ABSOLUTE ABEL SUMMABILITY

by

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Abstract

Let $f(z) = \sum_{0}^{\infty} a_n z^n$ be analytic in $|z|<1$ and let

$$V(f,\theta) = \int_{0}^{1} |f'(re^{i\theta})|dr.$$

The series $\sum_{0}^{\infty} a_n e^{in\theta}$ is said to be summable $|A|$ if $V(f,\theta)<\infty$. The concept of summability $|A|$ was introduced by Whittaker, and some results concerning summability $|A|$ and other connected subjects were obtained by Whittaker, Prasad, Zygmund, Mergelyan, Rudin and Piranian.

In this thesis we give some significant examples of functions for which $V(f,\theta) <\infty$ for almost all values of θ; for example areally mean p-valent functions.

We then construct a function $f(z)$ analytic in $|z|<1$ and continuous in $|z|\leq 1$ such that $V(f,\theta) = \infty$ a.e., and $\omega(t)$ the modulus of continuity of $f(e^{i\theta})$ satisfies the condition

$$\omega(t) = O\left(\frac{1}{\log \frac{1}{t}}\right)^{\beta} \qquad (o<\beta<\tfrac{1}{2}).$$

Note that one of Whittaker's result shows that as far as this result is concerned β cannot be replaced by any number greater than one.

Clearly, if $\Sigma a_n e^{in\theta}$ is summable $|A|$ it is also summable A (ordinary Abel summable) and therefore any Tauberian condition for summability A is also a Tauberian condition for summability $|A|$. We have proved that some of the well-known Tauberian conditions for summability A are also best possible for summability $|A|$.

Again if $f(z)=\Sigma a_n z^n$ is analytic in $|z|<1$ and of bounded characteristic, $V(f,\theta)$ may not be finite for any θ. It is however

proved that if $\gamma>1$ and

$$F(z) = \sum_{n=2}^{\infty} \frac{a_n z^n}{\sqrt{\log n}\,(\log \log n)^{\gamma}} \qquad (|z|<1)$$

then $V(F,\theta)<\infty$ a.e. Examples show that the index $\frac{1}{2}$ of $(\log n)$ above is "best possible".

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CHAPTER I

Introduction

Let $\{a_n\}_0^\infty$ be a complex sequence such that $\sum\limits_0^\infty a_n x^n$ converges for $0 \le x < 1$ and for such x let $\phi(x) = \sum\limits_0^\infty a_n x^n$. Then $\phi(x)$ maps $[0,1)$ onto some curve in the complex plane. The series $\sum\limits_0^\infty a_n$ is said to be absolutely Abel summable or summable $|A|$ if this curve is of finite length.* This is equivalent to saying that $\phi(x)$ is of uniformly bounded variation on $[0, \xi]$ for $0<\xi<1$; or

$$\int_0^1 |\phi'(x)| \, dx < \infty .$$

Clearly $\sum\limits_0^\infty a_n$ is summable $|A|$ only if $\lim\limits_{x\to 1-} \phi(x)$ exists finitely, since otherwise the curve considered above cannot be of finite length. Thus a series which is summable $|A|$ is also summable A (or Abel summable), but the converse is not necessarily true. When $\sum\limits_0^\infty a_n$ is summable $|A|$ and $\ell = \lim\limits_{x\to 1-} \phi(x)$, we write

$$\sum_0^\infty a_n = \ell \; (|A|)$$

and call ℓ the absolute Abel sum of $\sum\limits_0^\infty a_n$.

It is obvious that if $\sum\limits_0^\infty a_n$ is absolutely convergent then it is absolutely Abel summable and the ordinary sum of $\sum\limits_0^\infty a_n$ and the absolute Abel sum of $\sum\limits_0^\infty a_n$ are the same.

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* Note that multiple covering is taken account of.

However, unlike the case for ordinary Abel summability, convergence of $\sum_{0}^{\infty} a_n$ need not imply that the series is absolutely Abel summable. For example the series $\sum_{0}^{\infty} b_n$ with

$$b_n = \begin{cases} \dfrac{(-1)^{\log n/\log 2}}{\log n} & (n=2,2^2,2^3,\ldots) \\ 0 & (\text{otherwise}) \end{cases}$$

converges, but it is not absolutely Abel summable.

Now let us consider the function $f(z)$ analytic in the unit disc $U = \{|z|<1\}$ and suppose that

$$f(z) = \sum_{0}^{\infty} a_n z^n \qquad (|z|<1).$$

With $z = re^{i\theta}$ we have

$$f(re^{i\theta}) = \sum_{0}^{\infty} a_n r^n e^{in\theta} \qquad (0 \le r < 1).$$

We use the notation

$$(1.1) \qquad \begin{cases} w(f,r,\theta) = \int_0^r |f'(\rho e^{i\theta})|\, d\rho \\ w(f,1,\theta) = V(f,\theta) = \int_0^1 |f'(\rho e^{i\theta})|\, d\rho. \end{cases}$$

Evidently $V(f,\theta)$ is the total variation of f on the radius of U which terminates at the point $e^{i\theta}$, and geometrically speaking $V(f,\theta)$ is the length (finite or infinite) of the curve which is the image of this radius under f. If $V(f,\theta)$ is finite, the series $\sum_{0}^{\infty} a_n e^{in\theta}$ is absolutely Abel summable. $V(f,\theta)$ is called the radial variation of f. If f is bounded in U, by Fatou's Theorem [(3), p. 17], $\lim_{r\to 1-} f(re^{i\theta})$ exists finitely almost everywhere, but as we shall see later $V(f,\Theta)$ may not be finite for almost all values of Θ. In other words if $f \varepsilon H^{\infty}$, $\sum_{0}^{\infty} a_n e^{in\Theta}$ is summable A for almost all values of Θ but not necessarily summable $|A|$ for almost all values of Θ.

The idea of absolute Abel summability seems to have been first introduced by J.M. Whittaker [(15)]. He considered summability $|A|$ of Fourier Series and gave some sufficient conditions for such summability. He proved the following theorem.

Theorem 1.1. Let $f \varepsilon L^1$ $(0,2\pi)$ and have the Fourier Series

(1.2) $$\frac{ao}{2} + \sum_{n=1}^{\infty} (a_n \cos n\Theta + b_n \sin n\Theta).$$

Let

$$\phi(t) = \frac{f(\Theta + 2t) + f(\Theta - 2t) - 2\ell}{2}.$$

Then (1.2) has absolute Abel sum ℓ provided

$$\int_0^{\delta} \left|\frac{\phi(t)}{t}\right| dt$$

exists for some $\delta>0$.

In other words a Fourier series which converges in virtue of Dini's condition is absolutely Abel summable.

About the same time Prasad [(11)] gave other sufficient conditions for absolute Abel summability of a Fourier series. He proved the following theorem.

Theorem 1.2. In the notation of Theorem 1.1, (1.2) is absolutely Abel summable provided

(i) $\phi(t)$ is absolutely continuous in $(0,\delta)$ for some $\delta>0$;

(ii) $\int_0^{\delta} \left|\frac{\Phi(t)}{t^2}\right| dt$ exists for some $\delta>0$,

where $\Phi(t) = \int_0^t \phi(u)\, du.$

In this case the absolute Abel sum of (1.2) is ℓ.

It is pointed out by Prasad that (i) and Whittaker's condition are independent, but that Whittaker's condition is included in (ii), i.e. if Whittaker's condition is satisfied then so is (ii).

Later Prasad [(12)] obtained a number of other results. He proved the following theorems.

Theorem 1.3. In the notation of Theorem 1.1, (1.2) is absolutely Abel summable at θ_0 if there is some neighbourhood of θ_0 in which $f(\theta)$ is of bounded variation.

Theorem 1.4. In the notation of Theorem 1.1, given any integer $k\geq 1$, and $\gamma>1$

$$\sum_{n=N}^{\infty} \frac{a_n \cos n\theta + b_n \sin n\theta}{\log n \log_2 n \ldots (\log_k n)^{\gamma}}$$

is absolutely Abel summable for almost all θ, where $\log_1 n$ = $\log n$ and $\log_\nu n = \log(\log_{\nu-1} n)$ provided $\log_{\nu-1} n>0$; and N is taken large enough to ensure that all terms in the series are well defined.

The later paper of Prasad contains other results, but as they are not relevant to the problems we shall be concerned with, there seems little point in quoting them.

After these studies by Whittaker and Prasad of absolute Abel summability of Fourier series, Zygmund [(16)] obtained some results concerning absolute Abel summability of power series which are of intrinsic interest.

The "high indices" theorem of Hardy and Littlewood [(5), p. 173] asserts that if a series $\sum_0^\infty a_n$ with 'Hadamard gaps' (i.e. $a_n=0$ for $n_k<n<n_{k+1}$ where n_k is an increasing sequence of positive integers satisfying, for some fixed $q>1$, the relation $\frac{n_{k+1}}{n_k}>q$, $k=1,2,3\ldots$) is summable A, then it is convergent. Zygmund proved a parallel result for summability $|A|$ in the following form·

Theorem 1.5. Let the series $\sum_0^\infty a_n$ have Hadamard Gaps and let $f(z)=\sum_0^\infty a_n z^n$ be analytic in $|z|<1$; then if $\Sigma a_n e^{in\theta}$ is summable $|A|$, it is absolutely convergent. More over,

$$\sum_1^\infty |a_\nu| \leq A \int_0^1 |f'(re^{i\theta})|\, dr,$$

where A is a constant depending on f.

Zygmund also obtained an estimate for $w(f,r,\theta)$ as $r\to 1$ in the following form.

Theorem 1.6. Let $f(z)$ be analytic in $|z|<1$. Let E be the set of θ in $[0,2\pi]$ such that $f(z)$ has a finite angular limit at $e^{i\theta}$, i.e. $\lim f(z)$ exists finitely as $|z|\to 1$ in any stolz angle at $e^{i\theta}$, then

$$w(f,r,\theta) = \int_0^r |f'(\rho e^{i\theta})|\, d\rho = o\,\{\log^{\frac{1}{2}} (\tfrac{1}{1-r})\} \qquad (r\to 1)$$

almost everywhere in E.

Proof: It is known that [(17)]

$$g(\theta) = \left(\int_0^1 (1-\rho)\, |f'(\rho e^{i\theta})|^2\, d\rho\right)^{\frac{1}{2}}$$

is finite for almost every $\theta\varepsilon E$. For every such θ, Schwarz's inequality gives

$$\int_0^\rho |f'(Re^{i\theta})|dR \leq \{\int_0^\rho (1-R)|f'(Re^{i\theta})|^2 dR\}^{\frac{1}{2}} \{\int_0^\rho \frac{dR}{1-R}\}^{\frac{1}{2}}$$

$$= O\{\log^{\frac{1}{2}}(\frac{1}{1-\rho})\},$$

and it is immediate that O may be replaced by o. Thus we obtain

$$w(f,r,\theta) = \int_0^r |f'(Re^{i\theta})|dR = o\{\log^{\frac{1}{2}}(\frac{1}{1-r})\}$$

for almost every point θ at which $f(z)$ has a nontangential limit.

Zygmund then showed that the above result is best possible by proving the following theorem.

Theorem 1.7. For every function $\varepsilon(\rho)$, $(0 \leqslant \rho < 1)$ positive and tending to zero as $\rho \to 1$ there is a regular function $f(z)$, $|z|<1$ of the class H^2 (and so having a nontangential limit almost everywhere) such that

$$w(f, r,\theta) \neq O\{\varepsilon(\rho)\log^{\frac{1}{2}}(\frac{1}{1-r})\}$$

for almost every θ.

In the same paper Zygmund considered the effect of a random change of the signs of the co-efficients upon the behaviour of the function $V(f,\theta)$ and obtained the result:

Theorem 1.8. For every $0 \leq t \leq 1$, let

$$\phi_t(z) = \sum_{\nu=0}^{\infty} C_\nu \psi_\nu(z)\, z^\nu$$

where $\psi_0(t)$, $\psi_1(t)$ .... are Rademacher's functions. If the series

$$(1.3) \qquad \sum_{n=0}^{\infty} \{\sum_{\nu=2^n+1}^{2^{n+1}} |C_\nu|^2\}^{\frac{1}{2}}$$

converges, then for almost every t the expression $V(\phi_t,\Theta)$ is finite almost everywhere in Θ, and if the series (1.3) diverges then for almost every t the expression $V(\phi_t,\Theta)$ is infinite almost everywhere in Θ.

After Zygmund there seem to be no contributions to absolute Abel summability till a paper of Rudin [(13)] in which the following results are proved.

Theorem 1.9. There exists a function f(z) analytic and bounded in the unit disc $U(|z|<1)$, such that $V(f,\Theta) = \infty$ for almost all Θ, where $V(f,\Theta)$ has its usual meaning.

Theorem 1.10. There exists a Blaschke product B(z) such that $V(B,\Theta) = \infty$ for almost all Θ.

Theorem 1.11. There exists a function f, analytic in $|z|<1$ and continuous in $|z|\leq 1$ such that $V(f,\Theta) = \infty$ for almost all Θ.

The problem as to whether or not 'almost all' in Theorem 1.9, Theorem 1.10 and Theorem 1.11 can be replaced by 'all' is still outstanding. However, by considering the Riemann surface onto which $|z|<1$ is mapped by a function f(z) in H^∞ one sees that there are paths in $|z|<1$ going to boundary points on $|z|=1$ along which f(z) is of bounded variation. This is some evidence, but only very slight, in favour of a negative answer to the above problem of Rudin.

It follows from Theorem 1.9 that there is a function f(z) εH^∞ such that

$$\int_0^1 \int_0^{2\pi} |f'(re^{i\Theta})|\, r\,dr\,d\Theta = \infty$$

and this was proved prior to Rudin's work by Mergeylan [(9)].

Much more recently Piranian [(10)] constructed a Blaschke product for which the preceding integral result holds. That there is such a Blaschke product is of course also a consequence of Rudin's Theorem 1.10. However

Rudin's methods were non-constructive, and it is the constructive element in Piranian's work which makes it of significance.

Let F be a family of functions analytic in $|z|<1$. Then we call $\{\lambda_n\}$ a multiplier sequence for F (relative to absolute Abel summability), if whenever $f \in F$ and $f(z) = \sum_{o}^{\infty} a_n z^n$ $(|z|<1)$, then $\sum_{o}^{\infty} \lambda_n a_n e^{in\theta}$ is absolutely Abel summable for almost all θ. In this terminology we can express the result of Theorem 1.4, in a special case, as; if $\lambda_n = o$ $(n=o,1,2)$ and

$$\lambda_n = \frac{1}{\log n\ (\log\log n)^{\gamma}} \qquad (n = 3,4 \ldots\ldots;\gamma>1)$$

then (λ_n) is a multiplier sequence for H^1. We shall show later that in fact if k is a positive integer and $\lambda_n = o$ $(n=o, 1,2\ldots.N_o-1)$, $\gamma>1$,

$$\lambda_n = \{(\log n)^{\frac{1}{2}} \log_2 n \ldots .(\log_k n)^{\gamma}\}^{-1}$$

$(n=N_o, N_o+1,\ldots)$ where N_o is chosen (fixed) large enough to ensure that everything is well defined, then (λ_n) is a multiplier sequence for N, the set of functions analytic in $|z|<1$ and of bounded characteristic. This is an improvement of the result of Prasad.

As far as the index of $\log n$ in λ_n is concerned this result is best possible. If η $(o \leq \eta < \frac{1}{2})$ is given then $f(z) = \sum \frac{z^{2^n}}{n^{\frac{1}{2}+\eta}}$ is of bounded characteristic since in fact $f(z) \in H^2$, but $f^*(z) = \sum \frac{z^{2^n}}{n}$ is not absolutely Abel summable anywhere on $|z|=1$. This shows that we cannot replace $\frac{1}{2}$ by $\frac{1}{2}-\eta$, clearly somewhat more than this is true, but we shall deal with this later.

Of course in the result of Prasad or that alluded to above, one would like not specific forms of λ_n to be considered, but rather forms of λ_n satisfying some general condition that included the special forms. For example one would like smoothness conditions on $\phi(x)$ which together with perhaps a condition like $\int^{\infty} \frac{1}{x\phi(x)} dx < \infty$ would ensure that $\lambda_n = \frac{1}{\phi(n)}$ defines a multiplier sequence for N . We have not been able to obtain any such result although it would appear likely that there must be nontrivial results of this kind.

With any summability method one can consider associated Tauberian conditions, such a condition being one which together with summability ensures convergence. Since to be absolutely Abel summable is a stronger restriction on a series than ordinary Abel summability, any Tauberian condition for ordinary Abel summability is a fortiori, a Tauberian condition for absolute Abel summability. But one might imagine that such conditions could be weakened and still lead to ones for absolute Abel summability. As regards the status of the well known Littlewood's Tauberian condition [(5) p. 154] for ordinary Abel summability the following result was proved by Shapiro [(14)].

Theorem 1.12. Let $\eta (0<\eta<1)$ be given. Then there is a divergent series $\sum_{0}^{\infty} a_n$ with $a_n = O(\frac{1}{n^{1-\eta}})$ $(n\to\infty)$ which is absolutely Abel summable.

This shows that the index in Littlewood's condition is best possible relative to a Tauberian condition for absolute Abel summability. Later Kennedy and Szüsz [(8)] showed that in fact no weakening of Littlewood's condition at all was possible as far as a Tauberian condition for absolute Abel summability is concerned. They proved the following result.

Theorem 1.13. Let $\phi(n)>0$ $(n=0,1,2\ldots)$ and $\phi(n)\uparrow\infty(n\uparrow\infty)$. Then there is a divergent series $\sum_{0}^{\infty} a_n$ with $a_n = O\left(\frac{\phi(n)}{n}\right)$ $(n\to\infty)$ which is absolutely Abel summable.

If we define absolute Abel summability of order n, denoted by $(|A|, n)$ by the condition

$$(1.4) \qquad \int_0^1 \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_{n-2}} \int_0^{x_{n-1}} |f^{(n)}(x_n)|\,dx_n dx_{n-1} \cdots dx_1 < \infty,$$

then $(|A|, 1) = |A|$. However, we shall see that the result of Theorem 1.13 is equally applicable to summability $(|A|, n)$.

It is natural to consider other forms of Tauberian conditions for Abel summability and see to what extent they can be modified to give Tauberian conditions for summability $|A|$. There do not seem to be many such conditions, but one that is of interest is that of Fejer [(4), p. 817] . This says that if for some ϕ, $\sum_{n=0}^{\infty} a_n e^{in\phi}$ is Abel summable and $\Sigma n|a_n|^2<\infty$ then $\Sigma a_n e^{i\phi}$ converges. It follows easily that if $\Sigma n|a_n|^2<\infty$ then $\Sigma a_n e^{in\Theta}$ is Abel summable for almost all Θ, and so in fact $\Sigma a_n e^{in\Theta}$ converges for almost all Θ. There is a local form of this theorem which depends on the interpretation of $\pi \sum_{1}^{\infty} n|an|^2$ as an area integral,

$$\text{i.e.} \qquad \int_0^{\rho} \int_0^{2\pi} |f'(re^{i\Theta})|\,r d\Theta dr = \pi \sum_{1}^{\infty} n|a_n|^2 \rho^{2n}.$$

This form is: If $f(z) = \sum_{0}^{\infty} a_n z^n$ $(|z|<1)$ and for some α, $\phi< \alpha < \pi$ $f(z)$ maps the sector $\{z:z = re^{i\Theta}, |\Theta|<\alpha, 0\le r<1\}$ onto a Riemann surface of finite area and $a_n\to 0$ $(n\to\infty)$, then Σa_n is Abel summable. We shall see later that there are essentially no weaker forms of these conditions that give Tauberian conditions for absolute Abel summability.

CHAPTER 2

Absolute Abel summability relative to certain classes of analytic functions

Let

(2.1) $f(z) = \sum_{0}^{\infty} a_n z^n \quad (|z|<1)$

be analytic in the unit disc U ($|z|<1$) and let $V(f,\theta)$ be defined by (1.1) so that

$$V(f,\theta) = \int_0^1 |f'(re^{i\theta})|\,dr$$

and therefore by definition if for some θ $V(f,\theta)<\infty$, then $\sum_{0}^{\infty} a_n e^{in\theta}$ is summable $|A|$.

