{r}^{t} \mathbf{u}_{r+1}=\mathbf{u}_{r}^{t} \mathbf{A} \mathbf{u}_{r} \tag{2.4.5}
\end{equation*}
$$

give a measure of the coupling between the trial vectors, and a "Iinear prediction" of the first $t$ eigenvectors may be conveniently defined. At convergence, provided the final trial vectors $u$ are orthogonalised by $\mathbf{u}^{\mathrm{t}} \mathbf{u}=\mathbf{I}, \mathbf{B}$ will be the diagonal matrix of the $t$ highest eigenvectors, i.e. defining the $t$ lowest frequencies.

Subsequently, Rutishauser (1969) [25] proposed a more accurate interaction analysis step based on the eigenreduction of the matrix $\mathbf{B}$ (see discussion below). In addition, he showed that at each iteration stage, the component of the $(t+1)^{\text {th }}$ eigenvector is reduced in the $s^{\text {th }}$ trial vector by the factor $\left(\frac{\lambda_{t+1}}{\lambda_{j}}\right)$. Thus if $r$ eigenvectors are required, it is advisable to include 2 or 3 additional trial vectors to ensure satisfactory convergence.

Bronlund (1969) [26] reviewed Jenning's scheme in the context of the large structural eigenvalue problem, and noted that the need to form $\mathbf{A}$ destroyed banding in $\mathbf{K}$.

Recently, the method has been applied in the above context by Jennings and Orr (1971) [27] and Dong, Wolf and Peterson (1972) [28]. Both papers show that the need to form A explicitly could be avoided by carrying out the iterative relation

$$
\begin{equation*}
\dot{u}_{r}^{\prime}=L_{k}^{-1} M L_{k}^{-t} u_{r-1} \tag{2.4.6}
\end{equation*}
$$

in the 3 steps

$$
\begin{aligned}
\text { (i) solve for } \mathbf{v} \mathbf{L}_{\mathrm{k}}^{\mathrm{t}} \mathbf{v} & =\mathbf{u}_{\mathrm{r}-1} \quad \text { (back subst.) } \\
\text { (ii) form } & =\mathbf{M} \mathbf{v} \\
\text { (iii) solve for } \mathbf{u}_{r}^{\prime} \mathbf{L}_{\mathrm{k}} \mathbf{u}_{\mathrm{r}}^{\prime} & =\mathbf{w} \quad \text { (fwd subst.) }
\end{aligned}
$$

$L_{k}$ being the Choleski factorisation of the stiffness matrix.
The latter paper presents the $\mathbf{B}$-eigenreduction form of the interaction analysis by considering that a set of $t$ trial vectors specify a reduced set of $t$ generalised coordinates $z_{r}$ where

$$
\mathbf{x}=\mathbf{u}_{r} \mathbf{z}
$$

The projection of the full problem onto the subspace defined by these coordinates is given by

$$
\begin{equation*}
\lambda K_{r} \mathbf{z}_{r}=M_{r} \mathbf{z}_{r} \tag{2.4.9}
\end{equation*}
$$

where $\mathbf{K}_{r}=\mathbf{u}_{r}^{t} \mathbf{K} \mathbf{u}_{\mathbf{r}}, \quad \mathbf{M}_{\mathbf{r}}=\mathbf{u}_{\mathbf{r}}^{\mathbf{t}} \mathbf{M}_{\mathbf{r}} \mathbf{u}^{\text {. }}$. The above problem is eigenreduced to yield an orthogonal set of $t$ eigenvectors $\boldsymbol{O}_{r}$ within the subspace. These eigenvectors are used to define the optimally improved set of trial vectors

$$
\begin{equation*}
u_{r}=u_{r}^{\prime} \sigma_{r} \tag{2.4.10}
\end{equation*}
$$

in the full set of coordinate freedoms, which are input to the next iterative step. The above 'interaction analysis' is equivalent to the eigenreduction of the $\boldsymbol{B}$ matrix as proposed by Rutishauser. The proceedure is also known as subspace iteration.

### 2.4.4. S.I. with Semi-definite Stiffness Matrix

A problem clearly arises when rigid body freedoms are present in that the stiffness matrix is singular. The original method for overcoming this involved transformation to a set of generalised coordinates in which the rigid body freedoms appear explicitly and may be treated separately. A systematic procedure is given by Craig and Bampton [29]. However, the transformation stage tends to destroy banding and the approach is unsuitable for use in large problems.

A better method involves the addition of $\alpha \mathbf{M} \mathbf{X}$ to both sides of the eigenvalue equation to give [30]:

$$
\begin{equation*}
(\mathbf{K}+\alpha \mathbf{M}) \mathbf{X}=\left(w^{2}+\alpha\right) \mathbf{M} \mathbf{X} \tag{2.4.11}
\end{equation*}
$$

where $\alpha$ is a positive constant. The spectrum of eigenvalues is thus shifted by $\left(+\frac{1}{\alpha}\right)$ and the modified stiffness matrix $(\mathbf{K}+\alpha \mathbf{M})$ is non-singular. Direct convergence to the rigid modes, which have eigenvalues of $\left(+\frac{l}{\alpha}\right)$ is obtained.

### 2.4.5. Computational Aspects of S.I.

Simultaneous iteration involves simple matrix operations, is easy to program, and provides a convenient technique for extracting the first few eigenvalues and eigenvectors of a system.

The approach is suitable for large problems in that banding is retained. It is relatively easy to carry out the steps in equation (2.4.7) with $L_{k}$ and $M$ held in backing store if necessary. However, the overall stiffness matrix must be assembled to permit factorisation. Figenvectors of 'multiple root' eigenvalues are automatically
calculated.
Reference [27] gives core space and number of multiplication estimates as follows

$$
\begin{aligned}
& C=2 n b+3 n t+\frac{1}{2} t^{2} \\
& M=\left(3 n b t+212 n t^{2}+1_{6} t^{3}\right) c+\frac{1}{2} n b^{2}
\end{aligned}
$$

where $c$ is the number of iterations (usually around 6 or 7). The technique is less sensitive to bandwidth than Gupta's Sturm sequence program, but clearly loses advantage if $t$ becomes too large.

The method, as applied to the full-order sparse matrices, is compared with techniques developed in this thesis in Chapter 8. It is indeed utilised in some of these techniques at a 'component' level, and a routine SIMULI is briefly described in APPENDIX 3.

### 2.5. RAYLAETGH QU̇OTIFNT MINIMISATION METHODS

### 2.5.1. Introduction

The classical Rayleigh method for estimating the fundamental natural frequency of a system [5] involves evaluating the Rayleigh quotient

$$
\begin{equation*}
R(\mathbf{x})=\frac{\mathbf{x}^{t} \mathbf{K} \mathbf{x}}{\mathbf{x}^{t} \mathbf{M} \mathbf{x}} \tag{2.5.1}
\end{equation*}
$$

where $X$ is an assumed lowest mode shape. The basis of the approach is that the eigenvalues of $K X=\lambda M X$ are the stationary points of $R(X)$, the corresponding $X$ being equal to the eigenvector ( $c f$. Section 3.2).

Recently, algorithms from non-linear programming have been applied to solve this 'variational' form of the structural eigenproblem.

These algorithms essentially involve the iterative modification
of $\mathbf{X}$ to minimise $R(\mathbf{X})$. The lowest eigenvalue is obtained by unconstrained minimisation, while higher eigenvalues involve the side constraints that $\mathbf{X}$ must remain orthogonal to the j previously found eigenvectors $\boldsymbol{\phi}_{i},(i=1, j) i . e$.

$$
\phi_{i}^{t} M x=0 \quad(i=I, j)
$$

Mosi algorithms involve the gradient of $R$ which is easily computed from

$$
\nabla R(\mathbf{x})=\frac{2[\mathbf{K} \mathbf{X}-R(\mathbf{X}) \mathbf{M} \mathbf{x}]}{\left(\mathbf{X}^{t} \mathbf{M} \mathbf{X}\right)^{2}}
$$

Initially the method of steepest descent, where the local minimum search direction is the negative of the local gradient was employed. However, the approach is prone to poor convergence. Subsequently, the method of conjugate gradients has proved most successful.
2.5.2. The Method of Conjugate Gradients

The initial minimum search direction is set as in the above method, but successive directions are calculated from the local gradient to be orthogonal to previous search directions.

The Fletcher-Reeves method [31] was improved by Bradbury and Fletcher [32] and applied to the structural eigenproblem by Fox and Kapoor (1968) [33]. They utilised Rosen's gradient projection scheme [34] to confine the vector $X$ to the subspace orthogonal to previously found eigenvectors.

A problem in applying the conjugate gradient method is the homogeneous form of the Rayleigh quotient. Fox and Kapoor removed this indeterminacy by normalising a chosen element in $\mathbf{X}$ to unity after each step, thereby reducing the problem to ( $n-1$ ) independent variables. - The gradient projection scheme was complicated by this method.

Geradin (1971) [35] presented an improved algorithm which avoids refering to the length of the $X$ vector while building up the conjugate directions, and incorporates a scaling transformation to speed convergence. A further improved scheme has been reported by Fried (1972) [36].

### 2.5.3. Computational Aspects

The central advantage of gradient minimisation techniques is that not only do they operate directly on the general eigenproblem form, but that the mass and stiffness matrices need never be explicitly assembled. The matrix multiplications in equations (2.5.1) and (2.5.3) may simply use the elemental matrices,together with the appropriate global addresses, which are held on backing store.

The programming, although more involved than the simultaneous iteration method, is still relatively simple. Indeed standard gradient minimisation programs may be directly applicable. The technique is well suited for determining the few lowest frequencies of very large system. However, if higher frequencies are of interest, the eigenvectors of frequencies lower must first be located (as with S.I.).

According to recent experience [37] convergence difficulties are still encountered, the conditioning of the stiffness matrix being an important factor. In addition, the examples from Geradin show that a strong convergence criteria is required to avoid terminating at a 'local' minimum. It would thus appear that there is no simple way of sacrificing accuracy in pursuit of a faster solution.

No comparisons of this technique with other methods have been made by the author.
2.6. CONDENSATION TEĊHNIQUES

### 2.6.1. Introduction

It is well known that the number of freedoms required to describe the inertia distribution of a structure is less than that required for the stiffness distribution. Techniques have thus been developed to reduce the number of freedoms actually carried forward into dynamic analysis in the case of large problems. The elimination of variables inherent in this procedure implies that the 'condensed' problem is only approximate. The success of the process relies on the fact that, provided a good choice of retained freedoms is made, the behaviour of the structure at low frequencies is well approximated.

### 2.6.2. Basis of the Technique

The technique has its roots in Rayleigh's principle [38]
that "A first order error in an assumed mode shape results in a second order error in the natural frequency". Thus Rayleigh's method as mentioned in section 2.5.1. utilises one assumed mode shape, i.e. one freedom to define an approximate system model. The Rayleigh-Ritz method employs several deflection patterns which, in linear combination, hopefully furnish a better approximation. Indeed the finite element method itself is a generalisation of Rayleigh-Ritz using piecewise
assumed functions to represent a continuous media.
The underlying principle is that via a set of generalised coordinates, less in number than that in the full system (infinite for continuous systems), the lowest modes of the structure may be approximately represented.

### 2.6.3. Condensation in Large Problems

The traditional 'lumped-mass' approach to problems indeed involves the selection of a set of coordinates at which the inertia of the structure is considered to. act. However, even when this is employed, a more general technique is desirable for transforming to an equivalent problem with far fewer freedoms.

Of principal interest here is the approach pioneered independently by Irons [39, 40] ("eigenvalue economisers") and Guyan [13] ("Guyan reduction"). The full displacement vector $\mathbf{X}$ of order n is partitioned in $X_{m}$, containing $m$ master freedoms and $X_{s}$ containing ( $n-m$ ) slave freedoms. The master freedoms are to be retained, and must thus be capable of describing the low frequency behaviour of the structure. The partitioned equation of motion is thus

$$
\left[\begin{array}{ll}
\boldsymbol{K}_{\mathrm{mm}} & \boldsymbol{K}_{\mathrm{ms}}  \tag{2.6.1}\\
\boldsymbol{K}_{\mathrm{ms}}^{\mathrm{t}} & \mathbf{K}_{\mathrm{ss}}
\end{array}\right]-\lambda\left[\begin{array}{ll}
\boldsymbol{M}_{\mathrm{mm}} & \mathbf{M}_{\mathrm{ms}} \\
\mathbf{M}_{\mathrm{ms}}^{\mathrm{t}} & \mathbf{M}_{\mathrm{ss}}
\end{array}\right]\left[\begin{array}{l}
\mathbf{x}_{\mathrm{m}} \\
\mathbf{x}_{\mathrm{s}}
\end{array}\right]=\left[\begin{array}{l}
\mathbf{x}_{\mathrm{m}} \\
\mathbf{X}_{\mathrm{s}}
\end{array}\right]
$$

The slave variables must, in some way, be made dependent on the master variables. Irons and Guyan assumed that the slave displacements follow the static deformation patterns defined by the master displacements. Hence setting $\lambda=0$ in equation (2.6.1) and assuming that no external forces act at the slave freedoms, the second set yields:

$$
\begin{equation*}
\mathbf{x}_{\mathrm{s}}=-K_{\mathrm{ss}}^{-1} K_{m s}^{t} \mathbf{x}_{\mathrm{m}} \tag{2.6.2}
\end{equation*}
$$

The process is thus often known as STATIC.CONDENSATION. The transformation

$$
\left[\begin{array}{l}
x_{m}  \tag{2.6.3}\\
x_{s}
\end{array}\right]=\left[\begin{array}{cc}
\mathbf{I} & \\
-\mathbf{K}_{s s}^{-1} & K_{m s}^{t}
\end{array}\right] \mathbf{x}_{m}=\dot{T} x_{m}
$$

is thus defined to condense equation (2.6.1) to the form

$$
\begin{equation*}
\left(K_{m}-\lambda M_{m}\right) x_{m}=X_{m} \tag{2.6.4}
\end{equation*}
$$

where

$$
\begin{align*}
\mathbf{K}_{m}=\mathbf{T}^{t} \mathbf{K} \mathbf{T}= & \mathbf{K}_{\mathrm{mm}}-\mathbf{K}_{\mathrm{ms}} \mathbf{K}_{\mathrm{ss}}^{-1} \mathbf{K}_{\mathrm{ms}}^{\mathrm{t}} \\
\mathbf{M}_{\mathrm{m}}=\mathbf{T}^{\mathrm{t}} \mathbf{M} \mathbf{T}= & \mathbf{M}_{\mathrm{mm}}-\left(\mathbf{K}_{\mathrm{ms}} \mathbf{K}_{\mathrm{ss}}^{-1} \mathbf{M}_{\mathrm{ms}}^{\mathrm{t}}\right)^{\mathrm{t}} \\
& -\left(\mathbf{K}_{\mathrm{ms}} \mathbf{K}_{\mathrm{ss}}^{-1} \mathbf{M}_{\mathrm{ms}}^{\mathrm{t}}\right)+\left(\mathbf{K}_{\mathrm{ms}} \mathbf{K}_{\mathrm{ss}}^{-1} \mathbf{M}_{\mathrm{ss}} \mathbf{K}_{\mathrm{ss}}^{-1} \mathbf{K}_{\mathrm{ms}}^{\mathrm{t}}\right) \tag{2.6.6}
\end{align*}
$$

The title "mass condensation" was given to a formulation based on identical principles by Ramsden and Stoker [41]. Engineering intuition is used to pick out the master freedoms, for example at areas of high mass and reasonable flexibility. The effectiveness of the technique applied to plate structures has been demonstrated by Zienkiewicz et al [42].

### 2.6.4. The Interior Problem

A rigorous exposition of the assumptions involved in the above condensation process has been given by Wright and Miles [43] and Geradin [44]. The exact form of the master-slave relation from the second set of equation (2.6.1) is

$$
x_{s}=\left(I-\lambda K_{s s}^{-1} M_{s s}\right)^{-1} K_{s s}^{-1}\left(\lambda M_{m s}^{t}-K_{m s}\right) x_{m}
$$

which may be expanded to give

$$
\begin{equation*}
\mathbf{x}_{s}=\left[\mathbf{I}+\lambda \mathbf{K}_{s s}^{-1} \mathbf{M}_{\mathrm{SS}}^{\prime}+\lambda^{2}\left(\mathbf{K}_{\mathrm{ss}}^{-1} \mathbf{M}_{\mathrm{ss}}\right)+\ldots\right] \mathbf{K}_{\mathrm{ss}}^{-1}\left(\lambda \mathbf{M}_{\mathrm{ms}}^{\mathrm{t}}-\mathbf{K}_{\mathrm{ms}}\right) \mathbf{x}_{\mathrm{m}} \tag{2.6.8}
\end{equation*}
$$

If equation (2.6.8) is limited to first order in $\lambda$, substitution into the first set of equation (2.6.1) yields equation (2.6.4). For the above expansion to be valid, it is shown that if $\mu_{i}$ are the eigenvalues of the INTERIOR PROBIEM i.e.

$$
\begin{equation*}
\cdot\left(K_{S S}-\mu M_{S S}\right) X_{S}=0 \tag{2.6.9}
\end{equation*}
$$

(which is merely the fuil problem with master coordinates suppressed) we require

$$
\begin{equation*}
\frac{\lambda}{\mu_{1}}<1 \tag{2.6.10}
\end{equation*}
$$

The optimum choice of master coordinates is that which leads to the maximum fundamental eigenvalue of the interior problem.

Wright and Miles investigated the use of the second order approximation in equation (2.6.8). The resulting second order eigenproblem was expressed as a first order one of size 2 n . This scheme was found inferior to the first order approximation plus an improvement step involving the full order mass and stiffness matrices.

### 2.6.5. Computational Aspects

The above condensation procedure results in an approximate structural model with a shifted eigenbasis. By the theory of Section 3.2, the approximate eigenvalues are upper bounds on those of the full-order problem, but it is not possible to establish accuracy directly.

Wright and Miles, and Geradin considered the use of bound algorithms as to define limits of accuracy, however, the methods used involve the
first iterate of the approximate eigenvector using the full-order matrices. Thus the full-order stiffness matrix must be held in assembled form to permit factorisation. If this is to be carried out in practice one might well be advised to use a method which acts directly on the full-order matrices in the first place.

The condensed mass and stiffness matrices obtained in this technique are in general fully populated. To avoid large core requirements, the master freedoms should be kept small. Typically a reduction will be from 2000 freedoms to 150. Transformation methods are then highly suitable, although only the few lowest frequencies obtained are likely to be accurate.

It is finally important to note that while the condensation of the stiffness matrix is straightforward, that of the mass matrix is considerably more involved and can lead to large computing times. Thus if only a few frequencies are required, it would seem better to use a
direct full-order problem method.

### 2.6.6. Frequency. Dependent Condensation

This technique has recently been used [45] for the location of higher natural frequencies, for example where a known high frequency forcing function is present.

Essentially the procedure is to insert various trial $\lambda^{\prime}$ s in equation (2.6.7) to define a reducing transformation for each value. These trial $\lambda^{\prime}$ s are chosen around the range of interest. Clearly, if an eigenvalue of the condensed problem is exactly equal to the trial $\lambda$ assumed, the condensation procedure is 'exact' and $\lambda$ is an eigenvalue of the full-order problem.

It is felt that this approach offers no advantage over the direct Sturm sequence scanning method for large problems.

## CHAPTER 3

## WEINSTEETN'S METHOD

### 3.1. INTRODUCTION

The initial impetus for this work came from a study of Weinstein's method - a variational method for the solution of classical partial differential equation eigenvalue problems [15]. Certain features appeared to be of value in the context of large finite eigenproblems, in particular the application of Rayleigh's constraint theorem.

In this chapter, the basic variational properties of eigenvalues are summarised together with Rayleigh's theorem. Weinstein's method is recast in matrix terminology and its implications for large, but finite, eigenproblems discussed. Much of Weinsteins, and associated methods, are couched in ratherobscure mathematical terms. The emphasis here is not on rigorous proof, but to give an insight into the underlying principles.

### 3.2. THE VARIATIONAL CHARACTERISATION OF EIGENVALUES

### 3.2.1. The Recursive Characterisation of Eigenvalues

The special form of the linear eigenproblem (equation 1.4.10) is given by

$$
\begin{equation*}
(\mathbf{A}-\lambda \mathbf{I}) \mathbf{y}=0, \quad \lambda=\omega^{2} \tag{3.2.1}
\end{equation*}
$$

where $\mathbf{y}$ comprises a finite set of $n$ coordinates and $\mathbf{A}$ is symmetric, positive semi-definite. With eigenvectors $\boldsymbol{\psi}$ satisfying $\boldsymbol{\psi}^{t} \boldsymbol{\Psi}=I$, the transformation to normal coordinates

$$
\begin{equation*}
y=\Psi q \tag{3.2.2}
\end{equation*}
$$

applied to equation (3.2.1) produces the normal form:

$$
\begin{equation*}
(\Lambda-\lambda I) \mathbf{q}=0 \tag{3.2.3}
\end{equation*}
$$

where $\Lambda$ is the diagonal matrix of eigenvalues. The system potential and kinetic energies in $y$ coordinates are given respectively by the
quadratic forms:

$$
\begin{aligned}
& U=y^{t} \mathbf{A} \mathbf{y} \\
& T=y^{t} \mathbf{y}
\end{aligned}
$$

If the magnitude of the vector $y$ is fixed by the condition $y^{t} y=1$; then equation (3.2.1) may be written as moltyphed by $x^{+}$to give

$$
\begin{equation*}
\mathbf{y}^{\mathrm{t}} \mathbf{A} \mathbf{y}=\lambda=\sum_{i, j=1}^{n} a_{i, j} y_{i} y_{j} \tag{3.2.4}
\end{equation*}
$$

This defines an ellipsoid, in general skew w.r.t. the axes defined by the elements in $\mathbf{y}$. The normal form of the energies are

$$
U=\sum_{i=1}^{n} \lambda_{i} q_{i}^{2}, \quad T=\sum_{i=1}^{n} q_{i}^{2}
$$


hence defining $T=1$ as above gives the ellipse equation in the form

$$
\begin{equation*}
q^{t} \Lambda \dot{q}=\lambda=\sum_{i=1}^{n} \lambda_{i} q_{i}^{2} \tag{3.2.5}
\end{equation*}
$$

The normal coordinates $q_{i}$ are thus seen as defining the principal axes of the ellipsoid [47, p.81], and the eigenvalues are given by the values of $U$ when $y$ (or $q$ ) defines a principal axis, subject to $T=1$. Hence the eigenvalues of equation (3.2.1) are defined by the STATIONARY POINTS of $\mathrm{U}(\mathbf{y})$; a VARIATIONAL CHARACTERISATION.

Eigenvalues are normally thought of as stationary points of the RAYLETGH QUOTIFNT defined by

$$
\begin{equation*}
R(\mathbf{y})=\frac{\mathbf{y}^{t} \mathbf{A} \mathbf{y}}{\mathbf{y}^{t} \mathbf{y}} \tag{3.2.6}
\end{equation*}
$$

The form of $R$ is homogeneous in $\mathbf{y}$, hence specification of unique stationary values requires a condition such as $y^{t} y=1$.

The minimum of $R$ subject to this condition is given by $\mathbf{y}=\boldsymbol{\psi}_{I}$ and $R\left(\Psi_{1}\right)=\lambda_{1}$. For $y$ restricted to the subspace orthogonal to $\boldsymbol{\psi}_{1}$, the new minimum value of $R$ is obtained by $\boldsymbol{y}=\boldsymbol{\psi}_{2}$, ie. the second shortest principal axis.

The RECURSIVE CHARACTERISATION of eigenvalues thus states that "the ith. eigenvalue $\lambda_{i}$ and associated eigenvector $\psi_{i}$ of equation (3.2.1) are the minimum value and minimising vector of the Rayleigh quotient for all vectors orthogonal to the first (i - I) eigenvectors".

### 3.2.2. Rayleigh's Theorem

Originally presented in Rayleigh's "Theory of Sound" [38], Rayleigh's theorem concerns the effect of the imposition or removal of a constraint on a vibrating system:
"If one constraint is imposed upon a linearly elastic structure whose eigenvalues $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}$ are in ascending order, the eigenvalues of the constrained
structure $\lambda_{i}^{1}$ satisfy $\quad$.

$$
\begin{equation*}
\lambda_{i} \leqslant \lambda_{i}^{I} \leqslant \lambda_{i+1} \tag{3.2.7}
\end{equation*}
$$

For removal of one constraint, the relation is

$$
\begin{equation*}
\lambda_{i-1} \leqslant \lambda_{i}^{I} \leqslant \lambda_{i} \tag{3.2.8}
\end{equation*}
$$

The theorem may be illustrated directly for the fundamental eigenvalue of a system. Any constraint $\mathbf{P}$, where the constraint condition is that $p^{t} y=0$ is expressible as a linear combination of the eigenvectors. If the constraint is $\Psi_{i}^{t} \mathbf{y}=0$, $i \neq 1$, then
$\mathbf{y}=\boldsymbol{\psi}_{1}$ is permissible and the overall minimum of $R\left(\boldsymbol{\psi}_{1}\right)=\lambda_{1}$ is obtainable. However, if $\Psi_{l}^{t} \mathbf{y}=\dot{0}$, then by the recursive characterisation of eigenvalues, the new minimum of $R$ is $\lambda_{2}$. These represent the two limiting cases, hence any other constraint will produce a minimum in the range $\left[\lambda_{1} \lambda_{2}\right]$.

### 3.2.3. The Max-Min Characterisation of Eigenvalues

The previous section's discussion suggests that $\lambda_{2}$ may be defined as the maximum of all the possible minimum values of $R$ obtainable by application of one constraint. Often termed 'Courant's principle' $[46, p .31]$ this characterisation avoids explicitly utilising the lower eigenvectors:
"The $r$ th. eigenvalue $\lambda_{r}$ of a vibrating system is the maximum value that can be given to the minimum of the Rayleigh quotient by varying ( $r-1$ ) applied constraints".

Denoting ( $r-I$ ) constraints by $s_{i}^{t} \mathbf{y}=0,(i=1, r-I):-$

$$
\begin{equation*}
\lambda_{r}=\max \left(\mathbf{s}_{I} \ldots \ldots \mathbf{s}_{r-1}\right) \min R(y) \tag{3.2.9}
\end{equation*}
$$

This characterisation leads to the extension of Rayleigh's theorem. Consider a set of $m$ independent constraints $p_{j}$ applied to the system of equations(3.2.1). The rth. constrained system eigenvalue is thus given by

$$
\lambda_{r}^{m}=\max \left(\mathbf{s}_{1} \ldots \ldots \mathbf{s}_{r-1}, p_{I} \ldots \ldots \mathbf{p}_{m}\right) \min R(\mathbf{y})
$$

Clearly $\lambda_{r}^{m} \geqslant \lambda_{r}$ as additional constraints have been applied. However, if the $m$ constraints $p_{j}$ were free to vary, the maximum value $\lambda_{r}^{m}$ could achieve would be $\lambda_{r+m}$. Rayleigh's theorem for a general number of constraints is given as follows:
"If s arbitrary constraints are imposed upon a vibrating system with eigenvalues $\lambda_{1} \lambda_{2} \ldots . \lambda_{n}$ in ascending order, then the eigenvalues of the constrained system $\lambda_{r}^{m}$ must satisfy

$$
\begin{equation*}
\lambda_{r} \leqslant \lambda_{r}^{m} \leqslant \lambda_{r+s} \tag{3.2.10}
\end{equation*}
$$

### 3.2.4. The Rayleigh Ritz Method

The Rayleigh Ritz method (Section 2.6.2) is essentially a procedure for approximating continuously varying quantities via a finite number of assumed variation functions. In the present context, the reduction from an infinite, to a finite, number of unknown is equivalent to constraining the quantity in question. Hence in the case of a vibrating system where displacement is approximated by a set of assumed functions, the resulting eigenvalues are upper bounds according to Rayleigh's principle w.r.t. the exact eigenvalues. Increasing the number of freedoms produces monetonic convergence from above towards the exact value.

### 3.3. WEINSTETN'S METHOD

### 3.3.1. Introduction [15]

In contrast to the Rayleigh - Ritz method, Weinstein's method was introduced in 1935 to produce lower bounds on the true eigenvalues of
infinite (continuous) systems.
The technique involves weakening the boundary conditions on the actual problem until a soluble BASE PROBLEM is obtained; usually equivalent to the removal of an infinite series of constraints from the system. A finite series of constraints is then applied to the base problem to approximately reconstitute the original boundary conditions. A series of INTERMEDIATE PROBIEMS is thus defined. By Rayleigh's theorem, the eigenvalues of the intermediate problems converge from below the exact solutions.

The eigenvalues of the $r^{\text {th }}$. intermediate problem are defined by the Weinstein determinant of order $r$ which is constructed from a knowledge of the base system eigenvalues and eigenvectors together with the constraints.

The principle results are set out here in matrix terms for the case of a finite degree of freedom problem [15, Ch. 3].

### 3.3.2. The Resolvant Matrix

Considering equation (3.2.1) to specify a base problem with known solution, the $i^{\text {th. }}$ eigenvalue is given by

$$
\begin{equation*}
A \Psi_{i}=\lambda_{i} \Psi_{i} \tag{3.3.1}
\end{equation*}
$$

thus

$$
\begin{equation*}
(A-\lambda I) \psi_{i}=\left(\lambda_{i}-\lambda\right) \psi_{i} \tag{3.3.2}
\end{equation*}
$$

and defining the RESOLVANT MATRIX by

$$
H=(A-\lambda I)^{-1}
$$

we may write

$$
\begin{equation*}
H \psi_{i}=\frac{1}{\left(\lambda_{i}-\lambda\right)} \psi_{i} \tag{3.3.4}
\end{equation*}
$$

The eigenvectors of $H$ are $\boldsymbol{\psi}_{i}$, and the eigenvalues $\frac{1}{\lambda_{i}-\lambda}$. Clearly, $H$ is not defined for $\lambda=\lambda_{i}, i=1, n$.

A general vector $\mathbf{V}$ is expressible as a linear combination of the normalised eigenvectors:

$$
\begin{equation*}
v=\sum_{i=I}^{n}\left(v^{t} \psi_{i}\right) \psi_{i} \tag{3.3.5}
\end{equation*}
$$

hence

$$
\begin{equation*}
H v=\sum_{i=I}^{n} \frac{\left(v^{t} \Psi_{i}\right)}{\left(\lambda_{i}-\lambda\right)} \Psi_{i} \tag{3.3.6}
\end{equation*}
$$

In classical mechanics terms, if $\mathbf{v}$ is a general harmonic force amplitude vector at frequency $\sqrt{\lambda}, \mathbf{H}$ is the receptance matrix of the base problem, and the operation $\mathbf{H v}$ "resolves" the resulting displacements into normal coordinate directions. If $\mathbf{v}^{t} \boldsymbol{\psi}_{i}=0$ for a particular i, there is no generalised force in that mode, and the term is excluded from the series. Clearly, if $\lambda=\lambda_{i}$ and $\mathbf{v}^{t} \boldsymbol{\psi}_{i} \neq 0$, the condition of resonance occurs.

### 3.3.3. Weinstein Determinant for 1 Constraint

We now consider I displacement constraint applied to the base problem:

$$
\begin{equation*}
\mathbf{p}^{t} \mathbf{y}=0 \tag{3.3.7}
\end{equation*}
$$

This equation may be interpreted as a condition of zero work for a vector of harmonic forces $\mathbf{p}$ over the base system coordinates, hence the condition for free vibration of the constrained system may be written as

$$
\begin{equation*}
(H p)^{t} p=0 \tag{3.3.8}
\end{equation*}
$$

where $\mathbf{H p}$ defines the displacements caused by the constraint forces $\mathbf{p}$ The Weinstein determinant of order 1 is defined by

$$
\begin{equation*}
W_{1}(\lambda)=(H \mathbf{p})^{t} \mathbf{p}=\sum_{i=1}^{n} \frac{\left(\mathbf{p}^{t} \boldsymbol{\psi}_{i}\right)^{2}}{\left(\lambda_{i}-\lambda\right)} \tag{3.3.9}
\end{equation*}
$$

hence for free vibrations of the constrained system

$$
W_{1}(\lambda)=0
$$

FIGURE 3.1.

## TYPICAL PLOT OF $W_{1}(\lambda)$

CONSTPAINED SYSTEM EIGENVALUES

| COMPLETE | $\therefore$ RAISING |
| :---: | :---: |
| $\downarrow$ | $\downarrow$ |



A typical plot of $W_{I}(\lambda)$ is shown in Figure 3.1. The slope of the plot is positive always as indicated by

$$
\begin{equation*}
\frac{d W_{I}(\lambda)}{d \lambda}=\sum_{i=1}^{n} \frac{\left(p^{t} \Psi_{i}\right)^{2}}{\left(\lambda_{i}-\lambda\right)^{2}} \tag{3.3.10}
\end{equation*}
$$

The main properties of the Weinstein determinant are as follows:
(1) The zeros of $W_{1}(\lambda)$ are the eigenvalues of the constrained system which are not also eigenvalues of the base system.
(2) A pole at $W_{I}\left(\lambda_{i}\right)$ indicates the loss of a base system eigenvalue $\lambda_{i}$ (There will always be a computational pole at $\lambda=\lambda_{i}$ ).
A finite non-zero, limiting value, at $W_{1}\left(\lambda_{i}\right)$ indicates no change in base system multiplicity of $\lambda_{i}$.
A zero limiting value at $W_{l}\left(\lambda_{i}\right)$ indicates the increase by 1 of the multiplicity at $\lambda_{i}$.
(3) The interspacing of the constrained system eigenvalues $\lambda_{i}^{l}$ between those of the base system $\lambda_{i}$, is in accordance with Rayleigh's theorem. It may be verified that $\mathbf{p}_{I}=\boldsymbol{\Psi}_{I}$ ensures "complete raising of the lowest base system eigenvalue", as demanded by the recursive characterisation of eigenvalues (Section 3.2.2).
(4) The Weinstein criterion for "complete raising", i.e. that $\lambda_{i}^{l}=\lambda_{i+1}$ is that $W_{1}\left(\lambda_{i+1}-6\right)<0$ where $\varepsilon$ is a small perturbation.

### 3.3.4. Weinstein Determinant for Several Constraints

For a series of $m$ constraints attached to the base problem, it may be shown that for free vibration of the constrained system $W_{m}(\lambda)=0$, where $W_{m}(\lambda)$ is the Weinstein determinant for $m$ constraints defined by

$$
w_{m}(\lambda)=\left|\begin{array}{ccc}
\left(H p_{1}\right)^{t} p_{1} & & \\
\left(H p_{2}\right)^{t} p_{1} & \left(H p_{2}\right)^{t} \dot{p}_{2} & \text { SYMEERIC } \\
\vdots & & \ddots
\end{array}\right|
$$

The general term in the $m \times m$ determinental matrix is given by

$$
\begin{equation*}
W_{i j}^{m}=\sum_{k=1}^{n} \frac{\left(\mathbf{p}_{i}^{t} \boldsymbol{\Psi}_{k}\right)\left(\mathbf{p}_{j}^{t} \boldsymbol{\Psi}_{k}\right)}{\left(\lambda_{k}-\lambda\right)} \tag{3.3.12}
\end{equation*}
$$

The constrained system eigenvalues are obtained from the following rules; where the general base system eigenvalue $\lambda_{i}$ has multiplicity $p_{i}$. $W_{m}(\lambda)=0$ (order $r$ ), $\lambda=\lambda_{i}$ locates constrained system eigenvalues $W_{m}\left(\lambda_{i}\right)=$ POLE (order $r$ ) locates eigenvalue of multiplicity $\left(p_{i}-r\right)$. $W_{m}\left(\lambda_{i}\right)=$ FINITE, NON-ZERO ". $\quad$ " " $\quad$ " $W_{m}\left(\lambda_{i}\right)=$ ZERO (order $r$ ) " " " " $\quad$ ( $\left.p_{i}+r\right)$.

