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SUB-OPTIMAL CONTROL OF DISCRETE STOCHASTIC PROCESSES

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ABSTRACT

The problem of selecting a state feedback controller with a specified structure so as to minimize the expected value of quadratic cost for a discrete linear system disturbed by a zero-mean white noise disturbance is posed. This problem is solved by use of an iterative procedure. The existence properties of the solution and the convergence properties of the procedure are established. Numerical examples are considered to test the computational feasibility of the proposed procedure. It is then demonstrated that problems involving noise corrupted output feedback, problems involving dynamic compensators with fixed and tunable parameters, and team theoretic problems can be transformed into problems of the type treated.

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CONTENTS

CHAPTER 1

INTRODUCTION

1.1. Linear Quadratic Design, Suboptimal Controllers, and the Specific Optimal Approach

The Linear Quadratic approach to the design of feedback controllers for multivariable systems is well established [1]. It can be used for both the design of linear and of nonlinear systems. It is not, in general, possible to obtain feedback solutions for optimal nonlinear control problems. However, an optimal open loop control can be determined, the nonlinear equations can be linearized about the optimal operating point or trajectory, then by use of the linear quadratic results a feedback controller can be obtained [2]. This linear state feedback controller will ensure the behaviour of the nonlinear system remains near optimal.

The linear state feedback controllers obtained by solving the linear quadratic problem have many good features [1], guaranteed stability, good step responses, and insensitivity to noise and plant variations. Their chief disadvantage is their complexity. Normally not all the states can be measured and an estimator or observer is needed. For a stochastic problem the order of the estimator is that of the system. It is well known that by using frequency domain techniques one can obtain much simpler controllers which give good response. Unnecessary complexity is particularly burdensome in finite time problems where the time records of all the controller and estimator gains must be stored.

Moreover, if the system being controlled is geographically distributed, as a power system usually is, the requirements that the measurements be transmitted to a central location where the control calculations can be performed and that the control inputs be transmitted from the central location to the various actuators can cause severe telecommunications difficulties. For such systems it would be preferable to use local feedback, that is, feedback of measurements taken near the actuators. If such a controller could not provide adequate performance one might wish to try transmitting a few essential variables.

Thus a controller which has a worse performance, but which is easier and less costly to implement, may be more desirable than the optimal controller. Such controllers are termed Suboptimal Controllers. If the suboptimal controller is obtained by choosing the parameters in a specified controller structure optimally then the controller is called a Specific Optimal Controller. If the structure is well chosen a specific optimal controller which is easily implementable can give performance very near optimal. Such a design approach makes good use of both the human designers ability to identify good controller structures and the computer techniques for choosing parameters optimally.

Conventional nonlinear optimization techniques can be applied to the design of specific optimal controllers when the number of parameters is small. If the number of parameters is large then methods which take advantage of the structure and properties of the system must be used.

The problem considered in this thesis is that of optimally selecting the parameters in a linear controller of fixed structure so as to minimize a quadratic cost. The model is assumed to be linear and the time finite. Such problems occur when batch processes or grade changes are dealt with or when nonlinear trajectory optimization problems are linearised about the optimal trajectory. These problems can not, in general, be handled using frequency domain design procedures and the use of nonlinear optimization techniques is difficult and inefficient. A discrete formulation is appropriate as a digital implementation of the controller is necessary to cope with the storage of the parameter trajectories.

First, the problem in which only certain elements of the feedback gain matrix are allowed to be non-zero will be solved, by use of special structural relations which are derived in the thesis. Then, other linear compensator problems will be transformed into that form.

Although this thesis is primarily concerned with the finite time problem the theoretical results produced could be applied to the infinite time or steady state problem. Some brief remarks are made about this problem in the section on further work.

1.2. Problem Definition

Consider the linear discrete-time system

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{1.1}$$

where x_k is an n-vector termed state, and

$$E[x_0 x_0^T] = V_0 \tag{1.2}$$

By definition, E[ ] is the expectation operator. u_k is an m-vector termed control. w_k is an independent n-vector white noise process with zero mean and variance V_w. Associated with this linear system there is a quadratic cost of the form

$$L_k = \tfrac{1}{2} \sum_{i=k}^{N-1} [x_i^T Q x_i + u_i^T R u_i] + \tfrac{1}{2} x_N^T S_N x_N \tag{1.3}$$

where Q, R and S_N are positive semidefinite matrices. A, B, V_w, Q and

R can be either constant or time varying matrices. For reasons of notational simplicity they will not be given a time index.

The expected value of the quadratic cost over the time interval [0,N] will be denoted

$$J = E\{ L_0 \} \tag{1.4}$$

It is well known [1] that (1.4) is minimized when

$$u_k = \overset{*}{G}_k x_k \tag{1.5}$$

for $k = 0,1,\ldots\ldots,N-1$, where

$$\overset{*}{G}_k = -[R+B^T\overset{*}{S}_{k+1}B]^{-1}B^T\overset{*}{S}_{k+1}A \tag{1.6}$$

and

$$\overset{*}{S}_k = Q + A^T\overset{*}{S}_{k+1}A - A^T\overset{*}{S}_{k+1}B[R+B^T\overset{*}{S}_{k+1}B]^{-1}B^T\overset{*}{S}_{k+1}A \tag{1.7}$$

for $k = N-1,\ldots\ldots,0$, and $\overset{*}{S}_N = S_N$.

The policy of using the controls given by (1.5) will be termed the OPTIMAL CONTROL POLICY and be denoted

$$\overset{*}{\pi} = [\overset{*}{G}_k, k=0,1,\ldots\ldots,N-1] \tag{1.8}$$

The value the expected cost (1.4) takes when the Optimal Control Policy is to be used will be termed the OPTIMAL EXPECTED COST and be written

$$\overset{*}{J} = E[L_0 \,|\, \overset{*}{\pi}] \tag{1.9}$$

Control policies where the control is a linear, not necessarily optimal, transformation of the state will be considered.

A LINEAR CONTROL POLICY,

$$\pi = [G_i, i=0,\ldots\ldots,N-1] \tag{1.10}$$

is the policy of using the control actions

$$u_k = G_k x_k, \quad k=0,\ldots\ldots,N-1 \tag{1.11}$$

The value the expected cost (1.4) assumes when a particular Linear Control Policy, π, is used will be denoted

$$J(\pi) = E\left[L_0 \mid \pi\right] \tag{1.12}$$

The matrix G_k is termed a STATE FEEDBACK MATRIX.

Feedback Structure

In this thesis the problem of determining the best Linear Control Policy where certain elements in the state feedback matrix are constrained to be zero will be considered. It is thus useful to define the FEEDBACK STRUCTURE, σ, as the set of co-ordinates of the unconstrained elements in the State Feedback Matrix

$$\sigma = \{(i_1, j_1),(i_2, j_2),\ldots\ldots,(i_p, j_p)\} \tag{1.13}$$

A STRUCTURED FEEDBACK MATRIX

$$G_k^S = \left\{\Gamma_k(i,j)\right\} \tag{1.14}$$

is a State Feedback Matrix that satisfies

$$\Gamma_k(i,j) = 0 \qquad \text{if} \qquad (i,j) \notin \sigma \tag{1.15}$$

where σ is some specified feedback structure. The notation $\{\Gamma_k(i,j)\}$ means the elements of the matrix, G_k^S, are $\Gamma_k(i,j)$. The elements, $\Gamma_k(i,j)$, are termed gains, and those elements not constrained to be zero are termed the unconstrained gains. Thus p, defined by (1.13), is the total number of unconstrained gains.

A STRUCTURED CONTROL POLICY, π^S, is a Linear Control Policy (2.1) where all the State Feedback Matrices are Structured State Feedback Matrices,

$$\pi^S = [G_i^S, \quad i=0,\ldots\ldots,N-1] \tag{1.16}$$

By $J(\pi^S)$ is meant

$$J(\pi^S) = E[L_0 \mid \pi^S] \tag{1.17}$$

where π^S is a structured control policy. Thus $J(\pi^S)$ is a scalar function defined on the Np-dimensional Euclidean space where each of the feedback gains in a Structured Control Policy is taken as a coordinate.

Problem Statement

The principal problem dealt with in this thesis is that of finding the Structured Control Policy (1.16) which will minimize the expected cost (1.4). Using the notation that has been developed this may be written

$$\min_{\pi^S} J(\pi^S) \tag{1.18}$$

This problem will be termed the STRUCTURED CONTROL PROBLEM.

Additional Definitions

Some additional definitions are needed to solve the problem (1.18) and to compare the results produced with those produced by other authors.

If a structured control policy is used, (1.11) may be rewritten

$$u_k = \begin{bmatrix} u_k^1 \\ u_k^2 \\ \vdots \\ u_k^m \end{bmatrix} = \begin{bmatrix} g_k^{1T} x_k^1 \\ g_k^{2T} x_k^2 \\ \vdots \\ g_k^{mT} x_k^m \end{bmatrix} = G_k^S x_k \tag{1.19}$$

where, u_k^j, is a scalar called the jth. input or control j,

$$x_k^{T} = [\zeta_k^1 \quad \zeta_k^2 \ldots\ldots \zeta_k^n] \tag{1.20}$$

and ζ_k^i is a scalar termed state i,

$$x_k^{jT} = [\zeta_k^{\psi(j,1)} \quad \zeta_k^{\psi(j,2)} \ldots\ldots \zeta_k^{\psi(j,n_j)}] \tag{1.21}$$

for $j = 1,2,\ldots\ldots,m$. x_k^j is the nj-vector of those states that may be fed back to control j. Thus, the function $\psi(j,i)$ is a function whose domain of definition is the parts of integers (j,i) where $j \in \{1,2,\ldots\ldots,m\}$ and $i \in \{1,2,\ldots\ldots,n_j\}$ and whose range is the set $\{1,2,\ldots\ldots,n\}$. $\psi(j,i)$ is assumed to have the property

$$\psi(j,1) < \psi(j,2) < \ldots\ldots < \psi(j,n_j) \tag{1.22}$$

By definition

$$\ell(j) = \left\{ \psi(j,1),\ \psi(j,2),\ldots\ldots,\psi(j,n_j) \right\} \tag{1.23}$$

for $j = 1,2,\ldots.,m$.

If $i \in \ell(j)$ then one says state i is fed back to control j. $\ell(j)$ is a list containing the co-ordinates of those states that may be fed back to control j in ascending order.

$$(g_k^j)^T = [\gamma_k(j,1)\ \gamma_k(j,2)\ldots\ldots\gamma_k(j,n_j)] \tag{1.24}$$

for $j = 1,2,\ldots.,m$.

where $\gamma_k(j,i)$ is the gain associated with the feedback of state i to control j, thus

$$\gamma_k(j,i) = \Gamma_k(j,\psi(j,i)) \tag{1.25}$$

for $j = 1,2,\ldots..,m$ and $i = 1,2,\ldots..,n_j$.

If the notation

$$j x \ell(j) = \left\{(j,\psi(j,1)),\ (j,\psi(j,2)),\ldots..,(j,\psi(j,n_j))\right\} \tag{1.26}$$

is adopted then

$$\sigma = \left\{ j x \ell(j) \mid j=1,2,\ldots.,m \right\} \equiv [\ell(j) \mid j=1,2,\ldots,m] \tag{1.27}$$

is an alternative description of the Feedback Structure.

If it is to be emphasized that no elements of the State Feedback Matrix are constrained to be zero the word COMPLETE will be used (i.e. a Complete State Feedback Matrix or a Complete Linear Control Policy). If $p = mn$ then the Feedback Structure is Complete.

A PARTIAL STATE FEEDBACK MATRIX, G_k^P, is a structured state feedback matrix where the Feedback Structure satisfies

$$\ell(i) = \ell(j) \quad \text{for all } i,j \in \left\{1,2,\ldots\ldots,m\right\} \tag{1.28}$$

Thus it is a state feedback matrix where one or more columns are constrained to be zero.

A feedback structure which satisfies (1.28) will be termed a PARTIAL STATE FEEDBACK STRUCTURE. A structured control policy where the specified structure is a partial state feedback structure will be described as a PARTIAL STATE CONTROL POLICY and be written

$$\pi^P = [G_k^P, \ k=0,1,\ldots\ldots,N-1] \tag{1.29}$$

The problem of determining the Partial State Control Policy to minimise the expected cost, J, will be termed the PARTIAL STATE FEEDBACK PROBLEM and may be written

$$\underset{\pi^P}{\text{Min}} \ J(\pi^P) \tag{1.30}$$

When dealing with Partial State Feedback Structures it is useful to have the following definitions

$$n' = n_i \qquad i=1,2,\ldots\ldots,m \tag{1.31}$$

and

$$x'_k = x_k^i \qquad i=1,2,\ldots\ldots,m \tag{1.32}$$

Then (1.19) may be rewritten

$$u_k = G_k^P x_k = G'_k x'_k \tag{1.33}$$

where

$$G'_k = \begin{bmatrix} g_k^{1T} \\ g_k^{2T} \\ \vdots \\ g_k^{mT} \end{bmatrix} \tag{1.34}$$

It is convenient to restrict the state vector to be of the form

$$x_k = \begin{bmatrix} x'_k \\ \hline z'_k \end{bmatrix} \tag{1.35}$$

where x'_k is the n'-vector of those states that are available to be fed back, and z'_k is the (n-n')-vector of those states that are not fed back. By introducing this restriction no generality is lost. If the state vector is not in this form one may always reorder it so that it is.

Another problem considered in the literature is the Output Feedback Problem.

Assume an r-vector y_k termed the output exists and that

$$y_k = Cx_k + v_k \tag{1.36}$$

where v_k is an r-vector zero mean independent white noise process with a covariance matrix, V_v. If the control u_k is constrained to satisfy

$$u_k = K_k y_k \tag{1.37}$$

then the mxr matrix, K_k, is termed the OUTPUT FEEDBACK MATRIX, and the problem of finding the sequence $[K_k, k=0,\ldots\ldots,N-1]$ of Output Feedback Matrices which minimize J, (1.4), is the OUTPUT FEEDBACK PROBLEM.

The Partial State Feedback Problemø can be posed as an Output Feedback Problem by setting

$$v_k = 0 \qquad (1.39)$$

and

$$y_k = x'_k = Cx_k \qquad (1.39)$$

where

$$C = [I \quad 0] \qquad (1.40)$$

How Output Feedback Problems can be transformed into Partial State Feedback Problems will be shown in Chapter 7.

It should be noted that some authors use the word Incomplete for Partial or Output Feedback.

1.3. Literature Survey

The method of designing controllers, by first selecting a suitable controller structure then tuning the variable parameters to get good behaviour, is an old one. It is central to control systems design by simulation.

During the 40's and 50's methods of analytically choosing system parameters so as to minimize a squared error criterion were developed for single input, single output systems. This work is summarized in the book by Newton, Gould, and Kaiser [15].

By choosing a controller structure and taking the expectation of the cost function, stochastic control problems can be converted into static optimization problems if the controller gains are constant, or deterministic control problems if the gains are time varying. The resulting problems may, however, be very complex and require special techniques to solve them. A review of some recent results produced on this topic is given by Sims and Melsa [16].

Performance criteria other than expected cost have been suggested for specific optimal control problems. The min max criterion [17] might be preferred if poor behaviour for conditions that were unlikely to occur was unacceptable. Such criteria make the analysis more difficult and must, of course, produce a worse average behaviour than that produced by the controller which minimizes the expected cost.

This survey will be restricted to specific optimal solutions to linear quadratic control problems. The work that has been done in this area can be divided into three main categories:

A) Output or Partial State Feedback Control

B) Dynamic Linear Compensators of Fixed Dimension

C) Structured State Feedback

A) Output or Partial State Feedback Control

A problem of an output feedback type was first considered by Axsäter [18]. He dealt with a finite time continuous linear system with a white noise system disturbance. The control law, which was selected to minimize the expected cost, was a time varying linear combination of the noise free measurement vector. An algorithm was derived which will converge to an improved control law. Necessary conditions for optimality were derived and an optimal solution was proved to exist (i.e. all gains remain finite). A sufficient condition for the algorithm to converge to the optimal solution was established.

Output feedback control of discrete regulator systems was first considered by Cumming [3]. Under the assumption that measurement and system noise are independent zero-mean white noise processes, he derived a necessary condition for an output feedback matrix to minimize the expected cost per time interval. This necessary condition was used to produce an algorithm, which, given an initial output matrix that stabilizes the system, is guaranteed to improve the expected cost. Cumming also made the interesting observation that the value of state in the optimal control law is replaced by the best estimate of state given the available measurements in the optimal output feedback control.

Recently Ermer and Vandelinde [4] considered the discrete output feedback problem as described in Section 1.2 (the finite time version of the Cumming problem). They showed the solution to be one of the solutions to a two point boundary value problem. They suggested an algorithm for solving the two point boundary value problem but did not analyze its properties. Levine and Athans [9] have considered the infinite time, time invariant, output feedback problem for a continuous system where the feedback gains are constrained to be constant. The initial state is assumed to be uniformly distributed over a sphere in R^n centered at the origin, but the system is otherwise undisturbed. An algorithm is produced which will yield an improvement in cost at each iteration, given that an output feedback matrix which stabilizes the system is used as the starting point. This approach allows a constant output feedback controller to be designed without any knowledge of the underlying disturbance process. It has therefore created some applications interest [10]-[13].

Other publications dealing with output or partial state feedback control of continuous systems are [1] and [19]-[36].

B) Dynamic Linear Compensators of Fixed Dimension

This problem was first treated by Johansen [37]. He produced methods of computing solutions for stochastic finite time problems for both discrete and continuous cases where the controller is time varying and only the order of the compensator is fixed. His article contains many computational examples.

Other authors [38]-[46] have considered various problems where a dynamic compensator of fixed order is to be used.

C) Structured State Feedback

All previous work on structured state feedback has considered constant controllers for continuous systems which were not subject to system or measurement noise.

This problem was first considered by Dabke [47] and [48]. He considered min max cost and expected cost for a given initial state distribution. For both these costs the necessary condition for optimality is given as a simultaneous set of polynomial equations in the non-zero gains. Unfortunately, he only solved these necessary conditions for a problem involving one unconstrained gain and gives no means of solving these equations in general.

Martenson [49] produced a conjugate gradient algorithm for computing a structured feedback matrix with improved performance given that a structured feedback matrix which stabilizes the system is available as an initial value. The method is applied to two examples but the properties of the algorithm are not analyzed.

Jameson [50] also considers a method of computing the gradient of the cost function with respect to the gains in the structured feedback matrix. The cost functions he considers are quadratic cost with a fixed initial condition, the min max of the quadratic cost, and the min max of the worst comparison with the optimal control. The expression for the gradient can be set to zero and the equations solved for a simple example or the gradient can be used in a parameter optimization algorithm for more complex cases.

Fath [51] used an approach similar to that of Martenson and Jameson, but considered the problem where state alone was quadratically costed and G, the feedback matrix, was constrained so that HG $\leqslant$ L. Fath minimized the expected cost under the assumption that the initial conditions are uniformly distributed over a sphere centred at the origin.

Kosut [8] produced a necessary condition for minimizing the expected cost assuming the initial condition is uniformly distributed over a sphere centered at the origin. He further assumed that the cost of control matrix, R, is diagonal. Rather than using these necessary conditions to compute the optimal structured feedback matrix he proposed two suboptimal design approaches termed minimum error of excitation and minimum norm.

Brown and Vetter [52] expanded the expected cost function, for a given initial state distribution, as a Taylor series in the state feedback gains about the full state feedback optimal point. A suboptimal structured state feedback matrix was obtained by use of the second order sensitivity term.

Bengtsson and Lindahl [53] proposed that the gains of the structured feedback matrix be chosen so the modes of the resulting closed loop system are close to those of the system under optimal control. This design procedure requires that a weighting matrix, which gives a relative importance to the modes of the optimal system, be selected. They used this method to produce an output controller for a boiler model (5th. order) and a local feedback controller for a power system model (15th. order).

Isaksen and Payne [54] developed a method for computing suboptimal band structured feedback matrices for systems where the state transition matrix, A, has a diagonal band structure and there is no coupling between subsystems through the control or cost matrices. A suboptimal control is constructed for the complete system from a set of optimal controllers calculated for subsystems which possibly overlap. The method is used to produce a traffic responsive regulator for a 34 state model of a freeway.

1.4. The Outline of the Thesis with a Statement of the Contributions

In this thesis the discrete finite-time structured control problem is defined and solved. This problem has not previously been considered in the literature. Further it is shown that many problems involving noise corrupted outputs and dynamic compensators can be posed as structured control problems.

In Chapter 2 certain basic properties of linear systems controlled by linear control policies are derived. Lemma 1 contains a well known recursive formula for $E[x_k x_k^T]$. As this relation is usually derived under the added assumptions, that x_0 be zero mean and the distributions are Gaussian, a derivation is included to show the condition, that the system noise be an independent white noise process, is sufficient. Lemma 2 shows that the well known recursive relations for the expected value of quadratic cost associated with the optimal control policy hold in a slightly generalized form for any linear control policy. It is believed that this result has not been previously established.

Lemma 3 is the central result of the thesis. It states that the expected value of quadratic cost for a linear system controlled by a linear control policy can be expressed as a positive semidefinite quadratic in the gains of the structured state feedback matrix to be used on any time interval. This result is the basis for the procedures for computing the optimal structured control policy developed in Chapter 4. Lemma 4 gives a condition under which the quadratic form of Lemma 3 will be positive definite no matter what linear control policy is used. Lemma 4 is used to establish existence and convergence properties. Both the statement and the proof of Lemmas 3 and 4 are completely original.

In Chapter 3 the rules for choosing one structured or partial state feedback matrix optimally are stated with conditions under which the choice will be unique. Theorem 1, the rule for a structured feedback matrix follows directly from Lemma 3. In the proof of Theorem 2 the result of Theorem 1 is manipulated using partial state feedback properties. Special formulae involving matrices which may be of lower order than that of Theorem 1 result. Ermer and Vandelinde [4] have produced a result similar to Theorem 2. However, the proof stated here is original. Theorem 1 and Lemmas 5 and 6 are original both in statement and proof.

In Chapter 4 the single replacement rules of Chapter 3 are combined with the recursive relations for $E[x_k x_k^T]$ and expected cost to produce computational procedures for computing improved structured control policies. The convergence properties of these algorithms, both in cost and control policy, are established. The existence (in the sense that all gains remain finite), of the optimal structured control policy, and of the limiting control policies produced by the computational procedure, is proved. It is further shown that the optimal structured control policy and the limiting control policies are solutions to a certain two point boundary value problem, whose solutions are the set of singular points of the cost function. Methods of selecting initial control policies, that should produce convergence to the optimal, are proposed and discussed. The first two of these are original, the latter three are suitably modified versions of the suboptimal controls proposed by Kosut [8]. The computational procedures, theorems, and lemmas stated in this chapter are original as are all other comments not specifically attributed to another source.

In Chapter 5 the computational suitability of the proposed procedure is established. Two systems are analyzed. The results produced indicate that a rule of decreasing marginal returns with increasing controller complexity applies. Two heuristic methods of selecting good controller structures are suggested and tested. All the results and observations in this chapter are original.

In Chapter 6 the effect V_0 and V_w have on the optimal structured control policy is considered, as is the related problem of how one can select a suitable V_0 and V_w if the actual values are unknown. It is concluded $V_0 = I$ and $V_w = I$ is a reasonable choice if one wants a control that will have an acceptable response for a wide variety of conditions. A relation between the limiting behaviour as $N \to \infty$ of the optimal structured control policy and the solution to a similar discrete Levine and Athans [9] type problem is derived. This relation strengthens the rationale supporting both approaches. It means as well that techniques for computing solutions to the stochastic steady state problem (Cumming [3]) can be used for the deterministic steady state problem (Levine and Athans [9]) or vice versa. The observation that the partial state feedback optimal control replaces the unavailable states by the best estimate using the measurements available was first made by Cumming [3]. All other observations and results produced in Chapter 6 are original.

In Chapter 7 it is shown that problems involving noise corrupted output feedback, dynamic compensators with fixed and free parameters and team theoretic problems can be posed as structured control problems by suitable state and control augmentation. The problem of how to choose the initial state of a dynamic compensator is considered as well. The transformations and results produced in Chapter 7 are original. However other authors, [42] and [45], have considered similar approaches for problems where dynamic compensators of specified order are to be used.

In Chapter 8 two topics for further research are proposed and the results of the thesis are summarized. Some of the problems that remain to be solved for the related steady state structured control problem are mentioned and a computationally promising algorithm is stated. Next, problems related to the choice of a good feedback structure are briefly discussed. All results and observations produced in this Chapter are original.

CHAPTER 2

PROPERTIES OF LINEAR SYSTEMS CONTROLLED BY LINEAR STATE FEEDBACK

Three basic structural properties of Linear Systems controlled by Linear Control Policies are considered in this chapter. The linear system described by (1.1) with quadratic cost given by (1.3) controlled by a linear control policy (1.10) can be described as a recursive relation for $E[x_k x_k^T]$ in forward time and a recursive relation for quadratic cost in backward time. Further it will be shown that the expected cost (1.4) can be expressed as a quadratic function of the unconstrained gains on any time interval. These results will be used in later chapters to produce a computational procedure for obtaining the optimal structured control policy. Finally a useful property of a matrix introduced when J is expressed as a quadratic in the unconstrained gains on a time interval will be established.

Lemma 1:

If a linear system described by (1.1) is controlled using a linear control policy (1.10) then

$$V(k+1) = [A+BG_k]V(k)[A+BG_k]^T + V_w \tag{2.1}$$

where

$$V(k) = E[x_k x_k^T] = \left\{ \sigma_k(i,j) \right\} \tag{2.2}$$

for $k = 0,1,\ldots\ldots,N$.

$V(k)$ is a symmetric positive semidefinite matrix, with elements $\sigma_k(i,j)$.

Proof:

Substitution of (1.11) into (1.1) yields

$$x_{k+1} = [A+BG_k]x_k + w_k \tag{2.3}$$

Thus

$$E[x_{k+1}x_{k+1}^T] = E[[(A+BG_k)x_k+w_k][(A+BG_k)x_k+w_k]^T] \tag{2.4}$$

$$= E[(A+BG_k)x_kx_k^T(A+BG_k)^T]$$

$$+ E[(A+BG_k)x_kw_k^T]$$

$$+ E[w_kx_k^T(A+BG_k)^T]$$

$$+ E[w_kw_k^T] \tag{2.5}$$

Note that

$$E[(A+BG_k)x_kx_k^T(A+BG_k)^T] = (A+BG_k)E[x_kx_k^T](A+BG_k)^T$$

$$= (A+BG_k)V(k)(A+BG_k)^T \tag{2.6}$$

as

$$E[x_kx_k^T] = V(k)$$

by definition. As

$$E[(A+BG_k)x_kw_k^T] = (A+BG_k)E[x_kw_k^T] \tag{2.7}$$

and w_k is a zero mean white noise process which is uncorrelated with present state

$$E[x_k w_k^T] = 0 \qquad (2.8)$$

where 0 is the null matrix of appropriate size. Thus

$$E[(A+BG_k)x_k w_k^T] = 0 \qquad (2.9)$$

Considering the third term in (2.5) one sees that

$$E[w_k x_k^T (A+BG_k)^T] = E[(A+BG_k)x_k w_k^T]^T$$

$$= 0^T = 0 \qquad (2.10)$$

From the definition of covariance matrix and the fact that w_k is a zero mean process one finds that

$$E[w_k w_k^T] = V_w \qquad (2.11)$$

Substitution of (2.2), (2.6), (2.9), (2.10), and (2.11) into (2.5) yields

$$V(k+1) = [A+BG_k]V(k)[A+BG_k]^T + V_w \qquad (2.1)$$

V(k) is obviously symmetric by its definition (2.2). Further

$$a^T V(k)a = a^T E[x_k x_k^T]a = E[a^T x_k x_k^T a] = E[y^2] \geq 0 \qquad (2.12)$$

as $y = a^T x_k$ is a scalar. Thus V(k) is positive semidefinite as well.

Q.E.D.

Lemma 2:

If a linear system described by (1.1) is controlled using a linear control policy (1.10) then the expectation of the quadratic cost (1.3) can be expressed as

$$E[L_k] = \tfrac{1}{2}E[x_k^T S_k x_k] + \tfrac{1}{2}\sum_{i=k+1}^{N} tr[S_i V_w] \qquad (2.13)$$

or

$$E[L_k] = \tfrac{1}{2}tr[S_k V(k)] + \tfrac{1}{2}\sum_{i=k+1}^{N} tr[S_i V_w] \qquad (2.14)$$

for all k=0,.....,N-1,

where

$$S_i = Q + G_i^T R G_i + [A+BG_i]^T S_{i+1}[A+BG_i] \qquad (2.15)$$

and i=N-1,......,0.

S_i is positive semidefinite and symmetric as Q, R, and S_N are positive semidefinite and symmetric.

Corollary:

$$J = \tfrac{1}{2}tr[S_0 V_0] + \tfrac{1}{2}\sum_{i=1}^{N} tr[S_i V_w] \qquad (2.16)$$

Proof:

The proof will be by induction. Consider the case k = N-1. Take the expectation of (1.3) with k = N-1

$$E[L_{N-1}] = E[\tfrac{1}{2}x_{N-1}^T Q x_{N-1} + \tfrac{1}{2}u_{N-1}^T R u_{N-1} + \tfrac{1}{2}x_N^T S_N x_N] \qquad (2.17)$$

Substitution of (2.3) and (1.11), with k = N-1 in both these expressions, yields

$$E[L_{N-1}] = E[\tfrac{1}{2}x_{N-1}^T Q x_{N-1} + \tfrac{1}{2}x_{N-1}^T G_{N-1}^T R G_{N-1} x_{N-1}$$
$$+ \tfrac{1}{2}[(A+BG_{N-1})x_{N-1} + w_{N-1}]^T S_N [(A+BG_{N-1})x_{N-1} + w_{N-1}]]$$
$$(2.18)$$

$$= \tfrac{1}{2}E[x_{N-1}^T (Q + G_{N-1}^T R G_{N-1} + (A+BG_{N-1})^T S_N (A+BG_{N-1}))x_{N-1}]$$
$$+ E[x_{N-1}^T (A+BG_{N-1})^T S_N w_{N-1}] + \tfrac{1}{2}E[w_{N-1}^T S_N w_{N-1}]$$
$$(2.19)$$

As w_{N-1} is a zero mean white noise process uncorrelated with state x_{N-1}, as x_{N-1} is not dependent on w_{N-1} by (1.1).

$$E[w_{N-1}|x_{N-1}] = 0 \qquad (2.20)$$

where by this notation one means the conditional expectation of w_{N-1} given x_{N-1} occurs.

Then note that

$$E[x_{N-1}^T (A+BG_{N-1})^T S_N w_{N-1}] = E[E[x_{N-1}^T (A+BG_{N-1})^T S_N w_{N-1}|x_{N-1}]]$$
$$= E[x_{N-1}^T (A+BG_{N-1})^T S_N E[w_{N-1}|x_{N-1}]]$$
$$= 0 \qquad (2.21)$$

By use of the trace identity

$$E[x^T A x] = tr[AV] \qquad \text{where } V = E[xx^T] \tag{2.22}$$

and (2.11) one finds

$$E[w_{N-1}^T S_N w_{N-1}] = tr[S_N V_w] \tag{2.23}$$

The substitution of (2.15) with i = N-1, (2.21), and (2.23) into (2.19) yields

$$E[L_{N-1}] = \tfrac{1}{2} E[x_{N-1}^T S_{N-1} x_{N-1}] + \tfrac{1}{2} tr[S_N V_w] \tag{2.24}$$

Thus (2.13)holds if k = N-1. The second part of the proof is to show that it holds for k if it holds for k+1.

Taking the expectation of (1.3) gives

$$E[L_k] = E[\tfrac{1}{2} \sum_{i=k}^{N} (x_i^T Q x_i + u_i^T R u_i) + \tfrac{1}{2} x_N^T S_N x_N] \tag{2.25}$$

$$= E[\tfrac{1}{2} x_k^T Q x_k + \tfrac{1}{2} u_k^T R u_k] + E[L_{k+1}] \tag{2.26}$$

from (2.25) with k = k+1.

By assumption

$$E[L_{k+1}] = \tfrac{1}{2} E[x^T_{k+1} S_{k+1} x_{k+1}] + \tfrac{1}{2} \sum_{i=k+2}^{N} tr[S_i V_w] \tag{2.27}$$

Substitution of (2.27) into (2.26) and the additive property of expectation yields

$$E[L_k] = E[\tfrac{1}{2}x_k^TQx_k + \tfrac{1}{2}u_k^TRu_k + \tfrac{1}{2}x_k^TS_{k+1}x_{k+1}] + \tfrac{1}{2}\sum_{i=k+2}^{N} tr[S_iV_w] \qquad (2.28)$$

By repeating the argument used to get from equation (2.17) through to (2.24) one can show

$$E[\tfrac{1}{2}x_k^TQx_k + \tfrac{1}{2}u_k^TRu_k + \tfrac{1}{2}x_k^TS_{k+1}x_{k+1}]$$

$$= \tfrac{1}{2}E[x_k^TS_kx_k] + \tfrac{1}{2}tr[S_{k+1}V_w] \qquad (2.29)$$

The insertion of (2.29) into (2.28) yields

$$E[L_k] = \tfrac{1}{2}E[x_k^TS_kx_k] + \tfrac{1}{2}\sum_{i=k+1}^{N} tr[S_iV_w] \qquad (2.13)$$

As (2.13) holds for $k = N-1$ and holds for k if it holds for $k+1$, then it holds for

$$k=0,\ldots\ldots,N-1$$

By one of the trace identities (2.22) and the definition of $V(k)$, (2.2) on the first term of (2.13) one finds

$$E[L_k] = \tfrac{1}{2}tr[S_kV(k)] + \tfrac{1}{2}\sum_{i=k+1}^{N} tr[S_iV_w] \qquad (2.14)$$

The corollary is established by substituting (1.2) and (1.4) into (2.14) with $k = 0$.

Q.E.D.

Lemma 3:

If the state feedback matrix at time k of a Linear Control Policy, π, is a Structured State Feedback matrix, G_k^S, then the expected cost may be written

$$J(\pi) = \tfrac{1}{2}g_k^T F_k g_k + h_k^T g_k + c_k \tag{2.30}$$

where F_k, h_k, and c_k depend on G_i, $i=0,\ldots\ldots,k-1,k+1,\ldots\ldots,N-1$ only.

$$c_k = \tfrac{1}{2} \sum_{i=0}^{k-1} tr[[Q + G_i^T R G_i]V(i)] + \tfrac{1}{2} \sum_{i=k+1}^{N} tr[S_i V_w]$$

$$+ \tfrac{1}{2} tr[[Q + A^T S_{k+1} A]V(k)] \tag{2.31}$$

F_k is the (pxp) positive semi-definite symmetric matrix

$$F_k = \begin{bmatrix} r_{11}(k)V_{11}(k) & r_{12}(k)V_{12}(k) & \cdots\cdots\cdots & r_{1m}(k)V_{1m}(k) \\ r_{12}(k)V_{12}^T(k) & r_{22}(k)V_{22}(k) & \cdots\cdots\cdots & r_{2m}(k)V_{2m}(k) \\ \cdots & \cdots & \cdots & \cdots \\ r_{1m}(k)V_{1m}^T(k) & r_{2m}(k)V_{2m}^T(k) & \cdots\cdots\cdots & r_{mm}(k)V_{mm}(k) \end{bmatrix}$$

(2.32)

where

$$V_{ij}(k) = \begin{bmatrix} \sigma_k(\psi(i,1),\psi(j,1)) & \sigma_k(\psi(i,1),\psi(j,2)) \cdots\cdots & \sigma_k(\psi(i,1),\psi(j,n_j)) \\ \sigma_k(\psi(i,2),\psi(j,1)) & \sigma_k(\psi(i,2),\psi(j,2)) \cdots\cdots & \sigma_k(\psi(i,2),\psi(j,n_j)) \\ \cdots & \cdots & \cdots \\ \sigma_k(\psi(i,n_i),\psi(j,1)) & \sigma_k(\psi(i,n_i),\psi(j,2)) \cdots\cdots & \sigma_k(\psi(i,n_i),\psi(j,n_j)) \end{bmatrix} \tag{2.33}$$

and the elements $\sigma_k(i,j)$ are defined by equation (2.2) and $\psi(i,j)$ are defined by (1.21). The $r_{i,j}(k)$ are the elements of the symmetric matrix

$$\underline{R}(k) = \left\{ r_{ij}(k) \right\} = [R + B^T S_{k+1} B] \tag{2.34}$$

where S_{k+1} is defined by (2.15). g_k is the p-vector

$$g_k = \begin{bmatrix} g_k^1 \\ g_k^2 \\ \vdots \\ g_k^m \end{bmatrix} \tag{2.35}$$

where g_k^i, i=1,.....,m, are defined by (1.24) and (1.25) and h_k is the p-vector

$$h_k = \begin{bmatrix} V_1(k)A^T S_{k+1} b_1 \\ V_2(k)A^T S_{k+1} b_2 \\ \vdots \\ V_m(k)A^T S_{k+1} b_m \end{bmatrix} \tag{2.36}$$

where

$$V_i(k) = \begin{bmatrix} \sigma_k(\psi(i,1),1) & \sigma_k(\psi(i,1),2) & \cdots\cdots & \sigma_k(\psi(i,1),n) \\ \sigma_k(\psi(i,2),1) & \sigma_k(\psi(i,2),2) & \cdots\cdots & \sigma_k(\psi(i,2),n) \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_k(\psi(i,n_i),1) & \sigma_k(\psi(i,n_i),2) & \cdots\cdots & \sigma_k(\psi(i,n_i),n) \end{bmatrix} \tag{2.37}$$

and b_i, $i=1,2,\ldots,m$ are the columns of the matrix B defined in (1.1) thus

$$B = [b_1 b_2 \;\ldots\; b_m] \tag{2.38}$$

<u>Proof</u>:

Substitution of (1.3) into (1.4) yields

$$J = E[\tfrac{1}{2} \sum_{i=0}^{N-1} [x_i^T Q x_i + u_i^T R u_i] + \tfrac{1}{2} x_N^T S_N x_N] \tag{2.39}$$

Substitution of (1.11) into (2.39), the additive property of expectation and the use of (1.3) and (2.13) produces

$$J(\pi) = \tfrac{1}{2} E[x_k^T S_k x_k] + \tfrac{1}{2} \sum_{i=k+1}^{N} tr[S_i V_w] + \tfrac{1}{2} \sum_{i=0}^{k-1} tr[[Q + G_i^T R G_i] V(i)] \tag{2.40}$$

Examination of the recursive formula for S_i, (2.15), and the formula for V(i), (2.1), make it clear that only the first term of (2.40) is a function of G_k^S. This term will now be examined.

From the additive properties of expectation and (2.15) it is apparent that

$$E[x_k^T S_k x_k] = E[x_k^T[Q + A^T S_{k+1} A]x_k] + E[x_k^T G_k^{ST}[R + B^T S_{k+1} B]G_k^S x_k] + E[2x_k^T A^T S_{k+1} B G_k^S x_k] \tag{2.41}$$

By use of the definition of matrix multiplication, (1.19) may be rewritten

$$G_k^S x_k = \begin{bmatrix} g_k^{1T} x_k^1 \\ g_k^{2T} x_k^2 \\ \vdots \\ g_k^{mT} x_k^m \end{bmatrix} = \begin{bmatrix} (x_k^1)^T & 0 & \cdots & 0 \\ 0 & (x_k^2)^T & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & (x_k^m)^T \end{bmatrix} \begin{bmatrix} g_k^1 \\ g_k^2 \\ \vdots \\ g_k^m \end{bmatrix} \tag{2.42}$$

For reasons of notational convenience define

$$X_k = \begin{bmatrix} (x_k^1)^T & 0 & \cdots\cdots\cdots & 0 \\ 0 & (x_k^2)^T & \cdots\cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots\cdots & (x_k^m)^T \end{bmatrix} \tag{2.43}$$

Substitution of (2.35) and (2.43) into (2.42) produces

$$G_k^S x_k = X_k g_k \tag{2.44}$$

Substitution of (2.34) and (2.44) into the second term of (2.41) yields

$$E[x_k^T G_k^{ST}[R + B^T S_{k+1} B] G_k^S x_k] = g_k^T E[X_k^T \underline{R}(k) X_k] g_k \tag{2.45}$$

From the definitions (1.20), (1.21), (2.2), (2.32), (2.33), (2.34), (2.43), and the properties of expectation and multiplication of matrices it follows

$$E[X_k^T\underline{R}(k)X_k] = E\begin{bmatrix} x_k^1(x_k^1)^T r_{11}(k) & x_k^1(x_k^2)^T r_{12}(k) & \cdots\cdots & x_k^1(x_k^m)^T r_{1m}(k) \\ x_k^2(x_k^1)^T r_{12}(k) & x_k^2(x_k^2)^T r_{22}(k) & \cdots\cdots & x_k^2(x_k^m)^T r_{2m}(k) \\ \cdots & \cdots & \cdots & \cdots \\ x_k^m(x_k^1)^T r_{1m}(k) & x_k^m(x_k^2)^T r_{2m}(k) & \cdots\cdots & x_k^m(x_k^m)^T r_{mm}(k) \end{bmatrix}$$

$$= \begin{bmatrix} r_{11}(k)V_{11}(k) & r_{12}(k)V_{12}(k) & \cdots\cdots & r_{1m}(k)V_{1m}(k) \\ r_{12}(k)V_{12}^T(k) & r_{22}(k)V_{22}(k) & \cdots\cdots & r_{2m}(k)V_{2m}(k) \\ \cdots & \cdots & \cdots & \cdots \\ r_{1m}(k)V_{1m}^T(k) & r_{2m}(k)V_{2m}^T(k) & \cdots\cdots & r_{mm}(k)V_{mm}(k) \end{bmatrix}$$

$$= F_k \qquad (2.46)$$

Substitution of (2.46) into (2.45) produces

$$E[x_k^T G_K^{ST}[R + B^T S_{k+1} B] G_k^S x_k] = g_k^T F_k g_k \qquad (2.47)$$

Consider the variation in F_k with changes in the Linear Control Policy π. Note (2.34) and (2.15) imply that $\underline{R}(k)$ through S_{k+1} is a function of G_i, $i = k+1,\ldots\ldots,N-1$ only.

Note (2.1) implies that $V(k)$ is a function of G_i, $i=0,\ldots\ldots,k-1$ only. As $V_{ij}(k)$, $i=1,\ldots\ldots,m$, $j=1,\ldots\ldots,m$ are matrices composed of elements of $V(k)$, these matrices are functions of G_i, $i=0,\ldots\ldots,k-1$ only. Thus (2.32) implies that F_k is a function of G_i, $i=0,\ldots\ldots,k-1,k+1,\ldots\ldots,N-1$ only. F_k does <u>not</u> depend on the value chosen for G_k.