We consider below some functions defined by (2.1) which have $V(f,\theta)<\infty$ for almost all values of θ.

Theorem 2.1. If $\Sigma|a_n|<\infty$, then $V(f,\theta)<\infty$ for all values of θ.

Proof: By definition (1.1)

$$V(f,\theta) = \int_0^1 |f'(re^{i\theta})|\,dr$$

$$\leq \Sigma \int_0^1 n|a_n|r^{n-1}\,dr = \Sigma|a_n|<\infty \quad .$$

so we get, for all values of θ,

$$V(f,\theta) <\infty \, .$$

Theorem 2.2. If the area of the image of U ($|z|<1$) under f is finite, taking multiplicity into account then

$$V(f,\theta) <\infty \qquad \text{a.e.} \, .$$

Proof: By hypothesis

$$\int_{\rho=0}^{1} \int_{\Theta=0}^{2\pi} |f'(\rho e^{i\Theta})|^2 \rho d\rho d\Theta < \infty$$

so that

$$\int_{\rho=0}^{1} |f'(\rho e^{i\Theta})|^2 d\rho < \infty \qquad \text{a.e..}$$

Now by Schwarz's inequality we have

$$\int_{0}^{1} |f'(\rho e^{i\Theta})| d\rho \leq \left(\int_{0}^{1} |f'(\rho e^{i\Theta})|^2 d\rho\right)^{\frac{1}{2}}$$

$$< \infty \qquad \text{a.e.}$$

and so we get

$$V(f,\Theta) < \infty \qquad \text{a.e.}$$

and therefore $\sum_{0}^{\infty} a_n e^{in\Theta}$ is summable $|A|$ for almost all values of Θ.

Remark: Fejer [(4), p. 819] has shown that under the hypothesis of Theorem 2.2, the series (2.1) converges almost everywhere on the circumference $|z|=1$ and so in this case we have

(i) $\Sigma a_n e^{in\Theta}$ converges for almost all Θ.

(ii) $\Sigma a_n e^{in\Theta}$ is summable $|A|$ for almost all Θ.

It should however be remembered that in general (i) does not imply (ii) or vice-versa.

Theorem 2.3 (Hardy and Littlewood [(6)]). If $f^*(\Theta) = \lim_{r\to 1-} f(re^{i\Theta})$ and $f^* \varepsilon \text{Lip}\,\alpha$, $(0<\alpha<1)$, then $V(f,\Theta)$ is bounded. In fact the hypothesis implies $f'(re^{i\Theta}) = O\{(1-r)^{\alpha-1}\}$.

Proof: By Cauchy's Integral formula, we have

$$f'(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f^*(\phi)e^{i\phi}\, d\phi}{(e^{i\phi}-re^{i\theta})^2}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{f^*(\phi)\; e^{-2i\theta}e^{i\phi}}{(e^{i\phi}\; e^{-i\theta}-r)^2}\, d\phi$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-i\theta}\; f^*(\theta+\phi)e^{i\phi}\, d\phi}{(e^{i\phi} - r)^2}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{\{f^*(\theta+\phi)-f^*(\theta)\}e^{i(\phi-\theta)}\, d\phi}{(e^{i\phi} - r)^2},$$

and so we have

$$|f'(re^{i\theta})| \le \frac{1}{2\pi} \int_0^{2\pi} \frac{|f^*(\theta+\phi) - f^*(\theta)|\, d\phi}{|e^{i\phi} - r|^2}$$

$$= O\left(\int_0^{2\pi} \frac{|\phi|^\alpha\, d\phi}{|e^{i\phi}-r|^2}\right)$$

$$= O\left(\int_0^{\infty} \frac{|\phi|^\alpha\, d\phi}{(1-r)^2+\phi^2}\right).$$

If we put $\phi = (1-r)\tan\psi$ in the above integral, we get

$$|f'(re^{i\theta})| = O\left(\int_0^{\pi/2} (1-r)^{\alpha-1} \tan^\alpha\psi\, d\psi\right)$$

$$= O\,(1-r)^{\alpha-1}, \qquad (0<\alpha<1).$$

Now

$$V(f,\theta) = \int_0^1 |f'(re^{i\theta})|\, dr$$

$$< K \int_0^1 \frac{1}{(1-r)^{1-\alpha}}\, dr \qquad (0<\alpha<1)$$

where K is a constant

and hence

$$V(f,\theta) < \infty \text{ for all values of } \theta.$$

Theorem 2.4 (Rudin [(13)].). Let $B(z)$ be a Blaschke product given by

$$(2.2) \qquad B(z) = \prod_{n=1}^{\infty} \frac{b_n - z}{1-\bar{b}_n z} \frac{|b_n|}{b_n},$$

where

$$|b_n| < 1 \text{ and } \Sigma(1-|b_n|) < \infty.$$

If

$$(2.3) \qquad \sum_{n=1}^{\infty} (1-|b_n|) \log\left(\frac{1}{1-|b_n|}\right) < \infty,$$

then

$$\int_0^{2\pi} V(B,\theta)\, d\theta < \infty$$

and hence

$$V(B,\theta) < \infty \qquad \text{a.e.}.$$

Proof: Consider

$$g(z,b) = \frac{b-z}{1-\bar{b}z},$$

so that

$$g'(z,b) = \frac{1-|b|^2}{(1-\bar{b}z)^2}.$$

Now

$$\log B(z) = \sum_{n=1}^{\infty} \log \frac{|b_n|}{b_n} + \sum_{n=1}^{\infty} \log (g(z,b_n)).$$

By differentiating we get

$$\frac{B'(z)}{B(z)} = \sum_{n=1}^{\infty} \frac{g'(z,b_n)}{g(z,b_n)}$$

so that we have

$$|B'(z)| \leq \sum_{n=1}^{\infty} |g'(z,b_n)|$$

and so

$$\int_{\rho=o}^{1} \int_{\Theta=o}^{2\pi} |B'(\rho e^{i\Theta})| \rho d\rho d\Theta \leq \sum_{1}^{\infty} \int_{o}^{1} \int_{o}^{2\pi} |g'(\rho e^{i\Theta}, b_n)| \rho d\rho d\Theta$$

$$= \sum_{1}^{\infty} \int_{\rho=o}^{1} \int_{\Theta=o}^{2\pi} \frac{1-|b_n|^2}{|1-\bar{b}_n z|^2} \rho d\rho d\Theta .$$

Let $b_n = |b_n| e^{i\phi_n} = c_n e^{i\phi_n}$, so that

$$\int_{\rho=o}^{1} \int_{\Theta=o}^{2\pi} \frac{(1-c_n^2) \qquad \rho d\rho d\Theta}{(\rho^2 c_n^2 - 2\rho c_n \cos(\phi-\Theta) + 1)}$$

$$= \pi \frac{(1-|b_n|^2)}{|b_n|^2} \log \frac{1}{1-|b_n|^2} .$$

Now B(z) has no zeros at the origin and so we have $b_n > o$ for all n. Also we can assume that $o < |b_1| \leq |b_2| \leq |b_3| \leq \ldots \leq |b_n| \leq |b_{n+1}| \ldots$.

Thus we get

$$\int_{\rho=o}^{1} \int_{o}^{2\pi} |B'(\rho e^{i\Theta})| \rho d\rho d\Theta$$

$$\leq \pi \sum_{n=1}^{\infty} \frac{(1-|b_n|^2)}{|b_n|^2} \log \frac{1}{1-|b_n|^2}$$

$$\leq K \sum_{n=1}^{\infty} \{(1-|b_n|) \log \frac{1}{1-|b_n|}\},$$

where K is a constant.

By hypothesis

$$\sum_{n=1}^{\infty} (1-|b_n|) \log \frac{1}{(1-|b_n|)} < \infty$$

and so we get

$$\int_{\rho=0}^{1} \int_{\theta=0}^{2\pi} |B'(\rho e^{i\theta})| \rho d\rho d\theta < \infty ,$$

and so we have

$$\int_{\theta=0}^{2\pi} V(B,\theta)d\theta < \infty$$

so that $V(B,\theta) < \infty$ a.e.

and so the theorem is proved.

Theorem 2.5 (Piranian [(10)]). There exists a Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \left(\frac{b_n-z}{1-\bar{b}_n z}\right) \frac{|b_n|}{b_n}$$

such that

$$(2.4) \quad \sum_{n=1}^{\infty} (1-|b_n|) \log \frac{1}{1-|b_n|} = \infty$$

and

$$(2.5) \quad \int_{0}^{2\pi} V(B,\theta)d\theta = \infty .$$

Proof: First consider the function $\frac{a^n-z^n}{1-a^n z^n}$, where $2^{-1/n}<a<1$.

We write $a^n = \alpha$ and $z^n = \zeta$, and observe that for $o<\rho<\alpha$, the maximum and minimum values of $\left|\frac{\alpha-\zeta}{1-\alpha\zeta}\right|$ on the circle $|\zeta|=\rho$ are $\frac{\alpha+\rho}{1+\alpha\rho}$ and $\frac{\alpha-\rho}{1-\alpha\rho}$, respectively. The difference between the two moduli is $\frac{2\rho(1-\alpha^2)}{1-\alpha^2\rho^2}$. Therefore the function $\frac{a^n-z^n}{1-a^nz^n}$ whose 2n points of maximum and minimum modulus on the circle $|z|=r$ separate each other maps that circle onto a curve of length greater than $2n\ 2r^n\ \left(\frac{1-a^{2n}}{1-a^{2n}r^{2n}}\right)$ where $o<r<a$.

The integral of this quantity taken over the interval $3^{-1/n}<r<a$, is greater than $k_1 n(1-a)\ |\log n\ (1-a)|$, where k_1 is a constant independent of a and n.

We now consider the Blaschke product

$$(2.6) \qquad B(z) = \prod_k \left(\frac{a_k^{n_k} - z^{n_k}}{1-a_k^{n_k}z^{n_k}}\right) ,$$

where $o<a_k<1$ and $n_k\uparrow\infty \quad (k\uparrow\infty)$.

The product converges if

$$\Sigma n_k(1-a_k) < \infty ;$$

in particular if

$$n_k(1-a_k) = \frac{1}{k(\log k)^{3/2}}$$

$$k = 2,3,4..... \qquad .$$

From

$$\log(n_k(1-a_k)) = \log n_k + \log(1-a_k)$$

it follows that

$$\log(1-a_k) = -\log k - \frac{3}{2}\log\log k - \log n_k$$

and so we have

$$\Sigma\, n_k(1-a_k)\log\frac{1}{1-a_k}$$

$$= \sum_{k=2}^{\infty} \frac{1}{k(\log k)^{3/2}} \{\log k + \tfrac{3}{2}\log\log k + \log n_k\}$$

$$> \sum_{k=2}^{\infty} \frac{1}{k\sqrt{\log k}} .$$

$$= \infty .$$

Thus the Blaschke product defined by (2.6) satisfies the condition (2.4). We now want to prove that (2.5) is also true in this case.

Let the sequence $\{n_k\}$ increase sufficiently fast so that we obtain disjoint intervals $r_k<r<a_k$ such that

$$\int_{r_k}^{a_k}\int_0^{2\pi} |B'(re^{i\theta})|\, r d\theta dr$$

$$> K_2 n_k(1-a_k)\ |\log n_k(1-a_k)|$$

(where K_2 is a constant independent of k)

$$> K_2 \frac{1}{k(\log k)^{\frac{1}{2}}} .$$

Therefore summing over all such intervals we get

$$\int_0^1\int_0^{2\pi} |B'(re^{i\theta})|\, r d\theta dr$$

$$> \sum_{k=2}^{\infty}\int_{r_k}^{a_k}\int_0^{2\pi} |B'(re^{i\theta})|\, r d\theta dr$$

$$= \infty .$$

Thus (2.5) holds.

This completes the proof of the theorem.

<u>Theorem 2.6. Let $f(z) = z+a_2 z^2 + \ldots$ be univalent in $|z|<1$. Then</u>

$$V(f,\Theta) < \infty,$$

<u>for almost all values of Θ.</u>

<u>Remark</u>: We could consider this theorem as a corollary to Theorem 2.7. However we give the following proof as it is much simpler than that of Theorem 2.7.

<u>Proof</u>: Consider

$$g(z) = \frac{1}{f(z)} = \frac{1}{z} + b_o + b_1 z + b_2 z^2 + \ldots .$$

Then $g(z)$ is univalent in $o<|z|<1$, and therefore by the area theorem [(7), p. 3] we have

$$(2.7) \qquad \sum_{n=1}^{\infty} n\,|b_n|^2 \leq 1 .$$

Now

$$g'(z) = -\frac{1}{z^2} + b_1 + 2b_2 z + \ldots + nb_n z^{n-1} + \ldots ,$$

and so

$$\frac{1}{2\pi}\int_o^{2\pi} |g'(re^{i\Theta})|^2 \, d\Theta = \frac{1}{2\pi}\int_o^{2\pi} |g'(re^{i\Theta})|\,|\overline{g'(re^{i\Theta})}| \, d\Theta$$

$$= \frac{1}{r^4} + \sum_{n=1}^{\infty} n^2 |b_n|^2 \, r^{2n-2}$$

and so we have

$$\frac{1}{2\pi}\int_{\frac{1}{2}}^{1}\int_{0}^{2\pi}|g'(re^{i\theta})|^2 r\,d\theta\,dr = \left[-\tfrac{1}{2}r^{-2} + \sum_{1}^{\infty}\frac{n}{2}|b_n|^2 r^{2n}\right]_{\frac{1}{2}}^{1}$$

$$= \frac{3}{2} + \sum_{1}^{\infty}\frac{n|b_n|^2}{2}\left(1-\frac{1}{2^{2n}}\right)$$

$< \infty$ (by using (2.7)).

By substituting the value of g(z) in terms of f(z) in the above result we have

$$\int_{\frac{1}{2}}^{1}\int_{0}^{2\pi}\frac{|f'(re^{i\theta})|^2}{|f(re^{i\theta})|^4}\,r\,d\theta\,dr < \infty$$

and so

$$\int_{\frac{1}{2}}^{1}\int_{0}^{2\pi}\frac{|f'(re^{i\theta})|^2}{|f(re^{i\theta})|^4}\,d\theta\,dr < \infty . \tag{2.8}$$

Now since f(z) is univalent in $|z|<1$, it is of bounded characteristic and therefore for almost all values of θ, [(3) p.41]

$$\lim_{r\to 1-} f(re^{i\theta}) = f(e^{i\theta}) .$$

Let

$$E_n = \{\theta\epsilon\,[0,2\pi],\ |f(re^{i\theta})| \leqslant n \quad (0\leqslant r<1)\}$$

$(n = 1,2,3,4 \ldots\ldots)$

For sufficiently large n, $m(E_n)>0$.

Clearly for all n,

$$E_n \subset E_{n+1}$$

and so

$$\lim_{n\to\infty} m(E_n) = m(\bigcup_{n=1}^{\infty} E_n).$$

Now $\bigcup_{n=1}^{\infty} E_n$ contains all Θ such that $f(re^{i\Theta}) \to$ a finite limit as $r\to 1-$, and therefore

$$m(\bigcup_{n=1}^{\infty} E_n) = 2\pi .$$

Thus

$$\lim_{n\to\infty} m(E_n)=2\pi.$$

From (2.8) we obtain

$$\int_{\frac{1}{2}}^{1} \int_{E_n} \frac{|f'(re^{i\Theta})|^2}{|f(re^{i\Theta})|^4} d\Theta dr < \infty .$$

Since $|f(re^{i\Theta})| \leqslant n$ for $o\leqslant r<1$ and $\Theta\varepsilon E_n$, therefore we have

$$\int_{\frac{1}{2}}^{1} \int_{E_n} |f'(re^{i\Theta})|^2 d\Theta dr < \infty$$

and so

$$\int_{\frac{1}{2}}^{1} |f'(re^{i\Theta})|^2 dr < \infty$$

for almost all values of $\Theta\varepsilon E_n$ and for all n.

Now $\lim_{n\to\infty} m(E_n) = 2\pi$, and so we have

$$\int_{\frac{1}{2}}^{1} |f'(re^{i\theta})|^2 \, dr < \infty$$

for almost all values of θ (except perhaps those lying in a set of Lebesgue measure zero).

Thus we have

$$\int_{\frac{1}{2}}^{1} |f'(re^{i\theta})|^2 \, dr < \infty \qquad \text{a.e.} \quad ,$$

and so we get

$$\int_{0}^{1} |f'(re^{i\theta})|^2 \, dr < \infty \qquad \text{a.e.} \quad .$$

By Schwarz's inequality we get

$$V(f,\theta) = \int_0^1 |f'(re^{i\theta})| \, dr \leq \left(\int_0^1 |f'(re^{i\theta})|^2 dr\right)^{\frac{1}{2}} .$$

Hence

$$V(f,\theta) < \infty \qquad \text{a.e.} \quad .$$

This proves the theorem.

We shall now prove a result similar to that of Theorem 2.6, when $f(z)$ is p-valent in $|z|<1$. Before proceeding to prove the theorem, we give the definition of p-valent functions, and state some earlier results concerning them, which will be required for our proof.

Let $f(z)$ be regular in $U(|z|<1)$ and let $n(\omega)$ be the number of roots in U of the equation $f(z) = \omega$. Let

$$P(R) = \frac{1}{2\pi} \int_0^{2\pi} n(Re^{i\phi}) \, d\phi \quad ,$$

$$W(R) = \int_0^R p(\rho)\,d\rho^2 = \frac{1}{\pi}\int_0^{2\pi}\int_0^R n(\rho e^{i\phi})\,\rho\,d\rho\,d\phi .$$

The function $f(z)$ is said to be mean p-valent in U [(7) p. 23] if p is a positive number, and

(2.9) $$W(R) \leqslant pR^2 \qquad (o<R<\infty) .$$

Suppose that $f(z) = \sum_{o}^{\infty} a_n z^n$ is mean p-valent in $|z|<1$, and let

$$M(r,f) = \max_{|z|\leqslant r} |f(z)| \qquad (o<r<1)$$

and

$$\mu_q = \max_{\nu\leqslant q} |a_\nu| .$$

Then [(7) p. 31]

(2.10) $$M(r,f) < A(p)\,\mu_p\,(1-r)^{-2p} ,$$

where $o<r<1$, and [(7) p. 45]

(2.11) $$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^\lambda\,d\theta$$

$$\leqslant M(r_o,f)^\lambda + p\Lambda \int_{r_o}^{r} \frac{M(t,f)^\lambda}{t}\,dt$$

where $\lambda>o, \Lambda = \max(\lambda, \frac{\lambda^2}{2})$ and $(o<r_o<r<1)$.

If we further suppose that $\frac{1}{2}\leqslant r<1$, $o<\lambda\leqslant 2$, then [(7), p.46] there exists ρ such that $2r-1\leqslant\rho< r$ and

(2.12) $$\frac{1}{2\pi}\int_0^{2\pi} |f'(\rho e^{i\theta})|^2\,|f(\rho e^{i\theta})|^{\lambda-2}\,d\theta$$

$$\leq \quad \frac{4pM(r,f)^{\lambda}}{\lambda(1-r)} \quad .$$

Theorem 2.7. Let $f(z) = \sum_{n=o}^{\infty} a_n z^n$ be p-valent in $|z|<1$ i.e. f(z) satisfies the condition (2.9). Then for almost all values of θ ($o\leq\theta\leq2\pi$),

$$V(f,\theta) = \int_o^1 |f'(re^{i\theta})|dr < \infty \quad .$$

Proof: From (2.10) and (2.11), if we choose $r_o=\frac{1}{2}$, we have

$$\frac{1}{2\pi} \int_o^{2\pi} |f(re^{i\theta})|^{\lambda} d\theta \leq M(\tfrac{1}{2},f)^{\lambda}$$

$$+ \quad 2p \ \Lambda \int_{\frac{1}{2}}^{r} \{A(p)\mu_p \ (1-t)^{-2p}\}^{\lambda} \ dt$$

$$= M(\tfrac{1}{2}.f)^{\lambda} + 2p\Lambda \ \{A(p)\mu_p\}^{\lambda} \int_{\frac{1}{2}}^{r} (1-t)^{-2p\lambda} \ dt$$

$$= M(\tfrac{1}{2},f) \ + 2p\Lambda \ \{A(p) \ \mu_p\}^{\lambda} \left[\frac{(1-t)^{1-2p\lambda}(-1)}{1-2p\lambda}\right]_{\frac{1}{2}}^{r} \quad .$$

Therefore if $o<\lambda<\frac{1}{2p}$, then

$$\lim_{r\to1-} \quad \{\frac{1}{2\pi} \int_o^{2\pi} |f(re^{i\theta})|^{\lambda} d\theta\} < \infty \quad ,$$

which shows that $f(re^{i\theta})$ is of bounded characteristic and therefore $\lim_{r\to1-} f(re^{i\theta})$ exists finitely for almost all values of θ ($o\leq\theta\leq2\pi$).

