

# Ghost instabilities of cosmological models with vector fields nonminimally coupled to the curvature

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We prove that many cosmological models characterized by vectors nonminimally coupled to the curvature (such as the Turner-Widrow mechanism for the production of magnetic fields during inflation, and models of vector inflation or vector curvaton) contain ghosts. The ghosts are associated with the longitudinal vector polarization present in these models, and are found from studying the sign of the eigenvalues of the kinetic matrix for the physical perturbations. Ghosts introduce two main problems: (1) they make the theories ill-defined at the quantum level in the high energy/sub horizon regime (and create serious problems for finding a well behaved UV completion); (2) they create an instability already at the linearized level. This happens because the eigenvalue corresponding to the ghost crosses zero during the cosmological evolution. At this point the linearized equations for the perturbations become singular (we show that this happens for all the models mentioned above). We explicitly solve the equations in the simplest cases of a vector without vev in a FRW geometry, and of a vector with vev plus a cosmological constant, and we show that indeed the solutions of the linearized equations diverge when these equations become singular.

## I. INTRODUCTION

Although the WMAP measurements of the Cosmic Microwave Background (CMB) strongly support the inflationary paradigm [1], several studies pointed out some peculiar features in the data that seem at odd with the simplest inflationary predictions. These so called ‘anomalies’ include the low power in the quadrupole moment [2, 3, 4], the alignment of the lowest multipoles [5], a  $\sim 5^\circ$  cold spot with suppressed power [6], an asymmetry in power between the northern and southern ecliptic hemispheres [7], and broken rotational invariance [8].<sup>1</sup> The significance of some of these effects has increased in the latest studies [8, 10], based on the WMAP-5 years data. These observations have motivated a number of studies both on data analysis (how to construct estimators that can assess the degree of violation of statistical isotropy, see eg. [11]) and on theory (how to construct models that reproduce these features). One can for instance attempt to ascribe the departure from statistical isotropy to initial conditions at the onset of inflation, either on the background evolution [12, 13, 14, 15], or on a super-horizon isocurvature mode [16].<sup>2</sup>

Ref. [18] associates the super-horizon mode with a long cosmic string. In general, however, in these models the breaking of statistical isotropy is built-in as an initial or “boundary” condition, rather than being predicted from first principles (e.g., from a given lagrangian). A second major problem that the above models suffer is that inflation rapidly removes any initial background anisotropy [19], and blows to unobservably large scales any perturbation that was present at its onset. Therefore, these proposals require a minimal (and, therefore, tuned) amount of inflation. To improve over these two problems, one may consider introducing in the inflationary model some nonminimal ingredients that contrast the rapid isotropization caused by the inflaton. This has been realized through the addition of quadratic curvature invariants to the gravity action [20], with the use of the Kalb-Ramond axion [21], or of vector fields [22].<sup>3</sup>

The present work continues a series of previous papers in which we studied the stability of some of the models with vector fields. In Ref. [25], we presented a general discussion valid for three groups of models, characterized by (i) a potential  $V(A^2)$  for the vector [22], (ii) a fixed spatial norm of the vector, enforced by a lagrange multiplier [26], or (iii) a nonminimal coupling of the vector to the scalar curvature [27, 28, 29, 30]. We showed that all these models have ghost instabilities. Although few explicit computations were given in the Letter [25], we explained the physical reasons behind the instability: the terms that characterize the above models break the U(1) symmetry that would be otherwise associated to the vector field. This introduces an additional polarization (the longitudinal vector mode)

<sup>1</sup> Ref. [9] found an upper limit on the anisotropy, which is compatible with the result of [8].

<sup>2</sup> The super-horizon mode breaks the translational, and therefore can give rise to the asymmetry claimed in [7]; the Bianchi-I backgrounds considered in [12, 13, 14, 15] have instead planar symmetry, and have the correct structure to explain the violation of rotational invariance observed by [8]. General expressions for the correlation  $\langle a_{\ell m} a_{\ell' m'}^* \rangle$  between different multipoles were given in [12] for generic breaking of rotational invariance, and in [17] for generic breaking of translational invariance.

<sup>3</sup> While the models studied here have been proposed in the context of primordial inflation, vector fields with nonvanishing spatial vacuum expectation value (vev) have been also employed as sources of the late time time acceleration [23]. See also [24] for a different model of anisotropic dark energy.

that, in these models, turns out to be a ghost. In Ref. [31] we provided the explicit computations for the case (ii).

In the present work, we present explicit computations for the models in the group (iii). The first of such models [27] is a well known mechanism for the generation of magnetic fields. Among other things, it was pointed out there that, for the specific action

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{4}F^2 + \frac{1}{12}R A^2 \right] \quad (1)$$

the equations for the vector field (more precisely, those governing the vev, and the transverse polarizations) are the same as those of a scalar field minimally coupled to the curvature. This fact was also exploited in a series of recent papers, motivated by the WMAP anomalies. The work [28] studies inflation driven by  $N$  vector fields. The simplest realization of [28] is characterized by three mutually orthogonal vectors with equal vev: this provides a homogeneous and isotropic (Friedmann-Robertson-Walker, FRW) evolution; however, one can envisage the more complicated situation in which a large number  $N$  of fields is present, with random orientation. This provides a nearly isotropic expansion, with a naturally small,  $\mathcal{O}(1/\sqrt{N})$ , anisotropy. Ref. [29] provides a simpler version of the same idea, in which a single vector field plus an inflaton field (in a sense, replacing the average effect of the vectors) are present. Finally, Refs. [30] used the vector as a curvaton field, in order to produce a nearly scale invariant spectrum of primordial perturbations.

The presence of ghosts has not emerged in previous stability studies of such models, due their partial nature. The original work [27] studied the evolution of modes of the magnetic field, associated to the transverse photon polarizations, but did not discuss the role of the additional longitudinal mode. The results of [30] are based on the  $\delta N$  formalism, which computes the classical evolution of super-horizon modes, assuming that the quantum theory is under control (as it happens in the case of scalar field inflation, for which it was developed).<sup>4</sup> Ref. [32] studied the gravity waves (GW) in the model of vector inflation [28], assuming that the coupling of these modes with the other perturbations - which, for this model, is present already at the linearized level, see eq. (77) below - can be disregarded. Finally, Ref. [33] studied the linearized equations of motion for vector inflation either in the short wavelength, or in the long wavelength regime.<sup>5</sup> The study of the linearized equations alone does not allow to see whether a perturbation is a positive or negative energy mode, and for this reason the presence of ghosts does not appear in this analysis.<sup>6</sup> Moreover, as we show below, the linearized equations of motion for the perturbations become singular close to horizon crossing, in a regime where neither the long nor the short wavelength analyses apply. In Section VD, we provide a more detailed discussion of some claims made in [33].

It is important to stress that the ghost instability takes place also for those of the above models which have a FRW background. The presence of a ghost in these models is indeed due to the specific sign of the effective mass term  $M^2 = -R/6$  for the vector induced by the coupling to the curvature. Indeed, as we showed in [25], a negative mass squared for a vector results in a ghost, and not simply in a tachyon as in the scalar case. We stress that a massless vector has only the two transverse polarizations; it is therefore not surprising that the mass term controls the nature of the longitudinal mode. Another example on how the mass term controls the nature of a mode is given by the more complicated case of the graviton mass  $m_1^2 h_\mu^\mu h_\nu^\nu - m_2^2 h_{\mu\nu} h^{\mu\nu}$ . It is well known that, unless  $m_1 = m_2$  (namely, the Fierz-Pauli mass term [34]) the linearized spectrum of perturbations has a ghost.

The most obvious problem associated to a ghost is the instability of the vacuum. If a ghost is coupled to a normal field (and, in all the above theories, there are at least gravitational couplings), the vacuum will decay in ghost-nonghost excitations, with a rate which is UV divergent (since the final state quanta can have arbitrarily large momentum without violating energy conservation). To avoid the associated instantaneous vacuum decay, theories with ghosts are thought to be consistent only as effective theories, valid below some energy scale  $\Lambda$ . It has been shown in [35] that, a theory which has a ghost today coupled gravitationally to positive energy fields is phenomenologically viable only if  $\Lambda \lesssim \text{MeV}$ , otherwise we would see signatures of the vacuum decay in the diffuse  $\gamma$ -ray background. A stronger coupling would result in a tighter bound on  $\Lambda$ . All the above models discuss physics at much greater energy

<sup>4</sup> Moreover, we show below that also the classical evolution for the longitudinal mode leads to a divergence at super-horizon, when the total mass of the vector vanishes.

<sup>5</sup> In inflationary models, the wavelength of a perturbation grows nearly exponentially, while the Hubble rate  $H$  is nearly constant. Therefore, at sufficiently early times, the wavelength of any mode is smaller than the horizon scale  $H^{-1}$ , while the opposite is true at sufficiently late times. The moment at which the two scales are equal is dubbed ‘‘horizon crossing’’; we remark that perturbations of different size cross the horizon at different times (larger modes, exit the horizon earlier during inflation).

<sup>6</sup> Cf. the case of a scalar field, with lagrangian  $\mathcal{L} = \pm (\dot{\phi}^2 - m^2 \phi^2)/2$  (dot denotes derivative with respect to time, and we ignore spatial dimensions); the equation of motion is  $\ddot{\phi} + m^2 \phi = 0$  for either overall sign of the lagrangian, so that no instability appears from the equation. However, the minus sign in the kinetic term corresponds to a negative energy field (a ghost).

scales than MeV, so, a necessary (but, as we will see, not sufficient) condition for them to be consistent is that the ghost vanishes at late times. The easiest way to achieve this is to add a positive contribution  $m^2 > 0$  to the mass term,  $M^2 = -R/6 + m^2$ , and to require that  $m^2$  is lower than  $-R/6$  during inflation, but greater than it today. In fact, this is already built in in the models of vector inflation [28, 29], for which  $m^2$  controls the slow roll evolution of the vev of the vector. The same can be also done in the case of vector curvaton, where it is found that the mass provides an  $O(m^2/H^2)$  departure from scale invariance ( $H$  being the Hubble rate during inflation) [30]. For the case of photons, there are strong upper limits on the allowed value for  $m$  (see the limits on the PDG Particle Listing [36]). These limits are however less stringent than the present value of  $R/6 = O(10^{-33} \text{ eV})^2$ .

Even if it is straightforward to eliminate the ghost(s) from the present spectrum, this does not eliminate the instability of these models during or after the inflationary stage. We will comment on problems that arise at the nonlinear level in the Discussion Section VI. We stress here that the ghost instability of these models manifests itself already at the linearized level. The spectrum of the theory is obtained by computing the quadratic action for the dynamical perturbations  $Y_i$  around the given background. We do so by following, and generalizing to the present case, the exact same steps that are done in the standard computation of scalar field inflation to obtain the canonical modes of the system [37]. The quadratic action is characterized by a kinetic  $K_{ij} \dot{Y}_i^* \dot{Y}_j$  term, a mass term, and a mixed term (see eq. (24)). The eigenvalues of  $K$  control the nature of the physical modes (i.e., a mode is a positive / negative energy excitation if the corresponding eigenvalue is positive / negative). The kinetic matrix depends on background quantities, and thus on time. As a consequence, its eigenvalues depend on time, and the nature of a mode can change during the background evolution. From our computations, we find two different types of behavior, according to whether the vev of the vector is or is not vanishing:

- $\langle \vec{A} \rangle = 0$ : one eigenvalue of  $K$  is negative during most of the sub-horizon regime; it changes sign at some moment close to horizon crossing, without passing through zero (it does so by diverging). It crosses zero later on, when the total mass  $M^2$  vanishes.
- $\langle \vec{A} \rangle \neq 0$ : one eigenvalue of  $K$  starts positive, but crosses zero at some point close to horizon crossing, and remains negative for some amount of time.

The different behaviors are due to the fact that, for a nonvanishing vev, the perturbations of the vector are mixed with those of the metric at the linearized level (i.e. in the quadratic action), and this affects the spectrum of the theory. However, we see that the mixing with gravity does not remove the ghost. The case of vanishing vev applies to the computations of [27] and [30].<sup>7</sup> The case of a vector with a vev applies to the model of [29]. The vector inflation model of [28] is more complicated, since it contains an arbitrary number  $N$  of vectors (and, hence, of ghosts). We study the simplest realization, with three mutually orthogonal vectors. We find that, in this case, the model has three ghosts. Two eigenvalues behave as in the  $\langle \vec{A} \rangle = 0$  case just mentioned, while the third eigenvalue behaves as in the  $\langle \vec{A} \rangle \neq 0$  case. Hence, it appears that in this case only one linear combination of the ghosts is affected by the coupling to gravity.

As we show below, the system of linearized equations for the perturbations becomes singular when one of the eigenvalues of the kinetic matrix crosses zero. Correspondingly, we expect that the linearized solutions diverge at this moment. We explicitly solved the equations in the case of a single vector with no vev, and in the case of one vector with nonvanishing vev plus a cosmological constant. We did not solve the equations for the case of vector inflation (due to technical difficulties: the system contains 18 gauge invariant perturbations in its simplest realization). However we stress that, also for this model, we explicitly proved that (i) there are ghosts (which is by itself enough to pose serious doubts on any prediction obtained from this model; see the Discussion Section), and (ii) the equations for the perturbations become singular at some finite moments of time.

The plan of the paper is the following. In Section II, we review the basic mechanism for the models of generation of primordial magnetic fields, vector inflation, and vector curvaton. In Section III, we outline the computation of the quadratic action for the physical modes of the system. We also show there why vanishing eigenvalues of the kinetic matrix  $K$  result in singular linearized equations for the perturbations. In Section IV we present the computations for the  $\langle \vec{A} \rangle = 0$  case. We show that the theory has a ghost, and that the linearized perturbations diverge where the total mass of the vector vanishes. In Section V we study the case of nonvanishing vev. We first discuss the simplest

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<sup>7</sup> Notice, however, that for a vector curvaton the vev cannot be exactly zero if one wants to realize a violation of statistical isotropy of the perturbations. Indeed, to have a violation of statistical isotropy, one needs to “single out” different direction(s); this is precisely provided by the vev of the vector. We follow the computations of [30] which are performed under the assumption of zero or negligible vev. For a nonvanishing vev, the model of [30] belong to the second class of models just mentioned, and the ghost instability manifests itself in the second way just mentioned.

possibility of a vector and a cosmological constant. For this case, we both compute the quadratic action, and solve the linearized equations of the perturbations. Once again we verify that the modes diverge precisely when one eigenvalue of the kinetic matrix  $K$  crosses zero. We then study the case of one vector and one inflaton [29], and the simplest realization of vector inflation [28]. For these models we compute the kinetic term of the physical perturbations, and we study how the eigenvalues evolve in time. Conclusions and discussions are given in Section VI.

## II. REVIEW OF SOME MODELS WITH A $RA^2$ TERM

In this Section, we briefly review the reasons for introducing a nonminimal coupling of a vector field to the curvature. We start from the quadratic action of a vector field with a generic time-dependent mass:

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{4} F^2 - \frac{1}{2} M(t)^2 A^2 \right] \quad (2)$$

leading the equations of motion

$$\frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} F^{\mu\nu}) + M^2 A^\mu = 0 \quad (3)$$

Moreover, due to its antisymmetry, the field strength  $F_{\mu\nu}$  satisfies the identity

$$\partial_\mu F_{\rho\nu} + \partial_\nu F_{\mu\rho} + \partial_\rho F_{\nu\mu} = 0 \quad (4)$$

Turner and Widrow [27] suggested a mechanism for the generation of primordial magnetic fields during inflation, starting from the action (2).<sup>8</sup> In this mechanism, the vector field  $A_\mu$  is the electromagnetic field, and the mass term is proportional to the scalar curvature  $R$ . Following [27], we use conformal time  $\eta$ , defined by the line element  $ds^2 = a^2(\eta) (-d\eta^2 + d\vec{x}^2)$  in the present discussion, so that the electric and magnetic fields are ( $\epsilon_{123} = +1$ )

$$F_{i0} = a^2 E_i \quad , \quad F_{ij} = a^2 \epsilon_{ijk} B_k \quad (5)$$

The two equations (3) and (4) can be easily combined into an equation for the magnetic field [27]:

$$\left( \partial_\eta^2 - \partial_{\vec{x}}^2 + a^2 M^2 \right) \left( a^2 \vec{B} \right) = 0 \quad (6)$$

From this equation, we find the well known result that, in the massless case ( $M^2 = 0$ ), and in the large wavelength limit (i.e. negligible spatial gradient), the amplitude of a magnetic field decreases as  $\propto a^{-2}$ . Correspondingly, its energy density decreases as  $\propto a^{-4}$ . Consider instead the mass term [27]

$$M^2 = \xi R = 6 \xi \frac{a''}{a^3} \quad (7)$$

where  $\xi$  is a constant, and prime denotes derivative wrt  $\eta$ . The equation of motion for the magnetic field then becomes

$$\left( \partial_\eta^2 - \partial_{\vec{x}}^2 + 6 \xi \frac{a''}{a} \right) \left( a^2 \vec{B} \right) = 0 \quad (8)$$

During inflation,  $a = -1/(H\eta)$  where the Hubble rate  $H$  is nearly constant; inserting this into the equation of motion (and treating  $H$  as constant), we find that, in the large wavelength regime, the energy density of the magnetic field behaves as

$$\rho_B \propto \vec{B}^2 \propto a^{-5+\sqrt{1-48\xi}} \quad (9)$$

Therefore, for  $\xi < 0$  - corresponding to a negative  $M^2$  in the action (2) -  $\rho_B$  is less affected by the expansion with respect to the massless case.

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<sup>8</sup> More precisely, ref. [27] studied a more general action with a quadratic term of the type  $R_{\mu\nu} A^\mu A^\nu$  also included; we disregard this term in the present work.

It is instructive to compare the behavior of the magnetic field with respect to that of a massless scalar field coupled to the curvature, characterized by the action

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \xi_s R \phi^2 \right] \quad (10)$$

This leads to the equation of motion

$$\left( \partial_\eta^2 - \partial_{\bar{x}}^2 + (6\xi_s - 1) \frac{a''}{a} \right) (a \phi) = 0 \quad (11)$$

By comparing this equation with the analogous expression for the vector field<sup>9</sup>, it has been noted [27] that a vector field with  $\xi = -1/6$  behaves analogously to a scalar field minimally coupled to the curvature ( $\xi_s = 0$ ). Conversely, the standard vector field  $\xi = 0$  is analogous to a conformally coupled scalar,  $\xi_s = 1/6$  (therefore, no magnetic field is produced by the inflationary expansion in the standard case).

This analogy has been recently exploited in [28], that proposed a mechanism of inflation driven by a combination of  $N$  nonminimally coupled vector fields. The mass term of the vectors comprises of the coupling to the curvature plus a constant term:

$$S = \int d^4x \sqrt{-g} \sum_{a=1}^N \left[ -\frac{1}{4} F_{\mu\nu}^{(a)} F^{(a)\mu\nu} - \frac{1}{2} (\xi R + m^2) A_\mu^{(a)} A^{(a)\mu} \right] \quad (12)$$

The simplest case, and the one which has been most studied in [28], is characterized by three mutually orthogonal vectors with equal vev:

$$\langle A_\mu^{(1)} \rangle = (0, \mathcal{A}, 0, 0) \quad , \quad \langle A_\mu^{(2)} \rangle = (0, 0, \mathcal{A}, 0) \quad , \quad \langle A_\mu^{(3)} \rangle = (0, 0, 0, \mathcal{A}) \quad , \quad \mathcal{A} \equiv M_p a(t) B(t) \quad (13)$$

These sources allow for a FRW background, controlled by the equations of motion (we switch to physical time, and we denote by a dot derivative with respect to it)

$$\begin{aligned} H^2 - \frac{\dot{B}^2}{2} - \frac{1}{2} m^2 B^2 &= \frac{1+6\xi}{2} H B (2\dot{B} + H B) \\ \ddot{B} + 3H\dot{B} + m^2 B &= -(1+6\xi) B (\dot{H} + 2H^2) \end{aligned} \quad (14)$$

For  $\xi = -1/6$ , and upon the identification  $B = \phi / (\sqrt{3}M_p)$ , we recover the same equations as those of chaotic inflation driven by a minimally coupled scalar field  $\phi$ .

Another compelling feature of the proposal of [28] is that it can naturally give a small violation of isotropy. Indeed, for a large number  $N$  of vectors with random orientations and vev, one expects an almost isotropic expansion, with a deviation  $\Delta H/H = O(1/\sqrt{N})$  between the expansion rates of the different directions. Ref. [29] provides a slightly different mechanism in which a single nonminimally couple vector breaks the isotropy in one spatial direction, while a scalar field with greater energy density is responsible for the overall nearly isotropic expansion.

Vector fields with nonminimal coupling to the curvature have also been recently employed for the generation of a nearly scale invariant spectrum of perturbations [30]. Consider the action (12) for a single field ( $N = 1$ ) and with  $\xi = 1/6$ . Following the discussion of [30], we compute the evolution of the perturbations  $\delta A_\mu$  assuming that the vev  $\langle A_\mu \rangle$  can be neglected (see the remark we made about this in the Introduction). In this way, the perturbations of the vector do not mix with those of the metric at the linearized level. Moreover, with a negligible vector vev, we can study the evolution of  $\delta A_\mu$  in an unperturbed FRW background. In the present discussion, we only consider the transverse components of the perturbations,  $\delta A_\mu = (0, \delta \vec{A}^T)$ , with  $\partial_i \delta A_i^T = 0$ , since, as we will show in the following Section, there are serious problems with the longitudinal mode in these models. Due to the  $\xi = -1/6$  choice, the equation for this mode is identical to that of a minimally coupled curvaton scalar field:

$$\left[ \partial_\eta^2 - \partial_{\bar{x}}^2 + a^2 \left( m^2 - \frac{a''}{a^3} \right) \right] \delta A_i^T = 0 \quad (15)$$

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<sup>9</sup> It is appropriate to compare the behavior of  $a^2 \vec{B}$  with that of  $a \phi$ , since, due to the different structures of the kinetic terms, these are the canonically normalized fields in the two cases.

We proceed as in the scalar curvaton case. For simplicity, we assume a dS background; in momentum space, the Fourier transform of  $\delta A_i^T$  obeys to the equation

$$\left(\partial_\eta^2 + k^2 - \frac{2\mu^2}{\eta^2}\right)\delta\tilde{A}_i^T = 0 \quad , \quad \mu^2 \equiv 1 - \frac{m^2}{2H^2} \quad (16)$$

Among the two solutions of this equation, we chose the one that reduces to the adiabatic vacuum at early times, when the mode is deeply inside the horizon:

$$\delta\tilde{A}_i^T = \frac{\sqrt{\pi}}{2} \sqrt{|\eta|} H_{\frac{1}{2}}^{(1)}\sqrt{1+8\mu^2}(|k\eta|) \quad (17)$$

where  $H^{(1)}$  is the Hankel function of the first kind. Indeed, up to an irrelevant phase, this solution reduces to  $e^{-ik\eta}/\sqrt{2k}$  for  $|k\eta| \gg 1$ . In the opposite late time / super horizon regime ( $|k\eta| \ll 1$ ), from the expansion of the Hankel function, we then find the power spectrum for the transverse modes [30]

$$P_{T_i} \propto k^3 |\delta\tilde{A}_i^T|^2 \propto k^{3-\sqrt{1+8\mu^2}} = k^{\frac{2m^2}{3H^2} + \mathcal{O}\left(\frac{m^4}{H^4}\right)} \quad (18)$$

Namely  $m \ll H$  provides a small departure from scale invariance. We remark that this result follows from the solution (17). Indeed, eq. (16) has two solutions with two undetermined (and, in principle,  $k$ -dependent) integration constants. As it is customary, we chose the linear combinations of these solutions which reduces to the adiabatic vacuum in the early time sub-horizon regime. The phenomenological prediction (18) crucially relies on the fact that the theory must be under control in this regime.

For all the models we have reviewed, the discussion presented in this Section disregards the role of the longitudinal polarization of the vector field(s). In the next Sections we show that, for all the models discussed, this mode turns out to be a ghost.

### III. IDENTIFICATION OF GHOSTS, AND THEIR ASSOCIATED INSTABILITY

In the next two Sections, we will see that the models described in the previous Section have ghosts. We outline here the method we employ to find the ghosts, and the instability associated with them. The ghosts are among the physical excitations of the background geometries of the various models. Therefore, we need to compute the spectrum of these theories. To do so, we perturb the background solutions of a given model, and we expand the action at quadratic order in the perturbations. This is the free action for the perturbations (interaction among these fields come from expanding the initial action to higher orders), and the spectrum follows from the diagonalization of this action.

There are two main issues in this computation. The first one is associated to the gauge freedom that may be present in a theory. A gauge theory is in a sense a redundant formulation of a physical system, since different field configurations, related to each other by a (nonsingular) gauge transformation, describe the same physics. In the present context, the gauge freedom is the one associated with general coordinate transformations. We can expand a gauge transformation in a transformation acting on the background plus a transformation acting on the perturbations (loosely speaking, we decompose any gauge transformation into a “big transformation”, affecting the background, plus a “small” transformation, affecting the perturbations on a given background). In any explicit computation, one typically fixes the gauge freedom for the background and the perturbations separately. Concerning the latter step, one can choose a gauge that fixes the freedom completely (for instance, one can impose that some perturbations vanish; one needs to show that this choice can always be done, and that there is no residual freedom left); equivalently, one can find a set of gauge invariant linear combinations that do not change under the gauge transformation. Since the theory is gauge invariant, it must be possible to write down the equations of motion only in terms of the gauge invariant combinations, and solve those expressions. This is the method that we adopt in our explicit computations.

The second issue is that, even after removing the gauge redundancy, the remaining perturbations do not all necessarily describe dynamical degrees of freedom (e.g., physically propagating excitations). Modes that do not correspond to dynamical excitations enter in the action without time derivatives (up to boundary terms, which can be disregarded in a theory without boundaries). Example of nondynamical degrees of freedom are for instance the  $\delta A_0$  component of a vector, or the  $\delta g_{\mu 0}$  components of the metric. In the standard case (i.e., for  $\mathcal{L} \supset F_{\mu\nu}F^{\mu\nu}$ , and  $\mathcal{L} \supset R$ ), they enter in the action without time derivatives. In general, the quadratic action for the perturbations around a given background is formally of the type (in momentum space)

$$S = \int d^3k dt \left[ a_{ij} \dot{Y}_i^* \dot{Y}_j + \left( b_{ij} N_i^* \dot{Y}_j + \text{h.c.} \right) + c_{ij} N_i^* N_j + \left( d_{ij} \dot{Y}_i^* Y_j + \text{h.c.} \right) + e_{ij} Y_i^* Y_j + \left( f_{ij} N_i^* Y_j + \text{h.c.} \right) \right] \quad (19)$$

where  $Y_i$  are the dynamical modes, and  $N_i$  the nondynamical ones. The coefficients  $a_{ij}, \dots, f_{ij}$  depend on background quantities, and, for any given background solution, are functions of time. From the action (19), we find the linearized equations

$$\frac{\delta S}{\delta Y_i^*} = 0 \Rightarrow a_{ij} \ddot{Y}_j + \left[ \dot{a}_{ij} + d_{ij} - (d^\dagger)_{ij} \right] \dot{Y}_j + (b^\dagger)_{ij} \dot{N}_j + \left[ \dot{d}_{ij} - e_{ij} \right] Y_j + \left[ (\dot{b}^\dagger)_{ij} - (f^\dagger)_{ij} \right] N_j = 0 \quad (20)$$

$$\frac{\delta S}{\delta N_i^*} = 0 \Rightarrow c_{ij} N_j = -b_{ij} \dot{Y}_j - f_{ij} Y_j \quad (21)$$

We see explicitly that the equations of motion for the nondynamical modes are algebraic in them (they are constraint equations). As a consequence, the nondynamical modes do not introduce additional degrees of freedom, but are completely determined by the dynamical ones. We can obtain a system of differential equations for only the dynamical modes by inserting the solutions of eqs. (21) into equations (20):

$$K_{ij} \ddot{Y}_j + \left[ \dot{K}_{ij} + (\Lambda_{ij} - \text{h.c.}) \right] \dot{Y}_j + \left( \dot{\Lambda}_{ij} + \Omega_{ij}^2 \right) Y_j = 0 \quad (22)$$

where

$$\begin{aligned} K_{ij} &\equiv a_{ij} - (b^\dagger)_{ik} (c^{-1})_{km} b_{mj} \\ \Lambda_{ij} &\equiv d_{ij} - (b^\dagger)_{ik} (c^{-1})_{km} f_{mj} \\ \Omega_{ij}^2 &\equiv -e_{ij} + (f^\dagger)_{ik} (c^{-1})_{km} f_{mj} \end{aligned} \quad (23)$$

To obtain instead the quadratic action for the dynamical modes, we insert the solutions of eqs. (21) into the action (19):

$$S \rightarrow \int d^3k dt \left[ \dot{Y}_i^* K_{ij} \dot{Y}_j + \left( \dot{Y}_i^* \Lambda_{ij} Y_j + \text{h.c.} \right) - Y_i^* \Omega_{ij}^2 Y_j \right] \quad (24)$$

The extremization of this action provides exactly the above equations (22).