As R and S_{k+1} are positive semidefinite and symmetric, R(k) is positive semidefinite and symmetric. Thus (2.47) implies F_k is positive semidefinite. As V(k) is symmetric (2.32) and (2.33) make it apparent that F_k is symmetric.

If equation (2.44) is substituted into the last term of (2.41) one finds

$$E[2x_k^T A^T S_{k+1} B G_k^S x_k] = 2E[x_k^T A^T S_{k+1} B X_k] g_k = 2d^T g_k \qquad (2.48)$$

where by definition

$$d^T = E\,[x_k^T A^T S_{k+1} B X_k] \qquad (2.49)$$

Note that one may write

$$d^T = E[[(x_k^T A^T S_{k+1} b_1), (x_k^T A^T S_{k+1} b_2), \ldots, (x_k^T A^T S_{k+1} b_m)] X_k] \qquad (2.50)$$

where the b_i, $i=1,2,\ldots,m$ are the columns of B defined in (2.38).

The transpose of (2.50) is

$$d = E\left[\begin{bmatrix} x_k^1 & 0 & \ldots & 0 \\ 0 & x_k^2 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & x_k^3 \end{bmatrix}\begin{bmatrix} (x_k^T A^T S_{k+1} b_1) \\ (x_k^T A^T S_{k+1} b_2) \\ \vdots \\ (x_k^T A^T S_{K+1} b_m) \end{bmatrix}\right] = \begin{bmatrix} E[x_k^1 x_k^T] A^T S_{k+1} b_1 \\ E[x_k^2 x_k^T] A^T S_{k+1} b_2 \\ \vdots \\ E[x_k^m x_k^T] A^T S_{k+1} b_m \end{bmatrix} \qquad (2.51)$$

Use of definitions (1.20), (1.21), (2.2), (2.36) and (2.37) yields

$$d = \begin{bmatrix} V_1(k)A^T S_{k+1} b_1 \\ V_2(k)A^T S_{k+1} b_2 \\ \vdots \\ V_m(k)A^T S_{k+1} b_m \end{bmatrix} = h_k \qquad (2.52)$$

Substitution of (2.52) into (2.48) produces

$$E[2x_k^T A^T S_{k+1} B G_k^S x_k] = 2h_k^T g_k \qquad (2.53)$$

h_k is a function of the control policy π through $V_i(k)$ and S_{k+1}. As $V_i(k)$ is composed of elements of $V(k)$ it is a function of G_i, $i=0,\ldots,k-1$ only. S_{k+1} is a function of G_i, $i=k+1,\ldots,N-1$ only. Thus h_k is a function of G_i, $i=0,\ldots,k-1,k+1,\ldots,N-1$ only. h_k is not dependent on the value of G_k.

Substitution of (2.47) and (2.53) into (2.41) produces

$$E[x_k^T S_k x_k] = E[x_k^T[Q + A^T S_{k+1} A]x_k] + g_k^T F_k g_k + 2h_k^T g_k \qquad (2.54)$$

Substitution of (2.54) into (2.40) gives

$$J(\pi) = \tfrac{1}{2} g_k^T F_k g_k + h_k^T g_k + \tfrac{1}{2} E[x_k^T[Q + A^T S_{k+1} A]x_k] + \tfrac{1}{2} \sum_{i=k+1}^{N} tr[S_i V_W]$$

$$+ \tfrac{1}{2} \sum_{i=0}^{k-1} tr[[Q + G_i^T R G_i]V(i)] \qquad (2.55)$$

Define

$$c_k = \tfrac{1}{2}E[x_k^T[Q + A^TS_{k+1}A]x_k] + \tfrac{1}{2}\sum_{i=k+1}^{N} tr[S_iV_w]$$

$$+ \tfrac{1}{2}\sum_{i=0}^{k-1} tr[[Q + G_i^TRG_i]V(i)] \qquad (2.56)$$

Thus

$$c_k = \tfrac{1}{2}tr[[Q + A^TS_{k+1}A]V(k)] + \tfrac{1}{2}\sum_{i=k+1}^{N} tr[S_iV_w]$$

$$+ \tfrac{1}{2}\sum_{i=0}^{k-1} tr[[Q + G_i^TRG_i]V(i)] \qquad (2.31)$$

by use of the trace identity (2.22).

c_k is a function of; G_i, i=0,....,k-1 directly; S_i, i=k+1,....,N-1 and thus of G_i, i=k+1,.....,N-1; and V(i), i=0,....,k and thus of G_i, i=0,.....,k-1. Thus c_k is a function of G_i,i=0,....,k-1,k+1,....,N only. c_k is <u>not</u> dependent on G_k.

Substitution of (2.31) into (2.55) produces

$$J(\pi) = \tfrac{1}{2}g_k^TF_kg_k + h_k^Tg_k + c_k \qquad (2.30)$$

where F_k, h_k and c_k are functions of G_i,i=0,....,k-1,k+1,....,N, and F_k is positive semidefinite. Thus if G_i,i=0,....,k-1,k+1,.....,N are considered to be fixed, J is a quadratic form in g_k with constant coefficients which opens upwards. Thus the minimising value(s) can be found by differentiating (2.30) by g_k and setting the result equal to zero.

Q.E.D.

Lemma 4:

If R, V_0 and V_w are positive definite then for all Linear Control Policies,

$$g^T F_i g \geq \lambda g^T g \quad , \qquad i=0,\ldots,N-1 \qquad (2.57)$$

where λ is a positive constant whose value depends on the values of R, V_0, and V_w, only, and g is an arbitrary p-vector.

This condition can be interpreted as: the positive definiteness of F_i is bounded below.

Proof:

In the proof of Lemma 3 it was established that

$$g_k^T F_k g_k = E[x_k^T G_k^{ST}[R + B^T S_{k+1} B] G_k^S x_k] \qquad (2.47)$$

where G_k^S is a structured feedback matrix. As F_k is not a function of G_k any structured feedback matrix G_k^S may be inserted into the original Linear Control Policy without affecting F_k. Assume such a substitution is made.

Use of the additivity of expectation produces

$$g_k^T F_k g_k = E[x_k^T G_k^{ST} R G_k^S x_k] + E[x_k^T G_k^{ST} B^T S_{k+1} B G_k^S x_k] \qquad (2.58)$$

As S_{k+1} is positive semidefinite

$$E[x_k^T G_k^{ST} B^T S_{k+1} B G_k^S x_k] \geq 0 \qquad (2.59)$$

Thus

$$g_k^T F_k g_k \geq E[x_k^T G_k^{ST} R G_k^S x_k] \qquad (2.60)$$

If k=0 use of the trace identity (2.22) produces

$$g_0^T F_0 g_0 \geq tr[G_0^{ST} R G_0^S V_0] \tag{2.61}$$

If k=1,....,N-1 then substitution of (2.3) produces

$$g_k^T F_k g_k \geq E[w_k^T G_k^{ST} R G_k^S w_k] + 2E[w_k^T G_k^{ST} R G_k^S [A+BG_{k-1}^S] x_{k-1}]$$

$$+ E[x_{k-1}^T [A+BG_{k-1}^S]^T G_k^{ST} R G_k^S [A+BG_{k-1}^S] x_{k-1}] \tag{2.62}$$

As w_k and x_{k-1} are uncorrelated and R is positive definite

$$g_k^T F_k g_k \geq E[w_k^T G_k^{ST} R G_k^S w_k] \tag{2.63}$$

Use of the trace identity (2.22) then yields

$$g_k^T F_k g_k \geq tr[G_k^{ST} R G_k^S V_w] \tag{2.64}$$

for k=1,....,N-1.

As V_0 is a positive definite and symmetric matrix there is an orthogonal matrix T_0 which reduces V_0 to diagonal form

$$T_0^T V_0 T_0 = \Omega_0 \tag{2.65}$$

$$T_0^T T_0 = I \tag{2.66}$$

where Ω_0 is the diagonal matrix which has the eigenvalues of V_0 as the diagonal elements.

Similarly there is an orthogonal matrix T_w which reduces V_w to diagonal form

$$T_w^T V_w T_w = \Omega_w \qquad (2.67)$$

$$T_w^T T_w = I \qquad (2.68)$$

where Ω_w is the diagonal matrix which has the eigenvalues of V_w as the diagonal elements.

Note (2.65) and (2.67) may be rewritten as

$$V_0 = T_0 \Omega_0 T_0^T \qquad (2.69)$$

$$V_w = T_w \Omega_w T_w^T \qquad (2.70)$$

Substitution of (2.69) into (2.61) and (2.70) into (2.64) yields

$$g_0^T F_0 g_0 \geqslant tr[G_0^{ST} R G_0^S T_0 \Omega_0 T_0^T] \qquad (2.71)$$

or

$$g_k^T F_k g_k \geqslant tr[G_k^{ST} R G_k^S T_w \Omega_w T_w^T] \qquad (2.72)$$

for k=1,2,....,N-1.

Use of the identity

$$tr[AB] = tr[BA] \qquad (2.73)$$

where A and B are any two matrices so that the products are defined, gives

$$g_0^T F_0 g_0 \geq tr[\Omega_0 T_0^T G_0^{ST} R G_0^S T_0] \tag{2.74}$$

or

$$g_k^T F_k g_k \geq tr[\Omega_w T_w^T G_k^{ST} R G_k^S T_w] \tag{2.75}$$

for k=1,2,....,N-1.

Let λ_V denote the smallest of the eigenvalues of both V_0 and V_w. If V_w is time varying this can still be done as the number of matrices to be considered is finite. As both V_0 and V_w are positive definite $\lambda_V > 0$. R is positive definite thus $T_0^T G_0^T R G_0 T_0$ and $T_w^T G_i^T R G_i T_w$ are positive definite. By using the definition of trace one may deduce

$$g_0^T F_0 g_0 \geq \lambda_V tr[T_0^T G_0^{ST} R G_0^S T_0] \tag{2.76}$$

or

$$g_k^T F_k g_k \geq \lambda_V tr[T_w^T G_k^{ST} R G_k^S T_w] \tag{2.77}$$

for k=1,2,....,N-1.

Use of (2.66), (2.68) and (2.73) produces

$$g_k^T F_k g_k \geq \lambda_V tr[R G_k^S G_k^{ST}] \tag{2.78}$$

for k=0,1,....,N-1.

λ_R is defined to be the smallest eigen value of R. $\lambda_R > 0$ as R is positive definite. R is symmetric as well. Following the same line of proof as was used to get from (2.64) to (2.78) one can show

$$g_k^T F_k g_k \geq \lambda_V \lambda_R tr[G_k^S G_k^{ST}] \tag{2.79}$$

for k=0,1,....,N-1.

From (1.14), (1.15), (1.24), (1.25), and (2.35) it is

$$\mathrm{tr}[G_k^S G_k^{ST}] = \sum_{i=0}^{m} g_k^{iT} g_k^i = g_k^T g_k \qquad (2.80)$$

If one defines

$$\lambda = \lambda_V \lambda_R \qquad (2.81)$$

and substitutes this along with (2.80) into (2.79) one finds

$$g_k^T F_k g_k \geqslant \lambda g_k^T g_k \quad , \qquad (2.82)$$

for k=0,1,....,N-1.

As the Structural Feedback Matrix can always be chosen so that $g_k = g$ where g is an arbitrary p-vector.

$$g^T F_k g \geqslant \lambda g^T g \qquad (2.57)$$

for k=0,1,....,N-1.

Q.E.D.

CHAPTER 3

THE OPTIMAL CHOICE OF A SINGLE STRUCTURED STATE FEEDBACK MATRIX

By use of Lemma 3 the problem of how to choose a single structured state feedback matrix so as to minimize the expected cost, J, can be easily solved. If the structure is a Partial State Feedback Structure, then special formulae can be derived, which are similar in form to the relations for the optimal state feedback matrix, G_k^*. The partial state feedback result is analogous to those of Cumming [3] and of Ermer and Vandelinde [4]. In addition, conditions under which the optimal gains will be unique are considered.

The following theorem may be used to compute the optimal gains of the structured state feedback matrix.

Theorem 1:

The linear system (1.1) is assumed to be controlled by the Linear Control Policy, π. If the State Feedback Matrix at time k, G_k, is to be replaced by the Structured State Feedback Matrix, G_k^S, which minimises the expected cost, J, the p unconstrained gains of G_k^S may be computed by solving the p linear equations

$$F_k g_k = - h_k \tag{3.1}$$

where F_k is defined by (2.32), h_k by (2.36), and the relationship between g_k and G_k^S by (1.24), (1.25) and (2.35).

Proof:

As F_k is positive semidefinite and the G_i, i=0,1,....,k-1,k+1,...,N-1, are considered to be fixed, J, is a constant quadratic form in g_k which opens upwards. Thus the minimizing value(s) can be found by differentiating (2.30) with respect to g_k and setting the result equal to zero.

$$\frac{\partial J}{\partial g_k} = g_k^T F_k + h_k^T = 0 \qquad (3.2)$$

Thus J is minimised by any g_k that is a solution to

$$F_k g_k = - h_k \qquad (3.1)$$

Q.E.D.

Lemma 4 implies that if V_0, V_w, and R are positive definite all the F_k's will be positive definite and thus invertible which implies that the G_k^S's which minimise J, given that only one substitution is to be made, are unique. Now a less stringent sufficient condition for F_k to be invertable will be proved.

Lemma 5:

If V(k) and R(k) are positive definite then F_k is positive definite.

Proof:

Substitution of (2.34) into (2.47) produces

$$g_k^T F_k g_k = E[x_k^T G_k^{ST} \underline{R}(k) G_k^S x_k] \qquad (3.3)$$

Use of the trace indentity (2.22) yields

$$g_k^T F_k g_k = tr[G_k^{ST} \underline{R}(k) G_k^S V(k)] \qquad (3.4)$$

By use of the same line of proof as that to get from (2.64) to (2.82) one finds

$$g_k^T F_k g_k \geqslant C g_k^T g_k \qquad \text{where} \quad C > 0 \tag{2.57a}$$

As $g_k^T F_k g_k > 0$ for all $g_k \neq 0$. F_k is <u>Positive Definite</u> by definition. Q.E.D.

The Partial State Feedback Problem is of particular interest because if feedback is eliminated from a state, that state need not be measured or estimated. This results in a reduction in the cost of control. In this section the result of Theorem 1 is considered when the Structured Feedback Matrix, G_k^S is in fact of Partial Feedback Form, G_k^P.

Matrices A^1 of order (n x n') and A^2 of order (n x (n-n')) are defined such that

$$A = [A^1 \; A^2] \tag{3.5}$$

Substitution of (1.35) yields

$$Ax_k = A^1 x_k' + A^2 z_k' \tag{3.6}$$

Let

$$V_{x'}(k) = E[x_k'(x_k')^T] \tag{3.7}$$

and

$$V_{x'z'}(k) = E[x_k'(z_k')^T] \tag{3.8}$$

Then (1.35) and (2.2) imply

$$V(k) = \begin{bmatrix} V_{x'}(k) & V_{x'z'}(k) \\ V^T_{x'z'}(k) & V_{z'}(k) \end{bmatrix} \quad (3.9)$$

where

$$V_{z'}(k) = E[z'_k(z'_k)^T] \quad (3.10)$$

Theorem 2:

Given system (1.1) controlled by a linear control policy π (1.10), if the k-th element of π, G_k, is replaced by a Partial State Feedback Matrix, G^P_k, then the feedback gain matrix G'_k (1.34), that will minimize the expected cost J is

$$G'_k = -[R + B^TS_{k+1}B]^{-1}B^TS_{k+1}[A^1 + A^2V^T_{x'z'}(k)V^{-1}_{x'}(k)] \quad (3.11)$$

provided $[R + B^TS_{k+1}B]$ and $V_{x'}(k)$ are invertible, where S_{k+1} is defined by (2.15).

Proof:

By consideration of (1.20), (1.21), (1.32), (2.33) and (3.7) it is apparent that

$$V_{ij}(k) = E[x'_k(x'_k)^T] = V_{x'}(k), \qquad i,j=1,2,\dots,m \quad (3.12)$$

Substitution of (3.12) into (2.33) produces

$$F_k = \begin{bmatrix} r_{11}(k)V_{x'}(k) & r_{12}(k)V_{x'}(k) & \cdots\cdots & r_{1m}(k)V_{x'}(k) \\ r_{12}(k)V_{x'}(k) & r_{22}(k)V_{x'}(k) & \cdots\cdots & r_{2m}(k)V_{x'}(k) \\ \cdots & \cdots & \cdots & \cdots \\ r_{1m}(k)V_{x'}(k) & r_{2m}(k)V_{x'}(k) & \cdots\cdots & r_{mm}(k)V_{x'}(k) \end{bmatrix} \tag{3.13}$$

$\underline{R}(k) = [R + B^T S_{k+1} B]$ is invertible by assumption. $\underline{R}^{-1}(k)$ is symmetric as $\underline{R}(k)$ is symmetric, thus one may define

$$\underline{R}^{-1}(k) = \{\varphi_{ij}(k)\} \tag{3.14}$$

where

$$\varphi_{ij}(k) = \varphi_{ji}(k) \tag{3.15}$$

As $V_{x'}(k)$ is assumed invertible it may easily be verified by matrix multiplication that

$$F_k^{-1} = \begin{bmatrix} \varphi_{11}(k)V_{x'}^{-1}(k) & \varphi_{12}(k)V_{x'}^{-1}(k) & \cdots\cdots & \varphi_{1m}(k)V_{x'}^{-1}(k) \\ \varphi_{12}(k)V_{x'}^{-1}(k) & \varphi_{22}(k)V_{x'}^{-1}(k) & \cdots\cdots & \varphi_{2m}(k)V_{x'}^{-1}(k) \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{1m}(k)V_{x'}^{-1}(k) & \varphi_{2m}(k)V_{x'}^{-1}(k) & \cdots\cdots & \varphi_{mm}(k)V_{x'}^{-1}(k) \end{bmatrix} \tag{3.16}$$

As F_k^{-1} exists Theorem 1 implies that the unconstrained gains in G_k^P may be obtained by solving

$$g_k = - F_k^{-1} h_k \tag{3.17}$$

Use of (1.20), (1.21), (1.32), and (2.37) yields

$$V_i(k) = E[x_k'[(x_k')^T (z_k')^T]] \qquad i=1,2,\ldots,m \tag{3.18}$$

Thus

$$V_i(k) = [V_{x'}(k) \quad V_{x'z'}(k)] \qquad i=1,2,\ldots,m \tag{3.19}$$

by definition (3.7) and (3.8). Substitution of (3.19) into (2.36), then (2.36) and (3.16) into (3.17) produces

$$g_k = - \begin{bmatrix} \varphi_{11}(k)V_{x'}^{-1}(k) & \varphi_{12}(k)V_{x'}^{-1}(k) & \cdots & \varphi_{1m}(k)V_{x'}^{-1}(k) \\ \varphi_{12}(k)V_{x'}^{-1}(k) & \varphi_{22}(k)V_{x'}^{-1}(k) & \cdots & \varphi_{2m}(k)V_{x'}^{-1}(k) \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{1m}(k)V_{x'}^{-1}(k) & \varphi_{2m}(k)V_{x'}^{-1}(k) & \cdots & \varphi_{mm}(k)V_{x'}^{-1}(k) \end{bmatrix} \begin{bmatrix} [V_{x'} V_{x'z'}]A^T S_{k+1} b_1 \\ [V_{x'} V_{x'z'}]A^T S_{k+1} b_2 \\ \cdots \\ [V_{x'} V_{x'z'}]A^T S_{k+1} b_m \end{bmatrix}$$

(3.20)

Matrix multiplication and the substitution of the definition of g_k, (2.35) gives

$$\begin{bmatrix} g_k^1 \\ g_k^2 \\ \vdots \\ g_k^m \end{bmatrix} = - \begin{bmatrix} \sum_{i=1}^{m} \varphi_{1i}(k)[I \,(V_{x'}^{-1}(k)V_{x'z'}(k))]A^T S_{k+1} b_i \\ \sum_{i=1}^{m} \varphi_{2i}(k)[I \,(V_{x'}^{-1}(k)V_{x'z'}(k))]A^T S_{k+1} b_i \\ \vdots \\ \sum_{i=1}^{m} \varphi_{mi}(k)[I \,(V_{x'}^{-1}(k)V_{x'z'}(k))]A^T S_{k+1} b_i \end{bmatrix} \tag{3.21}$$

By taking the transpose of each g_k^i in (3.21) one finds

$$(g_k^j)^T = - \sum_{i=1}^{m} \varphi_{ji}(k) b_i^T S_{k+1} A \begin{bmatrix} I \\ \hline (V_{x'z'}(k))^T V_{x'}^{-1}(k) \end{bmatrix} \qquad (3.22)$$

Substitution of (3.5) and the appropriate matrix multiplication produces

$$(g_k^j)^T = - \sum_{i=1}^{m} \varphi_{ji}(k) b_i^T S_{k+1} [A^1 + A^2 (V_{x'z'}(k))^T V_{x'}^{-1}(k)] \qquad (3.23)$$

Substitution of (3.23) into (1.34) gives

$$G_k' = \begin{bmatrix} (g_k^1)^T \\ (g_k^2)^T \\ \vdots \\ (g_k^m)^T \end{bmatrix} = - \begin{bmatrix} \sum_{i=1}^{m} \varphi_{1i}(k) b_i^T S_{k+1} [A^1 + A^2 (V_{x'z'}(k))^T V_{x'}^{-1}(k)] \\ \sum_{i=1}^{m} \varphi_{2i}(k) b_i^T S_{k+1} [A^1 + A^2 (V_{x'z'}(k))^T V_{x'}^{-1}(k)] \\ \vdots \\ \sum_{i=1}^{m} \varphi_{mi}(k) b_i^T S_{k+1} [A^1 + A^2 (V_{x'z'}(k))^T V_{x'}^{-1}(k)] \end{bmatrix} \qquad (3.24)$$

The rules of matrix multiplication and the definitions of $\underline{R}^{-1}(k)$, (3.14) and B, (2.38) yield

$$G_k' = - \underline{R}^{-1}(k) B^T S_{k+1} [A^1 + A^2 (V_{x'z'}(k))^T V_{x'}^{-1}(k)] \qquad (3.25)$$

Substitution of (2.34) gives

$$G_k' = -[R + B^T S_{k+1} B]^{-1} B^T S_{k+1} [A^1 + A^2 (V_{x'z'}(k))^T V_{x'}^{-1}(k)] \tag{3.11}$$

Q.E.D.

Sufficient conditions for the inverses in (3.11) to exist are now examined.

Lemma 6:

A sufficient condition for $[R + B^T S_{k+1} B]$ to be invertible is that R be positive definite. A sufficient condition for $V_{x'}(k)$ to be invertible is that V_0^{11} and V_w^{11} are positive definite, where

$$V_0 = \begin{bmatrix} V_0^{11} & V_0^{12} \\ (V_0^{12})^T & V_0^{22} \end{bmatrix} \tag{3.26}$$

and

$$V_w = \begin{bmatrix} V_w^{11} & V_w^{12} \\ (V_w^{12})^T & V_w^{22} \end{bmatrix} \tag{3.27}$$

The partitioning in (3.26) and (3.27) is the same as that in (3.9).

Proof:

As S_{k+1} is positive semidefinite, if R is positive definite then

$[R + B^T S_{k+1} B]$ is positive definite and therefore invertible.

Define

$$W = \begin{bmatrix} W_{11} & W_{12} \\ (W_{12})^T & W_{22} \end{bmatrix} = [A + BG_{k-1}]V(k-1)[A + BG_{k-1}]^T \tag{3.28}$$

where the partitioning is the same as in (3.9).

As V(k-1), k=1,...,N are positive semidefinite, W is positive semidefinite and thus W_{11} must be positive semidefinite.

From (2.1), (3.9), (3.27), and (3.28) one may deduce

$$V_{x'}(k) = W_{11} + V_w^{11} \tag{3.29}$$

for k=1,2,....,N-1. As W_{11} is positive semidefinite and V_w^{11} is assumed positive definite, $V_{x'}(k)$ is positive definite for k=1,2,...,N-1.

$$V_{x'}(0) = V_0^{11} \tag{3.30}$$

by (1.2), and V_0^{11} is assumed to be invertible, thus the $V_{x'}(k)$, k=0,1,....,N are positive definite and therefore invertible.

CHAPTER 4

A METHOD FOR COMPUTING OPTIMAL STRUCTURED CONTROL POLICIES

4.1. Introduction

Theorem 1 of Chapter 3 suggests a method for computing a sequence of Structured Feedback Matrices. Start with a linear control policy, π. Change this policy one State Feedback Matrix at a time to a Structured Control Policy using Theorem 1 to evaluate which Structured Feedback Matrices would be best. Continue to change the policy one state feedback matrix at a time using Theorem 1. Each time a change is made an improvement in cost results (If the state feedback matrix replaced satisfied (3.1), the improvement will be zero). As cost is bounded below by zero, if state feedback matrices in the policy are changed in an ordered manner convergence in cost must occur.

It is shown that the Optimal Structured Control Policy and the limiting Structured Control Policy of the above computational procedure are always composed of finite gain elements if V_0, V_w and R are positive definite. The set of limiting Structured Control Policies is shown to either consist of one element or to be an uncountable connected set. Unfortunately, for certain systems, the limiting values produced depend on the initial linear control policy. In these cases the computational procedure does not always converge to the Optimal Structured Control Policy. The algorithm converges to a solution (or set of solutions) of a two point boundary value problem one of whose solutions is the optimal. Viewed in the parameter space, where each gain in a Structured Control Policy is a co-ordinate, the computational procedure may converge to a local minimum or a singular point. A simple example is given which illustrates these difficulties.

Choice of the initial linear control policy is important to ensure convergence to the optimal and to keep the computation time low. For Feedback structures where near optimal control is possible a good starting point is the Optimal Complete State Feedback Policy. A heuristic argument is given which explains why such a starting point should provide convergence to the optimal for such cases.

4.2. The Computational Procedures and Their Proof of Convergence

The changes in the State Feedback Matrices of the Linear Control Policy should be organised so that the number of computations and the computer storage requirement is minimised.

The evaluation of G_i using formula (3.1) of Theorem 1 requires a knowledge of S_{i+1} and V(i). S_{i+1} depends only on $G_j, j=i+1, i+2, \ldots, N-1$ and V(i) depends only on $G_j, j=0,1,\ldots,i-1$. If Control Policy substitutions were made in reverse time $(i=N-1, N-2, \ldots, 1, 0)$ then V(i), $i=1,2,\ldots,N-1$ could be evaluated using (2.1) before a set of substitutions $G_i, i=N-1, N-2, \ldots, 0$ were made. S_i could be calculated as each new control matrix $G_i, i=N-1, N-2, \ldots, 0$ was evaluated. This approach is described precisely by Computational Procedure A.

Definition

The replace symbol, $\longleftarrow$, when used in an expression such as $x \longleftarrow y$ means set the value of x equal to that of y.

Computational Procedure A:

1. Specify a linear control policy

$$\pi = [G_i, \ i=0,1,\ldots,N-1]$$

(for purposes of proof $k \leftarrow 0$ and $\pi(0) \leftarrow \pi$)

2. Let $i \leftarrow 0$ and $V(0) \leftarrow V_0$

3. $V(i+1) \leftarrow [A+BG_i]V(i)[A+BG_i]^T + V_w$

4. $i \leftarrow i+1$

5. If $i < N-1$ go to 3.

6. $G_i \leftarrow$ the structured state feedback matrix with unconstrained gains obtained by solving

$$F_i g_i = - h_1 \qquad (3.1)$$

where F_i and h_i are defined by (2.32) and (2.36) and are evaluated using the current values of $V(i)$ and S_{i+1}.
(for purposes of proof $k \leftarrow k+1$ $\pi(k) \leftarrow \pi$, the Linear Control Policy composed of the current values of G_i, $i=0,1,\ldots,N-1$)

7. $S_i \leftarrow Q + G_i^T R G_i + [A+BG_i]^T S_{i+1} [A+BG_i]$

8. If $i > 0$, $i \leftarrow i-1$ go to 6.

9. Go to 3.

For the partial state feedback case the result of Theorem 2 could replace that of Theorem 1 in step 6.

It may be noted that the calculation of S_i in reverse time followed by the substitution of the new G_i^S in the linear control policy in forward time would require only one new $V(i)$ to be calculated for each G_i replaced. This approach would be equally valid and have the same computational advantages as the procedure previously outlined.

The preceding procedure requires the calculation of (N-1) $V(i)$'s and N S_i's for each N changes in the control policy. If changes in control policy were computed in forward time, as the new $V(i)$ were calculated, as well as in backward time, as the new S_i were calculated, (2N-1) changes in control policy would be made for the calculation of (N-1)$V(i)$'s and N S_i's. If the improvement in cost on doing a forward and a backward time calculation is of the same order as doing two backward time calculations, this would reduce the total amount of computer time required to find a solution. Computational Procedure B is a method of doing such a calculation.

Computational Procedure B:

1. Specify a linear control policy

   $\pi = [G_i, i=0,1,\ldots,N-1]$

   (for purposes of proof $k \leftarrow 0$ $\pi(0) \leftarrow \pi$)

2. $i \leftarrow N$ $V(0) \leftarrow V_0$

3. $i \leftarrow i-1$

4. $S_i \leftarrow Q + G_i^T R G_i + [A+BG_i]^T S_{i+1} [A+BG_i]$

5. If $i > 0$ go to 3.

6. $G_i \leftarrow$ the structured state feedback matrix with unconstrained gains obtained by solving

$$F_i g_i = - h_i \tag{3.1}$$

   where F_i and h_i are defined by (2.32) and (2.36) and are evaluated by using the current values of $V(i)$ and S_{i+1}.

   (for purposes of proof $k \leftarrow k+1$ $\pi(k) \leftarrow \pi$)

7. $V(i+1) \leftarrow [A+BG_i]V(i)[A+BG_i]^T + V_w$

8. $i \leftarrow i+1$

9. If $i < N-1$ go to 6.

10. $G_i \leftarrow$ the structured state feedback matrix with unconstrained gains obtained by solving

$$F_i g_i = - h_i \tag{3.1}$$

   (for purposes of proof $k \leftarrow k+1$ $\pi(k) \leftarrow \pi$)

11. $S_i \leftarrow Q + G_i^T R G_i + [A+BG_i]^T S_{i+1} [A+BG_i]$

12. $i \leftarrow i-1$

13. If $i > 0$ go to 10.

14. Go to 6.

It can be seen that Computational Procedure A is a simpler and requires nmN fewer computer store locations than Computational Procedure B.

It will now be shown that these Computational Procedures always produce convergence in cost.

Notation:

Let $E[L_k \mid \pi(k)]$ mean the expected cost over the time interval k to N given that the Linear Control Policy $\pi(k)$ is used. Define

$$J(\pi(k)) = E[L_0 \mid \pi(k)] \tag{4.1}$$

By $F_i(k)$, $h_i(k)$, $g_i(k)$ etc. is meant the values F_i, h_i, g_i etc. assume if control policy $\pi(k)$ is used, where k is the total number of replacements of G_i that have been made using Computational Procedure A or B.

Theorem 3:

Computational Procedures A and B converge in cost. That is

$$\lim_{k \to \infty} J(\pi(k)) = \underline{J} \tag{4.2}$$

is defined.

Proof

Consider any $k \geq N$. The Linear Control Policy $\pi(k)$ will be a Structured Control Policy. By construction the Structured Control Policies $\pi(k)$ and $\pi(k+1)$ differ only in the element G_i. This is true for both Computational Procedures A and B. By use of (2.30) one can write the expected cost as

$$J(\pi(k)) = \tfrac{1}{2}g_i^T(k)F_i(k)g_i(k) + h_i^T(k)g_i(k) + c_i(k) \qquad (4.3)$$

$$J(\pi(k+1)) = \tfrac{1}{2}g_i^T(k+1)F_i(k+1)g_i(k+1) + h_i^T(k+1)g_i(k+1) + c_i(k+1) \qquad (4.4)$$

As F_i, h_i^T and c_i depend on G_i, $i=0,1,\ldots,k-1,k+1,\ldots,N-1$ only

$$F_i(k) = F_i(k+1)$$
$$h_i(k) = h_i(k+1) \qquad (4.5)$$
and
$$c_i(k) = c_i(k+1)$$

Thus

$$J(\pi(k+1)) = \tfrac{1}{2}g_i^T(k+1)F_i(k)g_i(k+1) + h_i^T(k)g_i(k+1) + c_i(k) \qquad (4.6)$$

$g_i(k+1)$ was chosen so that

$$F_i(k)g_i(k+1) = -h_i(k) \qquad (4.7)$$

which is the minimising value of g_i for the quadratic functions (4.3) and (4.6). Therefore

$$J(\pi(k)) > J(\pi(k+1))$$

with equality holding only if $g_i(k)$ satisfies

$$F_i(k)g_i(k) = -h_i(k) \qquad (4.8)$$

Thus $J(\pi(k))$, $k=N, N+1, \ldots$ is a monotonically decreasing sequence. The quadratic cost function L_0 defined by (1.3) is positive semi-definite by assumption, which implies $J(\pi(k))$ is bounded below by zero. Thus the sequence $J(\pi(k)), k=0,1, \ldots$ must converge [5,p.47]. Q.E.D.

4.3. The Optimal Structured Control Policy

A control policy can only be implemented if the gains are finite. Thus it is useful to have the following.

Definition:

A linear Control Policy will be said to EXIST if all its gains are finite.

If V_0, V_w and R are positive definite then the optimal structured control policy and the limiting structured control policies produced by the computational procedure can be shown to exist. To establish this result one needs the definition of the norm of a Linear Control Policy.

A Linear control policy can be considered as a point in the nmN space of gain elements. Similarly a structured control policy can be considered a point on the Np space of its free gain elements. It therefore makes sense to define norm in terms of the Euclidean norm on these spaces. That is the square root of the sum of the squares of the gain elements.

Definition:

The NORM of a Linear Control Policy, π, (written $|\pi|$) is

$$|\pi| = \left(\sum_{i=0}^{N-1} tr(G_i G_i^T)\right)^{\frac{1}{2}} \qquad (4.9)$$

which for a Structured Control Policy may be written

$$|\pi^S| = \left(\sum_{i=0}^{N-1} g_i^T g_i\right)^{\frac{1}{2}} \qquad (4.10)$$

This may be easily shown by use of (1.14), (1.15) (1.24), (1.25), (2.35) and the definition of trace.

Theorem 4:

If V_0, V_w and R are positive definite then the Optimal Structured Control Policy exists.

Theorem 5:

If V_0, V_w and R are positive definite then the Control Policies generated by Computational Procedures A and B satisfy

$$|\pi(k)| \leqslant C \qquad k=0,1,2,\ldots\ldots \qquad (4.11)$$

where C is a positive constant.

Corollary:

If V_0, V_w and R are positive definite then $\pi(k)$, $k=N,N+1,N+2,\ldots\ldots$ has at least one limit point, and all limit points exist.

These results follow directly from:

Lemma 7:

If V_0, V_w and R are positive definite then

$$J(\pi) \leqslant K \tag{4.12}$$

implies

$$|\pi| \leqslant C \tag{4.13}$$

where K is a positive constant and C is a positive constant related to K.

Proof:

The definition of $J(\pi)$, (1.12), (1.3), (1.11), and assumption (4.12) give

$$K \geqslant J(\pi) = E[\tfrac{1}{2} \sum_{i=0}^{N-1} x_i^T[Q+G_i^TRG_i]x_i + \tfrac{1}{2}x_N^TS_Nx_N] \tag{4.14}$$

and Q and S_N are positive semi-definite

$$K \geqslant J(\pi) \geqslant \tfrac{1}{2} \sum_{i=0}^{N-1} E[x_i^TG_i^TRG_ix_i] \tag{4.15}$$

Substitution of (1.1) with k=1,2,....,N-1 produces

$$K \geqslant \tfrac{1}{2}E[x_0^TG_0^TRG_0x_0] + \tfrac{1}{2} \sum_{i=1}^{N-1} E[w_i^TG_i^TRG_iw_i]$$

$$+ \sum_{i=1}^{N-1} E[w_i^TG_i^TRG_i[A+BG_{i-1}]x_{i-1}]$$

$$+\tfrac{1}{2} \sum_{i=1}^{N-1} E[x_{i-1}^T[A+BG_{i-1}]^TG_i^TRG_i[A+BG_{i-1}]x_{i-1}] \tag{4.16}$$

as R is positive definite

$$E[x_{i-1}^T[A+BG_{i-1}]^T G_i^T R G_i [A+BG_{i-1}]x_{i-1}] \geq 0 \qquad (4.17)$$

w_i is a zero mean white noise process uncorrelated with past or present state thus

$$E[w_i^T G_i^T R G_i [A+BG_{i-1}]x_{i-1}] = 0 \qquad (4.18)$$

Thus from (4.16) one can deduce

$$K \geq \tfrac{1}{2}E[x_0^T G_0^T R G_0 x_0] + \tfrac{1}{2}\sum_{i=0}^{N-1} E[w_i^T G_i^T R G_i w_i] \qquad (4.19)$$

Use of the trace identity (2.22) yields

$$K \geq \tfrac{1}{2}\mathrm{tr}[G_0^T R G_0 V_0] + \tfrac{1}{2}\sum_{i=0}^{N-1} \mathrm{tr}[G_i^T R G_i V_w] \qquad (4.20)$$

By use of the same argument that was used in the proof of Lemma 4 to get from equations (2.61) and (2.64) to (2.79) one finds

$$K \geq \tfrac{1}{2}\lambda_V \lambda_R (\sum_{i=0}^{N-1} \mathrm{tr}[G_i G_i^T]) \qquad (4.21)$$

λ_R is the smallest eigenvalue of R. $\lambda_R > 0$ as R is assumed positive definite. λ_V is the smallest number in the set of eigenvalues of V_0 and V_w. As both V_0 and V_w are assumed positive definite, $\lambda_V > 0$.

Substitution of (4.9) and the obvious algebraic manipulation yields

$$|\pi| \leqslant \sqrt{\frac{2K}{\lambda_V \lambda_R}} \tag{4.22}$$

Define

$$C = \sqrt{\frac{2K}{\lambda_V \lambda_R}} \tag{4.23}$$

as K, λ_V and λ_R are all positive C is positive. As λ_V and λ_R are both > 0, C is finite.

Q.E.D.

Proof of Theorem 4:

Consider any structured control policy, π^S, such that $|\pi^S|$ is finite. Associated with this control policy is a finite expected cost $J(\pi^S)$. This follows directly from (2.16). The optimal structured control policy, $\overset{*}{\pi}{}^S$, must satisfy

$$J(\overset{*}{\pi}{}^S) \leqslant J(\pi^S) \tag{4.24}$$

Then Lemma 7 states

$$|\overset{*}{\pi}{}^S| \leqslant C \tag{4.25}$$

If the sum of the squares of all the gains in a control policy is finite, then every gain must be finite. Thus $\overset{*}{\pi}{}^S$ exists.

Q.E.D.

<u>Proof of Theorem 5</u>:

It was shown in the proof of Theorem 3 that $J(\pi(i))$, $i=N,N+1,\ldots$ is a decreasing monic sequence. Thus

$$J(\pi(i)) \leq \max_{0\leq k\leq N} J(\pi(k)), \qquad i=0,1,2,\ldots \qquad (4.26)$$

and Lemma 7 yields

$$|\pi(i)| \leq C, \qquad i=0,1,2,\ldots \qquad (4.11)$$

$\pi(k), k=N,N+1,N+2,\ldots$ is an infinite sequence of structured feedback policies. If one considers each gain element to be a coordinate, it is an infinite sequence in an Np dimensional Euclidean space. (4.11) implies that it is an infinite sequence within the closed and bounded set $[\pi^S \,|\, |\pi^S| \leq C]$.

Thus it must have at least one limit point in the set [5, p.35]. As a closed set by definition contains all its limit points any other limit points must also lie within the set. Thus all limit points have finite gains and thus satisfy the definition of existence of a Linear Control Policy.

Q.E.D.

The Optimal Structured Control Policy exists and lies within the same set that contains the limit points of the computational procedure. One might hope that for all Linear Control Policies with which one might start the computational procedure there would be one limit point which would be the Optimal Structured Control Policy. This unfortunately is not necessarily true.

In discussing the nature of the limit points of the computational procedure the following definitions are useful.

Singular Point

The Structured Control Policy, π^1, will be termed a SINGULAR POINT of $J(\pi^S)$ if

$$\left.\frac{\partial J(\pi^S)}{\partial g_k}\right|_{\pi^S=\pi^1} = 0, \quad k=0,1,\ldots,N-1 \qquad (4.27)$$

Distance Measure

The definition of Norm of a Linear Control Policy (4.9) implies a measure of the distance between two control policies. If $\pi^1 = [G_k^1,\ k=0,1,\ldots,N-1]$ and $\pi^2 = [G_k^2,\ k=0,1,\ldots,N-1]$ then the DISTANCE between π^1 and π^2 is defined to be

$$|\ \pi^1 - \pi^2\ | = \left(\sum_{i=0}^{N-1} tr([G_i^1 - G_i^2]^T[G_i^1 - G_i^2])\right)^{\frac{1}{2}} \qquad (4.28)$$

ϵ-neighbourhood

An ϵ-NEIGHBOURHOOD (or neighbourhood) of a Linear Control Policy, π, is a set $N_\epsilon(\pi)$ consisting of all Linear Control Policies, π', such that $|\ \pi - \pi'\ | < \epsilon$.

(From Rudin [5, p.28]);

Absolute Maximum [minimum]

The function $J(\pi^S)$ takes on its ABSOLUTE MAXIMUM [MINIMUM] for the Structured Control Policy, π', if

$$J(\pi^S) \leqslant J(\pi') \tag{4.29}$$

$$[J(\pi^S) \geqslant J(\pi')] \tag{4.30}$$

for every Structured Control Policy, π^S.

(From Hadley [6, p.53])

If $J(\pi^S)$ is a constant then $J(\pi^S)$ takes on both its absolute maximum and its absolute minimum for any structured control policy π^S.

Strong Local Maximum [Minimum]

The function $J(\pi^S)$ is said to have a STRONG LOCAL MAXIMUM [MINIMUM] at π' if there exists an ϵ, $\epsilon > 0$ such that for all structured control policies, π^S, where $0 < | \pi' - \pi^S | < \epsilon$ then

$$J(\pi^S) < J(\pi') \tag{4.31}$$

$$[J(\pi^S) > J(\pi')] \tag{4.32}$$

(From Hadley [6, p.53])

Weak Local Maximum [Minimum]

The function $J(\pi^S)$ is said to have a WEAK LOCAL MAXIMUM [MINIMUM] at π' if it does not have a strong local maximum [minimum] at π' but there exists an $\epsilon > 0$ such that

$$J(\pi^S) \leqslant J(\pi') \tag{4.33}$$

$$[J(\pi^S) \geqslant J(\pi')] \tag{4.34}$$

for all π^S in the ϵ-neighbourhood of π', $N_\epsilon(\pi')$.