Let

$$(2.13)\quad E_n = \{\theta\epsilon[o,2\pi],\ \ |f(re^{i\theta})| \leqslant n\} \qquad (o\leqslant r<1)$$

$(n = 1,2,3,4 \ . \ . \ . \ . \ )$.

For sufficiently large n, $m(E_n)>o$.

Clearly $E_n \subset E_{n+1}$ for all n and so

$$\lim_{n\to\infty} m(E_n) = m(\bigcup_{n=1}^{\infty} E_n) \ .$$

Now $\bigcup_{n=1}^{\infty} E_n$ contains all θ such that $f(re^{i\theta}) \to$ a finite limit as $r\to 1-$, and therefore

$$m(\bigcup_{n=1}^{\infty} E_n) = 2\pi \ .$$

Thus

$$\lim_{n\to\infty} m(E_n) = 2\pi \ .$$

Now consider the integral

$$J_R(E_n) = \int_{2R-1}^{R} d\rho \int_{E_n} |f'(\rho e^{i\theta})| \ d\theta \qquad (\tfrac{1}{2}\leq R<1) \ .$$

By applying Schwarz's inequality to the inner integral we get

$$J_R(E_n) \leq \int_{2R-1}^{R} d\rho \qquad (\int_{E_n} |f'(\rho e^{i\theta})|^2 \ d\theta)^{\frac{1}{2}} \quad (m(E_n))^{\frac{1}{2}}.$$

Now applying Schwarz's inequality to the outer integral we get

$$(2.14)\; J_R(E_n) \leq (m(E_n))^{\frac{1}{2}} \left(\int_{2R-1}^{R} d\rho\right)^{\frac{1}{2}} \left(\int_{2R-1}^{R} d\rho \int_{E_n} |f'(\rho e^{i\Theta})|^2 \, d\Theta\right)^{\frac{1}{2}}.$$

$$= (m(E_n))^{\frac{1}{2}} (1-R)^{\frac{1}{2}} \left(\int_{2R-1}^{R} d\rho \int_{E_n} |f'(\rho e^{i\Theta})|^2 \, d\Theta\right)^{\frac{1}{2}} .$$

From (2.12) there exists a 'ρ_o' ($o<r<\rho_o<\frac{1+r}{2}$) such that

$$(2.12)' \quad \frac{1}{2\pi} \int_o^{2\pi} |f'(\rho_o e^{i\Theta})|^2 |f(\rho_o e^{i\Theta})|^{\lambda-2} \, d\Theta \leq \frac{8pM(\frac{1+r}{2}, f)^{\lambda}}{\lambda(1-r)} .$$

Now f(z) is mean p-valent in $|z|<1$ and therefore f(z) can have at most p zeros in $|z|<1$. Let them be $\alpha_1, \alpha_2, \ldots \alpha_k$ ($k\leq p$)* lying in $|z|<r_o<1$, and suppose $r>\frac{1}{2}$ is chosen near enough to 1 so that

$$|\alpha_i| < r_o < 2r-1 \qquad (i=1,2,3\ldots.k).$$

Again,

$$\{f(z)/(z-\alpha_1)(z-\alpha_2) \ldots\ldots\ldots\ldots (z-\alpha_k)\}^{\lambda-2}$$

is analytic in $|z|<1$, and so

$$\{f'(z)\}^2 \left\{\frac{f(z)}{(z-\alpha_1)\ldots\ldots(z-\alpha_k)}\right\}^{\lambda-2}$$

is analytic in $|z|<1$, and so

$$\frac{1}{2\pi} \int_o^{2\pi} \frac{|f'(\rho e^{i\Theta})|^2}{|f(\rho e^{i\Theta})|^{2-\lambda}} \{|\rho e^{i\Theta}-\alpha_1|\ldots\ldots|\rho e^{i\Theta}-\alpha_k|\}^{2-\lambda} \, d\Theta$$

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* Footnote: The following discussion includes the case when f(z) has no zeros at all in $|z|<1$, mutatis mutandis.

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \frac{|f'(\rho_0 e^{i\theta})|}{|f\cdot(\rho_0 e^{i\theta})|^{2-\lambda}} \{|\rho_0 e^{i\theta}-\alpha_1|\ldots\ldots|\rho_0 e^{i\theta}-\alpha_k|\}^{2-\lambda},$$

where $r_o \leq 2r - 1 \leq \rho \leq r$ and ρ_o is chosen so that (2.12)' holds. Hence for such ρ, assuming $o<\lambda\leq 2$, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{|f'(\rho e^{i\theta})|}{|f\ (\rho e^{i\theta})|^{2-\lambda}} \quad d\theta$$

$$\leq \frac{1}{2\pi}\frac{\{(1+\alpha_1)\ .\ .\ .\ .\ .\ (1+\alpha_k)^{2-\lambda}}{\{(r_o-\alpha_1).\ .\ .\ .\ .\ (r_o-\alpha_k)\}^{2-\lambda}} \int_0^{2\pi}\frac{|f'(\rho_0 e^{i\theta})|}{|f(\rho_0 e^{i\theta})|^{2-\lambda}}\ d\theta$$

$$\leq \frac{8p}{\lambda}\ \{\frac{(1+\alpha_1)\ .\ .\ .\ .\ .\ (1+\alpha_k)}{(r_o-\alpha_1)\ \ .\ .\ .\ .\ (r_o-\alpha_k)}\}^{2-\lambda}\ \frac{M(\frac{1+r}{2},\ f)^{\lambda}}{(1-r)}$$

$$< \ K \quad (1-r)^{-2p\lambda-1} \qquad \text{(by using (2.10))},$$

where $K = K(p,\lambda,\mu,f)$.

When $\theta\varepsilon E_n$, $|f(\rho e^{i\theta})|\leq n$ for $o\leq\rho<1$, and so the above inequality gives

$$\int_{E_n} |f'(\rho e^{i\theta})|^2\ d\theta \quad \leq \quad n^{2-\lambda}\ \ K(1-r)^{-2p\lambda-1},$$

for all ρ satisfying

$$r_o \leq 2r-1 \leq \rho \leq r \qquad (\text{and } r<1).$$

Thus we have

$$\int_{2r-1}^{r} d\rho \qquad \int_{E_n} |f'(\rho e^{i\theta})|^2\ d\theta \quad \leq n^{2-\lambda}\ K(1-r)^{-2p\lambda},$$

so that

$$(2.15)\qquad \left(\int_{2r-1}^{r} d\rho \int_{E_n} |f'(\rho e^{i\theta})|^2 \, d\theta\right)^{\frac{1}{2}} \leqslant n^{1-\frac{\lambda}{2}} K(1-r)^{-p\lambda} .$$

Now consider a sequence $\{R_\nu\}$ of positive integers such that

$$R_{\nu+1} = 2R_{\nu+2} - 1; \text{ with } R_1 = \tfrac{1}{2}$$

so that

$$R_\nu = \frac{1}{2} + \frac{1}{2^2} + \ldots\ldots \frac{1}{2^\nu} = 1 - \frac{1}{2^\nu}$$

$(\nu = 1, 2, 3, \ldots\ldots)$.

Let 'm' be the smallest positive integer such that

$$r_o < R_m = 1 - \frac{1}{2^m} ,$$

and so (2.15) is satisfied for

$$r = R_\nu \qquad (\nu = m+1, m+2, \ldots\ldots) .$$

From (2.14) and (2.15) we have for $\nu = m+1, m+2, \ldots$

$$J_{R_\nu}(E_n) \leqslant K'(1-R_\nu)^{\frac{1}{2}-p\lambda},$$

where $K'=K'(n,K)$.

Let $\alpha = \frac{1}{2}-p\lambda$ and choose λ so small that $\alpha>0$. Then with this choice of λ we have for $\nu = m+1, m+2, \ldots\ldots$.

$$J_{R_\nu}(E_n) \leq K' \frac{1}{2^{\nu\alpha}} \qquad (\alpha > o).$$

Hence for each integer n>o,

$$\sum_{\nu=m+1}^{\infty} J_{R_\nu}(E_n) < \infty ,$$

which gives

$$\int_{(1-\frac{1}{2^m})}^{1} \int_{E_n} |f'(\rho e^{i\Theta})| d\Theta d\rho < \infty ,$$

so that

$$\int_{1-(\frac{1}{2})^m}^{1} |f'(\rho e^{i\Theta})| d\rho < \infty ,$$

for almost all values of $\Theta \varepsilon E_n$ and since $\lim_{n\to\infty} m(E_n) = 2\pi$, we have

$$\int_{1-(\frac{1}{2})^m} |f'(\rho e^{i\Theta})| d\rho < \infty \qquad \text{a.e.}$$

Hence

$$V(f,\Theta) = \int_{o}^{1} |f'(\rho e^{i\Theta})| d\rho < \infty \qquad \text{a.e.}$$

This completes the proof of the theorem.

CHAPTER 3

Tauberian conditions for absolute Abel summability

We consider a function $f(z)$ analytic in the unit disc U ($|z|<1$) with $V(f,\theta)<\infty$ for some θ, where as defined in (1.1)

$$V(f,\theta) = \int_0^1 |f'(re^{i\theta})|\,dr .$$

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (|z|<1)$$

so that in this case $\sum_{n=0}^{\infty} a_n e^{in\theta}$ is summable $|A|$ and therefore summable A for those values of θ for which $V(f,\theta)<\infty$. If further

$$a_n = O\left(\frac{1}{n}\right) ,$$

then by Littlewood's Tauberian condition [(5), p. 154] $\sum_{n=0}^{\infty} a_n e^{in\theta}$ is convergent. From Theorem 1.12 and Theorem 1.13 it follows that as far as this conclusion is concerned, the condition $a_n = O\left(\frac{1}{n}\right)$ cannot be weakened.

Fejer's Tauberian condition [(4), p. 817] states that if for some θ

(i) $\sum_{n=0}^{\infty} a_n e^{in\theta}$ is summable A

and

(ii) $\sum_{n=1}^{\infty} n|a_n|^2 < \infty$,

then $\sum_{n=1}^{\infty} a_n e^{in\theta}$ is convergent.

In the next theorem we want to prove that like Littlewood's Tauberian conditon, Fejer's Tauberian condition is also best possible when $\sum_{n=o}^{\infty} a_n e^{in\theta}$ is summable $|A|$ or even summable $(|A|,n)$, where as defined by (1.4), $\sum_{o}^{\infty} a_n e^{in\theta}$ is said to be summable $(|A|,n)$ if

$$\int_o^1 \int_o^{r_1} \int_o^{r_2} \cdots \int_o^{r_{n-1}} |f^{(n)}(r_n e^{i\theta})| dr_1 \, dr_2 \ldots dr_n$$

$$< \infty \quad .$$

<u>Theorem 3.1.</u> <u>Given any positive sequence $\{\varepsilon_n\}_1^{\infty}$ such that $\varepsilon_n \downarrow o$ $(n\uparrow\infty)$, there exists a series $\sum_{n=1}^{\infty} a_n$ satisfying the following conditions</u>:

(i) $\sum_{n=1}^{\infty} a_n$ <u>is divergent</u>;

(ii) $\sum_{n=1}^{\infty} n \, \varepsilon_n |a_n|^2 < \infty$;

(iii) $\sum_{n=1}^{\infty} a_n e^{in\theta}$ <u>is summable $(|A|,2)$ for all θ.</u>

<u>Remark</u> 1: For simplicity we have restricted ourselves to the case n=2. However it should be clear from the proof that the result is true in general, i.e. for $(|A|,n)$, where n is a positive integer

<u>Proof</u>: Let $\varepsilon_n \downarrow o$ $(n\uparrow\infty)$ be such that

(3.1) $$o < \frac{1}{\sqrt{n}} < \varepsilon_n < \frac{1}{e} \qquad (n \geq no).$$

We can find a sequence $\{d_j\}_1^{\infty}$, $d_n \uparrow \infty$ $(n\uparrow\infty)$, and a subsequence $\{N_j\}_1^{\infty}$ of positive integers such that

$$\sum_{j=1}^{\infty} \varepsilon_{N_j} \, d_{N_j} < \infty \quad .$$

For example choose

$$d_n = \frac{1/\varepsilon_n}{\log 1/\varepsilon_n},$$

and then choose $\{N_j\}$ so that

$$(3.2) \qquad \sum_{j=1}^{\infty} \frac{1}{\log\left(\frac{1}{\varepsilon_{N_j}}\right)} < \infty .$$

Now choose $\{n_k\}_{k=1}^{\infty}$ a subsequence of $\{N_j\}$ such that

$$(3.3) \qquad n_{k+1} > 3n_k \qquad (k = 1,2,3\ldots).$$

From (3.2) we get

$$(3.4) \qquad \sum_{k=1}^{\infty} \frac{1}{\log\left(\frac{1}{\varepsilon_{n_k}}\right)} < \infty.$$

For large n, define m(n) to be the smallest positive integer such that

$$\frac{1}{m(n)} \frac{1}{\log\left(\frac{1}{\varepsilon_n}\right)} \leq \frac{1}{n} \frac{1}{\varepsilon_n} \qquad (n=1,2,3\ldots)$$

which gives

$$(3.5) \qquad \frac{1}{m(n)} \leq \frac{1}{n} \frac{1}{\varepsilon_n} \log\left(\frac{1}{\varepsilon_n}\right) \leq \frac{1}{m(n)-1} .$$

By using the inequality (3.1) we get

$$\frac{1}{m(n)} \leqslant \frac{1}{n} \sqrt{n} \log \sqrt{n} \to 0 \qquad (n\to\infty)$$

so that

$$(3.6) \qquad m(n) \to \infty \qquad (n\to\infty).$$

Also since m(n) is the smallest positive integer satisfying the inequality (3.5) we must have

$$(3.7) \qquad m(n) < n.$$

Let m_k be defined as $m(n_k)$, and so for the sequence $\{n_k\}$ we have

(3.5)' $$\frac{1}{m_k} \leq \frac{1}{n_k}\frac{1}{(\varepsilon_{n_k})}\log\left(\frac{1}{\varepsilon_{n_k}}\right) \leq \frac{1}{m_k-1}$$

(3.6)' $$m_k \to \infty \qquad (k\to\infty) ,$$

(3.7)' $$m_k < n_k .$$

Now consider the n^{th} Fejer polynomial

$$f_n(z) = \frac{1}{n} + \frac{z}{n-1} + \dots \frac{z^{n-1}}{1}$$

$$- \frac{z^{n+1}}{1} - \frac{z^{n+2}}{2} \dots - \frac{z^{2n}}{n} .$$

If $z = e^{i\theta}$, then

$$z^{-n}f_n(z) = - 2i(\sin\theta + \frac{\sin 2\theta}{2} + \dots \frac{\sin n\theta}{n})$$

and so by a known result [(1), p. 91]

(3.8) $$|f_n(z)| \leq K \ (|z|=1), \ (n=1,2,3\dots)$$

for some constant K. By the maximum modulus principle,

$$|f_n(z)| \leq K \ (|z|\leq 1, \ n=1,2,3\dots) .$$

For convenience we define $f_o(z) \equiv o$. Now for $n = 1,2 \dots$, $m=1,2\dots n$, let us define

$$F_{n,m}(z) = \frac{1}{n} + \frac{z}{n-1} + \frac{z^2}{n-2} + \dots + \frac{z^{n-m}}{m}$$

$$- \frac{z^{n+m}}{m} - \frac{z^{n+m+1}}{m+1} - \dots - \frac{z^{2n}}{n}$$

$$= F_n(z) - z^{n-m+1} F_{m-1}(z) ,$$

where for $m=1$, $F_{m-1}(z) \equiv o$.

We have from (3.8) that

$$|F_{n,m}(z)| < 2K$$

for $(|z| \leq 1)$, $(n=1,2,3\ldots;\ m=1,2,3\ \ldots n)$. In particular for the sequence $\{n_k\}$ we have

$$(3.9)\qquad |F_{n_k,m_k}(z)| < 2K$$

for $(|z| \leq 1)$, $(k=1,2,3\ldots)$.

Define

$$(3.10)\qquad f(z) = \sum_{k=1}^{\infty} \frac{z^{n_k} F_{n_k,m_k}(z)}{\log\left(\frac{1}{\varepsilon n_k}\right)} = \sum_{0}^{\infty} a_n z^n$$

where m_k, n_k satisfy (3.3), (3.4), (3.5)' (3.6)' and (3.7)'. Now the degree of the polynomial $z^{n_k} F_{n_k,m_k}(z)$ is $3n_k$, and the least degree of the non zero terms in $z^{n_{k+1}} F_{n_{k+1},m_{k+1}}(z)$ is n_{k+1} and since, by (3.3)

$$n_{k+1} > 3n_k \qquad (k = 1,2,3\ldots),$$

it follows that there is no overlap among the terms in the above sum for $f(z)$.

<u>Proof of (i)</u>. Let

$$S_n = \sum_{k=0}^{n} a_k$$

and then considering the definition of $F_{n_k,m_k}(z)$ we have

$$S_{2n_k-m_k} - S_{n_{k-1}}$$

$$= \left(\frac{1}{n_k} + \frac{1}{n_k-1} + \ldots \frac{1}{m_k}\right) \frac{1}{\log\left(\frac{1}{\varepsilon n_k}\right)} .$$

Now since m_k, $n_k \to \infty$ as $k \to \infty$, we have

$$(3.11) \quad \frac{1}{n_k} + \frac{1}{n_k - 1} + \ldots\ldots + \frac{1}{m_k} \sim \log \frac{n_k}{m_k} .$$

Also from (3.5)' we have

$$\log \frac{n_k}{m_k} \leq \log \left\{\frac{1}{\varepsilon_{n_k}} \log \left(\frac{1}{\varepsilon_{n_k}}\right)\right\} \leq \log \frac{n_k}{m_k - 1}$$

and so

$$\log \frac{n_k}{m_k} \leq \log \frac{1}{\varepsilon_{n_k}} + \log \log \frac{1}{\varepsilon_{n_k}} \leq \log \frac{n_k}{m_k - 1} ,$$

so that when $k \to \infty$, since $\varepsilon_{n_k} \to 0$, m_k, $n_k \to \infty$ we have

$$(3.12) \quad \log \frac{n_k}{m_k} \sim \log \frac{1}{\varepsilon_{n_k}} .$$

Thus, from (3.11) and (3.12), we obtain for large k

$$\frac{1}{n_k} + \frac{1}{n_k - 1} + \ldots \frac{1}{m_k} > \tfrac{1}{2} \log \frac{1}{\varepsilon_{n_k}}$$

and so we have for large values of k

$$S_{2n_k - m_k} - S_{n_k - 1} \geq \tfrac{1}{2}$$

Hence $\sum_{1}^{\infty} a_n$ diverges.

<u>Proof of (ii)</u>.

