

GROUND STATES FOR DIFFUSION DOMINATED FREE ENERGIES WITH LOGARITHMIC INTERACTION*

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Abstract. Replacing linear diffusion by a degenerate diffusion of porous medium type is known to regularize the classical two-dimensional parabolic-elliptic Keller–Segel model [V. Calvez and J. A. Carrillo, *J. Math. Pures Appl.* (9), 86 (2006), pp. 155–175]. The implications of nonlinear diffusion are that solutions exist globally and are uniformly bounded in time. We analyze the stationary case showing the existence of a unique, up to translation, global minimizer of the associated free energy. Furthermore, we prove that this global minimizer is a radially decreasing compactly supported continuous density function which is smooth inside its support, and it is characterized as the unique compactly supported stationary state of the evolution model. This unique profile is the clear candidate to describe the long time asymptotics of the diffusion dominated classical Keller–Segel model for general initial data.

Key words. ground states, free energy, logarithmic interaction, uniqueness

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1. Introduction. Ground states of free energies play a crucial role in the long time asymptotics of nonlinear aggregation diffusion models. These nonlocal partial differential equations (PDEs) are ubiquitous in the mathematical modeling of phenomena which involve a large number of particles. For instance, nonlocal drift-diffusion equations show up naturally in semiconductor modeling, bacterial chemotaxis, granular media, and many other areas; see [18, 9, 13] and the references therein. These equations are just based on two competing mechanisms, namely the attraction, modeled by a nonlocal force, and the repulsion, modeled by a nonlinear diffusion.

One of the archetypal models of nonlinear aggregation diffusion is the so-called classical parabolic-elliptic Keller–Segel model. This model was classically introduced as the simplest description for chemotactic bacteria movement in which the tendency of linear diffusion to spread repels the attraction due to the logarithmic kernel interaction in two dimensions. Although there is a large amount of literature in this field, many advances have been made in the last 10 years thanks to the combination of different ideas ranging from functional inequalities to entropy-entropy dissipation techniques passing through optimal transport. We refer the reader to [9, 8, 6, 11] and

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the references therein for some aspects of this *fair competition* case in which there is a well-defined critical mass. In fact, here a clear dichotomy arises: if the total mass of the system is less than the critical mass, then the long time asymptotics are described by a self-similar solution, while for a mass larger than the critical one, there is finite time blow-up. For the exact critical mass case, a detailed study has also been performed in [8, 6].

The existence of a well-defined critical mass can be generalized to more dimensions if one allows for degenerate diffusions. In fact, let us consider the evolution of the probability density ρ given by

$$(1.1) \quad \rho_t = \Delta \rho^m + \nabla \cdot (\rho(\nabla \mathcal{W} * \rho)) \quad \text{in } \mathbb{R}^d,$$

where $d \geq 2$, and with the homogeneous kernel $\mathcal{W}(x) = |x|^\alpha/\alpha$ with $-d < \alpha < 0$. By choosing $\alpha = 2 - d$, $d \geq 3$, and $m = 2 - 2/d$, it was shown in [7] that there exist a critical mass and an exact dichotomy as in the classical Keller–Segel model. This is based on the fact that these equations share a common structural setting, namely, they have a well-defined free energy functional given by

$$(1.2) \quad E(\rho) = \frac{1}{m-1} \int_{\mathbb{R}^d} \rho^m(x) dx + \frac{1}{2\alpha} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x-y|^\alpha \rho(x)\rho(y) dx dy,$$

and that the two terms in (1.2) scale equally by dilations in that particular case. Actually, this fact is also satisfied by all the *fair competition* cases in which $m = (d - \alpha)/d$. Therefore, there is a well-defined critical mass in all the fair competition cases by generalizing the arguments in [6]. While the analysis of the fair competition cases can be considered advanced, it is not so for both the *diffusion dominated* case, $m > (d - \alpha)/d$, and the *aggregation dominated* case, $m < (d - \alpha)/d$. Regarding the latter, recent results in [4] discriminate blow-up and global existence depending on the initial conditions and on the exponent m for the particular case of $\alpha = 2 - d$, $d \geq 3$. Other results in this direction also appear in a series of papers by Sugiyama [27, 28, 26]. However, in the diffusion dominated case, $m > (d - \alpha)/d$, there is little information about the long time asymptotics, seemingly due to the lack of confinement by the interaction kernel; see [4, 28]. It is actually proved that solutions exist globally, and that they are bounded uniformly in time without further information about their behavior at infinity. Existence of steady states in the case $\alpha = 2 - d$, $d \geq 3$, $m > 2 - 2/d$ has been analyzed in [4].

In this paper, we build upon the understanding of the long time asymptotics in the classical diffusion dominated case in two dimensions. Calvez and Carrillo proved in [10] that solutions corresponding to the classical diffusion dominated two-dimensional Keller Segel model,

$$(1.3) \quad \rho_t = \Delta \rho^m + \frac{1}{2\pi} \nabla \cdot (\rho(\nabla \log |x| * \rho)) \quad \text{in } \mathbb{R}^2, \quad \text{with } m > 1,$$

exist globally and are uniformly bounded in time. However, they were not able to clarify the long time asymptotics. Here we encounter once again the structural setting of a free energy functional given by

$$(1.4) \quad G[\rho] := \frac{1}{m-1} \int_{\mathbb{R}^2} \rho^m dx + \frac{1}{4\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log |x-y| \rho(x)\rho(y) dx dy.$$

Since the free energy $G[\rho]$ is decaying in time through the evolution of the flow associated to (1.3), one may expect convergence towards the possible (global) minimizers of

$G[\rho]$ over mass densities. Due to the translational invariance of (1.3), we will consider the set of admissible functions for the functional $G[\rho]$ as the set of *zero center-of-mass densities*

$$\mathcal{Y}_M := \left\{ \rho \in L^1_+(\mathbb{R}^2) \cap L^m(\mathbb{R}^2) : \|\rho\|_1 = M, \int_{\mathbb{R}^2} x\rho(x) dx = 0 \right\}$$

for a given *mass* $M > 0$.

This work is entirely devoted to showing the existence of a unique global minimizer of the free energy $G[\rho]$ in \mathcal{Y}_M . Furthermore, we will show that this global minimizer is a radially decreasing compactly supported continuous density function smooth inside its support, and that it is characterized as the unique (up to translation) compactly supported stationary state of the diffusion dominated Keller–Segel model (1.3) with $m > 1$. This unique profile is the clear candidate to describe the long time asymptotics of the evolution model (1.3) for general initial data, which will be treated elsewhere. Finally, we point out that for the model (1.1) with $d \geq 3$ and in the diffusion dominated case $m > 2 - 2/d$, this asymptotic regime is shown in [19, Theorem 5.6, Corollary 5.9] in the class of radially symmetric, continuous, and compactly supported initial data.

From the technical point of view, we cannot resort to classical concentration-compactness principles as used in [21, 22, 7, 2], which are closely related to homogeneous kernels as in (1.2). Actually, we take advantage of the logarithmic interaction kernel to show the confinement of the density in section 2. This is the basic building block in showing the existence of global minimizers that are radially decreasing due to symmetric decreasing rearrangement techniques. In section 3 we further identify them and show that they are compactly supported and smooth in their support using variational techniques. Finally, in section 4 a nonstandard application of the moving plane method shows that compactly supported stationary states of (1.3) coincide with the unique up to translation global minimizer of the previous sections.

2. Minimization of the free energy functional. Our goal is to minimize the functional $G[\rho]$ given by (1.4) defined on \mathcal{Y}_M for a given *mass* $M > 0$. Set $G[\rho] = H[\rho] + W[\rho]$, where

$$H[\rho] := \frac{1}{m-1} \int_{\mathbb{R}^2} \rho^m dx$$

is the *entropy functional*, defined on $L^m_+(\mathbb{R}^2)$, while

$$W[\rho] := -\frac{1}{2} \int_{\mathbb{R}^2} (\mathcal{K} * \rho)(x)\rho(x) dx$$

is the *interaction functional*, where

$$\mathcal{K}(x) := -\frac{1}{2\pi} \log|x|.$$

Let us first check that $W[\rho] > -\infty$ in this class. Notice that for each $\rho \in \mathcal{Y}_M$, Hölder's inequality implies that

$$\begin{aligned} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log^-|x-y| \rho(x)\rho(y) dx dy &= \int_{\mathbb{R}^2} \int_{|x-y| \leq 1} |\log|x-y|| \rho(x)\rho(y) dx dy \\ &\leq \|\rho\|_m \int_{\mathbb{R}^2} \left(\int_{|x-y| \leq 1} |\log|x-y||^{m'} dy \right)^{1/m'} \rho(x) dx \\ &\leq C M \|\rho\|_m, \end{aligned}$$

where $m' = m/(m-1)$ and C is a positive constant. Then, we have that $G[\rho] \in (-\infty, \infty]$ in \mathcal{Y}_M . Let us define the class of radial densities as

$$\mathcal{Y}_M^{rad} := \{ \rho \in L_+^1(\mathbb{R}^2) \cap L^m(\mathbb{R}^2) : \|\rho\|_1 = M, \rho = \rho^\# \},$$

where $\rho^\#$ is the *spherical decreasing rearrangement* of ρ ; see, for instance, [17, 3, 29] for the basic definitions and related properties.

THEOREM 2.1. *For any positive mass M , there exists a global radial minimizer $\rho_0 \in \mathcal{Y}_M^{rad}$ of the free energy functional G in \mathcal{Y}_M . Moreover, global minimizers satisfy $\rho_0 \log \rho_0 \in L^1(\mathbb{R}^2)$.*

Proof. We split the proof into three parts, proving first that global minimizers must be radial, and thus we can restrict our study to \mathcal{Y}_M^{rad} . We next show that the functional G is bounded from below in \mathcal{Y}_M^{rad} , and finally that the infimum is achieved in \mathcal{Y}_M^{rad} .

Step 1. *The candidates to be global minimizers of G are radial.* As soon as the interaction term $W[\rho]$ is finite, the interaction functional $W[\rho]$ decreases under rearrangement as proven in [12, Lemma 2], in the sense that

$$(2.1) \quad \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log|x-y| \rho(x)\rho(y) dx dy \geq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log|x-y| \rho^\#(x)\rho^\#(y) dx dy.$$

This shows that

$$\inf_{\mathcal{Y}_M} G = \inf_{\mathcal{Y}_M^{rad}} G.$$

Actually, all the minimizers of G in the class \mathcal{Y}_M must be radially decreasing, i.e., they lay in the class \mathcal{Y}_M^{rad} . Indeed, if ρ is a global minimizer in \mathcal{Y}_M , by inequality (2.1) we have that $\rho^\#$ is a radially decreasing global minimizer of G . Since the L^m -norms of ρ and $\rho^\#$ are equal, from $G[\rho] = G[\rho^\#]$ we deduce that

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log|x-y| \rho(x)\rho(y) dx dy = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log|x-y| \rho^\#(x)\rho^\#(y) dx dy.$$

Hence, using [12, Lemma 2] again, we find that ρ must be a translation of $\rho^\#$, that is, $\rho(x) = \rho^\#(x-y)$ for some $y \in \mathbb{R}^2$. Moreover, we have

$$\int_{\mathbb{R}^2} x\rho(x)dx = yM + \int_{\mathbb{R}^2} x\rho^\#(x)dx = yM,$$

and thus the zero center-of-mass condition holds if and only if $y = 0$, giving $\rho = \rho^\#$, namely, ρ is radially decreasing.

