NON-DEGENERACY OF WIENER FUNCTIONALS ARISING FROM ROUGH DIFFERENTIAL EQUATIONS

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Abstract. Malliavin Calculus is about Sobolev-type regularity of functionals on Wiener space, the main example being the Itô map obtained by solving stochastic differential equations. Rough path analysis is about strong regularity of solution to (possibly stochastic) differential equations. We combine arguments of both theories and discuss existence of a density for solutions to stochastic differential equations driven by a general class of non-degenerate Gaussian processes, including processes with sample path regularity worse than Brownian motion.

1. Introduction

It is basic question in probability theory whether a given stochastic process $\{Y_t : t \geq 0\}$ with values in some Euclidean space admits, at fixed positive times, a density with respect to Lebesgue measure. In a non-degenerate Markovian setting - ellipticity of the generator - an affirmative answer can be given using Weyl's lemma as discussed in McKean's classical 1969 text [23]. Around the same time, Hörmander's seminal work on hypoelliptic partial differential operators enabled probabilists to obtain criteria for existence (and smoothness) for densities of certain degenerate diffusions. This dependence on the theory of partial differential equations was removed when P. Malliavin devised a purely probabilistic approach, perfectly adapted to prove existence and smoothness of densities.

We recall some key ingredients of Malliavin's machinery, known as *Malliavin Calculus* or *stochastic calculus of variations*. Most of it can formulated in the setting of an abstract Wiener spaces $(W, \mathcal{H}, \mu)$. The concept is standard [3, 22, 24, 28] as is the notion of a *weakly non-degenerate* $\mathbb{R}^e$-valued Wiener functional φ which has the desirable property that the image measure $\varphi_* \mu$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^e$. (Functionals which are *non-degenerate* have a smooth density.) Precise definitions are given later on in the text.

Given these abstract tools, we turn to the standard Wiener space $C_0\left([0,1], \mathbb{R}^d\right)$ equipped with Wiener measure. From Itô's theory, we can realize diffusion processes by solving the stochastic differential equation

$$dY = \sum_{i=1}^{d} V_i(Y) \circ dB^i + V_0(Y)\, dt \equiv V(Y) \circ dB + V_0(Y)\, dt, \;\; Y(0) = y_0 \in \mathbb{R}^e$$

driven by a d-dimensional Brownian motion B along sufficiently well-behaved (drift - resp. diffusion) vector fields $V_0, ..., V_d$. The Itô-map $B \mapsto Y$ is notorious for its lack of strong regularity properties which rules out the use of any Fréchet-calculus.

Key words and phrases. Malliavin Calculus, Rough Paths Analysis.

On the positive side, it is smooth in a Sobolev sense on Wiener space ("*smooth in the sense of Malliavin*"). Under condition

$$(E): \text{ span}\,[V_1,...,V_d]_{y_0} = \mathcal{T}_{y_0}\mathbb{R}^e \cong \mathbb{R}^e;$$

or a less restrictive Hörmander's condition (allowing all Lie brackets of $V_0,...,V_d$ in spanning $\mathcal{T}_{y_0}\mathbb{R}^e$) the solution map $B \mapsto Y_t$, for $t>0$, is non-degenerate and one reaches the desired conclusion that Y_t has a (smooth) density. This line of reasoning due to P. Malliavin provides a direct probabilistic approach to the study of transition densities and has found applications from stochastic fluid dynamics to interest rate theory. It also shows that the Markovian structure is not essential and one can, for instance, adapt these ideas to study densities of Itô-diffusions as was done by Kusuoka and Stroock in [17].

Our interest lies in stochastic differential equations of type

$$dY = V(Y)\,dX + V_0(Y)\,dt \tag{1.1}$$

where X is a multi-dimensional Gaussian process. Such differential equations arise, for instance, in financial mathematics [2, 10, 11] or as model for studying ergodic properties of non-Markovian systems [12]. Assuming momentarily enough sample path regularity so that (1.1) makes sense path-by-path by Riemann-Stieltjes (or Young) integration the question whether or not Y_t admits a density is important and far from obvious. To the best of our knowledge, all results in that direction are restricted to fractional Brownian motion with Hurst parameter $H>1/2$. Existence of a density for Y_t, $t>0$, was established in [25] under condition (E). Smoothness results then appeared in [26], and under Hörmander's condition in [1].

The purpose of this paper is to give a first demonstration of the powerful interplay between Malliavin calculus and rough path analysis. After some remarks on $\mathcal{H}$-differentiability and a representation of the Malliavin covariance in terms of a 2D Young integral, we show that weak non-degeneracy (in the sense of Malliavin) of solutions to (1.1) at times $T>0$, and hence existence of a density, remains valid in an almost generic sense. Our assumptions are

- The vector fields $V=(V_1,...,V_d)$ at y_0 span $\mathcal{T}_{y_0}\mathbb{R}$, i.e. condition (E).
- The continuous, centered Gaussian driving signal X is such that the stochastic differential equation (1.1) makes sense as *rough differential equation* (RDE), [18, 21]. Applied in our context, this represents a unified framework which covers *at once* Gaussian signals with nice sample paths (such as fBM with $H>1/2$), Brownian motion, Gaussian (semi-)martingales, and last not least Gaussian signals with sample path regularity worse than Brownian motion provided there exists a sufficiently regular stochastic area[1].
- The Gaussian driving signal is sufficiently non-degenerate which is clearly needed to rule our examples such as $X\equiv 0$ or the Brownian bridge $X_t = B_t(\omega) - (t/T)\,B_T(\omega)$.

Smoothness of densities remains an open problem, some technical remarks about the difficulties involved are found in the last section.

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[1]For orientation, fractional Brownian motion is covered for any $H>1/3$.

