SEMI-MARKOV REPRESENTATIONS OF SOME STOCHASTIC POINT PROCESSES

by

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ABSTRACT

An orderly stochastic point process on the real line is shown to be characterised by the probabilities that each finite union of intervals contains no events. The equivalence of some definitions of stationarity for point processes and the non-equivalence of others are demonstrated. Various questions concerning the characterisation of the interval sequence by the counting distributions are discussed.

Many point processes can be represented by the sequence of times at which transitions occur in a semi-Markov process with general state space. The counting distributions of the point process are determined by the transition functions and initial distributions of the semi-Markov process. A fundamental relation between the synchronous and asynchronous stationary distributions of the semi-Markov process is used to relate the synchronous and asynchronous joint interval distributions. Some examples are considered, including non-orderly processes.

Applications to point processes with events of several types are considered, and Palm-Khintchine relations are derived. One such process, the bivariate Markov process of intervals, is examined in more detail, and sufficient conditions for the existence of a stationary distribution are given.

Simplifications which arise when the semi-Markov process is a Markov process with countable state space are discussed. A condition is given for all the serial correlations of intervals to be positive.

It is shown that the self-exciting process exists as a generalised Poisson cluster process. A clustering representation is derived also

for the point process generated by a random walk when the dominant tail of the step distribution is exponential. When both tails are exponential the cluster structure is that of a birth and death process. Markovian representations of the exponential Neyman-Scott process, the exponential self-exciting process and the double exponential random walk are given. The simpler interval properties of these processes are derived and some numerical values tabulated.

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CHAPTER 1

SOME PROPERTIES OF POINT PROCESSES

1.1. Motivation and Scope of the Thesis

The purpose of this thesis is to describe an approach to the theory of stochastic point processes on the real line $\mathbb{R}$ which is constructive and widely applicable. By identifying $\mathbb{R}$ with a time axis we may think of the process as evolving in time according to certain given probabilistic laws. The future of the process is assumed to depend on its past only through certain 'initial conditions'. These initial conditions must then form a stochastic process in continuous time, and this process must be Markovian. Lawrance (1970, 1971, 1972), uses this approach to discuss the properties of particular point processes and to derive conditions for stationarity. He defines the notions of 'arbitrary time', 'arbitrary event' and 'average event'. Intuitively an arbitrary time is a point of $\mathbb{R}$ chosen without knowledge of the process, an arbitrary event is an arbitrary time conditional on an event occurring at that time and an average event is an event whose serial number is randomly chosen. Rigorous definitions can be given using limiting procedures (Khintchine, 1955). Other methods, containing a heavy measure theoretic content, are adopted by Slivnyak (1962) and Ryll-Nardzewski (1961). Matthes (1963) gives an elegant treatment using marked point processes.

The analytical difficulties inherent in a general approach will be avoided by an appeal to the theory of semi-Markov processes. At some cost in generality, constructive definitions of the various types of initial conditions can be given. This programme will be carried out in Chapter 3 and some examples of the method will be discussed. A secondary motivation for this work is that the semi-Markov representation provides an explicit expression for the likelihood function. Some caution is needed here. The likelihood function is simple only when the state variable of the semi-Markov process is observed, either directly or because it is a function of the history of the process. In general this will not be so.

In Chapters 4 and 5 applications to multivariate processes, where the events are of several distinguishable types, will be considered. The important special case of the countable state semi-Markov process is discussed in Chapter 6. The self-exciting point process and the point process generated by a random walk, the subjects of Chapters 7 and 8 respectively, have simple semi-Markov representations in particular cases.

In the remainder of this chapter some aspects of the general theory of point processes will be reviewed. In Section 1.4 the complete intensity function is defined and some of its properties described. An important characterisation result is proved in Section 1.5. This is the only result in the thesis which extends immediately to processes defined in $\mathbb{R}^n$, or in more general spaces. In Chapter 2 various alternative definitions of stationarity are considered and some problems concerning the characterisation of the interval sequence are discussed.

1.2. Notational and Mathematical Conventions

The word 'point' in this thesis is used to denote any element of the real line. Those points which are also elements of a (realisation of) a stochastic point process are called events. To distinguish this meaning of the word event from its more general probabilistic meaning, the measurable sets of the underlying probability space are called σ-sets, or σ-events.

It is worth noting a few points of mathematical interpretation. If $\mathcal{W}$ is a probability space an element of $\mathcal{W}$ is denoted by ω, a random variable taking values in $\mathcal{W}$ by W and a probability measure on $\mathcal{W}$ by p(dω). The degenerate distribution which assigns probability one to the single element ω_o of $\mathcal{W}$ is denoted by $\delta_{\omega_o}^{d\omega}$. Functions are always taken to be measurable. In Chapters 3 and 4 derivatives are defined in the Radon-Nikodym sense. The absolute continuity of a function ensures the existence of a Radon-Nikodym derivative, although there may be sets of measure zero on which the function is not differentiable.

A table of symbols and notation is provided. Total consistency in notation has proved impracticable, but the major conventions of Chapter 3 are adhered to throughout the sequel. Propositions are always referred to by the full designation proposition x.z for the zth proposition of Chapter x. Equations are numbered consecutively within sections. The zth equation of Section x. y is referred to as (z) from within that section but as (x. y.z) from elsewhere. The class of integers is denoted by $\mathbb{Z}$, the non-negative integers by $\mathbb{Z}^+$ and the positive integers by $\mathbb{Z}^{++}$. Similar conventions apply to the real line $\mathbb{R}$ and to the minus sign used as a superscript. Thus $\mathbb{R}^-$ denotes the class of non-positive real numbers.

1.3. Some Facts about Point Processes

In this section we state a few salient results from the general theory of point processes which will be needed later and resolve some potential ambiguities of definition and terminology. More details are contained in Daley and Vere-Jones (1972). A heuristic treatment is given by Cox and Lewis (1966).

Each realisation of a stochastic point process on the real line is a countable, ordered subset $\{t_j : j \in \mathbb{Z}\}$ of $\mathbb{R}$. Then t_j is the time of the j'th event. It is assumed that $\{t_j\}$ has no finite limit points, so that $t_j \to \pm\infty$ as $j \to \pm\infty$. Conventionally we take $t_0 \leq 0 < t_1$. A point process $\mathcal{P}$ is a probability measure on the space of all such realisations. The (random) counting measure $\tilde{N}$ of the process is given by $\tilde{N}(A) = \# \{ j : t_j \in A\}$ for each bounded Borel subset A of $\mathbb{R}$. The underlying σ-field is the smallest σ-field which contains all sets $\mathcal{S}$ of the form

$$\mathcal{S} = \{ \tilde{N}(A_i) = k_i : i = 1, \ldots, n\} ,$$

where n, $k_1, \ldots, k_n$ are non-negative integers and $A_1, \ldots, A_n$ are bounded Borel subsets of $\mathbb{R}$. The process $\mathcal{P}$ is then characterised by its finite-dimensional distributions, i.e. the probabilities of the σ-sets $\mathcal{S}$ defined above. These in turn are determined by the joint distributions

$$\{ \text{Prob} [\tilde{N}(I_j) = k_j \ (j = 1, \ldots, n)] : n, \ k_1, \ldots, k_n \in \mathbb{Z}^+ \}$$

over the half-open binary rational intervals $I_j = [a2^{-k}, b2^{-\ell})$, where a, b, k, ℓ are integers.

The $\{t_j\}$ become random variables $\{T_j\}$. If, with probability one, $j \neq k$ implies that $T_j \neq T_k$ then $\mathcal{P}$ is said to be orderly. The nth moment measure of $\mathcal{P}$ is defined for disjoint bounded Borel sets $A_1, \ldots, A_n$ as

$$M_n\,(A_1, \ldots, A_n) = \quad E\left\{\tilde{N}(A_1), \ldots, \tilde{N}(A_n)\right\} \tag{1}$$

if this exists. If $M_n\,(dx_1, \ldots, dx_n) = m_n(x_1, \ldots, x_n)\,dx_1, \ldots, dx_n$, then m_n is called the nth moment density.

The process is completely stationary if all the finite-dimensional distributions are invariant under translation. It is simply stationary (Lawrance, 1970b) if Prob $\{\tilde{N}(I+t) = k\}$ is independent of t for all $k \in \mathbb{Z}^+$ and intervals I of $\mathbb{R}$. The first moment measure of a stationary process is $M_1(A) = \rho\,|A|$, where ρ, which may be infinite, is the rate of the process. If the process is also orderly then Korolyuk's theorem (Khintchine, 1955) gives

$$\rho = \lim_{h \to 0+} \left[\frac{1}{h} \text{Prob}\left\{\tilde{N}\,[t, t+h) \geq 1\right\}\right]. \tag{2}$$

The covariance density of a stationary, orderly point process is $M_2(dx_1 \times dx_2) - \rho^2\,dx_1\,dx_2$. It is a function of $x_1 - x_2$ alone, and as such has a Fourier transform called the counting spectrum. The function var $\{\tilde{N}(0, t)\}$ is the variance-time curve.

The interval sequence X_j is defined by $X_j = T_j - T_{j-1}$ $(j \in \mathbb{Z})$. In general this will not be a stationary sequence, even if $\mathcal{P}$ is completely stationary. However, rather difficult arguments (Slivnyak 1962; Ryll-Nardzewski, 1961) show that if $\mathcal{P}$ is completely stationary, is orderly and has finite rate then conditionally on $T_0 = 0$, the sequence $\{X_j\}$ is a stationary discrete time stochastic process. Since the σ-event $\{T_0 = 0\}$ has zero probability, the relevant conditional probabilities must be defined by a limiting operation, considering the σ-event $\{-h < T_0 \leq 0\}$ and letting $h \to 0+$. The interval properties of are just the properties of the stationary time-series $\{X_j\}$. In particular we can define the marginal and joint interval distributions, the serial correlations of intervals and their Fourier transform, the interval spectrum. The interval properties and counting properties are equivalent, but only through their full distributions. The second order properties alone are not equivalent.

The simplest point process is the Poisson process for which the counting measures of disjoint sets are independent. If the ~~process is orderly~~ first moment measure is absolutely continuous then (Gnedenko and Kovalenko, 1965) there must exist a rate function $\lambda(t) > 0$ such that the distribution of $\tilde{N}(A)$ is Poisson with mean $\int_A \lambda(u)du$, for every Borel set A. The stationary Poisson process has $\lambda(t) = \rho$, a constant. If the rate is a random function $\Lambda(t)$, then a new process, the doubly stochastic Poisson process is obtained. This process is completely stationary if and only if $\Lambda(t)$ is a stationary process.

The cluster processes form another important class. Suppose that each event of a process $\mathcal{P}_m$ of main events generates independently an almost surely finite collection (cluster) of D subsidiary events. These then suffer displacements $Z_1, \ldots, Z_D$, not necessarily positive, from the main event. The distribution of the cluster structure, i.e.

of the random variables $(D, Z_1, \ldots, Z_D)$ is the same for each cluster, and does not depend on $\mathcal{P}_m$. Distinct clusters have independent structures. Then the superposition of $\mathcal{P}_m$ with all the subsidiary events generated is a cluster process. When $\mathcal{P}_m$ is a stationary Poisson process we have a Poisson cluster process. If also the displacements Z_i $(i = 1, \ldots, D)$ are independent and identically distributed with a distribution which does not depend on D, we have a Neyman-Scott process (Vere-Jones, 1970). A cluster process is completely stationary if the process of main events is stationary.

A multivariate point process consists of events of finitely or countably many different, distinguishable types. Most of the concepts discussed above extend naturally to this case. Cox and Lewis (1972) and Milne (1971) give detailed discussions. The events of a particular type form a marginal process and the collection of all events without regard to type is the superposed process. The multivariate process is marginally orderly if each marginal process is orderly and is strongly orderly if the superposed process is orderly.

1.4. The Complete Intensity Function

A useful method of defining orderly but in general non-stationary point processes is to use the complete intensity function. This was discussed in the multivariate context by Cox and Lewis (1972). The history $\mathcal{H}_t$ of a point process $\mathcal{P}$ at time t is defined by

$$\mathcal{H}_t = \{ t_j : t_j \in \mathcal{P} \text{ and } t_j < t \} . \tag{1}$$

We do not exclude the possibility that $\mathcal{H}_t = \emptyset$ for some t. If $\mathcal{H}_0 = \emptyset$

with probability one then $\mathcal{P}$ is called <u>transient</u>. The complete intensity function $\lambda(t, \mathcal{H}_t)$ is given by

$$\lambda(t, \mathcal{H}_t) = \lim_{h \to 0+} \frac{1}{h} \operatorname{Prob}\left\{ \tilde{N}[t, t+h) \geq 1 \mid \mathcal{H}_t \right\}. \tag{2}$$

This may not exist. Moreover it is not always easy to demonstrate the existence of stationary point processes which have given complete intensity functions. However we do have

<u>Proposition 1.1</u>. If $\mathcal{H}_0$ has a specified distribution then there exists at most one orderly point process $\mathcal{P}^+$ in $t \geq 0$ which satisfies (2) for a given function $\lambda(.,.)$.

<u>Proof</u>. The joint interval distributions are determined by λ as follows:

$$\operatorname{Prob}(X_1 \leq x_1 \mid \mathcal{H}_0) = 1 - \exp\left\{ - \int_0^{x_1} \lambda(u, \mathcal{H}_u)\, du \right\},$$

where $\mathcal{H}_u = \mathcal{H}_0$ for $0 \leq u \leq x_1$; and for $n \geq 2$

$$\operatorname{Prob}(X_n \leq x_n \mid \mathcal{H}_0, X_1 = x_1, \ldots, X_{n-1} = x_{n-1}) = 1 - \exp\left\{ - \int_{t_{n-1}}^{t_n} \lambda(u, \mathcal{H}_u)\, du \right\},$$

where $\mathcal{H}_u = \mathcal{H}_0 \cup \{t_1, \ldots, t_{n-1}\}$ for $t_{n-1} < u \leq t_n$, and $t_i = x_1 + \ldots + x_i$. The joint interval distributions in turn determine the finite-dimensional distributions of counts. ⁎

<u>Corollary</u>. A transient point process is characterised by its complete intensity function.

The rate of $\mathcal{P}^+$ is $f(t) = E\{\lambda(t, \mathcal{H}_t)\}$ if this exists. We then have (cf. Leadbetter, 1971; Daley and Vere-Jones, 1972) that

$$E\{\tilde{N}[a,b)\} = \int_a^b f(u)\, du. \tag{3}$$

If $\mathcal{P}^+$ is stationary, f(t) = ρ, and (3) is Korolyuk's theorem (see equation 1.3.2) again.

The tractability of the likelihood function depends on the existence of a simple form for the complete intensity function. We shall use the complete intensity function in Chapter 7, and a multivariate version in Chapter 5.

1.5. A Fundamental Characterisation Result

The results given in this section simplify some of the proofs in Chapter 3. They are also of considerable interest in their own right. Although the statements and proofs given here apply only to processes defined on $\mathbb{R}$, extensions to processes defined in $\mathbb{R}^n$, or in any complete separable metric space require only minor modifications. Two related references should be noted. Mönch (1971) uses methods similar to ours but the statement of his theorem is weaker than our Proposition 1.4. More recently Kallenberg (1972) has given an elegant proof of Proposition 1.4 using Dynkin's extension theorem. Neither of these authors considers multivariate processes. Our techniques are based on Leadbetter (1968).

Proposition 1.2. Let $\mathcal{P}$ be an orderly point process. For n = 1, 2, 3, ..., i = 0, ±1, ±2, ... define random variables

$$\chi_{ni} = \begin{cases} 1 & \text{if } \check{N}\,[i/2^n, \ (i+1)/2^n) \geq 1\,, \\ 0 & \text{otherwise.} \end{cases}$$

Then the finite-dimensional distributions of $\mathcal{P}$ are completely determined by the joint distributions of the $\{\chi_{ni}\}$.

Proof. It will be shown that the joint distribution of $\{\tilde{N}(I_1), \ldots, \tilde{N}(I_k)\}$ is determined by the joint distributions of the $\{\chi_{ni}\}$, for any binary rational half-open intervals $I_1, \ldots, I_k$. The result follows from the fact that the $\{I_j\}$ form a base of the Borel sets (cf. Daley and Vere-Jones, 1972, theorem 2.5).

Let n_0 be such that $2^{n_0} a_j$, $2^{n_0} b_j$ are integers for $j = 1, \ldots, k$, where $I_j = [a_j, b_j)$. For each $n \geq n_0$, the interval $I_{ni} = [i/2^n, (i+1)/2^n)$ is said to be an n-component interval of I_j if $I_{ni} \subset I_j$. Then, for each $n \geq n_0$, I_j is the disjoint union of its n-component intervals. Let $m_1, \ldots, m_k \in Z^+$ and let E_n denote the σ-event

$$E_n = \left\{ \text{For each } j \ (1 \leq j \leq k), \ \sum_{i \in C_j} \chi_{ni} \geq m_j \right\},$$

where

$$C_j = \{ i : I_{ni} \text{ is an n-component interval of } A_j \}. \qquad (1)$$

Now $\text{Prob}(E_n)$ is determined by the joint distributions of the $\{\chi_{ni}\}$. Also, for $n \geq n_0$, $E_n \subset E_{n+1}$ and so, by orderliness,

$$\text{Prob}\left\{ \tilde{N}(A_j) \geq m_j \ (1 \leq j \leq k) \right\} = \text{Prob}(\lim E_n)$$

$$= \lim \text{Prob}(E_n).$$

Thus the joint distributions of the $\tilde{N}(A_j)$ are determined. ✱

The assumption of orderliness is essential to this result. However, the proof holds when the finite-dimensional distributions of are improper. The extension to multivariate processes is straightforward.

Proposition 1.3. Let $\mathcal{P}$ be a marginally orderly multivariate point process with $m < \infty$ types of event, and let the counting measures of the marginal processes be $\tilde{N}_\ell$ ($\ell = 1, \ldots, m$). For $n = 1, 2, \ldots$, and $i = 0, \pm 1, \pm 2, \ldots$ define random variables

$$\chi_{ni}^{(\ell)} = \begin{cases} 1 & \text{if } \tilde{N}_\ell\, [{}^{i}/2^n, \ {}^{(i+1)}/2^n) \geq 1 , \\ 0 & \text{otherwise.} \end{cases}$$

Then the finite-dimensional distributions of $\mathcal{P}$ are completely determined by the joint distributions of the $\chi_{ni}^{(\ell)}$.

Proof. We now define, for $m_j^{(\ell)} \in \mathbb{Z}^+$ $(1 \leq j \leq k,\ 1 \leq \ell \leq m)$,

$$E_n = \left\{ \text{For each } j, \ell\ (1 \leq j \leq k,\ 1 \leq \ell \leq m),\ \sum_{i \in C_j} \chi_{ni}^{(\ell)} \geq m_j^{(\ell)} \right. ,$$

where C_j is defined in (1) above, and proceed as before to show that the joint distributions of the $\tilde{N}_\ell(A_j)$ are determined. *

Returning to the univariate case, we have the fundamental

Proposition 1.4. Let $\mathcal{F}$ be the class of finite unions of half-open intervals and define the incidence probabilities of an orderly, univariate point process $\mathcal{P}$ with counting measure $\tilde{N}$ as $\{\text{Prob}\,[\tilde{N}(F) \neq 0] : F \in \mathcal{F}\}$. Then the finite-dimensional distributions of $\mathcal{P}$ are determined by the incidence probabilities of $\mathcal{P}$.

Proof. It is sufficient to show that the joint distribution of any finite collection $\mathcal{C}$ of the $\{\chi_{ni}\}$ defined in Proposition 1.2 is determined by the incidence probabilities. We can suppose without loss of generality that the elements of $\mathcal{C}$ refer to disjoint intervals $I_1, I_2, \ldots, I_k$. Then the joint distribution of the $\chi_{ni} \in \mathcal{C}$ is determined by the 2^k probabilities

$$p(i_1, \ldots, i_k) = \text{Prob}\left[\bigcap_{i_j=0} \{\tilde{N}(I_j) = 0\} , \bigcap_{i_j=1} \{\tilde{N}(I_j) \geq 1\} \right]. \tag{2}$$

We show by induction on $r = i_1 + \ldots + i_k$ that the incidence probabilities determine the $p(i_1, \ldots, i_k)$. We have

$$p(0, \ldots, 0) = \text{Prob}\left\{ \tilde{N}\left(\bigcup_{j=1}^{k} I_j \right) = 0 \right\} .$$

Suppose that $p(i_1, \ldots, i_k)$ is known for all $i_1, \ldots, i_k$ with $\sum i_j \leq r$. Then if $i_j = 1$ for $j \leq r+1$, $i_j = 0$ for $j > r+1$, we have

$$p(i_1, \ldots, i_k) = \text{Prob}\left\{ \tilde{N}\left(\bigcup_{j=r+2}^{k} I_j \right) = 0 \right\} - \sum_{\underset{\sim}{d} \in \mathcal{D}} p(\underset{\sim}{d}) ,$$

where $\mathcal{D}$ is the set of all k-tuples $(i_1, \ldots, i_k)$ of zero-one variables such that $i_j = 0$ $(j \geq r+2)$, and at least one $i_j = 0$ for $j \leq r+1$. A similar argument holds for any $p(i_1, \ldots, i_k)$ with $\sum i_j = r+1$. Hence the induction goes through. ✳

Again, the multivariate generalisation is immediate.

<u>Proposition 1.5.</u> Let $\mathcal{P}$ be a marginally orderly multivariate point process with $m < \infty$ types of event, and let the counting measures of the marginal processes be $\tilde{N}_\ell$ $(\ell = 1, \ldots, m)$. Let the class of joint incidence probabilities of $\mathcal{P}$ be defined as the class of all probabilities

$$\text{Prob}\left\{ \tilde{N}_1(F_1) + \tilde{N}_2(F_2) + \ldots + \tilde{N}_m(F_m) \neq 0 \right\} , \tag{3}$$

where each F_i is either the empty set or an element of $\mathcal{F}$. Then the finite-dimensional distributions of $\mathcal{P}$ are determined by its joint incidence probabilities.

<u>Proof.</u> It is sufficient to show that these probabilities determine the finite dimensional distributions of the $\{ X_{ni}^{(\ell)} \}$. The proof differs only notationally from that of Proposition 1.4. The analogue of $p(i_1, \ldots, i_k)$

defined in (2) above requires mk arguments $p(i_{\ell j})$ $(1 \leq \ell \leq m;$ $1 \leq j \leq k)$ and the induction is on $r = \sum \sum i_{\ell j}$. *

The meaning of Proposition 1.5 becomes clear when $\mathcal{P}$ is represented, as it may be, by a univariate process on the product space $\mathbb{R} \times \mathcal{M}$, where $\mathcal{M} = (1, \ldots, m)$. Proposition 1.5 then becomes merely an extension of Proposition 1.4 to a process defined in a larger space. As has been pointed out, many other such extensions are possible, but will not be considered here.

Corollary 1.6. Let $\mathcal{P}$ be a multivariate process with possibly a countable infinity of types of event. Suppose that the joint incidence probabilities (3) are known for every m. Then the finite-dimensional distributions of $\mathcal{P}$ are determined.

Proof. This is obvious, since any particular finite-dimensional distribution involves only finitely many types of event. *

CHAPTER 2

ON THE DEFINITION OF STATIONARITY

2. 1. INTRODUCTION

Complete stationarity of a univariate point process was defined in Chapter 1 to mean the invariance of all finite-dimensional distributions under translation. It is irrelevant whether these joint counting distributions are taken over Borel sets or over intervals. A formally weaker condition, kth order stationarity, holds if the joint distribution of $\{\tilde{N}(A_1), \ldots, \tilde{N}(A_k)\}$ is invariant under translation for all $\{A_1, \ldots, A_k\}$ belonging to a given class $\mathcal{A}_k$ of k-tuples of subsets of R. This definition is incomplete until $\mathcal{A}_k$ is specified. The ambiguity appears to have given rise to some confusion in the literature. In Section 2. 2 a generalisation of a construction due to Moran (1967) is used to show that when $\mathcal{A}_k = \{$ k-tuples of intervals $\}$, kth order stationarity does not imply complete stationarity, for any k. A different construction, given by Szasz (1970), could have been used to deduce the same result. However, Moran's construction and its generalisation lead to certain interesting problems concerning the interval sequence. These are discussed in Section 2. 3. Orderliness is assumed throughout this chapter.

The results of Section 1. 5 give

Proposition 2. 1. For each positive integer k, let $\mathcal{A}_k$ denote the class of k-tuples of finite unions of intervals. Then kth order stationarity as defined above and complete stationarity are equivalent.

Proof. It is obvious that complete stationarity implies kth order stationarity which in turn implies first order stationarity. It is therefore sufficient to show that first order stationarity implies complete stationarity. However, first order stationarity implies invariance of the incidence probabilities under translation, and it is clear from the proof of Proposition 1.2 that all finite dimensional distributions must then be invariant also.

2.2. A k-DIMENSIONAL QUASI-POISSON PROCESS

Moran (1967) constructs a point process $\mathcal{P}_2$ which is not a Poisson process, but which is such that $\tilde{N}[a, b)$ has a Poisson distribution with mean b-a for any a, b (a < b). His construction uses a bivariate exponential density of the form

$$f_2(x_1, x_2) = \exp\{-(x_1+x_2)\} + g(x_1, x_2), \qquad (1)$$

where $g(x_1, x_2)$ is defined to equal ε $(0 < \varepsilon < e^{-6})$ on the squares (0,2), (1,3), (2,1) and (3,0), and $-\varepsilon$ on the squares (0,3), (1,2), (2,0) and (3,1). Here (m, n) denotes the square $\{m \leq x_1 < m+1, n \leq x_2 < n+1\}$.

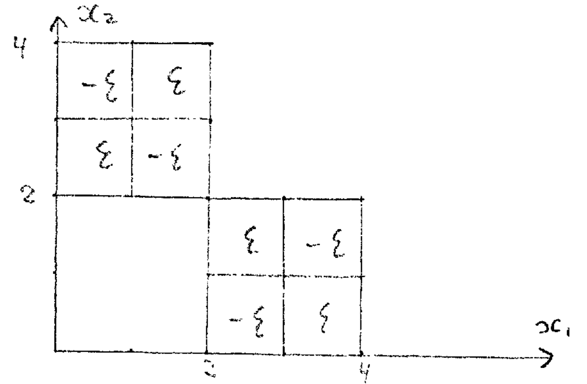

Figure 1. The function $g(x_1, x_2)$

The symmetry properties of g (Figure 1) ensure that if (X_1, X_2) has the joint density (1), then X_1 and X_2 are each marginally exponential with unit mean. Moreover $X_1 + X_2$ has the distribution, two-stage Erlangian, it would have if X_1 and X_2 were independent. This is because the integral of $g(x_1, x_2)$ over any set of the form $\{x_1 + x_2 \leq c\}$, $\{x_1 \leq c\}$ or $\{x_2 \leq c\}$ is zero. Moran constructs an interval sequence $\{\ldots, X_{-1}, X_0, X_1, \ldots\}$ such that, for each n, the pairs $\{X_{2n}, X_{2n+1}\}$ have the joint density (1), and such that successive pairs are mutually independent. The sequence is then imbedded in $\mathbb{R}$ so as to give a completely stationary point process which has the stated properties. Moran's process may be modified in several ways to define a process which is simply stationary but not completely stationary. For example the process $(\mathcal{P} \cap \mathbb{R}^{--}) \cup (\mathcal{P}_2 \cap \mathbb{R}^{+})$ has this property if $\mathcal{P}$ is a Poisson process of unit rate, independent of $\mathcal{P}_2$.

The process $\mathcal{P}_k$ constructed in this section is such that its joint counting distributions over any (k-1) contiguous intervals are the same as for a Poisson process of unit rate, but $\mathcal{P}_k$ is not a Poisson process. When $k = 2$, Moran's process is recovered.

<u>Proposition 2.2</u>. Let $k \geq 2$ be an integer. Then it is possible to construct a k-tuple of random variables $(X_1, \ldots, X_k)$ such that

(i) each X_i is exponential with unit mean,

(ii) the (k-1)-tuples $(X_1, \ldots, X_{k-1})$ and $(X_2, \ldots, X_k)$ are each (k-1)-tuples of independent random variables,

(iii) for each i $(0 \leq i \leq k-1)$ the (k-1)-tuples $(X_1, \ldots, X_i + X_{i+1}, \ldots, X_k)$ have the joint distributions that they would have if $(X_1, \ldots, X_k)$ were independent,

(iv) the random variables $(X_1, \ldots, X_k)$ are not mutually independent.