Step 2. *G is bounded from below in \mathcal{Y}_M^{rad} .* Here, we follow arguments from [10]. For any $\rho \in \mathcal{Y}_M$ such that

$$\rho \log \rho, \quad \rho \log(1 + |x|^2) \in L^1(\mathbb{R}^2),$$

the *logarithmic Hardy–Littlewood–Sobolev (HLS) inequality* [12] implies that there exists a constant $C(M) > 0$ such that

$$(2.2) \quad \int_{\mathbb{R}^2} \rho \log \rho dx \geq -\frac{2}{M} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log|x-y| \rho(x)\rho(y) dx dy - C(M).$$

Let us start by showing a bound from below in a restricted class of densities. Consider ρ in \mathcal{Y}_M^{rad} and first assume that ρ is continuous, with compact support. Applying (2.2) to (1.4) we have

$$(2.3) \quad \mathbb{G}[\rho] \geq -\frac{M}{8\pi}C(M) + \int_{\mathbb{R}^2} \left(\frac{\rho^m}{m-1} - \frac{M}{8\pi} \rho \log \rho \right) dx.$$

Now, let us choose $\theta \in (0, 1)$ and a value $\kappa_{\theta, m} > 1$ such that

$$\theta \frac{r^m}{m-1} - \frac{M}{8\pi} r \log r > 0 \quad \forall r > \kappa_{\theta, m}.$$

Then, we deduce

$$\begin{aligned} \int_{\mathbb{R}^2} \rho \log \rho dx &= \int_{\rho \leq 1} \rho \log \rho dx + \int_{1 < \rho \leq \kappa_{\theta, m}} \rho \log \rho dx + \int_{\rho > \kappa_{\theta, m}} \rho \log \rho dx \\ &\leq M \log \kappa_{\theta, m} + \frac{8\pi}{M(m-1)} \theta \int_{\mathbb{R}^2} \rho^m dx, \end{aligned}$$

and therefore,

$$\int_{\mathbb{R}^2} \left(\frac{\rho^m}{m-1} - \frac{M}{8\pi} \rho \log \rho \right) dx \geq -\frac{M^2}{8\pi} \log \kappa_{\theta, m} + \frac{1-\theta}{m-1} \int_{\mathbb{R}^2} \rho^m dx.$$

Hence, we infer from (2.3) that

$$(2.4) \quad \mathbb{G}[\rho] \geq -\frac{M}{8\pi} (M \log \kappa_{\theta, m} + C(M)) + \frac{1-\theta}{m-1} \int_{\mathbb{R}^2} \rho^m dx.$$

This bound from below being only dependent on the L^m -norm can be extended to \mathcal{Y}_M^{rad} by a density argument that we detail next. If $\rho \in \mathcal{Y}_M^{rad}$ is less regular, let us take a nondecreasing sequence of radially decreasing, compactly supported, continuous nonnegative functions $\tilde{\rho}_n$ converging strongly to ρ in $L^1(\mathbb{R}^2) \cap L^m(\mathbb{R}^2)$: such a choice is always possible, since we can approximate ρ first by smooth functions; then the sequence of their rearrangements satisfies the required conditions by the L^p -contraction property of the rearrangement map. If $\|\tilde{\rho}_n\|_1 = M_n$, let us construct the sequence

$$\rho_n := \frac{M}{M_n} \tilde{\rho}_n.$$

Thus $\rho_n \in \mathcal{Y}_M^{rad}$. Besides, from $M_n \nearrow M$, we get $\rho_n \nearrow \rho$ strongly in $L^1(\mathbb{R}^2) \cap L^m(\mathbb{R}^2)$, $\|\rho_n\|_1 = M$, and we can apply inequality (2.3) to deduce

$$(2.5) \quad \mathbb{G}(\rho_n) \geq -\frac{M}{8\pi} (M \log \kappa_{\theta, m} + C(M)) + \frac{1-\theta}{m-1} \int_{\mathbb{R}^2} \rho_n^m dx.$$

Hölder's inequality implies

$$\begin{aligned}
& \left| \int_{\mathbb{R}^2} \int_{|x-y| \leq 1} \log |x-y| (\rho_n(x)\rho_n(y) - \rho(x)\rho(y)) \, dx \, dy \right| \\
&= \left| \int_{\mathbb{R}^2} \int_{|x-y| \leq 1} \log |x-y| [(\rho_n(x) - \rho(x))\rho_n(y) + (\rho_n(y) - \rho(y))\rho(x)] \, dx \, dy \right| \\
&\leq \|\rho_n - \rho\|_m \int_{\mathbb{R}^2} \left[\left(\int_{|x-y| \leq 1} |\log |x-y||^{m'} \, dx \right)^{1/m'} \rho_n(y) \right] dy \\
&\quad + \|\rho_n - \rho\|_m \int_{\mathbb{R}^2} \left[\left(\int_{|x-y| \leq 1} |\log |x-y||^{m'} \, dy \right)^{1/m'} \rho(x) \right] dx \\
&= 2C M \|\rho_n - \rho\|_m \rightarrow 0.
\end{aligned}$$

Concerning the positive part of $\log |x-y|$, since ρ_n is a nondecreasing sequence converging to ρ , we have by the monotone convergence theorem that

$$\int_{\mathbb{R}^2} \int_{|x-y| > 1} \log^+ |x-y| \rho_n(x)\rho_n(y) \, dx \, dy \nearrow \int_{\mathbb{R}^2} \int_{|x-y| > 1} \log^+ |x-y| \rho(x)\rho(y) \, dx \, dy,$$

as $n \rightarrow \infty$, and thus $\mathbf{H}(\rho_n) \rightarrow \mathbf{H}(\rho)$, $\mathbf{W}(\rho_n) \rightarrow \mathbf{W}(\rho)$ as $n \rightarrow \infty$. Hence, we can pass to the limit in (2.5) and get (2.4) in \mathcal{Y}_M^{rad} .

Step 3. The infimum of \mathbf{G} is achieved in \mathcal{Y}_M^{rad} . Let

$$\mathcal{I} := \inf_{\rho \in \mathcal{Y}_M^{rad}} \mathbf{G}(\rho),$$

and let us choose a minimizing sequence of \mathbf{G} , i.e., a sequence $\{\rho_n\}_{n \in \mathbb{N}}$ in \mathcal{Y}_M^{rad} such that

$$(2.6) \quad \mathbf{G}[\rho_n] \rightarrow \mathcal{I} \quad \text{as } n \rightarrow \infty.$$

By the control of the functional \mathbf{G} in (2.4), we get that $\{\rho_n\}_{n \in \mathbb{N}}$ is bounded in $L^m(\mathbb{R}^2)$, and hence by (2.6) it follows that $\{\mathbf{W}[\rho_n]\}_{n \in \mathbb{N}}$ is bounded. In order to control the behavior at infinity, we follow arguments similar to those in [24] and [23, Proposition 7.10]. For all $R \geq 1$ and any $\rho \in L^1_+(\mathbb{R}^2) \cap L^m(\mathbb{R}^2)$, define the functional

$$(2.7) \quad \mathbf{W}_R[\rho] := \int_{\mathbb{R}^2} \int_{|x-y| > R} \log |x-y| \rho(x)\rho(y) \, dx \, dy.$$

By Hölder's inequality, we have

$$\begin{aligned}
(2.8) \quad \mathbf{W}_R[\rho] &= \int_{\mathbb{R}^2} \int_{|x-y| > R} \log |x-y| \rho(x)\rho(y) \, dx \, dy \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log |x-y| \rho(x)\rho(y) \, dx \, dy + \int_{\mathbb{R}^2} \int_{|x-y| \leq R} |\log |x-y|| \rho(x)\rho(y) \, dx \, dy \\
&\leq \mathbf{W}[\rho] + \|\rho\|_m \int_{\mathbb{R}^2} \left(\int_{|x-y| \leq R} |\log |x-y||^{m'} \, dy \right)^{1/m'} \rho(x) \, dx \leq \mathbf{W}[\rho] + C M \|\rho\|_m,
\end{aligned}$$

where $m' = m/(m-1)$. In particular, by (2.8) it follows that $\{W_R[\rho_n]\}_{n \in \mathbb{N}}$ is bounded. Now, let $x \in \mathbb{R}^2$ with $|x| \geq 1$ and notice that $\{y \in \mathbb{R}^2 : x \cdot y \leq 0\} \subset \{y \in \mathbb{R}^2 : |x-y| \geq 1\}$. Then, since ρ is nonnegative, for all $R \geq 1$ we get

$$(2.9) \quad W_1[\rho] \geq \int_{|x|>R} \int_{x \cdot y \leq 0} \log|x-y| \rho(x)\rho(y) dx dy.$$

Since $x \cdot y \leq 0$ implies $|x-y| \geq |x|$, we infer from (2.9) that

$$(2.10) \quad W_1[\rho] \geq \int_{|x|>R} \int_{x \cdot y \leq 0} \log|x| \rho(x)\rho(y) dx dy;$$

then if we assume $\rho = \rho^\#$, we find

$$(2.11) \quad W_1[\rho] \geq \log R \int_{|x|>R} \int_{x \cdot y \leq 0} \rho(x)\rho(y) dx dy = \frac{M \log R}{2} \int_{|x|>R} \rho(x) dx.$$

Thus the fact that $\{W[\rho_n]\}_{n \in \mathbb{N}}$ is bounded and (2.11) yield

$$(2.12) \quad \sup_{n \in \mathbb{N}} \int_{|x|>R} \rho_n(x) dx \leq \frac{C}{\log R} \xrightarrow{R \rightarrow \infty} 0,$$

that is, the so-called *confinement of the mass*. In order to check that $\{\rho_n\}$ is locally *equi-integrable*, we observe that for given $\varepsilon > 0$, setting

$$(2.13) \quad \mathbf{a} := \sup_{n \in \mathbb{N}} \|\rho_n\|_m < \infty$$

for any subset A of \mathbb{R}^N such that $|A| < \delta := (\varepsilon/\mathbf{a})^{m'}$ we have, by Hölder's inequality,

$$\int_A \rho_n dx \leq \mathbf{a} |A|^{\frac{m-1}{m}} < \varepsilon$$

for all $n \in \mathbb{N}$, that is, the sequence $\{\rho_n\}_{n \in \mathbb{N}}$ is equi-integrable. According to the Dunford–Pettis theorem using (2.12) and (2.13), there exists a function $\rho_0 \in L^1_+(\mathbb{R}^2) \cap L^m(\mathbb{R}^2)$ such that (up to subsequence)

$$(2.14) \quad \rho_n \rightharpoonup \rho_0 \quad \text{weakly in } L^1(\mathbb{R}^2) \cap L^m(\mathbb{R}^2)$$

and $\|\rho_0\|_1 = M$. Furthermore,

$$\|\rho_0\|_m \leq \liminf_{n \rightarrow \infty} \|\rho_n\|_m \leq C.$$

In particular, the interaction energy $W[\rho_0]$ of ρ_0 is bounded from below because the functional G is. Our aim is now to show that W is lower semicontinuous with respect to the $L^1 \cap L^m$ weak convergence, taking advantage of some arguments shown in [5]. Then fix $\varepsilon \in (0, 1)$, $R > 1$, and write

$$W[\rho_n] = A^\varepsilon[\rho_n] + B^\varepsilon[\rho_n] + W_R[\rho_n],$$

where

$$A^\varepsilon[\rho] := \int_{\mathbb{R}^2} \int_{|x-y| \leq \varepsilon} \log|x-y| \rho(x)\rho(y) dx dy,$$

$$B^\varepsilon[\rho] := \int_{\mathbb{R}^2} \int_{\varepsilon < |x-y| \leq R} \log|x-y| \rho(x)\rho(y) dx dy,$$

and the functional W_R is defined in (2.7). We notice that the same arguments used to prove inequality (2.8) yield

$$A^\varepsilon[\rho_n] \leq CM \|\rho_n\|_m \left(\int_0^\varepsilon r |\log r|^{m'} dr \right)^{1/m'};$$

then

$$(2.15) \quad A^\varepsilon[\rho_n] \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \text{ uniformly in } n.$$