2. Preliminaries on ODE and RDEs

2.1. **Controlled ordinary differential equations.** Consider the ordinary differential equations, driven by a smooth $\mathbb{R}^d$-valued signal $f = f(t)$ along sufficiently smooth and bounded vector fields $V = (V_1, ..., V_d)$ and a drift vector field V_0

$$dy = V(y)df + V_0(y)\,dt, \;\; y(t_0) = y_0 \in \mathbb{R}^e.$$

We call $U^f_{t\leftarrow t_0}(y_0) \equiv y_t$ the associated flow. Let J denote the Jacobian of U. It satisfies the ODE obtain by formal differentiation w.r.t. y_0. More specifically,

$$a \mapsto \left\{ \frac{d}{d\varepsilon} U^f_{t\leftarrow t_0}(y_0 + \varepsilon a) \right\}_{\varepsilon=0}$$

is a linear map from $\mathbb{R}^e \to \mathbb{R}^e$ and we let $J^f_{t\leftarrow t_0}(y_0)$ denote the corresponding $e \times e$ matrix. It is immediate to see that

$$\frac{d}{dt} J^f_{t\leftarrow t_0}(y_0) = \left[\frac{d}{dt} M^f\left(U^f_{t\leftarrow t_0}(y_0), t \right) \right] \cdot J^f_{t\leftarrow t_0}(y_0)$$

where $\cdot$ denotes matrix multiplication and

$$\frac{d}{dt} M^f(y,t) = \sum_{i=1}^d V_i'(y)\, \frac{d}{dt} f^i_t + V_0'(y)\,.$$

Note that $J^f_{t_2\leftarrow t_0} = J^f_{t_2\leftarrow t_1} \cdot J^f_{t_1\leftarrow t_0}$. We can consider Gateaux derivatives in the driving signal and define

$$D_h U^f_{t\leftarrow 0} = \left\{ \frac{d}{d\varepsilon} U^{f+\varepsilon h}_{t\leftarrow 0} \right\}_{\varepsilon=0}.$$

One sees that $D_h U^f_{t\leftarrow 0}$ satisfies a linear ODE and the variation of constants formula leads to

$$D_h U^f_{t\leftarrow 0}(y_0) = \int_0^t \sum_{i=1}^d J^f_{t\leftarrow s}\left(V_i\left(U^f_{s\leftarrow 0}\right)\right) dh^i_s.$$

2.2. **Rough differential equations.** Let $p \in (2,3)$. A geometric p-rough path $\mathbf{x}$ over $\mathbb{R}^d$ is a continuous path on $[0,T]$ with values in $G^2(\mathbb{R}^d)$, the step-2 nilpotent group over $\mathbb{R}^d$, of finite p-variation relative to d, the Carnot-Caratheodory metric on $G^2(\mathbb{R}^d)$, i.e.

$$\sup_n \sup_{t_1<\ldots<t_n} \sum_i d\left(\mathbf{x}_{t_i}, \mathbf{x}_{t_{i+1}}\right)^p < \infty.$$

Following [18, 6, 7, 9] we realize $G^2(\mathbb{R}^d)$ as the set of all $(a,b) \in \mathbb{R}^d \oplus \mathbb{R}^{d\times d}$ for which $Sym(b) \equiv a^{\otimes 2}/2$. (This point of view is natural: a smooth $\mathbb{R}^d$-valued path $x = (x^i_t)_{i=1,\ldots,d}$, enhanced with its iterated integrals $\int_0^t \int_0^s dx^i_u dx^j_s$, gives canonically rise to a $G^2(\mathbb{R}^d)$-valued path.) Given $(a,b) \in G^2(\mathbb{R}^d)$ one gets rid of the redundant $Sym(b)$ by $(a,b) \mapsto (a, b - a^{\otimes}/2) \in \mathbb{R}^d \oplus so(d)$. Applied to x enhanced with its iterated integrals over $[0,t]$ this amounts to look at the path x and its (signed) areas $\int_0^t (x^i_s - x^i_0)\, dx^j_s - \int_0^t \left(x^j_s - x^j_0\right) dx^i_s,\; i,j \in \{1,\ldots,d\}$. Without going into too much detail, the group structure on $G^2(\mathbb{R}^d)$ can be identified with the (truncated) tensor multiplication and is relevant as it allows to relate algebraically the path and area increments over adjacent intervals; the mapping $(a,b) \mapsto (a, b - a^{\otimes}/2)$ maps the Lie group $G^2(\mathbb{R}^d)$ to its Lie algebra; at last, the Carnot-Caratheodory metric is defined intrinsically as (left-)invariant metric on $G^2\left(\mathbb{R}^d\right)$ and satisfies $|a| + |b|^{1/2} \lesssim d\left((0,0),(a,b)\right) \lesssim |a| + |b|^{1/2}$.

One can then think of a geometric p-rough path $\mathbf{x}$ as a path $x:[0,T]\to\mathbb{R}^d$ enhanced with its iterated integrals (equivalently: area integrals) although the later need not make classical sense. For instance, *almost every* joint realization of Brownian motion and Lévy's area process is a geometric p-rough path [21, 6, 5]. The Lyons theory of rough paths [20, 21, 9] then gives deterministic meaning to the rough differential equation (RDE)

$$dy = V(y)\, d\mathbf{x}$$

for Lip^γ-vector fields (in the sense of Stein), $\gamma > p$. By considering the space-time rough path $\tilde{\mathbf{x}} = (t, \mathbf{x})$ and $\tilde{V} = (V_0, V_1, ..., V_d)$ this form is general enough to cover differential equations with drift[2]. The solution induces a flow $y_0 \mapsto U^{\mathbf{x}}_{t\leftarrow t_0}(y_0)$. The Jacobian $J^{\mathbf{x}}_{t\leftarrow t_0}$ of the flow exists and satisfies a linear RDE, as does the directional derivative

$$D_h U^{\mathbf{X}}_{t\leftarrow 0} = \left\{ \frac{d}{d\varepsilon} U^{T_{\varepsilon h}\mathbf{x}}_{t\leftarrow 0} \right\}_{\varepsilon=0}$$

for a smooth path h. If $\mathbf{x}$ arises from a smooth path x together with its iterated integrals the *translated rough path* $T_h\mathbf{x}$ is nothing but $x+h$ together with its iterated integrals. In the general case, we assume $h \in C^{q\text{-var}}$ with $1/p + 1/q > 1$, the translation $T_h\mathbf{x}$ can be written in terms of $\mathbf{x}$ and cross-integrals between $\pi_1(\mathbf{x}_{0,\cdot}) =: x$ and the perturbation h. (These integrals are well-defined Young-integrals.)

