Partial Pole Placement with Controller Optimization

Subashish Datta, Debraj Chakraborty and Balarko Chaudhuri

***Abstract*—An arbitrary subset $(n-m)$ of the (n) closed loop eigenvalues of an n^{th} order continuous time single input linear time invariant (LTI) system is to be placed using full state feedback, at pre-specified locations in the complex plane. The remaining m closed loop eigenvalues can be placed anywhere inside a pre-defined region in the complex plane. This region constraint on the unspecified poles is translated into an LMI constraint on the feedback gains through a convex inner approximation of the polynomial stability region. The closed loop locations for these m eigenvalues are optimized to obtain a minimum norm feedback gain vector. This reduces the controller effort leading to less expensive actuators required to be installed in the control system. The proposed algorithm is illustrated on a linearized model of a 4-machine, 2-area power system example.**

***Index Terms*—Linear systems, LMIs, Convex optimization, Control effort, Pole placement, Power systems.**

I. INTRODUCTION

All the closed loop poles of a controllable single input linear time invariant (LTI) system can be assigned arbitrary locations in the complex plane using full state feedback. However, in many applications, the control engineer is concerned with only a subset of the open loop poles (possibly because of their physical significance, instability, low damping and/or associated oscillations), and would like to move these undesired poles (henceforth called critical poles) to precise pre-specified locations inside some stability region. In these applications, typically, the remaining (non-critical) open loop poles are already stable and well damped and there are no obvious desired closed loop locations for them. It is considered enough if these well-behaved open loop poles do not lose their desirable properties in closed loop, or in other words, if these non-critical open loop poles continue to lie within some desired region of the complex plane in the closed loop. In such applications, since only a subset of the closed loop poles are specified, the extra degrees of freedom associated with the unspecified non-critical poles can be utilized to minimize the norm of feedback gain vector in full state feedback control systems.

Consider the problem of power oscillation damping controller design where low frequency electro-mechanical oscillations (0.1-3 Hz) are damped through expensive actuators (e.g. see [1]). State feedback approach has been used in the past to damp oscillations following large and small disturbances in power systems where the oscillatory behavior is dominated by a few poorly damped electromechanical modes with very little to zero influence from the other modes [2]. Hence, it is important to carefully place only those critical poles to ensure desired performance following disturbances. There is no need to worry about the remaining non-critical poles as long as their damping/settling times do not exceed those in open loop. In fact, it often turns out to be counter productive to relocate the non-critical poles or even force them to their open-loop positions. Due to the very nature of the non-critical modes, higher control efforts are required unless they are left alone to take their natural course. This results in an overall increase in the norm of the feedback gain vector and hence, costlier actuators.

In consideration of such applications, we formulate the optimization problem which will minimize the norm of feedback gain vector ensuring (i) the critical poles are moved to desired (precise) closed loop locations, and (ii) the non-critical poles remain stable in closed loop.

Additionally, it is often required that all non-critical poles should have a minimum settling time and/or damping ratio which implies that they should be located within some specific region of the complex plane in the closed loop. These requirements on the closed loop non-critical poles are translated into constraints in the coefficient space of the characteristic polynomial through an inner convex approximation of the polynomial stability region [3], [4]. In turn, these constraints define a linear matrix inequality (LMI) on the feedback controller gains. Thus the optimization problem mentioned above is solved with two types of constraints: (i) linear equality constraints arising out of the precise placement requirement of the critical closed loop poles, and (ii) LMI constraint arising out of the regional placement requirement of the closed loop non-critical poles.

Minimization of norm of the feedback gain vector with partial pole placement was introduced in [5], [6], the results of which are improved in this work through less conservative LMI stability region estimates in the polynomial coefficients space. Earlier work on minimum norm controller was reported in [7] where the Sylvester equation was used to simultaneously well condition the eigenvector matrix.

Various researchers (e.g. [7], [8], [9], [10] and [11]) have focused on finding numerically stable and efficient algorithms for multi-input multi-output (full) pole placement by minimizing the condition number of a related eigenvector matrix. Pole placement within arbitrary pre-specified subsets of the complex plane was studied by [12], [13] and references therein. It is well known that the set of polynomial coefficients corresponding to stable root locations, might not be convex (see [14], [15]). To overcome this non-convexity, in a series of papers (see [14], [3], [4] and the references therein) ellipsoidal inner approximations and LMI inner approximations for the polynomial coefficient stability region have been derived. Using a similar approach, [16] and [17] have designed fixed order stabilizing controllers for SISO polytopic plants with regional pole placement.