(From Hadley [6,p.54])

If $J(\pi^S)$ is constant over an ϵ-neighbourhood of π' then it has both a weak local maximum and a weak local minimum at π'.

It will be shown that all limit points of the computational procedure are solutions to a two point boundary value problem of which the optimal structured control policy is also a solution. All solutions to this two point boundary value problem are singular points of the function $J(\pi^S)$.

<u>Theorem 6</u>:

Consider the two point boundary value problem. Find a Structured Control Policy $\pi^S = [G_k^S,\ k=0,1,\dots,N-1]$ where

$$V(0) = V_0$$

$$V(k+1) = [A+BG_k^S]V(k)[A+BG_k^S]^T + V_w \tag{2.1}$$

for $k=0,1,\dots,N-1$,

S_N is given,

$$S_i = Q + G_i^{ST}RG_i^S + [A+BG_i^S]^T S_{i+1}[A+BG_i^S] \tag{2.15}$$

for $i=N-1,N-2,\dots,0$,

and all G_i^S, $i=0,1,\dots,N-1$ satisfy

$$F_i g_i = -h_i \tag{3.1}$$

where F_i is defined by (2.32), h_i is defined by (2.36), and the relationship between G_i^S and g_i is defined by (1.14), (1.15), (1.24), (1.25) and (2.35).

Then,

1. All Optimal Structured Control Policies are solutions of this two point boundary value problem.

2. The solutions of this two point boundary value problem are the singular points of the function $J(\pi^S)$.

3. No solution of this two point boundary value problem is a strong local maximum of $J(\pi^S)$.

4. If V_0, V_w and R are positive definite then solutions of this two point boundary value problem can not be absolute maxima nor can they be weak local maxima.

Proof:

It was shown in the proof of Lemma 3 (following (2.47)) that F_k is positive semidefinite. Therefore the functions

$$J(\pi^S) = \tfrac{1}{2}g_k^T F_k g_k + h_k^T g_k + c_k \qquad (2.30)$$

k=0,1,...,N-1 are convex in g_k [6, p.84]. Differentiation of (2.30) yields

$$\frac{\partial J(\pi^S)}{\partial g_k} = g_k^T F_k + h_k^T \qquad (4.35)$$

Thus if and only if

$$F_k g_k = -h_k, \qquad k=0,1,\ldots,N-1 \qquad (3.1)$$

does

$$\frac{\partial J(\pi^S)}{\partial g_k} = 0, \qquad k=0,1,\ldots,N-1 \qquad (4.27)$$

Thus as (3.1) holds for any solution to the two point boundary value problem, all solutions to the two point boundary value problem satisfy (4.27) and are thus termed singular points. As all singular points must satisfy (3.1) they are all solutions to the two point boundary value problem, 2.

The convexity of the functions (2.30) implies that if and only if π' satisfies (4.27) does $J(\pi')$ satisfy

$$J(\pi') \leq J(\pi'') \tag{4.36}$$

for all $\pi'' \in \Omega$ where

$$\Omega = [\, \pi^S \mid g_i = g_i', \; i=0,1,\ldots,j-1,j+1,\ldots,N-1; \quad g_j \neq g_j' \text{ where } j \in [0,1,\ldots,N-1] \text{ and } g_i', \; i=0,1,\ldots,N-1 \text{ are the unconstrained gain vectors, (2.35), of } \pi' \,]$$

An Optimal Structured Control Policy, $\overset{*}{\pi}{}^S$, is defined to be a global minimum. Thus

$$J(\overset{*}{\pi}{}^S) < J(\pi^S) \qquad \text{for all } \pi^S.$$

$\overset{*}{\pi}{}^S$ must then satisfy (4.36) which implies it satisfies (4.27), which implies it is a solution to the two point boundary value problem, 1.

Assume π' is a solution to the two point boundary value problem. Within any ϵ-neighbourhood of π' one can find a $\pi'' \in \Omega$. (4.36) contradicts (4.31). Thus a solution of the two point boundary value problem cannot be a strong local maximum, 3.

If V_0, V_w and R are positive definite then the F_k, $k=0,\ldots,N-1$ are positive definite by Lemma 4. Then the functions (2.30) are strictly convex in g_k [6, p.85], which implies

$$J(\pi') < J(\pi'') \qquad (4.37)$$

for all $\pi'' \in \Omega$, where π' is a solution of the two point boundary value problem. (4.37) contradicts (4.29) and (4.33). Thus π' cannot be an absolute maximum nor can it be a weak local maximum, 4.

Q.E.D.

It has been shown that if V_0, V_w and R are positive definite then all limit points of the computational procedure exist. It will now be shown that either there is one limit point or the set of limit points is uncountable and connected. All limit points satisfy the necessary condition stated in Theorem 6.

Theorem 7:

If V_0, V_w and R are positive definite, either $\pi(k)$, $k=N,N+1,\ldots,$ has one limit point or the set of limit points is an uncountable connected set. All limit points satisfy the two point boundary value problem defined in Theorem 6.

To establish this theorem a lemma is needed.

Lemma 8:

If R, V_0 and V_w are positive definite then

$$| g_i(k) - g_i(k+1) | \rightarrow 0 \text{ as } k \rightarrow \infty \qquad (4.38)$$

for $i=0,1,\ldots,N-1$.

Proof:

By construction the Structured Control Policies $\pi(k)$ and $\pi(k+1)$ differ only in one Structured Feedback Matrix G_i^S. Thus

$$g_j(k) = g_j(k+1) \qquad \text{for } j \neq i$$

which implies

$$| g_j(k) - g_j(k+1) | = 0 \qquad \text{if } j \neq i \tag{4.39}$$

To establish Lemma 8 it is only necessary to consider $| g_i(k) - g_i(k+1) |$.

Theorem 3 states that convergence in cost always occurs and

$$\lim_{k \to \infty} J(\pi(k)) = \underline{J} \tag{4.2}$$

It was also established that the sequence $J(\pi(k))$, $k=N,N+1,....$ is a monotonically decreasing sequence.

Define

$$D = J(\pi(N)) - \underline{J} \geqslant 0 \tag{4.40}$$

Addition and subtraction of terms and the use of the definition of $\underline{J}$ yields

$$D = \lim_{i \to \infty} \sum_{k=N}^{i} [J(\pi(k)) - J(\pi(k+1))] \tag{4.41}$$

Define

$$\delta_k = J(\pi(k)) - J(\pi(k+1)) \geqslant 0 \tag{4.42}$$

As V_0, V_w and R are positive definite $\ddot{F}_i(k)$ is invertible. (4.7) implies

$$g_i(k+1) = -\ddot{F}_i^{-1}(k)h_i(k) \tag{4.43}$$

Substitution of (4.43) into the expression for cost (4.6) produces

$$J(\pi(k+1)) = -\tfrac{1}{2}h_i^T(k)\ddot{F}_i^{-1}(k)h_i(k) + c_i(k) \tag{4.44}$$

Substitution of (4.3) and (4.44) into (4.42) gives

$$\delta_k = \tfrac{1}{2}g_i^T(k)\ddot{F}_i(k)g_i(k) + h_i^T(k)g_i(k) + \tfrac{1}{2}h_i^T(k)\ddot{F}_i^{-1}(k)h_i(k) \tag{4.45}$$

Define

$$\beta(k) = g_i(k) - g_i(k+1) \tag{4.46}$$

By use of (4.43) one finds

$$g_i(k) = \beta(k) - \ddot{F}_i^{-1}(k)h_i(k) \tag{4.47}$$

Substitution of (4.47) into (4.45) yields

$$\delta_k = \tfrac{1}{2}\beta^T(k)\ddot{F}_i(k)\beta(k) \tag{4.48}$$

As R, V_0 and V_w are positive definite Lemma 4 holds for any structured control policy. Setting $g = \beta(k)$ in (2.57) yields

$$\beta^T(k)\ddot{F}_i(k)\beta(k) \geqslant \lambda\beta^T(k)\beta(k) \tag{4.49}$$

Thus

$$\delta_k \geqslant \frac{\lambda}{2}\, \beta^T(k)\beta(k) \tag{4.50}$$

The definition of Euclidean Norm

$$|\, \beta(k)\, | = (\beta^T(k)\beta(k))^{\frac{1}{2}} \tag{4.51}$$

and (4.46) imply

$$|\, g_i(k) - g_i(k+1)\, | \leqslant \sqrt{\frac{2\delta_k}{\lambda}} \,, \tag{4.52}$$

As $\lim_{i\to\infty} \sum_{k=N}^{i} \delta_k = D$, $\delta_k \to 0$ as $k \to \infty$.

Thus (4.52) implies

$$|\, g_i(k) - g_i(k+1)\, | \to 0 \quad \text{as } k \to \infty \tag{4.38}$$

Q.E.D.

Proof of Theorem 7:

The proof of this theorem will be divided into three parts: a proof that either there is one limit point or the set of limit points is an uncountable set; a proof that if the set of limit points is uncountable then it is connected; and a proof that all limit points are solutions of the two point boundary value problem defined in Theorem 6.

A) There is one limit point or the set of limit points is uncountable

The Corollary of Theorem 5 states that at least one limit point exists. It will now be shown that if more than one limit point exists then the set of limit points is uncountable.

Define E to be the set of limit points (i.e. the subsequential limits) of the sequence of structured control policies $\pi(k)$, $k=N,N+1,\ldots$. E is a set in the space of Structured Control Policies. The definition of the limit point of a set is [5, p.28]

Limit Point

A Structured Control Policy π' is a LIMIT POINT of the set E if every neighbourhood of π' contains a structured control policy π^S such that $\pi^S \in E$.

As $\pi(k)$ and $\pi(k+1)$ for all $k > N$ differ only in one structured state feedback matrix G_i^S (4.28) reduces to

$$\left| \pi(k) - \pi(k+1) \right| = \left| g_i(k) - g_i(k+1) \right| \qquad (4.53)$$

Thus Lemma 8 implies

$$\left| \pi(k) - \pi(k+1) \right| \rightarrow 0 \text{ as } k \rightarrow \infty \qquad (4.54)$$

Assume a point $\pi' \in E$ exists such that π' is not a limit point of the set E and assume E contains more than one element. As π' is not a limit point of E and ϵ exists such that $N_\epsilon(\pi')$ contains no other points of the set E. As π' is a limit point of $\pi(k)$, $k=N,N+1,\ldots,N_{\epsilon/3}(\pi')$ contains an infinite subsequence of $\pi(k)$, $k=N,N+1,\ldots,$ [10, p.42, p.44].

Consider the set

$$S = [\pi^S \mid \tfrac{\epsilon}{3} \leqslant \left| \pi' - \pi^S \right| \leqslant \tfrac{2\epsilon}{3}] \subset N_\epsilon(\pi') \qquad (4.55)$$

$N_\epsilon(\pi')$ contains one member of E and that member is not within S. As S is a closed and bounded set it can only contain a finite number of elements of $\pi(k)$, $k=N,N+1,\ldots$, [5, p.35]. (4.54) implies there exists an M such that if $k > M$ then

$$|\ \pi(k) - \pi(k+1)\ | < \frac{\epsilon}{3} \qquad (4.56)$$

$N_{\epsilon/3}(\pi')$ contains an infinite subsequence of $\pi(k), k=M,M+1,\ldots.$ Let this subsequence be $\pi(k), k=M_1,M_2,M_3,\ldots.$ where $M_1 < M_2 < M_3 < \ldots.$ If $\pi(M_i+1) \in N_{\epsilon/3}(\pi')$, (4.56) implies $\pi(M_i+1) \in S$. As $\pi(k), k=N,N+1,\ldots..$ has only a finite number of points in S there exists a j such that $\pi(M_i+1) \in N_{\epsilon/3}(\pi')$ for all $i > j$, which implies $\pi(k) \in N_{\epsilon/3}(\pi')$ for all $k > M_j$. Therefore π' is the only limit point of $\pi(k), k=N,N+1,\ldots...$ Contradiction! It was assumed the set of limit points has more than one member. Thus if the set of limit points, E, has more than one member then all points $\pi' \in E$ are limit points of the set E.

A set of limit points is a closed set [5, p.45]. Therefore if E contains more than one element it must be by definition <u>perfect</u> [5, p.28]. Every non-empty perfect set in a Euclidean space is uncountable [5, p.36]. If E contains more than one point it is uncountable.

B) <u>If E is uncountable then it is connected</u>

The following definitions will be needed.

By $[\pi(k)]_N^\infty$ is meant the sequence $\pi(k), k=N,N+1,\ldots$.

<u>Interior Point</u> [5, p.28]

A point π' is an INTERIOR POINT of E if there is a neighbourhood $N_\epsilon(\pi')$ such that $N_\epsilon(\pi') \subset E$.

<u>Open Set</u> [5, p.28]

E is OPEN if every point of E is an interior point of E.

Connected Set [5, p.37]

A set E in a metric space $\mathscr{S}$ is said to be CONNECTED if there do not exist two disjoint open subsets I and J of $\mathscr{S}$ such that I intersects E and J intersects E, and $E \subset I \cap J$.

Here $\mathscr{S}$ is the metric space of all Structured Control Policies. Assume that E, the set of limit points of $[\pi(k)]_N^\infty$, is not connected and two disjoint open subsets I and J exist such that $E \subset I \cup J$ and I intersects E and J intersects E. Then

$$E = E \cap (I \cup J) = (E \cap I) \cup (E \cap J) \tag{4.57}$$

Define

$$\alpha = \text{glb} \left| \pi' - \pi \right| \tag{4.58}$$

for all $\pi' \epsilon (E \cap I)$ and $\pi \epsilon (E \cap J)$.

Select a sequence $\pi_i' \epsilon (E \cap I)$, i=1,2,3,.... and other sequence $\pi_i \epsilon (E \cap J)$, i=1,2,3,... such that

$$\left| \pi_1' - \pi_1 \right| < \alpha + 1$$

$$\left| \pi_2' - \pi_2 \right| < \alpha + 1/2$$

$$\left| \pi_3' - \pi_3 \right| < \alpha + 1/4$$

$$\vdots$$

$$\left| \pi_i' - \pi_i \right| < \alpha + 1/2^{i-1} \tag{4.59}$$

$$\vdots$$

$[\pi_i']_1^\infty$ is an infinite sequence within the closed and bounded set E [5, p.45]. Thus $[\pi_i']_1^\infty$ has at least one limit point $\underline{\pi}' \epsilon E$ [10, p.35]. (4.57) implies that either $\underline{\pi}' \epsilon (E \cap I)$ or $\underline{\pi}' \epsilon (E \cap J)$. $(E \cap I)$ and $(E \cap J)$ are disjoint as I and J are disjoint. Assume $\underline{\pi}' \epsilon (E \cap J) \subset J$. As all points in an open set are interior points there is a neighbourhood of $\underline{\pi}'$ which contains no elements of $(E \cap I)$. Thus $\underline{\pi}'$ can not be a limit point of $[\pi_i']_1^\infty \subset (E \cap I)$ [5, p.42 & 44]. The contradiction implies $\underline{\pi}' \epsilon (E \cap I)$. Similarly the infinite sequence $[\pi_i]_1^\infty$ has at least one limit point $\underline{\pi} \epsilon (E \cap J)$.

The construction of the sequence (4.59) implies that

$$| \underline{\pi}' - \underline{\pi} | = \alpha \tag{4.60}$$

As $\underline{\pi}' \epsilon I$ and as I is an open set there exists a neighbourbood $N_{\epsilon_1}(\underline{\pi}') \subset I$.

Construct the set

$$D = [\bigcup_{\pi' \epsilon (E \cap I)} [\pi \,||\, \pi' - \pi | < \epsilon_1/2]] \cap I \tag{4.61}$$

D is an open set [5, p.30] and

$$(E \cap I) \subset D \subset I \tag{4.62}$$

Similarly there exists a neighbourhood $N_{\epsilon_2}(\underline{\pi}) \subset J$. Construct the set

$$P = [\bigcup_{\pi' \epsilon (E \cap J)} [\pi \,||\, \pi' - \pi | < \epsilon_2/2]] \cap J \tag{4.63}$$

P is an open set and

$$(E \cap J) \subset P \subset J \tag{4.64}$$

Note that

$$\alpha > \epsilon_1 + \epsilon_2 \tag{4.65}$$

If this were not true one could find a point π such that $\pi \epsilon N_{\epsilon_1}(\underline{\pi}') \subset I$ and $\pi \epsilon N_{\epsilon_2}(\underline{\pi}) \subset J$ which would contradict the assumption that I and J are disjoint.

Define

$$d = \text{glb} \mid \pi' - \pi \mid \quad \text{for all } \pi' \epsilon D \text{ and } \pi \epsilon P \tag{4.66}$$

then

$$d > \frac{\epsilon_1 + \epsilon_2}{2} \tag{4.67}$$

for if this were not true one could find a $\pi' \epsilon (E \cap I)$ and $\pi \epsilon (E \cap J)$ such that

$$\mid \pi' - \pi \mid < \epsilon_1 + \epsilon_2 < \alpha \tag{4.68}$$

which would violate the definition of α, (4.58). Let

$$H = [\pi \mid \mid \pi \mid \leqslant C] \tag{4.69}$$

where

C is the positive constant defined in Theorem 5.

D' the complement of D, and

P' the complement of P are closed sets.

As H is a closed and bounded set

$$W = D' \cap P' \cap H \tag{4.70}$$

is a closed and bounded set which by construction contains no limit points of the sequence $[\pi(k)]_N^\infty$. Therefore W contains only a finite subsequence of $[\pi(k)]_N^\infty$.

Let $\pi(k)$, $k=M_1, M_2, M_3, \ldots$, where $M_1 < M_2 < M_3 < \ldots$, be the subsequence of $[\pi(k)]_N^\infty$ contained within D. Such an infinite subsequence must exist because D is an open set containing limit points of $[\pi(k)]_N^\infty$. For some M, $k > M$ implies

$$\left| \pi(k) - \pi(k+1) \right| < \frac{\epsilon_1 + \epsilon_2}{2} \tag{4.71}$$

This follows from (4.54). Thus if $M_i > M$, $\pi(M_i+1) \notin D$, (4.71) combined with (4.66) and (4.67) implies $\pi(M_i+1) \in W$. As W contains a finite subsequence of $[\pi(k)]_N^\infty$ there exists a j such that $i \geq j$ implies $\pi(M_i+1) \in D$. This implies that the subsequence $[\pi(k)]_{M_j}^\infty$ is contained in the set D. As $D \subset I$ and I and J are disjoint J can only contain a finite subsequence of $[\pi(k)]_N^\infty$. Thus J contains no limit points of the sequence $[\pi(k)]_N^\infty$, [5, p.42]. This contradicts the assumption that J intersects E, the set of limit points of $[\pi(k)]_N^\infty$. Thus E is a connected set.

C) <u>All limit points are solutions of the two-point boundary value problem of Theorem 6</u>

For simplicity only computational procedure A and the reverse time "pass" of computational procedure B will be considered. The proof for subsequences obtained from the forward time "pass" of computational procedure B differs only in the order in which changes of structured state feedback matrices occur.

Assume k is so selected that control policies $\pi(k)$ and $\pi(k+1)$ differ only in the Structured State Feedback Matrix G^S_{N-1}. Substitution of (4.43) into (4.52) produces

$$| g_{N-1}(k) + F^{-1}_{N-1}(k)h_{N-1}(k) | \leqslant \sqrt{\frac{2\delta_k}{\lambda}} \tag{4.72}$$

as $\delta_k \to 0$ as $k \to \infty$.

$$g_{N-1}(k) \to - F^{-1}_{N-1}(k)h_{N-1}(k) \text{ as } k \to \infty \tag{4.73}$$

and

$$\pi(k) \to \pi(k+1) \text{ as } k \to \infty \tag{4.54}$$

Similarly

$$| g_{N-2}(k+1) + F^{-1}_{N-2}(k+1)h_{N-2}(k+1) | < \sqrt{\frac{2\delta_{k+1}}{\lambda}} \tag{4.74}$$

which implies

$$g_{N-2}(k) \to - F^{-1}_{N-2}(k+1)h_{N-2}(k+1) \text{ as } k \to \infty \tag{4.75}$$

and

$$\pi(k) \to \pi(k+1) \to \pi(k+2) \text{ as } k \to \infty \tag{4.76}$$

Continuing in this fashion one finds

$$g_{N-1-i}(k+i) \rightarrow - F^{-1}_{N-1-i}(k+1)h_{N-1-i}(k+i) \text{ as } k \rightarrow \infty \qquad (4.77)$$

$i=0,1,\ldots,N-1$

and

$$\pi(k) \rightarrow \pi(k+1) \rightarrow \ldots\ldots \rightarrow \pi(k+N-1) \text{ as } k \rightarrow \infty \qquad (4.78)$$

Let $\pi(k)$, $k=M_1,M_2,M_3,\ldots.$ where $M_1 < M_2 < M_3 < \ldots..$ be a subsequence of $[\pi(k)]_k^{\infty}$ converging to $\underline{\pi}$. Let M_j differ from M_{j+1} in G_i^S. Add to the subsequence $\pi(k),k=M_1,M_2,M_3,\ldots.$ the elements $\pi(p),p=M_j-N+1+i,\ldots.,M_j-1,M_j+1,\ldots..,M_j+i+1$ for all $j=1,2,3,\ldots.$. (4.78) implies the addition of these elements will not affect the convergence of the subsequence. Let this new subsequence be denoted $\pi(k)$, $k=M_1',M_2',M_3',\ldots.$ where $M_1' < M_2' < M_3' < \ldots.$. At the limit of this subsequence $\underline{\pi}$ (4.77) implies

$$\underline{g}_i = - \underline{F}_i^{-1}\underline{h}_i \, , \qquad i=N-1,\ldots,0 \qquad (4.79)$$

where $\underline{g}_i$, $\underline{F}_i$ and $\underline{h}_i$ are the values g_i, F_i and h_i assume if control policy $\underline{\pi}$ is used. As $\underline{F}_i$ and $\underline{h}_i$ are calculated using V(i) and S_{i+1} which are computed using (2.1) and (2.15), $\underline{\pi}$ satisfies the two point boundary value problem of Theorem 6.

Q.E.D.

If the two point boundary value problem of Theorem 6 has a unique solution then the Computational Procedure will converge to the Optimal Structured Control Policy no matter what initial Linear

Control Policy $\pi(0)$ was used. Unfortunately this is not always the case.

Theorem 8:

The set of limit points of $[\pi(k)]_N^\infty$, E, and the cost associated with these limit points J, may depend on the initial control policy $\pi(0)$.

Proof:

The proof is by example. A simple system which has three solutions to the two point boundary value problem is

System A

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0.544721 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$V_0 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.75 \end{bmatrix} \qquad V_w = \begin{bmatrix} 0.25 & 0.136180 \\ 0.136180 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & -2.73676 \\ -2.73676 & 8 \end{bmatrix} \qquad R = 1$$

$$S_2 = \begin{bmatrix} 3 & -4 \\ -4 & 6 \end{bmatrix} \qquad N = 2 \qquad n_1 = 1$$

As N = 2, (2.16) becomes

$$J(\pi^S) = E[L_0] = \tfrac{1}{2}tr[S_0V_0] + \tfrac{1}{2}[S_1V_w] + \tfrac{1}{2}tr[S_2V_w] \qquad (4.80)$$

To compute $J(\pi^S)$ one must first calculate S_1 and S_0. As $n_1 = 1$ one may define

$$G_0 = [g_0 \quad 0] \quad \text{and} \quad G_1 = [g_1 \quad 0] \qquad (4.81)$$

Then

$$S_1 = Q + G_1^TRG_1 + [A+BG_1]^TS_2[A+BG_1] \qquad (4.82)$$

where

$$A+BG_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0.544721 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [g_1 \quad 0]$$

$$= \begin{bmatrix} g_1 & -1 \\ 1 & 0.544721 \end{bmatrix} \qquad (4.83)$$

Thus

$$S_1 = \begin{bmatrix} 1 & -2.73676 \\ -2.73676 & 8 \end{bmatrix} + [g_1 \quad 0][1]\begin{bmatrix} g_1 \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} g_1 & 1 \\ -1 & 0.544721 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} g_1 & -1 \\ 1 & 0.544721 \end{bmatrix}$$

$$= \begin{bmatrix} (4g_1^2 - 8g_1 + 7) & -(5.17893)g_1 + (4.53156) \\ -(5.17893)g_1 + (4.53156) & (17.13809) \end{bmatrix} \quad (4.84)$$

Similarly

$$S_0 = Q + G_0^T R G_0 + [A+BG_0]^T S_1 [A+BG_0] \quad (4.85)$$

$$= \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix} \quad (4.86)$$

where

$$s_{11} = 4g_1^2 g_0^2 - 8g_1 g_0^2 + 8g_0^2 - 2(5.17893)g_1 g_0 + 2(4.53156)g_1 + 18.13809 \quad (4.87)$$

$$s_{12} = -4g_1^2 g_0 + (5.17893)g_1 g_0 - (4.53156)g_0 + (5.17893)g_1 + 2.06715 \quad (4.88)$$

$$s_{22} = 4g_1^2 - (2.35786)g_1 + 15.14835 \quad (4.89)$$

Thus

$$\mathrm{tr}[S_0V_0] = 2g_1^2g_0^2 - 4g_1g_0^2 - 4g_1^2g_0 + 4g_0^2 + 3g_1^2 + (3.41054)g_1 + 22.49746 \tag{4.90}$$

$$\mathrm{tr}[S_1V_w] = g_1^2 - (3.41054)g_1 + 20.12231 \tag{4.91}$$

$$\mathrm{tr}[S_2V_w] = 5.66056 \tag{4.92}$$

Substitution of (4.90), (4.91) and (4.92) into (4.80) produces

$$J(\pi^S) = g_1^2g_0^2 - 2g_1g_0^2 - 2g_1^2g_0 + 2g_0^2 + 2g_1^2 + 24.14017 \tag{4.93}$$

which by (2.30) may be written

$$J(\pi^S) = \tfrac{1}{2}F_0(g_1)g_0^2 + h_0(g_1)g_0 + c_0(g_1) \tag{4.94}$$

or

$$J(\pi^S) = \tfrac{1}{2}F_1(g_0)g_1^2 + h_1(g_0)g_1 + c_1(g_0) \tag{4.95}$$

Thus

$$F_1(g_0) = 2g_0^2 - 4g_0 + 4 \tag{4.96}$$

which can also be found by using the relation

$$F_1(g_0) = [r_{11}(1)V_{11}(1)] \tag{4.97}$$

$$h_1(g_0) = -2g_0^2 \tag{4.98}$$

which can also be found by using the relation

$$h_1(g_0) = [V_1(1)A^T S_2 B] \tag{4.99}$$

and

$$c_1(g_0) = 2g_0^2 + 24.14017 \tag{4.100}$$

which can also be calculated using the relation

$$c_1(g_0) = \tfrac{1}{2}\text{tr}[[Q+G_0^T RG_0]V(0)] + \tfrac{1}{2}\text{tr}[S_2 V_w] + \tfrac{1}{2}\text{tr}[[Q+A^T S_2 A]V(1)] \tag{4.101}$$

Similarly one finds

$$F_0(g_1) = 2g_1^2 - 4g_1 + 4 \tag{4.102}$$

$$h_0(g_1) = -2g_1^2 \tag{4.103}$$

$$c_0(g_1) = 2g_1^2 + 24.14017 \tag{4.104}$$

The singular points of $J(\pi^S)$ may be found by taking the derivatives of $J(\pi^S)$ with respect to g_0 and g_1 and setting the resulting expressions equal to zero. Taking the derivative of $J(\pi^S)$ with respect to g_1 produces

$$\frac{\partial J(\pi^S)}{\partial g_1} = 2g_1(g_0^2 - 2g_0 + 2) - 2g_0^2 \tag{4.105}$$

thus

$$\frac{\partial J(\pi^S)}{\partial g_1} = 0 \qquad (4.106)$$

when

$$g_1 = \frac{g_0^2}{g_0^2 - 2g_0 + 2} \qquad (4.107)$$

Taking the derivative of $J(\pi^S)$ with respect to g_0 yields

$$\frac{\partial J(\pi^S)}{\partial g_0} = 0 \qquad (4.108)$$

when

$$g_0 = \frac{g_1^2}{g_1^2 - 2g_1 + 2} \qquad (4.109)$$

Substitution of (4.107) into (4.109) gives

$$g_0(g_0^4 - 5g_0^3 + 12g_0^2 - 16g_0 + 8) = 0 \qquad (4.110)$$

which when factored becomes

$$g_0(g_0 - 1)(g_0 - 2)(g_0^2 - 2g_0 + 4) = 0 \qquad (4.111)$$

Thus the singular points occur when $g_0 = 0$, $g_0 = 1$, and $g_0 = 2$ as $g_0^2 - 2g_0 + 4$ has no real roots. Substitution of these values into (4.107) yields the other coordinates of the singular points.

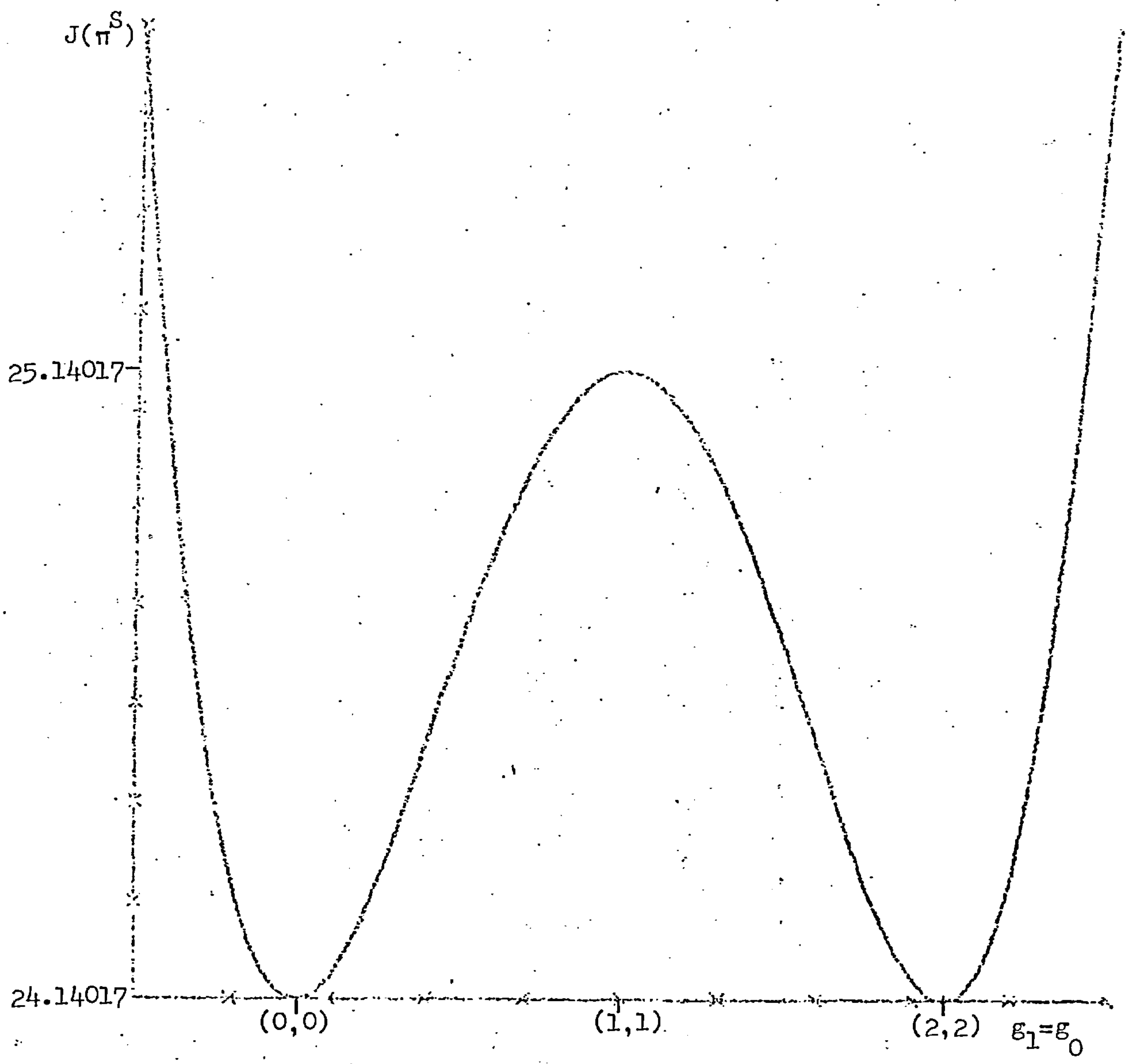

Figure 1: Plot of $J(\pi^S)$ along the line $g_1=g_0$ for System A.

$[g_0 = 0,\ g_1 = 0]$, $[g_0 = 1,\ g_1 = 1]$, $[g_0 = 2,\ g_1 = 2]$ are the singular points of $J(\pi^S)$. By Theorem 6 these points must be the solutions of the two point boundary value problem stated in that theorem.

By $J(\pi^S)\big|_{(g_0,g_1)}$ is meant the value $J(\pi^S)$ assumes if a control policy $\pi^S = [G_0 = [g_0 \quad 0]\ ;\ G_1 = [g_1 \quad 0]]$ is used.

$$J(\pi^S)\big|_{(0,0)} = 24.14017$$

$$J(\pi^S)\big|_{(1,1)} = 25.14017$$

$$J(\pi^S)\big|_{(2,2)} = 24.14017$$

Figure 1 is a plot of $J(\pi^S)$ along the line $g_1 = g_0$ from (-0.5, -0.5) to (2.5, 2.5), which implies that (0,0) and 2,2) are minima and (1,1) is a saddle point.

If one starts computational procedure A with an initial linear control policy cf the form

$$\pi(0) = [[1 \quad 0]\ ;\ [x \quad 0]] \tag{4.112}$$

it will converge to the limit point

$$\underline{\pi} = [[1 \quad 0]\ ;\ [1 \quad 0]] \tag{4.113}$$

in one step. $\underline{\pi}$ is the point (1,1) in the 2-dimensional Euclidean space where each of the feedback gains in a Structured Control Policy is taken as a coordinate.

Taking $\pi(0)$ as in (4.112) and following the computational procedure through to step 6 one finds.

$$g_1 = -F_1^{-1}(1)h_1(1) \tag{4.114}$$

By use of (4.96) and (4.98) this becomes

$$g_1 = \frac{-1}{2(1)^2 - 4(1) + 4}[-2(1)^2] = 1 \tag{4.115}$$

thus

$$\pi(1) = [[1 \quad 0] ; [1 \quad 0]] \tag{4.116}$$

Then the second iteration through to step 6 yields

$$g_0 = -F_0^{-1}(1)h_0(1) \tag{4.117}$$

which evaluated by use of (4.102) and (4.103) is

$$g_0 = \frac{-1}{2(1)^2 - 4(1) + 4}[-2(1)^2] = 1 \tag{4.118}$$

Thus

$$\pi(2) = [[1 \quad 0] ; [1 \quad 0]] \tag{4.119}$$

Similarly one finds

$$\pi(k) = \underline{\pi} \quad \text{for } k=3,4,5,.... \tag{4.120}$$

For initial control policies of the form (4.112) the system does not converge to a minimum. By suitable choices of $\pi(0)$ a computer implementation of computational procedure A [7] was made to yield all three solutions.

One might wonder if small changes in the parameters would greatly

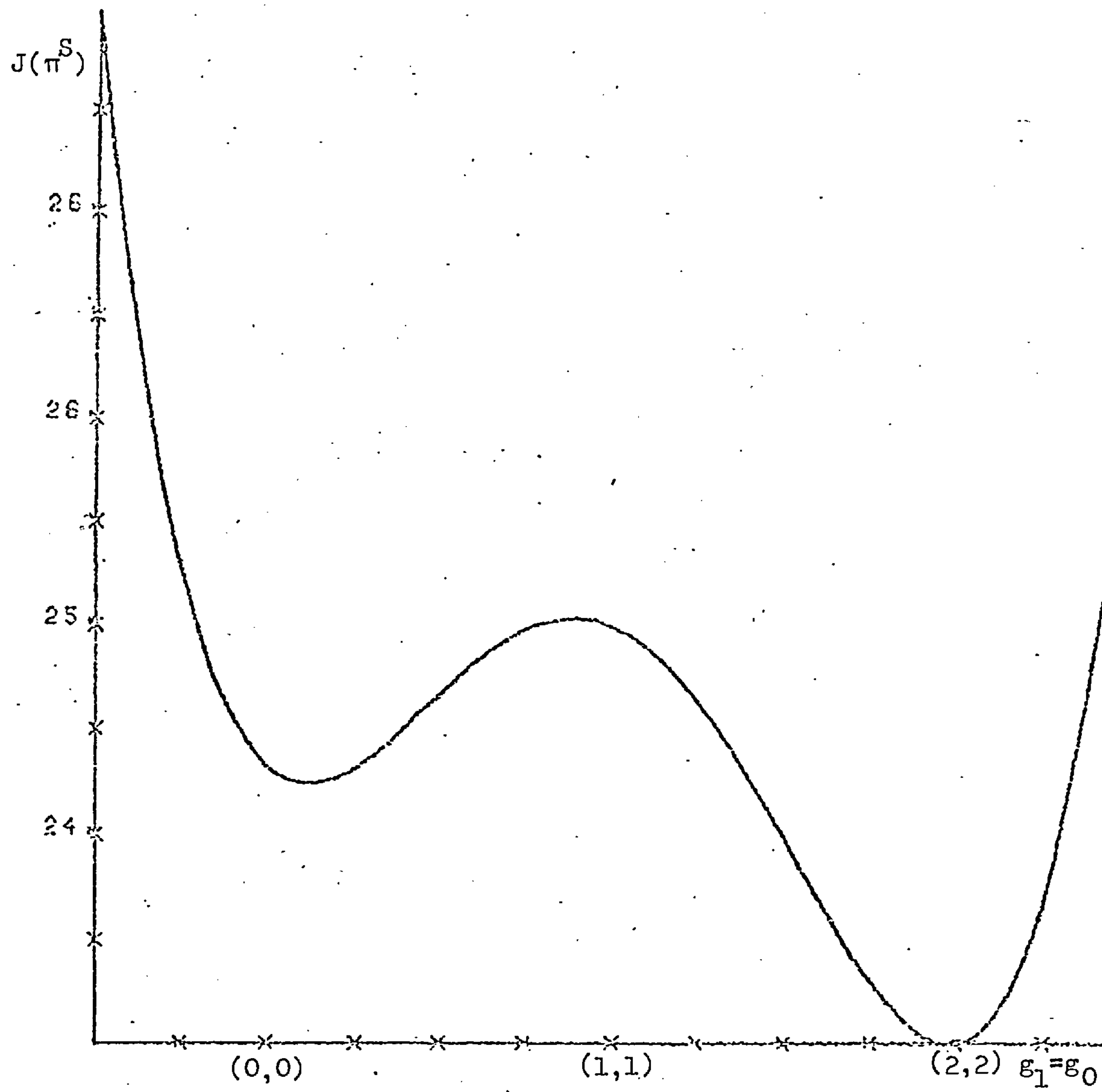

Figure 2: Plot of $J(\pi^S)$ for System B

(The numbers on the vertical axis have been rounded to two figures) alter the shape of the cost function. The values of the parameters in the system A were rounded off to test this. The System B is

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0.5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$V_0 = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.75 \end{bmatrix} \qquad V_w = \begin{bmatrix} 0.25 & 0.15 \\ 0.15 & 1 \end{bmatrix}$$

$$S_N = \begin{bmatrix} 3 & -4 \\ -4 & 6 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$$

$$R = 1 \qquad N = 2 \qquad n_1 = 1$$

Using the computer implementation of computational procedure A [7] an absolute minimum was found at

$$g_0 = 2.015 \qquad g_1 = 1.9687$$

$J(\pi^S) = 23.889$ at this point.

A strong local minimum was found at

$$g_0 = 0.13832 \qquad g_1 = 0.10864$$

$J(\pi^S) = 24.589$ at this point.

Figure 2 is a plot of $J(\pi^S)$ along the line $g_1 = g_0$ from (-0.5, -0.5) to (2.5, 2.5) for system B.

4.4. Choice of the Initial Linear Control Policy

Theorem 5 could be used to produce a bounded region in which the Optimal Structured Control Policy must exist. However, the high dimensionality of structured control policies for practical systems makes a thorough search of any such region impossible.

One must find an initial Linear Control Policy which produces convergence to the Optimal Structured Control Policy in most reasonable cases. Several possible methods of selecting an initial control policy will now be considered and compared.

A) The Optimal Complete State Feedback Policy

If the feedback structure is so chosen that the performance of the Optimal Structured Control Policy is close to the Optimal Control Policy then it is reasonable to assume that the sequences $[S_k]_{N-1}^{0}$ and $[V(k)]_{0}^{N}$ will be close for both policies. If $\pi(0)$ is taken to be the Optimal Control Policy one would expect that $\pi(N)$ would be very close to the Optimal Structured Control Policy.

Computational experience [7] has shown that this starting point produces convergence to the Optimal Structured Control Policy for $k < 10N$ [i.e. the first six digits of $J(\pi(k-N))$ are the same as $J(\pi(k))$] if the Optimal Structured Control Policy results in a system behaviour similar to that produced by the Optimal Control Policy.

B) Compute $\pi(0)$ by Assuming $V(k) = V_0$, $k=0,1,\ldots,N-1$

Assume $V(k) = V_0$ for $k=0,1,\ldots,N-1$ then calculate G_k^S, $k=N-1,N-2,\ldots,0$ backward in time by evaluating (3.1) and calculating S_k, $k=N-1,N-2,\ldots,1$ as these evaluations are made. The sequence of Structured State Feedback Matrices so calculated could be used as the Initial Linear Control Policy $\pi(0)$.

The calculation of $\pi(0)$ by this method would take approximately the same amount of computing time as that required to calculate the Optimal Control Policy. If this method is used convergence is usually not as quick as if A is used. Thus use of the Optimal Control Policy as $\pi(0)$ is to be preferred. Use of both starting points may however give the user greater confidence that the limiting control policy is in fact the optimal.

R.L. Kosut [8] suggests two methods of computing Suboptimal Structured State Feedback Matrices for the problem where the system is continuous and deterministic (i.e. $V_w = 0$), $V_0 = I$, R is diagonal, and a steady state solution is required. The methods are termed "Minimum Norm" and "Minimum Error Excitation". These approaches can be extended to produce Suboptimal Structured Control Policies for the discrete, stochastic finite-time problem considered here. These Suboptimal Structured Control Policies could then be used as initial Linear Control Policies for the computational procedures A or B.