Eq. (21) and (22) - or, equivalently, eqs. (20) and (21) - are the equations of motion for the perturbations. We can also obtain these equations in a different but equivalent form by simply expanding at first order in the perturbations the general equations of motion of the model. In general, this last procedure is technically simpler than expanding the action at second order in the perturbations, and extremizing it. However, while an action provides the equations of motion, in general the equations of motion do not fully determine the action. The quadratic action of the perturbations is necessary to quantize the perturbations, and provide their initial conditions.

The procedure just outlined formalizes the steps that are done in the standard case of scalar field inflation on a FRW background. Moreover, it extends it to an arbitrary number of dynamical and nondynamical fields. In the standard case, there are 5 scalar modes in the perturbations of the metric and of the scalar field. Two of them are eliminated by gauge fixing (or, equivalently, by the use of gauge invariant variables). Only 1 out of the 3 remaining modes is dynamical. While one can show that the action can be written solely in terms of gauge invariant quantities without using the equations of motion, the constraint equations are used to eliminate the nondynamical modes (for instance, the constraint equation (10.39) is used to obtain the final action (10.59) in [37]). The final step in the computation is to normalize this final mode so that the kinetic term of this final action is canonical.<sup>10</sup>

For a general problem, with more than one dynamical mode in the system, one needs to diagonalize the kinetic matrix  $K$ . If one eigenvalue of  $K$  is negative, the corresponding eigenmode is a ghost (a field with negative energy). This signals an instability of the vacuum, which decays in ghost-nonghost excitations with a rate which diverges in the UV. A second type of instability takes place whenever, due to the cosmological background evolution, an eigenvalue of  $K$  crosses zero, and  $K$  is noninvertible. Denote by  $t_*$  one of the moments at which this happens. The equations of motion become singular for  $t \rightarrow t_*$ , since the second derivative of the modes diverge in this limit. We expect that,

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<sup>10</sup> In the standard case, the canonically normalized scalar perturbation is the Mukhanov-Sasaki variable  $v$  [38]. We explicitly verified that, in the standard case of scalar field inflation, the action (24) obtained with the procedure described here coincides with the action (10.59) of [37].

correspondingly, the solutions of the linearized system diverge for  $t \rightarrow t_*$ .<sup>11</sup> In the next two Sections, we show that both types of instability occur in the models we are studying.

#### IV. GHOST INSTABILITY FOR $\langle A_\mu \rangle = 0$

We start our analysis from the case in which the vector field has no vev. This is the case for the computations presented in [27, 30] to study the evolution of the modes. On a technical level, the assumption of zero vev drastically simplifies the computation. Indeed, the actions we are studying (cf. eqs. (2) and (12)), are quadratic in the vectors; therefore, for  $\langle A_\mu \rangle = 0$ , the perturbations of the vector(s) are decoupled from those of any other field at the linearized level.<sup>12</sup> Therefore, the action for the vector field already starts at quadratic order in the perturbations, and we can simply study the evolution of  $\delta A_\mu$  in an unperturbed FRW background. The more involved case of  $\langle A_\mu \rangle \neq 0$  is studied in the following Section.

Consider the action

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{4} F^2 - \frac{1}{2} M^2 A^2 \right], \quad M^2 = -\frac{R}{6} + m^2 \quad (25)$$

in the FRW background,  $ds^2 = -dt^2 + a(t)^2 d\vec{x}^2$ . Parametrize the fluctuations of the vector field as  $A_\mu = (\alpha_0, \partial_i \alpha + \alpha_i^T)$ , where  $i = 1, 2, 3$ , and where the modes  $\alpha_i^T$  are transverse ( $\partial_i \alpha_i^T = 0$ ). These two polarizations are decoupled from the modes  $\{\alpha_0, \alpha\}$  (and also from each other) at the linearized level, and have been already studied in the previous Section (starting from eq. (15)). As we saw, they are well behaved in all regimes. Here we study the nature and the evolution of the other two perturbations.

It is well known that a massive vector has three physical degrees of freedom. The two transverse ones are encoded in  $\alpha_i^T$ . Therefore, the two perturbations  $\alpha_0$  and  $\alpha$  encode only one physical mode, which is the longitudinal polarization of the massive vector. This can be immediately seen from the equations of motion for the two perturbations following from (25), which, in Fourier space, read<sup>13</sup>

$$\alpha_0 = \frac{p^2}{p^2 + M^2} \dot{\alpha} \quad (26)$$

$$\ddot{\alpha} + \frac{(3p^2 + M^2)H + p^2 \frac{dM^2}{dt}}{p^2 + M^2} \dot{\alpha} + (p^2 + M^2) \alpha = 0 \quad (27)$$

where  $p = k/a$  is the physical momentum of the perturbation, and, as usual,  $H = \dot{a}/a$ . While eq. (27) is a second order differential equation, eq. (26) is an algebraic equation in  $\alpha_0$ . Therefore,  $\alpha_0$  does not introduce additional degrees of freedom, but it is completely determined once  $\alpha$  is known (compare these equations with the formal set of equations (20) and (21)).

We can also see this from the quadratic action for the perturbations. Inserting the decomposition  $A_\mu = (\alpha_0, \partial_i \alpha)$  in (25), and Fourier transforming the spatial coordinates, we find

$$S = \frac{1}{2} \int dt d^3k \left[ p^2 |\dot{\alpha}|^2 - p^2 M^2 |\alpha|^2 - p^2 (\alpha_0^* \dot{\alpha} + \text{h. c.}) + (p^2 + M^2) |\alpha_0|^2 \right] \quad (28)$$

The mode  $\alpha_0$  enters in the action without time derivatives, confirming that it is a nondynamical mode. We integrate it out: we compute its equation of motion from this action (namely, eq. (27) given above), we solve it, and we insert

<sup>11</sup> To integrate the system (22), one needs to invert the matrix  $K$ ; in this way one obtains a system of equations of the form  $\ddot{Y}_i = \dots$ , where the right hand sides contain only zero or first time derivatives of the unknowns. This system can be then integrated numerically. The inversion of  $K$  can be performed at all times  $t < t_*$ , and the resulting equations are regular until  $t_*$ ; however, the expression for the second derivatives blows up for  $t \rightarrow t_*$ , and we expect that the solutions diverge at this moment. This is precisely what happens in the two examples below in which we explicitly solve the linearized equations. This is also what happened in our previous study [31] of the model [26] which presents an identical instability.

<sup>12</sup> In the mechanism of [27], the full action is not quadratic in  $A_\mu$ , since the photon is obviously coupled to the current of charged fields. This coupling is important from reheating on, when the conductivity of the charged particles become high, and affects the evolution of the magnetic field. However, the presence of these fields can be disregarded during the inflationary stage, as done in [27].

<sup>13</sup> The first equation is  $\delta S / \delta A_0$ ; the second equation has been obtained by combining  $\delta S / \delta A_0$  and  $\delta S / \delta A_i$ .



the solution back into (28). This leads to the action for the longitudinal mode:<sup>14</sup>

$$S = \frac{1}{2} \int dt d^3k p^2 M^2 \left( \frac{|\dot{\alpha}|^2}{p^2 + M^2} - |\alpha|^2 \right) \quad (29)$$

The extremization of this action reproduces the equation of motion (27).<sup>15</sup> We remark that the lagrangian in (29) is proportional to the  $M^2$  term, and that, for  $p^2 > |M^2|$ , the longitudinal mode is a ghost (i.e. a field with negative energy, not simply a tachyon) whenever  $M^2 < 0$  [25, 31]. In our previous work [31] we provided two additional proofs of the fact that  $M^2 < 0$  leads to a ghost. The first is based on the direct computation of the propagator (we find that one residue at the pole of the propagator has the “wrong” sign for  $M^2 < 0$ ); the second on the Stuckelberg formalism (which is a convenient way to study the different polarizations of a massive vector or graviton; we rewrite  $A_\mu = A_\mu^T + \partial_\mu \phi / \sqrt{|M^2|}$ , with  $\partial^\mu A_\mu^T = 0$ ; it is then immediate to see that the field  $\phi$  is a ghost for  $M^2 < 0$ ).

The presence of a ghost signals the instability of the vacuum of the model, due to its (UV-divergent) decay in ghost-nonghost excitations. In addition, the equation of motion (27) indicates that the system may be unstable already at the linearized level, whenever  $\omega^2 \equiv p^2 + M^2$  or  $M^2$  vanish. Let us first discuss when this occurs. During inflation,  $m^2 \ll R$  for the models we are studying. As a consequence, during inflation,  $M^2 = m^2 - R/6 \simeq -2H^2 < 0$ , while  $\omega^2 = p^2 + M^2 \simeq p^2 - 2H^2$  goes from positive to negative. We denote by  $t_{\omega 1}$  the moment at which  $\omega^2$  vanishes; we note that this happens when the mode is close to horizon crossing, and that  $\omega^2$  remains negative from  $t_{\omega 1}$  until a time well after the end of inflation (since the mode is well outside the horizon,  $p \ll H$ , when inflation ends). After inflation,  $R/6 = \mathcal{O}(H^2)$  decreases as the universe expands, and it eventually drops below  $m^2$ . At this moment, which we denote by  $t_M$ , the mass term  $M^2$  goes from negative to positive. Therefore,  $\omega^2 = p^2 + M^2 > 0$  for all times  $t \geq t_M$ . Since we saw that  $\omega^2 < 0$  at the end of inflation, there is a moment after the end of inflation, and before  $t_M$ , at which  $\omega^2$  vanishes for the second time. We denote this moment by  $t_{\omega 2}$ . In summary,  $M^2$  vanishes at  $t = t_M$ , while  $\omega^2$  vanishes at  $t = t_{\omega 1}, t_{\omega 2}$ . Denoting by  $t_{\text{end}}$  the moment at which inflation finishes, and by  $t_0$  the present time, we have  $t_{\omega 1} < t_{\text{end}} < t_{\omega 2} < t_M < t_0$ .<sup>16</sup>

To see whether the linearized system diverges when either  $\omega^2$  or  $M^2$  vanish, we study the equation of motion (27) for a finite interval of time close to  $t_{\omega i}$  ( $i = 1, 2$ ) or  $t_M$ . We assume that the equation of state  $w$  of the source driving the expansion can be treated as constant in this interval (which is certainly true, provided the interval is not too extended). We also assume that  $-1 < w < 1/3$ . This is a rather general assumption, since it includes the following cases: inflation (with  $w \gtrsim -1$ ), coherent oscillations of the inflaton after inflation (which, for a quadratic inflaton potential, give an average  $w = 0$ ), matter domination (if  $m^2$  is sufficiently small, so that  $M^2$  vanishes at this stage), and also radiation domination (for which the equation of state is slightly smaller than  $1/3$ , due to the masses of the particles in the thermal bath, or the thermal trace anomaly).<sup>17</sup>

**Behavior of the linearized system for  $p^2 + M^2 \rightarrow 0$ .** The scale factor evolves as

$$a = a_{\omega i} \left( \frac{t}{t_{\omega i}} \right)^{\frac{2}{3(1+w)}} \quad (30)$$

where we recall that  $t_{\omega i}$  is either of the times at which  $\omega^2 = 0$ , and  $a_{\omega i}$  is the value of the scale factor at that time. The mass and frequency squared are given by

$$M^2 = m^2 - \frac{R}{6} = m^2 - \frac{2}{9t^2} \frac{1-3w}{(1+w)^2} \quad (31)$$

$$\omega^2 = p^2 + M^2 = p_{\omega i}^2 \left( \frac{t_{\omega i}}{t} \right)^{\frac{4}{3(1+w)}} + m^2 - \frac{2}{9t^2} \frac{1-3w}{(1+w)^2} \quad (32)$$

<sup>14</sup> We remark that, in going from the action (28) to the action (29), we are precisely following the general procedure that we outlined at a formal level in the previous Section.

<sup>15</sup> This equation, in conformal time, was already given in [30]; this confirms our algebra. The action for the longitudinal mode could be also written starting from this equation, up to an overall factor. The procedure described here provides the complete action. We also note that the Ref. [30] did not solve the equation at the moment in which the total mass of the vector vanishes; as we show here, the solution diverges at this moment.

<sup>16</sup> One may imagine that  $m$  is so small, so that  $t_M$  has not occurred yet. In this case the longitudinal polarization is still a ghost today; we disregard this possibility, due to the stringent limits on theories with ghosts found by [35] and discussed in the Introduction.

<sup>17</sup> The thermal trace anomaly is relevant for temperatures greater than the QCD phase transition, and gives  $1/3 - w = \mathcal{O}(10^{-3})$  [39]. Mass thresholds give  $1/3 - w = \mathcal{O}(10^{-2})$  when the temperature is close to the mass of a particle [40]. For temperatures below any Standard Model particle, the departure from  $w = 1/3$ , and  $R = 0$  can be neglected. However, as  $R$  drops to zero, there will be a moment in which  $M^2$  vanishes; we denote by  $w$  the value of the equation of state at this moment.

where  $p_{\omega_i}$  is the value of the physical momentum at  $t_{\omega_i}$ . We then find

$$\omega(t_{\omega_i}) = 0 \quad \Rightarrow \quad t_{\omega_i} = \frac{\sqrt{2}\sqrt{1-3w}}{3\sqrt{m^2+p_{\omega_i}^2}(1+w)} \quad (33)$$

We insert these expression for the momentum and the total mass in eq. (27), and we Taylor expand the resulting expression for  $t \simeq t_{\omega_i}$ . We find:

$$\ddot{\alpha} - \frac{\dot{\alpha}}{t-t_{\omega_i}} + C(t-t_{\omega_i})\alpha \approx 0 \quad (34)$$

where

$$C = \sqrt{\frac{2(m^2+p_{\omega_i}^2)}{1-3w}} [3m^2(1+w) + p_{\omega_i}^2(1+3w)] \quad (35)$$

Eq. (34) is solved by

$$\alpha \approx C_1 Ai' \left[ (-C)^{1/3} (t-t_{\omega_i}) \right] + C_2 Bi' \left[ (-C)^{1/3} (t-t_{\omega_i}) \right] \quad (36)$$

where  $C_{1,2}$  are two integration constants, and  $Ai'$  and  $Bi'$  the derivatives of the Airy functions  $Ai$  and  $Bi$ , respectively. These two solutions are regular at  $t = t_{\omega_i}$ , where they have the expansion series  $Ai', Bi' = \text{const.} + O(t-t_{\omega_i})^2$ . Since the linearized term is absent, we find that  $\alpha_0 \propto \dot{\alpha}/(t-t_{\omega_i})$  also remains finite as  $t = t_{\omega_i}$ .

**Behavior of the linearized system for  $M^2 \rightarrow 0$ .** The scale factor evolves as

$$a = a_M \left( \frac{t}{t_M} \right)^{\frac{2}{3(1+w)}} \quad (37)$$

where we recall that  $t_M$  is the times at which  $M^2 = 0$ , given by

$$t_M = \frac{\sqrt{2}}{3m} \frac{\sqrt{1-3w}}{1+w} \quad (38)$$

(cf. eq. (31)), and  $a_M$  is the value of the scale factor at  $t_M$ . We Taylor expand eq. (27) for  $t \simeq t_M$ :

$$\ddot{\alpha} + \frac{\dot{\alpha}}{t-t_M} + p_M^2 \alpha \approx 0 \quad (39)$$

where  $p_M$  is the value of the physical momentum of the mode at  $t_M$ . This equation is integrated to give

$$\alpha \approx C_1 J_0(p_M(t_M-t)) + C_2 Y_0(p_M(t_M-t)) \quad (40)$$

where  $J_0, Y_0$  are, respectively, the Bessel functions of the first and second kinds of order 0, and  $C_{1,2}$  are integration constants. While the  $J_0$  solution is regular at  $t = t_M$ , the  $Y_0$  solution has a logarithmic divergence. Correspondingly, the mode  $\alpha_0$  exhibits a linear divergence, as can be seen from equation (26).

The only way to avoid the singularity is to arrange the initial conditions so that  $C_2 = 0$ . This must be done for every mode (namely, for any comoving momentum  $k$ ) and for both the real and imaginary parts of the perturbations. We regard this as a completely unnatural assumption, since there is no reason why the initial conditions (set during inflation) should “know” about the singularity which is to occur later on when  $M^2$  vanishes.

In conclusion, the solutions of the linearized system remain finite when  $\omega^2 = p^2 + M^2 = 0$ , while they diverge when  $M^2 = 0$ . To verify this behavior, we performed a numerical evolution of eq. (27) for the specific case of  $w = 0$ ,  $p_M = 2$ . We then have  $t_{\omega_2} \simeq 0.058/m$  and  $t_M \simeq 0.47/m$ . We set the initial conditions  $\alpha = 1$ ,  $\dot{\alpha} = 0$  at  $t_M/100$ .<sup>18</sup> We plot in Figure 1 the resulting evolution of  $\alpha$  and  $\alpha_0$ . We see that the system is regular at  $t_{\omega_2}$ , while it diverges at  $t_M$  (we verified that  $\alpha_0$  diverges linearly, and, correspondingly,  $\alpha$  diverges logarithmically).

It is worth noting that the solutions diverge when the kinetic term of the longitudinal mode vanishes, cf. eq. (29). On the basis of what we wrote at the end of the previous Section, one should expect that this happens also for the cases in which the vector field(s) has a nonvanishing expectation value. We verify this explicitly in Section VB, for the simplest possible case of a vector field and a cosmological constant.

<sup>18</sup> The parameters of the evolution have no particular relevance, and have been chosen only for illustrative purposes. We have verified that other choices of parameters also confirm the behavior we have obtained analytically.

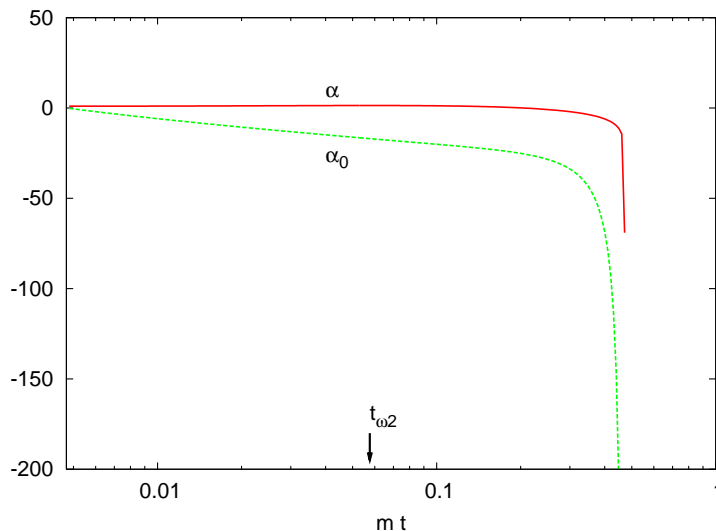


FIG. 1: Behavior of the linearized solutions representing the longitudinal mode. The system remains finite at the point  $t_{\omega i}$ , when the frequency vanishes, but it diverges at  $t_M$ , when the mass term  $M^2$  of the vector vanishes. See the main text for the parameters chosen.

## V. GHOST INSTABILITY FOR $\langle A_\mu \rangle \neq 0$

As we mentioned, when the vector field(s) has a vev, its perturbations mix with those of the metric already at the linearized level. The computation then becomes significantly harder than that discussed in the previous Section for the case of vanishing vev. Besides the increased number of perturbations, the spatial vev of the vector(s) breaks the isotropy of the background, so that, in general, one needs to study the cosmological perturbations of a non FRW space.

We start by studying the case of a single vector field with a vev along one spatial direction. The background has a 2d isotropy in the other two directions. In Subsection V A we review the formalism for dealing with the coupled system of perturbations of the vector and the metric in this context [12, 31]. By exploiting the symmetries of the background, the perturbations can be classified in two distinct subsets, which are decoupled from each other at the linearized level. We verified that one subset of perturbations, which includes the transverse vector modes, is well behaved, and, for brevity, we disregard it in the following discussions. The other subset, which we discuss in details, includes the longitudinal polarization of the vector.

In Subsection V B we apply this formalism to the simplest case of a single vector field plus a cosmological constant. In Subsection V C we study the case in which the cosmological constant is replaced by a scalar inflaton, which is the model proposed in [29]. Finally, in Subsection V D we study the model of vector inflation [28] (for which the formalism of Subsection V A does not apply; for this reason, this Subsection is a self-contained study).

### A. General formalism for a vector field in a 2d isotropic background

We review the general formalism for the study of perturbations [12, 31] for the case of a vector field with a nonvanishing vev along one spatial direction:

$$\begin{aligned} \langle A_\mu \rangle &= (0, M_p a(t) B_1(t), 0, 0) \\ ds^2 &= -dt^2 + a^2(t) dx^2 + b^2(t) (dy^2 + dz^2) \end{aligned} \quad (41)$$

The background geometry is a Bianchi-I background with 2d isotropy. The complete set of metric and vector field perturbations can be written in a way which exploits the residual 2d-isotropy of the system:

$$\delta g_{\mu\nu} = \begin{pmatrix} -2\Phi & a \partial_1 \chi & b (\partial_i B + B_i) \\ & -2a^2 \Psi & a b \partial_1 (\tilde{B} + \tilde{B}_i) \\ & & b^2 (-2\Sigma \delta_{ij} - 2 \partial_i \partial_j E - \partial_i E_i - \partial_j E_i) \end{pmatrix}$$

$$\delta A_\mu = (\alpha_0, \alpha_1, \partial_i \alpha + \alpha_i) \quad (42)$$

where  $i, j = 2, 3$  span the isotropic 2d-plane. The perturbations  $\{\Phi, \chi, B, \Psi, \tilde{B}, \Sigma, E, \alpha_0, \alpha_1, \alpha\}$  are 2d scalar perturbations, comprising one degree of freedom (d.o.f) each, and  $\{B_i, \tilde{B}_i, E_i, \alpha_i\}$  are 2d vector modes which satisfy the transversality condition ( $\partial_i B_i = \dots = 0$ ). Therefore, the 2d vector perturbations also comprise one d.o.f. each. Contrary to the 3d decomposition, there are no tensor modes, since there are no degrees of freedom left in a symmetric  $2 \times 2$  transverse and traceless matrix. The vector and scalar modes are decoupled at the linearized level (namely, in the linearized system of equations for the perturbations, or, equivalently, in their action, obtained by expanding the action for the model at quadratic order in the perturbations).

We need to eliminate the redundancy associated with the freedom of general coordinate transformations. We can do so by either choosing a gauge that completely fixes this freedom, as done in [12], or by combining the perturbations in gauge invariant modes [31]. We choose this second method, since it provides a nontrivial check on our algebra (since one needs to show that all the equations, and the action, can be written solely in terms of these combinations). Among different equivalent choices, we use the gauge invariant combinations [31]

$$\begin{aligned} \hat{\Phi} &= M_p \left[ \Phi + \left( \frac{\Sigma}{H_b} \right)^\bullet \right] \\ \hat{\Psi} &= M_p \left[ \Psi - \frac{H_a}{H_b} \Sigma + \frac{b}{a} \partial_1^2 \left( \tilde{B} + \frac{b}{a} E \right) \right] \\ \hat{B} &= -\frac{M_p}{b} \vec{\partial}_T^2 \left[ B - \frac{1}{b H_b} \Sigma + b \dot{E} \right] \\ \hat{\chi} &= -\frac{M_p}{a} \partial_1^2 \left[ \chi - \frac{1}{a H_b} \Sigma - a \left( \frac{b}{a} \left( \tilde{B} + \frac{b}{a} E \right) \right)^\bullet \right] \\ \hat{\alpha}_1 &= -\frac{1}{a} \left[ \alpha_1 + a M_p \frac{\dot{B}_1 + H_a B_1}{H_b} \Sigma - b M_p B_1 \partial_1^2 \left( \tilde{B} + \frac{b}{a} E \right) \right] \\ \hat{\alpha} &= -\frac{1}{a} \partial_1 \left[ \alpha - b M_p B_1 \partial_1 \left( \tilde{B} + \frac{b}{a} E \right) \right] \\ \hat{\alpha}_0 &= \frac{1}{a} \partial_1 \left[ \alpha_0 - a M_p B_1 \partial_1 \left( \frac{b}{a} \left( \tilde{B} + \frac{b}{a} E \right) \right)^\bullet \right] \end{aligned} \quad (43)$$

(where the dot and bullet denote time derivative,  $H_a \equiv \dot{a}/a$ ,  $H_b \equiv \dot{b}/b$ , and  $\vec{\partial}_T^2 \equiv \partial_2^2 + \partial_3^2$ ) in the 2d scalar sector, and

$$\begin{aligned} \hat{B}_i &= B_i + b \dot{E}_i \\ \hat{\tilde{B}}_i &= a \left( \tilde{B}_i + \frac{b}{a} E_i \right) \\ \hat{\alpha}_i &= \frac{\alpha_i}{M_p} \end{aligned} \quad (44)$$

in the 2d vector sector. If there is also one scalar field,  $\phi + \delta\phi$ , there is the additional 2d gauge invariant scalar mode

$$\hat{\delta\phi} = \delta\phi + \frac{\dot{\phi}}{H_b} \Sigma \quad (45)$$

We note that the gauge invariant modes have the following mass dimensions:  $[\hat{\tilde{B}}_i] = -1$ ,  $[\hat{B}_i] = [\hat{\alpha}_i] = 0$ ,  $[\hat{\Phi}] = [\hat{\Psi}] = [\hat{\alpha}_1] = [\hat{\alpha}] = [\hat{\delta\phi}] = 1$ ,  $[\hat{B}] = [\hat{\chi}] = [\hat{\alpha}_0] = 2$ .

The above gauge invariant combinations of the metric perturbations do not reduce to the ones which are commonly given in the literature [37] in the limit of isotropic background. However, we explicitly verified in [31] that our choice

and the more conventional one are equivalent. Our choice is motivated by the fact that it immediately identifies the nondynamical modes of the system: one can choose the gauge  $B = E = \Sigma = E_i = 0$ ,<sup>19</sup> in which the gauge invariant combinations  $\hat{\delta}g_{0\mu}$  and  $\hat{\delta}A_0$  reduce to the corresponding non dynamical perturbations  $\delta g_{0\mu}$  and  $A_0$ . These modes enter in the action without time derivatives, due to the specific form of the “kinetic terms”  $F^2$  and  $R$ . Therefore, we expect that also the gauge invariant combinations  $\hat{\delta}g_{0\mu}$  and  $\hat{\delta}A_0$  are nondynamical, as the explicit computations reported in the next Subsections confirm.

As we mentioned, the two subsets of modes (2d scalars vs 2d vectors) are decoupled from each other at the linearized level, and can be studied separately. We verified that, for the models of our interest, the 2d vector modes do not develop any instability. For brevity, we do not report these computations here, and we focus on the more problematic 2d scalar modes. In the next two Subsections we study the evolution of this system for the case of a single vector plus a cosmological constant or a scalar inflaton.

## B. One vector plus a cosmological constant

We study the simplest model in which a nonminimally coupled vector field with a spatial vev provides an anisotropic expansion. Besides the vector field, there is a vacuum energy  $V_0$  which is responsible for an overall accelerated expansion. The Subsection is divided into three parts. We first present the model, and discuss the background evolution. We then solve the linearized system of equations for the perturbations, and find that it diverges at some point close to horizon crossing. We finally compute the kinetic term  $K$  of the quadratic action of the perturbations, and show (i) that the divergence of the linearized system takes place precisely when one eigenvalue of  $K$  vanishes, and (ii) that one of the perturbations is a ghost for some moment of time.

### 1. The model and the background solution

The action of the model

$$S = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V_0 - \frac{1}{2} \left( m^2 - \frac{R}{6} \right) A_\mu A^\mu \right] \quad (46)$$

gives the equations of motion

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{M_p^2} \left[ -V_0 g_{\mu\nu} + T_{\mu\nu}^{(A)} \right] \\ T_{\mu\nu}^{(A)} &\equiv F_\mu^\sigma F_{\nu\sigma} - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g_{\mu\nu} + \left( m^2 - \frac{R}{6} \right) A_\mu A_\nu - \frac{1}{2} m^2 A_\alpha A^\alpha g_{\mu\nu} \\ &\quad - \frac{1}{6} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) A_\alpha A^\alpha - \frac{1}{6} (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) A_\alpha A^\alpha \\ \frac{1}{\sqrt{-g}} \partial_\mu \left[ \sqrt{-g} F^{\mu\nu} \right] - \left( m^2 - \frac{R}{6} \right) A^\nu &= 0 \end{aligned} \quad (47)$$

If we assume that the background solutions depend only on time, then the  $\nu$  component of the last of (47) gives  $A^0 = 0$ . For a homogeneous background, the vev of the vector is everywhere in the same direction; we choose the coordinates such that this direction coincides with the  $x$ -axis. We look for background solutions of the form (41), and we define

$$H_a \equiv \frac{\dot{a}}{a}, \quad H_b \equiv \frac{\dot{b}}{b}, \quad H \equiv \frac{H_a + 2H_b}{3}, \quad h \equiv \frac{H_b - H_a}{3} \quad (48)$$

---

<sup>19</sup> This was the choice made in [12], and it fixes completely the freedom associated to general coordinate transformations.