Proposition 1. *Let $\mathbf{X}$ be a geometric p-rough paths over $\mathbb{R}^d$ and $h \in C^q\left([0,1],\mathbb{R}^d\right)$ such that $1/p + 1/q > 1$. Then*

$$D_h U^{\mathbf{X}}_{t\leftarrow 0}(y_0) = \int_0^t \sum_{i=1}^d J^{\mathbf{X}}_{t\leftarrow s}\left(V_i\left(U^{\mathbf{X}}_{s\leftarrow 0}\right)\right) dh^i_s$$

where the right hand side is well-defined as Young integral.

Proof. At least when $\gamma > p+1$, both $J^{\mathbf{X}}_{t\leftarrow 0}$ and $D_h U^{\mathbf{X}}_{t\leftarrow 0}$ satisfy (jointly with $U^{\mathbf{X}}_{t\leftarrow 0}$) a RDE driven by $\mathbf{X}$. This is an application of Lyons' limit theorem and discussed in detail in [19, 20]. A little care is needed since the resulting vector fields now have linear growth. It suffices to rule out explosion so that the problem can be localized. The needed remark is that $J^{\mathbf{X}}_{t\leftarrow 0}$ also satisfy a linear RDE of form

$$dJ^{\mathbf{X}}_{t\leftarrow 0} = d\mathbf{M}^{\mathbf{X}} \cdot J^{\mathbf{X}}_{t\leftarrow 0}(y_0)$$

where $d\mathbf{M}^{\mathbf{X}} = V'\left(U^{\mathbf{X}}_{t\leftarrow 0}(y_0)\right) d\mathbf{X}$. Explosion can be ruled out by direct iterative expansion and estimates of the Einstein sum as in [18]. □

3. RDEs driven by Gaussian signals

We consider a continuous, centered Gaussian process with independent components $X = \left(X^1, ..., X^d\right)$ started at zero. This gives rise to an abstract Wiener space $(W, \mathcal{H}, \mu)$ where $W = \bar{\mathcal{H}} \subset C_0\left([0,T],\mathbb{R}^d\right)$. Note that $\mathcal{H} = \oplus_{i=1}^d \mathcal{H}^{(i)}$ and recall that elements of $\mathcal{H}$ are of form $h_t = \mathbb{E}\left(X_t \xi(h)\right)$ where $\xi(h)$ is a Gaussian

[2](1) If $\mathbf{x}$ lifts a $\mathbb{R}^d$-valued path x, then $\tilde{\mathbf{x}}$ is construct with cross-integrals of type $\int x dt, \int t dx$ all of which are canonically defined. (2) Including V_0 in the collection $\tilde{V}$ leads to suboptimal regularity requirements for V_0 which could be avoided by a direct analysis.

random variable. The ("reproducing kernel") Hilbert-structure on $\mathcal{H}$ is given by $\langle h, h' \rangle_{\mathcal{H}} := \mathbb{E}\left(\xi\left(h\right)\xi\left(h'\right)\right)$.

Existence of a Gaussian geometric p-rough path above X is tantamount to the existence of certain Lévy area integrals. From the point of view of Stieltjes integration the existence of Lévy's area for Brownian motion is a miracle. In fact, a subtle cancellation due to orthogonality of increments of Brownian motion is responsible for convergence and this suggests that processes with sufficiently fast decorrelation of their increments will also give rise to a stochastic Lévy area. The resulting technical conditions appears in [21] for instance. For Gaussian processes, a cleaner (and slightly weaker) condition can be given in terms of the *2D variation* properties of the covariance function $R\left(s,t\right) = \mathbb{E}\left(X_s \otimes X_t\right) = diag\left(R^{(1)}, ..., R^{(k)}\right)$.The assumption, writing $X_{t,t'} := X_{t'} - X_t$,

$$|R|^{\rho}_{\rho\text{-var};[0,T]^2} := \sup_{D=(t_i)} \sum_{i,j} \left|\mathbb{E}\left(X_{t_i,t_{i+1}} X_{t_j,t_{j+1}}\right)\right|^{\rho} < \infty$$

for $\rho < 2$ is known [8] to be sufficient (and essentially necessary) for the existence of a natural lift of X to a geometric p-rough path $\mathbf{X}$ for any $p > 2\rho$. Observe that the covariance of Brownian motion has finite ρ-variation with $\rho = 1$. As a more general example, a direct computation shows that fractional Brownian motion has finite ρ-variation with $\rho = 1/\left(2H\right)$.

The assumption of $|R|_{\rho\text{-var}} < \infty$ has other benefits, notably the following embedding theorem [8]. Since it is crucial for our purposes we repeat the short proof, as we apply it componentwise we can assume $d = 1$.