C) Minimum Norm

Let the initial Linear Control Policy $\pi(0)$ be the Structured Control Policy π^S that satisfies

$$\min_{\pi^S} \left| \overset{*}{\pi} - \pi^S \right| \tag{4.121}$$

where $\overset{*}{\pi}$ is the Optimal Control Policy. Obviously the projection of $\overset{*}{\pi}$ on to the pN Euclidean space of Structured Control Policies will be the solution to (4.121). Thus $\pi(0)$ is simply obtained by computing $\overset{*}{\pi} = [\overset{*}{G}_k, k=0,\ldots\ldots,N-1]$ and deleting those elements of the gain matrices, $\overset{*}{G}_k$, that are constrained to be zero.

As the cost function is continuous in the Linear Control Policies, nearness in norm does imply nearness in cost. This approach works well when the gain elements deleted are much smaller than those retained. However, if the gains deleted are of the same order of magnitude as those retained, the Structured Control Policy which differs from $\overset{*}{\pi}$ the least in norm may give very poor performance. In Chapter 5 an example is given of a stable system where the Structured Feedback Matrices, which differ the least from the Optimal State Feedback Matrices in norm, cause the system to become unstable. For the same system with the same feedback structure the Optimal Structured Control Policy gives acceptable performance.

Minimum Error Excitation

The concept of Minimum Error Excitation for a Discrete Stochastic System will now be developed. As in Chapter 1 the superscript '*' (i.e. $\overset{*}{x}_k$) will be used to denote the value a variable (including matrices) assumes if the Optimal Complete State Feedback Control Policy is used.

Then from (1.1) one gets

$$\overset{*}{x}_{k+1} = A\overset{*}{x}_k + B\overset{*}{u}_k + w_k \qquad (4.122)$$

$$\overset{*}{x}_0 = x_0 \qquad (4.123)$$

As the linear control policy (1.8) is used

$$\overset{*}{x}_{k+1} = (A + B\overset{*}{G}_k)\overset{*}{x}_k + w_k \qquad (4.124)$$

Similarly the transition equation associated with any structured control policy, π^S, is

$$x_{k+1}^S = (A + BG_k^S)x_k^S + w_k \tag{4.125}$$

$$x_0^S = x_0 \tag{4.126}$$

Consider now the difference between the optimal state $\overset{*}{x}_k$ and the state x_k^S that occurs if a structured control policy is used

$$e_k = x_k^S - \overset{*}{x}_k \tag{4.127}$$

Use of (4.124) and (4.125) gives

$$e_{k+1} = (A+BG_k^S)x_k^S - (A+B\overset{*}{G}_k)\overset{*}{x}_k \tag{4.128}$$

$$= (A+BG_k^S)(x_k^S-\overset{*}{x}_k) + B(G_k-\overset{*}{G}_k)\overset{*}{x}_k \tag{4.129}$$

Define

$$q_k = [G_k^S-\overset{*}{G}_k]\overset{*}{x}_k \tag{4.130}$$

q_k is termed the error excitation vector. It can be interpreted as the difference between the optimal input, $\overset{*}{u}_k$, and the input that would occur if the feedback matrix G_k^S was used when the system was in state $\overset{*}{x}_k$.

Substitution of (4.127) and (4.130) into (4.129) produces

$$e_{k+1} = (A+BG_k^S)e_k + Bq_k \tag{4.131}$$

From (4.123) and (4.126) and the definition of e_0 (4.127) it is apparent that

$$e_0 = 0 \tag{4.132}$$

From (4.131) and (4.132) it is apparent that if $q_k = 0$, $k=0,1,\ldots,N-1$ then $e_k = 0$ for $k=0,1,\ldots,N$. Further if one assumes $(A+BG_k^S)$ is stable, if q_k, $k=0,1,\ldots,N-1$ is kept small then e_k, $k=0,1,\ldots,N$ will be small, and the state trajectory using structured state feedback $[x_k^S]_0^N$ will be close to the optimal $[\overset{*}{x}_k]_0^N$. (1.3) implies that if this can be achieved then the Structured Control Policy will have an expected cost J close to the optimal expected cost $\overset{*}{J}$.

D) Minimum Error Excitation: the Direct Approach

A reasonable Structured Control Policy to use as the initial Linear Control Policy $\pi(0)$ is the one that minimizes

$$E_q = E[\sum_{k=0}^{N-1} q_k^T q_k] \tag{4.133}$$

Theorem 9:

The Structured Control Policy which minimizes E_q is a solution to

$$\overset{*}{V}_{ii}(k) g_k^i = \overset{*}{V}_i(k) \overset{*}{\delta}{}_k^i \tag{4.134}$$

for $i=1,2,\ldots,m$ and $k=0,1,\ldots,N-1$ where g_k^i is the vector of unconstrained gains in the i'th row of G_k^S defined by (1.24), $\overset{*}{V}_{ii}(k)$ and $\overset{*}{V}_i(k)$ are the values $V_{ii}(k)$ and $V_i(k)$, defined by (2.33) and (2.37), assume when the optimal control policy, $\overset{*}{\pi}$, is used, and

$$\overset{*}{G}_k = \begin{bmatrix} (\overset{*}{\delta}{}_k^1)^T \\ (\overset{*}{\delta}{}_k^2)^T \\ \vdots \\ (\overset{*}{\delta}{}_k^m)^T \end{bmatrix} \tag{4.135}$$

that is: $\overset{*}{\delta}{}^{i}_{k}$ is the i'th row vector of the Optimal Complete State Feedback Matrix $\overset{*}{G}_k$.

Corollary:

If the Feedback Structure is a Partial State Feedback Structure, then the Partial State Feedback Control Policy which minimizes E_q may be obtained by solving

$$G'_k = -[R+B^T\overset{*}{S}_{k+1}B]^{-1}B^T\overset{*}{S}_{k+1}[A^1 + A^2\overset{*}{V}{}^T_{x',z'}(k)\overset{*}{V}{}^{-1}_{x'}(k)] \qquad (4.136)$$

for k=N-1,N-2,....,0, provided the required inverses exist. $\overset{*}{S}_k$ is defined by (1.7), and $\overset{*}{V}_{x',z'}(k)$ and $\overset{*}{V}_{x'}(k)$ are the values $V_{x',z'}(k)$ and $V_{x'}(k)$ assume when $\overset{*}{\pi}$ is used.

Note that (4.136) is (3.11) with the values S_k and V(k) assume when the Optimal Complete State Feedback Control Policy, $\overset{*}{\pi}$, is used. Thus the Partial State Feedback Control Policy which minimizes E is composed of Partial State Feedback Matrices that are Optimal single replacements in $\overset{*}{\pi}$.

Proof:

Substitution of (4.130) into (4.133) yields

$$E_q = \sum_{k=0}^{N-1} E[q_k^T q_k] = \sum_{k=0}^{N-1} E[\overset{*}{x}{}^T_k[G^S_k - \overset{*}{G}_k]^T[G^S_k - \overset{*}{G}_k]\overset{*}{x}{}^T_k] \qquad (4.137)$$

Each G^S_k, k=0,1,....,N-1 affects only one additive term of the cost function. Thus, the total cost function can be minimized by choosing each G^S_k to minimize the term it affects.

Each term can be expanded to give

$$E[q_k^T q_k] = E[\overset{*}{x}_k^T \overset{*}{G}_k^T \overset{*}{G}_k \overset{*}{x}_k] - 2E[\overset{*}{x}_k^T \overset{*}{G}_k^T G_k^S \overset{*}{x}_k] + E[\overset{*}{x}_k^T (G_k^S)^T G_k^S \overset{*}{x}_k^T] \qquad (4.138)$$

This function is now in the form of (2.41). Therefore the same argument as was used to get from (2.41) to (2.54) may be used. Upon making the substitutions

$$x_k \leftarrow \overset{*}{x}_k$$

$[Q + A^T S_{k+1} A] \leftarrow \overset{*}{G}_k^T \overset{*}{G}_k$ in the first term of (2.41),

$[R + B^T S_{k+1} B] \leftarrow I$ in the second term of (2.41), and

$A^T S_{k+1} B \leftarrow -\overset{*}{G}_k^T$ in the third term of (2.41).

(2.54) becomes

$$E[q_k^T q_k] = E[\overset{*}{x}_k^T \overset{*}{G}_k^T \overset{*}{G}_k \overset{*}{x}_k] + g_k^T F_k' g_k + 2(h_k')^T g_k \qquad (4.139)$$

where g_k is defined by (2.35),

$$F_k' = \begin{bmatrix} \overset{*}{V}_{11}(k) & 0 & \cdots\cdots & 0 \\ 0 & \overset{*}{V}_{22}(k) & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & \overset{*}{V}_{mm}(k) \end{bmatrix} \qquad (4.140)$$

and

$$h'_k = - \begin{bmatrix} \overset{*}{V}_1(k)\overset{*}{\delta}{}^1_k \\ \overset{*}{V}_2(k)\overset{*}{\delta}{}^2_k \\ \vdots \\ \overset{*}{V}_k(k)\overset{*}{\delta}{}^m_k \end{bmatrix} \tag{4.141}$$

As (4.139) is a positive semidefinite quadratic form its minimum can be found by setting the derivative equal to zero. Thus

$$\frac{\partial E[q_k^T q_k]}{\partial g_k} = 2g_k^T F'_k + 2(h'_k)^T = 0 \tag{4.142}$$

Substitution of (2.35), (4.140), (4.141) and the obvious algebraic manipulation yields

$$\overset{*}{V}_{ii}(k)g^i_k = \overset{*}{V}_i(k)\overset{*}{\delta}{}^i_i \tag{4.134}$$

for i=1,2,....,m and k=0,1,...,N-1 as it must hold for every term in the sum (4.137).

If the feedback structure is a partial state feedback structure and $\overset{*}{V}_{x'}(k)$ is invertible for k=0,1,....,N-1, substitution of (4.134) into (1.34) and use of (3.12), (3.19), and definition (4.135) give

$$G'_k = \overset{*}{G}_k \left[\frac{I}{\overset{*}{V}{}^T_{x'z'}(k)\overset{*}{V}{}^{-1}_{x'}(k)} \right] \tag{4.143}$$

Substitution of (1.6) into (4.143) and use of (3.5) produces

$$G'_k = -[R+B^T\overset{*}{S}_{k+1}B]^{-1}B^T\overset{*}{S}_{k+1}[A^1 + A^2\overset{*}{V}^T_{x'z'}(k)\overset{*}{V}^{-1}_{x'}(k)] \qquad (4.136)$$

Q.E.D.

The Structured Control Policy that minimizes E_q is easily obtained by first solving (1.6) and (1.7) to obtain $\overset{*}{\pi}$ and then using (4.134) to obtain the unconstrained gain elements. The cost function E_q costs the deviations from the optimal control strictly by their amplitude. It would be more natural to use a cost function that costed deviations in control by the increase in expected quadratic cost (1.4) that was produced.

E) Minimum Error Excitation with Quadratic Cost

Theorem 10:

If a Linear Control Policy

$$\pi' = [G_i,\ i=0,\dots,N-1 \quad \text{where } G_i=\overset{*}{G}_i \text{ for } i\neq k] \qquad (4.144)$$

is used then

$$J(\pi') - J(\overset{*}{\pi}) = \tfrac{1}{2}E[\overset{*}{x}{}^T_k[G_k-\overset{*}{G}_k]^T[R+B^T\overset{*}{S}_{k+1}B][G_k-\overset{*}{G}_k]\overset{*}{x}_k] \qquad (4.145)$$

Proof:

From (1.3) and (2.31) one obtains

$$J = E[L_0] = \tfrac{1}{2}E[\sum_{i=0}^{k-1}[x_i^TQx_i + u_i^TRu_i]] + \tfrac{1}{2}E[x_k^TQx_k] + \tfrac{1}{2}E[u_k^TRu_k]$$

$$+ \tfrac{1}{2}E[x_{k+1}^T S_{k+1} x_{k+1}] + \tfrac{1}{2} \sum_{i=k+2}^{N} tr[S_i V_w] \qquad (4.146)$$

If the Optimal Complete State Feedback Control Policy $\overset{*}{\pi}$ is used (4.146) becomes

$$J(\overset{*}{\pi}) = \tfrac{1}{2}E[\sum_{i=0}^{k-1} (\overset{*}{x}{}_i^T Q \overset{*}{x}_i + \overset{*}{u}{}_i^T R \overset{*}{u}_i)] + \tfrac{1}{2}E[\overset{*}{x}{}_k^T Q \overset{*}{x}_k] + \tfrac{1}{2}E[\overset{*}{u}{}_k^T R \overset{*}{u}_k]$$

$$+ \tfrac{1}{2}E[\overset{*}{x}{}_{k+1}^T \overset{*}{S}_{k+1} \overset{*}{x}_{k+1}] + \tfrac{1}{2} \sum_{i=k+2}^{N} tr[\overset{*}{S}_i V_w] \qquad (4.147)$$

If the control policy π' is used (1.1), (1.11) and (4.144) imply

$$u_i = \overset{*}{u}_i \quad \text{for } i=0,1,\ldots,k-1 \qquad (4.148)$$

$$u_k = G_k \overset{*}{x}_k \qquad (4.149)$$

$$x_i = \overset{*}{x}_i \quad \text{for } i=0,1,\ldots,k \qquad (4.150)$$

and

$$x_{k+1} = A\overset{*}{x}_k + Bu_k + w_k \qquad (4.151)$$

Use of (4.144) and the substitution of (1.6) into (2.15) gives

$$S_k = \overset{*}{S}_k \quad \text{for } i=N,\ldots,k+1 \qquad (4.152)$$

By use of (4.146), (4.148), (4.150) and (4.152) one may express

$$J(\pi') = \tfrac{1}{2}E[\sum_{i=0}^{k-1} (\overset{*}{x}{}_i^T Q \overset{*}{x}_i + \overset{*}{u}{}_i^T R \overset{*}{u}_i)] + \tfrac{1}{2}E[\overset{*}{x}{}_k^T Q \overset{*}{x}_k] + \tfrac{1}{2}E[(u_k)^T R u_k]$$

$$+ \tfrac{1}{2}E[x_{k+1}^T \overset{*}{S}_{k+1} x_{k+1}] + \tfrac{1}{2} \sum_{i=k+2}^{N} tr[\overset{*}{S}_i V_w] \qquad (4.153)$$

Subtraction of (4.147) from (4.153) produces

$$J(\pi') - J(\overset{*}{\pi}) = \tfrac{1}{2}E[u_k^T R u_k + x_{k+1}^T \overset{*}{S}_{k+1} x_{k+1} - \overset{*}{u}{}_k^T R \overset{*}{u}_k - \overset{*}{x}{}_{k+1}^T \overset{*}{S}_{k+1} \overset{*}{x}_{k+1}] \qquad (4.154)$$

Substitution of (4.122) and (4.151) into (4.154) yields

$$J(\pi') - J(\overset{*}{\pi}) = \tfrac{1}{2}E[u_k^T R u_k + u_k^T B^T \overset{*}{S}_{k+1} B u_k - \overset{*}{u}{}_k^T R \overset{*}{u}_k - \overset{*}{u}{}_k^T B^T \overset{*}{S}_{k+1} B \overset{*}{u}_k$$

$$+ 2\overset{*}{x}{}_k^T A^T \overset{*}{S}_{k+1} B u_k - 2\overset{*}{x}{}_k^T A^T \overset{*}{S}_{k+1} B \overset{*}{u}_k]$$

$$+ E[[A\overset{*}{x}_k + Bu_k]^T \overset{*}{S}_{k+1} w_k + [A\overset{*}{x}_k + B\overset{*}{u}_k]^T \overset{*}{S}_{k+1} w_k] \qquad (4.155)$$

As $u_k = G_k \overset{*}{x}_k$ and $\overset{*}{u}_k = \overset{*}{G}_k \overset{*}{x}_k$ and $\overset{*}{x}_k$ is not correlated with w_k by assumption, the second term of (4.155) is zero. This, with the addition and subtraction of a term gives,

$$J(\pi') - J(\overset{*}{\pi}) = \tfrac{1}{2}E[u_k^T[R + B^T \overset{*}{S}_{k+1} B]u_k - u_k^T[R + B^T \overset{*}{S}_{k+1} B]\overset{*}{u}_k$$

$$+ \overset{*}{u}{}_k^T[R + B^T \overset{*}{S}_{k+1} B]u_k - \overset{*}{u}{}_k^T[R + B^T \overset{*}{S}_{k+1} B]\overset{*}{u}_k + 2\overset{*}{x}{}_k^T A^T \overset{*}{S}_{k+1} B[u_k - \overset{*}{u}_k]] \qquad (4.156)$$

From (1.5) and (1.6) one may obtain

$$- [R + B^T \overset{*}{S}_{k+1} B]\overset{*}{u}_k = B^T \overset{*}{S}_{k+1} A \overset{*}{x}_k \qquad (4.157)$$

Substitution of (4.157) into (4.156) and the combination of terms yields

$$J(\pi') - J(\overset{*}{\pi}) = \tfrac{1}{2}E([u_k - \overset{*}{u}_k]^T [R + B^T \overset{*}{S}_{k+1} B][u_k - \overset{*}{u}_k]) \qquad (4.158)$$

With the substitution of $u_k = G_k \overset{*}{x}_k$ and $\overset{*}{u}_k = \overset{*}{G}_k \overset{*}{x}_k$ (4.158) becomes

$$J(\pi') - J(\overset{*}{\pi}) = \tfrac{1}{2}E(\overset{*}{x}{}_k^T [G_k - \overset{*}{G}_k][R + B^T \overset{*}{S}_{k+1} B][G_k - \overset{*}{G}_k]\overset{*}{x}_k) \qquad (4.145)$$

Q.E.D.

Substitution of (4.130) into (4.145) produces

$$J(\pi) - J(\overset{*}{\pi}) = \tfrac{1}{2}E(q_k^T [R + B^T \overset{*}{S}_{k+1} B] q_k) \qquad (4.159)$$

It is thus reasonable to presume that the Structured Control Policy which minimizes

$$E_Q = \sum_{k=0}^{N-1} E[q_k^T [R + B^T \overset{*}{S}_{k+1} B] q_k] \qquad (4.160)$$

will produce near optimal control if the structural constraints are such that performance near to the Complete State Feedback Optimal can be obtained.

As E_Q is a sum of the increase in costs produced by single replacements in $\overset{*}{\pi}$ minimization of this cost function produces a Structured Control Policy which is composed of Structured State Feedback Matrices that are optimal single replacements in $\overset{*}{\pi}$.

Theorem 11:

The unconstrained gain vectors g_k, $k=0,1,\ldots,N-1$ (defined by (2.35)) of the Structured Control Policy that minimizes E_Q may be obtained by solving

$$\overset{*}{F}_k g_k = -\overset{*}{h}_k \tag{4.161}$$

where $\overset{*}{F}_k$ and $\overset{*}{h}_k$ are the values F_k and h_k assume when the Optimal Complete State Feedback Control Policy is used.

Corollary:

If $[R + B^T\overset{*}{S}_{k+1}B]$ and $\overset{*}{V}_{x'}(k)$ are invertible for $k=0,1,\ldots,N-1$ then there is a unique Partial State Feedback Control Policy which minimizes both E_q and E_Q.

Proof:

Substitution of (4.130) into (4.160) produces

$$\begin{aligned} E_Q &= \sum_{k=0}^{N-1} E[q_k^T[R + B^T\overset{*}{S}_{k+1}B]q_k] \\ &= \sum_{k=0}^{N-1} E[\overset{*}{x}{}_k^T[G_k^S - \overset{*}{G}_k]^T[R + B^T\overset{*}{S}_{k+1}B][G_k^S - \overset{*}{G}_k]\overset{*}{x}_k] \end{aligned} \tag{4.162}$$

Each Structured State Feedback Matrix, G_k^S, $k=0,1,\ldots,N-1$ affects only one additive term of the cost. Thus the total cost function can be minimized by selecting each G_k^S to minimize the term it affects. Substitution of (1.6) into a term of (4.162) yields

$$E[q_k^T[R + B^T\overset{*}{S}_{k+1}B]q_k] = E[\overset{*}{x}_k^T A^T \overset{*}{S}_{k+1}B[R + B^T\overset{*}{S}_{k+1}B]^{-1}B^T\overset{*}{S}_{k+1}A\overset{*}{x}_k]$$

$$+ 2E[\overset{*}{x}_k^T A^T \overset{*}{S}_{k+1}BG_k^{S*}x_k] + E[\overset{*}{x}_k^T[G_k^S]^T[R + B^T\overset{*}{S}_{k+1}B]G_k^{S*}x_k]$$

(4.163)

(4.163) is of the form of (2.41). Replacement of the first term,

$$E[x_k^T[Q + A^T S_{k+1}A]x_k] \leftarrow E[\overset{*}{x}_k^T A^T \overset{*}{S}_{k+1}B[R + B^T\overset{*}{S}_{k+1}B]^{-1}B^T\overset{*}{S}_{k+1}A\overset{*}{x}_k]$$

and the substitutions $x_k \leftarrow \overset{*}{x}_k$ and $S_{k+1} \leftarrow \overset{*}{S}_{k+1}$ allow the argument to be followed through to (2.54) which becomes

$$E[q_k^T[R + B^T\overset{*}{S}_{k+1}B]q_k] = E[\overset{*}{x}_k^T A^T \overset{*}{S}_{k+1}B[R + B^T\overset{*}{S}_{k+1}B]^{-1}B^T\overset{*}{S}_{k+1}A\overset{*}{x}_k]$$

$$+ g_k^T \overset{*}{F}_k g_k + 2\overset{*}{h}_k^T g_k \qquad (4.164)$$

As $[R + B^T\overset{*}{S}_{k+1}B]$ is positive semidefinite, (4.164) is a positive semidefinite quadratic form. Its minimum may be found by taking the derivative and setting it equal to zero.

$$\frac{\partial E}{\partial g_k}[q_k^T[R + B^T\overset{*}{S}_{k+1}B]q_k] = 2g_k^T\overset{*}{F}_k + 2\overset{*}{h}_k^T = 0 \qquad (4.165)$$

Therefore every unconstrained gain vector g_k of a Structured Control Policy that minimizes E_Q must be a solution to

$$\overset{*}{F}_k g_k = -\overset{*}{h}_k \tag{4.161}$$

If $[R + B^T\overset{*}{S}_{k+1}B]$ and $\overset{*}{V}_{x'}(k)$ are invertible one may follow the same argument that was used to establish Theorem 2 to obtain (4.136). The Optimal Control Policy $\overset{*}{\pi}$ is unique, thus the sequences of matrices $[\overset{*}{S}_k]_{N-1}^{0}$ and $[\overset{*}{V}(k)]_0^{N-1}$ are uniquely defined. As $[R+B^T\overset{*}{S}_{k+1}B]$ and $\overset{*}{V}_{x'}(k)$ are invertible (4.136) defines a unique G'_k for every $\overset{*}{S}_{k+1}$ and $\overset{*}{V}(k)$. Thus there is one Partial State Feedback Control Policy which minimizes both E_q and E_Q.

Q.E.D.

Let $\pi'(N)$ denote the value $\pi(N)$ takes when $\pi(0) = \overset{*}{\pi}$, and let $\pi(E_Q)$ denote the Structured Control Policy which minimizes E_Q. $\pi'(N)$ and $\pi(E_Q)$ are computed using the same formulae and similar values. In the computation of $\pi(E_Q)$ the values $\overset{*}{S}_k$, and $\overset{*}{V}(k)$, $k=0,1,\ldots.,N$ are used. Assuming Computational Procedure A is used then the computation of $\pi'(N)$ uses $\overset{*}{V}(k)$, $k=0,1,\ldots.,N-1$ but the S_k,$k=N-1,N-2,\ldots.,1$ are recalculated using the actual G_k^S in the control policy $\pi'(N)$. If Computational Procedure B is used then the computation of $\pi'(N)$ uses $\overset{*}{S}_k$, $k=1,2,\ldots,N$ but the $V(k)$,$k=1,2,\ldots.,N-1$ are calculated using the actual G_k^S in the control policy $\pi'(N)$. In either case $\pi'(N)$ can be said to be calculated using more information about the actual control that will be used than does the calculation of $\pi(E_Q)$. It is then reasonable to assume that $\pi'(N)$ will have a superior performance to $\pi(E_Q)$ in most cases.

Starting points A, C, D, and E all require the calculation of $\overset{*}{\pi}$. As the direct use of $\overset{*}{\pi}$ is the easiest starting point to implement and as it is anticipated that it will give the quickest convergence to the optimal in most cases, it is recommended. Use of some or all of the other starting points may give the user greater confidence that the limiting Structured Control Policy that results is the optimal.

CHAPTER 5

EXAMPLES

5.1. Introduction

In this chapter the feasibility of the proposed computational procedure will be established. It will be shown that simple controller structures can give near optimal performance. Experimental evidence will imply that a law of diminishing marginal returns for increasing controller complexity exists. The problem of how to choose a good feedback structure will be considered, as well.

Computational Procedure A, given in Chapter 4, is used to compute Optimal Structured Control Policies for two linear systems, and many possible feedback structures. Computational Procedure A is implemented in one of the programs in the computer aided design package DILPAC [7]. All computations were done on a PDP - 15 computer with 32K of 18 bit word core store and software multiply. Storage requirements limited the size of example that could be considered to seven states and fifty time intervals. For a seventh order system computation times of ten minutes were typical. Much bigger systems could be dealt with using larger and faster machines.

Choice of Feedback Structures

The problem of choosing the feedback structure containing p gain co-ordinates, which has the best performance, will be considered. Given a system with n states and m inputs there are $(nm)!/p!(nm - p)!$ feedback structures which contain p gain co-ordinates. Obviously it is impossible to test all structures unless nm is quite small. This problem has not been solved analytically. However, two useful heuristic methods for selecting good feedback structures will be evaluated. The concept of substructure is needed to present these methods.

A feedback structure, σ^1, is a SUBSTRUCTURE of a feedback structure, σ^2 if

$$\psi^1(j,i) \in \ell^2(j) \quad \text{for all} \quad i \in \left\{1,2,\ldots\ldots,n_j^1\right\}$$

and $\quad j \in \left\{1,2,\ldots\ldots,m\right\}$ where

$$\sigma^k = [\ell^k(j) \mid j=1,2,\ldots\ldots,m]$$

and

$$\ell^k(j) = \left\{\psi^k(j,1),\ \psi^k(j,2),\ldots\ldots\ldots,\psi^k(j,n_j^k)\right\}$$

for k=1,2

Two possible ways of selecting reasonable feedback structures are:

1. Given that a Linear Control Policy having a good performance, for example $\overset{*}{\pi}$, is available; order the feedback gain co-ordinates (i,j) by the magnitudes of the associated gains $|\Gamma_{(i,j)}|$. Select substructures containing p co-ordinates associated with the larger gain magnitudes.

2. If the performance of some feedback structures containing p+1 gains are available, rank these structures by performance. Choose the substructures of p gains common to two or more of the feedback structures having good performance.

When selecting a set of likely feedback structures these two approaches may of course be used in concert. The usefulness of these methods will be determined experimentally.

5.2. A Stable Fourth Order System

Assume

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{1.1}$$

where

$$A = \begin{bmatrix} 0.964 & 0.180 & 0.017 & 0.019 \\ -0.342 & 0.802 & 0.162 & 0.179 \\ 0.016 & 0.019 & 0.983 & 0.181 \\ 0.144 & 0.179 & -0.163 & 0.820 \end{bmatrix} \tag{5.1}$$

with eigenvalues

$$\lambda_{1,2} = 0.983 \pm j0.127 \tag{5.2}$$

$$|\lambda_{1,2}| = 0.991$$

and

$$\lambda_{3,4} = 0.801 \pm j0.201 \tag{5.3}$$

$$|\lambda_{3,4}| = 0.826$$

and

$$B = \begin{bmatrix} 0.019 & 0.001 \\ 0.180 & 0.019 \\ 0.005 & 0.019 \\ -0.054 & 0.181 \end{bmatrix} \tag{5.4}$$

The numbers have been rounded to three decimals for reasons of clarity. The value of the noise covariance matrix V_w is assumed to be

$$V_w = \begin{bmatrix} 0.01 & -0.01 & 0 & 0 \\ -0.01 & 0.02 & 0 & -0.01 \\ 0 & 0 & 0.01 & -0.01 \\ 0 & -0.01 & -0.01 & 0.025 \end{bmatrix} \tag{5.5}$$

The cost matrices Q, R, and S_N are chosen to be

$$Q = I \qquad R = 0.5\, I \qquad S_N = 5I \tag{5.6}$$

and the number of time intervals

$$N = 50 \tag{5.7}$$

In order to test the affect of V_0 on the optimal structured control policies two distinctly different V_0 matrices will be considered.

Case A: It is assumed that no knowledge is available about the initial state that is likely to occur. Thus

$$V_0 = I \tag{5.8}$$

is selected so a good average behaviour results.

Case B: It is assumed

$$V_0 = x_0 x_0^T + V_w \tag{5.9}$$

where

$$x_0^T = [10 \quad 0 \quad 10 \quad 0] \tag{5.10}$$

and

$$V_0 = \begin{bmatrix} 100.01 & -0.01 & 100.0 & 0 \\ -0.01 & 0.02 & 0 & -0.01 \\ 100.0 & 0 & 100.01 & -0.01 \\ 0 & -0.01 & -0.01 & 0.025 \end{bmatrix} \tag{5.11}$$

In this case one "almost" knows that a certain initial condition will occur. As the magnitude of the initial condition x_0 is much larger than the disturbances the initial condition dominates the cost. Feed forward control would work well. A feedforward control can be generated using the expected values of the states, therefore one would expect any feedback structure that does not constrain all the gains in one row of the state feedback matrix to be zero would be near optimal.

Tables 1 and 2 contain various feedback structures and there their performances for Case A and B respectively. Lists $\ell(1)$ and $\ell(2)$ define the feedback structure. p is the number of unconstrained gains in the feedback structure. I_t is the number of iterations required for convergence to the optimal structured control policy, where an iteration consists of N-1 evaluations of V(k), N evaluations of S_i and G_i^S and a calculation of the expected cost. J is the expected cost using the optimal structured control policy. $\pi(0) = \bar{\pi}$ in DILPAC [7]

$$\text{PIO, Percentage Increase Over Optimal} = \frac{J - \overset{*}{J}}{\overset{*}{J}} \times 100\% \tag{5.12}$$

where $\overset{*}{J}$ is the expected cost using the optimal control policy.

$$\text{PPI, Percentage of Possible Improvement} = \frac{J_u - J}{J_u - \overset{*}{J}} \times 100\% \qquad (5.13)$$

where J_u is the expected cost if no control is used.

Figure 3 contains the plots of PIO and PPI against the number of gains used, p, for the feedback structures producing lowest cost. Note the performance of the structured controllers is better for Case B than for Case A.

In Case B the Structured Control Policy was calculated using information about the initial condition that was likely to occur, thus the controller was tuned to deal with this initial condition well. One would expect the structured control policies calculated for Case B would have poor performance should an initial condition distant from $x_0^T = [10, 0, 10, 0]$ occur. In Case A the control policies are computed assuming any initial condition on a sphere centred at the origin has an equal probability of occurring. Thus, it is expected that the structured control policies computed for for Case A will give acceptable performance for any initial condition, but their performance for initial conditions close to $x_0^T = [10\ \ 0\ \ 10\ \ 0]$ will be inferior to those computed for Case B.

The presumption for Case B that all feedback structures which do not constrain one input to be zero will be near optimal is verified. The worst feedback structure of that type, $[\ell(1) = 4\ ;\ \ell(2) = 4]$ results in PIO = 15.4%. States 1 and 3 have large initial values and it can be noted that policies that include at least one of these states in $\ell(1)$ and in $\ell(2)$ produce the structures of lowest cost for a fixed number of unconstrained gains, p. In Case A the performance is averaged over all initial condition directions and there is no bias

TABLE 1

TABLE OF STRUCTURED CONTROLLER COSTS FOR THE

FOURTH ORDER SYSTEM FOR CASE A

Uncontrolled Cost, J_u = 56.482
Optimal Cost, $\overset{*}{J}$ = 8.969

Type	$\ell(1)$	$\ell(2)$	p	I_t	J	PIO%	PPI%
Structured	1,2,3,4	2,3,4	7	2	8.974	0.06	99.989
Structured	2,3,4	1,2,3,4,	7	4	9.139	1.90	99.642
Structured	1,2,4	1,2,3,4	7	4	9.214	2.68	99.484
Structured	1,2,3,4	1,3,4	7	4	9.273	3.39	99.360
Structured	1,2,3	1,2,3,4	7	5	9.454	5.41	98.98
Structured	1,2,3,4	1,2,4	7	6	9.800	9.26	98.25
Structured	1,3,4	1,2,3,4	7	10	10.11	12.7	97.60
Structured	1,2,3,4	1,2,3	7	11	10.15	13.2	97.51
Partial	2,3,4	2,3,4	6	4	9.153	2.05	99.612
Structured	1,2,4	2,3,4	6	4	9.217	2.76	99.478
Structured	1,2,3	2,3,4	6	5	9.459	5.46	98.97
Structured	1,2,3,4	3,4	6	5	9.489	5.80	98.91
Structured	2,4	1,2,3,4	6	5	9.545	6.44	98.79
Structured	1,2,3,4	2,4	6	6	9.891	10.3	98.06
Structured	1,3,4	2,3,4	6	11	10.13	14.1	97.56
Structured	1,2,3,4	2,3	6	5	10.30	14.8	97.20
Partial	1,2,4	1,2,4	6	7	10.36	15.2	97.07
Partial	1,3,4	1,3,4	6	7	10.97	22.3	95.79
Partial	1,2,3	1,2,3	6	8	11.78	31.3	94.08

TABLE 1 (Continued)

Type	ℓ(1)	ℓ(2)	p	I_t	J	PIO%	PPI%
Structured	2,4	2,3,4	5	5	9.583	6.85	98.71
Structured	1,2	2,3,4	5	4	9.588	6.90	98.70
Structured	2,3,4	3,4	5	5	9.632	7.39	98.60
Structured	1,2,4	3,4	5	5	9.733	8.52	98.39
Structured	2,3	2,3,4	5	6	9.919	10.6	98.00
Structured	2,3,4	2,4	5	6	10.07	12.3	97.68
Structured	3,4	2,3,4	5	7	10.32	15.1	97.16
Structured	2,3,4	2,3	5	7	10.50	17.1	96.78
Structured	1,4	2,3,4	5	6	10.50	17.1	96.78
Structured	1,2,4	2,4	5	9	10.67	19.0	96.42
Structured	1,2,4	2,3	5	7	10.76	20.0	96.23
Structured	2	2,3,4	4	6	10.07	12.3	97.7
Structured	1,2	3,4	4	5	10.10	12.6	97.6
Structured	2,4	3,4	4	6	10.18	13.2	97.5
Structured	2,3	3,4	4	7	10.55	17.6	96.7
Structured	4	2,3,4	4	5	10.60	18.2	96.6
Structured	1	2,3,4	4	7	11.10	23.8	95.5
Structured	-	1,2,3,4	4	1	11.20	24.9	95.3
Structured	1,2,3,4	-	4	1	11.34	26.4	95.0
Structured	2,3	2,4	4	10	11.43	27.4	94.8
Partial	3,4	3,4	4	7	11.44	27.5	94.8
Structured	2,4	1,3	4	11	11.70	30.5	94.3
Partial	2,4	2,4	4	17	11.84	32.0	94.0
Structured	3,4	1,2	4	9	12.35	37.7	92.9
Partial	1,2	1,2	4	8	12.37	38.0	92.8

TABLE 1 (Continued)

Type	ℓ(1)	ℓ(2)	p	I_t	J	PIO%	PPI%
Structured	1,3	2,4	4	12	12.39	38.2	92.8
Partial	1,3	1,3	4	36	26.58	196	62.9
Structured	2	3,4	3	6	10.67	19.0	96.4
Structured	-	2,3,4	3	3	11.21	25.0	95.3
Structured	2,3,4	-	3	-	11.45	27.7	94.8
Structured	1,2	4	3	8	11.83	31.9	94.0
Structured	2,4	3	3	9	12.00	33.8	93.6
Structured	2	2,4	3	16	12.44	38.7	92.7
Structured	1,2,4	-	3	-	12.50	39.4	92.6
Structured	4	3,4	3	-	12.66	41.2	92.2
Structured	-	1,2,4	3	8	12.72	41.8	92.1
Structured	2,4	4	3	15	12.82	42.9	91.9
Structured	2,4	2	3	14	13.15	46.6	91.2
Structured	1	2,4	3	12	13.17	46.8	91.2
Structured	-	1,3,4	3	9	13.41	49.5	90.7
Structured	1,2	3	3	11	13.72	52.9	90.0
Structured	1,3,4	-	3	-	13.87	54.6	89.7
Structured	1,2,3	-	3	10	14.16	57.9	89.1
Structured	1	3,4	3	-	14.18	58.1	89.0
Structured	-	1,2,3	3	-	15.32	70.8	86.6
Structured	2	4	2	14	13.63	52.0	90.2
Structured	-	2,4	2	12	13.65	52.2	90.1
Structured	2,4	-	2	10	13.99	56.9	89.4
Structured	-	3,4	2	7	14.25	58.9	88.9

TABLE 1 (Continued)

Type	$\ell(1)$	$\ell(2)$	p	I_t	J	PIO%	PPI%
Structured	3,4	-	2	7	14.52	61.9	88.3
Structured	1,2	-	2	7	14.57	62.4	88.2
Structured	4	2	2	13	15.12	68.6	87.1
Structured	2	3	2	11	16.19	80.5	84.8
Partial	2	2	2	15	17.34	93.4	82.4
Structured	1	4	2	11	18.12	102	80.7
Partial	4	4	2	12	18.15	102	80.7
Structured	-	1,4	2	12	18.31	104	80.3
Structured	1,4	-	2	9	18.58	107	79.8
Structured	1	3	2	20	29.87	233	56.0
Partial	1	1	2	26	31.27	249	53.1
Partial	3	3	2	78	46.23	415	21.6
Structured	2	-	1	9	19.15	113	78.6
Structured	-	2	1	11	19.49	117	77.9
Structured	-	4	1	10	19.96	123	76.9
Structured	4	-	1	11	20.63	130	75.5

TABLE 2

TABLE OF STRUCTURED CONTROLLER COSTS FOR THE FOURTH ORDER SYSTEM FOR CASE B

Uncontrolled Cost, J_u = 2172.8

Optimal Cost, $\overset{*}{J}$ = 559.3

Type	$\ell(1)$	$\ell(2)$	p	I_t	J	PI0%	PPI%
Structured	1,2,3,4	2,3,4	7	2	559.4	0.0179	99.993
Structured	1,2,4	1,2,3,4	7	3	559.4	0.0179	99.993
Structured	2,3,4	1,2,3,4	7	3	559.4	0.0179	99.993
Structured	1,2,3,4	1,3,4	7	3	559.5	0.0358	99.987
Structured	1,2,3,4	1,2,4	7	3	559.6	0.0536	99.981
Structured	1,2,3	1,2,3,4	7	3	559.6	0.0536	99.981
Structured	1,3,4	1,2,3,4	7	5	559.9	0.107	99.962
Structured	1,2,3,4	1,2,3	7	4	560.0	0.125	99.956
Partial	2,3,4	2,3,4	6	2	559.4	0.0179	99.993
Structured	1,2,4	2,3,4	6	3	559.4	0.0179	99.993
Structured	2,3,4	1,3,4	6	3	559.6	0.0536	99.981
Structured	1,2	1,2,3,4	6	-	559.7	0.0715	99.975
Partial	1,2,4	1,2,4	6	5	559.9	0.107	99.962
Structured	2,3	1,2,3,4	6	7	560.2	0.161	99.944
Partial	1,3,4	1,3,4	6	-	560.4	0.196	99.931
Partial	1,2,3	1,2,3	6	-	561.0	0.303	99.894
Structured	1,2,3,4	2,4	6	6	578.5	3.43	98.810
Structured	2,4	1,2,3,4	6	5	582.0	4.06	98.593
Structured	2,3,4	3,4	5	3	559.6	0.0536	99.981

TABLE 2 (Continued)

Type	ℓ(1)	ℓ(2)	p	I_t	J	PIO%	PPI%
Structured	1,2	2,3,4	5	3	559.7	0.0715	99.975
Structured	1,2,4	3,4	5	3	559.7	0.0715	99.975
Structured	1,4	2,3,4	5	4	559.9	0.107	99.962
Structured	3,4	2,3,4	5	5	560.0	0.125	99.956
Structured	1,2	1,2,4	5	5	560.1	0.143	99.950
Structured	1,2,4	1,4	5	5	560.2	0.161	99.944
Structured	2,3	2,3,4	5	5	560.3	0.179	99.938
Structured	2,3	1,3,4	5	5	560.5	0.214	99.925
Structured	1,2,4	2,4	5	8	578.9	3.50	98.785
Structured	2,4	2,3,4	5	6	582.0	4.06	98.593
Structured	1,2	3,4	4	-	559.9	0.107	99.962
Structured	1,2	1,4	4	5	560.4	0.196	99.931
Structured	2,3	3,4	4	5	560.6	0.233	99.919
Partial	3,4	3,4	4	-	560.6	0.233	99.919
Structured	1,4	3,4	4	6	560.9	0.286	99.900
Structured	3,4	1,2	4	-	561.1	0.321	99.888
Partial	1,2	1,2	4	-	561.1	0.321	99.888
Structured	3	2,3,4	4	6	561.2	0.340	99.882
Structured	3	1,3,4	4	9	562.7	0.429	99.789
Structured	1	2,3,4	4	23	563.8	0.805	99.721
Partial	1,3	1,3	4	-	571.1	2.11	99.268
Structured	2,4	3,4	4	7	582.4	4.13	98.568
Structured	2	2,3,4	4	8	583.3	4.29	98.512
Structured	1,2,3,4	-	4	1	632.8	13.14	95.444
Structured	-	1,2,3,4	4	1	656.6	17.4	93.969

TABLE 2 (Continued)

Type	$\ell(1)$	$\ell(2)$	p	I_t	J	PIO%	PPI%
Structured	3	3,4	3	7	563.1	0.680	99.764
Structured	1,2	1	3	7	564.1	0.859	99.702
Structured	1,2	3	3	17	565.3	1.07	99.628
Structured	1,4	3	3	23	565.5	1.11	99.615
Structured	1	3,4	3	24	566.6	1.31	99.547
Structured	1	1,4	3	21	567.0	1.38	99.522
Structured	2,3	3	3	16	567.0	1.38	99.522
Structured	1	2,3	3	23	568.3	1.61	99.442
Structured	2	3,4	3	10	583.8	4.38	98.481
Structured	1	3	2	-	578.3	3.40	98.82
Structured	3	1	2	-	582.9	4.22	98.54
Partial	1	1	2	-	583.1	4.26	98.52
Partial	3	3	2	-	596.8	6.70	97.68
Partial	2	2	2	-	618.8	10.6	96.31
Structured	2	4	2	16	621.7	11.2	96.13
Partial	4	4	2	-	645.7	15.4	94.65
Structured	3	-	1	27	674.9	20.7	92.84
Structured	2	-	1	11	679.6	21.5	92.54
Structured	1	-	1	31	679.8	21.5	92.53
Structured	-	1	1	34	682.3	22.0	92.38
Structured	-	3	1	70	688.1	23.0	92.02
Structured	-	2	1	15	698.6	24.9	91.37

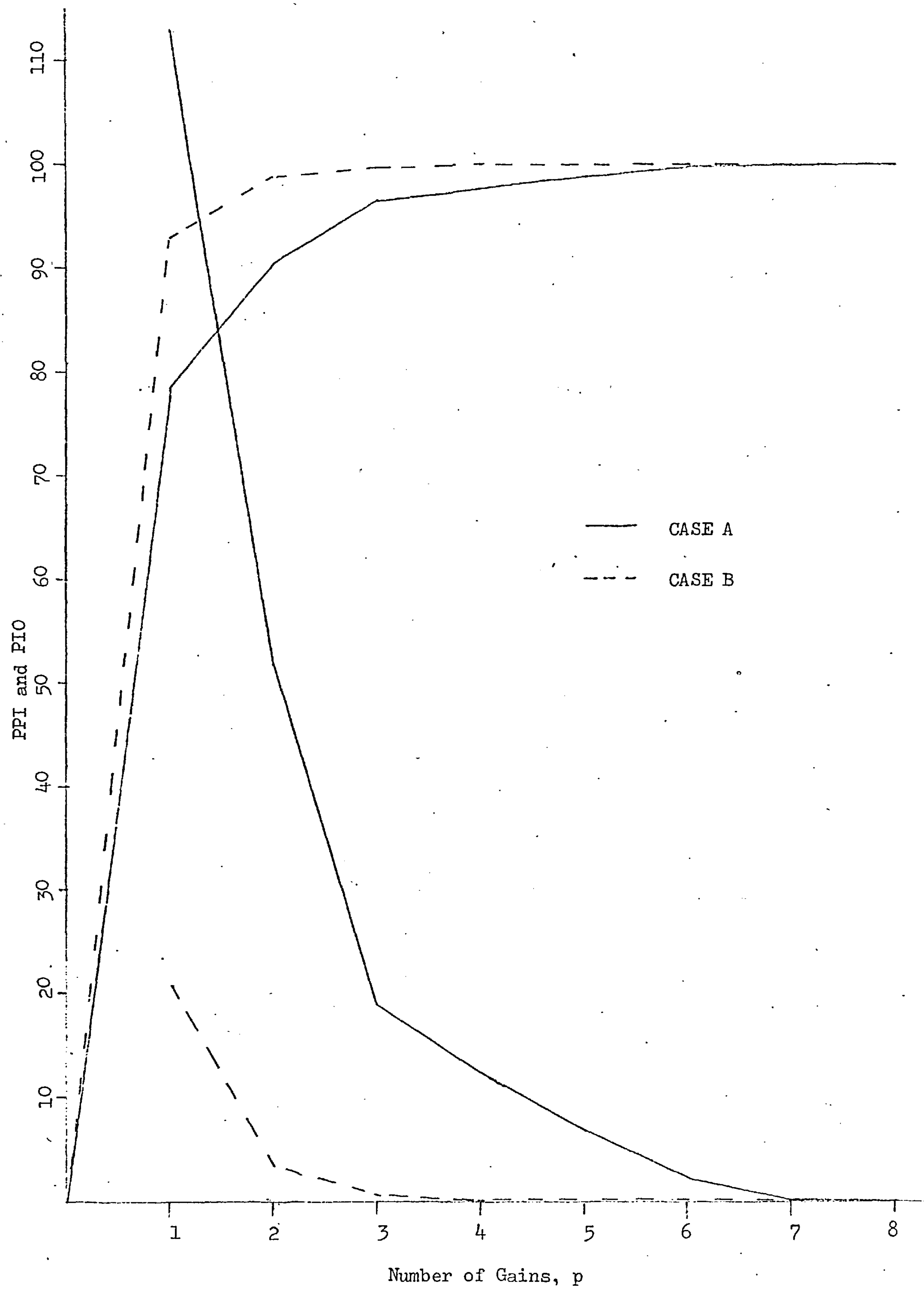

Figure 3: Plot of Performance vs. Complexity for the Fourth Order System

toward structures including states 1 and 3, in fact states 2 and 4 seem more important. It can be concluded that the performance of a given feedback structure in relation to other possible structures of equal complexity is influenced by V_0.