Observe that we can use the equi-integrability of the sequence $\{\rho_n\}$ and the fact that $\rho_n \rightharpoonup \rho_0$ weakly in $L^1_+(\mathbb{R}^2)$ to apply Lemma 2.3 in [5] and find

$$(2.16) \quad \rho_n \otimes \rho_n \rightharpoonup \rho_0 \otimes \rho_0 \quad \text{weakly in } L^1_+(\mathbb{R}^2 \times \mathbb{R}^2).$$

Then, since the function $\log|x-y|$ is bounded in $\{\varepsilon < |x-y| \leq R\}$ we have that

$$(2.17) \quad B^\varepsilon[\rho_n] \rightarrow \int_{\mathbb{R}^2} \int_{\varepsilon < |x-y| \leq R} \log|x-y| \rho_0(x) \rho_0(y) dx dy \quad \text{as } n \rightarrow \infty.$$

It remains then to get a bound from below of the last integral $W_R[\rho_n]$ for large n . In order to do this, we first point out that (2.16) implies that the sequence of densities $(\rho_n \otimes \rho_n)(x, y)$ converges to $(\rho_0 \otimes \rho_0)(x, y)$ in the weak-* sense as measures. Then using the fact that the function $\log^+|x-y|$ is bounded from below and obviously lower semicontinuous in the set $\{(x, y) : x \in \mathbb{R}^2, |x-y| > R\}$, inequality (5.1.15) in [1] gives

$$(2.18) \quad \liminf_{n \rightarrow \infty} W_R[\rho_n] \geq \int_{\mathbb{R}^2} \int_{|x-y| > R} \log|x-y| \rho_0(x) \rho_0(y) dx dy.$$

In particular, combining this last inequality with (2.8) we derive that

$$(\log|\cdot| * \rho_0)\rho_0 \in L^1(\mathbb{R}^2)$$

and $W[\rho_0]$ is finite. Now, using (2.17) and (2.18) we get

$$\begin{aligned} \liminf_{n \rightarrow \infty} W[\rho_n] &\geq \liminf_{n \rightarrow \infty} A^\varepsilon[\rho_n] \\ &\quad + \int_{\mathbb{R}^2} \int_{\varepsilon < |x-y| \leq R} \log|x-y| \rho_0(x) \rho_0(y) dx dy \\ &\quad + \int_{\mathbb{R}^2} \int_{|x-y| > R} \log|x-y| \rho_0(x) \rho_0(y) dx dy; \end{aligned}$$

thus letting $\varepsilon \rightarrow 0$, property (2.15) implies

$$\liminf_{n \rightarrow \infty} W[\rho_n] \geq W[\rho_0].$$

Using (2.14), this gives in turn

$$(2.19) \quad \mathcal{I} = \liminf_{n \rightarrow \infty} G[\rho_n] \geq H[\rho_0] + W[\rho_0].$$

By taking the rearrangement $\rho_0^\#$ of ρ_0 , since $W[\rho_0]$ is finite, inequality (2.1) implies

$$(2.20) \quad W[\rho_0] \geq W[\rho_0^\#],$$

and hence (2.19) gives

$$\mathcal{I} \geq \mathbf{G}[\rho_0^\#].$$

With this we have finished the proof of existence of global radial minimizers. Finally, we notice by (2.20) that $\mathbf{W}_1[\rho_0^\#]$ is finite, so for all $R \geq 1$ inequality (2.10) provides

$$\int_{|x|>R} \int_{x \cdot y \leq 0} \log |x| \rho_0^\#(x) \rho_0^\#(y) dx dy \leq \mathbf{W}_1[\rho_0^\#] < \infty,$$

that is

$$\frac{M}{2} \int_{|x|>R} \log |x| \rho_0^\#(x) dx \leq C,$$

namely, $\rho_0^\# \log(1 + |x|^2) \in L^1(\mathbb{R}^2)$. \square

Remark 2.2. Let us point out that the previous proof works in any dimension since the logarithmic HLS inequality holds with a constant that depends only on the dimension and the mass. We also emphasize that the use of the logarithmic potential is crucial here, since we do not know how to prove a quantitative confinement property when the Newtonian potential for dimensions larger than two is used instead.

3. Identification, regularity, and uniqueness of global minimizers. Our aim is to show a full characterization of any minimizer ρ_0 of the functional \mathbf{G} to relate them to the steady states to the two-dimensional Keller–Segel model. We first deduce the Euler–Lagrange conditions satisfied by critical points of the functional.

THEOREM 3.1. *Let $\rho_0 \in \mathcal{Y}_M$ be a global minimizer of the free energy functional \mathbf{G} defined in (1.4). Then ρ_0 satisfies*

$$(3.1) \quad \frac{m}{m-1} \rho_0^{m-1} - \mathcal{K} * \rho_0 = \mathcal{D}[\rho_0] \quad \text{a.e. in } \text{supp}(\rho_0)$$

and

$$(3.2) \quad \frac{m}{m-1} \rho_0^{m-1} - \mathcal{K} * \rho_0 \geq \mathcal{D}[\rho_0] \quad \text{a.e. outside } \text{supp}(\rho_0),$$

where

$$\mathcal{D}[\rho_0] = \frac{2}{M} \mathbf{G}[\rho_0] + \frac{m-2}{M(m-1)} \|\rho_0\|_m^m.$$

As a consequence, any global minimizer of \mathbf{G} verifies

$$(3.3) \quad \frac{m}{m-1} \rho_0^{m-1} = (\mathcal{K} * \rho_0 + \mathcal{D}[\rho_0])_+.$$

Proof. The technical difficulty here is to make good variations of the minimizer under the low available regularity conditions on ρ_0 obtained from Theorem 2.1, namely, $\rho_0 \in L^1_+ \cap L^m(\mathbb{R}^2)$ and $\rho_0 \log \rho_0 \in L^1(\mathbb{R}^2)$. We use some ideas from [25]. We first show (3.1). Let ρ_0 be a radially decreasing minimizer of \mathbf{G} . Taking any $\varepsilon > 0$ and a test function $\psi \in C_0^\infty(\mathbb{R}^2)$ such that $\psi(x) = \psi(-x)$, let us define the function

$$\varphi(x) = \left(\psi(x) - \frac{1}{M} \int_{\mathbb{R}^2} \psi(x) \rho_0(x) dx \right) \rho_0(x).$$

We point out that $\varphi \in L^1(\mathbb{R}^2) \cap L^m(\mathbb{R}^2)$, and in addition,

$$(3.4) \quad \int_{\mathbb{R}^2} \varphi(x) dx = \int_{\mathbb{R}^2} x\varphi(x) dx = 0,$$

and $\text{supp}(\varphi) \subseteq \text{supp}(\rho_0) =: E$. Moreover, for $\varepsilon < \varepsilon_0 := (2\|\psi\|_\infty)^{-1}$ we find

$$\rho_0 + \varepsilon\varphi = \rho_0(x) \left(1 + \varepsilon \left(\psi - \frac{1}{M} \int_{\mathbb{R}^2} \psi \rho_0 dx \right) \right) \geq \rho_0(x)(1 - 2\varepsilon\|\psi\|_\infty) \geq 0.$$

Due to (3.4), we have $\rho_0 + \varepsilon\varphi \in \mathcal{Y}_M$, and thus we can calculate the first variation $\frac{\delta \mathbf{G}}{\delta \varphi}(\rho_0)$ of the functional \mathbf{G} . Noting that $\text{supp}(\rho_0 + \varepsilon\varphi) \subseteq E$, we get

$$(3.5) \quad \frac{\mathbf{G}[\rho_0 + \varepsilon\varphi] - \mathbf{G}[\rho_0]}{\varepsilon} = \frac{1}{m-1} \int_{\dot{E}} \frac{(\rho_0 + \varepsilon\varphi)^m - \rho_0^m}{\varepsilon} dx - \int_{\mathbb{R}^2} \mathcal{K} * \rho_0 \varphi dx + \varepsilon \mathbf{W}[\varphi].$$

Using the first order Taylor expansion of $(\rho_0 + \varepsilon\varphi)^m$ at $\varepsilon = 0$, we have

$$\int_{\dot{E}} \frac{(\rho_0 + \varepsilon\varphi)^m - \rho_0^m}{\varepsilon} dx = m \int_0^1 \mathcal{G}_\varepsilon(t) dt,$$

where

$$\mathcal{G}_\varepsilon(t) := \int_{\dot{E}} |\rho_0 + \varepsilon t\varphi|^{m-2} (\rho_0 + \varepsilon t\varphi) \varphi dx.$$

By the definition of $\mathcal{G}_\varepsilon(t)$, it is obvious that for all $t \in [0, 1]$ and $\varepsilon < \varepsilon_0$, one has

$$|\mathcal{G}_\varepsilon(t)| \leq (\|\rho_0\|_m + \varepsilon_0 \|\varphi\|_m)^{m-1} \|\varphi\|_m \in L_t^1(0, 1),$$

and then Lebesgue's dominated convergence yields

$$(3.6) \quad \int_{\dot{E}_0} \frac{(\rho_0 + \varepsilon\varphi)^m - \rho_0^m}{\varepsilon} dx \xrightarrow{\varepsilon \rightarrow 0} m \int_{\mathbb{R}^2} \rho_0^{m-1} \varphi dx.$$

In addition, as $\rho_0(x) \log(1 + |x|^2) \in L^1(\mathbb{R}^2)$, the algebraic inequality

$$\log|x - y| \leq \frac{1}{2}(\log 2 + \log(1 + |x|^2)) + \log(1 + |y|^2)$$

and the estimate $|\varphi(x)| \leq 2\|\psi\|_\infty \rho_0(x)$ give $\mathbf{W}[\varphi] \leq C$. Therefore, using this last property and (3.6) to pass to the limit in (3.5) as $\varepsilon \rightarrow 0$, we obtain that

$$\int_{\mathbb{R}^2} \varphi \mathcal{F}(\rho_0) dx \geq 0,$$

since $\mathbf{G}[\rho_0 + \varepsilon\varphi] \geq \mathbf{G}[\rho_0]$, where

$$\mathcal{F}(\rho_0) := \frac{m}{m-1} \rho_0^{m-1} - \mathcal{K} * \rho_0 = \frac{m}{m-1} \rho_0^{m-1} + \frac{1}{2\pi} \log|x| * \rho_0.$$

Taking $-\psi$ instead of ψ , we finally obtain

$$(3.7) \quad \int_{\mathbb{R}^2} \varphi \mathcal{F}(\rho_0) dx = 0.$$

By the definition of φ , we conclude that

$$\int_{\mathbb{R}^2} [\mathcal{F}(\rho_0) - \mathcal{D}[\rho_0]] \rho_0 \psi \, dx = 0$$

for all even functions $\psi \in C_0^\infty(\mathbb{R}^2)$. Hence, we deduce

$$(3.8) \quad \frac{m}{m-1} \rho_0^{m-1} - \mathcal{K} * \rho_0 = \mathcal{D}[\rho_0] \quad \text{a.e. in } \{\rho_0 > 0\}.$$