The Weinstein criteria for complete raising of an eigenvalue by $m$ constraints, i.e. that $\lambda_{i}^{m}=\lambda_{i+m}$ is that the series of WEINSTEIN NUMBERS (the principal minors of $W_{m}(\lambda)$ with $W_{0}=1$ )

$$
w_{0}, w_{1}, w_{2}, \ldots \ldots, w_{m}
$$

evaluated at ( $\lambda_{i+m}-6$ ) must contain $k$ sign changes where $k$ is the first number satisfying $\lambda_{i+k}=\lambda_{i+m}$. (cf. extended Sturm sequence algorithm of Section 4.5.2). It may be readily verified that
$\mathbf{p}_{1}=\boldsymbol{\psi}_{1}, \ldots \ldots, \mathbf{p}_{\mathrm{m}}=\boldsymbol{\psi}_{\mathrm{m}}$ is one set of constraints which satisfies the criteria.
onlyset

### 3.4. SIGNIFICANCE OF WEINSTETN'S METHOD

Weinstein's method offers a means of determining the characteristics of one system by analysing another and applying constraints to the latter to reconstitute the former. In the context of the large eigenvalue problem the 'computationally convenient' base problem may be obtained by uncoupling the full problem into components, that is a number of smaller problems. Constraints may then be defined which recouple the components, and the properties of the fully assembled system determined via the Weinstein determinant of order $m$, instead of the fully assembled structural matrices. It is,however, unclear how the constrained system'eigenvectors are obtained.

It will be seen in the following chapter that this technique of uncoupling of a structure into components is the basis of Kron's eigenvalue procedure. The determination of eigenvectors is covered in this method.

In Weinstein's method for infinite systems [15, Ch. 7] it is significant that a finite number of constraints is employed in lieu of an infinite number to obtain lower bounds on the lower exact eigenvalues. The general concept of choosing constraints so as to produce the maximum raising of eigenvalues has strong computational advantages, especially where the lower eigenspectrum is of interest, as is generally the case in structural dynamics. This aspect is investigated in Section 6.4. as a means of introducing approximations into the Kron procedure.

## CHAPTER 4

## COMPONENT SYNTHESIS METHODS

### 4.1. INIRODUCTION

In this chapter, the general class of methods involving the eigenvalue analysis of structures via components is developed in a unified manner. In particular, Kron's method and Hurty's component mode method are established.

The underlying principle is the avoidance of the need to use the assembled mass and stiffness matrices of the structure. In addition, repetition within the structure may be utilised to reduce computer effort enabling larger problems to be solved.

The structure is considered to be built up from a number of component parts, also termed substructures, branches or sub-systems. Analysis of the components is initially performed, and the results used to synthesise the properties of the composite system.

Although this report is concerned with finite freedom problems, mention is made of infinite freedom problems both for completeness and in connection with the solution of non-algebraic eigenproblems.

Two basic approaches are identified, the component connection technique, which leads to $\mathrm{Kr}_{\text {ron's }}$ method, and the component release technique. A group of approximate methods is then introduced, which contains Hurty's method.

### 4.2. THE COMPONENT CONNECTION TTECHNIQUE

### 4.2.1. Introduction

The basic philosophy of this technique, as propounded by Kron [14.], is the 'tearing' of the composite system into several completely unconnected components. This constitutes the 'base' system. The properties of each component are expressed at its connection coordinates, and conditions of compatability and equilibrium invoked along the connection boundaries between components to obtain a frequency equation for the composite system. The order of this equation is usually substantially smaller than the fully assembled problem order.

### 4.2.2. The Dynamic Stiffness and Receptance Approaches

Discretisation by a displacement method is assumed. The undamped
equation of motion for a typical component of order $n_{0}$, which could, for example; be an assembly of finite elements, may be written as

$$
\begin{equation*}
\left(K_{0}-\lambda M_{0}\right) X_{0}=\mathbf{X}_{0}, \quad \lambda=\omega^{2} \tag{4.2.1}
\end{equation*}
$$

Let complete set of normalised eigenvectors be $\boldsymbol{\Phi}_{0}$, and associated diagonal matrix of eigenvalues be $\boldsymbol{\Lambda}_{0}$. The displacement vector may be partitioned according to:

$$
x_{o}=\left\{\begin{array}{lll}
{ }^{\circ} & & \\
x_{i} & { }_{x} \\
& & { }_{c}
\end{array}\right\}
$$

where ${ }^{\circ} X_{c}$ contains the set of $n_{o_{c}}$ connection coordinates. and ${ }^{\circ} X_{i}$ contains the set of $n_{o_{i}}\left(=n_{o}-n_{o_{c}}\right)$ internal coordinates.

The problem of expressing the properties of each component in terms of its connection coordinates may be approached in two ways:
(1) By the use of $S_{0}=\left(K_{0}-\lambda M_{0}\right)$, the component dynamic stiffness matrix.
(2) By the use of $\mathbf{R}_{0}=\mathbf{S}_{0}^{-1}$, the component receptance matrix.

This choice determines the conditions required to couple the components together, and the nature of the frequency determinant obtained. The following sections give concise parallel developments of the dual approaches to highlight the essential differences. It must be noted that all $\mathbf{S}, \mathbf{R}$ and $\mathbf{D}$ type matrices are functions of $\lambda$. Kron's method utilises the receptance approach for reasons that become clear.
4.2.3. The Base System Equations of Motion

|  | DYNAMIC STIFFNESS | RECEPTANCE |
| :---: | :---: | :---: |
| Component e. of.m. | $S_{0} \mathbf{x}_{0}=X_{0} \quad(4.2 .2)$ | $\mathbf{R}_{0} \mathbf{X}_{0} \doteq \mathrm{X}_{0}$ (4.2.3) |
| Normal forms: $D_{0}=\left(\Lambda_{0}-\lambda \mathbf{I}\right)$ | $S_{0}=\Phi_{0}^{-t} D_{0} \Phi_{0}^{-1}$ | $R_{0}=\Phi_{0} D_{0}^{-1} \Phi_{0}^{t}$ |
| $\left\lvert\, \begin{gathered}\text { Partitioned } \\ \text { e. of. } \mathrm{m} .\end{gathered}\right.$ |  |  |
| $\left\lvert\, \begin{gathered} \text { "Condensed" e. of. } \\ m . \\ \text { w.r.t. } x_{c} \\ \text { (Assume } X_{i}^{0}=0 \text { ) } \end{gathered}\right.$ | ${ }^{\circ} \mathbf{S}_{\mathrm{c}}{ }^{\circ} \mathrm{X}_{\mathrm{c}}={ }^{\circ} \mathrm{X}_{\mathrm{c}}$ | ${ }^{\circ} \mathbf{R}_{\mathrm{c}}{ }^{\circ} \mathrm{X}_{\mathrm{c}}={ }^{\circ} \mathrm{X}_{\mathrm{c}}$ |
| Condensed <br> Component <br> Matrices | $\begin{array}{r} { }^{\circ} \mathbf{S}_{\mathrm{c}}=\left[{ }^{\circ} \mathbf{S}_{\mathrm{cc}}-{ }^{\circ} \mathbf{S}_{\mathrm{Ci}}{ }^{\circ} \mathrm{S}_{i i}^{-I o} \mathbf{S}_{\mathrm{ic}}\right] \\ (4.2 .4) \end{array}$ | ${ }^{\circ} \mathbf{R}_{\mathrm{c}}={ }^{\circ} \mathbf{R}_{\mathrm{cc}}$ |
| Normal form of Condensed Matrices | $\boldsymbol{\phi}_{\mathrm{O}}^{-1}$ cannot be partitioned. No normal form exists. | Partition $\Phi_{o}=\left[\begin{array}{l} { }_{0} \Phi_{\mathrm{i}} \\ { }_{\mathrm{O}} \boldsymbol{\Phi}_{\mathrm{c}} \end{array}\right]$ ${ }^{\circ} \mathbf{R}_{\mathrm{c}}={ }^{0} \boldsymbol{\Phi}_{\mathrm{c}} \mathrm{D}_{\mathrm{o}}^{-I \mathrm{Io}} \boldsymbol{\Phi}_{\mathrm{c}}^{\mathrm{t}}$ $(4.2 .5)$ |
| ${ }^{\circ} S_{c}$ and ${ }^{0} R_{c}$ have poles coincident with: | Eigenvalues of component with its connection coordinates fixed. (RESTRAINED COMPONENT) | Eigenvalues of component with its connection coordinates free. (UNRESTRAINED COMPONENT) |
| Gather together <br> all condensed component e. of. m's in supermatrix equations. | $\mathbf{S}_{\mathrm{c}} \mathbf{x}_{\mathrm{c}}=\overline{\mathbf{x}}_{\mathrm{c}}(4.2 .6)$ | $\begin{aligned} & \mathbf{R}_{\mathrm{c}} \mathbf{X}_{\mathrm{c}}=\mathbf{X}_{\mathrm{c}} \\ & \mathbf{R}_{\mathrm{c}}^{\prime}=\boldsymbol{\Phi}_{\mathrm{c}} \mathbf{D}^{-1} \boldsymbol{\Phi}_{\mathrm{c}}^{\mathrm{t}} \end{aligned}$ |

Hence:

$$
\begin{aligned}
& \left.\mathbf{s}_{\mathrm{c}}=\Gamma^{1} \mathbf{s}_{c}{ }^{2} \mathbf{s}_{c} \ldots \ldots{ }^{j} \mathbf{s}_{c} \ldots{ }^{q} \mathbf{s}_{\mathrm{c}}\right] \\
& \left.\mathbf{R}_{\mathrm{c}}=\Gamma^{1} \mathbf{R}_{\mathrm{c}}{ }^{2} \mathbf{R}_{\mathrm{c}} \ldots \ldots{ }^{\mathrm{j}} \mathbf{R}_{\mathrm{c}} \ldots{ }^{\mathrm{q}} \mathbf{R}_{\mathrm{c}}\right\rfloor \\
& x_{c}=\left\{{ }^{l} x_{c}{ }^{2} x_{c} \ldots .^{j} x_{c} \ldots{ }^{q} x_{c}\right\} \\
& \mathbf{x}_{c}=\left\{{ }^{l} \mathbf{x}_{c}{ }^{2} \mathbf{x}_{\mathrm{c}} \ldots .^{j} \mathbf{x}_{c} \ldots{ }^{q} \mathbf{x}_{\mathrm{c}}\right\}
\end{aligned}
$$

where there are $q$ components in the structure.
In addition, the normal form of $\mathbf{R}_{\mathrm{c}}$ is available with

$$
\begin{aligned}
& D^{-1}=\left.\Gamma D_{I}^{-1} D_{2}^{-1} \ldots \ldots D_{j}^{-1} \ldots \ldots D_{q}^{-1}\right\rfloor \\
& \Phi_{c}=\Gamma I \\
&\left.\Phi_{c}^{2} \Phi_{c} \ldots{ }^{j} \Phi_{c} \cdots \cdots{ }^{q} \Phi_{c}\right]
\end{aligned}
$$

$\boldsymbol{\Phi}_{c}, \mathbf{S}_{\mathrm{c}}$ and $\mathbf{R}_{\mathrm{c}}$ are termed composite diagonal matrices, while $\mathrm{D}^{-1}$ is purely diagonal.

Let the order of the complete base system displacement vector $\mathbf{x}=\left\{\mathbf{x}_{\mathrm{c}} \mathbf{x}_{\mathrm{i}}\right\}$ be $\mathrm{n}_{\mathrm{t}}$ where

$$
n_{t}=\sum_{j=1}^{q} n_{0}^{j}
$$

and similarly let $X_{c}$ and $X_{i}$ be of order $n_{c}$ and $n_{i}$ respectively.

### 4.2.4. Transformation to the Composite System

Let there be $m$ equations of composite system comparability with the simple form $x_{i}=x_{j}$. A transformation may be defined:

$$
\begin{equation*}
x_{c}=T_{c} y_{c} \tag{4.2.8}
\end{equation*}
$$

where $\mathbf{y}_{\mathrm{c}}$ contains a set of $\ell$ connection freedoms in the composite
system. Where several component connection freedoms of the same sense coalesce, only one is carried into $\mathbf{y}_{c}$, hence $\mathbb{L} \leqslant m$. Internal freedoms are unchanged by the coupling up process and for completeness we may write

$$
\begin{equation*}
x_{i}=y_{i} \tag{4.2.9}
\end{equation*}
$$

The order of the composite system is thus given by $n=\left(n_{t}-m\right)$.
Equation (4.2.8) may be used to directly transform the dynamic stiffness equation (4.2.6):

$$
\begin{equation*}
\mathbf{T}_{\mathrm{c}}^{\mathrm{t}} \mathbf{S}_{\mathrm{s}} \mathbf{T}_{\mathrm{c}} \mathbf{y}_{\mathrm{c}}=\mathbf{T}_{\mathrm{c}}^{\mathrm{t}} \mathbf{X}_{\mathrm{c}}=\mathbf{Y}_{\mathrm{c}} \tag{4.2.10}
\end{equation*}
$$

where $\quad \mathbf{Y}_{c}$ is the set of forces at the composite system connection freedoms. The COMPOSITE SYSTEM CONDENSED DYNAMIC STIFFNESS MATRIX of order $\ell$ is defined by

$$
\begin{equation*}
\mathbf{S}=\mathbf{T}_{\mathrm{c}}^{\mathrm{t}} \mathbf{S}_{\mathrm{c}} \mathbf{T}_{\mathrm{c}} \tag{4.2.11}
\end{equation*}
$$

For free vibrations of the composite system, we require the two conditions to be satisfied:

$$
\begin{align*}
\mathbf{S} \mathbf{y}_{\mathrm{c}} & =0  \tag{4.2.12}\\
\mathbf{T}_{\mathrm{c}}^{\mathrm{t}} \mathbf{x}_{\mathrm{c}} & =0 \tag{4.2.13}
\end{align*}
$$

The latter implies that forces at the component connection freedoms must be internal, while the former provides a frequency determinental equation.

The force transformation contained in equation (4.2.10) cannot be used as it stands to transform the receptance equation (4.2.7). As shown in Section 5.2.2., a set of forces which satisfies the 'internal' condition is defined by a new transformation:

$$
\begin{equation*}
X_{c}=P_{c} \mathbf{c} \tag{4.2.14}
\end{equation*}
$$

$P_{c}$ is termed here the CONSTRAINT FORCE MATRIX, for each of the $m$ columns
of $\quad \mathbf{P}_{c}$ defines an internal set of forces in a constraint, the corresponding element in ( $\mathrm{C} \times \mathrm{x}$ ) being an undetermined multiplier. Furthermore it may be shown that the displacement compatability relation may be rewritten in the form

$$
\begin{equation*}
P_{c}^{t} x_{c}=0 \tag{4.2.15}
\end{equation*}
$$

Utilising the above 2 equations, equation (4.2.7) may be transformed to yield

$$
\begin{equation*}
\mathbf{P}_{\mathrm{c}}^{t} \mathbf{R}_{\mathrm{c}} \mathbf{P}_{\mathrm{c}} \mathbf{c}=\mathbf{R} \mathbf{c}=\mathbf{0} \tag{4.2.16}
\end{equation*}
$$

where $\mathbf{R}$ is the COMPOSITE SYSTEM CONDENSED RECEPTANCE MATRIX corresponding to the set of $m$ "constraint forces" in C.

A summary of this section is shown in the following table. In particular, the canonical form of $\mathbf{R}$ is given.

|  | DYNAMIC SITFFNESS | RECEPTANCE (KRON) |
| :---: | :---: | :---: |
| Transform to composite syster matrices | $\mathbf{S}=\mathbf{T}_{\mathrm{c}}^{\mathrm{t}} \mathbf{S}_{\mathrm{c}} \mathbf{T}_{\mathrm{c}}$ (4.2.17) | $\mathbf{R}=\mathbf{P}_{\mathrm{c}}^{t} \mathbf{R}_{\mathrm{c}} \mathbf{P}_{\mathrm{c}}$ (4.2.18) |
| Normal forms of $\mathbf{S}$ and $\mathbf{R}$ | No normal form | $\begin{array}{r} \mathbf{R}=\mathbf{P}_{\mathrm{c}}^{\mathrm{t}} \boldsymbol{\Phi}_{\mathrm{c}} \mathbf{D}^{-1} \boldsymbol{\Phi}_{\mathrm{o}}^{\mathrm{t}} \mathbf{P}_{\mathrm{c}} \\ (4.2 .19) \end{array}$ |
| Composite system unknowns | $\ell$ connection displacements in $\quad \mathbf{y}_{\mathrm{c}}$ | m connection forces in $\mathbf{c}$ |
| Frequency equation | $S_{c} y=0$ | $\mathbf{R c}=0$ |
| Conditions for free vibration | $\begin{aligned} \text { either } \quad \mathbf{y}_{c} & =0 \\ \text { or } \quad\|\mathbf{S}\| & =0 \end{aligned}$ | $\begin{aligned} \text { either } \quad \mathbf{C} & =\mathbf{0} \\ \text { or }\|\mathbf{R}\| & =0 \end{aligned}$ |
| Poles of <br> frequency deter- <br> minant coincide with | eigenvalues of restrained components | , eigenvalues of unrestrained components |

### 4.3. THE COMPONENT RETEASE TECHNIQUE

### 4.3.1. Introduction

The basic philosophy of this technique is to suppress the connection coordinates of the fully assembled 'composite'. system to generate independent restrained components. The composite system is regained by the release of these applied constraints.

The duel approaches of dynamic stiffness and receptance are again developed in parallel.
4.3.2. The Technigue Formulation

|  | DYNAMIC STIFFNESS | RECEPTANCE |
| :---: | :---: | :---: |
| Assembled e.of. m in partitioned form (composite system) |  | diag. matrices) |
| Set $\mathbf{y}_{\mathrm{c}}=\mathbf{0}$ to obtain base system e. of m. | $\begin{aligned} \mathbf{S}_{\mathbf{i}} \mathbf{X}_{\mathbf{i}} & =\mathbf{X}_{\mathbf{i}} \\ \text { where } & \\ \mathbf{S}_{\mathbf{i}} & =\mathbf{S}_{\mathbf{i} i} \end{aligned}$ | $\begin{gathered} \mathbf{R}_{i} \mathbf{X}_{i}=\mathbf{x}_{i} \\ \text { where } \\ \mathbf{R}_{i}=\left(\mathbf{R}_{i \mathrm{i}}-\mathbf{R}_{\mathrm{ic}} \mathbf{R}_{\mathrm{cc}}^{-1} \mathbf{R}_{\mathrm{ic}}^{\mathrm{t}}\right) \\ (4.3 .5) \end{gathered}$ |
| Normal forms of base system matrices | $S_{i}=\Phi_{i}^{-t} D_{i} \Phi_{i}^{-1}$ <br> (N.B. $\boldsymbol{\Phi}_{\mathrm{i}}$ is comp. diag., | $\begin{aligned} & R_{i}=\Phi_{i} \mathbf{D}_{\mathbf{i}}^{-1} \Phi_{i}^{t} \\ & \mathbf{D}_{i} \text { is diagonal) } \end{aligned}$ |
| Condense composite system e. of m by assuming $X_{i}=0$ <br> and eliminating $\mathbf{x}_{i}$ | $\begin{gathered} \mathbf{S} \mathbf{y}_{\mathrm{c}}=\mathbf{Y}_{\mathrm{c}}(4.3 .6) \\ \text { where } \\ \mathbf{S}=\left[\mathbf{S}_{\mathrm{cc}}-\mathbf{S}_{\mathrm{ic}}^{\mathrm{t}} \mathbf{S}_{\mathrm{ij}}^{-1} \mathbf{S}_{\mathrm{ic}}\right] \end{gathered}$ | $\mathbf{R}_{\mathrm{cc}} \mathbf{Y}_{\mathrm{c}}=\mathbf{y}_{\mathrm{c}}{ }_{(4.3 .8)}$ |
| Normal forms of $\mathbf{S}$ and $\mathbf{R}_{\mathrm{cc}}$ | $\begin{array}{r} \left.S_{c c}=\mathbf{S}_{i c}^{t} \boldsymbol{\Phi}_{\mathrm{i}} \mathbf{D}_{\mathrm{i}}^{-1} \boldsymbol{\Phi}_{\mathrm{i}}^{t} \mathbf{S}_{i \mathrm{ic}}\right] \\ (4.3 .9) \end{array}$ | $\mathbf{R}_{c c}$ cannot be expressed <br> in terms of the <br> normal form of $\mathbf{R}_{i}$ |
| Set $Y_{c}=$ for free vibration |  | $\begin{array}{rlrl} \text { either } & & \mathbf{y}_{\mathrm{c}} & =\mathbf{0} \\ \text { or } & \left\|\mathbf{R}_{\mathrm{cc}}\right\| & =\infty \end{array}$ |
|  | Poles of $\|\mathbf{S}\|$ are at restrained component eigenvalues. | Zeros of $\left\|\mathbf{R}_{\mathrm{cc}}\right\|$ are at restrained component eigenvalues. |
|  |  | (Clearly $\mathrm{R}_{\mathrm{cc}}=\mathrm{S}^{-1}$ ) |

### 4.4. COMPARISON FOR FINTTE SYSTEMS

Provided $m$ and $l$ are much less than $n$, the techniques described in the previous two sections have substantially reduced the effective order of the problem with no reduction in the number of freedoms and thus no inherent loss of accuracy. The penalty for this is that the final eigenvalue problem is no longer of the usual algebraic form. The determinants of $\mathbf{S}, \mathbf{R}, \mathbf{R}_{c c}$ are in general polynomial quotients in $\lambda$. The composite system equations of motion are non-algebraic eigenproblems to which the standard techniques of Chapter 2 are inapplicable. Instead, some form of frequency scan is required to locate values of $\lambda$ satisfying these equations, involving the setting up of the above matrices for a succession of trial values of $\lambda$.

From a consideration of the formulations of the two previous sections, it is clear that the 'natural' approach in the case of unrestrained components is receptance, and in the case of restrained components, dynamic stiffness. In both cases, the formation of the matrices $\mathbf{R}$ and $\mathbf{S}$ utilises the component eigenreductions, thus both speeding up the matrix set up for a given $\lambda$, and identifying the poles of the determinant. For finite system the remaining two approaches may be discarded as "misfits".

Although the size of $\mathbf{S}$ will often be smaller than that of $\mathbf{R}$, where more than two connection coordinates of the same sense coalesce at one point, the form of $\mathbf{R}$ (equation 4.2.19) is far more suitable for frequency scanning than that of $\mathbf{S}$ (equation 4.2.9) as in the latter $\mathbf{S}_{c c}$ and $\mathbf{S}_{i c}$ are still functions of $\lambda$. The receptance approach is thus to be preferred, except when $\ell \ll m$, and forms a powerful method for handling large structures in the minimum of core space without inherent loss of accuracy. The approach forms the basis of Kron's method, which is developed in detail in Chapter 5.

Returning to a more general view, the component connection technique involves the application of constraints to unrestrained components. Assuming the constraints to be applied sequentially, several intermediate systers are formed, and it is clear from Rayleigh's theorem that the eigenvalues of these systems converge monatonically towards those of the composite system from below (cf. Weinstein's method).

The component release technique on the other hand involves a
series of constraint releases and thus eigenvalues of the intermediate systems converge from above in monotonic fashion (cf. Rayleigh Ritz).

It is evident that both dynamic stiffness approaches reach the same composite system equation. In the component connection technique the process is essentially
(1) eliminate internal freedoms of component,
(2) assemble condensed component dynamic stiffness matrices. While in the component release method, the condensation process is carried out effectively on the assembled matrices.

### 4.5. INFINITE SYSTEMS

### 4.5.1. Comparison of Approaches

For completeness, it is useful to briefly discuss the case of infinite freedom systems. Such systems usually comprise structural elements which have dynamic stiffness or receptance terms as closed form functions of $\lambda$. Assembly is carried out by means of a finite set of displacement coordinates. Thus the resulting dynamic stiffness or receptance matrices constitute non-algebraic eigenproblems, as an infinite number of internal coordinates have implicitly been eliminated.

The implication of this is that determinant scanning methods are immediately necessary. Furthermore, the eigenreduction of the components is not a simple algebraic eigenproblem, and any finite set of modes used to represent the component would involve considerable effort and in any case are incomplete. The 'normal form' advantage of Kron's method is lost, and the dynamic stiffness formulation appears slightly superior in view of the possibility of smaller frequency determinants.

Simpson and Tabarrok (1968) [48] gave an example of an infinite freedom system analysed by the component connection dynamic stiffness approach. The natural frequencies were obtained by direct scanning of $|\mathbf{S}|$. It is interesting to note that their theoretical frequencies were lower than the experimental ones, the result expected when seen in the context of applying a finite set of constraints to couple an infinite freedom system.

The dynamic stiffness component release method has been developed extensively by Wittrick and Williams [49] for vibration and stability analysis of infinite systems.

The fundamental problem is to obtain frequencies which satisfy the non-algebraic eigenproblem

$$
\begin{equation*}
\mathbf{S}_{\mathrm{F}} \mathbf{x}_{\mathrm{F}}=\mathbf{0} \tag{4.4.1}
\end{equation*}
$$

where $\quad \mathbf{x}_{F}=\left\{\mathbf{x}_{i} \boldsymbol{y}_{c}\right\}$ (using the existing notation), and $S_{F}$ is the corresponding dynamic stiffness matrix. The straightforward scanning procedure involves triangulation at each trial $\lambda$ value to obtain the determinant of $\mathbf{S}_{F^{*}}$. After Gaussian triangulation of the first $n_{i}$ columns, the following form is obtained:

$$
\left[\begin{array}{ll}
s_{i i} & s_{i c} \\
0 & s
\end{array}\right]\left[\begin{array}{l}
x_{i} \\
y_{c}
\end{array}\right]=\left[\begin{array}{l}
0 \\
x_{c}
\end{array}\right]
$$

Clearly triangulation is implicit in the condensation of $\mathbf{S}_{\mathrm{F}}$ to $\mathbf{S}$, and triangulation of each column corresponds to the release of a coordinate. The advantage, in introducing the concept of components here is that the work involved in each factorisation may be reduced in the event of repeated components [50]. In addition, connection coordinates may be eliminated as "internal coordinates" of a new larger component, thus enabling the size of the working matrix to be kept small. This successive condensation process is identical to the well-known use of SUB-STRUCTURES in the efficient solution of static problems [12].

### 4.5.2. The Extended Sturm Sequence Algorithm

Of central importance to the development of the last method was the extension of the Sturm sequence scanning method (Section 2.3.3) to the non-algebraic eigenproblem by Wittrick and Williams [19]. In essence, the number of eigenvalues exceeded, J, by a given trial $\hat{\lambda}$ is given by

$$
\begin{equation*}
J=J_{0}=s\left[S_{F}(\lambda)\right] \tag{4.4.3}
\end{equation*}
$$

where $J_{0}$ is the number exceeded in the system with $\mathbf{x}_{F}=0$, and $s$ indicates 'sign count' as defined in Section 2.3.3. The algorithm is of great importance in Kron's method for scanning the matrix $\quad \mathbf{R}(\lambda)$
(Section 5.5.5). While a full derivation is contained in the above reference, the algorithm is considered here for the case of the condensed finite freedom dynamic stiffness matrix, $S(\lambda)$.

Consider $\quad S_{F}(\lambda)$ to be the dynamic stiffness matrix corresponding to a complete, finite, set of displacements $X_{F}$. From the Sturm sequence property, $s\left[S_{F}(\lambda)\right]$ equals the number of eigenvalues exceeded by $\lambda$. From equation (4.4.2), the number of eigenvalues exceeded in the base system defined by $\quad \boldsymbol{y}_{c}=0$ is clearly $s\left[\mathbf{S}_{i i}\right]$. Release of the $n_{c}$ coordinates in $\quad \mathbf{y}_{c}$ yields the fully factorised form

$$
\mathbf{S}_{F}^{\nabla}=\left[\begin{array}{cc}
\mathbf{S}_{\mathrm{ii}}^{\nabla} & \mathbf{s}_{\mathrm{ic}}^{\prime} \\
0 & \mathbf{S}^{\nabla}
\end{array}\right]
$$

and clearly

$$
\begin{equation*}
s[\mathbf{S}(\lambda)]=s\left[\mathbf{S}_{i i}(\lambda)\right]+s[\mathbf{S}(\lambda)] \tag{4.4.4}
\end{equation*}
$$

For a finite system,


$$
S_{i i}=\Phi_{i}^{-t} D_{i} \Phi_{i}^{-1}
$$

hence $J_{0}$ in equation (4.4.3) is given by $s\left[D_{i}(\lambda)\right]$, and we have

$$
\begin{equation*}
J=s\left[\mathbf{D}_{i}(\lambda)\right]+s[\mathbf{S}(\lambda)] \tag{4.4.5}
\end{equation*}
$$

### 4.6. APPROXIMATE COMPONENT METHODS (COMPONENT MODE METHODS)

### 4.6.1. Introduction

The methods discussed so far in this chapter have been exact in the sense that there is no approximation introduced other than the initial structural idealisation.

There is, however, a class of methods, known under the general title of component mode methods where the order of the problem is reduced by a Rayleigh - Ritz type approximation. In general, freedoms corresponding to high frequency component normal modes are discarded, and a set of
freedoms retained to give an approximate representation of the low frequency behaviour of the structure. The advantage of the approach is that the 'reduced' problem is expressed in the usual algebraic form, thus enabling standard algorithms to be employed.

The various methods, of which those due to Gladwell and Hurty are best known are a natural extension of the elimination method of Section 2.6. as shown in the following sub-section.

One of the methods, due to Craig and Bampton [55], is described in detail in APPENDIX 2, together with a computer program to efficiently implement the theory. lhis program is used to enable comparison of the component mode method with the forms of Kron's method developed in this report.

### 4.6.2. Elimination of Variables in Component Form

The assembled dynamic stiffness equation of motion (4.3.1) may be written in the form:

$$
\left\{\left[\begin{array}{ll}
\mathbf{K}_{i i} & \mathbf{K}_{i c}  \tag{4.6.1}\\
\mathbf{K}_{\mathbf{i c}}^{t} & \mathbf{K}_{c c}
\end{array}\right]-\lambda\left[\begin{array}{ll}
\mathbf{M}_{\mathbf{i i}} & \mathbf{M}_{\mathbf{i c}} \\
\mathbf{M}_{\mathbf{i c}}^{t} & \mathbf{M}_{c c}
\end{array}\right]\right\}\left[\begin{array}{l}
\mathbf{x}_{i} \\
\mathbf{y}_{c}
\end{array}\right]=\left[\begin{array}{l}
\mathbf{X}_{\mathbf{i}} \\
\mathbf{Y}_{c}
\end{array}\right]
$$

With $\mathbf{Y}_{c}$ identified as 'master freedoms' and $X_{i}$ as internal freedoms, equation (4.6.1) is synonymous with equation (2.6.1). The assumption that the interior freedoms. follow the static displacement patterns defined by the boundary freedom leads to the elimination method reducing transformation matrix $\mathbf{T}_{\mathrm{o}}$ :

$$
\left[\begin{array}{c}
\mathbf{x}_{i}  \tag{4.6.2}\\
\mathbf{y}_{\mathrm{c}}
\end{array}\right]=\left[\begin{array}{cc}
-\mathbf{K}_{i \mathrm{i}}^{-1} & \mathbf{K}_{\mathrm{ic}} \\
\mathbf{I}
\end{array}\right] \mathbf{y}_{\mathrm{c}}=\mathbf{T}_{\mathrm{o}} \mathbf{y}_{\mathrm{c}}
$$

No freedoms are carried forward from the internal coordinates. The "interior" eigenvalue problem:

$$
\begin{equation*}
\left(\mathbf{K}_{i i}-\lambda M_{i i}\right) \mathbf{x}_{i}=\mathbf{0} \tag{4.6.3}
\end{equation*}
$$

is clearly the set of restrained component eigenproblems, with $K$ ii
and $\mathrm{M}_{\text {ii }}$ composite diagonal matrices. The convergence condition for the first-order approximation (equation (2.6.10)) is now that the required composite system frequency must be less than the lowest natural frequency of all the restrained components.

The methods now briefly discussed are extensions of this basic approach, and may be characterised by their 'reducing transformation', which containsthe essential Rayleigh - Ritz approximation.

### 4.6.3. The Partial Orthogonalisation Technique [54]

In this simple method, a few of the lowest normal modes of each restrained component are calculated and gathered together in, $\overline{\boldsymbol{\Phi}}_{i}$ with generalised coordinates $\bar{q}_{i}$. These coordinates are used to describe the displacements of the internal coordinates, and the reducing transformation is given by

$$
\left[\begin{array}{l}
x_{i}  \tag{4.6.4}\\
y_{c}
\end{array}\right]=\left[\begin{array}{cc}
\bar{\Phi}_{i} & 0 \\
0 & I
\end{array}\right]\left[\begin{array}{c}
\bar{q}_{i} \\
y_{c}
\end{array}\right]
$$

The approximation is reasonable provided the normal modes can represent the composite system modes to a high degree. This is more likely for structures which are very rigid in the regions of the $\mathbf{y}_{c}$ coordinates.

### 4.6.4. The Branch Mode Method

Gladwell's branch mode method relies, as the name implies, on the structure having a branchlike topological configuration. Again, the lowest modes from, each branch are calculated, but here a branch normally consists of several components, one of which is free to distort, while the others are assumed either fixed or rigid; physical intuition is used to determine the better assumption. The reducing transformation may be written as:

$$
\left[\begin{array}{l}
\mathbf{x}_{i} \\
\mathbf{y}_{c}
\end{array}\right]=\left[\begin{array}{l}
\tilde{\boldsymbol{\Phi}}_{i} \\
\dot{\boldsymbol{\Phi}}_{c}
\end{array}\right] \dot{\mathbf{q}}
$$

where $\overline{\mathbf{q}}$ contains the amplitudes in the modes. In the special case where all branch modes are chosen so that 1 component vibrates while all the others are fixed, the transformation is equivalent to that of equation (4.6.4) with only the $\bar{q}_{i}$ freedoms included. This highlights the fact that the branch mode method implicitly assumes that the connection coordinates between any pair of components are rigid relative to each other.

### 4.6.5. The Component Mode Method

This group of methods is essentially the elimination method with a few low frequency restrained component normal modes includes from each component ( $\bar{\Phi}_{i}$ ). The reducing transformation is of the form:

$$
\left[\begin{array}{l}
x_{i} \\
y_{c}
\end{array}\right]=\left[\begin{array}{cc}
\bar{\Phi}_{i} & \Phi_{c} \\
0 & \mathbf{I}
\end{array}\right]\left[\begin{array}{l}
\bar{q}_{i} \\
\mathbf{y}_{c}
\end{array}\right]
$$

where the $\boldsymbol{\Phi}_{\mathrm{c}}$ are termed 'constraint modes' are are particularly important in the case of flexible boundaries. These modes are normally defined by the static approximation

$$
\begin{equation*}
\boldsymbol{\Phi}_{c}=-K_{i i}^{-1} K_{i c} \tag{4.6.7}
\end{equation*}
$$

Originally proposed by Hurty (1965) [52], rigid body modes were treated separately in the case of free-free components, with constraint modes calculated for redundant connection coordinates only. However, Craig and Bampton (1968) [55] utilised constraint modes for every connection freedom thus implicitly including any rigid freedoms. It is important to note that while the number of normal modes included is arbitrary, the number of constraint modes is fixed by the number of connection coordinates.