Figure 4 indicates that a law of diminishing marginal returns with increasing controller complexity applies in both cases. In Case A, 78.6% of the PPI is produced by using a controller which requires one measurement, one actuator, and one feedback gain to be stored. Use of a controller with three gains produced 96.4% of the possible improvement (PPI) and only a 19.0 PIO. In Case B this law of diminishing marginal returns is even more pronounced. The controller using one gain produced 92.94 PPI and the controller using three gains produced 99.764 PPI and a PIO of 0.680%.

Evaluation of the Heuristic Methods for Selecting Feedback Structures

In Tables 3 and 4 for Cases A and B respectively are listed: the best structures containing a fixed number of gains, p ; the gain co-ordinates they contain ordered in terms of the associated magnitudes in the steady state optimal state feedback matrix, $\overset{*}{G}$; and the ranking in terms of performance of the two best feedback structures containing p+1 gains of which the specified structure is a substructure.

For both cases A and B

$$\overset{*}{G} = \begin{bmatrix} -0.392 & -0.669 & -0.482 & -0.634 \\ -0.066 & -0.482 & -0.737 & -0.767 \end{bmatrix} \qquad (5.14)$$

Thus the ordering induced upon the co-ordinates of the state feedback matrix by the magnitudes of the gains in $\overset{*}{G}$ is

TABLE 3

EVALUATION OF THE HEURISTIC STRUCTURE SELECTION METHODS FOR CASE A

	Structure $\ell(1)$	$\ell(2)$	Ordered Gains	Rank by Cost of P+1 Structures
7	1,2,3,4	2,3,4	1,2,3,4,5,6,7	-
6	2,3,4,	2,3,4,	1,2,3,4,5,6	1 and 2
5	2,4	2,3,4	1,2,3,4,6	1 and 2
4	2	2,3,4	1,2,3,6	1 and 2
3	2	3,4	1,2,3	1 and 2
2	2	4	1,3	1 and 4
1	2	-	3	1 and 3

TABLE 4

EVALUATION OF THE HEURISTIC STRUCTURE SELECTION METHODS FOR CASE B

	Structure $\ell(1)$	$\ell(2)$	Ordered Gains	Rank by Cost of P+1 Structures
7	1,2,3,4	2,3,4	1,2,3,4,5,6,7	-
6	2,3,4	2,3,4	1,2,3,4,5,6,	1 and 2
5	2,3,4	3,4	1,2,3,4,5	1 and 3
4	1,2	3,4	1,2,3,7	2 and 3
3	3	3,4	1,2,5	3 and 4
2	1	3	2,7	3 and 4
1	3	-	2	2 and 4

Ordering	1	2	3	4	5	6	7	8
Co-ordinates	(2,4)	(2,3)	(1,2)	(1,4)	(1,3)	(2,2)	(1,1)	(2,1)

The optimal feedback structure, $\overset{*}{\sigma}_3$, containing 3 free gains for Case A is by Table 1

$$\overset{*}{\sigma}_3 = [\ell(1) = \{2\} ; \ell(2) = \{3,4\}] = \{(1,2),(2,3),(2,4)\}$$

Denoting these co-ordinates by the ordering induced upon the co-ordinates by $\overset{*}{G}$ one says $\overset{*}{\sigma}_3$ consists of the ordered gains, $\{1,2,3\}$.
The feedback structure $\overset{*}{\sigma}_3$ is a substructure of the feedback structure $[\ell(1) = \{2\} ; \ell(2) = \{2,3,4\}]$ and the feedback structure $[\ell(1) = \{1,2\} ; \ell(2) = \{3,4\}]$. As these feedback structures have the two lowest expected costs of the structures containing 4 unconstrained gains, $\overset{*}{\sigma}_3$ is termed a common substructure of the feedback structures containing p+1, 4, gains ranked 1 and 2 by cost. All other entries in Tables 3 and 4 are made in the same manner.

Table 3 indicates that the First Rule works reasonably well. Note, however, the presence of gain 6,(2,2), seems to be more important than the presence of gains 4,(1,4), or 5,(1,3), (5.14) indicates these gains differ little in magnitude. The presence of gain 3,(1,2), seems more important than the presence of gains 1,(2,4) or 2,(2,3), again these gains differ little in magnitude. One can infer that the size of the differences in gain magnitudes should be considered when choosing structures. The results for Case B, Table 4, also indicate that states associated with the larger gains in $\overset{*}{G}$ are more likely to be in the Optimal Structure than not. Gain 7,(1,1), appears in several optimal structures however. This can be attributed to the special nature of V_0 in Case B.

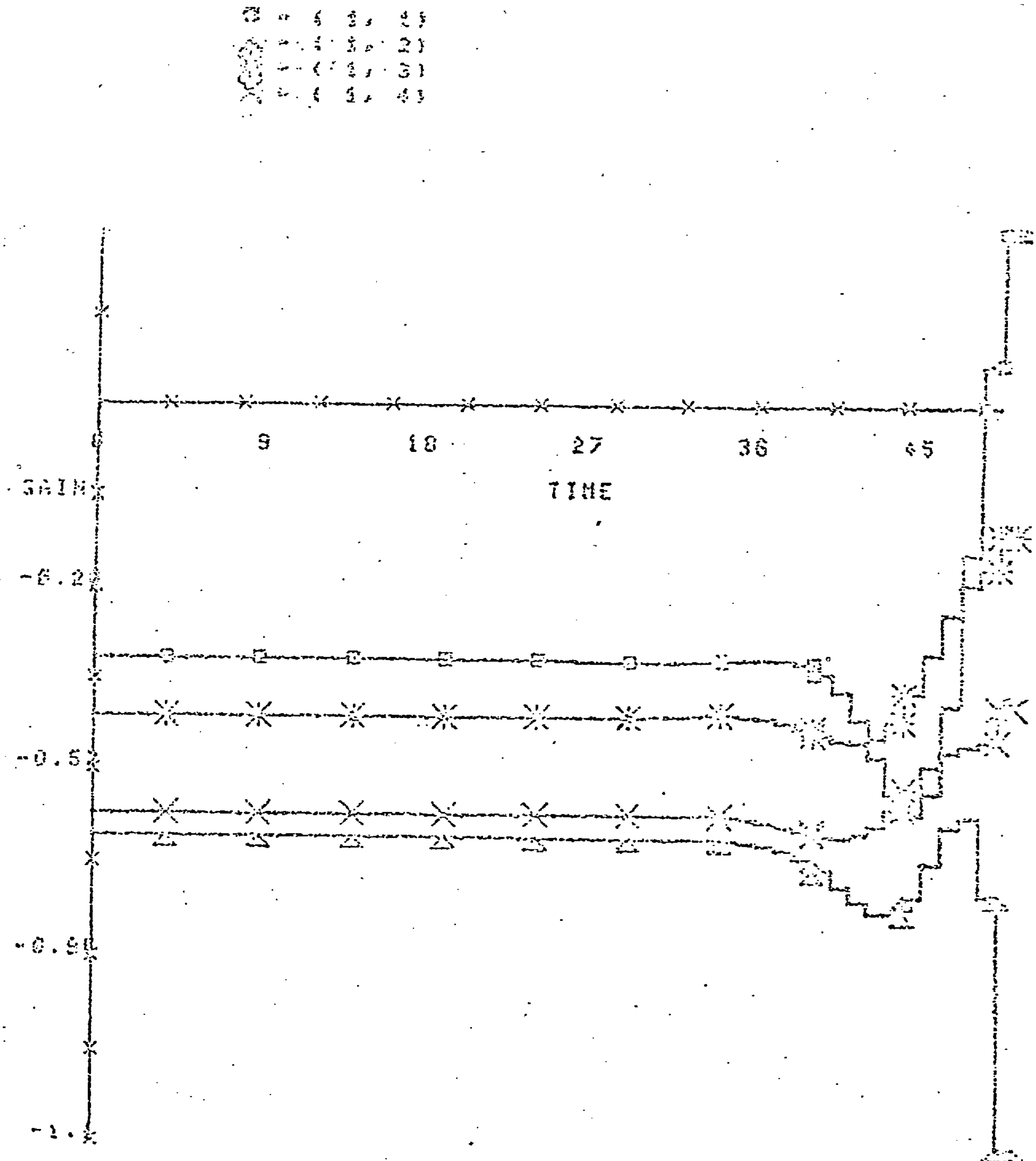

Figure 4: The Optimal Complete State Feedback Gains of the Fourth Order System for Input One.

(In this and all following gain plots the numbers shown on the vertical axis are rounded off to two figures)

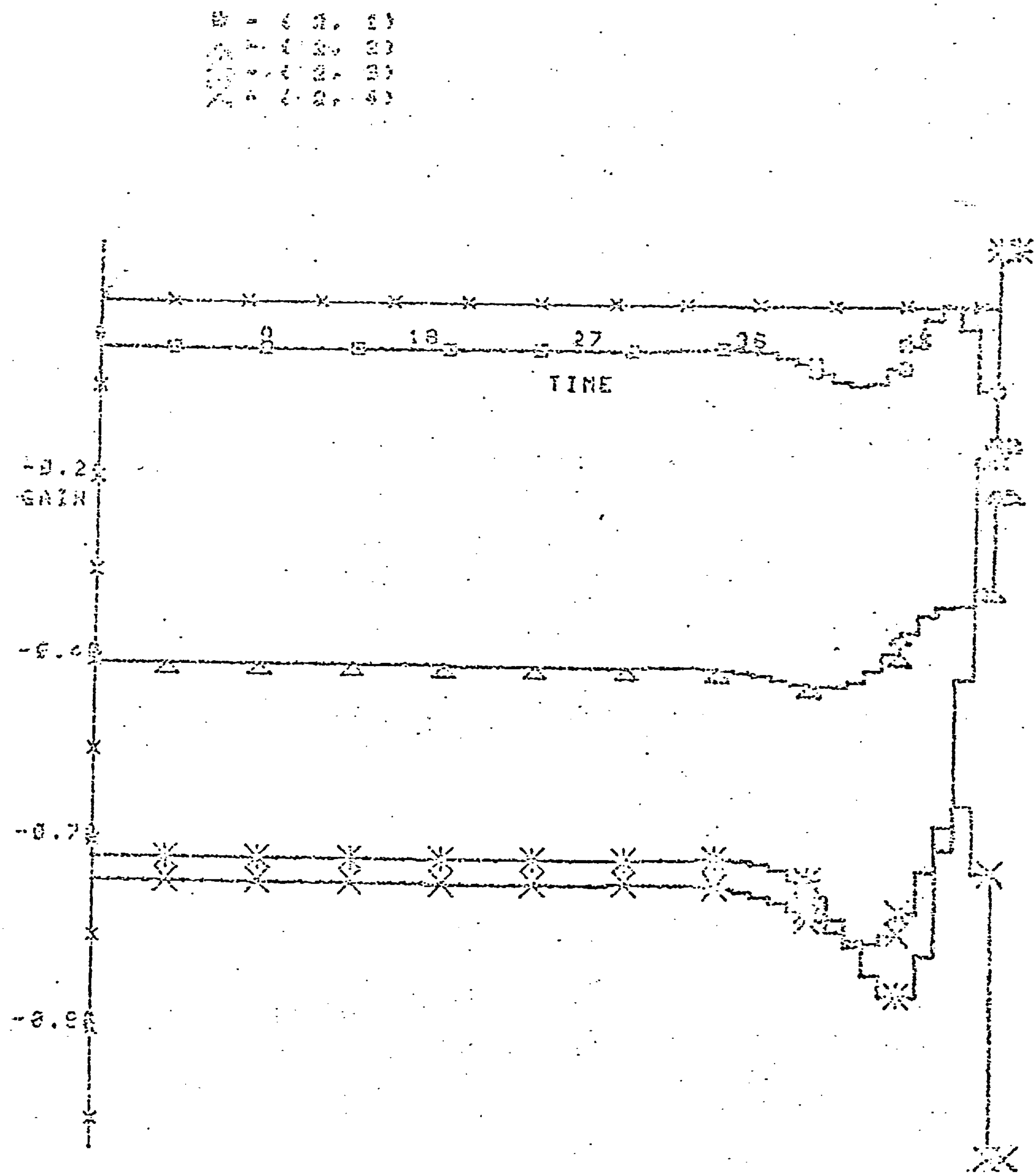

Figure 5: The Optimal Complete State Feedback Gains of the Fourth Order System for Input Two.

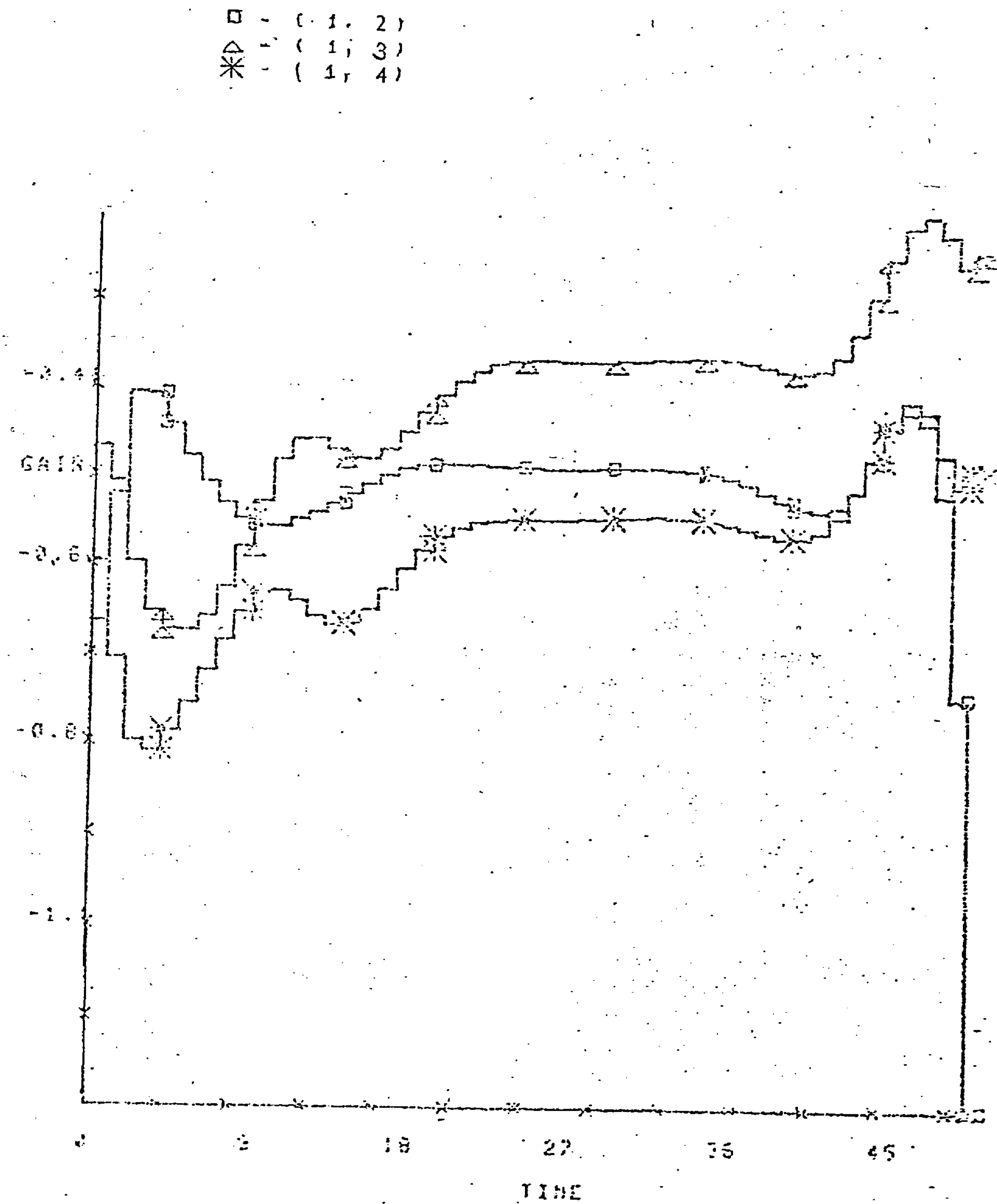

Figure 6: The Optimal Structured Feedback Gains for the First Input of the Fourth Order System for Case A with

$$\sigma = [\ell(1) = \{2,3,4\};\ \ell(2) = \{2,3,4\}]$$

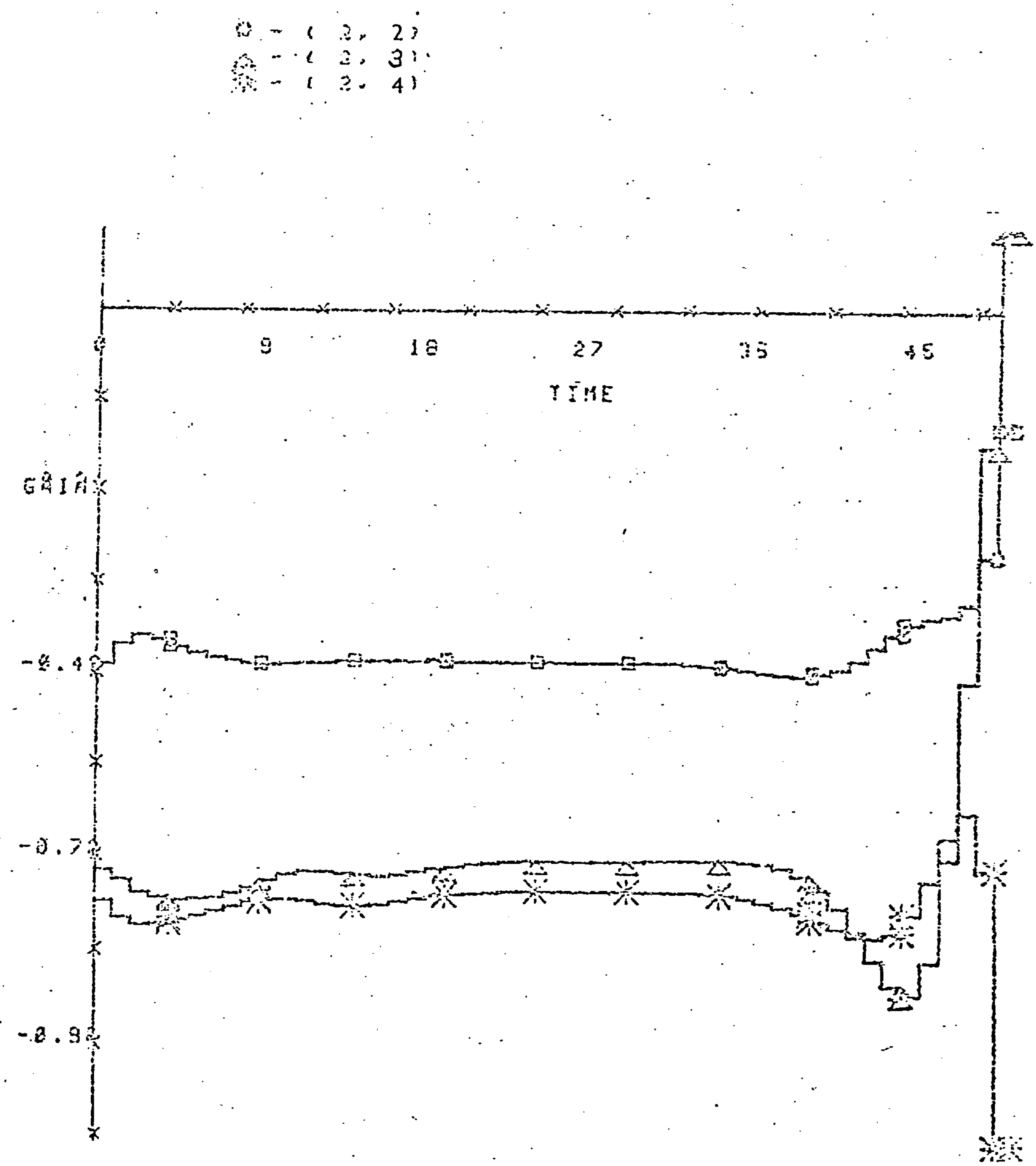

Figure 7: The Optimal Structured Feedback Gains for the Second Input Of The Fourth Order System for Case A with

$\sigma = [\ell(1) = \{2,3,4\};\ \ell(2) = \ 2,3,4\}]$

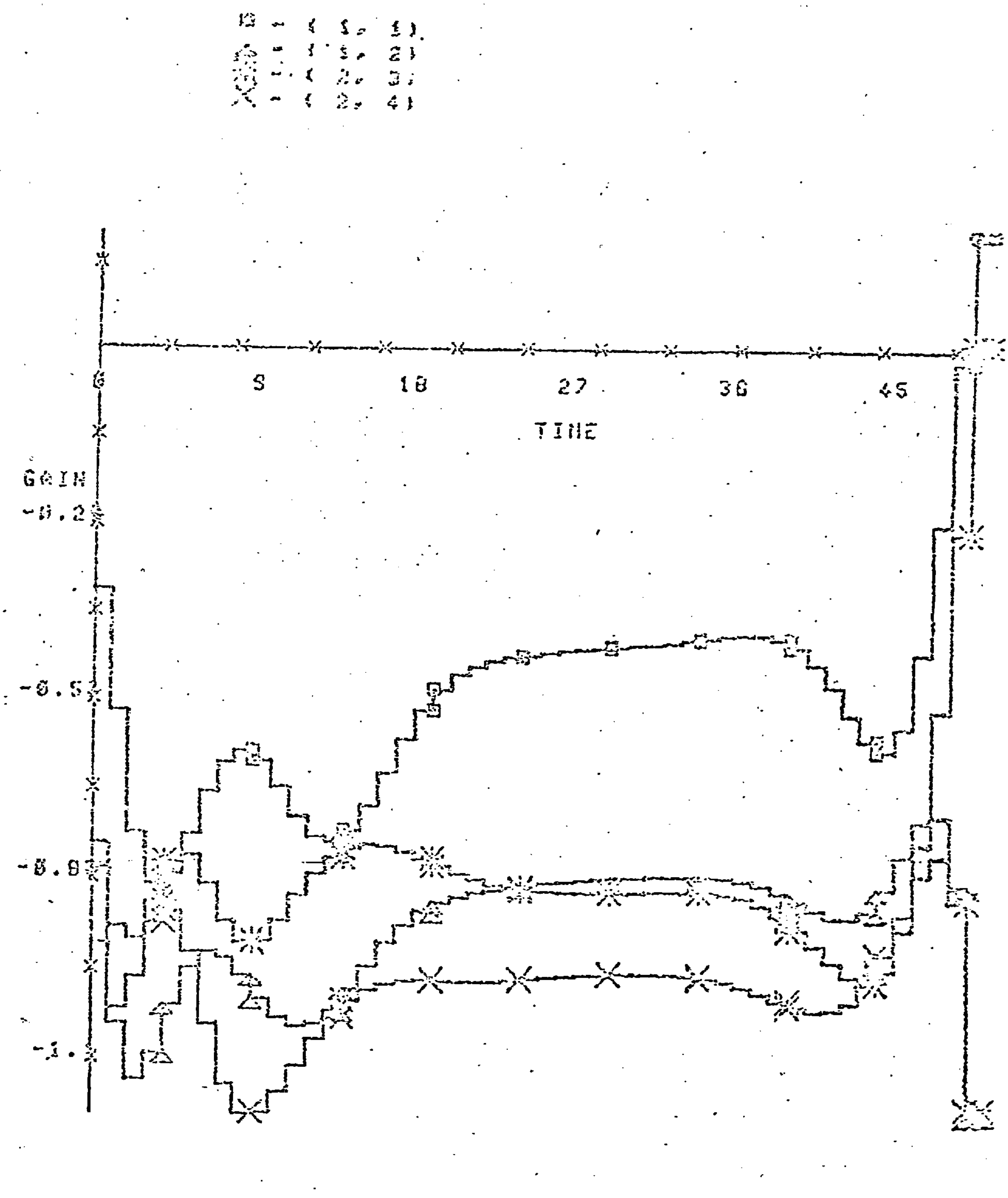

Figure 8: The Optimal Structured Feedback Gains of the Fourth Order System for Case A with

$$\sigma = [\ell(1) = \left\{1,2\right\};\ \ell(2) = \left\{3,4\right\}]$$

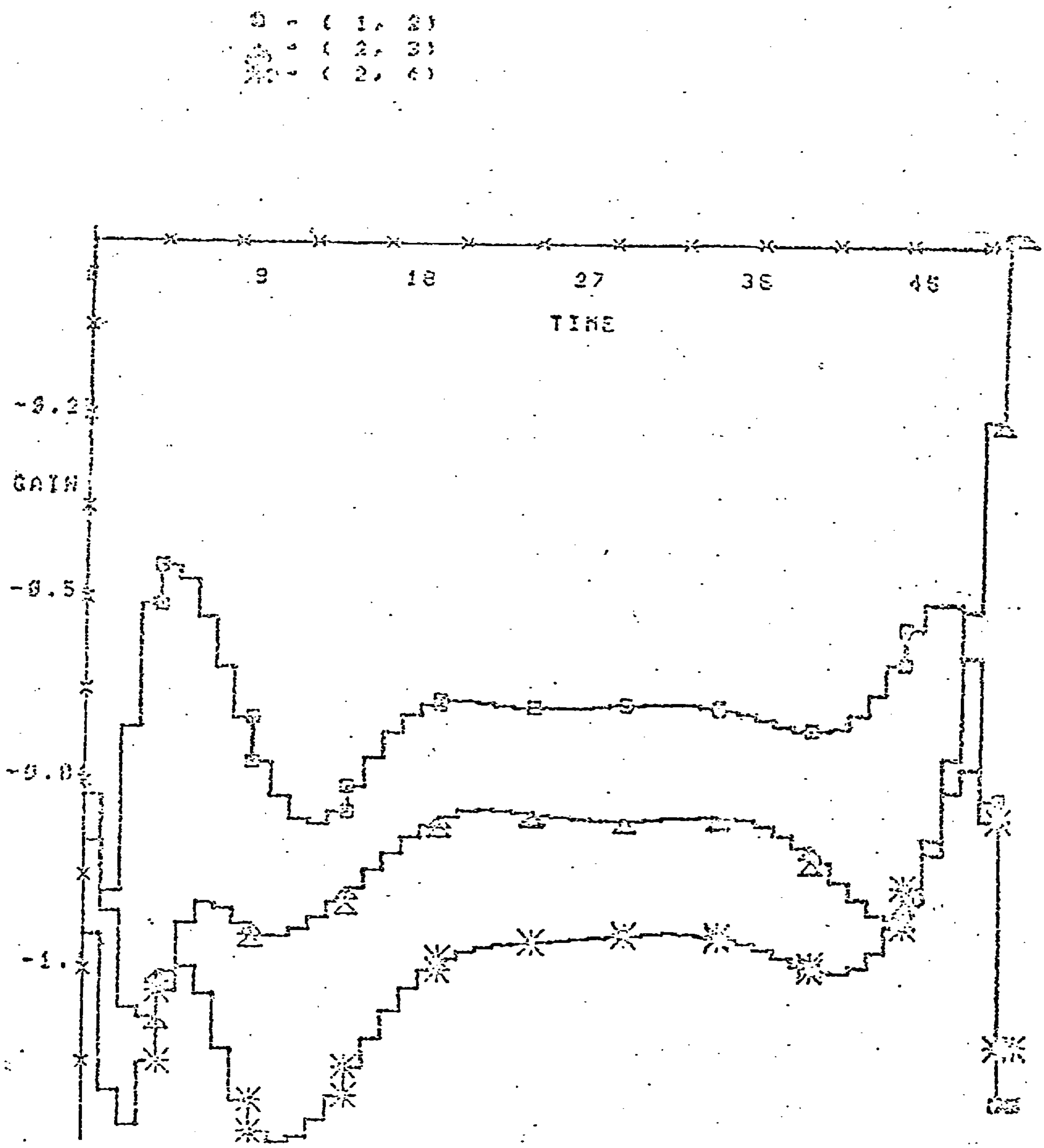

Figure 9: The Optimal Structured Feedback Gains of the Fourth Order System for Case A with

$$\sigma = [\ell(1) = \{2\};\ \ell(3) = \{3,4\}]$$

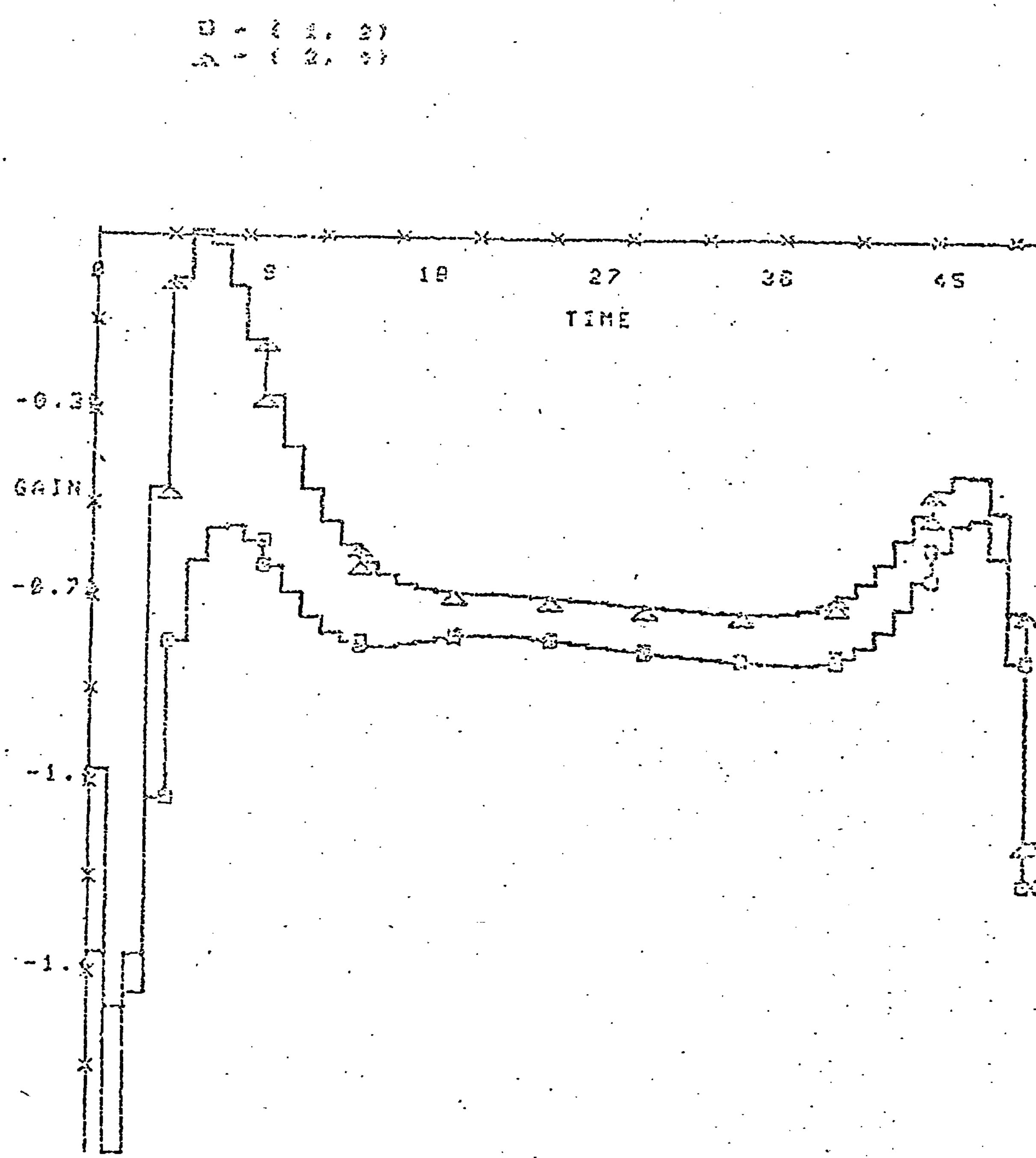

Figure 10: The Optimal Structured Feedback Gains of the Fourth Order System for Case A with

$\sigma = [\ell(1) = \{2\};\ \ell(2) = \{4\}]$

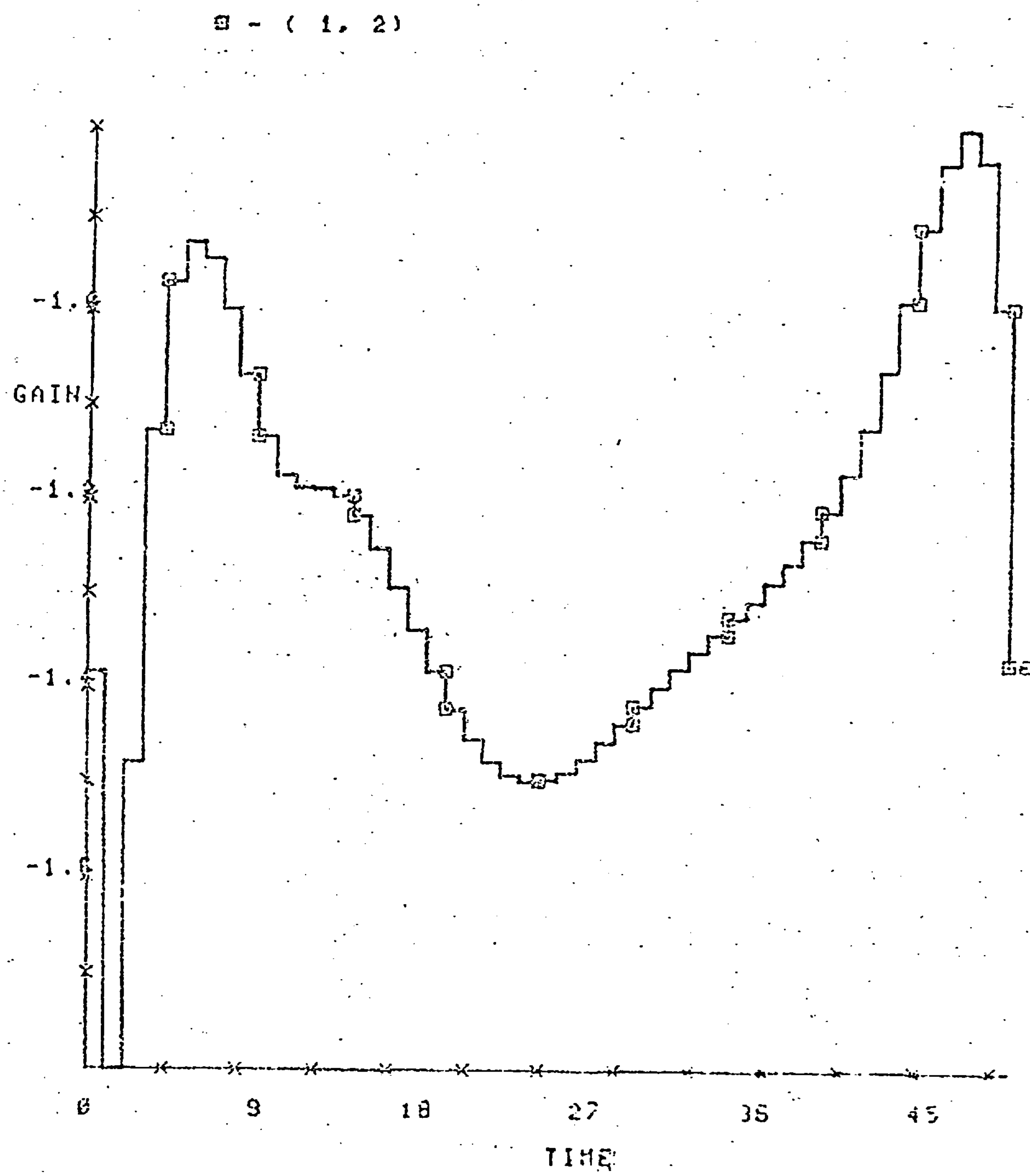

Figure 11: The Optimal Structured Feedback Gain of the Fourth Order System for Case A With $\sigma \doteq [\ell(1) = \left\{ 2 \right\}]$.