The remaining paper is organized as follows. In Section II-A, the problem is formulated after introducing some preliminary notations. Following [3] and [4], a procedure to find a stable convex LMI region in the polynomial coefficients

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Subashish Datta and Debraj Chakraborty are with the Department of Electrical Engineering, Indian Institute of Technology Bombay, Powai, Mumbai, India 400076. subashish@iitb.ac.in, dc@ee.iitb.ac.in

Balarko Chaudhuri is with the Department of Electronics and Electrical Engineering, Imperial College, London. b.chaudhuri@imperial.ac.uk

space is included in Section II-B. In Section III, an equivalent semidefinite program (SDP) is presented to obtain a minimum norm feedback gain vector. Numerical examples demonstrating the application of the proposed theory on a linearized model of a 4-machine, 2-area power system [18] are included in Section IV.

II. Preliminaries

A. Problem Formulation

Let us consider a continuous time LTI single-input system, with full state feedback control, defined by the following state space equations

$$\dot{x} = Ax + bu \; ; \; u = -k^T x \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and $k := \begin{bmatrix} k_1 & k_2 \cdots k_n \end{bmatrix}^T \in \mathbb{R}^n$. Assume that the pair (A,b) is controllable; then *all* the eigenvalues of the closed loop system

$$\dot{x} = (A - bk^T)x \tag{2}$$

can be placed at any arbitrary locations of the complex plane $\mathbb{C}$ through a unique choice of k.

However, in the applications of our interest, only a few critical closed loop eigenvalues are specified and the non-critical eigenvalues are allowed to assume any value in (or in a pre-specified subset of) the stable region of the complex plane. Let us denote $\{\mu_1,\mu_2,\ldots,\mu_m,\mu_{m+1},\ldots,\mu_n\}$, $(m \le n)$ as the n eigenvalues of A. Of these, $\{\mu_1,\mu_2,\ldots,\mu_m\}$ are non-critical and are not associated with any desired closed loop location whereas the remaining $(n-m)$ eigenvalues are critical and are required to be placed at $\{-\lambda_1,-\lambda_2,\ldots,-\lambda_{n-m}\}$ in closed loop. We will assume that the m non-critical eigenvalues of $(A-bk^T)$ are required to be located in some stable region $\mathbb{S}$ of the complex plane. Following [3], we will define $\mathbb{S}$ as follows:

$$\mathbb{S} = \left\{ s \in \mathbb{C} : \begin{bmatrix} 1 & s^* \end{bmatrix} \underbrace{\begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}}_{S} \begin{bmatrix} 1 \\ s \end{bmatrix} < 0 \right\} \tag{3}$$

where s^* denotes the complex conjugate of s and $S \in \mathbb{R}^{2\times 2}$. It has been shown that this region $\mathbb{S}$ can be used to represent some common stability regions in the complex plane (like arbitrary half planes and discs [3]), while being particularly useful for characterization of polynomial stability regions. The optimization problem can now be posed as follows:

Problem 1: Find $\inf \|k\|_2$ such that the eigenvalues of $(A-bk^T)$ have the following properties:

1) $(n-m)$ out of the total n eigenvalues are placed at $\{-\lambda_1,-\lambda_2,\ldots,-\lambda_{n-m}\}$.
2) remaining m eigenvalues are placed anywhere in $\mathbb{S}$.

Denote the unspecified closed loop poles of the system as $\{-p_1,-p_2,\ldots,-p_m\}$. Hence the characteristic equation of the closed loop system will be

$$\sigma(s) = \underbrace{\left[\prod_{j=1}^{m}(s+p_j)\right]}_{\alpha(s)} \underbrace{\left[\prod_{i=1}^{n-m}(s+\lambda_i)\right]}_{\beta(s)} \tag{4}$$

where $\alpha(s) := s^m + \alpha_{m-1}s^{m-1} + \ldots + \alpha_1 s + \alpha_0$ and $\beta(s) := s^{n-m} + \beta_{n-m-1}s^{n-m-1} + \ldots + \beta_1 s + \beta_0$. In (4), $\alpha(s)$ is a monic polynomial of unknown coefficients while $\beta(s)$ is a monic polynomial of known coefficients (completely defined from the problem specifications). The only requirement on $\alpha(s)$ is that the roots should be located in a pre-specified region $\mathbb{S} \subset \mathbb{C}$ defined in (3). Next, denote the set of all m^{th} degree monic polynomials with real coefficients as $\mathbb{R}[s]$ and define the set $C_s := \{\alpha(s) \in \mathbb{R}[s] : \text{ roots of } \alpha(s) \in \mathbb{S}\}$. Then, Problem 1 can be restated as follows:

Problem 2: Find $\inf \|k\|_2$, such that $(A-bk^T)$ has the following properties:

1) $(n-m)$ out of the total n eigenvalues are placed at $\{-\lambda_1,-\lambda_2,\ldots,-\lambda_{n-m}\}$.
2) the polynomial $\alpha(s) \in C_s$.

Our main objective is to convert Problem 2 into a semidefinite program (SDP). For this purpose, we first need to express constraints (1) and (2) above in terms of the problem unknowns i.e. $k_1, k_2, \ldots, k_n$, which is accomplished in Section III. It will be shown that constraint (1) is linear (and hence convex) in the unknowns. However, the set $C_s \subset \mathbb{R}[s]$ in Problem 2 above is not a convex set for $m \ge 3$ (see [15], [14]) and hence the optimization implied in Problem 2 is not convex for $m \ge 3$. To overcome this difficulty, we replace C_s with an inner convex approximation of C_s. For this purpose, we briefly discuss a result from [4] next.

B. LMI stability region in the polynomial coefficient space

Define $\widehat{\alpha}(s)$ as any polynomial in C_s and the coefficient vectors corresponding to $\widehat{\alpha}(s)$ and $\alpha(s)$ (defined in (4)) as $\widehat{\alpha} := [\widehat{\alpha}_0 \; \widehat{\alpha}_1 \ldots \widehat{\alpha}_{m-1}]^T \in \mathbb{R}^m$ and $\alpha := [\alpha_0 \; \alpha_1 \ldots \alpha_{m-1}]^T \in \mathbb{R}^m$ respectively. Further, let $\alpha_e := \begin{bmatrix} \alpha^T & 1 \end{bmatrix}^T \in \mathbb{R}^{m+1}$ and $\widehat{\alpha}_e := \begin{bmatrix} \widehat{\alpha}^T & 1 \end{bmatrix}^T \in \mathbb{R}^{m+1}$. Then, for a fixed $\widehat{\alpha}(s) \in C_s$ define the following set:

$$\mathbb{S}_{LMI} := \{\alpha(s) \in \mathbb{R}[s] : \alpha_e \widehat{\alpha}_e^T + \widehat{\alpha}_e \alpha_e^T - \Pi^T (S \otimes P)\Pi \ge 0 \text{ for some } P = P^T \in \mathbb{R}^{m\times m}\} \tag{5}$$

where $\otimes$ refers to the Kronecker product, ≥ 0 implies a positive semidefinite matrix, S as defined in (3), and $\Pi \in \mathbb{R}^{2m\times(m+1)}$ denotes a projection matrix given by

$$\Pi = \begin{bmatrix} 1 & & & 0 & \cdots & 0 \\ & \ddots & & 1 & & \\ & & 1 & & \ddots & \\ 0 & \cdots & 0 & & & 1 \end{bmatrix}^T_{(m+1)\times 2m}$$

It was shown in [3, Theorem 1] that for any given stable polynomial $\widehat{\alpha}(s) \in C_s$, the polynomial $\alpha(s) \in C_s$ if there exist a symmetric matrix $P \in \mathbb{R}^{m\times m}$ satisfying the matrix inequality $\alpha_e \widehat{\alpha}_e^T + \widehat{\alpha}_e \alpha_e^T - \Pi^T (S \otimes P)\Pi \ge 0$. For every fixed $\widehat{\alpha}(s)$, this result characterizes a subset of the stable polynomials and hence the set $\mathbb{S}_{LMI} \subseteq C_s$. Moreover the inequality in (5) is linear in the unknowns α_e and P, which makes this characterization convenient for convexifying Problem 2. This is achieved by replacing C_s with $\mathbb{S}_{LMI}$ in Problem 2.