Goldman (1969) [56] reported a variant utilising free-free normal modes and rigid body modes only. Like the branch mode method, a rigid boundary is assumed and application is limited to situations where this is a reasonable assumption.

Generalised methods utilising both restrained, and unrestrained component modes have been developed by MacNeal (1971) [57] and Benfield and
and Hruda (1971) [58]. These methods however are rather more complicated both to program and to use and involve a degree of intuition in the choosing of modes.

In all these methods, accuracy drops off sharply for higher composite system modes, and it is not in general easy to say whether a frequency is accurate (e.g. to within $1 \%$ ) or not. Hurty (1965) [53] produced a formula for estimating the error involved in each mode due to the truncation of higher modes, but it is of little practical help.
The component mode technique has found use particularly in the design of aerospace structures where modular configurations are commonplace.

### 4.7. SUMMARY OF COMPONENT METHODS

Component methods in general offer a means of economically handling large finite structures as a series of reasonably sized eigenproblems. In particular, use may be made of repeated components to reduce the work. A common need is also the synthesis of composite system characteristics from the results of practical vibration tests on component parts.

Provided all the component modes are used, Kron's method provides an excellent way of handling large systems with no inherent loss of accuracy. The computational application of the method, and its advantages are investigated fully in this thesis.

Where only the low frequency composite system eigenspectrum is of interest, the component mode approach offers a convenient means of eliminating unwanted freedoms, so reducing the problem to manageable size. However, it would appear from the literature that intuition must be used to decide when accuracy is likely to deteriorate

The question of introducing approximations into Kron's method has not been reported in the literature, and is investigated in this thesis. As a result, useful approximate methods for the description of the low frequency composite system are formulated and illustrated.

## CHAPTER 5

## KRON'S METHOD

### 5.1. INTRODUCTION

In the previous Chapter, Kron's method was introduced as the receptance approach in the component connection method. Here, the method is developed in detail. The displacement and force transformations are discussed and the Kron determinant examined. The calculation of eibenvalues and eigenvectors is discussed and the computational merits of the method outlined.

As originally proposed [14], Kron's method was couched in rather obscure electrical terminology. Simpson and Taborrok (1968) [48] gave a clear receptance formulation as indicated in the previous Chapter. and suggested a Newton's method algorithm for locating composite system eigenvalues. Brameller and Lo (1970) [59] utilised the escalator method for the eigenvalues. Simpson (1972) [60] gave a Lagrangian formulation together with a powerful eigenvalue algorithm based on the one proposed by Wittrick and Williams [19].

Further papers by Simpson have considered
(1) a Lagrangian derivation together with an alternative dual approach
in which the frequency determinant unknowns are displacements [61];
(2) the extension of the original formulation to the non-
-proportionaly damped eigenproblem [62].
These extensions are not considered in the present work.

### 5.2. THE CONSTRAINT TRANSFORMATION

5.2.1. The Displacement Transformation

The transformation of equation (4.2.8) of section 4.2.4. may be written:

$$
\mathbf{x}_{\mathbf{c}}=\left[\begin{array}{c}
\mathbf{x}_{c_{1}}  \tag{5.2.1}\\
\mathbf{x}_{c_{2}}
\end{array}\right]=\left[\begin{array}{c}
\mathbf{T}_{1} \\
\mathbf{T}_{2} \\
2
\end{array}\right] \cdot \mathbf{y}_{\mathrm{c}}
$$

where

## $\begin{aligned} X_{C_{I}}= & \text { the set of } \& \text { connection displacements to be carried through } \\ & \text { to the composite system. }\end{aligned}$

(DOMINANT DISPLACEMENTS)
$\begin{aligned} & X_{c_{2}}= \text { the set of } m \text { connection displacements not carried into } \\ & \text { the composite system. }\end{aligned}$
(DISCARDED DISPLACEMENTS)
In general, $\mathbf{T}_{1}(\ell \times \ell)$ may be constructed square non-singular, with $\mathbf{T}_{2}(\mathrm{~m} \times \ell)$ rectangular with $m \geqslant \ell$.

Constraints of the form $x_{i}=x_{j}$ are here termed SIMPIE CONSTRAINTS. By eppropriate arrangement we may write

$$
x_{c_{1}}=x_{c}
$$

so that

$$
T_{1}=I
$$

and $\mathbf{T}_{2}$ will also be a boolean matrix. If only 2 displacements coalesce at any point, $\quad X_{c_{2}}$ will be of length $\ell$, and by appropriate arrangement we may write

$$
\mathbf{T}_{2}=\mathbf{I}
$$

(For further discussion of the transformation, see [48]).
$\mathbf{y}_{c}$ may be eliminated in equation (5.2.1) to give an equation of the form

$$
\mathbf{P}_{\mathrm{c}}^{\mathrm{t}} \mathbf{x}_{\mathrm{c}}=0
$$

where

$$
\mathbf{P}_{\mathrm{c}}^{t}=\left[-\mathbf{T}_{2} \mathbf{T}_{1}^{-1}\right] \mathbf{I}
$$

may be defined for $\mathbf{T}_{I}$ non-singular. $\mathbf{P}_{c}^{t}$ is a ( $m \times n_{c}$ ) matrix, $n_{c}$ is the length of $X_{c}$. Each row represents one linearly independant holonomic constraint between the elements of the $\mathbf{X}_{c}$ vector, i.e.

$$
\mathbf{p}_{c_{i}}^{t} \mathbf{x}_{c}=0
$$

### 5.2.2. The Force Transformation

For free vibration of the composite system (Section 4.2.4) the condition was obtained:

$$
\begin{equation*}
\mathbf{T}_{\mathrm{c}}^{\mathrm{t}} \mathbf{X}_{\mathrm{c}}=0 \tag{5.2.3}
\end{equation*}
$$

It may be readily verified that a set of forces satisfying this is given by:

$$
\mathbf{X}_{\mathbf{c}}=\left[\begin{array}{l}
\mathbf{X}_{c_{1}} \\
\mathbf{X}_{c_{2}}
\end{array}\right]=\left[\begin{array}{cc}
-\mathbf{T}_{1}^{-t} & \mathbf{T}_{2}^{t} \\
\mathbf{I} & \mathbf{c}=\mathbf{P}_{\mathbf{c}} \mathbf{c}(5.2 .4) \\
\end{array}\right]
$$

where $X_{c}$ is partitioned comformably with $\left\{X_{c_{1}} \quad X_{c_{2}}\right\}$. Equation (5.2.4) expresses a force.transformation. Each column ${ }^{c_{1}}$ in ${ }^{2} \mathbf{P}_{c}$ is a set of forces in a particular constraint satisfying equilibrium, and the corresponding element in $\mathbf{C}(\mathrm{m} \times 1)$ is a 'generalised force coordinate' (or Lagrangian multiplier). Equation (5.2.3) implied that the forces in each constraint are internal, or self-equilibrating.

### 5.3. KRON'S DETERMINANT

Application of equations (5.2.1) and (5.2.4) to the base system receptance equation (4.2.7):

$$
\begin{equation*}
\Phi_{c} D^{-1}(\lambda) \Phi_{c}^{t} X_{c}=X_{c} \tag{5.3.1}
\end{equation*}
$$

leads to

$$
\mathbf{R}(\lambda) \mathbf{c}=\mathbf{0}
$$

where

$$
\begin{align*}
& \mathbf{R}(\lambda)=\mathbf{P}_{c}^{t} \boldsymbol{\Phi}_{\mathrm{c}} \mathbf{D}^{-1}(\lambda) \boldsymbol{\Phi}_{\mathrm{c}}^{t} \mathbf{P}_{\mathrm{c}} \\
& (m \times m) \quad\left(m \times n_{c}\right) \quad\left(n_{t} \times n_{t}\right) \quad\left(n_{c} \times m\right) \\
& \left(n_{c} \times n_{t}\right) \quad\left(n_{t} \times n_{c}\right) \text {, }
\end{align*}
$$

For non-trivial c vector, we obtain the frequency equation

$$
|R(\lambda)|=0
$$

known as the KRON DETERMINANT (of order m).

### 5.3.1. The Form of Kron's Determinant

Equation ( $5 \cdot 3.3$ ) may be re-written in the form

$$
\mathbf{R}(\lambda)=\mathbf{G}^{t} \mathbf{D}^{-1}(\lambda) \mathbf{G}
$$

where $G$ is an $\left(n_{t} \times m\right.$ ) matrix defined by

$$
\mathbf{G}=\boldsymbol{\Phi}_{\mathrm{c}}^{\mathrm{t}} \mathbf{P}_{\mathrm{c}}
$$

Each element in $\mathbf{G}$ is given by;

$$
\mathrm{g}_{\mathrm{kj}}=\left(\phi_{c_{k}}^{t} \mathbf{p}_{c_{j}}\right)
$$

Indeed the jth. column of $\mathbf{G}$ is the 'resolution' of the $j$ th. constraint vector in terms of the base system eigenvectors. Recalling that $D^{-1}(\lambda)$ has the simple diagonal form with $k t h$. term

$$
d_{\mathrm{kk}}=\left(\lambda_{k}^{0}-\lambda\right)^{-1}
$$

where $\lambda_{k}^{0}, k=1, n_{t}$ are the base system eigenvalues, the general term in Kron's determinant is given by

$$
\begin{equation*}
r_{i j}=\sum_{k=1}^{n_{t}} \frac{\left(\phi_{c_{k}}^{t} \mathbf{p}_{c_{i}}\right)\left(\phi_{c_{k}}^{t} \mathbf{p}_{c}\right)}{\left(\lambda_{k}^{0}-\lambda\right)}=\sum_{k=1}^{n_{t}} \frac{g_{k i} g_{k j}}{\left(\lambda_{k}^{o}-\lambda\right)} \tag{5.3.8}
\end{equation*}
$$

Clearly $\boldsymbol{R}(\lambda)$ is a symmetric ( $m \times m$ ) matrix, and in general a quotient of two polynomials in $\lambda$, the denominator being

$$
\begin{equation*}
\prod_{k=1}^{n_{t}}\left(\lambda_{k}^{0}-\lambda\right) \tag{5.3.9}
\end{equation*}
$$

In general, poles of $\mathbf{R}(\lambda)$ coincide with base system eigenvalues, while zeros of $\mathbf{R}(\lambda)$ coincide with composite system eigenvalues according to equation (5.3.4). Algorithms for locating composite system eigenvalues are considered in Section 5.5.

### 5.3.2. Equivalence to the Weinstein Determinant

By retaining the full eigenvectors $\boldsymbol{\Phi}$ and extending $\quad \mathbf{P}_{\mathrm{c}}$ to correspond to the full $\mathbf{X}$ vector, the constraints may be written as

$$
\mathbf{P}^{t} \mathbf{x}=\left[\begin{array}{ll}
0 & \mathbf{P}_{\mathrm{c}}^{\mathrm{t}}
\end{array}\right]\left[\begin{array}{l}
\mathbf{x}_{\mathrm{f}}  \tag{5.3.10}\\
\mathbf{x}_{\mathrm{c}}
\end{array}\right]=0
$$

and the general term in $\mathbf{G}$ becomes $\mathrm{g}_{\mathrm{ki}}=\left(\boldsymbol{\phi}_{\mathrm{k}}^{\mathrm{t}} \mathbf{p}_{\mathrm{i}}\right)$. By using this definition in equation (5.3.8), it is evident that the Kron determinant is identical to the finite freedom Weinstein determinant (see equation (3.3.12)). The properties of Section 3.3.4 thus apply equally to either.

### 5.4. THE CALCULATION OF EIGENVECTORS

### 5.4.1. Persistant Eigenvalues and Vectors

It is convenient to consider first the general case of a composite system eigenvalue $\lambda_{i}$ equal to a base system eigenvalue $\lambda_{k}^{0}$. Clearly $\mathbf{R}\left(\lambda_{i}\right)$ is not defined, but slightly perturbed arguments i.e. $\lambda_{i}+\varepsilon, \lambda_{i}+\varepsilon * 10$ where $\varepsilon \ll \lambda_{i}$ would enable the limiting value to be ascertained.

In general, it will be a superposition of zeros and poles which may be treated separately.

The multiplicity of the pole is equal to the number of base system eigenvalues 'lost' through application of the constraints. If the base system multiplicity of $\lambda_{k}^{0}$ was $p_{k}^{0}$, then by Rayleigh's theorem and ignoring any 'arrivals' from below $\ddot{\lambda}_{k}^{0}$, the PERSISTENT MUITIPLICITY of $\lambda_{i}$ is given by

$$
p_{i} \geqslant p_{k}^{\circ}-m \quad\left(p_{i_{p}} \geqslant 0\right)
$$

Such PERSISTENT EIGENVALUES correspond to an'infinite value of
$\left|R\left(\lambda_{i}\right)\right|$ and hence imply the condition:

$$
\begin{equation*}
c=0 \tag{5.4.1}
\end{equation*}
$$

This in turn implies that the imposition of the constraints have noeffect on the component modes corresponding to $\lambda_{k}^{\dot{j}}$, and that eigenvectors of persistent eigenvalues are simple linear combinations of the component modes corresponding to $\lambda_{k}^{0}$, such that the displacement transformation of equation 5.2.1. is satisfied.

### 5.4.2. Gained Eigenvalues and Vectors

Eigenvalues corresponding to roots of the numerator of $\mathbf{R}(\lambda)$, whether equal to a base system eigenvalue or not are termed here GAINED EIGENVALUES. The multiplicity of the eigenvalue is merely that of the of the $\operatorname{root}\left(p_{i_{G}}\right)$, hence the total multiplicity of $\lambda_{i}$ is in general given by

$$
p_{i}=p_{i_{p}}+p_{i_{G}}
$$

where $p_{i_{p}}=0$ always for $\lambda \neq \lambda_{k}^{0}$.
Corresponding to each gained eigenvalue $\lambda_{i}$, there will be $p_{i_{G}}$ non-trivial linearly independent, $\mathbf{C}$ vectors. $\mathbf{R}\left(\lambda_{i}\right)$ will be $p_{i_{G}}$-fold degenerate, and equation (5.3.2) may be partitioned:

$$
\left[\begin{array}{ll}
\mathbf{R}_{11} & \mathbf{R}_{12} \\
\mathbf{R}_{21} & \mathbf{R}_{22}
\end{array}\right]\left[\begin{array}{l}
\mathbf{C}^{\prime} \\
\mathbf{I}
\end{array}\right]=\left[\begin{array}{l}
0 \\
\mathbf{0}
\end{array}\right]
$$

where $\mathbf{R}_{11}$ is non-singular and $\quad \mathbf{R}_{22}$ is of order $p_{i_{G}}$. There are thus $p_{i_{G}}$ columns in $C$ where

$$
\mathbf{C}=\left[\begin{array}{llll}
\mathbf{c}_{1} & \mathbf{c}_{2} & \ldots & \left.\mathbf{c}_{j} \ldots\right]
\end{array}\right]=\left[\begin{array}{l}
\mathbf{C}^{\prime} \\
\mathbf{I}
\end{array}\right]
$$

From the first set in equation (5.4.3), $\quad \mathbf{C}^{\prime}$ is the solution to the
equation

$$
\begin{equation*}
\mathbf{R}_{11} \mathbf{C}^{\prime}=-\mathbf{R}_{12} \tag{5.4.4}
\end{equation*}
$$

thus enabling $C$ to be formed. Each column of $\mathbf{C}$ may then be used to generate the full eigenvector from the relation:

$$
\begin{equation*}
\mathbf{x}_{i j}=\Phi D^{-1}\left(\lambda_{i}\right) \Phi_{c}^{t} P_{c} c_{j} \tag{5.4.5}
\end{equation*}
$$

### 5.5. THE CALCULATION OF EIGENVALUES <br> 5.5.1. Introduction

As Kron's determinant is a non-linear function of $\lambda$, algebraic eigenvalue algorithms are inappropriate. Instead, frequency scanning must be used involving the evaluation of $|\mathbf{R}(\lambda)|$ for a succession of trial values of $\lambda$.

From Rayleigh's theorem, application of 1 constraint to the base system produces bounds on the constrained system eigenvalues of

$$
\begin{equation*}
\lambda_{i}^{o} \leqslant \lambda_{i}^{I} \leqslant \lambda_{i+1}^{0} \tag{5.5.1}
\end{equation*}
$$

and poles and zeros alternate along the $\lambda$ axis. However application of $m$ ( $>$ I) constraints yields the wide bounds

$$
\lambda_{i}^{0} \leqslant \lambda_{i}^{I} \leqslant \lambda_{i+m}^{0}
$$

and there is no set ordering of zeros and poles. A simple frequency scan of $|R(\lambda)|$ would require a fine mesh to reduce the risk of missing roots. The large number of determinant evaluations required would almost certainly lead to unacceptably long computer times.

### 5.5.2. Scalar Scanning

Proposed by Simpson and Tabarrok (1968) [48], the method involves partitioning equation (5.3.2) along its last row and column to give:

$$
\left[\begin{array}{ll}
\mathbf{r}_{s s} & \mathbf{r}_{s m} \\
\mathbf{r}_{s m}^{t} & \mathbf{r}_{m}
\end{array}\right]\left[\begin{array}{l}
\mathbf{c}_{s} \\
\mathbf{c}_{\mathrm{m}}
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right]
$$

Eliminating $\quad \mathbf{c}_{\mathrm{s}}$ leads to

$$
r_{m} c_{m}=0
$$

where

$$
\begin{equation*}
r_{\mathrm{m}}=\left(r_{\mathrm{mm}}-\mathbf{r}_{\mathrm{sm}} \mathbf{r}_{\mathrm{ss}}^{-1} \mathbf{r}_{\mathrm{sm}}\right) \tag{5.5.5}
\end{equation*}
$$

is a scalar function of $\lambda$ whose zeros are identical in general to those of $|\mathbf{R}(\lambda)|$. The poles of $r_{m}$ coincide with the zeros of $\left|\mathbf{r}_{S S}\right|$, i.e. the eigenvalues of the composite system with 1 constraint released. By equation (5.5.1) poles and zeros alternate, implying that the latter appear as simple roots. It is shown [48] that the component eigenvalues correspond to 'computational poles'. At the cost of inverting an ( $s \times s$ ) matrix (where $s=m-l$ ) for each trial $\lambda$, a simple expression for $\left(\frac{d_{r_{m}}}{d \lambda}\right)$ enables Newton convergence on the roots of $r_{m}$.

Clearly, the poles of $r_{m}$ are not known 'a priori' and a preliminary scan is advised. A further difficulty occurs when a simple pole and a zero of $r_{4}$ coincide, for example where the mth. constraint causes "zero raising" of an eigenvalue. A finite non-zero value results, and the eigenvalue must be detected by an auxiliary scan of $\left|r_{s S}\right|$. The algorithm is thus not entirely reliable.

### 5.5.3. The Escalator Method

The escalator method [63] was introduced for the solution of the algebraic eigenproblem, but is inefficient compared with more modern algorithms.

Brameller and Io [59] proposed its use in conjunction with the Kron determinant. One constraint is applied to the base system and the leading ( $1 \times 1$ ) sub-determinent of $\quad \mathbf{R}(\lambda)$ scanned enabling the calculation of the complete set of ( $n_{t}-1$ ) eigenvalues and vectors. These are then considered to form a new base problem, and a new determinent of order 1 is defined involving the second constraint.

The attraction of the approach is that the problem is reduced to a succession of simple 'l constraint scans' which all enjoy the property of alternating zeros and poles. However an enormous amount of useless 'intermediate' information is clearly generated. In addition, the complete set of vectors (which in contrast to the super-diagonal form of
the initial base system set axe in general full) are required to be held in backing store at each stage. The algorithm is thus wastefull both in time and core.

### 5.5.4. The Step-by-step Method

Developed by the present author, this method was used to calculate composite system eigenvalues and to study the "eigenvalue raising" effects of constraints.

The $m$ constraints are applied sequentially, the intermediate systems being characterised by successively larger leading sub-determinants of $\quad \mathbf{R}(\lambda)$. The eigenvalues of the $r$ th. system (i.e. r constraints applied) are determined from the ( $\mathrm{r} \times \mathrm{r}$ ) leading sub-matrix with the eigenvalues of the ( $r-1$ )th. system providing bounds according to equation (5-5.1). Thus
(i) any new eigenvalue must appear as a simple zero.
(ii) the multiplicity of an existing eigenvalue must satisfy

$$
p_{i}^{r-1}-1<p_{i}^{r}<p_{i}^{r-1}+1
$$

The procedure may be made selfchecking so that it is impossible to miss an eigenvalue. However, the process is inefficient in that the intermediate system eigenvalues are in general of no interest. Computational experiments have shown that for even very low $m$ values, the procedure of the following section is far more economical in time, and is thus to be preferred except where eigenvalue raising is to be studied.

### 5.5.5. The Extended Sturm Sequence Algorithm

Simpson (1972) [60] has proposed a modified form of Wittrick and Williams 'extended Sturm sequence algorithm' which was introduced in Section 4.5.2.. The arguments of that section may be conveniently extended to Kron's determinant.

The base system, corresponding to $\mathbf{c}=\mathbf{0}$, is characterised by $\mathbf{R}(\lambda)$ infinite, and from equation (5.3.5), the number of eigenvalues exceeded by a given $\lambda$ in the base system is $s[D(\lambda)]$, where $s$ indicates the SIGN COUNT.

As each element in $\mathbf{C}$ is released, a constraint is applied to the base system, and as a direct consequence of Rayleigh's theorem, there is
a possibility of one less eigenvalue being exceeded by $\lambda$. The loss of an eigenvalue is indicated by a change in sign between two consecutive principal minors of $\mathbf{R}(\lambda)$, hence the total number of eigenvalues lost on application of $m$ constraints is $s[\mathbf{R}(\lambda)]$.

The actual number of eigenvalues exceeded by $\lambda$ is thus given by

$$
J(\lambda)=s[D(\lambda)]-s[R(\lambda)]
$$

A full proof is given in Simpson [60]. In addition, he proposes a Newton algorithm for location of the roots of a scalar function of $\lambda$ as defined in Section (5.5.2). The algorithm involves inversion of a matrix of order ( $\mathrm{m}-1$ ) for each trial $\lambda$ and may thus be expensive in computer effort for large $m$ values.

This algorithm is not utilised in this work. Instead, a procedure for directly scanning $|R(\lambda)|$ has been developed, and is described in Section A1.3.3. It has been found far superior to the methods of the previous sections and is incorporated in the Kron's method programs.

In conclusion, the extended Sturm sequence algorithm is an extremely efficient way of solving the non-algebraic eigenproblem, and is the major reason why Kron's method is now highly competitive with other techniques for large structures.

### 5.6. CORE SPACE REQUIREMENTS

The principal advantage of Kron's method in the context of large problems is its relatively low demands on core space, thus allowing very large problems to be handled "in-core". The size of the composite system frequency matrix $\mathbf{R}(\lambda)$ is governed by the number of constraints $m$ and not by the order of the base system $n_{t}$. There is thus no implicit need to reduce the number of freedoms, hence very accurate results may be obtained; equivalent in fact to an analysis of the fully assembled structure with all freedoms retained. In this section, an approximate expression for the core space requirement is obtained, and the optimum way of defining the $\mathbf{R}(\lambda)$ matrix evolved.

The Kron determinant is fully defined by the matrices $\mathbf{G}$ and $\mathbf{D}$ according to equation (5.3.5). The general term (equation 5.3 .8 ) may be written

$$
\begin{equation*}
r_{i j}=\sum_{k=1}^{n_{t}} \frac{g_{k i} g_{k j}}{\left(\lambda_{k}^{o}-\lambda\right)}=\sum_{k=1}^{n_{t}} \frac{a_{i j}^{k}}{\left(\lambda_{k}^{0}-\lambda\right)} \tag{5.6.1}
\end{equation*}
$$

The components may be analysed sequentially with contributions to $\mathbf{G}$ and $\mathbf{R}$ stored on disc, thus space for one component only is required. With a typical component order of $n_{0}$, and assuming that a transformation method is required to obtain the complete eigenreduction, the core required is approximately $n_{o}^{2}$ locations.

Provided $m \leqslant n_{0}$ (approx.) these locations are available for the formation of the $\mathbf{R}(\lambda)$ matrix. In addition, storage must be available for the $D$ matrix ( $n_{t}$ locations) and the $G$ matrix, the requirement for which is not immediately obvious.

The formation of the $G$ matrix is indicated in equation (5.6.2) below.

$$
\begin{aligned}
& \left(n_{t} \times m\right) \\
& \left(n_{t} \times n_{c}\right) \\
& \left(n_{c} x m\right)
\end{aligned}
$$

Assuming all constraints only involve one coordinate from each of two adjacent component, then each constraint $\mathbf{p}_{c_{j}}$ will only have non-zero products with $0\left(2 n_{0}\right)$ eigenvectors. Thus in each column $\mathbf{g}_{i}$, there will be $2 n_{0}$ non-zero entries, and the total number of non-zero entries in
$G$ is $2 n_{0} \mathrm{~m}$.
Each term in $\quad \mathbf{R}(\lambda)$ represents the "linking" between two constraints, and will be non-zero only if the constraints refer to a common component. This determines the number of non-zero terms in each series for $r_{i j}$ as indicated in Fig. 5.1. An appropriate figure for the total number of non-zero $a_{i j}^{k}$ terms (equation 5.6.1) involved in $\quad \mathbf{R}(\lambda)$ may be obtained by assuming that each location in the lower triangle of $R(\bar{\lambda})$ has $n_{0}$. This gives $\left(n_{0} m^{2}\right) / 2$ non-zero $a_{i j}^{k}$ terms as against

FTGURE 5.1.

IIIUSTRATION OF THE DEPENDENCY OF CONSTRAINT LOCATION ON THE NUMBER OF NON-ZERO TERMS IN

EACH ELEMENT OF $R(\lambda)$
(1) Constraints i and jefer to the same components


Non-zero $g_{k i}$ and $g_{k j}$ terms coincide - maximum of $2 n_{o}$ non-zero terms in series for $r_{i j}$.
(2) Constraints i and irefer to one common component


Non-zero $g_{k i}$ and $g_{k j}$ coincide only when $k$ relates to component $B$, hence maximum of $n_{0}$ non-zero terms in series for $r_{i j}$.
(3) Constraints $i$, $j$ have no common component


All $g_{k i} \times g_{k j}$ products will be zero, so $r_{i j}=0$.
( $2 n_{0} m$ ) non-zero $g_{k i}$ terms. Thus for $m>4$, storage of the g-terms requires less core and is preferred, even though slightly more work is involved in the formation of $\quad \mathbf{R}(\lambda)$ for each trial $\lambda$. The, approximate total core requirement is given by


Clearly the minimisation of $n_{o}$ will have a favourable effect on $C_{\text {KRON }}$ but will inevitably lead to a larger $m$ value. It must be remembered that the effort involved in the full solution of $\quad \mathbf{R}(\lambda)$ is greater than for an algebraic eigenproblem of the same size.

The optimum subdivision of the composite structure will often be governed by the obvious advantages of repeated components. The question of how to take full advantage of repetition both in terms of components, and within the $G$ matrix is discussed in Chapter 7.

In the following Chapter, techniques which greatly reduce the above core requirement are introduced.

### 5.7. A SIMPLE EXAMPLE

A simple spring-mass system comprising 3 components is shown in Fig. 5.2. This example is selected to demonstrate the theory of Kron's method including the special case of persistent eigenvalues. The masses and spring stiffnesses are as indicated in the figure. The component eigenreduction may be verified as:
(a) THE COMPOSTTE SYSTEM

(b) THE ' BASE' SYSTEM (COMPONENTS A, B,C)


hence

$$
\mathbf{D}(\lambda)=\Gamma 0-\lambda, 2-\lambda, 0-\lambda, 1-\lambda, 0-\lambda, 3 / 2\rfloor
$$

and

$$
\left.\Phi \quad=\Gamma^{\mathrm{A}} \boldsymbol{\phi}^{\mathrm{B}} \boldsymbol{\Phi}^{\mathrm{C}} \boldsymbol{\Phi}\right\rfloor
$$

The two constraints are $x_{2}=x_{3}$ and $x_{4}=x_{5}$, hence defining

$$
\begin{aligned}
& x_{c_{1}}=x_{2} x_{4} \\
& x_{c_{2}}=x_{3} x_{5}
\end{aligned}
$$

equation (5.2.1) is written as

$$
\left[\begin{array}{l}
x_{c_{1}} \\
x_{c_{2}}
\end{array}\right]=\left[\begin{array}{l}
x_{2} \\
x_{4} \\
x_{3} \\
x_{5}
\end{array}\right]=\left[\begin{array}{cc}
1 & 0 \\
0 & 1 \\
\hdashline 1 & 0 \\
0 & 1
\end{array}\right]\left[\begin{array}{l}
y_{2} \\
y_{3}
\end{array}\right]=\left[\begin{array}{l}
T_{1} \\
T_{2}
\end{array}\right] \mathbf{y}_{c}
$$

Thus $\mathbf{T}_{1}=\mathbf{T}_{2}=\mathbf{I}$ in this case and we have,

$$
\mathbf{P}_{\mathrm{c}}=\left[\begin{array}{rr}
-1 & 0 \\
0 & -1 \\
1 & 0 \\
0 & 1
\end{array}\right]
$$

each column involving the set of internal forces in that constraint. The $G$ matrix is then calculated from equation (5.3.6):

$$
\begin{aligned}
& \mathbf{G}=\boldsymbol{\Phi}_{\mathrm{c}}^{\mathrm{t}} \mathrm{x} \cdot \mathbf{P}_{\mathrm{c}} \\
& {\left[\begin{array}{cc}
-\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} \\
0 & \frac{1}{\sqrt{3}} \\
0 & \frac{\sqrt{2}}{\sqrt{3}}
\end{array}\right]=\left[\begin{array}{cccc}
\frac{1}{\sqrt{2}} & 0 & 0 & 00 \\
-\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & -\frac{1}{2} & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{3}} \\
0 & 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}}
\end{array}\right]\left[\begin{array}{cc}
-1 & 0 \\
0 & -1 \\
1 & 0 \\
0 & 1
\end{array}\right]} \\
& \mathbf{R}(\lambda)=\mathbf{G}^{t} \mathbf{D}^{-1}(\lambda) \mathbf{G} \text { may be verified to be two } 2 \times 2 \text { matrix } \\
& R(\lambda)=\left[\begin{array}{cc}
{\left[\frac{3 / 4}{0-\lambda}+\frac{1 / 4}{1-\lambda}+\frac{1 / 2}{2-\lambda}\right]} & (S Y M M) \\
{\left[\frac{-1 / 4}{0-\lambda}+\frac{1 / 4}{1-\lambda}\right]} & {\left[\frac{7 / 12}{0-\lambda}+\frac{1 / 4}{0-\lambda}+\frac{2 / 3}{3 / 2-\lambda}\right]}
\end{array}\right]
\end{aligned}
$$

Rough plots of $r_{11}$ (ie. one constraint applied) and $\left(r_{11} \times r_{22}-r_{21}^{2}\right)$ (ie. both constraints applied) are shown in fig. 5.3. The eigenvalues of the intermediate and composite systems are tabulated below, emphasising Rayleigh's theorem:

## SIMPLE SPRING MASS SYSTEM

(a) PLOT OF T11 VERSUS (NOT TO SCALE)

(b) PLOT OF|R( $\lambda) \mid$ VERSOS $\lambda$ (NOT TO SCALE)


| $s[D(\lambda)]$ | 3 | 3 | 4 | 5 | 6 |
| :---: | :---: | :---: | :---: | :---: | :--- |
| $s[R(\lambda)]$ | 2 | 1 | 1 | 1 | 2 |
| $J$ | 1 | 2 | 3 | 4 | 4 |

BASE SYSTEM
INTERMEDIATE SYSTEM
COMPOSITE SYSTEM

* Component $C$ is unaffected by the first constraint and so its eigenvalues are merely carried through.

Note that in Fig. 5.3(b), the double pole at $\lambda=0$ indicates the loss of 2 eigenvalues (from the base system), while the finite non-zero values at $\lambda=1$ and $\lambda=\frac{3}{2}$ indicate 'no change', being the superposition of one pole and one zero. The $s[R(\lambda)]$ values in the table below this figure may be readily checked, confirming the operation of equation (5.5.7).

A single "zeroing" term is encountered on the leading diagonal of the factored form of each of $R\left(1_{3}\right), R(1+\epsilon)$ and $R\left(\frac{3}{2}+6\right)$, confirming the eigenvalues to be "gained". The corresponding eigenvectors are thus calculated by the equation (5.4.5) as:

| $\lambda$ | $1_{3}$ | 1 | $3_{2}$ |
| :---: | :---: | :---: | :---: |
| $y_{1}$ | 1 | -1 | 1 |
| $y_{2}$ | ${ }^{2} 3$ | 0 | $-\frac{1}{2}$ |
| $y_{3}$ | $-1_{3}$ | 1 | $1_{4}$ |
| $y_{4}$ | -1 | -1 | $-1_{8}$ |

As expected from Rayleighs principle, a perturbation of $\varepsilon=10^{-10}$ produced an error in the vector in the fifth decimal place.

However, no "zeroing" term is encountered using $\quad \mathbf{R}(0+\varepsilon)$ indicating a "persistent" eigenvalue. Clearly a linear combination of the 3 component rigid body modes satisfying the constraint transformation is $\left\{\begin{array}{llll}1 & 1 & 1 & 1\end{array}\right\}$ i.e. the composite system rigid body mode.

## CHAPTER 6

## APPROXIMATIONS IN KRON'S METHOD

### 6.1. INTRODUCTION

Kron's method, as established in Chapters 4 and 5, involves the application of $m$ constraints to a base system comprising $n_{t}$ degrees of freedom. Provided all the constraints are applied, and the full square set of eigenvectors and associated eigenvalues are employed from each component, results obtainable from an analysis of the Kron determinant will be 'exact', that is equivalent to an analysis of the fully assembled problem. In other words, the procedure involves no inherent loss of accuracy. This approach is termed the FULL KRON METHOD.

However, in most practical situations, only the lower composite system eigenspectrum is of interest, and it would clearly be advantageous to be able to obtain approximate results in this range with good savings in computer time and core space.

In this Chapter, methods of introducing approximations into Kron's method are developed, and their effect on the accuracy of the subsequent solutions discussed. Simple beam and plate bending examples provide numerical illustrations.

Two forms of approximation are introduced
(1) Reduction in the number of modal freedoms used to represent each component,
(2) Reduction in the number of constraints applied to couple the components.
The former is of the Rayleigh-Ritz type, resulting in the stiffness of each component being overestimated. The latter may be termed a Weinstein type approximation which tends to underestimate the stiffness at the component boundaries. When used simultaneously, a hybrid method results in which the opposing effects tend to cancel each other out.

It is emphasised that these approximate methods are intended to compliment the full Kron method and to further utilise the implicit low core space demands of the approach.