We know that $a_n = 0$ except when

$$(n_k \leq n \leq 3n_k) \qquad (k = 1, 2, 3 \ldots),$$

and

$$\varepsilon_{3n_k} < \varepsilon_{n_k} \qquad \text{(since } \varepsilon_n \downarrow 0\text{)}$$

so that we have,

$$\sum_{n=0}^{\infty} n\varepsilon_n \;|a_n|^2 = \sum_{k=1}^{\infty} \{\sum_{n_k\le n\le 3n_k} n\varepsilon_n|a_n|^2\}$$

$$\le \sum_{k=1}^{\infty} \varepsilon_{n_k} \left(\frac{6n_k}{\log \frac{1}{\varepsilon_{n_k}}}\right)^2 \{\frac{1}{n_k^2} + \cdots \frac{1}{m_k^2}\},$$

$$\le \sum_{k=1}^{\infty} \frac{6K\,\varepsilon_{n_k} \quad n_k}{(\log \frac{1}{\varepsilon_{n_k}})^2} \{\frac{1}{m_k} - \frac{1}{n_k}\} \quad \text{(for some constant K)}$$

$$\le \sum_{k=1}^{\infty} \frac{6K}{(\log \frac{1}{\varepsilon_{n_k}})} \quad \text{(by using (3.5)')}$$

$$< \infty \quad \text{(by using (3.4))}$$

Hence

$$\sum_{n=0}^{\infty} n\,\varepsilon_n|a_n|^2 < \infty$$

Proof of (iii)

From (3.10) we have

$$f'(z) = \sum_{k=1}^{\infty} \frac{n_k z^{n_k-1} F_{n_k,m_k}(z) + z^{n_k} F'_{n_k,m_k}(z)}{\log(\frac{1}{\varepsilon_{n_k}})}$$

Hence

$$f''(z) = \sum_{k=1}^{\infty} \frac{1}{\log(\frac{1}{\varepsilon_{n_k}})} \{ n_k(n_k-1)\, z^{(n_k-2)}F_{n_k,m_k}(z)$$

$$+ 2n_k z^{n_k-1} F'_{n_k,m_k}(z) + z^{n_k}F''_{n_k,m_k}(z)\} .$$

Now, since $F_{n_k,m_k}(z)$ is a polynomial of degree $2n_k$ bounded by $2K$ in $|z|\le 1$ (from (3.9)), we have by Bernstein's Theorem, [(1), p. 35] that

(3.11)' $\quad |F'_{n_k,m_k}(z)| < 4Kn_k(|z|\le 1)$.

Therefore $F'_{n_k,m_k}(z)$ is a polynomial of degree $(2n_k-1)$ bounded in $|z|\le 1$

by $4n_kK$, and so it follows again from Bernstein's Theorem, that

$$(3.12)' \qquad |F''_{n_k, m_k}(z)| < 4Kn_k(2n_k-1) \quad (|z| \le 1).$$

Thus for $0 \le r < 1$ we have by using (3.11)′ and (3.12)′ that, for all values of θ

$$|f''(re^{i\theta})| \le \sum_{k=1}^{\infty} \frac{2K}{\log(\frac{1}{\varepsilon_{n_k}})} \{n_k(n_k-1)r^{n_k-2}$$

$$+ \quad 2n_k \cdot 2n_k\, r^{n_k-1} + 2n_k(2n_k-1)r^{n_k}\}$$

and therefore for $0<\rho<1$ we have,

$$\int_0^{\rho} |f''(re^{i\theta})| \, dr \le \sum_{k=1}^{\infty} \frac{2K}{\log(\frac{1}{\varepsilon_{n_k}})} \{n_k\rho^{n_k-1}$$

$$+ 4n_k\rho^{n_k} + 2n_k \frac{(2n_k-1)}{n_k+1} \rho^{n_k+1}\}$$

$$< \sum_{k=1}^{\infty} \frac{2K}{\log(\frac{1}{\varepsilon_{n_k}})} \{n_k\rho^{n_k-1} + 4n_k\rho^{n_k-1} + 4n_k\rho^{n_k-1}\}$$

$$= \sum_{k=1}^{\infty} \frac{18K}{\log(\frac{1}{\varepsilon_{n_k}})} \{n_k\rho^{n_k-1}\}$$

so that

$$\int_0^1 \int_0^{\rho} |f''(re^{i\theta})| \, dr \, d\rho < \sum_{k=1}^{\infty} \frac{18K}{\log(\frac{1}{\varepsilon_{n_k}})}$$

$$< \infty \quad \text{(by (3.4))}.$$

Thus we have

$$\int_0^1 \int_0^{\rho} |f''(re^{i\theta})| \, dr \, d\rho \quad < \quad \infty \qquad \text{(for all values of } \theta).$$

Therefore $\sum_{n=0}^{\infty} a_n e^{in\theta}$ is summable $(|A|, 2)$ for all values of θ.

Remark 2: We notice that if in the above theorem we make the substitution

$$\frac{1}{\varepsilon_n} = \phi(n) ,$$

we get a positive sequence $\phi(n)\uparrow\infty$ $(n\uparrow\infty)$ with $\phi(n) < \sqrt{n}$ and it follows from (3.10) that

$$|a_n| \leq \frac{1}{\log\phi(n_k)} \frac{1}{m_k} \qquad n_k \leq n \leq 3n_k$$

$$k = 1,2,3....$$

and $a_n = 0$ for all other values of n.

By using the relation (3.5) we get

$$|a_n| \leq \frac{\phi(n_k)}{n_k} = \frac{3\phi(n_k)}{3n_k}$$

where $n_k < n < 3n_k$

so that

$$|a_n| \leq \frac{3\phi(n)}{n} \qquad (n_k \leq n \leq 3n_k)$$

and so we have in this case

$$a_n = 0\left(\frac{\phi(n)}{n}\right) .$$

Thus the above condition together with (i) and (ii) shows that Littlewood's Tauberian condition is also best possible when $\Sigma a_n e^{in\theta}$ is summable $(|A|,n)$.

Remark 3: In the proof of Theorem 3.1 one might, of course, consider an example of the kind introduced by Kennedy and Szüsz in the proof of their theorem, i.e. Theorem 1.13. However it would appear that when one does this, difficulties arise in dealing with the higher derivatives.

This example, in order to show that given $\phi(n)\uparrow\infty$ $(n\uparrow\infty)$, $a_n=O\left(\frac{\phi(n)}{n}\right)$ $(n\to\infty)$ is not a Tauberian condition for summability $|A|$, is the following

$$a_n = \begin{cases} \frac{k^2}{n} & \{k^2 n_k \leqslant n \leqslant (k^2+1)n_k\} \\ \frac{-k^2}{n} & \{(k^2+1)n_k \leqslant n \leqslant (k^2+2)n_k\} \\ 0 & \text{for all other values of } k \end{cases}$$

where $\{n_k\}$ is a sequence of positive integers such that

$$\phi(n_k) > k^2 \quad .$$

That this example satisfies the requirements follows from observing that Σa_n diverges, $f(z)$ is bounded, and $f(x)$ increases with x for $0 \leq x \leq 1$.

CHAPTER 4

Almost everywhere non-summability |A|

Suppose that

$$(4.1)\qquad f(z) = \sum_{0}^{\infty} a_n z^n \qquad (|z|<1)$$

is analytic and bounded in the unit disc $U = (|z|<1)$, and as defined by (1.1)

$$(4.2)\qquad V(f,\theta) = \int_{0}^{1} |f'(re^{i\theta})|\,dr.$$

By Fatou's Theorem [(3), p. 17], in this case $\lim_{r\to 1-} f(re^{i\theta})$ exists and is nonzero for almost all values of θ, unless $f(z)\equiv 0$. In other words $\sum_{0}^{\infty} a_n e^{in\theta}$ is summable A for almost all values of θ. But as we shall see later, for almost all values of θ, $\sum_{0}^{\infty} a_n e^{in\theta}$ may not be summable |A|. By definition $V(f,\theta) = \infty$ at all points where $\sum_{0}^{\infty} a_n e^{in\theta}$ is not summable | A|.

In this chapter we shall consider various classes of functions defined by (4.1) such that $V(f,\theta) = \infty$ for almost all values of θ.

Mergelyan [(9)] proved that there exists a function f(z), analytic and bounded in $|z|<1$, such that

$$\int_{r=0}^{1}\int_{\theta=0}^{2\pi} |f'(re^{i\theta})|\, r\,d\theta\,dr = \infty$$

and by using (4.2), the above result takes the form

$$\int_{0}^{2\pi} V(f,\theta)\, d\theta = \infty .$$

Rudin [(13)] has proved a proposition stronger than Mergelyan's, namely, that there exists a function f(z), analytic and bounded in $|z|<1$ and continuous in $|z|\leq 1$ such that $V(f,\theta) = \infty$ for almost all θ. In order to prove this result, Rudin first constructs a function $F(z)$ analytic and bounded in $|z|<1$, such that for almost all θ

$$V(F,\theta) < \infty \text{ and } \int_0^{2\pi} V(F,\theta)d\theta = \infty ,$$

and then obtains the required function in the form

$$f(z) = \sum_{k=1}^{\infty} C_k F(z^{n_k}) ,$$

where $\{n_k\}$ is some sequence of positive integers and $\{c_k\}$ is some sequence of positive numbers such that $\sum_{k=1}^{\infty} c_k < \infty$.

Both Mergelyan's and Rudin's arguments involve non-constructive steps and it was Piranian [(10)] who first gave two explcit constructions that prove Mergelyan's result. They are however inadequate to prove Rudin's result.

We are going to construct a class of functions $f(z) \varepsilon H^\infty$ $(|z|<1)$ for which Rudin's Theorem 1.9 and Theorem 1.10 hold.

Suppose that $f(z)$ is analytic and bounded in the unit disc U $(|z|<1)$. i.e. $f \varepsilon H^\infty$. If the Taylor series for $f(z)$ is absolutely convergent on $|z|=1$, then by Theorem 2.1 $V(f,\theta) < \infty$ for all values of θ. Thus if one wishes $V(f,\theta)$ to be infinite for some θ, where $f \varepsilon H^\infty$, then we must have $\sum_{n=0}^{\infty} |a_n| = \infty$ where $f(z) = \sum_0^\infty a_n z^n (|z|<1)$. One consequence that follows immediately from this is that we cannot have $V(f,\theta) = \infty$ for some θ and any $f\varepsilon H^\infty$ such that the series for $f(z)$ has Hadamard gaps. It is in fact well known that such an $f\varepsilon H^\infty$ if and only if its Taylor series converges absolutely on $|z| \leq 1$.

The lacunary nature of a series with Hadamard gaps usually means that its behaviour and those of its derivatives can be relatively easily estimated, so that in looking for particular examples, as in the situation we are considering, one tends first of all to find examples with Hadamard gaps if this is not excluded by general considerations. However, such considerations as above, exclude the use of such series in this case.

However, one can easily find an $f \varepsilon H^{\infty}$ for which $V(f,\Theta) = \infty$ for some Θ. All we have to do is to take $f(z)$ to be a Blaschke product whose zeros are sparse and lie on the ray $\arg z = \Theta$. In this case $\lim_{r \to 1-} f(re^{i\Theta})$ will not exist and so $\int_o^1 |f'(re^{i\Theta})| dr$, which is the length of the image of $[o, e^{i\Theta})$ by $f(z)$, must be infinite. It is clear that we can also find a Blaschke product $f(z)$ so that $V(f,\Theta_\nu) = \infty$ where $\Theta_1, \Theta_2, ..\Theta_\nu, ..\Theta_n$ are given real numbers in $[o, 2\pi]$. With a little bit more work one sees that one can do the same thing for any given countable set $\{\Theta_1, \Theta_2, . . \Theta_n . . . .\}$.

If one wishes to find an $f \varepsilon H^{\infty}$ for which $V(f,\Theta) = \infty$ for Θ in an uncountable set it appears that one has now to consider very much more sophisticated approaches to the problem. These lead in fact to functions in H^{∞} for which $V(f,\Theta) = \infty$ for almost all Θ.

The functions we are going to consider will in fact have one sided gaps and they are similar to those considered by Clunie [(2)] .

Let $f_o(z) \equiv g_o(z) \equiv 1$ and suppose $f_n(z)$ and $g_n(z)$ have been defined. We assume $\{\eta_n\}_1^{\infty}$ is a decreasing* sequence of positive numbers such that

$$(4.3) \qquad \sum_1^{\infty} \eta_n = \infty, \qquad \sum_1^{\infty} \eta_n^2 < \infty .$$

*This condition is not essential for the proof of the theorem, but we assume it in order to simplify the arguments.

$\{\nu_n\}_1^\infty$ is an increasing sequence of positive integers such that

$$(4.4) \qquad \nu_1 = 1 \ , \qquad \nu_{n+1} \gg \sum_{k=1}^{n} \nu_k \ ,$$

so that

$$\frac{\sum_{k=1}^{n} \nu_k}{\nu_{n+1}} \to 0 \qquad (n \to \infty) \ .$$

Define

$$(4.5) \qquad \begin{cases} f_{n+1}(z) = f_n(z) + \eta_{n+1} e^{i\phi_{n+1}} z^{\nu_{n+1}} g_n(z) \\ g_{n+1}(z) = -\eta_{n+1} f_n(z) + e^{i\phi_{n+1}} z^{\nu_{n+1}} g_n(z), \end{cases}$$

where $\{\phi_n\}_1^\infty$ is an arbitrary real sequence.

For $n \geq 0$ it follows that, on $|z|=1$, we have

$$|f_{n+1}(z)|^2 + |g_{n+1}(z)|^2 = (1+\eta_{n+1}{}^2)(|f_n(z)|^2 + |g_n(z)|^2) \ .$$

Hence for $n \geq 0$, we have on $|z| = 1$

$$(4.6) \quad |f_{n+1}(z)|^2 + |g_{n+1}(z)|^2 = 2(1+\eta_1^2)(1+\eta_2^2) \cdots (1+\eta_{n+1}{}^2)$$

$$(|f_0|^2 + |g_0|^2 = 2, \text{ by construction}).$$

Since $\sum_1^\infty \eta_k^2 < \infty$ (by (4.3)), the right-hand side of (4.6) is bounded for all $n \geq 0$, by a number which is independent of $\{\nu_n\}$ and $\{\phi_n\}$.

From the construction it also follows that the degree of $f_n(z)$ is at most $(\nu_1 + \nu_2 + \cdots \nu_n)$. It is also apparent that the terms of $\eta_{n+1} e^{i\phi_{n+1}} z^{\nu_{n+1}} g_n(z)$ are all of degree at least ν_{n+1} and since by (4.4) $\sum_{k=1}^{n} \nu_k \ll \nu_{n+1}$, it follows that $f_{n+1}(z)$ is obtained from $f_n(z)$ by adding on terms of degrees higher than that of $f_n(z)$ itself, i.e. $f_n(z)$ and $z^{\nu_{n+1}} g_n(z)$ have no terms of common index with respect to the dependence on z and from (4.6) $f_n(z)$ is bounded for all $n \geq o$ in $|z| \leq 1$. Thus we see that

$$(4.7) \qquad f_n(z) \to f(z) \qquad (n \to \infty),$$

where $f \varepsilon H^{\infty}$, the convergence being locally uniform in $|z|<1$. It also follows from (4.6) that the bound for f(z) is independent of the sequences $\{\nu_n\}$ and $\{\phi_n\}$ and depends only on the sequence $\{\eta_n\}$. Clearly we have

$$(4.8) \qquad f(z) = 1 + \sum_{k=o}^{\infty} \eta_{k+1} e^{i\phi_{k+1}} z^{\nu_{k+1}} g_k(z)$$

so that f(z) is analytic and bounded in $|z|<1$ and therefore by Fatou's theorem $\lim_{r\to 1-} f(re^{i\Theta})$ exists and is nonzero for almost all values of Θ. Let

$$(4.9) \qquad E_f = \{\Theta: \lim_{r\to 1-} f(re^{i\Theta}) \neq o\}$$

so that $$m(E_f) = 2\pi.$$

We write

$$(4.10) \qquad \begin{cases} z^{k_n} f_n(\frac{1}{z}) = F_n(z) \\ z^{k_n} g_n(\frac{1}{z}) = G_n(z) , \end{cases}$$

where $\{k_n\}$ is a sequence of positive integers to be specified later, with $k_o = o$, so that $G_o(z) \equiv F_o(z) \equiv 1$.

From (4.10) we have

$$F_{n+1}(z) = z^{(k_{n+1}-k_n)} F_n(z)$$
$$+ \eta_{n+1} e^{i\phi_{n+1}} z^{(k_{n+1} - k_n - \nu_{n+1})} G_n(z)$$

$$G_{n+1}(z) = -\eta_{n+1}\, z^{(k_{n+1}-k_n)} F_n(z) + e^{i\phi_{n+1}} e^{(k_{n+1}-k_n-\nu_{n+1})} G_n(z) .$$

Now we choose $\{k_n\}$ so that

$$k_{n+1}-k_n = \nu_{n+1} \qquad (n=0,1,2 \ldots) .$$

Clearly $k_n = \nu_1 + \nu_2 + \ldots \nu_n$, so that $F_n(z)$ and $G_n(z)$ are polynomials and we have

$$(4.11) \quad \begin{cases} F_{n+1}(z) = \eta_{n+1}\, e^{i\phi_{n+1}} G_n(z) + z^{\nu_{n+1}} F_n(z) \\ G_{n+1}(z) = e^{i\phi_{n+1}} G_n(z) - \eta_{n+1} z^{\nu_{n+1}} F_n(z) \end{cases}$$

and so on $|z|=1$,

$$|F_{n+1}(z)|^2 + |G_{n+1}(z)|^2 = (1+\eta_{n+1}^2)(|F_n(z)|^2 + |G_n(z)|^2)$$

$$= 2(1+\eta_1^2)(1+\eta_2^2) \ldots (1+\eta_{n+1}^2) .$$

Since $\sum_{k=1}^{\infty} \eta_k^2 < \infty$, by the maximum modulus principle $F_{n+1}(z)$ and $G_{n+1}(z)$ are bounded by M (say) in $|z|\leq 1$ for $n\geq 1$ and since $\{\nu_k\}$ satisfies (4.4), by an argument similar to the one used before, $z^{\nu_{n+1}} F_n(z)$ and $G_n(z)$ have no terms of common index with respect to the dependence on z. So we get

$$G_n(z) \to G(z) \qquad (n\to\infty),$$

where $G\in H^{\infty}$, the convergence being locally uniform in $|z|<1$ since $\sum_{k=1}^{\infty} \eta_k^2 < \infty$, and by Fatou's Theorem $\lim_{r\to 1-} G(re^{i\theta})$ exists and is non zero for almost all values of θ . i.e. if

(4.12) $$E_G = \{\Theta : \lim_{r\to 1-} G(re^{i\Theta}) \neq o\}$$

then

$$m(E_G) = 2\pi \ .$$

Clearly the bounds for $f_n(z)$, $g_n(z)$ $F_n(z)$ and $G_n(z)$ are all independent of $\{\phi_n\}$, and therefore to avoid unnecessary calculations, let us choose $\phi_n = o$ for all n, so that

(4.13) $$\begin{cases} f_{n+1}(z) = f_n(z) + \eta_{n+1} z^{\nu_{n+1}} g_n(z) \\ g_{n+1}(z) = -\eta_{n+1} f_n(z) + z^{\nu_{n+1}} g_n(z) \ , \end{cases}$$

(4.14) $$\begin{cases} F_{n+1}(z) = \eta_{n+1} G_n(z) + z^{\nu_{n+1}} F_n(z) \\ G_{n+1}(z) = G_n(z) - \eta_{n+1} z^{\nu_{n+1}} F_n(z) \ . \end{cases}$$

Then

(4.15) $$\begin{cases} f(z) = 1 + \sum_{k=o}^{\infty} \eta_{k+1} z^{\nu_{k+1}} g_k(z) \\ G(z) = 1 - \sum_{k=o}^{\infty} \eta_{k+1} z^{\nu_{k+1}} F_k(z) \ . \end{cases}$$

From the construction of $F_n(z)$ and $G_n(z)$ we also have on $|z|=1$, say at $z=e^{i\Theta}$

(4.16) $$\begin{aligned} |f_n(e^{-i\Theta})| &= |F_n(e^{i\Theta})| \leq M \text{ (for all n)} \\ |g_n(e^{-i\Theta})| &= |G_n(e^{i\Theta})| \leq M \text{ (for all n)} \ . \end{aligned}$$

After this preliminary discussion we shall prove the following two lemmas.