However to compute $\mathbb{S}_{LMI}$ explicitly we still need *a priori* a polynomial $\widehat{\alpha}(s) \in C_s$. This is referred to as the “central polynomial” in [3] and [4] where various domain dependent heuristics are provided for design choices for $\widehat{\alpha}(s)$. For our choice of $\widehat{\alpha}(s)$, we propose to use the polynomial formed out of the open loop non-critical poles as follows:

$$\widehat{\alpha}(s) = \left[\prod_{j=1}^{m}(s-\mu_j)\right] \tag{6}$$

Usually, the stable and adequately damped open loop eigenvalues are the ones classified as non-critical. Hence in most practical scenarios, $\mu_1, \mu_2, \ldots, \mu_m \in \mathbb{S}$ and hence $\widehat{\alpha}(s) \in C_s$. However, there are some situations (see Example 2 below) where all the m open loop non-critical poles (though stable and adequately damped) do not belong to the chosen stability region $\mathbb{S}$. As illustrated in Example 2, this might happen due to the limitations on the shapes of the stability regions constructible using (3). In such situations, one would have to heuristically choose the required number of poles, corresponding to the μ_i's outside $\mathbb{S}$, from the specified stability region. Specific design choices for such a case are discussed in Example 2.

Now, since $\mathbb{S}_{LMI} \subseteq C_s$ we can pose the following problem, which upper bounds the solution of Problem 2. It will be shown that Problem 3 is convex in $(k_1, k_2, \ldots, k_n)$.

Problem 3: Find $\inf \|k\|_2$ such that $(A - bk^T)$ has the following properties:

1) $(n-m)$ out of the total n eigenvalues are placed at $\{-\lambda_1, -\lambda_2, \ldots, -\lambda_{n-m}\}$.
2) the polynomial $\alpha(s) \in \mathbb{S}_{LMI}$.

III. Main Results

We show that Problem 3 can be formulated as an SDP. Let $a(s)$ be the open loop characteristic polynomial of (1) and define $a := \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1}\end{bmatrix}^T$ as its associated coefficient vector. Similarly define $\sigma := \begin{bmatrix}\sigma_0 & \sigma_1 & \cdots & \sigma_{n-1}\end{bmatrix}^T$ as the coefficient vector corresponding to the characteristic polynomial of system (2). Further define the controllability matrix $\mathscr{C} := \begin{bmatrix} b & Ab & A^2b & \cdots & A^{n-1}b\end{bmatrix}$ and

$$\mathscr{A} := \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

If the system is controllable, the closed loop eigenvalues (of $A - bk^T$) can be placed at any arbitrary locations in $\mathbb{C}$ and, the corresponding unique feedback gain vector k can be calculated from the following equation: $\mathscr{A}\mathscr{C}^T k + a = \sigma$ [19]. If we define $\bar{k} = \mathscr{A}\mathscr{C}^T k$ where $\bar{k} = [\bar{k}_1\ \bar{k}_2 \ldots \bar{k}_n]^T$, it follows that each $\bar{k}_i$ $(i = 1, \ldots, n)$ is a linear combination of $k_1, \ldots, k_n$ and hence σ can be written as

$$\sigma_j = \bar{k}_{j+1} + a_j \quad \text{for} \quad j = 0, 1, \ldots, n-1 \tag{7}$$

Recalling the expression for the required closed loop characteristic polynomial (4), the coefficients could be written as follows:

$$\begin{aligned} \sigma_{n-1} &= \beta_{n-m-1} + \alpha_{m-1} \\ \sigma_{n-2} &= \beta_{n-m-2} + \beta_{n-m-1}\alpha_{m-1} + \alpha_{m-2} \\ &\vdots \\ \sigma_2 &= \beta_0\alpha_2 + \beta_1\alpha_1 + \beta_2\alpha_0 \\ \sigma_1 &= \beta_0\alpha_1 + \beta_1\alpha_0 \\ \sigma_0 &= \beta_0\alpha_0 \end{aligned} \tag{8}$$

Since $(-\lambda_1, -\lambda_2, \ldots, -\lambda_{n-m})$ are specified by the designer, the coefficients $\beta_0, \beta_1, \ldots, \beta_{n-m-1}$ in (8) are known quantities. However, the non-critical poles $-p_1, -p_2, \ldots, -p_m$ are unspecified, so that $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ are unknown. First note that $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$ can be eliminated from equations (7) and (8) to get n linear equations:

$$\begin{aligned} \beta_{n-m-1} + \alpha_{m-1} &= \bar{k}_n + a_{n-1} \\ &\vdots \\ \beta_0\alpha_1 + \beta_1\alpha_0 &= \bar{k}_2 + a_1 \\ \beta_0\alpha_0 &= \bar{k}_1 + a_0 \end{aligned} \tag{9}$$

From (9), $(\alpha_0, \ldots, \alpha_{m-1})$ can be expressed in terms of m linear equations in $\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_n$. Let us represent this in following matrix form

$$\alpha = \mathscr{F}\bar{k} + g \tag{10}$$

where $\mathscr{F} \in \mathbb{R}^{m\times n}$, $g \in \mathbb{R}^m$.