Namely,  $H$  is the average expansion rate, while  $h$  parametrizes the departure from isotropy. The regime of small isotropy corresponds to  $h \ll H$ . In terms of these quantities, and for the background (41), equations (47) give <sup>20</sup>

$$\begin{aligned}
H^2 - h^2 &= \frac{V_0}{3M_p^2} + \frac{1}{6}\dot{B}_1^2 - \frac{2}{3}h B_1 \dot{B}_1 + \frac{1}{6}B_1^2 (m^2 - 4Hh + 5h^2) \\
\dot{h} + 3Hh &= \frac{1}{3}B_1^2 \left( \dot{H} - \frac{\dot{h}}{2} \right) + \frac{1}{3}\dot{B}_1^2 + \frac{1}{3}(2H - 5h) B_1 \dot{B}_1 + \frac{1}{3} \left( 3H^2 - \frac{11}{2}Hh + 5h^2 - m^2 \right) B_1^2 \\
2\dot{H} + 3H^2 + 3h^2 &= \frac{V_0}{M_p^2} - \frac{1}{2}\dot{B}_1^2 - \frac{1}{3}B_1 \left[ \ddot{B}_1 + (3H - 2h) \dot{B}_1 \right] + \frac{1}{6} (4Hh - 5h^2 + m^2) B_1^2 \\
\ddot{B}_1 + 3H \dot{B}_1 + (m^2 - 5h^2 - 2hH - 2\dot{h}) B_1 &= 0
\end{aligned} \tag{49}$$

One of the last three equations can be obtained from the other equations in (49) due to a nontrivial Bianchi identity. We see from the second of (49) that the anisotropy of the background is indeed supported by the vector vev  $B_1$ .

Below, when we study the perturbations of this model, we use the exact background equations (49). However, only for this discussion, we present the inflationary solution of (49) in the slow roll approximation (slow motion of  $B_1$  towards zero, so that the anisotropy is not quickly damped away) and for small anisotropy. Specifically, we disregard the time derivatives of  $H$ ,  $h$ , and look for expansion series solutions of the type

$$H = H_0 + c_H B_1^2 + \mathcal{O}(B_1^4) \quad , \quad h = c_h B_1^2 + \mathcal{O}(B_1^4) \quad , \quad \dot{B}_1 \simeq c_B H_0 B_1 \quad , \quad \text{with } B_1, c_B \ll 1 \tag{50}$$

We insert these expression in the last equation of (49) and expand in  $B_1$ . This gives  $3c_B H_0^2 + c_B^2 H_0^2 + m^2 = 0$ . Since,  $c_B \ll 1$ , this gives  $c_B = -\frac{m^2}{3H_0^2}$ . Therefore, slow roll requires  $m \ll H_0$ . We then insert (50) in the first of (49), expand in  $B_1$ , and neglect  $m$  vs.  $H_0$ . We find  $H_0 = \sqrt{V_0}/(\sqrt{3}M_p)$ ,  $c_H = m^2/(12H_0)$ . Finally, we insert (50) in the second and third of (49), expand in  $B_1$ , and again neglect  $m$  vs.  $H_0$ . These equations then become identical, and give  $c_h = H_0/3$ . Therefore,

$$H = H_0 + \frac{m^2}{12H_0} B_1^2 + \mathcal{O}(B_1^4) \quad , \quad h = \frac{H_0}{3} B_1^2 + \mathcal{O}(B_1^4) \quad , \quad \dot{B}_1 = -\frac{m^2}{3H_0} B_1 + \mathcal{O}(B_1^3) \tag{51}$$

where  $H_0$  is the expansion rate in absence of the vector field.

Before moving to the study of the perturbation, we note that the mass parameter  $m$  of the vector field must satisfy

$$B_1 H_0^2 \lesssim m^2 \ll H_0^2 \tag{52}$$

The second condition is due to the slow roll requirement, as we already pointed out. The first condition comes from requiring that the ‘‘mass term’’ for  $B_1$  in the last of (49) is positive, so that  $B_1$  evolves towards the origin (so that the anisotropy decreases during inflation rather than increasing).

## 2. Instability from the linearized equations

We now expand the field equations (47) at first order in the 2d scalar perturbations of the metric and the vector field. <sup>21</sup> We denote the resulting linearized equations as

$$\text{Eq}_{\mu\nu} : \delta \left( G_{\mu\nu} - \frac{1}{M_p^2} \left( -V_0 g_{\mu\nu} + T_{\mu\nu}^{(A)} \right) \right) = 0 \quad , \quad \text{Eq}_\mu : \delta \left( \frac{\delta S}{\delta A_\mu} \right) = 0 \tag{53}$$

We then Fourier transform these equations. Namely for each mode, we have

$$\delta(x) = \frac{1}{(2\pi)^3} \int d^3k \delta(k) e^{-ik_L x - ik_{T2} y - ik_{T3} z} \tag{54}$$

<sup>20</sup> The first of (49) is the 00 Einstein equation; the second is the linear combination of the (11)-(22) Einstein equations; the third equation is the linear combination of the (11) + 2 × (22) Einstein equations. Finally, the fourth equation is the  $x$ - component of the equations for the vector field (the 33 Einstein equation coincides with the 22 one, while the remaining equations are trivial).

<sup>21</sup> As we mentioned, the 2d vector perturbations are decoupled from the 2d scalar ones; we have verified that the 2d vectors do not develop any instability; for brevity, we do not present this computations here.

where  $\delta$  denotes any of the perturbations (we denote the perturbations in coordinate and momentum space with the same symbol). Modes with different comoving momenta are decoupled from each other at the linearized level. We denote by  $k_L$  the component of the comoving momentum in the  $x$ -direction, and by  $\vec{k}_T$  the component of the momentum in the  $y-z$  plane. The full comoving momentum is then given by  $k^2 = k_L^2 + \vec{k}_T^2$ . The physical momentum is instead

$$p^2 = p_L^2 + \vec{p}_T^2 = \frac{k_L^2}{a^2} + \frac{\vec{k}_T^2}{b^2} \quad (55)$$

The explicit expressions of the linearized equations for the scalar sector are given in equations (A1) of Appendix A. We could express these equations solely in terms of the gauge invariant modes defined in (43). This is a nontrivial check on our algebra. Here, we disregard some of the equations in (A1) which can be obtained from the remaining ones (due to Bianchi identities). The (complete) set of independent linearized equations which we choose to integrate is

$$\begin{aligned} \text{Eq}_{00} : & \left[ \left(1 - \frac{B_1^2}{3}\right) H + \left(1 + \frac{B_1^2}{6}\right) h + \frac{1}{6} \dot{B}_1 B_1 \right] \dot{\Psi} + \frac{1}{2} \left[ p_T^2 + \left(m^2 - \frac{p_T^2 + 2p_L^2}{6} + 5h^2 - 4hH\right) B_1^2 - 4h B_1 \dot{B}_1 + \dot{B}_1^2 \right] \hat{\Psi} \\ & + \left[ 3H^2 - \left(3 + \frac{5}{2} B_1^2\right) h^2 - \frac{1}{2} \dot{B}_1^2 + 2 \left(B_1 H + \dot{B}_1\right) B_1 h \right] \hat{\Phi} - \left[ \left(1 + \frac{B_1^2}{6}\right) (H + h) + \frac{1}{6} B_1 \dot{B}_1 \right] \hat{\chi} \\ & - \left[ \left(1 + \frac{B_1^2}{6}\right) \left(H - \frac{1}{2} h\right) + \frac{1}{6} B_1 \dot{B}_1 \right] \hat{B} - \left[ \left(\frac{1}{2} H - h\right) B_1 + \frac{1}{2} \dot{B}_1 \right] \hat{\alpha}_0 - \left(\frac{1}{2} \dot{B}_1 - B_1 h\right) \hat{\alpha}_1 \\ & + \left[ \frac{1}{2} \left(\frac{p_T^2}{3} - m^2 - 5h^2 + 4hH\right) B_1 + h \dot{B}_1 \right] \hat{\alpha}_1 = 0 \\ \text{Eq}_{01} : & \frac{1}{6} \left(\hat{\alpha}_1 - B_1 \dot{\Psi}\right) B_1 + \frac{B_1}{6} \left(B_1 H - 2B_1 h - 2\dot{B}_1\right) \hat{\Psi} + \frac{1}{6} \left(-B_1 H + 2h B_1 + \dot{B}_1\right) \hat{\alpha}_1 \\ & + \left(1 + \frac{B_1^2}{6}\right) \left(H + h + \frac{B_1 \dot{B}_1}{6 + B_1^2}\right) \hat{\Phi} + \left(\frac{p_T^2}{4p_L^2} + \frac{p_T^2}{24p_L^2} B_1^2 - \frac{1}{3} \mathcal{D}_{\chi\chi}\right) \hat{\chi} - \frac{6 + B_1^2}{24} \hat{B} - \frac{B_1}{3} \mathcal{D}_{\alpha_0\alpha_0} \hat{\alpha}_0 = 0 \\ \text{Eq}_{0i} : & \frac{1}{2} \left(1 - \frac{B_1^2}{6}\right) \dot{\Psi} + \frac{1}{6} B_1 \hat{\alpha}_1 + \left[\frac{B_1}{6} \left(H B_1 - 2\dot{B}_1\right) - \left(\frac{3}{2} + \frac{B_1^2}{12}\right) h\right] \hat{\Psi} + \left[\left(\frac{1}{3} H - \frac{7}{6} h\right) B_1 + \frac{2}{3} \dot{B}_1\right] \hat{\alpha}_1 \\ & + \left[\left(h - \frac{H}{2}\right) B_1 - \frac{\dot{B}_1}{2}\right] \hat{\alpha} + \left[-\left(\frac{1}{2} + \frac{B_1^2}{12}\right) h + \left(1 + \frac{B_1^2}{6}\right) H + \frac{1}{6} B_1 \dot{B}_1\right] \hat{\Phi} - \frac{1}{4} \left(1 + \frac{B_1^2}{6}\right) \hat{\chi} \\ & + \frac{p_L^2}{4p_T^2} \left(1 + \frac{B_1^2}{6}\right) \hat{B} = 0 \\ \text{Eq}_0 : & \hat{\alpha}_1 + \left(2h B_1 - H B_1 - \dot{B}_1\right) \hat{\Psi} + (H - 2h) \hat{\alpha}_1 + \frac{p_T^2}{p_L^2} \hat{\alpha} + \frac{p_T^2}{p_L^2} (H - 2h) \hat{\alpha} + \left(H B_1 - 2h B_1 + \dot{B}_1\right) \hat{\Phi} - \frac{2}{3B_1} \mathcal{D}_{\chi\chi} \hat{\chi} \\ & + \left(1 + \frac{p_T^2}{p_L^2} - \frac{2}{3} \mathcal{D}_{\alpha_0\alpha_0}\right) \hat{\alpha}_0 = 0 \\ \text{Eq}_{11} : & \frac{1}{2} B_1 \left(\ddot{\alpha}_1 - 2B_1 \ddot{\Psi}\right) - \left(3H B_1 + 2\dot{B}_1\right) B_1 \dot{\Psi} + \frac{1}{2} \left(-H B_1 + 8h B_1 - \dot{B}_1\right) \dot{\alpha}_1 + \left(\mathcal{M}_{\Psi\Psi} - p_T^2 B_1^2\right) \hat{\Psi} \\ & + \left(\mathcal{M}_{\Psi\alpha_1} + \frac{p_T^2}{2} B_1\right) \hat{\alpha}_1 + \left[\left(3 + \frac{1}{2} B_1^2\right) h + (3 - B_1^2) H + \frac{1}{2} B_1 \dot{B}_1\right] \hat{\Phi} \\ & + \left[\left(-\frac{9}{4} m^2 + \frac{p_T^2 + 2p_L^2}{4} - \frac{15}{4} h^2 + 3hH\right) B_1^2 + 3h B_1 \dot{B}_1 - 3 \left(\frac{p_T^2}{2} - \frac{V_0}{2M_p^2} + \frac{3}{2} h^2 - \frac{3}{2} H^2 + \frac{1}{4} \dot{B}_1^2\right)\right] \hat{\Phi} \\ & - \left(\frac{3}{2} - \frac{1}{4} B_1^2\right) \dot{\hat{B}} + \frac{1}{2} B_1^2 \dot{\hat{\chi}} + (2H - h) B_1^2 \hat{\chi} - \left[\left(\frac{9}{2} + \frac{1}{4} B_1^2\right) h + \left(\frac{9}{2} - \frac{5}{4} B_1^2\right) H + \frac{1}{2} B_1 \dot{B}_1\right] \hat{B} \\ & + \frac{3}{2} \left(2h B_1 - H B_1 - \dot{B}_1\right) \hat{\alpha}_0 = 0 \end{aligned}$$

$$\begin{aligned}
\text{Eq}_1 : \quad & \ddot{\hat{\alpha}}_1 - \frac{1}{3} B_1 \ddot{\hat{\Psi}} + 3H \dot{\hat{\alpha}}_1 + \left( \frac{8}{3} B_1 h - \frac{7}{3} B_1 H - \dot{B}_1 \right) \dot{\hat{\Psi}} + (\mathcal{M}_{\alpha_1 \alpha_1} + p_T^2) \hat{\alpha}_1 - \frac{B_1}{3} p_T^2 \hat{\Psi} \\
& + \left( \dot{B}_1 - 2 B_1 h \right) \dot{\hat{\Phi}} + \frac{B_1}{3} \left( \dot{B}_1 + \dot{\hat{\chi}} \right) + \dot{\hat{\alpha}}_0 + \frac{B_1}{3} (p^2 - 6m^2) \hat{\Phi} + \frac{1}{3} \left( 7 B_1 h + B_1 H - 3 \dot{B}_1 \right) \hat{B} + \frac{2}{3} (2H - h) B_1 \hat{\chi} \\
& - p_T^2 \hat{\alpha} + 2 (h + H) \hat{\alpha}_0 = 0
\end{aligned}$$

$$\text{Eq}_i : \quad \ddot{\hat{\alpha}} + 3 (H - 2h) \dot{\hat{\alpha}} + (\mathcal{M}_{\alpha\alpha} + p_L^2) \hat{\alpha} - p_L^2 \hat{\alpha}_1 + \frac{p_L^2}{p_T^2} \left( B_1 H - 2 B_1 h + \dot{B}_1 \right) \hat{B} + \dot{\hat{\alpha}}_0 + (2H - 4h) \hat{\alpha}_0 = 0 \quad (56)$$

where  $\mathcal{D}_{\chi\chi}$ ,  $\mathcal{D}_{\alpha_0\alpha_0}$ ,  $\mathcal{M}_{\Psi\Psi}$ ,  $\mathcal{M}_{\Psi\alpha_1}$ ,  $\mathcal{M}_{\alpha_1\alpha_1}$ ,  $\mathcal{M}_{\alpha\alpha}$  depend on the background quantities and are explicitly given in equations (A2) and (A3) in Appendix A. In writing (56), we have also made use of the physical momenta defined in eq. (55).

We now solve the system of equations (56) numerically. We start by noting that the first four equations in (56) contain at most a single time derivative for the perturbations, and do not contain any time derivative of the modes  $\{\hat{\Phi}, \hat{\chi}, \hat{B}, \hat{\alpha}_0\}$ ; these are the nondynamical modes of the system, and these first four equations can also be obtained by extremizing the quadratic action for the perturbations (eq. (63) below) with respect to these modes. These equations are precisely of the type (21) given above where we discussed the linearized equations at a formal level. Compare also with eq. (26) for the nondynamical mode  $\alpha_0$  in the simpler case studied in Section IV.

One can choose to solve these four constraint equations for  $\{\hat{\Phi}, \hat{\chi}, \hat{B}, \hat{\alpha}_0\}$ , and insert the solutions in the remaining equations in (56). In this way, we obtain a closed system of equations for the dynamical modes. A different but equivalent way to integrate the system (56) is to differentiate the constraint equations, so as to obtain differential equations for the nondynamical modes too. Combined with the remaining equations in (56), one then obtains a closed system that can be numerically integrated (the constraint equations need to be imposed as initial condition). In the first method mentioned, one obtains a system with fewer but more complicated equations. On the other hand, the second method results in a larger set of less complicated equations. Here, we adopt an ‘‘intermediate’’ method, which we have found to be convenient for the numerical integration. We solve the second of (56) for  $\hat{\chi}$ :

$$\begin{aligned}
\hat{\chi} = \frac{p_L^2}{\mathcal{D}} \left[ (6 + B_1^2) \hat{B} + 8 B_1 \mathcal{D}_{\alpha_0\alpha_0} \hat{\alpha}_0 + 4 \left( (H - 2h) B_1 - \dot{B}_1 \right) \hat{\alpha}_1 - 4 B_1 \left( (H - 2h) B_1 - 2 \dot{B}_1 \right) \hat{\Psi} \right. \\
\left. - 4 (6 + B_1^2) \mathcal{H} \hat{\Phi} - 4 B_1 \dot{\hat{\alpha}}_1 + 4 B_1^2 \dot{\hat{\Psi}} \right] \quad (57)
\end{aligned}$$

where the time dependent coefficients  $\mathcal{D}_{\alpha_0\alpha_0}$ ,  $\mathcal{D}$  and  $\mathcal{H}$  are given in equations (A2) and (A4) of Appendix A.

It is important to verify that the second of (56) can indeed be solved in terms of  $\hat{\chi}$ ; namely, that  $\mathcal{D} \neq 0$ . The easiest way to verify this is to use the slow-roll solutions (51) in the expression (A4) for  $\mathcal{D}$ , since we are working in a regime in which these slow-roll solutions are highly accurate. This gives

$$\mathcal{D} = 6p_T^2 + (p_T^2 - 24H^2 + 12m^2) B_1^2 + \mathcal{O}(B_1^4) \quad (58)$$

We can integrate out  $\hat{\chi}$  provided that  $\mathcal{D} \neq 0$ . We see that  $\mathcal{D}$  is positive at least as long as  $p_T \gtrsim 2H B_1$  (when the two terms become equal, the negative term dominates the  $\mathcal{O}(B_1^2)$  expression, and the expression cancels with the positive  $\mathcal{O}(B_1^0)$  term. At this point, the expansion series (58) breaks down). As we show below, the computation we are performing shows that the linearized perturbations blow up when  $p_L \approx \sqrt{12} H/B_1$ . Since integrating out  $\hat{\chi}$  is a step of this computation, our result can be trusted as long as  $p_T > 2H B_1$  when  $p_L \approx \sqrt{12} H/B_1$ . Namely, we consider modes for which  $p_T > (B_1^2/\sqrt{3}) p_L$  at that moment. This is not a strong restriction, since  $B_1 \ll 1$ .

We insert the solution (57) into the other equations in (56), and we differentiate the three remaining constraint equations (we stress that we do not lose any information, provided that these equations are imposed as initial conditions). In this way, we obtain a system of 6 differential equations, which can be formally written as

$$\mathcal{M}_\kappa \begin{pmatrix} \ddot{\hat{\Psi}} \\ \ddot{\hat{\alpha}}_1 \\ \ddot{\hat{\alpha}} \\ \dot{\hat{\Phi}} \\ \dot{\hat{B}} \\ \dot{\hat{\alpha}}_0 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{pmatrix}, \quad \mathcal{M}_\kappa \equiv \begin{pmatrix} \kappa_{11} & \kappa_{12} & 0 & \kappa_{14} & \kappa_{15} & \kappa_{16} \\ \kappa_{21} & \kappa_{22} & 0 & \kappa_{24} & \kappa_{25} & \kappa_{26} \\ 0 & 0 & 1 & 0 & 0 & 1 \\ \kappa_{41} & \kappa_{42} & 0 & \kappa_{44} & \kappa_{45} & \kappa_{46} \\ \kappa_{51} & \kappa_{52} & 0 & \kappa_{54} & \kappa_{55} & \kappa_{56} \\ \kappa_{61} & \kappa_{62} & \kappa_{63} & \kappa_{64} & \kappa_{65} & \kappa_{66} \end{pmatrix} \quad (59)$$



which correspond, respectively, to  $\text{Eq}_{11}$ ,  $\text{Eq}_1$ ,  $\text{Eq}_i$ , and to the dime derivatives of  $\text{Eq}_{00}$ ,  $\text{Eq}_{0i}$ , and  $\text{Eq}_0$ .

The elements of the matrix  $\mathcal{M}_\kappa$  depend on the background quantities. The functions on the right hand side of (59)  $f_i$  are expressed as linear combinations of  $\{\hat{\Psi}, \hat{\Psi}, \hat{\alpha}_1, \hat{\alpha}_1, \hat{\alpha}, \hat{\alpha}, \hat{\Phi}, \hat{B}, \hat{\alpha}_0\}$  whose coefficients also depend on the background quantities. The explicit expressions for the matrix  $\mathcal{M}_\kappa$  and the coefficients  $f_i$  are given in equations (A8) and (A11) of Appendix A.

We invert  $\mathcal{M}_\kappa$ , and integrate the system numerically. The determinant of  $\mathcal{M}_\kappa$  vanishes at some given time. As we approach that time,  $\mathcal{M}_\kappa^{-1}$  diverges. Correspondingly, the numerical solutions of the system also diverge.

By inserting the explicit expressions for the entries of  $\mathcal{M}_\kappa$  given in Appendix A, we find:

$$\det \mathcal{M}_\kappa = \frac{p_T^2 (6 + B_1^2)^2}{864\mathcal{D}} (h + H)^2 [72 \mathcal{D}_{\alpha_0\alpha_0} - 6 B_1^2 (3 - 2\mathcal{D}_{\alpha_0\alpha_0}) - B_1^4 (3 - 8\mathcal{D}_{\alpha_0\alpha_0})] \quad (60)$$

Using the expressions for  $\mathcal{D}_{\alpha_0\alpha_0}$  and  $\mathcal{D}$  given in (A2) and (A4), we then find that the determinant vanishes when

$$p_L^2 = p_{L*}^2 \equiv \frac{1}{B_1^2 (6 + B_1^2)} \left\{ -18 \left( 2m^2 - 4H^2 + 4h^2 + \dot{B}_1^2 \right) - \left( 6m^2 - 12H^2 - 72hH + 102h^2 + 7\dot{B}_1^2 \right) B_1^2 \right. \\ \left. - (4H^2 - 28hH + 31h^2) B_1^4 + 48hB_1\dot{B}_1 + 4(7h - 2H) B_1^3 \dot{B}_1 \right\} \quad (61)$$

This expression for  $p_{L*}$  is exact. Using the slow-roll solutions (51), we obtain the approximate expression:

$$p_{L*}^2 \simeq \frac{6(2H_0^2 - m^2)}{B_1^2} + \mathcal{O}(B_1^0) \quad (62)$$

To confirm that the linearized modes indeed blow up at  $p_{L*}$ , we integrate the system (59) for a specific choice of parameters (the parameters we use have no particular relevance, and are chosen only for illustrative purposes). We choose a small anisotropy,  $(h/H)_{\text{in}} = \mathcal{O}(B_{1,\text{in}}^2) \simeq 10^{-2}$ , and an initial value for the momentum,  $p_{\text{in}} \simeq 10^3 H_{\text{in}}$ , so that the mode is initially deeply inside the horizon. As we verify in Appendix B, the frequency for the modes is initially adiabatically evolving, so that we can choose the initial conditions for the dynamical modes  $\hat{\Psi}$ ,  $\hat{\alpha}_1$ ,  $\hat{\alpha}$  and their derivatives according to the adiabatic vacuum prescription. We then use the constraint equations (the first, the third, and the fourth of (56)), to provide the initial conditions for the three nondynamical modes  $\hat{\Phi}$ ,  $\hat{B}$ ,  $\hat{\alpha}_0$  of the system. The resulting evolution of the ‘‘relativistic Newtonian potential’’  $\hat{\Phi}$ , and of two other 2d scalar modes are shown in Figure 2. We see that the modes indeed diverge at some given time. We verified that  $p_L = p_{L*}$  at this moment. This is confirmed by the time evolution of one of the eigenvalue of the kinetic matrix of the dynamical perturbations (computed in VB3) shown in the left panel of the Figure. We see that the linearized solutions blow up precisely when the system (59), or equivalently, the kinetic matrix, becomes singular.