Lemma 4.1. Let $\theta \varepsilon E_G$, where E_G has been defined by (4.12). Then there exists a positive integer $n_o = n_o(\theta)$, and a $c(\theta) > o$ such that

$$|g_n(re^{-i\theta})| \geq c(\theta) > o,$$

where $n > n_o$ and $e^{-2/\nu_{n+1}} \leq r < 1.$

Proof: We know that for $\theta \varepsilon E_G$,

$$\lim_{r \to 1-} G(re^{i\theta}) = G(e^{i\theta}) \neq o.$$

Suppose that

$$(4.17) \qquad |G(e^{i\theta})| = 2\beta(\theta) > o.$$

There then exists a value $r_o = r_o(\theta)$, such that

$$(4.18) \qquad |G(re^{i\theta})| \geq \beta(\theta) \qquad (r_o < r < 1).$$

From (4.15) we have

$$|G(z) - G_n(z)| \leq \sum_{k=n+1}^{\infty} \eta_k \, |z|^{\nu_k} \, |F_{k-1}(z)| \qquad (|z| < 1).$$

From (4.16), $|F_n(z)| \leq M$ $(|z| < 1)$ for all n, and by (4.3) $o < \eta_{k+1} \leq \eta_k$ $(k \geq 1)$. Hence for $o \leq r < 1$, from (4.14) and (4.15),

$$|G(re^{i\theta}) - G_n(re^{i\theta})| \leq \eta_{n+1} M \sum_{k=n+1}^{\infty} r^{\nu_k}.$$

Take $r = e^{-1/\nu_{n+1}}$ and then

$$|G(re^{i\theta}) - G_n(re^{i\theta})| \leq \eta_{n+1} \, M \sum_{k=n+1} e^{-\frac{\nu_k}{\nu_{n+1}}}$$
$$\to o \qquad (n \to \infty)$$

since $\nu_{k+1} >> \nu_k$ for all k and $\eta_k \to o$ $(k \to \infty)$.

Therefore, from (4.18), it follows that for a given $\theta \varepsilon E_G$ and for all large 'n' we have

$$(4.19) \qquad |G_n(e^{\frac{-1}{\nu_{n+1}}} e^{i\theta})| \geq \beta(\theta)/2 .$$

Now $G_n(z)$ is a polynomial of degree $(\nu_1 + \nu_2 + \dots \nu_n)$ bounded by M in $|z| \leq 1$, therefore by Bernstein's Theorem

$$|G_n'(z)| \leq M (\nu_1 + \nu_2 + \dots \nu_n) \qquad (|z|<1).$$

Since $G_n(z)$ is analytic in $|z|<1$, we have for $o<r<1$,

$$G_n(e^{i\theta}) - G_n(re^{i\theta}) = \int_{re^{i\theta}}^{e^{i\theta}} G'_n(\zeta)\, d\zeta ,$$

so that for $o<r<1$ and for all n

$$|G_n(e^{i\theta}) - G_n(re^{i\theta})| \leq M(1-r)(\nu_1 + \nu_2 + \dots \nu_n).$$

Consider now $\theta \varepsilon E_G$ and those n for which (4.19) holds, and take $r=e^{-1/\nu_{n+1}}$ in the preceding inequality. This gives

$$|G_n(e^{i\theta}) - G_n(e^{\frac{-1}{\nu_{n+1}}} e^{i\theta})|$$

$$\leq M(1-e^{\frac{-1}{\nu_{n+1}}})(\nu_1 + \nu_2 + \dots \nu_n)$$

$$\leq M \frac{\nu_1 + \nu_2 + \dots \nu_n}{\nu_{n+1}}$$

$$\to o \qquad (n \to \infty),$$

and so for all large n, $n>n_1(\theta)$ say, we find from (4.19) that

$$(4.20) \quad |G_n(e^{i\theta})| \geq \frac{\beta(\theta)}{4} .$$

Now from (4.16) it follows that for $\theta \epsilon E_G$ and $n>n_1$,

$$(4.21) \quad |g_n(e^{-i\theta})| = |G_n(e^{i\theta})| \geq \frac{\beta(\theta)}{4} .$$

Again, since $g_n(z)$ is a polynomial of degree $(\nu_1 + \nu_2 + \dots\dots\dots\dots \nu_n)$ and bounded by M in $|z|\leq 1$, therefore by an argument similar to the one used for $G_n(z)$ to obtain (4.20) from (4.19), we conclude from (4.21) that there exists a positive integer $n_o>n_1$ such that for all $n\geq n_o$ and $e^{\frac{-2}{\nu_{n+1}}}\leq r < 1$

$$| g_n(re^{-i\theta}) | \geq \frac{\beta(\theta)}{8} > 0$$

where $\theta \epsilon E_G$.

cont/....

Writing

$$c(\Theta) = \frac{\beta(\Theta)}{8},$$

for $\Theta \in E_G$, and for all $n \geq n_o$, where n_o is a positive integer we get

$$|g_n(re^{-i\Theta})| \geq c(\Theta) > o.$$

This proves the lemma.

Lemma 4.2 Let f(z) be defined by (4.8) and we assume in addition to conditions (4.3) and (4.4) that

$$\frac{\nu_1 + \nu_2 + \ldots \quad \nu_n}{\nu_{n+1}} = o\ (\eta_{n+1})\ (n\to\infty).$$

If $\Theta \in E_G$ where E_G is defined by (4.12) then there exists a positive integer $N_o = N_o(\Theta)$ and a $d(\Theta) > o$ such that

$$\int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |f'(re^{-i\Theta})|\ dr \geq d(\Theta)\ \eta_{n+1} \qquad (n \geq N_o).$$

Proof: From (4.8) we have

$$f(z) = 1 + \sum_{k=1}^{\infty} \eta_{k+1}\ z^{\nu_{k+1}} g_k(z) \qquad (|z|<1).$$

By differentiating we get

$$f'(z) = \sum_{k=1}^{\infty} \eta_{k+1}\ (\nu_{k+1}\ z^{\nu_{k+1}-1}\ g_k(z) + z^{\nu_{k+1}}\ g'_k(z))$$

$$= \sum_{k=1}^{n-1} \eta_{k+1} \left(\nu_{k+1} z^{\nu_{k+1}-1} g_k(z) + z^{\nu_{k+1}} g'_k(z)\right)$$

$$+ \left(\eta_{n+1} \nu_{n+1} z^{\nu_{n+1}-1} g_n(z)\right) + \left(\eta_{n+1} z^{\nu_{n+1}} g_n'(z)\right)$$

$$+ \sum_{k=n+1}^{\infty} \eta_{k+1} \left(\nu_{k+1} z^{\nu_{k+1}-1} g_k(z) + z^{\nu_{k+1}} g'_k(z)\right),$$

where n is a positive integer satisfying $n>n_o$ with n_o as specified in Lemma 4.1.

Let $z=re^{-i\theta}$, where $o<r<1$, so that

$$f'(re^{-i\theta}) = \left[\sum_{k=1}^{n-1} \eta_{k+1}\{\nu_{k+1}(re^{-i\theta})^{\nu_{k+1}-1} g_k(re^{-i\theta}) + (re^{-i\theta})^{\nu_{k+1}} g'_k(re^{-i\theta})\}\right]$$

$$+ [\{\eta_{n+1}\nu_{n+1}(re^{-i\theta})^{\nu_{n+1}-1} g_n(re^{-i\theta})]$$

$$+ [\eta_{n+1}(re^{-i\theta})^{\nu_{n+1}} g_n'(re^{-i\theta})]$$

$$+ \left[\sum_{k=n+1}^{\infty} \eta_{k+1} \{\nu_{k+1}(re^{-i\theta})^{\nu_{k+1}-1} g_k(re^{-i\theta}) + (re^{-i\theta})^{\nu_{k+1}} g'_k(re^{-i\theta})\right]$$

$$= T_1 + T_2 + T_3 + T_4 \quad \text{(say)} \quad .$$

Since $g_k(z)$ is a polynomial of degree $\nu_1+\nu_2 + \dots \nu_k$ bounded by M in $|z|\leq 1$, we have

$$(4.21)' \quad \begin{cases} |g_k(z)| \leq M, \\ |g_k'(z)| \leq M(\nu_1+\nu_2 + \dots \nu_k) & \text{(By Bernstein's Theorem)} \\ \quad < M\nu_{k+1} & \text{(from (4.4))} \end{cases}$$

where $|z|\leq 1$, $k = 1,2,3,\dots$.

Now we have

$$|T_1| \leq \sum_{k=1}^{n-1} \eta_{k+1} \{\nu_{k+1}\, r^{\nu_{k+1}-1}\, |g_k(re^{-i\theta})| + r^{\nu_{k+1}} |g_k'(re^{-i\theta})|$$

By using (4.21)'we get

$$|T_1| \leq \sum_{k=1}^{n-1} 2M\eta_{k+1}\, \nu_{k+1}\, r^{\nu_{k+1}-1} \qquad (o<r<1)$$

so that

$$\int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |T_1|\, dr \leq \sum_{k=1}^{n-1} 2M\eta_{k+1} \Big[r^{\nu_{k+1}} \Big]_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}}$$

$$= 2M \sum_{k=1}^{n-1} \eta_{k+1} \left[e^{-\frac{\nu_{k+1}}{\nu_{n+1}}} - e^{\frac{-2(\nu_{k+1})}{\nu_{n+1}}} \right]$$

$$= 2M \sum_{k=1}^{n-1} \eta_{k+1}\, e^{-\frac{\nu_{k+1}}{\nu_{n+1}}} \left(1- e^{\frac{-\nu_{k+1}}{\nu_{n+1}}}\right).$$

Now $k<n$, so that by (4.4)

$$\nu_{k+1} << \nu_{n+1},$$

so that

$$\int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |T_1| dr \leqslant 2M \sum_{k=1}^{n-1} \eta_{k+1} \frac{\nu_{k+1}}{\nu_{n+1}} \left(1+O\left(\frac{\nu_{k+1}}{\nu_{n+1}}\right)\right) .$$

Again by (4.3), since $\{\eta_k\}$ is a positive decreasing sequence, therefore $o<\eta_k<\eta_1$ for all k. Thus we get

$$\int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |T_1| dr \leq M' \sum_{k=1}^{n-1} \left(\frac{\nu_{k+1}}{\nu_{n+1}}\right)$$

$$= \quad M' \quad \frac{\nu_1 + \nu_2 + \cdots \nu_n}{\nu_{n+1}}$$

$$= o\,(\eta_{n+1}) \qquad (n \to \infty),$$

since by the assumption of the lemma

$$\frac{\nu_1 + \nu_2 + \ldots \nu_n}{\nu_{n+1}} = o(\eta_{n+1}) \quad (n \to \infty).$$

Hence

$$\int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |T_1|\, dr = o(\eta_{n+1}) \qquad (n \to \infty) \quad .$$

We have

$$|T_2| = \eta_{n+1} \; r^{\nu_{n+1}-1} \; \nu_{n+1} \; |g_n(re^{-i\Theta})|.$$

We now suppose that $\Theta \varepsilon E_G$ and $e^{\frac{-2}{\nu_{n+1}}} \leq r < 1$, so that by Lemma 4.1 since we are assuming $n > n_o$,

$$|g_n(re^{-i\Theta})| \geq c(\Theta) > o \qquad (n \geq n_o)$$

and therefore

$$(4.22) \qquad \int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |T_2|\, dr \geq c(\Theta)\,(e^{-1} - e^{-2})\eta_{n+1} \quad (n \geq n_o)$$

$$\geq \frac{c(\Theta)}{12}\, \eta_{n+1} \qquad (n \geq n_o) \; .$$

Again

$$|T_3| = |\eta_{n+1}(re^{-i\Theta})^{\nu_{n+1}} \quad g_n'(re^{-i\Theta})|$$

$$\leq \eta_{n+1} r^{\nu_{n+1}} \quad M(\nu_1 + \nu_2 + \ldots \nu_n) \text{ (By (4.21)')}$$

$$\leq \eta_{n+1} \; r^{\nu_{n+1}-1} \quad M(\nu_1 + \nu_2 + \ldots . \nu_n) \quad (r<1),$$

so that

$$\int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |T_3| \, dr \leq M\eta_{n+1} \; (\nu_1 + \nu_2 + \ldots \nu_n) \quad \left[\frac{r^{\nu_{n+1}}}{\nu_{n+1}}\right]_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}}$$

$$= \quad M\eta_{n+1} \; \frac{\nu_1 + \nu_2 + \ldots . \nu_n}{\nu_{n+1}} \quad (e^{-1} - e^{-2})$$

$$= o \; (\eta_{n+1}) \qquad\qquad (n \to \infty) \; .$$

Now

$$|T_4| \leq \sum_{k=n+1}^{\infty} \eta_{k+1} \; \{\nu_{k+1} \; r^{\nu_{k+1}-1} \quad |g_k(re^{-i\Theta})|$$

$$+ \; r^{\nu_{k+1}} \; |g_k' \; (re^{-i\Theta})|\}$$

From (4.21)' we obtain

$$|T_4| \leq \sum_{k=n+1}^{\infty} \eta_{k+1} \; r^{\nu_{k+1}-1} \; (\nu_{k+1} M + r\nu_{k+1} \; M)$$

$$< \sum_{k=n+1}^{\infty} \eta_{k+1} \quad r^{\nu_{k+1}-1} \quad 2M\nu_{k+1} \quad (o<r<1),$$

so that

$$\int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |T_4| \, dr < 2M \sum_{k=n+1}^{\infty} \eta_{k+1} \Big[r^{\nu_{k+1}} \Big]_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}}$$

$$= 2M \sum_{k=n+1}^{\infty} \eta_{k+1} \left[e^{-\frac{\nu_{k+1}}{\nu_{n+1}}} - e^{-\frac{2\nu_{k+1}}{\nu_{n+1}}} \right]$$

$$= 2M \sum_{k=n+1}^{\infty} \eta_{k+1} \left[e^{-\frac{\nu_{k+1}}{\nu_{n+1}}} \left(1 - e^{-\frac{\nu_{k+1}}{\nu_{n+1}}}\right) \right].$$

In this case $k>n$, so that $\nu_{n+1} < \nu_{k+1}$ and therefore $\left(\frac{\nu_{k+1}}{\nu_{n+1}}\right) > 1.$

We shall therefore get for all such k

$$e^{\frac{\nu_{k+1}}{\nu_{n+1}}} \geq \left(\frac{\nu_{k+1}}{\nu_{n+1}}\right) > 1,$$

and so

$$0 < e^{-\frac{\nu_{k+1}}{\nu_{n+1}}} < 1,$$

and

$$e^{-\frac{\nu_{k+1}}{\nu_{n+1}}} < \frac{\nu_{n+1}}{\nu_{k+1}} .$$

Thus we obtain

$$\int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |T_4| \, dr \leq 2M \sum_{k=n+1}^{\infty} \eta_{k+1} \frac{\nu_{n+1}}{\nu_{k+1}} .$$

We know that

$$\frac{\nu_k}{\nu_{k+1}} \to 0 \qquad (k \to \infty).$$

Let

$$\frac{\nu_k}{\nu_{k+1}} \leq \alpha < 1, \quad (k>n)$$

so that

$$\frac{\nu_{n+1}}{\nu_{n+2}} \leq \alpha ,$$

$$\frac{\nu_{n+1}}{\nu_{n+3}} = \frac{\nu_{n+1}}{\nu_{n+2}} \frac{\nu_{n+2}}{\nu_{n+3}} \leq \alpha^2 < 1$$

$$\frac{\nu_{n+1}}{\nu_{n+k}} = \frac{\nu_{n+1}}{\nu_{n+2}} \frac{\nu_{n+2}}{\nu_{n+3}} \cdots \frac{\nu_{n+k-1}}{\nu_{n+k}} \leq \alpha^{k-1} .$$

Again since $\{\eta_k\}$ is a positive decreasing sequence we have

$$\eta_{k+1} \leq \eta_{n+1} \quad \text{for all } k>n .$$

Hence

$$\int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |T_4|\,dr \leq 2M\,\eta_{n+1} (\alpha + \alpha^2 + \alpha^3 + \ldots)$$

$$= 2M \frac{\alpha}{1-\alpha} \eta_{n+1} .$$

Now α can be chosen so that it tends to zero as $n\to\infty$ since

$$\frac{\nu_{n+1}}{\nu_{n+2}} \to 0 \qquad (n\to\infty) ,$$

so that

$$\int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |T_4|\,dr \leq 2M\,\eta_{n+1} \sum_{k=n+1}^{\infty} \frac{\nu_{n+1}}{\nu_{k+1}}$$

$$< \frac{2M\alpha}{1-\alpha}\eta_{n+1}$$

$$= o(\eta_{n+1}) \qquad (n\to\infty) .$$

We have now obtained

$$\int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |T_1|dr + \int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |T_2|dr + \int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |T_3|dr$$

(4.23)

$$= o(\eta_{n+1}) \qquad (n\to\infty) .$$

Therefore there exists a positive number N_1, such that

$$\int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} (|T_1| + |T_3| + |T_4|)dr$$

$$\leq \frac{c(\Theta)\eta_{n+1}}{24} \qquad (n \geq N_1).$$

Thus

$$\int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |f'(re^{-i\Theta})| dr \geq \int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |T_2| dr - \int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} (|T_1| + |T_3| + |T_4|) dr$$

$$< \frac{c(\Theta)}{24}\eta_{n+1}$$

for all $n \geq \max(n_o, N_1) = N_o$(say) .

Writing

$$d(\theta) = \frac{c(\theta)}{24} \text{ , we have}$$

$$(4.24) \qquad \int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |f'(re^{-i\theta})| \, dr \geq d(\theta)\eta_{n+1} \qquad (n \geq N_o);$$

where $d(\theta) > o$.

This proves the lemma.

Theorem 4.1: Let $f(z)$ be defined by (4.8). Then under the assumption of Lemma 4.2

$$\int_o^1 |f'(re^{i\theta})| \, dr = \infty$$

for all values of θ, except perhaps those lying in a set of Lebesgue measure zero.

Proof: Suppose $\phi \varepsilon E_G$ and $N \geq N_o = N_o(\phi)$, where N_o is defined in Lemma 4.2. Then by Lemma 4.2

$$\int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |f'(re^{-i\phi})| \, dr \geq d(\phi)\eta_{n+1} \qquad (n \geq N_o)$$

where $d(\phi) > o$.

Since $m(E_G) = 2\pi$ (by (4.12)), the above result is true for almost all values of ϕ. If we write $-\phi = \theta$, we obtain

$$\int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |f'(re^{i\theta})| \, dr \geq d'(\theta)\eta_{n+1} \qquad (n \geq N_o')$$

where $d'(\theta) = d(-\theta), N_o'(\theta) = N_o(-\theta)$ and the above result is true for almost all values of θ.

We deduce that for almost all values of θ,

$$\sum_{n=N_o'}^{\infty} \int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |f'(re^{i\theta})|\,dr \geq d'(\theta) \sum_{n=N_o'}^{\infty} \eta_{n+1}.$$

But by (4.3), we know that $\sum_{n=1}^{\infty} \eta_{n+1} = \infty$ and therefore $\sum_{n=N_o'}^{\infty} \eta_{n+1} = \infty$.

Hence for almost all values of θ

$$\sum_{n=N_o'}^{\infty} \int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |f'(re^{i\theta})|\,dr = \infty .$$

Clearly

$$\int_0^1 |f'(re^{i\theta})| \geq \sum_{n=N_o}^{\infty} \int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |f'(re^{i\theta})|\,dr$$

$$= \infty .$$

Thus for almost all values of θ,

$$\int_0^1 |f'(re^{i\theta})|\,dr = \infty.$$

This completes the proof of the theorem.

Now we want to construct a function H(z) which is analytic in $|z|<1$ and continuous in $|z|\leq 1$, and for which the result of the preceding theorem holds true.