Now $\alpha_0, \ldots, \alpha_{m-1}$ can be back-substituted in the set of n equations (9) to get $(n-m)$ linear equations in $(\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_n)$ which can be written in the form:

$$\mathscr{E}\bar{k} + h = \mathbf{0} \tag{11}$$

where $\mathscr{E} \in \mathbb{R}^{(n-m)\times n}$, $h \in \mathbb{R}^{n-m}$ and $\mathbf{0}$ is a zero vector of appropriate dimension. Using $\bar{k} = \mathscr{A}\mathscr{C}^T k$, we get the following set of equations:

$$\alpha = Fk + g \quad \text{and} \quad Ek + h = \mathbf{0} \tag{12}$$

where $F = \mathscr{F}\mathscr{A}\mathscr{C}^T \in \mathbb{R}^{m\times n}$ and $E = \mathscr{E}\mathscr{A}\mathscr{C}^T \in \mathbb{R}^{(n-m)\times n}$.

Corresponding to the relation $\alpha = Fk + g$, define α_e as

$$\alpha_e = \tilde{F}k + \tilde{g} \quad \text{where } \tilde{F} = \begin{bmatrix} F_{m\times n} \\ 0_{1\times n}\end{bmatrix} \text{ and } \tilde{g} = \begin{bmatrix} g \\ 1 \end{bmatrix} \tag{13}$$

Using (13) in the LMI defined in (5), we get

$$\tilde{F}k\widehat{\alpha}_e^T + \widehat{\alpha}_e k^T \tilde{F}^T + \tilde{g}\widehat{\alpha}_e^T + \widehat{\alpha}_e\tilde{g}^T - \Pi^T(S\otimes P)\Pi \geq 0 \tag{14}$$

Then the following result holds:

Theorem 1: For any fixed $\widehat{\alpha}(s) \in C_s$, if for some $k \in \mathbb{R}^n$ and for some $P = P^T \in \mathbb{R}^{m\times m}$, the relations (14) and $Ek + h = \mathbf{0}$ hold, then the eigenvalues of the matrix $(A - bk^T)$ satisfy the following properties:

1) $(n-m)$ out of the total n eigenvalues are $\{-\lambda_1, -\lambda_2, \ldots, -\lambda_{n-m}\}$.
2) the remaining m eigenvalues $-p_i \in \mathbb{S}$ for $i = 1, \ldots, m$.

Proof: Fix $\widehat{\alpha}(s) \in C_s$. Let some $k \in \mathbb{R}^n$ and $P = P^T \in \mathbb{R}^{m\times m}$ satisfy (14). Then $\alpha_e\widehat{\alpha}_e^T + \widehat{\alpha}_e\alpha_e^T - \Pi^T(S\otimes P)\Pi \geq 0$. Hence the polynomial $\alpha(s) \in \mathbb{S}_{LMI}$. But $\mathbb{S}_{LMI} \subseteq C_s$, so the roots of $\alpha(s)$ lie in $\mathbb{S}$. The $(n-m)$ equations $Ek+h=\mathbf{0}$ imply that the $(n-m)$ roots of polynomial $\beta(s)$ (see (4)) are placed at $\{-\lambda_1,\ldots,-\lambda_{n-m}\}$. ■

Theorem 1 defines the constraint set on the feedback gain vector k, which can be used to pose the feedback gain vector norm minimization problem as a SDP:

Problem 4: Find $\min_{P,k,\gamma}\gamma$ subject to

$$(i)\quad \gamma - k^T k \geq 0$$

$$(ii)\quad Ek+h=\mathbf{0}$$

$$(iii)\quad \tilde{F}k\widehat{\alpha}_e^T + \widehat{\alpha}_e k^T \tilde{F}^T + \tilde{g}\widehat{\alpha}_e^T + \widehat{\alpha}_e\tilde{g}^T - \Pi^T(S\otimes P)\Pi \geq 0$$

where $\gamma > 0$. Note that Problem 4 is an LMI constrained optimization with variables γ, k and P and can be solved by using solvers like *SeDuMi* [20] and its LMI interface [21] in MATLAB [22] environment.