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be any permutation of $(1, \ldots, k)$, and let $\sigma(\alpha) = \operatorname{sgn}(\alpha)$. For each such α, let S_α be the hypercube

$$S_\alpha = \{ x = (x_1, \ldots, x_k) : 2(\alpha_i - 1) < x_i < 2\alpha_i \quad (i = 1, \ldots, k) \} .$$

For each $j = (j_1, \ldots, j_k)$ with each j_i equal to 1 or 2, define the hypercube

$$T_j = \{ x = (x_1, \ldots, x_k) : j_i - 1 < x_i < j_i \quad (i = 1, \ldots, k) \} ,$$

and let $\tau(j) = +\varepsilon$ or $-\varepsilon$ according as $\sum j_i$ is even or odd. Then, if $x \in S_\alpha$ and if no x_i is integral, there exists a unique j such that $x - y \in T_j$, where $y_i = 2(\alpha_i - 1)$ $(i = 1, \ldots, k)$. Thus we can define a function $g : \mathbb{R}^k \to \mathbb{R}$ as follows. If, for some α, $x \in S_\alpha$ and no x_i is integral, then we set $g(x) = \sigma(\alpha)\, \tau(j)$, where j is defined above. Otherwise we take $g(x) = 0$.

Then $g(x)$ satisfies

$$\int_{x_i=0}^{\infty} g(x_1, \ldots, x_k)\, dx_i = 0 \quad (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k \in \mathbb{R}^+)$$

$$(1 \le i \le k) , \qquad (2)$$

$$\int_{x_{i-1}=0}^{y} \int_{x_i=0}^{y-x_{i-1}} g(x_1, \ldots, x_k)\, dx_i\, dx_{i-1} = 0$$

$$(x_1, \ldots, x_{i-2}, y, x_{i+1}, \ldots, x_k \in \mathbb{R}^+)$$

$$(2 \le i \le k). \qquad (3)$$

To prove (3) note that if $z = (z_1, \ldots, z_k)$ satisfies

$$z \in S_\alpha \cap \{ x : x_i + x_{i+1} < y \} ,$$

then z', formed by interchanging z_{i+1} and z_i in z, satisfies

$$z' \in S_{\alpha'} \cap \{ x : x_i + x_{i+1} < y \} ,$$

where $\sigma(\alpha) = -\sigma(\alpha')$ and $\tau\{j(z)\} = \tau\{j(z')\}$.

To prove (2) note that if $\underset{\sim}{z} \in S_{\underset{\sim}{\alpha}}$, then so does

$$\underset{\sim}{z}'' = (z_1, \ldots, z_{i-1}, 2(2\alpha_i - 1) - z_i, z_{i+1}, \ldots, z_k),$$

and that $\gamma\{\underset{\sim}{j}(\underset{\sim}{z}'')\} = -\gamma\{\underset{\sim}{j}(\underset{\sim}{z})\}$.

Now let

$$f_k(\underset{\sim}{x}) = \exp(-\sum_1^k x_i) + g(\underset{\sim}{x}) . \tag{4}$$

Provided $0 < \epsilon < \exp\{-k(k+1)\}$, $f_k(\underset{\sim}{x})$ is a joint density function with the required properties.

Remark. When k = 2, the function g(.) defined here is the same as that defined by Moran.

It is possible to define an infinite sequence $\{\ldots X_{-1}, X_0, X_1, \ldots\}$ such that for each m the joint distribution of the k-tuple $\{X_{mk+1}, \ldots, X_{(m+1)k}\}$ has the density $f_k(\underset{\sim}{x})$, and such that k-tuples corresponding to different values of m are independent. Then if

$$Y_m = \sum_{i=1}^{k} X_{mk+i} \quad (m = \ldots, -1, 0, 1, \ldots), \tag{5}$$

the Y_m are mutually independent, are identically distributed and have finite mean. Thus a stationary renewal process may be constructed from the Y_m in the usual way. The events of this process will be termed R-events.

The process $\mathcal{P}_k$ is now constructed by interpolating k-1 events between each pair of neighbouring R-events according to the joint density $f_k(\underset{\sim}{x})$ given by (4), conditioned on

$$\sum_{i=1}^{k} X_{mk+i} = Y_m .$$

Interpolations between different pairs of R-events are to be independent.

It is immediate from this construction that $\mathcal{P}_k$ is a completely stationary, orderly point process with unit rate, and that $\mathcal{P}_k$ is not a Poisson process.

Proposition 2.3. Suppose that for $k \geq 3$, the real numbers $a_1, \ldots, a_k$ satisfy $a_1 < a_2 < \ldots < a_k$. Then if

$$N_i = \tilde{N}\ [a_i,\ a_{i+1}) \qquad (i = 1, \ldots, k-1), \tag{6}$$

where $\tilde{N}$ is the counting measure of $\mathcal{P}_k$, the N_i are independently distributed, N_i having a Poisson distribution with mean $a_{i+1} - a_i$.

Proof. This is in two stages. We first consider the probability

$$\pi(\underset{\sim}{n}) = \text{Prob}(\tilde{N}^{(R)}\ [a_1, a_k) = 0,\ \ N_1 = n_1, \ldots, N_{k-1} = n_{k-1}), \tag{7}$$

where $\tilde{N}^{(R)}$ is the counting measure of the R-process. Clearly $\pi(\underset{\sim}{n}) = 0$ if $\sum n_i > k-1$. It will be shown by a detailed enumeration of particular cases that $\pi(\underset{\sim}{n})$ does not depend on ξ. In each case $\pi(\underset{\sim}{n})$ will be evaluated by conditioning on either the position $a_1 - u$ of the last R-event before a_1, or the position $a_k + v$ of the first R-event after a_k. The respective conditional probabilities will be denoted by $\pi_b(\underset{\sim}{n}, u)$ and $\pi_f(\underset{\sim}{n}, v)$. It is worth emphasizing that to show that $\pi(\underset{\sim}{n})$ does not depend on ξ it is sufficient to show that one of $\pi_b(\underset{\sim}{n}, u)$ and $\pi_f(\underset{\sim}{n}, v)$ does not depend on ξ. It is convenient to include in both conditioning σ-events the serial number m of the last R-event before a_1, and to denote the interval X_{mk+i} by Z_i $(i = 1, \ldots, k)$.

The various cases listed below include all possible values of $\underset{\sim}{n}$:

(i) $\sum n_i \leq k-2$;

(ii) for some i, $n_i \geq 3$;

(iiia) $n_1 = 2$;

(iiib) $n_{k-1} = 2$;

(iva) for some $s \geq 1$, $\sum_{i=1}^{s} n_i \geq s+1$;

(ivb) for some $r \geq 1$, $\sum_{i=k-r}^{k-1} n_i \geq r+1$;

(v) $n_i = 1$ for all i.

Case (i). If $\sum n_i \leq k-2$, then either the first event after a_k or the last event before a_1 (or both) must be interpolated events. In the former case $\pi_b(\underset{\sim}{n}, u)$, which depends only on the joint distribution of $(Z_1, \ldots, Z_{k-1})$, will not depend on ξ . In the latter case $\pi_f(\underset{\sim}{n}, v)$, which depends only on the joint distribution of $(Z_2, \ldots, Z_k)$, will not depend on ξ .

Case (ii). If $n_i \geq 3$, then for some j, $\pi_f(\underset{\sim}{n}, v)$ depends only on the joint distribution of $(Z_1, \ldots, Z_{j-1}, Z_j + Z_{j+1}, Z_{j+2}, \ldots, Z_k)$ and so does not depend on ξ .

Case (iiia). If $n_1 = 2$, then for some measurable subset S of $\mathbb{R}^{k-2}$ $\pi_f(\underset{\sim}{n}, v)$ can be expressed as

$$\pi_f(\underset{\sim}{n}, v) = \text{Prob}\left\{(Z_3, \ldots, Z_k) \in S, \sum_{3}^{k} Z_j \geq a_k + v - a_2, \sum_{2}^{k} Z_j < a_k + v - a_1\right\}$$

$$- \text{Prob}\left\{(Z_3, \ldots, Z_k) \in S, \sum_{3}^{k} Z_j \geq a_k + v - a_2, \sum_{1}^{k} Z_j < a_k + v - a_1\right\} \tag{8}$$

Neither component depends on ξ .

Case (iiib). Here there is a similar expression to (8) for $\pi_b(\underset{\sim}{n}, u)$.

Case (iva). Here we use induction on s, noting that for s = 1 (iva) reduces to (iiia) or to (ii). Suppose now that $\pi_f(\underset{\sim}{n}, r)$ does not depend on ξ if, for some r ($1 \leq r \leq s$),

$$\sum_{i=1}^{r} n_i \geq r + 1 .$$

We shall show that $\pi_f(\underset{\sim}{n}, v)$ does not depend on ξ if

$$\sum_{i=1}^{s+1} n_i \geq s + 2 .$$

If $n_{s+1} \geq 3$, then (ii) applies. If $n_{s+1} = 2$ and $n_s = 0$ ($s \geq 3$), or if $n_{s+1} \leq 1$, the induction hypothesis applies. If $n_{s+1} = 2$ and $n_s \geq 1$, then $n_s + n_{s+1} \geq 3$, and (ii) applies if the sth and (s+1)th intervals are pooled. The result now follows by subtraction, since it holds for all other $\underset{\sim}{n}$ with the same values of $n_1, \ldots, n_{s-1}, n_s + n_{s+1}, n_{s+2}, \ldots, n_{k-1}$. Hence the induction goes through, and $\pi_f(\underset{\sim}{n}, v)$ does not depend on ξ in (iva).

Case (ivb). A similar inductive argument may be applied to $\pi_b(\underset{\sim}{n}, u)$ in (ivb).

Case (v). Since $k \geq 3$, Case (iiia) may be applied to the pooled interval $[a_1, a_k)$ to show that Prob$\{\tilde{N}^{(R)}[a_1, a_k) = 0, \sum N_i = k-1\}$ does not depend on ξ. The result now follows by subtraction, since for any $\underset{\sim}{n}$ with $\sum n_i = k-1$, if (v) does not hold then one of (iva), (ivb) must hold.

This completes the first stage of the proof. The second stage is to evaluate probabilities of the form (7) when $N^{(R)}[a_1, a_k) \neq 0$. These may be determined by conditioning on the locations of all R-events within $[a_1, a_k)$, and using the independence of interpolations within different R-intervals to factorise the relevant conditional probabilities. Each factor then corresponds to a set of at most k-1 contiguous intervals $\{ [b_1, b_2), [b_2, b_3), \ldots, [b_{\ell-1}, b_\ell) \}$, where either b_1 or b_ℓ or possibly both are R-events. The joint distributions of counts over each such set may be evaluated by considering the same sequence of particular cases as above. Here, however, there is no need to condition further, i.e. on the positions of R-events outside $[a_1, a_k)$. The details, which are similar to those given in the first stage of the proof, are omitted. This completes the proof of Proposition 2.3. ✳

<u>Remarks</u> (i) This proof does not use the complete stationarity of $\mathcal{P}_k$, but the stationarity of the renewal process of R-events is needed.

(ii) For Moran's process (k=2) the proof given requires a slight modification. Specifically, Prob $\{ \tilde{N}^{(R)} [a_1, a_2) = 0, \; N_1 = 1 \}$ is evaluated by subtraction, since it is equal to

$$\text{Prob} \{ \tilde{N}^{(R)} [a_1, a_2) = 0 \} - \text{Prob} \{ \tilde{N}^{(R)} [a_1, a_2) = 0, \; N_1 = 0 \} ,$$

which does not depend on ξ.

(iii) The result of Proposition 2.3 is surprisingly strong. It cannot be proved without the rather indirect arguments given here. For example, if each $n_i = 1$, it is not possible to show directly that $\pi(\underset{\sim}{n})$ does not depend on ξ, since both $\pi_f(\underset{\sim}{n}, v)$ and $\pi_b(\underset{\sim}{n}, u)$ do depend on ξ.

The process $\mathcal{P}_k$ as constructed is completely stationary, but it may easily be modified to give a process which has its mth order joint counting distributions over intervals invariant under translation

$(2m \leq k)$, but which is not completely stationary. If $\mathcal{P}$ is a Poisson process of unit rate, then

$$Q_k = (\mathcal{P} \cap \mathbb{R}^{--}) \cup (\mathcal{P}_k \cap \mathbb{R}^{+})$$

is such a process. Alternatively the constant ε may be made a function of the serial number m of the R-interval, so that the interval sequence of $\mathcal{P}_k$ becomes non-stationary. As noted above, this does not affect the proof.

2.3. ON MORE GENERAL INTERVAL SEQUENCES

As the processes considered above are rather artificial, it is interesting to consider how far the construction may be generalised. There is a direct extension to 'quasi-renewal processes'.

Proposition 2.4. Let F be a distribution function with $F(0) = 0$ and finite mean which is not concentrated on fewer than 2k points of $\mathbb{R}$. Then there exists a (stationary or non-stationary) k-dimensional quasi-renewal process whose joint counting distributions over the (k-1) contiguous intervals $[a_1, a_2), \ldots, [a_{k-1}, a_k)$ are the same as those of the (stationary or non-stationary) renewal process with interval distribution F.

Proof. It is sufficient to construct the joint distribution of a k-tuple of random variables $(X_1, \ldots, X_k)$ which has the properties (ii), (iii) and (iv) of Proposition 2.2, with $\mathrm{Prob}(X_i \leq x) = F(x)$ $(i = 1, \ldots, k)$. If F has a continuous non-zero density function f, then the previous construction can be applied immediately for sufficiently small ε. Otherwise a little more work is needed. Let $A_1, \ldots, A_{2k}$ be disjoint Borel subsets of $\mathbb{R}$ such that for some $\delta > 0$, $F(A_i) > \delta$ $(i = 1, \ldots, 2k)$.

Then there exist positive measures m_i on the Borel subsets of A_i such that $m_i(A_i) = \delta$ $(i = 1, \ldots, k)$ and such that $F - m_i$ is a positive measure on A_i.

Let $B_j = A_{2j-1} \cup A_{2j}$ $(j = 1, \ldots, k)$ and for each permutation $\underset{\sim}{\alpha} = (\alpha_1, \ldots, \alpha_k)$ of $(1, \ldots, k)$ put

$$S_{\underset{\sim}{\alpha}} = \{ \underset{\sim}{x} : x_i \in B_{\alpha_i} \}$$

If $i_1, \ldots, i_k$ are distinct and $\underset{\sim}{i} = (i_1, \ldots, i_k)$, define

$$\tau(\underset{\sim}{i}) = \begin{cases} +1 \text{ if } \sum i_j \text{ is even,} \\ -1 \text{ if } \sum i_j \text{ is odd,} \end{cases}$$

and let $\mathcal{R}(\underset{\sim}{i})$ denote the rectangle $\mathcal{R}(\underset{\sim}{i}) = A_{i_1} \times \ldots \times A_{i_k}$. Define the signed measure G on $\mathbb{R}^k$ as follows. If, for some permutation $\underset{\sim}{\alpha}$, $\mathcal{R}(\underset{\sim}{i}) \subset S_{\underset{\sim}{\alpha}}$, then on $\mathcal{R}(\underset{\sim}{i})$

$$G = \operatorname{sgn}(\underset{\sim}{\alpha}) \, \tau(\underset{\sim}{i}) \, m_{i_1} \times \ldots \times m_{i_k} ,$$

and elsewhere $G \equiv 0$. Then $F(dx_1) \ldots F(dx_k) - G(dx_1, \ldots, dx_k)$ is a probability distribution on $\mathbb{R}^k$ which has the required properties. To see this note that if $\underset{\sim}{z} = (z_1, \ldots, z_k) \in \mathbb{R}^k$, then

$$G(dz_1, \ldots, dz_{k-1}, \mathbb{R}) = \ldots = G(\mathbb{R}, dz_2, \ldots, dz_k) = 0 ,$$

and that if $\underset{\sim}{z}'$ is obtained from $\underset{\sim}{z}$ by interchanging z_i and z_{i+1} for some i $(1 \leq i \leq k-1)$, then

$$G(d\underset{\sim}{z}) = -G(d\underset{\sim}{z}') .$$ *

Now let F^{i*} denote the ith convolution of F with itself $(i \geq 1)$. The sequence $\{ \ldots X_{-1}, X_0, X_1, \ldots \}$ will be called a two-dimensional quasi-renewal sequence if, for each $n \in Z$, $i \in Z^{++}$, the distribution of $X_{n+1} + \ldots + X_{n+i}$ is F^{i*}. The sequence will be called a k-dimensional quasi-renewal sequence (k > 2) if, for any $i_1, \ldots, i_{k-1} > 0$ and any n, the random variables

$$Y_j = X_{n+i_1+\ldots+i_{j-1}+1} + \ldots + X_{n+i_1+\ldots+i_j} \quad (1 \leq j \leq k-1) \qquad (1)$$

are independently distributed, Y_j having the distribution F^{i_j*}.

Proposition 2.5. The interval sequence of the k-dimensional quasi-renewal process constructed above is a k-dimensional quasi-renewal sequence. Conversely, if the mutually independent k-tuples $\{X_{mk+1}, \ldots, X_{(m+1)k}\}$ (m = ... -1, 0, 1, ...) are such that the sequence $\{X_n\}$ is a k-dimensional quasi-renewal sequence, then each such k-tuple must satisfy (ii) and (iii) of Proposition 2.2.

Proof. If all the elements of a k-tuple appear among the Y_j of (1), then at least two (consecutive) elements must appear in the same Y_j. In any case the mutual independence of the k-tuples ensures that the Y_j have the specified distributions.

The converse is established in three stages. First note that the X_n must be identically distributed. Then, by considering ℓ-tuples which straddle two of the given (independent) k-tuples, it may be proved inductively for $\ell \leq k-1$, that any ℓ-tuple of consecutive X_n is an ℓ-tuple of independent random variables. This proves (ii). Finally, (iii) is proved by considering the joint distribution of a k-tuple which straddles two of the given k-tuples.

The particular k-dimensional quasi-renewal sequences defined above are not stationary sequences. However they may easily be made so by randomising the serial number of the first interval. For stationary quasi-renewal sequences there is the result

Proposition 2.6. Let $\mathcal{P}$ be a completely stationary point process whose interval sequence is a two-dimensional quasi-renewal sequence. Then $\mathcal{P}$ is a two-dimensional quasi-renewal process.

Proof. This is immediate from the usual Palm-Khintchine formulae, which show that the distribution of a sum of consecutive asynchronous intervals may be expressed in terms of the distributions of sums of consecutive synchronous intervals. *

It seems likely that this result extends to k-dimensional quasi-renewal sequences. However we have not yet been able to prove this.

Lawrance (1972) has asked whether a process of Moran's type could have a non-zero serial correlation of intervals. The (negative) answer to this question is contained in the following amusing result.

Proposition 2.7. Let $\mathcal{P}$ be a second order quasi-renewal process of the type defined in Proposition 2.6. Then $\mathcal{P}$ has the same second order properties of counts and of intervals as the corresponding stationary renewal process.

Proof. The second order counting properties depend only on the rate of the process and on the covariance density. These depend only on the distributions of sums of consecutive intervals. For the covariances γ_m $(m = 1, 2, \ldots)$ of the (stationary) interval sequence, we have

$$\operatorname{var}(X_1+\ldots+X_{m+1}) - \operatorname{var}(X_1+\ldots+X_m) = \operatorname{var}(X_{m+1}) + 2\sum_{i=1}^{m} \gamma_i ,$$

giving a proof, by induction on m, that $\gamma_m = 0$. *

2.4. DISCUSSION

Many problems remain unsolved. Suppose that a sequence $(\ldots X_{-1}, X_0, X_1, \ldots)$ is defined to be quasi-stationary if the (univariate) distribution of the sum of any number of consecutive X_i is invariant under translation. Then the analogy with Proposition 2.6 strongly suggests that, provided $\mathbb{E}(X_i) < \infty$, the sequence $\{X_i\}$ may be imbedded in $\mathbb{R}$ to give a simply stationary point process. However an explicit construction when the sequence $\{X_i\}$ does not have an imbedded renewal process has not been found.

Conversely, it may be asked what conditions simple stationarity of a point process imposes on the interval sequence. It is known (Lawrance 1970 , also Chapter 4 in this thesis) that the Palm-Khintchine formulae hold, showing that the distribution of the sum of consecutive intervals starting from an arbitrary event, is invariant under translation. This does not necessarily imply quasi-stationarity of the interval sequence though. Here the distinction between an event at a specific (arbitrary) time, and an event with a specific (arbitrary) serial number, is important.

CHAPTER 3

THE SEMI-MARKOV MODEL FOR A POINT PROCESS

3.1. INTRODUCTION AND SUMMARY

The general semi-Markov process is used to present a measure-theoretic framework for a wide class of point processes. This enables the basic relations between synchronous and asynchronous distributions to be stated and proved in a simple form. It is shown how operations on the semi-Markov process may be used to construct more complicated point processes from simple ones. The method is best suited to processes which are defined by their interval properties. Any point process which evolves in time in such a way that its behaviour in (t, ∞) depends on its behaviour in $(-\infty, t]$ only through a process $\hat{W}_t$ of 'initial conditions' can be represented in this way. The fundamental theorem of Section 3.3 relates the synchronous and asynchronous stationary initial conditions. Some examples are considered briefly to illustrate the approach. These include processes with Markov-dependent intervals, some doubly stochastic processes and clustering processes. A generalised Palm-Khintchine formula is proved in Section 3.6, and extended to non-orderly processes in Section 3.7.

In Chapter 4 it will be shown how the semi-Markov construction may be used to tackle multivariate processes. The heavy dependence on the order properties of $\mathbb{R}$ precludes any simple extension to processes defined in $\mathbb{R}^n$, or more general spaces.

3.2. THE SEMI-MARKOV MODEL

Let $\{\ldots, W_{-1}, W_0, W_1, \ldots\}$ be a discrete parameter Markov chain with a quite general state space $\mathcal{W}$ and stationary transition function

$$P(A|\omega) = \text{Prob}(W_{i+1} \in A | W_i = \omega), \tag{1}$$

where $A \in \Omega_{\mathcal{W}}$, a σ-field of subsets of $\mathcal{W}$. It is assumed that W_0 has an initial distribution $q_0(.)$ defined on $\Omega_{\mathcal{W}}$. Conditionally on the entire realisation of this chain, the real, non-negative random variable X_{i+1} has distribution

$$F(x_{i+1}| \omega_i, \omega_{i+1}) = \text{Prob}(X_{i+1} \leq x_{i+1} | W_i = \omega_i, W_{i+1} = \omega_{i+1}), \tag{2}$$

and the $\{X_i\}$ are conditionally independent given the $\{W_i\}$. Usually only orderly point processes will be considered and it will accordingly be assumed that F satisfies

<u>Hypothesis H1</u>. With probability one the distributions $F(x|.,.)$ have no atoms at $x = 0$, i.e. with probability one $F(0| W_i, W_{i+1}) = 0$ for all i.

A standard technique, discussed in Section 3.4, enables most non-orderly processes to be analysed by considering an induced orderly process.

It follows from this construction that $\{W_i, X_i\}$ ($i = \ldots -1, 0, 1, \ldots$) is also a Markov chain, with state space $\mathcal{W} \times \mathbb{R}^+$, and transition function

$$G(A, x | \omega_0) = \int_{\omega_1 \in A} p(d\omega_1 | \omega_0)\, F(x | \omega_0, \omega_1) \tag{3}$$

defined on the usual product σ-field. A realisation of this Markov chain defines a realisation of the associated point process which has interval sequence $\{X_i\}$ $(i = \ldots -1, 0, 1, \ldots)$. Specifically, the associated point process with an event at the origin is defined by placing the nth and -nth events (for $n \geq 0$) at the points

$$T_n = \sum_{i=1}^{n} X_i \ ; \quad T_{-n} = - \sum_{i=1-n}^{0} X_i \ ,$$

respectively. A basic requirement is that the $\{T_i\}$ should have no finite limit points, so we assume

Hypothesis H2. With probability one, $T_n \to \infty$ and $T_{-n} \to -\infty$ as $n \to \infty$.

Let

$$N_t = \sup \{ n : T_n \leq t \} \ , \tag{4}$$

$$U_t = t - T_{N_t} \ . \tag{5}$$

Then with probability one, N_t and U_t are each right-continuous everywhere, with points of discontinuity only at events of the associated point process.

Hypothesis H3. The Markov chain $\{W_i\}$ has a unique stationary distribution $q(.)$ on Ω_W.

Thus $q(.)$ is the unique solution of

$$q(A) = \int_{\omega \in A} p(A \mid \omega) q(d\omega) = \int_{\omega \in \Omega_W} \int_{x=0}^{\infty} G(A, dx \mid \omega) \, q(d\omega) \ . \tag{6}$$

Then the augmented Markov chain $\{W_i, X_i\}$ also has a unique stationary distribution, given by

$$\overline{q}(d\omega, dx) = \int_{\omega_0 \in \mathcal{W}} q(d\omega_0)\, p(d\omega \mid \omega_0)\, F(dx \mid \omega_0, \omega) \,. \tag{7}$$

Proposition 3.1. If W_0 has the distribution q(.), then the interval sequence $\{\ldots, X_{-1}, X_0, X_1, \ldots\}$ is a strictly stationary process in discrete time.

Proof. The joint distributions of the $\{X_i, W_i\}$ are invariant under translation. *

For $t \in R$, define $\widetilde{W}_t = W_{N_t}$. Then we have the important result

Proposition 3.2. The process $\{\widetilde{W}_t, U_t\}$ is a well-defined Markov process in continuous time with state space $\mathcal{W} \times \mathbb{R}$. The process $\{U_t\}$ is separable.

Proof. The separability follows from the fact that $\{U_t\}$ has only finitely many discontinuities in bounded intervals. It is clearly possible to write down the joint distribution of $\{\widetilde{W}_{t_i}, U_{t_i}\}$ $(i = 1, \ldots, k)$ for any $k \in \mathbb{Z}^{++}$, $t_1 < \ldots < t_k \in R$, as a countable sum of terms corresponding to the number of events in $(t_i, t_{i+1}]$ $(i = 1, \ldots, k-1)$. The Markov property follows from the form of this distribution: the transition probabilities are exhibited in Proposition 3.4 below. *

Proposition 3.3. The finite-dimensional distributions of the associated point process are determined by the finite-dimensional distributions of the $\{U_t\}$.

Proof. Let $\widetilde{N}$ denote the counting measure of the associated point process and let R be a finite union of half-open intervals $[a_i, b_i)$. Then

$$\text{Prob}\{\widetilde{N}(R) \neq 0\} = \text{Prob}\{\inf_{t \in R} (U_t) = 0\} \,.$$

By separability the right hand side is determined by the joint distributions of the $\{U_t\}$. However, the assumption of orderliness (H1) and Proposition 1.2 ensure that knowledge of the left hand side for all R determines the finite-dimensional distributions of the process. ✳

We define the conditional interval survivor function

$$\mathcal{F}(x \mid \omega_0) = \int_{\omega_1 \in \mathcal{W}} p(d\omega_1 \mid \omega_0) \{1 - F(x \mid \omega_0, \omega_1)\} = \text{Prob}\{X_1 > x \mid W_0 = \omega_0\}. \tag{8}$$

The transition function $p_\gamma(d\omega, du \mid \omega_0, u_0)$ of $\{\tilde{W}_t, U_t\}$ can now be written down. We define

$$G_1(d\omega_1, du \mid \omega_0, u_0) = \frac{p(d\omega_1 \mid \omega_0) F(u_0 + du \mid \omega_0, \omega_1)}{\mathcal{F}(u_0 \mid \omega_0)}, \tag{9}$$

and, inductively, n-step analogues of G,

$$G_n(d\omega_1, u \mid \omega_0, u_0) = \int_{\omega \in \mathcal{W}} \int_{y=0-}^{u+} G_1(d\omega, dy \mid \omega_0, u_0) G_{n-1}(d\omega_1, u-y \mid \omega, 0). \tag{10}$$

Then

$$G_n(A, u \mid \omega_0, u_0) = \text{Prob}\left(\sum_{i=1}^{n} X_i \le u_0 + u,\ W_n \in A \mid W_0 = \omega_0,\ X_1 > u\right).$$

<u>Proposition 3.4</u>. The transition functions of the process $\{\tilde{W}_t, U_t\}$ are given by

$$P_\gamma(d\omega_1, du_1 \mid \omega_0, u_0) = \begin{cases} \delta_{\omega_0}^{d\omega_1} \delta_{u_0+\gamma}^{du_1} \dfrac{\mathcal{F}(u_0 + \gamma \mid \omega_0)}{\mathcal{F}(u_0 \mid \omega_0)} & (u_1 \ge \gamma), \quad (11) \\[2ex] \mathcal{F}(u_1 \mid \omega_1) \displaystyle\sum_{n=1}^{\infty} G_n(d\omega_1, \gamma - du_1 \mid \omega_0, u_0) & (u_1 < \gamma). \quad (12) \end{cases}$$

<u>Proof</u>. If $u_1 \ge \gamma$, then $u_1 = u_0 + \gamma$ and the interval in progress at

time t must still be in progress at time $t + \tau$. If $u_1 < \tau$, then n events can occur in $(t, t+\tau - u]$ $(n = 1, 2, \ldots)$ including an event at $t + \tau - u$, and the interval $(t + \tau - u, t + \tau]$ must be empty. *

3.3. THE STATIONARY SEMI-MARKOV PROCESS

The fundamental result proved below has appeared in several guises in the literature on semi-Markov processes. Çinlar (1969a) adopts an approach heavily dependent on the theory of functions. Orey (1961) regards a result essentially equivalent to Proposition 3.5 as too obvious to require proof. His formulation differs from ours, in that he assumes that the length of the (n+1)th interval is a deterministic function of W_n. However, as he points out, the general process can be treated by his methods if the state space is suitably extended. We have been unable to see how the dterministic assumption simplies the argument. For processes with countable state space, the result is an immediate corollary of certain renewal-type limit theorems. We refer to Çinlar (1969b) and Pyke and Schaufele (1964). Note that our condition H2 implies the strong regularity of Pyke and Schaufele (1964). We shall give a direct proof that the postulated stationary distribution is invariant. Such a proof is scarcely more difficult for the general semi-Markov process than it is for the renewal process considered by Doob (1948). It does not seem to have been given before in this form.