Now, we turn to the proof of (3.2). Let us take an even function $\psi \in C_0^\infty(\mathbb{R}^2)$ with $\psi \geq 0$ such that $\psi(x) \in [0, 1]$, and let us define the function

$$\varphi = \psi - \frac{\rho_0}{M} \int_{\mathbb{R}^2} \psi \, dx.$$

Then $\varphi \in L^1(\mathbb{R}^2) \cap L^m(\mathbb{R}^2)$ and

$$\int_{\mathbb{R}^2} \varphi(x) \, dx = \int_{\mathbb{R}^2} x \varphi(x) \, dx = 0.$$

In addition, denoting by $|\cdot|_N$ the N -dimensional Lebesgue measure, we have

$$\rho_0 + \varepsilon \varphi \geq \rho_0 \left(1 - \frac{\varepsilon}{M} \int_{\mathbb{R}^2} \psi \, dx \right) \geq \left(1 - \frac{\varepsilon}{M} |\text{supp } \psi|_N \right) \rho_0(x),$$

and then $\rho_0 + \varepsilon \varphi \geq 0$ for small ε in $\text{supp}(\rho_0)$ and outside since $\psi \geq 0$, and hence $\rho_0 + \varepsilon \varphi \in \mathcal{Y}_M$. Arguing as before, we obtain from (3.7)

$$\int_{\mathbb{R}^2} [\mathcal{F}(\rho_0) - \mathcal{D}[\rho_0]] \psi \, dx \geq 0$$

for all the functions ψ chosen as above, implying

$$\frac{m}{m-1} \rho_0^{m-1} \geq \mathcal{K} * \rho_0 + \mathcal{D}[\rho_0] \quad \text{a.e. outside } \text{supp}(\rho_0). \quad \square$$

Remark 3.2. Let us point out that inequality (3.2) is a consequence of the positivity and mass constraints on the class of possible minimizers, i.e., due to the fact that we are working with an optimization problem with convex constraints.

Actually, we can show many properties about the regularity of global radial minimizers to the free energy functional \mathcal{G} . Now, we give some information concerning the *asymptotic behavior* at infinity of the *logarithmic potential* of any density $\rho_0 \in \mathcal{Y}_M$, namely, the Newtonian potential

$$(3.9) \quad u(x) := (\mathcal{K} * \rho_0)(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y| \rho_0(y) \, dy.$$

The proof of the following result is contained in [14, Lemma 1.1]. Let $\rho \in \mathcal{Y}_M$. Then we have

$$(3.10) \quad \lim_{|x| \rightarrow \infty} \frac{u(x)}{\mathcal{K}(x)} = M.$$

With this further result in hand, we are now ready to give more information about the regularity of the radially decreasing minimizers of \mathcal{G} .

THEOREM 3.3. *All radially decreasing global minimizers of \mathcal{G} in \mathcal{Y}_M are compactly supported continuous functions in \mathbb{R}^2 and smooth inside their support.*

Proof. Let ρ_0 be a radially decreasing minimizer of G . Then there is a ball $B_R(0)$ such that $\{\rho_0 > 0\} = B_R(0)$. Let us consider the logarithmic potential of ρ_0 , namely, the function u defined in (3.9). As $\rho_0 \in L^m(\mathbb{R}^2)$, by [16, Lemma 9.9] we have $u \in W_{loc}^{2,m}(\mathbb{R}^2)$. By Morrey's theorem ($m > 1$), it follows that $u \in L_{loc}^\infty(\mathbb{R}^2)$, and by (3.8) we get

$$(3.11) \quad \frac{m}{m-1} \rho_0^{m-1} = u + C \quad \text{a.e. in } B_R(0).$$

Thus from the monotonicity of ρ_0 we deduce $\rho_0 \in L^\infty(\mathbb{R}^2)$. Hence [16, Lemma 4.1] implies $u \in C^1(\mathbb{R}^2)$. Now, for all $r > 0$ we define the *mass function* of ρ_0 ,

$$M_{\rho_0}(r) = \int_{B_r(0)} \rho_0(x) dx.$$

Take any $R_1 < R$ and consider the following boundary value problem:

$$(3.12) \quad \begin{cases} -\Delta v = \rho_0 & \text{in } B_{R_1}(0), \\ v(x) = u(R_1) & \text{on } \partial B_{R_1}(0). \end{cases}$$

The logarithmic potential (3.9) solves problem (3.12), whence $u = v$ on $\overline{B_{R_1}(0)}$. On the other hand, the solution of (3.12) can be written as in [29, 3]: if $r = |x| \in (0, R_1)$,

$$u(r) - u(R_1) = v(r) - v(R_1) = \frac{1}{4\pi} \int_{\pi r^2}^{\pi R_1^2} \frac{1}{s} \int_0^s \rho_0^*(\sigma) d\sigma ds,$$

where ρ_0^* is the *one-dimensional decreasing rearrangement* of ρ_0 . Differentiating we get

$$u'(r) = -\frac{1}{2\pi r} \int_0^{\pi r^2} \rho_0^*(\sigma) d\sigma = -\frac{1}{2\pi r} \int_{B_0(r)} \rho_0(x) dx = -\frac{M_{\rho_0}(r)}{2\pi r},$$

that is,

$$(3.13) \quad \frac{d}{dr}(\rho_0 * \mathcal{K})(r) = -\frac{M_{\rho_0}(r)}{2\pi r} \quad r > 0.$$

By identity (3.13) it follows that ρ_0 is *smooth* inside its support. Indeed, following some arguments of [19], first we observe that the function

$$f(r) := -\frac{M_{\rho_0}(r)}{2\pi r}$$

is continuous for $r > 0$ and $f(r) \rightarrow 0$ as $r \rightarrow 0$: indeed, we have

$$\lim_{r \rightarrow 0} f(r) = -\lim_{r \rightarrow 0} \frac{1}{r} \int_0^r t \rho_0(t) dt = -\lim_{r \rightarrow 0} r \rho_0(r) = 0.$$

Thus $u = \mathcal{K} * \rho_0$ is differentiable everywhere in the positive set $\{\rho_0 > 0\} = B_R(0)$ of ρ_0 . This property and (3.11) imply that ρ_0 is differentiable in $B_R(0)$, so $f(r)$ is twice differentiable. Then we can repeat this argument and conclude.

Let us prove that ρ_0 has *compact support*. There are two different ways to prove this property: one is based on the asymptotic behavior of the log-potential, and the

other relies on a pure ordinary differential equation (ODE) approach relating our global minimizers to nonlinear elliptic equations. We show both methods since they give complementary information. Concerning the first one, we simply use (3.10) to infer that $u(x) \sim MK(x) \rightarrow -\infty$ as $|x| \rightarrow \infty$; hence if (3.11) were satisfied for all x , for a sufficiently large R we would have $\rho_0 < 0$ for all $|x| > R$, which is a contradiction. Then ρ_0 must have compact support.

The other argument to prove that ρ_0 is compactly supported is the following. By contradiction, let us suppose that $\text{supp}(\rho_0) = \mathbb{R}^2$. Then (3.8) implies that the function $\theta := \rho_0^{m-1} \in L^{\frac{m}{m-1}}$ solves the problem

$$(3.14) \quad \begin{cases} -\Delta\theta = \frac{m-1}{m} \theta^{1/(m-1)} & \text{in } \mathbb{R}^2, \\ \theta \rightarrow 0 & \text{as } |x| \rightarrow \infty. \end{cases}$$

Since θ is radial, the first equation in (3.14) (which is an *Emden–Fowler type* equation) can be rewritten as

$$-(r\theta')' = \frac{m-1}{m} r \theta^{1/(m-1)}, \quad r > 0,$$

and with the change of variables $r = e^t$, $w(t) = \theta(e^t)$, the same equation reads

$$(3.15) \quad w''(t) + \frac{m-1}{m} e^{2t} w(t)^{\frac{1}{m-1}} = 0.$$

Now we can invoke [15, Corollary 1.2], since for all $a > 0$ we have

$$\int_a^\infty e^{2t} dt = \int_a^\infty t^{\frac{1}{m-1}} e^{2t} dt = +\infty.$$

We obtain that in both cases $m < 2$, $m > 2$ all the proper solutions to (3.15) are *oscillatory*, namely, they have a sequence of zeros tending to $+\infty$. But this contradicts the fact that θ is everywhere positive. The case $m = 2$ is even simpler. Indeed, in this case when θ satisfies the linear problem (recall that ρ_0 is smooth), we have

$$-(r\theta')' = \frac{r}{2} \theta, \quad r > 0,$$

and the condition $\theta \rightarrow 0$ as $r \rightarrow \infty$ obliges θ to have the form

$$\theta(r) = C J_0 \left(\frac{r}{\sqrt{2}} \right),$$

which is clearly oscillating, leading to a contradiction. Therefore, the support of ρ_0 must be compact. Finally, the Newtonian potential being smooth, together with (3.3), implies that the density ρ_0 is Hölder continuous in \mathbb{R}^2 with exponent $1/(m-1)$. \square

Remark 3.4. By (3.3) and arguing as in [7], we have that $\theta := \rho_0^{m-1}$ is the unique *classical solution* in $B(0, R)$, with zero boundary condition, to the elliptic equation

$$-\Delta\theta = \frac{m-1}{m} \theta^{1/(m-1)}.$$

Therefore, we can write θ in terms of a scaling of the solution ζ to the same problem in the unit ball, namely,

$$\rho_0(x) = R^{2(m-1)/(m-2)} \zeta \left(\frac{x}{R} \right).$$

With the above regularity of global minimizers, it is easy to show that the distributional gradient in \mathbb{R}^2 of ρ_0^m satisfies $\nabla \rho_0^m = \frac{m}{m-1} \rho_0 \nabla \rho_0^{m-1} = m \rho_0^{m-1} [\nabla \rho_0]_+$, with the last gradient being the classical gradient in its support. As a consequence,

$$(3.16) \quad \nabla \rho_0^m - \rho_0 \nabla (\mathcal{K} * \rho_0) = \rho_0 \left[\nabla \left(\frac{m}{m-1} \rho_0^{m-1} - \mathcal{K} * \rho_0 \right) \right]_+ = 0$$

in the sense of distributions. We have deduced the following result.

COROLLARY 3.5. *Global minimizers of the free energy functional \mathbf{G} are stationary solutions of the two-dimensional subcritical Keller–Segel model (1.3) in the distributional sense (3.16).*

Now, let us show the uniqueness of stationary states among the set of radially decreasing compactly supported smooth inside their support solutions. As a consequence, we conclude the uniqueness of global minimizers taking into account Corollary 3.5 and Theorems 2.1 and 3.3. With this aim, we briefly recall some of the main results contained in [19]. We first start with the definition of mass concentration.

DEFINITION 3.6. *Let $\rho_1, \rho_2 \in L^1_{loc}(\mathbb{R}^N)$, $N \geq 1$, be two radially symmetric functions on \mathbb{R}^N . We say that ρ_1 is less concentrated than ρ_2 , and we write $\rho_1 \prec \rho_2$ if for all $r > 0$ we get*

$$\int_{B_r(0)} \rho_1(x) dx \leq \int_{B_r(0)} \rho_2(x) dx.$$

The partial order relationship \prec is called *comparison of mass concentrations*. Of course, this definition can be suitably adapted if ρ_1, ρ_2 are radially symmetric and locally integrable functions on a ball B_R . Besides, if ρ_1 and ρ_2 are locally integrable on a general open set Ω , we say that ρ_1 is less concentrated than ρ_2 and we write again $\rho_1 \prec \rho_2$ simply if $\rho_1^\# \prec \rho_2^\#$.