Note 1: a) The above constraint set is always feasible since it is known that there is always at least one k which places the poles at arbitrary desired location. b) Since the above optimization deals with the coefficients of the closed loop characteristic polynomial, it is possible to place multiple number of poles at the same location in the complex plane. c) Theorem 1 is only sufficient in guaranteeing that the corresponding eigenvalues stay in $\mathbb{S}$. Consequently, in some cases, it might be possible to find a k which preserves the pole placement requirements but has a lower norm than the solution to Problem 4. Hence, a two step design procedure is suggested below to find a controller with maximum reduction in the norm.

Design Steps

1) Define a stability region $\mathbb{S}$ in the complex plane for the non-critical poles according to the requirement. Solve Problem 4 without considering the constraint (iii). If all the non-critical poles belong to $\mathbb{S}$ then stop; otherwise go to Step 2.
2) Form the nominal polynomial $\widehat{\alpha}(s)$ according to the equation (6). This step may require few trial and error iterations if all the non-critical poles do not belong to the chosen $\mathbb{S}$. Solve Problem 4.

IV. Numerical Examples

Example 1: Consider a continuous time, single input LTI system with

$$A=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\51&-10&-30&-10\end{bmatrix}\qquad b=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Eigenvalues of A are at $1,-3,-4\pm i$. We assume that the unstable pole is critical and needs to be placed at -1. The remaining 3 poles are assumed to be non-critical and are allowed to be placed arbitrarily to the left of a vertical line at -0.5 in the complex plane. Corresponding to this stability region the elements of S will be $s_{11}=1$, $s_{12}=1$ and $s_{22}=0$. Since all non-critical poles $(-3,-4\pm i)$ are in $\mathbb{S}$, the nominal polynomial $\widehat{\alpha}(s)$ can be formed according to (6) and it will be $\widehat{\alpha}(s)=s^3+11s^2+41s+51$.

According to the procedure given in Section III, the optimization problem is solved with *SeDuMi 1.05* [20] and its LMI interface [21] in MATLAB [22] environment. A comparison table is shown in Table I where Step 1 and Step 2 correspond to the solutions of Problem 4 without considering the constraint (iii) and with constraint (iii) respectively. The conventional partial pole placement step evaluates $\|k\|_2$ keeping the three non critical poles in their original locations. The percentage reductions in $\|k\|_2$ in Step 1 and Step 2 are compared with conventional partial pole placement. It is observed that a large reduction in $\|k\|_2$ is achieved in Step 1. However, the non-critical poles are in the unstable region. Hence the next design step is followed and it is observed that with constraint (iii), the percentage reduction in $\|k\|_2$ is about 57.3724%.

Table I
Comparison Table

Procedure	Closed loop poles	$\|k\|_2$	% Red. $\|k\|_2$	Sys. Cond.
Conven. Par. pole placement	$-4\pm 1i,\ -3,\ -1$	132.7253	–	Stable
Step 1 (without const. (iii))	$0\pm 6.4031i, -1, 1$	20	84.9312	Uns-table
Step 2 (with const. (iii))	$-2.5875\pm 0.2076i$, $-0.5028,\ -1$	56.5775	57.3724	Stable

Example 2: In this example, a linearized model of a 4-machine, 2-area power system [18] is considered. The 40^{th} order original model is reduced to a 10^{th} order equivalent system using balanced reduction. The modes (corresponding to the low frequency electro-mechanical modes) having damping ratio (ξ) less than 0.25 are classified as critical and hence need to be relocated such that their damping ratios increase beyond 0.25. Open loop pole locations and their damping ratios are given in Table II. It is observed that there are four critical poles, having damping ratio less than 0.25. The desired closed loop locations for these critical poles are chosen as:$-2\pm 6.9261i$ and $-2\pm 3.9352i$. We assume that the remaining 6 non-critical poles can assume any positions in the complex plane as long as their damping ratios are more than 0.25. According to [3], $\mathbb{S}$ can be chosen either as a half plane or a disc in $\mathbb{C}$. However a half plane would inadequately describe the cone corresponding to $\xi \geq 0.25$. Hence, $\mathbb{S}$ is chosen as a disc having center at $(-8,0)$ and radius 7.6 as an approximation to the cone illustrated in Fig. 1. The corresponding elements of matrix S would be $s_{11}=6.24$, $s_{12}=8$ and $s_{22}=1$.