From (4.5) we have

$$(4.25)\quad f_{n+1}(z) = 1 + \sum_{k=o}^{n} \eta_{k+1} z^{\nu_{k+1}} g_k(z).$$

Let $\{\lambda_n\}_1^\infty$, $\{\mu_n\}_1^\infty$ be sequences of positive integers such that for all n,

$$(4.25)'\quad \lambda_{n+1} > \mu_n > \lambda_n > n \text{ and } \sum_{n=1}^{\infty} \sum_{\lambda n}^{\mu n} \eta_{k+1} = \infty .$$

Let $\{\varepsilon_n\}_1^\infty$ be a decreasing sequence of positive numbers chosen so that

$$(4.26)\quad \sum_{n=1}^{\infty} \varepsilon_n < \infty$$

and

$$(4.27)\quad \sum_{n=1}^{\infty} \varepsilon_n \sum_{\lambda_n}^{\mu_n} \eta_{k+1} = \infty.$$

We now consider

$$(4.28)\quad H(z) = \sum_{n=1}^{\infty} \varepsilon_n \{\sum_{\lambda_n}^{\mu_n} \eta_{k+1} z^{\nu_{k+1}} g_k(z)\},$$

where the ν_k have been chosen to satisfy the following condition (4.4)' (and hence (4.4)).

$$(4.4)'\quad \frac{\nu_1 + \nu_2 + \dots \nu_p}{\nu_{p+1}} = o\ (\varepsilon_N \eta_{p+1})\ (\lambda_N \leq p \leq \mu_N; N\to\infty) .$$

Since by (4.6) and (4.25) for all n, $1+\sum_{k=0}^{n} \eta_{k+1} z^{\nu_{k+1}} g_k(z)$ is bounded for $|z|\leq 1$, we deduce that for all $n\geq 1$,

$$\sum_{k=0}^{n} \eta_{k+1} z^{\nu_{k+1}} g_k(z) \text{ is bounded by } k \text{ (say) for } |z|\leq 1.$$

Therefore

$$(4.29) \qquad \left| \sum_{\lambda_n}^{\mu_n} \eta_{k+1} z^{\nu_{k+1}} g_k(z) \right| \leq 2K \quad (|z|\leq 1),$$

for all λ_n, μ_n and for all n.

From (4.26) and (4.29) we conclude that the series of blocks for $H(z)$ in (4.28) converges uniformly in $|z|\leq 1$, and since each block is a polynomial, $H(z)$ is continuous in $|z|\leq 1$, and analytic in $|z|<1$.

Now we want to prove the following theorem.

Theorem 4.2. Let $H(z)$ be defined by (4.28). Then for almost all values of Θ,

$$\int_0^1 |H'(\rho e^{i\Theta})|\,d\rho = \infty .$$

Proof: Suppose p is a positive integer chosen so that

$$\lambda_N \leq p \leq \mu_N,$$

where N is a positive integer to be specified later.

We shall first consider

$$\int_{e^{-2/\nu_{p+1}}}^{e^{-1/\nu_{p+1}}} |H'(re^{i\Theta})|\,dr \qquad (\lambda_N \leq p \leq \mu_N).$$

From (4.28)

$$H'(z) = \sum_{n=1}^{\infty} \varepsilon_n \{ \sum_{k=\lambda_n}^{\mu_n} \eta_{k+1} (\nu_{k+1} z^{\nu_{k+1}-1} g_k(z) + z^{\nu_{k+1}} g_k'(z))\}$$

$$= \sum_{n=1}^{N-1} \varepsilon_n \{ \sum_{\lambda_n}^{\mu_n} \eta_{k+1} (\nu_{k+1} z^{(\nu_{k+1}-1)} g_k(z) + z^{\nu_{k+1}} g_k'(z))\}$$

$$+ \varepsilon_N \{ \sum_{\lambda_N}^{\mu_N} \eta_{k+1} (\nu_{k+1} z^{(\nu_{k+1}-1)} g_k(z) + z^{\nu_{k+1}} g_k'(z))\}$$

$$+ \sum_{n=N+1}^{\infty} \varepsilon_n \{ \sum_{\lambda_n}^{\mu_n} \eta_{k+1} (\nu_{k+1} z^{\nu_{k+1}-1} g_k(z) + z^{\nu_{k+1}} g_k'(z))\}$$

$$= \Bigg[\{ \sum_{n=1}^{N-1} \varepsilon_n \sum_{k=\lambda_n}^{\mu_n} (\eta_{k+1} (\nu_{k+1} z^{\nu_{k+1}-1} g_k(z) + z^{\nu_{k+1}} g_k'(z)))$$

$$+ \varepsilon_N \sum_{k=\lambda_N}^{p-1} \eta_{k+1} (\nu_{k+1} z^{\nu_{k+1}-1} g_k(z) + z^{\nu_{k+1}} g_k'(z))\} \Bigg]$$

$$+ \Big[\{ \varepsilon_N \eta_{p+1} \nu_{p+1} z^{\nu_{p+1}-1} g_p(z)\} \Big]$$

$$+ \Big[\{ \varepsilon_N \eta_{p+1} z^{\nu_{p+1}} g_p'(z)\} \Big]$$

$$+ \Bigg[\{ \varepsilon_N \sum_{p+1}^{\mu_N} \eta_{k+1} (\nu_{k+1} z^{\nu_{k+1}-1} g_k(z) + z^{\nu_{k+1}} g_k'(z))\}$$

$$+ \sum_{N+1}^{\infty} \varepsilon_n \sum_{k=\lambda_n}^{\mu_n} \eta_{k+1} (\nu_{k+1} z^{\nu_{k+1}-1} g_k(z) + z^{\nu_{k+1}} g_k'(z)) \Bigg]$$

$$= H_1 + H_2 + H_3 + H_4 \text{ (say).}$$

By using arguments similar to those used in Lemma 4.2 to obtain an estimate for

$$\int_{e^{-2/\nu_{n+1}}}^{e^{-1/\nu_{n+1}}} |Tj|\,dr, \qquad (j=1,2,3,4)$$

we shall get the following estimates for $\int_{e^{-2/\nu_{p+1}}}^{e^{-1/\nu_{p+1}}} |Hj|\,dr \quad (j=1,2,3,4)$:

Now

$$\int_{e^{-2/\nu_{p+1}}}^{e^{-1/\nu_{p+1}}} |H_1|\,dr \leq 2M \sum_{n=1}^{N-1} \varepsilon_n \sum_{\kappa=\lambda_n}^{\mu_n} \eta_{k+1} \left(\frac{\nu_{k+1}}{\nu_{p+1}} \left(1+O\left(\frac{\nu_{k+1}}{\nu_{p+1}}\right)\right)\right.$$

$$\left. + 2M \ \varepsilon_N \sum_{k=\lambda_N}^{p-1} \eta_{k+1} \left\{\frac{\nu_{k+1}}{\nu_{p+1}} \left(1+O\left(\frac{\nu_{k+1}}{\nu_{p+1}}\right)\right)\right\}.\right.$$

Since $\{\varepsilon_n\}$, $\{\eta_k\}$ are positive decreasing sequences, for all n and k

$$\varepsilon_n \leq \varepsilon_1$$

$$\eta_k \leq \eta_1 \qquad .$$

Also, in this case $k<p$, and therefore

$$\frac{\nu_{k+1}}{\nu_{p+1}} \to 0 \qquad (p\to\infty) \quad .$$

Hence

$$\int_{e^{-2/\nu_{p+1}}}^{e^{-1/\nu_{p+1}}} |H_1|\,dr \leq M' \ \frac{\nu_1 + \nu_2 + \ldots \ \nu_p}{\nu_{p+1}} \quad \left(1+O\left(\frac{\nu_p}{\nu_{p+1}}\right)\right)$$

$$= o(\eta_{p+1}\ \varepsilon_N) \qquad (\lambda_N \le p \le \mu_N\ ;\quad N\to\infty)$$

(from (4.4)' $\dfrac{\nu_1 + \nu_2 + \dots \nu_p}{\nu_{p+1}} = o(\varepsilon_N\ \eta_{p+1})$).

Again

$$\int_{e^{-2/\nu_{p+1}}}^{e^{-1/\nu_{p+1}}} |H_3|dr \le 2M''\varepsilon_N \left(\frac{\nu_1 + \nu_2 + \dots \nu_p}{\nu_{p+1}}\right) \eta_{p+1}$$

$$= o\ (\varepsilon_N\ \eta_{p+1}) \qquad (\lambda_N \le p \le \mu_N;\ N\to\infty)$$

since $(\nu_1 + \nu_2 + \dots \nu_p)/\nu_{p+1} \to o\ (p\to\infty)$.

Now

$$\int_{e^{-2/\nu_{p+1}}}^{e^{-1/\nu_{p+1}}} |H_4|dr \le \left(\varepsilon_N \sum_{p+1}^{\mu_N} \eta_{k+1} + \sum_{N+1}^{\infty} \varepsilon_n \sum_{\lambda_n}^{\mu_n} \eta_{k+1}\right)\left(2M' \frac{\nu_{p+1}}{\nu_{k+1}}\right)$$

$$\le 2M'\varepsilon_N\ \eta_{p+1} \sum_{p+1}^{\infty} \frac{\nu_{p+1}}{\nu_{k+1}}$$ (since the ε_n and η_n decrease with n

$$= o(\varepsilon_N \eta_{p+1}) \quad (\lambda_N \le p \le \mu_N;\ N\to\infty)$$

since $\dfrac{\nu_{p+1}}{\nu_{p+2}} \to o$ as $p\to\infty$.

Again, for $N > n_o$ by using Lemma 4.1 we have

$$\int_{e^{-2/\nu_{p+1}}}^{e^{-1/\nu_{p+1}}} |H_2|dr \ge \varepsilon_N \quad c(\theta)\ \eta_{p+1} \quad x \quad (e^{-1} - e^{-2})$$

$$\geq \frac{\varepsilon_N \eta_{p+1}}{12} c(\Theta) .$$

Hence if $\Theta \varepsilon E_G$

$$\int_{e^{-2/\nu_{p+1}}}^{e^{-1/\nu_{p+1}}} |H'(re^{-i\Theta})| \, dr \geqslant \varepsilon_N \eta_{p+1} \; q(\Theta) \; (N \geqslant N_o'')$$

where $q(\Theta) > 0$ and $N_o'' > n_o$.

Now

$$\int_o^1 |H'(re^{-i\Theta})| dr \geqslant \sum_{N=N_o''}^{\infty} \sum_{p=\lambda_N}^{\mu_N} \int_{e^{-2/\nu_{p+1}}}^{e^{-1/\nu_{p+1}}} |H'(re^{-i\Theta})| dr$$

$$\geqslant q(\Theta) \sum_{N=N_o''}^{\infty} \varepsilon_N \sum_{\lambda_N}^{\mu_N} \eta_{p+1}$$

$$= \infty \qquad (\text{since } \sum_{n=1}^{\infty} \varepsilon_n \sum_{\lambda_n}^{\mu_n} \eta_{k+1} = \infty).$$

Since $m(E_G) = 2\pi$ by changing Θ to $-\Theta$ we get for almost all values of

$$\int_o^1 |H'(re^{i\Theta})| dr = \infty.$$

This completes the proof of the theorem.

Let $H(z)$ be defined by (4.28), so that $H(z)$ is analytic in $|z|<1$ and continuous in $|z|\leq 1$. We now consider particular functions of this kind and obtain estimates for the modulus of continuity of $H(e^{i\Theta})$ in these cases. We shall give particular values to η_k, ν_k, λ_n, μ_n, and ε_n such that all the conditions of Theorem 4.2 are satisfied, and estimate the modulus of continuity of the corresponding $H(e^{i\Theta})$.

Let us choose

(i) $\eta_k = \frac{1}{k^\alpha} \quad (\frac{1}{2}<\alpha<1)$

(ii) $\nu_k = k^k$

(iii) $\begin{cases} \lambda_n = 2^n \\ \mu_n = 2^{n+1}-1 \end{cases}$

(iv) $\varepsilon_n = \frac{1}{2^{\gamma n}} \quad (\gamma>0,\ \alpha+\gamma<1)$.

From (i) it is obvious that $\{\eta_k\}$ is a decreasing sequence of positive numbers satisfying the conditions

$$\sum_{k=1}^{\infty} \eta_k = \infty, \qquad \Sigma\ \eta_k^2 = \sum_{k=1}^{\infty} \frac{1}{k^{2\alpha}} < \infty$$

so that (4.3) is satisfied.

Again from (ii), we have

$$\frac{\nu_1 + \nu_2 + \ldots\ldots \nu_p}{\nu_{p+1}} = \frac{1 + 2^2 + \ldots\ p^p}{(p+1)^{p+1}}$$

$$< \frac{p(p^p-1)}{(p-1)(p+1)^{p+1}}$$

$$< \frac{K}{p+1} ,$$

where K is a constant.

Now if $\lambda_N \leq p \leq \mu_N$, then from (iii) we get

$$\frac{1}{2^{N+1}} \leq \frac{1}{p+1} \leq \frac{1}{2^N+1}$$

which gives

$$\frac{1}{p+1} = \frac{1}{(p+1)^{\alpha}} \times \frac{1}{(p+1)^{\gamma}} \quad \frac{1}{(p+1)^{1-\alpha-\gamma}}$$

$$\leq \eta_{p+1} \quad \frac{1}{(2^N+1)^{\gamma}} \quad \frac{1}{(p+1)^{1-\alpha-\gamma}}$$

$$< \eta_{p+1} \quad \frac{1}{2^{N\gamma}} \quad \frac{1}{(p+1)^{1-\alpha-\gamma}}$$

$$= o(\eta_{p+1}\, \varepsilon_N) \qquad (\lambda_N \leq p \leq \mu_N;\ p \to \infty).$$

since $1-\alpha-\gamma > o$.

This gives

$$\frac{\nu_1 + \nu_2 \cdot \cdot \cdot \cdot \cdot \cdot \nu_p}{\nu_{p+1}} \to o \qquad (p \to \infty)$$

and if $\lambda_N \leq p \leq \mu_N$, then

$$\frac{\nu_1 + \nu_2 + \cdot \cdot \cdot \cdot \cdot \nu_p}{\nu_{p+1}} = o\,(\varepsilon_N\ \eta_{p+1})$$

$$(\lambda_N \leq p \leq \mu_N;\ p \to \infty).$$

Therefore ν_k given by (ii) satisfy (4.4)'.

From (i) and (iii) we get

$$\sum_{n=1}^{\infty} \sum_{\lambda_n'}^{\mu_n} \eta_{k+1} = \sum_{n=1}^{\infty} \sum_{2^n}^{2^{n+1}-1} \frac{1}{(k+1)^\alpha}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{(2^n+1)^\alpha} + \frac{1}{(2_n+2)^\alpha} + \cdots \frac{1}{2(^{n+1})^\alpha}\right)$$

$$\geqslant \sum_{n=1}^{\infty} \frac{2^n}{2^{(n+1)\alpha}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^\alpha} 2^{n(1-\alpha)}$$

$$= \infty \qquad (1-\alpha>0)$$

and so (4.25)' is satisfied.

Also from (iv) we get

$$\sum_{n=1}^{\infty} \varepsilon_n = \sum_{n=1}^{\infty} \frac{1}{2^{\gamma n}} = \frac{\frac{1}{2^\gamma}}{1-\frac{1}{2^\gamma}} < \infty ,$$

and from (i), (iii) and (iv) we get

$$\varepsilon_n \sum_{\lambda_n}^{\mu_n} \eta_{k+1} = \frac{1}{2^{n\gamma}} \sum_{2^n}^{2^{n+1}-1} \frac{1}{(k+1)^\alpha} ,$$

which gives

$$\varepsilon_n \sum_{\lambda_n}^{\mu_n} \eta_{k+1} \geqslant \frac{1}{2^{n\gamma}} \frac{2^n}{2^{(n+1)\alpha}} = \frac{2^{n(1-\alpha-\gamma)}}{2^\alpha}$$

for all values of n. This shows that

$$\sum_{n=1}^{\infty} \varepsilon_n \sum_{\lambda_n}^{\mu_n} \eta_{k+1} = \infty ,$$

since $1-\alpha-\gamma>o$, and so (4.26) and (4.27) are satisfied.

Thus with the choice of $\{\eta_k\}$, $\{\nu_k\}$, $\{\lambda_k\}$, $\{\mu_k\}$, $\{\varepsilon_k\}$ given by (i), (ii), (iii) and (iv) we notice that all the conditions of Theorem 4.2 are satisfied.

Let us consider

$$(4.30) \qquad h(z) = \sum_{n=1}^{\infty} \frac{1}{2^{n\gamma}} \sum_{2^n}^{2^{n+1}-1} \frac{1}{(k+1)^{\alpha}} z^{(k+1)^{(k+1)}} g_k(z);$$

where $\frac{1}{2}<\alpha<1$ and $\alpha+\gamma<1$.

Now h(z) is obtained from H(z) by substituting the values of η_k, ν_k, λ_n, μ_n, ε_n from (i), (ii), (iii) and (iv) respectively. Hence h(z) satisfies the conditions of Theorem 4.2, and so we have

$$\int_o^1 |h'(re^{i\theta})|dr = \infty ,$$

for almost all values of θ.

Now we prove the following theorem:

<u>Theorem 4.3.</u> <u>Given $\beta(o<\beta<\frac{1}{2})$, choose γ_o so that $\beta<\gamma_o<\frac{1}{2}$ and then choose α_o such that $\frac{1}{2}<\alpha_o<1$ and $\alpha_o+\gamma_o<1$. If h(z) is the function given in (4.30) with $\alpha=\alpha_o$ and $\gamma=\gamma_o$, then ω(t), the modulus of continuity of $h(e^{i\theta})$, satisfies the condition</u>

$$\omega(t) = O\{(\log \tfrac{1}{t})^{-\beta}\} \qquad (t\to 0_+)$$

Proof: We first consider the function H(z) defined by (4.28) and get an estimate for

$$|H(e^{i(\theta+t)}) - H(e^{i\theta})|$$

for small values of t. Then by substituting the values of η_k, ν_k, λ_n, μ_n, ε_n, from (i), (ii), (iii) and (iv), we shall obtain an estimate for

$$|h(e^{i(\theta+t)}) - h(e^{i\theta})|.$$

From (4.28) we have

$$|H(e^{i(\theta+t)}) - H(e^{i\theta})|$$

$$\leq \sum_{n=1}^{\infty} \varepsilon_n \{| \sum_{k=\lambda_n}^{\mu_n} \eta_{k+1} \{e^{i\nu_{k+1}(\theta+t)} g_k(e^{i(\theta+t)})$$

$$- e^{i\nu_{k+1}\theta} g_k(e^{i\theta}) \}|\}$$

$$= \left[\sum_{n=1}^{n_o} \varepsilon_n \; | \sum_{k=\lambda_n}^{\mu_n} \eta_{k+1} \{e^{i\nu_{k+1}(\theta+t)} g_k(e^{i(\theta+t)}) \right.$$

$$\left. - e^{i\nu_{k+1}\theta} g_k(e^{i\theta}) \}| \right]$$

$$+ \left[\sum_{n=n_o+1}^{\infty} \varepsilon_n \; | \sum_{k=\lambda_n}^{\mu_n} \eta_{k+1} \{e^{i\nu_{k+1}(\theta+t)} g_k(e^{i(\theta+t)}) \right.$$

$$\left. - e^{i\nu_{k+1}\theta} g_k(e^{i\theta}) \}| \right]$$

(where n_o is a positive integer, to be specified later.)

$$= \Sigma_1 + \Sigma_2 \quad \text{(say)}.$$

Now from (4.29),

$$\left|\sum_{k=\lambda_n}^{\mu_n} \eta_{k+1} \; z^{\nu_{k+1}} \; g_k(z)\right| \leqslant 2K \qquad (|z| \leq 1)$$

for all λ_n, μ_n, and for all n, so that

$$|\Sigma_2| \leq \sum_{n=n_o+1}^{\infty} \varepsilon_n \, 4K$$

$$= 4K \sum_{n=n_o+1}^{\infty} \varepsilon_n \quad .$$

Now

$$\Sigma_1 = \sum_{n=1}^{n_o} \varepsilon_n \left| \sum_{k=\lambda_n}^{\mu_n} \eta_{k+1} \{ e^{i\nu_{k+1}(\theta+t)} \; g_k(e^{i(\theta+t)}) \right.$$

$$- e^{i\nu_{k+1}\theta} \; g_k(e^{i(\theta+t)}) + e^{i\nu_{k+1}\theta} \; g_k(e^{i(\theta+t)})$$

$$\left. - e^{i\nu_{k+1}\theta} \; g_k(e^{i\theta})\}\right|$$

so that

$$\Sigma_1 \leq \sum_{n=1}^{n_o} \varepsilon_n \left| \sum_{k=\lambda_n}^{\mu_n} \eta_{k+1} \{e^{i\nu_{k+1}(\theta+t)} - e^{i\nu_{k+1}\theta}\} g_k(e^{i(\theta+t)}) \right|$$

$$+ \sum_{n=1}^{n_o} \varepsilon_n \left| \sum_{k=\lambda_n}^{\mu_n} \eta_{k+1} \; e^{i\nu_{k+1}\theta} \; (g_k(e^{i(\theta+t)}) - g_k(e^{i\theta}))\right|$$

$$\leq \sum_{n=1}^{n_o} \varepsilon_n \sum_{k=\lambda_n}^{\mu_n} \eta_{k+1} \; M \, |t| \, \nu_{k+1} \quad .$$

$$+ \sum_{n=1}^{n_o} \varepsilon_n \sum_{k=\lambda_n}^{\mu_n} \eta_{k+1} \; | \, g_k(e^{i(\theta+t)}) - g_k(e^{i\theta})| \, ,$$

since for all real t $|e^{it}-1| \leq |t|$.