Proposition 2. *Let R be the covariance of a real-valued centered Gaussian process. If R is of finite ρ-variation, then $\mathcal{H} \hookrightarrow C^{\rho\text{-}var}$. More, precisely, for all $h \in \mathcal{H}$,*

$$|h|_{\rho-var;[s,t]} \leq \sqrt{\langle h, h\rangle_{\mathcal{H}}} \sqrt{R_{\rho\text{-}var;[s,t]^2}}.$$

Proof. Every element $h \in \mathcal{H}$ can be written as $h_t = \mathbb{E}\left(ZX_t\right)$ for the Gaussian r.v. $Z = \xi\left(h\right)$. We may assume that $\langle h, h\rangle_{\mathcal{H}} = \mathbb{E}\left(Z^2_{\cdot}\right) = 1$. Let (t_j) be a subdivision of $[s,t]$ and write $|x|_{l^r} = \left(\sum_i x_i^r\right)^{1/r}$ for $r \geq 1$. If ρ' denote the Hölder conjugate of ρ we have

$$\begin{aligned}
\left(\sum_j \left|h_{t_j,t_{j+1}}\right|^{\rho}\right)^{1/\rho} &= \sup_{\beta,|\beta|_{l^{\rho'}}\leq 1} \sum_j \beta_j h_{t_j,t_{j+1}} = \sup_{\beta,|\beta|_{l^{\rho'}}\leq 1} \mathbb{E}\left(Z\sum_j \beta_j X_{t_j,t_{j+1}}\right) \\
&\leq \sup_{\beta,|\beta|_{l^{\rho'}}\leq 1} \sqrt{\sum_{j,k} \beta_j\beta_k \mathbb{E}\left(X_{t_j,t_{j+1}} X_{t_k,t_{k+1}}\right)} \\
&\leq \sup_{\beta,|\beta|_{l^{\rho'}}\leq 1} \sqrt{\left(\sum_{j,k} \left|\beta_j\right|^{\rho'}\left|\beta_k\right|^{\rho'}\right)^{\frac{1}{\rho'}} \left(\sum_{j,k}\left|\mathbb{E}\left(X_{t_j,t_{j+1}} X_{t_k,t_{k+1}}\right)\right|^{\rho}\right)^{\frac{1}{\rho}}} \\
&\leq \left(\sum_{j,k}\left|\mathbb{E}\left(X_{t_j,t_{j+1}} X_{t_k,t_{k+1}}\right)\right|^{\rho}\right)^{1/(2\rho)} \leq \sqrt{R_{\rho\text{-var};[s,t]^2}}.
\end{aligned}$$

Optimizing over all subdivision (t_j) of $[s,t]$, we obtain our result. □

One observes that for Brownian motion ($\rho = 1$) this embedding is sharp. Furthermore, the rough path translation $T_h\mathbf{X}$, which involves Young integrals, makes sense if $\rho < \rho^* = 3/2$. (This critical value comes from $1/\rho + 1/(2p) \sim 1/\rho + 1/(2\rho) = 3/(2\rho)$ and equating to 1.) The moral is that one can take deterministic directional derivatives in Cameron-Martin directions as long as $\rho < 3/2$. (En passant, we see that the effective tangent space to Gaussian RDE solutions is strictly bigger than the usual Cameron-Martin space as long as $\rho < 3/2$. In the special case of Stratonovich SDEs a related result predates rough path theory and goes back to Kusuoka [15]). For this reason we will assume

$$\rho < 3/2$$

for the remainder of this paper. This entails that we are dealing with geometric p-rough paths $\mathbf{X}$ for which we may assume

$$p \in (2\rho, 3).$$

Definition 1. [13], [24, Section 4.1.3], [30, Section 3.3] *Given an abstract Wiener space $(W, \mathcal{H}, \mu)$, a random variable (i.e. measurable map) $F : W \to \mathbb{R}$ is continuously $\mathcal{H}$-differentiable, in symbols $F \in C^1_{\mathcal{H}}$, if for μ-almost every ω, the map*

$$h \in \mathcal{H} \mapsto F(\omega + h)$$

is continuously Fréchet differentiable. A vector-valued r.v. $F = \left(F^1, ..., F^e\right) : W \to \mathbb{R}^e$ is continuously $\mathcal{H}$-differentiable iff each F^i is continuously $\mathcal{H}$-differentiable. In particular, μ-almost surely, $DF(\omega) = \left(DF^1(\omega), ..., DF^e(\omega)\right)$ is a linear bounded map from $\mathcal{H} \to \mathbb{R}^e$

Remark 1. *(1) The notion of continuous $\mathcal{H}$-differentiability was introduced in* [13] *and plays a fundamental role in the study of transformation of measure on Wiener space. Integrability properties of F and DF aside, $C^1_{\mathcal{H}}$-regularity is stronger than Malliavin differentiability in the usual sense. Indeed, by* [24, Thm 4.1.3] *(see also* [13], [30, Section 3.3]*) $C^1_{\mathcal{H}}$ implies $\mathbb{D}^{1,2}_{loc}$-regularity where the definition of $\mathbb{D}^{1,2}_{loc}$ is based on the commonly used Shigekawa Sobolev space $\mathbb{D}^{1,p}$. (Our notation here follows* [24, Sec. 1.2, 1.3.4]*). This remark will be important to us since it justifies the use of Bouleau-Hirsch's criterion (e.g.* [24, Section 2.1.2]*) for establishing absolute continuity of F (cf. proof of theorem 1).*
(2) Although not relevant to the sequel of this paper, it is interesting to compare $C^1_{\mathcal{H}}$-regularity with the Kusuoka-Stroock Sobolev spaces. Following [14, 16] *one defines $\tilde{\mathbb{D}}^{1,p}$ as the space of random-variables F which are (i) ray-absolutely-continuous (RAC) in the sense that for every $h \in \mathcal{H}$ there is an absolutely continuous version of the process $\{F(\omega + th) : t \in \mathbb{R}\}$; (ii) stochastically Gateaux differentiable (SGD) in the sense that there exists an $\mathcal{H}$-valued r.v. $\tilde{D}F$ such that for every $h \in \mathcal{H}$,*

$$(F(\omega + th) - F(\omega))/t \to \left\langle \tilde{D}F, h \right\rangle_{\mathcal{H}} \text{ as } t \to 0 \tag{3.1}$$

probability with respect to μ; and (iii) such that $F \in L^p$ and $\tilde{D}F \in L^p(\mathcal{H})$.
From Sugita [27] *it is known that $\tilde{\mathbb{D}}^{1,p} = \mathbb{D}^{1,p}$, at least for $p \in (1, \infty)$. Since $C^1_{\mathcal{H}}$-regularity is a local property it has nothing to say about the integrability property (iii) but it does imply a fortiori the regularity properties (i) and (ii). Indeed, (i) is trivially satisfied (without the need of h-dependent modifications!). As for (ii), $\tilde{D}F$ is given by the Fréchet differential $DF(\omega)$ of $h \in \mathcal{H} \mapsto F(\omega + h)$ and the convergence (3.1) holds not only in probability but μ-almost surely.*