### 3. Ghost from the quadratic action

In order to understand the physical reasons behind the instability we have just found, we compute the quadratic action for the perturbations. We expand the action (46) at quadratic order in the perturbations (42). The resulting action can be expressed solely in terms of the gauge invariant modes defined in (43)-(44). This provides a nontrivial check on our algebra. We also find that the action is the sum of two separate parts, one involving only the 2d scalar modes, and one the 2d vectors. We disregard this second piece in the following discussion. The action for the 2d

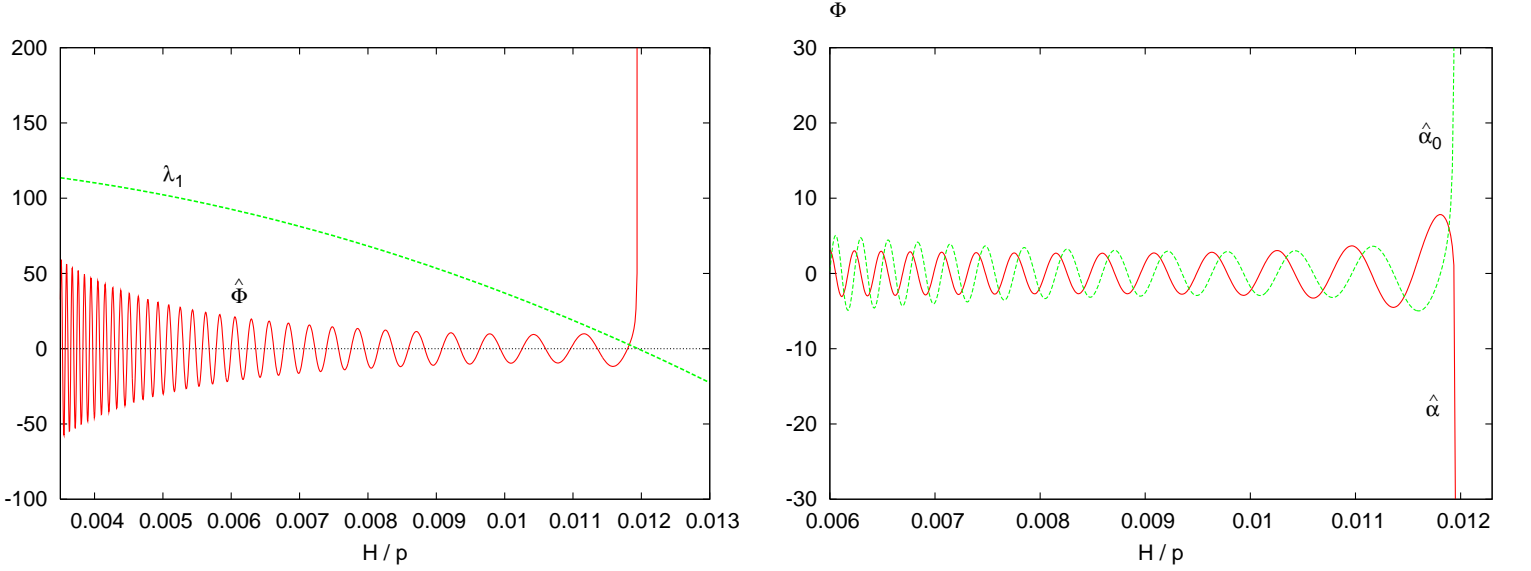


FIG. 2: Results from a numerical simulation with  $m^2 = 0.1 H_0 B_{\text{in}} = 0.1$ ,  $p_{L,\text{in}} = 400 H_{\text{in}}$ ,  $p_{T,\text{in}} = 900 H_{\text{in}}$ , (corresponding to  $H/p \simeq 10^{-3}$ ; the modes are initially in the adiabatic vacuum; only the final part of the evolution is shown in the two figures). Since  $H/p$  grows during inflation, we use this quantity as “time variable” in the Figure. Left panel: gauge invariant relativistic gravitational potential  $\hat{\Phi}$ . We show the real part in units of  $\hat{\Psi}_{\text{in}}$ . We also show the eigenvalue  $\lambda_1$  of the kinetic matrix (multiplied by  $3 \times 10^5$ , so that it is visible in the figure). We see that  $\hat{\Phi}$  diverges when  $\lambda_1 = 0$ . Right panel: real parts of the modes  $\hat{\alpha}_0$  (in units of  $100 H_0 \hat{\Psi}_{\text{in}}$ ) and  $\hat{\alpha}$  (in units of  $\hat{\Psi}_{\text{in}}$ ). Also these modes (as all the modes of the system) diverge when  $\lambda_1 = 0$ .

scalars reads

$$\begin{aligned}
S_{2\text{dS}} &= \frac{1}{3} \int d^3 k dt a b^2 \mathcal{L}_{2\text{dS}} \\
\mathcal{L}_{2\text{dS}} &= B_1^2 |\dot{\Psi}|^2 - \frac{B_1}{2} (\dot{\Psi} \dot{\alpha}_1^* + \text{h.c.}) + \frac{3p_T^2}{2p_L^2} |\dot{\alpha}|^2 + \frac{3}{2} |\dot{\alpha}_1|^2 - \frac{3}{2} \left(1 + \frac{1}{2} B_1^2\right) \left(H - 2h - \frac{2B_1 \dot{B}_1}{2 + B_1^2}\right) (\dot{\Psi}^* \hat{\Psi} + \text{h.c.}) \\
&\quad - \left(B_1 h - \frac{1}{2} B_1 H + \frac{1}{2} \dot{B}_1\right) (\dot{\Psi}^* \hat{\alpha}_1 + \text{h.c.}) - \left[\left(3 + \frac{1}{2} B_1^2\right) h + (3 - B_1^2) H + \frac{1}{2} B_1 \dot{B}_1\right] (\dot{\Psi}^* \hat{\Phi} + \text{h.c.}) \\
&\quad - \frac{B_1^2}{2} (\dot{\Psi}^* \hat{\chi} + \text{h.c.}) + \frac{1}{4} (6 - B_1^2) (\dot{\Psi}^* \hat{B} + \text{h.c.}) + \frac{3p_T^2}{2p_L^2} (H - 2h) (\dot{\alpha}^* \hat{\alpha} + \text{h.c.}) + \frac{3p_T^2}{2p_L^2} (\dot{\alpha}^* \hat{\alpha}_0 + \text{h.c.}) \\
&\quad - \frac{3}{2} (B_1 H - 2B_1 h + \dot{B}_1) (\dot{\alpha}_1^* \hat{\Psi} + \text{h.c.}) + \frac{3}{2} (H - 2h) (\dot{\alpha}_1^* \hat{\alpha}_1 + \text{h.c.}) + \frac{3}{2} (\dot{B}_1 - 2B_1 h) (\dot{\alpha}_1^* \hat{\Phi} + \text{h.c.}) \\
&\quad + \frac{B_1}{2} (\dot{\alpha}_1^* \hat{\chi} + \text{h.c.}) + \frac{B_1}{2} (\dot{\alpha}_1^* \hat{B} + \text{h.c.}) + \frac{3}{2} (\dot{\alpha}_1^* \hat{\alpha}_0 + \text{h.c.}) - (\mathcal{D}_{\Psi\Psi} + p_T^2 B_1^2) |\hat{\Psi}|^2 \\
&\quad - \left(\mathcal{D}_{\Psi\alpha_1} - \frac{1}{2} p_T^2 B_1\right) (\hat{\Psi}^* \hat{\alpha}_1 + \text{h.c.}) \\
&\quad + \left[\left(-\frac{3}{2} m^2 + \frac{2p_L^2 + p_T^2}{4} - \frac{15}{2} h^2 + 6hH\right) B_1^2 + 6hB_1 \dot{B}_1 - \frac{3}{2} (p_T^2 + \dot{B}_1^2)\right] (\hat{\Psi}^* \hat{\Phi} + \text{h.c.}) \\
&\quad - \left(hB_1^2 - \frac{1}{2} B_1^2 H + B_1 \dot{B}_1\right) (\hat{\Psi}^* \hat{\chi} + \text{h.c.}) - \left(\frac{9}{2} h + \frac{1}{4} (h - 2H) B_1^2 + B_1 \dot{B}_1\right) (\hat{\Psi}^* \hat{B} + \text{h.c.})
\end{aligned}$$

$$\begin{aligned}
& - \frac{3}{2} \left( B_1 H - 2B_1 h + \dot{B}_1 \right) \left( \hat{\Psi}^* \hat{\alpha}_0 + \text{h.c.} \right) - \left( \mathcal{D}_{\alpha\alpha} + \frac{3}{2} p_T^2 \right) |\hat{\alpha}|^2 + \frac{3}{2} p_T^2 \left( \hat{\alpha}^* \hat{\alpha}_1 + \text{h.c.} \right) \\
& + \frac{3}{2} \left( 2h B_1 - B_1 H - \dot{B}_1 \right) \left( \hat{\alpha}^* \hat{B} + \text{h.c.} \right) + \frac{3p_T^2}{2p_L^2} (H - 2h) \left( \hat{\alpha}^* \hat{\alpha}_0 + \text{h.c.} \right) - \left( \mathcal{D}_{\alpha_1\alpha_1} + \frac{3}{2} p_T^2 \right) |\hat{\alpha}_1|^2 \\
& + \left[ \left( \frac{3}{2} m^2 - \frac{p^2}{2} + \frac{15}{2} h^2 - 6hH \right) B_1 - 3h\dot{B}_1 \right] \left( \hat{\alpha}^* \hat{\Phi} + \text{h.c.} \right) - \frac{1}{2} \left( H B_1 - 2h B_1 - \dot{B}_1 \right) \left( \hat{\alpha}^* \hat{\chi} + \text{h.c.} \right) \\
& + \frac{1}{2} \left( 2H B_1 - 7h B_1 + 4\dot{B}_1 \right) \left( \hat{\alpha}_1^* \hat{B} + \text{h.c.} \right) + \frac{3}{2} (H - 2h) \left( \hat{\alpha}_1^* \hat{\alpha}_0 + \text{h.c.} \right) \\
& + \left[ \left( 9 + \frac{15}{2} B_1^2 \right) h^2 - 9H^2 + \frac{3}{2} \dot{B}_1^2 - 6h B_1 (H B_1 + \dot{B}_1) \right] |\hat{\Phi}|^2 + \frac{1}{2} (6 + B_1^2) \left( H + h + \frac{B_1 \dot{B}_1}{6 + B_1^2} \right) \left( \hat{\Phi}^* \hat{\chi} + \text{h.c.} \right) \\
& + \frac{1}{2} (6 + B_1^2) \left( H - \frac{1}{2} h + \frac{B_1 \dot{B}_1}{6 + B_1^2} \right) \left( \hat{\Phi}^* \hat{B} + \text{h.c.} \right) - \left( 3h B_1 - \frac{3}{2} H B_1 - \frac{3}{2} \dot{B}_1 \right) \left( \hat{\Phi}^* \hat{\alpha}_0 + \text{h.c.} \right) \\
& - \left( \mathcal{D}_{\chi\chi} - \frac{6 + 8B_1^2}{8p_L^2} p_T^2 \right) |\hat{\chi}|^2 - \frac{1}{8} (6 + B_1^2) \left( \hat{\chi}^* \hat{B} + \text{h.c.} \right) - \mathcal{D}_{\chi\alpha_0} \left( \hat{\chi}^* \hat{\alpha}_0 + \text{h.c.} \right) + \frac{p_L^2}{p_T^2} \frac{6 + B_1^2}{8} |\hat{B}|^2 \\
& - \left( \mathcal{D}_{\alpha_0\alpha_0} - \frac{3p^2}{2p_L^2} \right) |\hat{\alpha}_0|^2
\end{aligned} \tag{63}$$

where the coefficients  $\{\mathcal{D}_{\Psi\Psi}, \mathcal{D}_{\Psi\alpha_1}, \mathcal{D}_{\alpha\alpha}, \mathcal{D}_{\alpha_1\alpha_1}, \mathcal{D}_{\chi\chi}, \mathcal{D}_{\alpha_0\alpha_0}, \mathcal{D}_{\chi\alpha_0}\}$  depend on the background quantities (and, hence, are functions of time), and their explicit forms are given in equations (A2) of Appendix A. As a further check on our algebra, we explicitly verified that the extremization of this action with respect to the fields included in it reproduces the system of linearized equations (56).

In Section III we outlined at a formal level the computation that we are now performing for the model (46). We expressed the quadratic action in eq. (19), where we distinguished between the dynamical fields  $Y_i$  and the nondynamical ones  $N_i$ . The action (63) is the explicit form of the action (19) for the model we are studying. The variables  $\{\hat{\Phi}, \hat{\chi}, \hat{B}, \hat{\alpha}_0\}$  are the nondynamical modes  $\{N_i\}$ , since they enter into the action (63) without time derivatives. Starting from the action (63), we integrate out the nondynamical modes, and compute the action for the dynamical modes, following the same steps that lead from eq. (19) to eq. (24). In practice, we read the coefficients  $a_{ij}, \dots, f_{ij}$  from the action (63), by comparing it with the formal expression (19); we then compute the combinations  $K = a - b^\dagger c^{-1} b$ ,  $\Lambda = d - b^\dagger c^{-1} f$ ,  $\Omega^2 = -e + f^\dagger c^{-1} f$  (cf. eqs. (23)) that characterize the action of the dynamical modes. This computation is a straightforward algebraic manipulation; the resulting coefficients  $K_{ij}, \Lambda_{ij}, \Omega_{ij}^2$  are extremely lengthy, and we do not report them here.

The instability emerged from the linearized equations can be understood by studying the kinetic matrix  $K$ . In Figure 3 we show the three eigenvalues of this matrix, for the same numerical evolution (i.e., for the same parameters and initial conditions) as the one leading to Figure 2. We see that the two eigenvalues  $\lambda_{2,3}$  are always positive, indicating that the two corresponding eigenmodes are well behaved positive-energy fields. On the contrary, the eigenvalue  $\lambda_1$  vanishes close to horizon crossing. The system of linearized equations becomes singular at this point (cf. the formal equations (22); they are singular if the matrix  $K$  is noninvertible), and the linearized solutions diverge. We also see from the Figure that the eigenvalue  $\lambda_1$  is negative for some time after this moment. The corresponding eigenmode is a ghost in this time interval.

Although the exact expression for  $K$  is rather lengthy, it is possible to obtain a simple expansion series for its determinant, in the sub-horizon / early time limit:

$$\det \left( \frac{K}{a b^2} \right) = \frac{p_T^6 B_1^2}{96 p^6} - \frac{(2H_0^2 - m^2) p_T^6}{16 p_L^2 p^6} + \mathcal{O} \left( \frac{H_0^4}{p^4} \right) \tag{64}$$

To obtain this expression, we first expanded the exact expression for the determinant in a power series of the momenta; we then simplified each term in the series by using the slow roll solutions (51), and finally kept for each term only the leading expression for  $B_1^2 \ll 1$  (which corresponds to small anisotropy, since  $h/H = \mathcal{O}(B_1^2)$ ). The terms of  $\mathcal{O}(H_0^4/p^4)$  are parametrically suppressed with respect to the second term in (64) for  $H_0 \ll p$ . Therefore, the first two terms in (64) provide an accurate approximation of the determinant in the whole sub-horizon regime.

Eq. (64) shows that the determinant is positive at sufficiently early times, and it then becomes negative in the later part of the sub-horizon regime. This confirms the behavior of  $\lambda_1$  seen in Figure 3 (since  $\lambda_{2,3} > 0$ , the sign of the determinant coincides with that of  $\lambda_1$ ). The determinant vanishes when the two leading terms in (64) are (approximately) equal; this happens for  $p_L^2 \simeq 6(2H_0^2 - m^2)/B_1^2$ . Not surprisingly, this is precisely the approximate value (62) of  $p_L$  at which the linearized system (59) becomes singular.

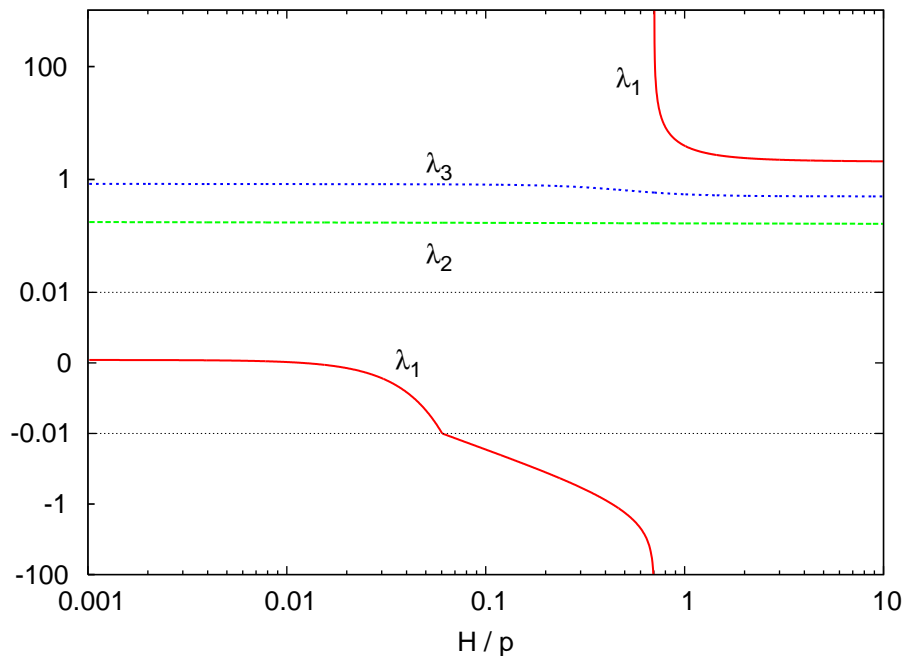


FIG. 3: Evolution of the eigenvalues of the kinetic matrix. The parameters and initial conditions are as in Figure 2. Due to the wide range spanned by the eigenvalues, in the  $y$  axis, we have used a linear scale inside the interval  $[-0.01, 0.01]$ , and a logarithmic scale outside.

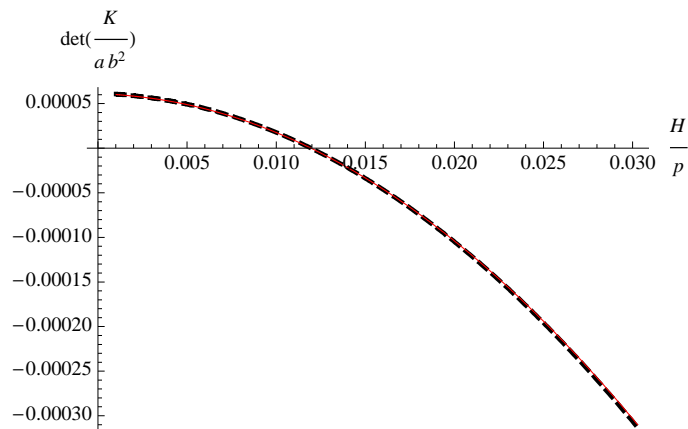


FIG. 4: Determinant of the kinetic matrix, for the same choice of parameters and initial conditions as in the previous Figure. Compared with the previous figure, we show a close up of early times, around the point where the determinant vanishes. The black dashed curve shows the exact determinant, while the red curve shows the approximate expression given in eq. (64).

In Figure 4, we compare the approximate expression for the determinant - the first two terms in (64) - with the exact one (not reported here). We see that the approximated expression is extremely accurate in the range of our interest.

### C. One vector plus a scalar inflaton

The model proposed in ref. [29] is very similar to the one we have studied in the previous Subsection, with the only difference that the vacuum energy that we have included is replaced there by a slowly rolling inflaton field. We expect that the same instability that we have found above is present also for this model. The study presented in this Subsection confirms this. We first briefly review the model and its slow roll solution. We then show that one of

the 2d scalar modes is a ghost. More precisely, as in the case studied above, the mode is well behaved in the deep UV regime, but it becomes a ghost close to horizon crossing. When this happens, the system of linearized equations becomes singular.

### 1. The model and the background solution

The model of ref. [29] is characterized by the action:

$$S = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) - \frac{1}{2} \left( m^2 - \frac{R}{6} \right) A_\mu A^\mu \right] \quad (65)$$

The inflaton field  $\phi$  replaces the vacuum energy  $V_0$  that we considered in the action (46). In this way, one can have a graceful exit from inflation. The field equations following from this action are

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{M_p^2} \left[ T_{\mu\nu}^{(\phi)} + T_{\mu\nu}^{(A)} \right] \quad , \quad T_{\mu\nu}^{(\phi)} \equiv \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left( \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi + V(\phi) \right) \\ \frac{1}{\sqrt{-g}} \partial_\mu \left[ \sqrt{-g} F^{\mu\nu} \right] - \left( m^2 - \frac{R}{6} \right) A^\nu &= 0 \quad , \quad \frac{1}{\sqrt{-g}} \partial_\mu \left[ \sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right] - V'(\phi) = 0 \end{aligned} \quad (66)$$

where  $T_{\mu\nu}^{(A)}$  is defined as in (47). The background solution is again of the form (41), giving

$$\begin{aligned} 3H^2 - 3h^2 &= \frac{1}{M_p^2} \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right) + \frac{1}{2} \dot{B}_1^2 - 2h B_1 \dot{B}_1 + \frac{1}{2} (m^2 + 5h^2 - 4hH) B_1^2 \\ \dot{h} + 3Hh &= \frac{1}{3} B_1^2 \left( \dot{H} - \frac{\dot{h}}{2} \right) + \frac{1}{3} \dot{B}_1^2 + \frac{1}{3} (2H - 5h) B_1 \dot{B}_1 + \frac{1}{3} \left( 3H^2 - \frac{11}{2} Hh + 5h^2 - m^2 \right) B_1^2 \\ 2\dot{H} + 3H^2 + 3h^2 &= \frac{1}{M_p^2} \left( -\frac{1}{2} \dot{\phi}^2 + V(\phi) \right) - \frac{1}{2} \dot{B}_1^2 - \frac{1}{3} B_1 \left[ \ddot{B}_1 + (3H - 2h) \dot{B}_1 \right] + \frac{1}{6} (4Hh - 5h^2 + m^2) B_1^2 \\ \ddot{B}_1 + 3H \dot{B}_1 + (m^2 - 5h^2 - 2hH - 2\dot{h}) B_1 &= 0 \\ \ddot{\phi} + 3H \dot{\phi} + V'(\phi) &= 0 \end{aligned} \quad (67)$$

where we have used the same combinations of Einstein equations we have written in (49).

We assume that the inflaton field is in the slow roll regime,  $\dot{\phi} \approx -V'(\phi)/3H$ , leading to inflation. For what concerns the evolution of the vector field, and of the anisotropy, we all the considerations done for the model (46) are valid also in the present case, with the only difference that the vacuum energy  $V_0$  is now replaced by the slowly decreasing potential energy of the inflaton. This leads to the slow roll and small anisotropy ( $B_1 \ll 1$ ) background evolution

$$\begin{aligned} H &\approx H_0 + \frac{m^2}{12H_0} B_1^2 + \mathcal{O}(B_1^4) \quad , \quad h \approx \frac{H_0}{3} B_1^2 + \mathcal{O}(B_1^4) \quad , \quad \dot{B}_1 \approx -\frac{m^2}{3H_0} B_1 + \mathcal{O}(B_1^3) \quad , \quad \dot{\phi} \approx -\frac{V'(\phi)}{3H_0} + \mathcal{O}(B_1^2) \\ H_0 &\equiv \frac{\sqrt{V(\phi)}}{\sqrt{3} M_p} \end{aligned} \quad (68)$$

### 2. Ghost instability from the quadratic action

We now study the perturbations of the background solution. We again decompose the perturbations as in (42), disregard the well behaved system of 2d vector modes, and construct the gauge invariant combinations (43) and (45).

We proceed as in VB3 by expanding the action of the model at quadratic order in the 2d scalar modes, and by

rewriting it solely in terms of the gauge invariant modes (which provides a nontrivial check on our algebra). We find

$$\begin{aligned}
S_{2\text{dS}} &= \frac{1}{2} \int d^3k dt a b^2 \mathcal{L}_{2\text{dS}} \\
\mathcal{L}_{2\text{dS}} &\supset |\delta\dot{\phi}|^2 + \frac{2}{3} B_1^2 |\dot{\Psi}|^2 + \frac{p_T^2}{p_L^2} |\dot{\hat{\alpha}}|^2 + |\dot{\hat{\alpha}}_1|^2 - \frac{1}{3} B_1 \left( \dot{\Psi}^* \hat{\alpha}_1 + \text{h.c.} \right) - \frac{\dot{\phi}}{M_p} \left( \delta\dot{\phi}^* \hat{\Phi} + \text{h.c.} \right) \\
&+ \frac{1}{3} \left( 2(B_1^2 - 3) H - (6 + B_1^2) h - B_1 \dot{B}_1 \right) \left( \dot{\Psi}^* \hat{\Phi} + \text{h.c.} \right) - \frac{B_1^2}{3} \left( \dot{\Psi}^* \hat{\chi} + \text{h.c.} \right) \\
&+ \left( 1 - \frac{B_1^2}{6} \right) \left( \dot{\Psi}^* \hat{B} + \text{h.c.} \right) + \frac{p_T^2}{p_L^2} \left( \dot{\hat{\alpha}}^* \hat{\alpha}_0 + \text{h.c.} \right) + \left( \dot{B}_1 - 2B_1 h \right) \left( \hat{\alpha}_1^* \hat{\Phi} + \text{h.c.} \right) \\
&+ \frac{B_1}{3} \left( \hat{\alpha}_1^* \hat{\chi} + \text{h.c.} \right) + \frac{B_1}{3} \left( \hat{\alpha}_1^* \hat{B} + \text{h.c.} \right) + \left( \hat{\alpha}_1^* \hat{\alpha}_0 + \text{h.c.} \right) \\
&+ \left( -6H^2 + (6 + 5B_1^2) h^2 + \dot{B}_1^2 - 4B_1 \left( \dot{B}_1 + H B_1 \right) h + \frac{\dot{\phi}^2}{M_p^2} \right) |\hat{\Phi}|^2 \\
&+ \frac{1}{3} \left( (6 + B_1^2) (H + h) + B_1 \dot{B}_1 \right) \left( \hat{\Phi}^* \hat{\chi} + \text{h.c.} \right) + \frac{1}{6} \left( (6 + B_1^2) (2H - h) + 2B_1 \dot{B}_1 \right) \left( \hat{B} \hat{\Phi}^* + \text{h.c.} \right) \\
&+ \left( \dot{B}_1 + (H - 2h) B_1 \right) \left( \hat{\Phi}^* \hat{\alpha}_0 + \text{h.c.} \right) + \left( \Delta + \frac{p_T^2}{12p_L^2} (6 + B_1^2) \right) |\hat{\chi}|^2 - \frac{1}{2} \left( 1 + \frac{B_1^2}{6} \right) \left( \hat{\chi}^* \hat{B} + \text{h.c.} \right) \\
&+ \frac{\Delta}{B_1} \left( \hat{\chi}^* \hat{\alpha}_0 + \text{h.c.} \right) + \frac{p_L^2}{12p_T^2} (6 + B_1^2) |\hat{B}|^2 + \left( \frac{\Delta}{B_1^2} + \frac{p^2}{p_L^2} \right) |\hat{\alpha}_0|^2 + \dots
\end{aligned} \tag{69}$$

where we have defined

$$\begin{aligned}
\Delta &= \frac{1}{p_L^2 (18 + 3B_1^2 + 2B_1^4)} \left\{ 3 \left( 6m^2 + 12h^2 - 12H^2 + 3\dot{B}_1^2 + 3\frac{\dot{\phi}^2}{M_p^2} \right) B_1^2 - 24h B_1^3 \dot{B}_1 + 2(2H - 7h) B_1^5 \dot{B}_1 \right. \\
&\quad \left. + 3 \left( m^2 + 17h^2 - 12hH - 2H^2 + \frac{7}{6} \dot{B}_1^2 + \frac{\dot{\phi}^2}{2M_p^2} \right) B_1^4 + \left( \frac{31}{2} h^2 - 14hH + 2H^2 \right) B_1^6 \right\}
\end{aligned}$$

In the action (69), we included only the terms that contribute to the kinetic matrix of the dynamical modes; the remaining terms, denoted by the dots, are omitted for brevity. Specifically, the terms included in (69) are those proportional to the coefficients  $a_{ij}$ ,  $b_{ij}$ , and  $c_{ij}$  in eq. (19), where the quadratic action is written at a formal level. These are the only terms entering in the kinetic matrix  $K$  of the dynamical modes, see eqs. (23) and (24). It is immediate to compute  $K$  from (69). However, the explicit entries of this matrix are very involved, and not illuminating. For this reason, we do not report them here. We can however compute the eigenvalues of this matrix numerically for any given choice of parameters (namely, for any background evolution, and momentum of the perturbation). In addition, it is possible to obtain a simple approximation for the determinant of  $K$

$$\det \left( \frac{K}{a b^2} \right) = \frac{p_T^6 B_1^2}{192 p^6} - \frac{(2H_0^2 - m^2) p_T^6}{32 p_L^2 p^6} + \mathcal{O} \left( \frac{H_0^4}{p^4} \right) \tag{70}$$

where  $H_0$  has been defined in eq. (68). This expression is accurate in the sub-horizon regime during inflation, in the limit of small anisotropy. More specifically, we obtained it using the same steps outlined after the analytic expression for the determinant of the model considered in the previous Subsection, eq. (64). It is worth noting that (70) differs from (64) only by an extra factor of 1/2; this is the original factor in the kinetic term of the inflaton (which is the only additional field in the model we are considering in this Subsection). This suggests that the perturbation  $\hat{\delta\phi}$  is decoupled from the other ones at leading order.

In figure 5 we present the results of a numerical evolution for a given set of parameters and initial conditions (starting from inflation, in the slow roll regime). The left panel shows the evolution of various background quantities. The right panel shows instead the determinant of the kinetic matrix for a mode with  $p_L = 100H$ ,  $p_T = 200H$  at the initial time (both the exact value, obtained from a numerical evaluation of the kinetic matrix, and the approximated value (70), are shown). We see that the determinant vanishes and becomes negative in the sub-horizon regime, when  $H/p \sim B$ .

In Figure 6 we show the evolution of the four eigenvalues of the kinetic matrix; notice that there is an additional dynamical mode, supported by the inflaton field, with respect to the model studied in the previous Subsection.

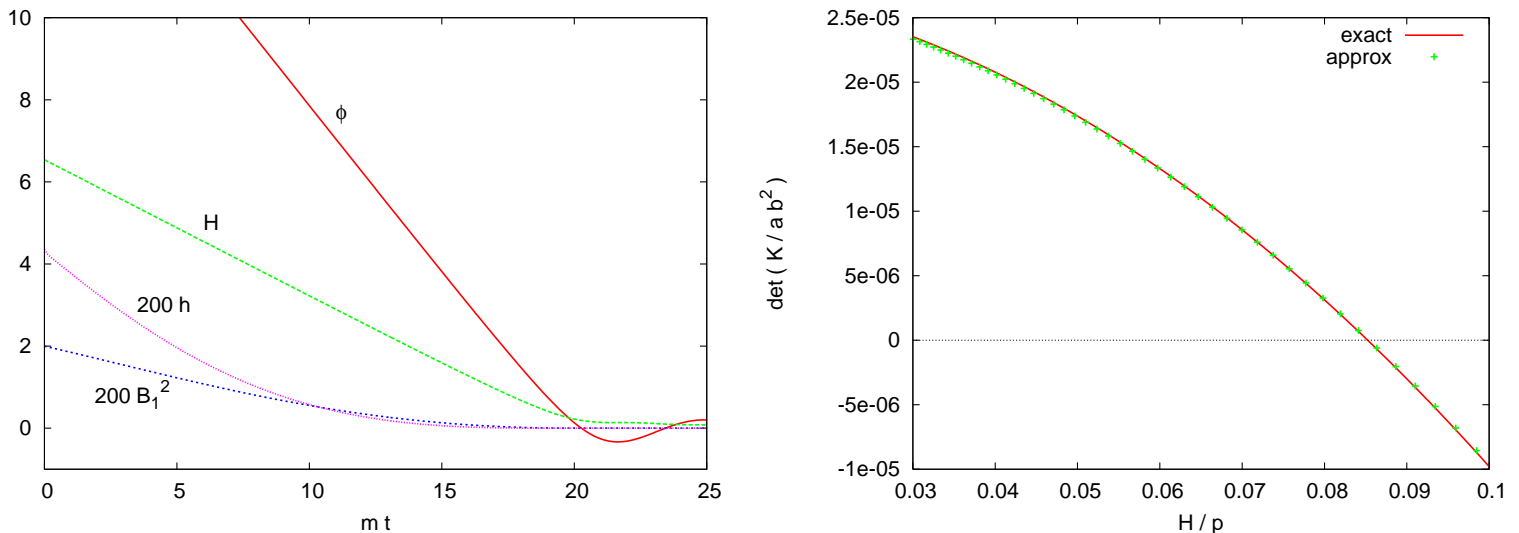


FIG. 5: Results from a numerical simulation starting at  $t = 0$  from  $\phi = 16$  (providing about 60 e-folds of inflationary expansion),  $B_1 = 0.1$  (providing a  $\sim B_1^2 \simeq 1\%$  anisotropy). More precisely, we have considered a massive inflaton potential, with the inflaton mass equal to  $m$ . Left panel: inflaton (in units of  $M_p$ ), hubble parameters (in units of  $m$ ), and dimensionless rescaled vector  $B_1$ . The anisotropic rate  $h$  and the vector are rescaled so that they are visible in the figure. Right panel: determinant of the kinetic matrix of the perturbations, for modes with initial momenta  $p_{L,\text{in}} = 100H_{\text{in}}$ ,  $p_{T,\text{in}} = 200H_{\text{in}}$ . The red curve shows the exact determinant, while the green points show the approximate expression given in eq. (70). The determinant vanishes at the time  $mt \simeq 0.16$ .