Finally, an economisation procedure in connection with the scanning of the Kron determinant, whether full or approximate, is introduced.

The computational advantages of these methods are discussed in detail in Chapter 8.

### 6.2. REDUCTION IN THE NUNBER OF COMPONENT MODELS

### 6.2.1. Introduction

The effect of including a reduced number of component modes while still applying the full set of $m$ constraints is now considered. The approximation is of the Rayleigh-Ritz type in that the composite system is constrained to vibrate with a reduced set of freedoms. Eigenvalues obtained are thus upper bounds on the 'full-solution' eigenvalues.

### 6.2.2. Truncation of Receptance Series

Initially, the approximation is discussed in terms of truncating the receptance series for each component.

Considering a typical component of order $n_{0}$, the modal matrix at the connection coordinates is partitioned according to

$$
{ }^{{ }^{\circ} \boldsymbol{\phi}_{c}}=\left[{ }^{0} \boldsymbol{\phi}_{\mathrm{c}_{1}}{ }^{\circ} \boldsymbol{\Phi}_{\mathrm{c}_{2}}\right]
$$

where ${ }^{\circ} \boldsymbol{\Phi}_{\mathrm{c}_{1}}$ contains the lowest $\mathrm{n}_{\mathrm{o}_{1}}$ modes and ${ }^{\circ} \boldsymbol{\phi}_{\mathrm{C}_{2}}$ contains the remainder. The full ${ }^{c_{1}}$ receptance matrix at ${ }^{\circ}{ }^{1}$ the $n_{o} \quad{ }_{c}{ }_{2}$ connection coordinates (Chapter 4, equation (4.2.5)) is

$$
{ }^{o} R_{c}={ }^{o} \Phi_{c} D_{o}^{-10} \Phi_{c}^{t}
$$

and introducing the partition of equation (6.2.1):

$$
\begin{align*}
{ }^{\circ} \mathbf{R}_{\mathrm{c}} & =\left[\begin{array}{cc}
{ }^{\circ} \boldsymbol{\Phi}_{\mathrm{C}_{1}}{ }^{\circ} \boldsymbol{\Phi}_{\mathrm{c}_{2}}
\end{array}\right]\left[\begin{array}{cc}
{ }^{\circ} \mathbf{D}_{1}^{-1} & 0 \\
0 & { }^{\circ} \mathbf{D}_{2}^{-1}
\end{array}\right]\left[\begin{array}{c}
{ }^{\circ} \boldsymbol{\Phi}_{\mathrm{c}_{1}}^{t} \\
{ }^{\circ} \boldsymbol{\Phi}_{\mathrm{c}_{2}}^{\mathrm{t}}
\end{array}\right] \\
& ={ }^{\circ} \boldsymbol{\Phi}_{\mathrm{c}_{1}}{ }^{\circ} \mathbf{D}_{1}^{-10} \boldsymbol{\Phi}_{\mathrm{c}_{1}}+{ }^{\circ} \boldsymbol{\Phi}_{\mathrm{c}_{2}}{ }^{\circ} \mathbf{D}_{2}^{-10} \boldsymbol{\Phi}_{\mathrm{c}_{2}} \tag{6.2.2}
\end{align*}
$$

where ${ }^{\circ} \mathbf{D}_{1}^{-1}$ involves the lowest $n_{o_{1}}$ eigenvalues. Each term in ${ }^{\circ} \mathbf{R}_{\mathrm{c}}$ is given by

$$
\dot{o}_{r_{p q}}=\sum_{k=1}^{n_{o}} \frac{{ }^{\circ} \oint_{p k} \cdot{ }^{\circ} \emptyset_{q k}}{\left({ }^{\circ} \lambda_{k}-\lambda\right)}
$$

where the component eigenvalues ${ }^{\circ} \lambda_{k}$ are arranged in ascending order. Inclusion of the lowest $\mathrm{n}_{\mathrm{o}}$ component modes yields the approximate receptance matrix

$$
\begin{equation*}
{ }^{\circ} R_{c_{1}}={ }^{\circ} \Phi_{c_{1}}{ }^{\circ} D_{1}^{-1 o} \Phi_{c_{1}} \tag{6.2.4}
\end{equation*}
$$

with each term a truncated receptance series (BK64, p. 215)

$$
\begin{equation*}
{ }^{\circ} r_{p q_{1}}=\sum_{k=1}^{n_{o_{1}}} \frac{{ }^{o} \oint_{p k} o^{o} \phi_{q k}}{\left({ }^{o} \lambda_{k}-\lambda\right)} \tag{6.2.5}
\end{equation*}
$$

with associated error

This approximation is reasonable provided ${ }^{\circ} \varepsilon_{p q_{1}} \ll{ }^{\circ} r_{p q_{1}}$, and clearly the number of terms that must be included in ${ }^{\circ}{ }_{r_{p q}}$ depends upon the convergence of the series in equation (6.2.3). Errors will be larger in general for diagonal terms, as all the terms in the series for ${ }^{\circ} \varepsilon_{p p_{7}}$ will be additive.

The terms included are intended to represent the component at low frequencies, thus $\lambda$ may be confined to the low end of the spectrum. If non-normalised eigenvectors $\bar{\Phi}_{0}$ are employed, the modal mass term appears explicitly in the receptance series:

$$
\begin{equation*}
o_{r_{p q}}=\sum_{k=1}^{n_{0}} \frac{\bar{\phi}_{p k} \bar{\phi}_{q k}}{o_{m_{k}}\left({ }^{0} \lambda_{k}-\lambda\right)} \tag{6.2.7}
\end{equation*}
$$

hence for $\lambda_{k} \gg \lambda$, the denominator may be approximated to ${ }^{\circ} m_{k}{ }^{\circ} \lambda_{k}$, which is equal to ${ }^{\circ} k_{k}$, the generalised stiffness. Convergence thus depends on the growth of the generalised stiffness terms, and the customary decrease in eigenvector amplitudes in the higher modes. However, it is known that receptance series for beams and plates in particular converge slowly for 'slopes' and higher order freedoms [64, Ch. 5].

In general, the generalised stiffness increases in a manner not too far removed from that of the eigenvalues. Hence a possible criteria would be to discard component modes corresponding to eigenvalues satisfying

$$
\begin{equation*}
\lambda_{\mathrm{k}}^{0}>\lambda_{\operatorname{MAX}} \times 10 \tag{6.2.8}
\end{equation*}
$$

where $\lambda_{\text {MAX }}$ is the maximum frequency square of interest. Continuing with the Kron formulation, the matrices ${ }^{\circ} \mathbf{D}_{1}$ and $\mathbf{O}_{\mathrm{C}_{7}}$ may be gathered for all components into the composite diagonal matrices $\mathbf{D}_{1}$ and $\Phi_{c_{1}}$, so that the approximate composite system receptance matrix is given by

$$
\begin{equation*}
\mathbf{R}_{1}=\mathbf{P}_{\mathrm{c}}^{\mathrm{t}} \Phi_{\mathrm{c}_{1}} \mathbf{D}_{1}^{-1} \Phi_{\mathrm{c}_{1}}^{\mathrm{t}} \mathbf{P}_{\mathrm{c}}=\mathbf{G}_{1}^{\mathrm{t}} \mathbf{D}_{1}^{-1} \mathbf{G}_{1} \tag{6.2.9}
\end{equation*}
$$

Each term in $\mathbf{R}_{I}$ is a truncated series:

$$
\begin{equation*}
I_{r_{i j}}=\sum_{k=1}^{n_{l}} \frac{E_{k i} g_{k, i}}{\left(\lambda_{k}^{o}-\lambda\right)} \tag{6.2.10}
\end{equation*}
$$

where there are $n_{1}$ base system eigenvalues $\lambda_{k}^{0}$ in $D_{1}$. For simple constraints, each term in $\mathbf{R}_{1}$ is a sum of receptance series from two adjacent components, e.g., $\left({ }^{a} r_{\mathrm{pq}_{1}}+{ }^{\mathrm{b}_{\mathrm{r}_{\text {st }}}}\right.$ ). Each series may satisfy a convergence criteria separately, but it is quite possible for the included parts to cancel to some extent whereas the errors $\left({ }^{a} \varepsilon_{p q_{1}}+{ }^{b} \varepsilon_{s t_{1}}\right)$ do not. The combined error may appear large.

In short, the extent of the effect of a reduction in modal freedoms when yiewed in terms of truncating component receptance series is unpredicable.

### 6.2.3. Effects of Truncation in Beam Examples

Two simple beam examples (Fig. 6.1), each comprising 2 components, were used to test the effect of truncation of receptance series.* Both examples involve the two constraints:-

$$
\begin{array}{ll}
x_{7}=x_{9} & \text { (lateral displacement continuity) } \\
x_{8}=x_{10} & \text { (slope continuity) }
\end{array}
$$

The natural frequencies of the components are shown in TABLE 6.1, while the first 4 composite system natural frequencies with all modes included are given in TABLE 6.2. The resulting percentage errors in the first 4 composite system natural frequencies for increasing degrees of truncation (starting with the highest mode) is shown in Fig. 6.2 for both examples.

In the case of the fixed-fixed beam, the loss in accuracy for frequencies $f_{2}$ and $f_{4}$ is quite reasonable. For example, the error in $f_{2}(=97.67 \mathrm{~Hz})$ only exceeds $1 \%$ when component modes at 776 Hz and above are omitted. However the accuracy of $f_{1}$ and $f_{3}$ decreases disasterously, even on the omission of component modes with frequency 2 orders of magnitude greater.

A similar pattern is evident with the cantilever beam, but here it is the second and fourth frequencies which lose accuracy extremely rapidly.

It is thus clear that in certain situations, omission of even one or two component modes can lead to unacceptable errors, and that truncation on the basis of equation 6.2 .8 appears impracticable.

### 6.2.4. The Cause of Large Errors

An investigation into the cantilever beam example illustrates the mechanism behind the large errors reeported in the previous sub-section. Here, the frequency determinant is given by

$$
|\mathbf{R}(\lambda)|=\left|\begin{array}{ll}
r_{11} & r_{21}  \tag{6.2.11}\\
r_{21} & r_{22}
\end{array}\right|=\left(r_{11} r_{22}-r_{21}^{2}\right)
$$

the elements $r_{i j}$ also being functions of $\lambda$. The roots of $|\boldsymbol{R}|$ at $\lambda_{1}$ and $\lambda_{3}$ involve the expression in equation (6.2.11) zeroing with $r_{11}, r_{22}$

* 2 dimensional, 4 degree of freedom beam finite elements were employed (APPENDIX 5).

FIGURE 6.1.

## SIMPLE 2 COMPONENT BEAM EXAMPLES

(a) FIXED -FIXED BEAM (8 ELEMENTS)


BASE SYSTEM
(b) CANPILEVER BEAM (8 ELEMENTS)




COMPONENT C

BASE SYSTEM

> In both examples
> $I=3.0^{\prime \prime} \quad E=107 \mathrm{Ib} / \mathrm{in}^{2}$
> $t=0.1^{11} \quad \nu=\frac{1}{3}$
> Unit breadth, $\rho=0.27 \mathrm{Ib} / \mathrm{in}^{3}$

TABLE 6.1.

| COMPONENT NATURAL <br> (BEAM FREQUENCIES <br> EXAMPLERS) |  |  |
| :---: | :---: | :---: |
|  | A and B | C |
|  |  |  |
| 1 | 22.26 | 0.0 |
| 2 | 139.66 | 0.0 |
| 3 | 393.61 | 141.79 |
| 4 | 776.51 | 392.86 |
| 5 | 1444.26 | 771.46 |
| 6 | 2319.50 | 1413.59 |
| 7 | 3677.17 | 2213.35 |
| 8 | 6033.46 | 3432.74 |
| 9. |  | 5716.37 |
| 10 |  | 6311.20 |

TABIE 6.2.

|  | EXACT COMPOSITE SYSTEM NATURAL FREQUENCIES <br> ( $\mathrm{H}_{z}$ ) |  |
| :---: | :---: | :---: |
|  | $\begin{gathered} \text { FIXED-FIXED } \\ \text { BEAM } \\ \hline \end{gathered}$ | CANTILEVER BEAM |
| 1 | 35.41 | 5.56 |
| 2 | 97.67 | 34.88 |
| 3 | 191.81 | 97.71 |
| 4 | 318.29 | 191.79 |


and $r_{21}$ of comparable magnitude. However, the roots at $\lambda_{2}$ and $\lambda_{4}$ result principally from the term $r_{22}$ zeroing, with $r_{21}$ small. This indicates that the slope continuity constraint is highly active, physically implying that a high bending moment is involved in the formation of the composite system mode. A high internal moment implies high curvatures in the region of the component junction. However, the component normal modes in Kronsmethod are calculated with free, unloaded connection coordinates, and thus such high curvatures will only be present in high component modes. Truncation of these modes seriously affects the ability of the components to represent the composite system mode.

As an illustration of this point, fig. 6.3 shows the first 4 composite system modes for the full number of component modes and for a truncation case. Note how the curvature in the region of the junction is reduced in the approximate modes corresponding to $\lambda_{2}$ and $\lambda_{4}$.

Numerically, the effect may be explained by examining the series for the $r_{22}$ terms at $\lambda_{2}$ :

$$
\begin{equation*}
r_{22}=\sum_{k=1}^{n_{t}} \frac{g_{k 2}^{2}}{\left(\lambda_{k}^{0}-\lambda_{2}\right)} \doteq 0 \tag{6.2.12}
\end{equation*}
$$

where $\lambda_{k}^{0}$, ( $k=1, n_{t}$ ) are the base system eigenvalues arranged in ascending magnitude order. If there are s base system eigenvalues less than $\lambda_{2}$ :

$$
\begin{equation*}
\sum_{k=1}^{s} \frac{g_{k 2}^{2}}{\left(\lambda_{2}^{0}-\lambda_{k}^{0}\right)} \doteqdot \sum_{k=s+1}^{n} \frac{g_{k 2}^{2}}{\left(\lambda_{k}^{0}-\lambda_{2}\right)} \tag{6.2.13}
\end{equation*}
$$

Fig. 6.4(a) illustrates the variation in the magnitude of the terms in the series for $r_{22}$ with $\lambda_{2}=0.480 \times 10^{5}$ (cf. exact $0.48017 \times 10^{5}$ ), while Fig. 6.4(b) plots the successive partial series summations against increasing number of terms included. It is clear that the series does not converge decisively, and with the cancelling out of the low frequency terms, even omission of the last 3 terms seriously affects the value of $r_{22^{\circ}}$. In such a case, the resulting, value of $r_{22}$ is negative, implying that a larger value of $\lambda_{2}$ is required to restore the:

> (not to scale)


$$
\begin{array}{r}
\mathrm{f}_{3}=97.71 \mathrm{~Hz} \\
\mathrm{f}_{3 \mathrm{~A}}=97.84 \mathrm{~Hz} \\
\text { (MODES IDENTICAL) }
\end{array}
$$


$f_{4}=191.79 \mathrm{~Hz}$
$f_{4 A}=212.23 \mathrm{~Hz}$

- E EXACT MODES

APPROXIMATE MODES

The $f_{i A}$ frequencies, and associated mode shapes (dotted lines) are for the approximate system with the lowest 5 modes included from component $A$, and the lowest 6 modes from component $C$.

(a) VARIATION IN THE SIZE OF TERMS IN SERIES FOR $r_{22}\left(\lambda_{2}\right)$

(b) VARTATION OF THE SUM OF THE SERIES FOR $r_{22}\left(\lambda_{2}\right)$ WITH NO. OF TERMS INCLUDED
balance in equation (6.2.13), as demanded by the Rayleigh-Ritz principle. To summarise, significant errors may be introduced by truncation where large internal connection forces arise, for example where a connection boundary between components is very flexible.

Goldman (1969) [56] reported a form of component mode method utilising rigid body modes and unrestrained component modes and remarked, without any details, that "quite large errors could be introduced in certain ill-conditioned circumstances".

### 6.3. USE OF STATIC MODES IN MODAL REDUCTION <br> 6.3.1. Static Constraint Modes

The attraction in being able to successfully utilise a much reduced number of component modes is three-fold. Firstly, the necessity to calculate the complete set of component modes is removed. Secondly, the core requirement for the matrix $G$ ( $2 n_{0} m$ locations) would be greatly reduced where the reduced number of component freedoms $n_{g_{0}}$ satisfies $n_{g} \ll n_{0}$. Finally, the series for each term in the ${ }_{0}$ Kron determinant would be reduced, thus speeding up scanning.

The problem is to define extra modes in addition to the lowest few component normal modes to provide the 'higher frequency freedom' associated with large constraint forces. This may be achieved by using static modes corresponding to unit forces in turn at the connection coordinates of the components, all other forces being zero. Hence for a component without rigid freedoms, the stiffness matrix is non-singular and the $n_{o_{c}}$ constraint modes ${ }^{\circ} \mathbf{Z}_{c}$ are calculated for $\lambda=0$ by

$$
\begin{equation*}
\mathbf{K}_{0}{ }^{\circ} \mathbf{Z}_{\mathrm{c}}=\mathbf{X}_{0} \tag{6.3.1}
\end{equation*}
$$

where

$$
x_{o}=\left\{{ }^{0} X_{i}{ }^{\circ} X_{c}\right\}=\left\{\begin{array}{ll}
0 & I
\end{array}\right\}
$$

The duality of this approach with the component mode method is again evident in that the latter uses constraint modes which correspond to unit displacements.

For a component with rigid body freedoms, the situation is a little more complicated. Three possibilities for forming constraint modes are:

> (i) use of $M_{o}$ in equation (6.3.1) in place of $K_{o}$
> (ii) combination of forces at connection coordinates to form self-equilibrating force systems
> (iii) use of $\left(K_{o}-\mu_{o}\right)$ (non-singular) in place of $K_{o}$

The first alternative, although used successfully, generated very 'high frequency' constraint modes with the result that extra normal modes are in general required. The second is undesirable in that it is not easily generalised, indeed it is impossible for the cantilever beam example unless additional 'dummy' connection coordinates are specified.

The final alternative has been used with complete success in the examples tried. The simultaneous iteration technique (S.I.) is employed for the few lowest component normal modes (see sub-section 7.3.1.). If i eigenvalues are so found, the additional trial vectors which are always used in S.I. yield approximate values for $\lambda_{i+1}$ and $\lambda_{i+2}$ etc.. A suitable choice for $\mu$ has been found to be

$$
\mu=\frac{\lambda_{i}+\lambda_{i+1}}{2}
$$

thus guaranteeing that ( $\mathbf{K}_{0}-\mu \mathbf{M}_{0}$ ) is non-singular.
It may be noted that the amount of work involved in the formation of the constraint modes is considerably less than for the same number of additional normal modes.

### 6.3.2. Orthogonalisation of Constraint Modes

The component receptance series must be formed from a set of orthogonal vectors, thus the $n_{0}$ constraint modes ${ }^{\circ} \boldsymbol{Z}_{c}$ must be orthogonalised w.r.t. the set ${ }^{c}$ of $n_{n}$ normal modes included ${ }^{\circ} \boldsymbol{\Phi}_{n}$.

This is carried out at'zero frequency', hence the resulting ortho-normal constraint modes may be termed static modes irrespective of the method by which they were generated. The Gramm-Schmidt process [46] is employed as indicated below.

The first constraint mode may be expressed as a linear combination of the complete set of component eigenvectors

$$
\phi_{\mathrm{k}}, \mathrm{k}=1, \quad \mathrm{n}_{\mathrm{o}} .
$$

$$
\begin{equation*}
\mathbf{z}_{c_{1}}=\sum_{k=1}^{n_{0}} \alpha_{k} \boldsymbol{\phi}_{k}=\sum_{k=1}^{n_{n_{0}}} \alpha_{k} \boldsymbol{\phi}_{k}+\bar{z}_{c_{1}} \tag{6.3.3}
\end{equation*}
$$

(The superscript o is dropped here for convenience.)

The orthogonality conditions

$$
\begin{equation*}
\phi_{k}^{t} \mathbf{K}_{0} \bar{z}_{c_{1}}=0, k=1, n_{n_{0}} \tag{6.3.4}
\end{equation*}
$$

are introduced, and with equation (6.3.3) yield the coefficients

$$
\begin{equation*}
\alpha_{k}=\frac{1}{\lambda_{k}} \phi_{k}^{t} K_{o} \mathbf{z}_{c_{1}} \tag{6.3.5}
\end{equation*}
$$

(Note that in the case where $\boldsymbol{\phi}_{k}$ is a rigid body mode, the orthogonality condition must refer to $\left.\mathbf{M}_{0}\right)$. Hence $\overline{\mathbf{z}}_{c_{1}}$ is
calculated from equation $(6.3 .3)$ and normalised such that

$$
\begin{equation*}
\overline{\mathbf{z}}_{\mathrm{c}_{1}}^{t} \mathbf{M}_{\circ} \overline{\mathbf{z}}_{\mathrm{c}_{1}}=1 \tag{6.3.6}
\end{equation*}
$$

Finally, the "pseudo-eigenvalue" for the mode is calculated from

$$
\begin{equation*}
\lambda_{c_{1}}=\overline{\mathbf{z}}_{c_{1}}^{t} K_{o} \overline{\mathbf{z}}_{c_{1}} \tag{6.3.7}
\end{equation*}
$$

Subsequently, $\overline{\mathbf{z}} c_{l}$ is treated as an additional normal mode and the process repeated for $\quad \mathbf{Z}_{c_{2}} . \quad$.

The process is $\quad 2$ reasonably fast, involving simple matrix operations. An advantage is that unnecessary constraint modes may be automatically discarded, further reducing the number of freedoms. The initial work done is calculated from

$$
c_{1}=z_{c_{i}}^{t} K_{o} z_{c_{i}}
$$

and the work done after orthogonalisation from

$$
c_{2}=\overline{\mathbf{z}}_{c_{i}}^{t} \mathbf{K}_{o} \overline{\mathbf{z}}_{c_{i}}
$$

Clearly if $\frac{c_{2}}{c_{1}} \ll 1$, the initial constraint mode contains a high
proportion ${ }_{l}$ of modes included so far, and is less likely to be of importance in the description of the component.
$\circ \overline{\mathbf{Z}}_{c}$, and ${ }^{\mathrm{c}_{\mathrm{o}}}$ orthogonalised constraint modes are gathered

$$
\boldsymbol{\Phi}_{o}=\left[{ }^{\circ} \boldsymbol{\Phi}_{\mathrm{n}}{ }^{\circ} \boldsymbol{Z}_{\mathrm{c}}\right]
$$

The total number of component modal freedoms is denoted by

$$
\begin{equation*}
n_{g_{0}}=n_{n_{0}}+n_{c_{0}} \tag{6.3.8}
\end{equation*}
$$

where $n_{n_{0}} \ll n_{0}$ (usually) and $n_{c_{0}}<n_{o_{c}}$.
The total number of modal freedoms in the base system is denoted by

$$
n_{g}=\sum_{i=1}^{q} n_{g_{0}}^{i}
$$

and for a worthwhile effective condensation of freedoms we require $n_{g} \ll n_{t}$.

It would seem sensible, on physical grounds, to include component normal modes up to a given reference $\lambda$ value for all components. If the maximum normal mode eigenvalue from each component is placed in a set $S$, the minimum and maximum of this set are designated $\lambda_{\text {min }}$ and $\lambda_{\text {max }}$ respectively. Thus $\lambda_{\text {min }}$ and $\lambda_{\text {max }}$ should ideally be close. An additional useful parameter is $\lambda_{c_{m i n}}$, the minimum constraint mode pseudoeigenvalue in the base ${ }^{c_{\text {min }}}$ system.

The technique described in this section is designated the MODAL APPROXIMATE KRON METHOD.

### 6.3.3. The Skew-Vector Effect

In general, the staticly orthogonalised constraint modes of the previous section violate the component mass matrix orthogonality condition i.e.

$$
\begin{align*}
z_{c_{i}}^{t} M_{o} \phi_{k} \neq 0 \quad i \neq k \quad i & =1, n_{c_{0}}  \tag{6.3.9}\\
k & =1, n_{g_{0}}
\end{align*}
$$

Away from $\dot{\lambda}^{\dot{\prime}}=0$, the constraint modes form a skew set of vectors in the space spanned by the strictly orthogonal set of $n_{o}$ component normal modes. Clearly the receptance series orthogonality condition is violated, and the eigenvalues obtained from an analysis of the Kron determinant will correspond to a different base problem.

Let $\mathbf{M}_{o_{\Delta}}$ be that part of the component mass matrix responsible for the coupling terms (assumed small) in the modal mass matrix. It may be shown that the first order change in an eigenvalue ' $\lambda_{i}$ due to $M_{O_{\Delta}}$, and assuming no change in eigenvector, is given by

$$
\lambda_{i_{\Delta}}=-\lambda_{i .} \phi_{i}^{t} M_{o_{\Delta}} \phi_{i}
$$

It is thus not possible to say in what sense the zero of the Kron determinant will be altered. However, the effect should be small provided $\quad \lambda \ll \lambda_{C_{\text {min }}}$. This effect is in addition to the Rayleigh--Ritz effect, which with consistent mass formulation yields eigenvalues which are upper bounds.

Fortunately, the eigenvector subsequently calculated includes only the Ráyleigh-Ritz effect amd the Rayleign quotient obtained from the generalised stiffness and mass terms $\lambda_{R_{Q_{i}}}$ will always be an upper bound.

The magnitude of the skew-vector effect may thus be investigated by defining the VARIATION NUMBER for each composite system eigenvalue by

$$
\begin{equation*}
V=\frac{\left(\sqrt{\lambda_{z_{j}}}-\sqrt{\lambda_{R Q_{i}}}\right)}{\sqrt{\lambda_{R Q_{j}}}} \times 100 \% \tag{6.3.11}
\end{equation*}
$$

### 6.3.4. Beam Examples

The importance of including constraint modes is illustrated with reference to the beam examples of Section 6.2. The composite system natural frequencies were calculated for 4 test cases, $a, b, c, d$ corresponding to the inclusion of an increasing number of component normal modes. Details of these cases appear in TABIE 6.3.* $f_{\text {MAX }}$ relates to the maximum component normal mode frequency included while $f_{c_{m i n}}$ relates to lowest constraint mode pseudo-frequency. Natural frequencies above this latter datum were found to be extremely inaccurate and are not presented.

On physical grounds, it is desirable to include component normal modes up to a common 'cutt-off' frequency, hence the choice of normal mode inclusions in the cantilever beam case (see TABIE 6.1).

The results are presented in Figures 6.5 and 6.6 to illustrate the convergence of natural frequency with increase in normal modes included. The locations of $f_{\max }$ and $f_{c_{m i n}}$ are indicated for each case. Convergence to the lower frequencies is extremely good, and takes place in order of ascending frequency monatonically from above as expected. In both examples, all approximate natural. frequencies below the relevant $f_{\max }$ are within $1 \%$ of the full solutions, while accuracy drops away between $f_{\max }$ and $f_{c_{\min }}$.

Figure 6.7 gives plots of frequency $\quad \min$ error and variation number for the 4 cases in the cantilever beam example. Comparison with figure 6.2(b) emphasises the importance of constraint modes in this example. Variation numbers are within $\pm 0.2 \%$ for natural frequencies below $f_{\text {max }}$. It would thus appear that 'zeros' of the Kron determinant may be safely used below the relevant $f_{\max }$. At higher

## TABLE 6.3.

## BEAM EXAMPLES - TEST CASE SPECIFICATTONS

| EXAMPLE: | Fixed-Fixed Beam |  |  |  | Cantilever Beam |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| CASE: | a | b | c | d | a | b | c | d |
| No. of Normal Modes | $(1,1)$ | $(2,2)$ | $(3,3)$ | $(4,4)$ | $(1,2)$ | $(2,3)$ | $(3,4)$ | $(4,5)$ |
| $\begin{gathered} \text { No. of } \\ \text { Constraint Modes } \end{gathered}$ | $(1,1)$ | $(1,1)$ | $(1,1)$ | $(1,1)$ | $(1,2)$ | $(1,2)$ | $(1,2)$ | $(1,2)$ |
| $\mathrm{f}_{\mathrm{MAX}} \quad\left(\mathrm{Hz}^{\prime}\right.$ | 22.26 | 139.7 | 393.6 | 776.5 | 22.26 | 141.8 | 393.6 | 776.5 |
| $f_{c_{M I N}}\left(H_{Z}\right)$ | 194.9 | 632.6 | 1365.4 | 2460.5 | 159.2 | 391.1 | 914.3 | 1690.3 |
|  | First number in above brackets refers to component $A$, second to component B . |  |  |  | First number in above brackets refers to component $A$, second to component $C$. |  |  |  |
| Full solution involves: | $(8,8)$ Normal modes |  |  |  | $(8,10)$ Normal modes |  |  |  |



CANTIIEVER BEAM EXAMPIP - PIOT OF COMPOSTTE SYSTHM NATURAL FREQUENCY VERSUS NO. OF COMPONENT NORMAL MODES INCLUDED INCORPORATING CONSTRAINT MODES


```
x INDICATES FREQUENCY ERROR (%)
- INDICATES VARIATION NO. (%)
```






$6^{\text {th }}$ frequency

frequencies, the "skew-vector" effect is more important, and much larger variation numbers are observed.

It may be noted that the 'unit shear force' constraint mode was omitted from the representation of component $A$ in all cases as its $\frac{c_{2}}{c_{1}}$ ratio was $O\left(10^{-4}\right)$. Inclusion of this mode was found to yield
marginal improvement in accuracy.

### 6.3.5. Plate Bending Example No. 1

The plate bending example in Figure 6.8 was used to further investigate the accuracy obtainable using constraint modes, and the extent of the skew-vector error. The example comprises 2 cantilever plate components to be coupled along the line XX. Conforming plate bending elements having 4 nodes and 4 freedoms per node were employed (APPENDIX 5). The base system thus comprises 96 freedoms, and connection involves 12 simple constraints. No use of symmetry was made in this illustrative example.

TABLE 6.4 gives the specifications of the test cases carried out. In cases A through $E$, all 12 constraint modes were included, while in $F$, apparently less important constraint modes were cut out, according to $\frac{c_{2}}{c_{1}} \ll 10^{-2}$ (Section 6.3.2).

The 'full solution' eigenvalues were established via the full Kron method and a direct assembly method. Tests were then carried out reducing the number of modal freedoms without constraint mode compensations. Loss of accuracy on a similar scale to that experienced with the beam examples of Section 6.2 .3 resulted.

The results of the test cases of TABLE 6.4 are presented in TABLE 6.5 include
(i) up to 8 composite system eigenvalues (as calculated by the Rayleigh quotient)
(ii) percentage eigenvalue error (w.r.t. full solutions)
(iii) variation number (\%)

The value and 'location' of the maximum component normal mode eigenvalue ( $\lambda_{\text {max }}$ ) and the minimum component constraint mode pseudo-eigenvalues $\left(\lambda_{c_{\text {min }}}\right)$ are also given.

For cases $A$ through $E$, the eigenvalue errors and variation numbers are in general within $1 \%$ below $\lambda_{\max }$, while between $\lambda_{\max }$ and $\lambda_{c_{m}}$ accuracy rapidly deteriorates and variation numbers can become $\frac{\min }{\mathrm{v}} \mathrm{ry}$ large.


Overall size of coupled plate $2^{\prime} \times 8^{\prime}$
Component orders $\quad n_{\dot{o}_{1}}=n_{0_{2}}=48$
Plate bending element dimensions $12^{\prime \prime} \times 12^{\prime \prime} \times 0.1^{\prime \prime}$
$E=10^{7} \mathrm{Ib} / \mathrm{in}^{2}, \quad v=\frac{1}{3} \quad, \rho=0.27 \mathrm{lb} / \mathrm{in}^{3}$
Order of base system $=96$
Number of constraints $=12$
Order of composite system $=84$

## TABIE 6.4.

PIATE BENDING EXAMPI.E NO. 1

SPECIFICATION OF TEST CASES

| CASE | FOR EACH COMPONENT |  |  |
| :---: | :---: | :---: | :---: |
|  | NO. OF NORMAL MODES INCLUDED | NO. OF CONSTRAINT MODES INCLUDED | TOTAL NO. OF COMPONENT FREEDOMS |
| A | 1 | 12 | 13 |
| B | 2 | 12 | 14 |
| C | 3 | 12 | 15 |
| D | 4 | 12 | 16 |
| E | 5 | 12 | 17 |
| F | 4 | 9* | 13 |
| ${ }^{*}$ CONSTRAINT MODES EXCLUDED IF $\frac{\mathrm{C}_{2}}{\mathrm{C}_{1}}<10^{-2}$ |  |  |  |

PLATE BENDING EXAMPLE NO. 1 - RESULIS OF TEST CASES

| CASE | A | B | C | D | E | F | FULL <br> SOLUTION |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| ETGENVALUE NO. | Pigenvalues (determined to 5 places) (Calculated from R.Q.) |  |  |  |  |  |  |
| $\begin{aligned} & 1 \\ & 2 \\ & 3 \\ & 4 \\ & 5 \\ & 6 \\ & 7 \\ & 8 \end{aligned}$ | $\begin{gathered} -\overline{74.979} \\ 586.77 \end{gathered}$ | $\begin{gathered} 74.979 \\ -541.80 \\ 586.76 \\ -\frac{2505.6}{7510.3}- \end{gathered}$ | $\begin{gathered} 74.934 \\ 541.80 \\ -568.87 \\ -2220.4 \\ 2455.0 \\ 6439.7 \\ 17646.0 \\ \hline \end{gathered}$ | 74.934 <br> 54.80 <br> 568.86 <br> 2220.4 <br> --240.3 <br> -6264.6 <br> 6477.7 <br> 13822.0 | $\begin{array}{\|c\|} \hline 74.933 \\ 541.80 \\ 568.84 \\ 220.4 \\ 2400.3 \\ 6113.6 \\ 6264.6 \\ \hline 13784.0 \end{array}$ | 74.937 541.80 576.28 2220.4 -2511.2 -26264.5 10215.0 21866.0 | $\begin{gathered} 74.931 \\ 54.80 \\ 568.84 \\ 2204.9 \\ 2389.6 \\ 611.4 \\ 6257.3 \\ 13247.0 \end{gathered}$ |
|  | Eigenvalue error (\% relative to full solution) |  |  |  |  |  |  |
| $\begin{aligned} & 1 \\ & 2 \\ & 3 \\ & 4 \\ & 4 \\ & 5 \\ & 6 \\ & 7 \\ & 8 \\ & \hline \end{aligned}$ | $\begin{gathered} -0.06 \overline{4} \\ -\underline{8.30} \end{gathered}$ | $\left[\begin{array}{c} 0.064 \\ 0.0 \\ 3.15 \\ -\frac{13.64}{214.3}- \end{array}\right]$ | $\begin{gathered} 0.004 \\ 0.0 \\ --\frac{0.005}{0.703}- \\ 2.74 \\ 5.37 \\ -182.0 \\ \hline \end{gathered}$ | 0.004 <br> 0.0 <br> 0.004 <br> 0.703 <br> --0.448 <br> 2.51 <br> 3.52 <br> - | $\begin{array}{\|c\|} 0.003 \\ 0.0 \\ 0.0 \\ 0.023 \\ 0.448 \\ 0.036 \\ --\frac{0.277}{4.13}- \\ \hline \end{array}$ | $\begin{gathered} 0.008 \\ 0.0 \\ 1.310 \\ 0.703 \\ ---\frac{5.090}{2.51} \\ 63.25 \\ 65.06 \\ \hline \end{gathered}$ |  |
|  | Variation no. (\%) |  |  |  |  |  |  |
| $\begin{aligned} & 1 \\ & 2 \\ & 3 \\ & 4 \\ & 5 \\ & 6 \\ & 7 \\ & 7 \end{aligned}$ | $\left.\begin{array}{r} -1.186 \\ -44.123 \end{array}\right]$ | $\left.\begin{array}{r} 1.215 \\ -0.075 \\ 46.551 \\ 4.054 \\ -=40.604 \end{array} \right\rvert\,$ | 0.045 <br> 0.005 <br> $--\frac{0.120}{2.453}$ <br> 24.281 <br> 55.441 <br> 12.171 | 0.052 <br> $-\quad 0.004$ <br> 0.141 <br> 2.860 <br> --0.421 <br> -0.218 <br> 63.928 <br> 20.074 | 0.004 <br> 0.001 <br> 0.014 <br> 0.143 <br> 0.631 <br> 0.360 <br> $--\frac{0.241}{1.492}$ | 1 -0.052 -0.003 0.039 2.863 -0.077 -0.218 29.542 -3.039 |  |
| $\begin{aligned} & \lambda_{\operatorname{MAX}}(--) \\ & \lambda_{C_{M I N}}(-) \end{aligned}$ | $\begin{aligned} & 29.734 \\ & 1226.6 \end{aligned}$ | $\begin{aligned} & 533.93 \\ & 4460.4 \end{aligned}$ | $\begin{array}{r} 1151.4 \\ 19792.0 \end{array}$ | $\begin{array}{r} 5727.0 \\ 27049.0 \end{array}$ | $\begin{array}{r} 9184.4 \\ 51525.0 \end{array}$ | $\begin{array}{r} 5727.0 \\ 41741.0 \end{array}$ |  |

PLATE BENDING FXXAMPLE NO. 1
CONVERGENCE OF COMPOSITE SYSTEM EIGENVALUES (5 TEST CASES)
(PIOT OF EIGENVALUE ERRORS VS. EIGENVALUE NUNBER)

| $\odot$ | CASE | $A$ |
| :---: | :---: | :---: |
| + | CASE | $B$ |
| $\Delta$ | CASE | $C$ |
| $\times$ | CASE | $D$ |
| $\square$ | CASE | E |



In case $B$, the first eigenvalue to exceed $\lambda_{c_{m i n}}$ was calculated to illustrate the total loss of accuracy expected. The excellent convergence of low composite system eigenvalues for increasing number of component freedoms is illustrated in Figure 6.9. A general accuracy criteria might be that
"component normal modes should be included up to the highest composite system $\lambda$ value of interest".
The results for case $F$, where only 9 out of the possible 12 constraint modes are included (omitted modes correspond to the 3 lateral displacements at the connection boundary nodes) show that while some composite system eigenvalues are unaltered, others are affected appreciably. Care must thus be taken when excluding constraint modes.