Proposition 3. *Let $\rho < 3/2$. For fixed $t \geq 0$, the $\mathbb{R}^e$-valued random variable*

$$\omega \mapsto U_{t\leftarrow 0}^{\mathbf{X}(\omega)}(y_0)$$

is continuously $\mathcal{H}$-differentiable.

Proof. Choose $p > 2\rho$ such that $1/p + 1/\rho > 1$. We may assume that $\mathbf{X}(\omega)$ has been defined so that $\mathbf{X}(\omega)$ is a geometric p-rough path for every $\omega \in W$. Let us also recall for $h \in \mathcal{H} \subset C^{\rho\text{-var}}$, the translation $T_h\mathbf{X}(\omega)$ can be written (for ω fixed!) in terms of $\mathbf{X}(\omega)$ and cross-integrals between $\pi_1(\mathbf{X}_{0,\cdot}) =: X \in C^{p\text{-var}}$ and h. (These integrals are well-defined Young-integrals.) Thanks to the definition of $\mathbf{X}(\omega)$ as the limit in probability of piecewise linear approximations to X and its iterated integrals (cf. [8]) and basic continuity properties of Young integrals we see that the event

$$\{\omega : \mathbf{X}(\omega + h) \equiv T_h\mathbf{X}(\omega) \text{ for all } h \in \mathcal{H}\} \tag{3.2}$$

has probability one. We show that $h \in \mathcal{H} \mapsto U_{t\leftarrow 0}^{\mathbf{X}(\omega+h)}(y_0)$ is continuously Fréchet differentiable for every ω in the above set of full measure. By basic facts of Fréchet theory, we must show (a) Gateaux differentiability and (b) continuity of the Gateaux differential.

Ad (a): Using $\mathbf{X}(\omega + g + h) \equiv T_gT_h\mathbf{X}(\omega)$ for $g, h \in \mathcal{H}$ it suffices to show Gateaux differentiability of $U_{t\leftarrow 0}^{\mathbf{X}(\omega+\cdot)}(y_0)$ at $0 \in \mathcal{H}$. For fixed t, define

$$Z_{i,s} \equiv J_{t\leftarrow s}^{\mathbf{X}}\left(V_i\left(U_{s\leftarrow 0}^{\mathbf{X}}\right)\right).$$

Note that $s \mapsto Z_{i,s}$ is of finite p-variation. We have, with implicit summation over i,

$$\begin{aligned}
\left|D_hU_{t\leftarrow 0}^{\mathbf{X}}(y_0)\right| &= \left|\int_0^t J_{t\leftarrow s}^{\mathbf{X}}\left(V_i\left(U_{s\leftarrow 0}^{\mathbf{X}}\right)\right)dh_s^i\right| \\
&= \left|\int_0^t Z_i dh^i\right| \\
&\leq c\left(|Z|_{p-var} + |Z(0)|\right) \times |h|_{\rho-var} \\
&\leq c\left(|Z|_{p-var} + |Z(0)|\right) \times |h|_{\mathcal{H}}.
\end{aligned}$$

Hence, the linear map $DU_{t\leftarrow 0}^{\mathbf{X}}(y_0) : h \mapsto D_hU_{t\leftarrow 0}^{\mathbf{X}}(y_0) \in \mathbb{R}^e$ is bounded and each component is an element of $\mathcal{H}^*$. We just showed that

$$h \mapsto \left\{\frac{d}{d\varepsilon}U_{t\leftarrow 0}^{T_{\varepsilon h}\mathbf{X}(\omega)}(y_0)\right\}_{\varepsilon=0} = \left\langle DU_{t\leftarrow 0}^{\mathbf{X}(\omega)}(y_0), h\right\rangle_{\mathcal{H}}$$

and hence

$$h \mapsto \left\{\frac{d}{d\varepsilon}U_{t\leftarrow 0}^{\mathbf{X}(\omega+\varepsilon h)}(y_0)\right\}_{\varepsilon=0} = \left\langle DU_{t\leftarrow 0}^{\mathbf{X}(\omega)}(y_0), h\right\rangle_{\mathcal{H}}$$

emphasizing again that $\mathbf{X}(\omega + h) \equiv T_h\mathbf{X}(\omega)$ almost surely for all $h \in \mathcal{H}$ simultaneously. Repeating the argument with $T_g\mathbf{X}(\omega) = \mathbf{X}(\omega + g)$ shows that the Gateaux differential of $U_{t\leftarrow 0}^{\mathbf{X}(\omega+\cdot)}$ at $g \in \mathcal{H}$ is given by

$$DU_{t\leftarrow 0}^{\mathbf{X}(\omega+g)} = DU_{t\leftarrow 0}^{T_g\mathbf{X}(\omega)}.$$

(b) It remains to be seen that $g \in \mathcal{H} \mapsto DU_{t\leftarrow 0}^{T_g\mathbf{X}(\omega)} \in L(\mathcal{H}, \mathbb{R}^e)$, the space of linear bounded maps equipped with operator norm, is continuous. To this end, assume