However, as for that case, one mode is a ghost for some time. We know that the system of linearized equations for the perturbations become singular when the eigenvalue  $\lambda_1$  crosses zero. We expect that also the solutions diverge at that point, analogously to the study of the previous Subsection.

#### D. Vector inflation

We now study the simplest realization of the idea of vector inflation proposed in [28]. It is characterized by three mutually orthogonal vector fields nonminimally coupled to the curvature. The three fields have equal vev, providing a FRW background. Despite the background solution is isotropic, the model suffers of the same ghost instability as the models studied above. The discussion is divided in two parts. We first introduce the model, and discuss the background evolution. We then study the spectrum of perturbations around this background solution.

##### 1. The model and the background solution

The action of the model is given in eq. (12), with  $\xi = -1/6$ , and  $N = 3$ :

$$S = \int d^4x \sqrt{-g} \sum_{a=1}^3 \left[ -\frac{1}{4} F_{\mu\nu}^{(a)} F^{(a)\mu\nu} - \frac{1}{2} \left( -\frac{R}{6} + m^2 \right) A_\mu^{(a)} A^{(a)\mu} \right] \quad (71)$$

The background vev of the three vectors is chosen as in eq. (13), while the background geometry is  $ds^2 =$

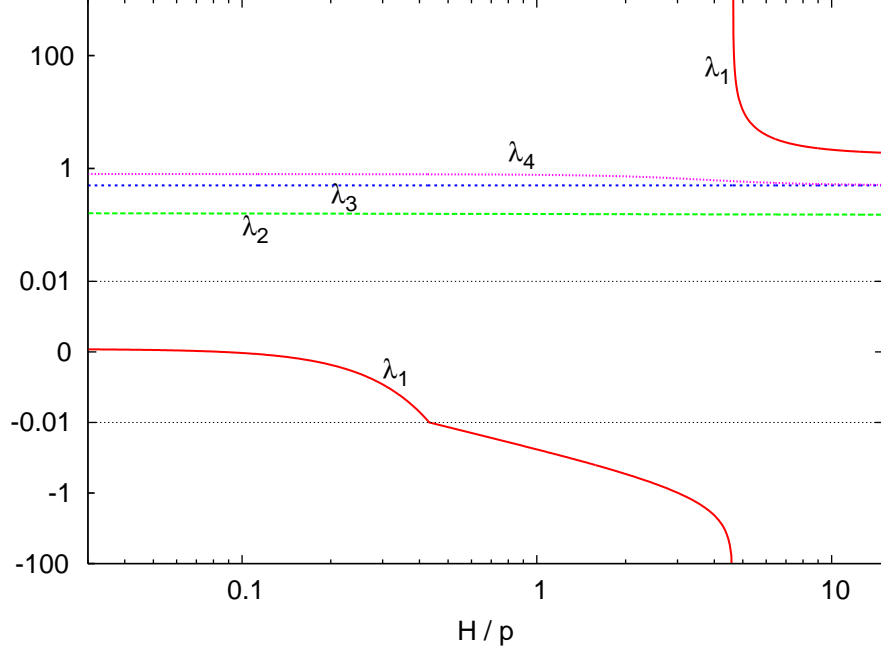


FIG. 6: Eigenvalues of the kinetic matrix, for the same choice of parameters and initial conditions as in the right panel of Figure 5. The eigenvalues  $\lambda_{2,3,4}$  are always positive, so that the sign and the behavior of the determinant are determined by  $\lambda_1$ . This eigenvalue is initially positive, crosses zero at  $H/p \simeq 0.082$  (see the previous figure), and diverges at  $H/p \simeq 4.6$ . In the  $y$  axis, we have used a linear scale inside the interval  $[-0.01, 0.01]$ , and a logarithmic scale outside.

$-dt^2 + a^2(t) d\vec{x}^2$ . The system is governed by the equations

$$\begin{aligned}
 G_{\mu\nu} &= \frac{1}{M_p^2} \sum_{a=1}^3 T_{\mu\nu}^{(a)} \\
 T_{\mu\nu}^{(a)} &\equiv F_{\mu}^{(a)\sigma} F_{\nu\sigma}^{(a)} - \frac{1}{4} F_{\alpha\beta}^{(a)} F^{(a)\alpha\beta} g_{\mu\nu} + \left(m^2 - \frac{R}{6}\right) A_{\mu}^{(a)} A_{\nu}^{(a)} - \frac{1}{2} m^2 A_{\alpha}^{(a)} A^{(a)\alpha} g_{\mu\nu} \\
 &\quad - \frac{1}{6} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right) A_{\alpha}^{(a)} A^{(a)\alpha} - \frac{1}{6} (g_{\mu\nu} \square - \nabla_{\mu} \nabla_{\nu}) A_{\alpha}^{(a)} A^{(a)\alpha} \\
 \frac{1}{\sqrt{-g}} \partial_{\mu} \left[\sqrt{-g} F^{(a)\mu\nu}\right] - \left(m^2 - \frac{R}{6}\right) A^{(a)\nu} &= 0
 \end{aligned} \tag{72}$$

For the background under considerations, the 00 Einstein equation and the  $i$ - $th$  spatial component of the equation for the  $i$ - $th$  vector give

$$\begin{aligned}
 H^2 &= \frac{1}{2} \left(\dot{B}^2 + m^2 B^2\right) \\
 \ddot{B} + 3H \dot{B} + m^2 B &= 0
 \end{aligned} \tag{73}$$

As always for a FRW geometry, the diagonal  $ii$  Einstein equations does not provide additional information (due to a nontrivial Bianchi identity). The remaining equations vanish identically. Upon the identification  $B = \phi_{\text{eff}} / (\sqrt{3}M_p)$ , we recover the same equations as those of chaotic inflation driven by a minimally coupled scalar field  $\phi_{\text{eff}}$  (where the suffix stands for “effective”) with potential  $V = m^2 \phi_{\text{eff}}^2 / 2$ . Inflation is therefore characterized by the slow roll evolution

$$H^2 \approx \frac{m^2}{2} B^2 \quad , \quad \dot{B} \approx -\frac{m^2}{3H} B \tag{74}$$



which applies as long as the slow roll conditions <sup>22</sup>

$$\begin{aligned}\epsilon &\equiv \frac{M_p^2}{2} \left( \frac{dV}{d\phi_{\text{eff}}} \right)^2 = \frac{2 M_p^2}{\phi_{\text{eff}}^2} = \frac{2}{3 B^2} \ll 1 \\ \eta &\equiv M_p^2 \frac{d^2 V}{d\phi_{\text{eff}}^2} = \frac{2 M_p^2}{\phi_{\text{eff}}^2} = \frac{2}{3 B^2} \ll 1\end{aligned}\tag{75}$$

are valid. After that,  $B$  performs damped coherent oscillations around  $B = 0$  (see for instance [42]):

$$B = \left( \frac{\sqrt{8}}{3 m t} + \mathcal{O}\left(\frac{1}{m^2 t^2}\right) \right) \sin(m t + \xi_0)\tag{76}$$

where  $\xi_0$  is a phase (irrelevant for this discussion). The coherent oscillations provide a stage of effective matter domination ( $w = 0$  once averaged over a few oscillations) [42]. It is expected that the vectors then decay into the visible matter (either perturbatively, or nonperturbatively) giving rise to the radiation dominated stage of cosmology.

From the slow roll conditions, we see that  $H^2 \gg m^2$  during most of the inflationary stage. At these times, the mass term for the vectors is negative,  $-R/6 + m^2 \simeq -2H^2 + m^2 < 0$ . However,  $R$  decreases as  $B$  rolls towards the origin, while  $m$  remains constant. We find numerically that  $-R/6 + m^2 = 0$  when  $B \simeq 1.048$ . This happens towards the end of inflation.

## 2. Ghost instability from the quadratic action

We now study the perturbations of the model (71) around the background solution just discussed. Since the background geometry is FRW, one may choose to adopt the standard decomposition of metric perturbations, and decompose them into scalar, vector, and tensor modes with respect to 3d spatial rotations [37]. While this is possible, one should however bear in mind that, contrary to the standard case, these three groups of modes cannot be studied separately even at the linearized level. Consider for instance the tensor perturbations  $h_{ij}^{TT}$ , introduced as the traceless ( $h_{ii}^{TT} = 0$ ), and transverse ( $\partial_j h_{ij}^{TT} = 0$ ) part of the spatial metric perturbations,  $\delta g_{ij} = a^2 h_{ij}^{TT}$  [37]. While in the case of scalar field inflation these modes are decoupled from the other perturbations, and can be studied separately, in the case under consideration they are coupled to the perturbations of the vector fields. Namely, we find the following coupling in the quadratic action  $\delta_2 S$  of the perturbations:

$$\delta_2 S \supset -\frac{M_p}{2} \int d^4 x a^2 \left[ \left( \dot{B} + H B \right) \delta \dot{A}_j^{(i)} + \left( 2H^2 + \dot{H} - m^2 \right) B \delta A_j^{(i)} + (i \leftrightarrow j) \right] h_{ij}^{TT}\tag{77}$$

As a consequence, the perturbations of the vector fields must be included in the linearized equations of motion for the tensor metric perturbations. In turns, the perturbations of the vector fields are coupled to the scalar and vector perturbations of the metric. One then finds that all the perturbations of the system need to be studied together, even at the linearized level. <sup>23</sup>

This makes the study extremely hard, and indeed no complete computation exists up to date. The metric has 10 perturbations, while each vector introduces 4 perturbations. Thus, in this simplest realization of vector inflation, one starts with a system of 22 coupled modes. Four perturbations can be removed by fixing the freedom of general coordinate transformations, leading to a system of 18 coupled modes. Not all these modes are dynamical. The nondynamical modes originate from the initial perturbations  $\delta g_{0\mu}$  and  $\delta A_0^{(a)}$ , which enter in the quadratic action of the perturbations without time derivatives. As we now show, it is possible to choose gauge invariant perturbations which associate a gauge invariant combination to each of the  $\delta g_{0\mu}$  and  $\delta A_0^{(a)}$  modes. These seven gauge invariant combinations are also nondynamical fields. We integrate them out as outlined in Section III, and obtain the quadratic

<sup>22</sup> See [41] for more detailed studies of initial and slow-roll conditions for vector inflation.

<sup>23</sup> Ref. [32] studied the tensor modes alone, arguing that the effect of the coupling to the  $\delta A_\mu^{(a)}$  perturbations can be ‘‘averaged away’’ in the limit of many vectors. However, each vector introduces a coupling of  $h_{ij}^{TT}$  with its own perturbations, and perturbations of different vectors cannot cancel against each other in the study of the spectrum (and, therefore, of the stability) of the theory.

action for the dynamical modes. From the study of the kinetic matrix of this system, we find that three modes become ghosts for some time during inflation.

Since the usual decomposition in scalar/vector/tensor modes does not provide decoupled sets of perturbations, we do not employ it, and simply decompose the perturbations as

$$g_{\mu\nu} = \begin{pmatrix} -1 - 2\Phi & a\chi_1 & a\chi_2 & a\chi_3 \\ a^2(1 - 2\Psi) & a^2\partial_1\partial_2 E_3 & a^2\partial_1\partial_3 E_2 & \\ & a^2(1 - 2\Sigma) & a^2\partial_2\partial_3 E_1 & \\ & & & a^2[1 - 2(T - \Psi - \Sigma)] \end{pmatrix}$$

$$A_\mu^{(a)} = A_\mu^{0(a)} + \alpha_\mu^{(a)} = a M_p B \delta_\mu^a + \alpha_\mu^{(a)} \quad (78)$$

where  $A_\mu^{0(a)}$  denotes background values of the vector fields. The parametrization of the  $\delta g_{33}$  component has chosen so that  $\delta g_i^i = -2T$ . Moreover, for algebraic convenience, some  $\delta g_{\mu\nu}$  entry has been written with spatial derivatives (in practice, we assume that the momentum of the modes - after Fourier transforming - has nonvanishing components in all three directions,  $k_x, k_y, k_z \neq 0$ ).

We need to fix the gauge freedom associated with general coordinate transformations. Consider the infinitesimal transformation  $x^\mu \rightarrow x^\mu + \xi^\mu$ , under which the perturbations of the metric and the vector field transform as  $\delta g_{\mu\nu} \rightarrow \delta g_{\mu\nu} - g_{\mu\nu}^{(0)} \xi^\alpha - g_{\mu\alpha}^{(0)} \xi_{,\nu}^\alpha - g_{\alpha\nu}^{(0)} \xi_{,\mu}^\alpha$ , and  $\delta A_\mu^{(a)} \rightarrow \delta A_\mu^{(a)} - A_{\mu,\alpha}^{(a)} \xi^\alpha - A_\alpha^{(a)} \xi_{,\mu}^\alpha$ . Due to the assumption of the modes we are studying, we need to consider infinitesimal transformations with nontrivial spatial dependence along all the three directions. We can therefore parametrize the transformation parameter as  $\xi^\mu = (\xi^0, \partial_1 \xi^1, \partial_2 \xi^2, \partial_3 \xi^3)$ . The explicit transformations of the modes in (78) are then

$$\begin{aligned} \Phi &\rightarrow \Phi - \partial_0 \xi^0, \quad \chi_i \rightarrow \chi_i + \frac{1}{a} \partial_i \xi^0 - a \partial_0 \partial_i \xi^i \quad (\text{no sum over } i) \\ E_1 &\rightarrow E_1 - \xi_2 - \xi_3, \quad E_2 \rightarrow E_2 - \xi_1 - \xi_3, \quad E_3 \rightarrow E_3 - \xi_1 - \xi_2, \\ \Psi &\rightarrow \Psi + H \xi^0 + \partial_1^2 \xi^1, \quad \Sigma \rightarrow \Sigma + H \xi^0 + \partial_2^2 \xi^2, \quad T \rightarrow T + 3H \xi^0 + \partial_i^2 \xi^i \\ \alpha_\mu^{(i)} &\rightarrow \alpha_\mu^{(i)} - \partial_0 (a M_p B) \xi^0 \delta_\mu^i - a M_p B \partial_\mu \partial_i \xi^i \quad (\text{no sum over } i) \end{aligned} \quad (79)$$

We consider the combinations

$$C^1 \equiv \frac{E_1 - E_2 - E_3}{2}, \quad C^2 \equiv \frac{E_2 - E_1 - E_3}{2}, \quad C^3 \equiv \frac{E_3 - E_1 - E_2}{2}, \quad C^0 \equiv \frac{1}{3H} [T - \partial_i^2 C^i] \quad (80)$$

that transform as  $C^\mu \rightarrow C^\mu + \xi^\mu$ . Then, the modes

$$\begin{aligned} \hat{\Phi} &\equiv M_p (\Phi + \partial_0 C^0) \\ \hat{\chi}_i &\equiv M_p \left( \chi_i - \frac{1}{a} \partial_i C^0 + a \partial_0 \partial_i C^i \right) \quad (\text{no sum over } i) \\ \hat{\Psi} &\equiv M_p (\Psi - H C^0 - \partial_1^2 C^1) \\ \hat{\Sigma} &\equiv M_p (\Sigma - H C^0 - \partial_2^2 C^2) \\ \hat{\alpha}_\mu^{(i)} &\equiv \alpha_\mu^{(i)} + \partial_0 (a M_p B) C^0 \delta_\mu^i + a M_p B \partial_\mu \partial_i C^i \quad (\text{no sum over } i) \end{aligned} \quad (81)$$

are gauge invariant (all these modes have mass dimension +1).

We expanded the metric and vector fields according to (78), and computed the quadratic action for the perturbations. We verified that the perturbations rearrange so that the quadratic action can be written solely in terms of the gauge invariant modes (81) (this is a nontrivial check on our algebra). In this way, we have eliminated the redundancy

associated to general coordinate transformations.<sup>24</sup> The action in Fourier space reads

$$\begin{aligned}
\delta_2 S &= \int d^3 k dt a^3 \mathcal{L} \\
\mathcal{L} &= \frac{1}{2} (2 + B^2) \left( |\dot{\Psi}|^2 + |\dot{\Sigma}|^2 \right) + \frac{1}{2a^2} \sum_{j,a=1}^3 |\dot{\hat{\alpha}}_j^{(a)}|^2 + \frac{1}{4} (2 + B^2) \left( \dot{\Psi}^* \dot{\Sigma} + \text{h.c.} \right) \\
&\quad - \frac{1}{4} (2 + B^2) \left[ i \dot{\Psi}^* (p_1 \hat{\chi}_1 - p_3 \hat{\chi}_3) + i \dot{\Sigma}^* (p_2 \hat{\chi}_2 - p_3 \hat{\chi}_3) + \text{h.c.} \right] \\
&\quad + \frac{1}{2a} \sum_{j,a=1}^3 \left( i p_j \dot{\hat{\alpha}}_j^{(a)*} \hat{\alpha}_0^{(a)} + \text{h.c.} \right) - \frac{1}{2a} \dot{B} \sum_{j=1}^3 \left( \dot{\hat{\alpha}}_j^{(j)*} \hat{\Phi} + \text{h.c.} \right) + \frac{B}{6a} \sum_{j,a=1}^3 \left( i p_j \hat{\chi}_j^* \dot{\hat{\alpha}}_a^{(a)} + \text{h.c.} \right) \\
&\quad - \frac{3}{2} m^2 B^2 |\hat{\Phi}|^2 - \frac{1}{2} \left( H B + \dot{B} \right) \sum_{j=1}^3 \left( i \hat{\Phi}^* p_j \hat{\alpha}_0^{(j)} + \text{h.c.} \right) + \frac{1}{2} \left[ B \dot{B} + (2 + B^2) H \right] \sum_{j=1}^3 \left( i \hat{\Phi}^* p_j \hat{\chi}_j + \text{h.c.} \right) \\
&\quad + \frac{1}{2} \left( p^2 + m^2 + H^2 - \frac{3}{2} m^2 B^2 \right) \sum_{a=1}^3 |\hat{\alpha}_0^{(a)}|^2 - \frac{B}{2} \left( H^2 + m^2 - \frac{3}{2} m^2 B^2 \right) \sum_{j=1}^3 \left( \hat{\alpha}_0^{(j)*} \hat{\chi}_j + \text{h.c.} \right) \\
&\quad + \frac{1}{4} \sum_{j=1}^3 \left[ p^2 + \frac{1}{2} (p^2 + 4m^2) B^2 - 3m^2 B^4 + 2H^2 B^2 \right] |\hat{\chi}_j|^2 - \frac{1}{8} (2 + B^2) \sum_{i,j=1}^3 p_i p_j \hat{\chi}_i \hat{\chi}_j^* + \dots \quad (82)
\end{aligned}$$

where we have used the physical momenta  $p_i \equiv k_i/a$ . Eq. (82) actually is not the full quadratic action of the perturbations, but some terms (denoted by dots) are omitted. Let us clarify this. We find that no time derivatives of the combinations  $\hat{\Phi}$ ,  $\hat{\chi}_i$ ,  $\hat{\alpha}_0^{(a)}$  enter in the quadratic action (neither in the terms shown here, nor in those omitted). These are the nondynamical gauge invariant modes of the system.<sup>25</sup> The remaining modes are dynamical. Eq. (82) is the action for all the gauge invariant modes of the system (both the dynamical, and the nondynamical ones). However, it contains only the terms that contribute to the kinetic matrix of the action for the dynamical modes, once the nondynamical modes have been integrated out.<sup>26</sup> These terms are given without any omission, nor approximation, so that eq. (82) contains all the necessary information for the exact computation of the kinetic matrix of the dynamical modes.

Before integrating out the nondynamical modes, we can easily see that 5 dynamical modes decouple from the remaining ones in the part of the action shown. There are 9 dynamical modes in the spatial perturbations of the three vector fields,  $\hat{\alpha}_j^{(a)}$ . These perturbations enter in the action (82) with a diagonal quadratic term (the second term,  $\propto |\hat{\alpha}_j^{(a)}|^2$ ). Then, they are coupled with the remaining modes only in the third line of (82)). We see that, out of the nine modes  $\hat{\alpha}_j^{(a)}$ , only the four linear combinations  $p_j \hat{\alpha}_j^{(a)}$  ( $a = 1, 2, 3$ ) and  $\dot{\hat{\alpha}}_j^{(j)}$  are involved in these couplings. The remaining 5 linear combinations are decoupled. Therefore, we can rotate the fields  $\hat{\alpha}_j^{(a)}$  into the coupled and

<sup>24</sup> Notice that the procedure we adopted is equivalent to choose the gauge  $E_1 = E_2 = E_3 = T = 0$ . It is easy to see from (79) that this choice can always be made, and it completely fixes the gauge freedom.

<sup>25</sup> Notice that they corresponds to the nondynamical perturbations  $\delta g_{0\mu}$  and  $\delta A_0^{(a)}$  in the gauge  $E_1 = E_2 = E_3 = T = 0$ .

<sup>26</sup> For clarity, compare with the formal discussion of Section III. The action (82) given here corresponds to the formal action (19), with only the terms proportional to the coefficients  $a_{ij}$ ,  $b_{ij}$ , and  $c_{ij}$  included. Those are the only terms necessary to compute the kinetic matrix  $K$  of the dynamical modes, cf. eqs. (23) and (24).

decoupled linear combinations:

$$\begin{aligned}
\hat{\alpha}_1^{(1)} &= \frac{p_{23}^2}{\sqrt{2} p^2} \hat{v}_1 + \frac{p_1}{p} \hat{v}_2 + \frac{p_{12} p_{23}}{p \sqrt{p_{12}^2 + p_{23}^2}} \hat{v}_8 - \frac{p_{13} p_{23}^2}{\sqrt{2} p^2 \sqrt{p_{12}^2 + p_{23}^2}} \hat{v}_9 \\
\hat{\alpha}_2^{(1)} &= -\frac{p_1 p_2}{\sqrt{2} p^2} \hat{v}_1 + \frac{p_2}{p} \hat{v}_2 + \frac{p_3}{p_{23}} \hat{v}_5 - \frac{p_1 p_2 p_{12}}{p_{23} p \sqrt{p_{12}^2 + p_{23}^2}} \hat{v}_8 + \frac{p_1 p_2 p_{13}}{\sqrt{2} p^2 \sqrt{p_{12}^2 + p_{23}^2}} \hat{v}_9 \\
\hat{\alpha}_3^{(1)} &= -\frac{p_1 p_3}{\sqrt{2} p^2} \hat{v}_1 + \frac{p_3}{p} \hat{v}_2 - \frac{p_2}{p_{23}} \hat{v}_5 - \frac{p_1 p_{12} p_3}{p_{23} p \sqrt{p_{12}^2 + p_{23}^2}} \hat{v}_8 + \frac{p_1 p_3 p_{13}}{\sqrt{2} p^2 \sqrt{p_{12}^2 + p_{23}^2}} \hat{v}_9 \\
\hat{\alpha}_1^{(2)} &= -\frac{p_1 p_2}{\sqrt{2} p^2} \hat{v}_1 + \frac{p_1}{p} \hat{v}_3 - \frac{p_3}{p_{13}} \hat{v}_6 - \frac{p_1 p_2 \sqrt{p_{12}^2 + p_{23}^2}}{\sqrt{2} p_{13} p^2} \hat{v}_9 \\
\hat{\alpha}_2^{(2)} &= \frac{p_{13}^2}{\sqrt{2} p^2} \hat{v}_1 + \frac{p_2}{p} \hat{v}_3 + \frac{p_{13} \sqrt{p_{12}^2 + p_{23}^2}}{\sqrt{2} p^2} \hat{v}_9 \\
\hat{\alpha}_3^{(2)} &= -\frac{p_2 p_3}{\sqrt{2} p^2} \hat{v}_1 + \frac{p_3}{p} \hat{v}_3 + \frac{p_1}{p_{13}} \hat{v}_6 - \frac{p_2 p_3 \sqrt{p_{12}^2 + p_{23}^2}}{\sqrt{2} p_{13} p^2} \hat{v}_9 \\
\hat{\alpha}_1^{(3)} &= -\frac{p_1 p_3}{\sqrt{2} p^2} \hat{v}_1 + \frac{p_1}{p} \hat{v}_4 + \frac{p_2}{p_{12}} \hat{v}_7 + \frac{p_1 p_3 p_{23}}{p_{12} p \sqrt{p_{12}^2 + p_{23}^2}} \hat{v}_8 + \frac{p_1 p_3 p_{13}}{\sqrt{2} p^2 \sqrt{p_{12}^2 + p_{23}^2}} \hat{v}_9 \\
\hat{\alpha}_2^{(3)} &= -\frac{p_2 p_3}{\sqrt{2} p^2} \hat{v}_1 + \frac{p_2}{p} \hat{v}_4 - \frac{p_1}{p_{12}} \hat{v}_7 + \frac{p_2 p_3 p_{23}}{p_{12} p \sqrt{p_{12}^2 + p_{23}^2}} \hat{v}_8 + \frac{p_2 p_3 p_{13}}{\sqrt{2} p^2 \sqrt{p_{12}^2 + p_{23}^2}} \hat{v}_9 \\
\hat{\alpha}_3^{(3)} &= \frac{p_{12}^2}{\sqrt{2} p^2} \hat{v}_1 + \frac{p_3}{p} \hat{v}_4 - \frac{p_{12} p_{23}}{p \sqrt{p_{12}^2 + p_{23}^2}} \hat{v}_8 - \frac{p_{12}^2 p_{13}}{\sqrt{2} p^2 \sqrt{p_{12}^2 + p_{23}^2}} \hat{v}_9
\end{aligned} \tag{83}$$

where for brevity we have defined  $p_{ij} \equiv \sqrt{p_i^2 + p_j^2}$ .

The combinations  $\hat{v}_1, \dots, \hat{v}_4$  are the coupled modes, while  $\hat{v}_5, \dots, \hat{v}_9$  are decoupled from the remaining perturbations. One can check that the matrix relating the vector  $\{\hat{\alpha}_1^{(1)}, \hat{\alpha}_2^{(1)}, \hat{\alpha}_3^{(1)}, \hat{\alpha}_1^{(2)}, \dots, \hat{\alpha}_3^{(3)}\}$  to the vector  $\{\hat{v}_1, \dots, \hat{v}_9\}$  is orthogonal. Therefore, the quadratic term  $\propto \sum_{j,a} |\hat{\alpha}_j^{(a)}|^2$  becomes  $\propto \sum_i |\hat{v}_i|^2$ . As a consequence, the modes  $\hat{v}_5, \dots, \hat{v}_9$  enter only through the term

$$\mathcal{L} \supset \frac{1}{2 a^2} \sum_{i=5}^9 |\dot{\hat{v}}_i|^2 \tag{84}$$

and are therefore decoupled from the other modes in the part of the action shown in (82) (we stress that this part is all we need to compute the kinetic term for the dynamical modes). Since the coefficient in front of each term in the sum (84) is positive, we see that these modes are well behaved (positive energy) excitations. Let us now focus on the coupled system of the remaining modes (which includes the remaining 6 dynamical modes, and the 7 nondynamical modes). The next step is to integrate out the nondynamical modes as outlined formally in equations from (19) to (24). Doing so, we end up with a rather long kinetic matrix for the six dynamical modes  $\{\hat{\Psi}, \hat{\Sigma}, \hat{v}_i\}$  ( $i = 1, 2, 3, 4$ ). We were unable to perform analytical simplifications of this matrix, as those leading to the expressions (64) and (70) for the simpler models studied above. However, we can evaluate this matrix numerically for any given choice of parameters.