Lemma. For any function $r(x, \omega)$ $(\mathbb{R} \times \mathcal{W} \to \mathbb{R}^+)$, any $a, b \in \mathbb{R}^+$, $\omega_0 \in \mathcal{W}$ and any measurable $A \subset \mathcal{W}$, we have

$$\int_{x=0}^{a}\int_{\omega\in A} r(x,\omega)\int_{u=0}^{\infty} G(d\omega, b+u-dx \mid \omega_0)du = \int_{x=0}^{a}\int_{\omega\in A} r(x,\omega)\int_{u=b-x}^{\infty} G(d\omega, du \mid \omega_0)dx. \tag{1}$$

Proof. This is a simple application of Fubini's theorem. Note that the inclusion or exclusion of the endpoints in the ranges of integration does not alter the value of either side of (1). ☀

Proposition 3.5. If (H1), (H2) and (H3) are satisfied, the Markov process $\{\tilde{W}_t, U_t\}$ $(t \in \mathbb{R})$ with state space $\mathcal{W} \times \mathbb{R}^+$ and transition functions given by (3.2.11) and (3.2.12) has an essentially unique stationary measure q given by

$$\tilde{q}(d\omega, du) = q(d\omega)\ \mathfrak{F}(u \mid \omega)\ du\ . \tag{2}$$

Proof. (i) Existence. Let $\tilde{q}_0(d\omega_0, du_0) = q(d\omega_0)\ \mathfrak{F}(u_0 \mid \omega_0)\ du_0$. We shall show that $\tilde{q}_\gamma(d\omega, du)$ is also given by (2). Now

$$\tilde{q}_\gamma(d\omega, du) = \int_{u_0=0}^{\infty}\int_{\omega_0\in\mathcal{W}} \tilde{q}_0(d\omega_0, du_0)p_\gamma(d\omega, du \mid \omega_0, u_0)\ . \tag{3}$$

If $u \geq \gamma$, substitution of (3.2.11) into (3) gives the result at once. If $u < \gamma$, (3) and (3.2.12) give

$$\tilde{q}_\gamma(d\omega, du) = \mathfrak{F}(u \mid \omega)\sum_{n=1}^{\infty}\int_{u_0=0}^{\infty}\int_{\omega_0\in\mathcal{W}} G_n(d\omega, \gamma-du \mid \omega_0, u_0)\,\mathfrak{F}(u_0 \mid \omega_0)q(d\omega_0)du_0 \tag{4}$$

We denote the corresponding expression with k replacing ∞ in the summation by $\tilde{q}_\gamma^{(k)}(d\omega, du)$. We show that

$$q(d\omega)\ \mathfrak{F}(u \mid \omega)du - \tilde{q}_\gamma^{(k)}(d\omega, du) = \mathfrak{F}(u \mid \omega)du \int_{\omega_0\in\mathcal{W}} G_k(d\omega, \gamma-u \mid \omega_0, 0)q(d\omega_0)\ , \tag{5}$$

in the sense that the integrals of the two sides over any measurable subset of $\mathcal{W} \times \mathbb{R}$ are equal. To prove (5) for k = 1, we write the left-hand side, using (3.2.6), as

$$\mathfrak{F}(u|\omega) \int_{\omega_0 \in \mathcal{W}} q(d\omega_0) \left\{ du \int_{y=0}^{\infty} G(d\omega, dy|\omega_0) - \int_{u_0=0}^{\infty} G(d\omega, \tau + u_0 - du|\omega_0) du_0 \right\}.$$

By the lemma, the second term in the bracket is

$$\int_{y=\tau-u}^{\infty} G(d\omega, dy|\omega_0).$$

To prove (5) inductively for a general k it is sufficient to show that

$$\mathfrak{F}(u|\omega) \int_{u_0=0}^{\infty} \int_{\omega_0 \in \mathcal{W}} G_{k+1}(d\omega, \tau - du|\omega_0, u_0) \mathfrak{F}(u_0|\omega_0) q(d\omega_0) du_0$$

$$= \mathfrak{F}(u|\omega) du \left\{ \int_{\omega_0 \in \mathcal{W}} G_k(d\omega, \tau - u|\omega_0, 0) q(d\omega_0) - \int_{\omega_0 \in \mathcal{W}} G_{k+1}(d\omega, \tau - u|\omega_0, 0) q(d\omega_0) \right\}.$$

This can be proved by expanding the terms containing G_{k+1} by the convolution formula (3.2.10) and applying the lemma to the right hand side.

By assumption H2 the right hand side of (5) tends to 0 as $k \to \infty$. Since $\tilde{q}_{\tau}^{(k)}(d\omega, du) \to \tilde{q}_{\tau}(d\omega, du)$ as $k \to \infty$, this proves existence.

<u>(ii) Uniqueness</u>. It follows from (3.2.11) that any stationary distribution for $\{\tilde{W}_t, U_t\}$ has the form

$$q'(d\omega)\, \mathfrak{F}(u|\omega)\, du .$$

If now we write down the equation corresponding to (4) for this postulated stationary distribution and allow $u \to \tau-$, the terms in which $n > 1$

become negligible compared to the first term by H1. After cancellation we see that q' must satisfy (3.2.6), and so, by H2, q' = q. ✳

Proposition 3.6. A process satisfying H1, H2 and H3 has a stationary probability distribution if and only if H4 below is satisfied.

Hypothesis H4. The mean interval length is finite, i.e.

$$\mu = \int_{\omega \in \mathcal{W}} \int_{u=0}^{\infty} q(d\omega)\ \mathcal{F}(u \mid \omega)du < \infty .$$

Proof. For then $\mu^{-1}\mathcal{F}(u \mid \omega)q(d\omega)du$ is a stationary probability distribution. ✳

We can now give definitions of synchronous and asynchronous realisations of the associated point process. For the process $\{\tilde{W}_t, U_t\}$ enables us to construct realisations with any prescribed initial conditions, i.e. not necessarily with an event at the origin. By a 'synchronous' process, with the origin at an 'average event' (cf. Lawrance 1971) is meant a process which has initial distribution

$$\tilde{q}_0(d\omega, du) = q(d\omega)\ \delta_0^{du} \quad . \tag{6}$$

An asynchronous process, with the origin at an 'arbitrary time', is obtained from the stationary initial distribution

$$\tilde{q}(d\omega, du) = \mu^{-1}\ q(d\omega)\ \mathcal{F}(u \mid \omega)du . \tag{7}$$

Then we have

Proposition 3.7. (i) The intervals of the synchronous process form a stationary discrete time process.

(ii) The asynchronous point process is completely stationary and has rate ρ.

Proof. Part (i) is Proposition 3.1 and Part (ii) follows from Propositions 3.3 and 3.6. ✳

Following Khintchine (1955) and others, we may define an 'arbitrary event' in the associated point process by taking the limit as $h \to 0+$ of the conditional distribution of $\{\tilde{W}_t, U_t\}$ at an arbitrary time, given that $U_t < h$.

Proposition 3.8. 'Average' and 'Arbitrary' event initial conditions are equivalent for processes satisfying all of H1 - H4.

Proof. This is immediate by dominated convergence, since

$\lim_{u \to 0} \mathfrak{F}(u \mid \omega) = 1$ for almost all ω. *

Note that orderliness is essential here. Without it we cannot sensibly define 'arbitrary' events. 'Average' events are still well-defined, however, and can be used to give an extended meaning to many of the Palm-Khintchine formulae discussed below.

3.4. OPERATIONS ON THE SEMI-MARKOV MODEL

Any point process has a formal representation as a semi-Markov process, as we can take $\mathcal{W}$ to be $(\mathbb{R}^+)^\infty$ and W_n the entire interval sequence up to the nth event, $W_n = \{\ldots X_{n-1}, X_n\}$. However, measure-theoretic difficulties arise in defining appropriate transition functions $p(d\omega_1 \mid \omega_0)$ and stationary measures $q(d\omega)$ on $\mathcal{W}$. These may be circumvented for particular processes by giving detailed constructions. It is helpful first to define some operations on the class of semi-Markov processes.

(A) Adjoining. Given a semi-Markov process $\{\tilde{W}_t, U_t\}$, it is often helpful to form a new semi-Markov process with larger state space but the same associated point process. We consider a few examples.

<u>(i) Independent Adjoining</u>. This is analogous to the independent marking discussed by Matthes (1963). Let $\{Z_i\}$ $(i = 0, \pm 1, \pm 2, \ldots)$ be any sequence of independent identically distributed random variables taking values in the space $\mathcal{Z}$ according to a probability measure $p_Z(d\omega_Z)$. Then if $\mathcal{W}^a = \mathcal{W} \times \mathcal{Z}$ and $W^a_n = \{W_n, Z_n\}$, $p(d\omega_1 \mid \omega_0)$ induces the Markov transition function $p^a(d\omega^a_1 \mid \omega^a_0)$ on $\mathcal{W}^a$, where

$$p^a(d\omega_1{}^a \mid \omega_0{}^a) = p(d\omega_1 \mid \omega_0)\, p_Z\, (d\omega_Z)\,.$$

The stationary distribution of this chain is

$$q^a(d\omega^a) = q(d\omega)\, p_Z(d\omega_Z)\,,$$

and of the continuous time process $\{\tilde{W}_t{}^a, U_t\}$ is

$$\tilde{q}^a(d\omega^a, du) = \rho \mathcal{F}(u \mid \omega)\, q(d\omega) du\, p_Z(d\omega_Z)\,.$$

Thus $\tilde{Z}_t = Z_{N_t}$ is independent of $\{\tilde{W}_t, U_t\}$ as would be expected.

<u>(ii) Adjoining the Next Interval and State</u>. Let $\mathcal{W}^a = \mathcal{W} \times \mathcal{W} \times \mathbb{R}^+$ and $W^a_n = (W_n, W_{n+1}, X_{n+1})$. The transition function is

$$p^a(d\omega_1{}^a \mid \omega_0{}^a) = p(d\omega_2 \mid \omega_1)\, F(dx_2 \mid \omega_1, \omega_2)\,.$$

The new conditional interval distribution is degenerate. The induced stationary distribution of $W_n{}^a$ is given by

$$q^a(d\omega_0{}^a) = q(d\omega_0)\, p(d\omega_1 \mid \omega_0)\, F(dx_1 \mid \omega_0, \omega_1)\,,$$

and that of $(\tilde{W}_t{}^a, U_t)$ is

$$\tilde{q}^a(d\omega^a, du) = \rho q(d\omega_0)\, p(d\omega_1 \mid \omega_0)\, F(dx_1 \mid \omega_0, \omega_1)\, I^u_{x_1}\, du\,,$$

$$\text{where } I^u_{x_1} = \begin{cases} 1 & (u \le x_1)\,, \\ 0 & (u > x_1)\,. \end{cases}$$

Note that the conditional distribution of the backward recurrence time U_t given $\tilde{W}_t^a$ is uniform over $[0, x_1)$, where x_1 is the length of the interval in progress at time t. Moreover, the unconditional distribution of the length of the interval in progress, obtained from (1) by integrating out ω_0, ω_1 and u, is $\rho x_1 F(dx_1)$, where F(.) is the synchronous marginal interval distribution. These are precisely the results obtained by heuristic length-biased sampling arguments (cf. Cox and Lewis 1966).

(iii) Adjoining the Previous State

Let $\mathcal{W}^a = \mathcal{W} \times \mathcal{W}$ and $W_n^a = \{W_{n-1}, W_n\}$. Then the transition function and stationary distribution are given by

$$p^a(d\omega_1^a \mid \omega_0^a) = p(d\omega_1^{1} \mid \omega_0^{1})\, \delta_{\omega_0^{1}}^{d\omega_1^{0}} , \tag{8}$$

where $W_n^a = \{W_n^0, W_n^1\}$,

$$q^a(d\omega_1^a) = q(d\omega_1^0)\, p(d\omega_1^{1} \mid \omega_1^0) , \tag{9}$$

$$\tilde{q}^a(d\omega_1^a, du) = q(d\omega_1^0)\, p(d\omega_1^{1} \mid \omega_1^0)\, \mathcal{F}(u \mid \omega_1^{1})du . \tag{10}$$

Thus the stationary distribution of W_n^a is obtained by giving the first component $W_n^0 = W_{n-1}$ the distribution $q(d\omega)$. The asynchronous stationary distribution is always obtained from the synchronous stationary distribution by Proposition 3.5.

(B) Filtering. Let $\mathcal{W}^f \subset \mathcal{W}$ be measurable, and suppose that entry into $\mathcal{W}^f$ occurs infinitely often with probability one. Then the filtered process is obtained by deleting all transitions except those into $\mathcal{W}^f$. Filtering of processes with countable state space has been discussed by Çinlar (1969b). The transition functions are given by

$$F^f(x \mid \omega_0^f, \omega_1^f)\, p(d\omega_1^f \mid \omega_0^f)$$

$$= \sum_{r=0}^{\infty} \int^{(r)}_{\substack{\bar{\omega}_1 \in \mathcal{W} \setminus \mathcal{W}^f \\ \vdots \\ \bar{\omega}_r \in \mathcal{W} \setminus \mathcal{W}^f}} \int^{(r+1)}_{\sum_0^r x_i \le x} p(d\bar{\omega}_1 \mid \omega_0^f)\, p(d\bar{\omega}_2 \mid \bar{\omega}_1) \ldots p(d\omega_1^f \mid \bar{\omega}_r)$$

$$\times\, F(dx_0 \mid \omega_0^f, \bar{\omega}_1)\, F(dx_1 \mid \bar{\omega}_1, \bar{\omega}_2) \ldots F(dx_r \mid \bar{\omega}_r, \omega_1^f)\,.$$

It is easy to verify that the stationary distribution on $\mathcal{W}^f$ is

$$q^f(d\omega^f) = \frac{q(d\omega^f)}{q(\mathcal{W}^f)}\,,$$

and that the intensity of the filtered process is $\rho\, q(d\omega^f)$. In general the asynchronous stationary distribution does not take a simple form.

If $\omega^f \in \mathcal{W}$ is an atom of $q(.)$, i.e. if $q(\{\omega^f\}) > 0$, and if $\mathcal{W}^f = \{\omega^f\}$, then the filtered process is a renewal process. This single-point filtering is a useful device when $\mathcal{W}$ is countable, but will not usually be possible otherwise.

<u>(C) Superposition.</u> Let $(\tilde{W}_t^{(a)}, U_t^{(a)})$ and $(\tilde{W}_t^{(b)}, U_t^{(b)})$ be two independent semi-Markov processes. The superposition can be represented as a semi-Markov process in the following way. Define

$$W_n^s = (I_n, W_n^{(a)}, W_n^{(b)}, V_n)\,,$$

where

$$I_n = \begin{cases} 0 \text{ if the nth event is of type A,} \\ 1 \text{ if the nth event is of type B,} \end{cases}$$

$W_n^{(a)}$ and $W_n^{(b)}$ define the states of the two processes just after the nth event (in the combined process), and V_n is the semi-synchronous backward recurrence time. The transition probabilities of the Markov chain $\{W_n^s\}$ involve the conditional interval distributions F_a and F_b of the two processes as well as the Markov transition probabilities. There are four possibilities for transitions $W_0^s \rightarrow W_1^s$, corresponding to the four values of (I_0, I_1):

(A) $I_0 = 0,\ I_1 = 0$. Here the transition probability is

$$p^s(0, d\omega_1^{(a)}, d\omega_1^{(b)}, dv_1 \mid 0, \omega_0^{(a)}, \omega_0^{(b)}, v_0)$$

$$= p_a(d\omega_1^{(a)} \mid \omega_0^{(a)})\, F_a(dv_1 - v_0 \mid \omega_0^{(a)}, \omega_1^{(a)})\, \delta_{\omega_0^{(b)}}^{d\omega_1^{(b)}}\, \frac{\mathcal{F}_b(v_1 \mid \omega_0^{(b)})}{\mathcal{F}_b(v_0 \mid \omega_0^{(b)})},$$

and the conditional distribution of X_1 is degenerate, i.e.

$$F(dx_1 \mid \omega_0^s, \omega_1^s) = \delta_{v_1 - v_0}^{dx_1} \quad ;$$

(B) $I_0 = 0,\ I_1 = 1$. Here we obtain

$$p^s(1, d\omega_1^{(a)}, d\omega_1^{(b)}, dv_1 \mid 0, \omega_0^{(a)}, \omega_0^{(b)}, v_0)$$

$$= p_b(d\omega_1^{(b)} \mid \omega_0^{(b)})\, \frac{F_b(dv_1 + v_0 \mid \omega_0^{(b)}, \omega_1^{(b)})}{\mathcal{F}_b(v_0 \mid \omega_0^{(b)})}\, \delta_{\omega_0^{(a)}}^{d\omega_1^{(a)}}\, \mathcal{F}_a(v_1 \mid \omega_0^{(a)}).$$

The conditional distribution of X_1 is then

$$F(dx_1 \mid \omega_0^s, \omega_1^s) = \delta_{v_1}^{dx_1}$$

Cases (C) and (D) corresponding to $I_0 = 1,\ I_1 = 0$ and $I_0 = 1,\ I_1 = 1$ can be deduced from (B) and (A) by exchanging A and B throughout. The synchronous stationary distribution is

$$q^s(0, d\omega^{(a)}, d\omega^{(b)}, dv) = \frac{\rho_a\rho_b}{\rho_a+\rho_b}\, q_a(d\omega^{(a)})\, q_b(d\omega^{(b)})\, \mathcal{F}_b(v\,|\,\omega^{(b)})dv,$$

$$q^s(1, d\omega^{(a)}, d\omega^{(b)}, dv) = \frac{\rho_a\rho_b}{\rho_a+\rho_b}\, q_a(d\omega^{(a)})\, q_b(d\omega^{(b)})\, \mathcal{F}_a(v\,|\,\omega^{(a)})dv\ .$$

The asynchronous stationary distribution q^s is given by

$$q^s(0, d\omega^{(a)}, d\omega^{(b)}, dv, du)$$

$$= \rho_a\rho_b q_a(d\omega^{(a)})q_b(d\omega^{(b)})\, \mathcal{F}_a(u\,|\,\omega^{(a)})\, \mathcal{F}_b(u+v\,|\,\omega^{(b)})du\ dv\ ,$$

with a similar result when I = 1.

3.5 SOME EXAMPLES

We now consider a few examples of point processes which can be given a semi-Markov representation.

(A) The Renewal Process. For the renewal process, $\mathcal{W}$ may be taken as a one-point set. The transition functions and stationary distribution of the Markov chain are trivial. Suppose that the interval distribution has density $\phi(x)$, survivor function $\Phi(x)$ and mean μ. The usual renewal density is

$$k_\phi(x) = \sum_{n=0}^{\infty} \phi^{n*}(x)\ , \tag{1}$$

where $\phi^{n*}(x)$ is the n-fold convolution of ϕ with itself, ($\phi^{0*}(x) = \delta_0^{dx}$).

We define also the delayed renewal density

$$\tilde{k}_\phi(x, u) = \int_{y=0}^{x} \frac{\phi(y+u)}{\Phi(u)}\, k_\phi(x-y)\, dy\ . \tag{2}$$

This is the renewal density of a modified process in which the component in use at $t = 0$ is known to have age u. The asynchronous process $\{U_t\}$ has transition density

$$p_\tau(u_1 \mid u_0) = \begin{cases} \tilde{k}_\phi(\tau - u_1 \mid u_0)\ \Phi(u_1) & (u_1 \leq \tau), \\ \delta_{u_0+\tau}^{du_1} \dfrac{\Phi(u_0+\tau)}{\Phi(u_0)} & (u_1 > \tau). \end{cases} \quad (3)$$

<u>(B) Countable Semi-Markov Processes.</u> These will be discussed in Chapter 6.

<u>(C) The Wold Process and Extensions.</u> Suppose that the state variable at the nth event is the length of the previous interval, so that $p(d\omega_n \mid \omega_{n-1}) = H(d\omega_n \mid \omega_{n-1})$ say, and $F(dx_n \mid \omega_{n-1}, \omega_n) = \delta_{\omega_n}^{dx_n}$. Then we have a representation of the Markov process of intervals discussed by Wold (1948a, 1948b), Cox (1955) and others. If $q(d\omega)$ is the stationary distribution for $\{W_n\}$, then the synchronous stationary interval distribution is

$$\int_{\omega_n \in \mathcal{W}} \int_{\omega_{n+1} \in \mathcal{W}} q(d\omega_n)\, p(d\omega_{n+1} \mid \omega_n)\, F(dx \mid \omega_n, \omega_{n+1}) = q(dx).$$

The asynchronous stationary initial conditions are given by

$$\tilde{q}(d\omega du) = \rho q(d\omega) \int_{x=u}^{\infty} \int_{\omega_1 \in \mathcal{W}} p(d\omega_1 \mid \omega)\, \delta_{\omega_1}^{dx}\, du$$

$$= \rho q(d\omega)\, \breve{\mathcal{F}}(u, \omega)\, du.$$

Similarly, the k-dependent renewal process has a representation with $\mathcal{W} = (R^+)^k$ and $W_n = (X_n, X_{n-1}, \ldots, X_{n-k+1})$.

<u>(D) The Doubly Stochastic Poisson Process with Markovian Rate.</u> Any semi-Markov representation of the general doubly stochastic Poisson

process must necessarily be rather complicated. However, when the rate process $\Lambda(t)$ is Markovian, there exists a representation with $\mathcal{W} = R^+$. For we can take $W_n = \Lambda(T_n)$. The transition functions and conditional interval distributions can be written down, but not usually in a simple form. Particular processes for which this approach might prove useful are the random hazard process studied by Gaver (1963) and Lawrance (1971), the shot noise process with exponential decay (cf. Chapter 6) and the Ornstein-Uhlenbeck process also discussed by Lawrance (1971).

(E) Clustering Processes. A fairly general treatment of cluster processes can be given from the semi-Markov viewpoint, but is extremely messy. We shall outline the approach without giving full details. It is assumed that a process of main events is given by a semi-Markov process $\{W_n, Z_n\}$. Each main event independently generates a cluster of subsidiary events, which are here assumed to follow the main event. The cluster structure is specified by the random variables $(M, Y_1, \ldots, Y_M)$ where M is the number of events generated (possibly zero), Y_1 is the distance from the main event to the first subsidiary event of the cluster and Y_i $(2 \leq i < M)$ are the distances between successive subsidiary events. We say that the cluster is operative at time t if the main event occurs at or before time t and the last subsidiary event occurs after time t. Then the state variable corresponding to the full process must include the state and the backward recurrence time of the main process, the number of operative clusters and a specification of the relevant history of each. The existence of a stationary distribution for the main process does not ensure the existence of a stationary cluster process without some conditions on the cluster structure.

3. 6. PALM-KHINTCHINE TYPE RESULTS

The semi-Markov representation allows a simple derivation of the so-called 'generalised Palm-Khintchine relations' connecting the joint distributions of synchronous and asynchronous intervals. We shall denote the synchronous sequence, measured from an 'arbitrary event', by $(X_1, X_2, \ldots)$, with joint distributions $F_1(dx_1)$, $F_2(dx_1, dx_2), \ldots$ The asynchronous sequence $(\tilde{X}_1, \tilde{X}_2, \ldots)$ measured from an arbitrary time will have joint distributions $\tilde{F}_1(dx_1)$, $\tilde{F}_2(dx_1, dx_2)$, etc. We assume that assumptions H1 - H4 are satisfied.

Proposition 3. 9. The joint distributions of n successive intervals under asynchronous and synchronous sampling are related by the equation

$$\tilde{F}_n(dx_1, dx_2, \ldots, dx_n) = \rho\, F_n\{(x_1, \infty), dx_2, \ldots, dx_n\} dx_1 . \tag{1}$$

Proof. The left hand side is

$$\rho \int_{u=0}^{\infty} \int_{\substack{\omega_0 \in \mathcal{W} \\ \vdots \\ \omega_n \in \mathcal{W}}}^{(n+1)} q(d\omega_0)\, \mathcal{F}(u \mid \omega_0) du\, p(d\omega_1 \mid \omega_0) \frac{F(u+dx_1 \mid \omega_0, \omega_1)}{\mathcal{F}(u \mid \omega_0)}$$

$$\times \prod_{i=2}^{n} p(d\omega_i \mid \omega_{i-1})\, F(dx_i \mid \omega_{i-1}, \omega_i)$$

$$= \rho\, dx_1 \int_{v=x_1}^{\infty} \int_{\substack{\omega_0 \in \mathcal{W} \\ \vdots \\ \omega_n \in \mathcal{W}}}^{(n+1)} q(d\omega_0) p(d\omega_1 \mid \omega_0)\, F(dv \mid \omega_0, \omega_1) \prod_{i=2}^{n} p(d\omega_i \mid \omega_{i-1}) F(dx_i \mid \omega_{i-1}, \omega_i)$$

(by the lemma to theorem 2. 5), and this is the same as the right-hand side. ✳

This result is the integral form of a relation conjectured in Lawrance (1971). In view of its importance we give an alternative, more intuitive, derivation using arguments from length-biased sampling. Suppose that we construct a synchronous realisation of the process and pick a point of $\mathbb{R}$ at random, independently of the process, to define the origin of the asynchronous process. Then the joint distribution of the length Y of the interval in which the origin falls and of the succeeding n-1 intervals $X_2, \ldots, X_n$ is

$$\frac{yF_n(dy, dx_2, \ldots, dx_n)}{\int^{(n)} y\, F_n(dy, dx_2, \ldots, dx_n)} = \rho\, y\, F_n(dy, dx_2, \ldots, dx_n)\ ,$$

since the chance of a random point falling in a particular interval is proportional to its length but is otherwise independent of the process. The position of this random origin within the interval is uniformly distributed over its length (0, y), so that

$$\tilde{F}_n(dx_1, \ldots, dx_n) = \int_{y=x_1}^{\infty} \frac{dx_1}{y}\, \rho\, y\, F_n(dy, dx_2, \ldots, dx_n)$$

$$= \rho\, F_n((x_1, \infty),\ dx_2, \ldots, dx_n) dx_1 .$$

All the usual moment formulae may be deduced from (1). For n = 1, (1) reduces to the well-known relation between the distributions of intervals and forward recurrence times.

A slightly stronger result than Proposition 3. 9 is needed for the work of Chapter 4.

Proposition 3.10. Let E be any set in the σ-field generated by $\{W_1, X_2, W_2, X_3, \ldots\}$, and let $\tilde{F}(dx, E)$, $F(dx, E)$ denote the improper asynchronous and synchronous distributions, i.e.

$$F(dx_1, E) = \text{Prob}\{\tilde{X}_1 \in dx_1, (\tilde{W}_1, X_2, \ldots) \in E\},$$

$$F(dx_1, E) = \text{Prob}\{X_1 \in dx_1, (W_1, X_2, \ldots) \in E\}.$$

Then $\tilde{F}(dx_1, E) = \rho F\{(x_1, \infty), E\} dx_1$.

Proof. This is the same as the proof of Proposition 3.9, except that the product $\prod_{i=2}^{n}$ is replaced by $\prod_{i=2}^{\infty}$ and the domain of integration $\{(\omega_1, \ldots, \omega_n) \in \mathcal{W}^n$ by $\{(\omega_1, x_2, \omega_2, \ldots) \in E\}$. *

It is also possible to derive relations involving both the forward and backward recurrence times. For example the result (Matthes, 1963)

$$\tilde{f}_2(u, v) = \rho f_1(u + v)$$

for the joint density of the forward and backward recurrence times holds if the density f_1 of F_1 exists.