Finally, it is noted that there is no consistant correlation between variation number and eigenvalue error.

### 6.4. REDUCFION IN THE NUMBER OF CONSTRAINIS

6.4.1. Introduction

In this section, a technique is presented which allows, in certain circumstances, a reduced set of constraints to be employed to connect components in Kron's method, while approximately retaining the low frequency composite system characteristics.

The attraction in being able to reduce the number of constraints which have to be applied is not only that the core requirement for

G ( $\simeq 2 n_{0} m$ ) is reduced but also that the size of $R$ is reduced. While the saving in core for $G$ is roughly linear in $m$, the saving in core space for $\mathbf{R}$, and in the computer effort required for the frequency scanning of $R$ vary as $m^{2}$. Verysignificant savings may thus be made.

If constraints are applied sequentially to the base system, a series of intermediate systems are formed. By Rayleigh's theorem, application of each constraint raises, or at least does not lower, the eigenvalues of the current intermediate system.

With 'simple' constraints, the pattern of "raising" of any eigenvalue is generally unpredicable, as indicated in Fig. 6.10(a). The principle of the technique is to express the $m$ simple constraints in generalised form by utilising ideas from Weinstein's method (Chapter 3), such that application of the first few generalised constraints produces the maximum raising of the lower eigenvalues (Fig. 6.IO(b)).
(a) TYPICAL ETGENVALTEE RATSTNG PAMPERN WITH SIMPLE CONSTRATNTS

(b) DESTRED EIGENVALUE RATSTING PATTIRRN FOR A 'LOW' ETGENVALUE

(c) $\frac{\text { LOCATION OF } 1 \text { PIRST } 2 E R 0^{\prime}, \lambda_{1}^{1}}{\text { IN PLOT OF } \Gamma_{11}(\lambda)}$


The technique is essentially limited to situations where a number of simple constraints link two adjacent components, that is where the value of $m$ is likely to be large anyway. A two-component plate bending example is included to illustrate the efficiency of the technïque.

It is assumed at this stage that the full set of component modes is employed. Hence the approximate eigenvalues obtained will be lower bounds on the exact ones.

### 6.4.2. Generalisation of Constraints

Assuming simple constraints, the composite system frequency equation in Kron's Method is given by

$$
\begin{equation*}
\mathbf{R} \mathbf{c}=\mathbf{P}_{\mathrm{c}}^{\mathrm{t}} \boldsymbol{\Phi}_{\mathrm{c}} \mathrm{D}^{-1} \boldsymbol{\Phi}_{\mathrm{c}}^{\mathrm{c}} \mathbf{P}_{\mathrm{c}} \mathbf{c}=\mathbf{0} \tag{6.4.1}
\end{equation*}
$$

where each column in $\mathbf{P}_{c}$ e.q. $\quad \mathbf{p}_{j}$ defines a set of internal forces in the jth. constraint. The technique involves the definition of a transformation,

$$
\begin{array}{ccc}
\mathbf{c}= & \mathbf{U} & \overline{\mathbf{c}} \\
(\mathrm{mxI}) & \left(\mathrm{mxm}_{g}\right) & \left(\mathrm{m}_{\mathrm{g}} \mathrm{xI}\right)
\end{array}
$$

where $\mathbf{U}$ defines $\mathrm{m}_{\mathrm{g}}$ independant linear combinations of the m simple constraints. Clearly $\mathrm{m}_{\mathrm{g}} \leqslant \mathrm{m}$. Hence a set of $\mathrm{m}_{\mathrm{g}}$ generalised constraints are defined by

$$
\begin{equation*}
\overline{\mathbf{P}}_{\mathrm{c}}=\mathbf{P}_{\mathrm{c}} \mathbf{U} \tag{6.4.3}
\end{equation*}
$$

and the 'reduced' Kron equation is defined by

$$
\begin{equation*}
\overline{\mathbf{R}} \overline{\mathbf{c}}=\overline{\mathbf{P}}_{c}^{t} \boldsymbol{\phi}_{c} \bar{D}^{-I} \boldsymbol{\phi}_{c}^{t} \overline{\mathbf{P}}_{c}=0 \tag{6.4.4}
\end{equation*}
$$

The problem is thus to define the $\mathrm{m}_{\mathrm{g}}$ generalised constraints in $\overline{\mathbf{P}}_{\mathrm{c}}$ such that the desired eigenvalue raising effect of Fig. 6.10 (b) is produced.

### 6.4.3. Natural Constraints

In Weinstein's method for infinite systems (Section 3.4) a finite
series of constraints is applied in lieu of an infinite series to obtain accurate lower bounds on the final system eigenvalues. This suggests the possibility of obtaining the same effect in finite systems having large numbers of constraints.

The basis of this method for defining generalised constraints is the important result (Section 3.3.3) that "if the first constraint is taken as the lowest eigenvector of the base problem, then the lowest eigenvalue is completely raised".

If the first constraint is defined by $\mathbf{p}_{1}$ the Kron determinental equation is given by

$$
\begin{equation*}
\sum_{\lambda=1}^{n_{t}} \frac{\left(\phi_{k}^{t} p_{1}\right)^{2}}{\left(\lambda_{k}^{o}-\lambda\right)}=0 \tag{6.4.5}
\end{equation*}
$$

From Weinstein, complete raising will occur if

$$
\begin{align*}
& \boldsymbol{\phi}_{1}^{t} \mathbf{p}_{1} \neq 0 \\
& \boldsymbol{\phi}_{k}^{t} \mathbf{p}_{1}=0 \quad(k=2, n) \tag{6.4.6}
\end{align*}
$$

Here, $\mathbf{p} l_{1}$ represents a set of internal constraint forces, hence equation (6.4.6) may be given a physical interpretation:
"For complete raising of $\lambda_{1}^{0}$ no work shall be done by the set of force $\mathbf{p}_{1}$ over the base system modes $\phi_{k},(k=2, n) "$.
alternatively
"For complete raising of $\lambda_{1}^{\circ}, \boldsymbol{\phi}_{I}$ must do the maximum amount of work over $\mathbf{p}_{1}^{\prime \prime}$.
This condition is clearly satisfied by $\mathbf{p}_{1}=\phi_{1}$. The choice of $\mathbf{p}_{2}=\boldsymbol{\phi}_{2}, \quad \mathbf{p}_{3}=\boldsymbol{\phi}_{3}$ then defines the series of constraints to be taken.

In our situation, $\mathbf{p}_{1}$ is almost certainly restricted to a subspace of the total space defined by the $n_{t}$ base system freedoms, corresponding to the connection coordinates. Hence the Kron equation is

$$
\begin{equation*}
\sum_{k=1}^{n_{t}} \frac{\left(\phi_{c_{k}}^{t} p_{c_{l}}\right)^{2}}{\left(\lambda_{k}^{o}-\lambda\right)}=0 \tag{6.4.7}
\end{equation*}
$$

The set of $n_{t}$ partial eigenvectors $\phi_{c_{k}}$ are Linearly dependent to degree $\left(n_{t}^{-} m\right)$, hence it is not possible in general to choose a $\mathbf{p}_{c_{~}}$ vector to satisfy the equivalent of equation 6.4.6. and produce ${ }_{c o m p}{ }^{c_{1}}$ te raising. Instead, $\lambda_{I}^{I}$ satisfies

$$
\lambda_{1}^{0} \leqslant \lambda_{1}^{I}<\lambda_{2}^{0}
$$

and is defined by (see Fig. 6.10(c))

$$
\begin{equation*}
\frac{\left(\phi_{c_{1}}^{\mathrm{t}} \mathbf{p}_{c_{1}}\right)^{2}}{\lambda_{1}^{I}-\lambda_{0}^{0}}=\sum_{k=2}^{n_{t}} \frac{\left(\phi_{c_{k}}^{t} \mathbf{p}_{c_{1}}\right)^{2}}{\lambda_{k}^{0}-\lambda_{1}^{1}} \tag{6.4.8}
\end{equation*}
$$

By inspection of this equation, it may be seen that choosing $p_{c_{1}}$ to
do the maximum amount of work over $\quad \boldsymbol{\phi}$ will permit a near do the maximum amount of work over maximum value of,$\lambda_{1}^{l}$. Hence choose

$$
\begin{equation*}
\mathbf{p}_{c_{1}}=\phi_{c_{1}} \tag{6.4.9}
\end{equation*}
$$

Additional constraints are then defined by $\overline{\mathbf{p}}_{c_{2}}=\boldsymbol{\phi}_{c_{2}}$ etc. Although not strictly necessary, it is convenient to form an orthogonal set of constraints by the Gramm-Schmidt scheme as follows: Normalise $\mathbf{p}_{c_{l}}$ initially by

$$
\begin{equation*}
p_{c_{1}}^{t} p_{c_{1}}=1 \tag{6.4.10}
\end{equation*}
$$

Then

$$
\begin{equation*}
\overline{\mathbf{p}}_{\mathrm{c}_{2}}=\mathbf{p}_{\mathrm{c}_{2}}+\alpha_{1} \mathbf{p}_{\mathrm{c}_{1}} \tag{6.4.11}
\end{equation*}
$$

Utilising the condition

$$
\begin{equation*}
\mathbf{p}_{c_{1}}^{t} \mathbf{p}_{c_{2}}=0 \tag{6.4.12}
\end{equation*}
$$

yields

$$
\alpha_{1}=\mathbf{p}_{c_{1}}^{t} \overline{\mathbf{p}}_{c_{2}}
$$

Hence $\quad \mathbf{p}_{c_{2}}$ is calculated from equation (6.4.11) and normalised by
$\mathbf{p}_{c_{2}}^{t} \mathbf{p}_{c_{2}}{ }^{c_{2}} \mathbf{1}$, and the process extended for $\overline{\mathbf{p}}_{c_{3}}$ etc..
The advantage of this orthogonalisation is that less important constraints where the ratio of post-orthogonalisation vector magnitude to that before orthogonalisation is less than a given tolerance may be cut out.

It is, of course, possible to generate a complete set of $m$ generalised constraints, however the intention is to stop short of this and use a reduced number of constraints. Such generalised constraints are here termed NATURAL CONSTRAINTS. This theory is one possible way of defining a reduced set of constraints which approximately represent the force patterns between components at low frequencies.

The technique has been developed here in general terms. An example is now given to illustrate the practical construction and effectiveness of natural constraints.

### 6.4.4. Plate Bending Example No. 2

The example (Fig. 6.11(a)) consists of 2 cantilever plate components to be connected along the line XX. The situation differs from that of plate example No. 1 in that there are a relatively large number of simple constraints ( $m=24$ ) coupling the components. The same 16 degree of freedom plate bending elements were employed, and the base system comprises 144 freedoms.

A set of $m_{g}(<m)$ natural constraints $P_{c}^{N}$ is defined using the partial eigenvectors of component $A$. The same set of constraints are applied to component $B$, hence the $\mathbf{G}$ matrix is defined by

$$
=\left[\begin{array}{l}
\mathbf{G}_{A} \\
\mathbf{G}_{B}
\end{array}\right]=\left[\begin{array}{cc}
\boldsymbol{\Phi}_{C}^{A} & 0 \\
0 & \boldsymbol{\phi}_{C}^{B}
\end{array}\right]\left[\begin{array}{r}
\mathbf{P}_{C}^{N} \\
-\mathbf{P}_{C}^{N}
\end{array}\right]=\left[\begin{array}{rr}
\boldsymbol{\phi}_{C}^{A} & \mathbf{P}_{C}^{N} \\
-\boldsymbol{\Phi}_{C}^{B} & \mathbf{P}_{C}^{N}
\end{array}\right]
$$

If orthogonal natural constraints are used, certain terms in $\mathbf{G}_{\mathrm{A}}$ will be zero. In this example, component $\mathrm{B}^{\prime}$ s eigenvector are merely a mirror image of those of component $A$, so the zeros will appear in $\mathbf{G}_{B}$ also. The natural frequencies of the components of course coincide.

## FTCURE 6.11.

(a) 2 COMPONENT PLATE BENDING EXAMPIE (NO.2)



COMPONETTT 1

$a=8^{\prime \prime}$
$t($ thickness $)=0.1^{11}$
$E=10^{7} 1 b / i n^{2}$
$b=7.2^{\prime \prime}$
$\mathrm{P}=0.27$
$\nu=\frac{1}{3}$
Component orders: $n_{o_{1}}=n_{o_{2}}=72$
Total number of base system freedoms: $n=144$
Number of simple constraints: $m=24$
(b) ILLUSTRATION OF SYMMETRIC AND ANTTSYMMETRTC NATURAL CONSTRAINTS


NO PARTITIONING
ANTISYMMETRIC
(w.r.t. boundary)


NO. OF NATURAL CONSTRATNIS APPLIED

Although the partial eigenvectors $\quad \phi_{c_{j}}^{A}$ may be used directly, it is physicaliy better to partition according $\begin{aligned} & \text { \# }\end{aligned}$

$$
\phi_{c_{i}}^{A}=\left\{\begin{array}{cc}
\hat{\phi}_{c_{i}}^{A} & \hat{\phi}_{c_{i}}^{A}
\end{array}\right\}
$$

where $\quad \hat{\boldsymbol{\phi}}_{c_{i}}^{A}$ corresponds to SYMMETRIC connection displacements and corresponds to ANTI-SYMMETRIC ones. If a mirror is considered to be placed along the line $X X$, symmetric displacements (here $w, \frac{\partial_{W}}{\partial y}$ ) do not appear to change sign in the mirror image, while anti-symmetric ones $\left(\frac{\partial w}{\partial x}, \frac{\partial^{2} w}{\partial x \partial y}\right)$ do. Each partial eigenvector thus produces 2 natural constraints in general (Fig. 6.11(b)).

The lowest 6 composite system natural frequencies were determined on application of $1,2,3, . \& . . ., 12$ natural constraints, while the full set of 24 constraints were used to generate exact frequencies.

The effect of the constraint applications on system natural frequencies is shown in (Fig. 6.12). As expected convergence is monotonic from below, and all frequencies are obtained to within $2 \%$ accuracy after 6 constraints. Convergence to composite system eigenvalues became progressively slower as frequency increased. As a result of the mirror imaged base system components, the base system natural frequencies are of multiplicity 2. One of these frequencies must in each case remain unchanged on application of the first constraint by Rayleigh's theorem.

To emphasise the advantages of a reduced number of constraints the core requirement and scanning time for the $\mathbf{R}$ matrix are compared for the cases of 6 natural constraints and 24 simple constraints. The lower triangle of $\mathbf{R}$ (fully populated) required 21 and 300 locations respectively, while the former's scanning time for the first 6 frequencies was a factor of 11.3 shorter. The time spent in generating the natural constraints was minimal.

### 6.4.5. Further Discussion

It is anticipated that the generalisation of constraints by the method of natural constraints is limited to situations where a large number of simple constraints perform the same topological
connection. Each constraint set in the base system may be treated separately with the possibility of a reduction from $m_{0}$ simple. constraints to $\mathrm{m}_{\mathrm{o}}$ generalised constraints.

Although the above example uses components which are merely mirror images of each other, the technique is immediately extendable to the coupling of unlike components.

Where the partial eigenvector shapes from all components to be joined along a given boundary are reasonably similar, then natural constraints may be generated with reference to either components. Alternatively, natural constraints could be generated for all components and a procedure devised to form an orthogonal set using the "lowest" few constraints from each. However, where a flexible component is to be attached to a relatively stiff component, for example the connection of an aircraft wing to the fuselage, it is essential to use the modes from the flexible component.

### 6.5. THE HYBRID APFROXIMATION METHOD

### 6.5.1. The Hybrid Method

The approximation techniques of Sections 6.3 and 6.4 may be used simultaneously to produce THE HYBRTD KRON APPROXIMATE METHOD for representation of the low composite system eigenspectrum. : The reduction in component mode numbers implies an incomplete description of displacement over the components, while the reduction in the number of constraints implies an incomplete description of the forces linking the components.

The resulting approximate eigenvalues must lie between those of the constituent approximations. Consider a base system comprising full sets of $n_{t}$ modal freedoms and $m$ constraints, and let the hybrid approximation
 If $\lambda_{i}{ }^{n}, m$ are the full solution eigenvalues, and ${ }_{i} \lambda_{i}{ }^{n},{ }^{m}, \lambda^{n^{n} t}{ }^{m} g$ are respectively, solutions with the modal approximation only and the constraint approximation only then we have

$$
\begin{equation*}
\lambda_{i}^{n^{\prime}},{ }^{m} g \leqslant \lambda_{i}^{n_{t}}, m \quad \leqslant \lambda_{i}^{n^{\prime}, m} \tag{6.5.1}
\end{equation*}
$$

According to Rayleigh's theorem, the hybrid solutions $\lambda^{n} g^{\prime, m} g$ must satisfy

$$
\lambda_{i}^{n_{t}}, m_{g} \leqslant \lambda_{i}^{n_{i}}, m_{g}
$$

as a reduction in modal freedoms is involved. Equally, they must satisfy

$$
\lambda_{i}^{n^{m} g} \leqslant \lambda_{i}^{n^{\prime}, m}
$$

as a reduction in the number of constraints is involved. Hence the desired result

$$
\lambda_{i}^{n_{t}, m^{m}} \leqslant \lambda_{i}^{n^{m} g} \leqslant \lambda_{i}^{n^{m}, m}
$$

is obtained. The hybrid solution is thus no worse than the least accurate of the constituent approximate results. Indeed, the effect of the constituent approximation is opposite, and thus tends to cancel out. However, convergence will no longer in general be monatonic and it is impossible to say whether the resulting eigenvalues are an upper or lower bound on the full solution eigenvalue.

In practical terms, the combination of the constituent approximations is intuitive, and enables the core space and time savings of both methods to be utilised. The procedure is indicated in Fig. 6.13 for a component of order $n_{0}$ with a connection boundary involving $m_{0}$ constraints.

Initially $n_{n_{0}}$ normal modes are obtained where usually $n_{n_{0}} \ll n_{0}$. Natural constraints are formed using all (or a subset) of these modes, number $m_{g_{0}}\left(<m_{0}\right)$ usually. These natural constraints specify the force patterns at the component boundary, hence it is logical to determine constraint modes which correspond to these patterns. Hence $n_{c_{o}}$ constraint modes are formed where $n_{c_{0}}<{ }^{m} \mathrm{~g}_{0^{\circ}}$ This has the advantage that the number of constraint modes is minimised.

The possibility then exists of utilising the constraint modes to define further natural constraints. Such constraints, when orthogonalised w.r.t. the existing natural constraints will be in general of high order, and the resulting increase in the number of constraints has not been found to justify the increase in accuracy obtained.

### 6.5.2. Application to Plate Bending Example No. 2

The hybrid method was employed to calculate the lowest 5 frequencies and modes of plate bending example no. 2 (Figure 6.11(a)) for various numbers of component modes as indicated in TABLE 6.6.

## FIG. 6.13

THE ORGANIZATION OF THE HYBRID APPROXIMATION METHOD
(Procedure For A Typical Component/Constraint Set)


TABLE 6.6

| APPROXIMATION CASE | FOR EACH COMPONENT |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
|  | NO. OF NORMAL MODES $n_{n_{0}}$ | NO. of natural CONSTRAINTS ${ }^{m} g_{0}$ | NO. OF CONSTRAIN MODES ${ }^{n}{ }^{c}$ 。 | TOTAI NO. OT DISPIACEMENT FREEDOMS $n_{g_{0}}=n_{n_{0}}+n_{c_{0}}$ |
| A | 2 | 4 | 4 | 6 |
| B | 3 | 6 | 6 | 9 |
| C | 4 | 8 | 8 | 12 |
| D | 5 | 10 | 10 | 15 |
| E | 6 | 12 | 12 | 1.8 |
| FULL SOLUTION | 72 | 24 SIMPLE CONSTRAINTS | 0 | 72 |

The convergence of the frequencies with increasing numbers of modal freedoms and constraints is indicated in Fig. 6.14. It may be seen that the first 5 frequencies are determined to within $\pm 2 \%$ in case C which utilises 12 out of 72 component freedoms and 8 out of 24 constraints. All frequencies obtained were greater or equal to those of Section 6.4.4 which correspond to the constraint approximation only.

The possibility of convergence from above and below is indicated, and indeed an increase in the number of modal freedoms/constraints may even cause a slight deterioration in accuracy (cf. mode 5, cases B and C). In this example, 4 out of 5 frequencies are "low" in case $E$, indicating that the constraint approximation is the more serious one.

The computational implications of the method are fully discussed in Chapter 8. However, it may be noted here that while the $G$ matrix for the full solution of this example is of dimensions ( $144 \times 24$ ), i.e. 3456 locations, the case $C$ here utilises a $G$ matrix of size ( $24 \times 8$ ), or 192 locations. The run time for case $C$ is approximately one-fifth of that for the full solution.


### 6.6. THE CONSTANT PART APPROXIMATION

This technique is a device for speeding up the scanning of
$|\mathbf{R}(\lambda)|$ when $\lambda$ is restricted to a low eigenvalue range, say
$0 \leqslant \lambda \leqslant \lambda_{1}$ It is applicable to either the full or approximate Kron methods.

Each term in $\quad \mathbf{R}(\lambda)$ is given by

$$
\begin{equation*}
r_{i j}=\sum_{k=1}^{n_{g}} \frac{g_{k i} g_{k j i}}{\lambda_{k}^{o}-\lambda} \tag{6.6.1}
\end{equation*}
$$

The terms in this series for which $\lambda_{k}^{\circ} \gg \lambda_{1}$ will remain essentially constant as $\lambda$ varies over the range defined above. Hence the series may be divided into a constant part $r_{i j}^{c}$, and a variable part $r_{i j}^{v}$ where

$$
\begin{equation*}
r_{i j}^{v}=\sum_{k=1}^{n_{1}} \frac{g_{k j} g_{k j j}}{\left(\lambda_{k}^{o}-\lambda\right)}, \quad r_{i j}^{c}=\sum_{k=n_{1}+1}^{n_{g}} \frac{g_{k j} g_{k j}}{\lambda_{k}^{o}} \tag{6.6.2}
\end{equation*}
$$

and

$$
r_{i j} \simeq r_{i j}^{v}+r_{i j}^{c}
$$

The terms included in $r_{i j}^{c}$ are those which satisfy

$$
\lambda_{k}^{0}>\bar{\lambda}
$$

where, for example, $\bar{\lambda}$ may be defined by

$$
\bar{\lambda}=\lambda_{1} \times 10^{2}
$$

In this case the maximum error induced in any term in $r_{i j}^{c}$ is $1 \%$. For a given $\bar{\lambda}$, a constant part of the $\mathbf{R}$ matrix may be formed for all time as $\mathbf{R}_{c}$. Thus

$$
\mathbf{R}(\lambda) \simeq \mathbf{R}_{\mathrm{v}}(\lambda)+\mathbf{R}_{\mathrm{c}}
$$

and for each trial $\lambda$ value, only $\mathbf{R}_{\mathrm{V}}(\lambda)$ need be evaluated. However storage will be required for $\mathbf{R}_{c}$ in addition to that for $\mathbf{R}(\lambda)$. Clearly the saving in time for each set up of $\mathbf{R}(\lambda)$ depends on the
factor ${ }^{n_{l}} / n_{g}$.
The effect of this "numerical" approximation will in general be to raise the composite system eigenvalue. The worst error may be assumed to lie in diagonal terms $r_{i i}^{c}$, as all terms therein are positive, hence the errors are additive. With $\lambda=0, r_{i i}^{c}$ will be an under-estimate of the true sum, hence the effect is analogous to a small truncation of the series (see Section 6.2).

A numerical illustration of the approximation is given in Section 8.5.

## CHAPTER 7

## THE COMPUTATIONAL IMPLEMENTATION OF KRON'S METHOD

### 7.1. SYSTEM ORGANISATION

### 7.1.1. The Systems EIG1 and EIG?

In this chapter, computational systems to implement both the full and approximate Kron methods, designated EIGl and EIG2 respectively, are briefly described. Both systems comprise an essentially similar suite of programs written in FORTRAN 4 and developed on the CDC 6400 computer at Imperial College, London University.

A detailed description of the systems, with particular emphasis on data supplied by the user and core space requirements, is contained in APPENDIX 1. Discussion in this chapter is thus limited to the overall system organisation and the major concepts and procedures used. The use of computer program symbols is limited to the appendix, where a reference list is included.

The computational efficiency of the systems is investigated in the following chapter.

### 7.1.2. System Features

The fundamental aim in the application of the theory was the minimisation of central memory requirements to permit as large a problem as possible to be handled effectively 'in core'. This philosophy requires that all major computing operations e.g., factorisation, matrix multiplication should be carried out 'in core', and that transfers to and from backing store should be minimised. Two major features are directed towards this aim.

Firstly, the attraction of Kron's method is perhaps greatest when the base system contains repetition, both in terms of components and linking constraints. In Section 7.2 the concept of a 'minimum data set' for the description of the base system is introduced. The computational systems are designed to use this minimum data set efficiently, in particular with respect to the formation of the $\mathbf{R}(\lambda)$ matrix.

Secondly, the nature of the technique allows the total process to be conveniently divided into four parts:

> Part l - The analysis of the base system Part 2 - Calculation of composite system eigenvalues Part 3 - Calculation of composite system eigenvectors Part 4 - Calculation of composite system generalised  mass and stiffness terms

This sub-division allows each part to be 'overlayed', that is, the program for each part is located in the same core area. The re-use of storage arrays for different purposes in each part further reduces overall core requirements. To facilitate this last feature, data produces by each part is written to disc store and only read back into core when required by subsequent parts. The basic system organisation and disc transfers for EIGI/EIG2 are shown in Figure 7.1.

### 7.1.3. The Overlay Structure

The overlay structure for EIGI/EIG2 is shown in Figure 7.2. The main overlay is resident in core throughout, and serves to call the primary overlays, record overall timings, and to control how far a given run proceeds. For example, if only natural frequencies are of interest, the systems may be terminated automatically after the completion of part 2.

The four primary overlays correspond to the four parts defined in the previous section. Once a primary overlay is completed, the following primary overlay and associated subroutines are loaded into the same core area. In addition, the first primary overlay calls upon two secondary overlays.

An important advantage of this subdivision is that it is simple to run each part as a self contained program with data transfers via permanent files. This may well be advantageous for large jobs and is a feature not so readily available with many other eigenvalue techniques.

### 7.2. STRUCTURAL SYSTEM CONCEPTS

### 7.2.1. The Minimum Data Set (M.D.S.)

The efficiency of Kron's method often depends largely on identifying a base system with repetition. This repetition is made use of in the computational systems by defining a 'minimum data set', necessary for the definition of the $\mathbf{R}(\lambda)$ matrix.

The base system may be thought of in schematic terms, as in figure 7.3(a); boxes represent components while arrows represent constraint sets. A

FIGURE 7.1.
SYSTEMS EIGI/SIG2 - BASIC ORGANISATION AND
USE OF DISC STORAGE

PART 1


constraint set is taken to be any number of constraints performing the same topological connection. However, to accommodate generalised constraints, a set should be further restricted to refer to one particular boundary region for each component involved.

It is inevitable that in any practical computer implementation, restrictions must be made. It is here assumed that all constraints are expressible as simple constraints initially i.e., $x_{i}=x_{j}$ (or $x_{i}=0$ ). This restriction still enables a large class of problems to be handled.

Fach component is assigned a unique identification (C1, $\mathrm{C}_{2}$ etc.), and likewise each constraint set (S1, S2 etc.). In the example of Figure 7.3(a), there are three unique components, in the sense that the remaining components are simply these translated and/or rotated in space. Each unique component selected is termed a PRIMARY COMPONENT (P.C.) and denoted by PC1, PC2, PC3 (Figure 7.3(b)).

The $\boldsymbol{R}(\lambda)$ matrix is defined by the equation

$$
\begin{equation*}
\mathbf{R}(\lambda)=\mathbf{G}^{t} \mathbf{D}^{-1}(\lambda) \mathbf{G} \tag{7.2.1}
\end{equation*}
$$

Clearly, the eigenvalues obtained from the primary component eigenreductions will be sufficient to fully define $\mathbf{D}(\lambda)$.

Each constraint set, in general, refers to two components, and defines a set of columns in the $\mathbf{G}$ : matrix. For a set of simple constraints connecting two components $A$ and $B$, these columns are defined by

$$
\left[\begin{array}{c}
\mathbf{G}_{A}  \tag{7.2.2}\\
\mathbf{G}_{B}
\end{array}\right]=\left[\begin{array}{cc}
A_{1} \boldsymbol{\Phi}_{c}^{t} & 0 \\
0 & { }^{B} \\
\boldsymbol{\phi}_{c}^{t}
\end{array}\right]\left[\begin{array}{c}
\mathbf{I} \\
-\mathbf{I}
\end{array}\right]
$$

The sub-matrix $G_{A}$ corresponds to the linking to the constraint set with component A, and is identified as a PARTIAL CONSTRAINT SEI (P.C.S.). Thus each constraint set in Figure $7.3(a)$ may be divided into two constituent partial constraint sets, and the unique set of these $G$ sub-matrices defined by PSI, PS2 etc.

Figure 7.3(b) is now the schematic for the MINIMUM DATA SET (M.D.S.) required to define $\mathbf{R}(\lambda)$. Each primary component contributes a set of eigenvalues for the definition of $\mathbf{D}(\lambda)$ and has at least one partial constraint set derived from its eigenvectors. Each partial constraint set

## FIGURE 7.3

(a) A TYPICAL BASE SYSTEM SCHEMATIC

(b) THE CORRESPONDING MINIMUM DATA SETT SCHEMATIC

$\begin{array}{ll}\text { PCI - PC3 } & \text { PRIMARY COMPONENTS } \\ \text { PSI }=P S 7 & \text { PARTIAL CONSTRAINT SETS }\end{array}$
(c) MTNIMUM DATA SET - TABULAR REPRESERTATION OF STORAGE REQUIREMENTS

| PRIMARY COMPONIENTS <br> (EIGENVALUE SETS) | PCl |  |  | PC 2 |  |  | PC3 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| PARTIAL CONSTRAINT SETS <br> ( G SUB-MATRICES) | PS1 | PS2 | PS3 | PS4 | PS5 | PS6 | PS7 |

contributes a unique sub-matrix of $\mathbf{G}$ (Figure 7.3(c)).
In general, each component will have some coordinate transformation to be applied to the partial constraint set matrices. In the interests of simplicity, this facility is not included in the current work.

### 7.2.2. A Simple Beam Example

To illustrate the procedure for defining the minimum data set, Figure 7.4(a) shows a simple free beam comprising four two-dimensional beam elements.

The first step is to break the composite system into components (Figure 7.4(b)). In the example, each element is considered to be a component, hence the four components are identical. To reconstitute the composite system, constraints are laid between the components. With the numbering system of Figure 7.4(b), these constraints are

$$
\begin{align*}
& x_{3}=x_{5} \\
& x_{4}=x_{6}
\end{align*}
$$

between the two left-hand component etc..
Figure 7.4(c) shows this base system in schematic form. Thus the components are identified C1, C2, C3, C4, and each of the constraint sets S1, S2, S3 comprises two constraints each. Those for Sl are given above in Equation (7-2.3).

The user is now in a position to extract the minimum data set schematic (Figure 7.4(d)) which in this case comprises one primary component and two partial constraint sets, the first of which refers to local component displacements $x_{3}$ and $x_{4}$, and the second to $x_{1}$ and $x_{2}$. Figure 7.4(e) finally gives a diagrammatic representation of the M.D.S. storagè requirements.

### 7.2.3. The Formation of the $\quad \mathbf{R}(\lambda)$ Matrix

The use of the minimum data set in the formation of the $\mathbf{R}(\lambda)$ matrix is illustrated by reference to the beam example of the previous section.