Optimization without constraint (iii) in Problem 4: The location of the closed loop poles and corresponding damping ratios are shown in Table III. The percentage reduction in norm of the feedback gain vector is compared with the conventional partial pole placement problem (see Table IV). It can be noticed that a substantial reduction in $\|k\|_2$ (76.2196%) is achieved in this step. However, the closed loop poles are not meeting the damping ratio requirement leading to unsatisfactory closed loop response. Hence design Step 2 is required.

Optimization with constraint (iii) in Problem 4: Since the non-critical poles -44.7511 and -0.0999 are not in the stability region $\mathbb{S}$ as defined above, (6) cannot be used to form $\widehat{\alpha}(s)$. Instead the six poles needed to create $\widehat{\alpha}(s)$ are formed out of the four non-critical open loop poles already within $\mathbb{S}$ $(-0.8960\pm 0.8730i, -11.6207, -0.7881)$;

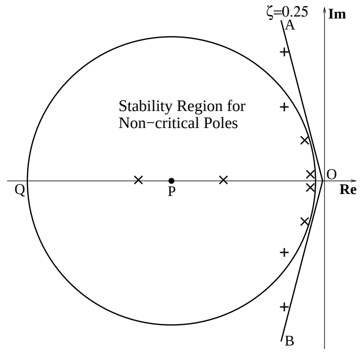

Figure 1. The cone AOB corresponds to the damping ratio $\zeta = 0.25$ region in the complex plane. The disc with center $P(-8,0)$ and radius $PQ = 7.6$ corresponds to the stability region for the non-critical poles. The $+$ marks are for critical poles and $\times$ marks are for non-critical poles in closed loop.

Table II
POLE LOCATIONS AND DAMPING RATIO TABLE

Open loop poles	Damping ratio (ξ)
$-0.5800 \pm 6.9241i$	**0.0834**
$-0.0467 \pm 3.9352i$	**0.0118**
$-0.8960 \pm 0.8730i$	0.7162
$-44.7511, -11.6207$	1
$-0.0999, -0.7881$	1

while the two non-critical open loop poles outside $\mathbb{S}$ $(-44.7511, -0.0999)$, are replaced (heuristically) with two new poles within $\mathbb{S}$ at -15 and -0.6. The resulting nominal polynomial would be $\widehat{\alpha}_e = [128.9886 \quad 546.0504 \quad 869.9103 \quad 677.7951 \quad 263.4905 \quad 29.8007 \quad 1]^T$. The location of the closed loop poles and corresponding damping ratios are shown in Table III and Fig. 1. It is observed that all modes are satisfying the damping ratio requirement. Furthermore, 74.5401% reduction in $\|k\|_2$ is achieved in this step.

Comparison of Actual Controller Effort: For completeness, a reduced order controller (comprising of a 10^{th} order observer and the state feedback gain vector k) for the full order plant (40^{th} order) is designed. The closed loop system is depicted in Fig. 2. To compare the controller effort (i.e. $\max_t \ |u(t)|$) between the conventional pole placement approach and the proposed approach, we present results for the following two cases:

1) the full order plant is driven with the controller using the k obtained with conventional partial pole placement approach.
2) the full order plant is driven with the controller using the k obtained by the proposed approach.

The maximum overshoot of the controller effort is compared through MATLAB simulation which is shown in Fig. 3. A substantial reduction (79.2569%) in maximum overshoot of the controller effort is observed. In addition, the output response of the full order plant for both cases is depicted in Fig. 4. This demonstrates that the proposed algorithm of minimizing $\|k\|_2$ reduces $\max_t \ |u(t)|$ effectively, while maintaining acceptable time domain performance.