3.7. NON-ORDERLY POINT PROCESSES

Although only orderly processes will be discussed in later chapters we shall show here how the methods developed in this chapter may be used to attack non-orderly processes. Specifically we shall show that Proposition 3.9 does not require orderliness. We shall use the standard device of collapsing the point process (Milne, 1971), replacing each multiple occurrence by a single occurrence at the same point to define an induced orderly point process. Some care is needed over the definition of the state variable W_n^* of the collapsed process. For our purposes it is sufficient to take $W_n^* = (\overline{W}_n, K_n)$, where $\overline{W}_n$ is the final state entered by the original process at each transition and K_n is the multiplicity of the transition. A discussion of non-orderly multivariate processes would require a larger state variable. Note that in continuous time $\tilde{W}_t = \tilde{\overline{W}}_t$, and that the conditional distributions $p(d\omega_n^* | \omega_{n-1}^*)$ and $F(dx | \omega_{n-1}^*, \omega_n^*)$ are independent of k_{n-1}. Thus $\{\overline{W}_n\}$ is itself a Markov chain.

Lemma. Suppose that a semi-Markov process $\{X_n, W_n\}$ satisfies H2, H3 and H4. Then the collapsed process satisfies all of H1 - 4.

Proof. This is straightforward. The stationary distribution of the chain $\{\overline{W}_n\}$ is $q(d\omega) = \alpha^{-1} \mathfrak{F}(0|\omega)\, q(d\omega)$, where $\alpha = \int q(d\omega)\, \mathfrak{F}(0|\omega) \cdot d\omega > 0$. The intensity of the collapsed process is $\alpha\rho$. ✳

Proposition 3.11. The relation between synchronous and asynchronous interval distributions given by Proposition 3.9 holds even if H3 is not satisfied.

Proof. We apply Proposition 3.10 to the collapsed process, noting that any event E of the σ-field generated by $(X_2, X_3, \ldots)$ belongs also to the σ-field generated by $(W_1, X_2, W_2, \ldots)$. We have, for $x_1 > 0$,

$$F(dx_1, E) = \alpha\, F^*(dx_1, E)$$

Hence

$$\tilde{F}(dx_1, E) = \tilde{F}^*(dx_1, E) = \alpha\rho F^* \{ (x_1, \infty), E \} \, dx_1$$

$$= \rho\, F \{ (x_1, \infty),\ E \} \, dx_1 .$$

3.8. DISCUSSION

It would be very convenient if a semi-Markov representation could be found for any stationary stochastic point process with finite rate. As has been pointed out in Section 3.4, a rigorous construction would require a rather deeper measure-theoretic background than has been considered here. The constructive approach given above is difficult to apply, for example, to the most general form of the doubly stochastic Poisson process.

A general restriction of the model is that it cannot apply to processes with infinite rate. For if $\rho = \infty$ then $E(X_n \mid W_{n-1}, W_n) = 0$ almost surely, for each n and thus $X_n = 0$ almost surely for each n, contradicting H1.

CHAPTER 4

SOME APPLICATIONS TO MULTIVARIATE POINT PROCESSES

4. 1. INTRODUCTION AND PRELIMINARY DEFINITIONS

The semi-Markov model provides a useful approach to the theory of multivariate point processes. In Section 4. 2, the results of Chapter 3 are used to derive a simple generalisation to multivariate processes of the usual formula for the forward recurrence time in a univariate process with known interval distribution. In Section 4. 3, this result is used to derive Palm-Khintchine formulae for multivariate processes. The concept of deterministic thinning introduced here appears to be new. For bivariate processes results of Milne (1971) and Wisniewski (1972, 1973) are recovered.

An m-variate semi-Markov process is a semi-Markov process $\{X_n, W_n\}$ whose state space $\mathcal{W}$ is partitioned into m measurable subsets $\mathcal{W}_1, \ldots, \mathcal{W}_m$. A transition into $\mathcal{W}_i$ defines an event of type i. It is assumed that all of the hypotheses H1 - 4 of Chapter 3 are satisfied. Thus the Markov chain $\{W_n\}$ has a unique stationary distribution $q(d\omega)$ which defines synchronous and asynchronous stationary initial conditions in the usual way. Moreover, the associated (multivariate) point process is strongly orderly and the rate ρ of the superposed process satisfies $0 < \rho < \infty$. Provided that $q(\mathcal{W}_i) > 0$ $(1 \leq i \leq m)$, the chain $\{W_n\}$ enters each $\mathcal{W}_i$ infinitely often, and so the marginal processes obtained by filtering by each $\mathcal{W}_i$ in turn are all well defined, with the transition functions given in Section 3. 4. If the original process is stationary each marginal process is also stationary and the rates ρ_i $(1 \leq i \leq m)$ satisfy $\sum \rho_i = \rho$. Synchronous and asynchronous realisations are defined as in Chapter 3. A semi-synchronous realisation, corresponding to an 'average' or 'arbitrary' event of specified type i at the origin, is obtained by conditioning the stationary distribution $q(d\omega)$ of W_0 on the σ-event $\{W_0 \in \mathcal{W}_i\}$.

Thus

$$q_i(d\omega) = \frac{q(d\omega)}{q(\mathcal{W}_i)} \qquad (\omega \in \mathcal{W}_i) . \tag{1}$$

The joint distributions of the synchronous forward recurrence times $\{V_{\cdot j} : 1 \leq j \leq m\}$, of the semi-synchronous forward recurrence times $\{V_{ij} : 1 \leq j \leq m\}$ and of the asynchronous forward recurrence times $\{V_{0j} : 1 \leq j \leq m\}$ may all be expressed in terms of the basic functions $q(.)$, $p(.)$ and $F(.\,|\,.,.)$. Note that the first suffix, i, denotes the type of event, if any, at the origin and the second suffix, j, the type of event to which the recurrence time is measured. There is a simple relation connecting the semi-synchronous and synchronous distributions of the process.

Proposition 4.1. Let E be any σ-set in the σ-field generated by $\{X_1, W_1, X_2, W_2, \ldots\}$, and let $p.(E)$ and $p_i(E)$ $(1 \leq i \leq m)$ denote the synchronous and semi-synchronous probabilities of E, respectively. Then

$$\rho\, p.(E) = \sum_{i=1}^{m} \rho_i\, p_i(E) . \tag{2}$$

Proof. We have

$$\rho\, p.(E) = \rho \int_{\omega \in \mathcal{W}} \text{Prob}(E|\omega)\, q(d\omega)$$

$$= \sum_{i=1}^{m} \rho\, q(\mathcal{W}_i) \int_{\omega \in \mathcal{W}_i} \text{Prob}(E|\omega) \frac{q(d\omega)}{q(\mathcal{W}_i)}$$

$$= \sum_{i=1}^{m} \rho_i\, p_i\,(E) .$$ *

4.2. A MULTIVARIATE RECURRENCE TIME RELATION

Let X_{ij} $(0 \leq i \leq m, \quad 1 \leq j \leq m)$ be defined as follows. Let $\underset{\sim}{\alpha} = (\alpha_1, \ldots, \alpha_m)$ be the unique permutation of $\mathcal{M} = (1, \ldots, m)$ such that

$$V_{i\alpha_1} < \ldots < V_{i\alpha_m} .$$

Define $X_{i1} = V_{i\alpha_1}$ and $X_{ij} = V_{i\alpha_j} - V_{i\alpha_{j-1}}$ $(j > 1)$. Suppose that the improper joint distributions of $(X_{i1}, \ldots, X_{im})$ corresponding to each $\underset{\sim}{\alpha}$ are $F_i^{(\underset{\sim}{\alpha})}(dx_1, \ldots, dx_m)$. Then Propositions 3.10 and 4.1 give, for the asynchronous distribution

$$F_0^{\underset{\sim}{\alpha}}(dx_1, \ldots, dx_m) = \rho\, F.^{(\underset{\sim}{\alpha})} \{(x_1, \infty),\ dx_2, \ldots, dx_m\}\, dx_1$$

$$= \sum_{i=1}^{m} \rho_i\, F_i^{(\underset{\sim}{\alpha})} \{(x_1, \infty),\ dx_2, \ldots, dx_m\}\, dx_1 . \tag{1}$$

Define the synchronous and asynchronous joint survivor functions

$$\mathcal{F}_i(v_1, \ldots, v_m) = \text{Prob}\{V_{ij} > v_i \ (1 \leq j \leq m)\} \quad (i = 0, 1, \ldots, m) . \tag{2}$$

Then equation (1) yields a simple relation among the $\mathcal{F}_i$. For clarity this is proved first for m = 2, and then generally.

Proposition 4.2. The function $\mathcal{F}_0(v_1, v_2)$ is absolutely continuous as a function of v_1 when v_2 is held fixed and vice versa. Moreover,

$$\left(\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2}\right) \mathcal{F}_0(v_1, v_2) = -\rho_1 \mathcal{F}_1(v_1, v_2) - \rho_2 \mathcal{F}_2(v_1, v_2) . \tag{3}$$

Proof. Here there are only two permutations of $\mathcal{M}$ say (1) = (1, 2) and (2) = (2, 1). If $v_1 \leq v_2$, the asynchronous bivariate survivor function is

$$\mathcal{F}_0(v_1, v_2) = \int_{x=v_1}^{\infty} \int_{y=(v_2-x)^+}^{\infty} F_0^{(1)}(dx, dy) + \int_{x=v_2}^{\infty} \int_{y=0}^{\infty} F_0^{(2)}(dx, dy) , \tag{4}$$

where $(v_2-x)^+$ denotes max $(v_2-x, 0)$. Substitution of (1) into (4) gives, for the coefficient of ρ_1 in $\mathcal{F}(v_1, v_2)$,

$$\int_{x=v_1}^{\infty} \int_{y=(v_2-x)^+}^{\infty} \int_{z=x}^{\infty} F_1^{(1)}(dz, dy)dx + \int_{x=v_2}^{\infty} \int_{y=0}^{\infty} \int_{z=x}^{\infty} F_1^{(2)}(dz, dy)dx.$$

For a fixed v_2, this is an integral with respect to v_1 of

$$- \int_{y=v_2-v_1}^{\infty} \int_{z=v_1}^{\infty} F_1^{(1)}(dz, dy) , \tag{5}$$

and, for a fixed v_1, it is an integral with respect to v_2 of

$$- \int_{y=0}^{v_2-v_1} \int_{z=v_2-y}^{\infty} F_1^{(1)}(dz, dy) - \int_{y=0}^{\infty} \int_{z=v_2}^{\infty} F_1^{(2)}(dz, dy) . \tag{6}$$

Combination of (5) and (6) after changing the order of integration gives the expression corresponding to (4) for the semi-synchronous survivor function $\mathcal{F}_1(v_1, v_2)$. The analogous result holds for the coefficient of ρ_2 and the case $v_1 \geq v_2$ may be dealt with similarly. *

This result immediately yields a number of moment formulae. For example, integration of (3) over the domain $0 \leq v_1, v_2 < \infty$ gives

$$E(V_{01}) + E(V_{02}) = \rho_1 E(V_{11} V_{12}) + \rho_2 E(V_{21} V_{22}), \qquad (7)$$

and integration of (3) over the domain $0 \leq v_2 < \infty$, $0 \leq u \leq v_2 < \infty$ gives

$$\frac{1}{2} E(V_{01}{}^2) + \frac{1}{2} E(V_{02}{}^2) + 2 E(V_{01} V_{02}) = \frac{1}{2} \sum_{i=1}^{2} \rho_i E\{V_{i1} V_{i2}(V_{i1}+V_{i2})\}, \qquad (8)$$

These give the formulae of Wisniewski (1972) quoted by Cox and Lewis (1972), on application of the usual univariate results of the form $E(V_{01}{}^2) = (1/3)\rho_1 E(V_{11}{}^3)$. They hold in the sense that if either side is finite, then the other side is also and they are equal.

In the general case we have

<u>Proposition 4.3.</u> The asynchronous and semi-synchronous joint distributions of forward recurrence times of an m-variate semi-Markov point process are related by the equation

$$\sum_{i=1}^{m} \frac{\partial}{\partial v_i} \mathcal{F}_0(v_1, \ldots, v_m) = - \sum_{i=1}^{m} \rho_i \mathcal{F}_i(v_1, \ldots, v_m). \qquad (9)$$

<u>Proof.</u> Let $\mathcal{A}$ denote the set of all permutations $\underset{\sim}{\alpha}$ of $\mathcal{M}$. Then for $0 \leq i \leq m$,

$$\mathcal{F}_i(v_1, \ldots, v_m) = \sum_{\underset{\sim}{\alpha} \in \mathcal{A}} \int_{\mathcal{R}} F_i^{(\underset{\sim}{\alpha})}(dx_1, \ldots, dx_m), \qquad (10)$$

where the domain of integration $\mathcal{R}$ is specified by the inequalities

$$x_j \geq 0 \qquad (1 \leq j \leq m),$$

$$x_1 + \ldots + x_j \geq v_{\alpha_j} \qquad (1 \leq j \leq m).$$

Using (1), we obtain

$$\mathcal{F}_0(v_1,\ldots,v_m) = \sum_{\underset{\sim}{\alpha}\in\mathcal{A}} \sum_{i=1}^{m} \rho_i K_i^{(\underset{\sim}{\alpha})}(v_1,\ldots,v_m), \tag{11}$$

where

$$K_i^{(\underset{\sim}{\alpha})}(v_1,\ldots,v_m) = \int_{\mathcal{S}} F_i^{(\underset{\sim}{\alpha})}(dx_1,\ldots,dx_m)dx_0 \quad (1 \le i \le m).$$

Here $\mathcal{S}$ denotes the region of $\mathbb{R}^{m+1}$ defined by the inequalities

$$x_i \ge 0 \quad (i = 2,\ldots,m),$$

$$x_0 > v_{\alpha_1},$$

$$x_1 > x_0,$$

$$\sum_{j=2}^{k} x_j + x_0 > v_{\alpha_k} \qquad (k = 2,\ldots,m).$$

If for some $j < k$, $v_{\alpha_j} > v_{\alpha_k}$, then the inequality including that v_{α_k} is redundant. Therefore the derivative of $K_i^{(\underset{\sim}{\alpha})}(v_1,\ldots,v_m)$ with respect to v_{α_k} is zero. Otherwise, say for $k \in \mathcal{M}_{\underset{\sim}{\alpha}} \subset \mathcal{M}$, we have

$$-\frac{\partial}{\partial v_{\alpha_k}} K_i^{(\underset{\sim}{\alpha})}(v_1,\ldots,v_m) = \int_{\mathcal{R}_k} F_i^{(\underset{\sim}{\alpha})}(dx_1,\ldots,dx_m), \tag{12}$$

where $\mathcal{R}_k$ is the region of $\mathbb{R}^m$ defined by the inequalities

$$x_i \ge 0 \qquad (i = 1,\ldots,m),$$

$$\sum_{1}^{k} x_j > v_{\alpha_k},$$

$$\sum_{j=\ell+1}^{k} x_j \le v_{\alpha_k} - v_{\alpha_\ell} \qquad (\ell = 1,\ldots,k-1),$$

$$\sum_{j=k+1}^{\ell} x_j > v_{\alpha_j} - v_{\alpha_k} \qquad (\ell = k+1, \ldots, m) .$$

The differentiation under the integral sign here is justified by the constant sign of the right hand side of (12), the Radon-Nikodym interpretation of derivatives and Fubini's theorem. Now $\mathcal{R}_k \subset \mathcal{R}$ for each $k \in \mathcal{M}_{\alpha}$. Also if $\underset{\sim}{x} \in \mathcal{R}$, then $\underset{\sim}{x} \in \mathcal{R}_k$ for precisely one $k \in \mathcal{M}_{\alpha}$. For, if

$$\theta_j = \sum_{\ell=1}^{j} x_\ell - v_{\alpha_j} \qquad (1 \leq j \leq m),$$

then $\underset{\sim}{x} \in \mathcal{R}_k$ if and only if $\theta_k - \theta_j \leq 0$ $(j \leq k)$ and $\theta_k - \theta_j > 0$ $(j > k)$. This holds for a unique $k \in \mathcal{M}$. In fact $k \in \mathcal{M}_{\alpha}$, for otherwise there would exist a $j < k$ with $v_{\alpha_j} > v_{\alpha_k}$, and so $\theta_j < \theta_k$. It follows that

$$-\sum_{k=1}^{m} \frac{\partial}{\partial v_{\alpha_k}} K_i^{(\alpha)}(v_1, \ldots, v_m) = \int_{\mathcal{R}} F_i^{(\alpha)}(dx_1, \ldots, dx_m) ,$$

which, together with (10) and (11), proves the Proposition. ✱

If $m = 2$ and the marginal processes are independent, then each joint survivor function factorises and

$$\frac{\partial}{\partial v_i} \mathcal{F}_0(v_1, v_2) = -\rho_i \mathcal{F}_i(v_1, v_2) \qquad (i = 1, 2) . \qquad (12)$$

An example of a process with dependent marginals for which (12) holds is given in Section 5.4.

4.3. MULTIVARIATE PALM-KHINTCHINE FORMULAE

Consider a stationary multivariate point process $\mathcal{P}$ with m different types of event and let $N_{0j}(0, t]$ denote the counting measure of the jth marginal. If $\mathcal{P}$ has a semi-Markov representation, then semi-synchronous counting measures $N_{ij}(0,t]$ $(1 \le i \le m)$, corresponding to events of type i at the origin may also be defined. Moreover, for each i, the joint distributions of $\{N_{ij}(0,t_j]\ (j = 1,\dots,m)\}$ can be expressed in terms of the transition functions and stationary distributions of the semi-Markov process. Thus we can define, for $m \ge 1$, $t_1,\dots,t_m \in \mathbb{R}$ $n_1, n_2, \dots \in \mathbb{Z}^+$, the probabilities

$$p_i(t_1,\dots,t_m | n_1,\dots,n_m) = \text{Prob}\{N_{ij}(0,t_j] = n_j \quad (1 \le j \le m)\} \tag{1}$$

for each $i = 0, 1, \dots, m$. In this section the relationship between these probabilities is investigated. From Proposition 4.3, replacing v_j by t_j, we have

$$\sum_{i=1}^{m} \frac{\partial}{\partial t_i} p_0(t_1,\dots,t_m | 0,\dots,0) = -\sum_{i=1}^{m} \rho_i\, p_i(t_1,\dots,t_m | 0,\dots,0), \tag{2}$$

since $V_{ij} > t_j$ if and only if $N_{ij}(0,t_j] = 0$. It might be thought that relations involving non-zero n_j would be more difficult to derive. However, by applying Proposition 4.3 to a modified process, we can recover the formulae given by Milne (1971) and generalise them to m-variate processes. The method used, which will be called <u>deterministic thinning</u>, will be considered first in relation to univariate processes.

Let $\mathcal{P}$ be a stationary, orderly univariate point process with finite rate ρ, and let the ordered sequence of events of $\mathcal{P}$ be $\dots E_{-1}, E_0, E_1, \dots$, where E_0 is the first event in $[0,\infty)$. For a given integer $k \ge 2$, let the random variable K be chosen independently of $\mathcal{P}$ so that

$$\text{Prob } (K = i) = {}^1/k \qquad\qquad (i = 0, 1, \ldots, k-1),$$

and let $\mathcal{P}(k)$ be the process

$$\mathcal{P}(k) = \{ E_i : K+i \text{ is a multiple of } k \} .$$

For example if $\mathcal{P}$ is a Poisson process then $\mathcal{P}(k)$ is a stationary renewal process with k-stage Erlangian interval distribution. Note that if an origin is taken at an arbitrary event of $\mathcal{P}$, or at an arbitrary time, the distribution of the number of events of $\mathcal{P}$ up to and including the first event of $\mathcal{P}(k)$ is uniform over $(1, \ldots, k)$, and independent of $\mathcal{P}$. It follows that $\mathcal{P}(k)$ is stationary, with rate ρ/k.

Let the asynchronous and synchronous survivor functions of $\mathcal{P}(k)$ be denoted by $\mathfrak{F}_i^{(k)}(v)$ for $i = 0, 1$ respetively, and let $\mathfrak{F}_i(v \mid j)$ denote the asynchronous and synchronous survivor functions for the sum of j consecutive intervals of $\mathcal{P}$. Then

$$\mathfrak{F}_0^{(k)}(v) = \frac{1}{k} \sum_{j=1}^{k} \mathfrak{F}_0 (v \mid j),$$

$$\mathfrak{F}_1^{(k)}(v) = \mathfrak{F}_1 (v \mid k) .$$

Hence the forward recurrence time formula

$$\tilde{F}(dx) = \rho \, F \{ (x, \infty) \} \, dx \qquad (3)$$

applied to $\mathcal{P}(k)$ gives

$$\frac{1}{k} \frac{\partial}{\partial v} \sum_{j=1}^{k} \mathfrak{F}_0 (v \mid j) = - \frac{\rho}{k} \mathfrak{F}_1 (v \mid k) . \qquad (4)$$

Thus we have, inductively on k, that

$$\frac{\partial}{\partial v} \{ \mathfrak{F}_0 (v \mid k) \} = \rho \{ \mathfrak{F}_1 (v \mid k-1) - \mathfrak{F}_1 (v \mid k) \},$$

or, in terms of $p_i(t \mid k) = \mathfrak{F}_i(t \mid k+1) - \mathfrak{F}_i(t \mid k)$,

$$\frac{\partial}{\partial t} p_0(t|k) = -\rho \left\{ p_1(t|k) - p_1(t|k-1) \right\} .$$

These are the well-known Palm-Khintchine relations. Note that the proof requires only simple stationarity of $\mathcal{P}$, for this implies simple stationarity of $\mathcal{P}(k)$, which is sufficient to derive (3) (cf. Lawrance, 1970).

The method extends to multivariate processes in an obvious way. Let $\mathcal{P}_m$ be an m-variate semi-Markov point process, and denote by $\mathcal{P}_m(k_1, \ldots, k_m) = \mathcal{P}_m(\underset{\sim}{k})$ the process obtained by independently thinning each marginal process in the way described above. It is important to note that $\mathcal{P}_m(\underset{\sim}{k})$ is also a semi-Markov point process. To see this, include in the state variable W_n of $\mathcal{P}$ terms $K_n^{(1)}, \ldots, K_n^{(m)}$ specifying the number of events in each marginal since the last undeleted event in that marginal. The transitions of the $\{K_i\}$ are to be determined by the transitions of $\mathcal{P}$ in the following way. If $W_n \in \mathcal{W}_i$, then

$$K_n^{(i)} = K_{n-1}^{(i)} + 1 \qquad (\text{mod } k_i),$$

$$K_n^{(j)} = K_{n-1}^{(j)} \qquad (j \neq i) .$$

The desired semi-Markov representation of $\mathcal{P}_m(\underset{\sim}{k})$ is obtained by filtering $\mathcal{P}_m$ by the set

$$\bigcup_{i=1}^{m} \left[\mathcal{W}_i \cap \{K^{(i)} = 0\} \right] .$$

Moreover, if $\mathcal{P}_m$ is stationary, then $\mathcal{P}_m(\underset{\sim}{k})$ is also stationary and the rate of the jth marginal of $\mathcal{P}_m(\underset{\sim}{k})$ is ρ_j/k_j.

Let the joint survivor functions of the forward recurrence times of $\mathcal{P}_m(\underset{\sim}{k})$ be denoted by

$$\mathcal{F}_i^{(\underset{\sim}{k})}(v_1, \ldots, v_m) \qquad (i = 0, 1, \ldots, m),$$

and the joint survivor functions for the sums of ℓ_j consecutive intervals in the jth marginal by

$$\mathcal{F}_i(v_1, \ldots, v_m \mid \ell_1, \ldots, \ell_m) \qquad (i = 0, 1, \ldots, m).$$

As usual the subscript i is zero for the asynchronous distributions, and otherwise denotes a semi-synchronous distribution. From Proposition 4.3 applied to $\mathcal{P}_m(\underset{\sim}{k})$,

$$\sum_{i=1}^{m} \frac{\partial}{\partial v_i} \mathcal{F}_0^{(\underset{\sim}{k})}(v_1, \ldots, v_m) = -\sum_{i=1}^{m} \frac{\rho_i}{k_i} \mathcal{F}_i^{(\underset{\sim}{k})}(v_1, \ldots, v_m). \tag{5}$$

However, it is easily seen that

$$\mathcal{F}_0^{(\underset{\sim}{k})}(v_1, \ldots, v_m) = \frac{1}{k_1 \ldots k_m} \sum_{j_\ell = 1 (\ell = 1, \ldots, m)}^{k_\ell} \mathcal{F}_0(v_1, \ldots, v_m \mid j_1, \ldots, j_m), \tag{6}$$

and that for $i = 1, \ldots, m$,

$$\mathcal{F}_i^{(\underset{\sim}{k})}(v_1, \ldots, v_m) = \frac{k_i}{k_1 \ldots k_m} \sum_{j_\ell = 1 (\ell \neq i)}^{m} \mathcal{F}_i(v_1, \ldots, v_m \mid j_1, \ldots, j_{i-1}, k_i, j_{i+1}, \ldots, j_m) \tag{7}$$

<u>Proposition 4.4.</u> Let D_i denote the different operator defined by

$$D_i f(\ell_1, \ldots, \ell_m) = f(\ell_1, \ldots, \ell_i, \ldots, \ell_m) - f(\ell_1, \ldots, \ell_i - 1, \ldots, \ell_m).$$

Then under the assumptions of Proposition 4.3,

$$\sum_{i=1}^{m} \frac{\partial}{\partial v_i} p_0(v_1, \ldots, v_m \mid \ell_1, \ldots, \ell_m) = -\sum_{i=1}^{m} \rho_i D_i p_i(v_1, \ldots, v_m \mid \ell_1, \ldots, \ell_m) \tag{8}$$

Proof. Note that the operators D_j commute with one another and with the operator $\partial/\partial v_i$ for each i. Now

$$\mathfrak{F}_i(v_1,\ldots,v_m \mid n_1,\ldots,n_m) = \sum_{\ell_j<n_j\ (j=1,\ldots,m)} p_i(v_1,\ldots,v_m \mid \ell_1,\ldots,\ell_m) .$$

Hence, using (5), (6) and (7) and defining $\underset{\sim}{\ell}+\underset{\sim}{1} = (\ell_1+1,\ldots,\ell_m+1)$, we have

$$\sum_{i=1}^{m} \frac{\partial}{\partial v_i} p_0(v_1,\ldots,v_m \mid \ell_1,\ldots,\ell_m)$$

$$= \sum_{i=1}^{m} \frac{\partial}{\partial v_i} (D_1,\ldots,D_m) \mathfrak{F}_0(v_1,\ldots,v_m \mid \ell_1+1,\ldots,\ell_m+1)$$

$$= (D_1,\ldots,D_m) \sum_{i=1}^{m} \frac{\partial}{\partial v_i} (D_1,\ldots,D_m) \left\{(\ell_1+1)\ldots(\ell_m+1)\, \mathfrak{F}_0^{\underset{\sim}{\ell}+\underset{\sim}{1}}(v_1,\ldots,v_m)\right\}$$

$$= -(D_1,\ldots,D_m)^2 \left\{(\ell_1+1)\ldots(\ell_m+1) \sum_{i=1}^{m} \frac{\rho_i}{\ell_i+1}\, \mathfrak{F}_i^{\underset{\sim}{\ell}+\underset{\sim}{1}}(v_1,\ldots,v_m)\right\}$$

$$= -(D_1,\ldots,D_m) \sum_{i=1}^{m} D_i \rho_i \mathfrak{F}_i(v_1,\ldots,v_m \mid \ell_1+1,\ldots,\ell_m+1)$$

$$= -\sum_{i=1}^{m} \rho_i D_i p_i(v_1,\ldots,v_m \mid \ell_1,\ldots,\ell_m) .$$

Throughout this proof any $\mathfrak{F}_i$ involving a zero ℓ_j, or any p_i involving a negative ℓ_j is given the value zero. *

For m = 2, this result is proved by Milne (1971) using orthodox sub-additivity arguments. The proof of Proposition 4.3 given here holds only for the class of semi-Markov point processes. However, the method used to deduce Proposition 4.4 from Proposition 4.3 is more general. It is not difficult to show that if $\mathcal{P}$ is an orderly point process in the usual sense, i.e. if the finite-dimensional distributions of $\mathcal{P}$ over disjoint Borel sets are consistently specified, then $\mathcal{P}(k)$ is also a point process. Moreover, this result extends to multivariate processes.

As noted by Milne, the relations given above do not enable the semi-synchronous distributions to be determined from the asynchronous distributions. Wisniewski (1972, 1973) has derived some relationships of that type for bivariate processes, but they all involve more complicated asynchronous distributions. For example it may be necessary to know the type of the event immediately preceding the origin. These relationships will not be considered here.