If $\rho(x, t)$ is a locally integrable function on \mathbb{R}^N for all times $t \geq 0$, we define the time dependant *mass function* of ρ as

$$(3.17) \quad M_\rho(r, t) = \int_{B_r(0)} \rho(x, t) dx.$$

If $\rho(x, t)$ is the solution to the evolution problem (1.3) where the initial data $\rho(x, 0)$ is a continuous, compactly supported, radially decreasing function, then it is easy to check (see [19]) that the mass function $M_\rho(r, t)$ satisfies, in the support $\{x : |x| < R(t)\}$ of $\rho(\cdot, t)$, the PDE

$$(3.18) \quad \frac{\partial M_\rho}{\partial t}(r, t) = 2\pi r \frac{\partial}{\partial r} \left(\left(\frac{1}{2\pi r} \frac{\partial M_\rho}{\partial r} \right)^m \right) + \left(\frac{1}{2\pi r} \frac{\partial M_\rho}{\partial r} \right) M_\rho.$$

Let us take a function $\rho(x, t)$ being C^1 in its positive set and $\rho(\cdot, t) \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ for each $t \geq 0$. We will say that ρ is a subsolution (resp., a supersolution) to (3.18) if the sign \leq (resp., the sign \geq) replaces the equal sign in (3.18).

In [19] the following result concerning the mass comparison is proved, which is readily seen to hold also in dimension $N = 2$.

PROPOSITION 3.7. *Assume that ρ_1, ρ_2 are, respectively, a subsolution and a supersolution to (3.18). Suppose that ρ_1, ρ_2 preserve the mass through time, i.e.,*

$$\int_{\mathbb{R}^2} \rho_1(x, t) dx = \int_{\mathbb{R}^2} \rho_1(x, 0) dx, \quad \int_{\mathbb{R}^2} \rho_2(x, t) dx = \int_{\mathbb{R}^2} \rho_2(x, 0) dx,$$

and that ρ_1 is less concentrated than ρ_2 at the initial time, namely,

$$\rho_1(x, 0) \prec \rho_2(x, 0).$$

Then the mass functions preserve the same order for all times:

$$\rho_1(x, t) \prec \rho_2(x, t) \quad \forall t \geq 0.$$

It is also possible to show a two-dimensional version of [19, Theorem 5.6], showing an exponential convergence of the mass function of the solution to (1.3) with a generic radial initial data to the mass function of a steady state having the same mass. Notice that the existence of a radially decreasing steady state with given mass M is guaranteed by Corollary 3.5 and Theorems 2.1 and 3.3.

THEOREM 3.8 (exponential convergence of the mass function). *Let $\rho(x, t)$ be the solution to (1.3) with initial data $\rho(x, 0) \geq 0$ being a continuous, radially decreasing, compactly supported function on \mathbb{R}^2 . Let ρ_0 be a radially decreasing steady state to (1.3) in the distributional sense (3.16) with mass M , where $M = \|\rho(x, 0)\|_1$. Then*

$$(3.19) \quad |M_\rho(r, t) - M_{\rho_0}(r)| \leq C e^{-\lambda t} \quad \forall r \geq 0,$$

where C depends on $\rho(x, 0)$, M , and m , and the rate λ depends only on M .

Proof. We briefly provide the main arguments of the proof, since it is totally analogous to the proof of [19, Theorem 5.6], to which the interested reader should refer. We can always assume that $\rho(0, 0) > 0$, as otherwise $\rho(0, t)$ will become positive in finite time (see [19, Corollary 5.5]). It is always possible to choose a small positive constant a such that

$$(3.20) \quad a^2 \rho_0(ax) \prec \rho(x, 0) \quad \text{and} \quad a^{-2} \rho_0(a^{-1}x) \succ \rho(x, 0).$$

For a given nonnegative function $\xi(x, t)$ which is differentiable in its positive set, let us introduce its velocity field $\vec{v}(x, t; \xi)$ through the formula

$$\vec{v}(x, t; \xi) = -\frac{m}{m-1} \nabla(\xi^{m-1}) + \nabla(\xi * \mathcal{K}).$$

It is possible to prove that if we consider the velocity field $\vec{v}(x; \rho_1^a)$ of $\rho_1^a(x) = a^2 \rho_0(ax)$, then its inward normal component

$$v(r) = \vec{v}(x; \rho_1^a) \cdot \left(-\frac{x}{r}\right) = \frac{m}{m-1} \frac{\partial}{\partial r} \rho_1^a - \frac{\partial}{\partial r} (\rho_1 * \mathcal{K})$$

satisfies, for all $a \in (0, 1)$, the estimate

$$v(r) \geq (1 - a^2(m-1))a^2 r \frac{M_{\rho_0}(ar)}{2\pi(ar)^2} \geq 0.$$

Since in the positive set of ρ_0 we have, for suitable positive constants C_1, C_2 ,

$$(3.21) \quad C_1 \leq \frac{M_{\rho_0}(ar)}{2\pi(ar)^2} \leq C_2,$$

by the previous estimate we find

$$v(r) \geq C_1(1 - a^2(m-1))a^2 r.$$

With the choice of a for which the two relations in (3.20) hold, we define the function

$$\phi(r, t) = k^2(t)\rho_0(k(t)r),$$

where we impose that the scaling factor $k(t)$ satisfies the following ODE with initial data $k(0) = a$:

$$k'(t) = C_1(k(t))^3(1 - (k(t))^{2(m-1)}).$$

Then one proves that $\phi(r, t)$ is a subsolution to (3.18) and that the following exponential estimate holds:

$$(3.22) \quad 0 \leq M_{\rho_0}(r) - M_\phi(r, t) \lesssim \exp(-2C_1(m-1)t).$$

Similarly, we construct a supersolution to (3.18) by taking into account the constant C_2 in (3.21) and defining the function

$$\eta(r, t) = k^2(t)\rho_0(k(t)r),$$

where $k(t)$ solves this time the following ODE with initial data $k(0) = 1/a$:

$$k'(t) = C_2(k(t))^3(1 - (k(t))^{2(m-1)}).$$

Then $\eta(r, t)$ is shown to be a supersolution to (3.18), whose mass function satisfies the estimate

$$(3.23) \quad 0 \leq M_\eta(r, t) - M_{\rho_0}(r) \lesssim \exp(-2C_2(m-1)t).$$

Now, from relations (3.20) we find $\phi(\cdot, 0) \prec \rho(\cdot, 0) \prec \eta(\cdot, 0)$, and thus by Proposition 3.7 we get

$$(3.24) \quad M_\phi(r, t) \leq M_\rho(r, t) \leq M_\eta(r, t)$$

for all $r, t > 0$. Then inequalities (3.22)-(3.23)-(3.24) yield

$$|M_\rho(r, t) - M_{\rho_0}(r)| \lesssim e^{-\lambda t}$$

where $\lambda = 2(m-1) \min\{C_1, C_2\}$. \square

As a consequence, the uniqueness of a radially decreasing steady state of a given mass M follows: in fact, if there were two such steady states $\rho_0, \bar{\rho}_0$, inequality (3.19) would ensure that

$$M_{\rho_0}(r) = M_{\bar{\rho}_0}(r),$$

and therefore differentiating we find $\rho_0 = \bar{\rho}_0$. Summarizing the results of the last two sections, we conclude with the following.

THEOREM 3.9 (uniqueness of global minimizers). *There is a unique global minimizer of the free energy functional G defined by (1.4) in \mathcal{Y}_M . Moreover, such a minimizer is the unique radially decreasing, compactly supported, and smooth in its support steady state of (1.3) in the distributional sense (3.16) characterized by (3.1)-(3.2).*

4. Symmetry of the steady states. The aim of this section is to establish the symmetry of any compactly supported steady state, and not only of global minimizers, which in turn will yield the uniqueness of compactly supported steady states. Consider a nonnegative density $\rho \in \mathcal{Y}_M$ and notice that, thanks to the fact that $\rho \log \rho$ and $\rho \log(1 + |x|^2)$ belong to $L^1(\mathbb{R}^2)$, the logarithmic potential associated to ρ , denoted in the rest of the paper by $u = \mathcal{K} * \rho$, is well defined. This is, for instance, a consequence of the logarithmic HLS inequality (2.2); see [12]. Let us specify the definition of steady state for the nonlinear diffusion Keller–Segel model (1.3) following [25, 4].

DEFINITION 4.1. *A nonnegative compactly supported density $\rho \in \mathcal{Y}_M$ is a stationary state for the evolution problem (1.3) if $\rho \in L^\infty(\mathbb{R}^2)$, $\rho^{m-1} \in W_{loc}^{1,m}(\mathbb{R}^2)$, and the couple (ρ, u) satisfies*

$$(4.1) \quad \Delta \rho^m = \frac{m}{m-1} \nabla \cdot (\rho \nabla \rho^{m-1}) = \nabla \cdot (\rho \nabla u) \text{ in } \mathbb{R}^2$$

in the distributional sense, with u being the Newtonian potential associated to ρ as in (3.9).

Let us first observe that the nonlinear term in the right-hand side of (4.1) makes sense for compactly supported steady states. Notice that the logarithmic potential $u = \mathcal{K} * \rho$ is an L^1 distributional solution of $-\Delta u = \rho$ with $\rho \in L^m(\mathbb{R}^2)$, $m > 1$. Thus, using the elliptic regularity theory [16, Lemma 9.9], we deduce that $u \in W_{loc}^{2,m}(\mathbb{R}^2)$ and, thanks to the Sobolev embedding in dimension 2, we have that $u \in C_{loc}^{1,\alpha}(\mathbb{R}^2)$ for some $\alpha > 0$. On the other hand, by the fact that $\rho \in L^m(\mathbb{R}^2)$, $m > 1$, and ∇u is locally bounded, we see that the left-hand side of (4.1) belongs to $W_{loc}^{-1,m'}(\mathbb{R}^2)$. Noticing that ρ^m is an L^1 distributional solution of (4.1) with datum in $W^{-1,m'}$ and that $\rho^m = 0$ on the boundary of a sufficiently large ball by the compact support hypothesis, we see that ρ^m is in fact a weak $W_{loc}^{1,m}$ solution of (4.1); cf. [20]. Sobolev embedding shows that both ρ^m and ρ belong to some Hölder space $C^{0,\alpha}(\mathbb{R}^2)$. Since $m > 1$ and $\rho^{m-1} \in W_{loc}^{1,m}(\mathbb{R}^2)$, we have $\nabla \rho^m = \frac{m}{m-1} \rho \nabla \rho^{m-1}$. We conclude that wherever ρ is positive, (4.1) can be interpreted as

$$\nabla \left(\frac{m}{m-1} \rho^{m-1} - u \right) = 0$$

in the sense of distributions in $\Omega = \text{supp}(\rho)$. Hence, the function $G(x) = \frac{m}{m-1} \rho^{m-1} - u(x)$ is constant in each connected component of Ω , and u satisfies the elliptic equation $-\Delta u = g(x, u)$ with the nonlinearity g given by

$$(4.2) \quad g(x, u) = \left(\frac{m-1}{m} \right)^{\frac{1}{m-1}} ((G(x) + u)^+)^{\frac{1}{m-1}}$$

for all $u \in \mathbb{R}$ and $x \in \Omega$. We are now ready to state our symmetry result.