$g_n \to_{n\to\infty} g$ in $\mathcal{H}$ (and hence in $C^{\rho\text{-var}}$). Continuity properties of the Young integral imply continuity of the translation operator viewed as map $h \in C^{\rho\text{-var}} \mapsto T_h\mathbf{X}(\omega)$ as p-rough path (see [21]) and so

$$T_{g_n}\mathbf{X}(\omega) \to T_g\mathbf{X}(\omega)$$

in p-variation rough path metric. To point here is that

$$\mathbf{x} \mapsto J^{\mathbf{x}}_{t\leftarrow\cdot} \text{ and } J^{\mathbf{x}}_{t\leftarrow\cdot}(V_i(U^{\mathbf{x}}_{\cdot\leftarrow 0})) \in C^{p\text{-var}}$$

depends continuously on $\mathbf{x}$ with respect to p-variation rough path metric: using the fact that $J^{\mathbf{x}}_{t\leftarrow\cdot}$ and $U^{\mathbf{x}}_{\cdot\leftarrow 0}$ both satisfy rough differential equations driven by $\mathbf{x}$ this is just a consequence of Lyons' limit theorem (the *universal limit theorem* of rough path theory). We apply this with $\mathbf{x} = \mathbf{X}(\omega)$ where ω remains a fixed element in (3.2). It follows that

$$\left\| DU^{T_{g_n}\mathbf{X}(\omega)}_{t\leftarrow 0} - DU^{T_g\mathbf{X}(\omega)}_{t\leftarrow 0}\right\|_{op} = \sup_{h:|h|_{\mathcal{H}}=1}\left| D_hU^{T_{g_n}\mathbf{X}(\omega)}_{t\leftarrow 0} - D_hU^{T_g\mathbf{X}(\omega)}_{t\leftarrow 0}\right|$$

and defining $Z^g_i(s) \equiv J^{T_g\mathbf{X}(\omega)}_{t\leftarrow s}\left(V_i\left(U^{T_g\mathbf{X}(\omega)}_{s\leftarrow 0}\right)\right)$, and similarly $Z^{g_n}_i(s)$, the same reasoning as in part (a) leads to the estimate

$$\left\| DU^{T_{g_n}\mathbf{X}(\omega)}_{t\leftarrow 0} - DU^{T_g\mathbf{X}(\omega)}_{t\leftarrow 0}\right\|_{op} \le c\left(|Z^{g_n} - Z^g|_{p\text{-var}} + |Z^{g_n}(0) - Z^g(0)|\right).$$

From the explanations just given this tends to zero as $n \to \infty$ which establishes continuity of the Gateaux differential, as required, and the proof is finished. □

Definition 2. [28, 24, 22] *Given a continuously $\mathcal{H}$-differentiable*[3] *r.v.* $F = (F^1, ..., F^e) : W \to \mathbb{R}^e$ *the Malliavin covariance matrix as the random matrix given by*

$$\sigma(\omega) := \left(\langle DF^i, DF^j\rangle_{\mathcal{H}}\right)_{i,j=1,...,e} \in \mathbb{R}^{e\times e}.$$

We call F weakly non-degenerate if $\det(\sigma) \neq 0$ *almost surely.*

We now give an integral representation of the Malliavin covariance matrix of $Y_t \equiv U^{\mathbf{X}(\omega)}_{t\leftarrow 0}(y_0)$, the solution to the RDE driven by $\mathbf{X}(\omega)$, the lift of $(X^1, ..., X^d)$, along vector fields $(V_1, ..., V_d)$, in terms of 2D Young integrals [31, 29, 8].

Proposition 4. *Let* $\sigma_t = \left(\left\langle DY^i_t, DY^j_t\right\rangle_{\mathcal{H}} : i,j = 1,...,e\right)$ *denote the Malliavin covariance matrix of* $Y_t \equiv U^{\mathbf{X}(\omega)}_{t\leftarrow 0}(y_0)$*, the RDE solution of* $dY = V(Y)\,d\mathbf{X}(\omega)$*. In the notation of section 2.2 we have*

$$\left(\left\langle DY^i_t, DY^j_t\right\rangle_{\mathcal{H}}\right)_{i,j=1,...,e} = \sum_{k=1}^d \int_0^t\int_0^t J^{\mathbf{X}}_{t\leftarrow s}(V_k(Y_s)) \otimes J^{\mathbf{X}}_{t\leftarrow s'}(V_k(Y_{s'}))\,dR^{(k)}(s,s')$$

Proof. Let $\left(h^{(k)}_n : n\right)$ be an ONB of $\mathcal{H}^{(k)}$. It follows that $\left(h^{(k)}_n : n = 1,2,...; k = 1,...,d\right)$ is an ONB of $\mathcal{H} = \oplus^d_{i=1}\mathcal{H}^{(k)}$ where we identify

$$h^{(1)}_n \in \mathcal{H}^{(1)} \equiv \begin{pmatrix} h^{(1)}_n \\ 0 \\ ... \\ 0 \end{pmatrix} \in \mathcal{H}$$

[3] $D^{1,p}_{loc}$ is enough to guarantee the existence of an $\mathcal{H}$-valued derivative.

and similarly for $k = 2, ..., d$. From Parseval's identity,

$$\begin{aligned}\sigma_t &= \left(\left\langle DY_t^i, DY_t^j \right\rangle_{\mathcal{H}}\right)_{i,j=1,\dots,e} \\ &= \sum_{n,k} \left\langle DY_t, h_n^{(k)} \right\rangle_{\mathcal{H}} \otimes \left\langle DY_t, h_n^{(k)} \right\rangle_{\mathcal{H}} \\ &= \sum_k \sum_n \int_0^t J_{t\leftarrow s}^{\mathbf{X}} \left(V_k\left(Y_s\right)\right) dh_{n,s}^{(k)} \otimes \int_0^t J_{t\leftarrow s}^{\mathbf{X}} \left(V_k\left(Y_s\right)\right) dh_{n,s}^{(k)} \\ &= \sum_k \int_0^t \int_0^t J_{t\leftarrow s}^{\mathbf{X}} \left(V_k\left(Y_s\right)\right) \otimes J_{t\leftarrow s'}^{\mathbf{X}} \left(V_k\left(Y_{s'}\right)\right) dR^{(k)}\left(s,s'\right).\end{aligned}$$