CHAPTER 5

THE BIVARIATE MARKOV PROCESS OF INTERVALS

5.1. INTRODUCTION

For the usual univariate renewal process, the backward recurrence time $\{U_t\}$ defines a Markov process in continuous time. A natural generalisation is to bivariate point processes which are such that the joint backward recurrence times $\{U_t^{(1)}, U_t^{(2)}\}$ form a Markov process. Cox and Lewis (1972) introduce such a process which they call the bivariate Markov process of intervals. Following Cox and Lewis, we define, for i = 1, 2,

$$\lambda_i(u_1, u_2) = \lim_{\delta \to 0} \frac{1}{\delta} \text{Prob}\left\{ \text{Event of type i in } [t, t+\delta) \;\middle|\; U_{t-0}^{j} = u_j \; (j=1,2) \right\}.$$

It is assumed that the process is well behaved, so that the λ_i exist and determine the finite time transition distributions. In fact (λ_1, λ_2) may be regarded as a vector complete intensity function for the process. Note that the λ_i are not functions of time. The point process will be completely stationary if and only if $\{U_t^{(1)}, U_t^{(2)}\}$ is a stationary process. In this chapter, conditions on the λ_i which ensure the existence of a stationary distribution are discussed and some of the simpler properties of the process are derived. Unfortunately the results are not very tractable.

5.2. EQUATIONS FOR THE STATIONARY DISTRIBUTION

If $\{U_t^{(1)}, U_t^{(2)}\}$ has a stationary distribution with joint density function $\tilde{q}(u_1, u_2)$, then, by considering transitions in a small interval

$[t, t+\delta)$, Cox and Lewis derive the equations

$$\frac{\partial \tilde{q}(u_1, u_2)}{\partial u_1} + \frac{\partial \tilde{q}(u_1, u_2)}{\partial u_2} = -\left\{\lambda_1(u_1, u_2) + \lambda_2(u_1, u_2)\right\} \tilde{q}(u_1, u_2), \quad (1)$$

$$\tilde{q}(0, u_2) = \int_{u_1=0}^{\infty} \lambda_1(u_1, u_2)\, \tilde{q}(u_1, u_2)\, du_1, \quad (2)$$

$$\tilde{q}(u_1, 0) = \int_{u_2=0}^{\infty} \lambda_2(u_1, u_2)\, \tilde{q}(u_1, u_2)\, du_2. \quad (3)$$

Setting $\lambda(u_1, u_2) = \lambda_1(u_1, u_2) + \lambda_2(u_1, u_2)$, we obtain for the general solution of (1),

$$\tilde{q}(u_1, u_2) = \hat{q}(u_1, u_2)\, g(u_1 - u_2), \quad (4)$$

where $g(.)$ is an arbitrary differentiable function and $\hat{q}(u_1, u_2)$ is the particular solution

$$\hat{q}(u_1, u_2) = \begin{cases} \exp\left\{-\int_0^{u_1} \lambda(t, t+u_2-u_1)dt\right\} & (u_1 \leq u_2), \\ \exp\left\{-\int_0^{u_2} \lambda(t+u_1-u_2, t)dt\right\} & (u_1 > u_2). \end{cases} \quad (5)$$

Let $g_1(x) = g(-x)$ if $x < 0$ and $g_2(x) = g(x)$ if $x \geq 0$. Then (2) and (3) give two simultaneous integral equations in g_1 and g_2. It does not seem possible to solve these analytically in general, though solutions corresponding to known results can be obtained if each λ_i is a function of u_i alone, or if each λ_i is a function of $\min(u_1, u_2)$ alone. These are the special cases of two independent renewal processes and the two-state semi-Markov process, respectively.

A solution may also be found if

$$\frac{\partial \lambda_1(u_1, u_2)}{\partial u_1} - \frac{\partial \lambda_2(u_1, u_2)}{\partial u_2} = 0 , \qquad (6)$$

for then there exists a differentiable function $\Lambda(u_1, u_2)$ for which

$$\lambda_i(u_1, u_2) = \frac{\partial \Lambda(u_1, u_2)}{\partial u_i} \qquad (i = 1, 2) .$$

In fact, provided the constant c can be chosen to make $\tilde{q}$ a probability density,

$$\tilde{q}(u_1, u_2) = c \exp \{ - \Lambda(u_1, u_2) \}$$

is a valid solution of (1), (2) and (3). In particular, if each λ_i is linear,

$$\lambda_i(u_1, u_2) = a_i u_1 + b_i u_2 + c_i \qquad (i = 1, 2) ,$$

then (6) holds if and only if $a_2 = b_1$, and then

$$\Lambda(u_1, u_2) = \frac{1}{2}(a_1 u_1{}^2 + b_2 u_2{}^2 + 2a_2 u_1 u_2) + c_1 u_1 + c_2 u_2 .$$

Here the stationary distribution is a truncated bivariate normal distribution.

In general the semi-synchronous backward recurrence times have joint densities

$$q_i(u_1, u_2) = \frac{\lambda_i(u_1, u_2)\, \tilde{q}(u_1, u_2)}{\int_{u_1=0}^{\infty} \int_{u_2=0}^{\infty} \lambda_i(u_1, u_2) \tilde{q}(u_1, u_2) du_1 du_2} \qquad (i = 1, 2) . \qquad (7)$$

The denominator of (7) is just the marginal intensity ρ_i. The densities of intervals in each marginal process may be found by integrating out the unwanted variable in (7). The densities of the semi-synchronous backward recurrence times U_{ij} $(j \neq i)$ may also be found from (7). Here as usual the first suffix denotes the type of event at the origin. The density of U_{ij} turns out to be

$$\frac{1}{\rho_i} g_i(u_j) \qquad (j \neq i) ,$$

giving a probabilistic meaning to the functions g_i introduced above.

5.3. THE ALTERNATING PROCESS

The work of Section 5.2 depends on the assumption of stationarity. It is clear that not all functions $\lambda_i(u_1, u_2)$ can define stationary bivariate Markov interval processes. For example if, for all u_1 and u_2,

$$\int_0^\infty \lambda(u_1+t, \; u_2+t) \, dt < \infty ,$$

then the process will eventually terminate. It is also possible to choose the λ_i so that the process explodes, giving infinitely many events in a finite interval. Other types of non-stationary behaviour may occur. Although necessary and sufficient conditions for the existence of a stationary distribution have not been found, in Section 5.5 a simple sufficient condition is given. As a preliminary another special process is considered here.

If $\lambda_i(u_1, u_2) = 0 \quad (u_i < u_{3-i}, \quad i = 1, 2,)$,

then the event types alternate, and the interval sequence $\{Z_0, Y_1, Z_1, Y_2, \ldots\}$ is a Markov chain governed by the transition densities

$$h_{21}(z, y) = \lambda_1(y+z, z) \exp \left\{ - \int_0^z \lambda_1(y+u, u) \, du \right\} ,$$

$$h_{12}(y, z) = \lambda_2(y, y+z) \exp \left\{ - \int_0^z \lambda_i(u, z+u)\, du \right\} \qquad (1)$$

Then Y_n and Z_n denote the nth intervals of type (1, 2) and (2, 1) respectively, and X_n will denote the nth interval without regard to type. A semi-Markov representation is obtained by taking $W_n = (\tau_n, X_n)$, where τ_n is the type of the nth event. If a stationary distribution $q(d\omega)$ does exist, then $q(\tau = i) = \frac{1}{2}$ (i = 1, 2) and

$$q(dx \mid \tau = 1) = g_{21}(x)\, dx\ , \quad q(dx \mid \tau = 2) = g_{12}(x)\, dx\ ,$$

where

$$g_{ij}(y) = \int_0^\infty h_{ij}(y, z)\, g_{ji}(z) dz \quad (i, j = 1, 2;\ \ i \neq j)\ . \qquad (2)$$

The conditional interval distribution is degenerate and the g_{ij} are also the stationary marginal interval densities. A stationary alternating process can be constructed if the means μ_{12} and μ_{21} of the g_{ij} are finite. The marginal intensities ρ_i take the common value

$$\rho_i = (\mu_{12} + \mu_{21})^{-1} \qquad (i = 1, 2)\ .$$

A sufficient condition for the existence of a stationary distribution on $\mathcal{W}$ is that Doeblin's condition should hold (Doob, 1953, p. 192). In our notation, this becomes

Condition D. There is a finite-valued measure ϕ on $\Omega_{\mathcal{W}}$ with $\phi(\mathcal{W}) > 0$, an integer $v \geq 1$ and a positive ε such that the ν-step transition probability $p^{(\nu)}(\xi, S)$ satisfies, for all $\xi \in \mathcal{W}$ and $S \in \Omega_{\mathcal{W}}$,

$$p^{(\nu)}(\xi, S) \leq 1 - \varepsilon \quad \text{if } \phi(S) \leq \varepsilon\ . \qquad (3)$$

Proposition 5.1. Suppose that there exists a function $\ell(x)$ $(x \in \mathbb{R}^+)$, non-zero on a set of positive measure, such that for all z

$$h_{12}(y,z) \geq \ell(y) . \tag{4}$$

Then Condition D holds for the chain $\{W_n\}$.

Proof. Take $\nu = 2$, $\epsilon = \frac{1}{2} \int_0^\infty \ell(x)\, dx$,

$$\phi\{\tau = 2,\ X \in (c,d)\} = \int_c^d \ell(x)\, dx ,$$

$$\phi\{\tau = 1,\ X \in (c,d) = \int_{z=c}^{d} \int_{y=0}^{\infty} h_{21}(z,y)\, \ell(y)\, dy\, dz .$$

*

It also follows from Doob's results that the stationary distribution is unique, for (4) implies that any set with positive ϕ-measure can be reached with positive probability from any point of $\mathcal{W}$. It can be shown from Doob (1953, Chapter 5, Theorem 7.5) that a central limit theorem holds for the distribution of the sum of n-consecutive intervals as $n \to \infty$, provided that for some $\delta > 0$,

$$\int z^{2+\delta}\, g_{ij}(z) < \infty \qquad (i \neq j\ ;\ \ i,j = 1,2) .$$

This result gives a simple proof of the asymptotic normality of N_t under the same conditions by the usual inversion argument (cf. Cox and Miller 1965, Chapter 9).

5.4. EXISTENCE OF A STATIONARY DISTRIBUTION FOR THE GENERAL PROCESS

A semi-Markov representation for the general bivariate Markov process of intervals may be derived. The state variable W is (τ, Y), where Y denotes the semi-synchronous backward recurrence time, $Y = U_{\tau, 3-\tau}$. The transition functions and conditional interval distributions are determined by the functions

$p_{ij}(u) = \text{Prob}(\Upsilon_1 = j \mid \Upsilon_0 = i,\ Y_0 = u) \qquad (i \neq j;\ i, j = 1, 2),$

$F_{ij}(x, u) = \text{Prob}(X_1 \leq x \mid \Upsilon_0 = i,\ \Upsilon_1 = j,\ Y_0 = u) \qquad (i \neq j;\ i, j = 1, 2).$

In terms of the λ_i, we have, for j = 1 and 2,

$$p_{1j}(u)\, f_{1j}(x, u) = \lambda_j(x, x+u) \exp\left\{ - \int_0^x \lambda(t, t+u)\, dt \right\}, \qquad (1)$$

$$p_{2j}(u)\, f_{2j}(x, u) = \lambda_j(x+u, x) \exp\left\{ - \int_0^x \lambda(t+u, t)\, dt \right\}, \qquad (2)$$

where $f_{ij}(x, u) = \partial / \partial x \left\{ F_{ij}(x, u) \right\}$ for each i and j. Conversely, the λ_j are determined by the p_{ij} and f_{ij} by formulae such as

$$\lambda_j(x, x+u) = \frac{p_{ij}(u)\, f_{ij}(x, u)}{\int_{t=x}^{\infty} \left\{ p_{11}(u) f_{11}(t, u) + p_{12}(u) f_{12}(t, u) \right\} dt} \qquad (3)$$

The integrals in (1) and (2) must tend to infinity with x, for otherwise the process might terminate.

The properties of the process might now be derived by an investigation of the processes $\{W_n\}$ and $\{U_t, \tilde{W}_t\}$. The continuous time process is clearly equivalent to the original defining process $\{U_t^{(1)}, U_t^{(2)}\}$. However there is a simpler approach which makes direct use of the structure of the point process. The nth event of the process is said to be a <u>last i-event</u> (i = 1, 2) if it is of type i and if the (n+1)th event is of the opposite type. The last events form a point process called the <u>imbedded process</u>.

<u>Proposition 5.2.</u> The imbedded process is an alternating bivariate

Markov interval process with transition densities $h_{ij}(x, y)$ given by

$$h_{ij}(x, y) = \bar{h}_{ij}(x, y)\, p_{ji}(x) \qquad (i \neq j\,;\quad i, j = 1, 2)\,, \tag{4}$$

where

$$\bar{h}_{ij}(x, y) = f_{ij}(x, y) + \int_{z=0}^{x} \bar{h}_{ij}(z, y)\, p_{jj}(y)\, f_{jj}(x-z, z)\,dz \tag{5}$$

Proof. The imbedded process is alternating, and the semi-synchronous backward recurrence time in the original process, measured to an event of the opposite type, must be measured to a 'last event' of that type. The equations (4) and (5) may be derived from an expression for $h_{ij}(x, y)$ as an infinite sum, or probabilistically. Note that $\bar{h}_{ij}(x, y)\,dx$ is the probability that some event, not necessarily a last event, occurs in $\{dx\}$ and that no last event occurs in $(0, x)$, given that a last i-event occurs at the origin and that the backward recurrence time at the origin is $U_{ij} = y$. ✱

It can now be shown that the condition H5 below is a sufficient condition for the interval sequence of the imbedded process to satisfy Doeblin's condition, and thus to have a stationary distribution.

Hypothesis H5. (i) The functions $\lambda_i(u_1, u_2)$ $(i = 1, 2)$ are bounded in any compact set.

(ii) There exist functions $\mu_i(u)$ $(i = 1, 2)$, non-negative, monotonic non-decreasing and not identically zero, such that $\lambda_i(u_1, u_2) \geq \mu_i(u_i)$ $(i = 1, 2)$.

Lemma 1. If H5 holds, then, for any $c > 0$, there exists an x such that for all u,

$$\mathfrak{F}_{ij}(x, u) = \int_{x}^{\infty} f_{ij}(y, u)\,dy < c \qquad (i \neq j\,;\quad i, u = 1, 2)\,.$$

Moreover, the moments of f_{ij} are bounded uniformly in u.

Proof. We have

$$\int_x^\infty \exp\left\{ - \int_0^y \lambda(t, t+u)\, dt \right\} \lambda_2(y, y+u)\, dy$$

$$\leq \exp\left\{ - \int_0^x \lambda(t, t+u) dt \right\} \int_x^\infty \exp\left\{ - \int_x^y \lambda_2(t, t+u) dt \right\} \lambda_2(y, y+u)\, dy$$

$$= \exp\left\{ - \int_0^x \lambda(t, t+u)\, dt \right\} .$$

From (1) and (2) it follows that

$$\frac{\mathcal{F}_{12}(x, u)}{1 - \mathcal{F}_{12}(x, u)} \leq \frac{\exp\left\{ - \int_0^x \lambda(t, t+u)\, dt \right\}}{\int_0^x \exp\left\{ - \int_0^y \lambda(t, t+u) dt \right\} \lambda_2(y, y+u)\, dy}$$

$$\leq \left[\int_0^x \mu_2(y) \exp\left\{ (x-y)\, \mu_2(y) \right\} dy \right]^{-1} , \qquad (6)$$

which is arbitrarily small for sufficiently large x. The first result follows immediately and the second by noting that if $\mu_2(y)$ is ultimately greater than ν (and there must be some $\nu > 0$ for which this is true), then the right hand side of (6) ultimately decreases at least as fast as $\exp(-\nu x)$. The proof for $\mathcal{F}_{21}$ is similar. *

Lemma 2. If H5 holds then there exists an x_1, and a function $\ell(x) \neq 0$, such that $h_{12}(x, u) > \ell(x)$ if $x > x_1$.

Proof. If $0 < c < 1$, then there exist x_0 and δ such that, for all u,

$$\int_0^{x_0-\delta} f_{12}(y,u)\, dy \geq c\ , \quad \mu_2(\delta) > 0\ .$$

Then, if $x \geq x_0$, two applications of (5) give

$$\bar{h}_{12}(x,u) \geq \int_{y=0}^{x} \bar{h}_{12}(y,u)\, p_{22}(y)\, f_{22}(x-y,y)\, dy$$

$$\geq \int_{y=0}^{x-\delta} f_{12}(y,u)\, p_{22}(y)\, f_{22}(x-y,y)\, dy$$

$$\geq c \inf_{0 \leq y \leq x-\delta} \left\{ p_{22}(y)\, f_{22}(x-y,y) \right\} .$$

However, for $0 \leq y \leq x-\delta$, we have

$$p_{22}(y)\, f_{22}(x-y,y) \geq \exp\left\{ - \int_0^{x-y} \lambda(t+y,t)\, dt \right\} \mu_2(x-y)$$

$$> \exp\left\{ - \int_0^{x-y} \lambda(t+y,t)\, dt \right\} \mu_2(\delta) > 0\ .$$

As this is a continuous function of y in $0 \leq y \leq x-\delta$ its infimum $\bar{\ell}(x)$ is also strictly positive. Then $h_{12}(x,y) \geq \bar{\ell}(x)\, p_{21}(x)$ which must also be strictly positive eventually. ✳

The results of Section 5.3, combined with the preceding lemmas give

<u>Proposition5.3.</u> If the condition H5 holds then the imbedded process has a unique stationary distribution in discrete time which generates a stationary distribution for the (continuous) backward recurrence time

process $\{U_{I,t}^{(1)}, U_{I,t}^{(2)}\}$ in the usual way.

The importance of this result lies in

Proposition 5.4. A bivariate Markov process of intervals is stationary if and only if its imbedded process is stationary.

Proof. The conditional distribution of $\{U_t^{(1)}, U_t^{(2)}\}$ given a full realisation of the imbedded process in $(-\infty, t]$ depends only on $U_{I,t}^{(1)}$ and $U_{I,t}^{(2)}$. The form of this conditional distribution, given $U_{It}^{(1)} = v_1$ and $U_{It}^{(2)} = v_2$ ($v_1 < v_2$ say) is

$$p(du_1, du_2 | v_1, v_2) = \delta_{v_1}^{du_1} \left\{ \frac{\mathcal{F}_{12}(v_1, v_2 - v_1)}{\mathcal{H}_{12}(v_1, v_1 - v_1)} \delta_{v_2}^{du_2} + \frac{\bar{h}_{12}(v_1 - u_2, v_2 - v_1) p_{22}(v_1 - u_2) \mathcal{F}_{22}(u_2, v_1 - u_2) du_2}{\mathcal{H}_{12}(v_1, v_2 - v_1)} \right\},$$

where $\mathcal{H}_{ij}$ and $\mathcal{F}_{ij}$ are the survivor functions corresponding to the densities h_{ij} and f_{ij}, repsectively. Thus $U_t^{(1)} = v_1$ and $U_t^{(2)}$ has a distribution with an atom of probability at v_2 and a density over $[0, v_1)$. If the imbedded process is stationary, with density $\tilde{q}_I(v_1, v_2)$, then the full process must also be stationary, with density

$$\tilde{q}(u_1, u_2)\, du_1 du_2 = \iint \tilde{q}_I(v_1, v_2)\, p(du_1, du_2 | v_1, v_2)\, dv_1\, dv_2 \,. \qquad (7)$$

The reverse implication is trivial. *

It can be shown by a little manipulation that (7) agrees with the density $\tilde{q}$ in Section 5. 1.

It is interesting to note that the stationary bivariate Markov process of intervals can have infinite rate. For example, suppose that

$$\lambda_1(u_1, u_2) = \alpha\,, \quad \lambda_2(u_1, u_2) = \begin{cases} \beta & (u_1 < u_2)\,, \\ \exp\{(\alpha+\beta)u_1\} & (u_1 \geq u_2)\,. \end{cases}$$

Then H5 holds. It is easily verified that the type 1 events are regeneration points for the process, and that the expected number of type 2 events in a synchronous interval of the type 1 process is infinite. It follows that $\rho_2 = \infty$.

We note that any stationary bivariate Markov interval process whose intensities satisfy the condition (5.2.6), i.e.

$$\frac{\partial \lambda_1(u_1, u_2)}{\partial u_2} = \frac{\partial \lambda_2(u_1, u_2)}{\partial u_1}\,,$$

when reversed in time, provides an example of a process whose forward recurrence time distributions satisfy the relations

$$\frac{\partial}{\partial v_i} \mathcal{F}_0(v_1, v_2) = -\rho_i\, \mathcal{F}_i(v_1, v_2) \qquad (i = 1, 2)$$

quoted in Chapter 4 (equation 4.2.12). More generally the defining equations (5.2.1) of the bivariate Markov interval process can be regarded as a (time-reversed) differential form of the relations (4.2.3) between the asynchronous and semi-synchronous forward recurrence time distributions. Thus, equations (5.2.1) hold for more general bivariate point processes. However, the λ_i, although defined in the same way as in Section 5.1, will not be complete intensity functions.

It does not seem possible to parameterise the general bivariate Markov process of intervals in a way which permits explicit determiniation of the interval distributions other than those given in Section 5.2. It is easy to write down expressions for the joint distribution of intervals in a marginal process, for example, but these involve the functions h_{ij}, and cannot be simplified unless a solution is found to equation (5.4.5).

CHAPTER 6

SEMI-MARKOV PROCESSES WITH COUNTABLE STATE SPACE

6. 1. INTRODUCTION

Semi-Markov processes with general state spaces were discussed in Chapter 3. In the remainder we shall principally be concerned with semi-Markov processes on a countable state space.

From our viewpoint the extra assumption makes little difference to the general theory, but it does allow a simpler notation. Moreover, the relative tractability of the processes examined in Chapters 6, 7 and 8 compared for example with those considered in Chapter 5, is due to the simplicity of $\mathcal{W}$. We shall not be concerned with the wealth of results about the classification of states, the existence of and speed of approach to limiting distributions and the analogues of the renewal equation. An extensive bibliography is given by Cheong, De Smit and Teugels (1972). Particular mention should also be made of the series of papers by Pyke (1961a, 1961b) and Pyke and Schaufele (1964, 1966), and of the paper by Çinlar (1969b). For the processes considered here, it will always be evident that all states intercommunicate and that both the imbedded Markov chain and the semi-Markov process itself have a unique stationary distribution.

In Section 6. 2, we consider some properties of the interval sequence when the semi-Markov process is in fact Markovian. The important concept of the stochastically monotone Markov chain is discussed in Section 6. 3. This was introduced by Daley (1968) and applied in Daley (196[illegible]) to the study of waiting times in a GI/G/1 queue. Related work has been done by Kalmykov (1962) and O'Brien (1972a, 1972b). The relevance to the theory of point processes appears to have passed unnoticed. We consider as an example the Neyman-Scott cluster process with exponential displacements. Two particular cases, corresponding to Poissonian and geometric cluster size distributions, are discussed.

The state variable of a countable semi-Markov process will be denoted by K, assumed to take values in $\mathbb{Z}^+$, and the transition probabilities and stationary distribution of the imbedded Markov chain by $\{p_{ij}\}$ and $\{q_i\}$, respectively. The conditional interval distribution $F(dx_1 | k_0, k_1)$ of X_1, given $K_0 = k_0$ and $K_1 = k_1$, will have mean $\mu_{k_0 k_1}$ say, so that the rate ρ of the process is given by

$$\rho^{-1} = \sum_{i,j} q_i \mu_{ij} p_{ij} , \tag{1}$$

assumed to be finite and positive. If there exists a sequence $\{g_k\}$ such that

$$F(x | k_0, k_1) = 1 - \exp\{-g_{k_0} x\} , \tag{2}$$

then the continuous time process

$$\tilde{K}_t = K_{N_t} \tag{3}$$

is Markovian with infinitesimal transition matrix $p_{ij}\ \delta t$ where, for each i,

$$\sum_j \tilde{p}_{ij} = 0 \quad , \quad \tilde{p}_{ij} = g_i \ p_{ij} \qquad (j \neq i) . \tag{4}$$

In the notation of Cox and Miller (1965, Chapter 4) $\tilde{p}_{ij} = q_{ij}$. It is easily verified that the stationary distribution $\tilde{q}_k$ of $\{\tilde{K}_t\}$ is given by

$$g_k \tilde{q}_k = \rho q_k \qquad (k \in \mathbb{Z}^+) . \tag{5}$$

It is not generally possible to reconstruct the process $\{K_n\}$ from $\{\tilde{K}_t\}$, because transitions from a state to itself are not recorded by $\{\tilde{K}_t\}$. A discussion of this point is given by Pyke (1961a, p. 1235).

6.2. PROPERTIES OF THE INTERVAL DISTRIBUTION WHEN $\tilde{K}_t$ IS MARKOVIAN

It is useful to collect a few results which hold when $\{\tilde{K}_t\}$ is Markovian, i.e. (6.1.2) is satisfied. This is a situation which often occurs in practice. The moments of the synchronous and asynchronous interval distributions can be determined quite simply. We have

$$\mathbb{E}(X) = \mathbb{E}\{\mathbb{E}(X|K)\} = \sum_{i=0}^{\infty} \frac{q_i}{g_i} = \frac{1}{\rho} , \tag{1}$$

$$\mathbb{E}(X^2) = \mathbb{E}\{\mathbb{E}(X^2|K)\} = 2 \sum_{i=0}^{\infty} \frac{q_i}{g_i^2} , \tag{2}$$

with similar results for the asynchronous intervals. The covariance sequence may be determined in a similar way, since

$$\gamma_n = \text{cov}\{X_1, X_{n+1}\} = \text{cov}\{\mathbb{E}(X_1|K_0), \mathbb{E}(X_{n+1}|K_n)\} . \tag{3}$$

Proposition 6.1. (i) If $\{\tilde{K}_t\}$ is a Markov process, the stationary synchronous and asynchronous interval distributions each have monotonically decreasing hazard functions.

(ii) The expected mean of the forward recurrence time distribution is greater than $1/\rho$.

Proof. The hazard $\lambda(x)$ for the synchronous distribution is

$$\lambda(x) = \frac{\sum_{i=0}^{\infty} q_i \, g_i \exp(-g_i \, x)}{\sum_{i=0}^{\infty} q_i \exp(-g_i \, x)} \cdot \tag{1}$$

Thus $\lambda(x)$ is differentiable and its derivative is negative, being of the form $E(G)^2 - E(G^2)$, where G is the random variable which takes the value g_i with probability proportional to $q_i \exp(-g_i \, x)$. A similar result holds for the asynchronous distribution. Using (6.1.5), we have

$$\mathbb{E}(\tilde{X}) = \sum_{i=0}^{\infty} \frac{\tilde{q}_i}{g_i} = \rho \sum_{i=0}^{\infty} \frac{q_i}{g_i^2} \geq \rho \left(\sum_{i=0}^{\infty} \frac{q_i}{g_i} \right)^2 = \mathbb{E}(X) ,$$

which proves (ii). ✳

Corollary. The coefficients of variation of synchronous and asynchronous intervals are greater than one.

Proof. This follows immediately from Proposition 6.1(i). ✳

The probability generating functions of the synchronous and asynchronous stationary distributions $\{q_i\}$ and $\{\tilde{q}_i\}$ are denoted by $Q(z)$ and $\tilde{Q}(z)$, respectively. It often happens that $g_i = a + bi$ $(i \in \mathbb{Z}^+)$ for some constants a and b. We then have the following easily verified relations connecting the synchronous and asynchronous interval survivor functions with Q and $\tilde{Q}$:

$$\mathcal{F}(x) = e^{-ax} \, Q(e^{-bx}) \, ; \tag{2}$$

$$\tilde{\mathcal{F}}(x) = e^{-ax} \, \tilde{Q}(e^{-bx}) \, ; \tag{3}$$

$$\rho \quad Q(z) = a \, \tilde{Q}(z) + bz \, \tilde{Q}'(z) \, , \tag{4}$$

the prime denoting differentiation with respect to z.

6.3. STOCHASTICALLY MONOTONE MARKOV CHAINS

Suppose that a Markov chain $\{K_n\}$ with real or countable state space is such that for each fixed k_1,

$$\text{Prob}(K_1 \leq k_1 \mid K_0 = k_0) \text{ is a non-increasing function of } k_0 . \tag{1}$$

Then it is easy to show inductively that a similar property holds for the n-step transitions, i.e. that for each fixed n and k_n

$$\text{Prob}(K_n \leq k_n \mid K_0 = k_0) \text{ is a non-increasing function of } k_0.$$

A Markov chain which satisfies (1) is called stochastically monotone. Daley (1968) gives a detailed discussion of the properties of such chains. A more general formulation which includes continuous time processes, but does not explicitly consider covariances, is given by Kalmykov (1962). Recently O'Brien (1972a, 1972b) has extended the concept to non-Markovian processes.