More specifically, we show the results of a numerical evolution starting from  $B_{in} = 10$ ,  $\dot{B}_{in} = -\sqrt{2} m/3$  according to the slow roll conditions (74), and with momentum of the modes initially satisfying  $p_1 = 100 H$ ,  $p_2 = 80 H$ ,  $p_3 = 60 H$  (these values have been chosen only for illustrative purposes and have no particular relevance, other than that we require the mode to be well inside the horizon initially; we verified that other choices of the momenta lead to the same conclusions as those shown here). We show in Figures 7 and 8 the evolution of the eigenvalues of the kinetic matrix (the second Figure is a close-up of the first one where the eigenvalue  $\lambda_3$  crosses zero) close to horizon crossing.<sup>27</sup> We see that two eigenvalues  $\lambda_{1,2}$  are initially negative; the corresponding eigenvalues are ghosts at these times. These eigenvalues become positive without crossing zero (they diverge). A third eigenvalue  $\lambda_3$  is initially positive, but it becomes negative crossing zero. It then diverges and becomes again positive. As the formal eqs. (22) show, the linearized system of equations becomes singular when this eigenvalue crosses zero. We expect that also the linearized solutions diverge at this point (cf. the explicit solutions given in the model of Subsection VB).

<sup>27</sup> The overall factor  $a^3$  appearing in (82) has not been included in this computation. This is irrelevant for our discussion, since this factor simply rescales all eigenvalues by a common positive number.

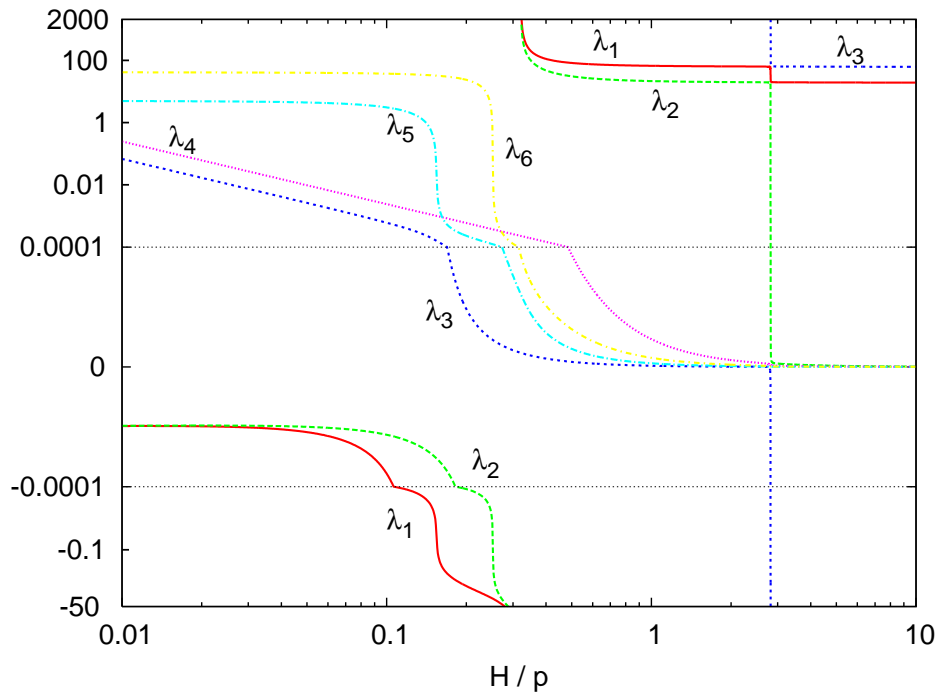


FIG. 7: Eigenvalues of the kinetic matrix for the dynamical and gauge invariant perturbations  $\hat{\Psi}, \hat{\Sigma}, \hat{v}_{1,\dots,4}$  for the model of vector inflation (71), and for one specific choice of initial conditions (see the main text). Since  $H/p$  increases with time during the stage shown, we use it as a “time variable” in this and the next Figure. The eigenmodes corresponding to  $\lambda_{1,2}$  are ghosts at the earliest times shown (low  $H/p$ ). The mode corresponding to  $\lambda_3$  becomes a ghost close to horizon crossing. This eigenvalues (and the determinant of the kinetic matrix) crosses zero at this point, signaling an instability of the system also at the linearized level. The kinetic matrix, and its eigenvalues, are dimensionless. Notice that we use linear units in the interval  $[-0.0001, 0.0001]$ , and logarithmic units outside.

The study so far concentrated on the nature of the modes at horizon crossing, in an inflationary regime for which the total mass term of the vectors was negative,  $-R/6 + m^2 \simeq -2H^2 + m^2 < 0$ . However, as we discussed after eq. (76),  $-R/6 + m^2$  vanishes towards the end of inflation. We studied the behavior of the eigenvalues of the kinetic matrix also around this point. We considered the same background evolution as for the previous plot, but smaller values of the momenta, so that  $H/p$  is not exponentially small at the times shown (we want to avoid that our results are affected by numerical inaccuracies). As shown in Figure 9, we find that two eigenvalues cross zero precisely when the total mass vanishes. Also at this point, the system of equations for the eigenvalues becomes singular. We expect that the linearized solutions diverge also at this point.

It is interesting to compare the behavior of the eigenvalues shown in these Figures with that obtained for the previous models. We find that the two eigenvalues  $\lambda_{1,2}$  behave precisely as in the case of zero vev studied in the previous Section: they are negative in the deep sub-horizon regime, they diverge close to horizon crossing, and they cross zero when the total mass vanishes. On the contrary, the eigenvalue  $\lambda_3$  behaves precisely as in the cases of a single vector with nonvanishing vev studied above (cf. Figures 3 and 6): it is positive in the deep sub-horizon regime, it crosses zero close to horizon crossing, and it remains negative for some time afterwards. It appears from these behaviors that the mixing with gravity affects only one linear combination of the three ghosts.

We conclude the present discussion with some remarks on the previous study [33] of perturbations of vector inflation. We already discussed in the Introduction while the computations of [33] - being limited to the linearized equations for the perturbations in either the sub-horizon or the super-horizon regime - cannot show the ghost instabilities found here. Here we want to reply to some specific comments on our previous works [25, 31] contained in [33]. Most of these remarks are answered by the fact that, in our previous works, we only provided arguments for the presence of ghosts in the model of vector inflation, deferring the explicit computation to the present work. It is mentioned in [33] that the ghost may be an artifact of having expanded the vector field in transverse and longitudinal part according to  $A_\mu^T + \partial_\mu \phi$ . This would introduce additional time derivatives, which could affect our findings. This Stuckelberg decomposition was introduced in [25, 31] only as the simplest way to elucidate the problem. The computations presented here do not introduce additional time derivatives in the parametrization of the perturbations, and lead to the same conclusions as the much simpler Stuckelberg analysis. It was also pointed out in [33] that the complete computation, with gravity

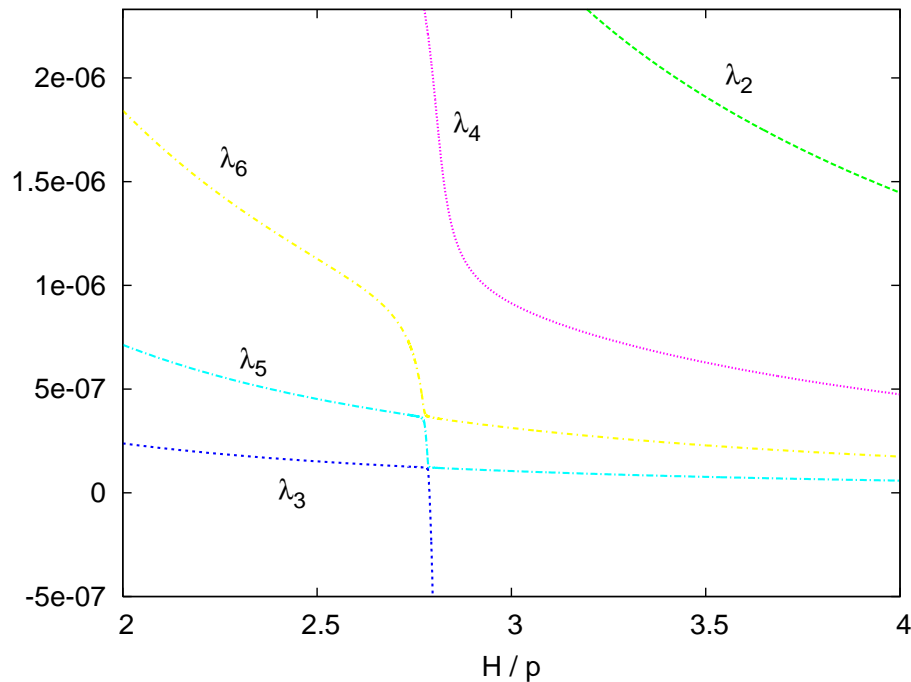


FIG. 8: Close up of the previous Figure where  $\lambda_3$  vanishes. A further close up shows that  $\lambda_5$  and  $\lambda_6$ , and, later,  $\lambda_3$  and  $\lambda_5$  do not cross each other, although they appear to do so in the Figure.

perturbations also included, was in order. This is precisely what is performed here. Another comment of [33] was that the model of vector inflation should only be regarded as an effective field theory, valid only at small energies. The (unknown) UV completion of this theory may be without ghosts. We agree with this claim.<sup>28</sup> This was precisely the point raised in our previous works. As we have seen here, the ghost instabilities appear during most of the sub-horizon regime, and close to horizon crossing. Any UV completion needs to be relevant at these stages. Therefore, the effective theory of vector inflation cannot be used for the study of cosmological perturbations in the sub-horizon regime. Since this regime is crucial to obtain phenomenological results, this invalidates any prediction of the effective model [28]. Finally, it was argued in [33] that the instability may simply be due to the growth of the scale factors, and to a wrong rescaling of fields. This is not the case, since the linearized system of equations - and, most likely, its solutions - diverges at some finite moments of time, while the scale factor remains finite.

## VI. DISCUSSION

Although the paradigm of inflation is well supported by the observational data, we still do not know the actual particle physics mechanism behind the inflationary expansion, and the generation of cosmological perturbations. It is customary to parametrize our ignorance in terms of scalar fields: they may be fundamental fields, or simply order parameters which provide an effective description of some degrees of freedom in the theory (for instance, the brane-antibrane separation in some string motivated models of inflation). However, it may well be possible that these two key elements of cosmology are due to higher spin fields. Vector fields are probably the simplest possibility after scalars. They can in principle leave a distinct signature from the scalar case, since a nonvanishing spatial vev of a vector breaks the isotropy of space. This can provide anisotropic expansion, and / or generate a spectrum of primordial perturbations that breaks statistical isotropy. The main obstacle faced by explicit realizations of this idea is that, in the standard case, vector fields decrease too quickly due to the expansion of the universe. Therefore, suitable modifications of the standard action need to be made.

Recently, a class of models was considered in which the vector is nonminimally coupled to the curvature,  $\mathcal{L} \supset$

<sup>28</sup> Apart from the fact that we are not aware of any well behaved UV completion of a theory with ghosts.

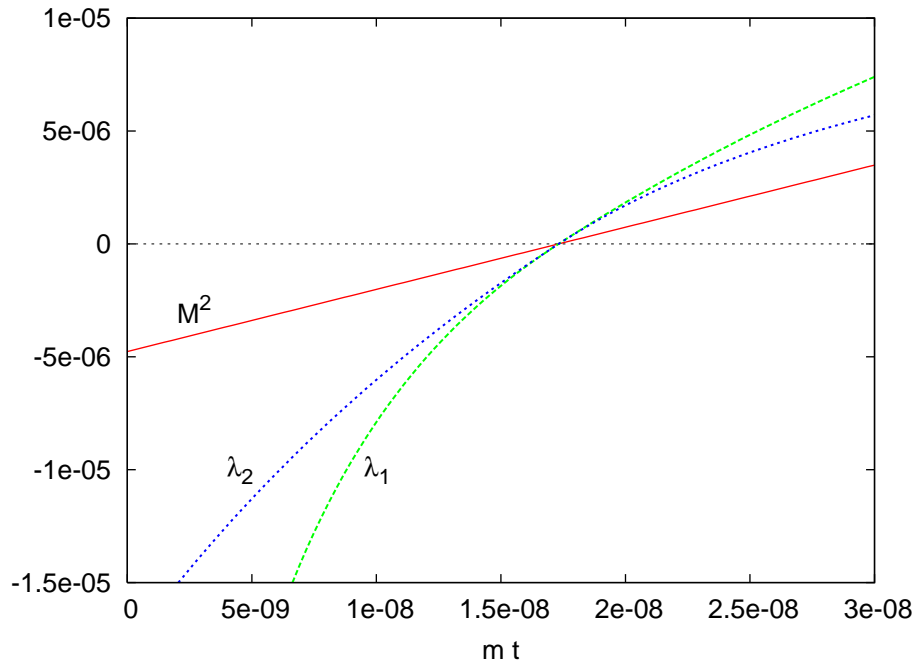


FIG. 9: Two eigenvalues vanish when the mass term  $M^2 = m^2 - R/6$  of the vector fields vanishes. The mass  $M^2$  is shown in units of  $m^2/300$ . We have  $B \simeq 1.048$  when the total mass vanishes (this occurs still during inflation). The mode has been chosen to be outside the horizon when the total mass vanishes ( $H/p \simeq 1872$  at the time shown, and  $k_1 : k_2 : k_3 = 10 : 8 : 4$ ).

$R/12 A^\mu A_\mu$ . Indeed, for this specific coupling, the vev of the vector evolves as that of a minimally coupled scalar field; this offers the possibility of realizing an inflationary background, with a controllable anisotropy [28, 29]. In addition, the transverse perturbations of the vector behave as the perturbations of a minimally coupled scalar; this is the basis of the mechanism of vector curvaton [30] for the generation of a nearly scale invariant spectrum of primordial perturbations. To our knowledge, the analogy between the  $R/12 A^\mu A_\mu$  coupling and the minimally coupled scalar field first appeared in the 1987 work by Turner and Widrow [27], where it was exploited for the generation of a primordial magnetic field during inflation. The renewed interest in this mechanism is mostly motivated by some features in the WMAP data that hint for a small break of statistical isotropy.

All of the mentioned works suggested new interesting mechanisms, and a complete check of the stability of these proposals was beyond their scope. It is tempting to assume that these models should be well behaved due to the strong analogy with the minimally coupled scalar field case. There is however a crucial difference between these two cases, and between the case of a minimally vs a nonminimally coupled vector; it is due to the longitudinal vector polarization. In the above works, the vector has a U(1) invariant kinetic term  $\mathcal{L} \supset -1/4 F_{\mu\nu} F^{\mu\nu}$ . If only this term was present, the vector would only have the two transverse polarizations. However, the nonminimal coupling to the curvature breaks the U(1) symmetry, and gives rise to an additional, longitudinal, polarization. The nature of this mode is controlled by the sign of the mass term, which, for these mechanisms to work, needs to be negative. For the scalar case, a negative mass squared means that the field is a tachyon; for a vector field, a negative mass squared means that the longitudinal polarization is a ghost.

Motivated by this initial consideration, we studied the stability of this class of theories. We did so by computing the free action for the dynamical (physically propagating) modes of such theories, around the background solutions considered in the various proposals. The sign of the eigenvalues of the kinetic matrix of these action indicates whether the corresponding eigenmode is a positive or negative energy excitation. Our computations confirmed that there is a ghost for each nonminimally coupled vector field in the model. As we already mentioned in the Introduction, theories with ghosts are consistent only as effective theories, valid below some energy scale  $\Lambda$ . Inflationary predictions heavily rely on the initial conditions; for instance, the vector curvaton mechanism of [30] results in a scale invariant spectrum because of the specific choice of initial adiabatic vacuum. This choice is made in the quantized theory for the perturbations, which is performed in the deep UV regime (energy  $\gg H^{-1}$ ) [37]. In presence of a cut-off, the initial adiabatic vacuum cannot be imposed at arbitrarily early times, and, depending on the precise numerical value of the cut-off, it may not be possible to impose it at all. This casts doubts on the phenomenological predictions obtained for such models.

In fact, all theories with an explicit mass  $M$  for the vector require a cut-off which makes them invalid at high energies, irrespectively of the sign of the mass term. We can see this based on the behavior of massive vector fields at high energies. The explicit mass breaks the gauge invariance in a hard way. It is well known that, in such case, the interactions of the longitudinal bosons violate unitarity at a scale which is parametrically set by  $M$ , leading to a quantum theory out of control. For the present models,  $M$  is the Hubble rate or below, so that the entire sub-horizon regime may be ill-defined. At high energies, the longitudinal mode will also interact strongly with the other fields in the theory (this will renormalize the coupling constants). Then, depending on exactly when this happens, the quantum theory of the perturbations may be out of control throughout the entire short wavelength regime. If this is the case, all initial conditions would become unjustified, and the theory would lose its predictive power. Although this problem is present for both positive and negative mass terms, a theory which has a hard vector mass and a ghost is more problematic than a theory with only a hard vector mass. The most immediate UV completion of a theory with a hard mass is through a higgs mechanism. The mass would be then due to the vev of a scalar field that becomes dynamical above the scale  $M$ . In this way the theory remains under control also in the short wavelength regime, and one can apply all the standard computations valid for scalar fields during inflation. However, if  $M^2$  needs to have the wrong sign, the scalar field in this UV completed theory needs to be a ghost. In fact, we are not aware of any well behaved UV completion of a theory with a ghost.

We stress that this instability is not related to the classical behavior of the solutions of the linearized equations for the perturbations, and it would be present even if the latter remained finite. However, we argued that, for the models considered here, also the linearized solutions diverge. This is a second instability that adds up to the one we have just discussed. This instability appears because the eigenvalues corresponding to the ghosts do not remain negative over the whole evolution, but change sign, and cross zero at some finite moment of time  $t_*$  (there are two such moments in the model of vector inflation considered here). We showed that the linearized equations are singular at  $t_*$ , and we expect that the linearized solutions diverge for  $t \rightarrow t_*$ . While it is possible that the divergency does not occur at the full nonlinear level, this instability also invalidates all the phenomenological signatures of these models which are based on the linearized computation (as for instance the primordial spectrum of perturbations). We solved the linearized equations only in the simplest cases of a vector field with no vev, and of a vector field with vev plus a cosmological constant. We did not solve them for the models [28, 29]. We have shown however that the linearized equations become singular also in these cases. It is important to stress that even if, due to some unexpected cancellation, the solutions to these equations would remain finite, this would not eliminate the ghost instability that we have discussed in the two previous paragraphs, and that we have proven to be present for all the models studied here.

We conclude the Discussion with some remarks on models different from those considered here, and in our previous works [25, 31]. The instability we pointed out motivates the study of such models, as for instance vector fields with nonstandard kinetic terms [26, 43, 45] (although, some of these proposals are also unstable), nonabelian vectors with nonminimal coupling to the curvature [46]<sup>29</sup>, spinors [47], or higher  $p$ -forms [21, 48, 49]<sup>30</sup>. Of particular interest, in our opinion, is a class of models characterized by a function of a scalar field multiplying the kinetic terms of the vectors,  $I(\phi) F^{\mu\nu} F_{\mu\nu}$ , but no potential term for the vector [45, 50]. U(1) invariance is preserved in these models, and the problematic longitudinal mode is absent. A complete study of the cosmological perturbations (conducted along the lines described here) is the next necessary step for obtaining the phenomenological predictions of these models.

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<sup>29</sup> The stability analysis performed here applies also to the nonabelian model of [46] if the vectors have no vev (since, in this case, the nonabelian structure does not affect the quadratic action for the perturbations); however, the case with a nonvanishing vev requires a separate investigation.

<sup>30</sup> The vector case is  $p = 1$ ; Ref. [49] generalized the arguments of [25, 31], and pointed out that also the  $p = 2$  case contains ghosts.



## APPENDIX A: DETAILS OF THE LINEARIZED COMPUTATION OF SUBSECTION VB

This appendix contains the details of the stability analysis of the model of Subsection VB. The explicit forms of the linearized equations (53) are

$$\begin{aligned}
\text{Eq}_{00} : & -\frac{2}{M_p} \left\{ \left[ \left(1 - \frac{1}{3}B_1^2\right) H + \left(1 + \frac{1}{6}B_1^2\right) h + \frac{1}{6}B_1 \dot{B}_1 \right] \dot{\Psi} \right. \\
& + \frac{1}{2} \left[ p_T^2 + \left( m^2 - \frac{p_T^2 + 2p_L^2}{6} + 5h^2 - 4hH \right) B_1^2 - 4hH B_1 \dot{B}_1 + \dot{B}_1^2 \right] \dot{\Psi} \\
& + \left[ 3H^2 - \left( 3 + \frac{5}{2}B_1^2 \right) h^2 - \frac{1}{2}\dot{B}_1^2 + 2 \left( B_1 H + \dot{B}_1 \right) B_1 h \right] \hat{\Phi} - \left[ \left( 1 + \frac{1}{6}B_1^2 \right) (H + h) + \frac{1}{6}B_1 \dot{B}_1 \right] \hat{\chi} \\
& - \left[ \left( 1 + \frac{1}{6}B_1^2 \right) \left( H - \frac{1}{2}h \right) + \frac{1}{6}B_1 \dot{B}_1 \right] \hat{B} - \left[ \left( \frac{1}{2}H - h \right) B_1 + \frac{1}{2}\dot{B}_1 \right] \hat{\alpha}_0 - \left( \frac{1}{2}\dot{B}_1 - B_1 h \right) \hat{\alpha}_1 \\
& \left. + \left[ \frac{1}{2} \left( \frac{1}{3}p^2 - m^2 - 5h^2 + 4hH \right) B_1 + h \dot{B}_1 \right] \hat{\alpha}_1 \right\} = 0 \\
\text{Eq}_{01} : & -2 \frac{i p_L a}{M_p} \left\{ \frac{1}{6} \left( \hat{\alpha}_1 - B_1 \dot{\Psi} \right) B_1 + \frac{B_1}{6} \left( B_1 H - 2B_1 h - 2\dot{B}_1 \right) \dot{\Psi} + \frac{1}{6} \left( -B_1 H + 2h B_1 + \dot{B}_1 \right) \hat{\alpha}_1 \right. \\
& \left. + \left( 1 + \frac{1}{6}B_1^2 \right) \left( H + h + \frac{B_1 \dot{B}_1}{6 + B_1^2} \right) \hat{\Phi} + \left( \frac{p_T^2}{4p_L^2} + \frac{p_T^2}{24p_L^2} B_1^2 - \frac{1}{3} \mathcal{D}_{\chi\chi} \right) \hat{\chi} - \frac{6 + B_1^2}{24} \hat{B} - \frac{B_1}{3} \mathcal{D}_{\alpha_0\alpha_0} \hat{\alpha}_0 \right\} = 0 \\
\text{Eq}_{0i} : & -2 \frac{i p_{Ti} b}{M_p} \left\{ \frac{1}{2} \left( 1 - \frac{B_1^2}{6} \right) \dot{\Psi} + \frac{1}{6} B_1 \hat{\alpha}_1 + \left[ \frac{B_1}{6} \left( H B_1 - 2\dot{B}_1 \right) - \left( \frac{3}{2} + \frac{B_1^2}{12} \right) h \right] \dot{\Psi} + \left[ \left( \frac{1}{3}H - \frac{7}{6}h \right) B_1 + \frac{2}{3}\dot{B}_1 \right] \hat{\alpha}_1 \right. \\
& + \left[ \left( h - \frac{1}{2}H \right) B_1 - \frac{1}{2}\dot{B}_1 \right] \hat{\alpha} + \left[ - \left( \frac{1}{2} + \frac{B_1^2}{12} \right) h + \left( 1 + \frac{1}{6}B_1^2 \right) H + \frac{1}{6}B_1 \dot{B}_1 \right] \hat{\Phi} - \frac{1}{4} \left( 1 + \frac{B_1^2}{6} \right) \hat{\chi} \\
& \left. + \frac{p_L^2}{4p_T^2} \left( 1 + \frac{1}{6}B_1^2 \right) \hat{B} \right\} = 0 \\
\text{Eq}_{11} : & \frac{2a^2}{3M_p} \left\{ \frac{1}{2} B_1 \left( \ddot{\alpha}_1 - 2B_1 \ddot{\Psi} \right) - \left( 3H B_1 + 2\dot{B}_1 \right) B_1 \dot{\Psi} + \frac{1}{2} \left( -H B_1 + 8h B_1 - \dot{B}_1 \right) \hat{\alpha}_1 + \left( \mathcal{M}_{\Psi\Psi} - p_T^2 B_1^2 \right) \dot{\Psi} \right. \\
& + \left( \mathcal{M}_{\Psi\alpha_1} + \frac{p_T^2}{2} B_1 \right) \hat{\alpha}_1 + \left[ \left( 3 + \frac{1}{2}B_1^2 \right) h + \left( 3 - B_1^2 \right) H + \frac{1}{2}B_1 \dot{B}_1 \right] \dot{\Phi} \\
& + \left[ \left( -\frac{9}{4} m^2 + \frac{p_T^2 + 2p_L^2}{4} - \frac{15}{4} h^2 + 3hH \right) B_1^2 + 3h B_1 \dot{B}_1 - 3 \left( \frac{p_T^2}{2} - \frac{V_0}{2M_p^2} + \frac{3}{2} h^2 - \frac{3}{2} H^2 + \frac{1}{4} \dot{B}_1^2 \right) \right] \hat{\Phi} \\
& - \left( \frac{3}{2} - \frac{1}{4} B_1^2 \right) \dot{B} + \frac{1}{2} B_1^2 \hat{\chi} + \left( 2H - h \right) B_1^2 \hat{\chi} - \left[ \left( \frac{9}{2} + \frac{1}{4} B_1^2 \right) h + \left( \frac{9}{2} - \frac{5}{4} B_1^2 \right) H + \frac{1}{2} B_1 \dot{B}_1 \right] \hat{B} \\
& \left. + \frac{3}{2} \left( 2h B_1 - H B_1 - \dot{B}_1 \right) \hat{\alpha}_0 \right\} = 0 \\
\text{Eq}_{1i} : & \frac{ab p_L p_{Ti}}{3M_p} \left\{ \frac{3}{p_L^2} \left[ \left( 2h - H \right) B_1 - \dot{B}_1 \right] \hat{\alpha}_1 + \mathcal{M}_{\Psi\alpha} \hat{\alpha} + \frac{6 + B_1^2}{4p_T^2} \dot{B} + \frac{6 + B_1^2}{4p_T^2} \left( 3H + 6h + 2 \frac{B_1 \dot{B}_1}{6 + B_1^2} \right) \hat{B} + \frac{6 + B_1^2}{4p_L^2} \hat{\chi} \right. \\
& \left. + \frac{6 + B_1^2}{4p_L^2} \left( 3H - 6h + 2 \frac{B_1 \dot{B}_1}{6 + B_1^2} \right) \hat{\chi} + B_1^2 \dot{\Psi} + \frac{1}{2} \left( 6 + B_1^2 \right) \hat{\Phi} - B_1 \hat{\alpha}_1 - \frac{3}{p_L^2} \left[ \left( H - 2h \right) B_1 + \dot{B}_1 \right] \hat{\alpha}_0 \right\} = 0
\end{aligned}$$

$$\begin{aligned}
\text{Eq}_{ij} : & \frac{b^2}{6M_p} \left\{ \left[ (6 - B_1^2) \ddot{\Psi} + \left[ (18 - B_1^2) H - (18 + B_1^2) h - 6B_1 \dot{B}_1 \right] \dot{\Psi} + [\mathcal{M}_{\Sigma\Psi} + (6 - B_1^2) p_T^2 - 2B_1^2 p_L^2] \Psi + 2B_1 \ddot{\alpha} \right. \right. \\
& + 2 \left[ (5H - 7h) B_1 + 5\dot{B}_1 \right] \dot{\alpha}_1 - (\mathcal{M}_{\Sigma\alpha_1} - 2p^2 B_1) \hat{\alpha}_1 + (6 + B_1^2) \left( 2H - h + 2\frac{B_1 \dot{B}_1}{6 + B_1^2} \right) \dot{\Phi} \\
& - \left[ (p^2 + 30h^2 - 24hH) B_1^2 - 24h B_1 \dot{B}_1 + 6 \left( p^2 + 6(h^2 - H^2) + \dot{B}_1^2 \right) \right] \hat{\Phi} - (6 + B_1^2) \dot{\chi} \\
& - (6 + B_1^2) \left( 3H - 3h + 2\frac{B_1 \dot{B}_1}{6 + B_1^2} \right) \hat{\chi} + 6 \left[ (H - 2h) B_1 + \dot{B}_1 \right] \hat{\alpha}_0 - (6 + B_1^2) \left( \dot{\hat{B}} + \left( 3H + 2\frac{B_1 \dot{B}_1}{6 + B_1^2} \right) \hat{B} \right) \right] \delta_{ij} \\
& + p_{Ti} p_{Tj} \left[ \frac{6 + B_1^2}{p_T^2} \left( \dot{\hat{B}} + \left( 3H + 2\frac{B_1 \dot{B}_1}{6 + B_1^2} \right) \hat{B} \right) - (6 - B_1^2) \hat{\Psi} + (6 + B_1^2) \hat{\Phi} - 2B_1 \hat{\alpha}_1 \right] \left. \right\} = 0 \\
\text{Eq}_0 : & i p_L \left\{ \dot{\alpha}_1 + (2h B_1 - H B_1 - \dot{B}_1) \hat{\Psi} + (H - 2h) \hat{\alpha}_1 + \frac{p_T^2}{p_L^2} \dot{\alpha} + \frac{p_T^2}{p_L^2} (H - 2h) \hat{\alpha} + (H B_1 - 2h B_1 + \dot{B}_1) \hat{\Phi} \right. \\
& \left. - \frac{2}{3B_1} \mathcal{D}_{\chi\chi} \hat{\chi} + \left( 1 + \frac{p_T^2}{p_L^2} - \frac{2}{3} \mathcal{D}_{\alpha_0\alpha_0} \right) \hat{\alpha}_0 \right\} = 0 \\
\text{Eq}_1 : & \frac{1}{a} \left\{ \ddot{\alpha}_1 - \frac{1}{3} B_1 \ddot{\Psi} + 3H \dot{\alpha}_1 + \left( \frac{8}{3} B_1 h - \frac{7}{3} B_1 H - \dot{B}_1 \right) \dot{\Psi} + (\mathcal{M}_{\alpha_1\alpha_1} + p_T^2) \hat{\alpha}_1 - \frac{B_1}{3} p_T^2 \hat{\Psi} \right. \\
& + (\dot{B}_1 - 2B_1 h) \dot{\hat{\Phi}} + \frac{B_1}{3} (\dot{\hat{B}} + \dot{\hat{\chi}}) + \dot{\alpha}_0 + \frac{B_1}{3} (p^2 - 6m^2) \hat{\Phi} + \frac{1}{3} (7B_1 h + B_1 H - 3\dot{B}_1) \hat{B} \\
& \left. + \frac{2}{3} (2H - h) B_1 \hat{\chi} - p_T^2 \hat{\alpha} + 2(h + H) \hat{\alpha}_0 \right\} = 0 \\
\text{Eq}_i : & \frac{p_{Ti}}{p_L b} \left\{ \ddot{\alpha} + 3(H - 2h) \dot{\alpha} + (\mathcal{M}_{\alpha\alpha} + p_L^2) \hat{\alpha} - p_L^2 \hat{\alpha}_1 + \frac{p_L^2}{p_T^2} (B_1 H - 2B_1 h + \dot{B}_1) \hat{B} + \dot{\alpha}_0 + (2H - 4h) \hat{\alpha}_0 \right\} = 0
\end{aligned} \tag{A1}$$

where the index  $i$  on the equations spans over the  $i = 2, 3$  isotropic spatial directions, and where the momenta are defined in eqs. (54) and (55).