Since $g_k(z)$ is a polynomial of degree $(\nu_1 + \nu_2 + \dots \nu_k)$, bounded by M in $(|z| \leq 1)$, by Bernstein's Theorem we have

$$|g_k'(z)| \leq M(\nu_1 + \nu_2 + \dots \nu_k) \qquad (|z| \leq 1).$$

Again $g_k(z)$ being analytic in $|z| \leq 1$, we have

$$|g_k(e^{i(\Theta+t)}) - g_k(e^{i\Theta})| = \left| \int_{e^{i\Theta}}^{e^{i(\Theta+t)}} g_k'(e^{i\phi})\, d\phi \right|$$

so that

$$|g_k(e^{i(\Theta+t)}) - g_k(e^{i\Theta})|$$

$$\leq M\,|t|\,(\nu_1 + \nu_2 + \dots \nu_k)$$

$$< M\,|t|\,\nu_{k+1} \qquad \text{(from (4.4))}.$$

Therefore

$$\Sigma_1 \leq 2M\,|t| \sum_{n=1}^{n_o} \epsilon_n \sum_{k=\lambda_n}^{\mu_n} \eta_{k+1}\, \nu_{k+1}$$

and so we obtain for small 't'

$$\Sigma_1 + \Sigma_2 \lessdot$$

$$2M\,|t| \sum_{n=1}^{n_o} \epsilon_n \sum_{k=\lambda_n}^{\mu_n} \eta_{k+1}\, \nu_{k+1} + 4K \sum_{n=n_o+1}^{\infty} \epsilon_n .$$

Now we substitute the values of n_{k+1}, ν_{k+1}, λ_n, μ_n and ε_n from (i), (ii), (iii) and (iv) and deduce that

$$|h(e^{i(\theta+t)}) - h(e^{i\theta})|$$

$$\leq 2M\,|t| \sum_{n=1}^{n_o} \frac{1}{2^{n\gamma}} \sum_{2^n}^{2^{n+1}-1} \frac{1}{(k+1)^{\alpha}} (k+1)^{k+1} + 4K \sum_{n_o+1}^{\infty} \frac{1}{2^{n\gamma}},$$

so that for $\tau > o$,

$$\omega(\tau) = \sup_{o \leq |t| \leq \tau} |h(e^{i(\theta+t)}) - h(e^{i\theta})|$$

$$\leq 2M\tau \sum_{n=1}^{n_o} \frac{1}{2^{n\gamma}} \sum_{2^n}^{2^{n+1}-1} (k+1)^{(k+1)-\alpha} + 4K' \frac{1}{2^{n_o\gamma}}$$

$$< 2M\tau \sum_{n=1}^{n_o} \frac{1}{2^{n\gamma}} 2^n (2^{n+1})^{(2^{n+1}-\alpha)} + 4K' \frac{1}{2^{n_o\gamma}}$$

$$= 2M\tau \sum_{n=1}^{n_o} \frac{1}{2^{n\gamma}} 2^n (2^{n+1})^{2^{n+1}} 2^{(n+1)(-\alpha)} + 4K' \frac{1}{2^{n_o\gamma}}$$

$$= 2M\tau\, 2^{-\alpha} \sum_{n=1}^{n_o} 2^{n(1-\alpha-\gamma)} (2^{n+1})^{2^{n+1}} + 4K' \frac{1}{2^{n_o\gamma}}$$

$$< 2M\tau\, 2^{-\alpha}\, n_o\, 2^{n_o(1-\alpha-\gamma)} (2^{n_o+1})^{2^{n_o+1}} + 4K' \left(\frac{1}{2^{n_o\gamma}}\right)$$

$$= 2M\, 2^{-\alpha} \left(\tau n_o\, 2^{n_o(1-\alpha-\gamma)} (2^{n_o+1})^{2^{n_o+1}} + K'' \frac{1}{2^{n_o\gamma}}\right).$$

Given τ, we want to choose n_o so that

$$\tau n_o\, 2^{n_o(1-\alpha-\gamma)} (2^{n_o+1})^{2^{n_o+1}} + K'' \frac{1}{2^{n_o\gamma}}$$

is sufficiently 'small'. In order to show that n_o can be so chosen and so complete the proof of the theorem, we require the next lemma.

Lemma 4.3 In the above notation, let $\zeta_o > o$ be the smallest $\zeta > o$ such that

$$\tau\, \zeta\, 2^{\zeta(1-\alpha-\gamma)} \, (2^{\zeta+1})^{2^{\zeta+1}} = \frac{1}{2^{\zeta\gamma}} .$$

Then if $n_o = [\zeta_o]$ and $o < \gamma' < \gamma$, we have

$$\tau\, n_o\, 2^{n_o(1-\alpha-\gamma)} \, (2^{n_o+1})^{2^{n_o+1}}$$

$$= O\{(\log \tfrac{1}{\tau})^{-\gamma'}\} \qquad (\tau \to 0+) .$$

Proof: By hypothesis of the lemma

$$\tau\, \zeta_o 2^{\zeta_o(1-\alpha-\gamma)} \, (2^{\zeta_o+1})^{2^{\zeta_o+1}} = \frac{1}{2^{\zeta_o\gamma}}$$

which gives

$$\tau\, .\, \zeta_o\, (2^{\zeta_o+1})^{2^{\zeta_o+1}} = \frac{1}{2^{\zeta_o(1-\alpha)}} .$$

Taking logarithm of both sides we get

$$\log \tau + \log \zeta_o + 2^{\zeta_o+1} \log 2^{\zeta_o+1} = -(1-\alpha)\, \zeta_o \log 2,$$

so that

$$\log \frac{1}{\tau} = \log\zeta_o + (1-\alpha)\, \zeta_o \log 2 + (\zeta_o+1) 2^{\zeta_o+1} \log 2$$

and therefore, since $\zeta_o \to \infty$ as $\tau \to o+$,

$$\log \frac{1}{\tau} \sim 2^{\zeta_o+1} \quad (\zeta_o+1) \log 2 \qquad (\tau \to 0+)$$

so that

$$\log \log \frac{1}{\tau} \sim \zeta_o \log 2 \qquad (\tau \to 0+).$$

Therefore for this value of ζ

$$(4.31) \quad \tau\zeta\, 2^{\zeta(1-\alpha-\gamma)} (2^{\zeta+1})^{2^{\zeta+1}} + \frac{K''}{2^{\zeta\gamma}} \leq K 2^{-\zeta_o\gamma} \leq K' \frac{1}{(\log \frac{1}{\tau})^{\gamma'}}$$

where K is a constant.

Suppose that

$$n_o = [\zeta_o].$$

If ζ_o is an integer, n_o satisfies (4.31), and if ζ_o is not an integer, we have

$$n_o < \zeta_o < n_o+1,$$

so that

$$n_o = \zeta_o - \delta \qquad (o<\delta<1)$$

and we have

$$\tau n_o\, 2^{n_o(1-\alpha-\gamma)} \quad (2^{n_o+1})^{2^{n_o+1}}$$

$$= \tau\ (\zeta_o-\delta)2^{(\zeta_o-\delta)(1-\alpha-\gamma)}\ 2^{(\zeta_o-\delta+1)2^{\zeta_o-\delta+1}}$$

$$=\tau\ \{\zeta_o 2^{\zeta_o(1-\alpha-\gamma)}\ 2^{(\zeta_o+1)\ 2^{\zeta_o+1-\delta}}\ 2^{-\delta(1-\alpha-\gamma)-\delta 2^{\zeta_o+\delta+1}}$$

$$-\ \ \delta\ 2^{(\zeta_o-\delta)(1-\alpha-\gamma)}\ 2^{(\zeta_o-\delta+1)2^{\zeta_o-\delta+1}}\}$$

$$=\ \tau\ \{\zeta_o\ 2^{\zeta_o(1-\alpha-\gamma)}\ 2^{(\zeta_o+1)2^{\zeta_o+1}(-\delta)}\ 2\ 2^{-\delta(1-\alpha-\gamma+2^{\zeta_o-\delta+1})}$$

$$-\ \delta\ \ 2^{(\zeta_o-\delta)(1-\alpha-\gamma)}\ \ 2^{(\zeta_o-\delta+1)\ 2^{\zeta_o-\delta+1}}\ \ \}$$

$$\leq\ \tau\ \ \zeta_o\ \ \ 2^{\zeta_o(1-\alpha-\gamma)}\ \ 2^{(\zeta_o+1)2^{\zeta_o+1}} \qquad \text{(since }\ \zeta_o>\delta>o)\ .$$

Also

$$\frac{1}{2^{n_o\gamma}} = \frac{1}{2^{(\zeta_o-\delta)\gamma}} = \frac{1}{2^{\zeta_o\gamma}}\ \ \frac{1}{2^{-\delta\gamma}}\ \ \frac{2}{2}\ ,$$

so that

$$\frac{1}{2^{n_o\gamma}} \leq \frac{2}{2^{\zeta_o\gamma}}\ . \qquad (1-\delta\gamma>o)\ .$$

Hence if $n_o = [\zeta_o]$

$$\tau n_o 2^{n_o(1-\alpha-\gamma)}\ (2^{n_o+1})^{2^{n_o+1}} + \frac{K''}{2^{n}\gamma} = O\ \{\frac{1}{(\log\frac{1}{\tau})^{\gamma'}}\}$$

and the lemma is proved.

From Lemma 4.3 it follows that if β is any positive number such that $o<\beta<\gamma<\frac{1}{2}$

then

$$\tau\, n_o\, 2^{n_o(1-\alpha-\gamma)}\, (2^{n_o+1}\cdot) + K'' \frac{1}{2^{n_o\gamma}}$$

$$= O\left(\frac{1}{\log \frac{1}{\tau}}\right)^{\beta} .$$

Therefore in order to obtain the result of the theorem, for a given $\beta<\frac{1}{2}$ we choose γ_o so that $\beta<\gamma_o<\frac{1}{2}$ and then choose α_o so that $\frac{1}{2}<\alpha_o<1$ and $\alpha_o + \gamma_o<1$. Then the function $h(z)$ given by (4.30) with $\alpha = \alpha_o$ and $\gamma = \gamma_o$ would satisfy the conditions of Theorem 4.3, where we now take $\gamma' = \beta$.

Theorem 4.4: Let us consider a function $\psi(z)$ analytic in $|z|<1$ and continuous in $|z|\leq 1$. Let

$$\psi(z) = \sum_{n=o}^{\infty} A_n z^n \qquad (|z|<1).$$

Suppose that

$$\omega(\tau) = O\left(\frac{1}{\log \frac{1}{\tau}}\right)^{1+\varepsilon} \qquad (\varepsilon>o),$$

where $\omega(\tau)$ is the modulus of continuity of $\psi(e^{i\Theta})$. Then

$$\sum_{o}^{\infty} A_n e^{in\Theta} \text{ is summable } |A| \text{ for all } \Theta\ (o\leq\Theta\leq 2\pi).$$

Proof: Given Θ $(o\leq\Theta\leq 2\pi)$ define for real t,

$$\phi_\Theta(t) = \left\{\frac{\psi(e^{i(\Theta+2t)}) + \psi(e^{i(\Theta-2t)}) - 2\psi(e^{i\Theta})}{2}\right\}.$$

Then

$$|\phi_\Theta(t)| \leq \left|\frac{\psi(e^{i(\Theta-2t)}) - \psi(e^{i\Theta})}{2}\right|$$

$$+ \left|\frac{\psi(e^{i(\Theta-2t)}) - \psi(e^{i\Theta})}{2}\right|$$

$$\leq K \left(\log \frac{1}{|t|}\right)^{-(1+\varepsilon)} \qquad \text{(by hypothesis)}$$

where K is a constant.

Therefore, if $\delta > o$,

$$\int_o^\delta \left|\frac{\phi_\Theta(t)}{t}\right| dt \leq K \int_o^\delta \left(\log \frac{1}{|t|}\right)^{-(1+\varepsilon)} dt$$

$$= K \left(\log \frac{1}{\delta}\right)^{-\varepsilon} < \infty$$

which shows that $\psi(e^{i\Theta})$ satisfies Dini's condition, and therefore from Whittaker's Theoreml.1 quoted in Chapter 1 the result of the theroem follows.

On account of the significance of the above result we give below the outline of the Whittaker's proof of Theroem 1.1, which states that if

$$\tfrac{1}{2} a_o + \sum_{n=1}^{\infty} (a_n \cos n\Theta + b_n \sin n\Theta)$$

is the Fourier series of a function $f(\Theta) \varepsilon L^1(-\pi,\pi)$, then the above series is summable $|A|$ with absolute Abel sum ℓ if

$$\int_0^\delta \left|\frac{\phi(t)}{t}\right| dt$$

exists for some $\delta > o$, where

$$\phi(t) = \frac{f(\theta+2t) + f(\theta-2t) - 2\ell}{2}.$$

Proof of Theorem 1.1 Let for $o \le x < 1$,

$$P(x) = \tfrac{1}{2} a_o + \sum_{n=1}^{\infty} x^n (a_n \cos n\theta + b_n \sin n\theta)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\alpha) \frac{1 - x^2}{1-2x \cos(\theta-\alpha)+x^2} d\alpha,$$

so that $P(x)$ is convergent for $o \le x < 1$.

Writing $\alpha = \theta + 2t$

$$Q(x) = P(x) - \ell = \frac{2}{\pi} \int_0^{\pi/2} \phi(t) \frac{1-x^2}{1-2x \cos 2t + x^2} dt.$$

The total variation of $Q(x)$ in $(0, x_1)$ is

$$\int_0^{x_1} |Q'(x)| dx = \frac{2}{\pi} \int_0^{x_1} \left| \int_0^{\pi/2} \phi(t) \frac{d}{dx} \left\{\frac{1-x^2}{1-2x\cos 2t + x^2}\right\} dt \right| dx$$

$$\le \frac{4}{\pi} \int_0^{\pi/2} |\phi(t)| \int_0^{x_1} \left| \frac{(1+x^2) \cos 2t - 2x}{(1-2x \cos 2t + x^2)^2} \right| dx$$

(Inverting the order of integration).

We Write

$$V(x_1, t) = 2\int_0^{x_1} \left| \frac{(1+x^2) \cos 2t - 2x}{(1-2x \cos 2t + x^2)^2} \right| dx$$

$$t_1 = \tfrac{1}{2} \operatorname{arc\,cos} \frac{2x_1}{1+x_1^2} = \operatorname{arc\,cot} x_1 - \frac{\pi}{4} \quad (o < t_1 < \tfrac{\pi}{4}).$$

Then if

$0 \leq t \leq t_1$ and $p(x) = \dfrac{1-x^2}{1-2x\cos 2t+x^2}$,

$$V(x_1,t) = 2\int_0^{x_1} \frac{(1+x^2)\cos 2t - 2x}{(1-2x\cos 2t + x^2)^2}\, dx$$

$$= \int_0^{x_1} p'(x)\, dx,$$

$$= p(x_1) - 1 < p(x_1)$$

$$\leq \frac{\sin 2t_1}{1-\cos 2t_1} = \cot t_1 < \frac{\pi}{2t_1} \leq \frac{\pi}{2t},$$

while if $t_1 \leq t \leq \pi/4$,

$$V(x_1,t) = \int_0^{\cot(t+\frac{\pi}{4})} p'(x)\, dx - \int_{\cot(t+\frac{\pi}{4})}^{x_1} p'(x)\, dx$$

$$= \frac{2}{\sin 2t} - 1 - p(x_1) < \frac{2}{\sin 2t} \leq \frac{\pi}{2t}.$$

Finally, if $\frac{\pi}{4} \leq t \leq \pi/2$

$$V(x,t) = -\int_0^{x_1} p'(x)\, dx < 1.$$

Thus

$$\int_0^{x_1} |Q'(x)|dx \leq \frac{2}{\pi} \left(\int_0^{t_1} + \int_{t_1}^{\pi/4} + \int_{\pi/4}^{\pi/2}\right) |\phi(t)|V(x_1,t)\, dt$$

$$< \int_0^{\pi/4} \left|\frac{\phi(t)}{t}\right| dt + \frac{2}{\pi} \int_{\pi/4}^{\pi/2} |\phi(t)|\, dt .$$

By hypothesis of Theorem 1.1,

$$\int_{0}^{\pi/4} \left|\frac{\phi(t)}{t}\right| dt < \infty$$

and therefore $Q(x)$ is of bounded variation in $[0,1)$.

Thus

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\,\theta)$$

is summable $|A|$ with absolute Abel sum ℓ.

This completes the proof of Theorem 1.1.

CHAPTER 5

Absolute Abel summability in a general setting

Let

$$f(z) = \Sigma a_n z^n \qquad (|z|<1)$$

be of bounded characteristic in $|z|<1$, so that $\lim_{r\to 1-} f(re^{i\theta})$ exists finitely for almost all values of θ, which implies that $\Sigma a_n e^{in\theta}$ is summable A for almost all values of θ. But in this case $\Sigma a_n e^{in\theta}$ may not be summable $|A|$ for any θ. To prove this we consider

$$(5.1) \qquad f(z) = \sum_{n=1}^{\infty} \frac{z^{2^n}}{n} \qquad (|z|<1).$$

Since $\Sigma \frac{1}{n^2} < \infty$ therefore $f(z) \in H^2$ $(|z|<1)$ and so $\lim_{r\to 1-} f(re^{i\theta})$ exists finitely a.e. i.e. $\sum_{n=1}^{\infty} \frac{1}{n}(e^{i\theta})^{2^n}$ is summable A a.e.. Also the series $\sum_{n=1}^{\infty} \frac{z^{2^n}}{n}$ has Hadamard gaps and therefore by the 'high indices' theorem of Hardy and Littlewood quoted in Chapter 1, $\Sigma \frac{1}{n} (e^{i\theta})^{2^n}$ is convergent for all values of θ, for which it is summable A. Therefore $\sum_{n=1}^{\infty} \frac{1}{n} (e^{i\theta})^{2^n}$ is convergent for almost all values of θ.

Again, since $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$, it follows by Zygmund's Theorem 1.5, quoted in chapter 1, that $\sum_{n=1}^{\infty} \frac{1}{n} (e^{i\theta})^{2^n}$ being lacunary, is not summable $|A|$ for any θ.

Thus the series $\sum_{n=1}^{\infty} \frac{(e^{i\theta})^{2^n}}{n}$ converges a.e. but is not summable $|A|$ for any θ.

Prasad in Theorem 1.4 quoted in Chapter 1 considered the function $f(\theta) \in L^1(o,2\pi)$ such that

$$f(\theta) \sim \tfrac{1}{2} a_o + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)$$

where

$$a_n = \frac{1}{2\pi} \int_o^{2\pi} \cos n\theta \, f(\theta) \, d\theta$$

$$b_n = \frac{1}{2\pi} \int_o^{2\pi} \sin n\theta \, f(\theta) \, d\theta$$

and proved that the series

$$\sum_{n=3}^{\infty} \frac{a_n \cos n\theta + b_n \sin n\theta}{\log n \, (\log\log n)^{\gamma}} \qquad (\gamma > 1)$$

(which is a Fourier series which converges almost everywhere) is summable $|A|$ for almost all values of θ.