We shall quote for future reference Daley (1968, Theorem 4, p. 311) which in his notation is

Theorem (Daley). If the strictly stationary discrete time process $\{X_n\}$ $(n = \ldots -1, 0, 1, \ldots)$ is Markovian with state space $\mathfrak{X} \subset \mathbb{R}$, with one-step transition function $p(\cdot|\cdot)$ which satisfies (1.1) (equivalent to our (1) above), if $\pi(\cdot)$ is an invariant probability measure on $\mathfrak{X}$ and if $f : \mathfrak{X} \to \mathbb{R}$ is a monotonic function quadratically integrable with respect to π, then the sequence

$$\{\gamma_n\} = \text{cov}\{[f(X_0), f(X_n)]\} \qquad (n = 0, 1, \ldots)$$

of serial covariances decreases monotonically to a non-negative limit γ_∞. If the invariant measure π for the process is unique, then $\gamma_\infty = 0$.

6.4. THE EXPONENTIAL NEYMAN-SCOTT PROCESS

As a first example of a point process generated by a countable Markov process we consider the Neyman-Scott cluster process obtained by taking a Poisson process $\{t_i\}$ of rate ν for the process of main events and allowing each main event independently to generate a random number D of subsidiary events which are then independently displaced in the positive direction from the main event. We shall assume throughout that the displacement distribution is exponential with parameter β. The probability generating function of D will be denoted by $P_D(z)$ and we take $E(D) < \infty$. We shall be concerned with the combined process of main and subsidiary events.

Let $\{\tilde{K}_t\}$ denote the total number of unexpired subsidiary events at time t, i.e. the total number of subsidiary events which themselves occur in (t, ∞), but which are generated by main events occurring in $(-\infty, t]$. Let $K_n = \tilde{K}_{T_n}$, the number of unexpired subsidiaries just after the nth process event. Then it is easily seen that $\{\tilde{K}_t\}$ and $\{K_n\}$ are respectively a Markov process in continuous time, and a Markov chain in discrete time. This follows from the lack of memory of the exponential distribution.

We consider first the discrete chain $\{K_n\}$. Let d_j = Prob(D = j). Then if $K_n = k$, the distribution of X_{n+1} is that of the minimum of k+1 independent exponential variables, k having parameter β and one having parameter ν. If the (n+1)th event is a main event then it may generate D descendants, giving $K_{n+1} = k + D$. If it is a subsidiary event, then $K_{n+1} = k-1$. In any case the conditional distribution $F(dx | k_0, k_1)$ is exponential with parameter $\nu + \beta k_0$. These results give

Proposition 6.2. The exponential Neyman-Scott process defined above has a semi-Markov representation with state space Z^+, transition probabilities $p_{ij} = 0$ if $j < i-1$,

$$p_{i,i-1} = \frac{\beta i}{\nu + \beta i},$$

$$p_{ij} = \frac{\nu}{\nu + \beta i} d_{j-i} \qquad (j \geq i), \tag{1}$$

and conditional interval distribution

$$F(x \mid i, j) = 1 - \exp\{-(\nu + \beta i) x\}. \tag{2}$$

It is clear that all states of the Markov chain communicate, so that there can be at most one stationary distribution. The existence of a stationary probability distribution is ensured by the fact that for large k, $\{K_n\}$ is stochastically dominated by a random walk with negative drift. An alternative proof can be constructed using the known stationarity of the point process to construct an appropriate stationary marked point process, along the lines of Matthes (1963). An analytic proof is given by Yang (1972), whose formulation includes as special cases both the Neyman-Scott process considered here and the exponential self-exciting process considered in Chapter 7. Our approach will be to exhibit the unique solution of the equilibrium equations and to use the existence results referred to above to deduce that this solution must be a probability distribution.

Proposition 6.3. The continuous time process $\{\tilde{K}_t\}$ has a stationary distribution $\{q_i\}$ whose probability generating function is

$$\tilde{Q}(z) = \exp\left\{ -\nu/\beta \int_0^{1-z} \frac{1 - P_D(1-x)}{x} dx \right\} \tag{3}$$

<u>Proof.</u> The equations for the equilibrium probabilities of $\{\tilde{K}_t\}$ are

$$(\nu + \beta j) q_j = \beta(j+1)\tilde{q}_{j+1} + \nu \sum_{i=0}^{j} d_{j-i}\, \tilde{q}_i \quad (j = 0, 1, \ldots) \tag{4}$$

Multiplication of (4) by z^j $(0 \leq z \leq 1)$ and summing over j gives

$$\nu \tilde{Q}(z) + \beta z\, \tilde{Q}'(z) = \beta\, \tilde{Q}'(z) + \nu P_D(z)\, \tilde{Q}(z) ,$$

which gives (3) on integration, since $\tilde{Q}(1) = 1$. ✱

As a check we can evaluate the asynchronous survivor function from (6. 2. 3) :

$$\mathfrak{z}(x) = e^{-\nu t}\, \tilde{Q}\, (e^{-\beta t}) . \tag{5}$$

Lawrance (1972) has derived the p. g. f. of counts, $E\left\{ z^{\tilde{N}(0,t)} \right\}$ for a general Neyman-Scott process, and it is easily verified that (5) is a special case of his results.

6. 5. SHOT NOISE PROCESSES AND PROPERTIES OF THE INTERVAL SEQUENCE

An important special case of the Neyman-Scott process occurs if D has a Poisson distribution with mean α/β. It can then be shown that the process of subsidiary events is equivalent to a shot noise doubly stochastic Poisson process with rate $\sum g(t-t_i)$, where $g(x) = \alpha e^{-\beta x}$ $(x > 0)$ and $\{t_i\}$ is the Poisson process of main events. This process for a general function g(.) is discussed by Bartlett (1964), Westcott (1971), Vere-Jones (1970)and Lawrance (1972). If the height α of each shot is an exponentially distributed random variable with mean θ, heights of different shots being independent of each other and of the $\{t_i\}$, then the distribution of D is geometric, with

$$P_D(z) = \frac{\beta}{\beta + \theta - \theta z} \quad . \tag{1}$$

The displacement distribution is, of course, still exponential with parameter β.

The serial correlations $\{\gamma_i\}$ for the interval sequence of the exponential Neyman-Scott process may be determined by the conditioning argument given in Section 6.2. They need not be positive in general. A simple example of a process with negative first serial correlation is obtained by taking $P_D(z) = z$, so that each main event generates precisely one subsidiary event, and $\nu << \beta$. However, if the cluster size distribution has sufficiently uniformly large hazard, we can assert positivity.

Proposition 6.4. Suppose that, for all k,

$$\frac{d_k}{\sum_{r=k}^{\infty} d_r} \geq \frac{\beta}{\beta + \nu} \quad . \tag{2}$$

Then the Markov chain $\{K_n\}$ is stochastically monotone and the covariance sequence $\{\gamma_m\}$ decreases monotonically to zero as $m \to \infty$.

Proof. It is sufficient to verify the stochastic monotonicity, since $E(X_i | K_i, K_{i-1}) = (\nu + \beta K_{i-1})^{-1}$ is a monotonic decreasing function of K_{i-1}, and Daley's theorem applies. We have from (2)

$$\{\nu + \beta(i-1)\} \; d_{j-i} \geq \beta \sum_{r=j-i+1}^{\infty} d_r \; ,$$

and so, for $j \geq i$,

$$\text{Prob}(K_1 \geq j | K_0 = i) - \text{Prob}(K_1 \geq j | K_0 = i-1)$$

$$= \frac{\nu}{\nu+\beta i} \sum_{r=j-i}^{\infty} d_r - \frac{\nu}{\nu+\beta(i-1)} \sum_{r=j-i+1}^{\infty} d_r$$

$$\geq 0.$$

For $j < i$, $\text{Prob}(K_1 \geq j | K_0 = i) = 1$. *

For geometrically distributed cluster sizes (2) takes the simple form

$$\theta \leq \nu \quad .$$

Explicit evaluation of $\tilde{Q}(z)$ is possible in the Poissonian and geometric cases. If $P_D(z) = e^{-\frac{\alpha}{\beta}(1-z)}$, we have, from Proposition 6. 3,

$$\log \{ \tilde{Q}(z) \} = -\frac{\nu}{\beta} \; \text{Ein} \left\{ \frac{\alpha}{\beta}(1-z) \right\},$$

where the function $\text{Ein}(u)$ given by

$$\text{Ein}(u) = \int_0^u \frac{1 - e^{-t}}{t} \, dt$$

is related to the exponential integral $E_1(u)$ tabulated by Abramowitz and Stegun (1964, Chapter 5) by the formula

$$\text{Ein}(u) = E_1(u) + \log u + \gamma ,$$

γ being Euler's constant.

If D has the geometric distribution (1), then we find

$$Q(z) = \left(\frac{\beta}{\beta+\theta-\theta z} \right)^{\nu/\beta+1} , \qquad \tilde{Q}(z) = \left(\frac{\beta}{\beta+\theta-\theta z} \right)^{\nu/\beta} . \qquad (3)$$

Thus, as would be expected from the form of the process, in equilibrium K_n is distributed as the sum of $\tilde{K}_t$ and an independent random variable with the distribution (1).

To conclude this Chapter we will show how the joint density of contiguous intervals may be determined in terms of $\mathcal{Q}$ and its derivatives. We restrict attention to the density $f_{XY}(x, y)$ of two contiguous intervals, though similar methods apply to higher order joint densities. We have

$$f_{XY}(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q_i p_{ij} (\nu + \beta i)(\nu + \beta j) \exp \{ -(\nu + \beta i)x - (\nu + \beta j)y \} .$$

The term $\nu + \beta i$ cancels with the denominator of p_{ij} and each term of the resulting series is a quadratic polynomial in i and j. After some manipulation we find that

$$f_{XY}(x, y) = e^{-\nu(x+y)} \Big[\beta^2 e^{-2\beta x - \beta y} \mathcal{Q}'' \{ e^{-\beta(x+y)} \}$$

$$+ \nu\beta \{ e^{\beta y} + P_D(e^{-\beta y}) \} e^{-\beta(x+y)} \mathcal{Q}' \{ e^{-\beta(x+y)} \}$$

$$+ \{ \nu^2 P_D(e^{-\beta y}) + \beta e^{-\beta y} P_D'(e^{-\beta y}) \} \mathcal{Q} \{ e^{-\beta(x+y)} \} \Big] .$$

In the geometric case we find from (1) and (3) that

$$\int_y^{\infty} f(x, y)dy = e^{-\nu(x+y)} \left\{ \frac{\beta}{\beta + \theta - \theta e^{-\beta(x+y)}} \right\}^{\nu/\beta + 1} \left\{ \frac{e^{-\beta x}(\beta + \nu)\theta}{\beta + \theta - \theta e^{-\beta(x+y)}} + \frac{\nu\beta}{\beta + \theta - \theta e^{-\beta y}} \right\} .$$

Numerical results for the simpler properties of the two particular Neyman-Scott processes considered here are given in the Appendix and compared with the corresponding results for processes discussed in Chapters 7 and 8 below.

CHAPTER 7

THE SELF-EXCITING PROCESS

7.1. MOTIVATION AND INTRODUCTION

Hawkes (1971a) has introduced a new class of point process which he calls self-exciting processes. These are formally defined by the requirement that the complete intensity function should be a linear function of the history of the process, so that

$$\lambda(t, \mathcal{H}_t) = \nu + \int_{-\infty}^{t} g(t-u)\, d\tilde{N}(u) . \qquad (1)$$

Here $\nu > 0$, $g(u) \geq 0$ for $u \geq 0$, and g is integrable with

$$0 < m = \int_{0}^{\infty} g(u)\, du < 1 . \qquad (2)$$

More recently, Jowett and Vere-Jones (1972) have pointed out that equations like (1) arise naturally in the theory of linear prediction.

The counting spectra of self-exciting processes and of related multivariate processes have been derived by Hawkes (1971a, 1971b, 1972) assuming stationarity. However, it is not absolutely clear from (1) that stationary self-exciting processes exist. In Section 7.2, we prove that they do exist, and show that the class of stationary self-exciting processes with finite rate is equivalent to a certain subclass of the class of generalised Poisson cluster processes. The existence part of this result has now been proved, independently of this work, by Hawkes (private communication). He also points out thaconnection with the generalised birth and death processes considered by Kendall (1949) and gives an equation for the generating functional.

Some consequences of the representation as a cluster process are considered in Section 7.3. When $g(x) = \alpha e^{-\beta x}$, there exists a simple Markovian representation. This is developed in the remaining sections of the chapter and is used to derive an expression for the p.g.f. of counts and to give a simple proof of the positivity of the interval serial correlations. A few possibilities for further work are outlined.

7.2. CHARACTERISATION OF THE STATIONARY SELF-EXCITING PROCESS

Lemma 1. If $\nu > 0$ and $0 < m < 1$, there exists a stationary, orderly point process with rate $\rho = \nu/(1-m)$ which has a complete intensity function of the form (7.1.1).

Proof. We can define inductively stationary, orderly point processes $\mathcal{P}_i$ $(i = 0, 1, \ldots)$ with respective counting measures $\tilde{N}_i$ $(i = 0, 1, \ldots)$ as follows. We take $\mathcal{P}_0$ to be a Poisson process with rate ν and $\mathcal{P}_1$ to be a cluster process which has $\mathcal{P}_0$ as the process of main events. A main event occurring at t_j generates a cluster of subsidiary events in a non-stationary Poisson process of rate $g(t-t_j)$ in $t > t_j$. Then (cf. Chapter 6) $\mathcal{P}_1$ is a Neyman-Scott cluster process. The distribution of the number of subsidiary events in a cluster is Poisson with mean m, and the independent displacements of each subsidiary event from the main event have density $g(x)/m$. We do not exclude the possibility of empty clusters and the events of $\mathcal{P}_0$ are not counted as events of $\mathcal{P}_1$.

For $i \geq 2$, P_i is the subsidiary process with P_{i-1} as the process of main events and the same cluster structure as that described above. It is clear that P_i is a well-defined stationary point process with rate $\rho_i = \nu m^i$ for all i and that the superposition $P_1 \cup \dots \cup P_i$ is both stationary and orderly. Now consider the superposition

$$\mathcal{S} = \bigcup_{i=0}^{\infty} P_i \,. \tag{1}$$

The convergence of $\sum \rho_i$ ensures that $\mathcal{S}$ is a well-defined point process. Moreover, $\mathcal{S}$ is stationary and orderly, and has rate

$$\rho = \sum_{i=0}^{\infty} \rho_i = \nu/(1-m) \,. \tag{2}$$

We show that the complete intensity function satisfies (7. 1. 1). For each $i \geq 1$, let $\mathcal{H}_t^{i-1}$ denote the history of P_{i-1} and define

$$\lambda_i(t, \mathcal{H}_t^{i-1}) = \lim_{\delta \to 0+} \frac{1}{\delta} \operatorname{Prob}\left\{ \tilde{N}_i [t, t+\delta) \geq 1 \mid \mathcal{H}_t^{i-1} \right\} . \tag{3}$$

Then, by the construction of P_i ,

$$\lambda_i(t, \mathcal{H}_t^{i-1}) = \int_{-\infty}^{t} g(t-u)\, d\tilde{N}_{i-1}(u) \qquad (i \geq 1) . \tag{4}$$

But

$$\lambda(t, \mathcal{H}_t) = \nu + \sum_{i=1}^{\infty} \lambda_i(t, \mathcal{H}_t^{i-1}) \,. \tag{5}$$

Equations (4) and (5) give

$$\lambda(t, \mathcal{H}_t) = \nu + \int_{-\infty}^{t} g(t-u)\, dN(u) \,.$$

The process $\mathcal{S}$ will be called the stationary iterated cluster process. Lemma 1 demonstrates the existence of stationary self-exciting processes. We now show that all such processes are iterated cluster processes. We note first that a transient iterated cluster process which still satisfies (7.1.1) in $t > 0$ may be constructed as before by taking $\mathcal{P}_0$ as a Poisson process in $t > 0$ only.

Lemma 2. The only transient self-exciting process corresponding to a given ν and $g(.)$ is the appropriate transient iterated cluster process.

Proof. This is immediate, since a transient point process is characterised by its complete intensity function. ⁎

Lemma 3. Let $A_1, \ldots, A_k$ be bounded Borel subsets of $[0, \infty)$. Then the joint distribution of $\{\tilde{N}(A_1 + \tau), \ldots, \tilde{N}(A_k + \tau)\}$ in a transient iterated cluster process tends as $\tau \to \infty$ to the joint distributions of $\{\tilde{N}(A_1), \ldots, \tilde{N}(A_k)\}$ for the stationary iterated cluster process.

Proof. Let $\mathcal{S}^+$ be a transient iterated cluster process. Let $\mathcal{S}^-$ be the iterated cluster process, independent of $\mathcal{S}^+$, obtained by taking $\mathcal{P}_0$ as a Poisson process in $t \leq 0$, and let $N^-(.)$ denote its counting measure. Then the superposition $\mathcal{S}^+ \cup \mathcal{S}^-$ is a stationary iterated cluster process. The complete intensity function of $\mathcal{S}^-$ satisfies

$$\lambda^-(t, \mathcal{H}_t^-) = \nu + \int_{-\infty}^{t} g(t-u)\, dN^-(u) \qquad (t \leq 0),$$

$$\lambda^-(t, \mathcal{H}_t^-) = \int_{-\infty}^{t} g(t-u)\, dN^-(u) \qquad (t > 0),$$

where $\mathcal{H}_t^-$ is the history function for $\mathcal{S}^-$. Thus

$$f^-(t) = \mathbb{E}\{\lambda^-(t, \mathcal{H}_t^-)\}$$

satisfies $f^-(t) = \rho$ in $t \leq 0$ and in $t > 0$

$$f^-(t) = \rho \int_t^\infty g(u)\, du + \int_0^t g(t-u)\, f^-(u)\, du . \qquad (7)$$

It is easily proved from this equation that $f^-(t) \to 0$ as $t \to \infty$. Since $N^-(.)$ is a non-negative integer valued random variable, and putting $B = A_1 \cup \ldots \cup A_k + \tau$ we obtain

$$\mathbb{E}\{N^-(B)\} = \int_B f^-(u)\, du \to 0 \qquad (\tau \to \infty)$$

it follows that $\text{Prob}\{N^-(B) > 0\} \to 0$ as $\tau \to \infty$. *

Lemma 4. If $m < 1$, the only stationary process with finite rate and complete intensity function given by (7.1.1) is the stationary iterated cluster process defined in Lemma 1.

Proof. Let $\mathcal{S}$ be such a process. We construct a new process $\mathcal{S}'$ as the superposition of two independent processes $\mathcal{S}^-$ and $\mathcal{S}^+$ as follows. We take $\mathcal{S}^+$ to be a transient iterated cluster process and $\mathcal{S}^-$ to be the unique process which has the same finite dimensional distributions as $\mathcal{S}$ in $t \geq 0$, and whose complete intensity function in $t > 0$ is given by

$$\lambda^-(t, \mathcal{H}_t^-) = \int_{-\infty}^t g(t-u)\, dN^-(u) .$$

Then, as in the proof of Lemma 3, it follows that the limiting distributions of $\mathcal{S}'$ as $\tau \to \infty$ are the same as the limiting distributions of $\mathcal{S}^+$ as $\tau \to \infty$, which, by Lemma 3, are just the distributions of the stationary iterated cluster process. But the complete intensity function of $\mathcal{S}'$ is the same as that of $\mathcal{S}$ in $t > 0$ and the two processes have the same distributions in $t \leq 0$. Thus the two processes have identical distributions in $t > 0$ also, and the result follows from the stationarity of $\mathcal{S}$. ✱

The preceding lemmas give immediately

<u>Proposition 7.1.</u> If $\nu > 0$ and the non-negative integrable function $g(u)$ has $m = \int_0^\infty g(u)\,du < 1$, then there exists precisely one stationary point process of finite rate whose complete intensity function satisfies (7.1.1). This process is an iterated cluster process and has rate $\nu/(1-m)$. The limiting distributions of the corresponding transient self-exciting process as $t \to \infty$ exist and are the same as the distributions of the stationary process.

7.3. THE ITERATED CLUSTER PROCESS AS A GENERALISED POISSON CLUSTER PROCESS

The events of $\mathcal{P}_0$ will be called immigrants. An event E_k of $\mathcal{P}_k$ is said to be a descendant of an event E_j of $\mathcal{P}_j$ $(j < k)$ if there exist events E_{j+1} of $\mathcal{P}_{j+1}, \ldots, E_{k-1}$ of $\mathcal{P}_{k-1}$, such that, for $j \leq i \leq k-1$, E_{i+1} is a member of the first generation (Neyman-Scott) cluster with main event E_i. It is clear that the total number S of descendants of a given event is distributed as the total number of offspring in a simple Galton-Watson branching process whose offspring distribution is Poisson with mean $m < 1$. Thus we have (Harris, 1963, Chapter 1) that S is finite with probability one, and

$$\mathbb{E}(S) = \frac{m}{1-m} < \infty , \tag{1}$$

$$\mathrm{var}(S) = \frac{m}{(1-m)^3} < \infty . \tag{2}$$

By grouping all the descendants of each immigrant, we obtain a representation of the iterated cluster process as a generalised Poisson cluster process. It follows from known results (Kerstan and Matthes, 1965; Westcott, 1971) that the stationary self-exciting process with finite rate is both infinitely divisible and mixing. The mixing property has already been verified implicitly in the proof of Proposition 7.1. The infinite divisibility follows from the Poisson clustering representation. It is worth noting that from the original definition (equation 7.1.1) it follows that the superposition of two independent self-exciting processes having the same g(.) is another self-exciting process.

A quantity of some interest in clustering processes is the total length L of a cluster. Here L is defined to be the distance between an immigrant and its last descendant. It is easy to write down a functional equation satisfied by the distribution function of L, but difficult to derive useful information from it. However, we do have the

Lemma. A necessary and sufficient condition for $\mathbb{E}(L) < \infty$ is that

$$\gamma = \int_0^\infty v\, g(v)\, dv < \infty .$$

Proof. The necessity is obvious. To prove sufficiency, note that, by conditioning on the configuration of the S descendants and using inequalities of the form $\max(Z_1, Z_2) \leq Z_1 + Z_2$ $(Z_1, Z_2 \geq 0)$, we have

$$E(L|S) \leq {}^{\gamma S}/m . \tag{3}$$

Hence

$$\mathbb{E}(L) \leq {}^{\gamma}/(1-m) \,. \qquad (4) \quad ⁎$$

If $\mathbb{E}(L) < \infty$ we can use a theorem of Lewis (1969) to prove the asymptotic normality of the counting distribution. Following Lewis, we define $H(u) = \mathbb{E}(N_u)$, where N_u is the number of descendants in $(0, u]$ of an immigrant at the origin. Then in our notation Lewis's theorem 3. 3 becomes

Theorem (Lewis). If $\mathbb{E}(S^2) < \infty$ and if

$$\frac{1}{\sqrt{x}} \int_0^x \left\{ \mathbb{E}(S) - H(u) \right\} du \to 0 \qquad (x \to \infty),$$

then the number of events in a transient Poisson cluster process is asymptotically normally distributed with mean $\nu t \,\mathbb{E}(S+1)$ and variance $\nu t \,\mathbb{E}(S+1)^2$.

Here it is easily verified that

$$\int_{u=0}^{x} \left\{ \mathbb{E}(S) - H(u) \right\} du < \mathbb{E}(LS) \,.$$

for all $x > 0$, and this is finite by (2) and (3). Substitution from (1) and (2) gives the asymptotic distribution of the transient self-exciting process to be normal with mean $\nu t/(1-m)$ and variance $\nu t/(1-m)^3$, provided $\gamma < \infty$. The stationary process has the same limiting distribution.

If $\gamma < \infty$, it is easy to verify that the asymptotic slope of the log-survivor function is $(-\nu)$, for standard methods can be used to show that the distribution of the number of operative clusters at an arbitrary time is Poisson with mean $\nu \,\mathbb{E}(L)$. Thus the posterior probability of there being any operative clusters at the start of an interval $[\tau, t+\tau)$ tends to zero as $\tau \to \infty$.

Finally, we note that the extension of this theory to multivariate self-exciting processes is straightforward. This provides an existence and characterisation theorem similar to Proposition 7.1 for the 'mutually exciting' processes considered by Hawkes (1971a, 1971b).

7.4. THE EXPONENTIAL SELF-EXCITING PROCESS

When $g(v) = \alpha e^{-\beta v}$ $(\alpha < \beta)$, a Markovian representation for the self-exciting process can be obtained in the same way as for the Neyman-Scott process discussed in Chapter 6. The only difference is that for the self-exciting process all events give rise to descendants. Although D will have a Poisson distribution with mean α/β, it is convenient to consider the more general processes obtained when D has a general distribution with p.g.f. $P_D(z)$ and mean $E(D) < 1$. We let $\tilde{K}_t$ denote the total number of first generation descendants of events in $(-\infty, t]$ that occur in (t, ∞), and let $K_n = \tilde{K}_{T_n}$. Then, as for the Neyman-Scott process, $\{\tilde{K}_t\}$ and $\{K_n\}$ have the Markov property. The counterpart of Proposition 6.1 is

Proposition 7.2. The exponential iterated cluster process has a semi-Markov representation with state space Z^+, transition probabilities

$$p_{ij} = \begin{cases} 0 \quad (j < i-1), \\ \dfrac{\beta i}{v+\beta i} d_{j-i+1} + \dfrac{v}{v+\beta i} d_{j-i} \quad (j \geq i-1), \end{cases} \qquad (1)$$

where $d_j = \text{Prob}(D = j)$ if $j \geq 0$ and $d_{-1} = 0$. The conditonal interval distribution is

$$F(x \mid i, j) = 1 - \exp\left\{ -(\nu + \beta i) x \right\} . \tag{2}$$

The general remarks following Proposition 6.2 apply here also. Moreover, the stationary distribution can be derived in a similar way.

Proposition 7.3. The stationary distribution $\{ \tilde{q}_i \}$ of the continuous time process $\{ \tilde{K}_t \}$ has p.g.f. $\tilde{Q}(z)$, where

$$\tilde{Q}(z) = \exp\left\{ -\frac{\nu}{\beta} \int_0^{1-z} \frac{1 - P_D(1-x)}{P_D(1-x) + x - 1} \, dx \right\} . \tag{3}$$

Proof. The equations for the equilibrium probabilities are

$$(\nu + \beta j) q_j = \sum_{i=0}^{j} d_{j-i} \tilde{q}_i + \beta \sum_{i=0}^{j+1} i \tilde{q}_i d_{j+1-i} . \tag{4}$$

Taking generating functions gives

$$\nu \tilde{Q}(z) + \beta z \tilde{Q}'(z) = \nu P_D(z) \tilde{Q}(z) + \beta \tilde{Q}'(z) P_D(z) ,$$

which on integration gives (3). ✱

Note. This theorem is a special case of a result proved in Yang (1972). Our interated cluster process is equivalent to a continuous time branching process with immigration. The only slightly anomalous feature is that each immigrant in our process splits immediately after entering the system. However, this can easily be dealt with by considering immigration with a random batch size, which is allowed in Yang's formulation. These remarks apply also in Chapter 8.

For the self-exciting process, (3) becomes

$$\tilde{Q}(z) = \exp\left\{ -\frac{\nu}{\beta} \int_0^{1-z} \frac{1 - e^{-(\alpha/\beta)x}}{e^{-(\alpha/\beta)x} + x - 1} \, dx \right\} . \tag{5}$$

We have not been able to express this integral in terms of known functions. However, it is easy to work out the moments of $\tilde{K}$. For example, using L'Hospital's rule, we obtain

$$\mathbb{E}(\tilde{K}) = \tilde{Q}'(1) = \frac{\alpha\nu}{\beta(\beta-\alpha)} ,$$

$$\operatorname{var}(\tilde{K}) = \tilde{Q}''(1) - \left\{ \tilde{Q}'(1) \right\}^2 + \tilde{Q}'(1) = \frac{\alpha\nu(2\beta-\alpha)}{2\beta(\beta-\alpha)^2} .$$

The asynchronous survivor function is given in terms of $\tilde{Q}$ by

$$\tilde{\mathcal{F}}(t) = e^{-\nu t}\,\tilde{Q}\,(e^{-\beta t}),$$

from (6.2.3).