Assuming that the eigenvalues of PCl are contained in $\mathbf{D}_{1}(\lambda)$, and that the $G$ sub-matrices corresponding to PSI and PS2 are given by $\mathbf{G}_{1}$ and $\mathbf{G}_{2}$ respectively, by reference to the base system schematic

## FIGURE 7.4.


(a.) 4 element free-free 20 Beam

(b) The base system (4 identical components)

(c) The base system schematic

$$
\mathrm{PS} 2 \rightarrow \mathrm{PCl} \leftarrow \mathrm{PSl}
$$

(d) The minimum data set schematic

| PRIMARY COMPONENTS <br> (EIGENVALUE SETS) | PC1 |  |
| :--- | :--- | :--- |
| PARTIAL CONSTRAINT SETS <br> (G SUB-MATRICES) | PS1 | PS2 |

(e) Storage for minimum data set
of Figure 7.4(c) it may be verified that:

$$
\mathbf{D}^{-1}=\Gamma \mathbf{D}_{1}^{-1} \mathbf{D}_{1}^{-1} \mathbf{D}_{1}^{-1} \mathbf{D}_{1}^{-1}, \downarrow
$$

and

$$
G=\left[\begin{array}{ccc}
G_{1} & 0 & 0  \tag{7.2.4}\\
-G_{2} & G_{1} & 0 \\
0 & -G_{2} & G_{1} \\
0 & 0 & -G_{2}
\end{array}\right]
$$

Forming the $\mathbf{R}(\lambda)$ matrix according to equation (7.2.1), a further advantage emerges. The $\mathbf{R}(\lambda)$ matrix is formed from a small number of sub-matrices or 'blocks' of the form $G_{i} D_{k}^{-1} G_{j}$. Thus

$$
R(\lambda)=\left[\begin{array}{ccc}
\left(A_{1}+A_{2}\right) & -A_{3}^{t} & 0 \\
-A_{3} & \left(A_{1}+A_{2}\right) & -A_{3}^{t} \\
0 & -A_{3} & \left(A_{1}+A_{2}\right)
\end{array}\right]
$$

where

$$
\begin{aligned}
& \mathbf{A}_{1}=\mathbf{G}_{1}^{t} \mathbf{D}_{1}^{-1} \mathbf{G}_{1} \\
& \mathbf{A}_{2}=\mathbf{G}_{2}^{t} \mathbf{D}_{1}^{-1} \mathbf{G}_{2} \\
& \mathbf{A}_{3}=\mathbf{G}_{1}^{t} \mathbf{D}_{1}^{-1} \mathbf{G}_{2}
\end{aligned}
$$

(N.B. All the $\mathbf{A}_{i}$ matrices, like the $\mathbf{D}_{\mathrm{k}}$ matrices are functions of $\lambda)$.

As the $\quad \mathbf{R}(\lambda)$ matrix must be constructed for every trial $\lambda$ during frequency scanning, it is highly desirable to make this process as efficient as possible. The identification of repeated blocks clearly reduces computer effort. The form of the $\mathbf{R}(\lambda)$ is fixed by the configuration of constraint sets in the base system, as each non-zero sub-matrix in $\mathbf{R}(\lambda)$ represents the linking between two constraint sets. Thus, once data describing the base system is specified, the form of $\mathbf{R}(\lambda)$ may be established for all time. The numbering of the
base system constraint sets, i.e. $\mathrm{Sl}, \mathrm{S} 2$ should be chosen to produce the minimum band in $\quad F(\lambda)$ to further reduce core requirements.

### 7.3. PRINCIPAL PROGRAM FEATUURES

7.3.1. Part 1 - Analysis of the Base System (Section A1.2)

Part I of the systems are responsible for forming
and storing the information required for the M.D.S.. The user directly supplies details of the primary components and their associated partial constraint sets. Both systems access a library of finite elements.

In EIGI, the full lower triangles of the primary component mass and stiffness matrices are stored to enable the extended Jacobi method to be used. The complete set of eigenvalues and eigenvectors are calculated, and the former, together with the $\mathbf{G}_{i}$ matrices are stored on disc.

In FIG2, the variable bandwidth scheme is used for the primary component mass and stiffness matrices, and simultaneous iteration is employed to calculate the lowest $n_{n}$ normal modes. This method is highly suitable in that the mass matrix is ${ }^{\circ}$ left in tact while the factorisation of the stiffness matrix is required anyway for the calculation of constraint modes. The user may reduce the number of constraints in any partial constraint set by converting to natural constraints. Subsequently, constraint modes corresponding to the constraints included, will be calculated and orthogonalised. Tolerances within the program control the cutting out of unwanted natural constraints or constraint modes. The parameters $\lambda_{\min }, \lambda_{\max }$ and $\lambda_{c_{\text {min }}}$ are set automatically within EIG2.

The principal core space requirement in part 1 corresponds to the component mass and stiffness matrices.. Although this will be less in the case of EIG2, additional space for the normal and constraint modes is required.

### 7.3.2. Part 2 - Calculation of Composite System Eigenvalues. (Section A1. 3)

Essentially, Part 2 is responsible for the formation and scanning of $|R(\lambda)|$. The M.D.S. information is loaded into core from disc and the data describing the base system in terms of primary components and partial constraint sets is read in. This information is all that is.
required for the definition of the form of $\mathbf{R}(\lambda)$, and data is then constructed for the economical set up of this matrix, which is held in variable bendwidth form.

A list of unique base system eigenvalues in ascending order is constructed and the composite system eigenvalues determined by use of the extended Sturm sequence algorithm (Section 4.5.2). This involves forming $\mathbf{R}(\lambda)$ for each trial value of $\lambda$ and obtaining its sign count via Choleski factorisation. The scanning procedure is fully described in Section Al. 3.3 of APPENDIX 1. Essentially, the multiplicity at each unique base system eigenvalue is determined, and the range between 2 adjacent base system eigenvalues investigated. Composite system eigenvalues in this range must appear as roots of $|\mathbf{R}(\lambda)|$. Bisection is used to isolate simple zeros whence more powerful algorithms may be used to home in on the root. The modified successive linear interpolation algorithm of Brent [65] has been used for this purpose (Section A.1.3.4).

The facility exists to calculate all composite system eigenvalues (in the case of EIGI), the lowest $n_{r}$ eigenvalues, or just those in a given range $\lambda_{1} \leqslant \lambda \leqslant \lambda_{2}$. The 'constant part' approximation may also be specified. In EIGI, $\bar{\lambda}$ is read in, while in EIG2 it is set equal to $\lambda_{\text {max }}$ so that the terms in the 'constant part' correspond only to constraint mode eigenvalues.

In EIG2 scanning is automatically halted once $\lambda_{\text {max }}$ is exceeded while in EIGI, the 'constant part' approximation is cancelled (if in use) if the current trial $\lambda$ comes within an order of $\bar{\lambda}$.

The principal core requirement in Part 2 is for the $\mathbf{R}(\lambda)$ matrix (stored in the same locations as the component mass and stiffness matrices), the 'minimum data set', and the lists of unique base system eigenvalues and multiplicities.

### 7.3.3. Part 3 - Calculation of Composite System Eigenvectors (Section A1.4)

It is convenient from a system organisation point of view to calculate all the eigenvalues required and then to proceed to the vector calculations. The user may then obtain those eigenvectors which appear of interest.

Part 3 essentially implements the theory of Section 5.4 for the calculation of eigenvectors of both persistant and gained eigenvalues.

Multiple eigenvectors corresponding to the same eigenvalue are calculated.

The primary component modes are recalled from disc to enable conversion of the modal eigenvector to physical displacements, the core space used for the $\mathbf{R}(\lambda)$ matrix being reused. Additional core space is required only for a pair of modal and physical eigenvectors.

### 7.3.4. Part 4 - Calculation of Generalised Mass and Stiffness Terms

The calculation of the generalised terms corresponding to each composite system eigenvector is straightforward, the primary component mass and stiffness matrices being recalled from disc.

The ratio of generalised stiffness to generalised mass yields $\lambda_{R Q}$ for that mode, and the variation number (Section 6.3.3.) is automatically calculated in the case of HIG2.

## CHAPTER 8

## THE COMPUTATIONAL EFFICIENCY OF KRON'S METHOD

### 8.1. INTRODUCTION

In this chapter, the full and approximate Kron methods are reviewed and their computational efficiency investigated. Comparisons are made with direct assembly methods and with the component mode method described in APPENDIX 2.

The capability of Kron's method to economically analyse repetitive structures and to conveniently handle displacement boundary conditions is illustrated with reference to a stiffened plate example.

### 8.2. THE FULL KRON METHOD

8.2.1. General Review

Where accurate eigenvalues are required over a wide range of $\lambda$, methods involving no inherent approximation, save that due to the initial discretisation, are necessary. For large order systems, transformation methöds are unsuitable and possible techniques remaining include
(i) Gupta's sturm sequence method [22] (Section 2.3.4)
(ii) The Full Kron Method (Chapter 5)

The full Kron method is the only practical technique for handling large structures in the piecewise manner, yet retaining full accuracy. There is no need to assemble the complete structural mass or stiffness matrices, and provided the number of constraints, $m$, required to couple the components is much less than the assembled problem size, the technique places relatively low demands on core space (Section 5.6). The composite system eigenvalue problem is non-algebraic, but may be conveniently solved via the extended Sturm sequence algorithm, (Section 5.5.5). The approach is thus competitive computer time-wise.

The piecewise approach allows the recognition of repeated components and constraint sets. The effort involved in the analysis of the components, and of the composite system frequency matrix may thus be minimised (Section 7.2).

### 8.2.2. Computational Requirements

The efficiency of a particular method may be judged on the basis of
computer time and core space requirements. For the former, it is usually sufficient to obtain an estimate of the number of multiplication operations involved.

The convenient implementation of the Kron method has been discussed in general terms in Chapter 7, and individual core requirements are formulated in APPFNDIX 1. These core requirements are summarised here in TABLF 8.2(a). All symbols may be referenced in the glossary. The difficulty whereby core space is reused for several purposes is overcome by the specification of three major areas, $C_{1}, C_{2}$ and $C_{3}$. The overall maximum requirement; established in equation (8.2.1), is highly dependent on $n_{0}$ (here taken to be the maximum component order). However it is independent of $r$, the number of composite system eigenvalues/ eigenvectors required. The minimisation of $m_{p s}$, the number of partial constraint sets, is particularly beneficial. The $R(\lambda)$ matrix is stored in variable bandwidth form, and the expression for $C_{R}$ utilises an 'average semi-bandwidth $d$ '. The requirements of the extended Jacobi transformation method are contained in TABLE 8.1.

Expressions for the number of multiplications involved are shown in TABLE $8.2(b)^{\dagger}$. The expression for the scanning of $\mathbf{R}(\lambda)$ deserves comment. Assuming $n_{0}$ non-zero terms in each series for $r_{i j}$, the number of multiplications per matrix formation is $2 m \mathrm{an}_{0}$. Factorisation of $R(\lambda)$ involves a further md ${ }^{2}$ multiplication. $A s 2 n_{0}>d$ usually, the set up time exceeds factorisation time. Recognition of repeated blocks within $\mathbf{R}(\lambda)$ (Section 7.2.3) and the 'constant part approximation' (Section 6.6) are techniques by which the time taken for this operation may be greatly reduced and a factor of $1_{2}$ is included here in view of these features. The major effort is in the component eigenreductions (proportional to $n_{o}$ and to $n_{p c}$ ) and in the scanning of the $\mathbf{R}(\lambda)$ matrix (proportional to $m$ and $r$ ). The work involved in the formation of the $G$ matrix terms is ignored.

### 8.2.3. Comparison with Gupta's Method

A theoretical comparison of computational requirements between Kron's method and Gupta's method is given for the 2-dimensional frame structure, of figure 8.1(a). Each member is considered to comprise 2 axial-flexural beam elements possessing six freedoms each. The fully
\# In all such expression, the effort involved in forming mass and stiffness matrices is neglected.

TABLE 8.1.

COMPUTATIONAI REQUIREMENTS FOR DIRECT
ASSEMBLY METHODS

| METHOD | APPROXIMATE CORE REQUIREMENTS (WORDS) | $\begin{gathered} \text { APPROXIMATE } \\ \text { NUMBER OF } \\ \text { MULTIPIICATIONS } \end{gathered}$ | COMMENTS |
| :---: | :---: | :---: | :---: |
| $\begin{gathered} \text { Extended } \\ \text { Jacobi } \\ \text { Transformation } \\ \text { Method } \end{gathered}$ | $\begin{aligned} c_{J}= & \left(n^{2}+2 n\right) \\ & (8.1 .1) \end{aligned}$ | $\begin{aligned} & \mathrm{m}_{\mathrm{J}}=6 \mathrm{~cm}^{3} \\ & \text { (8.1.2) } \end{aligned}$ | Unsuitable for n > 150 <br> $\mathrm{c}=$ No. of iterations (typically $c=6$ ) Produces complete set of eigenvalues and eigenvectors |
| Gupta's <br> Sturm <br> Sequence <br> Method | $\begin{aligned} c_{G}= & 5 n b \\ & (8.1 .3) \end{aligned}$ | $m_{G}=25 \mathrm{nb}^{2} \mathrm{r}$ (8.1.4) | ```2nb}\mp@subsup{}{}{2}\mathrm{ nultiplications for examining each trial \lambda. Average of 12 trial \lambda's per root located.``` |
| Simultaneous Iteration | $\begin{aligned} & c_{S I}= \\ & 2 m b+3 n t+\frac{1}{2} t^{2} \\ & \quad \text { (8.1.5) } \end{aligned}$ | $m_{S I}={ }_{2}^{1} \mathrm{nb}^{2}+$ <br> $\left[3 n b t+{ }_{2} n t^{2}+{ }_{6}{ }_{6} t^{3}\right] c$ <br> (8.1.6) | $\begin{aligned} & t=\text { No. of trial } \\ & \text { vectors } \\ & c=\text { No. of iterations } \\ & \text { Typically } \\ & t=r+3 \\ & c=7 \end{aligned}$ |

TABLE 8.2.

COMPUTATIONAL REQUIREMENTS OF THE FULL KRON METHOD
(a) Core Space

|  | AREA | $\begin{array}{\|l} \text { IOCAL } \\ \text { SYMBOI } \end{array}$ | NO. OF LOCATIONS (WORDS) | USE |
| :---: | :---: | :---: | :---: | :---: |
| PART 1 | ${ }^{1}$ | ${ }^{\mathrm{c} K M}$ | $\left(n_{0}{ }^{2}+n_{0}\right)$ | Full lower triangular storage of component $\mathrm{K}_{\mathrm{o}}, \mathrm{M}_{\mathrm{o}}$. Eigenvectors occupy same space. |
| PART 2 | $c_{1}$ $c_{2}$ $c_{3}$ | $\left\{\begin{array}{c} c_{\mathrm{R}} \\ \mathrm{c}_{\mathrm{MDS}} \\ { }^{\mathrm{c}_{\mathrm{E}}} \end{array}\right.$ | $\begin{gathered} 2 m d \\ n_{p c} n_{o}+m_{p s}^{m} n_{o} \\ 3 n_{t} \end{gathered}$ | R matrix (+ constant part space) Minimum data set. <br> List of unique eigenvalues, multiplicities, base multiplicities |
| PART 3 | $c_{1}$ $c_{2}$ $c_{3}$ | $c_{R} / c_{\text {KM }}$ $c_{\text {MDS }}$ $c_{V}$ | $2 n_{t}$ | R matrix/component eigenvectors <br> c vector, modal and physical composite system eigenvector. |
| PART 4 | $c_{1}$ $c_{3}$ | $\left\lvert\, \begin{aligned} & { }^{\mathrm{C} \mathrm{KM}} \\ & { }_{\mathrm{V}} \mathrm{~V} \end{aligned}\right.$ |  | Component $\mathbf{K}_{0}, \mathbf{M}_{0}$ <br> Current composite system eigenvector + gen. mass and stiffness terms. |
| OVERALI <br> MAXIMUM REQUIREMENT <br> Wher | ${ }^{c} \mathrm{~K}$ <br> cho | $\begin{equation*} =[ \tag{8.2.1} \end{equation*}$ <br> ice exi | $\left.\begin{array}{l} \left.n_{o}^{2}+n_{o}\right) \\ o r \\ \text { ord } \end{array}\right]+\left[\dot{n}_{p c^{n}}\right.$ <br> sts, the larger re | $\left.n_{0}+m_{p s} m_{0} n_{0}\right]+\left[3 n_{t}\right]$ <br> quirement must be used. |

## TABLE 8.2. (Continued)

(b) Number of Multiplications

|  | SYMBOL | NUMBER OF MUITS. | COMMENT |
| :---: | :---: | :---: | :---: |
| PART 1 | $m_{1}$ | $\left(6 \mathrm{cn}{ }_{0}^{3}\right) \mathrm{n}_{\mathrm{pc}}$ | Eigenreduction of each primary component by extended Jacobi method |
| PART 2 | $\mathrm{m}_{2}$ | $\operatorname{l2r}\left(\mathrm{mdn}_{0}+\mathrm{md}^{2}\right)$ | Formation and factorisation of each $R(\lambda)$. (Average of 12 trial $\lambda$ 's per roo't assumed) |
| PART 3 | ${ }^{m}$ | $\begin{aligned} & r\left(m d n_{o}+m d^{2}+2 m d\right. \\ & \quad+2 n m_{0}+n_{t} n_{0} \end{aligned}$ | Formation, factorisation and deflation of $\mathbf{R}\left(\lambda_{i}\right)$ <br> Solution for $\mathbf{C}$ <br> Formation of eigenvector $x_{i}$ |
| PART 4 | $\mathrm{m}_{4}$ | $2 r n_{0}\left(n_{0}+1\right) q$ | Formation of generalised mass and stiffness terms |

(a) 2-DIMENSIONAL FRAME STROCTURE

$\mathrm{n}=180 \quad \mathrm{~b}=20$
(b) THE BASE SYSTHM ' 1 ' (KRON'S METHOD) $\left(n_{t}=198, m=18\right)$


$n_{0}=72 \quad b_{0}=17$

$$
n_{0}=63 \quad b_{0}=17
$$

Base System Schematic

(c) KRON'S METHOD EASE SYSTEM NO. $2 \quad\left(n_{t}=225, \quad m=45\right)$

$n_{0}=36$
$b_{0}=10$
$n_{0}=45$
$b_{0}=10$
Base System Schematic


$$
\begin{array}{ll}
n_{p c}=2 & n_{c}=6 \\
m_{p s}=3 & m_{s}=5
\end{array}
$$

assembled structural matrices used in Gupta's method are of order 180, with a semi-bandwidth of 20. No use of symmetry is made in this illustrative example. The general requirements for Gupta's method are contained in TABLE 8.1.

Two possible base systems are shown for use in Kron's method. The first (Figure 8.l(b)) utilises two large primary components while the value of $m$ is kept low. The second (Figure 8.1(c)) utilises two small primary components, however the value of $m$ is somewhat larger. Base system schematics are shown for both cases.

TABLE 8.3(a) summarises the approximate core requirements of Gupta's method and the two Kron systems. The core advantages of the latter are clear, and in particular, the second Kron system demonstrates the advantage of keeping down the component size. Indeed with the core space allocation scheme of TABLE 8.2(a), the minimum core space is roughly obtained for $m \simeq 2 n_{0}$. The low core requirements for the minimum data sets emphasise the advantages of recognising repetition in the base system. It may be noted that the core required for part 2, i.e. scanning of $R(\lambda)$ is respectively 2.84 K and 3.34 K . This program stage may thus be efficiently executed in a small core -partition.

The number of multiplications required by the methods are summarised in TABLE 8.3(b), r being left as a parameter. Figure 8.2 shows a plot of multiplications versus $r$ for the 3 cases considered. The Kron method clearly does a good deal of work once and for all at the component eigenreduction stage, while the work in Gupta's method varies linearly with $r$. Thus if $r<5$, Gupta is faster, but for $r \geqslant 5$, 'Kron plus base system 2 ' is superior. If $r \geqslant 36$, then the greater amount of initial work in 'Kron plus base system l' pays off via the scanning of a smaller $\boldsymbol{R}(\lambda)$ matrix.

### 8.2.4. General Conclusions

To summarise, the core requirements of $K_{r o n ' s ~ m e t h o d ~ a r e ~}^{\text {n }}$ certainly less than those of Gupta, while the time requirements may also be. There is no essential difference in the accuracy attainable by the two methods. However, the accuracy of the composite system representation in Kron's method is clearly controlled by the accuracy of the component eigenreductions. Although a detailed error analysis is not attempted in the current work, it may be that with

## TABLE 8.3.

COMPUTATIONAL REQUIREMENTS FOR FULL KRON/STURM SEQUENCE METHODS
(a) Core Requirements

(b) Number of Multiplications

|  | FULL KRON METHOD |  | GUPTA'S S.S. METHOD |
| :---: | :---: | :---: | :---: |
|  | Base System 1 | Base System 2 |  |
| $\mathrm{m}_{1}$ | 22,438 K | 5,781 K |  |
| $\mathrm{m}_{2}$ | 146 rK | 569 rK |  |
| $\mathrm{m}_{3}$ | 16 rK | 61 rK |  |
| ${ }^{\text {m }}$ KRON | ( $22,438+162 r) K$ | (5,781x630r)K |  |



NO. OF EIGENVAIUES/EIGENVECTORS REQUIRED
repetition, the accumulation of roundoff error is less in Kron's method when compared with the large order matrix factorisations required in Gupta's method.

There is no inherent topological restriction in Kron's method, however care must be taken to choose a base system to suit requirements. If a large number of composite system eigenvalues are required, keep m low, while if only a few are required, keep the component size small. A physically sensible choice of components will help avoid any possible ill--conditioning.

A practical example of the use of the full Kron method is given in Section 8.4.

### 8.3. THE APPROXIMATE KRON METHODS (A.K.M.)

### 8.3.1. General Review

Where only the lower eigenspectrum of a structure is of interest, the approximate Kron methods introduced in Chapter 6 may be used. The advantages of the full Kron method (Section 8.2) are retained, while approximate results may be obtained with large savings in computer resources.

The MODAL APPROXIMATE KRON MEIHOD involves the use of a small number of component normal modes plus constraint modes to represent the component. Simultaneous iteration for these normal modes is fast and utilises banding in the component mass and stiffness matrices. The size of the minimum data set is reduced, and likewise the effort in forming the corresponding $\mathbf{R}(\lambda)$ matrix.

Where components are to be connected along continuous boundaries, the possibility exists of reducing the number of constraints along a that boundary, in addition to the above approximation; the HYBRID APPROXIMATE KRON METHOD resulting. The core required for the minimum data set is further reduced both due to the reduced number of generalized constraints and the reduced number of constraint modes (Section 6.5). However, of greatest importance is the reduction in order of the $\quad \mathbf{R}(\lambda)$ matrix implying a considerable decrease in scanning effort.

### 8.3.2. Computational Requirements

Approximate expressions for core space requirements and numbers of multiplications are given in TABLE 8.4 for the approximate Kron methods. Provision is made for a reduced number of component modal freedoms,

## TABLE 8.4.

COMPUTATIONAL REQUIREMENTS OF THE APPROXIMATE KRON
METHOD
(a) Core Space

|  | AREA | LOCAL SYMBOL | NO. OF LOCATIONS | USE |
| :---: | :---: | :---: | :---: | :---: |
| PART 1 | $\begin{aligned} & c_{1} \\ & c_{2} \\ & c_{3} \end{aligned}$ | $\begin{aligned} & \mathrm{c}_{\mathrm{KM}} \\ & \mathrm{c}_{\mathrm{M}} \\ & \mathrm{c}_{\mathrm{W}} \end{aligned}$ | $\begin{gathered} 2 n_{0} b_{0} \\ n_{0} n_{g o} \\ \text { (working space) } \end{gathered}$ | $K_{o}, M_{o} \text { matrices }$ <br> Normal and constraint modes <br> Interaction matrix/generalised constraint sets etc. |
| PART 2 | $\begin{aligned} & c_{1} \\ & c_{2} \\ & c_{3} \end{aligned}$ | $\left\{\begin{array}{l} c_{\mathrm{R}} \\ c_{\mathrm{MDS}} \\ c_{\mathrm{E}} \end{array}\right.$ | $\begin{gathered} 2 m_{g}{ }^{d} g \\ \mathrm{n}_{\mathrm{pc}} \mathrm{n}_{\mathrm{go}}+\mathrm{m}_{\mathrm{pc}} \mathrm{~m}_{\mathrm{go}} \mathrm{n}_{\mathrm{go}} \\ 3 \mathrm{n}_{\mathrm{g}} \end{gathered}$ | ```R matrix (+ constant part) Minimum data set Unique eigenvalues, multiplicities base multiplicities``` |
| PART 3 | $\begin{aligned} & c_{1} \\ & c_{2} \\ & c_{3} \end{aligned}$ | $\begin{aligned} & c_{\mathrm{R}} / \mathrm{c}_{\mathrm{M}} \\ & \mathrm{c}_{\mathrm{MDS}} \\ & \mathrm{c}_{\mathrm{V}} \end{aligned}$ | $2 n_{t}$ | $\mathbf{R}$ matrix/component modes <br> (as part 2) <br> C vector/modal eigenvector/ physical eigenvector |
| PART 4 | $c_{1}$ $c_{3}$ | $c_{\mathrm{KM}}$ $c_{\mathrm{V}}$ |  | (as part 1) <br> Current composite system eigenvector + generalised mass and stiffness terms |
| OVERALL MAXIMUM REQUIREMENT | $\bar{c}_{K}=\left[\begin{array}{c} 2 n_{0} b_{0}  \tag{8.3.1}\\ o r \\ 2 m_{g} d_{g} \end{array}\right]+\left[\begin{array}{c} n_{p o} n_{g o}{ }_{0}^{+m}{ }_{p s}{ }^{m}{ }_{g o} n_{g o} \\ \text { or }_{0} \\ n_{0}{ }_{g o} \end{array}\right]+\left[\begin{array}{c} 2 n_{t} \\ o r \\ 3 n_{g} \end{array}\right]$ <br> Where choice exists, the larger requirement must be used |  |  |  |

TABLTE 8.4. (Continued)
(b) Number of Multiplications

|  | SYMBOL | NUMBER OF MULITIPLICATIONS | COMMENT |
| :---: | :---: | :---: | :---: |
| PART I | $\mathrm{m}_{1}$ | $\left\{\begin{array}{l} n_{p c}^{\left[n_{o} b_{o}^{2}+\right.} \\ \quad\left(3 n_{o} b_{o} t+{ }_{2}^{5} n_{0} t^{2}+1_{6} t^{3}\right) c \\ \left.\quad+8 m_{g o} n_{0} b_{o}\right] \end{array}\right.$ | $\left.\begin{array}{l}\text { Factorisation of } K_{o} \\ \text { Simult. Iteration } \\ \begin{array}{l}\text { Solution for and } \\ \text { orthogonalisation of } \\ \text { constraint modes }\end{array}\end{array}\right\}$For each <br> primary <br> component |
| PART 2 | $\mathrm{m}_{2}$ | $\operatorname{l2r}\left(m_{g} d_{g} g_{0}+m_{g} d_{g}^{2}\right)$ | As for full Kron |
| PART 3 | ${ }^{m}$ | $\begin{aligned} & r\left(m_{g \cdot g}^{d} n_{g o}+m_{g} d_{g}^{2}\right. \\ & \quad+2 m_{g}^{d} g \\ & \left.\quad+2 n_{g}^{m} g_{g o}+n_{t} n_{g o}\right) \end{aligned}$ | As for full Kron |
| PART 4 | $\mathrm{m}_{4}$ | $2 \mathrm{~m} \mathrm{n}_{0}\left(\mathrm{n}_{0}+1\right) \mathrm{q}$ | As for full Kron |

$n_{g_{0}}<n_{0}$, and a reduced number of constraints in constraint sets,
$m_{g_{0}}<m_{0}$. The order of the $R(\lambda)$ matrix is given by $m_{g}(<m)$ with semi-bandwidth $d_{g}$. The component mass and stiffness matrices are assumed to have semi-bandwidth $b_{0}$. The work required to form any generalised constraints is neglected. The general requirements for simultaneous iteration [27] are recorded in TABLE 8.1.

### 8.3.3. The 2-D Frame Example

TABLE 8.5 sets out the theoretical requirements of the MODAL APPROXIMATE KRON METHOD applied to the 2-dimensional frame introduced in Section 8.2.3. It is assumed that five component normal modes are used in all components and that all constraint modes are included. Thus primary component $I$ in base system $I$ requires storage for five normal modes and 18 constraint modes.

Here, base system 1 is probably a better choice in that the total number of generalised freedoms retained $\left(n_{g}\right)$ is 51 out of a total base system order ( $n_{t}$ ) of 198. In base system 2, the respective figures are 120 and 225 , due to the large number of constraint modes being carried.

The savings in core and computing effort compared with the full Kron method (TABLE 8.3) are evident, the overall core requirements being reduced by about one third. Note that with base system $I$, the scanning of $\mathbf{R}(\lambda)$ may be carried out in just over IK words of core, compared to 2.8 K for a full Kron method. The savings associated with the minimum data set are most marked.

Comparison of the figures for $M$, in TABLES 8.3 and 8.5 illustrates the enormous savings in not having to form a complete component eigenreduction. Thus for base system 2 and $r=10$, the total effort is reduced by a factor of 10 on using the approximate method.

Simultaneous iteration (S.I.) (Section 2.4.3) is a fast convenient method for obtaining the lowest eigenvalues of large order matrices with no inherent loss of accuracy. The requirements of S.I. for the frame are included in TABLE 8.5 for $r=5$ and $r=10$ to permit general comparisons.

While the core space advantages of the Kron approach are again exemplified, the effort involved is comparable for low $r$. Thus provided the core is available for S.I., the superior accuracy makes this approach more attractive. However, the effort involved in S.I. increases more

## TABIE 8.5.

## 2-D FRAME EXAMPLE

COMPUTATIONAL REQUIREMENTS FOR APPROXIMATE KRON/DIRECT SIMUITANEOUS ITERATION METHODS
(a) Core Requirements

| APPROXIMATE KRON METHOD |  |  |  | SIMULTANEOUS ITERATION |
| :---: | :---: | :---: | :---: | :---: |
|  |  | BASE SYSTEM 1 | BASE SYSTEM 2 |  |
| $\begin{aligned} & c_{1} \\ & c_{2} \end{aligned}$ | $\begin{aligned} & \mathrm{c}_{\mathrm{KM}} \\ & \mathrm{c}_{\mathrm{M}} \end{aligned}$ | $\begin{aligned} & 2.488 \mathrm{~K} * \\ & 1.449 \mathrm{~K} * \end{aligned}$ | $\begin{aligned} & 0.900 \mathrm{~K} \\ & 0.828 \mathrm{~K} \text { * } \end{aligned}$ | $c_{S I}=[7.5+0.52(r+3)] \mathrm{K}$ |
| $c_{1}$ $c_{2}$ $c_{3}$ | $\begin{aligned} & c_{R} \\ & c_{\mathrm{MDS}} \\ & \mathrm{c}_{\mathrm{E}} \end{aligned}$ | $\begin{aligned} & 0.324 \mathrm{~K} \\ & 0.577 \mathrm{~K} \\ & 0.111 \mathrm{~K} \end{aligned}$ | $\begin{aligned} & 1.530 \mathrm{~K} * \\ & 0.577 \mathrm{~K} \\ & 0.111 \mathrm{~K} \end{aligned}$ | $\text { For } \begin{aligned} \mathrm{r} & =5 \\ \mathrm{c}_{\mathrm{SI}} & =11.52 \mathrm{~K} \end{aligned}$ |
| $c_{3}$ | ${ }^{\text {c }}$ V | 0.396 K * | 0.450 K * | For $r=10$ |
| ${ }^{c_{\mathrm{KRON}}}$ |  | 4.293K | $\underline{2.808 \mathrm{~K}}$ |  |

* Indicates Contribution to Maximum Core Space Demand
(b) Number of Multiplications

than linearly with $r$, as does the core requirement. Hence for very large structures with a good degree of repetition, and where $r$ is reasonably large, the approximate Kron method would, in all likelihood, offer far more economical approximate solutions.


### 8.3.4. Comparison with the Component Mode Method (C.M.M.)

A convenient implementation of a form of the component mode method (System EIG3) is described in APPENDIX 2.

A detailed theoretical comparison with the approximate Kron method is difficult as the choice of base system and number of component modes included will depend on the particular problem circumstances. However they are both component-wise methods for reducing the number of freedoms in large eigenvalue problems. The computational requirements will be largely similar, and both may utilise repetition in some form.

Whereas the number of connection freedoms may often be less than the number of constraints, the additional normal mode freedoms used in the component mode method can lead to composite system matrices of larger order. However, if component normal modes are to be included up to a pre-determined cut-off value, then fewer 'higher frequency' fixed constraint normal modes in the C.M.M. will in general be required. Furthermore, the component eigenproblem is of smaller order.

Physically, the fixed constraint component normal modes of the C.M.M. may be expected to well represent composite system behaviour when connection boundaries are stiff, for example when $m$ is large. By the same token, the free connection coordinate normal modes utilised in the Kron approach will be superior where boundaries are flexible, for example when $m$ is small. However the hybrid approximate Kron method may be used in situations where $m$ is large to good advantage.

A practical comparison of the methods is given in the example of the following section.

### 8.4. A STIFFFNFD PLATE EXAMPLE

8.4.1. Introduction

The stiffening of plate structures via the attachment of ribs is common practice in many engineering branches. Wi.th regard to vibrations, the stiffening may be employed to raise the lowest structural frequencies, or indeed to ensure that there are no natural frequencies close to a higher known forcing frequency.

FIGURE 8.3.
EXXARLE

— Typical beam element


Plate element dimensions $12^{\prime \prime} \times 12^{\prime \prime} \times 0.2^{\prime \prime}$
Beam element dimensions $12^{\prime \prime} \times 0.1^{\prime \prime} \times 0.1^{\prime \prime}$ ( $=$ area of A/2)
Number of freedoms in full problem $=304$

Semi-bandwidth $=23$

As the main intention is to provide a practical comparison of methods described in this thesis, the example is kept particularly simple. The structure comprises a cantilever plate with beam supports at regular intervals. The beams are of solid cross section, and have stiffness only in the lateral plate direction. This stiffness is kept low to avoid frequency bunching (Figure 8.3). The arrangement permits the use of repetition to be illustrated.

The structure has an axis of symmetry XX. Here the half--problem is analysed for symmetric modes, that is with $\frac{\partial_{w}}{\partial y}$ and $\frac{\partial^{2} w}{\partial x^{\partial} y}$ constrained along XX. A further feature of interest is the way in which these boundary conditions are handled by the Kron method approach.

The problem is then briefly reconsidered assuming a simply supported end YY, destroying symmetry. The advantages of Kron's method are further exemplified.

The methods considered in the following discussion are
(i) The Full Kron Method (Program EIGI)
(ii) The Approximate Kron Method (Hybrid version) (Program EIG2) (iii) The Component Mode Method (Program EIG3):

### 8.4.2. Half-Structure Base System Specifications

## Kron's Method

Details of the base system for use with both the full and approximate Kron methods are contained in Figure 8.4.

Leaving aside the symmetric boundary conditions to be applied along XX, the half-structure may be torn along the beam centre-lines to produce three identical components. The one primary component PCI and associated partial constraint sets PSI to PS4 are indicated in Figure 8.4(a). The division of the constraints into symmetric and anti-symmetric sets is not only a pre-requisite for defining generalised constraints. For it is an important feature of Kron's method that boundary conditions, such as those along $X X X$, may be treated as constraints appiied to the base system. Thus in the base system schematic of Figure 8.4(b), constraint set $S 5$ embodies constraints of the form $\frac{\partial w}{\partial y}=0$ etc..

In the full Kron method, a full eigenreduction of a 64-freedom component is required, and with each constraint set comprising 8 constraints, the order of the composite system frequency matrix is 40.