We shall consider similar results for functions $f(z)$ which are analytic in $|z| < 1$ and of bounded characteristic. The next theorem is weaker than what we shall eventually prove, and the method of proof is to a considerable extent due to Prasad [(12) p. 416)].

Theorem 5.1. Let

$$f(z) = \sum_{n=3}^{\infty} a_n z^n \qquad (|z| < 1)$$

be analytic in $|z|<1$ and of bounded characteristic. If we define

$$(5.2) \qquad F_\gamma(z) = \sum_{n=3}^{\infty} \frac{a_n z^n}{\log n\,(\log\log n)^\gamma} \qquad (\gamma>1),$$

then

$$\sum_{n=3}^{\infty} \frac{a_n e^{in\Theta}}{\log n\,(\log\log n)^\gamma}$$

is summable $|A|$ a.e. .

In order to prove the above theorem, we need the following lemma.

Lemma 5.1. Let $f(z)$ be analytic in $|z|<1$, and of bounded characteristic. Then for almost all values of Θ,

$$\int_0^r |f'(\rho e^{i\Theta})|\,d\rho < K(\Theta)\log\left(\frac{1}{1-r}\right) \qquad (o<r<1).$$

Proof: Since $f(z)$ is analytic in $|z|<1$, and of bounded characteristic, therefore for almost all values of Θ,

$$(5.3) \qquad \lim_{r\to 1-} f(re^{i\Theta}) = f(e^{i\Theta}) \quad .$$

Further, for any $\varepsilon>o$, $f(z)$ tends uniformly to $f(e^{i\Theta})$ as $z\to e^{i\Theta}$ inside an angular domain of opening $\pi-\varepsilon$ having vertex at $e^{i\Theta}$ and bisected by the radius drawn to $e^{i\Theta}$. Let this angular domain be denoted by Λ. Then given a positive number δ, there exists a K_1, such that if $\zeta\varepsilon\Lambda\cap\{|\zeta-e^{i\Theta}|\le\delta\}$,

$$|f(\zeta)| < K_1 .$$

Let ρ be a positive number such that $1-\frac{\delta}{2}<\rho<1$, and c be a circle of radius $\frac{1-\rho}{2}$ with centre at $\rho e^{i\theta}$, so that if $0<\varepsilon<\frac{\pi}{2}$ $c \subset \Lambda \cap \{|\zeta-e^{i\theta}| \leq \delta\}$.

By Cauchy's integral formula we have

$$f'(\rho e^{i\theta}) = \frac{1}{2\pi i} \int_c \frac{f(\zeta)}{(\zeta-\rho e^{i\theta})^2} \, d\zeta ,$$

so that for a 'θ' satisfying (5.3), we have

$$|f'(\rho e^{i\theta})| \leq 2K_1/_{1-\rho} \qquad (1-\tfrac{\delta}{2}<\rho<1)$$

which gives

$$|f'(\rho e^{i\theta})| = O(\tfrac{1}{1-\rho}) \qquad (\rho\to 1-)$$

and therefore

$$\int_0^r |f'(\rho e^{i\theta})| d\rho = O(\log \tfrac{1}{1-r}) \qquad (r\to 1-) .$$

Hence

$$\int_0^r |f'(\rho e^{i\theta})| \, d\rho \leq K(\theta) \log \frac{1}{1-r} .$$

Thus for a value of θ for which (5.3) holds, we have

$$\int_0^r |f'(\rho e^{i\theta})| \, d\rho \leq K(\theta) \log \frac{1}{1-r} ,$$

and so we have

$$\int_0^r |f'(\rho e^{i\theta})|\, d\rho \leq K(\theta) \log \frac{1}{1-r} \qquad \text{a.e.}$$

This proves the Lemma.

Proof of Theorem 5.1 .

We know that for $\alpha > o$

(5.4) $$\int_0^\infty e^{-nt}\, t^{\alpha-1}\, dt = \frac{\Gamma(\alpha)}{n^\alpha} .$$

If we put $e^{-t} = u$, we shall have

(5.5) $$\int_0^1 u^{n-1} (\log \frac{1}{u})^{\alpha-1} du = \frac{\Gamma(\alpha)}{n^\alpha} .$$

Again from (5.2) for $o<\rho<1$

(5.6) $$F_\gamma(\rho e^{i\theta}) = \sum_{n=3}^\infty \left(\frac{a_n e^{in\theta} \rho^n}{\log n\, (\log \log n)^\gamma}\right) .$$

Consider for $o<\rho<1$

(5.7) $$\int_0^1 f(\rho u e^{i\theta}) (\log \frac{1}{u})^{\alpha-1} \frac{du}{u}$$

$$= \int_0^1 \left(\sum_{n=3}^\infty a_n e^{in\theta} \rho^n u^n\right) (\log \frac{1}{u})^{\alpha-1} \frac{du}{u}$$

$$= \sum_{n=3}^\infty a_n e^{in\theta} \rho^n \frac{\Gamma(\alpha)}{n^\alpha} \qquad \text{(by using (5.5)).}$$

Again, for $\beta > -1$,

$$\int_0^\infty \frac{d\alpha\, \alpha^\beta}{\Gamma(\alpha)} \int_0^1 u^{n-1} (\log \frac{1}{u})^{\alpha-1} du$$

$$= \int_0^\infty \frac{d\alpha\ \alpha^\beta}{\Gamma(\alpha)} \quad \frac{\Gamma(\alpha)}{n^\alpha}$$

$$= \int_0^\infty e^{-\alpha \log n} \quad \alpha^\beta \quad d\alpha$$

$$= \Gamma(\beta+1) \quad \frac{1}{(\log n)^{\beta+1}} \qquad \text{(by using (5.4))}.$$

Similarly, for $\gamma > 0$,

$$(5.8) \quad \int_0^\infty \frac{d\beta\ \beta^{\gamma-1}}{\Gamma(\beta+1)} \int_0^\infty \frac{d\alpha\ \alpha^\beta}{\Gamma(\alpha)} \int_0^1 u^{n-1} \quad (\log \frac{1}{u})^{\alpha-1}\ du$$

$$= \int_0^\infty \frac{1}{(\log n)^{\beta+1}} \quad \beta^{\gamma-1} \quad d\beta$$

$$= \int_0^\infty e^{-(\beta+1)\log\log n} \quad \beta^{\gamma-1} \quad d\beta$$

$$= \frac{\Gamma(\gamma)}{\log n\ (\log\log n)^\gamma} .$$

Therefore from (5.6), (5.7) and (5.8) we get

$$F_\gamma(\rho e^{i\theta}) = \frac{1}{\Gamma(\gamma)} \int_0^\infty \frac{d\beta\ \beta^{\gamma-1}}{\Gamma(\beta+1)} \int_0^\infty \frac{d\alpha\ \alpha^\beta}{\Gamma(\alpha)} \int_0^1 f\ (\rho u e^{i\theta})\ (\log \frac{1}{u})^{\alpha-1} \frac{du}{u}$$

and therefore

$$(5.9) \quad \int_0^r |F'_\gamma(\rho e^{i\theta})|\,d\rho$$

$$\leq \frac{1}{\Gamma(\gamma)} \int_0^\infty \frac{d\beta\ \beta^{\gamma-1}}{\Gamma(\beta+1)} \int_0^\infty \frac{d\alpha\ \alpha^\beta}{\Gamma(\alpha)} \int_0^1 \int_0^r |f'(\rho u e^{i\theta})|(\log \frac{1}{u})^{\alpha-1}\ du\,d\rho$$

$$= \frac{1}{\Gamma(\gamma)} \int_0^\infty \frac{d\beta\ \beta^{\gamma-1}}{\Gamma(\beta+1)} \int_0^\infty \frac{d\alpha\ \alpha^\beta}{\Gamma(\alpha)} \int_0^1 \frac{du(\log \frac{1}{u})^{\alpha-1}}{u} \int_0^r |f'(\rho u e^{i\Theta})| u d\rho \quad .$$

Now from the hypothesis of the theorem

$$|f'(\varepsilon e^{i\Theta})| = O(\varepsilon^2) \qquad (\varepsilon \to o_+)$$

which gives that

$$\int_0^\varepsilon |f'(\rho e^{i\Theta})|\ d\rho = O\ (\varepsilon^3) \qquad (\varepsilon \to o_+) \quad ,$$

so that by Lemma 5.1, since $f(z)$ is of bounded characteristic, we have

$$\int_0^r |f'(\rho e^{i\Theta})| d\rho \leq K(\Theta)\ (\log \frac{1}{1-r} - r - \frac{r^2}{2}) \qquad \text{a.e.} \quad ,$$

so that

$$\int_0^r |f'(\rho u e^{i\Theta})| u d\rho \leq K(\Theta)\ (\log \frac{1}{1-ru} - ru - \frac{r^2u^2}{2}) \qquad \text{a.e.}$$

$$= K(\Theta) \sum_{n=3}^\infty \frac{r^n u^n}{n} \qquad \text{a.e.} .$$

Hence for almost all values of Θ,

$$\int_0^r |F'_\gamma(\rho e^{i\Theta})| d\rho$$

$$\leq \frac{K(\Theta)}{\Gamma(\gamma)} \int_0^\infty \frac{d\beta\ \beta^{\gamma-1}}{\Gamma(\beta+1)} \int_0^\infty \frac{d\alpha\alpha^\beta}{\Gamma(\alpha)} \int_0^1 \frac{du(\log \frac{1}{u})^{\alpha-1}}{u} \sum_{n=3}^\infty \frac{r^n u^n}{n}$$

$$= K(\Theta) \sum_{n=3}^\infty \frac{r^n}{n} \frac{1}{\Gamma(\gamma)} \int_0^\infty \frac{d\beta\ \beta^{\gamma-1}}{\Gamma(\beta+1)} \int_0^\infty \frac{d\alpha\ \alpha^\beta}{\Gamma(\alpha)} \int_0^1 u^{n-1}(\log \frac{1}{u})^{\alpha-1}\ du$$

$$= K(\Theta) \sum_{n=3}^{\infty} \frac{r^n}{n} \frac{1}{\log n} \frac{1}{(\log\log n)^{\gamma}} .$$

Consequently, for $\gamma>1$,

$$\int_0^1 |F'_{\gamma}(\rho e^{i\Theta})|d\rho < \infty \qquad \text{a.e.},$$

since for $\gamma>1$

$$\sum_{n=3}^{\infty} \frac{1}{n} \frac{1}{\log n} \frac{1}{(\log\log n)^{\gamma}} < \infty .$$

Therefore

$$\sum_{n=3}^{\infty} \frac{a_n e^{in\Theta}}{\log n\,(\log\log n)^{\gamma}}$$

is summable $|A|$ for almost all values of Θ.

This completes the proof of the theorem.

Clearly the result of the above theorem depends on the estimate of $\int_0^r |f'(\rho u e^{i\Theta})|u d\rho$ on the right hand side of (5.9) given by Lemma 5.1. In the following theorem we shall obtain a result stronger than that of Theroem 5.1 by using Zygmund's Theorem 1.6 quoted in chapter 1, instead of Lemma 5.1 to get the estimate of $\int_0^r |f'(\rho u e^{i\Theta})|u d\rho$.

Theorem 5.2. Let $f(z) = \sum_{n=3}^{\infty} a_n z^n$ $(|z|<1)$ be analytic in $|z|<1$ and of bounded characteristic (so that $\lim_{r\to 1-} f(re^{i\Theta}) = f(e^{i\Theta})$ a.e.) then

$$\sum_{n=3}^{\infty} \frac{a_n e^{in\Theta}}{\sqrt{\log n}\,(\log\log n)^{\gamma}} \qquad (\gamma>1)$$

is summable $|A|$ for almost all values of Θ $(0\leq\Theta\leq 2\pi)$.

<u>Proof:</u> Let for $\gamma>1$,

$$F_\gamma(z) = \sum_{n=3}^{\infty} \frac{a_n z^n}{\sqrt{(\log n)}(\log\log n)^\gamma} \qquad (|z|<1).$$

By using arguments similar to those used in Theorem 5.1 to obtain (5.9) we shall have

$$\int_0^r |F'_\gamma(\rho e^{i\theta})|\, d\rho$$

$$\leq \frac{1}{\Gamma(\gamma)} \int_0^\infty \frac{d\beta\ \beta^{\gamma-1}}{\Gamma(\beta+\frac{1}{2})} \int_0^\infty \frac{d\alpha\ \alpha^{\beta-\frac{1}{2}}}{\Gamma(\alpha)} \int_0^1 (\log \frac{1}{u})^{\alpha-1} du \int_0^r |f'(\rho u e^{i\theta})|\, d\rho .$$

From Theorem 1.6 we have for almost all θ,

$$\int_0^r |f'(\rho e^{i\theta})| d\rho = o((\log \frac{1}{1-r})^{\frac{1}{2}}) \qquad (r\to 1-) .$$

In what follows we shall consider only those θ for which the above is true.

Now

$$f(\rho e^{i\theta}) = O(\rho^3) \qquad (\rho\to o_+)$$

so that

$$f'(\rho e^{i\theta}) = O(\rho^2) \qquad (\rho\to o_+)$$

and therefore

$$\int_0^r |f'(\rho e^{i\theta})|\, d\rho = O(r^3) \qquad (r\to o_+) .$$

Hence for $o \leq r < 1$,

$$\int_o^r |f'(\rho e^{i\Theta})| d\rho \leq K(\Theta) \left(\log \frac{1}{1-r} - r - \frac{r^2}{2} - \frac{r^3}{3} - \frac{r^4}{4} - \frac{r^5}{5}\right)^{\frac{1}{2}}.$$

Thus

$$(5.10) \quad \int_o^r |F'_\gamma (\rho e^{i\Theta})| d\rho$$

$$\leq \frac{1}{\Gamma(\gamma)} \int_o^\infty \frac{d\beta \ \beta^{\gamma-1}}{\Gamma(\beta+\frac{1}{2})} \int_o^\infty \frac{d\alpha \ \alpha^{\beta-\frac{1}{2}}}{\Gamma(\alpha)} \int_o^1 (\log \frac{1}{u})^{\alpha-1} \frac{du}{u} \quad x$$

$$K \left\{\log \left(\frac{1}{1-ur}\right) - ur - \frac{u^2 r^2}{2} - \frac{u^3 r^3}{3} - \frac{u^4 r^4}{4} - \frac{u^5 r^5}{5}\right\}^{\frac{1}{2}}.$$

To complete the proof of the theorem we need the next lemma.

Lemma 5.2. Let

$$\phi(z) = \frac{z^6}{6} + \frac{z^7}{7} + \ldots \qquad (|z| < 1)$$

and

$$\psi(z) = \sum_{n=3}^{\infty} \frac{1}{n\sqrt{\log n}} z^n .$$

Then for $o < r < 1$

$$\phi(r) \leq A^2 \{\psi(r)\}^2,$$

where A is a constant.

<u>Proof:</u> Let

$$\{\psi(r)\}^2 = \sum_{n=6}^{\infty} A_n r^n \qquad (0<r<1),$$

so that for $n \geqslant 6$,

$$A_n = \sum_{\substack{j+k=n \\ j,k\geq 3}} \frac{1}{j(\log j)^{\frac{1}{2}}} \cdot \frac{1}{k(\log k)^{\frac{1}{2}}}$$

$$= \frac{1}{3}\cdot\frac{1}{\sqrt{\log 3}} \quad \frac{1}{(n-3)\sqrt{\log(n-3)}} + \ldots \frac{1}{j\sqrt{\log j}} \quad \frac{1}{(n-j)\sqrt{\log(n-j}}$$

$$+ \ldots\ldots\ldots$$

Considering the sum of terms for which $j<\frac{n}{6}$, we get

$$A_n \geqslant \frac{1}{3\sqrt{\log 3}} \cdot \frac{1}{(n-3)\sqrt{\log(n-3)}} + \ldots\ldots$$

$$\ldots + \frac{1}{\frac{n}{6}\sqrt{\log\frac{n}{6}}} \quad \frac{1}{\frac{5n}{6}\sqrt{\log\frac{5n}{6}}}$$

$$\geq \frac{1}{n\sqrt{\log n}} \left(\frac{1}{3\sqrt{\log 3}} + \frac{1}{4\sqrt{\log 4}} + \ldots\ldots \frac{1}{\frac{n}{6}\sqrt{\log(\frac{n}{6})}}\right)$$

$$\geq \frac{1}{n\sqrt{\log n}} \quad \frac{1}{\sqrt{\log n}} \left(\frac{1}{3} + \frac{1}{4} + \ldots\ldots\ldots \frac{6}{n}\right)$$

$$\sim \frac{1}{n \quad \log n} \log n \qquad (n\to\infty)$$

which shows that for $n \geqslant 6$,

$$\frac{1}{n} \leq K_1 \quad A_n,$$

where K_1 is a constant.

Therefore for $o<r<1$ we have

$$\sum_{6}^{\infty} \frac{1}{n} r^n \leq K_1 (\sum_{6}^{\infty} A_n r^n) .$$

Writing $K_1 = A^2$ we get the result of the lemma.

Proof of Theorem 5.2.

By using the above lemma, (5.10) gives

$$\int_o^r |F'_\gamma(\rho e^{i\theta})| d\rho \leq \frac{K'}{\Gamma(\gamma)} \int_o^\infty \frac{d\beta \ \beta^{\gamma-1}}{\Gamma(\beta+\frac{1}{2})} \int_o^\infty \frac{d\alpha \ \alpha^{\beta-\frac{1}{2}}}{\Gamma(\alpha)} \times$$

$$\int_o^1 \sum_{n=3}^{\infty} \frac{1}{n(\log n)^{\frac{1}{2}}} u^{n-1} (\log \frac{1}{u})^{\alpha-1} du$$

(where K' is a constant).

$$= \frac{K'}{\Gamma(\gamma)} \sum_{n=3}^{\infty} \frac{r^n}{n(\log n)^{\frac{1}{2}}} \int_o^\infty \frac{d\beta \ \beta^{\gamma-1}}{\Gamma(\beta+\frac{1}{2})} \int_o^\infty \frac{d\alpha \ \alpha^{\beta-\frac{1}{2}}}{\Gamma(\alpha)} \int_o^1 u^{n-1} (\log \frac{1}{u})^{\alpha-1} du$$

So that from (5.4) and (5.5) we get

$$\int_o^r |F'_\gamma(\rho e^{i\theta})| d\rho \leq K'' \sum_{n=3}^{\infty} \frac{r^n}{n \log n (\log \log n)^\gamma} ,$$

which gives

$$\int_o^1 |F'_\gamma(\rho e^{i\theta})| d\rho < \infty ,$$

since $\sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)^\gamma} < \infty$

This completes the proof of the theorem.

Remark 1. From the proofs of Theorem 5.1 and Theorem 5.2 it is clear that the theorems would still hold true if $(\log\log n)^{\gamma}$ occuring in the series of $F_{\gamma}(z)$ is replaced by $\log_2 n \log_3 n \ldots (\log_p n)^{\gamma}$ where p is a positive integer,

$\log_p(n) = \log\log_{p-1}(n)$, $\log_1 n = \log n$,

$n \geq N$ where N is a positive integer chosen so large that $\log_p(N)$ is well defined, and $\gamma > 1$.

Remark 2. The index $\frac{1}{2}$ of $(\log n)$ in Theorem 5.2 is best possible. To prove this we consider

$$f(z) = \sum_{n=1}^{\infty} \frac{z^{2^n}}{n^{\frac{1}{2}+\eta}} \qquad (|z|<1)$$

where $0<\eta<\frac{1}{2}$. Since $\frac{1}{n^{1+2\eta}} < \infty$ therefore $f(z) \in H^2$ and consequently $f(z)$ satisfies all the conditions of Theorem 5.2. Also from (5.1) we know that if

$$F(z) = \sum_{n=1}^{\infty} \left(\frac{z^{2^n}}{n}\right) \qquad (|z|<1)$$

then $V(F,\theta) = \infty$ a.e..

This shows that the index $(\frac{1}{2})$ of $(\log n)$ cannot be replaced by $\frac{1}{2}-\eta$ for any $\eta\,(0<\eta<\frac{1}{2})$.

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