7.5. THE ASYNCHRONOUS COUNTING DISTRIBUTION

In this section we derive an expression for the p.g.f. $\tilde{P}(z,t) = \mathbb{E}\left\{ z^{\tilde{N}(0,t)} \right\}$ of counts in an arbitrary interval $(0,t]$. As will be seen, the result is not particularly tractable. We start by considering the generating function $P_1(z,t)$ of the number $N_1(0,t]$ of events in $(0,t]$, due to an immigrant at 0 which is known to have precisely one first generation descendant in $(0,\infty)$. We obtain a functional equation for $P_1(z,t)$ by conditioning on the time of the first event, if any, in $(0,t]$. This gives

$$P_1(z,t) = e^{-\beta t} + z \sum_{i=0}^{\infty} d_i \int_0^t \beta e^{-\beta(t-x)} \left\{ P_1(z,x) \right\}^i dx ,$$

i.e.

$$P_1(z,t) = e^{-\beta t} + z \int_0^t \beta e^{-\beta(t-x)} P_D\left\{ P_1(z,x) \right\} dx. \qquad (1)$$

This equation can be solved by differentiation with respect to t. For the right hand side is clearly differentiable, and so

$$\frac{\partial P_1(z,t)}{\partial t} = \beta z\, P_D\left\{P_1(z,t)\right\} - \beta\, P_1(z,t)\,. \tag{2}$$

For each fixed z with $0 < z < 1$, this is an ordinary differential equation in P_1. By considering the sign of the derivative, we can deduce that $P_1(z,t)$ decreases monotonically from its initial value $P_1(z,0) = 1$ to a limiting value which is the solution of the equation

$$P_1(z,\infty) = z\, P_D\left\{P_1(z,\infty)\right\}. \tag{3}$$

Equation (3) is the functional equation satisfied by the total number of descendants, including the initial individual, in a simple Galton-Watson process. Thus the formal solution of (2), given by

$$\beta t = -\int_0^{1-P_1} \frac{du}{zP_D(1-u) - (1-u)}\,, \tag{4}$$

does determine $P_1(z,t)$ as a function of z and t.

We now consider the contribution to $\tilde{N}(0,t)$ from immigrants who enter the process in $(0,t)$ and their descendants. The number I of such immigrants has a Poisson distribution with mean νt, and conditionally on I they are independently and uniformly distributed over $(0,t)$. Each immigrant gives rise independently to a total number N_I of events in $(0,t)$. We determine the p.g.f. $P_I(z,t)$ in terms of P_1. The number of first generation descendants of an immigrant at $t-x\,(0 \le x < t)$ has p.g.f. P_D, and each gives rise independently to a number of descendants in $(t-x,t)$ distributed as $N_1(0,x)$. Since x is uniformly distributed over $(0,t)$, $P_I(z,t)$ is given by

$$P_I(z,t) = \frac{z}{t}\int_0^t \sum_{i=0}^{\infty} d_i \left\{P_1(z,x)\right\}^i dx$$

$$= \frac{z}{t}\int_0^t P_D\left\{P_1(z,x)\right\} dx.$$

The p.g.f. of the total innovatory component in (0, t) is

$$P(z,t) = \sum_{i=0}^{\infty} e^{-\nu t} \frac{(\nu t)^i}{i!} \left\{P_I(z,t)\right\}^i$$

$$= \exp\left[-\nu t + \nu z \int_0^t P_D\left\{P_1(z,x)\right\} dx\right]$$

$$= \exp\left[-\nu\int_{x=0}^t \left\{1 - P_1(z,x)\right\} dx - \frac{\nu}{\beta}\left\{1 - P_1(z,t)\right\}\right], \qquad (5)$$

from (3).

Finally we note that the asynchronous count $\tilde{N}(0,t)$ is the sum of the innovatory component and K independent random variables each with p.g.f. $P_1(z,t)$, where K has the stationary distribution $\{\tilde{q}_k\}$. Thus, we obtain

$$\tilde{P}(z,t) = P(z,t)\,\tilde{Q}\left\{P_1(z,t)\right\}, \qquad (6)$$

where P_1 and P are given by (4) and (5) above, and $\tilde{Q}$ by (7.4.3.

7.6. PROPERTIES OF THE INTERVAL SEQUENCE

The intensity ρ of the exponential self-exciting process may be obtained from the formula $\rho = \sum(\nu + \beta i)\, q_i$, or from the general existence theorem. Thus

$$\rho = \frac{\nu\beta}{\beta - \alpha} \tag{1}$$

The p.g.f. of the synchronous stationary distribution $\{q_i\}$ is given by

$$\rho\, \mathcal{Q}(z) = \nu \tilde{\mathcal{Q}}(z) + \beta z \tilde{\mathcal{Q}}'(z)\,. \tag{2}$$

It is worth making a few remarks concerning computation. Although the results of Sections 7.4 and 7.5 do give explicit expressions for the p.g.f., these are not particularly suited to numerical work. However, the form of the equilibrium equations does permit a simple iterative solution. For the synchronous probabilities $\{q_i\}$, we have

$$q_0 = p_{00}\, q_0 + p_{10}\, q_1\,,$$

$$q_1 = p_{01}\, q_0 + p_{11}\, q_1 + p_{21}\, q_2\,, \text{ etc.}$$

Thus the ratios q_i/q_0 $(i = 1, 2, \ldots)$ may be determined by simple back substitution. The value of q_0 may be obtained from Proposition 7.3 (substituting $(z = 0)$ or it may be approximated by taking $\sum_{i=0}^{N} q_i \sim 1$ for some large N. An important proviso is that the stationary distribution should not be spread over too many states. These remarks apply also to the Neyman-Scott processes discussed in Chapter 6.

We shall not attempt to derive analytic results for the joint distributions of intervals. We shall merely point out that it is possible to obtain expressions for the joint density $f_{X,Y}(x, y)$ of two contiguous synchronous intervals in terms of $\mathcal{Q}$ and its derivatives (cf. Section 6.5).It does not seem possible to obtain simple results for the serial correlations. However, we do have

<u>Proposition 7.4</u>. The serial correlations $\{\gamma_m\}$ $(m = 1, 2, \ldots)$

of the synchronous interval sequence decrease monotonically to zero as $m \to \infty$.

<u>Proof</u>. It is sufficient to show that the Markov chain $\{p_{ij}\}$ is stochastically monotone. We have immediately from Proposition 7.2 that for any k and $i \geq 1$,

$$\mathrm{Prob}(K_1 \geq k \mid K_0 = i) \geq \sum_{k-i+1}^{\infty} d_j \geq \mathrm{Prob}(K_1 \geq k \mid K_0 = i-1),$$

proving the result. *

7.7. CONCLUDING REMARKS

As has been seen, the interval properties of even the simplest self-exciting process are not particularly tractable. It is not possible to obtain explicit results for a general function g(.). However, the method of stages can be used when g(.) has a special Erlangian form, and a countable semi-Markov representation obtained. The simple iterative method of Section 7.6 is not available for numerical solution. For a general function g(.) it is possible to obtain an integral equation for the p.g.f. $\tilde{P}(z, t)$ of $\tilde{N}(0, t)$ which might allow a numerical solution (Hawkes, private communication).

In the exponential case the complete intensity function $\lambda(t, \mathcal{H}_t)$ is itself a Markov process. For $\lambda - \nu$ decays exponentially and jumps by α whenever an event occurs. The stationary state equations for λ can be solved iteratively for $\nu < \lambda \leq \nu + \alpha$, $\nu + \alpha \leq \lambda \leq \nu + 2\alpha, \ldots$, but only by successive integration. The approach we have given seems superior.

CHAPTER 8

A RANDOM WALK POINT PROCESS

8.1. INTRODUCTION AND PRELIMINARIES

A random walk with positive steps generates a renewal process. Daley (1970) has discussed the point process generated by a random walk with the two-sided step distribution with density

$$f_s(x) = p\, f_2(-x)\, I(-\infty, 0) + q\, f_1(x)\, I(0, \infty)\ , \tag{1}$$

where $0 < p = 1-q < 1$, f_1 and f_2 are probability densities on $\mathbb{R}^+$ and $I(A)$ denotes the indicator function of the set A. It is assumed that f_1 and f_2 have finite means $1/\beta$ and $1/\alpha$ respectively, and that the walk has positive drift, so that

$$\mu = \frac{q}{\beta} - \frac{p}{\alpha} > 0\ . \tag{2}$$

Daley derived an integral equation satisfied by the p. g. f. of counts in a Borel set A and gave an explicit solution when A is an interval and the step distribution has two exponential tails, i. e.

$$f_1(x) = \beta\, e^{-\beta x}\ , \tag{3}$$

$$f_2(x) = \alpha\, e^{-\alpha x}\ . \tag{4}$$

He obtained the marginal distribution of intervals in the stationary point process and the joint distribution of two contiguous intervals.

A different approach based on the technique of ladder variables used by Feller (1966, Chapter XII) is adopted here. We show that when (3) holds there is a representation of the walk as a Poisson cluster process. Moreover, there are simple formulae for the interval distribution and counting spectrum of this process. It is shown that when both (3) and (4) hold the cluster structure is that of a linear birth and death process. There is then a simple Markovian representation which gives an easy proof of the positivity of the interval correlations and algorithms for their calculation.

We imagine the random walk to be performed by a particle which jumps from a point Y_m to another point Y_{m+1} at the (m+1)th epoch. Words such as 'between', 'before' and 'after' refer to the ordering of the $\{Y_m\}$ induced by the random walk, not to the usual ordering of $\mathbb{R}$. The path is the sequence $\{\ldots Y_{-1}, Y_0, Y_1, \ldots\}$. The point process $\mathcal{P} = \{\ldots T_{-1}, T_0, T_1, \ldots\}$ consists of the $\{Y_m\}$ reordered so that $\ldots < T_{-1} < T_0 < T_1 < \ldots$. Usually we take $T_0 = Y_0 = 0$. In general it is not possible to reconstruct the path from the sequence $\{T_n\}$, or from the interval sequence $\{X_i\}$, where as usual $X_i = T_i - T_{i-1}$.

It follows from (2) that the ascending ladder points (successive maxima) of the walk form a non-terminating renewal process. The renewal measure of the ladder process is denoted by ψ. For a random walk which starts at the origin, Feller (1966, Chapter XII, p. 311) gives the

Duality lemma. The measure ψ admits of two interpretations. For $I \subset \mathbb{R}^+$,

(a) $\psi(I)$ is the expected number of ladder points in I ;

(b) $\psi(I)$ is the expected number of visits to I prior to the first entry into $\mathbb{R}^-$.

If the first step is known to be positive then the expected number of visits to I prior to the first entry into $\mathbb{R}^-$ is given by

$$\psi_+(I) = \psi(I)/q \; . \qquad (5)$$

8.2. STEP DISTRIBUTIONS WITH POSITIVE EXPONENTIAL TAIL

Proposition 8.1. If (8.1.2) and (8.1.3) hold, then P is a Poisson cluster process. The ladder points are the main events and the cluster corresponding to the ith ladder point is the set of points visited between the ith and (i+1)th ladder epochs.

Proof. Note that the subsidiary events of a cluster lie to the left of the main event. From the lack of memory of the exponential distribution, it follows immediately that the ladder points form a Poisson process and that sections of the path between successive ladder epochs are mutually independent. ※

Corollary. If $Y_0 = 0$, then that part of the future path $\{Y_i : i > 0\}$ which lies in $t > 0$ generates a stationary point process in $\mathbb{R}^+$.

If $t \in \mathbb{R}$ and $i \in \mathbb{Z}$, then i is said to be a positive or negative crossing of t if $Y_i \leq t < Y_{i+1}$ or $Y_{i+1} \leq t < Y_i$, respectively. The multiplicity $\tilde{M}_t$ at time t is the number of negative crossings of t. Then the positive drift ensures that $\tilde{M}_t$ is finite and that the number of positive crossings of t is $\tilde{M}_t + 1$.

Proposition 8.2. If (8.1.2) and (8.1.3) hold, then

(i) conditionally on $\tilde{M}_t = m$, the $m+1$ sections into which the path is divided by successive positive crossings of t are mutually independent,

(ii) the distribution of $\tilde{M}_t$ is geometric, with $\text{Prob}(M_t = m) = r^m(1-r)$

(m = 0, 1, . . .), where $r = {}^{p\beta}/(q\alpha)$.

Proof. Part (i) is immediate from the lack of memory of the exponential distribution. Hence, to prove (ii), it is sufficient to show that the probability of at least one negative crossing of t is r. Equivalently, we show that

$$\text{Prob}(\exists\, i > 0 : Y_i < 0 \mid Y_0 = 0,\ Y_1 > 0) = r.$$

For $u \geq 0$, let

$$L(u) = \text{Prob}(\exists\, i > 0 : Y_1, \ldots, Y_{i-1} > 0,\ Y_i < -u \mid Y_0 = 0,\ Y_1 > 0).$$

Then (cf. Feller op. cit. p. 387) we have

$$L(u) = \int_{x=0}^{\infty} \psi_+(dx)\, p\, \mathcal{F}_2(u+x)\ ,$$

where $\mathcal{F}_2$ is the survivor function of f_2. Since the ladder points form a Poisson process, the duality lemma gives $\psi(I) = \beta|I|$ and $\psi_+(I) = \beta|I|/q$. Hence

$$L(0) = \frac{p\beta}{q\alpha} = r\ .$$

Corollary. $L(u)/L(0) = \alpha \int_{x=0}^{\infty} \mathcal{F}_2(u+x)dx.$

Thus, conditionally on i being a negative crossing of t, the distribution of $t - Y_{i+1}$ is the same as the recurrence time distribution in a stationary renewal process with interval density f_2. We denote the survivor function of this recurrence time by $\tilde{\mathcal{F}}_2$.

Proposition 8.2 and its corollary permit a simple derivation of the interval distribution of the point process. We first determine the survivor function

$$\tilde{\mathcal{F}}(x) = \text{Prob}\{\tilde{N}(t-x, t] = 0\},$$

by conditioning on the multiplicity $\tilde{M}_t$. Suppose that $\tilde{M}_t = m$ and that the (m+1) positive crossings of t and the m negative crossings are $i_1, \ldots, i_{m+1}$ and $j_1, \ldots, j_m$, respectively. Then $\tilde{N}(t-x, t] = 0$ if and only if

$$t - Y_{i_k} > x \qquad (k = 1, \ldots, m+1),$$

$$t - Y_{j_k+1} > x \qquad (k = 1, \ldots, m).$$

These random variables are mutually independent, with survivor functions $e^{-\beta x}$ and $\tilde{\mathcal{F}}_2(x)$, respectively. Thus

$$\text{Prob}\{\tilde{N}(t-x, t] = 0 \mid \tilde{M}_t = m\} = (e^{-\beta x})^{m+1} \{\tilde{\mathcal{F}}_2(x)\}^m .$$

Using Proposition 8.2(ii) to remove the conditioning on m, we obtain

Proposition 8.3. If (8.1.2) and (8.1.3) hold then the survivor function of the forward recurrence time is given by

$$\tilde{\mathcal{F}}(x) = \frac{(1-r)\, e^{-\beta x}}{1 - r\, e^{-\beta x}\, \tilde{\mathcal{F}}_2(x)} .$$

Since the rate of the process is $\rho = 1/\mu$, the usual recurrence time formula for stationary point processes gives the

Corollary. The marginal interval distribution has survivor function

$$\mathcal{F}(x) = \frac{q(1-r)^2\, e^{-\beta x} \{\beta + r\alpha e^{-\beta x}\, \tilde{\mathcal{F}}_2(x)\}}{\beta \{1 - r\, e^{-\beta x}\, \tilde{\mathcal{F}}_2(x)\}^2} .$$

8.3. DOUBLE EXPONENTIAL STEP DISTRIBUTION

If both (8.1.3) and (8.1.4) hold, there is a strengthening of Proposition 8.2, *viz*

Proposition 8.4. If $f_1(x) = \beta e^{-\beta x}$, $f_2(x) = \alpha e^{-\alpha x}$, then

(i) conditionally on $\tilde{M}_t = m$, the 2m+2 sections into which the path is divided by successive crossings of t are mutually independent;

(ii) $\tilde{M}_t$ is a Markov process.

Proof. Part (i) is again immediate from the lack of memory of the exponential distribution. Thus, conditionally on $\tilde{M}_t = m$, that part of the path which lies in $\tau < t$ is independent of that part in $\tau > t$. It follows that the values of $\tilde{M}_\tau$ in $\tau < t$ are conditionally independent of those in $\tau > t$, given $\tilde{M}_t$. This proves Part (ii).

To conform with our general notation for imbedded Markov processes, we now let $\tilde{K}_t = \tilde{M}_t$. We also define the process $\{K_n\}$ by the usual relation

$$K_n = \tilde{K}_{T_n} .$$

Then K_n is the multiplicity just after the nth event. Also, $\{K_n\}$ is a Markov chain, and, for each n, the joint distributions of $\{K_i, X_i\}$ in $i > n$ depend on their joint distributions in $i \leq n$ only through K_n. Thus $\{K_i, X_i\}$ is a semi-Markov representation of the point process.

We now consider the transition probabilities of $\{K_i, X_i\}$. Letting $\gamma = \beta/(\alpha+\beta)$, we have

Proposition 8.5. The transition probabilities of $\{K_n\}$ are given by

$$p_{k\ell} = 0 \qquad (\ell \neq k-1,\ k,\ k+1),$$

$$p_{k,k-1} = \frac{\alpha q\ \ k}{(\alpha+\beta)(k+\gamma)}, \quad p_{k,k+1} = \frac{\beta p\ (k+1)}{(\alpha+\beta)(k+\gamma)},$$

$$p_{k,k} = \frac{(\alpha p+\beta q)\ k + \beta q}{(\alpha+\beta)(k+\gamma)} . \tag{1}$$

The conditional interval density is

$$f(x_1 \mid k_0, k_1) = (\alpha+\beta)(k_0+\gamma) \exp\{-(\alpha+\beta)(k_0+\gamma)x_1\} .$$

Proof. Consider a typical section of the path from a positive to the next negative crossing of T_n. There are 4 ways in which this section may include an event at T_n+x but no event in (T_n, T_n+x), see Figure 2. Note that it is irrelevant whether the event at T_n is on the upper or lower part of the section. In (a) and (c), T_n+x is the first point in the

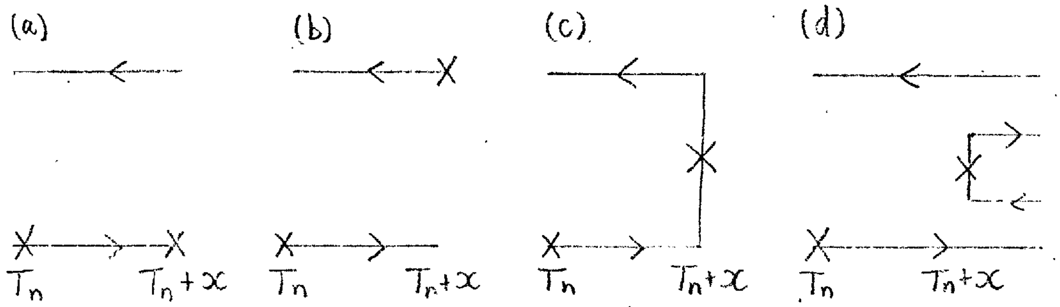

Figure 2. The transitions of $\{K_n\}$.

section visited by the particle. In (b) and (c) it is the last but in (d) it is neither the first nor the last.

In (a) we require the improper density that the particle visits $T_n + x$ on its first step, then takes a positive step and ultimately returns past T_n, visiting no point in (T_n, T_n+x). This density is to be conditional on the first step being positive and on ultimate return to t. This gives a term

$$q\beta e^{-\beta x}\, qre^{-\alpha x}/(qr) = q\beta e^{-(\alpha+\beta)x} .$$

Similarly for (b) and (c) we obtain terms $p\alpha\, e^{-(\alpha+\beta)x}$ and $q\alpha e^{-(\alpha+\beta)x}$, respectively. In (d) the appropriate σ-event is that the particle overshoots T_n+x on its first step, ultimately returns to T_n+x, then takes a positive step and finally returns to t, visiting no point in (t, t+x). This gives a term

$$q\, e^{-\beta x}\, r\, \alpha\, q\, r\, e^{-\alpha x}/(rq) = \beta\, p\, e^{-(\alpha+\beta)x} .$$

For the final section of the path, which does not return to T_n, only (a) and (d) are possible and the appropriate conditional densities are $\beta q\, e^{-\beta x}$ and $\beta p\, e^{-\beta x}$, respectively.

If the (n+1)th event of the point process occurs at T_n+x, there must be precisely one section of the path for which one of (a), (b), (c) or (d) holds, and the remaining sections must contain no event in (T_n, T_n+x). Moreover, if $K_n = k$, then $K_{n+1} = k$, k, k-1 and k+1 in (a), (b), (c) and (d), respectively. The proof is completed by appealing to Proposition 8.4(i) and noting that the probability of a section containing no event in (t, t+x] is $\exp\{-(\alpha+\beta)x\}$. ✱

<u>Proposition 8.6.</u> The synchronous stationary distribution is given by

$$q_k = \frac{q}{\beta}(1-r)^2\,(\alpha+\beta)\,(k+\gamma)r^k , \tag{2}$$

with p.g.f.

$$Q(z) = \frac{q}{\beta}(1-r)^2\,\frac{(\beta+\alpha r z)}{(1-rz)^2} . \tag{3}$$

<u>Proof.</u> It is easy to verify that the distribution (2) is invariant under the transition probabilities (1). Alternatively, (3) may be deduced from Proposition 8.2 and the relation (6.2.4) between the synchronous and asynchronous stationary distributions. ✱

8.4. SOME PROPERTIES OF THE INTERVAL SEQUENCE

The survivor function for a synchronous interval is

$$\mathcal{F}(x) = \sum_{k=0}^{\infty} q_k \exp\{-(\alpha+\beta)(k+\gamma)x\} .$$

We find that this agrees with Daley's (1970) result

$$\mathcal{F}(x) = \frac{(q\alpha - p\beta)^2 \, (q e^{\beta x} + p e^{-\alpha x})}{(q\alpha \, e^{\beta x} - p\beta \, e^{-\alpha x})^2}, \tag{1}$$

which is a special case of the corollary to Proposition 8.3 above. The moments of the interval distribution may be found from (1), or by using

$$\mathbb{E}(X^m) = \mathbb{E}\left\{ \mathbb{E}(X^m \mid K) \right\} = \sum_{k=0}^{\infty} \frac{q_k \; m!}{\left\{ (\alpha+\beta)(k+\gamma) \right\}^m}$$

After substituting for q_k and simplifying, we obtain for the first two moments

$$\mathbb{E}(X) = q/\beta - p/\alpha ,$$

as it must, and

$$\mathbb{E}(X^2) = \frac{2q(1-r)^2}{\beta(\alpha+\beta)} \; \Phi(r, 1, \gamma) ,$$

where, following Daley again, Φ is defined in Erdelyi, Magnus, Oberhettinger and Tricomi (1953, p. 27) as

$$\Phi(r, m, \gamma) = \sum_{k=0}^{\infty} \frac{r^k}{(k+\gamma)^m} . \tag{2}$$

In general Φ does not appear to have been tabulated, though, as Daley points out, it is a special case of Gauss's hypergeometric function. A simple expression is available in one case however, for

$$\Phi(r, 1, {}^1/2) = r^{-\frac{1}{2}} \log \left\{ \frac{1 + r^{1/2}}{1 - r^{1/2}} \right\} ,$$

The results of Section 6.2 ensure that the marginal interval distribution has a hazard function which decreases to β as $x \to \infty$, and a coefficient of variation greater than one. The serial correlations of the interval sequence may be obtained using similar methods. For, if $\{p_{ij}^{(m)}\}$ are the m-step transition probabilities of $\{p_{ij}\}$, $p_{ij}^{(m)} = 0$ if $|j-i| > m$ and hence

$$E(X_1 X_{m+1}) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{q_i\, p_{ij}^{(m)}}{(\alpha+\beta)^2 (i+\gamma)(j+\gamma)} \tag{3}$$

may be expressed as a single infinite series. Since $p_{ij}^{(m)}$ is a rational function of i if j-i is fixed it is possible to express the serial correlation γ_m of lag m in terms of $\Phi(r, s, \gamma)$ $(s = 1, \ldots, m+1)$. The algebra becomes tedious even for small m however, and numerical results can be obtained quite easily from (3). Explicitly for m=1 we have

$$E(X_1 X_2) = \frac{q(1-r)^2}{\beta(\alpha+\beta)^2} \left\{ \frac{\alpha\beta(q-p)}{\alpha+\beta} \Phi(r, 2, \gamma) + \frac{3\alpha\beta+\alpha^2 p+\beta^2 q}{\alpha+\beta} \Phi(r, 1, \gamma) - 2q\alpha \right\},$$

which agrees with a result of Daley (1970), and for m = 2,

$$E(X_1 X_3) = \frac{q(1-r)^2}{\beta(\alpha+\beta)^3} \sum_{k=0}^{\infty} r^k \left[\frac{2\beta^2 p^2 (k+1)(k+2)}{(k+\gamma)(k+\gamma+1)(k+\gamma+2)} \right.$$

$$+ \frac{\{(k+1)\beta q + k\alpha p\}^2}{(k+\gamma)^3} + \frac{\beta p(\beta p+2\alpha p)(k+1)^2 + 2\beta^2 pq(k+1)(k+2)}{(k+\gamma)(k+\gamma+1)^2}$$

$$\left. + \frac{\beta p(2\beta q+\alpha q)(k+1)^2 + 2\alpha\beta p^2 k(k+1)}{(k+\gamma)^2 (k+\gamma+1)} \right].$$

After reduction by partial fractions this yields

$$E(X_1 X_3) = \frac{(1-r)^2 q}{\beta(\alpha+\beta)^5} \left\{ A\, \Phi(r, 3, \gamma) + B\, \Phi(r, 2, \gamma) + C\, \Phi(r, 1, \gamma) + D \right\},$$

where

$$A = \alpha^2\beta^2(p-q)^2 ,$$

$$B = \alpha\beta \{\beta^2(-3pq+2q^2) + \alpha\beta(-4p^2+4q^2) + a^2(-2p^2+3pq)\} ,$$

$$C = \beta^4 q^2+\alpha\beta^3(-p^2+9pq+6q^2) + \alpha^2\beta^2(6p^2+22pq+6q^2)$$

$$+ \alpha^3\beta(6p^2+9pq-q^2) + \alpha^4 p^2 ,$$

$$-D = (\alpha+\beta) \{\alpha\beta^2(pq+4q^2) + \alpha^2\beta(5q^2+11pq) + \alpha^3(6pq-q^2)\} .$$

There is the usual qualitative result for the behaviour of the serial correlations.

Proposition 8.7. The serial correlations $\{\gamma_m : m=1, 2, \ldots\}$ of the interval sequence of the double exponential random walk point process are positive for all m and decrease monotonically to zero as $m \to \infty$.

Proof. We have $E(X_n | K_{n-1}, K_n) = f(K_{n-1})$, where $f(x) = \{(\alpha+\beta)x+\beta\}^{-1}$ is a monotone function of x. Thus, by the theorem of Daley quoted in Section 6.3 above, it is sufficient to show that the Markov chain $\{K_n\}$ is stochastically monotone, i.e. that

$$\text{Prob}(K_1 \leq k_1 | K_0 = k_0) \leq \text{Prob}(K_1 \leq k_1 | K_0 = k_0-1) \qquad (4)$$

for all k_0 and k_1. If $k_0 \leq k_1$, the right hand side of (4) is unity, whereas if $k_0 > k_1+1$, the left hand side is zero. If $k_0 = k_1+1$, then (4) is

$$\frac{\alpha(k_1+1)q}{(\alpha+\beta)(k_1+\gamma+1)} \leq \frac{\alpha k_1+\beta(k_1+1)q}{(\alpha+\beta)(k_1+\gamma)} ,$$

which always holds for $k_1 \geq 1$. ✳

8.5. THE CLUSTER STRUCTURE OF THE DOUBLE EXPONENTIAL PROCESS

For the double exponential random walk there is a pleasing representation of the within cluster structure.

Proposition 8. 8. If (8. 1. 2), (8. 1. 3) and (8. 1. 4) hold, then the structure of the clusters defined in Proposition 8. 1 is that of a linear birth and death process, reversed in time, in which a single individual dies in a small interval δt and simultaneously reproduces to form 0, 1 or 2 similar individuals with respective probabilities $\alpha q \delta t$, $(\alpha p+\beta q)\delta t$, $\beta p \delta t$.