Table III
POLE LOCATIONS AND DAMPING RATIO TABLE

Closed loop poles (without const. (iii))	Damping ratio (ξ)	Closed loop poles (with const. (iii))	Damping ratio (ξ)
$-2 \pm 6.9261i$	0.2774	$-2 \pm 6.9261i$	0.2774
$-2 \pm 3.9352i$	0.4530	$-2 \pm 3.9352i$	0.4530
$-5.6749 \pm 4.7914i$	0.7640	$-0.9515 \pm 2.2154i$	0.3946
$-0.3006 \pm 2.0094i$	**0.1420**	$-0.4176 \pm 0.3151i$	0.7982
$-0.4338 \pm 0.3987i$	0.7362	$-9.8734, -5.0861$	1

Table IV
COMPARISON TABLE

Procedure	$\|k\|_2$	% Red. in $\|k\|_2$	Remark
Conven. partial pole placement	235.3568	–	Satisfactory
Step 1 (without constraint (iii))	55.9687	76.2196	Not Satisfactory
Step 2 (with constraint (iii))	59.9216	74.5401	Satisfactory

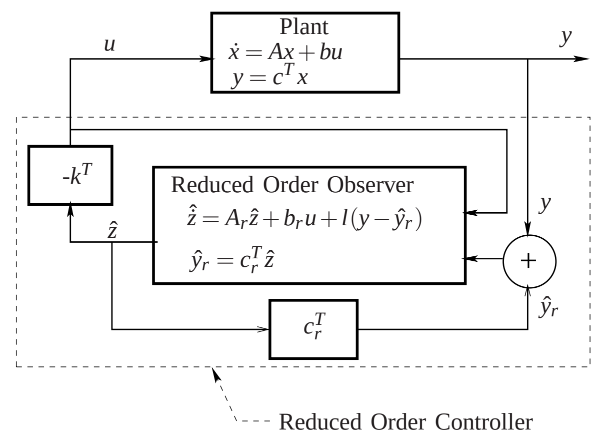

Figure 2. Closed loop controller-observer system. Here $\hat{z}$ and $\hat{y}_r$ denote the reduced order observer state and output respectively. A_r, b_r, and c_r denote the reduced order system matrices. l is the reduced order observer gain vector.

V. CONCLUSION

It is shown that two different types of pole placement constraints for critical and non-critical poles can be formulated in terms of the state feedback controller gains. Due to non-convexity of the region C_s corresponding to the region $\mathbb{S}$, an LMI stability region is constructed inside C_s. This enables the

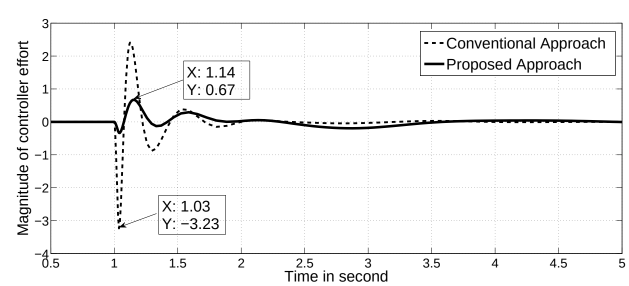

Figure 3. Comparison between the maximum overshoot of the controller effort. The magnitude of maximum overshoot of the controller effort (u) for conventional partial pole placement approach (dashed line) and proposed approach - Step 2 (solid line) is 3.23 and 0.67 respectively. The percentage reduction in the magnitude of maximum overshoot of u is about 79.25% in proposed approach.

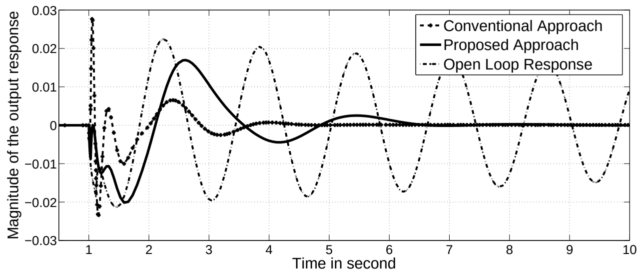

Figure 4. Output response of the 4-machine, 2-area power system driven by the controller using the feedback gain vector k obtained in conventional partial pole placement approach and proposed approach - Step 2 (solid line). The highly oscillatory response corresponds to the open loop system response. The simulation is done in MATLAB Simulink. The open loop system and the closed loop system are excited with a step input of step length 0.05 second.

formulation of an equivalent SDP over feedback controllers. Similarly, other relevant controller and closed loop characteristics, like closed loop sensitivity and controller H_∞ norm can be likewise optimized, and are topics of current research. It would also be interesting to extend the current technique to multi-input systems.

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