THEOREM 4.2. *Let $\rho \in \mathcal{Y}_M$ be any nonnegative compactly supported stationary state. Then ρ is radially symmetric about the origin.*

The proof of Theorem 4.2 will be achieved thanks to a nonstandard moving plane type argument for u , especially thanks to a precise decay estimate at infinity and a symmetry property for the function G introduced above. This result is known in the corresponding range of nonlinearities in larger dimensions. Here, the main technical difficulty is to deal with the logarithmic behavior of the Newtonian potential in two dimensions.

First of all, we need to prove a precise decay estimate for u : we already know, thanks to (3.10), that $u(x)$ behaves like $-M \log |x|$ when $|x|$ is large, but unfortunately

this is not enough for our purposes. Let us assume that $\text{supp}(\rho) \subset B_{r_o}(0)$, i.e., $\rho(x) = 0$ for any $|x| > r_o$: we can refine the asymptotic behavior of u as given by the following.

PROPOSITION 4.3. *There are $C_1, C_2 > 0$ such that for all $|x| \geq 2r_o$,*

$$(4.3) \quad |u(x) - MK(x)| \leq C_1 r_o^2 |x|^{-2}$$

and

$$(4.4) \quad |\nabla(u(x) - MK(x))| \leq C_2 r_o^2 |x|^{-3}$$

hold.

Proof. First, notice that

$$|u(x) - MK(x)| = \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} (\log|x-y| - \log|x|) \rho(y) dy \right|.$$

We can proceed essentially as in the proof of [25, Lemma 1]. Notice that for $|x| \geq 2r_o$ we have $\text{supp}(\rho) \subset \{|y| \leq \frac{|x|}{2}\}$. Thus, thanks to the homogeneity of the derivatives of the kernel $\mathcal{K}(x)$ and the zero center-of-mass condition, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^2} (\mathcal{K}(x-y) - \mathcal{K}(x)) \rho(y) dy \right| &= \left| \int_{\mathbb{R}^2} \left(\frac{1}{2} \nabla^2 \mathcal{K}(x - \sigma y) y \cdot y - \nabla \mathcal{K}(x) \cdot y \right) \rho(y) dy \right| \\ &\leq \frac{1}{2} \int_{|y| \leq \frac{|x|}{2}} |\nabla^2 \mathcal{K}(x - \sigma y)| |y|^2 \rho(y) dy \\ &= \frac{1}{2} \int_{|y| \leq \frac{|x|}{2}} \left| \nabla^2 \mathcal{K} \left(\frac{x - \sigma y}{|x - \sigma y|} \right) \right| \frac{|y|^2}{|x - \sigma y|^2} \rho(y) dy \\ &\leq \frac{1}{2} \left(\sup_{\mathbb{S}^1} |\nabla^2 \mathcal{K}| \right) \int_{|y| \leq \frac{|x|}{2}} \frac{|y|^2}{(|x| - |y|)^2} \rho(y) dy \\ &\leq \frac{2}{|x|^2} \left(\sup_{\mathbb{S}^1} |\nabla^2 \mathcal{K}| \right) \int_{|y| \leq \frac{|x|}{2}} |y|^2 \rho(y) dy \\ &\leq 2Mr_o^2 \left(\sup_{\mathbb{S}^1} |\nabla^2 \mathcal{K}| \right) |x|^{-2} \end{aligned}$$

due to a simple Taylor expansion for some $0 < \sigma(y) < 1$, leading to the desired estimate (4.3). Replacing the kernel $-\log|x|$ with its x_1 derivative $-x_1/|x|^2$ (which is homogeneous of degree -1), the same proof leads to (4.4). \square

We will now start a moving plane type procedure in order to establish that the solution is even with respect to the first variable. Then, thanks to the rotational invariance, we will deduce that u is even with respect to any plane through the origin, which in turn means that u is in fact radial. Hence, let $x_\lambda = \sigma_\lambda(x) = (2\lambda - x_1, x_2)$ be the reflection of $x \in \Sigma_\lambda = \{x_1 < \lambda\}$ with respect to the plane $T_\lambda = \{x_1 = \lambda\}$, and let $u_\lambda(x) := u(x_\lambda) = u(2\lambda - x_1, x_2)$ be the corresponding reflection of u .

Thanks to the strict monotonicity of our kernel, we will start by showing that well away from the x_2 axis the difference of u and u_λ is nonpositive as shown by the following.

LEMMA 4.4. *If $\lambda < -r_o$, then $u_\lambda(x) \geq u(x)$ for any $x \in \Sigma_\lambda$.*

Proof. From (4.3), for $|x| \geq 2r_o$ and $|x_\lambda| \geq 2r_o$ we find

$$(4.5) \quad u(x) - u_\lambda(x) \leq \frac{M}{2\pi} \log \frac{|x_\lambda|}{|x|} + Cr_o^2 (|x|^{-2} + |x_\lambda|^{-2}).$$

For $x \in \Sigma_\lambda$ and $\lambda < 0$, we observe that $|x| - \lambda \leq |x_\lambda| \leq |x|$. Hence, we get

$$(4.6) \quad \limsup_{|x| \rightarrow \infty} [u(x) - u_\lambda(x)] \leq 0 \text{ for any } x \in \Sigma_\lambda,$$

while $u = u_\lambda$ on T_λ by definition. Observe that since ρ is $C^{0,\alpha}$, by Schauder estimates, u and u_λ satisfy, respectively,

$$-\Delta u(x) = \rho(x), \quad -\Delta u_\lambda(x) = \rho(x_\lambda)$$

in the classical pointwise sense. Moreover, since for $|x| > r$ we have that $\rho(x) = 0$ while $\rho(x_\lambda) \geq 0$ always, by (4.6) we finally obtain

$$(4.7) \quad \Delta u \geq \Delta u_\lambda \text{ in } \Sigma_\lambda, \quad u \leq u_\lambda \text{ on } \partial\Sigma_\lambda, \quad \limsup_{|x| \rightarrow \infty} [u - u_\lambda] \leq 0.$$

By (4.7) and the classical comparison principle, we find $u \leq u_\lambda$ on Σ_λ . \square

Next, we will show that the same property of Lemma 4.4, for a fixed negative λ , is true outside a sufficiently large ball.

LEMMA 4.5. *For any $\lambda < 0$, there exists $R_\lambda > 0$ such that $u_\lambda(x) \geq u(x)$ for any $x \in (B(0, R_\lambda))^c \cap \Sigma_\lambda$.*

Proof. For any $x \in \Sigma_\lambda$, we have, as above, $|x| - \lambda \leq |x_\lambda| \leq |x|$, and thus,

$$\lim_{|x| \rightarrow \infty} \frac{|x_\lambda|^2}{|x|^2} = 1.$$

This easily implies that there exists $R_{1,\lambda}$ sufficiently large such that

$$(4.8) \quad |x_\lambda| \leq |x| \leq 2|x_\lambda| \quad \text{for } |x| \geq R_{1,\lambda}.$$

Using (4.5) in view of (4.8) for $x_1 < -R_{1,\lambda}$, and if, furthermore, we assume $x_1 < \lambda - \frac{5C\pi r_o^2}{M|\lambda|}$, we have

$$\begin{aligned} u(x) - u_\lambda(x) &\leq \frac{M}{2\pi} \log \frac{|x_\lambda|}{|x|} + Cr_o^2(|x|^{-2} + |x_\lambda|^{-2}) \\ &\leq \frac{M}{4\pi} \log \frac{|x_\lambda|^2}{|x|^2} + 5Cr_o^2|x|^{-2} \\ &= \frac{M}{4\pi} \log \left(1 + \frac{4\lambda(\lambda - x_1)}{|x|^2} \right) + 5Cr_o^2|x|^{-2} \\ &\leq \frac{M\lambda(\lambda - x_1) + 5C\pi r_o^2}{|x|^2} < 0. \end{aligned}$$

In particular we have found that

$$(4.9) \quad u(x) - u_\lambda(x) < 0 \quad \text{for } x_1 < \mu_{1,\lambda} := \min \left\{ -R_{1,\lambda}, \lambda - \frac{5C\pi r_o^2}{M|\lambda|} \right\}.$$

Observe that by continuity and (4.9) we have that $u(x) - u_\lambda(x) \leq 0$ on $x_1 = \mu_{1,\lambda}$. Notice that if $\mu_{1,\lambda} \leq x_1 < \lambda$ and $|x_2| \geq 2r_o$, we can apply (4.4) in order to get

$$(4.10) \quad |u_{x_1}(x) - MK_{x_1}(x)| \leq Cr_o^2|x|^{-3}.$$

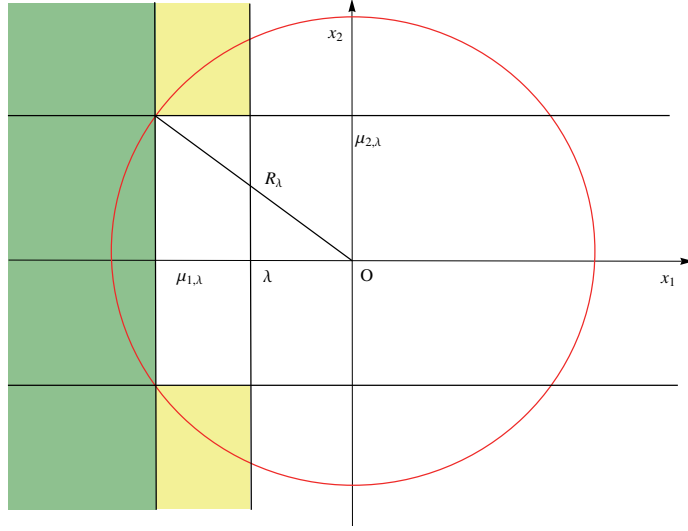


FIG. 1. Illustration for the estimates of Lemma 4.5.