Proof. This is essentially the same as Proposition 8. 2, except that here we are concerned with sections of the path between negative and positive crossings. There are four possible configurations corresponding to the occurrence of an event, see Figure 3. The approriate probabilities

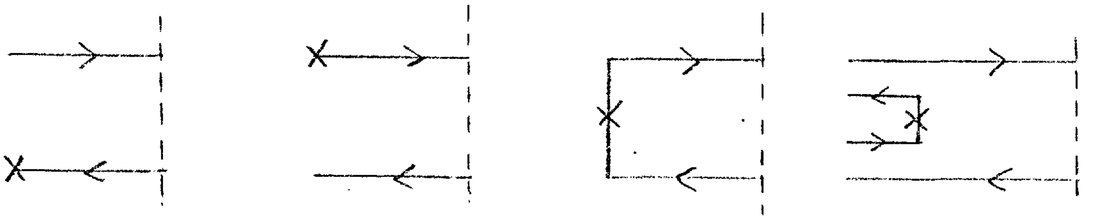

Figure 3. The cluster structure.

can be written down as before. For example in the first case we obtain $e^{-\alpha x}$ p. 1. $e^{-\beta x}$. ✳

Proposition 8. 8 permits a treatment of the asynchronous counting distribution based on the corresponding results of Chapter 7. We have shown that the two-sided random walk may be represented as a reversed time iterated cluster process, in which the process of centres has rate β, the parameter of the lifetime distribution (corresponding to the 'β' of Chapter 7) is $\alpha+\beta$, and the offspring distribution has p. g. f.

$$P_D(z) = (\alpha+\beta)^{-1}\left\{ \alpha q + (\alpha p+\beta q)z + \beta p z^2 \right\}. \qquad (1)$$

Here we can evaluate explicitly the p. g. f. $P_1(z,t)$ of $N[-t, 0)$ given by (7.5.4), which becomes

$$(\alpha+\beta)t = -\int_{P_1}^{1} \frac{du}{z\ P_D(u) - u} \tag{2}$$

From the general remarks following (7.5.4), we see that the denominator of (2) is always negative in the region of interest. There must therefore be a factorisation

$$z\ P_D(u) - u = c(u - \phi_1)(\phi_2 - u)\ , \tag{3}$$

where $\phi_2 > 1$, $\phi_1 < 1$ and $c > 0$ are independent of u. In fact ϕ_1 and ϕ_2 are the roots of

$$\beta pzu^2 + \{(\alpha p+\beta q) z - \alpha - \beta\} u + \alpha qz = 0\ . \tag{4}$$

After expansion by partial fractions and integration, we obtain

$$P_1(z,t) = \frac{\phi_1(\phi_2-1)\exp\{(\phi_2-\phi_1)\beta pzt\} + \phi_2(1-\phi_1)}{(\phi_2-1)\exp\{(\phi_2-\phi_1)\beta pzt\} + 1-\phi_1}\ . \tag{5}$$

To calculate the total p. g. f. $\tilde{P}(z,t)$, we can proceed as in Chapter 7, but there is one important difference. Here the immigrants do not split immediately on entry. However an immigrant (main event) at $(-t + x)$ generates no subsidiary events with probability q, and $N_1(0,x)$ subsidiary events with probability p. Thus the total innovatory component is

$$P(z,t) = \sum_{i=0}^{\infty} e^{-\beta t} \frac{(\beta t)^i}{i!} \left[z \left\{ q + \frac{p}{t} \int_0^t P_1(z,x)dx \right\} \right]^i \tag{6}$$

After performing the integration in (6), simplifying and substituting (5) and (6) into (7.5.5), noting that here $\tilde{Q}(z) = (1-r)/(1-rz)$, we find that

$$P(z,t) = \frac{(\alpha q-\beta p)(\phi_2-\phi_1)\exp\{-\beta t(1-qz)+\beta pz\phi_2 t\}}{(\phi_2-1)(\alpha q-\beta p\phi_1)\exp\{\beta pz(\phi_2-\phi_1)t\}+(1-\phi_1)(\alpha q-\beta p\phi_2)}. \quad (7)$$

This agrees with Daley's result. It must be remembered that in this expression ϕ_1 and ϕ_2 are functions of z, so that expansion of (7) as a power series in z is not easy.

8.6. THE CROSS-INTENSITY FUNCTION AND MOMENT MEASURES

Finally we consider the moment measures of the process. The second order intensity function is of course closely related to the 'renewal function' for the random walk. This has been studied, for example by Feller (1966, Chapter 11). However, in random walk contexts, interest usually centres on the distributions and epochs of the ascending and descending ladder variables. Here we shall be concerned with the full process and it will be seen that the representation as a cluster process does simplify the arguments. At first however we consider a step distribution with general densities f_1 and f_2, and use methods based on the regenerative character of the random walk.

Define intensity functions $m_1(t)$ and $m_2(t)$ $(t > 0)$ as follows

$$m_1(t)dt = \text{Prob}\ \{\exists n > 0 : Y_n \in (t, t+dt) \mid Y_0 = 0\}, \quad (1)$$

$$m_2(t)dt = \text{Prob}\ \{\exists n > 0 : Y_n \in (-t, -t+dt) \mid Y_0 = 0\}. \quad (2)$$

The existence of m_1 and m_2 will usually follow from the existence of the renewal density for the ladder process. In the particular cases we consider below the cluster process representation automatically gives a proof.

<u>Proposition 8.9.</u> The functions m_1 and m_2 satisfy the equations

$$m_1(t) = qf_1(t) + q \int_0^t m_1(u)f_1(t-u)du + p \int_{u=t}^{\infty} m_1(u)f_2(u-t)du +$$

$$+ q \int_{u=0}^{\infty} m_2(u)f_1(t+u)du , \qquad (3)$$

$$m_2(t) = pf_2(t) + p \int_0^t m_2(u)f_2(t-u)du + q \int_{u=t}^{\infty} m_2(u)f_1(u-t)du +$$

$$+ p \int_{u=0}^{\infty} m_1(u)f_2(t+u)du. \qquad (4)$$

Proof. If $t > 0$, then $Y_n = t$ if and only if (i) $Y_1 = t$, (ii) $0 \leq Y_{n-1} < t$ and $Y_n = t$; (iii) $Y_{n-1} \geq t$ and $Y_n = t$, or (iv) $Y_{n-1} < 0$ and $Y_n = t$. This gives (3) and a similar decomposition gives (4). *

In general, there is no simple solution of these equations. However, when the positive tail is exponential, we have

Proposition 8.10. Suppose that $f_1(x) = \beta e^{-\beta x}$ and that the walk has positive drift $1/\rho$. Then $m_1(t) \equiv \rho$ and the Laplace transform $m_2^*(s)$ of $m_2(t)$ is given by

$$m_2^*(s) \left\{ 1 - pf_2^*(s) - \frac{q\beta}{\beta - s} \right\} = pf_2^*(s) + \frac{q(\rho-\beta)}{s-\beta} + \frac{p\rho}{s} \left\{ 1 - f_2^*(s) \right\} \qquad (5)$$

Proof. That $m_1(t) = \rho$ follows immediately from the corollary to Proposition 8.1. Substitution into (3), without taking transforms, gives $m_2^*(\beta) = (\rho-\beta)/\beta$. Taking transforms in (4) and substituting for $m_2^*(\beta)$, we obtain (5).

Corollary. For the two-sided exponential walk,

$$m_1(u) = \frac{\alpha\beta}{q\alpha - p\beta}, \quad m_2(u) = \frac{pq(\alpha+\beta)^2}{q\alpha - p\beta} \exp\left\{-(q\alpha - p\beta)u\right\}. \tag{6}$$

It is worth noting that the existence of m_1 and m_2 for the random walk point process ensures the existence of finite, absolutely continuous moment measures of all orders. For we let $\mathcal{Q}_k$ be the set of all permutations $\underset{\sim}{\alpha} = (\alpha_1, \ldots, \alpha_k)$ of $(1, \ldots, k)$ and define $m(t) = m_1(t)$ $(t > 0)$ and $m(t) = m_2(-t)$ $(t < 0)$. Then, if $t_1 < \ldots < t_k$, the particle may visit the points $t_1, \ldots, t_k$ in any order. Thus we have, for the kth moment measure,

$$M^{(k)}(dt_1 \times \ldots \times dt_k) = \rho \sum_{\underset{\sim}{\alpha} \in \mathcal{Q}_k} \prod_{j=2}^{k} m(t_{\alpha_j} - t_{\alpha_{j-1}}) dt_1 \ldots dt_k .$$

APPENDIX

SOME NUMERICAL VALUES

Some numerical results are given for the simpler processes discussed in Chapter 6 to 9. The four processes considered are the two Neyman-Scott cluster processes with exponential displacements when the cluster size has Poisson and geometric distributions, respectively the exponential self-exciting process and the random walk point process with double exponential step distribution. The rate of each process is set to unity, leaving two parameters which can be varied separately in each case. Tables 1. 1, 2. 1, 3. 1 and 4. 1 give the values of the third parameter for specified values of two given parameters for each process.

The interval properties were calculated from the synchronous and asynchronous stationary distributions of the imbedded Markov process. For all except the random walk process, for which a simple explicit solution is known, the synchronous stationary distribution was calculated by the iterative procedure discussed in Section 7.5 Some of the values obtained for the geometric shot-noise process were checked with the terms in the negative binomial distribution (6.5.3). The serial correlations of the interval sequence were found by iterating the transition matrix of the imbedded Markov chain. The transition matrix and the stationary distribution were truncated after 40 states. This placed some restriction on the values of the parameters for which results could be obtained. However, the method used to calculate the stationary distribution ensured that this was not sensitive to the point at which the transition matrix was truncated.

The coefficients of variation of intervals and the asymptotic slopes of the variance-time curves are exhibited in Tables 1.2, 2.2, 3.2 and 4.2. The ratio of these two quantities, which is known to equal

$$1 + 2 \sum_{k=1}^{\infty} \gamma_k \, ,$$

where $\{\gamma_k\}$ are the serial correlations of intervals (Cox, 1962, p. 134) is also given. This ratio gives a useful measure of the departure from a renewal process, particularly when the $\{\gamma_i\}$ are all positive. Tables 1.3, 2.3, 3.3 and 4.3 give values of γ_1, γ_2 and γ_{10}. The mean and variance of the forward recurrence time are displayed in Tables 1.4, 2.4, 3.4 and 4.4.

For all four processes the coefficient of variation of intervals and the asymptotic dispersion of counts are known to be greater than one. Moreover, the mean forward recurrence time is also greater than unity. The serial correlations $\{\gamma_i\}$ of the random walk process and the self-exciting process decrease monotonically to zero. It is interesting to see from the numerical values given that the serial correlations decrease quite slowly, i.e. that

$$\frac{\gamma_1}{\sum \gamma_k}$$

is usually quite small. The greatest dispersion of counts for moderate values of the parameters seems to be exhibited by the random walk point process, followed by the self-exciting process. A direct comparison of the results for the two Neyman-Scott processes is illuminating. A Poisson cluster size distribution gives smaller dispersion and lower serial correlations than a geometric cluster size distribution when the parameters of the process are the same. This would be expected from the shot-noise interpretations of the two processes, since there is an extra source of random variation in the latter process due to the randomness of the shot height.

1. NEYMAN-SCOTT PROCESS WITH POISSON CLUSTER SIZES

Here the Poisson process of main events has rate ν , the distribution of the cluster size is Poisson with mean α/β and the displacement of the subsidiary events has density $\beta e^{-\beta x}$.

We take $\rho = \frac{\nu}{1 + \alpha/\beta} = 1$.

TABLE 1.1. Values of ν for specified α and β

β \ α	5.0	2.0	1.0	0.5	0.2	0.1
5.0	.500	.714	.833	.909	.962	.980
2.0	.286	.500	.667	.800	.909	.952
1.0	.167	.333	.500	.667	.833	.909
0.5			.333	.500	.714	.833
0.2			.167	.286	.500	.667

TABLE 1.2. Dispersion of Counts and of Intervals

The first entry in each cell is the asymptotic slope, $\lim_{t \to \infty} \left[t^{-1} \operatorname{var} \left\{ \tilde{N}(0,t) \right\} \right]$, of the variance time curve, the second entry is var(X), the variance of a synchronous interval, and the third entry is the ratio of these quantities,

β \ α	5.0	2.0	1.0	0.5	0.2	0.1
5.0	2.500	1.686	1.367	1.191	1.078	1.040
	2.718	1.675	1.335	1.167	1.067	1.033
	.920	1.007	1.024	1.021	1.010	1.007
2.0	4.214	2.500	1.833	1.450	1.191	1.098
	4.761	2.402	1.682	1.337	1.134	1.067
	.885	1.041	1.090	1.085	1.050	1.029
1.0	6.833	3.667	2.500	1.833	1.367	1.191
	7.649	3.262	2.051	1.508	1.201	1.100
	.893	1.124	1.219	1.216	1.138	1.083
0.5			3.667	2.500	1.686	1.367
			2.466	1.675	1.263	1.132
			1.487	1.493	1.335	1.208
0.2			6.833	4.214	2.500	1.833
			3.041	1.816	1.304	1.155
			2.247	2.320	1.917	1.587

TABLE 1.3. Serial Correlations of the Interval Sequence

The entries in each cell are the serial correlations γ_1, γ_2 and γ_{10} of lags 1, 2 and 10, respectively, for the synchronous interval sequence.

β \ α	5.0	2.0	1.0	0.5	0.2	0.1
5.0	-0.047	-0.001	.009	.008	.005	.002
	.004	.004	.002	.002	.001	.000
	.000	.000	.000	.000	.000	.000
2.0	-0.052	-0.001	.025	.027	.017	.010
	-0.022	.011	.012	.009	.015	.003
	.000	.000	.000	.000	.000	.000
1.0	-0.016	.014	.048	.053	.035	.021
	-0.036	.012	.027	.026	.017	.010
	-0.002	.000	.000	.000	.000	.000
0.5			.083	.088	.059	.036
			.048	.052	.036	.023
			.003	.003	.002	.001
0.2			.146	.137	.087	.053
			.087	.098	.067	.042
			.017	.019	.014	.001

TABLE 1.4. Mean and Variance of the Recurrence Time

The first entry in each cell is the mean, $E(\tilde{X}_1)$, and the second entry is the variance, $var(\tilde{X}_1)$, of the forward recurrence time.

β \ α	5.0	2.0	1.0	0.5	0.2	0.1
5.0	1.859	1.337	1.168	1.084	1.033	1.017
	3.935	1.935	1.428	1.204	1.079	1.039
2.0	2.881	1.701	1.341	1.168	1.067	1.033
	11.472	3.708	2.116	1.500	1.187	1.091
1.0	4.324	2.131	1.526	1.254	1.100	1.050
	31.511	7.307	3.263	1.930	1.327	1.157
0.5			1.733	1.338	1.132	1.066
			5.408	2.568	1.507	1.237
0.2			2.021	1.408	1.152	1.077
			11.168	3.584	1.702	1.320

2. THE NEYMAN-SCOTT PROCESS WITH GEOMETRIC CLUSTER SIZES

Here the Poisson process of main events has rate ν, the distribution of the cluster size is geometric with mean α/β and the displacement of the subsidiary events has density $\beta e^{-\beta x}$.

We take $\rho \equiv \frac{\nu}{1 + \alpha/\beta} = 1$.

TABLE 2.1. Values of ν for specified α and β

β \ α	5.0	2.0	1.0	0.5	0.2	0.1
5.0	.500	.714	.833	.909	.962	.980
2.0		.500	.667	.800	.909	.952
1.0			.500	.667	.833	.909
0.5			.333	.500	.714	.833
0.2					.500	.667

TABLE 2.2 Dispersion of Counts and of Intervals

The first entry in each cell is the asymptotic slope, $\lim_{t \to \infty} \left[t^{-1} \operatorname{var} \left\{ \tilde{N}(0,t) \right\} \right]$, of the variance time curve, the second entry is var(X), the variance of a synchronous interval and the third entry is the ratio of these quantities.

β \ α	5.0	2.0	1.0	0.5	0.2	0.1
5.0	3.000	1.800	1.400	1.200	1.080	1.040
	2.752	1.684	1.338	1.168	1.067	1.033
	1.090	1.069	1.046	1.027	1.012	1.007
2.0		3.000	2.000	1.500	1.200	1.100
		2.468	1.705	1.344	1.135	1.067
		1.216	1.173	1.116	1.057	1.031
1.0			3.000	2.000	1.400	1.200
			2.142	1.538	1.206	1.102
			1.401	1.300	1.161	1.089
0.5			5.000	3.000	1.800	1.400
			2.729	1.773	1.284	1.138
			1.832	1.692	1.402	1.230
0.2					3.000	2.000
					1.375	1.177
					2.182	1.699

TABLE 2.3 Serial Correlations of Intervals

The entries in each cell are the serial correlations γ_1, γ_2 and γ_{10} of lags 1, 2 and 10, respectively, for the synchronous interval sequence.

β \ α	5.0	2.0	1.0	0.5	0.2	0.1
5.0	.029	.026	.018	.011	.005	.003
	.009	.006	.004	.002	.001	.000
	.000	.000	.000	.000	.000	.000
2.0		.062	.053	.037	.019	.010
		.024	.019	.013	.006	.003
		.000	.000	.000	.000	.000
1.0			.094	.073	.040	.022
			.045	.035	.020	.011
			.001	.018	.000	.000
0.5			.141	.118	.068	.039
			.082	.072	.043	.025
			.006	.004	.002	.017
0.2					.106	.060
					.084	.049
					.018	.040

TABLE 2.4. Mean and Variance of the Recurrence Time

The first entry in each cell is the mean, $E(\tilde{X}_1)$, and the second entry is the variance, $var(\tilde{X}_1)$, of the forward recurrence time.

β \ α	5.0	2.0	1.0	0.5	0.2	0.1
5.0	1.876	1.342	1.169	1.084	1.033	1.017
	3.948	1.938	1.429	1.204	1.079	1.039
2.0		1.734	1.352	1.172	1.068	1.034
		3.758	2.132	1.505	1.187	1.091
1.0			1.571	1.269	1.103	1.051
			3.374	1.961	1.332	1.158
0.5			1.864	1.386	1.142	1.069
			5.990	2.736	1.536	1.244
0.2					1.187	1.089
					1.854	1.361

3. THE EXPONENTIAL SELF-EXCITING PROCESS

Here the process has a complete intensity function of the form

$$\lambda(t, \mathcal{H}_t) = \nu + \int_{-\infty}^{t} \alpha e^{-\beta(t-u)} dN(u) ,$$

and $\alpha < \beta$.

We take $\rho = \dfrac{\nu}{1 - \alpha/\beta} = 1$.

TABLE 3.1. Values of ν for specified α and β

β \ α	.8	.7	.6	.5	.4	.3	.2
.9	.111	.222	.333	.444	.556	.667	.778
.8		.125	.250	.375	.500	.625	.750
.7			.143	.285	.429	.571	.714
.6				.167	.333	.500	.667
.5					.200	.400	.600
.4						.250	.500
.3							.333

TABLE 3.2. Dispersion of Counts and of Intervals

The first entry in each cell is the asymptotic slope, $\lim_{t \to \infty} \left[t^{-1} \text{var} \left\{ \tilde{N}(0,t) \right\} \right]$, of the variance time curve, the second entry is var(X), the variance of a synchronous interval, and the third entry is the ratio of these quantities.

β \ α	.8	.7	.6	.5	.4	.3	.2
.9	81.000	20.250	9.000	5.063	3.240	2.250	1.653
	11.635	5.001	3.107	2.255	1.780	1.480	1.272
	6.962	4.049	2.897	2.245	1.820	1.520	1.300
.8		64.000	16.000	7.111	4.000	2.560	1.778
		9.628	4.140	2.607	1.923	1.541	1.297
		6.647	3.865	2.728	2.080	1.661	1.371
.7			49.000	12.250	5.444	3.062	1.960
			7.700	3.346	2.157	1.628	1.330
			6.364	3.661	2.524	1.881	1.474
.6				36.000	9.000	4.000	2.250
				5.893	2.638	1.766	1.374
				6.109	3.412	2.265	1.638
.5					25.000	6.250	2.778
					4.272	2.037	1.441
					5.852	3.068	1.928
.4						16.000	4.000
						2.920	1.563
						5.479	2.559
.3							9.000
							1.920
							4.688

TABLE 3. 3. Serial Correlations of the Interval Sequence

The entries in each cell are the serial correlations γ_1, γ_2 and γ_{10} of lags 1, 2 and 10, respectively, for the synchronous interval sequence.

β \ α	.8	.7	.6	.5	.4	.3	.2
.9	.204	.197	.176	.149	.119	.088	.058
	.148	.140	.121	.099	.076	.054	.034
	.058	.042	.027	.015	.007	.003	.001
.8		.209	.198	.170	.136	.100	.065
		.153	.141	.117	.091	.064	.040
		.059	.041	.024	.012	.005	.026
.7			.216	.196	.158	.116	.075
			.159	.141	.110	.078	.048
			.060	.039	.021	.009	.003
.6				.223	.188	.137	.087
				.166	.137	.097	.059
				.062	.037	.017	.006
.5					.227	.169	.104
					.172	.126	.076
					.064	.032	.012
.4						.221	.131
						.173	.101
						.065	.025
.3							.184
							.152
							.059

TABLE 3.4 Mean and Variance of the Recurrence Time

The first entry in each cell is the mean, $E(\tilde{X}_1)$, and the second entry is the variance, $\mathrm{var}(\tilde{X}_1)$, of the forward recurrence time.

	.8	.7	.6	.5	.4	.3	.2
.9	6.318	3.000	2.054	1.628	1.390	1.240	1.136
	70.751	16.230	6.988	3.952	2.605	1.892	1.467
.8		5.314	2.570	1.804	1.461	1.271	1.149
		53.413	12.131	5.283	3.050	2.058	1.528
.7			4.350	2.173	1.579	1.314	1.165
			38.300	8.666	3.870	2.311	1.612
.6				3.447	1.819	1.383	1.187
				25.569	5.869	2.754	1.732
.5					2.636	1.519	1.220
					15.442	3.760	1.926
.4						1.960	1.281
						8.152	2.319
.3							1.460
							3.737

4. THE RANDOM WALK WITH DOUBLE EXPONENTIAL STEP DISTRIBUTION

Here the density of a step is

$$q\,\beta\, e^{-\beta x}\, I(0,\infty) + p\,\alpha\, e^{\alpha x}\, I(-\infty, 0)\ ,$$

where $p + q = 1$, $q/\beta - p/\alpha > 0$.

We take $\rho \equiv q/\beta - p/\alpha$.

TABLE 4.1. Values of α and r for specified q and β

Here $r = p\beta/q\beta$ is the asynchronous probability of at least one return to the origin. The first entry in the Table is α and the second entry is r.

β \ q	.9	.8	.7	.6	.5	.4	.3
.8	.800						
	.111						
.7	.350	1.400					
	.222	.125					
.6	.200	.600	1.800				
	.333	.250	.143				
.5	.125	.333	.750	2.000			
	.444	.375	.286	.167			
.4	.080	.200	.400	.600	2.000		
	.556	.500	.429	.333	.200		
.3	.050	.120	.300	.400	.750	1.800	
	.667	.625	.571	.500	.400	.250	
.2	.029	.067	.120	.200	.333	.600	1.400
	.778	.750	.714	.667	.600	.500	.333

TABLE 4. 2. Dispersion of Counts and of Intervals

The first entry in each cell is the asymptotic slope, $\lim_{t \to \infty} \left[t^{-1} \operatorname{var} \left\{ \tilde{N}(0,t) \right\} \right]$, of the variance time curve, the second entry is var(X), the variance of a synchronous interval, and the third entry is the ratio of these quantities.

β \ q	.9	.8	.7	.6	.5	.4	.3
.8	2.125						
	1.311						
	1.621						
.7	4.306	2.469					
	1.453	1.584					
	2.964	1.559					
.6	9.000	4.556	3.074				
	1.630	1.747	1.946				
	5.521	2.608	1.580				
.5	19.000	9.000	5.667	4.000			
	1.851	1.964	2.139	2.435			
	10.265	4.582	2.649	1.643			
.4	41.500	19.000	11.500	7.750	5.500		
	2.137	2.252	2.419	2.681	3.129		
	19.420	8.437	4.754	2.891	1.758		
.3	99.000	44.556	26.407	17.333	11.889	8.259	
	2.529	2.649	2.817	3.067	3.468	4.182	
	39.146	16.820	9.374	5.651	3.428	1.975	
.2	289.000	129.000	75.667	49.000	33.000	22.333	14.714
	3.122	3.250	3.426	3.680	4.067	4.718	5.971
	92.569	39.692	22.086	13.315	8.114	4.734	2.464

TABLE 4. 3. Serial Correlations of the Interval Sequence

The entries in each cell are the serial correlations γ_1, γ_2 and γ_{10} of lags 1, 2 and 10, respectively, for the synchronous interval sequence.

β \ q	.9	.8	.7	.6	.5	.4	.3
.8	.080						
	.057						
	.006						
.7	.128	.095					
	.106	.057					
	.033	.004					
.6	.170	.149	.102				
	.150	.111	.058				
	.070	.024	.004				
.5	.210	.193	.162	.104			
	.193	.159	.115	.061			
	.113	.134	.023	.005			
.4	.249	.235	.212	.172	.105		
	.234	.204	.166	.120	.066		
	.158	.098	.057	.027	.008		
.3	.288	.276	.257	.227	.179	.105	
	.274	.248	.216	.177	.129	.072	
	.205	.146	.102	.066	.036	.013	
.2	.328	.317	.302	.278	.242	.188	.107
	.317	.294	.267	.234	.193	.142	.080
	.256	.201	.157	.118	.083	.051	.022

TABLE 4.4. Mean and Variance of the Recurrence Time

The first entry in each cell is the mean, $E(\tilde{X}_1)$, and the second entry is the variance, $var(\tilde{X}_1)$, of the forward recurrence time.

β \ q	.9	.8	.7	.6	.5	.4	.3
.8	1.155						
	1.479						
.7	1.226	1.292					
	1.795	1.931					
.6	1.315	1.373	1.473				
	2.240	2.409	2.621				
.5	1.425	1.482	1.570	1.718			
	2.899	3.114	3.399	3.749			
.4	1.568	1.626	1.710	1.840	2.064		
	3.953	4.243	4.629	5.143	5.782		
.3	1.764	1.824	1.908	2.034	2.234	2.591	
	5.839	6.263	6.827	7.594	8.648	10.025	
.2	2.061	2.125	2.213	2.340	2.534	2.860	3.485
	9.942	10.656	11.604	12.901	14.739	17.429	21.339

TABLE OF NOTATION

Notation used only in the Chapter where it is defined has been omitted.

(i) General Notation

$\mathbb{R}$ the real numbers

$\mathbb{R}^n$ n-fold Cartesian product of $\mathbb{R}$

$\mathbb{R}^{--}$ the strictly negative real numbers

$\mathbb{R}^{-}$ the non-positive real numbers

$\mathbb{R}^{+}$ the non-negative real numbers

$\mathbb{R}^{++}$ the strictly positive real numbers

$\mathbb{Z}$ the integers. The superscript notation used for subsets of $\mathbb{R}$ is also used for subsets of $\mathbb{Z}$.

$\#(A)$ cardinal of a set A

$|A|$ Lebesgue measure of a set A

$\int^{(n)}$ n-fold integral (e.g. over $\mathbb{R}^n$)

$\emptyset$ empty set

sgn(α) is +1 or -1 according as the permutation $\underset{\sim}{\alpha}$ is even or odd

p.g.f. probability generating function

$\delta_{\omega_0}^{d\omega}$ the degenerate probability distribution concentrated at ω_0

$\mathbb{E}(X)$ expectation of the random variable X

(ii) Notation Specific to the Thesis

The Section in which the symbol is first used is given.

$\tilde{N}(.)$ the (asynchronous) counting measure of a point process;

$$\tilde{N}[a,b] = \tilde{N}([a,b])\text{ etc.}$$

$\{X_j\}$	the synchronous interval sequence	(1.3), (3.6)
T_n	time of the nth event after the origin	(1.3), (3.2)
$\mathcal{H}_t$	history of a point process at time t	(1.4)
$\mathcal{W}$	state space of a semi-Markov process	(3.2)
ω	element of	(3.2)
$\Omega_{\mathcal{W}}$	the σ-field of measurable subsets of $\mathcal{W}$	(3.2)
W, W_n	state variable of a semi-Markov process	(3.2)
$p(d\omega_1 \mid \omega_0)$	transition function of the Markov chain $\{W_n\}$	(3.2)
$q(d\omega)$	synchronous stationary distribution on $\mathcal{W}$	(3.2)
N_t	number of events in $(0, t)$	(3.2)
$\{\tilde{W}_t, U_t\}$	continuous time semi-Markov process	(3.2)
$F(dx \mid \omega_0, \omega_1)$	conditional interval distribution	(3.2)
$\mathcal{F}(x \mid \omega_0)$	conditional interval survivor function	(3.2)
μ	mean of a synchronous interval	(3.3)
$F_1(dx_1), F_2(dx_1, dx_2)\ldots$	synchronous interval distributions	(3.6)
$\tilde{F}_1(dx_1), \tilde{F}_2(dx_1, dx_2)\ldots$	asynchronous interval distributions	(3.6)
$V_{\cdot j}$	multivariate synchronous forward recurrence times	(4.1)
$V_{ij} (i>0)$	multivariate semi-synchronous forward recurrence times	(4.1)
V_{0j}	multivariate asynchronous forward recurrence times	(4.1)
K, K_n	state variable of a countable imbedded Markov chain	(6.1)
$\{\tilde{K}_t\}$	countable state Markov process	
$\{p_{ij}\}$	transition matrix of $\{K_n\}$	(6.1)
$\{q_i\}$	synchronous stationary distribution of $\{K_n\}$	(6.1)
$Q, \tilde{Q}$	p.g.f.'s of $\{q_i\}$, $\{\tilde{q}_i\}$, respectively	(6.2)

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