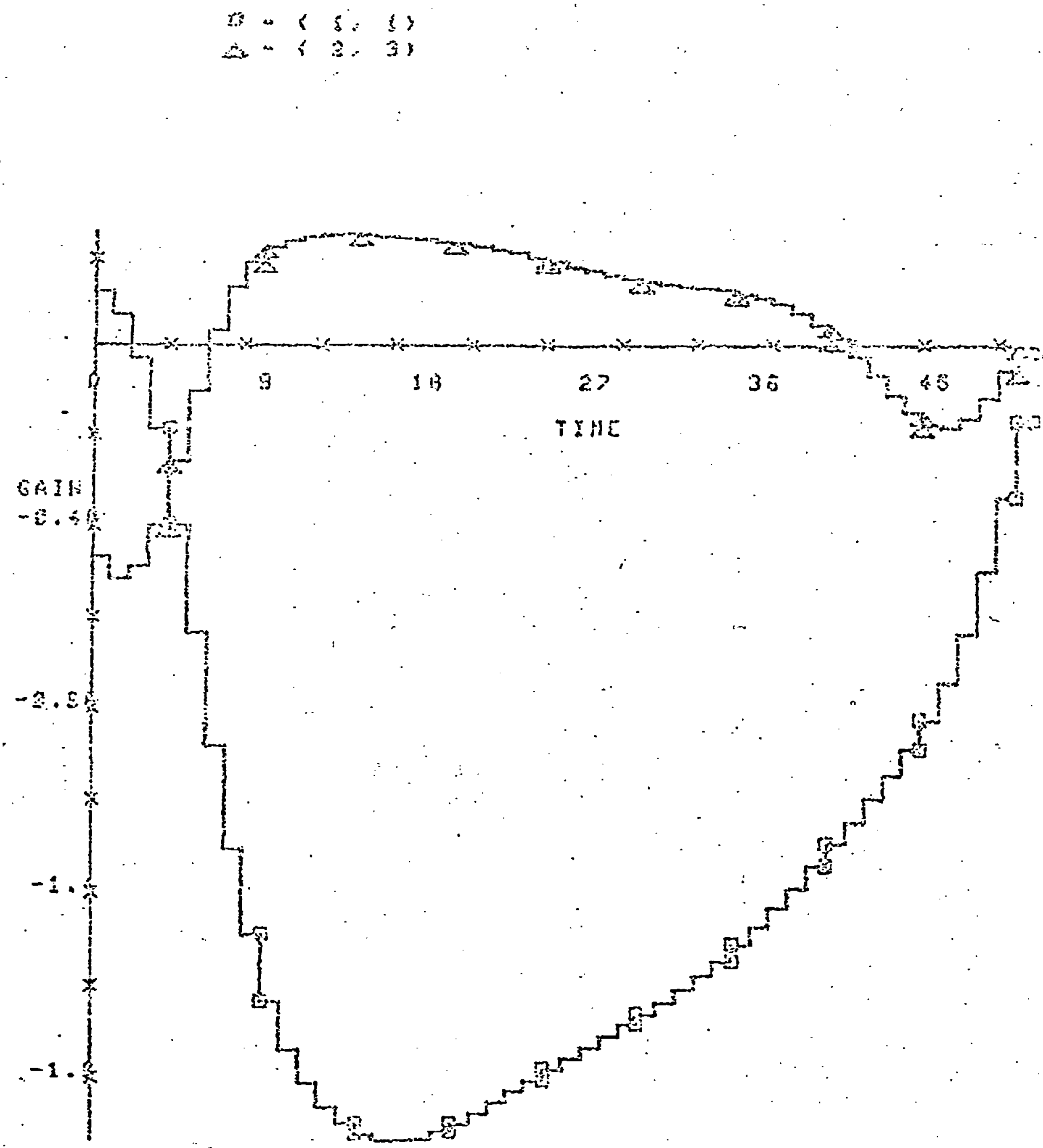

Figure 12: The Optimal Structured Feedback Gains of the Fourth Order System for Case A with

$$\sigma = [\ell(1) = \{1\};\ \ell(2) = \{3\}]$$

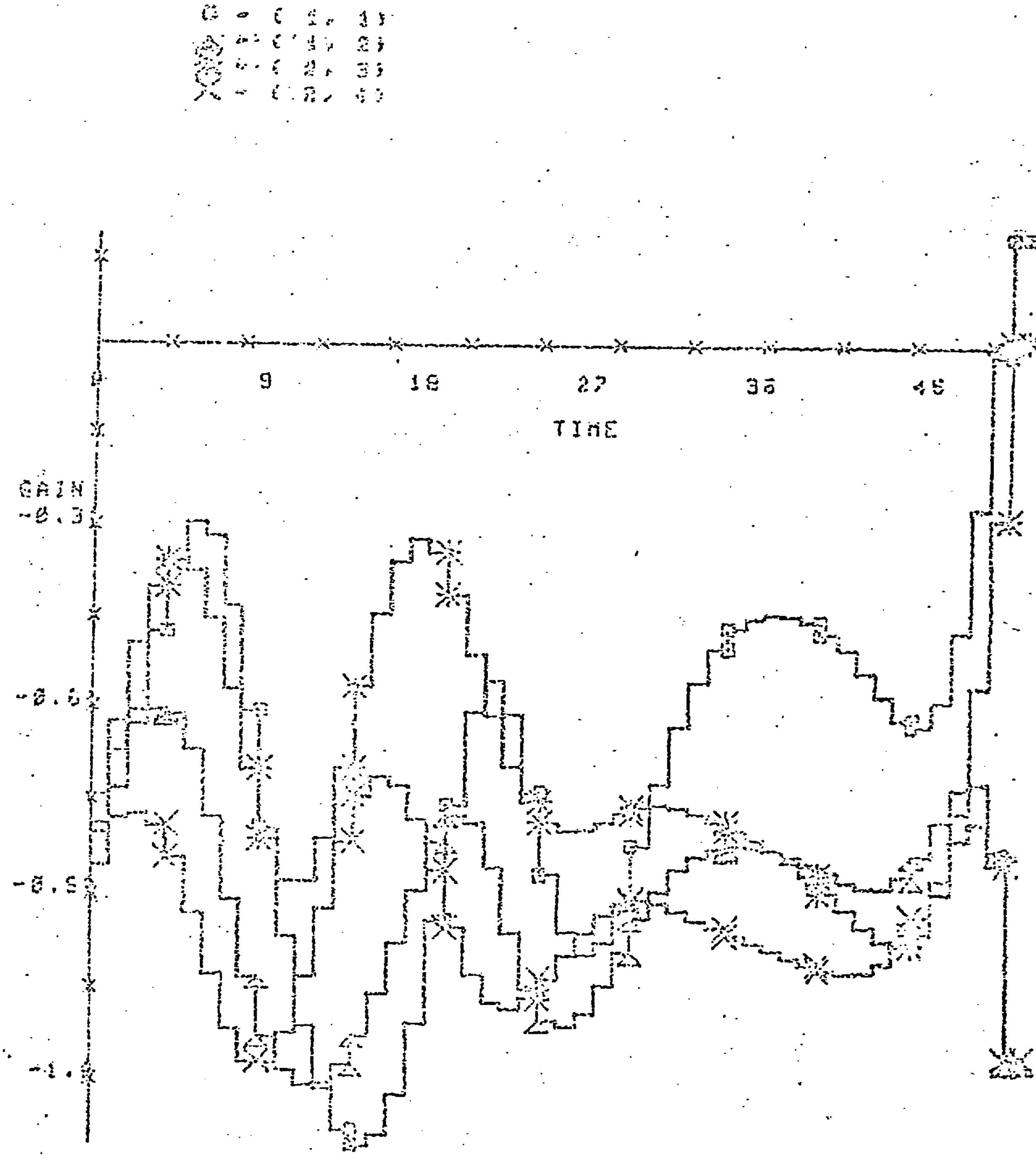

Figure 13: The Optimal Structured Feedback Gains of the Fourth Order System for Case B With

$$\sigma = [\ell(1) = \{1,2\};\ \ell(2) = \{3,4\}]$$

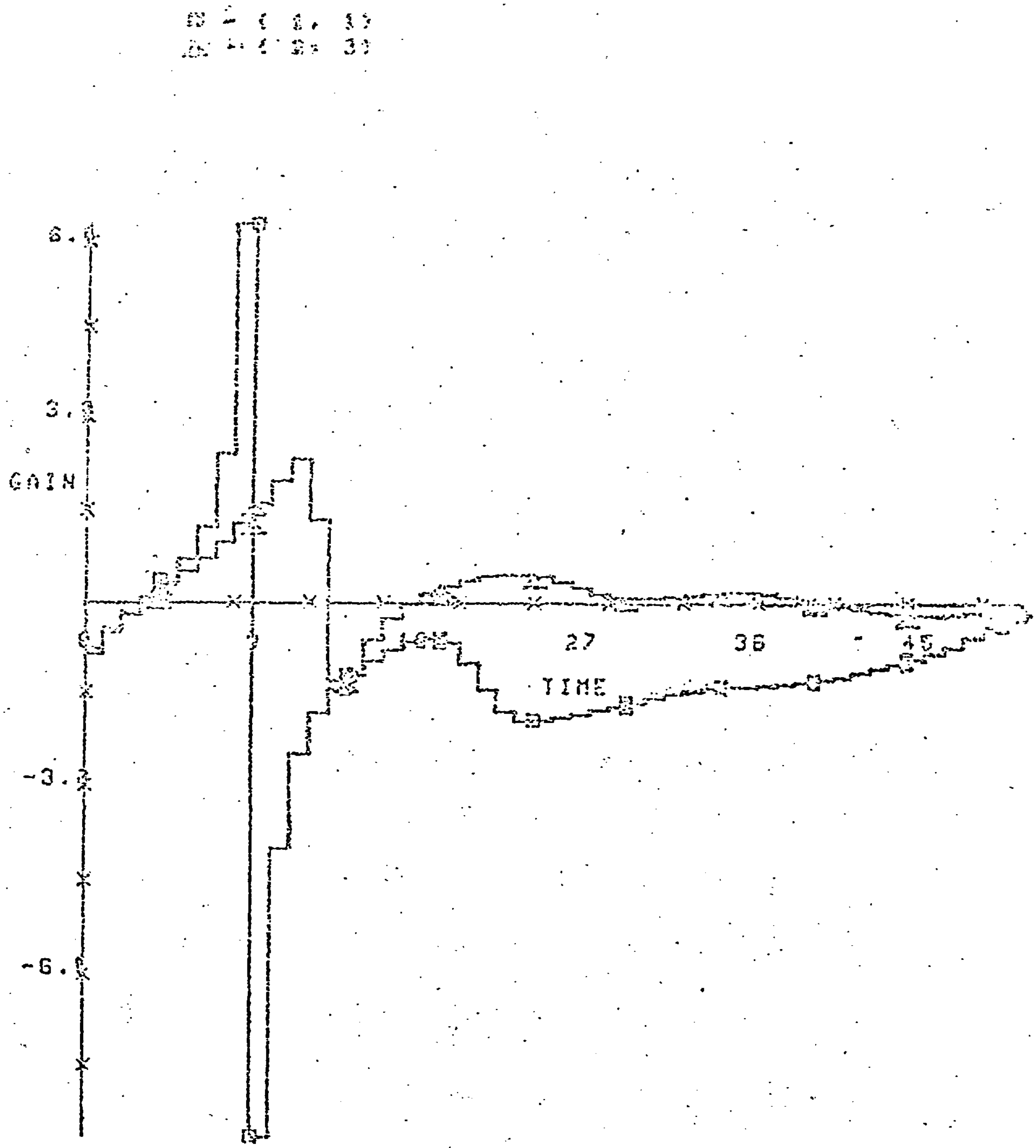

Figure 14: The Optimal Structured Feedback Gains of the Fourth Order System for Case B With

$$\sigma = [\ell(1) = \{1\};\ \ell(2) = \{3\}]$$

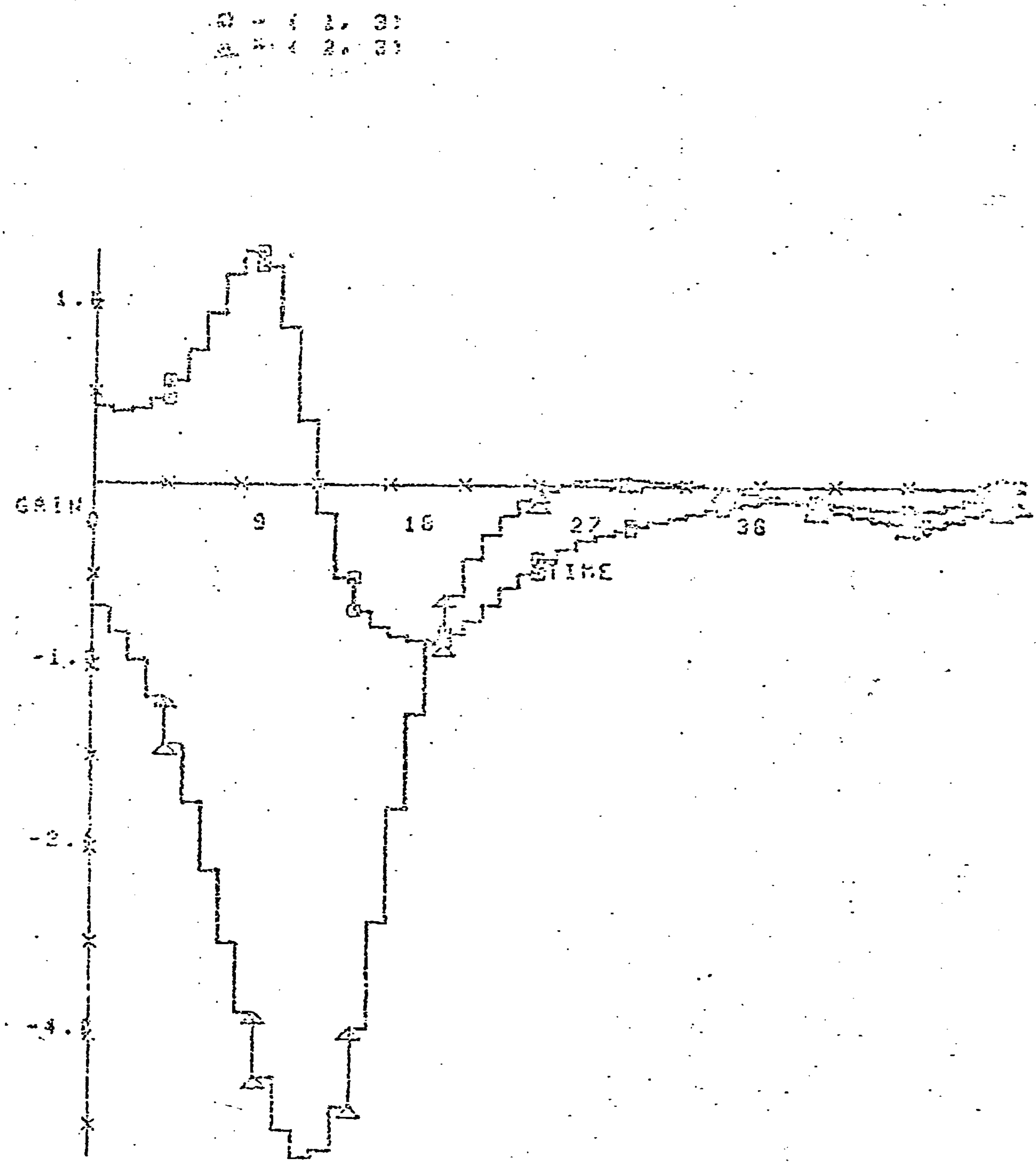

Figure 15: A Non-Optimal Solution to the Two Point Boundary Value Problem Associated With The Fourth Order System for Case A With $\sigma = [\ell(1) = \left\{3\right\};\ \ell(2) = \left\{3\right\}]$

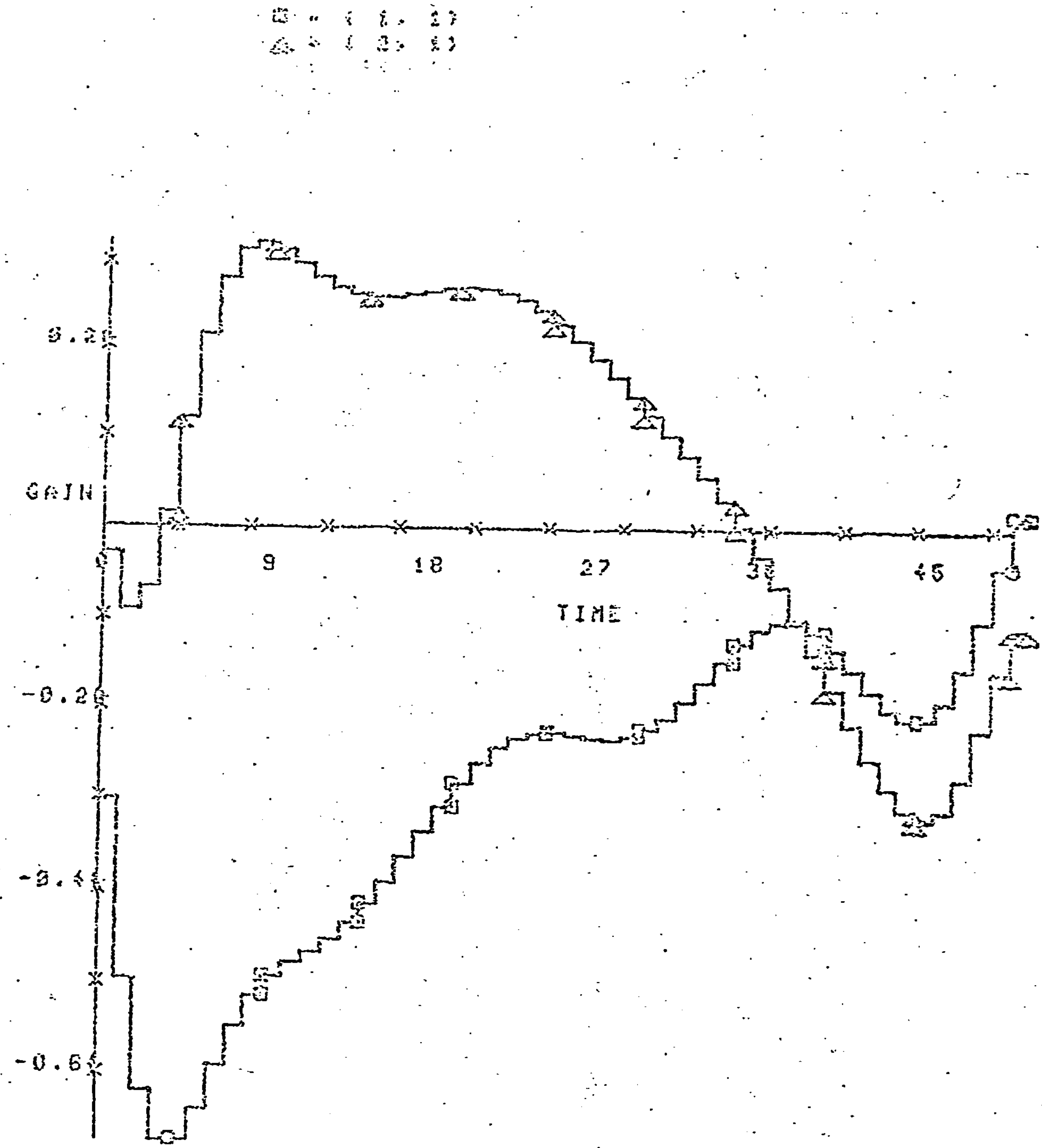

Figure 16: The Optimal Structured Feedback Gains of the Fourth Order System for Case A with

$$\sigma = [\ell(1) = \{3\};\ \ell(2) = \{3\}]$$

Rule two is also justified. In case A the results are particularly striking. To use this rule successfully one must evaluate enough feedback structures to be sure one has found the four structures containing p+1 gains that are lowest in cost.

The Optimal Structured Gain Trajectory Plots

Figures 4 and 5 contain the plots of the optimal complete state feedback gains against time. All the gains settle to their steady states within this time interval.

For Case A the plots of the unconstrained gains vs. time for some good feedback structures are contained in Figures 6 and 7, Figure 8, Figure 9 , Figure 10, and Figure 11. The figures are arranged in order of decreasing controller complexity. Since the gains are the solution of a two boundary value problem the time variations occur both near the starting time and near the terminal time. Note the time variations increase as the controller complexity decreases. The gain trajectory, depicted in Figure 11, which is associated with the simplest feedback structure $[\ell(1) = \left\{ 2 \right\}]$, fails to reach a steady state during the time interval.

Figures 13 and 14 are the plots of the controller gain trajectories for two good feedback structures for Case B. The effect the two different values of V_0 have on the optimal structured control policy for the feedback structure $[\ell(1) = \{1,2\}\ ;\ \ell(2) = \{3,4\}]$ may be ascertained by comparing Figures 8 and 13. In Case B, Figure 13, no steady state is reached and the gains appear periodic. This can be attributed to their "following" the expected value of state induced by the expected initial condition. Figures 12 and 14 contain the optimal structured control policies for the structure $[\ell(1) = \{1\}\ ;\ \ell(2) = \{3\}]$ for Cases A and B respectively. While this structure provides good control for Case B (PIO = 3.4%) for Case A the PIO is 233%. Thus V_0 can have a large effect on both the gain trajectories and the choice of feedback structure.

A Non-Optimal Solution to the Two Point Boundary Value Problem.

When Computational Procedure A was used with a feedback structure $[\ell(1) = \{3\}\ ,\ \ell(2) = \{3\}]$ and with $\pi(0) = \overset{*}{\pi}$, the procedure converged in 52 iterations to the control policy illustrated in Figure 15. The expected cost, J, associated with this Structured Control policy is 58.28. As the uncontrolled cost, J_u, is 56.48 this control policy is worse than no control at all. When the linear control policy, $\pi = [G_i = 0 \mid i=0,1,\ldots,N-1]$, was used as the initial control policy the computational procedure converged in 92 iterations to the structured control policy depicted in Figure 16. The control policy of Figure 16 has an expected cost of 46.23. When Initial Condition B given in section 4.4 was used the computational procedure converged to the structured control policy of Figure 16 in 78 iterations. It therefore seems likely that the structured control policy of Figure 16 is the Optimal Structured Control Policy. The Optimal Control Policy does not make a good starting point for the structure [$\{3\}$; $\{3\}$] because little control is possible using this feedback structure. Thus the sequences $[S_k]_{N-1}^{0}$ and $[V(k)]_{0}^{N}$ associated with the optimal structured control policy will be far different from the sequences associated with $\overset{*}{\pi}$. This result implies one must find an initial linear control policy which results in a system behaviour near that of the optimal structured control policy, to ensure that the computational procedure will converge to the optimal structured control policy.

An Unstable Minimum Norm Controller

It was mentioned in section 4.4, Part C if the gains deleted are of the same magnitude as those retained the performance of the minimum norm controller may be very poor. If the feedback structure $[\ell(1) = \{1,3\}\ ;\ \ell(2) = \{1,3\}]$ is specified then the structured state feedback matrix that differs the least from $\overset{*}{G}$ in Norm is

$$G^S = \begin{bmatrix} -0.392 & 0 & -0.482 & 0 \\ -0.066 & 0 & -0.737 & 0 \end{bmatrix} \tag{5.15}$$

(5.15) is obtained from (5.14) by deleting those gains constrained to be zero. The eigenvalues of the closed loop system, $A + BG^S$, are

$$\lambda_{1,2} = 0.9802 \pm j\,0.2356 \quad |\lambda_{1,2}| = 1.008$$

$$\lambda_{3,4} = 0.7923 \pm j\,0.2200 \quad |\lambda_{3,4}| = 0.822$$

Thus the minimum norm controller has made a stable system unstable.

The optimal steady state structured state feedback matrix computed using the algorithm given in Chapter 8 is

$$G^S = \begin{bmatrix} -1.419 & 0 & -1.646 & 0 \\ -1.217 & 0 & 2.572 & 0 \end{bmatrix} \qquad (5.16)$$

The closed loop eigenvalues are

$$\lambda_{1,2} = 0.8538 \pm j\,0.2360 \quad |\lambda_{1,2}| = 0.886$$

$$\lambda_{3,4} = 0.9193 \pm j\,0.1719 \quad |\lambda_{3,4}| = 0.935$$

5.3. An Unstable Seventh Order System

Assume

$$x_{k+1} = Ax_k + Bu_k + w_k \qquad (1.1)$$

where

$$A = \begin{bmatrix} 0.604 & 0.197 & 0.027 & 0 & 0.002 & -1\text{x}10^{-4} & 0.002 \\ -0.027 & 0.998 & 0.250 & 0 & 0.031 & -0.002 & 0.029 \\ -0.197 & -0.027 & 0.998 & 0 & 0.248 & -0.026 & 0.219 \\ -0.002 & 0.221 & 0.029 & 0.779 & 0.224 & -1\text{x}10^{-4} & 0.002 \\ 0 & 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.219 & 0.755 & 0.193 \\ 0 & 0 & 0 & 0 & -0.264 & -0.193 & 0.755 \end{bmatrix}$$

(5.17)

and

$$B = \begin{bmatrix} 0.002 & 1x10^{-4} & 0.002 \\ 0.034 & 0.003 & 0.034 \\ 0.281 & 0.031 & 0.279 \\ 0.039 & 0.029 & 0.231 \\ 0.250 & 0.250 & 0 \\ 0.083 & 0.083 & 0.026 \\ -0.009 & -0.009 & 0.219 \end{bmatrix} \quad (5.18)$$

The eigenvalues of the A matrix (5.17) are

$$\begin{aligned} \lambda_1 &= 1.012 \\ \lambda_2 &= 1.0 \\ \lambda_3 &= 0.9744 \\ \lambda_4 &= 0.7788 \\ \lambda_{5,6} &= 0.7546 \pm j\ 0.1927 \quad |\lambda_{5,6}| = 0.778 \\ \lambda_7 &= 0.6137 \end{aligned} \quad (5.19)$$

It is assumed

$$V_0 = I, \quad V_w = (0.1)I, \quad Q = I, \quad S_N = I, \quad R = I \quad (5.20)$$

and $N \doteq 50.$ (5.21)

Table 5 lists the structures for which optimal structured control policies were computed and tha resulting performance. The structures are grouped by the number of unconstrained gains they contain and arranged in order of increasing cost. The column headings have the

TABLE 5

STRUCTURED CONTROLLER COSTS FOR A

SEVENTH ORDER SYSTEM

Uncontrolled Cost, J_u = 2637.5
Optimal Cost, $\overset{*}{J}$ = 79.26

p	n_i	$\ell(1)$	$\ell(2)$	$\ell(3)$	I_t	J	PIO%	PPI%
19	6,6,7	1,2,3,5,6,7	1,2,3,4,5,6	1,2,3,4,5,6,7	2	79.26	~ 0	~ 100
17	5,5,7	1,2,3,5,7	2,3,4,5,6,	1,2,3,4,5,6,7	2	79.32	0.0758	99.997
14	5,3,6	1,2,3,5,7	2,5,6	1,2,3,4,6,7	3	79.85	0.745	99.976
14	7,0,7	1,2,3,4,5,6,7	-	1,2,3,4,5,6,7	1	86.87	9.60	99.702
14	7,7,0	1,2,3,4,5,6,7	1,2,3,4,5,6,7	-	1	94.74	19.5	99.394
14	0,7,7	-	1,2,3,4,5,6,7	1,2,3,4,5,6,7	1	95.12	20.0	99.380
12	4,3,5	1,2,3,5	2,5,6	1,2,3,4,7	3	80.19	1.17	99.964
11	4,2,5	1,2,3,5	5,6	1,2,3,4,7	4	80.88	2.04	99.936
9	3,1,5	2,3,5	5	1,2,3,4,7	3	81.35	2.64	99.918
9	4,1,4	1,2,3,5	5	1,2,3,7	4	81.81	3.22	99.900
8	3,1,4	2,3,5	5	2,3,4,7	4	81.95	3.39	99.894
8	3,1,4	2,3,5	5	1,2,3,7	3	82.02	3.48	99.892
8	4,1,3	1,2,3,5	5	2,3,7	4	82.13	3.62	99.887
8	3,2,3	2,3,5	5,6	2,3,7	4	82.36	3.91	99.878
8	3,1,4	2,3,5	5	1,2,3,4	4	84.37	6.45	99.800
8	4,1,3	1,2,3,5	5	2,3,4	4	84.41	6.50	99.798
8	4,1,3	1,2,3,5	5	1,2,3	4	85.21	7.51	99.767
8	2,1,5	3,5	5	1,2,3,4,7	4	85.58	7.97	99.752
8	4,1,3	1,2,3,5	5	3,4,7	4	85.65	8.06	99.750
8	3,1,4	2,3,5	5	1,3,4,7	4	86.14	8.68	99.731

TABLE 5 (Continued)

p	n_i	$\ell(1)$	$\ell(2)$	$\ell(3)$	I_t	J	PIO%	PPI%
8	3,1,4	1,3,5	5	2,3,4,7	4	86.24	8.81	99.727
8	3,2,3	3,5,6	5,6	3,4,7	7	104.2	31.5	99.025
7	3,1,3	2,3,5	5	2,3,7	4	83.61	4.23	99.869
7	3,1,3	2,3,5	5	2,3,4	4	84.78	6.96	99.784
7	4,1,2	1,2,3,5	5	2,3	4	85.40	7.75	99.759
7	3,1,3	2,3,5	5	1,2,3	4	85.42	7.77	99.759
7	3,1,3	2,3,5	5	3,4,7	4	86.29	8.87	99.725
7	2,1,4	3,5	5	2,3,4,7	5	86.35	8.95	99.722
7	2,1,4	2,3	5	2,3,4,7	5	87.93	10.94	99.661
7	2,1,4	3,5	5	1,2,3,4	5	89.14	12.47	99.613
7	7,0,0	1,2,3,4,5,6,7	-	-	1	110.4	39.29	98.78
7	1,2,4	2	5,6	1,3,4,7	8	110.6	39.54	98.77
7	0,7,0	-	1,2,3,4,5,6,7	-	1	208.7	163	94.94
7	0,0,7	-	-	1,2,3,4,5,6,7	1	246.6	211	93.46
6	3,1,2	2,3,5	5	2,3	4	85.78	8.23	99.745
6	2,1,3	3,5	5	2,3,7	4	87.03	9.80	99.696
6	3,1,2	2,3,5	5	3,7	5	87.05	9.83	99.695
6	3,0,3	2,3,5	-	2,3,7	3	88.82	12.1	99.626
6	2,1,3	2,3	5	2,3,7	5	89.05	12.6	99.617
6	2,1,3	3,5	5	2,3,4	5	89.75	13.2	99.589
6	3,1,2	2,3,5	5	3,4	5	90.93	14.7	99.543
6	2,1,3	2,3	5	2,3,4	5	91.03	14.8	99.540
6	2,1,3	2,5	5	2,3,7	4	91.57	15.5	99.518
6	3,1,2	2,3,5	5	2,7	5	93.07	17.4	99.460
6	2,1,3	3,5	5	3,4,7	7	104.8	32.2	99.001
5	2,1,2	3,5	5	2,3	5	90.88	14.7	99.545

TABLE 5 (Continued)

p	n_i	$\ell(1)$	$\ell(2)$	$\ell(3)$	I_t	J	PIO%	PPI%
5	3,0,2	2,3,5	-	2,3	4	92.14	16.3	99.496
5	3,1,1	2,3,5	5	3	6	92.17	16.3	99.495
5	2,1,2	2,3	5	2,3	5	92.45	16.6	99.484
5	1,1,3	3	5	2,3,7	6	93.18	17.6	99.455
5	3,0,2	2,3,5	-	3,7	5	93.42	17.9	99.446
5	2,0,3	3,5	-	2,3,7	5	94.77	19.6	99.393
5	2,1,2	2,3	5	3,7	5	95.03	19.9	99.383
5	2,0,3	2,3	-	2,3,7	7	124.9	57.6	98.22
4	1,1,2	3	5	2,3	5	97.08	22.5	99.303
4	2,0,2	3,5	-	2,3	6	99.03	24.9	99.227
4	1,1,3	5	5	2,3	5	99.36	25.4	99.214
4	2,1,1	2,3	5	3	6	99.83	26.0	99.195
4	3,0,1	2,3,5	-	3	6	99.85	26.0	99.195
4	0,1,3	-	5	2,3,7	4	100.8	27.2	99.158
4	3,1,0	2,3,5	5	-	4	101.0	27.4	99.150
4	2,1,1	2,3	5	7	5	104.0	31.2	99.032
4	1,1,2	3	5	2,7	7	109.2	37.8	98.83
4	2,1,1	3,5	5	2	9	113.6	43.3	98.66
4	2,0,2	3,5	-	3,7	11	127.2	60.5	98.13
4	1,1,2	3	5	3,7	9	127.2	60.5	98.13
4	2,1,1	3,5	5	3	10	128.6	62.3	98.07
3	1,0,2	5	-	2,3	5	104.8	32.2	99.01
3	0,1,2	-	5	2,3	4	107.3	35.4	98.90
3	2,1,0	2,3	5	-	3	110.4	39.3	98.78
3	3,0,0	2,3,5	-	-	4	114.1	44.0	98.64
3	1,1,1	3	5	2	6	119.8	51.1	98.42

TABLE 5 (Continued)

p	n_i	$\ell(1)$	$\ell(2)$	$\ell(3)$	I_t	J	PIO%	PPI%
3	1,1,1	3	5	3	6	135.4	70.8	97.81
3	2,0,1	3,5	-	3	8	138.1	74.2	97.70
3	2,0,1	2,3	-	3	7	139.4	75.9	97.65
3	1,0,2	3	-	2,3	31	198.9	151	95.32
3	0,1,2	-	5	2,7	23	414.7	423	86.89
2	1,0,1	5	-	3	9	141.2	78.1	97.57
2	0,1,1	-	5	3	6	144.2	81.9	97.46
2	1,1,0	3	5	-	5	146.7	85.1	97.36
2	2,0,0	2,3	-	-	5	163.6	106	96.70
2	2,0,0	3,5	-	-	7	164.1	107	96.68
2	0,0,2	-	-	2,3	6	302.1	281	91.29
2	1,0,1	3	-	3	9	331.4	318	90.14
1	1,0,0	3	-	-	7	346.7	337	89.55
1	0,0,1	-	-	3	4	463.4	485	84.98
1	1,0,0	5	-	-	14	1454	1734	46.26
1	0,1,0	-	5	-	12	1469	1753	45.68
1	1,0,0	2	-	-	20	2488	3039	5.843

TABLE 6

EVALUATION OF THE HEURISTIC STRUCTURE SELECTION

METHODS FOR A SEVENTH ORDER SYSTEM

p	Structure				Gains Used		Rank by Cost of p+1 Structures
	n_i	$\ell(1)$	$\ell(2)$	$\ell(3)$	$\overset{*}{G}$ Ordering	p=7 Ordering	
8	3,1,4	2,3,4	5	2,3,4,7	1,2,3,4,5,6,7,11		
7	3,1,3	2,3,5	5	2,3,7	1,2,3,4,5,6,7	1,2,3,4,5,6,7	1 and 2
6	3,1,2	2,3,5	5	2,3	1,2,3,4,5,6	1,2,3,4,5,6	1 and 2
5	2,1,2	3,5	5	2,3	1,2,3,4,6	1,2,3,4,6	1 and 2
4	1,1,2	3	5	2,3	1,2,3,6	1,3,4,6	1 and 4
3	1,0,2	5	-	2,3	1,4,6	1,2,6	2 and 3
2	1,0,1	5	-	3	1,4	1,2	1 and 7
1	1,0,0	3	-	-	2	3	3 and 4

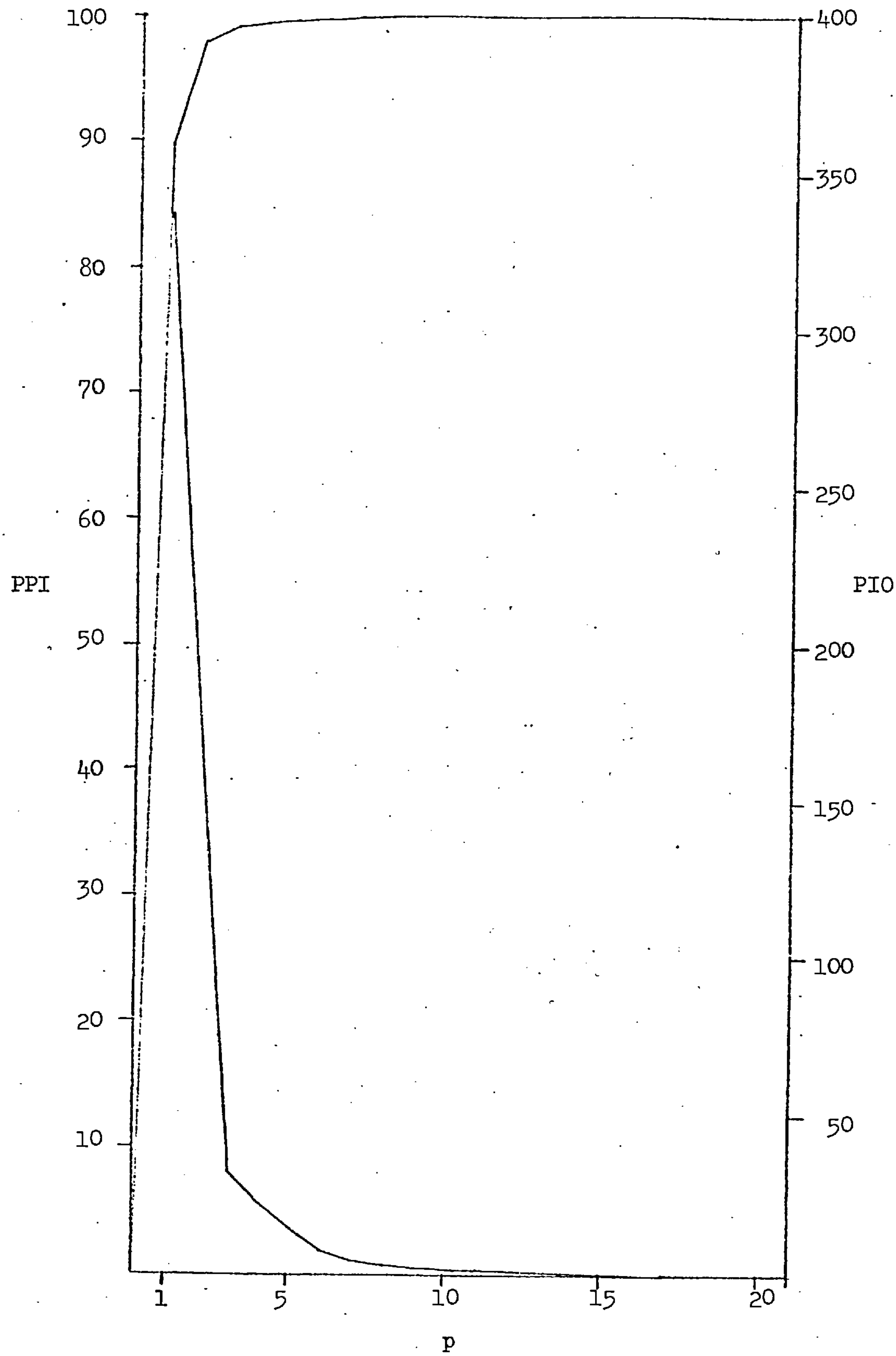

Figure 17: Plot of PPI and PIO Against p For The Best Feedback Structures of the Seventh Order System.

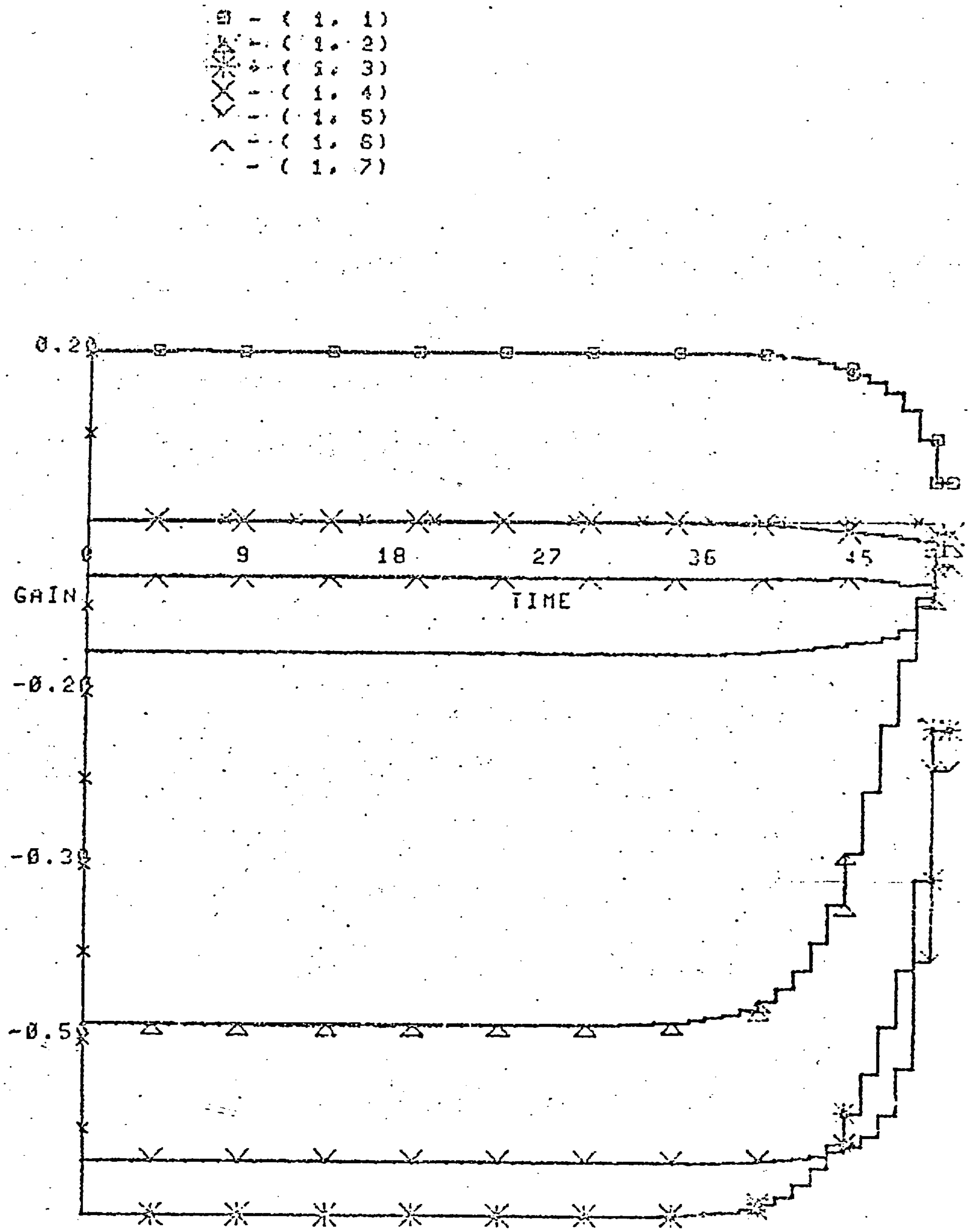

Figure 18: The Optimal Complete State Feedback Gains of the Seventh Order System for Input One.