Utilising the Hybrid Approximate Kron Method; use of 4 component

## ANALYSIS OF HALF STRUCIURE

## IDEALISATION OF SYSTEM FOR KRON'S METHOD

(a) Definition of Primary Component and Partial Constraint Sets


PCl contains 64 displacement freedoms.
Each partial constraint set comprises 8 displacement freedoms.
(b) Base System Schematic


## FIGURE 8.4. (Continued)

(c) Base System Data

| COMPONENT | CORRESPONDING <br> PRIMARI COMPONENT |
| :---: | :---: |
| Cl | PCl |
| C 2 | PCl |
| C 3 | $\mathrm{PC2}$ |


| CONSTRAINT | PARTIAL CONSTRAINT SET |  |
| :---: | :---: | :---: |
|  | 1 | 2 |
| SII | PSI | PS3 |
| S2 | PS2 | PS4 |
| S3 | PS1 | PS3 |
| S4 | PS2 | PS4 |
| S5 | PS2 | - |


16. boundary freedoms


$$
w, \frac{\partial w}{\partial y} \text { set to zero. }
$$

COMPONENT 2 (IREP=1)
Order for normal mode analysis $=32$
Number of constraint modes $=32$

COMPONENT 3 (IREP=0)
Order for normal mode analysis $=40$ Number of constraint modes $=16$
normal modes (for example) will generate 4 generalised constraints in each set, and a total of 16 static constraint modes for the primary component. The order of the composite system frequency matrix will thus be reduced to 20 . In such a case, the symmetric boundary condition, like component continuity, will only be approximately satisfied.

## Gomponent Mode Method

The base system for the component mode method is indicated in Figure 8.5. In contrast with the Kron approach, boundary conditions must be incorporated prior to component analysis. Thus all three components must be analysed for normal and constraint modes. However, use of the IREP parameter in program EIG3 (Section A2.4) avoids having to re-set up the mass and stiffness matrices for component 2.

With the numbers of fixed constraint normal modes taken from the components as 3, 2, 2 respectively, the total order of the composite system problem is given by

```
32 (boundary freedoms) + 7 (normal mode freedoms) = 39
```


### 8.4.3. Symmetric Modes - Test Cases

The basic set of 7 test cases, identified A through I, are detailed in TABIE 8.6. The first relates to the full Kron method, while the last two relate to the component mode method.

The first five symmetric modes were evaluated in each case. "The eigenvalues from run A, validated by a direct assembly eigensolution, are here taken to be the 'exact' solution. Percentage eigenvalue errors are given for the remaining test cases. The mode shapes are indicated in Figure 8.6.

The computational requirements of each case are also indicated in TABLE 8.7, as are comparative core requirements for Gupta's Sturm sequence and Jenning's simultaneous iteration methods.

### 8.4.4. Symmetric Modes - Results

## The Full Kron Method (Run A)

Although the lowest five symmetric modes were computed, there is of course no 'Iow frequency' restriction with program EIGI. The maximum core requirement of 7 K words compares favourably with the 17 K required by Gupta.

The eigenvalues were determined to five places by simple bisection

STIFFENED PLATE EXAMPLE - SYMMETRIC MODES (HALF STRUCTURE)

| PROGRAM: | EIGI | EIG2 |  |  |  | EIG3 |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| CASE: | A | B | C | D | E | F | G |
| DETAILS: | $\begin{aligned} n_{0} & =64 \\ m & =40 \end{aligned}$ | $n_{n o}=2$ $n_{c o}=4$ $m_{g}=10$ | $n_{n o}=3$ $n_{c o}=6$ $m_{g}=15$ | $\begin{aligned} & n_{\mathrm{no}}=4 \\ & n_{\mathrm{co}}=8 \\ & \mathrm{~m}_{\mathrm{g}}=20 \end{aligned}$ | $\begin{aligned} & n_{\mathrm{no}}=5 \\ & \mathrm{n}_{\mathrm{co}}=10 \\ & \mathrm{~m}_{\mathrm{g}}=25 \end{aligned}$ | $\begin{aligned} n_{n o} & =2 / 1 / 1 \\ n^{\prime} & =36\end{aligned}$ | $\begin{aligned} n_{n 0} & =3 / 2 / 2 \\ n^{\prime} & =39\end{aligned}$ |
| EIGEN- 1. <br> VALUE 2. <br> ERRORS 3. <br> $(\%)$ 4. <br>  5. | ' EXACT EIGENVALUES' | $\begin{gathered} -3.543 \\ -1.573 \\ -1.631 \\ \hline-4.465 \\ +46.1 \end{gathered}$ | $\begin{gathered} -3.531 \\ -1.364 \\ -0.863 \\ -2.806 \\ \hline-0.822 \end{gathered}$ | $\begin{aligned} & -3.054 \\ & -1.233 \\ & -0.794 \\ & -2.235 \\ & -0.637 \end{aligned}$ | $\begin{aligned} & -1.197 \\ & -0.804 \\ & -0.689 \\ & -1.110 \\ & -0.499 \end{aligned}$ | $\begin{aligned} & 0.154 \\ & 0.176 \\ & 0.087 \\ & 4.64 \\ & 7.71 \end{aligned}$ | $\begin{aligned} & 0.0 \\ & 0.041 \\ & 0.070 \\ & 0.238 \\ & 3.82 \end{aligned}$ |
| VARIATION 1. <br> NOS. (\%) 2. <br> (EIG2 3. <br> ONLY) 4. <br>  5. |  | $\begin{gathered} -0.003 \\ -0.007 \\ -0.212 \\ \hline-3.708 \\ -6.627 \end{gathered}$ | $\begin{array}{r} -0.003 \\ 0.026 \\ 0.184 \\ -1.423 \\ \hline 0.020 \end{array}$ | $\begin{array}{r} -0.002 \\ 0.011 \\ 0.317 \\ -0.756 \\ 0.156 \end{array}$ | $\begin{aligned} & -0.003 \\ & -0.003 \\ & -0.035 \\ & -0.461 \\ & -0.055 \end{aligned}$ |  |  |

Indicates Location of Highest Component Mode Eigenvalue

STIFFENED PLATE EXAMPLE - SYMMETRIC MODES (HALF STRUCTURE) - COMPUTATIONAL REQUIREMENTS


* Indicates Contribution to Maximum Core Requirement


## STIFFENED PLATE EXAMPLE

SYMMETRIC MODES OF HALF STRUCITURE



root location, an average of 13 bisections being required.
The Hybrid Approximate Kron Method (Runs B, C, D, E)
The four test cases here correspond to an increasing number of component normal modes ( $n_{n}$ ) included. In the hybrid method this reflects itself in increasing numbers of constraint modes and generalised constraints. Eigenvalue convergence is fairly slow, however with $n_{n o}=5$, errors are within about $1 \%$. Variation numbers below $\lambda_{\max }$ are small. As observed in the example of Section 6.5, accuracy deteriorates rapidly once $\lambda_{\max }$ is exceeded. It is interesting to note that convergence is from below, that is, the approximation due to generalised constraints is dominant.

The core space and computer time predictions of Section 8.3 are borne out. The very low time demands of runs $B$ and $C$ suggest that the technique may be used for fast prediction of rough frequencies.

As in Run A, eigenvalues were determined to 5 places by an average of 13 simple bisections.

## The Component Mode Method (Runs F, G)

These two cases correspond to the inclusion of $2,1,1$ and $3,2,2$ component normal modes respectively. The accuracy obtained for the lowest modes is superior to that from the hybrid approximate Kron method, possibly because with stiff connection boundaries it is the better suited method in this situation. The computational requirements are generally similar to that of the previous method.

### 8.5. IMPROVED SCANNING FFFICIENCY IN KRON'S METHOD

Two methods of reducing the total effort involved in scanning the $\mathbf{R}(\lambda)$ matrix, evaluated with reference to the stiffened plate example, are
(i) use of Brent's linear interpolation root location algorithm (Section Al.3.4)
(ii) use of the 'constant part' approximation (Section 6.6).

### 8.5.1. The Full Kron Method

In Part 2 of run A (TABLE 8.7), a total of 77 formation/ factorisations of the matrix $\mathbf{R}(\lambda)$ were required in the location of the first. five symmetric mode eigenvalues. Out of these, 13 were used prior
to locating ranges containing simple zeros. The location of the zeros within these ranges then took an average of an additional 13 each.

Use of Brent's linear interpolation algorithm reduced this average to 9 , and the time taken for part 2 was reduced from 41.3 secs to 31.5 secs, a saving of $24 \%$. Identical eigenvalues (to 5 places) were obtained.

The constant part approximation reduces the formation time for $\mathbf{R}(\lambda)$, although eigenvalue accuracy suffers. With $\bar{\lambda}=10^{5}$, only the lowest 8 component eigenvalues remained 'active', yet the loss of accuracy as indicated below was extremely small:

| Eigenvalue No. \% Eigenvalue Error ${ }^{*}$ <br> 1 0.0 <br> 2 0.002 <br> 3 0.026 <br> 4 0.579 <br> 5 0.029 |
| :--- |
| w.r.t. values obtained in run A. |

The corresponding loss in eigenvector accuracy is likewise small. As predicted in Section 6.6, the approximate eigenvalues obtained are consistently high . The time saving obtained is impressive. Using the linear interpolation algorithm, the part 2 time was reduced from 31.5 secs to 12.8 secs, a saving of $59 \%$. ${ }^{+}$

However, it must be remembered that while the linear interpolation algorithm may be used at all times, the advantages of the constant part approximation recedes as the range of interest of $\lambda$ rises.

### 8.5.2. The Approximate Kron Method

The above techniques were implemented for the case $D$ in TABIE 8.6. Again, use of the linear interpolation algorithm reduced the average number of formation/factorisations of the $\mathbf{R}(\lambda)$ matrix from 13 to 9 for each root location. The time taken for part 2 in this case was reduced from 14.9 secs to 13.9 secs, a saving of $20.4 \%$.
$\dagger$ The total time for all 4 parts is thus reduced by $29.6 \%$ to 67.7 secs.

Use of the constant part approximation, with $\bar{\lambda}$ automatically set to twice the highest component normal mode eigenvalue, $\lambda_{\text {max }}$, produced significant savings with small (additional) loss in eigenvalue accuracy. The time of 3.9 secs was reduced to 2.3 secs, a saving of $41 \%$, while the percentage change in eigenvalues w.r.t. those obtained in run $D$ are as follows:

| Eigenvalue No. | \% Eigenvalue Change |
| :---: | :---: |
| 1 | 0.006 |
| 2 | 0.051 |
| 3 | 1.060 |
| 4 | 1.649 |
| 5 | 0.714 |

The overall run time was reduced by $13.8 \%$ to 16.2 seconds. As a range of interest of $\lambda$ is always to be confined below $\lambda_{\text {max }}$ in practice, use of the constant part approximation thus appears very attractive.

### 8.6. ANALYSIS OF THE COMPIETE STIFFENED PLATE

To further illustrate the advantages of repetition in Kron's method, the problem of the stiffened plate is reconsidered with the end YY (Figure 8.3) simple supported. Symmetry is thus destroyed, and analysis of the full structure ( 300 freedoms) is required.

The base system adopted for Kron's method is shown in Figure 8.7 Again, only one primary component is defined. The simple support boundary condition is handled by defining an additional partial constraint set, PS5, corresponding to the ' $w$ ' freedoms along side 2 of the primary component (Figure 8.4a). [In the case of the full Kron method, the set PSI could simply be split into 2 sets comprising ' $w$ ' and ' $\frac{\partial w '}{\partial x}$ ' freedoms respectively].

### 8.6.1. The Full Kron Method

Run $H$ (TABLE 8.8) relates to the full Kron method with linear interpolation and the constant part approximation $\left(\bar{\lambda}=10^{5}\right)$. The total number of constraints involved is 84. The first 5 eigenvalues were

## FIGURE 8.7.

ANALYSIS OF FUULL STRUCTURE (SIMPLY SUPPORTED END)
IDEALISATION OF SYSTEM FOR KRON'S METHOD
(a) Base System Schematic
(b) Base System Data


| CONSTRAINT | PARTIAL CONSTRAINT SETS |  |
| :---: | :---: | :---: |
| SET | 1 | 2 |
| S1 | PS1 | PS3 |
| S2 | PS2 | PS4 |
| S3 | PS1 | PS3 |
| S4 | PS2 | PS4 |
| S5 | PS1 | PS3 |
| S6 | PS2 | PS4 |
| S7 | PS1 | PS3 |
| S8 | PS2 | PS4 |
| S9 | PS1 | PS3 |
| S10 | PS2 | PS4 |
| S11 | PS5 | - |

## TABLE 8.8

ANALYSIS OF THE COMPLETE STIFFENED PTATE

|  | H | I |
| :---: | :---: | :---: |
|  | FULL KRON METHOD $\begin{aligned} & \bar{\lambda}=10^{5} \\ & m=84 \end{aligned}$ | HYBRID APPROXTMATE <br> KRON METHOD $\begin{aligned} & n_{\mathrm{no}}=4 \\ & \mathrm{n}_{\mathrm{co}}=8 \\ & \mathrm{~m}_{\mathrm{g}}=44 \end{aligned}$ |
| EIGENVALUES |  |  |
| 1. <br> 2. <br> 3. <br> 4. <br> 5. | $\begin{gathered} 338.20 \\ 399.71 \\ 565.82 \\ 927.70 \\ 1633.6 \end{gathered}$ | $\begin{array}{cc} 328.40 & (-2.90) \\ 391.45 & (-2.07) \\ 559.24 & (-1.16) \\ 919.31 & (-0.90) \\ 1615.1 & (-1.13) \end{array}$ |
| CORE REQUIREMENTS (K words) |  |  |
|  | $\begin{aligned} \mathrm{c}_{\mathrm{KM}} & =4.288^{*} \\ \mathrm{c}_{\mathrm{MDS}} & =2.368^{*} \\ \mathrm{c}_{\mathrm{R}} & =1.778 \\ c_{\mathrm{V}} & =1.152^{*} \end{aligned}$ | $\begin{aligned} & \mathrm{c}_{\mathrm{KM}}=2.240^{*} \\ & \mathrm{c}_{\mathrm{M}}=1.536 * \\ & \mathrm{c}_{\mathrm{MDS}}=0.420 \\ & \mathrm{c}_{\mathrm{R}}=0.478 \\ & \mathrm{c}_{\mathrm{V}}^{\prime}=0.768 * \end{aligned}$ |
|  | $\mathrm{c}_{\mathrm{KRON}}=7.808^{\circ}$ | $c_{\text {KRON }}=4.544$ |
| CP TIME (6600 seconds) |  |  |
| PART 1 <br> PART 2 <br> PART 3 <br> PART 4 | $\begin{array}{r} 47.7 \\ 30.5 \\ 8.3 \\ 7.4 \end{array}$ | $\begin{array}{r} 13.3 \\ 9.4 \\ 2.4 \\ 1.5 \end{array}$ |
| TOTAL | 93.9 | 26.6 |

* INDICATES CONTRIBUTION TO MAXIMUM REQUIREMENT

| Uses linear <br> Interpolation | YES | NO |
| :--- | :---: | :---: |
| Uses the C.P. <br> Approximation | YES | NO |

obtained. Direct solution verified that the eigenvalue error induced by the constant part approximation was in all cases less than $0.1 \%$. Clearly the doubling of the problem size had a relatively small effect on computational requirements. Core required was 7.218 K words (cf. 6.976 in run $A$ of TABLE 8.7) while $C P$ time required was 93.9 seconds (cf. 67.7 seconds in section 8.5 .1 ). In contrast, the requirements of Gupta's method would have been roughly doubled.
8.6.2. The Hybrid Approximate Kron Method

Use of the hybrid approximate Kron method to obtain fast eigenvalue estimates was tested by means of run $I$ in TABLE 8.8. Four component normal modes were included. Linear interpolation and the constant part were not used, thus comparison may be made directly with run $D$ of TABLE 8.7. While the partial constraint sets PS1 to PS4 were expressed in generalised form, the four constraints in PS5 were retained as simple constraints so that the simply supported boundary condition was enforced in full. The total number of constraints involved was thus 44.

Again, the computational requirements do not greatly exceed those for the half-problem. The core is increased from about 3.9 K words to 4.544 K words, while the time is increased from 18.8 secs to 26.6 secs. The eigenvalues and mode shapes obtained are reasonable guides as to the 'exact' system properties, although additional constraints are probably required to reduce the error on the two lowest eigenvalues. As with Gupta's method, the requirement of direct simultaneous iteration would roughly double for the full problem.

A possible base system for the component mode method is shown in Figure 8.8. It is interesting to note that with either 3 or 2 normal modes per component, the order of the composite system matrices is 94 , that is more than twice the 44 'constraint freedoms' utilised in run $I$.

In general terms; it is thus clear that the larger the degree of repetition, the more attractive both the full and approximate Kron methods become.

FIGURE 8.8.
ANALYSIS OF FULL STRUCTURE (SIMPLY SUPPORTED END) IDEALISATION OF SYSTEM FOR COMPONENT MODE METHOD

Connection Normal coordinate freedoms
mode

| freedoms |  |
| :---: | :--- |
| 3 | IREP $=0$ |

2 IREP $=1$

16

2 IREP $=2$

2
IREP $=2$

2
IREP $=2$

16
.3
IREP $=0$

Total ordex of composite
system $n=9$

## CHAPTER 9

## CONCLUSIONS AND FURTHER RESEARCH

### 9.1. THE FULL KRON METHOD

Hitherto, Kron's method had received scant attention in the field of structural eigenvalue problems. However, the recent publicity given to the method by Simpson, and in particular his proposition of the extended sturm sequence algorithm for scanning has opened the door for its establishment as a competitive algorithm.

In this work, the basic approach is paralleled with the above work, and the practical implementation on a digital computer and use thereof has been successfully investigated.

It has been established that Kron's method is particularly suitable where
(i) the value of $m$ is small
(ii) repeated components and constraint sets may be identified.

The tremendous core space advantages of the approach enable maximum use to be made of computing resources, either to solve extremely large problems efficiently 'in-core', or to confine middle range problems to a relatively small partition, a useful feature in a time sharing environment.

In assessing the role that Kron's method should play in the field of eigenvalue algorithms, several facts stand out. The effort involved in implementing the method is probably greater than for direct assembly techniques. User effort is greater in that the components and connection data must be specified, but this is a light task compared to the initial structural idealisation. The technique is particularly suitable when a large number of composite system eigenvalues/eigenvectors are required, and of course when relatively high frequencies are of interest. Finally, the ability to carry out the analysis in several stages, with the possibility of checking and restarting after each stage, is an additional attraction when analysing extremely large order structures.

### 9.2. THE APPROXIMATE KRON METHOD

A major original contribution of the current research has been to establish techniques which further utilise the advantages inherent in

Kron's method to economically obtain approximate low frequency composite system eigenvalues and eigenvectors.

A degree of judgement is required on behalf of the user, however the approaches developed are extremely versatile. Where few connection coordinates are involved, relatively few static constraint modes will be required to compensate for the excluded component modal freedoms. A Rayleigh-Ritz type approximation is thus introduced so that the resulting eigenvalues are upper bounds. Where many connection coordinates are involved along a stiff boundary, more component normal modes may be required but alternative economies are available via a reduced set of generalised constraints and associated static constraint modes. In this hybrid approach, the error induced by the modal freedom approximation and the constraint approximation tends to cancel out.

As it is often only the lowest modes of very large order structures that are required, the economical approximate implementation of the Kron method would thus appear attractive for many practical situations.

### 9.3. FURTHER RESEARCH

In many ways, the research carried out for this thesis has been a preliminary study. There is scope for further detailed analysis of the mathematical basis for the approximate techniques introduced and for improved methods for selecting generalised constraints. Also, any piece of numerical analysis should ultimately be underwritten by error analysis.

A particularly desirable development would be a means of establishing bounds on the results obtained from the approximate methods. The approach described in Section 2.6.5 is inapplicable in the case of component techniques in that the fully assembled structural stiffness matrix is at no stage formed, and alternative methods must be derived.

The computer programs developed, while far from optimum do form a basis for the efficient implementation of Kron's method. Algorithms for the efficient location of the roots of the $\boldsymbol{R}(\lambda)$ matrix will be just one area deserving further attention.

It is, however, in the field of applications that the author feels the most exciting prospects to lie. The various techniques have been tested here on limited examples, and a full evaluation on truly large problems is of the highest priority.

### 9.4. SOME APPLICARIONS

### 9.4.1. Typical Structures

The global dynamic analysis of aircraft structures is perhaps the classic example of a component structure. With the main fuselage, tail section and wings identified as principal components, it is clear that the value of $m$ may be kept low. Component modes and frequencies are often available from practical vibration tests. It will be particularly important to include constraint modes for the wings if only a few free-free wing modes are included.

A common natural frequency problem is that of turbo-generator foundations. Such structures often embody a high degree of repetition, again permitting Kron's method to be used to advantage.

### 9.4.2. Sophisticated Constraints

There is no theoretical restriction on the form of the constraints relating to component displacements, provided that a non-singular
$\mathbf{T}_{1}$ matrix (Section 5.2.1) may be formed. A general constraint connecting components A and B may be written as

$$
\sum_{i=1}^{n_{A}} c_{i}^{A} x_{i}^{A}+\sum_{j=1}^{n_{B}} c_{j}^{B} x_{j}^{B}=0
$$

where $x_{i}^{A}, x_{j}^{B}$ are displacement freedoms and $c_{i}^{A}, c_{j}^{B}$ are coefficients.
A useful application of the above form of constraint occurs where there is a displacement freedom mismatch between components, perhaps due to a different form of idealisation. A simple illustration is indicated in Figure 9.1, a possible constraint equation being

$$
\begin{equation*}
x_{1}=\frac{\ell}{L} x_{3}+\left(1-\frac{l}{L}\right) x_{2} \tag{9.4.1}
\end{equation*}
$$



### 9.4.3. Modification Analysis

An important advantage of any component-wise technique is that the effect of small modifications to the structure may be evaluated efficiently. Only those components altered need to be re-analysed, and provided modifications are small, the original normal modes may be used for example as input to the simultaneous iteration routine to rapidly converge to the new ones.

Again, assuming small or local modifications, the effect on the overall structure modes and frequencies are likely to be minor. It would not seem a difficult task to arrange for the original eigenvalues to be used as first approximations to the roots of the new $\quad \mathbf{R}(\lambda)$ matrix to minimise scanning effort.

Finally, it is the authors belief that with the trend to larger and more comple $x$ structural systems, Kron's method of piecewise eigenvalue analysis will take its place in the ever growing field of eigenvalue algorithms.

## APPENDIX 1

## THE COMPUTIATIONAL SYSTEMS EIGI AND EIG2 (KRON'S METHOD)

## Al.1. INTRODUCTION

In this appendix, the systems EIG1 and EIG2 for the implementation of the full and approximate Kron methods respectively are described. Particular emphasis is placed on the user supplied data and principal core storage requirements. Sections Al. 2 to Al. 5 deal with the four constituent parts.

The major computer program symbols (capital letters) necessary for general understanding and use of the systems are recorded at the end of this appendix in three reference tables:

TABLE Al. 1 List of input data,
TABLE Al. 2 Major program variables
TABLE Al. 3 Fixed parameters.
The input data is classified as either 'data' (D) or 'parameters' (P), the latter being used to define or control a process. Program variables are set within the systems and record information about the problem being processed. Fixed parameters are defined in DATA statements in the main overlay and are not normally varied. They include tolerances and iteration limits.

Large parts of the systems are identical and in general, discussion relates to the more complex system EIG2. To indicate differences, the following symbols are used:

$$
\dagger \text { applies to EIGl only }
$$

* applies to EIG2 only.


## Al. 2. PART I - THE ANALYSIS OF THE BASE SYSTEM

A1.2.1. PROGRAM COMPNT (OVERTAY 1.0)
This program is responsible for the analysis of the base system
 is shown in Figure Al.1. **

The run title, together with the parameter IMOD and the number of primary components NPC are read in, and the two secondary overlays called in sequence for each P.C.. On completion of the analysis of each P.C., information concerning it is stored and the variables NTP
$\therefore$ All flowcharts appear at the end of this appendix.
and NGP are updated.

## A1.2.2. PROGRAM SETKM (OVERLAY 1, I)

Responsible for forming the primary component mass and stiffness matrices, the flowchart is shown in Figure AI.2. The data describing the primary component is read in including NO, the order, and in the case. of EIG2 the parameter ITYP.

In EIGI, the component mass and stiffness matrices are stored in full lower triangular form while the variable bandwidth scheme is used in EIG2. The respective core requirements are thus

$$
\begin{equation*}
{ }^{1} C_{1}=N O *(N O+1) \quad{ }^{2} C_{1}=3 * N O+2 * B O * N O \tag{Al.I}
\end{equation*}
$$

where BO is an average semi-bandwidth.
Both version access a small library of finite elements FELIB, but may be easily modified to accept any particular element or library.

## A1.2.3. PROGRAM MODES (OVERIAY 1,2) (EIGI Version)

This program is responsible for eigenreducing the component matrices and storing the M.D.S. information on disc. A flowchart is show in Figure Al. 3.

The extended Jacobi method (the routine DIAG, APPFNDIX 4) is used to form the complete set of NO modes and eigenvalues for the primary component which are located in the area previously used for the mass and stiffness matrices.

The number of partial constraint sets, MCS, relating to the current P.C. is then read in. For each P.C.S., the MO connection coordinates are read as local displacement numbers, and the $G_{i}$ matrix, which is simply the appropriate eigenvector partition is stored on disc 10 together with the eigenvalues. The dimensions of each $G_{i}$ matrix is NO by MO, hence for a total of NPC primary components and MPS partial constraint sets the M.D.S. core requirement is given by:

$$
\begin{equation*}
{ }^{I_{C}}=\sum_{k=1}^{N P C} \mathrm{NO}_{k}+\sum_{i=1}^{\mathrm{MPS}} \quad \mathrm{NO}_{i} * \mathrm{MO}_{i} \tag{A1.2}
\end{equation*}
$$

This core space is not infact required until Part 2 .
On completion of each P.C.S. information concerning it is stored, and the variable $M$ is updated.

Al.2.4. PROGRAM MODES (OVERLAY 1,2) (EIG2 Version)
The EIG2 version of PROGRAM MODES (figure Al.4) is considerably more complex than the EIGI version in that it contains facilities for calculating constraint modes and generalised constraints, thus permitting a reduction in the number of component freedoms and/or linking constraints.

Simultaneous iteration (the routine SIMUIT, APPENDIX 3) is used to calculate the first NN normal modes of the primary component. In the case of a free component, a multiple of $\mathbf{M}_{o}$ is added on to
$K_{o}$ prior to factorisation.
SUBROUTINE CSTRNT reads MCS and the data concerning each P.C.S. i.e. MO, MGO, ICST and ISET. Any new natural constraint sets are calculated, as are constraint modes corresponding to all constraints. On completion of each P.C.S. the variables $M$ and MG are updated.

Where natural constraints are requested by ICST $=2$ SUBROUTINE GCON is called. The number and nature of the nodal freedoms involved must be identical along the connection boundary, and the 'normal' and 'tangential' freedoms must be specified by the user at a typical node. The magnitude ratio of each orthogonalised constraint is calculated and if less than TLIN, the natural constraint is discarded.

SUBROUTINE CMODE forms an ortho-normal set of NM constraint modes and calculates their associated pseudo-eigenvalues. The total number of freedoms for the current P.C. is given by $N G O=N N+N M$. The set of NGO modes must be stored in addition to the component mass and stiffnéss matrices, and require a space of

$$
\begin{equation*}
{ }^{2} C_{2}=N O * N G O \tag{Al.3}
\end{equation*}
$$

Further space for the interaction matrix (S.I.) and any natural constraint sets may be neglected.

Finally, SUBROUTINE GTERMS forms and stores the $G_{i}$ matrices for each P.C.S.. Here the size of each $\mathbf{G}_{i}$ is NGO*MGO, hence the M.D.S. core requirement is given by

$$
\begin{align*}
{ }^{2} \mathrm{C}_{3}= & \sum_{i=1}^{\mathrm{NPC}} \mathrm{NGO}_{i}+\sum_{j=1}^{\mathrm{MPS}} \mathrm{NGO}_{j} * \mathrm{MGO}_{j}  \tag{Al.4}\\
& \text { (eigenvalues) } \quad\left(\mathrm{G}_{i} \text { matrices }\right)
\end{align*}
$$

The parameters EMIN, EMAX corresponding to $\lambda_{\min }, \lambda_{\max }$ are updated during SUBROUTINE MODES.

## A1.2.5. Core Requirements for Part 1

The total requirement for EIGI is given by ( ${ }^{1} C_{1}$ ) while for EIG2 it is ( ${ }^{2} \mathrm{C}_{1}+{ }^{2} \mathrm{C}_{2}$ ) (Equations Al.l, Al.3). With NO large (e.g. > 100), the latter requirement can show good savings over the former.

## A1.3. PART 2-CALCULATION OF THE COMPOSITE SYSTEM EIGENVALUES

## Al.3.1. PROGRAM EIGVAL (Overlay 2,0)

The flowchart for this program is shown in Figure Al.5. Initially, the parameters IVAL and ICON plus any associated data are read in. IVAL controls the number of eigenvalues located i.e. either the NR lowest or those in the range $E 1<\lambda<E_{2}$. ICON controls whether the 'constant part' approximation is used. If it is, $\bar{\lambda}$ is set equal to $2 \times$ EMAX in EIG2 while $\bar{\lambda}$ must be specified by the user in EIGI. Terms in the 'constant part' are thus defined by $\lambda_{i}>\bar{\lambda}_{\text {. }}$

SUBROUTINE INPUT reads the M.D.S. information from disc store in the core space previously used for $K{ }_{e}$ and $\mathbf{M}_{e}$ in part I. The data describing the base system in terms of P.C.'s and P.C.S.'s is then read in, allowing SUBROUTINE SETUP to establish the form of $\mathbf{R}(\lambda)$ and data to facilitate its economical automated formation. The variables $M M$ and $L$ are set here. $M M$ is the order of $\mathbf{R}(\lambda)$ and is equal to the number of constraints applied to the base system. The variable bandwidth storage scheme is used for $\mathbf{R}(\lambda)$, hence $L$ is the length of the vector containing the off-diagonal terms. The core space required for $\mathbf{R}(\lambda)$ is given by

$$
\begin{equation*}
C_{4}=3 * M M+2 * L \tag{Al.5}
\end{equation*}
$$

the additional set of diagonal and off-diagonal arrays being required
for storage of the 'constant part' of $R(\lambda)$. The arrays used to store $K_{o}$ and $\mathbf{M}_{0}$ in Part 1 are re-used for this purpose thus allowing a value of $M M$ close to the maximum allowable NO value.

SUBROUIINE EARRAY utilises the base system data to construct an ordered list of unique base system eigenvalues and associated multiplicities for the primary component eigenvalues over the range of interest of $\lambda$.

Control is finally passed to SUBROUTINE SVAL which locates the composite system eigenvalues by use of the extended Sturm sequence scanning algorithm (See Section Al.3.3.).

## A1.3.2. The Description of the Base System

The data description of the base system components and constraint sets is listed in TABIE AI.I. It is important to note that the unique identification number assigned to each component (e.g. Cl, C2,...) is given by the location in the ITC array. Similarly the first set of entries in the arrays IDA, MTA, IDB, MTB correspond to constraint set Sl etc.

The simple beam example of Section 7.2.2. is here used to illustrate the data required (derived from Figure 7.4(c)).

| SYMBOL | VALUES | COMMENTS |
| :---: | :---: | :---: |
| NC | 4 | No. of components |
| ITC(NC) | $\left\{\begin{array}{llll}1 & 1 & 1 & 1\end{array}\right\}$ | P.C. types |
|  | (Cl) (C2) (C3) (C4) | Identifiers |
| MS | 3 | No. of constraint sets |
| IDA(MS) | $\left\{\begin{array}{lll}1 & 2 & 3\end{array}\right\}$ | Component ident. nos. |
| MTA(MS) | $\left\{\begin{array}{lll}1 & 1 & 1\end{array}\right\}$ | P.C.S. types |
| IDB(MS) | $\left\{\begin{array}{lll}2 & 3 & 4\end{array}\right\}$ | Component ident. nos. |
| MTB (MS) | $\left\{\begin{array}{lll}2 & 2 & 2\end{array}\right\}$ | P.C.S. types |
|  | (S1) (S2) (S3) | Identifiers |

## A1.3.3. SUBROUTINE SVAL

This routine implements the extended Sturm sequence scanning algorithm (Section 4.6.4), which essentially enables the number of eigenvalues exceeded by any given $\lambda$ to be established, thus allowing any or all of the eigenvalues to be converged upon.

For. a given argument, $\lambda$, the number of eigenvalues exceeded is obtained by calling SUBROUTINE KOUNT which in turn calls SUBROUTINE RTERMS to form $\mathbf{R}(\lambda)$, and SUBROUITNE TRIAD to form the Cholesky decomposition of $\quad \mathbf{R}(\lambda)$.

SUBROUTINE SVAL is thus primarily concerned with the scanning procedure, and a flowchart is shown in Figure Al. 6 .
$\mathbf{R}(\lambda)$ is singular at any base system (primary) eigenvalue $\lambda_{i}^{0}$, hence to establish the composite system multiplicity, it is necessary to evaluate the number of eigenvalues exceeded at $\lambda_{i}^{O}-\varepsilon$ and $\lambda_{i}^{O}+\varepsilon$, where is a small perturbation.

Between each pair of primary eigenvalues is deemed to lie an a 'principal range', for example:

$$
\lambda_{i}^{0}+\varepsilon \leqslant \lambda \leqslant \lambda_{i+1}^{0}-
$$

(see Figure Al.7a).
Composite system eigenvalues occurring within a principal range must appear as roots of $\quad \mathbf{R}(\lambda)$. (Secondary eigenvalues).

The scanning starts as the lowest primary eigenvalue and the procedure
(1) establish multiplicity at a primary eigenvalue
(2) locate roots in following principal range
is repeated henceforth.
Each principal range is repeatedly bisected, thus producing several. sub-ranges, until a root of order unity is isolated in a sub-range. In this event, control is passed to SUBROUTINE ZFRO where interpolation methods may be used to speed convergence (see Section Al.3.4). Care is taken to recognise convergence to a multiple root.

The bounds for other sub-ranges produced by the bisection are stored in a working table, provided that there are roots within the upper sub--range. This procedure, which is illustrated in Figure A.1.7(b) ensures that the information obtained by each trial $\lambda$ evaluation is not wasted.

In EIG2, scanning is halted once an eigenvalue is located which
(a) TYPICAL|R( $\lambda$ )|PLOT INDICATING PRINCIPAL RANGES


PRINCIPAL RANGES
(b) PROCEDURE FOR LOCATION OF ROOTS IN PRINCIPAL RANGE


exceeds EMAX, and a warning message is printed out.
In EIGI, the 'constant part' approximation corresponding to ICON $=1$ is discontinued. (if in use) by simply setting ICON $=0$ when an upper limit on a principal range is encountered which satisfies

$$
\mathrm{EU}>\mathrm{EBAR} / 10.0
$$

## A1.3.4. The Location of Simple Roots

The well-known technique of simple bisection was utilised throughout most of the developed work, principally because it is a 'safe' algorithm to use.

However, investigations into more efficient algorithms were made at a late stage with attention focussing on the technique of modified. successive linear interpolation as proposed by Peters and Wilkinson [20]. In essence, the value of $|\mathbf{R}(\lambda)|$ is evaluated at each trial $\lambda$ (involving just a little more work than that required for establishing a sign count) enabling linear interpolation to be used for the new root estimate. Unacceptable iterates and slow convergence are avoided by combining linear interpolation with bisection.