Only the 2d scalar perturbations are included in this computation. More in general, if we include both the 2d scalar and 2d vector modes, the perturbed equations carrying  $i$  or  $j$  indices split in two separate parts, one for the 2d scalar, and one for the 2d vector modes. Although we have not written the 2d vector parts of the above equations, we have explicitly checked that they are decoupled from the 2d scalar parts. The 2d vector parts are not related to the instability we have demonstrated in the main text; therefore we do not discuss them here. Another property to be noted in the system (A1) is that not all the equations are independent. Using the perturbed Bianchi identities, it is possible to obtain eqs.  $\text{Eq}_{ij}$ ,  $\text{Eq}_{1i}$  from the remaining ones in (A1). This remaining equations are the set of linearized equations (56) that we have chosen to solve in the main text.<sup>31</sup>

For brevity, we have grouped some of the extended terms that depend on the background quantities, which appear both in the action and in the linearized equations. We have denoted them with calligraphic letters  $\mathcal{D}$ ,  $\mathcal{M}$ . These

<sup>31</sup> One could have chosen a different but equivalent subset of independent equations. Our choice is related to the fact that the linearized equations of motion obtained by extremizing the quadratic action (63) precisely gives the set of equations (56).

terms are

$$\begin{aligned}
\mathcal{D}_{\Psi\Psi} &= \frac{1}{18+3B_1^2+2B_1^4} \left[ \left( \frac{3}{2}m^2 + \frac{195}{8}h^2 - \frac{39}{2}hH \right) B_1^6 + 18 \left( 3m^2 - 6H^2 - \frac{51}{4}h^2 + 15hH \right) B_1^2 - \frac{39}{2}h\dot{B}_1 B_1^5 \right. \\
&\quad \left. + \frac{9}{2}(6h-7H)\dot{B}_1 B_1^3 + 27(8h-3H)B_1\dot{B}_1 + 27 \left( 3H^2 - 3h^2 - \frac{5}{2}\dot{B}_1^2 \right) \right. \\
&\quad \left. + 3 \left( 6m^2 + \frac{51}{4}h^2 - \frac{51}{4}H^2 + \frac{13}{8}\dot{B}_1^2 \right) B_1^4 \right] \\
\mathcal{D}_{\Psi\alpha_1} &= \frac{-1}{18+3B_1^2+2B_1^4} \left[ \left( \frac{45}{4}h^2 - 9hH \right) B_1^5 + 27(2h-H)\dot{B}_1 + 3(H-5h)\dot{B}_1 B_1^4 - \frac{9}{2}(6h+H)\dot{B}_1 B_1^2 \right. \\
&\quad \left. + 9 \left( 3m^2 - 6h^2 + 12hH - 9H^2 + \frac{3}{2}\dot{B}_1^2 \right) B_1 + 3 \left( \frac{3}{2}m^2 + \frac{39}{2}h^2 - 12hH - \frac{9}{2}H^2 + \frac{7}{4}\dot{B}_1^2 \right) B_1^3 \right] \\
\mathcal{D}_{\alpha\alpha} &= \frac{3p_T^2}{p_L^2(18+3B_1^2+2B_1^4)} \left[ \left( \frac{15}{4}h^2 - 3hH \right) B_1^4 - 12hB_1\dot{B}_1 + (2H-7h)\dot{B}_1 B_1^3 \right. \\
&\quad \left. + 9 \left( m^2 - 2h^2 + 4hH - 3H^2 + \frac{1}{2}\dot{B}_1^2 \right) + \left( \frac{3}{2}m^2 + \frac{39}{2}h^2 - 12hH - \frac{9}{2}H^2 + \frac{7}{4}\dot{B}_1^2 \right) B_1^2 \right] \\
\mathcal{D}_{\alpha_1\alpha_1} &= \frac{p_L^2}{p_T^2} \mathcal{D}_{\alpha\alpha} \\
\mathcal{D}_{\chi\chi} &= \frac{-1}{p_L^2(18+3B_1^2+2B_1^4)} \left[ \left( \frac{93}{4}h^2 - 21hH + 3H^2 \right) B_1^6 - 36h\dot{B}_1 B_1^3 + 3(2H-7h)\dot{B}_1 B_1^5 \right. \\
&\quad \left. + 9 \left( 3m^2 + 6h^2 - 6H^2 + \frac{3}{2}\dot{B}_1^2 \right) B_1^2 + 3 \left( \frac{3}{2}m^2 + \frac{51}{2}h^2 - 18hH - 3H^2 + \frac{7}{4}\dot{B}_1^2 \right) B_1^4 \right] \\
\mathcal{D}_{\alpha_0\alpha_0} &= \frac{1}{B_1^2} \mathcal{D}_{\chi\chi} \\
\mathcal{D}_{\chi\alpha_0} &= \frac{1}{B_1} \mathcal{D}_{\chi\chi} \tag{A2}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{M}_{\Psi\Psi} &= \frac{3}{18+3B_1^2+2B_1^4} \left[ (m^2 + 5h^2 - 4hH) B_1^6 - 4hB_1^5\dot{B}_1 - 2(4H-19h)B_1^3\dot{B}_1 + 24(H-2h)B_1\dot{B}_1 + 3\dot{B}_1^2 \right. \\
&\quad \left. + \left( \frac{11}{2}m^2 - \frac{91}{2}h^2 + 46hH - 12H^2 + \dot{B}_1^2 \right) B_1^4 - \left( 3m^2 + 21h^2 + 12hH - 36H^2 + \frac{23}{2}\dot{B}_1^2 \right) B_1^2 \right] \\
\mathcal{M}_{\Psi\alpha_1} &= \frac{1}{18+3B_1^2+2B_1^4} \left[ -(2m^2 + 5h^2 - 4hH) B_1^5 - 36(H-2h)\dot{B}_1 + 4(H-h)B_1^4\dot{B}_1 + 6(3H-16h)B_1^2\dot{B}_1 \right. \\
&\quad \left. - 9 \left( m^2 - 13h^2 + 12hH - 2H^2 - \frac{1}{9}\dot{B}_1^2 \right) B_1^3 + 18 \left( m^2 + h^2 + 4hH - 6H^2 + \frac{5}{3}\dot{B}_1^2 \right) B_1 \right] \\
\mathcal{M}_{\Psi\alpha} &= \frac{3}{2p_L^2(18+3B_1^2+2B_1^4)} \left[ -3(4H-5h)hB_1^5 + 4(H-5h)B_1^4\dot{B}_1 - 36(H-2h)\dot{B}_1 - 6(6h+H)B_1^2\dot{B}_1 \right. \\
&\quad \left. + 36 \left( m^2 - 2h^2 + 4hH - 3H^2 + \frac{1}{2}\dot{B}_1^2 \right) B_1 + 6 \left( m^2 + 13h^2 - 8hH - 3H^2 + \frac{7}{6}\dot{B}_1^2 \right) B_1^3 \right]
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_{\alpha_1\alpha_1} &= \frac{2}{18 + 3B_1^2 + 2B_1^4} \left[ (m^2 + 10h^2 - 8hH) B_1^4 - 8h B_1^3 \dot{B}_1 - 6(2H - 5h) B_1 \dot{B}_1 + 9 \left( m^2 - 5h^2 + 4hH - \frac{2}{3} \dot{B}_1^2 \right) \right. \\
&\quad \left. + 3 \left( \frac{5}{2} m^2 - \frac{13}{2} h^2 + 10hH - 6H^2 + \dot{B}_1^2 \right) B_1^2 \right] \\
\mathcal{M}_{\alpha\alpha} &= \frac{2}{18 + 3B_1^2 + 2B_1^4} \left[ (m^2 + 22h^2 - 14hH) B_1^4 - 8h B_1^3 \dot{B}_1 - 6(2H - 5h) B_1 \dot{B}_1 + 9 \left( m^2 + 7h^2 - 2hH - \frac{2}{3} \dot{B}_1^2 \right) \right. \\
&\quad \left. + 3 \left( \frac{5}{2} m^2 - \frac{1}{2} h^2 + 7hH - 6H^2 + \dot{B}_1^2 \right) B_1^2 \right] \\
\mathcal{M}_{\Sigma\Psi} &= \frac{12}{18 + 3B_1^2 + 2B_1^4} \left[ (m^2 + 5h^2 - 4hH) B_1^6 - 4h B_1^5 \dot{B}_1 + 2(7h - 4H) B_1^3 \dot{B}_1 + 6(7h - 2H) B_1 \dot{B}_1 - 15 \dot{B}_1^2 \right. \\
&\quad \left. + \left( 15m^2 - 57h^2 + 60hH - 18H^2 - \frac{5}{2} \dot{B}_1^2 \right) B_1^2 + \left( \frac{11}{2} m^2 - \frac{1}{2} h^2 + 10hH - 12H^2 + \dot{B}_1^2 \right) B_1^4 \right] \\
\mathcal{M}_{\Sigma\alpha_1} &= \frac{4}{18 + 3B_1^2 + 2B_1^4} \left[ 2 \left( m^2 + \frac{5}{2} h^2 - 2hH \right) B_1^5 + \frac{3}{2} (h - 6H) B_1^2 \dot{B}_1 + 9(7h - 2H) \dot{B}_1 + (2H - 11h) B_1^4 \dot{B}_1 \right. \\
&\quad \left. + 36 \left( m^2 - \frac{7}{2} h^2 + 4hH - \frac{3}{2} H^2 + \frac{1}{6} \dot{B}_1^2 \right) B_1 + 9 \left( m^2 + 2h^2 - 2H^2 + \frac{5}{9} \dot{B}_1^2 \right) B_1^3 \right] \quad (\text{A3})
\end{aligned}$$

As we have discussed in the main text, we proceed by solving the second of (56) for the mode  $\hat{\chi}$ , and inserting the solution back into the rest of the equations. Next, we differentiate equations Eq<sub>00</sub>, Eq<sub>0i</sub>, Eq<sub>0</sub> (with the solution for  $\hat{\chi}$  given in (57) inserted in them) in order to obtain first order differential equations for the modes  $\hat{\Phi}$ ,  $\hat{B}$ ,  $\hat{\alpha}_0$ . Combined with Eq<sub>11</sub>, Eq<sub>1</sub>, Eq<sub>i</sub> (again with the solution (57) inserted in them), these equations form the set of equations to be solved numerically, summarized in matrix form in (59). Here we give the detailed expressions of the terms appearing in this system of equations.

We first define the following useful combinations of background dependent terms which frequently appear in the linearized system:

$$\begin{aligned}
\mathcal{D} &= (6 + B_1^2) p_T^2 - 8p_L^2 \mathcal{D}_{\chi\chi} \\
\mathcal{H} &= H + h + \frac{B_1 \dot{B}_1}{6 + B_1^2} \quad (\text{A4})
\end{aligned}$$

where  $\mathcal{D}_{\chi\chi}$  is defined in (A2). The time derivative of  $\mathcal{D}$  and  $\mathcal{H}$  are also useful, which is explicitly given by

$$\begin{aligned}
\dot{\mathcal{D}} &= -2p_T^2 (6 + B_1^2) (H + h) + 16p_L^2 (H - 2h) \mathcal{D}_{\chi\chi} + 2p_T^2 B_1 \dot{B}_1 - 8p_L^2 \dot{\mathcal{D}}_{\chi\chi} \\
\dot{\mathcal{H}} &= -2 \frac{B_1 \dot{B}_1}{6 + B_1^2} \mathcal{H} + \dot{H} + \dot{h} + 2(H + h) \frac{B_1 \dot{B}_1}{6 + B_1^2} + \frac{\dot{B}_1^2 + B_1 \ddot{B}_1}{6 + B_1^2} \quad (\text{A5})
\end{aligned}$$

where  $\ddot{B}_1$ ,  $\dot{H}$ ,  $\dot{h}$  are obtained from (49):

$$\begin{aligned}\dot{H} &= \frac{2}{18 + 3B_1^2 + 2B_1^4} \left[ \left( m^2 - \frac{35}{4}h^2 + 7hH - 3H^2 \right) B_1^4 + 12h B_1 \dot{B}_1 + (7h - 2H) B_1^3 \dot{B}_1 - \frac{9}{2} \left( 6h^2 + \dot{B}_1^2 \right) \right. \\ &\quad \left. - \frac{3}{2} \left( 18h^2 - 12hH + \frac{7}{6}\dot{B}_1^2 \right) B_1^2 \right] \\ \dot{h} &= \frac{1}{18 + 3B_1^2 + 2B_1^4} \left[ 3(2H - 5h) h B_1^4 + 8h B_1^3 \dot{B}_1 + 6(2H - 5h) B_1 \dot{B}_1 + 6 \left( \dot{B}_1^2 - 9hH \right) \right. \\ &\quad \left. - 6 \left( m^2 - 2h^2 + \frac{11}{2}hH - 3H^2 + \frac{1}{2}\dot{B}_1^2 \right) B_1^2 \right] \\ \ddot{B}_1 &= \frac{1}{18 + 3B_1^2 + 2B_1^4} \left[ -2(m^2 + 10h^2 - 8hH) B_1^5 + 2(8h - 3H) B_1^4 \dot{B}_1 - 54H \dot{B}_1 + 15(H - 4h) B_1^2 \dot{B}_1 \right. \\ &\quad \left. - 18 \left( m^2 - 5h^2 + 4hH - \frac{2}{3}\dot{B}_1^2 \right) B_1 - 15 \left( m^2 - \frac{13}{5}h^2 + 4hH - \frac{12}{5}H^2 + \frac{2}{5}\dot{B}_1^2 \right) B_1^3 \right] \quad (\text{A6})\end{aligned}$$

We also need the explicit expressions for  $\dot{\mathcal{D}}_{\chi\chi}$  and  $\dot{\mathcal{D}}_{\alpha_0\alpha_0}$ :

$$\begin{aligned}\dot{\mathcal{D}}_{\chi\chi} &= 2 \left[ H - 2h - \frac{3 + 4B_1^2}{18 + 3B_1^2 + 2B_1^4} B_1 \dot{B}_1 \right] \mathcal{D}_{\chi\chi} \\ &\quad - \frac{9}{p_L^2 (18 + 3B_1^2 + 2B_1^4)} \left[ 2 \left( \frac{31}{4}h^2 - 7hH + H^2 \right) B_1^5 \dot{B}_1 - 12h B_1^2 \dot{B}_1^2 + \frac{5}{3} (2H - 7h) B_1^4 \dot{B}_1^2 \right. \\ &\quad \left. + 6 \left( m^2 + 2h^2 - 2H^2 + \frac{1}{2}\dot{B}_1^2 \right) B_1 \dot{B}_1 + 4 \left( \frac{m^2}{2} + \frac{17}{2}h^2 - 6hH - H^2 + \frac{7}{12}\dot{B}_1^2 \right) B_1^3 \dot{B}_1 \right. \\ &\quad \left. - 4\dot{h} B_1^3 \dot{B}_1 + \left( \frac{2}{3}\dot{H} - \frac{7}{3}\dot{h} \right) B_1^5 \dot{B}_1 + \left( \frac{31}{6}h\dot{h} - \frac{7}{3} \left( H\dot{h} + h\dot{H} \right) + \frac{2}{3}H\dot{H} \right) B_1^6 - 4h B_1^3 \ddot{B}_1 \right. \\ &\quad \left. + \left( \frac{2}{3}H - \frac{7}{3}h \right) B_1^5 \ddot{B}_1 + \left( 12(h\dot{h} - H\dot{H}) + 3\dot{B}_1 \ddot{B}_1 \right) B_1^2 \right. \\ &\quad \left. + \left( 17h\dot{h} - 6 \left( H\dot{h} + h\dot{H} \right) - 2H\dot{H} + \frac{7}{6}\dot{B}_1 \ddot{B}_1 \right) B_1^4 \right] \\ \dot{\mathcal{D}}_{\alpha_0\alpha_0} &= \frac{1}{B_1^2} \left( \dot{\mathcal{D}}_{\chi\chi} - 2\frac{\dot{B}_1}{B_1} \mathcal{D}_{\chi\chi} \right) \quad (\text{A7})\end{aligned}$$

Now we give the explicit form of the matrix  $\mathcal{M}_\kappa$  used in (59):

$$\begin{aligned}\kappa_{11} &= -B_1^2 + \frac{2p_L^2 B_1^4}{\mathcal{D}} \\ \kappa_{12} &= \frac{B_1}{2} - \frac{2p_L^2 B_1^3}{\mathcal{D}} \\ \kappa_{14} &= \left( 3 + \frac{B_1^2}{2} \right) h + (3 - B_1^2) H + \frac{1}{2} B_1 \dot{B}_1 - \frac{2p_L^2 B_1^2 (6 + B_1^2)}{\mathcal{D}} \mathcal{H} \\ \kappa_{15} &= -\frac{3}{2} + \frac{B_1^2}{4} + \frac{p_L^2 B_1^2 (6 + B_1^2)}{2\mathcal{D}} \\ \kappa_{16} &= \frac{4p_L^2 B_1^3 \mathcal{D}_{\alpha_0\alpha_0}}{\mathcal{D}}\end{aligned}$$

$$\begin{aligned}
\kappa_{21} &= -\frac{B_1}{3} + \frac{4p_L^2 B_1^3}{3\mathcal{D}} \\
\kappa_{22} &= 1 - \frac{4p_L^2 B_1^2}{3\mathcal{D}} \\
\kappa_{24} &= -2B_1 h + \dot{B}_1 - \frac{4p_L^2 B_1 (6 + B_1^2)}{3\mathcal{D}} \mathcal{H} \\
\kappa_{25} &= \frac{1}{3} B_1 \left( 1 + \frac{p_L^2 (6 + B_1^2)}{\mathcal{D}} \right) \\
\kappa_{26} &= 1 + \frac{8p_L^2 B_1^2 \mathcal{D}_{\alpha_0 \alpha_0}}{3\mathcal{D}} \\
\kappa_{41} &= \left( 1 + \frac{B_1^2}{6} \right) h + \left( 1 - \frac{B_1^2}{3} \right) H + \frac{1}{6} B_1 \dot{B}_1 - \frac{2p_L^2 B_1^2 (6 + B_1^2)}{3\mathcal{D}} \mathcal{H} \\
\kappa_{42} &= B_1 h - \frac{\dot{B}_1}{2} + \frac{2p_L^2 B_1 (6 + B_1^2)}{3\mathcal{D}} \mathcal{H} \\
\kappa_{44} &= 3H^2 - \left( 3 + \frac{5}{2} B_1^2 \right) h^2 - \frac{1}{2} \dot{B}_1^2 + 2B_1 (B_1 H + \dot{B}_1) h + \frac{2p_L^2 (6 + B_1^2)^2}{3\mathcal{D}} \mathcal{H}^2 \\
\kappa_{45} &= -\frac{1}{2} \left[ \left( 1 + \frac{B_1^2}{6} \right) (2H - h) + \frac{1}{3} B_1 \dot{B}_1 + \frac{p_L^2 (6 + B_1^2)^2}{3\mathcal{D}} \mathcal{H} \right] \\
\kappa_{46} &= B_1 \left( h - \frac{H}{2} \right) - \frac{\dot{B}_1}{2} - \frac{4p_L^2 B_1 (6 + B_1^2)}{3\mathcal{D}} \mathcal{H} \mathcal{D}_{\alpha_0 \alpha_0} \\
\kappa_{51} &= \frac{1}{2} \left( 1 - \frac{1}{6} B_1^2 \right) - \frac{p_L^2 B_1^2 (6 + B_1^2)}{6\mathcal{D}} \\
\kappa_{52} &= \frac{1}{6} \left( 1 + \frac{p_L^2 (6 + B_1^2)}{\mathcal{D}} \right) B_1 \\
\kappa_{54} &= \frac{1}{12} (6 + B_1^2) (2H - h) + \frac{1}{6} B_1 \dot{B}_1 + \frac{p_L^2 (6 + B_1^2)^2}{6\mathcal{D}} \mathcal{H} \\
\kappa_{55} &= -\frac{p_L^4 (6 + B_1^2)}{3p_T^2 \mathcal{D}} \mathcal{D}_{\chi\chi} \\
\kappa_{56} &= -\frac{p_L^2 B_1 (6 + B_1^2)}{3\mathcal{D}} \mathcal{D}_{\alpha_0 \alpha_0} \\
\kappa_{61} &= -\frac{8p_L^2 B_1}{3\mathcal{D}} \mathcal{D}_{\chi\chi} \\
\kappa_{62} &= 1 + \frac{8p_L^2}{3\mathcal{D}} \mathcal{D}_{\chi\chi} \\
\kappa_{63} &= \frac{p_T^2}{p_L^2} \\
\kappa_{64} &= B_1 (H - 2h) + \dot{B}_1 + \frac{8p_L^2 (6 + B_1^2)}{3B_1 \mathcal{D}} \mathcal{H} \mathcal{D}_{\chi\chi} \\
\kappa_{65} &= -\frac{2p_L^2 (6 + B_1^2)}{3B_1 \mathcal{D}} \mathcal{D}_{\chi\chi} \\
\kappa_{66} &= 1 + \frac{p_T^2}{p_L^2} - \frac{2p_T^2 (6 + B_1^2)}{3\mathcal{D}} \mathcal{D}_{\alpha_0 \alpha_0} \tag{A8}
\end{aligned}$$

Finally, we explicitly write down the right hand side of (59) involving the functions  $f_i$  where  $i = 1 \dots 6$ . Each  $f_i$  is a linear combination of the variables

$$F_i \equiv \{\dot{\Psi}, \dot{\hat{\alpha}}_1, \dot{\hat{\alpha}}, \hat{\Psi}, \hat{\alpha}_1, \hat{\alpha}, \hat{\Phi}, \hat{B}, \hat{\alpha}_0\} \tag{A9}$$

The explicit forms of the functions  $f_1, \dots, f_6$  can then be expressed as

$$f_i = \sum_j \mathcal{A}_{ij} F_j \quad (\text{A10})$$

The coefficients  $\mathcal{A}_{ij}$  depend entirely on the background quantities. They are given by

$$\begin{aligned} \mathcal{A}_{11} &= \left(2\dot{B}_1 + 3H B_1\right) B_1 - \frac{2p_L^2 B_1^4}{\mathcal{D}} \left(H + 4h + 4\frac{\dot{B}_1}{B_1} - \frac{\dot{\mathcal{D}}}{\mathcal{D}}\right) \\ \mathcal{A}_{12} &= \left(\frac{H}{2} - 4h\right) B_1 + \frac{\dot{B}_1}{2} + \frac{2p_L^2 B_1^3}{\mathcal{D}} \left(H + 4h + 2\frac{\dot{B}_1}{B_1} - \frac{\dot{\mathcal{D}}}{\mathcal{D}}\right) \\ \mathcal{A}_{13} &= 0 \\ \mathcal{A}_{14} &= p_T^2 B_1^2 - \mathcal{M}_{\Psi\Psi} + \frac{4p_L^2 B_1^4}{\mathcal{D}} \left[\frac{1}{2}(\dot{H} - 2\dot{h}) + \left(H + h - \frac{\dot{\mathcal{D}}}{2\mathcal{D}}\right) \left(H - 2h - 2\frac{\dot{B}_1}{B_1}\right) - \frac{\dot{B}_1^2}{B_1^2} - \frac{\ddot{B}_1}{B_1} + (H - 2h) \frac{\dot{B}_1}{B_1}\right] \\ \mathcal{A}_{15} &= -\frac{p_T^2}{2} B_1 - \mathcal{M}_{\Psi\alpha_1} - \frac{2p_L^2 B_1^3}{\mathcal{D}} \left[\dot{H} - 2\dot{h} + 2\left(H + h - \frac{\dot{\mathcal{D}}}{2\mathcal{D}}\right) \left(H - 2h - \frac{\dot{B}_1}{B_1}\right) + (H - 2h) \frac{\dot{B}_1}{B_1} - \frac{\ddot{B}_1}{B_1}\right] \\ \mathcal{A}_{16} &= 0 \\ \mathcal{A}_{17} &= \frac{3}{2} \left(p_T^2 - \frac{V_0}{M_p^2} + 3h^2 - 3H^2 + \frac{\dot{B}_1^2}{2}\right) + \frac{1}{4} (9m^2 - 2p_L^2 - p_T^2 + 15h^2 - 12hH) B_1^2 - 3h B_1 \dot{B}_1 \\ &\quad + \frac{2p_L^2 B_1^2 (6 + B_1^2)}{\mathcal{D}} \left[\mathcal{H} + 2\frac{B_1 \dot{B}_1}{6 + B_1^2} \mathcal{H} + 2\left(H + h - \frac{\dot{\mathcal{D}}}{2\mathcal{D}}\right) \mathcal{H}\right] \\ \mathcal{A}_{18} &= \frac{9}{2} \left(1 + \frac{B_1^2}{18}\right) h + \frac{9}{2} \left(1 - \frac{5B_1^2}{18}\right) H + \frac{1}{2} B_1 \dot{B}_1 - p_L^2 (6 + B_1^2) \left(\mathcal{H} - \frac{\dot{\mathcal{D}}}{2\mathcal{D}}\right) \frac{B_1^2}{\mathcal{D}} \\ \mathcal{A}_{19} &= \frac{3B_1}{2} \left(H - 2h + \frac{\dot{B}_1}{B_1}\right) - \frac{4p_L^2 B_1^3}{\mathcal{D}} \mathcal{D}_{\alpha_0\alpha_0} \left[2H + 2h + \frac{\dot{B}_1}{B_1} + \frac{\dot{\mathcal{D}}_{\alpha_0\alpha_0}}{\mathcal{D}_{\alpha_0\alpha_0}} - \frac{\dot{\mathcal{D}}}{\mathcal{D}}\right] \\ \\ \mathcal{A}_{21} &= \frac{1}{3} (7H - 8h) B_1 + \dot{B}_1 - \frac{4p_L^2}{3} \left(H + 4h + 4\frac{\dot{B}_1}{B_1} - \frac{\dot{\mathcal{D}}}{\mathcal{D}}\right) \frac{B_1^3}{\mathcal{D}} \\ \mathcal{A}_{22} &= -3H + \frac{4p_L^2}{3} \left(H + 4h + 2\frac{\dot{B}_1}{B_1} - \frac{\dot{\mathcal{D}}}{\mathcal{D}}\right) \frac{B_1^2}{\mathcal{D}} \\ \mathcal{A}_{23} &= 0 \\ \mathcal{A}_{24} &= \frac{p_T^2 B_1}{3} - \frac{8p_L^2 B_1^3}{3\mathcal{D}} \left[\left(2H + 2h - \frac{\dot{\mathcal{D}}}{\mathcal{D}}\right) \left(h - \frac{H}{2} + \frac{\dot{B}_1}{B_1}\right) + \dot{h} - \frac{\dot{H}}{2} - (H - 2h) \frac{\dot{B}_1}{B_1} + \frac{\dot{B}_1^2}{B_1^2} + \frac{\ddot{B}_1}{B_1}\right] \\ \mathcal{A}_{25} &= -p_T^2 - \mathcal{M}_{\alpha_1\alpha_1} - \frac{4p_L^2 B_1^2}{3\mathcal{D}} \left[\left(2H + 2h - \frac{\dot{\mathcal{D}}}{\mathcal{D}}\right) \left(H - 2h - \frac{\dot{B}_1}{B_1}\right) + \dot{H} - 2\dot{h} + (H - 2h) \frac{\dot{B}_1}{B_1} - \frac{\ddot{B}_1}{B_1}\right] \\ \mathcal{A}_{26} &= p_T^2 \\ \mathcal{A}_{27} &= \left(2m^2 - \frac{p^2}{3}\right) B_1 + \frac{4p_L^2 B_1 (6 + B_1^2)}{3\mathcal{D}} \left[2\left(H + h - \frac{\dot{\mathcal{D}}}{2\mathcal{D}} + \frac{B_1 \dot{B}_1}{6 + B_1^2}\right) \mathcal{H} + \dot{\mathcal{H}}\right] \\ \mathcal{A}_{28} &= -\left(\frac{H}{3} + \frac{7h}{3}\right) B_1 + \dot{B}_1 - \frac{2p_L^2 B_1 (6 + B_1^2)}{3\mathcal{D}} \left(\mathcal{H} - \frac{\dot{\mathcal{D}}}{2\mathcal{D}}\right) \\ \mathcal{A}_{29} &= -2(H + h) - \frac{16p_L^2 B_1^2 \mathcal{D}_{\alpha_0\alpha_0}}{3\mathcal{D}} \left[H + h - \frac{\dot{\mathcal{D}}}{2\mathcal{D}} + \frac{\dot{B}_1}{2B_1} + \frac{\dot{\mathcal{D}}_{\alpha_0\alpha_0}}{2\mathcal{D}_{\alpha_0\alpha_0}}\right] \end{aligned}$$