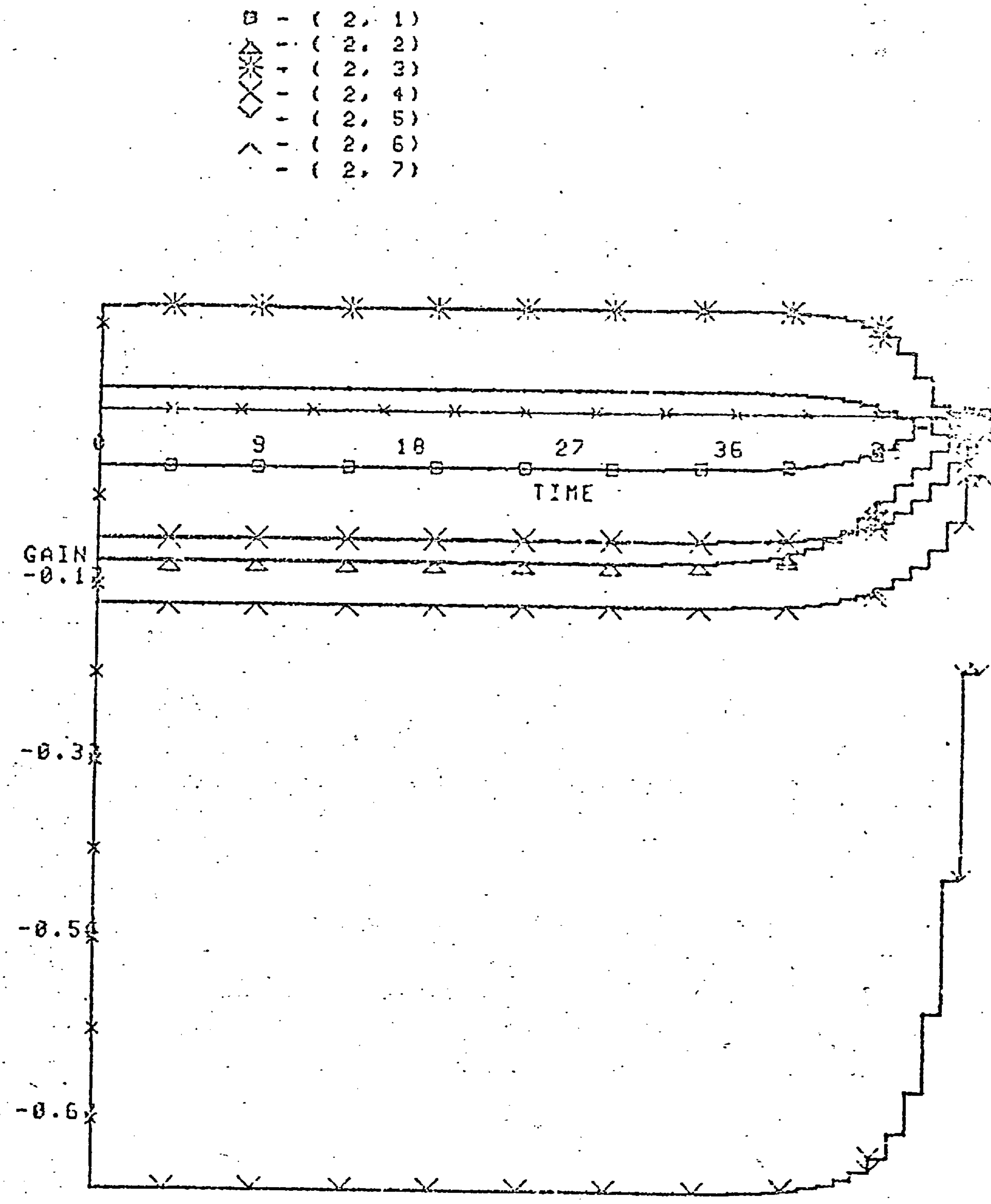

Figure 19: The Optimal Complete State Feedback Gains of the Seventh Order System for Input Two.

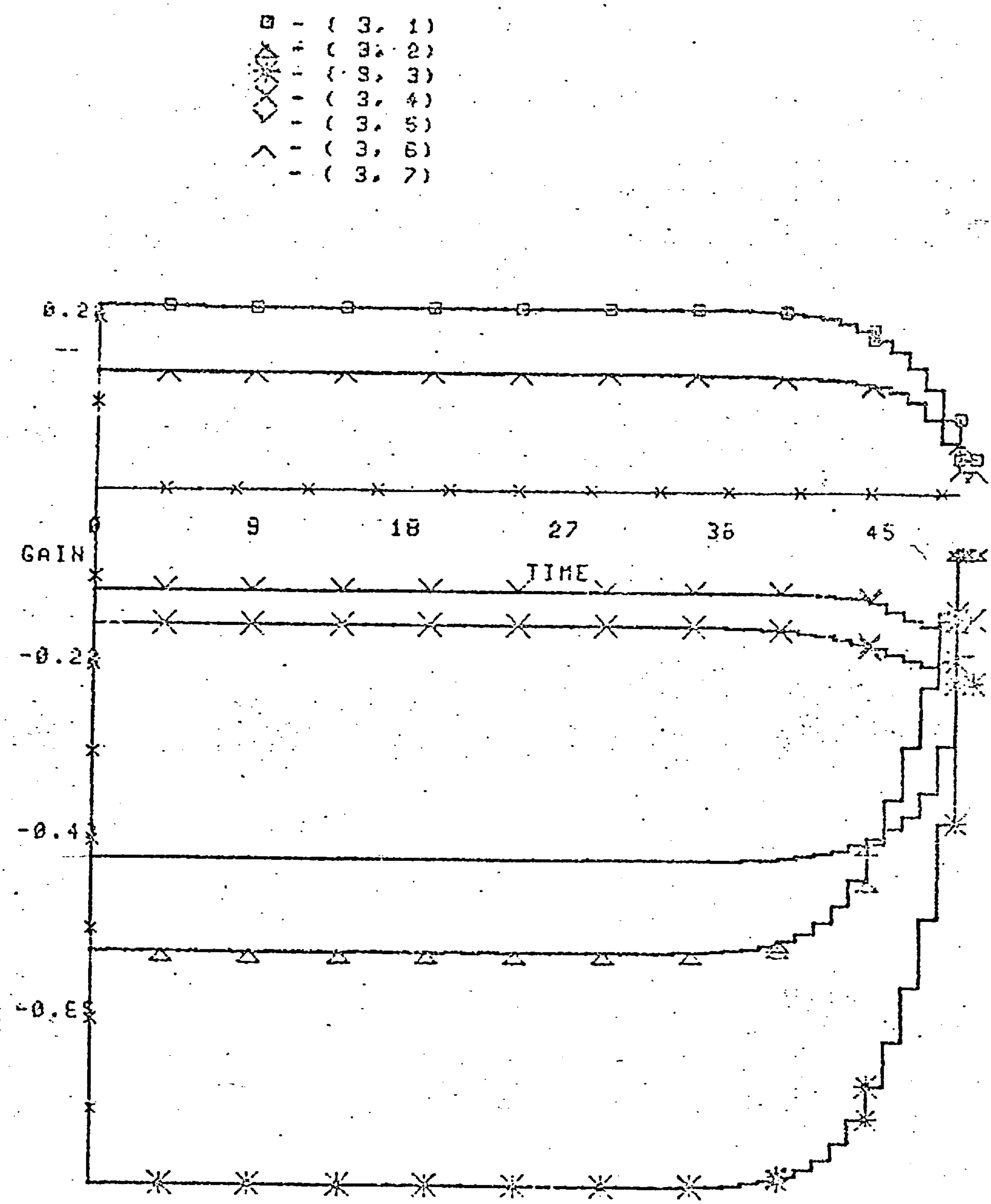

Figure 20: The Optimal Complete State Feedback Gains of the Seventh Order System for Input Three

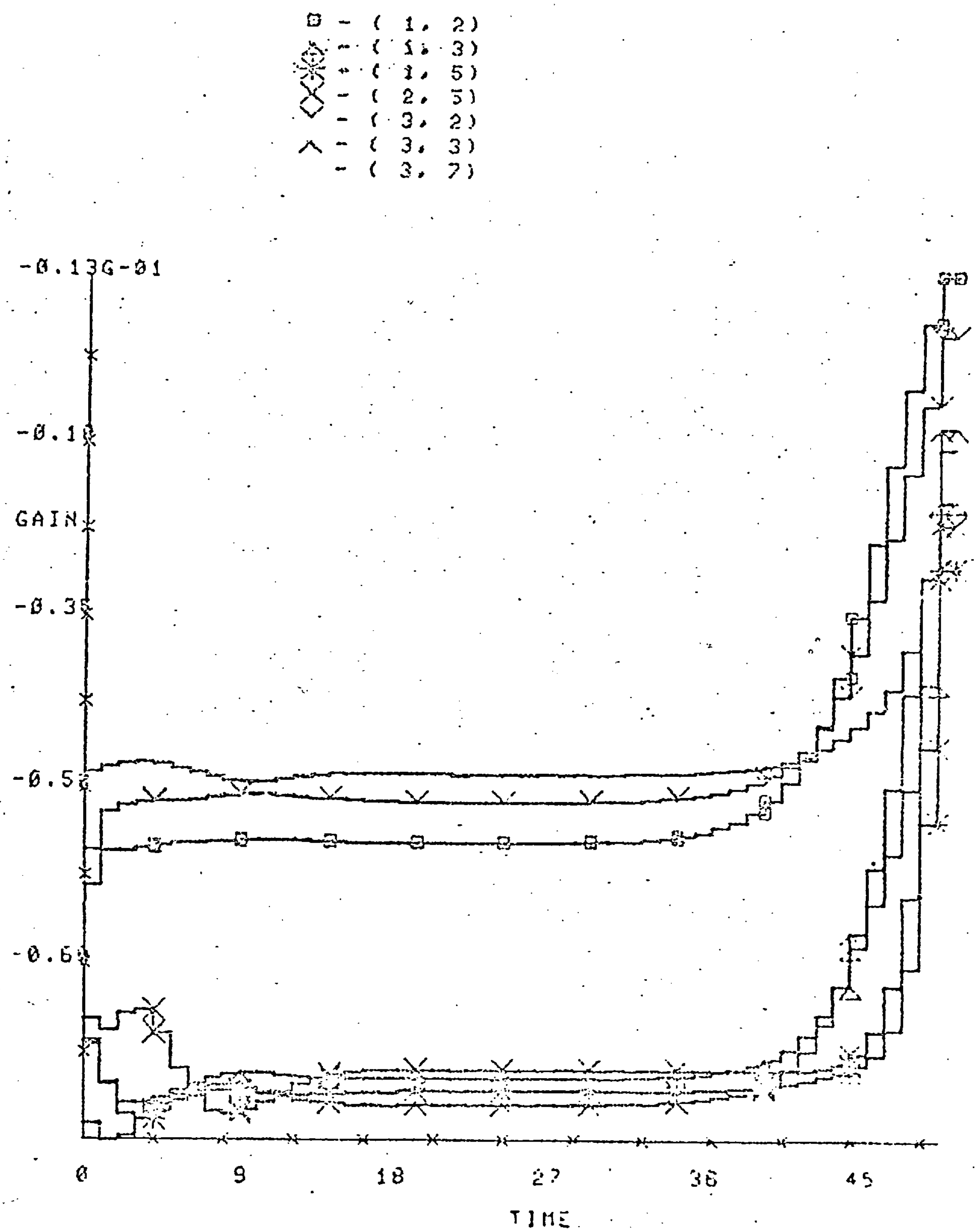

Figure 21: The Optimal Structured Feedback Gains of the Seventh Order System With $\sigma = [\ell(1) = \{2,3,5\};\ \ell(2) = \{5\};\ \ell(3) = \{2,3,7\}]$

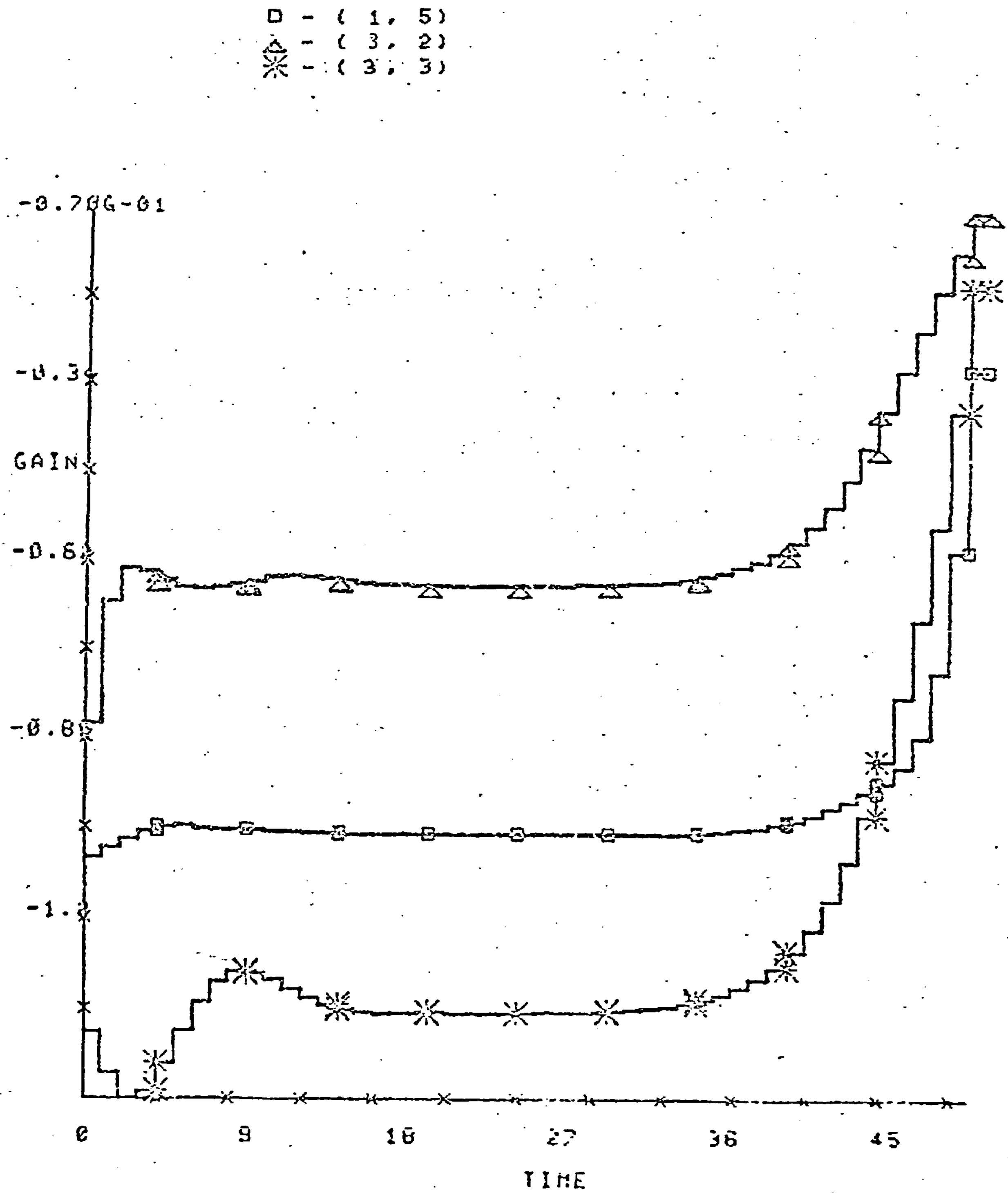

Figure 22: The Optimal Structured Feedback Gains of the Seventh Order System With

$\sigma = [\ell(1) = \{5\}; \quad \ell(3) = \{2,3\}]$

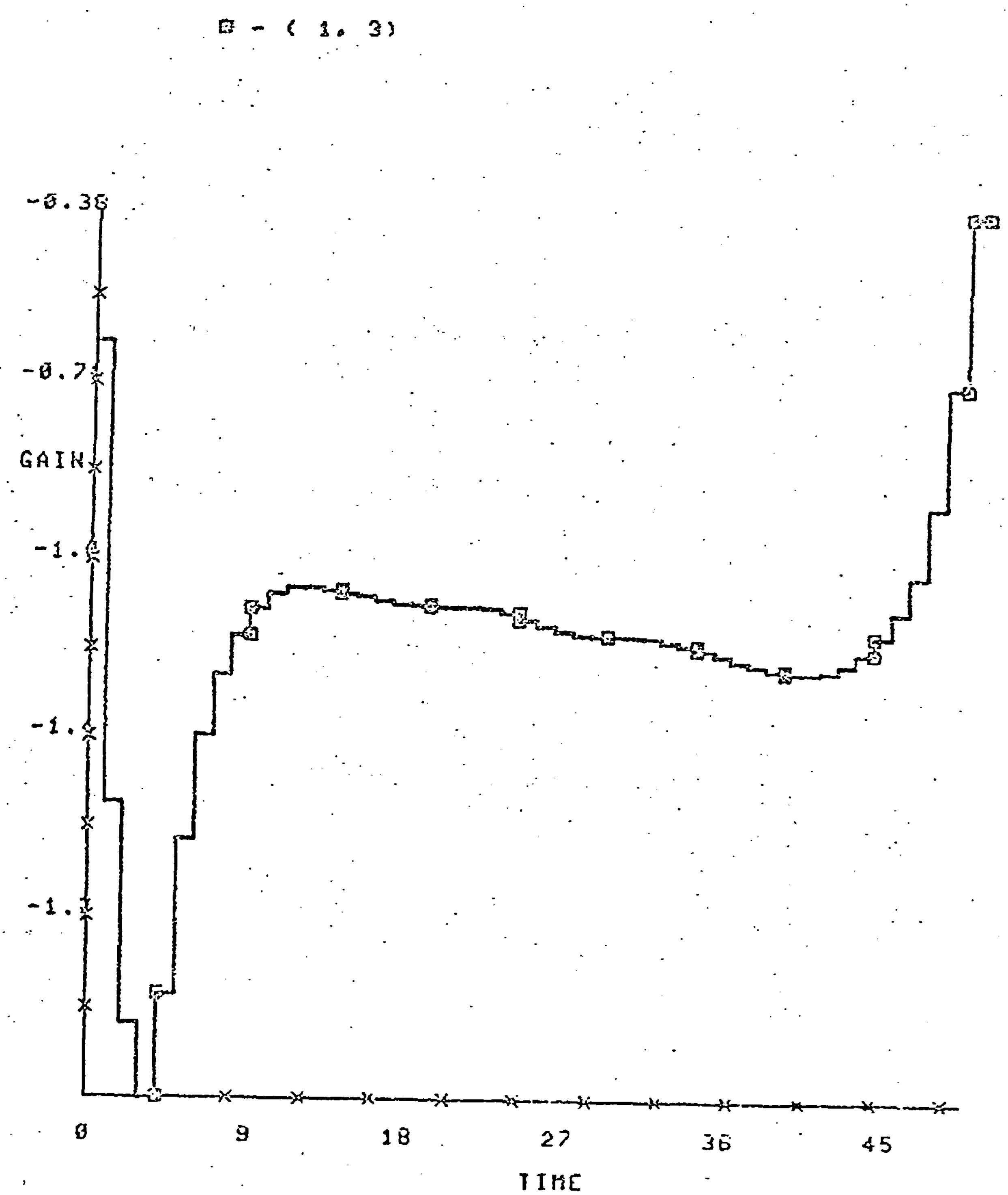

Figure 23: The Optimal Structured Feedback Gains of the Seventh Order System with $\sigma = \{(1,3)\}$

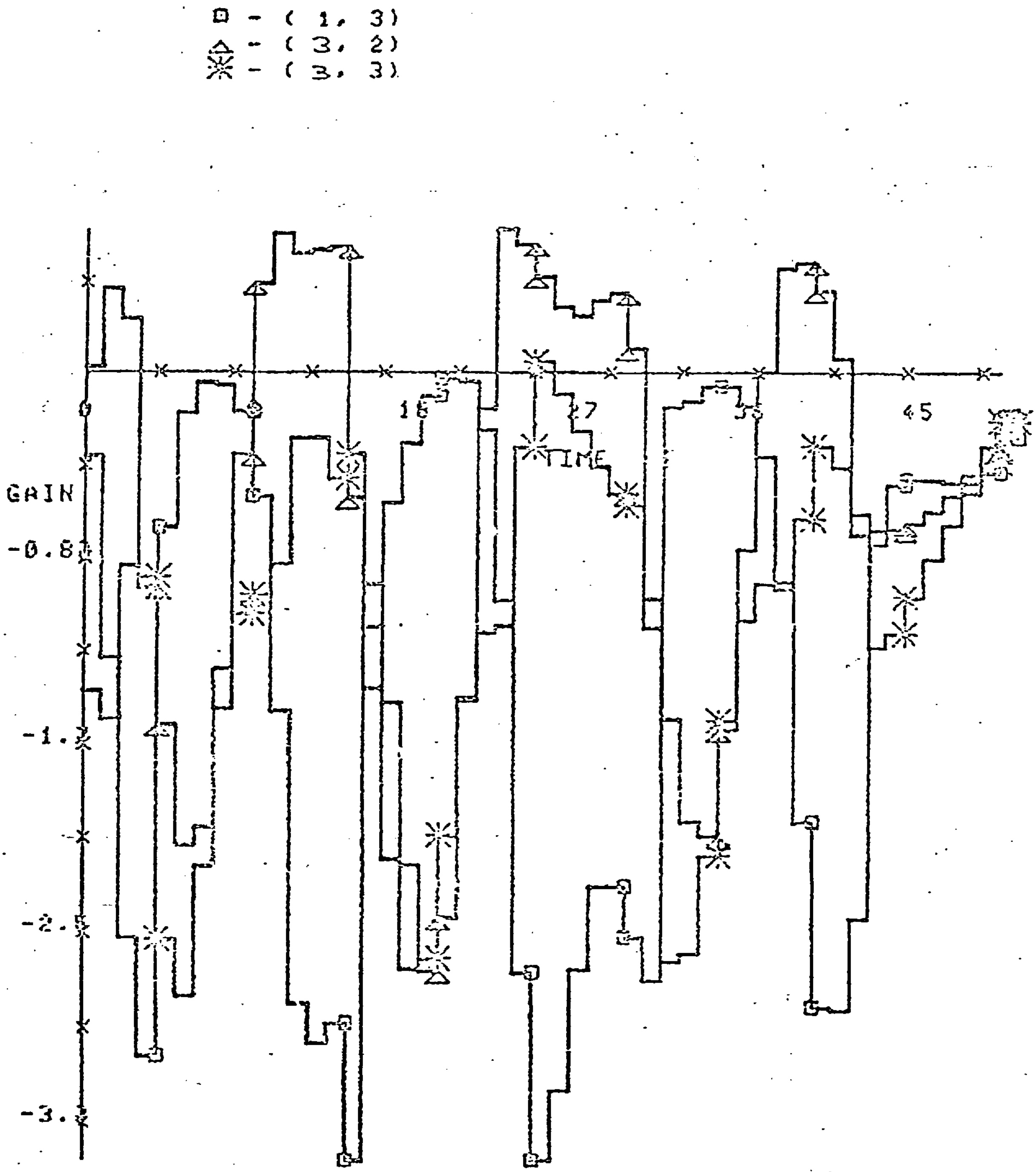

Figure 24: The Optimal Structured Feedback Gains of the Seventh Order System With

$$\sigma = \left\{ (1,3),(3,2),(3,3) \right\}$$

same meaning as in Tables 1 and 2, but an extra column head n_i is included. This column contains, from left to right, the n_1, n_2, and n_3 for each feedback structure that was used.

Figures 18, 19, and 20 contain the optimal complete state feedback gain trajectories for inputs one, two, and three respectively. As can be clearly seen there is a wide variation in the magnitudes of the gains. This suggests Method 1 can be successfully applied using the ordering induced on the gain co-ordinates by $\overset{*}{G}$. When feedback structures containing only those p co-ordinates corresponding to the p largest gains in the $\overset{*}{G}$ matrix were tried it was found that with $p \geq 7$ there was little increase in cost over the use of $\overset{*}{\pi}$. Several likely 8 gain structures were considered to suggest other promising 7 gain structures. Then a careful search for the best structures with $p = 7, 6, \ldots, 1$ was made using a combination of Methods 1 and 2. Figure 17 is the plot of PPI and PIO against p for these best structures It shows that the improvement in performance obtained by use of an extra gains decreases as the number of gains increases. The shapes of the curves in Figure 17 are quite similar to those of Figure 3.

There seems very little incentive to use a controller containing more than 7 gains. The optimal structured control policy associated with the best 7 gain structure [$\ell(1) = \{2,3,5\}$; $\ell(2) = \{5\}$; $\ell(3) = \{2,3,7\}$] is plotted in Figure 21. It produces only a 4.23% increase in cost over optimal, and 99.869 of the PPI.

A controller containing 3 gains and requiring only three measurements and two inputs produces control which could still be termed near optimal with a PIO = 32.2%. The optimal gain trajectories for this feedback structure are plotted in Figure 22.

The best controller requiring 2 gains produces a 78.1% PIO. This however is 97.57 PPI. The large PPI figure can be attributed to the uncontrolled system being unstable.

Figure 23 is the plot of the optimal controller gain trajectory for the best single gain structure, $[\ell(1) = \{3\}]$ (i.e. $n_2 = n_3 = 0$). This results in an expected cost 337% greater than $\overset{*}{J}$, but it does stabilize the system. The optimal structured state feedback matrix when k = 20 is

$$G_{20}^{S} = \begin{bmatrix} 0 & 0 & -1.136 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (5.22)$$

The eigenvalues of $A + BG_{20}^{S}$ are

$$\lambda_1 = 1.0$$

$$\lambda_{2,3} = 0.8292 \pm j\,0.1239 \quad |\lambda_{2,3}| = 0.8384$$

$$\lambda_{4,5} = 0.7856 \pm j\,0.2910 \quad |\lambda_{4,5}| = 0.8314$$

$$\lambda_6 = 0.7788$$

$$\lambda_7 = 0.5601$$

Thus if simple stability is all that is required a controller using one gain, one measurement, and one actuator would suffice.

Evaluation of the Proposed Structure Selection Methods

Table 6 contains the evaluation of the proposed structure selection methods for this example. This table is similar to tables 3 and 4 and the entries were constructed in the same manner as those of tables 3 and 4.

In addition to ordering the gain co-ordinates by the ordering implied by $\overset{*}{G}$ they have been ordered by the magnitudes of the optimal gains of the feedback structure which gives lowest expected cost for p = 7. This ordering can be seen to be more useful than the $\overset{*}{G}$ ordering when p = 2. The optimal steady state feedback matrix is

$$\overset{*}{G} = \begin{bmatrix} 0.197 & -0.569 & -0.780 & -0.002 & -0.720 & -0.064 & -0.150 \\ -0.054 & -0.143 & 0.102 & -0.122 & -0.730 & -0.184 & 0.021 \\ 0.231 & -0.564 & -0.845 & -0.163 & -0.122 & 0.147 & -0.451 \end{bmatrix} \qquad (5.23)$$

The ordering induced on the gain co-ordinates is

Ordering	1	2	3	5	6	7	8	9	10	11	12	13	14	15	16
Co-ordinate	3,3	1,3	2,5	1,2	3,2	3,7	3,1	1,1	2,6	3,4	1,7	3,6	2,2	3,5	2,4

Ordering	17	18	19	20	21	4
Co-ordinate	2,3	1,6	2,1	2,7	1,4	1,5

For the feedback structure [$\ell(1) = \{2,3,5\}$; $\ell(2) = \{5\}$; $\ell(3) = \{2,3,7\}$] the structured state feedback matrix of the optimal structured control policy, at time k = 20, is

$$G_{20}^{S} = \begin{bmatrix} 0 & -0.572 & -0.798 & 0 & -0.811 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.790 & 0 & 0 \\ 0 & -0.533 & -0.824 & 0 & 0 & 0 & -0.506 \end{bmatrix} \qquad (5.24)$$

The ordering induced upon the gain co-ordinates is

Ordering	1	2	3	4	5	6	7
Co-ordinate	3,3	1,5	1,3	2,5	1,2	3,2	3,7

Both methods of selecting feedback structures work well, but Method 2 does not seem to work as well as Method 1 when $p \leqslant 2$. It was found that Method 2 could be usefully used in a negative sense. That is substructures of a structure that has a high cost, need not be tested.

An Optimal Structured Control Policy with a Periodic Behaviour

The optimal structured control policy of the feedback structure

$$\sigma' = [\ell(1) = \left\{ 3 \right\} ; \ \ell(3) = \left\{ 2,3 \right\}] \qquad (5.25)$$

depicted in Figure 24, can be clearly seen to have a periodic nature. The expected cost J associated with it is 198.9. With $\pi(0) = \overset{*}{\pi}$ it took 31 iterations to converge. It has been shown that the system can be stabilized using a constant gain feedback structure $[\ell(1) = \left\{ 3 \right\}]$. As this feedback structure is a substructure of σ' the system can be stabilized using a constant structured feedback matrix with feedback structure σ'.

This result implies that the optimal structured feedback matrices may not settle to steady state values in the middle of a long time interval, even when the system can be stabilized using a constant structured feedback matrix.

5.4. Conclusions

It can be concluded for some systems near optimal controls can be obtained which require fewer gains, measurements, and actuators than the optimal. It can be further concluded for some systems, each additional gain added to the compensator will produce successively smaller improvement in performance. For the two systems which have been considered, it can also be noted that the plots of PIO vs. p could be well approximated by a negative exponential function.

It can also be concluded that the heuristic methods of choosing good controller structures are helpful. However, intuition and judgement are still necessary to decide how many trials should be made, in what proportions each method should be used, and to which linear control policy method one should be applied. Physical insight is of course always helpful.

It can also be observed that the number of iterations necessary to compute the optimal structured control policy increases as the number of gains decreases. This could be explained by the system behaviour, when the optimal structured control policy, $\overset{*}{\pi}^{S}$, is used, being increasingly different from the system behaviour when the optimal control policy, $\overset{*}{\pi}$, is used, thus producing sequences $[S_k]_0^{N-1}$ and $[V(k)]_0^N$ increasingly different from the sequences $[\overset{*}{S}_k]_0^{N-1}$ and $[\overset{*}{V}(k)]_0^N$. If structures are being evaluated by working down from a complex structure to a simpler one then optimal structured control policies for feedback structures of which the present structure is a substructure are available. The system behaviour associated with the $\overset{*}{\pi}^{S}$, of a feedback structure, containing only one (or a few) more gain(s) than the substructure, should be similar to the system behaviour associated with the $\overset{*}{\pi}^{S}$ of a substructure. If such a $\overset{*}{\pi}^{S}$ were used as $\pi(0)$ when the $\overset{*}{\pi}^{S}$ of the substructure was computed then, in all probability, computation time would be reduced.

CHAPTER 6

THE IMPORTANCE OF V_0 AND V_w

6.1. Introduction

The formulae used for the computation of the optimal structured control policy require a knowledge of V_0 and V_w. These matrices need not be known when frequency domain techniques, or the Linear Quadratic Optimal Approach are used to design a controller.

When using frequency domain design techniques deterministic models can be used for systems containing stochastic disturbances as these disturbances can be handled implicitly. The frequency response of the closed loop system is rolled off at a frequency well below the lowest frequency in the noise spectrum. The nature of the stochastic disturbances or even the band of frequencies they occupy need not be known precisely.

There is no means of handling disturbances implicitly when time domain design techniques are used. By happy circumstance the optimal control policy for a linear system with quadratic cost is optimal for any initial condition and any zero mean white noise disturbance. This is not true for an optimal structured control policy. To get the best possible performance one must have knowledge of the disturbance processes and initial states that are likely to occur. V_0 and V_w can be interpreted as weighting matrices for the initial conditions and disturbances respectively.

The need for information about V_0 and V_w could be considered a disadvantage of the design procedure proposed in this thesis. In this chapter the problem of how V_0 and V_w may be selected to produce acceptable control when the actual V_0 and V_w are unknown will be considered. Levine and Athans [9] have considered the problem of designing partial state feedback regulators for systems where the initial state is unknown. Their problem formulation will be shown to be closely related to the infinite time version of the structured control problem, defined by (1.18), if $V_w = I$. This will suggest $V_0 = V_w = I$ is a suitable choice for these matrices in the absence of other information.

The effect V_0 and V_w have on $\overset{*}{\pi}{}^{S}$ will be considered in an heuristic manner. This will further substantiate the reasonableness of the choice $V_0 = V_w = I$ if a robust controller is desired.

In the next section the numerical problems associated with singular V_0 and V_w will be considered. It will be suggested small diagonal terms be added to singular V_0 and V_w so as to make them positive definite. This should produce a more robust controller design as well as solving the numerical difficulties.

The arguments of sections 6.3 and 6.4 of this chapter are intuitive and are not mathematically justified. It is hoped that they will be helpful none the less.

6.2. Disturbances Uniformly Distributed Over a Sphere

Levine and Athans [9] have proposed computing partial state feedback controllers for continuous regulatory systems under the assumptions $V_0 = I$ and $V_w = 0$ where the partial state feedback matrix is constrained to be constant. The assumption $V_0 = I$ is equivalent to assuming the possible initial states are uniformly distributed over a sphere in R^n centred at the origin. As all possible initial state directions are weighted uniformly, the resulting control policy should produce acceptable control for any initial state. The computational evidence available [10 - 13] suggests this approach works well.

A result that will connect a discrete structured control problem of the Levine and Athans type with the infinite time version of the problem (1.18) will now be developed.

Theorem 12:

Let problems A and B be defined as

A) $$\min_{G^S} J' \tag{6.1}$$

where

$$J' = E[\tfrac{1}{2} \sum_{k=0}^{\infty} x_k^T Q x_k + u_k^T R u_k] \tag{6.2}$$

$$x_{k+1} = Ax_k + Bu_k \tag{6.3}$$

$$E[x_0 x_0^T] = V \tag{6.4}$$

and

$$u_k = G^S x_k \tag{6.5}$$

where G^S is a constant structured feedback matrix.

Assume, $\overset{*}{G}{}^S$, the solution to (6.1) exists and $[A+B\overset{*}{G}{}^S]$ is asymptotically stable.

B) $$\min_{\pi^S(N)} J(\pi^S(N)) \tag{6.6}$$

for N=1,2,3,...., where

$$J(\pi^S(N)) = E[\tfrac{1}{2} \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i + \tfrac{1}{2} x_N^T S_N x_N] \tag{6.7}$$

$$x_{k+1} = Ax_k + Bu_k + w_k \tag{1.1}$$

$$E[x_0 x_0^T] = V_0 \tag{1.2}$$

$$u_k = G_k^S x_k \qquad (1.19)$$

$$\pi^S(N) = [G_k^S,\ k=0,1,\ldots,N-1] \qquad (6.8)$$

and w_k is a zero mean white noise process with covariance V.

Let

$$\overset{*}{\pi}{}^S(N) = [G_k(N), k=0,1,2,\ldots,N-1] \qquad (6.9)$$

N=1,2,3,....., be the solutions to (6.6). If

$$\lim_{N\to\infty} G_k(N) \to \underline{G} \qquad (6.10)$$

for all k such that

$$\sqrt{N} < k < N - \sqrt{N} \qquad (6.11)$$

and $[A+B\underline{G}]$ is asymptotically stable then $\underline{G}$ is a solution to A.

<u>Proof</u>:

Consider problem A. As $w_k = 0$ implies $V_w = 0$ use of (2.16) produces

$$J' = \tfrac{1}{2}\mathrm{tr}[S_0 V] \qquad (6.12)$$

As $A+B\overset{*}{G}{}^S$ is assumed asymptotically stable no new restriction is added to problem A if it is assumed $A+BG^S$ is asymptotically stable. This assumption and (2.15) yield

$$S_0 = Q + G^{ST}RG^S + [A+BG^S]^T S_0 [A+BG^S] \qquad (6.13)$$

Substituting S for S_0 problem A becomes

$$\min_{G^S} \tfrac{1}{2}\mathrm{tr}[SV] \tag{6.14}$$

where

$$S = Q + G^{ST}RG^S + [A+BG^S]^T S[A + BG^S] \tag{6.15}$$

Define the set of finite time control policies

$$\pi(N) = [G_k = \overset{*}{G}{}^S, k=0,1,\ldots,N-1] \tag{6.16}$$

If this control policy is applied to the system of problem B the associated expected cost is by use of (2.15) and (2.16)

$$J(\pi(N)) = \tfrac{1}{2}\mathrm{tr}[S_0V_0] + \tfrac{1}{2}\sum_{k=1}^{N} \mathrm{tr}[S_kV] \tag{6.17}$$

where

$$S_k = Q + \overset{*}{G}{}^{ST}R\overset{*}{G}{}^S + [A + B\overset{*}{G}{}^S]^T S_{k+1}[A + B\overset{*}{G}{}^S] \tag{6.18}$$

for k=N-1,N-2,....,0

Consider the solutions to problem B, (6.9). By use of (2.15) and (2.16) the expected cost may be expressed

$$J(\overset{*}{\pi}{}^S(N)) = \tfrac{1}{2}\mathrm{tr}[\underline{S}_0V_0] + \tfrac{1}{2}\sum_{k=1}^{N} \mathrm{tr}[\underline{S}_kV] \tag{6.19}$$

where $\underline{S}_N = S_N$

and

$$\underline{S}_k = Q + G_k^T(N)RG_k(N) + [A + BG_k(N)]^T\underline{S}_{k+1}[A + BG_k(N)] \tag{6.20}$$

for k=N-1,N-2,....,0. Define

$$C = \lim_{N\to\infty} \frac{J(\pi(N)) - J(\overset{*}{\pi}{}^{S}(N))}{N} \tag{6.21}$$

As $\overset{*}{\pi}{}^{S}(N)$ is a solution to (6.6)

$$C \geqslant 0 \tag{6.22}$$

Define k_1 and k_2 to be the largest integers such that

$$k_1 \leqslant \sqrt{N} \tag{6.23}$$

and

$$k_2 \leqslant N - 2\sqrt{N} \tag{6.24}$$

For $N > 10$, $k_2 > k_1$ thus by use of (6.17) and (6.18) one may express C in the form

$$C = H_1 + H_2 + H_3 + H_4 \tag{6.25}$$

where

$$H_1 = \lim_{N\to\infty} \frac{tr[S_0V_0] - tr[\underline{S}_0V_0]}{2N} \tag{6.26}$$

$$H_2 = \lim_{N\to\infty} \frac{1}{2N} \sum_{k=1}^{k_1} tr[S_kV] - tr[\underline{S}_kV] \tag{6.27}$$

$$H_3 = \lim_{N\to\infty} \frac{1}{2N} \sum_{k=k_1+1}^{k_2} tr[S_kV] - tr[\underline{S}_kV] \tag{6.28}$$

and

$$H_4 = \lim_{N\to\infty} \frac{1}{2N} \sum_{k=k_2+1}^{N} tr[S_k V] - tr[\underline{S}_k V] \qquad (6.29)$$

As $[A + B\overset{*}{G}S]$ is asymptotically stable $tr[S_0 V_0]$ is finite, thus

$$\lim_{N\to\infty} \frac{tr[S_0 V_0]}{2N} = 0 \qquad (6.30)$$

The positive semidefiniteness of $\underline{S}_0$ and V_0 implies $tr[\underline{S}_0 V_0] \geqslant 0$ therefore

$$\lim_{N\to\infty} - \frac{tr[\underline{S}_0 V_0]}{2N} \leqslant 0 \qquad (6.31)$$

which implies that

$$H_1 \leqslant 0 \qquad (6.32)$$

As $[A + B\overset{*}{G}S]$ is asymptotically stable $tr[S_k V]$ is bounded above by some constant, K, thus

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{k=1}^{k_1} tr[S_k V] \leqslant \lim_{N\to\infty} \frac{\sqrt{N} K}{2N} = 0 \qquad (6.33)$$

and

$$\lim_{N\to\infty} \frac{1}{2N} \sum_{k=k_2+1}^{N} tr[S_k V] \leqslant \lim_{N\to\infty} \frac{2\sqrt{N} K}{2N} = 0 \qquad (6.34)$$

This with the positive semidefiniteness of $\underline{S}_k$ and V imply that

$$H_2 \leqslant 0 \qquad (6.35)$$

and

$$H_4 \leqslant 0 \tag{6.36}$$

As $(N-k_2) \to \infty$ as $N \to \infty$, (6.18) (6.24), and the asymptotic stability of $[A + B\overset{*}{G}{}^S]$ imply

$$S_k \to S' \quad \text{as} \quad N \to \infty \tag{6.37}$$

where $k \leqslant k_2$ and

$$S' = Q + \overset{*}{G}{}^{ST}R\overset{*}{G}{}^S + [A + B\overset{*}{G}{}^S]^T S'[A + B\overset{*}{G}{}^S] \tag{6.38}$$

(6.10), (6.20), (6.23), (6.24) and the asymptotic stability of $[A + B\underline{G}]$ yield

$$\underline{S}_k \to \underline{S} \quad \text{as} \quad N \to \infty \tag{6.39}$$

for $k_1 < k < k_2$ where

$$\underline{S} = Q + \underline{G}^T R\underline{G} + [A + B\underline{G}]^T \underline{S}[A + B\underline{G}] \tag{6.40}$$

By use of (6.37) and (6.39) one can deduce

$$H_3 = \tfrac{1}{2}(\text{tr}[S'V] - \text{tr}[\underline{S}V]) \tag{6.41}$$

As $\overset{*}{G}{}^S$ is known to be a solution to the problem defined by (6.14) and (6.15)

$$\tfrac{1}{2}\text{tr}[S'V] \leqslant \tfrac{1}{2}\text{tr}[\underline{S}V] \tag{6.42}$$

thus

$$H_3 \leqslant 0 \tag{6.43}$$

(6.22), (6.25), (6.32), (6.35), (6.36), and (6.43) imply

$$C = 0 \tag{6.44}$$

and

$$H_i = 0 \tag{6.45}$$

for i = 1,2,3,4. Thus (6.41) implies

$$\tfrac{1}{2}\mathrm{tr}[S'V] = \tfrac{1}{2}\mathrm{tr}[\underline{S}V] \tag{6.46}$$

$\underline{G}$ satisfies (6.15) which is immediate from (6.40). As $\overset{*}{G}{}^S$ is a solution to problem A by assumption and $\underline{G}$ produces the same expected cost, which is apparent from (6.46) and (6.14), $\underline{G}$ is a solution to problem A.

Q.E.D.

Corollary:

$$J(\pi(N)) \rightarrow J(\overset{*}{\pi}{}^S(N)) \quad \text{as} \quad N \rightarrow \infty \tag{6.47}$$

This follows directly from (6.44) and the definition of C, (6.21).

If $V_w = I$ and the solutions exhibit the required steady state behaviour then the steady state value of the structured state feedback matrix will be a solution to a Levine and Athans type problem. It is reasonable to expect such a solution will produce robust control. This suggests that the choice, $V_w = V_0 = I$, for the finite time problem will probably produce a robust control policy. This choice will be intuitively justified in the next section.

6.3. The Effect of V_0 and V_w on $\overset{*}{\pi}^S$

Examination of the equation (2.16) reveals that multiplication of both V_0 and V_w by any constant k will result in the scaling of the expected cost, J, by a constant k irrespective of the linear control policy used. Thus the structured control policy that minimizes J for a given V_0 and V_w will minimize the expected cost if $E[x_0 x_0^T] = kV_0$ and the noise covariance matrix is kV_w. Therefore one may conclude: the actual amplitude of the noise does not affect the optimal structured control policy, $\overset{*}{\pi}^S$.

It is reasonable to assume it is the cross correlation information contained in the sequence $[V(k)]_0^{N-1}$ that affects $\overset{*}{\pi}^S$. The complexity of the expressions makes a complete analysis very difficult. The validity of this assumption will be tested by comparing the optimal controls for the complete and the partial state feedback cases. Both the initial state and the disturbance will be assumed to have zero mean Gaussian probability distributions.