In the practical tests discussed in Chapter 8 however, a similar algorithm due to Brent [65] was incorporated into the computational systems.

## A1.4. PART 3-CALCULATION OF COMPOSITE SYSTEM ETGENVECTORS

The flow chart for the controlling PROGRAM EIGVEC is shown in Fig. Al. 8.

SUBROUTINE SET assigns unique displacement and modal freedom numbers to the base system components, and sets the variables $N T$ and $N G$.

For each unique eigenvalue of interest $\lambda_{i}$, multiplicity $p_{i}, \mathbf{R}\left(\lambda_{i}\right)$ is formed. SUBROUTINE TRIAD now simultaneously factorises this matrix and removes row/columns corresponding to a "zero" diagonal term. The resulting factorisation is thus of the non-singular matrix $\quad \mathbf{R}_{11}$. The number of "zeros" encountered is equal to the 'gain multiplicity', MZ.

Eigenvectors corresponding to persistent eigenvalues are calculated by SUBROUPINE PVEC, with the 'persistent multiplicity' given by $M P=p_{i}-M Z$.

Each eigenvector corresponding to a gained eigenvalue is calculated
by the sequence of routines:
(i) RHS - forms a column of the $\mathbf{R}_{12}$ matrix
(ii) SLVE . . forward and backward substitution for ć.
(iii) FILL - forms complete $C$ vector
(iv) RVEC - forms physical eigenvector $X_{i}$ (involves re-reading component modes from disc store).

Completed eigenvectors are written to disc 10.
The component eigenvectors are stored in place of $\quad \mathbf{R}\left(\lambda_{i}\right)$, thus the only additional core requirements over and above that for Part 2 is for the modal and physical eigenvectors:

$$
\begin{equation*}
{ }^{I_{C_{5}}}=2 * N T \quad{ }^{2} C_{5}=N T+N G \tag{Al.6}
\end{equation*}
$$

A1.5. PART 4-CALCULATION OF GENERALISED MASS AND STIFFNESS TFRMS
The flow chart for PROGRAM GENKM is shown in Figure Al.9. The contribution to the generalised mass and stiffness terms is calculated for each component individually, and added in to the accumulating terms. The ratio of generalised stiffness to generalised mass yields the (Rayleigh quotient) eigenvalue for that mode.

Program listings for the computational systems described in this appendix are to be incorporated in an Aeronautical Structures. Departmental Report.

FIGURE AI. 1
PART 1 - OVERLAY ( 1,0 ) - PROGRAM COMPNT


## FIGURE AI.?

PART 1 - OVFRTAY (1, 1) PROGRAM SETKM

$\dagger$ Different Subroutine Versions for EIGl

FIGURE AI. 3
PART 1 - OVERLAY (2,I) - PROGRAM MODES (ETGI)
(Called for each primary component)


FIGURE AI. 4
PART' 1 - OVERLAY ( 2,1 ) PROGRAM MODES (EIG2)


## FIGURE Al. 5

PART 2-OVERLAY $(2,0)$ - PROGRAM EIGVAI


## FIGURE AI. 6

## SUBROUTINE SVAL

(Extended Sturm Sequence Scanning Algorithm)


FIGURE A1. 6 (Continued)


## FIGURE AI. 6 (Continued)

The Analysis of a Principal
Range
5

DEFINE SUB-RANGE DATA
FROM WORKING TABLE
(ie. $X I \leqslant X \leqslant X U$ )
$\left\{\begin{array}{ll}\mathrm{XI}, & \mathrm{KI} \\ \mathrm{XU}, & \mathrm{KU}\end{array}\right\}$

SER NO. OF ROOTS
IN SUB-RANGE


FIGURE A1. 8
PART 3-OVERIAY $(3,0)$ - PROGRAM ETGVEC


FIGURE AI. 9

## PART 4 - OVERLAY $(4,0)$ - PROGRAM GENKM



TABLE AI. 1

LIST OF INPUT DATA

| COMMENI | PROGRAM SYMBOL | CLASS | USE/MEANING |
| :---: | :---: | :---: | :---: |
| PART 1 | TITTE <br> IMOD <br> NPC <br> Data <br> NO <br> ITYP <br> NN <br> MCS <br> MO <br> MGO <br> ICST <br> ISET <br> MA (MO) | D P D Scribe D $P$ $D$ $D$ $D$ $D$ $P$ $P$ $D$ | Title of job <br> No. of parts to be executed $(=1,2,3, \text { or } 4)$ <br> No. of primary components <br> omponent. First card always contains <br> Primary component order <br> $=0($ fixed component) $=1$ (free component) <br> No. of component normal modes required <br> No. of partial constraint sets for current P.C. <br> No. of connection freedoms in current P.C.S. <br> No. of generalised connection freedoms required <br> P.C.S. type $0=$ normal, $1=$ generalised constraints (read in), $2=$ natural constraints <br> G.C.S. identifier $=0$ implies a new set <br> Local connect. coord. nos for P.C.S. |
| $\xrightarrow{\text { PART } 2}$ | IVAI <br> NR <br> E1 <br> $\mathrm{E}_{2}$ <br> ICON <br> FBAR | P <br> D <br> D <br> D <br> P <br> D | $\begin{aligned} & \text { Control parameter for comp. system } \\ & \text { eigenvalue } \\ & \begin{aligned} \text { Calculation } & =0 \text { calculates all } \\ & =1 \text { lowest } N R \text { calculated } \\ & =2 E 1 \leqslant \lambda \leqslant E 2 \text { calcld. } \end{aligned} \end{aligned}$ <br> Constant part approximation control $\begin{aligned} \text { parameter } & =0 \text { no approximation } \\ & =1 \lambda_{i}^{\circ}>\text { EBAR defines } \\ & =\frac{1}{\text { constant part }} \end{aligned}$ |

TABLEEA.I (Continued)

| COMMENT | PROGRAM SYMBOL | CLASS | USE/MEANING |
| :---: | :---: | :---: | :---: |
| PART 2 (contd) | $\begin{gathered} \mathrm{NC} \\ \operatorname{ITC}(\mathrm{NC}) \\ \mathrm{MS} \\ I D A(M S) \\ M T A(M S) \\ I D B(M S) \\ M T B(M S) \end{gathered}$ | $\begin{aligned} & D \\ & D \\ & D \\ & D \\ & D \\ & D \\ & D \end{aligned}$ | No. of components in base system <br> P.C. type for each component <br> No. of constraint sets in base system <br> $\left.\left.\begin{array}{l}\text { Component ident. nos } \\ \text { P.C.S. types } \\ \text { Component ident. nos } \\ \text { P.C.S. types }\end{array}\right\} \begin{array}{c}\text { Dominant } \\ \text { half }\end{array}\right\} \begin{aligned} & \text { Discarded } \\ & \text { half }\end{aligned}$ |
| PART 3 | IVEC | D | No. of eigenvectors required |

TABLEAI. 2
LIST OF MAIN PROGRAM VARTABLES

| COMMENT | NAME | SETP IN ROUTINE | USE/MEANING |
| :---: | :---: | :---: | :---: |
| PART 1: | NTP <br> NGP <br> NGO <br> EMIN <br> EMAX <br> M <br> MG <br> MPS <br> MGS <br> NM <br> NLG | COMPNT COMPNT MODES MODES MODES CSTRNT CSTRNT CSTRNT GCON CMODE GTERMS | Sum of P.C. displacement freedoms <br> Sum of P.C. generalised freedoms <br> P.C. generalised order <br> Of the set comprising the highest normal mode eigenvalues from all <br> P.C.'s, EMIN is the minimum, EMAX is the maximum <br> Total no. of connection coords specified for all P.C.S.'s <br> Total no. of generalised constraints specified for all P.C.S.'s <br> Total no. of P.C.S.'s <br> Total no. of G.C.S.'s specified <br> No. of ortho-normal constraint modes included for a given P.C. <br> Total no. of $\mathbf{G}$ terms written to disc, hence defines length of array required to store them in-core in part 2. |
| PART 2: | MM <br> L <br> NE <br> NV | SETUP <br> SETUP <br> EARRAY <br> EARRAY | Order of the matrix $\mathbf{R}(\lambda)$ <br> Length of vector containing off-diagonal terms for $\mathbf{R}(\lambda)$ <br> Total no. of entries in the lists of unique eigenvalues and associated multip licities <br> Total no. of eigenvalues contained in above list |
| PART 3: | NT <br> NG | SET <br> SET | Total no. of displmt. freedoms in base system <br> Total no. of generalised freedoms in base system |

## TABLE AI. 3

## LIST OF FIXED PARAMETERS

The fixed parameters relating to the routine SIMULT (S.I.) in EIG2, and to the routine DIAG (E.J.) in EIG. are described along with these routines in APPENDICES 3 and 4. respectively.

| COMMENT | NAME | $\begin{aligned} & \text { ROUTINE } \\ & \text { USED IN } \end{aligned}$ | MEANING/USE | TYPICAL <br> VALUE |
| :---: | :---: | :---: | :---: | :---: |
| * | FAC | MODES | Used to provide a non-singular stiffness matrix for the S.I. process in the case of a free component by $\mathbf{K}_{o}=\left(\mathbf{K}_{0}+F A C * \mathbf{M}_{0}\right)$ | $10^{3}$ |
| * | TOL | TRIAK | Tolerance for recognising a zero diagonal term on factorisation | $10^{6}$ |
| * | TLM | CMODE | Tolerance for inclusion testing of constraint modes | $10^{-2}$ |
| * | TLIN | GCON | Tolerance for inclusion testing of generalised constraints | $10^{-2}$ |
|  | TLU | EARRAY | Tolerance for separation of close eigenvalues | $10^{-4}$ |
|  | ILIX | $\begin{aligned} & \text { SVAL } \\ & \text { ZERO } \\ & \text { EIGVEC } \end{aligned}$ | Tolerance for use in setting scanning algorithm arguments, location of roots, perturbation of primary eigenvalue arguments | $10^{-4}$ |
|  | TLZ | TRIAD | Tolerance for recognising zero diagonal terms on triangulation | $10^{-7}$ |
|  | MIN | ZERO | Minimum no. of simple bisections | 7 |
|  | MAX | ZERO | Maximum no. of bisections + interpolation steps | 15 |

## APPENDIX 2

## THE COMPONENT MODE METHOD AND SYSTEM EIG3

## A2.1. GENERAL DESCRTPTITON

The Component Mode Method has been discussed in a general context in Section 4.6.5.

Here, the method of Craig and Bampton [55], requiring the calculation of fixed constraint normal modes and constraint modes for each component is extended to a non-diagonal mass matrix.

Use of the simultaneous iteration technique both for component normal modes and composite system normal modes enables the procedure to be programmed in an extremely concise way. Details of the program EIG3 are included in this Appendix.

Numerical results obtained are compared with results from Kron's method in Chapter 6.

A reference list of nomenclature is contained in TABIE A2.2.

## A2.2. THE COMPONENT ANALYSIS

Initially the structure is subdivided into components by restraining the connection freedoms along component boundaries. Each component may however be considered separately from the rest of the structure. A typical component is now considered.

Let the full displacement vector be partitioned into connection freedoms $\quad \mathbf{x}_{c}$ and internal freedoms $\mathbf{x}_{i}$, and the undamped equation of motion partitioned accordingly:

$$
\left\{\left[\begin{array}{ll}
\mathbf{K}_{i i} & \mathbf{K}_{i c}  \tag{A2.2.1}\\
\mathbf{K}_{c i} & \mathbf{K}_{c c}
\end{array}\right]-\lambda\left[\begin{array}{cc}
\mathbf{M}_{i i} & \mathbf{M}_{\mathbf{i c}} \\
\mathbf{M}_{c i} & \mathbf{M}_{c c}
\end{array}\right]\right\}\left[\begin{array}{l}
\mathbf{x}_{i} \\
\mathbf{x}_{c}
\end{array}\right]=\left[\begin{array}{l}
\mathbf{X}_{i} \\
\mathbf{X}_{c}
\end{array}\right]
$$

A2.2.1. Normal Modes
The 'fixed constraint' normal modes are found for $\quad \mathbf{X}_{\mathrm{c}}=\mathbf{0}$ and $\mathbf{X}_{\mathbf{i}}=\mathbf{0}$, that is by the eigenreduction of:

$$
\begin{equation*}
\left(\mathbf{K}_{i i}-\lambda \mathbf{M}_{i i}\right) \mathbf{x}_{i}=0 \tag{A2.2.2}
\end{equation*}
$$

In general only the few lowest modes are found; $\boldsymbol{\Phi}_{\mathrm{N}}$ with generalised coordinates $\mathbf{q}_{\mathrm{N}}$.

## A2.2.2. Constraint Modes

Constraint modes are found by imposing a unit displacement in turn at the connection coordinates while all other connection coordinates are fixed. Static modes are used, hence with $\lambda=0$, the first set of equation (A2.2.1) gives

$$
\begin{equation*}
\mathbf{K}_{i i} \mathbf{x}_{i}+\mathbf{K}_{i c} X_{c}=\mathbf{0} \tag{2}
\end{equation*}
$$

If the set of imposed connection coordinate displacement patterns are gathered together in the unit matrix $I$, the corresponding constraint modes $\boldsymbol{\Phi}_{\mathrm{c}}$ are obtained by the solution to the equation:

$$
\begin{equation*}
K_{i i} \Phi_{c}=-K_{i c} \tag{A2.2.4}
\end{equation*}
$$

It may be noted here that the number of normal modes $n_{n}$ is arbitrary while the number of constraint modes $n_{c}$ is fixed.

## A2.2.3. The Component Coordinate Transformation

The transformation to generalised coordinates is defined by

$$
x_{0}=\left[\begin{array}{l}
x_{i}  \tag{A2.2.5}\\
x_{c}
\end{array}\right]=\left[\begin{array}{ll}
\Phi_{N} & \Phi_{c} \\
0 & I
\end{array}\right]\left[\begin{array}{l}
q_{N} \\
x_{c}
\end{array}\right]=T \mathbf{q}
$$

and utilising this to transform the equation (A2.2.1) results in

$$
\begin{equation*}
\left(\mathbf{K}_{c}-\lambda \mathbf{M}_{c}\right) \mathbf{q}=\mathbf{Q} \tag{A2.2.6}
\end{equation*}
$$

where

$$
\begin{aligned}
& \mathbf{K}_{c}=\mathbf{T}^{\mathrm{t}} \mathbf{K}_{0} \\
& \mathbf{M}_{\mathrm{c}}=\mathbf{T}^{{ }^{t} \mathbf{M}_{o}} \\
& \mathbf{Q}=\mathbf{T}^{\mathrm{t}} \mathbf{X}_{\mathrm{o}}
\end{aligned}
$$

For non-diagonal $K_{0}$ and $M_{o}$ (component stiffness and mass matrices), the form of $\mathbf{K}_{c}$ and $\quad \mathbf{M}_{c}$ is identical - here shown for $\quad \mathbf{K}_{c}$ :

$$
K_{c}=\left[\begin{array}{lc}
\left(\Phi_{N}^{t} K_{i i} \Phi_{N}\right) & (S Y M M .) \\
\left(\Phi_{c}^{t} K_{i i} \Phi_{N}+K_{c i} \Phi_{N}\right) & \left(K_{c c}+\Phi_{c}^{t} K_{c i}^{t}+K_{c i} \Phi_{c}+\Phi_{c} K_{i i} \Phi_{c}\right)
\end{array}\right]
$$

(A2.2.7)

By virtue of the orthogonality of the normal modes, and with suitable normalisation:

$$
\begin{aligned}
& \boldsymbol{\Phi}_{\mathrm{N}}^{\mathrm{t}} \mathbf{K}_{\mathrm{ii}} \boldsymbol{\Phi}_{\mathrm{N}}=\boldsymbol{\Lambda}_{\mathrm{N}} \\
& \boldsymbol{\phi}_{\mathrm{N}}^{\mathrm{t}} \mathbf{M}_{\mathrm{ij}} \boldsymbol{\phi}_{\mathrm{N}}=\mathbf{I}
\end{aligned}
$$

where $\Lambda_{N}$ is the diagonal matrix of normal mode eigenvalues.
However, by equation (A2.2.4), the off diagonal block in $\mathbf{K}_{c}$ is zero, and the second diagonal block simplifies so that equation (A2.2.6) may be written as:

$$
\left\{\left[\begin{array}{ll}
\Lambda_{\mathrm{N}} & 0  \tag{A2.2.8}\\
0 & \overline{\mathbf{K}}_{\mathrm{cc}}
\end{array}\right]-\lambda\left[\begin{array}{cc}
\mathbf{I} & \overline{\mathbf{M}}_{\mathrm{cN}}^{t} \\
\overline{\mathbf{M}}_{\mathrm{cN}} & \overline{\mathbf{M}}_{\mathrm{cc}}
\end{array}\right]\right\}\left[\begin{array}{l}
\mathbf{q}_{N} \\
\mathbf{x}_{\mathrm{c}}
\end{array}\right]=\left[\begin{array}{c}
0 \\
\mathbf{X}_{\mathrm{c}}
\end{array}\right]
$$

where

$$
\begin{align*}
& \bar{K}_{c c}=\mathbf{K}_{c c}+\mathbf{K}_{c i} \Phi_{c}  \tag{A2.2.9}\\
& \bar{M}_{c c}=M_{c c}+M_{c i} \Phi_{c}+\phi_{c}^{t} M_{c i}^{t}+\Phi_{c}^{t} \mathbf{M}_{i i} \Phi_{c} \text { (A2.2.10) } \\
& \bar{M}_{c N}=\Phi_{c}^{t} M_{i i} \Phi_{N}+M_{c i} \Phi_{N} \tag{A2.2.11}
\end{align*}
$$

## A2.2.4. Use of SimuItaneous Iteration

The usual method for solving equation (A2.2.4) is to form the Cholesky factorisation of $K_{i i}$, and to perform forward and backward substitution for each R.H.S..

The factorised stiffness matrix is thus readily available for use in a simultaneous iteration process for the normal modes. S.I. is thus very convenient, involving forward and backward substitution again and matrix products with the mass matrix $M_{i i}$, provided only a small fraction of the normal modes are required.

A2.3. ASSEMBLY AND SOLJTIION

## A2.3.1. The Assembled Composite System Matrices

$A$ consideration of two components, $A$ and $B$, is sufficient to indicate the assembled system matrices. Component normal coordinates are carried through as independent freedoms, while the connection coordinates are related to the global set of connection coordinates by a boolean transformation, ensuring displacement compatibility between components.

Considering ${ }^{A} X_{c}={ }^{B} X_{c}=X_{c}$, the composite system matrix eigenvalue equation is as follows: $\left(\mathbf{X}_{c}=\mathbf{0}\right)$

Provided only a fraction of the normal modes from each component is included, the order of the above eigenvalue problem is considerably smaller than the fully assembled problem. In addition, the partitioning and general form of equation (A2.2.12) indicates that a variable bandwidth storage method would.place relatively low demands on core.

The general composite system eigenvalue equation is denoted by

$$
\begin{equation*}
\left(K^{\prime}-\lambda M^{\prime}\right) \quad q^{\prime}=0 \tag{A2.2.13}
\end{equation*}
$$

## A2.3.2. Solution of the Composite System Eigenproblem

The composite system eigenproblem is of 'algebraic' form and may be solved by standard method, however transformation methods would greatly increase the core requirement. Equation (A2.2.12) describes an approximate system in which higher mode freedoms are not included, thus only the lower eigenspectrum is of practical interest. Clearly, the factorisation of the stiffness matrix $\mathbf{K}^{\prime}$ is a very light task and so simultaneous iteration is again highly suited.

In a situation where a fair number of eigenvalues are required to good accuracy, or just intermediate ones, the Sturm sequence method may be used. Clearly, only factorisation of the last $n_{b}$ rows of ( $K^{\prime} \cdot-\lambda M^{\prime}$ ) is required for each trial $\lambda$ (where $n_{b}$ is the number of global connection coordinates). The sign count so obtained is added to the number of component normal mode eigenvalues exceeded by $\lambda$ to obtain the full sign count.

## A2.3.3. Eigenvector Calculation

The full composite system eigenvectors are obtained from the generalised eigenvectors by application of equation (A2.2.5) for each component, thus obtaining the displacement at the internal freedoms.

## A2.4. THE COMPUTATIONAL SYSTEM EIG3

The theory of the previous sections is applied in the program system EIG3. The three part organisation and use of disc storage is indicated in Fig. A2.1, while the overlay structure is shown in Fig. A2.2.

Table: A2.3 lists the input data and other principal program symbols respectively at the end of this appendix.

The three constituent parts are now briefly discussed.

## A2.4.1. Part l - The Analysis of the Components

The flowchart for PROGRAM COMPNI is show in Fig. A2.3. No attempt is made to define a minimum data set as in EIG2, however the parameter IREP (see Table A2.3) is introduced to save work in cases where there
is a form of repetition between components.
PROGRAM SETKM, responsible for forming $K_{o}$ and $M_{o}$ is identical to that of $\mathrm{EIG} \mathrm{G}_{2}$, using variable bandwidth storage.

PROGRAM MODES is flowcharted in Fig. A2.4. 'K ${ }_{0}$ and $M_{0}$ are
first partitioned conformably with $\quad \mathbf{x}_{0}=\left\{\mathbf{x}_{i} \mathbf{x}_{c}\right\}$, and the last NC rows are held in full, where NC is the dimension of $X_{c}$. The spaces containing $K_{c i}$ and $M_{c i}$ are useful in the subsequent computation of $\quad \bar{K}_{c c}, \quad{ }_{c i}^{\bar{M}_{c c}}$ and ${ }^{c i} \bar{M}_{c N}$ once the normal and constraint modes have been calculated.

Where suppression of the connection freedoms does not render $\mathbf{K}_{\text {ii }}$ non-singular, additional internal freedoms may be designated connection freedoms to remove rigid modes.

A2.4.2. Part 2 - The Composite System Assembly and Analysis
The flowchart for PROGRAM VALVEC is shown in Fig. A2.5. The various component matrices are read back from disc where they were stored during part l, and assembled into composite system matrices of the general form:


The dimension $N B$ is read in at the start of the run, while NA is calculated during part l. Variable bandwidth storage enables large problems to be held in a reasonable core space, provided NB is not too large. The storage areas used for $\mathbf{K}_{\circ}$ and $\mathbf{M}_{\circ}$ in part 1 are reused for
$\mathbf{K}^{\prime}$ and $\mathbf{M}^{\prime}$. Similarly, the routines for factorisation, S.I., mass matrix multiplication etc., used during part 1 are simply reused during part 2.

## A2.4.3. Part 3- Full Eigenvectors. Generalised Mass and Stiffness Terms

The flowchart for PROGRAM CMPVEC is shown in Fig. A2.6. For each component in turn, the local eigenvector is calculated, and the contribution towards the composite system generalised terms calculated.

## A2.5. APPLITCATION TO PLATE BENDING EXAMPLE (NO.1)

The program EIG3 was tested using the plate bending example introduced in Section 6.3.5. The results obtained by including 2, 3 and 4 'fixed constraint' normal modes (cases A, B, C respectively) from each component are shown in TABLE A2.1. 12 'unit displacement' constraint modes are included in all cases. Hence the orders of the composite systems are given by

| Case A | $2+2+12=16$ |
| :--- | :--- |
| Case B | $3+3+12=18$ |
| Case C | $4+4+12=20$ |

The lowest six eigenvalues, obtained to five places, are presented together with percentage error with respect to the full solution.

The results validate the operation of the progran and indicate the excellent representation of the low composite systen eigenspectrum. It was noted that, in general, accuracy dropped off rapidly once the highest component normal mode eigenvalue had been exceeded.

These results may be compared with those obtained by the approximate Kron method (TABLE 6.5). While accuracy of the lowest eigenvalues is generally better with the latter approach, the accuracy fall-off occurs earlier. This is expected as the 'fixed constraint' normal modes extend to a higher $\lambda$ value than an equivalent number of 'free connection coordinate modes' as used in Kron's method.

## TABLIE A2. 1

PLATE BENDING EXAMPLE (NO. 1)
RESULTS FROM TEST CASES

|  | A | B | C | FULL |
| :---: | :---: | :---: | :---: | :---: |
|  | Composite System Eigenvalues |  |  | SOLuTion |
| 1 | 75.046 | 74.947 | 74.947 | 74.931 |
| 2 | 544.41 | 544.26 | 542.22 | 541.80 |
| 3 | 570.24 | 568.94 | 568.88 | 568.84 |
| 4 | 2323.3 | 2238.5 | 2222.4 | 2204.9 |
| 5 | 2398.0 | 2394.0 | 2391.4 | 2389.6 |
| 6 | 6909.1 | 6258.3 | 6184.7 | 6111.4 |
| \% Errors w.r.t. full solution |  |  |  |  |
| 1 | 0.153 | 0.021 | 0.021 |  |
| 2 | 0.482 | 0.454 | 0.077 |  |
| 3 | 0.246 | 0.018 | 0.007 |  |
| 4 | 5.41 | 1.52 | 0.794 |  |
| 5 | 0.352 | 0.184 | 0.075 |  |
| 6 | 13.05 | 2.40 | 1.20 |  |


| First 4 'fixed constraint' component eigenvalues are |  |
| :---: | :---: |
| 1 | 1228.5 |
| 2 | 3205.4 |
| 3 | 9478.5 |
| 4 | 17159.0 |

## EIG3 - BASIC ORGANISATION AND DISC TRANSFERS




FIGURE A2. 3.
PROGRAM COMPNT - OVERLAY $(1,0)$


FIGURE A2.4.


FIGURE A2.5.


FIGURE A2. 6.
PROGRAM CMPVEC - OVERLAY $(3,0)$



## TABLE A2. 3

## COMPUTER PROGRAM NOMENCLATURE FOR EIG3

LIST OF INPUT DATA


[^0]
## APPERDIX 3

## SUBROUTINE SIMUIT (SIMULTANEOUS ITERATION METHOD)

## A3.1. SUBROUTINE SIMULT

The theory of simultaneous iteration has been dealt with in Section 2.3. A flowchart of the routine SIMULI is show here in Figure A3.1.

The three fixed parameters used by the routine are as follows: $N A V=$ number of additional trial vectors to be used above NN (typically 3)
LIM $=$ the maximum number of iterations permissable (typically 10)
TLS $=$ a tolerance for convergence testing (typically $10^{-5}$ )
The stiffness matrix is factorised prior to entering SIMULT. The interaction analysis utilises the off-diagonal terms in the interaction matrix to provide a measure of the coupling between trial vectors. Convergence is obtained when all the first NN diagonal terms in the interaction matrix are within a small tolerance of the previous iteration's values. This tolerance is given by the latest diagonal term $x$ TLS.

FIGURE A3.1.
SUBROUTINE SINULT


## APPENDIX 4

## SUBROUTINE DIAG (EXTENDED JACOBI METHOD)

## A4.1. BASIC THEORY

The extended Jacobi method involves the conversion of the structure mass 'and stiffness matrices simultaneously to diagonal form via a series of similarity transformations:

$$
\begin{aligned}
\mathbf{P}_{\mathbf{r}}^{t} \mathbf{K}_{\mathbf{r}-1} \mathbf{P}_{\mathbf{r}}=\mathbf{K}_{\mathbf{r}} & \mathbf{K}_{\mathrm{r}} \rightarrow \mathbf{K} \quad \text { (diag) as } \mathbf{r} \rightarrow \infty \\
\mathbf{P}_{\mathrm{r}}^{t} \mathbf{M}_{\mathrm{r}-1} \mathbf{P}_{\mathbf{r}}=\mathbf{M}_{\mathbf{r}} & \mathbf{M}_{\mathbf{r}} \rightarrow \mathbf{m} \quad \text { (diag) as } \mathrm{r} \rightarrow \infty
\end{aligned}
$$

The technique is useful where a complete set of eigenvalues and eigenvectors are required for fairly small problems.

Each transformation step zeros a pair of off-diagonal lower triangle terms: e.g., for the ( $i, j$ ) termis (non-zero terms in $P_{r}$ indicated).


The condition that $k_{i j}=m_{i j}=0$ enables $p_{1}$ and $p_{2}$ to be calculated by

$$
\begin{equation*}
p_{1}=\frac{-b \pm \sqrt{\left(b^{2}-4 a c\right)}}{2 a} \tag{A4.2}
\end{equation*}
$$

where

$$
\begin{aligned}
& a=k_{j j} m_{i j}-m_{j j} k_{i j} \\
& b=k_{j j} m_{i i}-m_{j j} k_{i j} \\
& c=k_{i j} m_{i i}-m_{i j} k_{i i}
\end{aligned}
$$

whence

$$
\begin{equation*}
p_{2}=\frac{-\left(k_{i, j}+p_{1} k_{j j}\right)}{\left(k_{i i}+p_{2} k_{i j}\right)}=-\frac{\alpha}{\beta} \tag{A4.3}
\end{equation*}
$$

The minus sign is used consistently in the numerator of equation A4.2. Care must be taken to recognise ill-conditioning in this numerator. The following special cases are also recognised

| if | $a=0$ | $p_{1}=-\frac{c}{b}$ |
| :--- | :--- | :--- |
| if | $a=b=0$ | $p_{1}$ is indeterminate |
| if | $\beta=0$, | an alternative expression for equation | (A4.3) involving mass terms should be used.

The procedure adopted is to work through the off-diagonal terms in a columnwise manner. In general, however, previously zeroed locations are disturbed by subsequent transformation, and several 'passes' are required until all the off-diagonal terms are within some acceptable tolerance. The transformed mass and stiffness matrices remain symmetric at each stage. Both zero and coincident eigenvalues and associated vectors are obtained by the technique.

If the total number of transformations used to obtain a 'converged' state is $s$, the eigenvalues are given by $\lambda_{i}=k_{i i} / m_{i i}$ while the eigenvectors are formed from the transformation matrices by

$$
\begin{equation*}
\phi=\prod_{r=1}^{s} P_{r} \tag{A4.4}
\end{equation*}
$$

A normalisation procedure is incorporated on each transformation. If $p_{1}$ or $p_{2}$ exceed unity modulus, the appropriate column of $\quad \mathbf{P}_{r}$ is factored by $\frac{1}{p_{1}}$ or $\frac{1}{p_{2}}$. At each stage it is thus only necessary to save the terms ${ }^{p_{1}} p_{1},{ }^{\prime} p_{2}, p_{3}, p_{4}$ where $p_{3}$ and $p_{4}$ are the diagonal terms in the columns containing $p_{2}$ and $p_{1}$ respectively.

## A4.2. SUBROUTINE DIAG

SUBROUTINE DIAG and associated subroutines are responsible for carrying out the theory of the previous section. DIAG initially reads in the control parameters listed below:

| TOL | tolerance for off-diagonal term testing |
| :---: | :---: |
| MAXK | ```maximum number of complete 'passes' allowable (typically 8-convergence usually obtained in 5 or 6)``` |
| IC | a write parameter - used as a debugging aid (normally 0 ) |
| IN | eigenvector normalisation parameter |
|  | $=0$ no normalisation |
|  | $=1$ w.r.t. largest term |
|  | $\begin{aligned} & =2 \text { such that } \phi_{i}^{t} \phi_{i}=1 \\ & =3 \text { such that } \quad \phi_{i}^{t} M \phi_{\dot{j}}=1 \end{aligned}$ |
| IW | number of eigenvalues and eigenvectors required as printed output (starting with the lowest). |

DIAG accesses the structure mass and stiffness matrices and the problem order via COMMON blocks. The matrices must be stored in triangular columnwise form. The transformations are then carried out, the values $p_{1}, p_{2}, p_{3}, p_{4}$ being stored on a scratch disc file. A test for convergence is made at the end of each 'pass', the criteria being that all off-diagonal terms must be less than a tolerance based on the smallest diagonal terms. As a zeroing leading diagonal stiffness matrix term is possible, a minimum diagonal term is set within the program. At convergence, the eigenvalues are calculated.

Control is passed to SUBROUTINE MULT which recovers the $p_{i}$ terms from disc store and forms up the eigenvector matrix in the same core locations as utilised by the input mass and stiffness matrices. Finally SUBROUIINE WRIT prints the eigenvalues and eigenvectors in ascending order.

## APPENDIX 5

## BEAM AND PLATE FINITE FLEMEFITS

Brief details of the beam and plate bending finite elements utilised in conjunction with the computational systemsare included in this appendix.

## A5.1. 2-DIMENSIONAL BEAM ELEMENT

The beam element used included flexural stiffness in one plane only and is rigid in extension (Figure A5.I(a)). Hermitian polynomials corresponding to the four element freedoms and expressed in terms of the local coordinate $q$ define the displacement interpolation functions:

$$
\begin{aligned}
& w_{1}=\frac{1}{4}\left(2-3 q+q^{3}\right) \\
& w_{2}=\frac{1}{8}\left(1-q-q^{2}+q^{3}\right) \\
& w_{3}=\frac{1}{4}\left(2+3 q-q^{3}\right)
\end{aligned}
$$

Ignoring the effect of shear flexibility, there is one non-zero shear component i.e., es. Assuming linear elasticity with Youngs modulus $E$, and density per unit volume $\rho$, straightforward application of the displacement method produces an element stiffness matrix given by

$$
\hat{\mathbf{k}}_{g}=\frac{E I}{\ell^{3}}\left[\begin{array}{cccc}
12 & & \text { Symm. } \\
6 \ell & 4 l^{2} & & \\
-12 & -6 \ell & 12 & \\
6 \ell & 2 l^{2} & -6 l & 4 l^{2}
\end{array}\right] \quad, I=\frac{t^{3}}{12}
$$

and ignoring rotary inertia, a kinematically consistent mass matrix given by

$$
\mathbf{m}_{g}=\frac{\rho t \ell}{420}\left[\begin{array}{rrrr}
156 & & \text { Symm } \\
22 l & 4 l^{2} & & \\
54 & 13 \ell & 156 & \\
-13 l & -3 l^{2} & -22 l & 4 l^{2}
\end{array}\right]
$$

(assuming unit depth).

## A5.2. RECTANGULAR PLATE BENDING FWLEMENT

Initial tests were carried out using a. 12 degree of freedom unconforming rectangular plate bending element (see for example [66]). Possessing $w, \frac{\partial w}{\partial x}, \frac{\partial_{w}}{\partial y}$ freedoms at each node, the total number of freedoms is insufficient to define a complete cubic displacement polynomial over the element, and normal slope continuity between adjacent elements is not enforced. This non-confirming property leads to erratic convergence as grid size is refined.

Such behaviour was observed in some of the test examples, and it was decided to employ a 16 degree of freedom conforming element instead [67] The conforming properties assure that convergence to the exact natural frequency is monatonically from above [68].

The element is shown in Figure A5.2, the twist ( $\left(\frac{\partial^{2} w}{\partial x \partial y}\right)$ being the additional freedom included at each of the four nodes. The element was programmed utilizing Hermitian polynomials, in terms of local coordinates $q_{1}, q_{2}$, and Gaussian integration. Constant thickness is assumed.

Both elements employed were validated against documented test case.
; FIGURE A5.1.
2-DIMENSIONAL BEAM ELEMENT

$<$ FIGURE A5.2.
16 DEGREE OF FREEDOM RECTANGULAR
PLATE BENDING ELEMENT


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[^0]:    not required if $\operatorname{IREP}=1$ )
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