Recalling that $\frac{\partial}{\partial x_1} u_\lambda(x) = -u_{x_1}(2\lambda - x_1, x_2)$, choosing $|x_2| \geq \max\{2r_o, R_{1,\lambda}\}$, we can apply (4.10) and (4.8) in order to deduce that

$$\begin{aligned}
 \frac{\partial}{\partial x_1} (u_\lambda(x) - u(x)) &\leq M \left(\frac{x_1}{2\pi|x|^2} + \frac{2\lambda - x_1}{2\pi|x_\lambda|^2} \right) + Cr_o^2(|x|^{-3} + |x_\lambda|^{-3}) \\
 &\leq M \frac{\lambda}{\pi|x|^2} + M(2\lambda - x_1) \left(\frac{1}{2\pi|x_\lambda|^2} - \frac{1}{2\pi|x|^2} \right) + 9Cr_o^2|x|^{-3} \\
 &\leq M \frac{\lambda}{\pi|x|^2} + M \frac{4\lambda(2\lambda - x_1)(x_1 - \lambda)}{2\pi|x_\lambda|^2|x|^2} + 9Cr_o^2|x|^{-3} \\
 &\leq M \frac{\lambda}{\pi|x|^2} + M \frac{8\lambda(2\lambda - \mu_{1,\lambda})(\mu_{1,\lambda} - \lambda)}{\pi|x|^4} + 9Cr_o^2|x|^{-3} \\
 &= M \frac{\lambda}{\pi|x|^2} \left(1 + O\left(\frac{1}{|x|}\right) \right) < 0
 \end{aligned}$$

if, furthermore, $|x_2| > R_{2,\lambda}$ sufficiently large. Thus, choosing $\mu_{2,\lambda} = \max\{2r_o; R_{1,\lambda}; R_{2,\lambda}\}$, we get

$$(u(x) - u_\lambda(x))_{x_1} < 0 \quad \text{for any } \mu_{1,\lambda} \leq x_1 < \lambda \quad \text{and } |x_2| \geq \mu_{2,\lambda},$$

while $u \leq u_\lambda$ for $x_1 = \mu_{1,\lambda}$. This in turn implies that

$$(4.11) \quad u(x) - u_\lambda(x) < 0 \quad \text{for any } \mu_{1,\lambda} \leq x_1 < \lambda \quad \text{and } |x_2| \geq \mu_{2,\lambda}.$$

Finally, from (4.9) and (4.11), choosing $R_\lambda := \sqrt{\mu_{1,\lambda}^2 + \mu_{2,\lambda}^2}$, we obtain $u(x) - u_\lambda(x) \leq 0$ for any $x \in (B(0, R_\lambda))^c \cap \Sigma_\lambda$. The proof of this lemma is illustrated in Figure 1. \square

Thanks to Lemma 4.4 and to the fact that $\sigma_\lambda(\overline{\Omega \cap \Sigma_\lambda})$ is empty for $\lambda < -r_o$, the following quantity is well defined:

$$(4.12) \quad \Lambda := \sup\{\lambda < 0 : u_\lambda(x) > u(x) \text{ for any } x \in \Sigma_\lambda \text{ and } \sigma_\lambda(\overline{\Omega \cap \Sigma_\lambda}) \subset \overline{\Omega}\}.$$

Moreover, by the continuity with respect to λ and the fact that Σ_λ is a decreasing set-valued function of λ , we see that Λ is in fact attained. Our aim is then to show that $\Lambda = 0$. However, since our problem is not autonomous (notice the x dependence of the nonlinearity $g(x, u)$ given by (4.2)), we cannot proceed with a standard moving plane argument, and we need to recall from [25] an important reflection property of the function $G(x)$ introduced above.

LEMMA 4.6. $G(x) = G(x_\lambda)$ for any $x \in \Sigma_\lambda$ and $\lambda \leq \Lambda$.

Proof. Without loss of generality, we will prove the statement for $x \in \Omega \cap \Sigma_\lambda$ and derive by continuity the property up to $\overline{\Omega \cap \Sigma_\lambda}$. By (4.12) we have that $\sigma_\lambda(\Omega \cap \Sigma_\lambda) \subset \overline{\Omega}$ and, as the reflection σ_λ is a homeomorphism and sends interior points to interior points of Ω , we get that $\sigma_\lambda(\Omega \cap \Sigma_\lambda) \subset \Omega$. Now, we will prove that $\sigma_\lambda(\omega \cap \Sigma_\lambda) \subset \omega$ for any component ω of Ω and for $\lambda \leq \Lambda$. Let us fix any $x \in \omega$ and let $\mu = x_1$. Then $\sigma_\mu(x) = x$, $x \in \Sigma_\lambda$ for any $\lambda > \mu$, and $\sigma_\lambda(x) \in \Omega$ for any $\mu \leq \lambda \leq \Lambda$. Since $\sigma_\mu(x) = x \in \omega$ and the range of $\sigma_\lambda(x)$, for $\mu \leq \lambda \leq \Lambda$, is a line segment wholly inside Ω , we necessarily have that $x_\lambda \in \omega$ for any $\mu \leq \lambda \leq \Lambda$, which in turn implies that $G(x) = G(x_\lambda)$. \square

We are now ready to prove Theorem 4.2.

4.1. Proof of Theorem 4.2. We will proceed with a moving plane argument for $u(x) := (\mathcal{K} * \rho)(x)$. Recalling the definition of Λ given in (4.12) as the maximal negative parameter λ for which the reflection u_λ is larger than u , we want to prove that $\Lambda = 0$. So let us argue by contradiction and suppose that $\Lambda < 0$. Recall that u satisfies $-\Delta u = g(x, u)$ with the nonlinearity $g(x, u)$ in (4.2) being nonnegative and increasing with respect to u . By direct computation and thanks to Lemma 4.6, we also get

$$-\Delta u_\Lambda = g(x_\Lambda, u_\Lambda) = g(x, u_\Lambda).$$

Now, by the continuity with respect to λ and x we have that $u_\Lambda(x) \geq u(x)$ for $x \in \Sigma_\Lambda$. This implies that $g(x, u) \leq g(x_\Lambda, u_\Lambda)$ for any $x \in \Sigma_\Lambda$, i.e., $\Delta u \geq \Delta u_\Lambda$. By the strong comparison principle, we infer that either $u \equiv u_\Lambda$ or $u < u_\Lambda$ on Σ_Λ . But the case $u \equiv u_\Lambda$ can be ruled as follows. Since u satisfies

$$(4.13) \quad -\Delta u = \rho,$$

we have that $\rho(x) = \rho(x_\Lambda)$, and thus by the zero center-of-mass condition for ρ and through the change of variables $(x'_1, x'_2) = (2\Lambda - x_1, x_2)$, we get

$$0 = \int_{\mathbb{R}^2} x_1 \rho(x) dx = \int_{\mathbb{R}^2} (2\Lambda - x'_1) \rho(2\Lambda - x'_1, x'_2) dx' = \int_{\mathbb{R}^2} (2\Lambda - x'_1) \rho(x'_1, x'_2) = 2\Lambda M;$$

therefore $\Lambda = 0$, a contradiction. Hence $u < u_\Lambda$ on Σ_Λ .

For small $\varepsilon > 0$ with $\Lambda + \varepsilon < 0$, let us consider the cap $\Sigma_{\Lambda+\varepsilon}$ with corresponding reflection $u_{\Lambda+\varepsilon}$. We will proceed by dividing Σ_Λ into four subsets as illustrated in Figure 2. Let $R_{\Lambda+\varepsilon} > 2r_o$ be as in Lemma 4.5, and consider the concentric balls $B(0, 2r_o)$ and $B(0, R_{\Lambda+\varepsilon})$. We divide the cap $\Sigma_{\Lambda+\varepsilon}$ into four subsets given by

$$A_1 := \overline{B(0, R_{\Lambda+\varepsilon})}^c \cap \Sigma_{\Lambda+\varepsilon}; \quad A_2 := \overline{B(0, 2r_o)} \cap \Sigma_{\Lambda-\varepsilon};$$

$$A_3 := \overline{B(0, 2r_o)} \cap (\Sigma_{\Lambda+\varepsilon} \setminus \Sigma_{\Lambda-\varepsilon}); \quad A_4 := (B(0, R_{\Lambda+\varepsilon}) \setminus \overline{B(0, 2r_o)}) \cap \Sigma_{\Lambda+\varepsilon}.$$

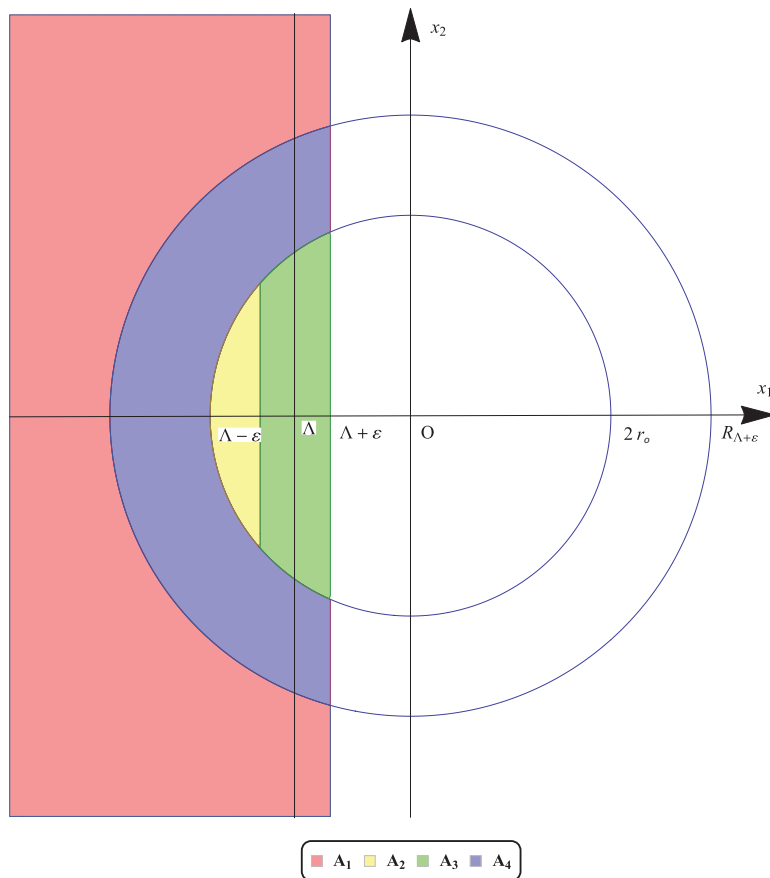


FIG. 2. Illustration of the division of sets for the proof of Theorem 4.2.

On the set A_1 , we can apply Lemma 4.5 to get that $u_{\Lambda+\varepsilon}(x) \geq u(x)$ for all $x \in A_1$. Observe that the set A_2 is compact; then by continuity the fact that $u_{\Lambda}(x) > u(x)$ implies that $u_{\Lambda}(x) \geq u(x) + 2\sigma$ for some small $\sigma > 0$ and for all $x \in A_2$. Thus, by the continuity with respect to λ , we see that $u_{\Lambda+\varepsilon}(x) \geq u(x) + \sigma$ for all $x \in A_2$ for ε small enough, which means

$$u_{\Lambda+\varepsilon}(x) \geq u(x) \quad \forall x \in A_2.$$

On the set A_3 we need to argue as follows. Since $u < u_{\Lambda}$ on Σ_{Λ} and $u = u_{\Lambda}$ on T_{Λ} , by Hopf's lemma we know that $\partial_{\nu}(u - u_{\Lambda}) > 0$ on T_{Λ} . But $\partial_{\nu}(u - u_{\Lambda}) = 2\partial_{\nu}u = 2u_{x_1} > 0$ on T_{Λ} . In particular, there exists a constant σ_o such that $u_{x_1} \geq \sigma_o > 0$ on $T_{\Lambda} \cap B(0, 2r_o)$, which implies that there exists $\varepsilon > 0$ small enough such that $u_{x_1} \geq \sigma_o/2$ on A_3 by continuity. Then we can show that

$$\begin{aligned} u_{\Lambda+\varepsilon}(x) - u(x) &= u(2(\Lambda + \varepsilon) - x_1, x_2) - u(x_1, x_2) \\ &= \int_{x_1}^{2(\Lambda+\varepsilon)-x_1} u_{x_1}(s, x_2) ds \geq \sigma_o(\Lambda + \varepsilon - x_1) \geq 0 \end{aligned}$$

on A_3 . Finally, for any $x \in A_4$ we have $\rho(x_{\Lambda+\varepsilon}) \geq 0$ and $\rho(x) = 0$ because $|x| > 2r_o$. Moreover, from the sign condition proved on the other A_j , $j = 1, \dots, 3$, and the fact

that $u_{\Lambda+\varepsilon} = u$ on $T_{\Lambda+\varepsilon}$, we have $u_{\Lambda+\varepsilon} \geq u$ on ∂A_4 . Thus

$$\Delta u \leq \Delta u_{\Lambda+\varepsilon} \text{ in } A_4, \quad u \leq u_{\Lambda+\varepsilon} \text{ on } \partial A_4.$$