Combining equations (1.33) and (3.11) produces the optimal partial state feedback control,

$$u_k = - [R + B^T S_{k+1} B] B^T S_{k+1} [A^1 + A^2 V^T_{x'z'}(k) V^{-1}_{x'}(k)] x'_k \tag{6.48}$$

If x_0 and $[w_k]_0^{N-1}$ are uncorrelated zero mean Gaussian processes and a linear control policy is used then $[x_k]_0^N$ are zero mean Gaussian processes whose co-variance matrices $[V(k)]_0^N$ are given by (2.1). The conditional distribution of z'_k given x'_k has mean

$$E [z'_k / x'_k] = V^T_{x'z'}(k) V^{-1}_{z'}(k) x'_k \tag{6.49}$$

and covariance matrix [16, p.29]

$$V_{z'/x'}(k) = V_{z'}(k) - V^T_{x'z'}(k)V^{-1}_{z'}(k)V_{x'z'}(k) \qquad (6.50)$$

Substitution of (6.49) into (6.48) yields

$$u_k = -[R + B^T S_{k+1} B]^{-1} B^T S_{k+1} [A^1 x'_k + A^2 E[z'_k / x'_k]] \qquad (6.51)$$

By use of (1.5), (1.6), (1.35) and (3.5) the optimal control may be expressed as

$$u_k = -[R + B^T \overset{*}{S}_{k+1} B]^{-1} B^T \overset{*}{S}_{k+1} [A^1 x'_k + A^2 z'_k] \qquad (6.52)$$

As $\overset{*}{S}_{k+1}$ and S_{k+1} are both quadratic cost matrices dependent on $[G_i]_{k+1}^{N-1}$, the optimal partial state feedback control differs from the optimal control in z'_k being replaced by its expected value given x'_k the available information (Cumming [3]).

From (6.50) it is apparent that the larger are the terms of $V_{x'z'}(k)$, the smaller will be the terms of $V_{z'/x'}(k)$. (6.51) and (6.52) imply if $V_{z'/x'}(k)$ is small then the optimal partial state feedback control will be close to the optimal control, on average. From (2.1) it is apparent that the larger the elements of V_0^{12} and V_w^{12} are (defined by (3.26) and (3.27)), the larger will be the elements of the sequence $[V_{z'/x'}(k)]_0^{N-1}$ and the better will be the performance of the optimal partial state feedback control policy when compared to the optimal control policy.

Similarly from (6.50) it is apparent if $V_{z'}(k) \ll V_{x'}(k)$ then the terms of $V_{Z'/x'}(k)$ must be comparatively small and partial state feedback control would work well. From (2.1) it is apparent that the smaller V_0^{22} and V_w^{22} are, the smaller will be the terms in the sequence $[V_{z'}(k)]_0^{N-1}$ which implies the better will be the relative performance of the optimal partial state feedback control policy.

If $V_0 = V_w = I$ then all disturbance directions in R^n are equally likely to occur. If V_0^{12} and V_w^{12} are large or $V_0^{22} < V_0^{11}$ and $V_w^{22} < V_w^{11}$ then certain possible directions of disturbance are more likely to occur than others.

The optimal structured control policy will be tuned to handle best the disturbance directions which are most heavily weighed (have a high probability of occurring) by the given V_0 and V_w. As the behaviour of the system when subject to a disturbance along a direction which is lightly weighted has a small effect on the expected cost, the performance when such disturbances occur may be poor. If V_0 and V_w were to contain errors or drift so that a disturbance direction associated with a poor performance was given a larger probability of occurring the system performance would be degraded.

The assumption $V_0 = V_w = I$ gives equal weight to all disturbance directions thus the performance will be acceptable no matter what disturbance occurs. The control policy so calculated will give adequate control irrespective of the actual disturbance weightings. It is a safe assumption to make when V_0 and V_w are unknown.

6.4. Singular V_0 and V_w

The condition that V_0 and V_w be positive definite was only a part of a sufficient condition for F_k to be invertable. For some problems F_k will still be invertable and a unique solution for g_k will exist. However, singular F_k presents no great problem as solutions to (3.1) still exist. The solution

$$g_k = - F_k^+ h_k \tag{6.54}$$

where F_k^+ is the pseudo inverse of F_k, is of particular interest as it is the solution of minimum norm, $| g_k |$. If a singularity exists for all F_k on the optimal trajectory then the number of gains in the feedback structure may possibly be reduced without producing any increase in expected cost.

Singular V_0 and V_w mean that x_0 and $[w_k]_0^{N-1}$ are constrained to lie in certain subspaces of R^n. If unmodelled disturbances occur with components in the subspaces which were not weighted when π^{*S} was computed, poor behaviour may result. It is probably safer to add small quantities to the diagonal elements of V_0 and V_w and thus ensure all disturbance directions are given some probability of occurring. This will make V_0 and V_w positive definite. As V_0 and V_w are probably only known to one or two significant figures these added quantities could be of the order of significance and would not affect the accuracy of the solution. Such an addition would produce a control that would probably be more robust than the one produced using singular V_0 and V_w and would eliminate the possibility of numerical difficulties associated with singular F_k.

CHAPTER 7

THE USE OF STATE AND CONTROL AUGMENTATION TO STUDY MORE GENERAL FEEDBACK STRUCTURES

7.1. Introduction

Most problems which deal with linear quadratic systems disturbed by white noise that are to be controlled using a linear compensator of fixed structure can be transformed into a structured state feedback problem by suitable augmentation of the state and control vectors.

In this chapter a set of problems of increasing complexity will be transformed into structured state feedback problems concluding with a team theoretic type problem. Both instantaneous and delayed measurement equations will be considered. To avoid repetition the simpler problems will be discussed using only the instantaneous measurement equation.

The analysis will go no further than showing which A, B, Q, R, S_N, V_O, V_w and the structure on G_k^S that could be used. All these matrices will have special structures which should be used if the specific optimal controller is to be calculated efficiently.

The problem of how to choose the initial state of a dynamic compensator optimally will also be considered.

7.2. Feedback of Noise Corrupted Outputs

Consider the Output Feedback Problem defined in Section 1.2. It will be assumed that the measurement noise, v_k, is uncorrelated with the process noise, w_k. Some of the elements of K_k may be constrained to be zero.

Substitution of (1.1) into (1.36) produces

$$y_{k+1} = CAx_k + CBu_k + Cw_k + v_{k+1} \tag{7.1}$$

Define

$$\bar{x}_k = \begin{bmatrix} y_k \\ x_k \end{bmatrix} \qquad (7.2)$$

and

$$\bar{w}_k = \begin{bmatrix} Cw_k + v_{k+1} \\ w_k \end{bmatrix} \qquad (7.3)$$

As

$$E[w_k v_k^T] = 0 \qquad (7.4)$$

by assumption. $\bar{w}_k$ is a white noise process. (1.1), (7.1) and (7.2) imply

$$\bar{x}_{k+1} = \bar{A}\bar{x}_k + \bar{B}u_k + \bar{w}_k \qquad (7.5)$$

where

$$\bar{A} = \begin{bmatrix} 0 & CA \\ 0 & A \end{bmatrix} \qquad (7.6)$$

and

$$\bar{B} = \begin{bmatrix} CB \\ B \end{bmatrix} \qquad (7.7)$$

Define

$$\bar{V}_w = E[\bar{w}_k \bar{w}_k^T] \qquad (7.8)$$

Substitution of (7.3) and use of the definitions of V_w and V_v, combined with the fact that w_k and v_k are white noise processes, produces

$$\bar{V}_w = \begin{bmatrix} CV_wC^T + V_v & CV_w \\ V_wC^T & V_w \end{bmatrix} \tag{7.9}$$

Similarly define

$$\bar{V}_0 = E[\bar{x}_0\bar{x}_0^T] \tag{7.10}$$

Use of (1.36) and (7.2) produces

$$\bar{V}_0 = E\left[\begin{bmatrix} Cx_0 + v_0 \\ x_0 \end{bmatrix}\begin{bmatrix} x_0^TC^T + v_0^T & x_0^T \end{bmatrix}\right] \tag{7.11}$$

v_k is an independent white noise vector.
This and the definition of V_0, (1.2), imply

$$\bar{V}_0 = \begin{bmatrix} CV_0C^T + V_v & CV_0 \\ V_0C^T & V_v \end{bmatrix} \tag{7.12}$$

If one selects

$$\bar{R} = R \tag{7.13}$$

$$\bar{Q} = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} \tag{7.14}$$

and

$$\bar{S}_N = \begin{bmatrix} 0 & 0 \\ 0 & S_N \end{bmatrix} \tag{7.15}$$

then

$$\bar{L}_0 = \tfrac{1}{2} \sum_{i=0}^{N-1} [\bar{x}_i^T Q \bar{x}_i + \bar{u}_i^T R u_i] + \tfrac{1}{2}\bar{x}_N^T \bar{S}_N \bar{x}_N \tag{7.16}$$

$$= \tfrac{1}{2} \sum_{i=0}^{N-1} [x_i^T Q x_i + u_i^T R u_i] + \tfrac{1}{2} x_N^T S_N x_N$$

$$= L_0 \tag{7.17}$$

as desired. (1.37) may be rewritten

$$u_k = \bar{G}_k^S \bar{x}_k \tag{7.18}$$

where

$$\bar{G}_k^S = [K_k \quad 0] \tag{7.19}$$

The computational procedures of Chapter 4 can be used to solve the structured control problem defined by (7.5), (7.16), and (7.18) which will yield the sequence $[K_k]_0^{N-1}$ that is the solution to the original output feedback problem.

In the conventional output feedback problem no elements of K_k are constrained to be zero. This problem has been transformed into a partial state feedback problem. In Section 1.1 it was established that partial state feedback problems can be posed as output feedback problems. Therefore these two problems are equivalent.

7.3. Feedback From a Predefined Dynamic System

Consider a linear system defined by (1.1) and (1.36) with associated quadratic cost (1.3). The output is fed into a compensator defined by

$$z_{k+1} = Dz_k + Ey_k \tag{7.20}$$

and the control input is constrained to be of the form

$$u_k = H_k z_k + K_k y_k \tag{7.21}$$

where some of the elements of H_k and K_k may be constrained to be zero. The sequences $[H_k]_0^{N-1}$ and $[K_k]_0^{N-1}$ are to be chosen to minimize J.

The case $K_k = 0$ is a special case of the more general problem. However, if this case is considered directly, an augmented linear system of lower dimension can be used. This simpler case will be dealt with first.

Case $K_k = 0$

(7.21) now becomes

$$u_k = H_k z_k \tag{7.22}$$

It will be assumed that C is of full rank and the state has been suitably transformed so that

$$y_k = Cx_k = x_k' \tag{1.39}$$

where x_k' is defined by (1.35). (1.35), (1.36), (1.39), and (7.20) yield

$$z_{k+1} = Dz_k + [E \quad 0]x_k + Ev_k \tag{7.23}$$

Define

$$\bar{x}_k = \begin{bmatrix} z_k \\ x_k \end{bmatrix}. \tag{7.24}$$

and

$$\bar{w}_k = \begin{bmatrix} Ev_k \\ w_k \end{bmatrix} \tag{7.25}$$

$\bar{w}_k$ is a white noise process even if v_k and w_k are correlated. Define

$$V_{vw} = E[v_k w_k^T] \tag{7.26}$$

By combining (1.1) and (7.23) one can write

$$\bar{x}_{k+1} = \overline{Ax}_k + \bar{B}u_k + \bar{w}_k \tag{7.27}$$

where

$$\bar{A} = \left[\begin{array}{c:c:c} D & E & 0 \\ \hdashline 0 & \multicolumn{2}{c}{A} \end{array}\right] \tag{7.28}$$

and

$$\bar{B} = \begin{bmatrix} 0 \\ B \end{bmatrix} \tag{7.29}$$

Define

$$\bar{V}_w = E[\bar{w}_k \bar{w}_k^T] \tag{7.30}$$

(7.25), (7.26), and the definitions of V_v and V_w yield

$$\bar{V}_w = \begin{bmatrix} EV_vE^T & EV_{vw} \\ V_{vw}^T E^T & V_w \end{bmatrix} \tag{7.31}$$

Again define

$$\bar{V}_0 = E[\bar{x}_0\bar{x}_0^T] \tag{7.32}$$

Let

$$V_z(0) = E[z_0 z_0^T] \tag{7.33}$$

and

$$V_{zx}(0) = E[z_0 x_0^T] \tag{7.34}$$

(1.2), (7.24), (7.33), and (7.34) allow (7.32) to be rewritten

$$\bar{V}_0 = \begin{bmatrix} V_z(0) & V_{zx}(0) \\ V_{zx}^T(0) & V_0 \end{bmatrix} \tag{7.35}$$

$V_z(0)$ and $V_{zx}(0)$ may either be specified by the problem definition or be treated as parameters that are to be chosen optimally. The latter possibility will be dealt with when the transformation to a structured state feedback problem has been completed.

The quadratic cost associated with the linear system (7.27),

$$\bar{L}_0 = \tfrac{1}{2} \sum_{k=0}^{N-1} [\bar{x}_k^T \bar{Q} \bar{x}_k + u_k^T \bar{R} u_k] + \tfrac{1}{2} \bar{x}_N^T \bar{S}_N \bar{x}_N \tag{7.36}$$

satisfies

$$\bar{L}_0 = L_0 \tag{7.37}$$

when

$$\bar{R} = R \tag{7.38}$$

$$\bar{Q} = \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} \tag{7.39}$$

and

$$\bar{S}_N = \begin{bmatrix} 0 & 0 \\ 0 & S_N \end{bmatrix} \quad (7.40)$$

The combination of (7.22) and (7.24) produces

$$u_k = \bar{G}_k^S \bar{x}_k \quad (7.41)$$

where

$$\bar{G}_k^S = [H_k \quad 0] \quad (7.42)$$

defines the feedback structure to be used. (7.27), (7.36), (7.41) define a structured state feedback problem whose solution will yield the solution to the original problem defined by (1.1), (1.4), (1.36), (7.20) and (7.22).

The Possible Design Parameters $V_z(0)$ and $V_{zx}(0)$

The construction of the physical system (7.20) models will determine whether $V_z(0)$ and $V_{zx}(0)$ are fixed or can be assigned values so as to maximize performance. Their values are determined by z_0 the initial state of the compensator system. Assume z_0 is a stochastic process such that

$$z_0 = \underline{z} + \tilde{z} \quad (7.43)$$

where

$$\underline{z} = E[z_0] \quad (7.44)$$

and $\tilde{z}$ is a zero mean stochastic process with a covariance matrix

$$V_{\tilde{z}} = E[\tilde{z}\tilde{z}^T] \quad (7.45)$$

(7.33), (7.43), (7.44) and (7.45) imply

$$V_z(0) = \underline{z}\underline{z}^T + V_{\tilde{z}} \tag{7.46}$$

Theorem 13

If the sequence of Feedback Matrices $[H_k]_0^{N-1}$ is used to control the linear system defined by (1.1), (1.36), (7.20), and (7.22), the selection of z_0 such that

$$\tilde{z} = 0 \tag{7.47}$$

and

$$S_z(0)z_0 = - S_{zx}(0)\underline{x} \tag{7.48}$$

will minimize J, defined by (1.4), where

$$\underline{x} = E[x_0] \tag{7.49}$$

$\overline{S}_0$ is the cost matrix defined for the augmented linear system (7.27) by (2.15).

$$\overline{S}_0 = \begin{bmatrix} S_z(0) & S_{zx}(0) \\ S_{zx}^T(0) & S_0 \end{bmatrix} \tag{7.50}$$

where the partitioning is that implied by (7.24).

Corollary:

If x_0 is zero mean then $z_0 = 0$ will minimize J.

Proof:

From (2.13) it is apparent that $\overline{x}_0$, and thus z_0, only affects the term $\frac{1}{2}E[\overline{x}_0^T\overline{S}_0\overline{x}_0]$ of the expected cost. Substitution of (7.24) and (7.50) into this term produces

$$\tfrac{1}{2}E[\bar{x}_0^T \bar{S}_0 \bar{x}_0] = \tfrac{1}{2}E[z_0^T S_z(0) z_0 + 2z_0^T S_{zx}(0) x_0 + x_0^T S_0 x_0] \tag{7.51}$$

Let

$$x_0 = \underline{x} + \tilde{x} \tag{7.52}$$

where $\underline{x}$ is defined by (7.49) and $\tilde{x}$ is a zero mean stochastic process.

Substitution of (7.43) and (7.52) into (7.51) plus the expansion of terms yield

$$\begin{aligned}\tfrac{1}{2}E[\bar{x}_0^T \bar{S}_0 \bar{x}_0] = \tfrac{1}{2}E[&\underline{z}^T S_z(0)\underline{z} + 2\underline{z}^T S_z(0)\tilde{z} + \tilde{z}^T S_z(0)\tilde{z} \\ &+ 2\underline{z}^T S_{zx}(0)\underline{x} + 2\underline{z}^T S_{zx}(0)\tilde{x} + 2\tilde{z}^T S_{zx}(0)\underline{x} \\ &+ 2\tilde{z}^T S_{zx}(0)\tilde{x} + \underline{x}^T S_0 \underline{x} + 2\tilde{x}^T S_0 \underline{x} + \tilde{x}^T S_0 \tilde{x}]\end{aligned} \tag{7.53}$$

$\tilde{x}$ is a zero mean stochastic process and there is no physical means by which $\tilde{z}$ can be forced to be correlated with it. Thus

$$E[\tilde{z}^T S_{zx}(0)\tilde{x}] = 0 \tag{7.54}$$

By use of this and the fact both $\tilde{x}$ and $\tilde{z}$ are zero mean (7.53) becomes

$$\begin{aligned}\tfrac{1}{2}E[\bar{x}_0^T \bar{S}_0 \bar{x}_0] = &\tfrac{1}{2}\underline{z}^T S_z(0)\underline{z} + \tfrac{1}{2}E[\tilde{z}^T S_z(0)\tilde{z}] + \underline{z}^T S_{zx}(0)\underline{x} + \tfrac{1}{2}\underline{x}^T S_0 \underline{x} \\ &+ \tfrac{1}{2}E[\tilde{x}^T S_0 \tilde{x}]\end{aligned} \tag{7.55}$$

As $\tilde{z}$ only affects the term $\tfrac{1}{2}E[\tilde{z}^T S_z(0)\tilde{z}]$ of the expected cost J and $S_z(0)$ is positive semidefinite J will be minimized if one sets

$$\tilde{z} = 0 \qquad (7.47)$$

$\underline{z}$ only affects the term (7.55) of J which is a positive semidefinite quadratic form in $\underline{z}$, therefore the value of $\underline{z}$ which minimizes J can be found by the differentiation of J by $\underline{z}$ and the setting of the derivative to zero.

$$\frac{\partial J}{\partial \underline{z}} = \underline{z}^T S_z(0) + \underline{x} S_{zx}^T(0) = 0 \qquad (7.56)$$

(7.43) and (7.47) imply

$$z_0 = \underline{z} \qquad (7.57)$$

When this is substituted in to into (7.56) and the transpose is taken

$$S_z(0) z_0 = - S_{zx}(0)\underline{x} \qquad (7.48)$$

results.

Q.E.D.

If $S_z(0)$ is invertable, (7.48) yields

$$z_0 = - S_z^{-1}(0) S_{zx}(0)\underline{x} \qquad (7.58)$$

Substitution of (7.49) and (7.58) into (7.33) and (7.34) yields

$$V_z(0) = S_z^{-1}(0) S_{zx}(0) \underline{xx}^T S_{zx}(0) S_z^{-1}(0) \qquad (7.59)$$

$$V_{zx}(0) = - S_z^{-1}(0) S_{zx}(0) \underline{xx}^T \qquad (7.60)$$

Note the optimal choice of $V_z(0)$ and $V_{zx}(0)$ depends on $\bar{S}_0$ and thus on the Structured Control policy. However, the optimal structured control policy depends on $\bar{V}_0$ whose value is determined by the choice of $V_z(0)$ and $V_{zx}(0)$.

If $\underline{x} = 0$ then the optimal choice is $V_z(0) = 0$ and $V_{zx}(0) = 0$ irrespective of the value of $[H_k]_0^{N-1}$. If $\underline{x} \neq 0$ then an analytic solution to the problem of choosing $V_z(0)$, $V_{zx}(0)$, and $[H_k]_0^{N-1}$ optimally does not appear possible. However, a computational procedure can be constructed for selecting $V_z(0)$, $V_{zx}(0)$, and $[H_k]_0^{N-1}$ which will show an improvement in cost at each step.

1. Select an initial $V_z(0)$ and $V_{zx}(0)$
2. Use one of the computational procedures of Section 4.2 to evaluate the optimal structured control policy for (7.27)
3. Use (7.48) to compute z_0. Compute an new $V_z(0)$ and $V_{zx}(0)$.
4. Go to 2.

As J is bounded below by zero, convergence in cost must occur.

$\tilde{z} = 0$ was selected so as to give the smallest quadratic cost rather than to produce a control that would be robust. The addition to $V_z(0)$ of a small term of the form ϵI might give the resulting control strategy a better performance if small unmodelled disturbances should occur. This does not imply however, that there should be any attempt to insert a small noise vector into the actual compensator.

$K_k \neq 0$ and $H_k \neq 0$

It will now be assumed that the control, u_k, is constrained to satisfy (7.21). To handle this case it will be necessary to include both z_k and y_k in the augmented state vector,

$$\bar{x}_k = \begin{bmatrix} z_k \\ y_k \\ x_k \end{bmatrix} \qquad (7.61)$$

The combination of equations (1.1), (7.1), (7.21) and (7.61) produces

$$\bar{x}_{k+1} = \overline{Ax}_k + \bar{B}u_k + \bar{w}_k \qquad (7.62)$$

where

$$\bar{A} = \begin{bmatrix} D & E & 0 \\ 0 & 0 & CA \\ 0 & 0 & A \end{bmatrix} \qquad (7.63)$$

$$\bar{B} = \begin{bmatrix} 0 \\ CB \\ B \end{bmatrix} \qquad (7.64)$$

and

$$\bar{w}_k = \begin{bmatrix} 0 \\ Cw_k + v_{k+1} \\ w_k \end{bmatrix} \qquad (7.65)$$

It is again necessary to restrict w_k and v_k to be uncorrelated so that $\bar{w}_k$ is a white zero mean stochastic process.

By definition

$$\bar{V}_w = E[\bar{w}_k \bar{w}_k^T] \qquad (7.66)$$

Substitution of (7.65) into (7.66) and use of the facts v_k and w_k are zero mean white noise processes with covariance matrices V_v and V_w

respectively produce

$$\overline{V}_w = \begin{bmatrix} 0 & 0 & 0 \\ 0 & CV_wC^T+V_v & CV_w \\ 0 & V_wC^T & V_w \end{bmatrix} \tag{7.67}$$

To avoid degredation of controller performance should there be small unmodelled noises in the compensator, it might be best to insert ϵI in place of the top left hand zero matrix in $\overline{V}_w$.

$$\overline{V}_0 = E[\overline{x}_0\overline{x}_0^T] \tag{7.68}$$

by definition. From (1.36), the definitions of V_v, V_0 (1.2), and $V_{zx}(0)$, (7.34), and the fact v_k is a white noise process and is assumed uncorrelated with state, it follows that

$$E[z_0y_0^T] = V_{zx}(0)C^T \tag{7.69}$$

$$E[y_0y_0^T] = CV_0C^T + V_v \tag{7.70}$$

and

$$E[y_0x_0^T] = CV_0 \tag{7.71}$$

Then (1.2), (7.33), (7.34), (7.61), (7.68), (7.69), (7.70) and (7.71) yield

$$\overline{V}_0 = \begin{bmatrix} V_z(0) & V_{zx}(0)C^T & V_{zx}(0) \\ CV_{zx}^T(0) & CV_0C^T + V_w & CV_0 \\ V_{zx}^T(0) & V_0C^T & V_0 \end{bmatrix} \tag{7.72}$$

Again a similar analysis to that of Theorem 13 can be made if the initial state of the compensator z_0 can be chosen freely.

If

$$\bar{R} = R \tag{7.73}$$

$$\bar{Q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Q \end{bmatrix} \tag{7.74}$$

and

$$\bar{S}_N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & S_N \end{bmatrix} \tag{7.75}$$

then

$$\bar{L}_0 = \tfrac{1}{2}\left(\sum_{k=0}^{N-1} [\bar{x}_k^T \bar{Q} \bar{x}_k + u_k^T \bar{R} u_k] + \bar{x}_N^T \bar{S}_N \bar{x}_N \right) \tag{7.76}$$

$$= L_0 \tag{7.77}$$

Equation (7.21) may be rewritten

$$u_k = \bar{G}_k^S \bar{x}_k \tag{7.78}$$

where

$$\bar{G}_k^S = [H_k \,\vdots\, K_k \,\vdots\, 0] \tag{7.79}$$

Thus the solution of the structured control problem defined by (7.62), (7.76), and (7.78) will yield the sequences of matrices $[H_k]_0^{N-1}$ and $[K_k]_0^{N-1}$ which will minimize J for the linear system and compensator defined by (1.1), (1.36), (7.20) and (7.21).

However, it may be desirable to modify $\overline{Q}$ so that the compensator states z_k are costed to ensure they remain within certain acceptable levels.

7.4 Compensators of Fixed Structure in which Some of the Parameters may be "Tuned"

The matrices D and E in the compensator (7.20) will now be assumed to contain predetermined elements (constant or time varying) and other elements which are to be chosen so as to minimize J. Thus

$$D = D^f + D_k^v \tag{7.80}$$

and

$$E = E^f + E_k^v \tag{7.81}$$

where D^f and E^f are the matrices containing all the predetermined elements. If there are no predetermined elements then D^f and E^f equal zero. D_k^v and E_k^v are matrices containing zeros and the elements of D and E respectively which are to be chosen optimally.

By defining $\overline{x}_k$ as in (7.61) and using (7.80) and (7.81), (7.20) can be rewritten

$$z_{k+1} = [D^f \quad E^f \quad 0]\overline{x}_k + I[D_k^v \quad E_k^v \quad 0]\overline{x}_k \tag{7.82}$$

Define

$$u_k' = [D_k^v \quad E_k^v \quad 0]\overline{x}_k \tag{7.83}$$

and

$$\overline{u}_k = \begin{bmatrix} u_k' \\ u_k \end{bmatrix} \tag{7.84}$$

If $\bar{w}_k$ is defined by (7.65), equations (1.1), (7.1), and (7.82) may be combined to yield

$$\bar{x}_{k+1} = \overline{Ax}_k + \overline{Bu}_k + \bar{w}_k \tag{7.85}$$

where

$$\bar{A} = \begin{bmatrix} D^f & E^f & 0 \\ 0 & 0 & CA \\ 0 & 0 & A \end{bmatrix} \tag{7.86}$$

and

$$\bar{B} = \begin{bmatrix} I & 0 \\ 0 & CB \\ 0 & B \end{bmatrix} \tag{7.87}$$

As $\bar{x}_k$ and $\bar{w}_k$ are defined by (7.61) and (7.65) respectively $\bar{V}_0$ and $\bar{V}_w$ are given by (7.72) and (7.67) respectively and the remarks made following these equations still apply. If

$$\bar{R} = \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} \tag{7.88}$$

and $\bar{Q}$ and $\bar{S}_N$ are defined by (7.74) and (7.75) respectively then

$$\bar{L}_0 = \frac{1}{2} \sum_{k=0}^{N-1} [\bar{x}_k^T \bar{Q} \bar{x}_k + \bar{u}_k^T \bar{R} \bar{u}_k] + \frac{1}{2} \bar{x}_N^T \bar{S}_N \bar{x}_N \tag{7.89}$$

$$= L_0 \tag{7.90}$$

Equations (7.21), (7.61), (7.83), and (7.84) may be combined to yield

$$\bar{u}_k = \bar{G}_k^S \bar{x}_k \tag{7.91}$$

where

$$\bar{G}_k^S = \begin{bmatrix} D_k^v & E_k^v & 0 \\ H_k & K_k & 0 \end{bmatrix} \tag{7.92}$$

Thus equations (7.85), (7.89) and (7.91) define a structured control problem whose solution will give the sequences of matrices $[D_k^v]_0^{N-1}$, $[E_k^v]_0^{N-1}$, $[H_k]_0^{N-1}$, and $[K_k]_0^{N-1}$ which will minimize the expected cost (1.4) for the system defined by (1.1), (1.36), (7.20), (7.21), (7.80) and (7.81).

If no structure is imposed on the matrices D_k^v, E_k^v, H_k, and K_k the linear system defined by (7.20) and (7.21) will have more free parameters than the transfer function: thus the optimal solution will not be unique. If the original cost function (1.4) is retained and a unique solution is desired then the linear system equations (7.20) and (7.21) must be constrained to be in some suitable canonical form. However, even if D_k^v, E_k^v, H_k, and K_k are constrained to satisfy some canonical form either the gains in D_k^v or E_k^v or the values of the compensator state z_k may become unacceptably large unless $\bar{R}$ and $\bar{Q}$ are modified so that u_k' and z_k are costed.

Delayed Measurement

If the output equation should be of the form

$$y_k = Cx_{k-1} + v_k \tag{7.93}$$

rather than the form of (1.36), then (7.93) should replace (7.1) in the manipulations of the preceding development.

With $\bar{x}_k$ defined by (7.61) and $\bar{u}_k$ defined by (7.84), equations (1.1), (7.93), and (7.82) may be combined to yield

$$\bar{x}_{k+1} = \overline{A}\bar{x}_k + \overline{B}\bar{u}_k + \bar{w}_k \tag{7.94}$$

where

$$\overline{A} = \begin{bmatrix} D^f & E^f & 0 \\ 0 & 0 & C \\ 0 & 0 & A \end{bmatrix} \tag{7.95}$$

$$\overline{B} = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & B \end{bmatrix} \tag{7.96}$$

and

$$\bar{w}_k = \begin{bmatrix} 0 \\ v_k \\ w_k \end{bmatrix} \tag{7.97}$$

$\bar{w}_k$ is a zero mean white noise process as v_k and w_k are. v_k and w_k may be correlated. By definition

$$\overline{V}_w = E[\bar{w}_k \bar{w}_k^T] \tag{7.98}$$

By combining (7.26), (7.97), (7.98), and the definitions of V_v and V_w one gets

$$\overline{V}_w = \begin{bmatrix} 0 & 0 & 0 \\ 0 & V_v & V_{vw} \\ 0 & V_{vw} & V_w \end{bmatrix} \tag{7.99}$$

Define

$$\overline{V}_0 = E[\overline{x}_0\overline{x}_0^T] \quad (7.100)$$

$$V_{zy}(0) = E[z_0y_0^T] \quad (7.101)$$

$$V_y(0) = E[z_0y_0^T] \quad (7.102)$$

and

$$V_{yx}(0) = E[y_0x_0^T] \quad (7.103)$$

The combination of these and the definitions of $V_z(0)$, (7.33), $V_{zx}(0)$, (7.34), and V_0, (1.2), produces

$$\overline{V}_0 = \begin{bmatrix} V_z(0) & V_{zy}(0) & V_{zx}(0) \\ V_{zy}^T(0) & V_y(0) & V_{yx}(0) \\ V_{zx}^T(0) & V_{yx}^T(0) & V_0 \end{bmatrix} \quad (7.104)$$

One can now proceed as in the instantaneous measurement case (i.e. output equation (1.36) from equation (7.88) through the discussion that follows (7.92). The remarks regarding a robust controller and the choice of the initial state of the compensator are again appropriate.

7.5. Team Theoretic Problems

Assume the control vector, u_k, is broken into a set of ℓ subvectors, U_k^ℓ, of dimension M_i termed subcontrols, where $i = 1,2,\ldots,\ell$.

$$u_k = \begin{bmatrix} U_k^1 \\ U_k^2 \\ \vdots \\ U_k^\ell \end{bmatrix} \quad (7.105)$$

where

$$\sum_{i=1}^{\ell} M_i = m \qquad (7.106)$$

Each subcontrol (i) has a set of measurements, y_k^i, available to it. The value of each sub control is determined by the output of a linear system of fixed structure, termed the subcontroller, into which the available measurements are fed. The free parameters of the subcontrollers are to be so selected that the expected cost (1.4) for the linear system (1.1) is minimized. This is termed a team theoretic problem as each subcontrol can be viewed as a player in a team.

Let

$$y_k^i = C^i x_k + v_k^i \qquad (7.107)$$

where y_k^i is an r_i-vector, and v_k^i is an r_i-vector of zero mean white noise processes with covariance matrix V_v^i, v_k^i is uncorrelated with w_k. Subcontroller i is defined by

$$z_k^i = D_k^i z_k^i + E_k^i y_k^i \qquad (7.108)$$

$$U_k^i = H_k^i z_k^i + K_k^i y_k^i \qquad (7.109)$$

where

$$D_k^i = D_f^i + D_v^i(k) \qquad (7.110)$$

and

$$E_k^i = E_f^i + E_v^i(k) \qquad (7.111)$$

D_f^i and E_f^i are fixed, possibly time varying, matrices. H_k^i, K_k^i, $D_v^i(k)$, and $E_v^i(k)$ are structured matrices containing the parameters of the subcontroller that are to be selected to minimize J, (1.4).

If one now defined

$$z_k = \begin{bmatrix} z_k^1 \\ z_k^2 \\ \vdots \\ z_k^{\ell} \end{bmatrix} \tag{7.112}$$

$$y_k = \begin{bmatrix} y_k^1 \\ y_k^2 \\ \vdots \\ y_k^{\ell} \end{bmatrix} \tag{7.113}$$

and

$$v_k = \begin{bmatrix} v_k^1 \\ v_k^2 \\ \vdots \\ v_k^{\ell} \end{bmatrix} \tag{7.114}$$

by use of (7.107) one can obtain

$$y_k = Cx_k + v_k \tag{1.36}$$

where

$$C = \begin{bmatrix} C^1 \\ C^2 \\ \vdots \\ C^{\ell} \end{bmatrix} \tag{7.115}$$

V_v the covariance matrix of v_k can be expressed

$$V_v = \begin{bmatrix} V_v^1 & V_v^{12} & \cdots\cdots & V_v^{1\ell} \\ (V_v^{12})^T & V_v^2 & \cdots\cdots & V_v^{2\ell} \\ \cdots & \cdots & \cdots & \cdots \\ (V_v^{1\ell})^T & (V_v^{2\ell})^T & \cdots\cdots & V_v^{\ell} \end{bmatrix} \quad (7.116)$$

where

$$V_v^{ij} = E[v_k^i v_k^{jT}] \quad (7.117)$$

By use of (7.108) one obtains

$$z_{k+1} = Dz_k + Ey_k \quad (7.20)$$

where

$$D = \begin{bmatrix} D_k^1 & 0 & \cdots\cdots & 0 \\ 0 & D_k^2 & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & D_k^{\ell} \end{bmatrix} \quad (7.118)$$

and

$$E = \begin{bmatrix} E_k^1 & 0 & \cdots\cdots & 0 \\ 0 & E_k^2 & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & E_k^{\ell} \end{bmatrix} \quad (7.119)$$

One gets by use of (7.110)

$$D = D^f + D_k^v \quad (7.80)$$

where

$$D^f = \begin{bmatrix} D_f^1 & 0 & \cdots\cdots & 0 \\ 0 & D_f^2 & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & D_f^{\ell} \end{bmatrix} \qquad (7.120)$$

and

$$D_k^v = \begin{bmatrix} D_v^1(k) & 0 & \cdots\cdots & 0 \\ 0 & D_v^2(k) & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & D_v^{\ell}(k) \end{bmatrix} \qquad (7.121)$$

(7.111) may be used to obtain

$$E = E^f + E_k^v \qquad (7.81)$$

where

$$E^f = \begin{bmatrix} E_f^1 & 0 & \cdots\cdots & 0 \\ 0 & E_f^2 & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & E_f^{\ell} \end{bmatrix} \qquad (7.122)$$

and

$$E_k^v = \begin{bmatrix} E_v^1(k) & 0 & \cdots\cdots & 0 \\ 0 & E_v^2(k) & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & E_v^{\ell}(k) \end{bmatrix} \qquad (7.123)$$

By use of (7.109) one obtains

$$u_k = H_k z_k + K_k y_k \tag{7.21}$$

where

$$H_k = \begin{bmatrix} H_k^1 & 0 & \cdots\cdots & 0 \\ 0 & H_k^2 & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & H_k^\ell \end{bmatrix} \tag{7.124}$$

and

$$K_k \quad \begin{bmatrix} K_k^1 & 0 & \cdots\cdots & 0 \\ 0 & K_k^2 & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & K_k^\ell \end{bmatrix} \tag{7.125}$$

The team theoretic problem has now been transformed into a problem with a compensator of fixed structure in which some of the parameters may be tuned. This problem was dealt with in section 7.4.

If equation (7.107) should be replaced by an equation of delayed measurement form, the problem can be transformed, in exactly the same manner, to the delayed measurement form of the problem dealt with in section 7.4.

CHAPTER 8

PROBLEMS FOR FURTHER RESEARCH AND A SUMMARY OF RESULTS

8.1. Problems for Further Research

The topic of Specific Optimal Linear Controls for Linear Quadratic Systems is not yet fully developed and many interesting problems remain. Two of the problems, which the results of this thesis suggest, are the computation of constant controllers for infinite time, time invariant, systems, and the problem of choosing a "good" or the "best" feedback structure of a given level of complexity.

A) The Infinite Time Problem

The results of Chapter 5 suggest that if the system is time invariant then the controller gains settle to steady state values in the centre of a long time interval in many cases. This concept is expressed precisely by (6.10) and (6.11). Many long term regulatory problems exist where the controller can be implemented easily and cheaply if the feedback gains are constant.

The conditions under which the steady state property, stated in (6.10) and (6.11), occurs remain to be established. An example in Chapter 5 implied that the condition that $[A+BG^S]$ can be made stable is not sufficient to guarantee the steady state property will hold.

It would also be useful to determine the conditions under which the infinite time problem has a solution which produces a finite cost per interval. The conditions where the Levine and Athans type problem has a solution but the property (6.10) and (6.11) does not hold would also be of interest.

If it is assumed (6.10) and (6.11) hold, use of the finite time algorithms to solve steady state problems would still be wasteful of computer time and store. The following algorithm has been used successfully and produced results that were similar to those produced by the finite time computational procedure. It seems to be quick and requires little store.

A Computational Procedure for Computing the Steady State Structured Feedback Matrix

1. $V \leftarrow V_0, \quad S \leftarrow S_0$

2. $$F \leftarrow \begin{bmatrix} r_{11}V_{11} & r_{12}V_{12} \cdots\cdots & r_{1m}V_{1m} \\ r_{12}V_{12}^T & r_{22}V_{22} \cdots\cdots & r_{2m}V_{2m} \\ \cdots & \cdots & \cdots \\ r_{1m}V_{1m}^T & r_{2m}V_{2m}^T \cdots\cdots & r_{mm}V_{mm} \end{bmatrix}$$

where

$$r_{ij} = [R + B^TSB]$$

and

$$V_{ij} = \begin{bmatrix} \sigma(\psi(i,1),\ \psi(j,1)) & \sigma(\psi(i,1),\psi(j,2)) \ldots & \sigma(\psi(i,1),\psi(j,n_j)) \\ \sigma(\psi(i,2),\ \psi(j,1)) & \sigma(\psi(i,2),\psi(j,2)) \ldots & \sigma(\psi(i,2),\psi(j,n_j)) \\ \cdots & \cdots & \cdots \\ \sigma(\psi(i,n_i),\psi(j,1)) & \sigma(\psi(i,n_i),\psi(j,2)) \ldots & \sigma(\psi(i,n_i),\psi(j,n_j)) \end{bmatrix}$$

where

$$\left\{\sigma(i,j)\right\} = V$$

3. $$h \leftarrow \begin{bmatrix} V_1A^TSb_1 \\ V_2A^TSb_2 \\ \vdots \\ V_mA^TSb_m \end{bmatrix}$$

where

$$V_i = \begin{bmatrix} \sigma(\psi(i,1),1) & \sigma(\psi(i,1),2) & \cdots\cdots & \sigma(\psi(i,1),n) \\ \sigma(\psi(i,2),1) & \sigma(\psi(i,2),2) & \cdots\cdots & \sigma(\psi(i,2),n) \\ \cdots & \cdots & \cdots & \cdots \\ \sigma(\psi(i,n_i),1) & \sigma(\psi(i,n_i),2) & \cdots\cdots & \sigma(\psi(i,n_i),n) \end{bmatrix}$$

4. $G^S \leftarrow$ the structured feedback matrix with unconstrained gains, g, obtained by solving

$$Fg = h$$

5. $V \leftarrow [A + BG^S]V[A + BG^S]^T + V_w$

6. $S \leftarrow Q + G^{ST}RG^S + [A + BG^S]^T S[A + BG^S]$

7. Go to 2

The convergence properties and the nature of the G^S to which this computational procedure converges remain to be established. The computational efficiency of this procedure should be compared with that of the direct parameter optimization techniques and the Cumming Algorithm [3]. The Cummings Algorithm was developed for the output feedback problem but the replacement of the output feedback necessary condition by (3.1) would allow it to be used on structured control problems.

B) <u>Choice of Feedback Structure</u>

Use of the heuristic structure selection methods, discussed in Chapter 5, requires many trials and therefore a lot of computing time to find the "best" structure for a given number of feedback gains. It would be useful to have a more direct method of determining whether a given structure was good or bad.

This would probably require an understanding of the controllability properties of a feedback structure. A method for determining whether a system could be made stable using a given feedback structure would also be useful. Some method of relating controllability to specific disturbance and cost structures would be necessary.

8.2. Summary of Results

A problem of specific optimal control, the optimal choice of a Structured Control Policy has been treated in this thesis. Certain basic properties of linear systems controlled by linear state feedback were derived and used to establish a rule for choosing one structured feedback matrix optimally.

This rule was then used in a computational procedure for determining the Optimal Structured Control Policy. The convergence properties of the computational procedure were evaluated. It was found that not all structured control policies that may be produced are optimal. A good starting point is necessary to ensure convergence to the optimal and methods of selecting suitable starting points were discussed.

Computational results were obtained for two systems which demonstrated that linear systems of a reasonable size can be handled. These results also indicated that a rule of decreasing marginal returns with increasing controller complexity applies.

The problem of how to make a suitable choice of V_0 and V_w when there was little available information was discussed. It was demonstrated that most problems, where a linear system is to be controlled by a linear compensator of fixed structure so as to minimize the expected value of a quadratic cost, can be posed as structured feedback problems. Some of the design problems relating to these more general problems were discussed.

The relation of the finite time results to the infinite time problem were briefly dealt with. In conclusion some unsolved problems of structured state feedback were mentioned.

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