For the last step we used that

$$\sum_n \int_0^T f dh_n \int_0^T g dh_n = \int_0^T \int_0^T f\left(s\right) g\left(t\right) dR\left(s,t\right)$$

whenever $f = f\left(t,\omega\right)$ and g are such that the integrals are a.s. well-defined Young-integrals. The proof is a consequence of $R\left(s,t\right) = \mathbb{E}\left(X_s X_t\right)$ and the L^2-expansion of the Gaussian process X,

$$X\left(t\right) = \sum_n \xi\left(h_n\right) h_n\left(t\right)$$

where $\xi\left(h_n\right)$ form an IID family of standard Gaussians. □

Two special cases are worth considering: in the case of Brownian motion $dR\left(s,s'\right)$ is a Dirac measure on the diagonal $\{s = s'\}$) and the double integral reduces to a (well-known) single integral expression; in the case of fractional Brownian motion with $H > 1/2$ it suffices to take the $\partial^2/\left(\partial s \partial t\right)$ derivative of $R_H\left(s,t\right) = \left(t^{2H} + s^{2H} - |t-s|^{2H}\right)/2$ to see that

$$dR_H\left(s,s'\right) \sim |t-s|^{2H-2} dsdt$$

which is integrable iff $2H - 2 > -1$ or $H > 1/2$. (The resulting double-integral representation of the Malliavin covariance is also well-known and appears, for instance, in [25, 26, 1].)

4. Existence of a density for Gaussian RDEs

We remain in the framework of the previous sections, $Y_t\left(\omega\right) \equiv U_{t\leftarrow 0}^{\mathbf{X}(\omega)}\left(y_0\right)$ denotes the (random) RDE solution driven a Gaussian rough path $\mathbf{X}$, the natural lift of a continuous, centered Gaussian process with independent components $X = \left(X^1, ..., X^d\right)$ started at zero. Under the **standing assumption** of finite ρ-variation of the covariance, $\rho < 3/2$, we know that $\mathbf{X}\left(\omega\right)$ is a.s. a geometric p-rough path for $p \in \left(2\rho, 3\right)$. Recall that this means that $\mathbf{X}$ can be viewed as path in $G^2\left(\mathbb{R}^d\right)$, the step-2 nilpotent group over $\mathbb{R}^d$, of finite p-variation relative to the Carnot-Caratheodory metric on $G^2\left(\mathbb{R}^d\right)$.

Condition 1. ***Ellipticity assumption on the vector fields:*** *The vector fields $V_1, ..., V_d$ span the tangent space at y_0.*

Condition 2. ***Non-degeneracy of the Gaussian process on*** $[0,T]$***:*** *Fix* $T>0$*. We assume that for any smooth* $f=(f_1,\dots,f_d):[0,T]\to\mathbb{R}^d$

$$\left(\int_0^T f dh \equiv \sum_{k=1}^d \int_0^T f_k dh^k = 0 \forall h \in \mathcal{H}\right) \implies f \equiv 0.$$

Note that non-degeneracy on $[0,T]$ implies non-degeneracy on $[0,t]$ for any $t\in(0,T]$. It is instructive to see how this condition rules out the Brownian bridge returning to the origin at time T or earlier; a Brownian bridge which returns to zero after time T is allowed. The following lemma contains a few ramifications concerning condition 2. Since $\mathcal{H}=\oplus_{k=1}^d\mathcal{H}^{(k)}$ there is no loss in generality in assuming $d=1$.

Lemma 1. *(i) The requirement that* f *is smooth above can be relaxed to* $f\in C^{p\text{-}var}$ *for* $p>2\rho$ *small enough.*
(ii) The requirement that $\int f dh = 0 \forall h\in\mathcal{H}$ *can be relaxed to the the quantifier "for all* h *in some orthonormal basis of* $\mathcal{H}$*".*
(iii) The non-degeneracy condition 2 is equivalent to saying that for all smooth $f\neq 0$*, the zero-mean Gaussian random variable* $\int_0^T f dX$ *(which exists as Young integral or via integration-by-parts) has positive definite variance.)*
(iv) The non-degeneracy condition 2 is equivalent to saying that for all times $0<t_1<\dots<t_n<T$ *the covariance matrix of* $(X_{t_1},\dots,X_{t_n})$*, that is,*

$$(R(t_i,t_j))_{i,j=1,\dots,d}$$

is (strictly) positive definite.

Proof. (i) Continuity properties of the Young integral, noting that $1/p+1/\rho>1$ for $p>2\rho$ small enough, and weak compactness of the unit ball in $\mathcal{H}$. (ii) Any $h\in\mathcal{H}$ can be written as limit in $\mathcal{H}$ of $h^{[n]}\equiv\sum_{k=1}^n\langle h_k,h\rangle h_k$ as $n\to\infty$ when (h_k) is an ONB for $\mathcal{H}$. From Proposition 2 it follows that h is also the limit of $h^{[n]}$ in ρ-variation topology. Conclude with continuity of the Young integral. (iii) Every element $h\in\mathcal{H}$ can be written as $h_t=\mathbb{E}(ZX_t)$ with $Z\in\xi(\mathcal{H})$. By taking L^2-limits one easily justifies the formal computation

$$dh_t=\mathbb{E}\left(\dot{X}_t Z\right)dt \implies \int_0^T f dh = \mathbb{E}\left[\left(\int_0^T f dX\right)Z\right]$$

and our condition is equivalent to saying that

$$\int_0^T f dX \perp \xi(\mathcal{H}).$$

On the other hand, it is clear that $\int f dX$ itself is an element of the 1st Wiener Itô Chaos $\xi(\mathcal{H})$ and so must be 0 in L^2. In other words, saying that $\int f dh = 0$ for all $h\in\mathcal{H}$ says precisely that

$$\mathrm{Var}\left[\int_0^T f dX\right]=\left|\int_0^T f dX\right|_{L^2}^2=0.$$

Conversely, assume that

$$\mathrm{Var}\left[\int_0^T f dX\right]=0.$$

Then $\int f dX = 0$ with probability 1 and by the Cameron-Martin theorem, for any $h \in \mathcal{H}$,

$$\int_0^T f d(X+h) = 0$$

with probability one which implies $\int f dh = 0$ for all $h \in \mathcal{H}$. (iv) Suffices to note that

$$\int_0^T \int_0^T f(s) f(t) dR(s,t) = \sum_n \left| \int_0^T f(s) dh_n \right|^2$$