By the comparison principle, we then conclude that $u_{\Lambda+\varepsilon}(x) \geq u(x)$ for all $x \in A_4$. We have thus proved that $u_{\Lambda+\varepsilon}(x) \geq u(x)$ for any $x \in \Sigma_{\Lambda+\varepsilon}$. Then, in order to contradict the maximality of Λ as given by (4.12), it only remains to prove that $\sigma_{\Lambda+\varepsilon}(\overline{\Omega \cap \Sigma_{\Lambda+\varepsilon}}) \subset \overline{\Omega}$. Arguing by contradiction, if this were not true, there would exist two sequences $\varepsilon_k \rightarrow 0$ and $y_k \in \overline{\Omega \cap \Sigma_{\Lambda+\varepsilon_k}}$ such that $x_k := \sigma_{\Lambda+\varepsilon_k}(y_k) \notin \overline{\Omega}$. Without loss of generality, due to Ω being compact, we can suppose that $y_k \in \Omega \cap \Sigma_{\Lambda+\varepsilon_k}$, $y_k \rightarrow \bar{y}$ and $x_k \rightarrow \bar{x}$ as $k \rightarrow \infty$. This implies that $\bar{x} = \sigma_{\Lambda}(\bar{y})$ and $\bar{x} \notin \Omega$, while $\bar{y} \in \overline{\Omega \cap \Sigma_{\Lambda}}$. But by continuity and by the definition of Λ in (4.12), we have that necessarily $\bar{x} \in \overline{\Omega}$, and therefore $\bar{x} \in \partial\Omega$, i.e., $\rho(\bar{x}) = 0$. Lemma 4.6 implies that

$$u(\bar{x}) = -G(\bar{x}) = -G(\sigma_{\Lambda}(\bar{y})) = -G(\bar{y}) = u(\bar{y}) - \frac{m}{m-1} \rho^{m-1}(\bar{y}) \leq u(\bar{y}) = u_{\Lambda}(\bar{x}) \leq u(\bar{x}).$$

From this we deduce that $\bar{x} = \sigma_{\Lambda}(\bar{x})$ and also $\bar{x} \in T_{\Lambda} = \partial\Sigma_{\Lambda}$ which, by Hopf's lemma, implies that $u_{x_1}(\bar{x}) > 0$. Moreover, $\sigma_{\Lambda}(\bar{x}) = \bar{x}$, which in turn gives that $\bar{x} = \bar{y}$. Now, since $y_k \rightarrow \bar{x}$, for sufficiently large k we have that $u_{x_1}(y_k) \geq \frac{1}{2}u_{x_1}(\bar{x}) > 0$. In particular, since $y_k \in \Omega$ and $\rho = \frac{m-1}{m}(C+u)^{\frac{1}{m-1}}$, there exists $\tau > 0$ independent of k such that $\rho(y_k + \sigma e_1) \geq \rho(y_k) > 0$ for any k large and $0 \leq \sigma \leq \tau$, with $e_1 = (1, 0)$ as usual. This means that $y_k + \sigma e_1 \in \Omega$ for $\sigma \in [0, \tau]$. But observe that by definition we deduce that

$$\sigma_{\Lambda+\varepsilon_k}(y_k) = 2(\Lambda + \varepsilon_k - y_{k,1})e_1 + y_k$$

with $\varepsilon_k \searrow 0$ and $\Lambda + \varepsilon_k > y_{k,1} \rightarrow \Lambda$, which implies that $\sigma_{\Lambda+\varepsilon_k}(y_k) \in \Omega$ for k sufficiently large, contradicting our assumption.

Hence $\Lambda = 0$, which implies that $u(x_1, x_2) \leq u(-x_1, x_2)$ for any $x_1 \leq 0$. Repeating the same arguments in the opposite direction, we reach $u(x_1, x_2) \geq u(-x_1, x_2)$ for any $x_1 \geq 0$. But since

$$u(x_1, x_2) \leq u(-x_1, x_2) \leq u(-(-x_1), x_2) = u(x_1, x_2),$$

we obtain that u is even in x_1 . By rotational invariance, u is even with respect to any hyperplane through the origin and hence radially symmetric. Hence, (4.13) gives the radial symmetry of ρ too. \square

Remark 4.7 (properties of compactly supported stationary states). Let ρ be a stationary state to (1.3) with compact support. Then, Theorem 4.2 tells us that ρ is radial around its center of mass, assumed to be zero without loss of generality, so we have that the equation

$$(4.14) \quad \frac{m}{m-1} \rho^{m-1} = u + C$$

holds for some constant C . Arguing as in Theorem 3.3, we find that ρ satisfies the equation

$$\frac{d}{dr}(\rho * \mathcal{K})(r) = \frac{du}{dr} = -\frac{M_{\rho}(r)}{2\pi r},$$

where M_{ρ} is the mass function (3.17) of ρ . But then, a simple differentiation of (4.14) gives

$$\frac{m}{m-1} \frac{d}{dr} \rho^{m-1} = -\frac{M_{\rho}(r)}{2\pi r},$$

and thus ρ is radially decreasing. The same argument as in the proof of Theorem 3.3 shows that ρ is smooth inside its support.

THEOREM 4.8. *There is a unique up to translation compactly supported steady state to (1.3) with mass M . Moreover, such steady state is radially decreasing, continuous, and smooth in its support, and it coincides (up to translation) with the global minimizer of the free energy functional G in \mathcal{Y}_M .*

Proof. By Theorem 4.2 and Remark 4.7 we have that any compactly supported steady state ρ is a radially symmetric decreasing continuous function smooth in its support. Then Theorem 3.9 provides its uniqueness and identifies it with the global minimizer of G in the class \mathcal{Y}_M . \square

REFERENCES

- [1] L. AMBROSIO, N. GIGLI, AND G. SAVARÉ, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, 2nd ed., Lectures in Math. ETH Zürich, Birkhäuser Verlag, Basel, 2008.
- [2] J. BEDROSSIAN, *Global minimizers for free energies of subcritical aggregation equations with degenerate diffusion*, Appl. Math. Lett., 24 (2011), pp. 1927–1932.
- [3] C. BENNETT AND R. SHARPLEY, *Interpolation of Operators*, Pure Appl. Math. 129, Academic Press, Boston, MA, 1988.
- [4] S. BIAN AND J.-G. LIU, *Dynamic and steady states for multi-dimensional Keller-Segel model with diffusion exponent $m > 0$* , Comm. Math. Phys., 323 (2013), pp. 1017–1070.
- [5] A. BLANCHET, V. CALVEZ, AND J. A. CARRILLO, *Convergence of the mass-transport steepest descent scheme for the subcritical Patlak-Keller-Segel model*, SIAM J. Numer. Anal., 46 (2008), pp. 691–721.
- [6] A. BLANCHET, E. A. CARLEN, AND J. A. CARRILLO, *Functional inequalities, thick tails and asymptotics for the critical mass Patlak-Keller-Segel model*, J. Funct. Anal., 262 (2012), pp. 2142–2230.
- [7] A. BLANCHET, J. A. CARRILLO, AND P. LAURENÇOT, *Critical mass for a Patlak-Keller-Segel model with degenerate diffusion in higher dimensions*, Calc. Var. Partial Differential Equations, 35 (2009), pp. 133–168.
- [8] A. BLANCHET, J. A. CARRILLO, AND N. MASMOUDI, *Infinite time aggregation for the critical Patlak-Keller-Segel model in \mathbb{R}^2* , Comm. Pure Appl. Math., 61 (2008), pp. 1449–1481.
- [9] A. BLANCHET, J. DOLBEAULT, AND B. PERTHAME, *Two-dimensional Keller-Segel model: Optimal critical mass and qualitative properties of the solutions*, Electron. J. Differential Equations, 2006 (2006), pp. 1–32. (electronic).
- [10] V. CALVEZ AND J. A. CARRILLO, *Volume effects in the Keller-Segel model: Energy estimates preventing blow-up*, J. Math. Pures Appl. (9), 86 (2006), pp. 155–175.
- [11] J. F. CAMPOS AND J. DOLBEAULT, *Asymptotic estimates for the parabolic-elliptic Keller-Segel model in the plane*, Comm. Partial Differential Equations, 39 (2014), pp. 806–841.
- [12] E. CARLEN AND M. LOSS, *Competing symmetries, the logarithmic HLS inequality and Onofri’s inequality on S^n* , Geom. Funct. Anal., 2 (1992), pp. 90–104.
- [13] J. A. CARRILLO, R. J. MCCANN, AND C. VILLANI, *Kinetic equilibration rates for granular media and related equations: Entropy dissipation and mass transportation estimates*, Rev. Mat. Iberoamericana, 19 (2003), pp. 971–1018.
- [14] D. CHAE AND G. TARANTELO, *On planar selfdual electroweak vortices*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 21 (2004), pp. 187–207.
- [15] Z. DOŠLÁ AND N. PARTSVANIA, *Oscillation theorems for second order nonlinear differential equations*, Nonlinear Anal., 71 (2009), pp. e1649–e1658.
- [16] D. GILBARG AND N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Classics in Math., Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
- [17] G. H. HARDY, J. E. LITTLEWOOD, AND G. PÓLYA, *Inequalities*, 2nd ed., Cambridge University Press, Cambridge, UK, 1952.
- [18] A. JÜNGEL, *Quasi-hydrodynamic Semiconductor Equations*, Progr. Nonlinear Differential Equations Appl., 41, Birkhäuser Verlag, Basel, 2001.
- [19] I. KIM AND Y. YAO, *The Patlak-Keller-Segel model and its variations: Properties of solutions via maximum principle*, SIAM J. Math. Anal., 44 (2012), pp. 568–602.
- [20] J. LERAY AND J.-L. LIONS, *Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder*, Bull. Soc. Math. France, 93 (1965), pp. 97–107.

- [21] P.-L. LIONS, *Minimization problems in $L^1(\mathbf{R}^3)$* , J. Funct. Anal., 41 (1981), pp. 236–275.
- [22] P.-L. LIONS, *The concentration-compactness principle in the calculus of variations. The locally compact case. I*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 1 (1984), pp. 109–145.
- [23] R. J. MCCANN, *A Convexity Principle for Interacting Gases and Equilibrium Crystals*, Ph.D. thesis, Department of Mathematics, Princeton University, Princeton, NJ, 1994.
- [24] R. J. MCCANN, *A convexity principle for interacting gases*, Adv. Math., 128 (1997), pp. 153–179.
- [25] G. STRÖHMER, *Stationary states and moving planes*, in Parabolic and Navier-Stokes Equations. Part 2, Banach Center Publ. 81, Polish Acad. Sci. Inst. Math., Warsaw, 2008, pp. 501–513.
- [26] Y. SUGIYAMA, *Global existence in sub-critical cases and finite time blow-up in super-critical cases to degenerate Keller-Segel systems*, Differential Integral Equations, 19 (2006), pp. 841–876.
- [27] Y. SUGIYAMA, *Application of the best constant of the Sobolev inequality to degenerate Keller-Segel models*, Adv. Differential Equations, 12 (2007), pp. 121–144.
- [28] Y. SUGIYAMA, *Time global existence and asymptotic behavior of solutions to degenerate quasi-linear parabolic systems of chemotaxis*, Differential Integral Equations, 20 (2007), pp. 133–180.
- [29] G. TALENTI, *Elliptic equations and rearrangements*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 3 (1976), pp. 697–718.