$$\begin{aligned}
\mathcal{A}_{31} &= \mathcal{A}_{32} = \mathcal{A}_{34} = \mathcal{A}_{37} = 0 \\
\mathcal{A}_{33} &= -3 (H - 2h) \\
\mathcal{A}_{35} &= p_L^2 \\
\mathcal{A}_{36} &= -p_L^2 - \mathcal{M}_{\alpha\alpha} \\
\mathcal{A}_{38} &= -\frac{p_L^2}{p_T^2} \left( (H - 2h) B_1 + \dot{B}_1 \right) \\
\mathcal{A}_{39} &= -2 (H - 2h)
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_{41} &= \left( \frac{2}{3}H - \frac{1}{3}h \right) B_1 \dot{B}_1 - \frac{2}{3}\dot{B}_1^2 - \frac{1}{6}B_1 \ddot{B}_1 - \frac{p_T^2}{2} - \left( m^2 - \frac{2p_L^2 + p_T^2}{6} + 5h^2 - 4hH \right) \frac{B_1^2}{2} + 2h B_1 \dot{B}_1 \\
&\quad - \left( 1 + \frac{B_1^2}{6} \right) \dot{h} - \left( 1 - \frac{B_1^2}{3} \right) \dot{H} + \frac{2p_L^2 B_1^2 (6 + B_1^2)}{3\mathcal{D}} \left[ \left( -3H + 6h + 4\frac{\dot{B}_1}{B_1} - \frac{\dot{\mathcal{D}}}{\mathcal{D}} + 2\frac{B_1 \dot{B}_1}{6 + B_1^2} \right) \mathcal{H} + \dot{\mathcal{H}} \right]
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_{42} &= \frac{1}{2} \left( m^2 - \frac{p^2}{3} + 5h^2 - 4hH - 2\dot{h} \right) B_1 - 2h \dot{B}_1 + \frac{1}{2} \ddot{B}_1 \\
&\quad - \frac{2p_L^2 B_1 (6 + B_1^2)}{3\mathcal{D}} \left[ \left( -3H + 6h + 2\frac{\dot{B}_1}{B_1} - \frac{\dot{\mathcal{D}}}{\mathcal{D}} + 2\frac{B_1 \dot{B}_1}{6 + B_1^2} \right) \mathcal{H} + \dot{\mathcal{H}} \right]
\end{aligned}$$

$$\mathcal{A}_{43} = 0$$

$$\begin{aligned}
\mathcal{A}_{44} &= (H + h) p_T^2 - \left[ \frac{2p_L^2 + p_T^2}{6} H - \frac{4p_L^2 - p_T^2}{6} h - 2(\dot{H}h + H\dot{h}) + 5h\dot{h} \right] B_1^2 - \left[ m^2 - \frac{2p_L^2 + p_T^2}{6} - 4Hh - 2\dot{h} + 5h^2 \right] B_1 \dot{B}_1 \\
&\quad - \dot{B}_1 \ddot{B}_1 + 2(\dot{B}_1^2 + B_1 \ddot{B}_1) h + \frac{2p_L^2 B_1^2 (6 + B_1^2)}{3\mathcal{D}} \left( 2h - H + 2\frac{\dot{B}_1}{B_1} \right) \left[ \left( 4h - 2H - \frac{\dot{\mathcal{D}}}{\mathcal{D}} + 2\frac{B_1 \dot{B}_1}{6 + B_1^2} \right) \mathcal{H} + \dot{\mathcal{H}} \right] \\
&\quad + \frac{2p_L^2 B_1^2 (6 + B_1^2)}{3\mathcal{D}} \mathcal{H} \left[ 2\dot{h} - \dot{H} + 2\frac{\dot{B}_1^2}{B_1^2} + 2(2h - H) \frac{\dot{B}_1}{B_1} + 2\frac{\ddot{B}_1}{B_1} \right]
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_{45} &= -h \ddot{B}_1 + \left[ \frac{1}{2} \left( m^2 - \frac{p^2}{3} \right) + \frac{5}{2} h^2 - 2hH - \dot{h} \right] \dot{B}_1 + \left[ \left( \frac{p^2}{3} - 2\dot{h} \right) H + \left( \frac{-2p_L^2 + p_T^2}{3} + 5\dot{h} - 2\dot{H} \right) h \right] B_1 \\
&\quad + \frac{2p_L^2 B_1 (6 + B_1^2)}{3\mathcal{D}} \left( 2h - H + \frac{\dot{B}_1}{B_1} \right) \left[ \left( 2H - 4h + \frac{\dot{\mathcal{D}}}{\mathcal{D}} - 2\frac{B_1 \dot{B}_1}{6 + B_1^2} \right) \mathcal{H} - \dot{\mathcal{H}} \right] \\
&\quad - \frac{2p_L^2 B_1 (6 + B_1^2)}{3\mathcal{D}} \mathcal{H} \left[ 2\dot{h} - \dot{H} + (2h - H) \frac{\dot{B}_1}{B_1} + \frac{\ddot{B}_1}{B_1} \right]
\end{aligned}$$

$$\mathcal{A}_{46} = 0$$

$$\begin{aligned}
\mathcal{A}_{47} &= -6H \dot{H} + \dot{B}_1 \ddot{B}_1 - 2h \left( 2H B_1 \dot{B}_1 + B_1^2 \dot{H} + \dot{B}_1^2 + B_1 \ddot{B}_1 \right) - 2B_1 \left( B_1 H + \dot{B}_1 \right) \dot{h} + (6 + 5B_1^2) h \dot{h} + 5B_1 \dot{B}_1 h^2 \\
&\quad + \frac{4p_L^2 (6 + B_1^2)^2}{3\mathcal{D}} \mathcal{H} \left[ \left( H - 2h + \frac{\dot{\mathcal{D}}}{2\mathcal{D}} - 2\frac{B_1 \dot{B}_1}{6 + B_1^2} \right) \mathcal{H} - \dot{\mathcal{H}} \right]
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_{48} &= \frac{1}{6} \left[ (2H - h) B_1 \dot{B}_1 + \dot{B}_1^2 + B_1 \ddot{B}_1 \right] - \frac{1}{12} (6 + B_1^2) (\dot{h} - 2\dot{H}) - \frac{p_L^2 (6 + B_1^2)^2}{3\mathcal{D}} (H - 2h) \mathcal{H} \\
&\quad + \frac{p_L^2 (6 + B_1^2)^2}{6\mathcal{D}} \left[ \left( 4\frac{B_1 \dot{B}_1}{6 + B_1^2} - \frac{\dot{\mathcal{D}}}{\mathcal{D}} \right) \mathcal{H} + \dot{\mathcal{H}} \right]
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_{49} &= \frac{1}{2} \ddot{B}_1 + \frac{1}{2} (H - 2h) \dot{B}_1 + \frac{1}{2} (\dot{H} - 2\dot{h}) B_1 \\
&\quad - \frac{8p_L^2 B_1 (6 + B_1^2)}{3\mathcal{D}} \mathcal{D}_{\alpha_0 \alpha_0} \left[ \mathcal{H} \left( H - 2h + \frac{\dot{\mathcal{D}}}{2\mathcal{D}} - \frac{\dot{B}_1}{2B_1} - \frac{\dot{\mathcal{D}}_{\alpha_0 \alpha_0}}{2\mathcal{D}_{\alpha_0 \alpha_0}} \right) - \frac{1}{2} \dot{\mathcal{H}} - \frac{B_1 \dot{B}_1}{6 + B_1^2} \mathcal{H} \right]
\end{aligned}$$



$$\begin{aligned}
\mathcal{A}_{51} &= -\frac{1}{6} B_1 \left( B_1 H - 3 \dot{B}_1 \right) + \frac{1}{2} \left( 3 + \frac{B_1^2}{6} \right) h + \frac{p_L^2 B_1^2 (6 + B_1^2)}{3\mathcal{D}} \left[ 3h - \frac{3}{2} H + \frac{B_1 \dot{B}_1}{6 + B_1^2} + 2 \frac{\dot{B}_1}{B_1} - \frac{\dot{\mathcal{D}}}{2\mathcal{D}} \right] \\
\mathcal{A}_{52} &= -\frac{B_1}{6} (2H - 7h) - \frac{5}{6} \dot{B}_1 + \frac{p_L^2 B_1 (6 + B_1^2)}{3\mathcal{D}} \left[ \frac{3}{2} H - 3h - \frac{B_1 \dot{B}_1}{6 + B_1^2} - \frac{\dot{B}_1}{B_1} + \frac{\dot{\mathcal{D}}}{2\mathcal{D}} \right] \\
\mathcal{A}_{53} &= B_1 \left( \frac{H}{2} - h \right) + \frac{\dot{B}_1}{2} \\
\mathcal{A}_{54} &= \frac{1}{6} h B_1 \dot{B}_1 - \frac{1}{6} \left( B_1 H - 2 \dot{B}_1 \right) \dot{B}_1 - \frac{1}{6} B_1 \left( H \dot{B}_1 + B_1 \dot{H} - 2 \ddot{B}_1 \right) + \frac{1}{2} \left( 3 + \frac{1}{6} B_1^2 \right) \dot{h} \\
&\quad + \frac{p_L^2 B_1^2 (6 + B_1^2)}{3\mathcal{D}} \left( 2h - H + 2 \frac{\dot{B}_1}{B_1} \right) \left( 2h - H + \frac{B_1 \dot{B}_1}{6 + B_1^2} - \frac{\dot{\mathcal{D}}}{2\mathcal{D}} \right) \\
&\quad + \frac{p_L^2 B_1^2 (6 + B_1^2)}{6\mathcal{D}} \left[ 2\dot{h} - \dot{H} + 2(2h - H) \frac{\dot{B}_1}{B_1} + 2 \frac{\dot{B}_1^2}{B_1^2} + 2 \frac{\ddot{B}_1}{B_1} \right] \\
\mathcal{A}_{55} &= \frac{1}{6} (7h - 2H) \dot{B}_1 + \frac{1}{6} (7\dot{h} - 2\dot{H}) B_1 - \frac{2}{3} \ddot{B}_1 - \frac{p_L^2 B_1 (6 + B_1^2)}{3\mathcal{D}} \left( 2h - H + \frac{B_1 \dot{B}_1}{6 + B_1^2} - \frac{\dot{\mathcal{D}}}{2\mathcal{D}} \right) \left( 2h - H + \frac{\dot{B}_1}{B_1} \right) \\
&\quad - \frac{p_L^2 B_1 (6 + B_1^2)}{6\mathcal{D}} \left[ 2\dot{h} - \dot{H} + (2h - H) \frac{\dot{B}_1}{B_1} + \frac{\ddot{B}_1}{B_1} \right] \\
\mathcal{A}_{56} &= \left( \frac{\dot{H}}{2} - \dot{h} \right) B_1 + \left( \frac{H}{2} - h \right) \dot{B}_1 + \frac{\ddot{B}_1}{2} \\
\mathcal{A}_{57} &= \frac{1}{6} \left[ B_1 (h - 2H) - \dot{B}_1 \right] \dot{B}_1 - \frac{1}{6} B_1 \ddot{B}_1 + \frac{1}{12} (6 + B_1^2) (\dot{h} - 2\dot{H}) + \frac{p_L^2 (6 + B_1^2)^2}{3\mathcal{D}} \left[ H - 2h - 2 \frac{B_1 \dot{B}_1}{6 + B_1^2} + \frac{\dot{\mathcal{D}}}{2\mathcal{D}} - \frac{\dot{\mathcal{H}}}{2\mathcal{H}} \right] \mathcal{H} \\
\mathcal{A}_{58} &= \frac{p_L^4 (6 + B_1^2) \mathcal{D}_{xx}}{3p_T^2 \mathcal{D}^2} \left[ 12h (\mathcal{D} + 4p_L^2 \mathcal{D}_{xx}) - 16p_L^2 \frac{B_1 \dot{B}_1}{6 + B_1^2} \mathcal{D}_{xx} + (6 + B_1^2) p_T^2 \frac{\dot{\mathcal{D}}_{xx}}{\mathcal{D}_{xx}} \right] \\
\mathcal{A}_{59} &= \frac{p_L^2 B_1 \mathcal{D}_{\alpha_0 \alpha_0}}{3\mathcal{D}^2} \left[ p_T^2 B_1^3 \dot{B}_1 + 12 (3p_T^2 - 4p_L^2 \mathcal{D}_{xx}) \frac{\dot{B}_1}{B_1} + 12 (p_T^2 - 2p_L^2 \mathcal{D}_{xx}) B_1 \dot{B}_1 + p_T^2 \left( 6h + \frac{\dot{\mathcal{D}}_{\alpha_0 \alpha_0}}{\mathcal{D}_{\alpha_0 \alpha_0}} \right) B_1^4 \right. \\
&\quad \left. + 12 \left( 18p_T^2 h + 4p_L^2 \dot{\mathcal{D}}_{xx} + (3p_T^2 - 4p_L^2 \mathcal{D}_{xx}) \frac{\dot{\mathcal{D}}_{\alpha_0 \alpha_0}}{\mathcal{D}_{\alpha_0 \alpha_0}} \right) + 4 \left( 18p_T^2 h + 2p_L^2 \dot{\mathcal{D}}_{xx} + (3p_T^2 - 2p_L^2 \mathcal{D}_{xx}) \frac{\dot{\mathcal{D}}_{\alpha_0 \alpha_0}}{\mathcal{D}_{\alpha_0 \alpha_0}} \right) B_1^2 \right] \\
\mathcal{A}_{61} &= -\frac{4p_T^4}{3\mathcal{D}^2} \left\{ \frac{3}{4} (2h - H) B_1^5 - \frac{3}{4} \dot{B}_1 B_1^4 - 9 \left( 3 - 4 \frac{p_L^2}{p_T^2} \mathcal{D}_{xx} \right) \dot{B}_1 - \left( 9 - 10 \frac{p_L^2}{p_T^2} \mathcal{D}_{xx} \right) B_1^2 \dot{B}_1 + 2h \left( 9 - 20 \frac{p_L^2}{p_T^2} \mathcal{D}_{xx} \right) B_1^3 \right. \\
&\quad + \left[ -9 (3 + B_1^2) + 14 (6 + B_1^2) \frac{p_L^2}{p_T^2} \mathcal{D}_{xx} - 64 \frac{p_L^4}{p_T^4} \mathcal{D}_{xx}^2 \right] H B_1 + 2 \left[ 27 - 120 \frac{p_L^2}{p_T^2} \mathcal{D}_{xx} + 64 \frac{p_L^4}{p_T^4} \mathcal{D}_{xx}^2 \right] h B_1 \\
&\quad \left. - 2 (6 + B_1^2) B_1 \frac{p_L^2}{p_T^2} \dot{\mathcal{D}}_{xx} \right\} \\
\mathcal{A}_{62} &= -\frac{4p_T^4}{3B_1 \mathcal{D}^2} \left\{ \frac{3}{4} (H - 2h) B_1^5 + 4 \frac{p_L^2}{p_T^2} \left( 3 - 4 \frac{p_L^2}{p_T^2} \mathcal{D}_{xx} \right) \mathcal{D}_{xx} \dot{B}_1 - 2 \frac{p_L^2}{p_T^2} \mathcal{D}_{xx} B_1^2 \dot{B}_1 \right. \\
&\quad - 2 \left[ 9 (3 + B_1^2) - 20 (6 + B_1^2) \frac{p_L^2}{p_T^2} \mathcal{D}_{xx} + 64 \frac{p_L^4}{p_T^4} \mathcal{D}_{xx}^2 \right] h B_1 + \left[ 9 (3 + B_1^2) - 14 (6 + B_1^2) \frac{p_L^2}{p_T^2} \mathcal{D}_{xx} + 64 \frac{p_L^4}{p_T^4} \mathcal{D}_{xx}^2 \right] H B_1 \\
&\quad \left. + 2 (6 + B_1^2) \frac{p_L^2}{p_T^2} B_1 \dot{\mathcal{D}}_{xx} \right\} \\
\mathcal{A}_{63} &= -\frac{p_T^2}{p_L^2} (H - 8h)
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_{64} &= \ddot{B}_1 + (H - 2h) \dot{B}_1 + (\dot{H} - 2\dot{h}) B_1 - \frac{16p_L^2 B_1}{3\mathcal{D}} \mathcal{D}_{\text{xx}} \left( 2h - H + 2 \frac{\dot{B}_1}{B_1} \right) \left( H - 2h + \frac{\dot{B}_1}{2B_1} - \frac{\dot{\mathcal{D}}_{\text{xx}}}{2\mathcal{D}_{\text{xx}}} + \frac{\dot{\mathcal{D}}}{2\mathcal{D}} \right) \\
&\quad + \frac{8p_L^2 B_1}{3\mathcal{D}} \mathcal{D}_{\text{xx}} \left( 2 \frac{\dot{B}_1^2}{B_1^2} + 2\dot{h} - \dot{H} + 2(2h - H) \frac{\dot{B}_1}{B_1} + 2 \frac{\ddot{B}_1}{B_1} \right) \\
\mathcal{A}_{65} &= 2\dot{h} - \dot{H} + \frac{16p_L^2}{3\mathcal{D}} \mathcal{D}_{\text{xx}} \left( 2h - H + \frac{\dot{B}_1}{B_1} \right) \left( H - 2h + \frac{\dot{B}_1}{2B_1} - \frac{\dot{\mathcal{D}}_{\text{xx}}}{2\mathcal{D}_{\text{xx}}} + \frac{\dot{\mathcal{D}}}{2\mathcal{D}} \right) \\
&\quad - \frac{8p_L^2}{3\mathcal{D}} \mathcal{D}_{\text{xx}} \left( 2\dot{h} - \dot{H} + (2h - H) \frac{\dot{B}_1}{B_1} + \frac{\ddot{B}_1}{B_1} \right) \\
\mathcal{A}_{66} &= -\frac{p_T^2}{p_L^2} \left( \dot{H} - 2\dot{h} - 6hH + 12h^2 \right) \\
\mathcal{A}_{67} &= -\ddot{B}_1 + (2h - H) \dot{B}_1 + (2\dot{h} - \dot{H}) B_1 + \frac{16p_L^2 (6 + B_1^2)}{3B_1 \mathcal{D}} \mathcal{D}_{\text{xx}} \mathcal{H} \left[ H - 2h - \frac{B_1 \dot{B}_1}{6 + B_1^2} + \frac{\dot{B}_1}{2B_1} - \frac{\dot{\mathcal{D}}_{\text{xx}}}{2\mathcal{D}_{\text{xx}}} + \frac{\dot{\mathcal{D}}}{2\mathcal{D}} - \frac{\dot{\mathcal{H}}}{2\mathcal{H}} \right] \\
\mathcal{A}_{68} &= -\frac{2p_L^2 p_T^2}{3B_1^2 \mathcal{D}^2} \mathcal{D}_{\text{xx}} \left\{ \left[ B_1^4 + 4 \left( 3 + 2 \frac{p_L^2}{p_T^2} \mathcal{D}_{\text{xx}} \right) B_1^2 + 12 \left( 3 - 4 \frac{p_L^2}{p_T^2} \mathcal{D}_{\text{xx}} \right) \right] \dot{B}_1 - B_1 (6 + B_1^2)^2 \left( 6h + \frac{\dot{\mathcal{D}}}{\mathcal{D}} \right) \right\} \\
\mathcal{A}_{69} &= \frac{2p_T^6}{3p_L^2 \mathcal{D}^2} \left\{ 3 \left[ 3B_1^4 + 36 (3 + B_1^2) - 16 \frac{p_L^2}{p_T^2} (6 + B_1^2) \mathcal{D}_{\text{xx}} \left( 3 - \frac{p_L^2}{p_T^2} \mathcal{D}_{\alpha_0 \alpha_0} \right) + 192 \frac{p_L^4}{p_T^4} \mathcal{D}_{\text{xx}}^2 \right] h \right. \\
&\quad + \frac{p_L^2}{p_T^2} \mathcal{D}_{\alpha_0 \alpha_0} \left[ -16 \frac{p_L^2}{p_T^2} \mathcal{D}_{\text{xx}} B_1 \dot{B}_1 + \left( 8 \frac{p_L^2}{p_T^2} \mathcal{D}_{\text{xx}} + B_1^2 \right) \frac{\dot{\mathcal{D}}_{\alpha_0 \alpha_0}}{\mathcal{D}_{\alpha_0 \alpha_0}} B_1^2 + 4(3 + B_1^2) \left( 3 - 4 \frac{p_L^2}{p_T^2} \mathcal{D}_{\text{xx}} \right) \frac{\dot{\mathcal{D}}_{\alpha_0 \alpha_0}}{\mathcal{D}_{\alpha_0 \alpha_0}} \right. \\
&\quad \left. \left. + 8(6 + B_1^2) \frac{p_L^2}{p_T^2} \dot{\mathcal{D}}_{\text{xx}} \right] \right\} \tag{A11}
\end{aligned}$$

Thus, we have the full form of the equation (59), which we integrate numerically.

## APPENDIX B: EARLY TIME CANONICAL ACTION AND INITIAL CONDITIONS

We discuss here how we set the initial conditions for the perturbations entering in the linearized system (59). As in the standard case [37], the initial conditions follow from the quantization of the quadratic action for the dynamical modes. As we show in Section III, the quadratic action - formally written in eq. (24) - is obtained by integrating the nondynamical fields out of the quadratic action for the perturbations - formally written in eq. (19).

The quadratic action of the 2d scalar perturbations of the model (46) is given in eq. (63). The corresponding action for the dynamical modes is extremely lengthy, and we do not explicitly write it here. Fortunately, to set the initial conditions for the perturbations we only need the leading terms in an early time expansion of this action. We are interested in the phenomenologically relevant case of moderate anisotropy ( $B_1 \ll 1$ ), for which  $H_a \simeq H_b$  are nearly constant. Therefore, the two scale factors  $a$  and  $b$  grow nearly exponentially with time, and  $p/H$  (where  $p$  is either the longitudinal or the transverse component, and  $H$  is either  $H_a$  or  $H_b$ ) is exponentially large in the asymptotic past. Therefore, the early time expansion of the action coincides with the sub-horizon  $p/H \gg 1$  expansion, exactly as in the standard case.

Specifically, we first compute the exact matrices  $K$ ,  $\Lambda$ ,  $\Omega^2$  that form the action for the dynamical modes (cf. eq. (23)), and then expand them for  $p/H \gg 1$ . Since the resulting expressions are still quite involved, we further expand them for  $B_1 \ll 1$ . This procedure is legitimate provided that

$$\frac{H}{p} \ll B_1 \ll 1 \tag{B1}$$

which, as we remarked, is always true in the case of small anisotropy, and for sufficiently early times. At leading order, we obtain the expression

$$S_{\text{can}} \simeq \frac{1}{2} \int d^3k dt \left\{ |\dot{H}_+|^2 - p^2 |H_+|^2 + |\dot{\Delta}_+|^2 - p^2 |\Delta_+|^2 + |\dot{\Delta}_-|^2 - p^2 |\Delta_-|^2 \right\} \tag{B2}$$

where the modes  $H_+$ ,  $\Delta_+$ , and  $\Delta_-$  are related to the original perturbations by

$$\begin{aligned}
\hat{\Psi} &= \frac{1}{\sqrt{ab^2}} \left\{ \left( \frac{\sqrt{2}p^2}{p_T^2} - \frac{3H_0^4 (p_L^2 + 10p_T^2) - m^2 p_T^2 (6H_0^2 - m^2)}{18\sqrt{2}H_0^4 p_T^2} B_1^2 \right) H_+ - \left( 1 - \frac{m^2}{3H_0^2} \right) \frac{p}{p_T} B_1 \Delta_+ \right. \\
&\quad \left. + \frac{3H_0^2 (4p_L^4 + 7p_L^2 p_T^2 + p_T^4) - m^2 p_T^2 p^2}{6\sqrt{6}H_0^2 p_T^4} B_1^2 \Delta_- \right\} \\
\hat{\alpha} &= \frac{1}{\sqrt{ab^2}} \left\{ \left( \frac{p}{2p_T} - \frac{3H_0^4 (p_L^2 - 5p_T^2) (p_L^2 + 3p_T^2) + 4H_0^2 m^2 p_T^2 (p_L^2 + 7p_T^2) - 4m^4 p_T^4}{144H_0^4 p_T^3 p} B_1^2 \right) \Delta_+ \right. \\
&\quad \left. + \left( \frac{\sqrt{6}}{B_1} - \frac{p_T^2 - 3p_L^2}{8\sqrt{6}p_T^2} B_1 \right) \Delta_- \right\} \\
\hat{\alpha}_1 &= \frac{1}{\sqrt{ab^2}} \left\{ -\frac{p}{\sqrt{2}p_T} \Delta_+ + \left( \frac{\sqrt{6}}{B_1} - \frac{3H_0^2 (p_L^2 - 3p_T^2) + 8m^2 p_T^2}{24\sqrt{6}H_0^2 p_T^2} B_1 \right) \Delta_- \right\} \tag{B3}
\end{aligned}$$

The modes  $H_+$ ,  $\Delta_+$ , and  $\Delta_-$  are the canonical variables of the system (they are the analogous of the Mukhanov-Sasaki [38] variable  $v$  in the standard case of scalar field isotropic inflation). As in the standard case, their early times frequency is given by the momentum  $p$ , up to  $O(H/p)$  subdominant corrections. Since the momentum changes adiabatically at early times ( $\dot{p}/p^2 = O(H/p)$ ), we can set the initial conditions for the canonical modes according to the adiabatic vacuum prescription, precisely as done in the standard case [37]:

$$H_{+,in} = \Delta_{+,in} = \Delta_{-,in} = \frac{1}{\sqrt{2p}} \quad , \quad \dot{H}_{+,in} = \dot{\Delta}_{+,in} = \dot{\Delta}_{-,in} = -i\sqrt{\frac{p}{2}} \tag{B4}$$

which are  $O(H/p)$  accurate. From eqs. (B3) and (B4) we thus obtain the initial conditions for  $\{\hat{\Psi}, \hat{\alpha}, \hat{\alpha}_1\}$  and their time derivatives. Finally, the first, third, and fourth of eqs. (56) provide the initial conditions for the nondynamical modes  $\hat{\Phi}, \hat{\chi}$ , and  $\hat{\alpha}_0$ . In this way, we have the initial conditions for all the modes of the system (59).

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