For the last step we used that

$$\sum_n \int_0^T f dh_n \int_0^T g dh_n = \int_0^T \int_0^T f(s) g(t) dR(s,t)$$

which is a consequence of $R(s,t) = \mathbb{E}(X_s X_t)$ and the L^2-expansion $X(t) = \sum_n \xi(h_n) h_n(t)$. (v) Left to the reader. (vi) Obvious. □

Remark 2. *The variance of $\int f dX$ can written as 2D Young integral,*

$$\int_{[0,T]^2} f_s f_t dR(s,t).$$

To put the following result in context, recall that the covariance of Brownian motion, $(s,t) \mapsto \min(s,t)$, has finite 1-variation with $\rho = 1$. For fractional Brownian motion with Hurst parameter H one can take $\rho = 1/2H$. The following result then applies to fractional Brownian driving signals with $H > 1/3$.

Theorem 1. *Let $\mathbf{X}$ be natural lift of a continuous, centered Gaussian process with independent components $X = (X^1, ..., X^d)$, with finite ρ-variation of the covariance, $\rho < 3/2$ and non-degenerate in the sense of condition 2. Let $V = (V_1, ..., V_d)$ be a collection of Lip^3-vector fields on $\mathbb{R}^e$ which satisfy the ellipticity condition 1. Then the solution to the (random) RDE*

$$dY = V(Y) d\mathbf{X},\ Y(0) = y_0 \in \mathbb{R}^e$$

admits a density at all times $t \in (0,T]$ with respect to Lebesgue measure on $\mathbb{R}^e$.

Proof. Fix $t \in (0,T]$. From Proposition 3 we know that $U_{t \leftarrow 0}^{\mathbf{X}(\omega)} y_0 = Y_t$ is continuously $\mathcal{H}$-differentiable. By a well-known criterion due to Bouleau-Hirsch[4] the proof is reduced to show a.s. invertibility of the Malliavin covariance matrix

$$\sigma_t = \left(\left\langle DY_t^i, DY_t^j \right\rangle_{\mathcal{H}} \right)_{i,j=1,\dots,e} \in \mathbb{R}^{e \times e}.$$

Assume there exists a (random) vector $v \in \mathbb{R}^e$ which annihilates the quadratic form σ_t. Then[5]

$$0 = v^T \sigma_t v = \left| \sum_{i=1}^e v_i DY_t^i \right|_{\mathcal{H}}^2 \quad \text{and so } v^T DY_t \equiv \sum_{i=1}^e v_i DY_t^i \in 0 \in \mathcal{H}.$$

By propositions 1 and 3,

$$\forall h \in \mathcal{H} : v^T D_h Y_t = \int_0^t \sum_{j=1}^d v^T J_{t \leftarrow s}^{\mathbf{X}} (V_j(Y_s)) dh_s^j = 0 \tag{4.1}$$

[4]Combine the result of [Nualart, 4.1.3] and [Nualart, section 2].

[5]Upper T denotes the transpose of a vector or matrix.

where the last integral makes sense as Young integral since the (continuous) integrand has finite p-variation regularity. Noting that the non-degeneracy condition on $[0,T]$ implies the same non-degeneracy condition on $[0,t]$ we see that the integrand in (4.1) must be zero on $[0,t]$ and evaluation at time 0 shows that for all $j=1,...,d,$

$$v^T J^{\mathbf{X}}_{t\leftarrow 0}\left(V_j\left(y_0\right)\right)=0.$$

It follows that the vector $v^T J^{\mathbf{X}}_{t\leftarrow 0}$ is orthogonal to $V_j\left(y_0\right), \; j=1,...,d$ and hence zero. Since $J^{\mathbf{X}}_{t\leftarrow 0}$ is invertible we see that $v=0$. The proof is finished. □

The reader may be curious to hear about smoothness in this context. Adapting standard arguments would require $L^p\left(\Omega\right)$ estimates on the Jacobian of the flow $J^{\mathbf{X}(\omega)}_{t\leftarrow 0}$. Using the fact that it satisfies a linear RDE, $dJ^{\mathbf{X}}_{t\leftarrow t_0}\left(y_0\right)=d\mathbf{M}^{\mathbf{X}}\cdot J^{\mathbf{X}}_{t\leftarrow t_0}\left(y_0\right)$, with $d\mathbf{M}^{\mathbf{X}}=V'\left(Y\right)d\mathbf{X}$ one can see that

$$\log\left|J^{\mathbf{X}}_{t\leftarrow t_0}\left(y_0\right)\right|=O\left(\left\|\mathbf{X}\right\|^p_{p\text{-var}}\right). \tag{4.2}$$

(This estimate appears in [26] for $p<2$ but can be seen [18, 9] to hold for all $p\geq 1$. We believe it to be optimal.) Using the Gauss tail of the homogenous p-variation norm of Gaussian rough paths (see [7, 8]) we see that L^q-estimates for all $q<\infty$ hold true when $p<2$ and this underlies to density results of [26, 1]. On the other hand, for $p>2$ one cannot obtain L^q-estimates from (4.2) and further probabilistic input will be needed.

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